

Valuation of Convertible Bonds

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Chapter 1

Introduction

1.1 Convertible Bond: Definition and Classification

A convertible bond in a *narrow sense* refers to a bond which can be converted into a firm's common shares at a predetermined number at the bondholder's decision. Convertible bonds are hybrid financial instruments with complex features, because they have characteristics of both debts and equities, and usually several equity options are embedded in this kind of contracts. The optimality of the conversion decision depends on equity price, future interest rate and default probability of the issuer. The decision making can be further complicated by the fact that most convertible bonds have call provisions allowing the bond issuer to call the bond back at a predetermined call price. Similar to a straight bond, the convertible bondholder receives coupon and principal payments. The *broad definition* of a convertible bond covers also e.g. mandatory convertibles, where the issuer can force the conversion if the stock price lies below a certain level.

The options embedded in a convertible bond can greatly affect the value of the bond. Definition 1.1.1 gives a description of different conversion and call rights and the convertible bonds can thus be classified according to the option features.

Definition 1.1.1. *American-style conversion right* gives its owner the right to convert a bond into γ shares at any time t before or at maturity T of the contract. The constant $\gamma \in \mathbb{R}_+$ is referred to as the *conversion ratio*. While *European-style conversion right* can only be exercised at maturity T . If the firm defaults before maturity, the conversion value is zero. *American-style call right* refers to the case where issuer can buy back the bonds any time during the life of the debt contract at a given *call level* H , which can be time- and stock-price-dependent. Whereas in the case of *European-style call right* the bond seller can only buy back the bonds at maturity. A *European-style (callable and) convertible bond* can only be converted (or called) at maturity T while an *American-style (callable and) convertible bond* can be converted or called at any time during the life of the debt.

There are numerous research on different types of convertible bonds. One example is *mandatory convertible bonds*, which belong to the family of European-style convertible bonds, where both bondholder and issuer own conversion rights. The holder will exercise the conversion right if the stock price lies above an upper strike level, whereas the issuer can force the conversion if the stock price lies below a lower strike level. In other words, the bondholder is subject to the downside risk of the stock, while he can also participate (usually partially) in the upside potential of the stock at maturity. Mandatory convertible bonds have been studied by Ammann and Seiz (2006) who examine the empirical pricing and hedging of them. They decompose the bond into four components: a long call, a short put, par value and coupon payments. In their pricing model, simple Black-Scholes formula is used for the valuation of the option component, the volatility is assumed to be constant and credit spreads are only considered for the valuation of coupons. It means that no default risk is considered for the payoff at maturity only the coupons are considered to be risky, therefore there is no comprehensive treatment of the default risk.

The American-style callable and convertible bond¹ has attracted the most research attention due to its exposure to both credit and market risk and the corresponding optimal conversion and call strategies. The bondholder receives coupons plus the return of principal at maturity, given that the issuer (usually the shareholder) does not default on the obligations. Moreover, prior to the maturity the bondholder has the right to convert the bond into a given number of stocks. On the other hand, the bond is also callable by the issuer, i.e. the bondholder can be enforced to surrender the bond to the issuer for a previously agreed price. In the context of the structural model the arbitrage free pricing problem was first treated by Brennan and Schwarz (1977) and Ingersoll (1977). Recent articles of Sirbu, Pilovsky and Schreve (2004) and Kallsen and Kühn (2005) treat the optimal behavior of the contract partners more rigorously. In McConnell and Schwarz (1986) and Tsiveriotis and Fernandes (1998) credit spread is incorporated for discounting the bond component. This approach is implemented and tested empirically by Ammann, Kind and Wilde (2003) for the French convertible bond market. More recently, the so-called equity-to-credit reduced-form model is developed e.g. in Bielecki, Crèpey, Jeanblanc and Rutkowski (2007) and Kühn and van Schaik (2008) to model the interplay of credit risk and equity risk for convertible bonds. In Bielecki et al. (2007) the valuation of callable and convertible bond is explicitly related to the defaultable game option.

1.2 Modeling Approaches and Main Results

Convertible bonds are exposed to different sources of randomness: interest rate, equity and default risk. Empirical research indicates that firms that issue convertible bonds often tend to be highly leveraged, the default risk may play a significant role. Moreover,

¹In praxis it is simply called *callable and convertible bond*.

the equity and default risk cannot be treated independently and their interplay must be modeled explicitly. In the following we will summarize the modeling approaches and the main results achieved in this thesis.

Default risk models can be categorized into two fundamental classes: firm's value models or *structural models*, and *reduced-form* or default-rate models. In the structural model, one constructs a stochastic process of the firm's value which indirectly leads to default, while in the reduced-form model the default process is modeled directly. In the structural models default risk depends mainly on the stochastic evolution of the asset value and default occurs when the random variable describing the firm's value is insufficient for repayment of debt. For example, by the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period. Instead of asking why the firm defaults, in the reduced-form model formulation, the intensity of the default process is modeled exogenously by using both market-wide as well as firm-specific factors, such as stock prices. The default intensities, like the stock volatilities cannot be observed directly either, but explicit pricing formulas and/or algorithms, which are derived by imposing absence of arbitrage conditions, can be inverted to find estimates for them.

1.2.1 Structural approach

While both approaches have certain shortcomings, the strength of the structural approach is that it provides economical explanation of the capital structure decision, default triggering, influence of dividend payments and of the behaviors of debtor and creditor. It describes why a firm defaults and it allows for the description of the strategies of the debtor and creditor. Especially for complex contracts where the strategic behaviors of the debtor and the creditor play an important role, structural models are well suited for the analysis of the relative powers of shareholders and creditors and the questions of optimal capital structure design and risk management. Moreover, the structural approach allows for an integrated model of equity and default risk through common dependence on stochastic variables.

In this thesis, we first adopt a structural approach where the Vasiček-model is applied to incorporate interest rate risk into the firm's value process which follows a geometric Brownian motion. A default is triggered when the firm's value hits a low boundary. Within the structural approach we will discuss the problem of no-arbitrage prices and fair coupon payments for bonds with conversion rights. The idea is the following: Consider a firm that is financed by both equity and debt. In periods where the value of the firm increases the bondholders might want to participate in this growth. For example, this can be achieved by converting debt into a certain number of shares. If such a conversion

is valid the equity holders are short of call options. One can limit the upside potential of the payoff through a call provision such that equity holders have the right to buy back the bonds at a fixed price. Convertible bonds put this idea into practice by giving the bondholder the right to convert the debt into equity with a prescribed conversion ratio at prescribed times or time periods. A concrete example is the European-style callable and convertible bond. The holder of a convertible bond has the possibility to participate in the growth potential of the terminal value of the firm, but in exchange he receives lower coupons than for the otherwise identical non-convertible bond.

In the case of American conversion rights, meaning that conversion is allowed at any time during the life of the contract, and by existence of a call provision for the issuer this leads to a problem of optimal stopping for both bondholder and issuer. Therefore when we compute the no-arbitrage price of such a contract, we have to take into account the aspect of strategic optimal behaviors which are the study focus of this thesis. Based on the results of Kifer (2000) and Kallsen and Kühn (2005) we show that the optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. The assumption that the call value is always larger than the conversion value prior to maturity T and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution. Furthermore, the no-arbitrage price can be approximated numerically by means of backward induction. In absence of interest rate risk, the recursion procedure is carried out on the Cox-Ross-Rubinstein binomial lattice. To incorporate the influence of the interest rate risk, we use a combination of an analytical approach and a binomial tree approach developed by Menkveld and Vorst (1998) where the interest rate is Gaussian and correlation between the interest rate process and the firm's value process is explicitly modeled. We show that the influence of interest rate risk is small. This can be explained by the fact that the volatility of the interest process is in comparison with that of the firm's value process relatively low and, moreover, both parties have the possibility for early exercise.

In practice it is often a difficult problem to calibrate a given model to the available data. Here one major drawback of the structural model is that it specifies a certain firm's value process. As the firm's value, however, is not always observable, e.g. due to incomplete information, determining the volatility of this process is a non-trivial problem. In this thesis, we circumvent this problem by applying the uncertain volatility model of Avellaneda, Levy and Parás (1995) and combining it with the results of Kallsen and Kühn (2005) on game option in incomplete market to derive certain pricing bounds for convertible bonds. Hereby we only know that the volatility of the firm's value process lies between two extreme values. The bondholder selects the stopping time which maximizes the expected

payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimistic estimate from the aspect of the bondholder. The optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound of the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This expectation is also the most pessimistic one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived. Numerically, to make the computation tractable a constant interest rate is assumed. The pricing bounds can be calculated with recursions on a recombining trinomial tree developed by Avellaneda et al. (1995). It can be shown that due to the complex structure and early exercise possibility a callable and convertible bond has narrower bounds than a simple debt contract. One reason is that the former contract combines short and long option positions which have varying convexity and concavity of the value function. In the approach of Avellaneda et al. (1995), however, the selection of the minimum or maximum of the volatility for the valuation depends on the convexity of the valuation function. Moreover, both parties can decide when they exercise. Therefore each of them must bear the strategy of the other party in mind, and consequently the pricing bound is narrowed.

Modeling of the American-style callable and convertible bond as a defaultable game option within structural approach has been studied by Sirbu et al. (2004) and further developed in a companion paper of Sirbu and Schreve (2006). In their models the volatility of the firm's value and the interest rate are constant. The bond earns continuously a stream of coupon at a fixed rate. The dynamic of the firm's value does not follow a geometric Brownian motion, but a more general one-dimensional diffusion due to the fixed rate of coupon payment. Default occurs if the firm's value falls to zero which means both equity and bond have zero recovery. The no-arbitrage price of the bond is characterized as the result of a two-person zero-sum game. Viscosity solution concept is used to determine the no-arbitrage price and optimal stopping strategies. Our model differs from theirs mainly by allowing non-zero recovery rate of the bond and default occurs if the firm's value hit a low but positive boundary. The dynamic of the firm's value follows a geometric Brownian motion which means that the underlying process, the evolution of the firm's value, does not depend on the solution of the game option. Therefore the results of Kifer (2000) can be applied to the valuation of the bond. Simple recursion with a binomial tree can be used to derive the value of the bond and the optimal strategies. Moreover, stochastic interest rate and uncertain volatility can be incorporated into our model.

1.2.2 Reduced-form approach

Sometimes the true complex nature of the capital structure of the firm and information asymmetry make it hard to model the firm's value and the capital structure. In this case the reduced-form model is a more proper approach for the study of convertible bonds.

Stock prices, credit spreads and implied volatilities of options are used as model inputs. In this thesis the stock price is described by a jump diffusion. It jumps to zero at the time of default. In order to describe the interplay of the equity risk and the default risk of the issuer, we adopt a parsimonious, intensity-based default model, in which the default intensity is modeled as a function of the pre-default stock price. This assumes, in effect, that the equity price contains sufficient information to predict the default event. To make the combined effect of the default and equity risk of the underlying tractable, it is assumed that the default intensity has two values, one is the normal default rate, and the other one is much higher if the current stock price falls beneath a certain boundary. Thus, during the life time of the bond, the more time the stock price spends below the boundary, the higher the default risk. This model has certain similarity with some structural models, e.g. in the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period.

Within the intensity-based default model, we first analyze mandatory convertible bonds, which are contracts of European-style. The coupon rate of a mandatory convertible bond is usually higher than the dividend rate of the stock. At maturity it converts mandatorily into a number of stocks if the stock price lies below a lower strike level. The holder will exercise the conversion right if the stock price lies above an upper strike level. They are issued by the firms to raise capital, usually in times when the placement of new equities are not advantageous. Empirical research indicates that firms that issue mandatory convertibles tend to be highly leveraged. In some literature it is argued that, due to the offsetting nature of the embedded option spread, a change in volatility has only an unnoticeable effect on the mandatory convertible value. Therefore, the influence of the volatility on the price is limited. But we show that if the default intensity is explicitly linked to the stock price, the impact of the volatility can no longer be neglected.

In the case of American conversion and call rights, there are two sources of risks which are essential for the valuation, one stemming from the randomness of prices, the other stemming from the randomness of the termination time, namely the contract can be stopped by call, conversion and default. In the intensity-based default model the default time is modeled as the time of the first jump of a Poisson process and it is not adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by the pre-default stock price process. To price a defaultable contingent claim we need not only the information about the evolution of the pre-default stock price but also the knowledge whether default has occurred or not which is described by the filtration $(\mathcal{H}_t)_{t \in [0, T]}$. The filtration $(\mathcal{G}_t)_{t \in [0, T]}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, contains the full information and is larger than the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. This problem can be circumvented with specific modeling of the default time, e.g. Lando (1998) shows that if the time of default is modeled as the first jump of a Poisson process with random intensity, which is called doubly stochastic Poisson process or Cox process and under some measurable conditions, the expectations with respect to \mathcal{G}_t can be reduced to the

expectation with respect to \mathcal{F}_t . With the help of the filtration reduction we move to the fictitious default-free market in which cash flows are discounted according to the modified discount factor which is the sum of the risk free discount factor and the default intensity. Hence the results of the game option in the default-free setting can be extended to the defaultable game option in the intensity model². The embedded option rights owned by both of the bondholder and the issuer can be exercised optimally according to the well developed theory on the game option. The optimization problem is not approximated with recursions on a tree as in the case of the structural approach, it is formulated and solved with help of the theory of doubly reflected backward stochastic differential equations (BSDE) which is a more general approach developed by Cvitanić and Karatzas (1996). The parabolic partial differential equation (PDE) related to the doubly reflected BSDE is provided by Cvitanić and Ma (2001) and it can be solved with finite-difference methods. Furthermore, pricing bound is derived under rational optimal behavior, if the stock volatility is assumed to lie in a certain interval.

Defaultable game option and its application to callable and convertible bonds within reduced-form model have been studied in Bielecki, Crèpey, Jeanblanc and Rutkowski (2006) and Bielecki et al. (2007). They consider a primary market composed of the savings account and two primary risky assets: defaultable stock and credit default swap with the stock as reference entity. In our model, instead of credit default swap contract we assume zero-coupon risky bonds are traded in the market. They and the callable and convertible bonds default at the same time. Another difference is that we formulate the default event according to Lando (1998), where the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity, which is called Cox process. The reduction of filtration from $(\mathcal{G}_t)_{t \in [0, T]}$ to $(\mathcal{F}_t)_{t \in [0, T]}$ is applied for the derivation of the no-arbitrage price of the bond. It simplifies the calculations. Some complex contract features of the callable and convertible bond treated by Bielecki et al. (2007) are not investigation subjects of our model, instead we focus on the uncertain volatility of the stock and the derivation of the no-arbitrage pricing bounds.

1.3 Structure of the Thesis

The remainder of the thesis is structured as follows. From Chapter 2 to Chapter 5 convertible bonds are treated within structural approach. Chapter 2 introduces the model framework of the structural approach: market assumptions, dynamics of the interest rate and firm's value processes, capital structure and the default mechanism are established. The Vasiček-model is applied to incorporate interest rate risk into the firm's value pro-

²In the structural approach, the default time is a predictable stopping time, and adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by the firm value process, thus the discounted payoff of the convertible bond is adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Therefore we can apply the results on game option developed by Kifer (2000) directly to derive the unique no-arbitrage value and the optimal strategies.

cess which follows a geometric Brownian motion. The model covers both the firm specific default risk and the market interest rate risk and correlation of them. Moreover the contract features of a straight coupon bond are described and closed form solution of the no-arbitrage value is derived. European-style convertible bonds are studied in Chapter 3. They are essentially a straight bond with an embedded down and out call option if the bond is non-callable or a call spread if the bond is callable. Closed form solutions are presented. Chapter 4 focuses on the American-style callable and convertible bond: its contract feature and the decomposition into a straight bond and a game option component. The optimal strategies and the formulation and solution of the optimization problem are first presented with constant interest rate, then the interest rate risk is incorporated. Furthermore, a closely related contract form, the Bermudan-style callable and convertible bond is discussed. In Chapter 5 uncertain volatilities of the firm value are introduced and pricing bounds are derived for both European- and American-style convertible bonds.

Throughout Chapter 6 to Chapter 8 the convertible bonds are dealt within reduced-form approach, where stock price, credit spreads and implied volatilities of options are used as model inputs for the valuation. Chapter 6 describes the intensity-based default model. According to Lando (1998) the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity. The stock price is modeled as a jump diffusion. It jumps to zero at the default. The default intensity is modeled as a function of the pre-default stock price. Reduction of filtration is introduced. In Chapter 7 the mandatory convertible bond is studied while Chapter 8 is dedicated to the American-style callable and convertible bond, the formulation of the optimal strategies and the solution of the optimization problem with the doubly reflected BSDE. Chapter 9 concludes the thesis.

Chapter 2

Model Framework Structural Approach

In the structural approach, firm's value is modeled by a diffusion process. Default occurs if the firm's value is insufficient for repayment of the debt according to some prescribed rules. The liability of the firm can be characterized as contingent claim on the firm's value.

The origin of the structural approach goes back to Black and Scholes (1973) and Merton (1974). These models assume that a default can only occur at the maturity of the debt, therefore the debt value can be characterized as a European contingent claim on the firm's value. It is extended by Black and Cox (1976) to allow for defaults before the maturity of the debt if the firm's value hits a certain boundary, which is also called first passage model. In this case the debt value is a contingent claim on the firm's asset which has similar payoffs as in case of a barrier option. Longstaff and Schwartz (1995) extend the first passage model by allowing interest rate to be stochastic and correlated with the firm's value process. Semi-closed-form solutions are derived for defaultable bonds. Another, similar but mathematically simpler approach is developed by Briys and de Varenne (1997), where a default is triggered when the T -forward price of the firm's value hits a lower barrier. Further extension of the first passage model is carried out by Zhou (1997). It is assumed that the firm's value follows a jump-diffusion process. The aim of the introduction of jumps in the firm value process is to capture the feature of the sudden default of the firm. These are representative models and there are numerous literature with extensions to the original firm's value approach. A survey of the various models is beyond the scope of this thesis. The structural approach finds its application in the praxis. It is e.g. implemented in a commercial model package marketed by KMV corporation.

The aforementioned structural models all assume a competitive capital market where the borrowing and lending interest rate are the same and the trading takes place without any restrictions. There is no constraint for short-sales of all assets, no cost for bankruptcy and no tax differential for equity and debt. Thus the Modigliani-Miller theorem is valid, i.e.

the value of the firm is invariant to its capital structure. For example, in Merton (1974), Section V, the validity of the Modigliani-Miller theorem in the presence of bankruptcy is proved explicitly.

Our model is a first passage model and the model assumptions are made mainly according to Briys and de Varenne (1997) and Bielecki and Rutkowski (2004)¹, with some slight modifications. The model covers both the firm specific default risk and the market interest rate risk and correlation of them. The remainder of the chapter is organized as follows: Section 2.1 summarizes the general market assumptions. The dynamics of the interest rate and firm's value are given in Section 2.2 and 2.3. The default mechanism is described in Section 2.4. The distribution of the default time and the joint distribution of the firm's terminal value and the default probability which are useful for the further calculations are derived in Section 2.5. The valuation formula for a straight coupon bond is derived in Section 2.6

2.1 Market Assumptions

We adopt the standard assumptions in structural models:

- The financial market is frictionless, which means there are no transactions costs, bankruptcy costs and taxes, and all securities in the market are arbitrarily divisible.
- Every individual can buy or sell as much of any security as he wishes without affecting the market price.
- Risk-free assets earn the instantaneous risk-free interest rate.
- One can borrow and lend at the same interest rate and take short positions in any securities.
- The Modigliani-Miller theorem is valid, i.e. the firm's value is independent of the capital structure of the firm. In particular, the value of the firm does not change at the time of conversion and is reduced by the amount of the call price paid to the bondholder at the time of the call.
- Trading takes place continuously.

Under these assumptions, financial markets are complete and frictionless, according to Harrison and Kreps (1979) there exists a unique probability measure P^* under which the continuously discounted price of any security is a P^* -martingale.

¹See, Section 3.4 of their book.

2.2 Dynamic of the Risk-free Interest Rate

In the literature, there exist different approaches for modeling of the interest rate risk. We adopt the *bond price approach*, where the dynamics of a family of bond prices, usually the zero coupon bond prices, are modeled exogenously. The interest rate dynamics can be derived endogenously. Let us fix a time interval $[t_0, T^*]$, and let $B(t, T)$ stand for the price of a zero coupon bond at time $t_0 \leq t \leq T$, where $T \leq T^*$ is the maturity time of the bond. The payment at maturity is normalized to one monetary unit, formally,

$$B(T, T) = 1, P^* - \text{a.s. } \forall T \in [t_0, T^*].$$

Definition 2.2.1. $B(t, T)$ is driven by an n -dimensional standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^*)$,

$$dB(t, T) = B(t, T) (r(t) dt + b(t, T) dW^*(t)), \quad (2.1)$$

where $W^*(t) = (W_1^*(t), \dots, W_n^*(t))^\top \in \mathbb{R}^n$ denotes an n -dimensional Brownian motion with respect to the martingale measure P^* . $b(t, T)$ describes the volatility of the zero coupon bond, which is a time dependent deterministic function and must satisfy the following conditions

- at the maturity date the volatility should be zero,

$$b(T, T) = (b_1(T, T), \dots, b_n(T, T))^\top = 0, \in \mathbb{R}^n, \quad \forall T \in [t_0, T^*]^2$$

- for each $t \in [t_0, T]$, $b(t, T)$ is square integrable with respect to t ,

$$\int_0^T \|b(u, T)\|^2 du := \int_0^T \sum_{j=1}^n b_j(u, T)^2 du < \infty$$

- for each $t \in [t_0, T]$, $b(t, T)$ is differentiable with respect to T .

The solution of Equation (2.1) can be expressed as

$$B(t, T) = B(t_0, T) \exp \left\{ \int_{t_0}^t (r(u) - \frac{1}{2} \|b(u, T)\|^2) du + \int_{t_0}^t b(u, T) dW^*(u) \right\}. \quad (2.2)$$

The term structure of the spot interest rate can be derived endogenously according to the bond dynamic defined by Equation (2.1)³. The corresponding conform spot rate is normally distributed, therefore, it is also called *n-factor Gaussian term structure model*. Due to its analytical tractability, the Gaussian term structure is widely applied.

² \top denotes the transpose of the matrix

³Details can be found, e.g. in Sandmann (2000), Chapter 10.

Although there exists a positive possibility that negative spot rates will be generated, but the probability that such situation occurs can be minimized through proper parameter choices. Moreover, Gaussian term structures can be easily integrated with Black and Scholes (1973) model to value stock option under stochastic interest rate.

A prominent example of Gaussian term structure is the *Vasiček-model*, in its simplest form a *one-factor* mean-reverting model which has received broad application. In this case $W^*(t)$ denotes a 1-dimensional Brownian motion. The volatility of the zero coupon bond has the following form

$$b(t, T) = \frac{\sigma_r}{b_r}(1 - e^{-b_r(T-t)}),$$

with constant speed of mean reverting factor $b_r > 0$ and constant volatility $\sigma_r > 0$. This specification of volatility satisfies all conditions in definition 2.2.1. Accordingly, the conform short rate follows an Ornstein–Uhlenbeck process,

$$dr(t) = (a_r - b_r r(t))dt + \sigma_r dW_1^*(t), \quad (2.3)$$

where a_r is a constant, $W_1^*(t)$ is a 1-dimensional standard Brownian motion under the martingale measure P^* , and it governs the movement of the interest rate. W_1^* and W^* move in opposite direction, i.e. $dW_1^*(t) = -dW^*(t)$ because the increase of the interest rate causes the reduction of the zero bond price. The short rate is pulled to the long-run mean $\frac{a_r}{b_r}$ at a speed rate of b_r .

2.3 Dynamic of the Firm's value

the Vasiček-model is applied to incorporate interest rate risk into the process of the firm's value. The interest rate r_t is governed under the martingale measure P^* by Equation (2.3). Equation (2.1) describing the value of a default free zero coupon bond $B(t, T)$ can be reformulated as⁴

$$dB(t, T) = B(t, T)(r_t dt - b(t, T)dW_1^*(t)) \quad (2.4)$$

The firm's value V is assumed to follow a geometric Brownian motion under the martingale measure P^* of the form

$$\frac{dV_t}{V_t} = (r_t - \kappa)dt + \sigma_V(\rho dW_1^*(t) + \sqrt{1 - \rho^2}dW_2^*(t)) \quad (2.5)$$

where $W_2^*(t)$ is a 1-dimensional standard Brownian motion, independent of $W_1^*(t)$ and

⁴Instead of $W^*(t)$, here we let $W_1^*(t)$ govern the movement of the risk-free bond price with the purpose to emphasize the impact of the interest rate risk and its correlation with the firm's value.

$\rho \in [-1, 1]$ is the correlation coefficient between the interest rate and the firm's value. The volatility $\sigma_V > 0$ and the payout rate κ are assumed to be constant. The amount $\kappa V_t dt$ is used to pay coupons and dividends.

Under the martingale measure P^* the no-arbitrage price of a contingent claim is derived as expected discounted payoff, but in the case of stochastic discount factor the calculation can be quite complicated. It has been shown in the literature that the calculation can be simplified if the T -forward risk adjusted martingale measure P^T is applied.

Definition 2.3.1. A T -forward risk adjusted martingale measure P^T on (Ω, \mathcal{F}_T) is equivalent to P^* and the Radon-Nikodým derivative is given by the formula

$$\frac{dP^T}{dP^*} = \frac{\exp\{-\int_0^T r(u)du\}}{\mathbb{E}_{P^*}\left[\exp\{-\int_0^T r(u)du\}\right]} = \frac{\exp\{-\int_0^T r(u)du\}}{B(0, T)},$$

and when restricted to the σ -field \mathcal{F}_t ,

$$\frac{dP^T}{dP^*}|_{\mathcal{F}_t} := \mathbb{E}_{P^*}\left[\frac{\exp\{-\int_0^T r(u)du\}}{B(0, T)}\middle|\mathcal{F}_t\right] = \frac{\exp\{-\int_0^t r(u)du\}B(t, T)}{B(0, T)}.$$

Especially for Gaussian term structure model, when the zero bond price is given by Equation (2.4), an explicit density function exists. Namely,

$$\frac{dP^T}{dP^*}|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2}\int_0^t b^2(u, T)du - \int_0^t b(u, T)dW_1^*(u)\right\}$$

Furthermore,

$$W_1^T(t) = W_1^*(t) + \int_0^t b(u, T)du \quad (2.6)$$

follows a standard Brownian motion under the forward measure P^T .

Thus the forward price of the firm's value $F_V(t, T) := V_t/B(t, T)$ satisfies the following dynamics under the T -forward risk adjusted martingale measure P^T ⁵,

$$\begin{aligned} \frac{dF_V(t, T)}{F_V(t, T)} &= -\kappa dt + (\rho\sigma_V + b(t, T))dW_1^T(t) + \sigma_V\sqrt{1-\rho^2}dW_2^*(t) \\ &= -\kappa dt + \sigma_F(t, T)dW^T(t), \end{aligned} \quad (2.7)$$

where $W_1^T(t)$ is given by Equation (2.6) and

$$\sigma_F^2(t, T) = \int_0^t \left(\sigma_V^2 + 2\rho\sigma_V b(u, T) + b^2(u, T)\right)du, \quad (2.8)$$

⁵The dynamic of the forward firm value is derived by application of Itô's Lemma.

and $W^T(t)$ is a 1-dimensional standard Brownian motion that arises from the independent Brownian motions $W_1^T(t)$ and $W_2^*(t)$ ⁶ by the following equality in law $aW_1^T(t) + bW_2^*(t) \sim \sqrt{a^2 + b^2}W^T(t)$, where a, b are constant. Thus the auxiliary process

$$F_V^\kappa(t, T) := F_V(t, T)e^{\kappa t} \quad (2.9)$$

is a martingale under P^T and is log-normally distributed. Specifically, we have

$$dF_V^\kappa(t, T) = F_V^\kappa(t, T) \cdot \sigma_F(t, T)dW^T(t). \quad (2.10)$$

According to Equation (2.5) a constant payout rate of κ is assumed, and $\kappa V_t dt$ is the sum of the continuous coupon and dividend payments. Thus the firm's value $F_V(t, T)$ is not a martingale under the T -forward risk adjusted martingale measure P^T , but after compensated with the payout, the auxiliary process $F_V^\kappa(t, T)$ is a martingale under P^T .

2.4 Capital Structure and Default Mechanism

The equity price may drop at time of conversion, as the equity-holders may own a smaller portion of the equity after bondholders convert their holdings and become new equity-holders. To capture this effect, we assume that until time of conversion, at time t , the firm's asset consists of m identical stocks with value S_t and of n identical bonds with value D_t , thus

$$V_t = m \cdot S_t + n \cdot D_t.$$

The bonds can be straight bond or any kind of convertible bond with European- or American-style conversion and/or call right. Especially, at time $t = 0$, the initial firm's value satisfies

$$V_0 = m \cdot S_0 + n \cdot D_0. \quad (2.11)$$

Moreover, we set the principal that the firm must pay back at maturity T to be L for each bond and assume that bondholders are protected by a safety covenant that allows them to trigger early default. The firm defaults as soon as its value hits a prescribed barrier ν_t , and the default time τ is defined in a standard way by

$$\tau = \inf \{t > 0 : V_t \leq \nu_t\}. \quad (2.12)$$

Assumption 2.4.1. The default barrier ν_t at time t is supposed to be a fixed quantity K with $0 < K \leq nL$ discounted with the default-free zero coupon bond $B(t, T)$ and compensated with the effect due to the payout of coupons and dividends. The value of

⁶The independence of $W_1^T(t)$ and $W_2^*(t)$ is due to the assumption that $W_1^*(t)$ and $W_2^*(t)$ are independent and this property remains after the change of measure acted on $W_1^*(t)$.

the default barrier depends on the discount factor and the payout rate,

$$\nu_t = \begin{cases} KB(t, T)e^{-\kappa t} & t < T \\ nL & t = T. \end{cases} \quad (2.13)$$

Since interest rates are stochastic in this setting the default barrier ν_t is stochastic as well. But if ν_t is expressed in forward price and compensated with the payout it equals K which is a constant. Combined with the forward price of the firm's value this specification is mathematically convenient because it eases the further calculations and enables closed-form solutions of the no-arbitrage prices of the straight and European-style (callable and) convertible bond. This default mechanism is also economical reasonable as the barrier and the firm's value move with the same trend. Furthermore, it ensures that the discounted rebate payment to the bondholders is always smaller than the discounted principal. In this case the forward value of the barrier can be computed as

$$\frac{KB(t, T)e^{-\kappa t}}{B(t, T)} = Ke^{-\kappa t}.$$

2.5 Default Probability

The default time defined by Equations (2.12) and (2.13) can be further calculated as

$$\begin{aligned} \tau &:= \inf\{t > 0, V_t \leq \nu_t\} = \inf\left\{t > 0, \frac{V_t}{B(t, T)}e^{\kappa t} \leq K\right\} \\ &= \inf\{t > 0, F_V^\kappa(t, T) \leq K\} = \inf\{t > 0, y_t \leq \ln K\} \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} y_t &:= \ln F_V^\kappa(t, T) \\ &= \ln F_V^\kappa(0, T) - \frac{1}{2} \int_0^t \sigma_F^2(t, T) du + \int_0^t \sigma_F(t, T) dW_u^T. \end{aligned}$$

Define

$$F_0 := F_V^\kappa(0, T) = V_0/B(0, T),$$

$$y_0 := \ln F_0.$$

To eliminate the time-dependence in the volatility and consider the following deterministic time change. The time changed Brownian motion has the volatility $\sigma = 1$, and the time is scaled to A_t , and satisfies the following relationship

$$y_t = \tilde{y}_{A_t}$$

with

$$A_t := \int_0^t \sigma_F^2(u, T) du = \int_0^t (\sigma_V^2 + 2\rho\sigma_V b(u, T) + b^2(u, T)) du. \quad (2.15)$$

Let A^{-1} stand for the inverse time change, define $\tilde{y}_t := y_{A_t^{-1}}$, then

$$\tilde{y}_t = y_0 + Z_t - \frac{1}{2}t$$

where Z_t is a standard Brownian motion in the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_{A_t^{-1}}$.⁷ For the default time τ in Equation (2.14) we have

$$\{\tau > t\} = \{\tilde{\tau} > A_t\}$$

where

$$\tilde{\tau} := \inf\{t \geq 0, \tilde{y}_t \leq \ln K\}. \quad (2.16)$$

The distribution of the firm's value V_T given that the firm survives can be transformed similarly as

$$\begin{aligned} P[V_T \geq x, \tau \geq T] &= P\left[\frac{V_T}{B(T, T)} e^{\kappa T} \geq \frac{x}{B(T, T)} e^{\kappa T}, \tau \geq T\right] \\ &= P[F_V^\kappa(T, T) \geq x e^{\kappa T}, \tau \geq T] \\ &= P[y_T \geq \ln x + \kappa T, \tau \geq T] \\ &= P[\tilde{y}_T \geq \ln x + \kappa T, \tilde{\tau} \geq A_T] \end{aligned} \quad (2.17)$$

where we used that $B(T, T) = 1$.

Remark 2.5.1. For the calculation of Equations (2.16) and (2.17) we need the following distribution laws, which can be found in Musiela and Rutkowski (1998), p. 470. Let $X_t = \nu t + \sigma W_t$ denote a Brownian motion with drift and denote its minimum up to time t by m_t , and the first hitting time of $a \leq 0$ by $\tau_a := \inf\{t \geq 0, X_t \leq a\}$. Then we have

$$P[\tau_a \leq t] = P[m_t \leq a] = N\left(\frac{a - \nu t}{\sigma\sqrt{t}}\right) + e^{2\nu a\sigma^{-2}} N\left(\frac{a + \nu t}{\sigma\sqrt{t}}\right), \quad (2.18)$$

$$P[X_t \geq b, m_t \geq a] = N\left(\frac{-b + \nu t}{\sigma\sqrt{t}}\right) - e^{2\nu a\sigma^{-2}} N\left(\frac{2a - b + \nu t}{\sigma\sqrt{t}}\right), \quad (2.19)$$

for $b \geq a$, where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

⁷See Revuz and Yor (1991) for details.

Setting $\nu = -1/2$, $\sigma = 1$, $t = A_t$, $T = A_T$, $a = \ln(K/F_0)$, $b = \ln(x/F_0) + \kappa T$ in Equations (2.18) and (2.19), and after some calculations we obtain the default probability and the terminal distribution of the firm's value given that there is no pre-maturity default

$$P[\tau \leq t] = N(d_1(t)) + \frac{F_0}{K} N(d_2(t)) \quad (2.20)$$

$$\text{and } P[V_T \geq x, \tau \geq T] = N(d_3(x, T)) - \frac{F_0}{K} N(d_4(x, T)) \quad (2.21)$$

with

$$\begin{aligned} d_1(t) &:= \frac{\ln \frac{K}{F_0} + \frac{1}{2} A_t}{\sqrt{A_t}}, & d_2(t) &:= d_1(t) - \sqrt{A_t}, \\ d_3(x, t) &:= \frac{\ln \frac{F_0}{x} - \kappa t - \frac{1}{2} A_t}{\sqrt{A_t}}, & d_4(x, t) &:= \frac{2 \ln K - \ln(F_0 x) - \kappa t - \frac{1}{2} A_t}{\sqrt{A_t}}. \end{aligned} \quad (2.22)$$

where A_t is defined by Equation (2.15).

Accordingly the survival probability is

$$P[\tau > t] = N(-d_1(t)) - \frac{F_0}{K} N(d_2(t)), \quad (2.23)$$

and it shows that due to the specific choice of the random barrier, the stochastic interest rate and the payout rate κ have no influence on the default time distribution in this situation. Another distribution needed for the later calculations is

$$\begin{aligned} &P[V_T \leq x, \tau > T] \\ &= P[\tau > T] - P[V_T > x, \tau > T] \\ &= \left(N(-d_1(T)) - \frac{F_0}{K} N(d_2(T)) \right) - \left(N(d_3(x, T)) - \frac{F_0}{K} N(d_4(x, T)) \right). \end{aligned} \quad (2.24)$$

2.6 Straight Coupon Bond

Before describing convertible bonds in detail, we first study a straight coupon bond, i.e. a non-convertible and non-callable coupon bond. In praxis coupons are usually paid at discrete equally spaced time points. For calculation purpose we assume that the coupons are paid out continuously with a constant rate of c , till maturity T or default time τ , given that the firm's value is above the level η_t , $t \in [0, T]$ with

$$\eta_t = wB(t, T)e^{-\kappa t},$$

where w is a constant. For mathematical convenience η_t is defined in the similar manner as the default barrier ν_t . The assumption on the mechanism of the coupon payments is

to solve a technical problem and to make the computation tractable. The amount $\kappa V_t dt$ is used to pay coupons and dividends. Each bondholder receives the coupon payment $c dt$, and the total amount of the coupons is $n \cdot c dt$. The remaining amount $\kappa V_t dt - n \cdot c dt$ is used to pay dividends. Because the payout rate κ in the model is held constant, by lower firm's value the total payout may not suffice to pay the coupons. However, the shareholders in our model are not allowed and not able to raise short term credit to pay the coupons. The assumption is also economically reasonable, as in praxis there exist such coupon bonds. The firm can interrupt the coupon payments in the case that the firm does not operate properly and the firm's value is too low. If there is no default till maturity T the bondholders receive at maturity $\min\left(L, \frac{V_T}{n}\right)$ for each bond. In the case of an early default, the residual of the firm's value is divided among the bondholders and a rebate of $\frac{V_\tau}{n}$ will be paid to each bond at default time τ . Applying Equation (2.23), one can calculate the no-arbitrage value of the coupons and rebate payment at default. The no-arbitrage value or price of a claim can be derived as the expected discounted value under the martingale measure P^* or the discounted expected value under the T -forward risk adjusted martingale measure P^T .

The *no-arbitrage value of the accumulated coupons* amounts to

$$\begin{aligned} & c \int_0^T B(0, s) P^T[V_s > \eta_s, \tau > s] ds \\ &= c \int_0^T B(0, s) \left\{ N(d_3(w, s)) - \frac{F_0}{K} N(d_4(w, s)) \right\} ds \end{aligned}$$

where for derivation of the equality Equation (2.21) is applied.

The *no-arbitrage value of the rebate payment* in the case of an early default is

$$B(0, T) \int_0^T \frac{K}{n} e^{-\kappa \tau} dP^T[\tau \leq t] = \frac{1}{n} B(0, T) (K J_1 + F_0 J_2)$$

with

$$J_1 = \int_0^T e^{-\kappa s} dN(d_1(s)) \quad J_2 = \int_0^T e^{-\kappa s} dN(d_2(s)).$$

The *no-arbitrage value of the payment at maturity* is the sum of two components

$$B(0, T) \mathbb{E}_{P^T}[L \mathbf{1}_{\{V_T > nL, \tau > T\}}] + B(0, T) \mathbb{E}_{P^T}\left[\frac{V_T}{n} \mathbf{1}_{\{V_T \leq nL, \tau > T\}}\right].$$

Applying Equations (2.21) and (2.24) the following results can be derived

$$\mathbb{E}_{P^T}[L \mathbf{1}_{\{V_T > nL, \tau > T\}}] = L \left[N(d_3(nL, T)) - \frac{F_0}{K} N(d_4(nL, T)) \right],$$

and

$$\begin{aligned}
& \mathbb{E}_{P^T} \left[\frac{V_T}{n} \mathbf{1}_{\{V_T \leq nL, \tau > T\}} \right] \\
&= - \int_{Ke^{-\kappa T}}^{nL} \frac{x}{n} dN \left(\frac{\ln \frac{F_0}{x} - \kappa T - \frac{1}{2} A_T}{\sqrt{A_T}} \right) \\
&\quad + \int_{Ke^{-\kappa T}}^{nL} \frac{x F_0}{n K} dN \left(\frac{2 \ln K - \ln(F_0 x) - \kappa T - \frac{1}{2} A_T}{\sqrt{A_T}} \right) \\
&= \frac{F_0}{ne^{\kappa T}} [N(-d_2(T)) - N(d_5(nL, T))] + \frac{K}{ne^{\kappa T}} [N(d_6(nL, T)) - N(d_1(T))],
\end{aligned}$$

where d_5 and d_6 are defined in Equation (2.27), and the second equality is derived with the aid of the following integrations

$$\begin{aligned}
\int_0^y x dN \left(\frac{\ln x + a}{b} \right) &= e^{\frac{1}{2}b^2 - a} N \left(\frac{\ln y + a - b^2}{b} \right), \\
\int_0^y x dN \left(\frac{-\ln x + a}{b} \right) &= e^{\frac{1}{2}b^2 + a} N \left(\frac{-\ln y + a + b^2}{b} \right).
\end{aligned}$$

To sum up, the *no-arbitrage value of a (single) straight coupon bond* equals

$$\begin{aligned}
SB(0) &= c \int_0^T B(0, s) \left\{ N(d_3(w, s)) - \frac{F_0}{K} N(d_4(w, s)) \right\} ds + \frac{1}{n} B(0, T) (K J_1 + F_0 J_2) \\
&\quad + B(0, T) \cdot \left\{ L \left[N(d_3(nL, T)) - \frac{F_0}{K} N(d_4(nL, T)) \right] \right. \\
&\quad \quad + \frac{F_0}{ne^{\kappa T}} [N(-d_2(T)) - N(d_5(nL, T))] \\
&\quad \quad \left. + \frac{K}{ne^{\kappa T}} [N(d_6(nL, T)) - N(d_1(T))] \right\} \tag{2.25}
\end{aligned}$$

where

$$\begin{aligned}
d_1(t) &:= \frac{\ln \frac{K}{F_0} + \frac{1}{2} A_t}{\sqrt{A_t}}, & d_2(t) &:= d_1(t) - \sqrt{A_t}, \\
d_3(x, t) &:= \frac{\ln \frac{F_0}{x} - \kappa t - \frac{1}{2} A_t}{\sqrt{A_t}}, & d_4(x, t) &:= \frac{2 \ln K - \ln(F_0 x) - \kappa t - \frac{1}{2} A_t}{\sqrt{A_t}}, \\
d_5(x, t) &:= d_3(x, t) + \sqrt{A_t}, & d_6(x, t) &:= d_4(x, t) + \sqrt{A_t}
\end{aligned} \tag{2.26}$$

$$J_1 := \int_0^T e^{-\kappa s} dN(d_1(s)), \quad J_2 := \int_0^T e^{-\kappa s} dN(d_2(s)) \quad (2.27)$$

and A_t is defined by Equation (2.15). The coupon payments and the rebate payment have no explicit solutions and thus the integrals in the first term of the right hand side (*rhs*) of Equation (2.25), J_1 and J_2 have to be integrated numerically.

Example 2.6.1. As a concrete numerical example with initial flat term structure and the initial interest rate equal to the long run mean we compute the no-arbitrage prices of straight coupon bonds with parameters $T = 8$, $\sigma_V = 0.2$, $b = 0.1$, $V_0 = 1000$, $L = 100$, $K = 400$, $w = 1300$, $m = 10$, $n = 8$, $r_0 = 0.06$.⁸

The bond prices can be derived with Equations (2.25) and (2.27). Set $\sigma_r = 0$, interest rate risk is neglected. The results for straight coupon bonds with and without interest rate risk are listed in Table 2.1.

		$\sigma_r = 0.01$			$\sigma_r = 0.02$			$\sigma_r = 0$
κ	c	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	
0.02	0	58.60	57.77	56.98	58.94	57.30	55.77	57.93
0.04	0	56.62	55.67	54.79	57.04	55.15	53.53	55.85
0.04	2	65.69	64.33	63.13	66.24	63.57	61.42	64.60
0.04	4	74.76	73.00	71.47	75.45	72.00	69.32	73.35

Table 2.1: No-arbitrage prices of straight bonds, with and without interest rate risk

Table 2.1 shows that depending on negative or positive correlation of interest rate and firm's value process, interest rate risk may increase or decrease the prices of the straight bonds. The reason is that increasing correlation ρ between the interest rate process and the firm's value process causes increasing volatility of the forward prices of the firm's value which can be verified by Equation (2.15). The coupons will only be paid out, if the firm's value is above a certain level η_t , and the coupon payment terminates as soon as the default barrier is touched. The value of the coupons can rise or fall with volatility, depending on the choice of the level η_t . In our example the level is chosen below the initial firm's value, therefore the value of the coupons decreases in volatility. The redemption of the principal part of a straight bond consists of a long position in the principal and a short position of a down and out put with rebate paid at the hitting time τ . Because the value of the latter position increases, therefore the value of the redemption falls with the increasing volatility. In total, the value of the non-convertible bond decreases in ρ . This

⁸The choice of w and in combination with the discount factor and κ , make the level η_t fluctuate around 800, which is lower than the initial value of the firm.

effect is amplified by a larger interest rate volatility. In comparison with the case there is no interest rate risk, the price of the straight bond is higher in the case that the interest rate and firm's value process are negatively correlated and vice versa. In accordance with the intuition, the numerical results demonstrate that the straight bond is more valuable by higher coupons and thus lower dividend payments.

Chapter 3

European-style Convertible Bond

A *European-style convertible bond* entitles its holder to receiving coupons plus the principal at maturity, given that the issuer does not default on the obligations. Moreover, at maturity the bondholder has the right to convert the bond into a given number of shares. To limit the upside potential of the payoff, a call provision may be incorporated to provide the equity holders with the right to buy back the bond for a previously agreed price. This type of contract is called the *European-style callable and convertible bond*.

The chapter is organized as follows: Section 3.1 shows that a European-style convertible bond is essentially a straight bond with an embedded down and out call option. Closed-form solution for the valuation is derived. The European-style callable and convertible bond can be decomposed into a bond component and a component consisting of down and out call option spread. Its valuation formula is given in Section 3.2. In Example 3.1.1 and 3.2.1 no-arbitrage prices of the bonds are calculated for given conversion ratios while in Example 3.2.3 the no-arbitrage conversion ratios are computed for given initial values of the bonds.

3.1 Conversion at Maturity

By a European-style convertible bond conversion can only take place at the maturity date of the contract. According to the assumption on the capital structure made in Section 2.4, the asset of the firm consists of m shares and n bonds. Moreover, we assume that all n bonds are converted at the same time, and there is no partial conversion. The parameters n and m in this model describe the ratio of equity and debt. Together with the conversion ratio γ , they determine how the firm value will be divided among the shareholder and bondholder if conversion happens.

The bondholder has the right but not the obligation to convert. Each bond can be converted into γ shares. In the case of conversion, the number of shares amounts $m + \gamma n$,

and the conversion value for each bond would be $\frac{\gamma V_T}{m + \gamma n}$. The bondholder will only exercise the conversion right if $\frac{\gamma V_T}{m + \gamma n} > L$. Therefore, given no premature default, the bondholder receives at maturity the maximum of L and $\frac{\gamma V_T}{m + \gamma n}$ for each bond. Compared to an otherwise identical straight bond the convertible bond has an extra payment of $\left(\frac{\gamma V_T}{m + \gamma n} - L\right)^+$ ¹ which is a European call option given no default on the firm value process.

Thus the no-arbitrage price of a European-style convertible bond $CB(0)$ at time $t = 0$ can be expressed as the sum of the price of an otherwise identical straight bond and the discounted expected value of the conversion right, $CR(0)$, i.e.

$$CB(0) = SB(0) + CR(0) \quad (3.1)$$

with

$$CR(0) := B(0, T) \mathbb{E}_{PT} \left[\left(\frac{\gamma V_T}{m + \gamma n} - L \right)^+ \mathbf{1}_{\{V_T > nL, \tau > T\}} \right]$$

The price of a straight bond $SB(0)$ has been derived in section 2.6, and can be solved with Equation (2.25).

For the calculation of the no-arbitrage price of the conversion right we need the following result which is derived with the help of Equation (2.21)

$$\mathbb{E}_{PT} [V_T \mathbf{1}_{\{V_T > y, \tau > T\}}] = \int_y^\infty x d \left(-N(d_3(x, T)) + \frac{F_0}{K} N(d_4(x, T)) \right).$$

Finally the discounted expected value of the conversion right, $CR(0)$ can be solved with

$$\begin{aligned} CR(0) = B(0, T) & \frac{\gamma}{m + \gamma n} \left(\frac{F_0}{e^{\kappa T}} N(d_5(\tilde{L}, T)) - \frac{K}{e^{\kappa T}} N(d_6(\tilde{L}, T)) \right) \\ & - B(0, T) L \left(N(d_3(\tilde{L}, T)) - \frac{F_0}{K} N(d_4(\tilde{L}, T)) \right) \end{aligned} \quad (3.2)$$

where $\tilde{L} := \left(n + \frac{m}{\gamma}\right) L$, d_3 , d_4 , d_5 , and d_6 are defined in Equation (2.27).

Example 3.1.1. (Continuation of Example 2.6.1) The same model parameters as in Example 2.6.1 are assumed. The initial term structure is flat and the parameters are $T = 8$, $\sigma_V = 0.2$, $\sigma_r = 0.02$, $b = 0.1$, $V = 1000$, $L = 100$, $K = 400$, $w = 1300$, $\gamma = 2$, $m = 10$, $n = 8$, $c = 2$, $r_0 = 0.06$.

¹ $(x)^+ := \max[x, 0]$

κ	$SB(0)$		$CR(0)$		$CB(0)$	
	$\rho = -0.5$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0.5$
0.02	68.15	63.67	14.72	18.79	82.87	82.46
0.03	67.28	62.62	11.63	15.55	78.91	78.17
0.04	66.24	61.43	9.05	12.77	75.30	74.19

Table 3.1: No-arbitrage prices of European-style convertible bonds

The results in Table 3.1 show that due to the specific choice of the random barrier, the payout rate κ has no influence on the distribution of default time, which can be verified by Equation (2.20), but rebate payment decreases in κ . Therefore the value of the straight bond $SB(0)$ decreases in payout rate κ . Meanwhile, the value of conversion right $CR(0)$ decreases when κ rises. It is quite intuitive as the firm value at maturity declines if more dividends are paid out. The total effect is that the value of a European convertible bond decreases in the payout rate κ .

Increasing correlation ρ between the interest rate and the firm's value causes increasing volatility of the forward price of the firm's value. The default probability rises in volatility, which results in a reduction of the value of the straight bond $SB(0)$. But on the other side, the value of conversion right $CR(0)$ increases in volatility, therefore the total effect is not monotonic. The influence of the interest rate risk on the price of the convertible bond is relatively small which is recognized by the value of the convertible bond, i.e. the numbers listed in the columns under $CB(0)$ in Table 3.1. The reason is that in the example the volatility of the interest rate is much smaller than that of the firm's value.

Remark 3.1.2. For the model parameters chosen in Example 3.1.1, due to the offsetting nature of the value of the straight bond and conversion right, the value of the European-style convertible bond is insensitive to the change of volatility. With $\kappa = 0.02$ and $\sigma_V = 0.2$ the price of $CB(0)$ e.g. equals 82.46. If the volatility of the firm value is raised to $\sigma_V = 0.4$, the price is 81.92, and it changes only slightly.

3.2 Conversion and Call at Maturity

In a contract with conversion rights the equity holder is short of call options. The upside potential of the payoff can be limited through a call provision which provides equity holders the right to buy back each bond at a fixed price H . The bondholder will exercise the conversion right if $\frac{\gamma V_T}{m + \gamma n} > L$, but the conversion value is capped by H . Thus if $V_T > \frac{m + \gamma n}{\gamma} H$ the convertible bond with call provision will no longer profit from the upside potential of the firm value. Therefore given no default, the extra payment

additional to an otherwise identical straight bond amounts to

$$\left(\frac{\gamma V_T}{m + \gamma n} - L\right)^+ - \left(\frac{\gamma V_T}{m + \gamma n} - H\right)^+$$

which is a European call spread on the firm's value.

Thus the no-arbitrage value of a European callable and convertible bond $CCB(0)$ at time $t = 0$ is given as the sum of the value of a straight bond plus the value of the capped conversion right, $CCR(0)$,

$$CCB(0) = SB(0) + CCR(0), \quad (3.3)$$

with

$$CCR(0) := B(0, T) \mathbb{E}_{PT} \left[\left\{ \left(\frac{\gamma V_T}{m + \gamma n} - L\right)^+ - \left(\frac{\gamma V_T}{m + \gamma n} - H\right)^+ \right\} \mathbf{1}_{\{V_T > nL, \tau > T\}} \right].$$

The price of a straight bond $SB(0)$ has been derived in section 2.6, and can be solved with Equation (2.25). The value of $CCR(0)$ can be derived with the same method as by calculation of $CR(0)$,

$$\begin{aligned} CCR(0) = B(0, T) & \left\{ -L \left[N(d_3(\tilde{L}, T)) - \frac{F_0}{K} N(d_4(\tilde{L}, T)) \right] \right. \\ & + H \left[N(d_3(\tilde{H}, T)) - \frac{F_0}{K} N(d_4(\tilde{H}, T)) \right] \\ & + \frac{\gamma}{m + \gamma n} \left[\frac{F_0}{e^{\kappa T}} \left(N(d_5(\tilde{L}, T)) - N(d_5(\tilde{H}, T)) \right) \right. \\ & \left. \left. - \frac{K}{e^{\kappa T}} \left(N(d_6(\tilde{L}, T)) - N(d_6(\tilde{H}, T)) \right) \right] \right\} \quad (3.4) \end{aligned}$$

with $\tilde{H} := \left(n + \frac{m}{\gamma}\right) H$, d_3 , d_4 , d_5 , and d_6 are defined in Equation (2.27).

Example 3.2.1. (Continuation of Example 3.1.1) The same model parameters as in Example 3.1.1 are assumed. The initial term structure is flat and the parameters are $T = 8$, $\sigma_V = 0.2$, $\sigma_r = 0.02$, $b = 0.1$, $V = 1000$, $L = 100$, $K = 400$, $w = 1300$, $\gamma = 2$, $m = 10$, $n = 8$, $c = 2$, $r_0 = 0.06$, $H = 150$.

Table 3.2 summarizes the values of the capped conversion right $CCR(0)$ and the prices of the callable and convertible bond $CCB(0)$. And for comparison reason, the prices of

	$SB(0)$		$CCR(0)$		$CCB(0)$		$CB(0)$	
κ	$\rho = -0.5$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0.5$	$\rho = -0.5$	$\rho = 0.5$
0.02	68.15	63.67	8.99	8.85	77.14	72.52	82.87	82.46
0.03	67.28	62.62	7.44	7.67	74.72	70.29	78.91	78.17
0.04	66.24	61.43	6.05	6.58	72.29	68.00	75.30	74.19

Table 3.2: No-arbitrage prices of European-style callable and convertible bonds

the otherwise identical convertible but non-callable bond are also listed in Table 3.2. One can see that the prices are reduced substantially through the call provision with a call price of $H = 150$, which is 1.5 times of the principal.

The value of the capped conversion right $CCR(0)$ decreases as κ rises. The impact of the correlation ρ on the value of $CCR(0)$ is relative small but not monotonic and depends on the value of κ . Positive or negative ρ may increase or decrease the volatility of the forward price of the firm's value. The option component $CCR(0)$ is a call spread and its sensitivity to the change of volatility is not monotonic, and depending on other factors $CCR(0)$ may increase or decrease in volatility. In our example in the case that $\kappa = 0.03$ and $\kappa = 0.04$, higher volatility results in larger value of the $CCR(0)$, while by $\kappa = 0.02$ the effect is reversed. The influence of the interest rate risk is relatively small which is recognized by the results listed in the columns under $CCR(0)$ in Table 3.2.

Increasing correlation ρ between the interest rate process and the firm's value causes higher default probability, subsequently smaller value of the straight bond $SB(0)$, while ρ has relative small effect on the value of the capped conversion right $CCR(0)$. Therefore, in our example, the total effect is that the interest risk that positively correlated with the firm's value process reduces the value of the callable and convertible bond $CCB(0)$.

In Examples 3.1.1 and 3.2.1, the initial firm value V_0 , the principal of the debt L , the number of the shares m , the number of the bonds n and the conversion ratio γ are given exogenously, hence, the no-arbitrage bond price can be calculated explicitly. Subsequently the initial equity price S_0 can be determined endogenously via the assumption on the capital structure made in section 2.4

$$V_0 = m \cdot S_0 + n \cdot D_0,$$

where D_0 stands for the price of convertible bond $CB(0)$ or callable and convertible bond $CCB(0)$. The results are summarized in Table 3.3.

Example 3.2.2. (Continuation of Example 3.1.1 and 3.2.1) The same model parameters are assumed. The initial term structure is flat and the parameters are $T = 8$, $\sigma_V =$

0.2, $\sigma_r = 0.02$, $\rho = 0.5$, $b = 0.1$, $V = 1000$, $L = 100$, $K = 400$, $w = 1300$, $\gamma = 2$, $m = 10$, $n = 8$, $c = 2$, $r_0 = 0.06$, $H = 150$.

κ	$CB(0)$	$S(0)$	$CCB(0)$	$S(0)$
0.02	82.46	34.04	72.52	41.98
0.03	78.17	37.47	70.29	43.77
0.04	74.19	40.64	68.00	45.60

Table 3.3: No-arbitrage prices of S_0 under positive correlation $\rho = 0.5$

The empirical relevance of Example 3.2.2 could be that a firm is established to finance a project with equities and convertible bonds. The initial capital demand and the features of the convertible bond, e.g. the conversion ratio, the principal and coupons, with or without call provision, are given as model parameter, the no-arbitrage value of the shares can be derived and used as the emission price.

There can also be situations that a firm wants to expand and finance a further project with convertible debt. Suppose that till expansion the firm is solely financed with equity and the share price is given. And given the principal and coupons, conversion and call features, the task is to find a no-arbitrage conversion ratio which does not change the value of the shares at the issuance time of the the bond. Within our model framework the no-arbitrage conversion ratio can be determined and it is illustrated with Example 3.2.3.

Example 3.2.3. Till expansion the firm is financed solely with equity, the number of shares is n and the total value of equity amounts to E_0 . The firm issues convertible bonds to finance the expansion of a total amount of $V_0 - E_0$. The convertible bond has a maturity of $T = 8$ years, a principal of $L = 100$ and an annual coupon of $c = 2$, and there are n such bonds. In one case it is assumed that there is no call provision, while in another case the conversion value is capped at $H = 250$. The task is thus to find the conversion ratio such that the emission price of each bond equals 100. Initial term structure is flat, the model parameters are $V_0 = 1000$, $\sigma_V = 0.2$, $\sigma_r = 0.02$, $b = 0.1$, $L = 100$, $K = 400$, $w = 1300$, $r_0 = 0.06$. The no-arbitrage conversion ratios for two different capital structures are listed in 3.4.

In Example 3.2.3, the share and debt price are the same for different cases with $S_0 = 50$ and $S_0 = 100$. The results in Table 3.4 demonstrate that by the same initial share and bond price, the no-arbitrage conversion ratio is higher if the debt ratio is higher. The conversion ratio of the callable and convertible bond is more sensitive to the change of the debt ratio than the convertible but non-callable bond. Positive correlation of interest rate risk and firm's value process reduces the conversion ratio of the convertible but non-callable bond, while by the callable and convertible bond the effect reverses. This effect

	$E_0 = 500, m = 10, n = 5$		$E_0 = 300, m = 6, n = 7$	
ρ	<i>CB</i>	<i>CCB</i>	<i>CB</i>	<i>CCB</i>
-0.5	2.00	2.28	2.16	2.79
0.5	1.92	2.74	2.16	5.25

Table 3.4: No-arbitrage conversion ratios

is stronger if the firm is higher leveraged.

Chapter 4

American-style Convertible Bond

In Chapter 3 we deal with the case that conversion and call can only take place at maturity. We find that the debt can be decomposed into a bond component and an option component. The no-arbitrage price of the option component is solely determined by the firm's value at maturity. For a more flexible and realistic contract, call and conversion rights are considered to be *American-style*, granting continuous exercise opportunities for both bond- and shareholder. Closely related to the American-style is the *Bermudan-style* conversion and call rights, which can only be exercised at certain discrete time points during the life of the contract.

In practice, bonds with American-style conversion and call options are named *callable and convertible bond*. In the following, sometimes we use this term without explicitly referring of American-style. A callable and convertible bond entitles its holder to receiving coupons plus the principal at maturity, given that the issuer does not prematurely default on the obligations. Moreover, prior to the maturity date the bondholder has the right to convert the bond into a given number of shares. While on the other hand, the issuer can enforce the bondholder to surrender the bond for a previously agreed price. It is essentially a straight bond with an embedded option. Thus, it tends to offer a lower coupon rate. Two sources of risks are related to the optimal investment in the callable and convertible bond, one stemming from the randomness of firm's value, and the other stemming from the randomness of the termination time, namely the contract can be stopped by call, conversion and default.

After the inception of the contract, the bondholder's aim is to exercise the conversion option in order to maximize the value of the bond. The issuer will call the bond if he can reissue a bond with lower debt cost. Another incentive of the issuer to call a bond is to limit the bondholder's participation in rising stock prices. Such considerations lead to the problem of optimal stopping for both bond- and shareholder where certain aspects of strategic behaviors play an important role. The problem of callable and convertible bond can generically be reduced to the pricing problem of so-called *game options*.

Modeling of callable and convertible bond as a defaultable game option within structural approach has been studied by Sirbu et al. (2004) and further developed in a companion paper of Sirbu and Schreve (2006). They assume that the firm's value comprises the equity in the form of a single stock, and a single callable and convertible bond. The volatility of the firm's value is constant. The bond earns continuously a coupon at a fixed rate while the dividends are paid at a rate which is a fixed fraction of the equity value. The interest rate is also assumed to be constant. In their model the dynamic of the firm's value does not follow a geometric Brownian motion, but a more general one-dimensional diffusion due to the fixed rate of coupon payment. Default occurs if the firm's value falls to zero which is caused by the coupon payment. According to this default mechanism both equity and bond have zero recovery. In the first paper, they assume that the bond is perpetual, i.e. it never matures and can only be terminated by conversion, call or default. In the second paper the bond has finite maturity while the other assumptions remain unchanged. The determination of the optimal call and conversion strategies is characterized as a optimal stopping game between the equity- and bondholder. Viscosity solution concept is used to determine the no-arbitrage price and optimal stopping strategies. They show that if the coupon rate is below the interest rate times the call price, then conversion should precede call. On the other hand, if the dividend rate times the call price is below the coupon rate, call should precede conversion.

Our model differs from theirs mainly by allowing non-zero recovery rate of the bond and default occurs if the firm's value hits a lower but positive boundary. The dynamic of the firm's value follows a geometric Brownian motion which means the underlying process, the evolution of the firm's value, does not depend on the solution of the game option. Therefore the results of Kifer (2000) can be applied to the callable and convertible bond. Simple recursion with a binomial tree can be used to derive the value of the bond and the optimal strategies. Moreover, stochastic interest rate can be incorporated into our model.

The remainder of the chapter is structured as follows. Section 4.1 gives a formal description of the contract feature of the callable and convertible bond. The theoretical fundamental for the pricing of the game option is summarized in Section 4.2.1, and the optimal stopping times are derived in Section 4.2.2. Given the optimal strategies, the callable and convertible bond is valued by means of a tractable recursion method, we first assume that there is no interest rate risk, and in particular, the interest rate is assumed to be constant in Section 4.3. Then the results are extended in Section 4.5 to the case with stochastic interest rate. Section 4.4 gives a description of the contract feature of a Bermudan-style callable and convertible bond. The valuation is carried out with the similar recursion as in the case of American-style contracts. The only difference is that the conversion and call payoff is zero on dates when conversion and call are not allowed.

4.1 Contract Feature

In the following we assume that the bond matures at time $T \in \mathbb{R}_+$. The same features of coupon payments and default mechanism as in the case of European convertible bonds are proposed for the American-style contract. The coupons are paid out continuously with a constant rate of c , given that the firm's value is above the level η_t . The contract terminates either at maturity T or, in case of premature default, at the default time τ , which is the first hitting time of the barrier ν_t by the firm's value. Moreover, the contract stops also by conversion or call. The bondholder can stop and convert the bond into equities according to the prescribed conversion ratio γ . The conversion time of the bondholder is denoted as $\tau_b \in [0, \tau]$. The shareholder can stop and buy back the bond at a price given by the maximum of the deterministic call level H_t and the current conversion price. This ensures that the payoff by call is never lower than the conversion payoff. This assumption makes the aspect of game option relevant and interesting for the valuation of callable convertible bonds. The call time of the seller is denoted as $\tau_s \in [0, \tau]$.

4.1.1 Discounted payoff

First, we introduce the notation $\beta(s, t) = \exp\{-\int_s^t r(u)du\}$ which is the discount factor, where $r(t)$ is the instantaneous risk-free interest rate. The discounted payoff of a callable and convertible bond can be distinguished in four cases.

- (i) Let $\tau_b < \tau_s \leq T$, such that the contract begins at time 0 and is stopped and converted by the bondholder. In this case, the discounted payoff $conv(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through conversion

$$\begin{aligned} conv(0) = & c \int_0^{\tau_b \wedge \tau} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \frac{\nu_\tau}{n} \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq \tau_b\}} \\ & + \beta(0, \tau_b) \mathbf{1}_{\{\tau_b < \tau\}} \left(\frac{\gamma V_{\tau_b}}{m + \gamma n} \right). \end{aligned} \quad (4.1)$$

- (ii) Let $\tau_s < \tau_b \leq T$, such that the contract is bought back by the shareholder before the bondholder converts. In this case, the discounted payoff $call(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through call,

$$\begin{aligned} call(0) = & c \int_0^{\tau_s \wedge \tau} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \frac{\nu_\tau}{n} \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq \tau_s\}} \\ & + \beta(0, \tau_s) \mathbf{1}_{\{\tau_s < \tau\}} \max \left\{ H_{\tau_s}, \frac{\gamma V_{\tau_s}}{m + \gamma n} \right\}. \end{aligned} \quad (4.2)$$

(iii) If $\tau_s = \tau_b < T$ the discounted payoff of the bond equals the smaller value, i.e. the discounted payoff with conversion.

(iv) For $\tau_b \geq T$ and $\tau_s \geq T$, the discounted payoff of a callable and convertible bond at time 0 is

$$\begin{aligned} term(0) &= c \int_0^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \frac{\nu_\tau}{n} \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq T\}} \\ &\quad + \beta(0, T) \mathbf{1}_{\{T < \tau\}} \max \left\{ \frac{\gamma V_T}{m + \gamma n}, \min \left\{ \frac{V_T}{n}, L \right\} \right\}. \end{aligned}$$

Note that $\frac{V_T}{n} > \frac{\gamma V_T}{m + \gamma n}$ since $n, m \in \mathbb{N}_+$ and $\gamma \in \mathbb{R}_+$. Hence in the case $\frac{V_T}{n} \leq L$ the bondholder would not convert and

$$\mathbf{1}_{\{V_T \leq nL\}} \max \left\{ \frac{\gamma V_T}{m + \gamma n}, \min \left\{ \frac{V_T}{n}, L \right\} \right\} = \frac{V_T}{n}.$$

Thus, in the case (iv), we can rewrite the discounted payoff $term(0)$ as

$$\begin{aligned} term(0) &= c \int_0^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \frac{\nu_\tau}{n} \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq T\}} \\ &\quad + \beta(0, T) \mathbf{1}_{\{T < \tau, V_T > nL\}} \max \left\{ \frac{\gamma V_T}{m + \gamma n}, L \right\} + \beta(0, T) \mathbf{1}_{\{T < \tau, V_T \leq nL\}} \frac{V_T}{n}. \end{aligned} \quad (4.3)$$

Denote the minimum of conversion and call time by $\zeta = \tau_s \wedge \tau_b$. Then, all in all, the discounted payoff of a callable and convertible bond $ccb(0)$ is given as the sum of the payoffs in the former four cases and amounts to

$$\begin{aligned} cbb(0) &= \mathbf{1}_{\{\zeta < \tau\}} \left(c \int_0^{\zeta \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\zeta = \tau_s < \tau_b \leq T\}} \beta(0, \zeta) \max \left\{ H_\zeta, \frac{\gamma V_\zeta}{m + \gamma n} \right\} \right. \\ &\quad \left. + \mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} \beta(0, \zeta) \frac{\gamma V_\zeta}{m + \gamma n} + \mathbf{1}_{\{\zeta = T\}} \beta(0, \zeta) \max \left\{ \frac{\gamma V_T}{m + \gamma n}, L \right\} \right) \\ &\quad + \mathbf{1}_{\{\tau \leq \zeta\}} \left(c \int_0^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) \frac{\nu_\tau}{n} \right. \\ &\quad \left. + \mathbf{1}_{\{T < \tau\}} \beta(0, T) \min \left\{ \frac{V_T}{n}, L \right\} \right). \end{aligned} \quad (4.4)$$

4.1.2 Decomposition of the payoff

Same as in the case of European convertible bond, the American-style callable and convertible bond can also be decomposed into a straight bond component and an option component. We can reformulate $ccb(0)$ in Equation 4.4 as follows

$$\begin{aligned}
ccb(0) &= \mathbf{1}_{\{\zeta < \tau\}} \beta(0, \zeta) \left(\mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} \frac{\gamma V_\zeta}{m + \gamma n} + \mathbf{1}_{\{\zeta = \tau_s < \tau_b \leq T\}} \max \left\{ H_\zeta, \frac{\gamma V_\zeta}{m + \gamma n} \right\} \right. \\
&\quad \left. + \mathbf{1}_{\{\zeta = T\}} \max \left\{ \frac{\gamma V_T}{m + \gamma n}, L \right\} \right) \\
&\quad + \underbrace{\left(c \int_0^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) \frac{\nu_\tau}{n} + \mathbf{1}_{\{T < \tau\}} \beta(0, T) \min \left\{ \frac{V_T}{n}, L \right\} \right)}_{=d(0)} \\
&\quad - \mathbf{1}_{\{\zeta < \tau\}} \underbrace{\left(c \int_\zeta^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) \frac{\nu_\tau}{n} + \mathbf{1}_{\{T < \tau\}} \beta(0, T) \min \left\{ \frac{V_T}{n}, L \right\} \right)}_{:=\beta(0, \zeta) \phi_\zeta}.
\end{aligned}$$

Since $\frac{V_T}{n} \geq L$ if $\zeta \leq T$,¹ thus the following decomposition can be achieved, which enables us to investigate the pure effect caused by the conversion and call rights.

Theorem 4.1.1. The payoff of a callable and convertible bond can be decomposed into a straight bond $d(0)$ and a defaultable game option component $g(0)$.

$$ccb(0) = d(0) + g(0) \quad (4.5)$$

with

$$d(0) := c \int_0^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) \frac{\nu_\tau}{n} + \mathbf{1}_{\{T < \tau\}} \beta(0, T) \min \left\{ \frac{V_T}{n}, L \right\},$$

and

$$\begin{aligned}
g(0) &:= \mathbf{1}_{\{\zeta < \tau\}} \beta(0, \zeta) \left\{ \mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} \left(\frac{\gamma V_\zeta}{m + \gamma n} - \phi_\zeta \right) \right. \\
&\quad \left. + \mathbf{1}_{\{\zeta = \tau_s < \tau_b \leq T\}} \left(\max \left\{ H_\zeta, \frac{\gamma V_\zeta}{m + \gamma n} \right\} - \phi_\zeta \right) + \mathbf{1}_{\{\zeta = T\}} \left(\frac{\gamma V_T}{m + \gamma n} - L \right)^+ \right\},
\end{aligned}$$

¹Otherwise the bondholder would not make use of his conversion right.

where

$$\phi_\zeta := c \int_\zeta^{\tau \wedge T} \beta(0, s) \mathbf{1}_{\{V_s > \eta_s\}} ds + \mathbf{1}_{\{\tau \leq T\}} \beta(\zeta, \tau) \frac{\nu_\tau}{n} + \mathbf{1}_{\{T < \tau\}} \beta(\zeta, T) \min \left\{ \frac{V_T}{n}, L \right\} \quad (4.6)$$

is the discounted value (discounted to time ζ) of the sum of the remaining coupon payments and the principal payment of a straight coupon bond given that it has not defaulted till time ζ .

4.2 Optimal Strategies

After the inception of the contract, the bondholder's aim is to maximize the value of the bond by means of optimal exercise of the conversion right. The incentive of the issuer to call a bond is to limit the bondholder's participation in rising stock prices. The embedded option rights owned by both of the bondholder and issuer can be treated with the well developed theories on the game option.

4.2.1 Game option

In this section we summarize the valuation problem of game options² and highlight some important results derived by Kifer (2000), Kallsen and Kühn (2004) and Kallsen and Kühn (2005).

Definition 4.2.1. Let $T \in \mathbb{R}_+$. Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$. A *game option* is a contract between a seller A and a buyer B which enables A to terminate it and B to exercise it at any time $t \in [0, T]$ up to the maturity date T . If B exercises at time t , he obtains from A the payment X_t . If A terminates the contract at time t before it is exercised by B , then he has to pay B the amount Y_t , where X_t and Y_t are two stochastic processes which are adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$, and satisfy the following condition

$$X_t \leq Y_t, \quad \text{for } t \in [0, T], \quad \text{and} \quad X_T = Y_T. \quad (4.7)$$

Moreover, if the seller A terminates and the buyer B exercises at the same time, A only has to pay the lower value X_t . Loosely speaking, the seller must pay certain penalty if he terminates the contract before the buyer exercises it.

Game options include both American and European options as special cases. Formally, if we set $Y_t = \infty$ for $t \in [0, T)$, then we obtain an American option. A European option is obtained by setting $X_t = 0$ for $t \in [0, T)$ and X_T is a nonnegative \mathcal{F}_T -measurable

²In Kifer (2000) and Kallsen and Kühn (2005) game options are alternatively also called game contingent claims, but we will only use the term game option.

random variable.

If the seller A selects a stopping time τ_A as termination time and the buyer B chooses a stopping time τ_B as exercise time, then A promises to pay B at time $\tau_A \wedge \tau_B$ the amount

$$g(\tau_A, \tau_B) := X_{\tau_B} \mathbf{1}_{\{\tau_B \leq \tau_A\}} + Y_{\tau_A} \mathbf{1}_{\{\tau_A < \tau_B\}}, \quad (4.8)$$

which denotes the payoff of a game option.

The aim of the buyer B is to maximize the payoff $g(\tau_A, \tau_B)$, while the seller A tends to minimize the payoff. It is proved in the literature that under a martingale measure P^* ³, the optimal strategy for the buyer is therefore to select the stopping time which maximizes his expected discounted payoff given the minimizing strategy of the seller, while the seller will choose the stopping time that minimizes the expected discounted payoff given the maximizing strategy of the buyer. This *max-min strategy* of the buyer leads to the lower value of the game option, whereas the *min-max strategy* of the seller leads to the upper value of the game option. In a complete market the condition described by Equation (4.7) ensures that the lower value equals the upper value such that there exists a solution for the pricing problem of a game option.

The existence and uniqueness of the no-arbitrage price in a complete market where the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ is generated by a standard one-dimensional Brownian motion is proved in Kifer (2000), Theorem 3.1. The no-arbitrage price of a game option equals $G(0)$,

$$\begin{aligned} G(0) &= \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \mathbb{E}_{P^*} [e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \\ &= \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \mathbb{E}_{P^*} [e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \end{aligned} \quad (4.9)$$

where \mathcal{F}_{0T} is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$. After the inception of the contract, the value process $G(t)$, $t \in (0, T]$ satisfies

$$\begin{aligned} G(t) &= \text{esssup}_{\tau_B \in \mathcal{F}_{tT}} \text{essinf}_{\tau_A \in \mathcal{F}_{tT}} \mathbb{E}_{P^*} [e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t] \\ &= \text{essinf}_{\tau_A \in \mathcal{F}_{tT}} \text{esssup}_{\tau_B \in \mathcal{F}_{tT}} \mathbb{E}_{P^*} [e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t]. \end{aligned} \quad (4.10)$$

Where \mathcal{F}_{tT} is the set of stopping times with values in $[t, T]$. Further, the optimal stop-

³In *complete* market, the equivalent martingale measure P^* is unique, while in *incomplete* market a martingale measure P^* can be chosen with some hedging arguments.

ping times for the seller A and buyer B respectively are

$$\begin{aligned}\tau_A^* &= \inf\{t \in [0, T] \mid e^{-rt}Y_t \leq G(t)\} \\ \tau_B^* &= \inf\{t \in [0, T] \mid e^{-rt}X_t \geq G(t)\}.\end{aligned}\tag{4.11}$$

It is optimal for the seller A to buy back the option as soon as the current exercise value $e^{-rt}Y_t$ is equal to or smaller than the value function $G(t)$, while the optimal strategy for the buyer B is to exercise the option as soon as the current exercise value $e^{-rt}X_t$ is equal to or greater than the value function $G(t)$.

Kallsen and Kühn (2004) and Kallsen and Kühn (2005) study the game option in incomplete market. The authors assume that the game option can be traded together with the other primary assets during the entire contract period $[0, T]$, which means that the payoff processes X and Y may depend on the market price process of the game option G . If the condition $X_t \leq G_t \leq Y_t$ for $0 \leq t \leq T$ is satisfied and the lower payoff X_t is bounded, i.e. it cannot take the value infinity, Theorem 2.9 of Kallsen and Kühn (2005) states that the *max-min* strategy of the buyer and the *min-max* strategy of the seller can be applied and the similar result as in the case of complete market can be achieved. $G(t)$ is an arbitrage-free price process if and only if it is a semi-martingale and satisfies

$$\begin{aligned}G(t) &= \operatorname{esssup}_{\tau_B \in \mathcal{F}_{tT}} \operatorname{essinf}_{\tau_A \in \mathcal{F}_{tT}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t] \\ &= \operatorname{essinf}_{\tau_A \in \mathcal{F}_{tT}} \operatorname{esssup}_{\tau_B \in \mathcal{F}_{tT}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B) | \mathcal{F}_t]\end{aligned}\tag{4.12}$$

for some $Q \in \mathcal{Q}$. The optimal stopping times τ_A^* and τ_B^* can also be described with Equation (4.11). The only difference is that $G(t)$ is derived under the expectation of some $Q \in \mathcal{Q}$.

A possible martingale measure Q is derived in Kallsen and Kühn (2004) in the following way. The underlying securities are assumed to be governed by some objective probability measure P . In incomplete market the derivation of the no-arbitrage price of the game option can no longer be done independently of the market agent's preference. A unique price can only be derived under stronger assumptions. They use the *neutral derivative pricing rule* which relies on both utility maximization and market clearing condition of the game option market⁴. The investors maximize their expected utility of financial gains. They may have different risk aversion parameters but they behave quite identically in the sense that all of them have the same form of utility function. Thus all of the investors can be summarized as a representative investor who has the aggregated utility function in the same form of the individual utility functions. The aggregated risk aversion parameter can be specified explicitly. The market clearing condition requires that the optimal portfolio of the representative investor contains no contingent claims. Under these quite strong

⁴No clearing of the underlying risky assets is required

assumptions the neutral pricing measure Q can be derived. The unique arbitrage free price process of the game option is then recovered as the value of a zero-sum Dynkin game under the neutral pricing measure Q .

4.2.2 Optimal stopping and no-arbitrage value of callable and convertible bond

The discounted conversion value of the callable and convertible bond, described with Equation (4.1), contains expressions about default times. As in the structural approach, the default time is a predictable stopping time, and adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by the firm's value. Thus the discounted conversion value is adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. And the same is valid for the discounted call value and the discounted terminal payoff, described with Equations (4.2) and (4.3) respectively. Moreover, the call value is always larger than the conversion value for $t < T$, and they coincide at maturity T . Hence, the payoffs in the case of conversion and call satisfy the requirements on the payoffs of the game option. Furthermore, the market in our structural approach is assumed to be complete. Therefore the theory on game option developed by Kifer (2000) can be applied to derive the unique no-arbitrage value and the optimal strategies.

Proposition 4.2.2. Plugging the payoff functions $ccb(0)$ in Equation (4.9), the unique no-arbitrage price $CCB(0)$ at time $t = 0$ of the callable and convertible bond is given by

$$CCB(0) = \sup_{\tau_b \in \mathcal{F}_{0T}} \inf_{\tau_s \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[ccb(0)] = \inf_{\tau_s \in \mathcal{F}_{0T}} \sup_{\tau_b \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[ccb(0)]. \quad (4.13)$$

After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[ccb(0) | \mathcal{F}_t] \\ &= \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \mathbb{E}_{P^*}[ccb(0) | \mathcal{F}_t]. \end{aligned} \quad (4.14)$$

The optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. The assumption that the call value is always larger than the conversion value prior to the maturity and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution.

Furthermore, the optimal stopping times for the equity holder and bondholder respectively

are

$$\begin{aligned}\tau_b^* &= \inf\{t \in [0, T] \mid conv(0) \geq CCB(t)\} \\ \tau_s^* &= \inf\{t \in [0, T] \mid call(0) \leq CCB(t)\}.\end{aligned}\tag{4.15}$$

It is optimal to convert as soon as the current conversion value is equal to or larger than the value function $CCB(t)$, while the optimal strategy for the issuer is to call the bond as soon as the current call value is equal to or smaller than the value function $CCB(t)$.

Remark 4.2.3. The no-arbitrage value of the callable and convertible bond and the optimal stopping times described by Equation (4.13) and (4.15) incorporate also the case of stochastic interest rate. Kifer (2000) assumes that the interest rate is constant, but this assumption is not necessary, because game option is essentially a zero-sum Dynkin stopping game and the *min-max* and *max-min* strategies are also valid for the stochastic discount factor. For details, see e.g. Kifer (2000) and Cvitanic and Karatzas (1996).

In section 4.1.2 it has been shown that the callable and convertible bond can be decomposed into a straight bond and a game option component

$$ccb(0) = d(0) + g(0).$$

Therefore the no-arbitrage price of the callable and convertible bond can also be derived in the following way

$$CCB(0) = \mathbb{E}_{P^*}[d(0)] + \mathbb{E}_{P^*}[g(0)].$$

The no-arbitrage price of the game option component $G(0)$ equals

$$\begin{aligned}G(0) &:= \mathbb{E}_{P^*}[g(0)] \\ &= \sup_{\tau_b \in \mathcal{F}_{0T}} \inf_{\tau_s \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[g(0)] = \inf_{\tau_s \in \mathcal{F}_{0T}} \sup_{\tau_b \in \mathcal{F}_{0T}} \mathbb{E}_{P^*}[g(0)].\end{aligned}\tag{4.16}$$

4.3 Deterministic Interest Rates

In general, closed-form solutions of the optimization problems stated in Equations (4.13) and (4.16) are not available. One alternative solution is to approximate the continuous time problem with a discrete time one. The no-arbitrage value of the callable and convertible bond can then be derived by a recursion formula. In order to focus on the recursion procedure, we assume in the first step that the interest rate is constant. Theorem 2.1 of Kifer (2000) illustrates the recursion method for the game option and the optimal stopping strategies of both counterparts. The discretization method and its convergence is proved in Proposition 3.2 of the same paper. We will apply and adapt this recursion

method to determine the no-arbitrage value and optimal stopping times of the callable and convertible bond.

4.3.1 Discretization and recursion schema

The time interval $[0, T]$ is discretized into N equidistant time steps $0 = t_0 < t_1 < \dots < t_N = T$, with $t_i - t_{i-1} = \Delta$. Assume that the bondholder does not receive the coupon for the period in which the bond is converted, while receives the dividends for the converted shares, though. If the bond is called, coupon will be paid. $CCB(t_n)$, the recursion value of the callable and convertible bond at time t_n , can be derived by means of the *max-min* or *min-max* recursion, illustrated in Figure 4.1 and 4.2. Note that in complete markets the *max-min* strategy leads to the same value as the *min-max* strategy. Hence it does not matter whether we carry out the recursion according to the strategy of the bondholder or that of the shareholder.

For $n = 0, 1, \dots, N - 1$,

$$CCB(t_n) = \begin{cases} \min \left\{ e^{-rt_n} \max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\}, \right. \\ \left. \max \left\{ e^{-rt_n} \frac{\gamma V_{t_n}^+}{m + \gamma n}, \mathbb{E}_{P^*} [CCB(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ e^{-rt_n} \frac{V_{t_n}^+}{n} & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases} \quad (4.17)$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases} \quad (4.18)$$

Figure 4.1: *Min-max* recursion callable and convertible bond, strategy of the issuer

where $V_{t_n}^+$ is the firm's value just before payout and ν_{t_n} is the default barrier. The discretized coupon c_{t_n} equals $c\Delta$, and will only be paid out if the firm's value is above certain level, i.e. $V_{t_n}^+ > \eta_{t_n}$, therefore c_{t_n} is path-dependent.

Furthermore, for each $i = 0, 1, \dots, N - 1$, the rational conversion time after time t_i equals

$$\tau_b^*(t_i) = \min \left\{ t_k \in \{t_i, \dots, t_{N-1}\} \mid e^{-rt_k} \frac{\gamma V_{t_k}^+}{m + \gamma n} = CCB(t_k) \right\},$$

For $n = 0, 1, \dots, N - 1$,

$$CCB(t_n) = \begin{cases} \max \left\{ e^{-rt_n} \frac{\gamma V_{t_n}^+}{m + \gamma n}, \min \left\{ e^{-rt_n} \max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\}, \right. \right. \\ \left. \left. \mathbb{E}_{P^*} [CCB(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ e^{-rt_n} \frac{V_{t_n}^+}{n} & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases} \quad (4.19)$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases} \quad (4.20)$$

Figure 4.2: *Max-min* recursion callable and convertible bond, strategy of the bondholder

the rational call time after time t_i equals

$$\tau_s^*(t_i) = \min \left\{ t_k \in \{t_i, \dots, t_{N-1}\} \mid e^{-rt_k} \max \left\{ H + c_{t_k}, \frac{\gamma V_{t_k}^+}{m + \gamma n} \right\} = CCB(t_k) \right\}.$$

Therefore at time t_k , it is optimal to convert or call when the current conversion or call value equals the payoff function $CCB(t_k)$.

Remark 4.3.1. For convenience of notation, the call value H is assumed be constant, but the same recursion formulas also hold in the case of a deterministic and time dependent call level $H(t)$. In that case H has to be replaced by $H(t_n^+)$ in the above formulas.

Analogously, the no-arbitrage value of the pure *game option component* $G(t_n)$ at time t_n can be derived through the recursion shown in Figure 4.3 with ϕ_{t_n} as discretized value defined by Equation (4.6).

4.3.2 Implementation with binomial tree

As the firm's value in our structural model follows a geometric Brownian motion, in absence of interest rate risk, it can be approximated by the Cox-Ross-Rubinstein model. The time interval $[0, T]$ is divided in N subintervals of equal lengths, the distance between two periods is $\Delta = T/N$. The stochastic evolution of the firm's value is then modeled by

$$V(i, j) = V(0)u^j d^{i-j} \hat{\kappa}^i, \quad \text{for all } j = 0, \dots, i, \quad i = 1, \dots, N, \quad (4.21)$$

For $n = 0, 1, \dots, N - 1$,

$$G(t_n) = \begin{cases} \min \left\{ e^{-rt_n} \left(\max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\} - \phi_{t_n} \right), \right. \\ \left. \max \left\{ e^{-rt_n} \left(\frac{\gamma V_{t_n}^+}{m + \gamma n} - \phi_{t_n} \right), \mathbb{E}_{P^*} [G(t_{n+1}) | \mathcal{F}_{t_n}] \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ 0 & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases}$$

or

$$G(t_n) = \begin{cases} \max \left\{ e^{-rt_n} \left(\frac{\gamma V_{t_n}^+}{m + \gamma n} - \phi_{t_n} \right), \right. \\ \left. \min \left\{ e^{-rt_n} \left(\max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\} - \phi_{t_n} \right), \right. \right. \\ \left. \left. \mathbb{E}_{P^*} [G(t_{n+1}) | \mathcal{F}_{t_n}] \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ 0 & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases}$$

and

$$G(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T+}}{m + \gamma n} - L - c_N, 0 \right\} & \text{if } V_{T+} > n(L + c_N) \\ 0 & \text{if } V_{T+} \leq n(L + c_N) \end{cases}$$

Figure 4.3: *Max-min* and *min-max* recursion game option component

and

$$u = e^{\sigma_V \sqrt{\Delta}}, \quad d = e^{-\sigma_V \sqrt{\Delta}}, \quad \hat{\kappa} = e^{-\kappa \Delta},$$

where $V(i, j)$ denotes the firm's value at time t_i after j up movements, and less the amount to be paid out. And according to Equation (4.21), the firm's value just before the payment equals $\frac{V(i, j)}{\hat{\kappa}}$, and the total amount to be paid out at time t_i is $V(i, j) \frac{\hat{\kappa}}{1 - \hat{\kappa}}$. We see that u, d and $\hat{\kappa}$ are time and state independent. The equivalent martingale measure P^* exists if the periodical discount factor $d < 1 + \hat{r} = e^{r\Delta} < u$. The transition probability is given by

$$p^* := \frac{1 + \hat{r} - d}{u - d}.$$

Concretely, the recursion procedure of the *min-max* strategy⁵ of the issuer of a callable and convertible bond, described by Equations (4.17) and (4.18), can be implemented within the Cox-Ross-Rubinstein model with Algorithm I (Figure 4.4). And the recursion of the best strategy of the game option component is given in Algorithm II (Figure 4.5).

⁵The algorithm of *max-min* strategy can be written in the similar way, therefore we omit this case.

The no-arbitrage price of the callable and convertible bond is then given by $CCB(0, 0)$ while the no-arbitrage value of the game option component is given by $G(0, 0)$.

$$\begin{aligned}
& \text{for } j = 0, 1, \dots, N, \\
& \quad \text{if } \frac{V(N, j)}{\hat{\kappa}} > nL + nc_{N, j}, \\
& \quad \quad \text{then } CCB(N, j) = \max \left\{ \frac{\gamma}{m + \gamma n} \cdot \frac{V(N, j)}{\hat{\kappa}}, L + c_{N, j} \right\} \\
& \quad \quad \text{else, } CCB(N, j) = \frac{V(N, j)}{n \cdot \hat{\kappa}} \\
& \text{for } i = N - 1, \dots, 0, \\
& \quad \text{for } j = i, \dots, 0, \\
& \quad \quad \text{if } V(i, j) > K, \text{ then} \\
& \quad \quad \quad CCB(i, j) = \min \left\{ \max \left[H + c_{i, j}, \frac{\gamma}{m + \gamma n} \cdot \frac{V(i, j)}{\hat{\kappa}} \right], \right. \\
& \quad \quad \quad \max \left[\frac{\gamma}{m + \gamma n} \cdot \frac{V(i, j)}{\hat{\kappa}}, \frac{1}{1 + \hat{r}} \left(p^* \cdot CCB(i + 1, j) \right. \right. \\
& \quad \quad \quad \left. \left. + (1 - p^*) \cdot CCB(i + 1, j + 1) \right) + c_{i, j} \right] \left. \right\}, \\
& \quad \quad \quad \text{else, } CCB(i, j) = \frac{V(i, j)}{n \cdot \hat{\kappa}}
\end{aligned}$$

Figure 4.4: Algorithm I: *Min-max* recursion American-style callable and convertible bond

The first loop in Algorithm I (Figure 4.4) and II (Figure 4.5) determines the optimal strategy and thus the optimal terminal value $CCB(N, j)$ or $G(N, j)$ respectively. While the second loop in the both algorithms determines the value of $CCB(i, j)$ or $G(i, j)$ according to the *min-max* strategy at node (i, j) of the tree. $D(i, j)$ in Algorithm II denotes the time discretized value of the sum of the remaining coupon payments and the principal payment of a straight coupon bond given by Equation (4.6). The value of each $CCB(i, j)$ is stored in a data matrix, and the event of conversion, call or continuation of the contract is recorded for each node (i, j) . Then given the development, i.e. the path of the firm's value $V(i, j)$, the bondholder and issuer can determine their optimal stopping times by moving forward alongside the tree. At the time the contract is terminated, i.e. converted, called or default at the node (i, j) , $CCB(i, j)$ is then the discounted payoff of the callable and convertible bond for this realization of the firm's value.

$$\begin{aligned}
& \text{for } j = 0, 1, \dots, N, \\
& \quad \text{if } \frac{V(N, j)}{\hat{\kappa}} > nL + nc_{N, j}, \\
& \quad \quad \text{then } G(N, j) = \max \left\{ \frac{\gamma}{m + \gamma n} \cdot \frac{V(N, i)}{1 - \hat{\kappa}} - L - c_{N, j}, 0 \right\} \\
& \quad \quad \text{else, } G(N, j) = 0 \\
& \text{for } i = N - 1, \dots, 0, \\
& \quad \text{for } j = i, \dots, 0, \\
& \quad \quad \text{if } V(i, j) > K, \text{ then} \\
& \quad \quad \quad G(i, j) = \min \left\{ \max \left[H + c_{i, j}, \frac{\gamma}{m + \gamma n} \cdot \frac{V(i, j)}{\hat{\kappa}} \right] - D(i, j), \right. \\
& \quad \quad \quad \max \left[\frac{\gamma}{m + \gamma n} \cdot \frac{V(i, j)}{\hat{\kappa}} - D(i, j), \frac{1}{1 + \hat{r}} \left(p^* \cdot G(i + 1, j) \right. \right. \\
& \quad \quad \quad \left. \left. + (1 - p^*) \cdot G(i + 1, j + 1) \right) \right] \left. \right\}, \\
& \quad \quad \quad \text{else, } G(i, j) = 0
\end{aligned}$$

Figure 4.5: Algorithm II: *Min-max* recursion game option component

4.3.3 Influences of model parameters illustrated with a numerical example

The no-arbitrage value of the callable and convertible bond is affected by the randomness of the firm's value, and the randomness of the termination time. It is a complex contract and influenced by a number of parameters: e.g. the value of coupon and principal, default barrier, volatility of the firm's value, conversion ratio, call level, maturity, etc. The firm's value in total follows a diffusion process, while the bond and equity value are results of a strategic game, which are not simple diffusion processes. Change of one parameter causes simultaneous changes of the value of bond and equity. For example, intuitively, the increment of the conversion ratio causes the rise of conversion value, thus the rise of the bond price, but at the same time the reduction of the equity value, and consequently the decline of the conversion value. The direction and quantity of the total effect cannot be determined without numerical evaluation. Moreover, to design a meaningful callable and convertible bond, the parameters should in accordance with each other. The situation such that the bond will be converted or called immediately after the start of the contract, should not happen. In the following, we will illustrate the influences of the model parameters and their interactions with a close study of a numerical example.

Example 4.3.2. As an explicit numerical example we choose the following parameters: $T = 8$, $\sigma_V = 0.2$, $r = 0.06$, $V(0) = 1000$, $K = 400$, $\omega = 1300$, $L = 100$, $\gamma = 1.5$, $m = 10$, $n = 8$, $H = 120$.

		$\sigma_V = 0.2$				$\sigma_V = 0.4$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	48.01	26.85	74.86	40.11
0.04	2	65.15	8.65	73.80	40.96	52.38	20.41	72.79	41.77
0.04	3	69.83	7.82	77.65	37.88	56.39	18.72	75.12	39.91
0.04	4	74.50	6.99	81.50	34.80	60.40	17.03	77.44	38.06

Table 4.1: Influence of the volatility of the firm's value and coupons on the no-arbitrage price of the callable and convertible bond (384 steps)

Remark 4.3.3. Within the example all results, except for the results in Table 4.2, are derived with $\Delta = 1/48$, which approximately corresponds to a weekly valuation. By $\Delta = 1/48$ and a maturity of $T = 8$ it corresponds to a tree with 384 steps.

The results in Table 4.1 are derived for different payout ratios κ and coupons c ⁶. They illustrate first that the value of the game option component decreases when coupon payment rises. The reason is that the value of the remaining coupon and principal payment defined by Equation (4.6) can be thought as the strike of the game option, which is an increasing function of coupon rate c , and the value of the game option component decreases in strike. The large price difference of $G(0)$ in the case $\kappa = 0$, $c = 0$, to the case $\kappa = 0.04$, $c = 2$ is due to the increment of payout ratio and coupon rate. Both factors together result in a large drop of the value of $G(0)$. The second effect shown by Table 4.1 is that the more volatile the firm's value, the larger the default probability, hence the smaller the value of straight bond. But on the other side the game option component $G(0)$ becomes more valuable. In our example, the value of the callable and convertible bond which is the sum of the both components decreases in volatility⁷.

The stability of the recursion is demonstrated with Table 4.2. The recursions are carried out alongside trees with different steps for $\sigma_V = 0.2$ and $\sigma_V = 0.4$. We can see that the numerical results stabilized at $\Delta = 1/48$. Further refinements ($\Delta = 1/100$ and $\Delta = 1/250$) of the tree do not change the numerical results considerably while much more time are needed for the calculation. Therefore, for the further calculations in this example Δ is always set to be $1/48$.

⁶The coupons are to be paid if the firm's value is above $\eta_t = \omega \cdot e^{-r(T-t)} e^{-\kappa t}$, The default barrier is $\nu_t = K \cdot e^{-r(T-t)} e^{-\kappa t}$

⁷In Example 4.3.2, the value of the callable and convertible bond increases in volatility, but one cannot argue it generally, as it depends also on other factors e.g. default barrier and maturity.

	$\sigma_V = 0.2$			$\sigma_V = 0.4$		
Δ	$SB(0)$	$CCB(0)$	$G(0)$	$SB(0)$	$CCB(0)$	$G(0)$
1	69.37	77.00	7.64	54.30	73.83	19.54
1/12	69.82	77.65	7.82	56.14	75.12	18.96
1/48	69.83	77.64	7.81	56.39	75.12	18.72
1/100	69.83	77.64	7.81	56.45	75.11	18.66
1/250	69.83	77.64	7.81	56.51	75.12	18.61

Table 4.2: Stability of the recursion

		$\gamma = 1.5$				$\gamma = 2$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	59.40	22.96	82.35	34.12
0.04	2	65.15	8.65	73.80	40.96	65.15	13.12	78.27	37.38
0.04	3	69.83	7.82	77.65	37.88	69.83	11.71	81.54	34.77
0.04	4	74.50	6.99	81.50	34.80	74.50	10.39	84.90	32.08

Table 4.3: Influence of the conversion ratio on the no-arbitrage price of the callable and convertible bond (384 steps)

Table 4.3 has the same structure as Table 4.1 and shows the influence of the conversion ratio γ on $G(0)$ and $CCB(0)$. The volatility of the firm's value is kept to be constant, i.e. $\sigma_V = 0.2$. The change of conversion ratio γ does not affect the price of the straight coupon bond and it only changes the value of $G(0)$. The increase of γ from 1.5 to 2.0 makes the game option component more valuable, thus in total the callable and convertible bond more valuable⁸. The case by $\kappa = 0.04$, $c = 2$ and $\gamma = 2$ is not a good contract design. As with $CCB(0) = 78.27$, and $S(0) = 37.38$, the initial price of the bond is almost equal to the initial conversion value, which means that the conversion may take place very quickly after the inception of the contract, because a slight increase of the firm's value will make conversion the optimal choice of the bondholder. Usually it is not the intention of the issuer to issue a bond which will be converted or called immediately after the inception of the contract.

Table 4.4 is also structured in the same way as Tables 4.3 and 4.1. It demonstrates the influence of the maturity T on $G(0)$ and $CCB(0)$. The volatility of the firm's value and conversion ratio are $\sigma_V = 0.2$ and $\gamma = 1.5$. Comparing the case $T = 8$ with $T = 6$, we observe that the straight bond is more valuable with shorter maturity, because the default probability is lower and by positive interest rate the principal is more valuable if it is paid earlier. The game option component $G(0)$ is less valuable in the case of shorter maturity. It is due to two effects: first, shorter maturity means less conversion

⁸Again we cannot take it as a general result, as it depends also on the parameters m and n .

		$T = 8$				$T = 6$			
κ	c	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$	$SB(0)$	$G(0)$	$CCB(0)$	$S(0)$
0	0	59.40	16.92	76.32	38.94	67.21	11.91	79.12	36.71
0.04	2	65.15	8.65	73.80	40.96	71.75	6.22	77.97	37.62
0.04	3	69.83	7.82	77.65	37.88	75.57	5.75	81.33	34.94
0.04	4	74.50	6.99	81.50	34.80	74.50	5.27	84.67	32.26

Table 4.4: Influence of the maturity on the no-arbitrage price of the callable and convertible bond (384 steps)

chances for the bondholder, and secondly, an increase of the value of the straight bond reduces the value of the equity thus the conversion value. The reduction of $G(0)$ may in turn increase the value of equity, here the final result is that reduction in maturity increases the value of the callable and convertible bond $CCB(0)$.

κ	c	$\omega = 0, \sigma_V = 0.2$	$\omega = 0.04, \sigma_V = 0.2$	$\omega = 0, \sigma_V = 0.4$	$\omega = 0.04, \sigma_V = 0.4$
0	0	16.92	14.84	26.85	24.40
0.04	2	8.65	7.67	20.41	19.00
0.04	4	6.99	3.64	17.03	13.58

Table 4.5: Influence of the call level on the no-arbitrage price of the game option component (384 steps)

The value of the game option component can be restricted when the call level is reduced. This effect is confirmed by the results in Table 4.5. The reduction of the call level is achieved by making the call level to be time dependent

$$H(t) = e^{-\omega(T-t)H}, \quad \omega \geq 0. \quad (4.22)$$

The value of $H(t)$ increases in time and reaches H at maturity T . By $\omega = 0$, the call level reaches its maximum and is a constant H . The impact of the call level on the no-arbitrage price of game option component is stronger in the case of higher coupon rate c and lower volatility of the firm's value σ_V . Finally, we compare the value of the European and American conversion and call rights.⁹ The model parameters are assumed to be the same for both cases. For the results in the column $G(0)^1$ the call level is set to be constant with $H = 120$, while by $G(0)^2$, the call level is time dependent and defined according to Equation (4.22) with $\omega = 0.04$ and $H = 120$.

⁹As we have shown that both European and American convertible bond can be decomposed into a straight coupon bond and an option component, therefore the price difference of European and American convertible bond is solely determined by the characteristic of the option components.

		$\sigma_V = 0.2$				$\sigma_V = 0.4$			
κ	c	$G(0)^1$	$G(0)^2$	$CR(0)$	$CCR(0)$	$G(0)^1$	$G(0)^2$	$CR(0)$	$CCR(0)$
0	0	16.92	14.84	7.12	3.40	26.85	24.40	17.91	3.82
0.04	2	8.65	7.67	5.46	1.87	20.41	19.00	12.38	2.92
0.04	4	6.99	3.64	3.46	1.87	17.03	13.58	12.38	2.92

Table 4.6: Comparison European- and American-style conversion and call rights (384 steps)

Table 4.6 illustrates that the game option component $G(0)^1$ and $G(0)^2$ are much more valuable than the European callable conversion right $CCR(0)$. The reason is that the latter is capped by H at maturity regardless of the firm's value while the call value of the former is the maximum of H and the conversion value till maturity and equals the conversion value at maturity. The value of $G(0)^1$ and $G(0)^2$ are also larger than the European non-callable conversion right $CR(0)$. The only exception is that $G(0)^2$ differs only slightly from $CR(0)$ for a higher coupon of $c = 4$. Both $G(0)$ and $CR(0)$ are sensitive to changes of volatility σ_V , while $CCR(0)$ varies only slightly by a relative large change of σ_V .

4.4 Bermudan-style Convertible Bond

Closely related to the American-style is the Bermudan-style conversion and call rights, which can only be exercised at certain discrete time points during the lifetime of the contract. For derivation of the no-arbitrage value of the Bermudan-style callable and convertible bond we only need to modify the recursion schema displayed in Figures 4.1 and 4.2 such that on dates t_n when conversion and call are not allowed¹⁰

$$CCB(t_n) = \mathbb{E}_{P^*}[CCB(t_{n+1})|\mathcal{F}_{t_n}] + e^{-rt_n}c_{t_n}.$$

Assume that conversion and call are allowed only on M equidistant discrete time points. The time interval $[0, T]$ is discretized into N equally distanced time steps $0 = t_0 < t_1 < \dots < t_N = T$, and N is chosen such that $N/M = \delta$, and δ is an integer, the conversion and call can only take place at time points $0 < t_\delta < t_{2\delta} \dots < t_{M\delta} = T$.

The modified recursion procedure for the *max-min* strategy¹¹ thus can be written as, for $n = 0, 1, \dots, N - 1$ and $m = 1, \dots, M - 1$,

¹⁰See, Kifer (2000), p. 461 and Wilmott (2006) Vol. 3, p.1245.

¹¹The modification is the same for both *max-min* and *min-max* strategy.

$$CCB(t_n) = \begin{cases} \max \left\{ e^{-rt_n} \frac{\gamma V_{t_n}^+}{m + \gamma n}, \right. \\ \min \left\{ e^{-rt_n} \max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\}, \right. \\ \left. \mathbb{E}_{P^*} [CCB(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n}, n = \delta m \\ \mathbb{E}_{P^*} [CCB(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} & \text{if } V_{t_n}^+ > \nu_{t_n}, n \neq \delta m \\ e^{-rt_n} \frac{V_{t_n}^+}{n} & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases}$$

Figure 4.6: *Max-min* recursion Bermudan-style callable and convertible bond

Furthermore, for each $i = 1, \dots, M-1$, the rational conversion time after time $t_{i\delta}$ equals

$$\tau_b^*(t_{i\delta}) = \min \left\{ t_k \in \{t_{i\delta}, \dots, t_{(M-1)\delta}\} \mid e^{-rt_k} \frac{\gamma V_{t_k}^+}{m + \gamma n} = CCB(t_k) \right\},$$

the rational call time after time $t_{i\delta}$ equals

$$\tau_s^*(t_{i\delta}) = \min \left\{ t_k \in \{t_{i\delta}, \dots, t_{(M-1)\delta}\} \mid e^{-rt_k} \max \left\{ H + c_{t_k}, \frac{\gamma V_{t_k}^+}{m + \gamma n} \right\} = CCB(t_k) \right\}.$$

Example 4.4.1. (Continuation of Example 4.3.2) The parameters are set to be the same as in Example 4.3.2: $T = 8$, $r = 0.06$, $V(0) = 1000$, $\sigma_V = 0.2$, $K = 300$, $L = 100$, $\gamma = 1.5$, $m = 10$, $n = 8$, $H = 120$. The results are derived for different payout ratios κ and coupons c . In the first case the call level is constant with $H = 120$, while in the second case the call level is time dependent and defined according to Equation (4.22), where $\omega = 0.04$.

The no-arbitrage prices of the game option component of the American- (Am)¹² and Bermudan- (BmM and BmY) style callable and convertible bond are summarized in Ta-

¹²The solution of the no-arbitrage value of an American style game option is achieved by approximating the continuous time problem with a discrete time one, hence strictly it is also the no-arbitrage value of a

		$\omega = 0$			$\omega = 0.04$		
κ	c	$G(0)\text{Am}$	$G(0)\text{BmM}$	$G(0)\text{BmY}$	$G(0)\text{Am}$	$G(0)\text{BmM}$	$G(0)\text{BmY}$
0	0	16.92	17.41	17.44	14.84	15.10	24.40
0.04	2	8.65	8.66	8.52	7.67	7.84	8.23
0.04	3	7.82	7.89	7.96	5.68	5.93	6.83
0.04	4	6.99	7.11	7.43	3.64	3.92	5.15

Table 4.7: Comparison American- and Bermudan-style conversion and call rights (384 steps)

ble 4.7. The values of $G(0)\text{BmM}$ and $G(0)\text{BmY}$ are derived under the condition that the conversion and call are only allowed on a fixed date of each month or year. Interestingly, in our example, $G(0)\text{BmM}$ and $G(0)\text{BmY}$ are more valuable than their American pendant $G(0)\text{Am}$ in almost all cases. The only exception is $\kappa = 0.04$ and $c = 2$, where $G(0)\text{Am}$ is larger than $G(0)\text{BmY}$. The reason is that the value of a game option is determined by strategies of both contract partners. If the bondholder has less chances to convert, this means also that the shareholder has less chances to call and thus to control the maximization strategy of the bondholder. Thus we cannot argue generally that the Bermudan-style contract is always more or less valuable than the American one. Their price differences are more evident, if the call level is low ($\omega = 0.04$), coupons are high ($c = 4$) and less exercise dates are allowed.

4.5 Stochastic Interest Rate

4.5.1 Recursion schema

In this section we solve the optimization problems stated in Equations (4.13) and (4.16) by allowing stochastic interest rate. Similar as in Section 4.3, the continuous time problem is approximated with a discrete time one and the no-arbitrage value is derived by a recursive formula. We discretize the forward price of the firm's value process modeled in Section 2.3. Accordingly, the call level and coupons are adjusted to the forward value. The recursion is carried out on the T -forward adjusted values, see Figure 4.7, where $F_V(t_n^+, T)$ is the forward price of the firm's value just before payout and $CCB_F(t_n)$ is the T -forward value of the callable and convertible bond at time t_n . At the terminal date T , $F_V(T, T) = V_T$ thus $CCB_F(T) = CCB(T)$. ν_{t_n} is the default barrier. The coupon c_{t_n} will only be paid out if the firm's value is above certain level, i.e. $V_{t_n^+} > \eta_{t_n}$. The no-arbitrage price of the callable and convertible bond equals $B(0, T)CCB_F(0)$.

Bermudan-style contract, but with much more exercise chances than the other contracts where conversion and call are only allowed monthly or yearly.

For $n = 0, 1, \dots, N - 1$,

$$CCB_F(t_n) = \begin{cases} \min \left\{ \max \left\{ \frac{H + c_{t_n}}{B(t_n, T)}, \frac{\gamma F_V(t_n^+, T)}{m + \gamma n} \right\}, \right. \\ \left. \max \left\{ \frac{\gamma F_V(t_n^+, T)}{m + \gamma n}, \right. \right. \\ \left. \left. \mathbb{E}_{P^T}[CCB_F(t_{n+1}) | \mathcal{F}_{t_n}] + \frac{c_{t_n}}{B(t_n, T)} \right\} \right\} & \text{if } F_V(t_n^+, T) > \nu_{t_n} \\ \frac{F_V(t_n^+, T)}{n} & \text{if } F_V(t_n^+, T) \leq \nu_{t_n} \end{cases} \quad (4.23)$$

and

$$CCB(T) = \begin{cases} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases} \quad (4.24)$$

Figure 4.7: *Min-max* recursion callable and convertible bond, T -forward value

4.5.2 Some conditional expectations

The recursion formula, Equation (4.23) contains both $F_V(t_n, T)$ and $B(t_n, T)$ as variables. In order to circumvent a two-dimensional tree, we solve $CCB_F(t_n, T)$ as conditional expectation given $F_V(t_n, T)$. To achieve the analytical closed-form solution, we first explore the relationship between $F_V(t, T)$ and $B(t, T)$.

According to the assumptions on the firm's value made in Section 2.3, under P^T the auxiliary forward price of the firm's value $F_V^\kappa(t, T)$ and the T -forward price of the default free zero coupon bond $F_B(t, s, T) := \frac{B(t, s)}{B(t, T)}$, $t \leq s < T$ are both martingales, and satisfy

$$\begin{aligned} dF_V^\kappa(t, T) &= F_V^\kappa(t, T) \cdot \sigma_F(t, T) dW_t^T. \\ dF_B(t, s, T) &= F_B(t, s, T) \cdot \sigma_B(t, s, T) dZ_t^T \end{aligned}$$

with

$$\begin{aligned} \sigma_F^2(t, T) &= \int_0^t \sigma_V^2 + 2\rho\sigma_V b(u, T) + b^2(u, T) du \\ \sigma_B^2(t, s, T) &= \int_0^t (b(u, s) - b(u, T))^2 du \end{aligned}$$

and

$$b(t, s) = \frac{\sigma_r}{b}(1 - e^{-b(s-t)}).$$

W_t^T and Z_t^T are two correlated standard Brownian motion with constant coefficient of correlation equals ρ .

Hence $F_V^\kappa(t, T)$ and $F_B(t, t, T) = \frac{B(t, t)}{B(t, T)} = \frac{1}{B(t, T)}$ are bivariate normally distributed and have the following variances, expectations and covariances¹³

$$\begin{aligned} \sigma_1^2 &:= \mathbb{V}_{PT}[\ln F_V^\kappa(t, T)] &= \int_0^t (\sigma_V^2 + 2\rho\sigma_V b(s, T) + b^2(s, T)) ds \\ \sigma_2^2 &:= \mathbb{V}_{PT}[\ln F_B(t, t, T)] &= \frac{1}{2b^3}(1 - e^{-2bt})b(t, T)^2 \\ \mu_1 &:= \mathbb{E}_{PT}[\ln F_V^\kappa(t, T)] &= \ln F_V^\kappa(0, T) - \frac{1}{2}\sigma_1^2 \\ \mu_2 &:= \mathbb{E}_{PT}[\ln B(t, T)] &= \mathbb{E}[-\ln F_B(t, t, T)] = \ln \frac{B(0, T)}{B(0, t)} + \frac{1}{2}\sigma_2^2 \end{aligned}$$

and

$$\begin{aligned} \gamma &:= \text{Cov}_{PT}(\ln F_V^\kappa(t, T), \ln B(t, T)) \\ &= -\text{Cov}_{PT}(\ln F_V^\kappa(t, T), \ln F_B(t, T)) \\ &= \int_0^t \left(\rho\sigma_V(b(u, T) - b(u, t)) + (b(u, t)b(u, T) - b(u, t)^2) \right) du. \end{aligned}$$

Given these relationships the expectation and variance of $\ln B(t, T)$ conditional on the forward price of the firm's value can be derived with the following formulas

$$\mu_3 := \mathbb{E}\left[\ln B(t, T) \mid \ln F_V^\kappa(t, T) = \bar{w}\right] = \mu_2 + \frac{\gamma}{\sigma_1^2}(\ln \bar{w} - \mu_1), \quad (4.25)$$

$$\sigma_3^2 := \mathbb{V}\left[\ln B(t, T) \mid \ln F_V^\kappa(t, T) = \bar{w}\right] = \sigma_2^2 - \frac{\gamma^2}{\sigma_1^2}. \quad (4.26)$$

Therefore, conditional on $\ln F_V^\kappa(t, T) = \bar{w}$ the random variable $\ln(B(t, T))$ equals

$$\ln B(t, T) \left(\ln F_V^\kappa(t, T) = \bar{w} \right) = \mu_3 + \sigma_3 x$$

where x is a standard normal random variable. Thus the following conditional expecta-

¹³For details see Menkveld and Vorst (2000).

tion can be derived after some elementary integration

$$\mathbb{E}\left[\frac{1}{B(t,T)} \mid \ln F_V^\kappa(t,T) = \bar{w}\right] = \exp\left(-\mu_3 + \frac{1}{2}\sigma_3^2\right) \quad (4.27)$$

$$\begin{aligned} \mathbb{E}\left[\left(\frac{p}{B(t,T)} - q\right)^+ \mid \ln F_V^\kappa(t,T) = \bar{w}\right] &= \int_{-\infty}^h \left(\frac{p}{e^{\mu_3 + \sigma_3 \cdot x}} - q\right) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= p \cdot e^{-\mu_3 + \frac{\sigma_3^2}{2}} N(h + \sigma_3) - q \cdot N(h) \end{aligned} \quad (4.28)$$

with $h = (\ln(p/q) - \mu_3)/\sigma_3$ for some $p, q \in \mathbb{R}_+$. Here, $N(\cdot)$ denotes the cumulative distribution function of a standard normal distribution.

4.5.3 Implementation with binomial tree

For the implementation of the recursion schema displayed in Figure 4.7 we apply the method developed by Menkveld and Vorst (1998) which is a combination of an analytical approach and a one-dimensional binomial tree approach. A simple recombining binomial tree for the forward price $F_V(t, T) := V_t/B(t, T)$ of the firm's value can be constructed with the trick that the interval $[0, T]$ is not divided into periods of equal length, but into periods of equal volatility. Recursion is then carried out alongside the T -forward risk adjusted tree. The interval $[0, T]$ is divided into periods $0 = t_0 < t_1 < \dots < t_N = T$ of equal volatility

$$\sigma_F^N := \frac{1}{N} \int_0^T (\sigma_V^2 + 2\rho\sigma_V b(s, T) + b^2(s, T)) ds.$$

The stochastic evolution of the forward price of the firm's value is then modeled by

$$F_V(n, j) = F(0)u^j d^{n-j} \hat{\kappa}_n, \quad \forall j = 0, \dots, n, \quad n = 1, \dots, N$$

with $F(0) = V(0)/B(0, T)$ and

$$u = e^{\sigma_F^N}, \quad d = e^{-\sigma_F^N}, \quad \hat{\kappa}_n = e^{-\kappa \Delta_n}, \quad \Delta_n = t_n - t_{n-1},$$

where $F_V(n, j)$ denotes the forward price of the firm's value after payout, at time t_n after j up-movements. $F(0)$ is the initial forward price of the firm's value. The expressions show that u and d are time and state independent. $\hat{\kappa}_n$ is time dependent as the time steps are no longer of equal length. The (time dependent) coupon payment is given by $c(n) = c\Delta_n$. The forward martingale measure P^T exists because $d < 1 < u$ and the transition probability is given by

$$p^T := \frac{1-d}{u-d}.$$

Thus the conditional expectation in the recursion schema can be calculated as

$$EV(n, j) := p^T \cdot CCB_F(n+1, j) + (1 - p^T) \cdot CCB_F(n+1, j+1)$$

The forward price of the firm's value at time t_n after j up movements and just before payout is

$$F_V(n+, j) := \frac{F_V(n, j)}{1 - \hat{\kappa}_n}.$$

At each node (n, j) we calculate the expected value of the *min-max* strategy under the measure P^T conditional on the available information $F_V(n, j)$. The calculation is tedious but can be solved analytically. We make first some simplifications of the notations which are only used for the calculation of $CCB_F(n, j)$. $H(n, j)$ and $c(n, j)$ are written as H and c , and

$$CV := \frac{\gamma F_V(n+, j)}{m + \gamma n} \quad EV := EV(n, j)$$

which are conversion and simple recursion value. According to the recursion formula Equation (4.23),

$$\begin{aligned} CCB_F(n, j) &= \min \left\{ \max \left\{ \frac{H+c}{B(t_n, T)}, CV \right\}, \max \left\{ CV, EV + \frac{c}{B(t_n, T)} \right\} \right\} \\ &= \min \left\{ \left[\frac{H+c}{B(t_n, T)} - CV \right]^+ + CV, \left[EV + \frac{c}{B(t_n, T)} - CV \right]^+ + CV \right\} \\ &= CV + \left[EV + \frac{c}{B(t_n, T)} - CV \right]^+ \\ &\quad - \left[\frac{H}{B(t_n, T)} - EV \right]^+ \mathbf{1}_{\left\{ \frac{H+c}{B(t_n, T)} > CV \right\}} \mathbf{1}_{\left\{ EV + \frac{c}{B(t_n, T)} > CV \right\}}. \end{aligned} \quad (4.29)$$

Equation (4.29) can be further calculated in two cases.

(i) $CV \leq EV$

$$CCB_F(n, j) = EV + \frac{c}{B(t_n, T)} - \left[\frac{H}{B(t_n, T)} - EV \right]^+ \quad (4.30)$$

because in this case the second term of Equation (4.29) is certainly positive and $\frac{H}{B(t_n, T)} > CV$ includes also the case $\frac{H+c}{B(t_n, T)} > CV$.

(ii) $CV > EV$

$$\begin{aligned} CCB_F(n, j) &= CV + \left[\frac{c}{B(t_n, T)} - (CV - EV) \right]^+ \\ &\quad - \left[\frac{H}{B(t_n, T)} - EV \right]^+ \mathbf{1}_{\{B(t_n, T) > MIN\}} \end{aligned} \quad (4.31)$$

where

$$MIN := \min \left[\frac{H}{EV}, \frac{H+c}{CV}, \frac{c}{CV-EV} \right].$$

According to the conditional expectations given in Equations (4.27) and (4.28), the analytical solution of Equations (4.30) and (4.31) can be derived as conditional expectations given $F_V^\kappa(n, j) = F_V(n, j)e^{\kappa t_n} = \bar{w}$.

(i) $CV \leq EV$

$$CCB_F(n, j) = EV + c \cdot \exp \left[-\mu_3 + \frac{\sigma_3^2}{2} \right] - H \cdot \exp \left[-\mu_3 + \frac{\sigma_3^2}{2} \right] N(h_1 + \sigma_3) + EV \cdot N(h_1)$$

where

$$h_1 := \frac{\ln \frac{H}{EV} - \mu_3}{\sigma_3}.$$

(ii) $CV > EV$

$$\begin{aligned} CCB_F(n, j) &= CV + c \cdot \exp \left[-\mu_3 + \frac{\sigma_3^2}{2} \right] \cdot N(h_2 + \sigma_3) - (CV - EV)N(h_2) \\ &\quad + H \cdot \exp \left[-\mu_3 + \frac{\sigma_3^2}{2} \right] \cdot N(h_3 + \sigma_3) - EVN(h_3) \end{aligned}$$

where

$$\begin{aligned} h_2 &:= \frac{\ln \frac{c}{CV-EV} - \mu_3}{\sigma_3} \\ h_3 &:= \frac{\ln MIN - \mu_3}{\sigma_3}. \end{aligned}$$

And μ_3 and σ_3 have been defined in Equations (4.25) and (4.26).

In the following numerical example we compute the no-arbitrage price of a callable and convertible bond with stochastic interest rates.

Example 4.5.1. The initial term structure is flat, choose $T = 8$, $\sigma_V = 0.2$, $K = 400$, $\omega = 1300$, $\sigma_r = 0.02$, $b = 0.1$, $V(0) = 1000$, $L = 100$, $K = 400$, $m = 10$, $n = 8$, $H = 120$, $\gamma = 1.5$, $r_0 = 0.06$.¹⁴ The recursions are carried out alongside a tree with 384 steps.

The no-arbitrage prices of a straight bond, a callable and convertible bond and the game option component in American-style with and without stochastic interest rates are presented in Table 4.8. “Non” stands for no interest rate risk, “-0.5” and “0.5” give the

¹⁴The default barrier is $\eta_t = KB(t, T)e^{-\kappa t}$, the same assumption as by European callable and convertible bond. The coupons are to be paid if the firm's value is above $\eta_t = \omega B(t, T)e^{-\kappa t}$.

correlation coefficient of the interest rate and firm's value. The values are derived for different payout and coupon combinations.

		G(0)			CCB(0)			SB(0)		
κ	c	Non	-0.5	0.5	Non	-0.5	0.5	Non	-0.5	0.5
0	0	16.92	15.80	19.07	76.32	76.00	76.49	59.40	60.21	57.41
0.04	2	8.65	7.42	9.97	73.80	73.97	73.40	65.15	66.56	62.35
0.04	4	6.99	6.03	8.88	81.05	82.09	80.29	74.50	76.06	71.41

Table 4.8: No-arbitrage prices of the non-convertible bond, callable and convertible bond and game option component in American-style with stochastic interest rate (384 steps)

Increasing correlation between the interest rate and the firm's value causes increasing volatility of the forward price of the firm's value. The default probability rises with increasing volatility, which results in a reduction of the value of the straight bond $SB(0)$. But on the other side, the value of the game option component $G(0)$ increases in volatility. Therefore in general the total effect is uncertain, in our concrete example the total value declines with increasing correlation. Moreover, the influence of the interest rate risk is relatively small which is recognized by the value of the convertible bond, the results listed in the columns under $CCB(0)$.

Chapter 5

Uncertain Volatility of Firm Value

In practice it is often a difficult problem to calibrate a model to the available data. Here one major drawback of the structural model approach is that it specifies a certain firm's value process. As the firm's value, however, is not always observable, e.g. due to incomplete information, determining the volatility of this process is a non-trivial problem. Moreover, the interest rate risk and the uncertainty about the correlation of the interest rate and firm value process are other contributors to the uncertainty of the volatility.

To relax the assumption of constant volatility of the firm's value, one can specify volatility as a particular function of the firm's value, or model volatility itself with a stochastic process. However, specification of a reasonable model for the volatility dynamics and precise estimation of the parameters would be a difficult task. We circumvent these problems by assuming that the volatility of the firm's value process lies between two extreme values. The volatility is no longer assumed to be constant or a function of underlying and time. It is instead assumed to lie between two extreme values σ_{\min} and σ_{\max} , which can be viewed as a confidence interval for the future volatilities. This assumption is less stringent compared to the approaches where the volatility is modeled as a function of the underlying or as a stochastic process. It needs also less parameter inputs.

Valuation of European-style convertible bonds in this setting can be solved with e.g. the PDE approach proposed independently by Avellaneda et al. (1995) and Lyons (1995). The no-arbitrage pricing bound is derived in the following way: at each time and given the firm's value, the volatility is selected dynamically from the two values σ_{\min} and σ_{\max} in a way that always the one with the worse effect on the value of the convertible bond from the aspect of bondholder or shareholder is chosen, thus to determine the no arbitrage bound. Pricing bound of a European-style convertible bond can also be derived with the probabilistic approach proposed e.g. by Frey (2000). The author shows that by applying time change for continuous martingales, the problem is equivalent to optimal stopping of a corresponding American-style derivative with partial exercise feature under constant volatility, i.e. the optimal stopping time is confined in a time window. One can then

use numerical methods for the pricing of American type securities to solve the valuation problem.

We treat the American-style callable and convertible bond with uncertain volatility by applying the model of Avellaneda et al. (1995) and combining it with the results of Kallsen and Kühn (2005) on game option in incomplete market such that certain pricing bounds can be derived. The bondholder selects the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimist estimate from the aspect of the bondholder. The optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound for the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder and the expectation is also the most pessimist one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived. Same as in case of European convertible bonds the volatility is selected dynamically from the two values σ_{\min} and σ_{\max} in a way that always the one with the worse effect, thus the most pessimist pricing measure is chosen.

The remainder of the chapter is structured as follows. Section 5.1 summarized the solution concepts for models with uncertain volatility. In Section 5.2 they are applied for computing pricing bounds for a European convertible bond. Section 5.3 studies the pricing bounds of an American-style callable and convertible bond.

5.1 Uncertain Volatility Solution Concept

5.1.1 PDE approach

The *uncertain volatility* model on a single asset is first proposed independently by Avellaneda et al. (1995) and Lyons (1995). It is an extension of the Black-Scholes framework to deal with the biased estimate of the historical volatility or the smile effect of the implied volatility¹. Avellaneda et al. (1995) study the case of derivatives written on a single underlying asset. The volatility of the asset is not assumed to be a constant or a function of the underlying or rather stochastic. Instead, it is only assumed to lie between two extreme values σ_{\min} and σ_{\max} , which can be viewed as a confidence interval for the future volatilities. This assumption is less stringent compared to other approaches and it needs also less parameter inputs. The derivation of a no-arbitrage pricing bound is based on a super-hedging strategy which is a worst case estimation. At each (t, x) the volatility is selected dynamically from the two values σ_{\min} and σ_{\max} in a way that always the one with the worse effect on the value of the derivative from aspect of seller or buyer is chosen.

¹The volatility implied from the traded options, plotted as a function of the strike price, often exhibits a specific *U-shape*, which is referred to as the *smile effect*.

For a given martingale measure Q , suppose the stock price evolves according to the following dynamic

$$dS_t = S_t(rdt + \sigma_t dW_t^*),$$

where, for simplification the interest rate r is assumed to be constant. The super-hedge, i.e. the worst case scenario leads to the solution of a non-linear PDE, which is called Black-Scholes-Barenblatt equation

$$\frac{\partial f}{\partial t} + r \left(S \frac{\partial f}{\partial S} - f \right) + \frac{1}{2} \Sigma^2 \left[\frac{\partial^2 f}{\partial S^2} \right] S^2 \frac{\partial^2 f}{\partial S^2} = 0, \quad (5.1)$$

with terminal value $f(S, T) = F(S)$, and $\Sigma^2[x]$ stands for a volatility parameter which depends on x , the convexity of function f . For example, the super-hedge price for the seller of a call option can be obtained by setting

$$\Sigma^2[x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \geq 0 \\ \sigma_{\min}^2 & \text{else.} \end{cases}$$

The authors provide also a simple algorithm for solving the equation by a trinomial tree and prove the convergence of this discrete scheme. In case of vanilla European options, the pricing bounds can be derived simply with the Black-Scholes equations using the extreme values of the volatility parameter, thus the nonlinear solution is reduced to the linear Black-Scholes solution.

Lyons (1995) treats the case of derivatives written on multiple assets. The volatility is assumed to lie in some convex region depending on the prices of the underlying and time. Same as Avellaneda et al. (1995), the volatility matrix is chosen such that the worst effect on the derivative is achieved. However, vanilla European options written on multi-assets, in general, cannot be derived simply by using the extreme values of the volatility parameter. Moreover, it is only possible under particular conditions to reduce the nonlinear solution to the linear Black-Scholes solution.

5.1.2 Probabilistic approach

In one-dimensional case, Frey (2000) shows that by applying time change for continuous martingales, the super-hedge of a European type derivative under uncertain volatility is equivalent to optimal stopping of a corresponding American type derivative with partial exercise feature under constant volatility, i.e. the optimal stopping time is confined in a time window. One can then use numerical methods for the pricing of American type securities to solve the super-delta-hedge problem. We summarize the idea and result. For details of proof see Frey (2000).

The forward price F_t satisfies under the forward martingale measure P^T the stochastic differential equation

$$dF_t = F_t \sigma_t dW_t,$$

or equivalently

$$\ln F_t = \ln F_0 - \int_0^t \frac{1}{2} \sigma_u^2 du + \int_0^t \sigma_u dW_u.$$

Applying the deterministic time change

$$A_t := \int_0^t \sigma_u^2 du,$$

and let A^{-1} stand for the inverse time change, define $\tilde{F}_t := F_{A_t^{-1}}$, then given F_0 ,

$$\tilde{F}_t = F_0 + Z_t - \frac{1}{2}t$$

where Z_t is a standard Brownian motion with $\sigma = 1$ in the time changed filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_{A_t^{-1}}$.² Therefore,

$$\sup_{\sigma \in [\sigma_{\min}, \sigma_{\max}]} E_{P^T}[f(F_T) | \mathcal{F}_0] = \sup_{\tau \in \mathcal{T}_{[\tau_1, \tau_2]}} E_{P^T}[f(\tilde{F}_T) | \tilde{\mathcal{F}}_0]. \quad (5.2)$$

with $\tau_1 = \int_0^T \sigma_{\min}(u)^2 du$, $\tau_2 = \int_0^T \sigma_{\max}(u)^2 du$ and $\mathcal{T}_{[\tau_1, \tau_2]}$ is a set of stopping times with respect to the filtration $\{\tilde{\mathcal{F}}_u\}_{0 \leq u \leq A_T}$.

5.2 Pricing Bounds European-style Convertible Bond

To make the computation tractable, we make some simplifications on the firm's value process described in Section 2.3 and the default mechanism defined in Section 2.4. The interest rate r , the payout rate κ and the default barrier K are assumed to be constant. The volatility of the firm's value lies between two extreme values σ_{\min} and σ_{\max} which are two constant. The firm's value process can thus be described with the following diffusion process

$$dV_t = V_t((r - \kappa)dt + \sigma_t dW_t)$$

and

$$\sigma_{\min} \leq \sigma_t \leq \sigma_{\max}.$$

As an example we examine the upper and lower bound of a European convertible but

²See Revuz and Yor (1991) for details on deterministic time change of Brownian motion.

non-callable bond³, its no arbitrage price should lie between the bounds

$$CB^+(0) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q [cb(0)] \quad (5.3)$$

and

$$CB^-(0) = \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q [cb(0)], \quad (5.4)$$

where

$$\begin{aligned} cb(0) = & \int_0^{\tau \wedge T} c \cdot e^{-rs} ds + \frac{K}{n} \cdot e^{-r\tau} \mathbf{1}_{\{\tau < T\}} \\ & + e^{-rT} \mathbf{1}_{\{T < \tau, V_T > nL\}} \max \left\{ \frac{\gamma V_T}{m + \gamma n}, L \right\} + e^{-rT} \mathbf{1}_{\{T < \tau, V_T \leq nL\}} \frac{V_T}{n}, \end{aligned}$$

and \mathcal{Q} is the family of equivalent martingale measures.

According to Avellaneda et al. (1995), the upper and lower bound $CB^+(0)$ and $CB^-(0)$ can be obtained by solving the Black-Scholes-Barenblatt equation

$$\frac{\partial CB}{\partial t} + (r - \kappa) \left(V \frac{\partial CB}{\partial V} - CB \right) + \frac{1}{2} \Sigma^2 \left[\frac{\partial^2 CB}{\partial V^2} \right] V^2 \frac{\partial^2 CB}{\partial V^2} - c = 0 \quad \text{for } V > K \quad (5.5)$$

with terminal value

$$CB(T, V_T) = \max \left\{ \frac{\gamma V_T}{m + \gamma n}, L \right\} \mathbf{1}_{V_T > nL} + \frac{V_T}{n} \mathbf{1}_{V_T \leq nL},$$

and boundary condition

$$CB(t, K) = e^{-rt} \frac{K}{n}.$$

$\Sigma^2[x]$ stands for a volatility parameter which depends on x . $CCB^+(0)$ is derived by setting

$$\Sigma^2[x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \geq 0 \\ \sigma_{\min}^2 & \text{else} \end{cases} \quad (5.6)$$

$CB^-(0)$ is derived by setting

$$\Sigma^2[x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \leq 0 \\ \sigma_{\min}^2 & \text{else} \end{cases} \quad (5.7)$$

³The upper and lower bound of a European callable and convertible bond can be derived in the same way, we only need to change the terminal value.

Avellaneda et al. (1995) provide also a simple algorithm for solving the Black-Scholes-Barenblatt equation by a trinomial tree. According to this discretization, Equation 5.5 can be solved in the following way. The time interval $[0, T]$ is divided in N subintervals of equal lengths. The distance between two periods is $\Delta = T/N$. After each period Δ , the firm's value will go up, in the middle way, or down, and then has the corresponding value

$$V_{t_{n+1}} = u \cdot V_{t_n}, \quad V_{t_{n+1}} = m \cdot V_{t_n}, \quad V_{t_{n+1}} = d \cdot V_{t_n},$$

where

$$u = e^{\sigma_{\max} \sqrt{\Delta} + (r - \kappa) \Delta}, \quad m = e^{(r - \kappa) \Delta}, \quad d = e^{-\sigma_{\max} \sqrt{\Delta} + (r - \kappa) \Delta}.$$

The so constructed tree is recombining because $m^2 = u \cdot d$. The stochastic evolution of the firm's value is then modeled by

$$V(n, j) = V(0) \cdot e^{j \cdot \sigma_{\max} \sqrt{\Delta} + n \cdot (r - \kappa) \Delta}, \quad \forall j = 0, \dots, 2n, \quad n = 1, \dots, N,$$

where $V(n, j)$ denotes the firm's value at time $t_n := n\Delta$ in state j . At time t_{n+1} there are three possible nodes conditional on (n, j) : in case of an up-movement we have $(n+1, j+1)$, in case of a down-movement $(n+1, j-1)$ and in case of the middle way $(n+1, j)$. Thus higher j indicates a higher firm's value at time t_n . $V(0)$ is the initial firm's value. The transition probability for the up- and down-movement is, respectively, given by

$$\begin{aligned} p_u(p) &:= p \cdot \left(1 - \frac{\sigma_{\max} \sqrt{\Delta}}{2} \right) \\ p_d(p) &= p \cdot \left(1 + \frac{\sigma_{\max} \sqrt{\Delta}}{2} \right) \\ p_m(p) &= 1 - 2p \end{aligned}$$

where the parameter p varies in the range $\sigma_{\min}^2 / (2\sigma_{\max}^2) \leq p \leq 1/2$.⁴ This condition ensures that the uncertain volatility σ takes values such that $\sigma_{\min} \leq \sigma \leq \sigma_{\max}$. The trinomial tree has one degree of freedom at each node, thus the choice of risk-adjusted probabilities is not unique. This freedom is used to model heteroskedasticity. For $p = 1/2$, highest probabilities are assigned to the extreme nodes u and d which yields the largest volatility. While for $p = \sigma_{\min}^2 / (2\sigma_{\max}^2)$ highest probability is assigned to center node m , thus the lowest volatility is achieved. Therefore, by fixing u , d and m and allowing the risk-adjusted probabilities to vary over a one-dimensional set, a range of variances within the volatility band $[\sigma_{\min}, \sigma_{\max}]$ can be modeled.

⁴The transition probabilities depend on p because otherwise we would have a deterministic volatility model.

For each node (n, j) , the upper and lower bound can be calculated as

$$CB^+(n, j) = c\Delta + e^{-r\Delta} \text{Sup}_p [p_u(p)CB^+(n+1, j+1) + p_m(p)CB^+(n+1, j) + p_d(p)CB^+(n+1, j-1)] \quad (5.8)$$

and

$$CB^-(n, j) = c\Delta + e^{-r\Delta} \text{Inf}_p [p_u(p)CB^-(n+1, j+1) + p_m(p)CB^-(n+1, j) + p_d(p)CB^-(n+1, j-1)]. \quad (5.9)$$

Equations(5.8) and (5.9) can be further written in more explicit form

$$CB^+(n, j) = c\Delta + e^{-r\Delta} \begin{cases} CB^+(n+1, j) + \frac{1}{2}Z^+(n+1, j) & \text{if } Z^+(n+1, j) > 0 \\ CB^+(n+1, j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}Z^+(n+1, j) & \text{if } Z^+(n+1, j) \leq 0 \end{cases}$$

and

$$CB^-(n, j) = c\Delta + e^{-r\Delta} \begin{cases} CB^-(n+1, j) + \frac{1}{2}Z^-(n+1, j) & \text{if } Z^-(n+1, j) < 0 \\ CB^-(n+1, j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}Z^-(n+1, j) & \text{if } Z^-(n+1, j) \geq 0 \end{cases}$$

where $Z^+(n+1, j)$ and $Z^-(n+1, j)$ are the approximations of the second-derivative operator Σ^2 in Equation (5.5) and are defined as

$$Z^\pm(n+1, j) := \left(1 - \frac{\sigma_{\max}\sqrt{\Delta}}{2}\right)CB^\pm(n+1, j+1) + \left(1 + \frac{\sigma_{\max}\sqrt{\Delta}}{2}\right)CB^\pm(n+1, j-1) - 2CB^\pm(n+1, j).$$

Example 5.2.1. As a concrete numerical example, we set $T = 8$, $V_0 = 1000$, $L = 100$, $K = 300$, $m = 10$, $n = 8$, $\gamma = 2$, $r = 0.06$.

κ	c	$\sigma_V \in [0.2, 0.4]$		$\sigma_V = 0.2$	$\sigma_V = 0.4$
0.04	2	70.20	79.05	75.72	73.33
0.04	3	73.50	83.96	80.52	76.75
0.04	4	76.72	88.94	85.32	80.16

Table 5.1: Pricing bounds for European convertible bonds with uncertain volatility (384 steps)

Table 5.1 shows that the upper and lower bound of a European convertible bond cannot be derived by using the extreme value of the volatilities. Because it has a mixed convexity,

κ	c	$\sigma_V \in [0.2, 0.4]$		$\sigma_V = 0.2$	$\sigma_V = 0.4$
0.04	2	56.29	72.91	71.70	57.24
0.04	3	59.66	77.86	76.50	60.66
0.04	4	63.01	82.88	81.30	64.08

Table 5.2: Pricing bounds European callable and convertible bonds with uncertain volatility (384 steps)

and the Black-Scholes-Barenblatt equation selects the volatility path that generates the best or worst estimation. The upper and lower bound of a European callable and convertible bond are shown in Table 5.2. They differ only slightly from the prices calculated with the extreme volatilities σ_{\max} and σ_{\min} . The reason is that although the European callable and convertible bond has mixed convexity, but the value of the conversion right is capped with H , and the default probability plays a more important role by the valuation.

Remark 5.2.2. Sometimes the volatility bound is time dependent, for example one can estimate a narrow bound for the near future, but the long-term volatility is hard to estimate and would have a wider interval. In this case, and suppose there are no coupons, the probabilistic approach developed by Frey (2000) would be simple to deal with. Through the time change the process is no longer time dependent and thus simpler to discretize, and the recursion is easy to carry out.

5.3 Pricing Bounds American-style Convertible Bond

The relax of the assumption of deterministic volatility and the adoption of the uncertain volatility introduce market incompleteness. There would be a set of possible equivalent martingale measures which are compatible with the no arbitrage requirement. The holder and issuer of an American callable and convertible bond must not only decide their optimal stopping strategies but also the proper pricing measure.

This problem has been considered by Kallsen and Kühn (2005) in context of game option in incomplete market. Theorem 2.2 of their paper tells us that: suppose that only a *buy-and-hold* strategy is allowed in the game option, while the underlying risky asset and the savings account can be traded dynamically, the set of initial no-arbitrage prices is determined by super hedging and lies in the interval $[G_{low}(0), G_{up}(0)]$ with

$$G_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \quad (5.10)$$

$$G_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[e^{-r(\tau_A \wedge \tau_B)} g(\tau_A, \tau_B)] \quad (5.11)$$

where \mathcal{Q} is the family of equivalent martingale measures, \mathcal{F}_{0T} is the set of stopping

times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$, and $g(\tau_A, \tau_B)$ is defined in Section 4.2.1 by Equation (4.8). The bondholder selects the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, and the expectation is taken with the most pessimistic estimate from the aspect of the bondholder. The optimal strategy of the bondholder and his choice of the pricing measure determine the lower bound of the no-arbitrage price. Whereas the issuer chooses the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This expectation is also the most pessimistic one but from the aspect of the issuer, thus the upper bound of the no-arbitrage price can be derived.

Suppose that the callable and convertible bond is not traded dynamically, applying the results from the theory of game option which are given in Equations (5.10) and (5.11), the set of initial no-arbitrage prices can be determined. It is given by the interval $[CCB_{low}(0), CCB_{up}(0)]$ with

$$CCB_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] \quad (5.12)$$

and

$$CCB_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] \quad (5.13)$$

where \mathcal{Q} is the family of equivalent martingale measures, \mathcal{F}_{0T} is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$, $ccb(0)$ is defined in Section 4.1.1 by Equation (4.4). The upper and lower bound $CCB_{up}(0)$ and $CCB_{low}(0)$ can be derived by solving Equations (5.13) and (5.12) which can be approximated with the recursions demonstrated in Figures 5.1 and 5.2.

Applying the trinomial tree developed by Avellaneda et al. (1995), the expectation in the recursions can be further written in a more explicit form. Define

$$EV^+(t_n) := \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[CCB_{up}(t_{n+1}) | \mathcal{F}_{t_n}]$$

and

$$EV^-(t_n) := \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[CCB_{up}(t_{n+1}) | \mathcal{F}_{t_n}],$$

at each node (n, j)

$$EV^+(n, j) = \begin{cases} CCB_{up}(n+1, j) + \frac{1}{2}Z^+(n+1, j) & \text{if } Z^+(n+1, j) > 0 \\ CCB_{up}(n+1, j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2}Z^+(n+1, j) & \text{if } Z^+(n+1, j) \leq 0 \end{cases}$$

For $n = 0, 1, \dots, N - 1$,

$$CCB_{up}(t_n) = \begin{cases} \min \left\{ e^{-rt_n} \max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\}, \right. \\ \max \left\{ e^{-rt_n} \frac{\gamma V_{t_n}^+}{m + \gamma n}, \right. \\ \left. \left. \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q [CCB_{up}(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ e^{-rt_n} \frac{V_{t_n}^+}{n} & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases}$$

Figure 5.1: Recursion: upper bound for callable and convertible bond by uncertain volatility of the firm's value

and

$$EV^-(n, j) = \begin{cases} CCB_{low}(n + 1, j) + \frac{1}{2} Z^-(n + 1, j) & \text{if } Z^-(n + 1, j) < 0 \\ CCB_{low}(n + 1, j) + \frac{\sigma_{\min}^2}{2\sigma_{\max}^2} Z^-(n + 1, j) & \text{if } Z^-(n + 1, j) \geq 0 \end{cases}$$

where $Z^+(n + 1, j)$ and $Z^-(n + 1, j)$ are the approximations of the second-derivative and are defined as

$$Z^+(n + 1, j) := \left(1 - \frac{\sigma_{\max} \sqrt{\Delta}}{2}\right) CCB_{up}(n + 1, j + 1) + \left(1 + \frac{\sigma_{\max} \sqrt{\Delta}}{2}\right) CCB_{up}(n + 1, j - 1) - 2CCB_{up}(n + 1, j)$$

$$Z^-(n + 1, j) := \left(1 - \frac{\sigma_{\max} \sqrt{\Delta}}{2}\right) CCB_{low}(n + 1, j + 1) + \left(1 + \frac{\sigma_{\max} \sqrt{\Delta}}{2}\right) CCB_{low}(n + 1, j - 1) - 2CCB_{low}(n + 1, j).$$

We show the influence of the uncertain volatility with a numerical example.

Example 5.3.1. Let $T = 8$, $\sigma_{\min} = 0.2$, $\sigma_{\max} = 0.4$, $V = 1000$, $L = 100$, $K = 300$, $m = 10$, $n = 8$, $H = 120$, $\gamma = 1.5$, $r = 0.06$. In Table 5.3 the call level is kept

For $n = 0, 1, \dots, N - 1$,

$$CCB_{low}(t_n) = \begin{cases} \max \left\{ e^{-rt_n} \frac{\gamma V_{t_n}^+}{m + \gamma n}, \min \left\{ e^{-rt_n} \max \left\{ H + c_{t_n}, \frac{\gamma V_{t_n}^+}{m + \gamma n} \right\}, \right. \right. \\ \left. \left. \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q [CCB_{low}(t_{n+1}) | \mathcal{F}_{t_n}] + e^{-rt_n} c_{t_n} \right\} \right\} & \text{if } V_{t_n}^+ > \nu_{t_n} \\ e^{-rt_n} \frac{V_{t_n}^+}{n} & \text{if } V_{t_n}^+ \leq \nu_{t_n} \end{cases}$$

and

$$CCB(T) = \begin{cases} e^{-rT} \max \left\{ \frac{\gamma V_{T^+}}{m + \gamma n}, L + c_{t_N} \right\} & \text{if } V_{T^+} > n(L + c_{t_N}) \\ e^{-rT} \frac{V_{T^+}}{n} & \text{if } V_{T^+} \leq n(L + c_{t_N}) \end{cases}$$

Figure 5.2: Recursion: lower bound for callable and convertible bond by uncertain volatility of the firm's value

constant with H while in Table 5.4 the call level is time dependent with

$$H(t) = e^{-w(T-t)} H, \quad w = 0.04.$$

κ	c	$\sigma_V \in [0.2, 0.4]$ Am	$\sigma_V \in [0.2, 0.4]$ BmY	$\sigma_V = 0.2$	$\sigma_V = 0.4$		
0	0	73.68	78.67	74.00	80.80	76.33	74.90
0.04	2	69.20	75.70	69.16	76.91	73.81	71.60
0.04	3	71.23	79.22	71.49	81.11	77.66	73.35
0.04	4	73.20	82.94	73.84	85.45	81.50	75.08

Table 5.3: Pricing bounds for American callable and convertible bond with uncertain volatility and constant call level H (384 steps)

The pricing bounds for American- and Bermudan-style callable and convertible bonds with uncertain volatility which lies in the interval $[0.2, 0.4]$ are summarized in Tables 5.3 and 5.4. Am and BmY are abbreviations for American and Bermudan-style callable and convertible bond where the latter can only be exercised on the last day of a year. These price bounds are compared with the results if they are calculated with the extreme values of the volatility. Since we chose a relatively wide range of volatilities, $\sigma_{\min} = 0.2$ and $\sigma_{\max} = 0.4$, the price differential of the lower and upper bound is relatively large. Moreover, the lower (upper) bounds are smaller (larger) than the results calculated with extreme volatilities. The Bermudan-style contract has almost the same lower bound as its American-style pendant, while its upper bound is considerably higher, e.g. in Table 5.3

for $\kappa = 0.04$ and $c = 4$ the price bounds are $[73.20, 82.94]$ in the American case and $[73.84, 85.45]$ for the Bermudan case. The difference in upper bounds is more evident.

κ	c	$\sigma_V \in [0.2, 0.4]$ Am		$\sigma_V \in [0.2, 0.4]$ BmY		$\sigma_V = 0.2$	$\sigma_V = 0.4$
0	0	71.97	75.06	72.56	78.90	74.25	72.37
0.04	2	69.04	73.56	69.13	76.00	72.83	70.57
0.04	3	70.26	76.24	70.91	79.38	75.51	71.49
0.04	4	71.20	78.94	72.47	82.56	78.16	72.41

Table 5.4: Pricing bounds for American callable and convertible bond with uncertain volatility and time dependent call level $H(t)$ (384 steps)

The reduction of the call level is achieved in Table 5.4 by making it time dependent. Comparing the results in Tables 5.3 and 5.4, we see that both lower and upper bound are lower in Table 5.4. It is intuitive as the callable and convertible bond is less valuable by a lower call level. The reduction of the call level has larger impact on the upper bound. For example, for $\kappa = 0.04$ and $c = 4$, in American case, the lower bound goes from 73.20 to 71.20 while the upper bound drops from 82.94 to 78.94.

From Section 4.1.2 we know that the callable and convertible bond can be decomposed into a straight bond and a game option component. We could make a naïve computation: calculate the price of the straight bond with σ_{\max} (σ_{\min}) and the price of game option component with σ_{\min} (σ_{\max}), add them together and compare the $sum1$ ($sum2$) with the lower (upper) bound of the callable and convertible bond. The results are listed in Table 5.5.

κ	c	$\sigma_V \in [0.2, 0.4]$ Am		$Sum1$	$Sum2$
0	0	71.97	75.06	64.93	86.25
0.04	2	69.04	73.56	61.03	85.56
0.04	3	70.26	76.24	64.21	88.55
0.04	4	71.20	78.94	67.39	91.53

Table 5.5: Comparison between no-arbitrage pricing bounds and “naïve” bounds

We see that the “naïve” lower bounds $Sum1$ are smaller than the no-arbitrage lower bounds, while the “naïve” upper bounds $Sum2$ are larger than the no-arbitrage upper bounds. It confirms that the callable and convertible bond must be calculated as an entity. One reason is that it contains positions with varying convexity and concavity. In the approach of Avellaneda et al. (1995), however, the selection of the minimum or maximum of the volatility for the valuation depends on the convexity of the entire portfolio. Moreover, both parties can decide when they exercise, therefore each of them must bear the strategy of the other party in mind and the decision is made on the expected value of the aggregated positions.

Chapter 6

Model Framework Reduced Form Approach

In the former chapters convertible bonds are treated within structural approach. The firm's value is modeled as a diffusion process and the liability and equity of the firm are characterized as contingent claims of the firm's value. The liability can be different types of convertible bonds, an interesting case is the American-style callable and convertible bond, where the optimal strategies of the counterparts play an important role and the prices of the liability and equity are results of strategic optimal stopping. Our idealized model has been well-suited and convenient for the analysis of the relative powers of bond- and shareholders and the illustration of the optimal strategies. However, sometimes the true complex nature of the capital structure of the firm and information asymmetry make it hard to model the firm's value and the capital structure. Often the firm's value cannot be observed continuously. Furthermore, if the full set of the liabilities from different creditors of a real firm is to be modeled, the structural model will soon be unfeasible. In this case the reduced-form model is a more proper approach for the study of convertible bonds, and the traded stock price should be used as primary model input.

Instead of asking why the firm defaults, reduced form models treat default as an unpredictable event governed by an exogenous default rate or intensity process. Reduced form models go back to Jarrow and Turnbull (1995), the authors consider the simplest case where the default is driven by a Poisson process with constant intensity. The constant intensity is relaxed in Madan and Unal (1998), and the default arrival rate is characterized as responsive to abnormal equity returns. In Duffie and Singleton (1999) random intensity of the default time is treated with recursive methods and affine model of default is introduced. In Lando (1998) the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity, which is called doubly stochastic Poisson process or Cox process. Since then the intensity-based reduced-form credit risk modeling literature has enjoyed remarkable development. Surveys of the literature are provided e.g. by Duffie and Singleton (2003), Bielecki and Rutkowski (2004) and Schönbucher (2003).

Within the reduced-form approach, stock price, credit spreads and implied volatilities of options are used as model inputs for pricing of convertible bonds. The reason is that stock is a traded asset, credit spreads and implied volatilities are parameters which can be estimated from the market data. One of the early models is proposed by Davis and Lischka (1999). They construct a model framework that incorporate Black-Scholes stock price, Gaussian stochastic interest rate and stochastic default intensity driven by a Brownian motion that also governs the movement of the stock price. It is called *two-and-a-half factors model* and has found its application in the industry. A similar model has been developed by Ayache, Forsyth and Vetzal (2003). Linetsky (2006) and Duffie and Singleton (2003)(p.206ff) model the default intensity as a negative power function of the underlying stock price. In Linetsky (2006) closed-form solutions in form of spectral expansions are derived for European-style derivative securities which are exposed to equity and credit risk simultaneously. Duffie and Singleton (2003) value a callable and convertible bond with the intensity-based default model. In Bielecki et al. (2007) and Kovalov and Linetsky (2008) the default intensity is modeled as a deterministic function of the underlying stock price. The valuation of callable and convertible bond is explicitly related to the defaultable game option and BSDE or PDE is applied to solve the optimization problem.

In order to describe the interplay of the equity risk and the default risk of the issuer, we adopt a parsimonious, intensity-based model, in which the default intensity is modeled as a function of the pre-default stock price. This assumes, in effect, that the pre-default stock price contains sufficient information to judge the credit quality of the firm. To make the combined effect of the default and equity risk of the underlying tractable, it is assumed that the default intensity has two values, one is the normal default rate, and the other one is much higher if the current stock price falls beneath a certain boundary. Thus, during the life time of the bond, the more time the stock price spends below the boundary, the higher the default risk. In this setting, default intensity is strongly influenced by the stock price but they are not perfectly correlated. This model has certain similarity with some structural models. For example, in the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period. However, different as in the case of structural models where the default time is predictable, by reduced form models the default is a sudden event and it is a further source of risk other than the price risks. It may bring incompleteness to the market if there is no defaultable security traded in the market.

6.1 Intensity-based Default Model

In the following we will formulate the default event according to Lando (1998), where the time of default is modeled directly as the time of the first jump of a Poisson process with

random intensity, which is called Cox process.

6.1.1 Inhomogenous poisson processes

An inhomogeneous Poisson process N with intensity function $h(t) > 0$ is a non-decreasing, integer-valued process with independent increments. $N_0 = 0$ and the probability of n jumps in $[s, t]$ is

$$P[N_t - N_s = n] = \frac{1}{n!} \left(\int_s^t h(u) du \right)^n \exp \left\{ - \int_s^t h(u) du \right\}.$$

In particular, the probability of no jumps in $[s, t]$ equals

$$P[N_t - N_s = 0] = \exp \left\{ - \int_s^t h(u) du \right\}.$$

The first jump time of N is

$$\tau = \inf \left\{ t \geq 0 : \int_0^t h(u) du \geq E_1 \right\}.$$

where E_1 is an exponentially distributed random variable with parameter 1.

The compensated Poisson process M_t , $0 \leq t \leq T$

$$M_t := N_t - \int_0^t h(u) du, \quad t \geq 0$$

is a martingale with respect to the filtration $(\mathcal{F}_t^N)_{t \in [0, T]}$ generated by the process N_t , $0 \leq t \leq T$.

6.1.2 Cox process and default time

A Cox process is a generalization of the Poisson process in which the intensity is allowed to be random but in such a way that if it is conditional on a particular realization $h(\cdot, \omega)$ of the intensity, the jump process becomes an inhomogeneous Poisson process with intensity $h(s, \omega)$. The random intensity is often characterized as a function of the current level of a set of state variables

$$h(s, \omega) = h(X_s).$$

X is an \mathbb{R}^d -valued stochastic process in the filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in [0, T]}, Q)$. And $h : \mathbb{R}^d \rightarrow [0, \infty)$ is a nonnegative, continuous function. According to this construction the Cox process has the following properties

$$\mathbb{E}_Q[dN] = h(t)dt$$

and given the realization (path) of the intensity h ,

$$\begin{aligned} P[N_t - N_s = n] &= \mathbb{E}_Q \left[P[N_t - N_s = n] | h \right] \\ &= \mathbb{E}_Q \left[\frac{1}{n!} \left(\int_s^t h(u) du \right)^n \exp \left\{ - \int_s^t h(u) du \right\} \right]. \end{aligned}$$

In particular, the probability of no jumps in $[s, t]$ equals

$$P[N_t - N_s = 0] = \mathbb{E}_Q \left[\exp \left\{ - \int_s^t h(u) du \right\} \right]. \quad (6.1)$$

Lando (1998) models the default time as the first jump time of a Cox process with intensity process $h(X_t)$,

$$\tau = \inf \left\{ t \geq 0 : \int_0^t h(X_s) ds \geq E_1 \right\}.$$

where E_1 is an exponentially distributed random variable with parameter 1. The state variables X may include information about stock price, risk-free interest rate and other economical relevant factors which can predict the likelihood of default. Given that a firm survives till time t , its default probability within the next small time interval Δt equals $h(X_t)\Delta t + o(\Delta t)$. According to Equation (6.1) the survival probability of a firm thus equals

$$P[\tau > t] = \mathbb{E}_Q \left[\exp \left\{ - \int_0^t h(u) du \right\} \right].$$

6.2 Defaultable Stock Price Dynamics

In the Black and Scholes (1973) economy, it is assumed that, in the absence of default risk, the stock price is driven by an n -dimensional standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^*)$. Ω is a set which contains all states of the world, and P^* is the risk neutral probability measure. \mathcal{F} is a σ -algebra of subsets of Ω , and \mathcal{F}_t contains all information about the stock price till time t . The filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$ is a family of σ -algebras and describes the information structure on the stock market, and T denotes a fixed finite time horizon. The dynamics of the stock can be described by the following stochastic differential equation (SDE),

$$dS_t = S_t (r(t) dt + \sigma_t dW^*(t)) \quad (6.2)$$

where $r(t) > 0$ is the risk free instantaneous interest rate and the volatility of the stock price $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}^n$ is an n -dimensional bounded, deterministic function. $\{W^*(t)\}_{t \in [0, T]}$ is a n -dimensional standard Brownian motion under the martingale mea-

sure P^* . Solving the differential Equation (6.2), we obtain

$$S_t = S_0 \exp \left\{ \int_0^t (r(u) - \frac{1}{2} \|\sigma_u\|^2) du + \int_0^t \sigma_u dW^*(u) \right\},$$

where S_0 is the initial stock price.

The literature on stock options usually model the firm's stock price as geometric Brownian motions and preclude the possibility of default. Whereas modeling of default event and credit spread is an essential task of study on corporate bond. Apart from convertible bonds there are also other hybrid products which have both the characteristics of equity and debt. Facing these problems, the two strands of research have merged recently. Default risk is integrated in the diffusion of the stock prices. In the reduced-form framework, one specifies the default intensity as a decreasing function of the underlying stock price. The default event is modeled as the first jump time of a doubly stochastic Poisson process. For example, Linetsky (2006) and Duffie and Singleton (2003) (p.206ff) model the default intensity as a negative power function of the underlying stock price. This assumes, in effect, that the equity price conveys sufficient information for the prediction of the default probability.

In the following, the dynamic of the defaultable stock prices will be introduced. The Brownian motion which governs the movement of the stock prices is assumed to be 1-dimensional¹. The model framework is established according to Linetsky (2006).

Assumption 6.2.1. A filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, Q)$ where $\mathbb{G} := \{\mathcal{G}_t\}_{t \in [0, T]}$ is assumed. It supports a 1-dimensional Brownian motion $\{W_t, t \geq 0\}$, and an exponentially distributed random variable E_1 with parameter 1. The random variable E_1 is independent of the Brownian motion W . The stock price process S is subject to default. The pre-default stock price is denoted as \tilde{S}_t . The default intensity is specified as a decreasing function of the underlying stock price, and is denoted as $h(\tilde{S})$ where $h : \mathbb{R} \rightarrow [0, \infty)$ is a nonnegative, continuous function. The default time τ is modeled as

$$\tau = \inf \left\{ t \geq 0 : \int_0^t h(\tilde{S}_u) du \geq E_1 \right\}. \quad (6.3)$$

It corresponds to the first jump time of a doubly stochastic Poisson process with intensity $h(\tilde{S}_t)$. Take an equivalent martingale measure Q as given. Under Q , the pre-default stock price \tilde{S}_t is a diffusion process solving the following stochastic differential equation

$$d\tilde{S}_t = (r_t + h(\tilde{S}_t))\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t, \quad (6.4)$$

¹It is a rough approximation of the reality but it makes the computation tractable and closed-form solution can be derived.

where r_t is the risk-free instantaneous interest rate and σ_t is the volatility of the pre-default stock price. Furthermore, it is assumed that if the firm defaults the stock price jumps to zero. Therefore the price process of the defaultable stock S follows the jump diffusion

$$dS_t = S_{t-}(r_t dt + \sigma_t dW_t - dM_t), \quad (6.5)$$

with

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} h(\tilde{S}_u) du,$$

which is a martingale with respect to the filtration \mathbb{G} .

Assumption 6.2.2. In particular, we assume that the intensity function $h(\tilde{S}_t)$ has two values

$$h(\tilde{S}_t) = \begin{cases} a & \text{if } \tilde{S}_t \leq K \\ b & \text{if } \tilde{S}_t > K \end{cases} \quad (6.6)$$

where a , b and K are constant and $a > b > 0$.

The firm has a normal default intensity b . If the firm is in trouble, i.e. the stock price is lower than the constant level K , it has a higher default rate a . Thus, during the life time of the bond, the more time the stock price spends below the boundary, the higher the default risk. Thus, the default intensity is strongly influenced by the stock price but they are not perfectly correlated. Moreover, this model has certain similarity with some structural models, e.g. in the first-passage approach, the firm defaults immediately when its value falls below the boundary, while in the excursion approach, the firm defaults if it reaches and remains below the default threshold for a certain period.

Linetsky (2006) and Duffie and Singleton (2003)(p.206ff) model the default intensity as a negative power function of the underlying stock price. In Linetsky (2006) closed-form solutions in form of spectral expansions are derived for bonds and stock options. The expansions contain several special functions and integration of them. In both papers, the parameters of the negative power function are chosen in the way that, there is a small region, if the stock price is above it, the default probability is quite low. As soon as the stock price goes below this region, the default probability rises dramatically. Therefore our simple assumption can be seen as an approximation of the power function modeling.

6.3 Information Structure and Filtration Reduction

At first, we will explain the information structure due to the interplay of the stock and default risk. According to assumption 6.2.1 on the stock price and the default intensity, the information about the aforementioned two risks is contained in the full-filtration \mathbb{G} ,

which is composed of two sub-filtrations

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H},$$

where $\mathbb{G} := \{\mathcal{G}_t\}_{t \in [0, T]}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. $\mathcal{F}_t = \sigma\{\tilde{S}_s : 0 \leq s \leq t\}$ contains information about the evolution of the pre-default stock price \tilde{S}_t . In our model the default intensity $h(\tilde{S}_t)$ depends only on the pre-default stock price \tilde{S}_t , and there are no other state variables involved, therefore, the information about the likelihood of the default is given by \mathcal{F}_t . $\mathcal{H}_t = \sigma\{\mathbf{1}_{\tau \leq s} : 0 \leq s \leq t\}$ holds the information whether there has been a default till time t . $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ then corresponds to knowing the evolution of the stock price up to time t and whether default has occurred or not. E_1 is independent of sigma field \mathcal{F}_T and $\mathcal{H}_t \subseteq \sigma(E_1)$. In this information setting,

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t \vee \sigma(E_1). \quad (6.7)$$

Under such construction of filtration, it has been shown in Lando (1998) that, under some measurable conditions, the expectations with respect to \mathcal{G}_t can be reduced to the expectation with respect to \mathcal{F}_t . There are three basic components for the valuation of default contingent claims: *promised payment* X at expiry, *a stream of payments* $Y_s \mathbf{1}_{\tau > s}$ which stops when default occurs and *recovery payment* Z_τ at time of default. In particular for convertible bonds the expiry time can be the maturity date T , the conversion or call time τ_b or τ_s , which is written as $\tilde{T} = \tau_b \wedge \tau_s \wedge T$. For a given equivalent martingale measure Q , the expected value of these three basic components are:

$$\mathbb{E}_Q \left[\exp \left(- \int_t^{\tilde{T}} r_s ds \right) X \mathbf{1}_{\tau > T} \middle| \mathcal{G}_t \right] = \mathbf{1}_{\tau > t} \mathbb{E}_Q \left[\exp \left(- \int_t^{\tilde{T}} (r_s + h_s) ds \right) X \middle| \mathcal{F}_t \right], \quad (6.8)$$

$$\mathbb{E}_Q \left[\int_t^{\tilde{T}} Y_s \mathbf{1}_{\tau > s} \exp \left(- \int_t^s r_u du \right) ds \middle| \mathcal{G}_t \right] = \mathbf{1}_{\tau > t} \mathbb{E}_Q \left[\int_t^{\tilde{T}} Y_s \exp \left(- \int_t^s (r_u + h_u) du \right) ds \middle| \mathcal{F}_t \right], \quad (6.9)$$

and

$$\mathbb{E}_Q \left[\exp \left(- \int_t^\tau r_s ds \right) Z_\tau \middle| \mathcal{G}_t \right] = \mathbf{1}_{\tau > t} \mathbb{E}_Q \left[\int_t^{\tilde{T}} Z_s h_s \exp \left(- \int_t^s (r_u + h_u) du \right) ds \middle| \mathcal{F}_t \right], \quad (6.10)$$

Where X is $\mathcal{F}_{\tilde{T}}$ measurable², i.e. $X \in \mathcal{F}_{\tilde{T}}$. Y and Z are adapted processes, i.e. Y_t and Z_t are measurable for each $t \in [0, \tilde{T}]$. h_u is the abbreviation of $h(\tilde{S}_u)$ and stands for the default intensity. The *lhs* (left hand sides) of Equations (6.8), (6.9) and (6.10) show that, in the original market subject to default risk, cash flows are discounted according to the risk free discount factor $\exp(-\int_s^t r_u du)$. With the help of filtration reduction we

²Note that τ_b and τ_s can be any time in the interval $[0, T]$. The measurable condition is satisfied because conversion and call payoff are adapted processes.

move to the fictitious default-free market in which cash flows are discounted according to the modified discount factor $\exp(-\int_s^t(r_u + h_u)du)$. This effect is demonstrated by the *rhs* (right hand sides) of Equations (6.8), (6.9) and (6.10).

Remark 6.3.1. If the market is complete, e.g. the defaultable stock and defaultable discount bond with maturity T are tradeable, there exists a unique martingale measure P^* for the valuation. In incomplete market, the equivalent martingale measure Q can e.g. be the so-called minimal martingale measure introduced by Föllmer and Schweizer (1990) or the minimal entropy martingale measure proposed by Frittelli (2000). The former measure emerges from the mean-variance optimal hedging strategy which minimizes the variance between the random payoff and the terminal wealth generated from a self-financing strategy. Whereas the latter minimizes the relative entropy to the original objective measure P . Both measures have the nice property that zero risk premium is associated with default timing risk, i.e. the risk-neutral intensity under Q remains the same as the original intensity under P . Details about these results can be found e.g. in Blanchet-Scalliet, El Karoui and Martellin (2005).

Chapter 7

Mandatory Convertible Bond

Mandatory convertibles are equity-linked hybrid securities. The coupon rate of a mandatory convertible is usually higher than the dividend rate of the stock. Given no default, at maturity the bond converts mandatorily into a number of stocks if the stock price lies below a lower strike level. The holder will exercise the conversion right if the stock price lies above an upper strike level. Typically the bondholder is subject to the full downside risk of the stock, while he can only participate partially in the upside potential of the stock. Usually they have a maturity of 3-5 years. They are issued by the firms to raise capital, usually in times when the placement of new stocks are not advantageous. Empirically, it can be observed that firms that issue mandatory convertibles tend to be highly leveraged. They intend to improve the future ratings by issuance of mandatory convertibles.

In some literature it is argued that, due to the offsetting nature of the embedded option spread, a change in volatility has only unnoticeable effects on the value of the mandatory conversion. Therefore, the influence of the volatility on the no-arbitrage price is limited. But in the following we will show that if the default intensity is explicitly linked to the stock price, the impact of the volatility can no longer be neglected.

The remainder of the chapter is structured as follows. We start in section 7.1 with a description of the contract feature and in section 7.2 the mandatory convertible bond is valued in a default-free complete market. Section 7.3 aims to treat the joint effect of equity and default risk. Finally, section 7.4 relaxes the assumption of constant volatility and no-arbitrage pricing bound is determined.

7.1 Contract Feature

A typical payoff of mandatory convertible bond at the maturity is shown by figure (7.1). Formally the payoff at maturity can be summarized as $\max\{\min\{\gamma_1 S_T, L\}, \gamma_2 S_T\}$, with conversion ratio $\frac{L}{K_l} =: \gamma_1 > \gamma_2 := \frac{L}{K_u}$. The payoff can be further decomposed into 1

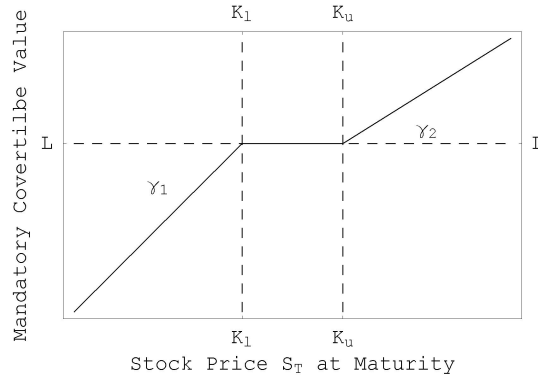


Figure 7.1: Payoff of mandatory convertible bond at maturity

long position in the principal L , γ_1 short position of put with lower strike K_l , and γ_2 long position of call with upper strike K_u . Zero recovery of the bond is assumed¹, thus the total payoff of the bond is the sum of the mandatory conversion value and the coupon payments. Assume that the bondholder receives coupons at discrete time points $0 = t_0 < t_i < \dots < t_N = T$, and the coupon rate is constant, therefore the discounted payoff of a mandatory convertible coupon bond at time t amounts to

$$\begin{aligned}
 mcb(t) &= c \cdot \sum_{i=[t]+1}^N \beta(t, t_i) \mathbf{1}_{\{t_i < \tau\}} + \beta(t, T) \max\{\min\{\gamma_1 S_T, L\}, \gamma_2 S_T\} \mathbf{1}_{\{T < \tau\}} \quad (7.1) \\
 &= c \cdot \sum_{i=[t]+1}^N \beta(t, t_i) \mathbf{1}_{\{t_i < \tau\}} + \beta(t, T) \{L - [L - \gamma_1 S_T]^+ + [\gamma_2 S_T - L]^+\} \mathbf{1}_{\{T < \tau\}},
 \end{aligned}$$

where c is the coupon rate, L is the principal and $\beta(s, t) = \exp\{-\int_s^t r(u) du\}$ is the discount factor, where $r(t)$ is the risk-free instantaneous interest rate, $[t]$ denotes the integer part of t , and $[x]^+$ stands for $\max[x, 0]$. Its no-arbitrage price under the equivalent martingale measure Q is

$$MCB(t) = \mathbb{E}_Q[mcb(t)]. \quad (7.2)$$

7.2 Default-free Market

The mandatory convertible bond is exposed to equity, interest and default risk. At the first step, we ignore the default risk and value the mandatory convertible bond in a traditional Black-Scholes model with constant interest rate r . The no-arbitrage price of

¹It is an economical reasonable assumption because the mandatory bonds are junior debt with low priority.

a mandatory convertible bond amounts to

$$\begin{aligned}
 MCB(t) = & c \sum_{i=[t]+1}^N e^{-r(t_i-t)} + e^{-r(T-t)} L \left\{ N\left(-d_1\left(\frac{L}{\gamma_1}, t\right)\right) - N\left(-d_1\left(\frac{L}{\gamma_2}, t\right)\right) \right\} \\
 & + \gamma_1 S_t N\left(d_2\left(\frac{L}{\gamma_1}, t\right)\right) + \gamma_2 S_t N\left(-d_2\left(\frac{L}{\gamma_2}, t\right)\right),
 \end{aligned} \tag{7.3}$$

where

$$\begin{aligned}
 d_1(x, t) & := \frac{\ln x - \ln S_t - r(T-t) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\
 d_2(x, t) & := d_1 - \sigma\sqrt{T},
 \end{aligned}$$

and $N(\cdot)$ denotes the cumulative normal distribution.

Example 7.2.1. The prices of different bonds with parameters $T = 4$, $S_0 = 100$, $L = 100$, $K_l = 100$, $r = 0.06$, $c = 6$ are shown in the figure 7.2.

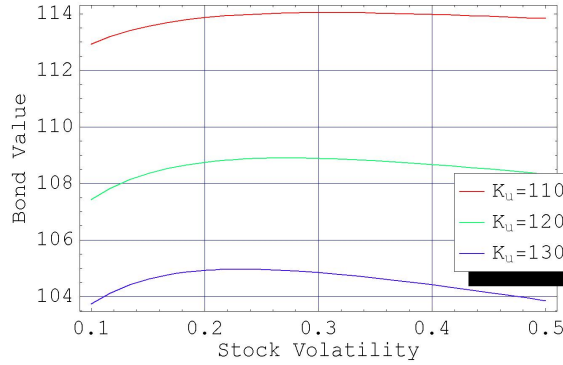


Figure 7.2: Value of mandatory convertible bond by different stock volatilities and different upper strike prices

We can observe that the price of the mandatory convertible bond is not monotonic to the change of the volatilities, and sensitive to the choice of the upper strike price. In this setting, the argument is justified that due to the offsetting nature of the embedded option spread, a change in volatility has only a slight effect on the no-arbitrage value of the mandatory convertible bond. But the situation will change if the default risk, especially the combined effect of default risk and equity risk is taken into account.

7.3 Default Risk

The combined effect of default risk and equity risk of the underlying will be demonstrated with a parsimonious model. The price dynamic of a defaultable stock is modeled in Section 6.2, i.e. according to the Assumptions 6.2.1 and 6.2.2. Moreover, constant stock volatility and interest rate are assumed.

Under an equivalent martingale measure Q the pre-default stock price \tilde{S}_t follows a diffusion process solving the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t &= (r + h(\tilde{S}_t))\tilde{S}_t dt + \sigma\tilde{S}_t dW_t \\ &= (r + a - (a - b)\mathbf{1}_{\{\tilde{S}_t > K\}})\tilde{S}_t dt + \sigma\tilde{S}_t dW_t \end{aligned} \quad (7.4)$$

The price process of the defaultable stock S follows the jump diffusion

$$dS_t = S_{t-}(r dt + \sigma dW_t - dM_t). \quad (7.5)$$

The bond defaults at the time the stock price jumps to zero. Moreover, zero recovery of the bond is assumed. In the following sections we will calculate the no-arbitrage price of a mandatory convertible bond which payoff is described with Equation (7.1).

7.3.1 Change of measure

For derivation of the expected value of a mandatory convertible bond written on a defaultable stock we need the survival distribution and joint distribution of survival probability and terminal value at the maturity. First we define the auxiliary process

$$Y_t := \frac{\ln \tilde{S}_t}{\sigma} = \left(\frac{r + a - \frac{\sigma^2}{2} - (a - b)\mathbf{1}_{\{Y_u > h\}}}{\sigma} \right) dt + dW_t \quad (7.6)$$

where $h := \frac{\ln K}{\sigma}$.

Girsanov transform is used to remove the drift term of \tilde{S}_t . The relationship between the original and new probability measure is

$$Q|_{\mathcal{G}_t} = Z_t \cdot \tilde{Q}|_{\mathcal{G}_t} \quad (7.7)$$

with

$$Z_t = \exp \left(\int_0^t \left(z - \frac{f(Y_u)}{\sigma} \right) d\tilde{W}_u - \frac{1}{2} \int_0^t \left(z - \frac{f(Y_u)}{\sigma} \right)^2 du \right)$$

where

$$\begin{aligned} z &:= \frac{r + a - \frac{\sigma^2}{2}}{\sigma} \\ f(Y_t) &:= (a - b)\mathbf{1}_{\{Y_t > h\}}. \end{aligned}$$

\tilde{W}_t is a standard Brownian motion under \tilde{Q} and satisfies

$$\tilde{W}_t = W_t + \int_0^t \left(z - \frac{f(Y_u)}{\sigma} \right) du.$$

Under the new measure \tilde{Q} the auxiliary process Y_t is a Brownian motion without drift.

The Tanaka formula states that for $d \in \mathbb{R}$ and a standard Brownian motion B_t ,

$$(B_t - d)^+ = (-d)^+ + \int_0^t \mathbf{1}_{\{B_s > d\}} dB_s + \frac{1}{2} L_t^d$$

where L_t^d is the local time of a standard Brownian motion at the level d

$$L_t^d := \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|B_s - d| \leq \epsilon\}} ds.$$

Lemma 7.3.1. (Atlan, Geman and Yor (2006)) Using the Tanaka formula, the martingale Z_t can be expressed in terms of Brownian motion at time t and its local and occupation time till time t .

$$Z_t = \exp(2\lambda(-d_K)^+) \phi(\tilde{W}_t) \exp(\lambda L_t^{d_K}) \exp(-\alpha_+ \Gamma_t^{(d_K, +)} - \alpha_- \Gamma_t^{(d_K, -)}) \quad (7.8)$$

with

$$\begin{aligned} d_K &:= \frac{\ln \frac{K}{S_0}}{\sigma}, & \lambda &:= \frac{a - b}{2\sigma} \\ \alpha_+ &:= 2\lambda^2 + \frac{z^2}{2} - 2\lambda z, & \alpha_- &:= \frac{z^2}{2} \\ \phi(x) &:= \exp(zx - 2\lambda(x - d_K)^+), \end{aligned}$$

and

$$\Gamma_t^{(d, +)} := \int_0^t \mathbf{1}_{\{B_s \geq d\}} ds, \quad \Gamma_t^{(d, -)} := \int_0^t \mathbf{1}_{\{B_s \leq d\}} ds \quad (7.9)$$

denote the time spent by the standard Brownian motion till time t in interval $[d, \infty)$ and $(-\infty, d]$ respectively, which are the occupation times.

7.3.2 Valuation of coupons

The survival probability till time t under the original equivalent martingale measure Q can be expressed as

$$P[t < \tau] = \mathbb{E}_Q \left[e^{-\int_0^t h(\tilde{S}_u) du} \right] = \mathbb{E}_Q \left[e^{-\int_0^t a - (a-b) \mathbf{1}_{\{Y_t > h\}} du} \right].$$

Under the new measure \tilde{Q}

$$P[t < \tau] = \mathbb{E}_{\tilde{Q}} \left[Z_t \cdot e^{-\int_0^t a - (a-b) \mathbf{1}_{\{Y_t > h\}} du} \right].$$

After inserting equation(7.8) and some simple calculations we obtain,

$$P[t < \tau] = e^{2\lambda(-d_K)^+ - at} \mathbb{E}_{\tilde{Q}} \left[\phi(\tilde{W}_t) \exp(\lambda L_t^{d_K}) \exp(-\tilde{\alpha}_+ \Gamma_t^{(d_K, +)} - \alpha_- \Gamma_t^{(d_K, -)}) \right] \quad (7.10)$$

with

$$\tilde{\alpha}_+ = \alpha_+ - (a - b).$$

We assume first that $S_0 > K$, thus $d_K < 0$ ². It means that, at the inception of the contract, the stock price lies above the critical level and the firm is not in trouble. The calculation of equation (7.10) can be decomposed into two cases:

- (I) the level K is *never touched* during the life of the contract,
- (II) the level K is *touched* during the life of the contract.

Thus equation (7.10) can be expressed as

$$Surv(t) := P[t < \tau] = SurvI(t) + SurvII(t) \quad (7.11)$$

with

$$SurvI(t) := e^{-2\lambda d_K - (\alpha_+ + b)t} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_t > d_K\}} \phi(\tilde{W}_t) \right]$$

and

$$SurvII(t) := e^{-2\lambda d_K - at} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_t \leq d_K\}} \phi(\tilde{W}_t) \exp(\lambda L_t^{d_K}) \exp(-\tilde{\alpha}_+ \Gamma_t^{(d_K, +)} - \alpha_- \Gamma_t^{(d_K, -)}) \right]$$

where $\phi(x) = \exp(zx - 2\lambda(x - d_K)^+)$ and m_t denotes the minimum of the standard Brownian motion \tilde{W} till time t .

Lemma 7.3.2. According to the joint density of Brownian motion and its running maxima (see Proposition 8.1 Karatzas and Shreve (1991), p.95) and the symmetric of the Brownian motion, the joint density of Brownian motion at time t and its minimum till

²The case $d_K > 0$ can be solved in the similar way.

time t , is given as, for $y \leq 0$ and $x \geq y$,

$$P[W_t \in dx, m_t \in dy] = \frac{2(x-2y)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y-x)^2}{2t}\right) dx dy.$$

After integration with respect to x we obtain,

$$\begin{aligned} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_t > d_K\}} \phi(\tilde{W}_t) \right] &= \int_{-\infty}^{-\frac{d_K}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \exp(2\lambda d_K) \cdot \exp\left(-\frac{y^2}{2} + (2\lambda\sqrt{t} - z\sqrt{t})y\right) dy \\ &\quad - \int_{-\infty}^{\frac{d_K}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \exp(2(z-\lambda)d_K) \cdot \exp\left(-\frac{y^2}{2} + (2\lambda\sqrt{t} - z\sqrt{t})y\right) dy. \end{aligned}$$

Then integrate with respect to y , we obtain

$$\begin{aligned} SurvI(t) &= \exp\left(\left(\frac{(2\lambda-z)^2}{2} - \alpha_+ - b\right)t\right) \\ &\quad \times \{N(d_1(t, d_K)) - \exp(2(z-2\lambda)d_K) \cdot N(d_1(t, -d_K))\} \end{aligned} \quad (7.12)$$

with

$$d_1(t, x) := (z-2\lambda)\sqrt{t} - \frac{x}{\sqrt{t}}$$

and $N(\cdot)$ denote the cumulative distribution function of a standard normal distribution.

For derivation of the value of $SurvII(t)$ we use the result stated in the following Lemma.

Lemma 7.3.3. (Atlan et al. (2006) Proposition II.15) For a standard Brownian motion W_t , any function $\phi \in L^1(\mathbb{R})$, $L_t^{d_K}$ denotes its local time at level $d_K \leq 0$, m_t denotes its minimum, $\Gamma_t^{d_K,+}$ and $\Gamma_t^{d_K,-}$ denote the times spent above and below the level d_K till time t . The laplace transform of the function

$$g(t) := \mathbb{E} \left[\mathbf{1}_{\{m_t \leq d_K\}} \phi(W_t) \exp(\lambda L_t^{d_K}) \exp(\alpha_+ \Gamma_t^{d_K,+} - \alpha_- \Gamma_t^{d_K,-}) \right]$$

with respect to the maturity time t is given as

$$\begin{aligned} \hat{g}(\theta) &:= \int_0^\infty e^{-\theta t} g(t) dt \\ &= \frac{2e^{d_K \sqrt{2(\theta+\alpha_+)}} \int_0^\infty e^{-x \sqrt{2(\theta+\alpha_+)}} \phi(d_K+x) dx + \int_0^\infty e^{-x \sqrt{2(\theta+\alpha_-)}} \phi(d_K-x) dx}{\sqrt{2(\theta+\alpha_+)} + \sqrt{2(\theta+\alpha_-)} - 2\lambda}. \end{aligned} \quad (7.13)$$

Insert $\phi(x) = \exp(zx - 2\lambda(x-d_K)^+)$ in the former equation and after some elementary

calculations the Laplace transform of $SurvII(t)$ can be stated as

$$\widehat{SurvII}(\theta) = 2 \left(\frac{S_0}{K} \right)^{\frac{2\lambda - z - \sqrt{2(\theta + b + \alpha_+)}}{\sigma}} \frac{\frac{1}{\sqrt{2(\theta + b + \alpha_+) + 2\lambda - z}} + \frac{1}{\sqrt{2(\theta + a + \alpha_-) + z}}}{\sqrt{2(\theta + b + \alpha_+) + 2\lambda - z} + \sqrt{2(\theta + a + \alpha_-) - 2\lambda}}, \quad (7.14)$$

where θ must be sufficient large to satisfies the both conditions

$$\begin{aligned} \theta &> \frac{(z - 2\lambda)^2}{2} - (\alpha_+ + b) \\ \theta &> \frac{z^2}{2} - (\alpha_- + a). \end{aligned}$$

The survival probability $SurvII(t)$ can be derived by inverting $\widehat{SurvII}(\theta)$ numerically.

Notation 7.3.4. We use the EULER algorithm introduced by Abate and Whitt (1995) for inversion of Laplace transform throughout the paper. It is a Fourier-series method and Euler summation is employed to accelerate the convergence.

Proposition 7.3.5. In summary, for $d_k \leq 0$, the no-arbitrage value of coupon payments is,

$$\mathbb{E}_Q \left[c \cdot \sum_{i=1}^N e^{-rt_i} \mathbf{1}_{\{t_i < \tau\}} \right] = c \cdot \sum_{i=1}^N e^{-rt_i} Surv(t_i) \quad (7.15)$$

where $Surv(t_i)$ can be computed with equations(7.11), (7.12) and (7.14).

7.3.3 Valuation of terminal payment

The expected value of the payment of mandatory convertible bond at maturity is composed of three parts, the principal payment, γ_1 short position of put with lower strike K_l and γ_2 long position of call with upper strike K_u . The following calculation is carried out for the case that the following three conditions are satisfied,

- $d_K < 0$, the stock price lies above the critical level and the firm is not in trouble.
- $S_0 > K_l$, at inception of the contract, the put component is out of money.
- $S_0 < K_u$, at inception of the contract, the call component is out of money.

Value of principal

Proposition 7.3.6. The expected value of the principal payment is,

$$e^{-rT} \mathbb{E}_Q \left[e^{-\int_0^T h(\tilde{S}_u) du} L \right] = e^{-rT} L \cdot Surv(T) \quad (7.16)$$

where $Surv(T)$ can be computed with help of equations(7.11), (7.12) and (7.14).

Value of put component

Denote the expected value of *put component* as

$$\begin{aligned} PC(T) &:= e^{-rT} \mathbb{E}_Q \left[e^{-\int_0^T h(\tilde{S}_u) du} (L - \gamma_1 \tilde{S}_T)^+ \right] \\ &= e^{-rT} \mathbb{E}_{\tilde{Q}} \left[Z_T \cdot e^{-\int_0^T h(\tilde{S}_u) du} (L - \gamma_1 \tilde{S}_T)^+ \right]. \end{aligned} \quad (7.17)$$

The valuation of equation (7.17) is decomposed into two cases:

- (I) the level K is *never touched* during the life of the contract,
- (II) the level K is *touched* during the life of the contract.

Thus equation (7.17) can be expressed as

$$PC(T) = PCI(T) + PCII(T) \quad (7.18)$$

with

$$PCI(T) := e^{-2\lambda d_K - r(\alpha_+ + b)T} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_T > d_K\}} \phi(\tilde{W}_T) \right]$$

and

$$PCII(T) := e^{-2\lambda d_K - (a+r)T} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_T \leq d_K\}} \phi(\tilde{W}_t) \exp(\lambda L_T^{d_K}) \exp(-\tilde{\alpha}_+ \Gamma_T^{(d_K, +)} - \alpha_- \Gamma_T^{(d_K, -)}) \right]$$

where $\phi(x) = e^{zx - 2\lambda(x - d_K)^+} (L - \gamma_1 S_0 e^{\sigma x})^+$.

Proposition 7.3.7. Under assumptions that $d_K < 0$ and $S_0 > K_l$,

$$\begin{aligned} PCI(T) = & L \cdot K_1(T) \cdot K_2(T) \left\{ (N(d_1(T, d_K)) - N(d_1(T, d_{L_1}))) \right. \\ & \left. - K_4(N(d_1(T, -d_K)) - N(d_1(T, -d_{\gamma_1}))) \right\} \\ & - \gamma_1 S \cdot K_3(T) \left\{ (N(d_2(T, d_K)) - N(d_2(T, d_{L_1}))) \right. \\ & \left. - K_5(N(d_2(T, -d_K)) - N(d_2(T, -d_{\gamma_1}))) \right\}, \end{aligned} \quad (7.19)$$

where

$$\begin{aligned}
d_{L_1} &:= \frac{\ln \frac{K_1}{S_0}}{\sigma} & d_{\gamma_1} &:= \frac{\ln \frac{K^2}{K_1 S_0}}{\sigma} \\
d_1(t, x) &:= (z - 2\lambda)\sqrt{t} - \frac{x}{\sqrt{t}} & d_2(t, x) &:= (z - 2\lambda + \sigma)\sqrt{t} - \frac{x}{\sqrt{t}} \\
K_1(t) &:= \exp(-(\alpha_+ + b)t) & K_2(t) &:= \exp\left(\frac{(2\lambda - z)^2 t}{2}\right) \\
K_3(t) &:= \exp\left(\frac{(2\lambda - z - \sigma)^2 t}{2}\right) & K_4 &:= \exp(2(z - 2\lambda)d_K) \\
K_5 &:= \exp(2(z - 2\lambda + \sigma)d_K).
\end{aligned}$$

the Laplace Transform of $PCII(t)$ is,

$$\widehat{PCII}(\theta) = \frac{2M(\theta)(Z_1(\theta) - Z_2(\theta) + Z_3(\theta) - Z_4(\theta))}{N(\theta)} \quad (7.20)$$

where

$$\begin{aligned}
\mu &:= r + b + \alpha_+, & \nu &:= r + a + \alpha_-. \\
M(\theta) &= \left(\frac{K}{S_0}\right)^{\frac{z - 2\lambda + \sqrt{2(\theta + \mu)}}{\sigma}} & N(\theta) &= \sqrt{2(\theta + \mu)} + \sqrt{2(\theta + \nu)} - 2\lambda \\
Z_1(\theta) &= \frac{L}{\sqrt{2(\theta + \mu)} + 2\lambda - z} \left(1 - \left(\frac{\gamma_1 K}{L}\right)^{\frac{2\lambda - z + \sqrt{2(\theta + \mu)}}{\sigma}}\right) \\
Z_2(\theta) &= \frac{\gamma_1 K}{\sqrt{2(\theta + \mu)} + 2\lambda - z - \sigma} \left(1 - \left(\frac{\gamma_1 K}{L}\right)^{\frac{2\lambda - z - \sigma + \sqrt{2(\theta + \mu)}}{\sigma}}\right) \\
Z_3(\theta) &= \frac{L}{\sqrt{2(\theta + \nu)} + z} \\
Z_4(\theta) &= \frac{\gamma_1 K}{\sqrt{2(\theta + \nu)} + z + \sigma}.
\end{aligned}$$

Value of call component

Denote the expected value of *call component* as

$$\begin{aligned}
CC(T) &:= e^{-rT} \mathbb{E}_Q \left[e^{-\int_0^T h(\tilde{S}_u) du} (\gamma_2 \tilde{S}_T - L)^+ \right] \\
&= e^{-rT} \mathbb{E}_{\tilde{Q}} \left[Z_T \cdot e^{-\int_0^T h(\tilde{S}_u) du} (\gamma_2 \tilde{S}_T - L)^+ \right].
\end{aligned} \quad (7.21)$$

The valuation of equation (7.21) is decomposed into two cases:

- (I) the level K is *never touched* during the life of the contract,
- (II) the level K is *touched* during the life of the contract.

Thus equation (7.21) can be expressed as

$$CC(T) = CCI(T) + CCII(T) \quad (7.22)$$

with

$$CCI(T) := e^{-2\lambda d_K - r(\alpha_+ + b)T} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_T > d_K\}} \phi(\tilde{W}_T) \right]$$

and

$$CCII(T) := e^{-2\lambda d_K - (a+r)T} \mathbb{E}_{\tilde{Q}} \left[\mathbf{1}_{\{m_T \leq d_K\}} \phi(\tilde{W}_t) \exp(\lambda L_T^{d_K}) \exp(-\tilde{\alpha}_+ \Gamma_T^{(d_K, +)} - \alpha_- \Gamma_T^{(d_K, -)}) \right]$$

where $\phi(x) = e^{zx - 2\lambda(x - d_K)^+} (\gamma_2 S_0 e^{\sigma x} - L)^+$.

Proposition 7.3.8. Under assumptions that $d_K < 0$ and $S_0 < K_u$,

$$\begin{aligned} CCI(T) = & L \cdot K_1(T) \cdot K_2(T) \left\{ N(d_1(T, d_{L_2})) - K_4 \cdot N(d_1(T, -d_{\gamma_2})) \right\} \\ & - \gamma_2 S_0 \cdot K_3(T) \left\{ N(d_2(T, d_{L_2})) - K_5 \cdot N(d_2(T, -d_{\gamma_2})) \right\}, \end{aligned} \quad (7.23)$$

where $K_1(T)$, $K_2(T)$, $K_3(T)$, K_4 , K_5 , $d_1(t, x)$ and $d_2(t, x)$ are defined in equation(7.19) and

$$d_{L_2} := \frac{\ln \frac{K_u}{S_0}}{\sigma} \quad d_{\gamma_2} := \frac{\ln \frac{K^2}{K_u S_0}}{\sigma}.$$

the Laplace Transform of $CCII(t)$ is,

$$\widehat{CCII}(\theta) = \frac{2M(\theta)(Z_5(\theta) - Z_6(\theta))}{N(\theta)} \quad (7.24)$$

where $M(\theta)$ and $N(\theta)$ are defined in equation(7.20) and

$$\begin{aligned} Z_5(\theta) &= \frac{\gamma_2 K}{\sqrt{2(\theta + \mu)} + 2\lambda - z - \sigma} \cdot \left(\frac{\gamma_2 K}{L} \right)^{\frac{2\lambda - z - \sigma + \sqrt{2(\theta + \mu)}}{\sigma}} \\ Z_6(\theta) &= \frac{L}{\sqrt{2(\theta + \mu)} + 2\lambda - z} \left(\frac{\gamma_2 K}{L} \right)^{\frac{2\lambda - z + \sqrt{2(\theta + \mu)}}{\sigma}}. \end{aligned}$$

7.3.4 Numerical example

Example 7.3.9. As a concrete numerical we compute the prices of different mandatory convertible bonds with parameters $T = 4$, $r = 0.06$, $a = 0.5$, $b = 0.02$, $K = 60$, $S_0 = 100$, $K_l = 100$, $L = 100$, and $c = 6$. The results are compared with the default-free case and summarized in table(7.1)

	$K_u = 120$		$K_u = 130$		$K_u = 140$	
σ	default-free	defaultable	default-free	defaultable	default-free	defaultable
0.2	108.75	106.64	104.93	102.33	102.07	99.04
0.3	108.89	106.47	104.85	102.07	101.67	98.58
0.4	108.67	105.41	104.42	100.80	100.97	97.05
0.5	108.33	104.11	103.86	99.26	100.17	95.27

Table 7.1: No-arbitrage prices of mandatory convertible bond without and with default risk

The results in table 7.1 show that default risk reduces the price of the mandatory convertible bond. The influence of the stock volatility on the price is no longer limited if default risk is considered. For example, in default free case, by $K_u = 120$, the price of the mandatory convertible bond is 108.75 if σ equals 0.2, and it amounts 108.33 if σ equals 0.5. The price difference is 0.42, which is quite small. But by consideration of default risk, the price difference amounts to 2.53, and can no longer be neglected. Therefore, the argument in some literature, that due to the offsetting nature of the embedded option spread, a change in volatility has only a minor effect on the mandatory convertible value cannot be justified if the default intensity is explicitly linked to the stock price.

7.4 Default Risk and Uncertain Volatility

Suppose that the seller and buyer relax the assumption of constant volatility by the valuation and adopt the *uncertain volatility* approach to super-hedge the position³. Define the price of the mandatory convertible bond at time t as

$$\begin{aligned}
 J_t &:= \mathbb{E}_Q[mcb(t)] \\
 &= 1_{\tau > t} \sum_{i=[t]+1}^N \mathbb{E}_Q \left[\exp \left(- \int_t^{t_i} (r + h(\tilde{S}_s)) ds \right) \cdot c \mid \mathcal{F}_t \right] \\
 &\quad + 1_{\tau > t} \mathbb{E}_Q \left[\exp \left(- \int_t^T (r + h(\tilde{S}_s)) ds \right) \Phi(\tilde{S}_T) \mid \mathcal{F}_t \right]
 \end{aligned}$$

³In this case the default risk are linked to the equity price, the probabilistic approach proposed by Frey (2000) does not work, because we can no longer achieve constant volatility by applying time change for continuous martingales. We can only apply the PDE approach.

where

$$\Phi(\tilde{S}_T) := L - [L - \gamma_1 \tilde{S}_T]^+ + [\gamma_2 \tilde{S}_T - L]^+.$$

Applying Black-Scholes-Barenblatt equation, the pricing bounds of J_t can be expressed with the following PDE on *non-coupon dates*,

$$\frac{\partial J_t}{\partial t} + \frac{1}{2} \Sigma^2 \left[\frac{\partial^2 J_t}{\partial \tilde{S}_t^2} \right] \tilde{S}_t^2 \frac{\partial^2 J_t}{\partial \tilde{S}_t^2} + (r + h(\tilde{S}_t)) \tilde{S}_t \frac{\partial J_t}{\partial \tilde{S}_t} - (r + h(\tilde{S}_t)) J_t = 0, \quad (7.25)$$

the lower bound can be achieved by setting

$$\Sigma^2 [x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \leq 0 \\ \sigma_{\min}^2 & \text{else,} \end{cases}$$

while the upper bound can be derived with

$$\Sigma^2 [x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \geq 0 \\ \sigma_{\min}^2 & \text{else,} \end{cases}$$

and on *coupon dates* t_c

$$J_{t_c^-} = J_{t_c^+} + c.$$

where t_c^- and t_c^+ are the time just before and after the coupon payment respectively. In the following Example 7.4.1 explicit finite-difference method is applied for the numerical solution.

Example 7.4.1. The volatility of stock is supposed to lie within the interval $[0.2, 0.4]$. The other model parameters are the same as in Example 7.3.9, with $T = 4$, $r = 0.06$, $K = 60$, $S_0 = 100$, $K_l = 100$, $L = 100$, and $c = 6$. The no-arbitrage pricing bounds are listed in table 7.2.

	$a = 0.5, b = 0.02$			$a = 0, b = 0$		
K_u	lower	upper	spread	lower	upper	spread
120	104.05	108.78	4.73	106.91	110.80	3.89
130	98.86	105.14	6.28	102.15	107.55	5.40
140	94.74	102.27	7.53	98.37	105.00	6.63

Table 7.2: No-arbitrage pricing bounds mandatory convertible bonds with stock price volatility lies within the interval $[0.2, 0.4]$.

Results in table 7.2 show the no-arbitrage pricing bounds due to uncertainty about the stock volatility. Explicit modeling of default risk enlarges the price spread.

7.5 Summary

A mandatory convertible bond can be considered as a straight bond with embedded put and call option. It is exposed to equity, interest and default risk. The focus of our study is the joint effect of default and equity risk on the valuation of mandatory convertible bond. We adopt a parsimonious model and assume that the default intensity can only have two constant values. A normal default rate, which is relative low, but if the stock price falls beneath a critical boundary, the default intensity is much higher. Laplace transform of the price of mandatory convertible bond is derived. Numerical example shows that if the default risk is incorporated, the influence of the stock volatility is no longer negligible for the valuation. Finally, we drop the assumption of constant volatility which is one of the critical assumptions in the option valuation, and derive the pricing bound by application of uncertain volatility approach.

Chapter 8

American-style Convertible Bond

In Chapter 4 and Chapter 5 the American-style callable and convertible bond has been studied within the structural approach. Strategical optimal behavior of the bond- and shareholder has been the focus of the investigation. In Kifer (2000) the existence and uniqueness of the no-arbitrage price of a game option is derived for a underlying process which follows a Brownian diffusion and the payoffs of the game option are adapted to the filtration generated by the underlying process. In the structural approach, the firm's value follows a geometric Brownian motion and the default time is a predictable stopping time, thus the payoffs of the convertible bond are adapted to the filtration generated by the firm's value. Thus, we can apply the results on game option developed by Kifer (2000) to derive the unique no-arbitrage value and the optimal strategies of the callable and convertible bond. The optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. Furthermore, the no-arbitrage price can be approximated numerically by means of backward induction on a recombining binomial tree.

Within the reduced-form approach, stock price, credit spreads and implied volatilities of options are used as model inputs. The price of a defaultable stock is described by a jump diffusion (See Section 6.2). The default is an unpredictable event governed by an exogenous default rate or intensity process. It is not adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by the pre-default stock prices which follows a Brownian diffusion. The price of a defaultable stock S is adapted to a larger filtration $(\mathcal{G}_t)_{t \in [0, T]}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ which contains the information about the evolution of the pre-default stock prices and the knowledge whether default has occurred or not. We apply the results of Kallsen and Kühn (2005) to derive the unique no-arbitrage value and the optimal strategies. Because their results are derived for more general stochastic processes which include the jump diffusion process. Within the reduced-form approach, the *max-min* and *min-max* strategies are still valid for the callable and convertible bond but they are derived with respect to

the filtration $(\mathcal{G}_t)_{t \in [0, T]}$. In Section 6.2.1 it has been shown that if the time of default is modeled as the first jump of a Cox process and under some measurable conditions, the expectations with respect to \mathcal{G}_t can be reduced to the expectation with respect to \mathcal{F}_t . Further calculations can thus be simplified. The results of doubly reflected backward stochastic differential equations (BSDE) for continuous diffusions developed by Cvitanić and Karatzas (1996) can be used for computing the no-arbitrage price.

One of the early models on callable and convertible bond within reduced-form approach is proposed by Davis and Lischka (1999). They construct a model framework that incorporate Black-Scholes stock price, Gaussian stochastic interest rate and stochastic default intensity driven by a Brownian motion that also governs the movement of the stock price. To derive the price of a callable and convertible bond the continuous processes are approximated with a multi-dimensional tree. It is called *two-and-a-half factors model* and has found its application in the industry. A similar model has been developed by Ayache et al. (2003). But in both models the game option character of the contract is not considered. Defaultable game option and its application to callable and convertible bonds within reduced-form model have been studied in Bielecki et al. (2006) and Bielecki et al. (2007). Some complex contract features of the callable and convertible bond are treated in the latter paper. The approach in this thesis differs from theirs in that we formulate the default event directly as the time of the first jump of a Poisson process with random intensity. The derivation of the further results is simpler. Furthermore, instead of the contract features such as no-call period or delayed call we focus on the uncertain volatility of the stock and the derivation of the no-arbitrage pricing bounds.

The remainder of the chapter is structured as follows. We start in section 8.1 and 8.3 with a description of the contract feature of the callable and convertible bond and its expected discounted payoff. Section 8.2 describes the optimal strategies. Section 8.4 summarized some results of BSDE which are closely related to financial market. Section 8.5 formulates the solution of callable and convertible bond as doubly reflected BSDE and 8.6 solves the problem numerically. Section 8.7 treats the case that there is uncertainty about the volatility of the stock price.

8.1 Contract Feature

The contract feature of the American-style callable and convertible bond has been described in Section 4.1. The payoff of the bond within the reduced-form model differs from that within a structural model only at one point that the former use stock price as input while by the latter the firm's value is the model input¹. The bondholder can stop

¹For ease of reading we give a complete description of the payoff and accept that there are some repetitions of Section 4.1

and convert the bond into stocks according to the prescribed conversion ratio γ . The conversion time of the bondholder is $\tau_b \in [0, \tau]$, where τ is the default time. The issuer which is often the shareholder can stop and buy back the bond for a price given by the maximum of call level H and the current conversion price, where H can be constant or time dependent. The call time of the seller is $\tau_s \in [0, \tau]$.

The payoff of a defaultable callable and convertible bond can be distinguished in four cases. The principal of the bond is L , R_t stands for the recovery process, S_t is the stock price at time t and c the coupon rate.

- (i) Let $\tau_b < \tau_s \leq T$, such that the contract begins at time 0 and is stopped and converted by the bondholder. In this case, the discounted payoff $ccb(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through conversion

$$conv(0) = c \int_0^{\tau_b \wedge \tau} \beta(0, s) ds + R_\tau \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq \tau_b\}} + \beta(0, \tau_b) \mathbf{1}_{\{\tau_b < \tau\}} \gamma S_{\tau_b}.$$

- (ii) Let $\tau_s < \tau_b \leq T$, such that the contract is bought back by the issuer before the bondholder converts. In this case, the discounted payoff $call(0)$ of the callable and convertible bond at time 0 is composed of the accumulated coupon payments and the payoff through call,

$$call(0) = c \int_0^{\tau_s \wedge \tau} \beta(0, s) ds + R_\tau \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq \tau_s\}} + \beta(0, \tau_s) \mathbf{1}_{\{\tau_s < \tau\}} \max[H, \gamma S_{\tau_s}].$$

- (iii) If $\tau_s = \tau_b < T$ the discounted payoff of the bond equals the smaller value, i.e. the discounted payoff with conversion.

- (iv) For $\tau_b \geq T$ and $\tau_s \geq T$, the discounted payoff of a callable and convertible bond at time 0 is

$$term(0) = c \int_0^{\tau \wedge T} \beta(0, s) ds + R_\tau \cdot \beta(0, \tau) \mathbf{1}_{\{\tau \leq T\}} + \beta(0, T) \mathbf{1}_{\{T < \tau\}} \max[\gamma S_T, L].$$

Denote the minimum of conversion and call time by $\zeta = \tau_s \wedge \tau_b$. Then, the discounted payoff of a callable and convertible bond in all four cases can be expressed with one

equation,

$$\begin{aligned}
cbb(0) &:= \mathbf{1}_{\{\zeta < \tau\}} \left(c \int_0^{\zeta \wedge T} \beta(0, s) ds + \mathbf{1}_{\{\zeta = \tau_s < \tau_b \leq T\}} \beta(0, \zeta) \max \{H, \gamma S_\zeta\} \right. \\
&\quad \left. + \mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} \beta(0, \zeta) \gamma S_\zeta + \mathbf{1}_{\{\zeta = T\}} \beta(0, T) \gamma S_T \right) \\
&+ \mathbf{1}_{\{\tau \leq \zeta\}} \left(c \int_0^{\tau \wedge T} \beta(0, s) ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) R_\tau + \mathbf{1}_{\{T < \tau\}} \beta(0, T) L \right).
\end{aligned} \tag{8.1}$$

Theorem 8.1.1. Same as within the structural model, the payoff of a callable and convertible bond can be decomposed into a straight bond and a defaultable game option component $g(0)$.

$$ccb(0) = d(0) + g(0) \tag{8.2}$$

with

$$d(0) := c \int_0^{\tau \wedge T} \beta(0, s) ds + \mathbf{1}_{\{\tau \leq T\}} \beta(0, \tau) R_\tau + \mathbf{1}_{\{T < \tau\}} \beta(0, T) L$$

and

$$\begin{aligned}
g(0) &:= \mathbf{1}_{\{\zeta < \tau\}} \beta(0, \zeta) \left\{ \mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} (\gamma S_\zeta - \phi_\zeta) \right. \\
&\quad \left. + \mathbf{1}_{\{\zeta = \tau_s < \tau_b \leq T\}} (\max \{H_\zeta, \gamma S_\zeta\} - \phi_\zeta) + \mathbf{1}_{\{\zeta = T\}} (\gamma S_T - L)^+ \right\}.
\end{aligned}$$

where

$$\phi_\zeta := c \int_\zeta^{\tau \wedge T} \beta(0, s) ds + \mathbf{1}_{\{\tau \leq T\}} \beta(\zeta, \tau) R_\tau + \mathbf{1}_{\{T < \tau\}} \beta(\zeta, T) L \tag{8.3}$$

is the discounted value (discounted to time ζ) of the sum of the remaining coupon payments and the principal payment of a straight coupon bond given that it has not defaulted till time ζ .

8.2 Optimal Strategies

As the call value is strictly larger than the conversion value prior to maturity and they are the same at the maturity, thus, we can apply the the theories of game option developed by Kallsen and Kühn (2005). Within the reduced-form approach, the *max-min* and *min-max* strategies are still valid for the callable and convertible bond but they are derived with respect to the filtration $(\mathcal{G}_t)_{t \in [0, T]}$. The optimal strategy for the bondholder is to select the stopping time which maximizes the expected payoff given the minimizing strategy of the issuer, while the issuer will choose the stopping time that minimizes the expected payoff given the maximizing strategy of the bondholder. This *max-min strategy* of the bondholder leads to the lower value of the convertible bond, whereas the *min-max strategy* of the issuer leads to the upper value of the convertible bond. The assumption

that the call value is always larger than the conversion value prior to the maturity and they are the same at maturity T ensures that the lower value equals the upper value such that there exists a unique solution.

Under an equivalent martingale measure Q , the no-arbitrage price of the callable and convertible bond at the inception of the contract, $CCB(0)$ is given by

$$CCB(0) = \sup_{\tau_b \in \mathcal{G}_{0T}} \inf_{\tau_s \in \mathcal{G}_{0T}} \mathbb{E}_Q[ccb(0)|\mathcal{G}_0] = \inf_{\tau_s \in \mathcal{G}_{0T}} \sup_{\tau_b \in \mathcal{G}_{0T}} \mathbb{E}_Q[ccb(0)|\mathcal{G}_0]. \quad (8.4)$$

where \mathcal{G}_{0T} is the set of stopping times with respect to the filtration $\{\mathcal{G}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$. After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \text{esssup}_{\tau_b \in \mathcal{G}_{tT}} \text{essinf}_{\tau_s \in \mathcal{G}_{tT}} \mathbb{E}_Q[ccb(0)|\mathcal{G}_t] \\ &= \text{essinf}_{\tau_s \in \mathcal{G}_{tT}} \text{esssup}_{\tau_b \in \mathcal{G}_{tT}} \mathbb{E}_Q[ccb(0)|\mathcal{G}_t]. \end{aligned} \quad (8.5)$$

where \mathcal{G}_{tT} is the set of stopping times with respect to the filtration $\{\mathcal{G}_u\}_{t \leq u \leq T}$ with values in $[t, T]$. Furthermore, the optimal stopping times for the equity holder and bondholder respectively are

$$\begin{aligned} \tau_b^* &= \inf\{t \in [0, T] \mid conv(0) \geq CCB(t)\} \\ \tau_s^* &= \inf\{t \in [0, T] \mid call(0) \leq CCB(t)\}. \end{aligned} \quad (8.6)$$

It is optimal to convert as soon as the current conversion value is equal to or larger than the value function $CCB(t)$, while the optimal strategy for the issuer is to call the bond as soon as the current call value is equal to or smaller than the value function $CCB(t)$.

In general, the optimization problem formulated via equation(8.4) has no closed-form solution.² After the reduction of the filtration from $(\mathcal{G}_t)_{t \in [0, T]}$ to $(\mathcal{F}_t)_{t \in [0, T]}$ the no-arbitrage value can be formulated as adapted solution of backward stochastic differential equations (BSDE) with two reflecting barriers. In Section 8.4 we give a brief summary of the results on BSDE which are closely related to the financial market. At first we show the reduction of the filtration.

8.3 Expected Payoff

Applying the methodology of filtration reduction described in Section 6.3 expected payoffs related to a callable and convertible bond have simple and explicit expressions. For a given

²The continuous time problem can be approximated with a discrete time one and the no-arbitrage price of the callable and convertible bond can then be derived e.g. by recursion alongside the branches of a tree. But the dynamic of the stock price is modeled as jump diffusion with varying drifts and it is sometimes difficult to construct a recombining tree especially if the uncertain volatility is considered. Therefore we need to solve it with the help of BSDE.

equivalent martingale measure Q , the no-arbitrage price of a straight coupon bond with face value L , constant continuous coupon rate c , maturity T and a constant recovery amount R upon default time τ is

$$\begin{aligned} D(t) &= \mathbf{1}_{\tau > t} E_Q \left[\exp \left(- \int_t^T (r_s + h_s) ds \right) L \middle| \mathcal{F}_t \right] \\ &\quad + \mathbf{1}_{\tau > t} E_Q \left[\int_t^T (c + R \cdot h_s) \cdot \exp \left(- \int_t^s (r_u + h_u) du \right) ds \middle| \mathcal{F}_t \right]. \end{aligned} \quad (8.7)$$

In the fictitious default-free market, the sum of the discounted cash flows in equation(8.7) corresponds to a default-free coupon bond with face value L and variable coupon rate $\bar{c} + R \cdot h_s$. The modified discount factor amounts $\exp(-\int_s^t (r_u + h_u) du)$. At the inception of the contract, $t = 0$, the expression can be simplified to

$$\begin{aligned} D(0) &= E_Q \left[\exp \left(- \int_0^T (r_s + h_s) ds \right) L \right] \\ &\quad + E_Q \left[\int_0^T (\bar{c} + R \cdot h_s) \cdot \exp \left(- \int_0^s (r_u + h_u) du \right) ds \right]. \end{aligned} \quad (8.8)$$

where $E_Q[\cdot]$ is an abbreviation for $E_Q[\cdot | \mathcal{F}_0]$.

Equations (8.4) and (8.5) can be reformulated as

$$CCB(0) = \sup_{\tau_b \in \mathcal{F}_{0T}} \inf_{\tau_s \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0) | \mathcal{F}_0] = \inf_{\tau_s \in \mathcal{F}_{0T}} \sup_{\tau_b \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0) | \mathcal{F}_0]. \quad (8.9)$$

where \mathcal{F}_{0T} is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{0 \leq u \leq T}$ with values in $[0, T]$. After the inception of the contract, the value process $CCB(t)$ satisfies

$$\begin{aligned} CCB(t) &= \mathbf{1}_{\tau > t} \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \mathbb{E}_Q[ccb(0) | \mathcal{F}_t] \\ &= \mathbf{1}_{\tau > t} \text{essinf}_{\tau_s \in \mathcal{F}_{tT}} \text{esssup}_{\tau_b \in \mathcal{F}_{tT}} \mathbb{E}_Q[ccb(0) | \mathcal{F}_t]. \end{aligned} \quad (8.10)$$

where \mathcal{F}_{tT} is the set of stopping times with respect to the filtration $\{\mathcal{F}_u\}_{t \leq u \leq T}$ with values in $[t, T]$, and

$$\begin{aligned} E_Q[ccb(0) | \mathcal{F}_t] &= \mathbf{1}_{\{\tau > t\}} E_Q \left[\int_t^{\zeta \wedge T} (\bar{c} + R \cdot h_s) \cdot \exp \left(- \int_t^s (r_u + h_u) du \right) ds \right. \\ &\quad + \mathbf{1}_{\{\zeta = \tau_b < \tau_s < T\}} \exp \left(- \int_t^{\zeta} (r_s + h_s) ds \right) \gamma \tilde{S}_{\zeta} \\ &\quad + \mathbf{1}_{\{\zeta = \tau_s < \tau_b < T\}} \exp \left(- \int_t^{\zeta} (r_s + h_s) ds \right) \max[H, \gamma \tilde{S}_{\zeta}] \\ &\quad \left. + \mathbf{1}_{\{\zeta = T\}} \exp \left(- \int_t^T (r_s + h_s) ds \right) \max[L, \gamma \tilde{S}_T] \middle| \mathcal{F}_t \right]. \end{aligned}$$

8.4 Excursion: Backward Stochastic Differential Equations

The study of non-linear BSDE is initiated by Pardoux and Peng (1990). The authors prove existence and uniqueness of the solution under suitable assumptions on the coefficient and the terminal value of the BSDE. Since then it has been recognized that the theory of BSDE is a useful tool to formulate and study many problems in finance, e.g. hedging and pricing of European contingent claims, see El Karoui and Quenez (1997). Further studies are carried out in El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) to BSDE's with reflection, i.e., the solution is forced to stay above a given stochastic process. Existence and uniqueness of the solution is proved. Moreover they show that in a special case the solution is the value function of a mixed optimal stopping and optimal stochastic control problem. Concrete examples are pricing of American option in complete and incomplete market. These results are further generalized in Cvitanić and Karatzas (1996) to the case of two reflecting barrier processes, i.e. the solution process of the BSDE has to remain between the prescribed upper- and lower-boundary processes. They prove the existence of the solution and show that the solution coincides with the value of a Dynkin game, therefore establish the uniqueness of the solution. There are numerous studies on theory and numerics of BSDE's. A comprehensive review will go out of the range of our study. We will only summarize the results closely related to financial market, especially the game option.

8.4.1 Existence and uniqueness

The existence and uniqueness of the backward stochastic differential equation was first treated in Pardoux and Peng (1990).

Definition 8.4.1. Let $T \in \mathbb{R}_+$. Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$. The filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ is generated by a d -dimensional Brownian motion W . Consider the following BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t^\top dW_t, \quad Y_T = \xi, \quad (8.11)$$

or equivalently

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s^\top dW_s$$

where

- The terminal value ξ is an n -dimensional \mathcal{F}_T -measurable square integrable random vector.
- f maps $\Omega \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{d \times n}$ into \mathbb{R}^n . f is assumed to be $\mathcal{P} \otimes \mathcal{B}^n \otimes \mathcal{B}^{d \times n}$ measurable. \mathcal{P} denotes σ -algebra of \mathcal{F}_t -progressively measurable subsets of $\Omega \times$

\mathbb{R}_+ . Moreover f is uniformly Lipschitz, i.e. there exists $C > 0$ such that $dt \times dP$ a.s. for all y_1, z_1, y_2, z_2

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

- Y and Z are \mathbb{R}^n and $\mathbb{R}^{d \times n}$ valued progressively measurable processes and Y is continuous. Z^\top denotes the transpose of the matrix Z .
- f is called the *driver* of the BSDE.

There *exists* a *unique* pair of adapted process (Y, Z) satisfies equation (8.11).

8.4.2 Comparison theorem

Let (f^1, ξ^1) and (f^2, ξ^2) be two pairs of driver and terminal value of two BSDE's, and (Y^1, Z^1) and (Y^2, Z^2) be the associated solutions. Suppose that $\xi^1 \geq \xi^2$ P a.s., and $\delta_2 f_t := f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2) \geq 0$ $dt \times dP$ a.s.. Then we have $Y^1 \geq Y^2$ P a.s.. Moreover the comparison is strict, i.e. on the event $\{Y_t^1 = Y_t^2\}$, we have $\xi^1 = \xi^2$, $f^1(s, Y_s^2, Z_s^2) = f^2(s, Y_s^2, Z_s^2)$ $dt \times dP$ a.s. and $Y_s^1 = Y_s^2$, $t \leq s \leq T$ a.s.. The comparison theorem is e.g. useful for calculation of upper bound of contingent claim in incomplete market.

8.4.3 Forward backward stochastic differential equation

A well-investigated class of BSDE's is of the following form, it is also called forward backward stochastic differential equation (FBSDE)

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s^\top dW_s$$

where g and f are deterministic functions and X satisfies the following SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s)^\top dW_s$$

where b and σ are measurable functions. The adapted solution of Y is associated to the solution of a quasi-linear parabolic PDE

$$\begin{cases} u_t + \frac{1}{2} \text{tr}\{\sigma \sigma^\top u_{xx}\} + bu_x + f(t, x, u, u_x \sigma) = 0 \\ u(T, x) = g(x). \end{cases} \quad (8.12)$$

The explicit expression of the solution (Y, Z) is

$$Y_t = u(t, X_t), \quad Z_t = \partial_x u(t, X_t) \sigma(t, X_t).$$

8.4.4 Financial market

Consider a *complete market* there are $n + 1$ primary assets which are denoted by the vector $S = (S^0, S^1, \dots, S^n)^\top$. S^0 is a non-risky asset and has the following price dynamic

$$dS_t^0 = S_t^0 r_t dt$$

r_t is the deterministic interest rate. The price process for S^i , $i \in (1, \dots, n)$ is modeled by the linear SDE driven by an n -dimensional Brownian motion W , defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$,

$$dS_t^i = S_t^i \left(b_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j \right).$$

P is the objective probability measure. Assume that the number of risky assets equals the dimension of the Brownian motion³. By absence of arbitrage there exists an n -dimensional bounded and progressively measurable vector θ such that

$$b_t - r_t \mathbf{1} = \sigma_t \theta_t, \quad dt \times dP \quad a.s.,$$

where $\mathbf{1}$ denotes n -dimensional unit vector. σ_t is an $n \times n$ matrix and is assumed to have full rank. θ is called the premium of the market risk. Under these assumptions the market is complete.

For hedge of a European contingent claim in complete market a self-financing and replicating portfolio can be builded. At time t the trading strategy $\phi_t = (\phi_t^1, \dots, \phi_t^n)^\top$ can be decided. And under the assumption of self-financing the investment in the risk-less asset must satisfy $\phi_t^0 S_t^0 = V_t - \sum_{i=1}^n \phi_t^i S_t^i$. Therefore the value of the self-financing portfolio has the following dynamic

$$\begin{aligned} dV_t &= r_t V_t dt + \pi_t^\top (b_t - r_t \mathbf{1}) dt + \pi_t^\top \sigma_t dW_t \\ &= r_t V_t dt + \pi_t^\top \sigma_t (dW_t + \theta_t dt). \end{aligned}$$

The vector $\pi_t = (\pi_t^1, \dots, \pi_t^n)^\top$ with $\pi_t^i = \phi_t^i S_t^i$ denotes the amount of the money invested in risky assets i at time t . In expression of BSDE

$$V_t = \xi + \int_t^T f(s, V_s, Z_s) ds - \int_t^T Z_s^\top dW_s,$$

where ξ is the terminal value of contingent claim, $Z_t^\top = \pi_t^\top \sigma_t$ and

$$f(t, y, z) = -r_t y - z_t^\top \theta_t. \quad (8.13)$$

³This assumption and the full rank of volatility matrix ensure the completeness of the market

The driver in equation (8.13) is a *linear* function of y and z .

Non-linear BSDE can arise in *incomplete market*. A simple example is that the borrowing interest rate is higher than lending. Denote the borrowing interest rate as R_t . The amount of money borrowed at time t equals to $(V_t - \sum_{i=1}^n \pi_t^i)^-$, where $(x)^-$ denotes $\min\{x, 0\}$. The dynamic of the portfolio is

$$dV_t = r_t V_t dt + \pi_t^\top \sigma_t \theta_t dt + \pi_t^\top \sigma_t dW_t - (R_t - r_t) \left(V_t - \sum_{i=1}^n \pi_t^i \right)^- dt.$$

The value process and trading strategy can also be summarized in expression of BSDE

$$V_t = \xi + \int_t^T f(s, V_s, Z_s) ds - \int_t^T Z_s^\top dW_s,$$

but in this case the driver is sub-linear

$$f(t, y, z) = -r_t y - z_t^\top \theta_t + (R_t - r_t)(y - \mathbf{1}^\top (\sigma_t^\top)^{-1} z)^-. \quad (8.14)$$

The non-linear term (the third term) depends on both y and z . Due to the existence and uniqueness theorem of BSDE unique price process and dynamic hedge can be determined. Usually BSDE has no closed-form solution. The value can be derived with the help of the solution of a quasi-linear parabolic PDE according to equation (8.12) or with numerical simulations.

Remark 8.4.2. For hedge of a European contingent claim in complete market no essential gain can be achieved by introducing the BSDE concept, but it is a useful tool to deal with market incompleteness.

8.5 Hedging and Optimal Stopping Characterized as BSDE with Two Reflecting Barriers

In general, the optimization problem formulated via Equation (8.9) has no closed-form solution. Cvitanić and Karatzas (1996) show that, the no-arbitrage value can be formulated as adapted solution of backward stochastic differential equations (BSDE) with two reflecting barriers. The proper BSDE for valuation of callable and convertible bond will be derived via hedging arguments. It has been shown in literatures that the most significant risk factor for a typical convertible bond is the equity price subject to default risk. Interest rate risk is usually a secondary consideration. Therefore we assume that the default-free interest rate is deterministic. Another hypothesis which make the hedge possible, requires that two kinds of risky assets are traded in the market:

8.5. HEDGING AND OPTIMAL STOPPING CHARACTERIZED AS BSDE WITH TWO REFLECTIONS

- *defaultable stock*, with its dynamic described by equation(7.5),
- *defaultable zero-coupon bond with zero recovery*, based on the assumption of absence of interest rate risk, its dynamic can be expressed as

$$d\bar{B}_t = \bar{B}_{t-}(r_t dt - dM_t), \quad (8.15)$$

with

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} h(\tilde{S}_u) du ,$$

equivalently, the pre-default bond price \tilde{B}_t satisfies

$$d\tilde{B}_t = (r_t + h(\tilde{S}_t))\tilde{B}_t dt.$$

The bond holder pays the price, which is a non-random amount at time zero and is entitled to the cumulative coupon payments and the lump-sum settlement at conversion or call time, or at default. While the issuer receives the price, but must provide the aforementioned random payments to the bondholder. The issuer's objective is to hedge his short position by trading in the market in such a way as to make the necessary payments and still be solvent at the termination of the contract, almost surely. The price process of the callable and convertible bond is then associated with the following hedging strategy, with investment in risky zero bonds and stock,

$$dCCB(t) + (\bar{c} + R \cdot h_t)dt = (r_t + h_t)CCB(t)dt - dK^+(t) + dK^-(t) + \pi_t \sigma_t dW_t, \quad (8.16)$$

where $K^+(t)$ and $K^-(t)$ are two continuous, increasing and adapted processes satisfy

$$\int_0^T (CCB(t) - CV(t))dK^+(t) = \int_0^T (CCB(t) - Call(t))dK^-(t) = 0$$

where π_t denotes the amount of money invested in the risky stock, $CV(t)$ the conversion value, $Call(t)$ the call value.

Proposition 8.5.1. In standard expression of BSDE,

$$\begin{cases} CCB(t) = g(\tilde{S}_T) + \int_t^T f(s, \tilde{S}_s, CCB(s))ds - \int_t^T Z_s dW_s + \int_t^T dK_s^+ - \int_t^T dK_s^- \\ CV(\tilde{S}_t) \leq CCB(t) \leq Call(\tilde{S}_t) \quad \forall 0 \leq t \leq T \\ \int_0^T (CCB(s) - CV(\tilde{S}_s))dK_s^+ = \int_0^T (Call(\tilde{S}_s) - CCB(s))dK_s^- = 0 \end{cases} \quad (8.17)$$

with

$$\begin{aligned} d\tilde{S}_t &= (r_t + h(\tilde{S}_t))\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \\ f(t, CCB(t)) &= (\bar{c} + R \cdot h_t) - (r_t + h_t)CCB(t). \end{aligned}$$

where $Z_t = \pi_t \sigma_t$, and $f(t, CCB(t))$ is the *driver*.

The value process of the convertible bond is forced to stay between the upper- and lower-boundary, which are the call and conversion value respectively. This effect is achieved through the two reflection processes $K^+(t)$, and $K^-(t)$, which push the value process of the callable and convertible bond upward or downward to prevent the boundary crossing. The "push" is minimal in the sense that it will only be carried out in the case that $CCB(t) = CV(t)$ or $CCB(t) = Call(t)$. According to Cvitanić and Karatzas (1996), the existence and uniqueness of the solution of equation (8.17) is ensured, if additional to the general conditions on terminal value and the driver defined in definition 8.4.1, the following conditions are satisfied

- K^+ and K^- are continuous, increasing and adapted processes.
- CV and $Call$ are two continuous, progressively measurable processes and satisfy

$$CV(t) < Call(t), \quad \forall 0 \leq t \leq T \quad \text{and} \quad CV(T) \leq \xi \leq Call(T) \quad a.s.$$

Having formulated the no-arbitrage value of the callable and convertible bond as solution of BSDE with two reflecting barriers, our next task is to derive numerical solutions.

Remark 8.5.2. According to our assumptions, the bondholder can only exchange the bond against stock of one prescribed firm. However, BSDE with two reflecting barriers usually encompasses the more general case, where the bondholder can convert the bond into a basket of risky stocks, i.e. Z can be \mathbb{R}^d , $d \geq 1$ valued and the hedge portfolio contains positions in d different risky stocks.

8.6 Numerical Solution

There are basically two types of schemes for solving BSDE's. The first type is the numerical solution of a parabolic PDE related to the BSDE and the second type of algorithms works backwards and treats the stochastic problem directly via simulation. For financial problems with few random factors, the associated PDE provided by Cvitanić and Ma (2001) can be solved with finite-difference methods. For callable and convertible bond with more than three risky stock as underlying, a direct treatment with Monte Carlo method is a better method. A recursion algorithm is provided e.g. in Chassagneux (2007). Equation (8.17) belongs to a well-investigated class of BSDE's in a Markovian framework, the FBSDE.

Proposition 8.6.1. According to Cvitanić and Ma (2001) the solution of equation (8.17)

is associated with the following PDE, which is called the obstacles problem,

$$\begin{cases} (Call - CV) \wedge \{(u - Call) \vee -[u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + (r + h_t)xu_x + f(t, x, u)]\} = 0 \\ u(T, x) = g(x). \end{cases} \quad (8.18)$$

For simplicity of the notations, x stands for \tilde{S} and h_t the default intensity $h(\tilde{S}_t)$. The driver $f(t, x, u) = (c + R \cdot h_t) - (r_t + h_t)u$. The explicit expression of the solution (CCB, Z) is

$$CCB(t) = u(t, x_t), \quad Z_t = \partial_x u(t, x_t)\sigma(t, x_t).$$

Here, we will not give an exact mathematical definition of the obstacle problem, and discuss the existence and uniqueness of its solution, for details see Cvitanić and Ma (2001). We apply explicit finite difference method for derivation of the numerical solution, i.e. we work step by step down the grid. Finite difference methods can be thought as a generalization of the binomial concept and is more flexible. In the finite-difference methods the grid is fixed but parameters change to reflect a changing diffusion. At first, we derive the value \tilde{u}_i^k backwardly from the next time period, then compare it with the payoffs by conversion or call. If \tilde{u}_i^k is greater or lesser than the call or conversion value, it will be replaced by the call or conversion value respectively. For each time step k and stock step i ,

$$u_i^k = \min[Call, \max[CV, \tilde{u}_i^k]].$$

Example 8.6.2. As an illustrative example we compute the no-arbitrage price of a defaultable callable and convertible bond. The default intensity is modelled as piecewise constant function of the pre-default stock price.

$$h(\tilde{S}_t) = \begin{cases} a & \text{if } \tilde{S}_t \leq K \\ b & \text{if } \tilde{S}_t > K \end{cases}$$

In default case, the stock value jumps to zero, while the bond has a constant recovery rate of $R = 30\%$ of the face value. The convertible value is $CV_t = \gamma\tilde{S}_t$, and the call value is always larger than the convertible value and amounts $Call_t = \max[H, \gamma\tilde{S}_t]$. The model parameters are given as $T = 4$, $r = 0.06$, $S_0 = 70$, $a = 0.5$, $b = 0.02$, $K = 30$, $L = 100$, $c = 3$, $\gamma = 1.2$. The no-arbitrage values by different stock volatilities and the comparison with the default free case are summarized in table 8.1. The stability of numeric is ensured by proper choice and combination of the steps for the stock price and time.

The results in table 8.1 show that, in default free case, the price of callable and convertible bond increases in volatility. But if default risk is considered and the default intensity is explicitly linked to the stock price, the price increases at first with increasing volatility then decreases after the volatility exceeds a certain value. The increasing volatility increase the conversion value but it also increases the default probability.

	$H = 110$		$H = 120$		$H = 130$	
σ	defaultable	default-free	defaultable	default-free	defaultable	default-free
0.1	95.02	96.52	96.59	97.73	97.51	98.36
0.2	97.34	99.21	99.56	101.45	101.11	102.94
0.3	98.33	100.88	101.32	103.99	103.45	106.32
0.4	97.85	101.96	101.25	105.68	103.70	108.65
0.5	96.85	102.65	100.33	106.84	102.91	110.21

Table 8.1: No-arbitrage prices of American-style callable and convertible bond without and with default risk by reduced-form approach

8.7 Uncertain Volatility

Suppose that the seller and buyer relax the assumption of constant volatility by the valuation and adopt the assumption of *uncertain volatility*. In this case the market is incomplete, i.e. there is no unique price of market risk, there is a set of possible equivalent martingale measures which are compatible with the no arbitrage requirement.

Proposition 8.7.1. Suppose that only a *buy-and-hold* strategy is allowed in the callable and convertible bond, while only the risky stock and defaultable zero-coupon bond can be traded dynamically. The set of initial no-arbitrage prices is determined by super hedging and lies in the interval $[CCB_{low}(0), CCB_{up}(0)]$ with

$$CCB_{low}(0) = \sup_{\tau_B \in \mathcal{F}_{0T}} \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] = \inf_{\tau_A \in \mathcal{F}_{0T}} \inf_{Q \in \mathcal{Q}} \sup_{\tau_B \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0)], \quad (8.19)$$

$$CCB_{up}(0) = \inf_{\tau_A \in \mathcal{F}_{0T}} \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[ccb(0)] = \sup_{\tau_B \in \mathcal{F}_{0T}} \sup_{Q \in \mathcal{Q}} \inf_{\tau_A \in \mathcal{F}_{0T}} \mathbb{E}_Q[ccb(0)], \quad (8.20)$$

where \mathcal{Q} is the family of equivalent martingale measures.

Proof 8.7.2. Applying theorem 2.2 of Kallsen and Kühn (2005).

The lower and upper bound are derived under the most pessimistic expectations of the buyer and seller respectively.

Theorem 8.7.3. Combine proposition 8.6.1 with proposition 8.7.1. The solution of equation (8.19) and (8.20) is associated with the following PDE

$$\begin{cases} (Call - CV) \wedge \left\{ (u - Call) \vee - \left[u_t + \frac{1}{2} \Sigma^2[u_{xx}] x^2 u_{xx} + (r + h_s) x u_x + f(t, x, u) \right] \right\} = 0 \\ u(T, x) = g(x). \end{cases} \quad (8.21)$$

where $\Sigma^2[x]$ stands for a volatility parameter which depends on x . CCB_{low} is derived

by setting

$$\Sigma^2 [x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \leq 0 \\ \sigma_{\min}^2 & \text{else} \end{cases}$$

and CCB_{up} is derived by setting

$$\Sigma^2 [x] = \begin{cases} \sigma_{\max}^2 & \text{if } x \geq 0 \\ \sigma_{\min}^2 & \text{else} \end{cases}$$

Example 8.7.4. The volatility of stock is supposed to lie within the interval $[0.2, 0.4]$. The other model parameters are the same as in example 8.6.2, with $T = 4$, $R = 30\%$, $r = 0.06$, $K = 30$, $S_0 = 70$, $L = 100$, and $c = 3$. The bid and ask prices are listed in Table 8.2.

	$a = 0.5, b = 0.02$			$a = 0, b = 0$		
H	lower	upper	spread	buyer	lower	upper
120	99.19	102.97	3.79	101.45	105.68	4.22
130	100.76	105.69	4.94	102.91	108.65	5.73
140	101.64	107.75	6.11	103.70	110.85	7.15
150	102.15	109.17	7.02	104.11	112.36	8.25

Table 8.2: No-arbitrage pricing bounds with stock price volatility lies within the interval $[0.2, 0.4]$, reduced-form approach

Default risk reduces the price but in contrast to example 7.4.1, explicit modeling of default risk does *not* enlarge the price spread. The reason is that default risk brings varying convexity and concavity to the value function. Moreover, both parties can decide when they exercise. Therefore each of them must bear the strategy of the other party in mind. The pricing bound is not only determined by the default risk and volatility but also depends on the optimal exercises.

8.8 Summary

The exposure of callable and convertible bonds to both credit and equity risk and the corresponding optimal conversion and call strategies build the focus of our study. Same as in case of mandatory convertible bond, the interplay between equity and credit risk is taken into account by adopting an intensity-based default model in which the risk-neutral default intensity is linked to the equity price. The embedded option rights owned by both of the bondholder and issuer is treated by the well developed theories on the Dynkin

game and can be solved with help of the associated doubly reflected backward stochastic differential equations (BSDE). Valuation of callable and convertible bond as defaultable game option has been proposed by Bielecki et al. (2007). But our model framework is more simple and we give pricing bounds for uncertain stock volatility.

Chapter 9

Conclusion

Firms raise capital by issuing debt, equity and hybrid instruments. Convertible bonds, usually with call provision, are an important example of the hybrid instrument. Issuance of callable and convertible bonds is closely related to the aim of a firm to increase the value of debt or to achieve a lower coupon level than that of a simple coupon bond. In order to study the no-arbitrage value of conversion and call we adopted first an idealized firm value model where the firm issues only stocks and convertible bonds and the value of the firm is the aggregate value of both. We discussed the case when conversion and call can only take place at maturity. The value of conversion and call is then equivalent to a European call spread. More interesting and more relevant for practical applications is the American-style conversion and call right. The optimal conversion and call times and the value of the convertible and callable bond were derived with the aid of the game option theory. We then extended the results by integrating stochastic interest rates. Finally, we discussed the problem of uncertain volatility of the firm value, e.g. due to incomplete information. We derived pricing bounds for callable and convertible bonds under the assumption that the volatility of the firm value process lies between two extreme values. The pricing bounds can be improved if a narrower confidence interval of the volatility of the firm value is available, or we need more market information and/or more knowledge of the risk preferences of the bond- and shareholder.

The example studied in this thesis assumes that the interest rate follows the Vasiček model and the firm's value evolves according to a geometric Brownian motion. Within this setting we first derived the no-arbitrage values of European callable and convertible bonds. The example shows that the no-arbitrage price is essentially determined by the terminal firm's value, the conversion ratio, the call price and the payout ratio. The influence of the interest rate is relatively small, because in the example, and also in practice, the volatility of the firm value is much larger than that of the interest rate. The influence of stochastic interest rates in the case of American-style convertible and callable bonds is also not prominent in this context because its volatility is relatively low and moreover there are early exercise possibilities of both contract sides.

Our idealized firm value model illustrates how the optimal strategies work and what are the important underlying factors. For practical use other features have to be taken into account. For example, a firm issues usually several different kinds of debt with different priorities. Convertible bonds are usually junior debt. The mutual dependence of the different debts and stocks must also be modeled. For pricing purpose it may be more convenient to model the stock price process directly because the firm's value is not directly observable. In this case the reduced-form model is a more proper approach for the study of convertible bonds. Within the intensity-based default model, we first analyzed mandatory convertible bonds, which are contracts of European-style then the American-style callable and convertible bond. We studied the interplay of the equity risk and the default risk of the issuer within a parsimonious, intensity-based default model, in which the default intensity is modeled as a function of the pre-default stock price. Within the reduced-form approach, the max-min and min-max strategies are still valid for the American-style callable and convertible bond. BSDE and the associated PDE were used for the calculation of the no-arbitrage price and pricing bounds if uncertain volatility of the stock price is assumed.

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