Jump processes with variable scaling parameters

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Preface

The leitmotif of this thesis is non-locality. More precisely, it deals with a class of pure jump processes with space-dependent jump measures and with related non-local integro-differential operators. Such objects arise in different contexts for example in partial differential equations. They have drawn increasing interest also by practioneers in the last years. This thesis wants to enlighten some important aspects of this theory. Each chapter contains results which are each of a different flavor.

Chapter 1 describes the general framework and surveys some results. Chapters 2 and 3 both study questions of regularity of solutions of non-local integro-differential operators – the first one by analytic means, i.e. pseudodifferential operator techniques, the second one by probabilistic means. Finally, chapter 4 studies Markov chain approximations of related jump processes. All chapters are self-contained. The results of Chapter 4 are published in [HK07] while the results of Chapter 2 and Chapter 3 are accepted for publication, see [AH08] and [HK09].

This appears to be the place for some brief personal words: First of all, I want to express sincere thanks to my advisor Moritz Kaßmann for fruitful and close collaboration. His enthusiasm has lit me the way through this thesis. I also want to thank my second advisor Karl-Theodor Sturm for his kind support in the last years. I always felt at home and fully integrated within his group. I also want to thank all members of his group for the friendly atmosphere. In particular I owe many a recreative tea-break to Kathrin Bacher, Ann-Kathrin Jarecki, Nicolas Juillet and Hendrik Weber. I would also like to thank Helmut Abels for the interesting collaboration which crystalized itself in Chapter 2 and Zhen-Qing Chen and Takashi Kumagai for helpful discussions especially on the results of Chapter 4.

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1 Introduction

We give here a survey-style introduction to the topics of this thesis. Our main aim is twofold: On one hand we want to motivate the interest in jump processes with varying jump measures. On the other hand we want to put our results in the later chapters in the right framework. Our emphasis here is on regularity theory and on processes with state space $\mathbb{R}^d$.

1.1 A motivation: From Brownian motion and the Laplacian to diffusions and elliptic operators

The classical example per se to exhibit the fruitful interplay between analysis and probability are Brownian motion on $\mathbb{R}^d$ and the Laplacian. It already provides insight into many important aspects of the theory, though the full force of the methods becomes not visible until turning to general diffusion processes and related second order operators with varying coefficients. It is a similar transition from spatially homogeneous Lévy jump processes and their generators to spatially inhomogeneous jump processes which lies at the foundation of this thesis. A good reference is for example Karatzas-Shreve [KS91]. For the theory of diffusions we additionally refer to Stroock-Varadhan [SV06] and Bass [Bas98], for the theory of elliptic partial differential equations with irregular coefficients to Han-Lin [HL97].

The Laplacian on $\mathbb{R}^d$ is the constant coefficient second order partial differential operator $\Delta = \sum_{i=1}^d \partial_{ii}^2$. It is invariant under translations and rotations. A two times differentiable function u is called harmonic on an open set $\Omega \subset \mathbb{R}^d$ if it solves $\Delta u(x) = 0$ for all $x \in \mathbb{R}^d$. Harmonic functions enjoy many beautiful properties. They are characterized by the so-called mean value property: u is harmonic on Ω if and only if for any ball $B(x_0, r)$ with $\overline{B(x_0, r)} \subset \Omega$ the value of u in x_0 is the mean over the sphere $S(x_0, r) = \partial B(x_0, r)$:

$$u(x_0) = \frac{1}{\operatorname{vol} S(x_0, r)} \int_{S(x_0,r)} u(x)\, d\sigma(x). \tag{1.1}$$

Here σ denotes the volume measure on the sphere. More generally, one has for any $x \in B(x_0, r)$ the Poisson kernel representation

$$u(x) = \frac{r^{d-2}(r^2 - |x - x_0|^2)}{\operatorname{vol} S(x_0, r)} \int_{S(x_0,r)} \frac{f(y)}{|x - y|^d} d\sigma(y). \tag{1.2}$$

(1.2) provides the explicit and unique solution to the Dirichlet problem $\Delta u(x) = 0$ for $x \in B(x_0, r)$, $\lim_{y\to x} u(y) = f(x)$ for $x \in S(x_0, r)$ with boundary data $f \in C(S(x_0, r))$. Furthermore one obtains from (1.2) on one hand immediately the maximum principle: A non-constant function u which is harmonic in Ω attains no maximum or minimum in Ω. On the other hand it implies Harnack's inequality[1]

[1] See Kaßmann [Kas07b] for a survey on Harnack inequalities.

(HI) *There exists a constant $c > 0$ with the following property: If a function u is non-negative and harmonic on the ball $B(x_0, r)$ then*

$$\sup_{x \in B(x_0, r/2)} u(x) \leq c \inf_{x \in B(x_0, r/2)} u(x).$$

Here the constant depends only on the dimension d and can be read of directly from (1.2). If u is harmonic on Ω it is smooth there.

Let us turn to probability: Standard Brownian motion in $\mathbb{R}^d$ starting in $x \in \mathbb{R}^d$ is a stochastic process W_t whose increments $W_{t+s} - W_t$ are normally distributed with mean 0 and covariance $s \cdot \mathrm{Id}$. By Itô's formula $u(W_t) - u(W_0) - \int_0^t \Delta u(W_s) ds$ is a $\mathbb{P}^x$-martingale for all $x \in \mathbb{R}^d$. By optional stopping we get for a function u which is harmonic in an open set Ω

$$u(x) = \mathbb{E}^x\big(u(W_{\tau(\Omega')})\big). \tag{1.3}$$

Here $\tau(\Omega')$ denotes the first time the process exits a set Ω' which is relative compact in Ω. Taking into account that Brownian motion is rotationally invariant with respect to its starting point the distribution of $W_{\tau(B(x_0,r))}$ is the uniform distribution on $S(x_0, r)$. Therefore setting $\Omega' = B(x_0, r)$ we recover the mean value property 1.1. In the same way one sees that the Poisson kernel appearing in (1.2) is precisely the distribution of W_t starting in x at the time it first exits $B(x_0, r)$. In fact, many other analytic objects related to the Laplacian can be expressed in terms of Brownian motion.

There are different ways to generalize the Laplacian by introducing space-dependent coefficients. One straight forward ansatz is to consider second order operators

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} a_{ij}(x) \partial_i \partial_j u(x) \tag{1.4}$$

where the $a_{ij} \colon \mathbb{R}^d \to \mathbb{R}$ are bounded measurable functions.

Stroock and Varadhan [SV06] introduced with the concept of martingale problems a strong tool to relate such operators to Markov processes. Let $\mathcal{D}([0,\infty), \mathbb{R}^d)$ be the space of all càdlàg paths in $\mathbb{R}^d$ i.e. right continuous paths which have left limits. Let X_t the coordinate process. A family $(\mathbb{P}^x)_{x \in \mathbb{R}^d}$ of probability measures on $\mathscr{D}([0,\infty), \mathbb{R}^d)$ is called a solution of the martingale problem associated to $\mathcal{L}$ if for any $x \in \mathbb{R}^d$ it holds that $X_0 = x$ $\mathbb{P}^x$-almost surely and for any $u \in C^2(\mathbb{R}^d)$

$$f(X_t) - f(X_0) - \int_0^t (\mathcal{L}u)(X_s)\, ds$$

is a local $\mathbb{P}^x$-martingale. If there exists a unique solution one says that the martingale problem is well-posed – in this case the solution is a strong Markov family.

For an operator $\mathcal{L}$ as above there exists always a strong Markov solution with continuous paths if $A(x) = (a_{ij}(x))$ is uniformly elliptic, i.e. for all $x \in \mathbb{R}^d$

$$\langle \xi, A(x)\xi \rangle \geq c\, |\xi|^2 .$$

Uniqueness needs additional assumptions in the case $d \geq 3$. It holds for example if the functions a_{ij} are continuous on $\mathbb{R}^d \setminus \{0\}$.

Assume now that $(X_t, \mathbb{P}^x)$ is a strong Markov solution of the martingale problem associated to $\mathcal{L}$. Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C^2(\Omega)$ satisfy $\mathcal{L}u(x) = 0$ for all $x \in \Omega$ – we will also say that u is $\mathcal{L}$-harmonic on Ω. As in the case of Brownian motion u then satisfies a mean value property

$$u(x) = \mathbb{E}^x\big(u(X_{\tau(\Omega')})\big) \tag{1.5}$$

where again $\Omega' \Subset \Omega$. (1.5) already makes sense if u is bounded and measurable and one can use this to define the notion of functions harmonic with respect to X_t. In this setting Krylov and Safonov [KS79] show that $\mathcal{L}$-harmonic functions satisfy the corresponding Harnack inequality (HI) as in the translation-invariant case of Brownian motion. It is also possible to derive a-priori Hölder estimates of the following type:

(HC) *There exist $C > 0$ and $\gamma \in (0,1)$ with the following property: If a function u is bounded and harmonic on $B(x_0, r)$ then*

$$|u(x) - u(y)| \le C \|u\|_\infty R^{-\gamma} \cdot |x - y|^\gamma .$$

Note, that the Harnack inequality (HI) implies (HC) by an iteration argument which is due to Moser.

Operators as (1.4) have some disadvantages when we aim at allowing for irregular coefficients. Namely, the range of $\mathcal{L}$ depends drastically on the regularity of the coefficients a_{ij} and is for example unstable with respect to even small perturbations in the L^∞-norm. A pathway which evades this problem emanates from the concept of weak solutions. Also, this approach opens up the possibility to apply Hilbert space methods. If a function u is harmonic in Ω then testing with $v \in C_0^\infty(\Omega) = 0$ yields

$$0 = \int_\Omega (\Delta u)(x) v(x)\, dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

The right hand of this equations makes sense already for $u \in H^1(\Omega)$. Such u will then be called weakly harmonic on Ω.

One can now introduce bounded measurable coefficients $a_{ij} \colon \mathbb{R}^d \to \mathbb{R}$ with $a_{ij}(x) = a_{ji}(x)$ where the matrix-valued function $A(x) = (a_{ij}(x))$ is again uniformly elliptic and study weak solutions $u \in H^1(\Omega)$ of the divergence-form equation $\operatorname{div} A(x)\nabla u = 0$ in the following sense: Let $\mathcal{E} \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the symmetric bilinear form

$$\mathcal{E}(u, v) = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x) dx.$$

Then a function $u \in H^1_{\mathrm{loc}}(\Omega)$ is called $\mathcal{E}$-harmonic on Ω if $\mathcal{E}(u, v) = 0$ for all $v \in C_0^\infty(\Omega)$. The celebrated localization techniques of deGeorgi, Nash and Moser then show that the set of bounded $\mathcal{E}$-harmonic functions still satisfies a-priori Hölder estimates (HC) and the Harnack inequality (HI).

1.2 Lévy processes and their generators

Dropping the assumption of Gaussian distributed increments in the definition of Brownian motion leads to a much wider class of space- and time-homogeneous Feller processes, so-called Lévy processes. In a sense which becomes apparent in the next section they can be seen as local models of time-homogeneous Feller processes. A comprehensive reference is Sato [Sat99], we additionally refer the reader to Berg-Forst [BF75], Bertoin [Ber96], Applebaum [App04] and Kyprianou [Kyp06].

By definition a Lévy process starting in $x \in \mathbb{R}^d$ is a stochastically continuous process X_t with $X_0 = x$ almost surely such that its increments $X_{t+s} - X_t$ are independent and identically distributed. There always exists a modification of such a process with càdlàg paths, and we will hence assume this here. Basic examples are Brownian motion, the compound Poisson process and the Cauchy process.

Let X_t be a Lévy process. Then for any time t the distribution μ_t of X_t in $\mathbb{R}^d$ is infinitely divisible. Moreover, μ_t is a convolution semigroup and therefore $\widehat{\mu}_t(\xi) = e^{-t\psi(\xi)}$ by the Theorem of Schönberg. The characteristic exponent $\psi\colon \mathbb{R}^d \to \mathbb{C}$ is a continuous negative-definite function with $\psi(0) = 0$. Conversely, any such continuous negative-definite function gives rise to a convolution semigroup and hence a Lévy process. Moreover, the characteristic exponents of Lévy processes are continuous negative-definite functions and can be expressed by the Lévy-Khintchine formula

$$\psi(\xi) = A\xi \cdot \xi + ib \cdot \xi + \int_{\mathbb{R}^d} \left(1 - e^{ix\cdot\xi} + i1_{\{|x|\le 1\}} x \cdot \xi\right) \nu(dx)$$

where A is a nonnegative definite matrix, $b \in \mathbb{R}^d$ and ν a Lévy measure, i.e. a Borel measure von $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(dh) < \infty$. The triple (A, b, ν) – which is unique – is called the Lévy characteristic of (X_t). By the Lévy-Itô decomposition, X_t is the independent sum of a Brownian motion with covariance matrix A, a deterministic drift, a compound Poisson process and a pure jump martingale. The last two parts are described by ν. In the case $\nu(\mathbb{R}^d) = \infty$ the jumping times of the process are dense in $\mathbb{R}^+$ for almost any path.

The generator $\mathcal{L}$ of a Lévy process with Lévy characteristic (A, b, ν) operates on the Banach space $C_b^2(\mathbb{R}^d)$ of twice differentiable functions with all derivatives of order up to 2 bounded as

$$\begin{aligned}\mathcal{L}u(x) = \sum_{i,j=1}^{d} a_{ij}\partial_{ij}u(x) + \sum_{i=1}^{d} b_i \partial_i u(x) \\ + \int_{\mathbb{R}^d} \left(u(x+h) - u(x) - 1_{\{|h|\le 1\}} h \cdot \nabla u(x)\right) \nu(dh).\end{aligned}$$

In the Fourier space, $\mathcal{L}$ acts as multiplication with $-\psi(\xi)$ and can hence be viewed as pseudodifferential operator with symbol $-\psi(\xi)$. Moreover, because of its translation-invariance $\mathcal{L}$ acts as convolution operators on the space of compactly supported distributions.

1.3 General Markov processes

We have seen in the first section that one can, under mild assumptions on the coefficients, associate to a second-order elliptic partial differential operator in non-divergence form a diffusion process via an appropriate martingale problem. A similar approach can also be used in a general framework to relate an linear operator to a strong Markov process, see Ethier-Kurtz [EK86] for a comprehensive study.

Given an operator $\mathcal{L}$ defined on a dense subset of $\mathscr{D}(\mathcal{L}) \subset C(\mathbb{R}^d)$ one can formulate the martingale problem for $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ in the same fashion as in Section 1.1. More precisely, the a probability measure $\mathbb{P}^x$ on the space $\mathcal{D}([0,\infty), \mathbb{R}^d)$ of càdlàg paths is a solution to the martingale problem to $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ if $X_0 = 0$ $\mathbb{P}^x$-almost surely and

$$f(X_t) - f(X_0) - \int_0^t (\mathcal{L}u)(X_s)\,ds$$

is a local $\mathbb{P}^x$-martingale. If the martingale problem for $(\mathcal{L}, \mathscr{D}(\mathcal{L}))$ is well-posed i.e. has for all $x \in \mathbb{R}^d$ a unique solution the corresponding process has the strong Markov property.

The class of operators which we want to study via the martingale problem can be motivated as follows: Let $\mathcal{L}$ be the generator of a Feller process. Then it satisfies the positive maximum principle by the Theorem of Hille and Yoshida. A result of Courrège tells us that such operators locally look like generators of Lévy processes:

Theorem 1.1 ([Cou66]) *Let $\mathcal{L}$ be a linear operator mapping $C_0^\infty(\mathbb{R}^d)$ to $C(\mathbb{R}^d)$ and satisfying the positive maximum principle. Then*

$$\mathcal{L}(u) = \sum_{i,j=1}^d a_{ij}(x)\partial_i\partial_j u(x) + \sum_{i=1}^d b_i(x)\partial_i u(x) + c(x)u(x) \qquad (1.6)$$
$$+ \int \big(u(x+h) - u(x) - 1_{\{|h|<1\}}\nabla u(x)\big)\nu(x,dh)$$

where $(a_{ij}(x))$ is positive definite, $c(x) \geq 0$ and $\nu(x,\cdot)$ is a family of Lévy measures.

Again it is possible to view operators (1.6) as pseudodifferential operators. The symbols $\psi(x,\xi)$ have, by the Theorem of Lévy-Khinchine, the property that $\psi(x,\xi)$ is negative definite for all $x \in \mathbb{R}^d$. This perspective lies at the heart of the work of Hoh, Jacob and others, see [Jac01], [Jac02], [Jac05]. Regularity results in this framework may be found in [Jac94], [Jac92], [Jac93]. Hoh has also developed a pseudodifferential calculus and studied the martingale problem calculus for such operators [Hoh98a]. Another approach to the martingale problem for operators can be found in Komatsu [Kom73].

Generalizing diffusions with generators in divergence form leads to the concept of Dirichlet forms, see [FŌT94]. Shortly, let $\mathcal{F} \subset L^2(\mathbb{R}^d)$ be a dense subspace. Then a Dirichlet form $\mathcal{E}$ is a positive, symmetric and closed bilinear form on $\mathcal{F}$ which is sub-markovian. $\mathcal{E}$ is called regular if $\mathcal{F} \cap C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ with respect to the L^2 norm as well as dense in $C_c(\mathbb{R}^d)$ with respect to the supremum norm. A statement somehow in the same spirit as the Theorem of Courrège is the Beurling-Deny formula:

Theorem 1.2 ([FŌT94, Theorem 3.2.1]) *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $\mathcal{D}(\mathcal{E})$. Then there exists a unique decomposition*

$$\mathcal{E}(u,v) = \mathcal{E}^{(c)}(u,v) + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)k(x,y)dx\,dy + \int\limits_{\mathbb{R}^d} u(x)v(x)j(x)\,dx.$$

Here $\mathcal{E}^{(c)}$ is a regular local Dirichlet form and $k\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ and $j\colon \mathbb{R}^d \to \mathbb{R}^+$ are measurable functions.

Regular Dirichlet forms can be associated to Hunt processes, a class of Markov processes.

One can generalize the strong (with respect to an operator $\mathcal{L}$, probabilistic (with respect to a Markov process) and the weak (with respect to testing with a Dirichlet form) notion of harmonic functions of the first section in an obvious way. These different notions can be related to each other, see Chen [Che09].

1.4 Non-local operators and jump processes

The general class of Markov processes and their generators which we have described in the past section can always be decomposed in a local, diffusive part and a non-local one. There is a highly evolved theory for purely local operators. In the center of this thesis is the other extreme case, purely non-local operators and the related Markov processes. We concentrate on processes which have a infinite intensity of small jumps, i.e. where the jump measure has a singularity in 0, and on the state space $\mathbb{R}^d$, but the setting makes sense also in more general state spaces, see for example Chen-Kumagai [CK03] for d-sets in $\mathbb{R}^d$. In the presence of local and non-local components, i.e. for jump-diffusions, the diffusion often dominates in questions of regularity; for results in this framework see for example Kolokoltsov [Kol00] or [CK09].

Our setup can be outlined as follows: On one hand we study operators which are for $\alpha \in (0,1)$ of the form

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} \big(u(x) - u(x+h)\big)\nu(x,dh), \quad u \in C_b^2(\mathbb{R}^d) \tag{1.7}$$

and for $\alpha \in [1,2)$ of the form

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} \big(u(x) - u(x+h) - 1_{\{|h|<1\}}\nabla u(x)\cdot h\big)\nu(x,dh), \quad u \in C_b^2(\mathbb{R}^d). \tag{1.8}$$

Here $\nu(x,\cdot)$ is a family of Lévy measures with $\sup_x \int_{\mathbb{R}^d} \big(1 \wedge |h|^2\big)\nu(x,dh) < \infty$ and $\nu(x,\mathbb{R}^d) = \infty$ for all $x \in \mathbb{R}^d$. Obviously, by Taylor's theorem $\mathcal{L}$ maps $C_b^2(\mathbb{R}^d)$ to $B(\mathbb{R}^d)$. One then takes the approach via the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. A typical assumption for us is that $\nu(x,\cdot)$ is absolutely continuous, $\nu(x,dh) = n(x,h)dh$ and that there exist $c, R_0 > 0$, $0 < \alpha \leq \beta < 2$ with

$$c^{-1}\,|h|^{-d-\alpha} \leq n(x,h) \leq c\,|h|^{-d-\beta} \qquad \text{for } |h| \leq R_0. \tag{1.9}$$

For many results pointwise bounds as (1.9) are not really necessary. The behavior of the process is dominated by the integrated singularity of ν in 0.

The divergence-form approach circles around purely non-local Dirichlet forms of the type

$$\mathcal{E}(u,v) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big)k(x,y)\,dx\,dy \tag{1.10}$$

where $k\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ is a symmetric measurable function. Again we assume that the singularity of k near the diagonal is at least of order $-d-\alpha$, i.e. that there are $c, R_0 > 0$

and $0 < \alpha \leq \beta < 2$ with

$$c^{-1}\,|x-y|^{-d-\alpha} \leq k(x,y) \leq c\,|x-y|^{-d-\beta} \qquad \text{for } |x-y| \leq R_0, \tag{1.11}$$

$$k(x,y) \leq c \qquad \text{for } |x-y| \geq R_0. \tag{1.12}$$

Then $(\mathcal{E}, H^{\beta/2}(\mathbb{R}^d))$ is a regular Dirichlet form, and we can associate to it a Hunt process X_t. The intensity X_t jumps from x to y then is proportional to $k(x,y)$.

We distinguish the case where we can choose $\alpha = \beta$ in (1.9) resp. (1.11) and call this the fixed order case. We will see in the sequel that in many respects fixed order operators have similar properties as elliptic second-order operators. All other cases are called of variable order – and here interesting phenomena occur.

Let us first describe some regularity results in the translation-invariant case $\nu(x,dh) = \nu(dh)$. Then the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$ is well-posed, the solution being the Lévy process generated by $\mathcal{L}$. The most basic example – and the model at least for the fixed order case – is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ which generates the rotationally symmetric α-stable Lévy process with characteristic exponent $|\xi|^\alpha$ resp. Lévy measure $\nu(dh) = c(d,\alpha)\,|h|^{-d-\alpha}$. Already M. Riesz [Rie38] has calculated the Poisson kernel of the ball for $-(-\Delta)^{\alpha/2}$ and noted that $-(\Delta)^{\alpha/2}$-harmonic or shortly α-harmonic functions also satisfy a Harnack inequality of the following type:

(HI$_{\text{nl}}$) *There exists a constant $c > 0$ with the following property: If a function u is bounded and nonnegative on $\mathbb{R}^d$ and α-harmonic on the ball $B(x_0, r)$ then*

$$\sup_{x \in B(x_0, r/2)} u(x) \leq c \inf_{x \in B(x_0, r/2)} u(x).$$

In applications, one often makes use of the fact that the constant c appearing here is scale-independent.[2] Due to the non-locality of the operator the boundedness and non-negativity of u on $\mathbb{R}^d$ is essential. In fact, one can construct a bounded function u which is α-harmonic on the ball $B_1(0)$ with $u(0) = 0$ which is nonnegative on $B_1(0)$, see Kaßmann [Kas07a]. Also, it is not possible to iterate (HI$_{\text{nl}}$) in the spirit of the Moser iteration to get (HC) for α-harmonic functions. Nonetheless, the fractional Laplacian is a elliptic pseudodifferential operator of Hörmander type and hence α-harmonic functions are smooth on their domain of α-harmonicity. M. Itô [Itô66] shows by analytic means the equivalence of the mean value property of α-harmonic functions and the α-harmonicity. The corresponding Riesz potential plays an important role in potential theory, see for example Blumenthal-Getoor [BG68] or Bliedtner-Hansen [BH86].

Regularity of harmonic functions for generators of symmetric, but not necessarily rotationally symmetric α-stable Lévy processes, so-called anisotropic fractional Laplacians is studied by for example by Bogdan and Sztonyk. The corresponding Lévy measures are symmetric and homogeneous of degree α, i.e. $\nu(B) = \nu(-B)$ and $\nu(B) = r^\alpha \nu(rB)$ for any Borel sets B. Because of its homogenity, there exists a finite Borel measure μ on the sphere S^{d-1} such that

$$\nu(B) = \int_0^\infty \int_{S^{d-1}} 1_B(r\sigma)\,\mu(d\sigma)\,r^{-d-\alpha}\,dr.$$

[2] There are also generalizations of (HI$_{\text{nl}}$) in which the constant c depends on the radius r. In particular, c might explode for $r \to 0$. Such a result is proven by Bass and Kaßmann [BK05a] for a class of variable-order operators resp. processes.

As a consequence, for any $c > 0$ the process is invariant under the scaling $c^{-1/\alpha}X_{ct}$. Let $d \geq 2$. One can precisely characterize the set of symmetric α-stable Lévy measures which satisfy ($\mathrm{HI_{nl}}$), see [BS05] for absolutely continuous measures and [BS07] for the general case. Namely, ($\mathrm{HI_{nl}}$) is equivalent to the relative Kato condition

$$\int_{B(y,1/2)} |x-y|^{\alpha-d}\,\nu(dx) \leq c\nu(B(y,\tfrac{1}{2})) \qquad \text{for all } |y| > 1.$$

In [Szt09] it is shown that under the condition that the Lévy measure ν is a γ-measure on S^{d-1}, i.e. $\nu(B(x,r)) \leq cr^{\gamma}$ for all $|x| = 1$ and $r \in (0,1/2)$, with $\gamma > d-\alpha$, the following a-priori Hölder estimate ($\mathrm{HC_{nl}}$) holds

($\mathrm{HC_{nl}}$) *There exist $C > 0$ and $\gamma \in (0,1)$ with the following property: If a function u is harmonic on the ball $B(x_0,R)$ and bounded on $\mathbb{R}^d$ then for any $x,y \in B(x_0,R/2)$*

$$|u(x)-u(y)| \leq C\,\|u\|_\infty\,R^{-\gamma}\cdot|x-y|^{\gamma}\,.$$

Similar to the non-local Harnack inequality ($\mathrm{HI_{nl}}$) we have now to assume that u is bounded on the whole space. If $\gamma > 1+d-\alpha$ it is also possible to derive corresponding a-priori estimates on the C^1-norm of harmonic functions. This result applies particularly to the case of the fractional Laplacian.

In a completely other spirit is a result by Picard-Savona [PS00]. Here it is shown for – not necessarily α-stable – Lévy measures which behave asymptotically around 0 like a non-degenerate α-stable Lévy measure that the regularity of a function which is harmonic with respect to the Lévy measure in Ω depends on the accessibility of the complement of Ω. More precisely, there exists for any $k \in \mathbb{N}$ a constant $N_k > 0$ such that u is k-times continuously differentiable in $x \in \Omega$ if the Lévy can not reach $\mathbb{R}^d \setminus \Omega$ with less than N_k jumps.

The results of Chapter 2

Chapter 2 picks up the question of smoothness of $\mathcal{L}$-harmonic functions in a much broader context: Assume that the Lévy measure ν is absolutely continuous with smooth density n such that all derivatives of n are square-integrable on $\mathbb{R}^d \setminus B_1(0)$ and in addition

$$\liminf_{r\to 0}\inf_{\omega\in S^{d-1}} r^{\alpha-2}\int_{|h|\leq r} |\omega\cdot h|^2\; n(h)\,dh > 0. \tag{1.13}$$

Then, as Theorem 2.1 states, functions $u \in H^{-\infty}(\mathbb{R}^d)$ are smooth on domains where they are $\mathcal{L}$-harmonic. In contrast to the regularity results mentioned before we do not assume an upper bound on the Lévy measure near 0. In particular, our result covers Lévy measures of variable order, where scaling arguments do not work anymore. As a consequence one can construct examples where Harnack's inequality ($\mathrm{HI_{nl}}$) fails but functions are still smooth on domains where they are harmonic. (1.13) excludes Lévy measures which are degenerate in the sense that their support near 0 lies in a proper subspace of $\mathbb{R}^d$. This condition is for example satisfied if $n(h) \geq c\,|h|^{-d-\alpha}$ near 0 or for non-degenerated smooth α-stable Lévy measures. By Theorem 2.2 the smoothness of the Lévy measure is a necessary condition.

Let us now drop the assumption of translation invariance. Studies of the martingale problem for non-local perturbations of elliptic second-order operators go back to Komatsu [Kom73]. In [Kom84a] he also considers small perturbations of generators of α-stable Lévy processes. Bass' study [Bas88] of the martingale problem for non-local operators (1.8) also includes examples of variable order operators. Existence for example holds as soon as $\mathcal{L}$ maps $C^2(\mathbb{R}^d)$ to the space of uniformly continuous functions. One basic class of examples are stable-like[3] operators with jump measure $\nu(x, dh) = c_{\alpha(x)} |h|^{-d-\alpha(x)} dh$ where $\alpha\colon \mathbb{R}^d \to (0, 2)$, $0 < \inf \alpha(x) \le \sup \alpha(x) < 2$ and $c_{\alpha(x)}$ is chosen such that the symbol of $\mathcal{L}$ is $-|\xi|^{\alpha(x)}$. If α is Lipschitz-continuous one can solve a stochastic differential equation driven by the rotationally symmetric α-stable Lévy process to prove that the martingale problem is well-posed. Bass shows that this also holds if α is merely Dini continuous. For other results and extensions see also Mikulevičius and Pragarauskas [MP90], [MP92] and Bass and Tang [BT09] and the references therein. An approach to uniqueness in the martingale problem by pseudo-differential calculus with non-classical negative definite symbols can be found in Hoh, see [Hoh94], [Hoh95] and [Hoh98a].

A major step in the regularity theory for non-local operators is the work of Bass and Levin [BL02a]. They show that if $\nu(x, dh) = n(x, h)dh$ where $n\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ is measurable with $n(x, h) = n(x, -h)$ and

$$c^{-1} |h|^{-d-\alpha} \le n(x, h) \le c\,|h|^{-d-\alpha} \qquad \text{for all } x, h \in \mathbb{R}^d \tag{1.14}$$

then also the Harnack inequality ($\mathrm{HI_{nl}}$) and Hölder a-priori estimates ($\mathrm{HC_{nl}}$) hold.

Both results of [BL02a] are proven by probabilistic techniques and need only existence – and not uniqueness[4] – of a strong Markov solution of the martingale problem for $(\mathcal{L}, C_b^2(\mathbb{R}^d))$. The machinery behind the proofs – in both cases a clever iteration argument – works in a much wider setting of purely non-local Markov processes. The necessary properties can be stated in terms of the stochastic process. This is demonstrated for the Harnack inequality by Song and Vondraček [SV04], where for example also relativistic α-stable Lévy measures and mixed stable processes are covered. For ($\mathrm{HC_{nl}}$) a generalization is worked out by Bass and Kaßmann in [BK05b]. As an example they establish ($\mathrm{HC_{nl}}$) for stable-like processes with Dini continuous exponent $\alpha(x)$. Hoh [Hoh08] gives the sufficient criteria for ($\mathrm{HC_{nl}}$) in terms of characteristic exponents, c.f. the discussion in the previous section. Bass and Chen apply the techniques in the context of solutions of stochastic differential equations driven by rotational symmetric α-stable processes [BC09]. There are also parabolic versions of both proofs, see for example Chen and Kumagai [CK03]. A purely analytic proof which makes use of so-called growth lemmas can be found in the work of Silvestre [Sil06]. An interesting feature in this approach is that for stable-like operators no regularity of the function α is needed. For fixed-order ($\alpha = \beta$) Dirichlet forms $\mathcal{E}$ which satisfy (1.11) and (1.12) Kaßmann [Kas09] has developed analytic techniques in the spirit of deGeorgi, Nash and Moser to prove ($\mathrm{HC_{nl}}$) for weakly harmonic functions.

The Bass-Levin proof of ($\mathrm{HC_{nl}}$) only depends on two properties of the Markov process X_t: First, one needs an upper bound of the probability of exiting a ball by jumping outside of a bigger, concentric ball of the following type:

[3] Some authors also speak of stable-like processes in the case of pure jump Markov processes whose jump measures are comparable to those of an rotationally invariant α-stable process in the sense of (1.14).

[4] In fact, a-priori Hölder estimates on harmonic functions are a useful tool for proving uniqueness, as it is indicated in chapter 3.

(*) *There exists $c > 0$ and $\gamma \in (0,1)$ such that for any $R > 2r > 0$ and any $x \in \mathbb{R}^d$*

$$\mathbb{P}^x\big(X_{\tau_{B(x,r)}} \notin B(x,R)\big) \leq c\frac{r^\gamma}{R^\gamma}.$$

Assumption (*) – which is only necessary for r and R small – is not very restrictive and needs essentially a lower bound on the jump measure in sufficiently many directions near 0. Really at the heart of the proof of ($\mathrm{HC_{nl}}$) is a hitting-time estimate in the spirit of Krylov-Safanov [KS79]:

(KS) *There exist $c > 0$ and $\delta \in (0,1/2)$ such that for any measurable subset $A \subset B(x,r)$ with $|A| > \delta\,|B(x,r)|$ and any $y \in B(x,r/2)$ the probability of hitting A before exiting the ball $B(x,r)$ under $\mathbb{P}^y$ is bounded from below by c:*

$$\mathbb{P}^y(T_A < \tau_{B(x,r)}) \geq c. \tag{1.15}$$

Here, $|A|$ denotes the Lebesgue measure of A. (KS) is only needed for small radii r as well. It is a condition which demands the small jumps of the process to be "isotropic" enough. The constant δ might also depend on a and r.[5]

The results of chapter 3

It is an interesting question whether one can prove a-priori continuity estimates for harmonic functions in situations where (KS) does not hold anymore, but where the lower bound in (1.16) decreases for to 0 for $r \to 0$. We call this degenerate hitting time estimates of Krylov-Safonov type. We address this issue in chapter 3. As it turns out, this is indeed possible in some cases, but the a-priori control on the continuity module one gets is weaker than Hölder. Anyhow, such controls are still sufficient for many applications of ($\mathrm{HC_{nl}}$).

More precisely, we use the following degenerated Krylov-Safonov estimate:

(degKS) *There exist $N \geq 0$, $\delta \in (0,1/2)$ and a function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$\liminf_{r\to 0} \varphi(r)\cdot|\log r|\cdot\prod_{i=0}^{N}\log^i(|\log r|) > 0$$

such that the following holds: For any measurable subset $A \subset B(x,r)$ with $|A| > \delta\,|B(x,r)|$ and any $y \in B(x,r/2)$ we have:

$$\mathbb{P}^y(T_A < \tau_{B(x,r)}) \geq \varphi(r). \tag{1.16}$$

Here $\log^i$ denotes the i-times iterated logarithm with the convention that $\log^0(x) = 1$.

Theorem 3.2 can be phrased in the following way: Given a pure-jump strong Markov process X_t satisfying (*) and (degKS), the following a-priori continuity estimate for functions which are harmonic with respect to X_t holds:

[5] It is sufficient for the proof of ($\mathrm{HC_{nl}}$) to work that each $A \subset B(x,r)$ has a compact subset that is hit with probability $\geq c$ before exiting $B(x,r)$.

(C) *For any $R > 0$ there exists a function $\vartheta\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{r\to 0} \vartheta(r) = 0$ such that the following holds: If a function u is harmonic on the ball $B(x_0, R)$ and bounded on $\mathbb{R}^d$ then for any $x, y \in B(x_0, R/2)$*

$$|u(x) - u(y)| \leq \vartheta(|x-y|)\, \|u\|_\infty\,.$$

We formulate this result in chapter 3 in terms of the jump kernel $\nu(x, dh)$ for strong Markov solutions of the martingale problem for operators of type (1.4). We also construct an example which satisfies (degKS) but not (KS) in Theorem 3.1 – in addition, ($\mathrm{HI}_{\mathrm{nl}}$) is not satisfied here.

The above stated result is also interesting if one relates it to the work of Bass, Barlow, Chen and Kaßmann. Namely, they construct in [BBCK09] for any $0 < \alpha < \beta < 1$ a variable order kernel k such that there exists a non-continuous function which is harmonic with respect to the corresponding Hunt process. Notably, our regularity result can be applied to variable order kernels for which the growth near the diagonal differs at most at a logarithmic scale, see Theorem 3.3.

The results of Chapter 4

Finally let us turn to a question of quite different flavor: Is it possible to approximate non-local processes as in the previous section by somehow "simpler" Markov processes? For example, interesting canditates are Markov chains which are more suitable for numerical simulations. Such results for diffusions have been derived in the non-divergence form case by Stroock and Varadhan [SV06] and in the divergence form case by Stroock and Zheng [SZ97]. Bass and Kumagai [BK08] have similar results for Markov chains of unbounded range where again the limit process is a diffusion.

In the last chapter of this thesis we approach this question for a jump process X_t corresponding to non-local Dirichlet form with measurable, symmetric kernel $k\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ which satisfies

$$c\,|x-y|^{-d-\alpha} \leq k(x,y) \leq c^{-1}\,|x-y|^{-d-\alpha} \qquad \text{for all } x, y \in \mathbb{R}^d.$$

We construct a sequence of unbounded Markov chains whose state-spaces are grids getting finer and finer such that, after specifying adequate starting points, they approximate X_t. See Theorem 4.3 for a precise statement. We also give sufficient conditions for tightness in Theorem 4.20. The focus lies here on allowing for as much as anisotropy as possible. For example, the Markov chains are not required to connect any two points of the state space directly by one jump, but only by a finite number of jumps.

Let us finally mention also that Bass, Kumagai and Uemura recently have studied weak convergence of Markov chains to jump-diffusions [BKU08].

2 On hypoellipticity of generators of Lévy processes

2.1 Introduction

Hypoellipticity of elliptic partial differential operators or, more generally, pseudodifferential operators is one of the classical topics in the theory of partial differential equations. Briefly, an operator $\mathcal{L}$ is called hypoelliptic if the singular support of u is contained in the singular support of $\mathcal{L}u$ for all u in the domain of $\mathcal{L}$. In particular, hypoellipticity comprises the C^∞-regularity of functions on their domains of $\mathcal{L}$-harmonicity where we call $u\colon \mathbb{R}^d \to \mathbb{R}$ $\mathcal{L}$-harmonic on Ω if $\mathcal{L}u = 0$ on Ω.

This analytic notion has, if $\mathcal{L}$ generates a strong Markov process $(X_t)_{t\geq 0}$ in an appropriate way, a probabilistic counter-part. More precisely, a bounded measurable function u is said to be harmonic on Ω with respect to $(X_t)_{t\geq 0}$ if $u(X_{t\wedge\tau_\Omega})$ is, for all $x \in \mathbb{R}^d$, a local $\mathbb{P}^x$-martingale. Here τ_Ω denotes the first exit time of Ω and $\mathbb{P}^x$ is the probability measure under which the process starts in x, i.e. $X_0 = x$ a.s.. If $(X_t)_{t\geq 0}$ is a Brownian motion, this yields the mean value property of classical harmonic functions. In fact, harmonicity with respect to a reasonable Markov process can always be defined by a generalized mean-value property, see for example [BH86]. Functions harmonic in this sense play an important role in the potential theory of Markov processes. This motivates an increasing interest for example in questions of regularity of these operators by probabilists. Since, by the Theorem of Courrège [Cou66], generators of Feller processes are pseudodifferential operators with continuous negative definite symbols as described for example in [Jac01, Jac02, Jac05] or [Hoh98a], generally they do not fit in the framework of classical symbol classes, for example the Hörmander class $S^m_{1,0}$.

Regularity of functions which are harmonic with respect to jump processes has been an object of intense studies in the last years. Let us mention here for example [PS00], [BL02a], [BK05b], [SU07], [HK09], [Szt09], [Hoh08], [Kas09]. Most of these papers deal with a-priori continuity estimates in the broader context of processes with space-dependent jump measures. They rely on a delicate interplay between lower and upper bounds on the jump measures, i.e., they deal with fixed order operators where the small jumps are in principle comparable to those of an rotationally symmetric α-stable Lévy process. The variable order case is far more difficult as it is for example emphasized by a counter-example in [BBCK09].

In this chapter we concentrate on stochastic processes with stationary and independent increments, so-called Lévy processes. Their generators are translation-invariant and map $C_0^\infty(\mathbb{R}^d)$ continuously to $C^\infty(\mathbb{R}^d)$. Hence they act as convolution with a distribution. We show that in this case essentially smoothness and a lower bound on the Lévy measure are enough to yield smoothness of harmonic functions.

Let us give a precise formulation of our results: Let ν be a Lévy measure, i.e., a non-negative Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \min(1, |h|^2)\nu(dh) < \infty$.

Moreover, we assume that ν is absolutely continuous with respect to the Lebesgue measure with a density n satisfying the following assumptions:

(A1) $n \in C^\infty(\mathbb{R}^d \setminus \{0\})$ and $n|_{\mathbb{R}^d \setminus B_1(0)} \in H^\infty(\mathbb{R}^d \setminus B_1(0))$.

(A2) There exists $r_0, c > 0$, $\alpha \in (0,2)$ such that for all $\omega \in S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, $0 < r \leq r_0$:

$$\int_{|h|\leq r} |h \cdot \omega|^2 \, \nu(dh) \geq cr^{2-\alpha}. \tag{2.1}$$

(A1) ensures that n is smooth on $\mathbb{R}^d \setminus \{0\}$ with all its derivatives square-integrable away from 0. Note also, that (A2) only assumes a lower bound on the growth of ν near 0. For example, (A2) holds if $n(h) \geq c\,|h|^{-d-\alpha}$ near 0. The generator of the associated Lévy process L is on $C_b^2(\mathbb{R}^d)$ given by

$$Lu(x) = \int_{\mathbb{R}^d} \left(u(x+h) - u(x) - \frac{h \cdot \nabla u(x)}{1 + |x|^2} \right) \nu(dx). \tag{2.2}$$

It acts in the Fourier space as multiplication operator with the continuous negative-definite function associated to ν by the Lévy-Khinchine formula, cf. (2.5) below. Moreover, it is of order 2 on certain weighted Sobolev spaces $H^{\psi,s}(\mathbb{R}^d)$, see Section 2.2 for precise definitions.

We say that L is locally hypoelliptic with respect to $H = H^{-\infty}(\mathbb{R}^d)$ or $H = H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, if for any $f \in H$ and a distribution $u \in \mathscr{S}'(\mathbb{R}^d)$ such that $Lu = f$ in $\mathbb{R}^d$ and $U \subset \mathbb{R}^d$ open we have

$$f|_U \in C^\infty(U) \quad \Rightarrow \quad u|_U \in C^\infty(U).$$

The translation invariance of L implies that local hypoellipticity of L on $H^{s_0}(\mathbb{R}^d)$ for some $s_0 \in \mathbb{R}$ entails local hypoellipticity on $H^{-\infty}(\mathbb{R}^d)$, cf. Lemma 2.6 below. Therefore we will call L sometimes simply locally hypoelliptic in this case.

Our main results now reads as follows:

Theorem 2.1 *Let ν be an absolutely continuous Lévy measure with density n that satisfies (A1)–(A2). Then the generator of the pure-jump Lévy process L given by (2.2) is locally hypoelliptic on $H^{-\infty}(\mathbb{R}^d)$.*

Moreover, in the case that ν is a compactly supported Lévy measure satisfying (2.1) it is also necessary that ν is smooth on $\mathbb{R}^d \setminus \{0\}$. More precisely, we have

Theorem 2.2 *Let ν be a compactly supported Lévy measure that satisfies (A2). Assume furthermore that L is locally hypoelliptic on $H^{-\infty}(\mathbb{R}^d)$. Then ν is absolutely continuous with a density which is smooth on $\mathbb{R}^d \setminus \{0\}$.*

Note that a compactly supported Lévy measure with smooth density on $\mathbb{R}^d \setminus \{0\}$ automatically satisfies (A1). Hence (A1) is also necessary in that case.

We want to finish this section by some examples.

Let $\alpha \in (0,2)$ and $f\colon S^{d-1} \to \mathbb{R}^+$ be a smooth function such that the support of f is not contained in any proper subspace of $\mathbb{R}^d$. We set $\nu(dh) = |h|^{-d-\alpha} f(h/|h|)$. Then ν

satisfies (A1) and (A2) and therefore the generator of the associated symmetric α-stable Lévy process is hypoelliptic.

It is also interesting to remark the following: In [BK05a] there is given a counter-example of a Lévy process which does not admit a scale-invariant Harnack inequality. One can modify this construction in an obvious way such that our results apply. Hence in this example the related Lévy process has still a hypoelliptic generator.

2.2 Prerequisites

We start by recalling some basic concepts. Details can be found for example in [BF75], [Sat99] and [Jac01, Jac02, Jac05].

We denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space, by $\mathscr{D}'(\mathbb{R}^d)$ the space of distributions, i.e. the topological dual of $C_0^\infty(\mathbb{R}^d)$, by $\mathscr{E}'(\mathbb{R}^d)$ the space of compactly supported distributions, and by $\mathscr{S}'(\mathbb{R}^d)$ the space of tempered distributions, i.e., the dual of $\mathscr{S}(\mathbb{R}^d)$. Moreover, let $H^s(\mathbb{R}^d)$ be the usual L^2-Sobolev space of order $s \in \mathbb{R}$. Furthermore we set $H^\infty(\mathbb{R}^d) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$ and $H^{-\infty}(\mathbb{R}^d) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$. We also write $\mathscr{F}$ for the Fourier transform and denote $\widehat{u} = \mathscr{F}[u]$. Note that by the Paley-Wiener Theorem [Hör03, Theorem 7.3.1] $\mathscr{E}'(\mathbb{R}^d) \subset H^{-\infty}(\mathbb{R}^d)$.

A function $\phi\colon \mathbb{R}^d \to \mathbb{C}$ is called negative definite if the matrix $(\phi(\xi_i) + \overline{\phi(\xi_j)} - \phi(\xi_i - \xi_j))_{i,j=1}^k$ is positive definite for each choice of $k \in \mathbb{N}$, $\xi_1, \dots \xi_k \in \mathbb{R}^d$. Then ϕ satisfies for example $\phi(\xi) = \overline{\phi(-\xi)}$, $\Re\phi(\xi) \geq \phi(0)$ and the Peetre-type inequality

$$\frac{(1+|\phi(\xi)|)^s}{(1+|\phi(\eta)|)^s} \leq 2^{|s|}(1+|\phi(\xi-\eta)|)^{|s|}. \tag{2.3}$$

Note also the estimate:

$$|\phi(\xi)-\phi(\eta)| \leq 4(1+|\phi(\xi-\eta)|)(1+\sqrt{\operatorname{Re}\phi(\xi)}). \tag{2.4}$$

This follows from the third inequality of Lemma 3.6.21 in [Jac01] which implies

$$\begin{aligned} |\phi(\xi)-\phi(\eta)| &\leq |\phi(\xi)-\phi(\eta)+\phi(\eta-\xi)| + |\phi(\eta-\xi)| \\ &\leq 2\sqrt{\operatorname{Re}\phi(\eta-\xi)}\sqrt{\operatorname{Re}\phi(\xi)} + |\phi(\eta-\xi)|\,(1+\sqrt{\operatorname{Re}\phi(\xi)}). \end{aligned}$$

If ϕ is locally bounded we have in addition $|\phi(\xi)| \leq c(1+|\xi|^2)$. The set of continuous negative definite functions $CN(\mathbb{R}^d)$ is a convex cone closed in the topology of uniform convergence on compact sets. Each $\phi \in CN(\mathbb{R}^d)$ has the unique Lévy-Khinchine representation

$$\phi(\xi) = b + A\xi\cdot\xi + i\xi\cdot\gamma + \int_{\mathbb{R}^d} \left(1 - e^{ih\cdot\xi} + \frac{ih\cdot\xi}{1+|h|^2}\right)\nu(dh). \tag{2.5}$$

Here, $b \geq 0$, A is a positive definite matrix, $\gamma \in \mathbb{R}^d$ and ν is a Lévy measure, i.e., a non-negative Borel measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and $\int(1\wedge|h|^2)\nu(dh) < \infty$.

By the Theorem of Schönberg [BF75, Thm. 7.8], the elements $\phi \in CN(\mathbb{R}^d)$ satisfying $\phi(0) = 0$ are in one-to-one correspondence with Lévy processes $(X_t)_{t\geq 0}$. More precisely, the Fourier transform of the distribution of X_t in $\mathbb{R}^d$ is given by $e^{-t\phi(\xi)}$, on the other hand the generator of $(X_t)_{t\geq 0}$ is the pseudodifferential operator with symbol $-\phi(\xi)$

$$-\phi(D_x)u(x) = -\mathscr{F}_{\xi\mapsto x}\left[\phi(\xi)\widehat{u}(\xi)\right] = -\int_{\mathbb{R}^d} e^{ix\cdot\xi}\phi(\xi)\widehat{u}(\xi)đ\xi,$$

where $đ\xi = (2\pi)^{-d}d\xi$.

An important example for continuous negative definite functions are the functions $\xi \mapsto |\xi|^\alpha$ where $\alpha \in (0,2]$. The corresponding Lévy processes are the rotationally invariant α-stable Lévy processes for $\alpha \in (0,2)$ and in particular a Brownian motion for $\alpha = 2$.

In our framework it is useful to introduce weighted (or anisotropic) Sobolev spaces tailored on the operators we consider here. We fix a continuous negative definite reference function $\psi\colon \mathbb{R}^d \to \mathbb{C}$ which satisfies $\lim_{|\xi|\to\infty}|\psi(\xi)| = \infty$ and set

$$H^{\psi,s}(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) : \|u\|_{\psi,s} := \left\|(1+|\psi(\cdot)|)^{s/2}\widehat{u}\right\|_{L^2(\mathbb{R}^d)} < \infty\right\}.$$

Define also for an open set $\Omega \subset \mathbb{R}^d$

$$H^{\psi,s}_{\mathrm{loc}}(\Omega) = \left\{u\colon \Omega \to \mathbb{R} : \chi u \in H^{\psi,s}(\mathbb{R}^d) \text{ for all } \chi \in C_0^\infty(\Omega)\right\}.$$

We have $H^s(\mathbb{R}^d) = H^{|\xi|^2,s}(\mathbb{R}^d)$ and $H^s(\mathbb{R}^d) \subset H^{\psi,s}(\mathbb{R}^d)$ due to $|\psi(s)| \leq C(1+|\xi|^2)$. Note also, that $(H^{\psi,s}(\mathbb{R}^d))^* \cong H^{\psi,-s}(\mathbb{R}^d)$: u is in $H^{\psi,s}(\mathbb{R}^d)$ if and only if $(u,v)_{L^2} \leq c\,\|v\|_{\psi,-s}$ for all $v \in H^{\psi,-s}(\mathbb{R}^d)$.

Let $\phi \in CN(\mathbb{R}^d)$ satisfy the following conditions:

(S1) There exists $\kappa_1 > 0$ such that $|\phi(\xi)| \leq \kappa_1(1+|\psi(\xi)|)$.

(S2) There exist $\kappa_2, r_0 > 0$ such that $|\phi(\xi)| \geq \kappa_2\,|\psi(\xi)|$ if $|\xi| \geq r_0$.

Then, by (S1), $\phi(D_x)$ maps $H^{\psi,s+2}(\mathbb{R}^d)$ continuously to $H^{\psi,s}(\mathbb{R}^d)$.

Theorem 2.3 *Let ϕ satisfy (S1) and (S2). Let $t \in \mathbb{R}$ and $f \in H^{\psi,t}(\mathbb{R}^d)$. If $u \in H^{-\infty}(\mathbb{R}^d)$ is a solution of $\phi(D_x)u = f$ in $\mathscr{S}'(\mathbb{R}^d)$, then $u \in H^{\psi,t+2}(\mathbb{R}^d)$.*

Proof: Without loss of generality we may assume $f \in L^2(\mathbb{R}^d)$. Then $\phi\widehat{u} \in L^2(\mathbb{R}^d)$, $\widehat{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and $\lim_{|\xi|\to\infty}|\phi(\xi)| = \infty$ imply $(1+|\phi|)\widehat{u} \in L^2(\mathbb{R}^d)$. Thus (S2) implies the statement of the Theorem. ■

Moreover, the commutator $[\phi(D_x),\chi]$ of $\phi(D_x)$ and the multiplication with $\chi \in C_0^\infty(\mathbb{R}^d)$ acts with order 1 in $H^{\psi,-\infty}(\mathbb{R}^d)$, i.e.:

Lemma 2.4 *Let ϕ satisfy (S1) and (S2), $t \in \mathbb{R}$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then for all $u \in H^{\psi,t+1}(\mathbb{R}^d)$ we have*

$$\|[\phi(D_x),\chi]u\|_{t,\psi} \leq c\,\|u\|_{t+1,\psi}.$$

Proof: Let $u, v \in \mathscr{S}(\mathbb{R}^d)$. Then on one hand we have

$$\mathscr{F}\big(\big[\phi(D_x),\chi\big]u\big)(\xi) = \int \widehat{\chi}(\xi-\eta)\big(\phi(\xi)-\phi(\eta)\big)\widehat{u}(\eta)\,d\eta.$$

By the Theorem of Plancherel, (S1), (2.4) and (2.3) we estimate:

$$\begin{aligned}
\left|(\phi(D_x),\chi]u,v)_{L^2(\Omega)}\right| &\le \iint |\widehat{\chi}(\xi-\eta)|\,|\phi(\xi)-\phi(\eta)|\,|\widehat{u}(\eta)|\,|\widehat{v}(\xi)|\,d\eta\,d\xi \\
&\le C\iint |\widehat{\chi}(\xi-\eta)|\,(1+|\xi-\eta|^2)(1+|\psi(\xi)|)^{1/2}\,|\widehat{u}(\eta)|\,|\widehat{v}(\xi)|\;d\eta\,d\xi \\
&= C\iint |\widehat{\chi}(\xi-\eta)|\,(1+|\xi-\eta|^2)\left(\frac{1+|\psi(\xi)|}{1+|\psi(\eta)|}\right)^{(t+1)/2} \\
&\qquad\cdot(1+|\psi(\eta)|)^{(t+1)/2}\,|\widehat{u}(\eta)|\,(1+|\psi(\xi)|)^{-t/2}\,|\widehat{v}(\xi)|\;d\eta\,d\xi \\
&\le C\iint |\widehat{\chi}(\xi-\eta)|\,(1+|\xi-\eta|^2)^{(|t|+3)/2}(1+|\psi(\eta)|)^{(t+1)/2}\,|\widehat{u}(\eta)| \\
&\qquad\cdot(1+|\psi(\xi)|)^{-t/2}\,|\widehat{v}(\xi)|\;d\eta\,d\xi \\
&\le C\big\|(1+|\xi|^2)^{(|t|+3)/2}\widehat{\chi}(\xi)\big\|_{L_1(\mathbb{R}^d)}\,\|u\|_{\psi,t+1}\,\|v\|_{\psi,-t} \le C\,\|u\|_{\psi,t+1}\,\|v\|_{\psi,-t}\,.
\end{aligned}$$

The assertion now follows by continuity and the characterization of $H^{\psi,s}(\mathbb{R}^d)$ as dual of $H^{\psi,-s}(\mathbb{R}^d)$. ■

A direct application of this commutator estimate yields local regularity of the following type:

Theorem 2.5 *Let ϕ satisfy (S1) and (S2). If $\chi \in C_0^\infty(\mathbb{R}^d)$, $t \in \mathbb{R}$, $f \in L^2(\mathbb{R}^d)$ with $\chi f \in H^{\psi,t}(\mathbb{R}^d)$ and $u \in H^{\psi,t+1}(\mathbb{R}^d)$ solves $\phi(D_x)u = f$, then $\chi u \in H^{\psi,t+2}(\mathbb{R}^d)$.*

Unfortunately, we cannot expect to iterate this result without additional assumptions as it is illustrated by Theorem 2.2.

We finish this section by showing that the notion of local hypoellipticity with respect to $H^s(\mathbb{R}^d)$ is independent of $s \in \mathbb{R}$.

Lemma 2.6 *Let $s_0 \in \mathbb{R}$. If L is locally hypoelliptic with respect to $H^{s_0}(\mathbb{R}^d)$, then L is locally hypoelliptic with respect to $H^{-\infty}(\mathbb{R}^d)$.*

Proof: Let L be locally hypoelliptic with respect to $H^{s_0}(\mathbb{R}^d)$. In order to prove that L is locally hypoelliptic with respect to $H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, let $f \in H^s(\mathbb{R}^d)$, $u \in \mathscr{S}'(\mathbb{R}^d)$ with $Lu = f$ and $U \subset \mathbb{R}^d$ open such that $f|_U \in C^\infty(U)$. Then $f' = \langle D_x\rangle^{s_0-s} f \in H^s(\mathbb{R}^d)$, where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ and $\langle D_x\rangle^s$ denotes the pseudodifferential operator with symbol $\langle\xi\rangle^s$. Moreover, $Lu' = f'$ with $u' = \langle D_x\rangle^{s_0-s}u$ since L commutes with $\langle D_x\rangle^{s_0-s}$. Since $\langle\xi\rangle^{s_0-s} \in S^{s_0-s}_{1,0}(\mathbb{R}^d\times\mathbb{R}^d)$, $\langle D_x\rangle^{s_0-s}$ is pseudo-local, i.e., $\operatorname{sing\,supp} f' = \operatorname{sing\,supp}\langle D_x\rangle^{s_0-s} f \subseteq \operatorname{sing\,supp} f$, cf. e.g. [Hör05, Theorem 18.1.16]. Hence $f'|_U \in C^\infty(U)$ and therefore $u'|_U \in C^\infty(U)$ due to the local hypoellipticity with respect to $H^{s_0}(\mathbb{R}^d)$. Finally, since $\langle D_x\rangle^{s-s_0}$ is pseudo-local too, $\operatorname{sing\,supp} u = \operatorname{sing\,supp}\langle D_x\rangle^{s-s_0}u' \subseteq \operatorname{sing\,supp} u'$. Thus $u|_U \in C^\infty(U)$. This shows that L is locally hypoelliptic with respect to $H^s(\mathbb{R}^d)$ for any $s \in \mathbb{R}$. Hence L is locally hypoelliptic with respect to $H^{-\infty}(\mathbb{R}^d)$. ■

2.3 Proof of Theorem 2.1

Let ψ be the continuous negative definite function associated by (2.5) to the pure-jump Lévy process with Lévy measure ν and let L be its generator. The real part of ψ is

$$\operatorname{Re}\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(h\cdot\xi))\nu(dh). \tag{2.6}$$

Lemma 2.7 *Assume (A2). Then there exists some $c > 0$ such that $1 + |\psi(\xi)| \geq c\,|\xi|^\alpha$.*

Proof: Using (A2) and the inequality $1 - \cos x \geq \frac{x^2}{4}$ for $|x| \leq \frac{1}{2}$, we estimate for all $|\xi| \geq (2r_1)^{-1}$

$$\begin{aligned} |\psi(\xi)| \geq \operatorname{Re}\psi(\xi) &\geq \int_{|h|\leq(2|\xi|)^{-1}} (1-\cos(h\cdot\xi))\nu(dh) \\ &\geq \frac{|\xi|^2}{4}\int_{|h|\leq(2|\xi|)^{-1}} \left|h\cdot|\xi|^{-1}\xi\right|^2 \nu(dh) \geq c\,|\xi|^\alpha . \end{aligned}$$

■

Therefore for all $s > 0$ the anisotropic Sobolev space $H^{\psi,s}(\mathbb{R}^d)$ is continuously embedded in $H^{\alpha s/2}(\mathbb{R}^d)$.

Observe that by (A2) the asymptotic behavior of $|\psi(\xi)|$ for $|\xi| \to \infty$ remains unchanged if one cuts off the large jumps of ν in the following sense: Fix for $r > 0$ a function $\rho_r \in C_0^\infty(B_{2r}(0))$ with $0 \leq \rho_r \leq 1$ and $\rho_r \equiv 1$ on $B_r(0)$. Then ψ can be decomposed as $\psi = \psi_{r,\text{long}} + \psi_{r,\text{short}}$ where

$$\begin{aligned} \psi_{r,\text{long}}(\xi) &= \int_{\mathbb{R}^d}\Big(1 - e^{ih\cdot\xi})(1-\rho_r(h)\Big)\,\nu(dh), \\ \psi_{r,\text{short}}(\xi) &= \int_{\mathbb{R}^d}\left(1 - e^{ih\cdot\xi} + \frac{ih\cdot\xi}{1+|h|^2}\right)\rho_r(h)\nu(dh) + i\xi\cdot\int_{\mathbb{R}^d}\frac{h(1-\rho_r(h))}{1+|h|^2}\nu(dh). \end{aligned}$$

Because $\psi_{r,\text{long}}$ is bounded, Lemma 2.7 implies that $\psi_{r,\text{short}}$ satisfies (S1) and (S2). Note also that the operator associated to $\psi_{r,\text{short}}$ is $2r$-local in the sense that

$$\operatorname{supp}\psi_{r,\text{short}}(D_x)u \subset B_{2r}(\operatorname{supp} u) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \operatorname{supp} u) < 2r\}$$

For the following we also assume that $\nu(dh) = n(h)dh$ and the density n satisfies (A1). The key step in our argument is the following regularity result.

Lemma 2.8 *Let $\Omega \subset \mathbb{R}^d$ be open. If $f \in L^2(\mathbb{R}^d)$ with $f|_\Omega \in H^{\psi,t}_{\text{loc}}(\Omega)$ and $u \in H^{\psi,1}(\mathbb{R}^d)$ with $u|_\Omega \in H^{\psi,t+1}_{\text{loc}}(\Omega)$ solve $\psi(D_x)u = f$ in $\mathscr{S}'(\mathbb{R}^d)$, then $u|_\Omega \in H^{\psi,t+2}_{\text{loc}}(\Omega)$.*

Proof: Let $\chi_1 \in C_0^\infty(\Omega)$. We fix $r > 0$ such that $4r < \operatorname{dist}(\mathbb{R}^d \setminus \Omega, \operatorname{supp}\chi_1)$ and choose a cut-off function $\chi_2 \in C_0^\infty(\Omega)$ with $\chi_2 \equiv 1$ on $B_{4r}(\operatorname{supp}\chi_1)$. If $\psi_{r,\text{short}}$ and $\psi_{r,\text{long}}$ are as above, then $\chi_2 u$ solves

$$\psi_{r,\text{short}}(D_x)(\chi_2 u) = f - \psi_{r,\text{long}}(D_x)u - \psi_{r,\text{short}}(D_x)\big((1-\chi_2)u\big) = \widetilde{f} \quad \text{in } \mathscr{S}'(\mathbb{R}^d),$$

where $\widetilde{f}$ is in $L^2(\mathbb{R}^d)$. By (A1), $\psi_{r,\text{long}}(D_x)u$ is the sum of a convolution of u with $(1-\rho_r)n \in H^\infty(\mathbb{R}^d)$ – which is smooth – and a constant multiple of u. Since $\operatorname{supp}(1-\chi_2)u \subset$

$\mathbb{R}^d \setminus B_{4r}(\operatorname{supp}\chi_1)$ the support of $\psi_{r,\text{short}}(D_x)\big((1-\chi_2)u\big)$ is contained in $\mathbb{R}^d \setminus \operatorname{supp}\chi_1$. Hence $\chi_1\widetilde{f} \in H^{\psi,t}(\mathbb{R}^d)$, and Theorem 2.5 yields $\chi_1\chi_2 u = \chi_1 u \in H^{\psi,t+2}(\mathbb{R}^d)$. ■

Proof of Theorem 2.1: Because of Lemma 2.6 it is sufficient to prove that L is locally hypoelliptic with respect to $L^2(\mathbb{R}^d)$. To this end let $\Omega \subset \mathbb{R}^d$ and let $u \in \mathscr{S}'(\mathbb{R}^d)$ be a solution of $\psi(D_x)u = f$ in $\mathscr{S}'(\mathbb{R}^d)$ with $f \in L^2(\mathbb{R}^d)$ and $f|_\Omega \in C^\infty(\Omega)$. Moreover, let $\chi \in C^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ on $\mathbb{R}^d \setminus B_2(0)$ and $\chi \equiv 0$ on $B_1(0)$ and let $u_1 = \chi(D_x)u$. Then $u = u_1 + u_2$ with $u_2 \in C^\infty(\mathbb{R}^d)$ and $u_1 \in \mathscr{E}'(\mathbb{R}^d) \hookrightarrow H^{-\infty}(\mathbb{R}^d)$ solves

$$Lu_1 = f_1 := \chi(D_x)f \qquad \text{in } \mathscr{S}'(\mathbb{R}^d).$$

Here $f_1 \in L^2(\mathbb{R}^d)$ by Plancherel's theorem and $\operatorname{sing\,supp} f_1 = \operatorname{sing\,supp} f_2$ because of [Hör05, Theorem 18.1.16]. Since $f_1 \in H^{\psi,\infty}_{\text{loc}}(\Omega)$, iterating Lemma 2.8 implies $u_1 \in H^{\psi,\infty}_{\text{loc}}(\Omega)$. By Lemma 2.7 we have $u_1 \in H^{\infty}_{\text{loc}}(\Omega)$ and therefore, by Sobolev embedding, $u_1 \in C^\infty(\Omega)$. Since $u = u_1 + u_2$ with $u_2 \in C^\infty(\mathbb{R}^d)$, this finishes the proof. ■

2.4 Proof of Theorem 2.2

The proof of Theorem 2.2 is based on the results of Chapter 16 in [Hör05], which will be summarized below. Let us first fix some notation: If E is a set, then $\operatorname{ch} E$ denotes its convex hull. The supporting function of a convex, compact subset $E \subset \mathbb{R}^d$ is given by

$$H_E(x) = \sup_{y \in E} x \cdot y$$

where $H_\emptyset \equiv -\infty$ by definition. This gives a one-to-one correspondence between convex compact subsets and the set $\mathcal{H}$ of convex, subadditive, positively homogenous functions. For each $u \in \mathscr{E}'(\mathbb{R}^d)$ we denote by $\mathcal{H}(u) \subset \mathcal{H}$ the same set as defined in [Hör05, Definition 16.3.2]. We omit the precise definition at this point since it is a bit involved and not needed for our purposes. In the following we will only use some of the properties of $\mathcal{H}(u)$, which we summarize now.

Theorem 2.9 *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and let H be the supporting function of* $\operatorname{ch}\operatorname{sing\,supp} u$. *Then*

$$H(x) = \sup_{h \in \mathcal{H}(u)} h(x).$$

The latter theorem coincides with [Hör05, Theorem 16.3.4].

Theorem 2.10 *Let $u \in \mathcal{E}'(\mathbb{R}^d)$ and let $h \in \mathcal{H}(u)$. Then there is some $w \in \mathcal{E}'(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$ with* $\operatorname{sing\,supp} w = \{0\}$ *such that h is the supporting function of* $\operatorname{ch}\operatorname{sing\,supp} u * w$.

The statement of the theorem is just the first statement of [Hör05, Theorem 16.3.13] with the only difference that the statement is formulated with $w \in \mathcal{E}'(\mathbb{R}^d)$ only. That indeed there is some $w \in \mathcal{E}'(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$ with the stated properties is shown in the proof of [Hör05, Theorem 16.3.13].

Finally, we note that u is called invertible if $-\infty \notin \mathcal{H}(u)$, cf. [Hör05, Definition 16.3.12]. The following condition for u not to be invertible will be used several times:

Theorem 2.11 *Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$. Then the following statements are equivalent:*

1. $-\infty \in \mathcal{H}(\mu)$
2. *For every $x \in \mathbb{R}^d$ there is some $w \in C^0(\mathbb{R}^d) \setminus C^1(\mathbb{R}^d)$ such that* $\operatorname{sing\,supp} w = \{x\}$ *and $\mu * w \in C^\infty(\mathbb{R}^d)$.*
3. *There is some $w \in \mathcal{E}'(\mathbb{R}^d)$ such that $\mu * w \in C^\infty(\mathbb{R}^d)$ but $w \notin C^\infty(\mathbb{R}^d)$.*

The latter theorem coincides with [Hör05, Theorem 16.3.9].

Theorem 2.12 *Let $\mu \in \mathcal{E}'(\mathbb{R}^d)$. Then the following statements are equivalent:*

1. *$u \in \mathcal{D}'(\mathbb{R}^d)$ and $\mu * u \in C^\infty(\mathbb{R}^d)$ implies $u \in C^\infty(\mathbb{R}^d)$.*
2. *μ is hypoelliptic in the sense of [Hör05], i.e., μ is invertible and*

$$\frac{|\operatorname{Im}\zeta|}{\log|\zeta|} \to_{|\zeta|\to\infty} \infty \qquad \text{on } \{\zeta \in \mathbb{C}^d : \hat{\mu}(\zeta) = 0\}.$$

3. *There is some $E \in \mathcal{E}'(\mathbb{R}^d)$ such that $E * \mu - \delta \in C^\infty(\mathbb{R}^d)$ and* $\operatorname{sing\,supp} E = -\operatorname{sing\,supp} \mu$.

Proof: The theorem follows directly from equivalent conditions (i),(ii), and (v) of [Hör05, Theorem 16.6.5], where we note that hypoellipticity is defined in Definition 16.6.4 of [Hör05]. ■

Let us remark the following: If ν is a Lévy measure, then the associated operator $L\colon C_0^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ is linear, translation invariant and continuous and can therefore be written as convolution with a distribution $\mu \in \mathscr{D}'(\mathbb{R}^d)$, cf. [Hör05, Theorem 4.2.1]. Moreover, for $u \in C_0^\infty(\mathbb{R}^d)$ and $x \notin \operatorname{supp} u$ it follows that

$$Lu(x) = \int_{\mathbb{R}^d} \left(u(x+h) - u(x) - \frac{h \cdot \nabla u(x)}{1+|x|^2} \right) \nu(dh) = \int_{\mathbb{R}^d} u(x+h)\nu(dh) = \tilde{\nu} * u,$$

where $\tilde{\nu}$ denotes the reflection of ν i.e., $\langle \tilde{\nu}, \varphi \rangle := -\langle \nu, \tilde{\varphi} \rangle$ and $\tilde{\varphi}(x) = \varphi(-x)$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Thus μ and $\tilde{\nu}$ agree on $\mathbb{R}^d \setminus \{0\}$. As a consequence $\operatorname{supp}\mu = -\operatorname{supp}\nu$ is compact and $\operatorname{sing\,supp}\mu \setminus \{0\} = -\operatorname{sing\,supp}\nu \setminus \{0\}$. The following proposition relates local hypoellipticity for L as we have defined it above and to hypoellipticity of μ in the sense of Hörmander, cf. Theorem 2.12:

Proposition 2.13 *Let L be a Lévy operator that is locally hypoelliptic and satisfies (A2). Then for any $u \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in C^\infty(\mathbb{R}^d)$ such that $Lu = f$ we have $u \in C^\infty(\mathbb{R}^d)$.*

Proof: Let $M > 0$ be such that $\operatorname{supp}\mu \subseteq \overline{B_M(0)}$. In order to show that $u \in C^\infty(\mathbb{R}^d)$ it is sufficient to show that $u|_{B_R(0)} \in C^\infty(B_R(0))$ for any $R > 0$. Therefore let $R > 0$ be arbitrary and let $\eta \in C_0^\infty(\mathbb{R}^d)$ such that $\eta \equiv 1$ on $B_{R+M}(0)$. Then

$$Lu(x) = \mu * u(x) = \mu * (\eta u)(x) \qquad \text{for all } x \in B_R(0),$$

where $\eta u \in \mathcal{E}'(\mathbb{R}^d)$. Now there is some $s \in \mathbb{R}$ such that $\eta u \in H^s(\mathbb{R}^d)$. Thus $L(\eta u) = f' \in H^{s-2}(\mathbb{R}^d)$ since $|\psi(\xi)| \le C(1+|\xi|)^2$ for every continuous negative definite function

ψ. Moreover, $f'|_{B_R(0)} = Lu|_{B_R(0)} = f|_{B_R(0)} \in C^\infty(B_R(0))$, which implies $\eta u|_{B_R(0)} \in C^\infty(B_R(0))$ because of the local hypoellipticity of L. Since $R > 0$ was arbitrary we obtain $u \in C^\infty(\mathbb{R}^d)$. ■

Proof of Theorem 2.2: First of all, because of Proposition 2.13, the first statement of Theorem 2.12 is true. Therefore there is a parametrix $E \in \mathscr{E}'(\mathbb{R}^d)$ such that

$$k = E * \mu - \delta \in C^\infty(\mathbb{R}^d) \quad \text{and} \quad \operatorname{sing\,supp} E = \operatorname{sing\,supp} \mu.$$

Here even $k \in C_0^\infty(\mathbb{R}^d)$ since E and μ have compact support. Since μ is in turn a parametrix for E, E is also hypoelliptic due to Theorem 2.12 again. In particular this implies that E is invertible, i.e., $-\infty \notin \mathcal{H}(E)$.

Next we show $\mathcal{H}(E) = \{0\}$. To this end let $h \in \mathcal{H}(E)$. By Theorem 2.10 there is some $w \in \mathscr{E}'(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ with $\operatorname{sing\,supp} w = \{0\}$ such that h is the supporting function of $\operatorname{ch\,sing\,supp} E * w$. In particular, $w \in L^2(\mathbb{R}^d)$. Next let $v := \mathscr{F}^{-1}\left[\eta(\xi)\psi(\xi)^{-1}\hat{f}(\xi)\right]$, where $\eta \in C^\infty(\mathbb{R}^d)$ such that $\eta(\xi) = 1$ for $|\xi| \geq \rho + 1$ and $\eta(\xi) = 0$ for $|\xi| \leq \rho$ where ρ is as in (S2). Then $v \in H^{\psi,2}(\mathbb{R}^d)$ and

$$Lv = f + k' \qquad \text{where } k' \in C^\infty(\mathbb{R}^d).$$

Now, if $u = E * w$, then $\mu * (u - v) = (k + k') * w \in C_0^\infty(\mathbb{R}^d)$. Thus $u - v \in H^\infty(\mathbb{R}^d)$ and therefore $u \in H^{\psi,2}(\mathbb{R}^d)$. Now, because convolution with μ is by assumption locally hypoelliptic we have

$$\operatorname{sing\,supp} E * w \subseteq \operatorname{sing\,supp} \mu * E * w = \operatorname{sing\,supp}(w + k * w) = \operatorname{sing\,supp} w = \{0\}\,.$$

As noted above $-\infty \notin \mathcal{H}(E)$, and therefore h cannot be the supporting function of $\emptyset$. We conclude $\operatorname{ch\,sing\,supp} E * w = \{0\}$ which implies $h \equiv 0$. This shows $\mathcal{H}(E) = \{0\}$.

Thus the supporting function of $\operatorname{ch\,sing\,supp} E$ is $H(x) = \sup_{h \in \mathcal{H}(E)} h(x) = 0$ and finally

$$\{0\} = \operatorname{ch\,sing\,supp} E = \operatorname{sing\,supp} E = -\operatorname{sing\,supp} \mu.$$

This completes the proof. ■

3 A-priori continuity estimates for $\mathcal{L}$-harmonic functions

3.1 Introduction

Regularity of solutions to differential equations is closely related to qualitative properties of the corresponding Markov process. A good example is the modern theory of fully nonlinear partial differential equations of second order which came to real life after Hölder a-priori estimates for solutions to elliptic and parabolic second order equations with irregular coefficients were established [CC95]. The derivation of these a-priori estimates was first based on hitting time estimates for diffusion processes [KS79].

In the last years these regularity results which are by now classical for local diffusion operators have been investigated for nonlocal operators and related jump processes. In this article we discuss continuity a-priori estimates for functions which are harmonic with respect to nonlocal integro-differential operators, respectively Markov jump processes. In comparison with existing results on Hölder a-priori estimates we need to impose only weak conditions on the jump kernels.

The main tool used in previous proofs of Hölder regularity for functions harmonic with respect to Markov processes is to show that for all $r < 1/2$, $A \subset B(x_0, r)$ satisfying $|A| \geq \frac{1}{2}\,|B(x_0, r)|$ and for all $y \in B(x_0, \frac{r}{2})$

$$\mathbb{P}^y \left(T_A < \tau_{B(x_0,r)}\right) \geq c > 0\,. \tag{3.1}$$

Here, T and τ denote entry and exit times, respectively, and $|A|$ the Lebesgue measure of the measurable set A. Estimate (3.1) is at the heart of [KS79] and is basically a probabilistic reformulation of what is known as growth lemmas, see [Lan98]. In this work our main goal is to extend [BK05b] and to prove a-priori continuity estimates in situations where (3.1) fails, see Theorem 3.2 and Theorem 3.1.

With the help of Theorem 3.2 we are able to establish the Feller property for a certain class of Markov processes, see Theorem 3.3. It is interesting to compare this result to Theorem 1.9 of [BBCK09] where it is shown that the martingale problem may fail under slightly weaker conditions. One aim of the present work is to shed more light into this area of research.

Let us be more precise and present our results. Let $\nu = \{\nu(x, \cdot)\}_{x\in\mathbb{R}^d}$ be a family of Lévy measures satisfying

$$\sup_{x\in\mathbb{R}^d} \int\limits_{\mathbb{R}^d} \min\left(|h|^2, 1\right) \nu(x, dh) < \infty\,.$$

For $u \in C_b^2(\mathbb{R}^d)$ set

$$\mathscr{L}u(x) = \int\limits_{\mathbb{R}^d\setminus\{0\}} \left(u(x + h) - u(x) - \mathbb{1}_{\{|h|<1\}}\langle h, \nabla u(x)\rangle\right) \nu(x, dh). \tag{3.2}$$

Fix some $\delta < \frac{1}{2}$ and define

$$S(x,r) = \int\limits_{|h|\geq r} \nu(x,dh) \ ,$$
$$L(x,r) = S(x,r) + \frac{1}{r}\Big| \int\limits_{1\geq|h|\geq r} h \, \nu(x,dh)\Big| + \frac{1}{r^2} \int\limits_{|h|<r} |h|^2 \ \nu(x,dh) \ ,$$
$$N(x,r) = \inf\left\{\nu(x,M) : M \subset B(0,2r), |M| \geq \delta\, |B(0,r)|\right\} .$$

We will need the following assumptions:

(A) There is a strong Markov process $(X_t, \mathbb{P}^x)$ having right continuous paths with left limits such that $u(X_t) - u(x) - \int_0^t \mathscr{L}u(X_s)\, ds$ is a $\mathbb{P}^x$-martingale for all $u \in C_b^2(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$.

(B1) $\sup\limits_{x\in\mathbb{R}^d} L(x,1) < \infty$.

(B2) There exist $\kappa_1 > 0$ and $\sigma > 0$ such that for all $x \in \mathbb{R}^d$, $r \in (0,1/2)$, $1 < \lambda < \frac{1}{r}$

$$S(x,\lambda r) \leq \kappa_1 \lambda^{-\sigma} S(x,r) \, .$$

(B3) There exists $\kappa_2 > 0$ such that for all $x, y \in \mathbb{R}^d$, $r \in (0,1/2)$, $|x-y| < 2r$

$$N(x,r) \geq \frac{\kappa_2}{|\ln r|} L(y,r/2).$$

Assumptions (A), (B1) and (B2) are mild and also appear in [BK05b]. Our central assumption is (B3) which differs significantly from Assumption 2.1 (b) in [BK05b]. It allows for a certain degeneracy which we focus on in the present work. At the end of this section we discuss some examples where (B1) through (B3) are satisfied.

Given an integro-differential operator $\mathscr{L}$ of type (3.2) we call functions $u\colon \mathbb{R}^d \to \mathbb{R}$ harmonic with respect to $\mathscr{L}$ or simply $\mathscr{L}-$harmonic in an open set $D \subset \mathbb{R}^d$ if for any open set $D' \Subset D$ the process $u(X_{s\wedge\tau_{D'}})$ is a $\mathbb{P}^x$-martingale. This definition of harmonicity ensures that functions $u \in C_b^2(\mathbb{R}^d)$ satisfying $\mathscr{L}u(x) = 0$ for $x \in D$ are indeed $\mathscr{L}$-harmonic in D.

Define the local modulus of continuity of a function u on the ball $B(x_0,R)$ as follows:

$$\omega_u(t;x_0,R) = \sup_{\substack{x,y\in B(x_0,R)\\ |x-y|<t}} |u(x) - u(y)| \ .$$

Let us introduce two kinds of a-priori estimates.

($\mathrm{HC_{nl}}$) The *Hölder continuity a-priori estimate* ($\mathrm{HC_{nl}}$) holds if for every $R \in (0,1)$ there exist $c > 0$ and $\gamma \in (0,1)$ such that for any bounded function $u\colon \mathbb{R}^d \to \mathbb{R}$ which is $\mathscr{L}$-harmonic in a ball $B(x_0,R)$ we have

$$\omega_u(t;x_0,R/2) \leq ct^\gamma \left\|u\right\|_\infty \quad \forall\, t > 0 \ .$$

(C) The *continuity a-priori estimate* (C) holds if for every $R \in (0,1)$ there exists a function $\vartheta\colon (0,1) \to \mathbb{R}_+$ with $\lim_{t\to 0} \vartheta(t) = 0$ such that for every bounded function $u\colon \mathbb{R}^d \to \mathbb{R}$ which is $\mathscr{L}$-harmonic in a ball $B(x_0,R)$ we have

$$\omega_u(t;x_0,R/2) \le \|u\|_\infty \vartheta(t) \quad \forall\, t \in (0,1)\,.$$

Clearly, ($\mathrm{HC_{nl}}$) implies (C) by the choice $\vartheta(t) = ct^\gamma$. (C) guarantees that the set of all functions which are $\mathscr{L}$-harmonic in $B(x_0,R)$ is compact in $C(B(x_0,R/2))$. (C) is often the minimal condition that is needed, for example when dealing with nonlinear elliptic operators satisfying so called natural growth conditions. A local analog of ($\mathrm{HC_{nl}}$) was established by DeGiorgi [DG57] and Nash [Nas58] for weak solutions to $\mathrm{div}(A(\cdot)\nabla u) = 0$ and later by Krylov-Safonov for diffusion equations in non-divergence form. We refer to the end of this section for a short discussion about known results in the case of jump processes.

As mentioned above we prove our main results under assumptions where uniform hitting time estimates as (3.1) do not hold necessarily. We illustrate this phenomenon for a fixed Lévy measure $\nu(x,dh) = \nu(dh)$. More precisely, we have the following result.

Theorem 3.1 *There exists a Lévy measure ν satisfying (A), (B1), (B2), (B3) and sequences $r_n \to 0$, $A_n \subset B(0,r_n)$ satisfying $|A_n| \ge \frac{5}{8}|B(0,r_n)|$ such that*

$$\mathbb{P}^0\left(T_{A_n} < \tau_{B(0,r_n)}\right) \to 0 \quad \textit{for } n \to \infty\,. \tag{3.3}$$

Note that (A) is automatically satisfied when considering a fixed Lévy measure. In light of (3.3) regularity of harmonic functions or resolvents under our assumptions is an interesting and subtle question. Our main result reads as follows.

Theorem 3.2 *Assume that ν satisfies assumptions (A), (B1), (B2), (B3). Then for each $R \in (0,1/2)$ there is $c > 0$ such that for all bounded functions $u\colon \mathbb{R}^d \to \mathbb{R}$ being $\mathscr{L}$-harmonic on $B(x_0,R)$ its modulus of continuity on $B(x_0,R/2)$ satisfies*

$$\omega_u(t;x_0,R/2) \le c\,\|u\|_\infty\, |\ln t|^{-\rho} \quad \forall t \in (0,1/2)\,. \tag{3.4}$$

The constant $\rho > 0$ depends only on the constants appearing in (B2) and (B3). In particular, for each $p > 1/\rho$, u is p-Dini continuous, i.e.

$$\lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{\omega_u(t;x_0,R/2)^p}{t}\, dt < c\,.$$

Assumptions (B1), (B2) and (B3) are applicable to cases where the following two phenomena might appear simultaneously.

(i) For given $M \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$ the mapping $x \mapsto \nu(x,M)$ might be discontinuous.

(ii) For given $x \in \mathbb{R}^d$ the measure $\nu(x,\cdot)$ might not be almost symmetric, i.e. the quantity $\inf\limits_{M\subset B_r(0)\setminus\{0\}} \frac{\nu(x,M)}{\nu(x,-M)}$ might be zero for all $r > 0$.

Once Theorem 3.2 is established it is not too difficult to determine a Feller semigroup corresponding to $\nu(x, dh)$. For this purpose it is not necessary to have ($\mathrm{HC_{nl}}$), see also [Kom88]. Any uniform control over the modulus of continuity for the resolvents is good enough.

In the following result we apply our method in the framework of Dirichlet forms.

Theorem 3.3 *Define a regular Dirichlet-form* $(\mathscr{E}, D(\mathscr{E}))$ *by*

$$\begin{aligned} \mathscr{E}(u,v) &= \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)k(x,y)dx\,dy\,, \\ D(\mathscr{E}) &= \overline{C_c^{0,1}(\mathbb{R}^d)}^{\mathscr{E}_1}\,. \end{aligned} \tag{3.5}$$

where $k\colon \mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ *is measurable and satisfies* $k(x,y)=k(y,x)$ *and*

$$c_0|x-y|^{-d-\alpha} \le k(x,y) \le c_1 \ln\left(\tfrac{3}{|x-y|}\right)|x-y|^{-d-\alpha} \quad \text{for } |x-y|\le 1\,, \tag{3.6}$$

$$0 \le k(x,y) \le c_2|x-y|^{-d-\gamma} \quad \text{for } |x-y|>1\,, \tag{3.7}$$

with $\alpha\in(0,1)$, $c_0,c_1,c_2,\gamma>0$. *Then the restriction of the corresponding semi-group to* $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ *can be extended to a Feller semigroup* (T_t) *on* $C_\infty(\mathbb{R}^d)$.

Here $C_c^{0,1}(\mathbb{R}^d)$ is the space of all Lipschitz-continuous functions with compact support and $\overline{C_c^{0,1}(\mathbb{R}^d)}^{\mathscr{E}_1}$ denotes the closure of this space with respect to the norm $\mathscr{E}_1=\mathscr{E}(\cdot,\cdot)+\|\cdot\|^2_{L^2}$. The tuple $(\mathscr{E}, D(\mathscr{E}))$ is indeed a regular Dirichlet form as it can be proved like in Example 1.2.4 of [FŌT94]. For more information on Dirichlet forms, the corresponding Hunt process and other related objects we refer the reader to [FŌT94].

In light of Theorem 1.9 in [BBCK09] it is an interesting task to further weaken assumption (B3). An integrability test suggests that continuity estimates break down under an assumption of the type $N(x,r)\ge \frac{\kappa_2}{|\log r|^{1+\varepsilon}}L(y,r/2)$ for some $\varepsilon>0$. By our techniques we can get quite close to this if we replace (B3) by

(B3') There exists $\kappa_2>0$, $r_0>0$ and $M\in\mathbb{N}$ such that for all $x,y\in\mathbb{R}^d$, $r\in(0,r_0)$, $|x-y|<2r$

$$N(x,r)\ge \frac{\kappa_2}{\Psi(r)}L(y,r/2),$$

where $\Psi(r)=|\log r|\prod_{k=1}^M \log^k(|\log r|)$ and $\log^k=\log\circ\log\circ\ldots\circ\log$ denotes the $(k-1)$-times iterated logarithm.

Corollary 3.4 *Assume (A), (B1), (B2) and (B3'). Then (C) holds.*

We outline the proof of this result at the end of Section 3.4. Note that the logarithm in (3.6) can be replaced by the more general function Ψ without affecting Theorem 3.3.

Related results and examples

We close this section with a short overview on related results and some examples. Komatsu establishes a-priori estimates in [Kom88] and [Kom95] in the case $\nu(x,dh) = a(x)|h|^{-d-\alpha}\,dh$ and $0 < c_0 \leq a(x) \leq c_1$. ($\mathrm{HC_{nl}}$) is proved by Bass and Levin [BL02a] in the case where $\nu(x,dh)$ is absolutely continuous with density $n(x,h)$ satisfying $n(x,h) = n(x,-h)$ and $c_0\,|h|^{-d-\alpha} \leq n(x,h) \leq c_1\,|h|^{-d-\alpha}$ with $\alpha \in (0,2)$, see also [SV04]. ($\mathrm{HC_{nl}}$) is also studied with probabilistic methods by Bass and Kaßmann in [BK05b] not assuming ν to have a density. In [Sil06], Silvestre uses methods of partial differential equations to show ($\mathrm{HC_{nl}}$) in a similar context. Recently, the celebrated analytic methods of DeGiorgi, Nash and Moser were extended to non-local Dirichlet forms [Kas09]. For symmetric jump processes corresponding to operators of type (3.24) with $\nu(x,dh) = n(x,h)dh$, $n(x,h) = n(x+h,-h)$, ($\mathrm{HC_{nl}}$) is established by Bass and Levin in [BL02b] on the lattice and by Chen and Kumagai in [CK03] for quite general state spaces under the assumption $c_0\,|h|^{-d-\alpha} \leq n(x,h) \leq c_1\,|h|^{-d-\alpha}$, $\alpha \in (0,2)$. Schilling and Uemura [SU07] derive ($\mathrm{HC_{nl}}$) for such kernels allowing for certain mild perturbations for large h. [BKK09] and [HK07] apply ($\mathrm{HC_{nl}}$) in order to prove convergence of approximation schemes for symmetric jump processes.

Concerning the Feller property, substantial work has been carried out using methods from the theory of partial differential and pseudo differential operators by Jacob [Jac92, Jac94], Hoh [Hoh94, Hoh98b] and others, see [Jac05] for references. Different from our context, the main assumption there is that given $M \in \mathscr{B}(\mathbb{R}^d \setminus \{0\})$, the mapping $x \mapsto \nu(x,M)$ is smooth.

Finally, let us give an example where assumptions (B1) through (B3) are satisfied. Let us mention that all examples of kernels $\nu(x,dh)$ from the literature that satisfy (A) and lead to ($\mathrm{HC_{nl}}$) are covered by our assumptions.

Example 3.5 *Let $\alpha \in (0,2)$, $2 > r_0 > 1$ and $\nu(x,dh) = n(x,h)dh$. Suppose*

$$c_0\,|h|^{-d-\alpha} \leq n(x,h) \leq c_1\,|h|^{-d-\alpha}\log\Big(\frac{3}{|h|}\Big) \qquad \textit{for } |h| \leq r_0\,,$$
$$\inf_{x\in\mathbb{R}^d} S(x,1) < \infty\,.$$

Then ν satisfies (B1)–(B3).

The example above indicates that large jumps have no substantial influence on our result. Note that (B1) through (B3) do not require $n(x,h)$ to be continuous neither in x nor h. Furthermore it includes cases which are not covered by earlier contributions since they all deal with what we call almost symmetric measures.

Definition 3.6 *Let μ be a Lévy-measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\mu(\mathbb{R}^d \setminus \{0\}) = \infty$. We say that μ is almost rotationally invariant at 0 if*

$$\exists c > 0 : \qquad \liminf_{r\to\infty} \inf_{M\subset B_r(0)\setminus\{0\}} \frac{\mu(M)}{\mu(\rho(M))} > c. \tag{3.8}$$

for any rotation ρ about the origin. We say that μ is almost symmetric at 0 if

$$\liminf_{r\to\infty} \inf_{M\subset B_r(0)\setminus\{0\}} \frac{\mu(M)}{\mu(-M)} > 0\,. \tag{3.9}$$

It is clear that one can choose $\nu(x,dh) = n(x,h)dh$ as in example 3.5 leading to measures $\nu(x,\cdot)$ which are neither almost symmetric nor almost rotationally invariant. Choose $d = 2$, $M = \{(x,y) \in \mathbb{R}^2; x > 0, y > 0\}$ and set

$$n(x,h) = \Big\{|h|^{-3} + |h|^{-3} \ln\big(\frac{3}{|h|}\big) \mathbb{1}_{\{h \in M\}}(h)\Big\} \mathbb{1}_{\{|h| \le 1\}}(h)$$

Then the measure $\nu(x,\cdot) = \nu(\cdot)$ is a Lévy-measure satisfying (B1) through (B3) but it is not almost symmetric. In section 3.5 we discuss similar examples where $\nu(x,\cdot)$ depends on $x \in \mathbb{R}^d$ non-continuously.

3.2 Preliminaries

We denote the open ball in $\mathbb{R}^d$ with center x and radius r by $B(x,r)$ or $B_r(x)$, the characteristic function of a set $A \subset \mathbb{R}^d$ by $\mathbb{1}_A$ and the Lebesgue measure of a Borel set A by $|A|$. Define the function spaces

$$\begin{aligned} C_\infty(\mathbb{R}^d) &= \Big\{u \in C(\mathbb{R}^d) : \lim_{|x|\to\infty} u(x) = 0\Big\}, \\ C_b^k(\mathbb{R}^d) &= \Big\{u \in C^k(\mathbb{R}^d) : \text{all derivatives up to order } k \text{ bounded}\Big\}, \\ C_c^k(\mathbb{R}^d) &= \Big\{u \in C^k(\mathbb{R}^d) : \operatorname{supp} u \text{ compact}\Big\}. \end{aligned}$$

The following lemma will be essential when proving properties of certain anisotropic Lévy processes. Let us define for $a, \rho \in (0,1)$ the following sets.

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2; |y| \ge |x|^a, x^2 + y^2 < 1\}, \\ E_\rho &= A \cap \{(x,y) \in \mathbb{R}^2; \sqrt{x^2+y^2} \ge \rho\}, \\ F_\rho &= A \cap \{(x,y) \in \mathbb{R}^2; x \ge \rho\}. \end{aligned}$$

Lemma 3.7 *Let $g \colon \mathbb{R}^2 \to \mathbb{R}$ be invariant under rotations, i.e. $g(x,y) = f(r)$ where $r = \sqrt{x^2+y^2}$ and $f \colon \mathbb{R}_+ \to \mathbb{R}$. Then*

$$\iint\limits_{E_\rho} g(x,y)\, dy\, dx = \mathcal{O}\Big(\int_\rho^1 r^{1/a} f(r)\, dr\Big) \qquad \text{for } \rho \to 0. \tag{3.10}$$

Let $\beta \in (0,2)$, $\beta \neq 1/a - 1$. Asymptotically for $\rho \to 0$ we then obtain

$$\iint\limits_{E_\rho} \ln\big(\tfrac{1}{\sqrt{x^2+y^2}}\big)\big(\sqrt{x^2+y^2}\big)^{-2-\beta}\, dy\, dx = \mathcal{O}\left(\ln(\tfrac{1}{\rho})\rho^{\frac{1}{a}-1-\beta}\right), \tag{3.11}$$

$$\iint\limits_{F_\rho} \ln\big(\tfrac{1}{\sqrt{x^2+y^2}}\big)\big(\sqrt{x^2+y^2}\big)^{-2-\beta}\, dy\, dx = \mathcal{O}\left(\ln(\tfrac{1}{\rho})\rho^{1-a-a\beta}\right), \tag{3.12}$$

Proof: Let us prove (3.10) first. Using polar coordinates (r,θ) instead of Euclidean coordinates (x,y) we obtain

$$\iint\limits_{E_\rho} g(x,y)\, dy\, dx = 4\int_\rho^1 \int_{\phi(r)}^{\frac{\pi}{2}} r f(r)\, d\theta\, dr = 4\int_\rho^1 \big(\frac{\pi}{2} - \phi(r)\big) r f(r)\, dr, \tag{3.13}$$

where $\phi(r)$ is the unique angle satisfying

$$1)\quad \frac{\pi}{4} \le \phi(r) \le \frac{\pi}{2} \qquad \text{and } 2) \quad r^a \cos^a(\phi(r)) = r \sin(\phi(r)) \,.$$

Note that 2) is equivalent to

$$\frac{\pi}{2} - \phi(r) = r^{\frac{1}{a}-1} \sin^{1/a}(\phi(r)) \frac{\frac{\pi}{2} - \phi(r)}{\cos(\phi(r))} \,.$$

Since both functions, $\sin^{\frac{1}{a}}(x)$ and $(\frac{\pi}{2} - x)/\cos(x)$ are bounded from above and below by positive constants for $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$ which is the range of $\phi(r)$ we obtain

$$\iint\limits_{E_\rho} g(x,y)\, dy\, dx \approx \int_\rho^1 r^{\frac{1}{a}-1} r f(r)\, dr \,, \tag{3.14}$$

which proves (3.10). Next, we prove (3.11). Using integration by parts we derive

$$\begin{aligned} \int_\rho^1 \ln(\tfrac{1}{r}) r^{\frac{1}{a}-2-\beta}\, dr &= \frac{1}{\frac{1}{a}-1-\beta} \ln(\tfrac{1}{r}) r^{\frac{1}{a}-1-\beta}\Big|_\rho^1 + \frac{1}{\frac{1}{a}-1-\beta} \int_\rho^1 r^{\frac{1}{a}-2-\beta}\, dr \\ &= \frac{-1}{\frac{1}{a}-1-\beta} \Big\{ \ln(\tfrac{1}{\rho}) \rho^{\frac{1}{a}-1-\beta} - \frac{1}{\frac{1}{a}-1-\beta} (1-\rho^{\frac{1}{a}-1-\beta}) \Big\} \,, \end{aligned}$$

which proves (3.11). In order to prove (3.12) note that $F_\rho \subset E_{\sqrt{\rho^2+\rho^{2a}}}$. Together with (3.11) this implies

$$\begin{aligned} \iint\limits_{F_\rho} \ln(\tfrac{1}{\sqrt{x^2+y^2}}) (\sqrt{x^2+y^2})^{-2-\beta}\, dy\, dx &\le c \ln(\tfrac{1}{\sqrt{\rho^2+\rho^{2a}}}) (\sqrt{\rho^2+\rho^{2a}})^{\frac{1}{a}-1-\beta} \\ &\le \mathcal{O}\left(\ln(\tfrac{1}{\rho}) \rho^{1-a-a\beta} \right) \qquad \text{for } \rho \to 0 \,. \end{aligned}$$

Concerning the lower estimate in (3.11) we observe

$$\begin{aligned} &\iint\limits_{F_\rho} \ln(\tfrac{1}{\sqrt{x^2+y^2}}) (\sqrt{x^2+y^2})^{-2-\beta}\, dy\, dx \\ &\qquad \ge c \int_\rho^{2\rho} \int_{x^a}^{2(x^a)} \ln(\tfrac{1}{\sqrt{y^2+y^2}}) (\sqrt{y^2+y^2})^{-2-\beta}\, dy\, dx \\ &\qquad \ge c \int_\rho^{2\rho} \ln(\tfrac{1}{x^a}) (x^a)^{-1-\beta}\, dx \ge ca \ln(\tfrac{1}{\rho}) \rho^{1-a-a\beta} \,, \end{aligned}$$

which proves (3.12). ■

From now on $(X_t, \mathbb{P}^x)$ will always be a strong Markov process associated to $\nu(x, dh)$ by assumption (A). Let $\Delta X_t = X_t - X_{t-}$ be the jump of X_t at time t and, for a Borel set A, let τ_A be the first exit time of A, T_A the first hitting time. Recall that a function $u\colon \mathbb{R}^d \to \mathbb{R}$ is said to be *harmonic with respect to* $\mathscr{L}$ in an open set $U \subset \mathbb{R}^d$, if for any open set $U' \Subset U$ the process $u(X_{s\wedge\tau_{U'}})$ is a $\mathbb{P}^x$-martingale.

Let us state and prove several technical lemmas. As in [BL02a, Prop. 2.3] one can prove that $(\nu(x, x-dh), dt)$ is a Lévy system for X_t:

Lemma 3.8 *Suppose (A) holds. For disjoint Borel sets $A, B \subset \mathbb{R}^d$ and bounded stopping times S*

$$\mathbb{E}^{x_0} \sum_{s \le S} \mathbb{1}_{\{X_{s-} \in A, X_s \in B\}} = \mathbb{E}^{x_0} \int_0^S \mathbb{1}_A(X_s)\nu(X_s, B - X_s)\, ds.$$

We will now estimate some probabilities which play a crucial role in the proof of Theorem 3.2. Set

$$\overline{L}(x_0, r) = \sup_{x \in B(x_0,r)} L(x, r).$$

The proofs of the following results can be found in [BK05b].

Lemma 3.9 *Assume that (A), (B1) hold. Then there exists a constant $\kappa_3 > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0, 1/2)$*

$$\mathbb{P}^{x_0}(\tau_{B(x_0,r)} < t) \le \kappa_3 t \overline{L}(x_0, r)\, .$$

Lemma 3.10 *Assume that (A), (B1) and (B2) hold. For a Borel set B and $r \in (0, 1/2)$ let $U = \inf\{t : |\Delta X_t| \ge r\}$ be the time of the first jump greater than r. Then, for all $1 < \lambda < \frac{1}{r}$ and $x \in \mathbb{R}^d$ we have*

$$\mathbb{P}^x\left(|\Delta X_{U \wedge \tau_B}| \ge \lambda r\right) \le \kappa_1 \lambda^{-\sigma}\, ,$$

where κ_1 is the constant in (B2).

Lemma 3.11 *Assume that (A), (B1) and (B3) hold. Then there exists a constant $\kappa_4 > 0$ such that for $r \in (0, 1/2)$, $A \subset B(x_0, r)$, $|A| \ge \delta\, |B(x_0, r)|$ and $y \in B(x_0, \frac{r}{2})$*

$$\mathbb{P}^y\left(T_A < \tau_{B(x_0,r)}\right) \ge \frac{\kappa_4}{|\ln r|}\, .$$

Due to the special form of (B3) the assertion of Lemma 3.11 is considerably weaker than the corresponding result in [BK05b].

Lemma 3.12 *Assume (A) and $\lim_{r\to 0} \inf_{x \in \mathbb{R}^d} S(x, r) = \infty$. Then there exist a function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r \to 0} \vartheta(r) = 0$ and $r_0 > 0$ such that*

$$\mathbb{E}^y \tau_{B(x,r)} \le \vartheta(r) \qquad \forall\, x, y \in \mathbb{R}^d, r_0 > r > 0\, .$$

Proof: We follow the proof of Lemma 3.4 in [BK05a]. Let U be the time of the first jump greater than $2r$. Note that $\tau_{B(x,r)} \le U$ and that, because of the additional assumption on $S(x, r)$, there exists a function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ and $r_0 > 0$ with $\lim_{r\to 0} \vartheta(r) = 0$ and $S(x, 2r) \ge \frac{1}{2}\vartheta(r)^{-1}$ for all $r < r_0$. Assume $r < r_0$. If $\mathbb{P}^y(U \le \vartheta(r)) \le \frac{1}{2}$, then by the Lévy system identity

$$\begin{aligned}\mathbb{P}^y(U \le \vartheta(r)) &= \mathbb{E}^y \sum_{s \le U \wedge \vartheta(r)} \mathbb{1}_{\{|\Delta X_s| > 2r\}} = \mathbb{E}^y \int_0^{U \wedge \vartheta(r)} S(X_s, 2r) ds \\ &\ge \frac{1}{2}\vartheta(r)^{-1}\mathbb{E}^y(U \wedge \vartheta(r)) \ge \frac{1}{2}\mathbb{P}^y(U > \vartheta(r)) \ge \frac{1}{4}.\end{aligned}$$

Therefore in any case $\mathbb{P}^y(U \leq \vartheta(r)) \geq \frac{1}{4}$. Now let θ_t be the Markov shift operator. For $m \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{P}^y(U > (m+1)\vartheta(r)) \leq \mathbb{P}^y(U > m\vartheta(r), U \circ \theta_{m\vartheta(r)} > \vartheta(r)) \\
&= \mathbb{E}^y\big(\mathbb{P}_{X_{m\vartheta(r)}}(U > \vartheta(r); U > m\vartheta(r))\big) \leq \frac{1}{2}\mathbb{P}^y(U > m\vartheta(r)) \leq \ldots \leq 2^{-(m+1)}.
\end{aligned}$$

Hence we see $\mathbb{E}^y U \leq 4\vartheta(r)$ completing the proof. ∎

3.3 Degeneration of hitting time estimates

The aim of this section is to prove Theorem 3.1. Thereby, we show that our assumptions made in Theorem 3.2 allows for cases where (3.1) does not hold. The construction below is similar to the class of examples given in section 5 of [BK05a] but not included in that class. However, a generalized Harnack inequality would fail for our example, too.

Proof of Theorem 3.1: Let $0 < \alpha < \beta < 1$ and $a = (1 + \beta - \alpha)^{-1}$. In particular we have $a \in (0, 1)$ and $\frac{1}{a} - 1 - \beta = -\alpha$. As in section 3.2 set

$$A = \{(h_1, h_2) \in \mathbb{R}^2; |h_2| \geq |h_1|^a, (h_1)^2 + (h_2)^2 < 1\}.$$

Let $\nu(dh) = n(h)dh$ be the symmetric Lévy measure with density

$$n(h) = |h|^{-2-\alpha} + \mathbb{1}_A(h)\, |\ln |h||\, |h|^{-2-\beta}. \tag{3.15}$$

Observe that for a Lévy measure the martingale problem always has a unique solution, the Lévy process (X_t) with Lévy characteristic $(0, 0, \nu)$, see for example [Sat99]. An application of (3.10) shows

$$S(r) = S(x, r) = \mathcal{O}(r^{-\alpha} \ln \tfrac{1}{r}) \quad \text{as } r \to 0.$$

Furthermore we have

$$\begin{aligned}
&\iint\limits_{A \cap \{|h| \leq r\}} |\ln |h||\, |h|^{-\beta}\, dh \\
&\leq 4 \int_0^r \int_0^{r^{1/a}} (-\ln \sqrt{(h_1)^2 + (h_2)^2})(\sqrt{(h_1)^2 + (h_2)^2})^{-\beta}\, dh_1\, dh_2 \\
&\leq 4 \int_0^r \int_0^{r^{1/a}} (-(h_2)^{-\beta} \ln h_2)\, dh_1\, dh_2 \leq c r^{1 + \frac{1}{a} - \beta} \ln(\tfrac{1}{r}).
\end{aligned}$$

Together with symmetry of the measure we obtain

$$L(r) = L(x, r) \leq c r^{-\alpha} \ln(\tfrac{1}{r}).$$

Clearly, $|A \cap B(0, r)| / |B(0, r)|$ tends to 0 for $r \to 0$, hence

$$N(x, r) = N(r) = \mathcal{O}(r^{-\alpha}).$$

Altogether, ν satisfies the assumptions of Theorem 3.2. Set $B_r = B(0,r)$ and define $T(r) = r^\alpha / \sqrt{\ln(\frac{1}{r})}$. We will prove

$$\lim_{r\to 0} \mathbb{P}^0(\sup_{s\le T(r)} |X_s| < r) = 0\,, \tag{3.16}$$

$$\lim_{r\to 0} \mathbb{P}^0(\sup_{s\le T(r)} \left|X_s^1\right| > \frac{r}{16}) = 0\,, \tag{3.17}$$

where $X_t = (X_t^1, X_t^2)$ but X_t^1 and X_t^2 are not necessarily independent. Let r_n be an arbitrary sequence in $\mathbb{R}_+$ with $r_n \to 0$. Together, (3.16) and (3.17) mean that, in the limit $r_n \to 0$ the process has left $B(0,r_n)$ up to time $T(r_n)$ but has moved right or left not further than the distance $r/16$. (3.3) follows after choosing $A_n \subset B(0,r_n) \setminus \{(x,y) \in B(0,r_n), \frac{-r_n}{16} \le x \le \frac{r_n}{16}\}$ large enough.

Let us first prove (3.16). Note the following: Let μ be a Lévy measure, Y_t the associated pure-jump Lévy process and $B \subset \mathbb{R}^d$ a Borel set. Then, for any time T, the quantity $\sum_{s\le T} \mathbb{1}_{\{\Delta Y_s \in B\}}$, i.e. the number of jumps in B of the Lévy process before T, is Poisson distributed with parameter $T\mu(B)$. Using (3.10) and our choice of a show

$$I_1(r) = \nu(\{|h| > 2r\}) = \mathcal{O}(r^{-\alpha}\ln(\tfrac{1}{r})) \quad \text{for } r \to 0\,.$$

In particular $T(r)I_1(r)$ tends to ∞ for $r \to 0$. Thus the probability of a jump of size greater than $2r$ and therefore exiting an r-ball before the time $T(r)$ tends to 1 for $r \to 0$. This proves (3.16).

Next we write (X_t) as the sum of two independent Lévy processes (Y_t) and (Z_t) with Lévy measures $\nu_Y(dh) = |h|^{-2-\alpha}\, dh$ and $\nu_Z(dh) = \mathbb{1}_A(h)\,|h|^{-2-\beta}\ln(\frac{1}{|h|})$. (Y_t) is a rotationally invariant α-stable process. Hence we get by scaling

$$\lim_{r\to 0} \mathbb{P}^0(\sup_{s\le T(r)} |Y_s| > r/32) = \lim_{r\to 0} \mathbb{P}^0(\sup_{s\le(-\ln r)^{-1/2}} |Y_s| > 1/32) = 0\,. \tag{3.18}$$

Lemma 3.7 implies for $r \to 0$

$$I_2(r) = \nu_Z(\{|h_1| > r/32\}) = \mathcal{O}(r^{1-a-a\beta}\ln(\tfrac{1}{r}))\,.$$

Because of $1 - a - a\beta = -\alpha/(1+\beta-\alpha) > -\alpha$ we have $T(r)I_2(r) \to 0$ for $r \to 0$, which is the expected number of times Z_s^1 has jumps greater than $r/32$. In other words:

$$\lim_{r\to 0} \mathbb{P}^0(\sup_{s\le T(r)} \left|\Delta Z_s^1\right| > r/32) = 0\,. \tag{3.19}$$

It remains to handle the small jumps of (Z_t^1). For this we remove all jumps with $(\Delta Z_t) \in \{|h_1| > r/32\}$ and obtain a Lévy process (W_t) with Lévy measure

$$\nu_W(dh) = \mathbb{1}_{\{|h_1|\le r/32\}}\nu_Z(dh)\,.$$

Note that (W_t) has bounded jumps and therefore moments of all orders. Hence (W_t^1) is a martingale. We apply Doob's inequality and estimate

$$\mathbb{P}^0(\sup_{s\le T(r)} \left|W_s^1\right| > r/16) \le \frac{4\mathbb{E}^0(W_{T(r)}^1)^2}{(r/16)^2}\,.$$

Here $\mathbb{E}^0(W_t^1)^2 = tI_3(r)$, where

$$I_3(r) = \int\limits_{|h_1| \le r/16} (h_1)^2 \, \nu(dh) = 4 \int_0^{r/16} \int_{|h_1|^a}^1 (h_1)^2 \, |h|^{-2-\beta} \ln(\tfrac{1}{|h|}) \, dh_2 \, dh_1$$
$$\le 4 \int_0^{r/16} (h_1)^2 \int_{|h_1|^a}^1 |h_2|^{-2-\beta} \ln(\tfrac{1}{|h_2|}) \, dh_2 \, dh_1 \le c r^{3-a-a\beta} \, .$$

We obtain

$$\lim_{r \to 0} \mathbb{P}^0 (\sup_{s \le T(r)} \left| W_s^1 \right| > r/16) = 0 \, . \tag{3.20}$$

Combining (3.18), (3.19) and (3.20) we see that, starting in 0, the probability that (X_s^1) leaves the interval $(-r/8, r/8)$ before time $T(r)$ tends to 0 for $r \to 0$. Assertion (3.17) is proved. The proof of Theorem 3.1 is complete. ■

3.4 Continuity of $\mathscr{L}$-harmonic functions

The aim of this section is to prove Theorem 3.2 and Corollary 3.4.

Proof of Theorem 3.2: We will use an alteration of the method worked out in [BK05b] and prove the continuity of u in $z_1 \in B(z_0, \frac{R}{2})$ by an induction argument. The logarithmic degeneration in Lemma 3.11 requires a subtle change of the argument given in [BK05b]. Set $K = \|u\|_\infty$ and define furthermore

$$r_n = \theta_2 4^{-n} \, ,$$

where we select $\theta_2 = R/32$, in particular $B(z_1, 2r_1) \subset B(z_0, \frac{3R}{4})$. We write $B_n = B(z_1, r_n)$, $\tau_n = \tau_{B_n}$ and

$$M_n = \sup_{x \in B_n} u(x) \, , \qquad m_n = \inf_{x \in B_n} u(x) \, .$$

We will show

$$M_n - m_n \le s_n \tag{3.21}$$

for all n where s_n is a series decreasing monotone to 0. In our case

$$s_n = \theta_1 n^{-\rho}$$

will do the job, where $\theta_1 > 2K$ and $1 > \rho > 0$ will be specified later. Here the role of the upper bound on ρ is only to keep notation simple.

Let us assume for a moment that (3.21) holds already for $1, \dots, n$. Choose arbitrary $y, z \in B_{n+1}$ and define

$$A_n = \left\{ x \in B_n : u(x) \le \frac{M_n + m_n}{2} \right\} \, .$$

Without loss of generality suppose $|A_n| \geq \frac{1}{2}|B_n|$ (otherwise we look at the function $K-u$). Let $D \subset A_n$ compact with $|D| \geq \delta |B_n|$. By the $\mathscr{L}$-harmonicity of u in $B(x_0, R)$ we get

$$\begin{aligned} u(z)-u(y) &= \mathbb{E}^z\left(u\left(X_{\tau_n\wedge T_D}\right)-u(y)\right) \\ &= \mathbb{E}^z\left(u\left(X_{\tau_n\wedge T_D}\right)-u(y); T_D<\tau_n, X_{\tau_n}\in B_{n-1}\setminus B_n\right) \\ &\quad + \mathbb{E}^z\left(u\left(X_{\tau_n\wedge T_D}\right)-u(y); T_D>\tau_n, X_{\tau_n}\in B_{n-1}\setminus B_n\right) \\ &\quad + \sum_{i=1}^{n-2}\mathbb{E}^z\left(u\left(X_{\tau_n\wedge T_D}\right)-u(y); X_{\tau_n}\in B_{n-i-1}\setminus B_{n-i}\right) \\ &\quad + \mathbb{E}^z\left(u\left(X_{\tau_n\wedge T_D}\right)-u(y); X_{\tau_n}\notin B_1\right) \\ &=: I_1+I_2+I_3+I_4. \end{aligned}$$

Set

$$p_n = \mathbb{P}^z(T_D<\tau_n).$$

Then, by definition of A_n and (3.21), we derive the estimates

$$\begin{aligned} I_1 &\leq \left(\frac{M_n+m_n}{2}-m_n\right)\mathbb{P}^z(T_D<\tau_n) = \frac{1}{2}s_n p_n\,, \\ I_2 &\leq s_{n-1}(1-p_n)\,. \end{aligned}$$

To handle I_3 and I_4 we have to look at the probabilities

$$F_j = \mathbb{P}^z\left(X_{\tau_n}\notin B_{n-j}\right)\,.$$

The event defining F_j can only take place, if the process (X_t) has no jumps larger than $2r_n$ for $t<\tau_n$ and jumps at least $r_{n-j}-r_n$ at time τ_n. So by Lemma 3.10 it follows:

$$\begin{aligned} F_j &\leq \mathbb{P}^z\left(|\Delta X_{\tau_n}|\geq r_{n-j}-r_n, \sup_{s<\tau_n}|\Delta X_s|\leq 2r_n\right) \leq \kappa_1\left(\frac{2r_n}{r_{n-j}-r_n}\right)^{\sigma} \\ &\leq \kappa_1 3^{\sigma}4^{-j\sigma} = c_1 4^{-j\sigma}\,, \end{aligned}$$

Again, we use our hypothesis (3.21) as well as summation by parts and obtain

$$\begin{aligned} I_3 &\leq \sum_{i=1}^{n-2}s_{n-i-1}(F_i-F_{i-1}) = s_1F_{n-2}-s_{n-2}F_0+\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i \\ &\leq s_1F_{n-2}+\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i \leq \theta_1c_14^{-\sigma(n-2)}+\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i\,. \end{aligned}$$

Finally we estimate

$$I_4 \leq 2KF_{n-1} \leq \theta_1c_14^{-\sigma(n-1)}.$$

In consequence, we have

$$
\begin{aligned}
u(z)-u(y) &\leq s_{n+1}\left[\frac{s_n}{s_{n+1}}\cdot\frac{p_n}{2}+\frac{s_{n-1}}{s_{n+1}}(1-p_n)+\frac{1}{s_{n+1}}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i\right.\\
&\qquad\left.+\frac{\theta_1 c_2}{s_{n+1}}4^{-n\sigma}\right]\\
&\leq s_{n+1}\left[-\frac{p_n}{2}\cdot\frac{s_{n-1}}{s_{n+1}}+\left(1+\frac{2}{n-1}\right)^{\rho}\right.\\
&\qquad\left.+\frac{(n+1)^{\rho}}{\theta_1}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})F_i+c_2\,(n+1)^{\rho}\,4^{-n\sigma}\right].
\end{aligned}
\tag{3.22}
$$

Lemma 3.11 implies

$$p_n \geq \frac{\kappa_4}{|\ln r_n|}=\frac{\kappa_4}{|\ln\theta_2-n\ln 4|}\geq\frac{\kappa_4}{|\ln\theta_2|+n\ln 4}.$$

s_{n-1}/s_{n+1} is bounded from below by 1, thus the first term in (3.22) is bounded from above by

$$-\frac{c_4}{|\ln\theta_2|+n\ln 4}\,.$$

Moreover the second term (3.22) behaves for $n\to\infty$ as

$$1+\frac{2\rho}{n-1}+\mathcal{O}\left(\frac{1}{(n-1)^2}\right),$$

The most laborious part is estimating the sum in (3.22):

$$
\begin{aligned}
\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})\,F_i &\leq \sum_{i=1}^{\lceil (n-3)/2\rceil}(s_{n-i-1}-s_{n-i})\,F_i+\sum_{i=\lceil (n-3)/2\rceil}^{n-3}(s_{n-i-1}-s_{n-i})\,F_i\\
&\leq \left(s_{\lceil n/2\rceil -1}-s_{\lceil n/2\rceil}\right)\sum_{i=1}^{\infty}F_i+F_{\lceil (n-3)/2\rceil}\sum_{i=1}^{\infty}(s_i-s_{i+1})\,.
\end{aligned}
$$

Here both series converge. Finally an easy application of the mean value theorem yields

$$s_k-s_{k+1}\leq\theta_1\rho k^{-\rho-1},$$

and therefore there exist $c_5, c_6>0$ with

$$\frac{1}{s_{n+1}}\sum_{i=1}^{n-3}(s_{n-i-1}-s_{n-i})\,F_i\leq\frac{c_5\rho}{n-1}+c_6(n+1)^{\rho}4^{-\sigma n/2}.$$

Altogether we have

$$u(z)-u(y)\leq s_{n+1}\left(1-\frac{c_4}{|\ln\theta_2|+n\ln 4}+\frac{c_5\rho}{n-1}+\frac{c_7}{(n-1)^2}+c_8 4^{-\sigma n/3}\right)\,. \tag{3.23}$$

Note that the constants in (3.23) are independent of the choice of y, z, ρ, θ_1 and θ_2. Therefore the estimate in (3.23) gives us also an upper bound for $M_{n+1} - m_{n+1}$. Next, select ρ small enough and then n_0 large enough such that for all $n > n_0$

$$1 - \frac{1}{3} \cdot \frac{c_4}{|\ln \theta_2| + n \ln 4} + \frac{c_5 \rho}{n-1} + \frac{c_7}{(n-1)^2} + c_8 4^{-\sigma n} < 1 \,.$$

Finally, choose θ_1 in such a way that

$$M_n - m_n \leq 2K \leq s_n \qquad \forall 1 \leq n \leq n_0 \,.$$

Now, (3.21) holds for all n. Moreover, looking carefully over the preceding proof we see, that ρ and θ_2 only depend on R and not on u. Consequently we might choose θ_1 proportional to $\|u\|_\infty$. Therefore the modulus of continuity of u on $B(x_0, R/2)$ is bounded from above by $C \|u\|_\infty (-\ln t)^{-\rho}$. We now take into account that the integral

$$\int_0^{1/2} \frac{dt}{t(-\ln t)^\eta}$$

exists for $\eta > 1$. Hence u is p-Dini continuous for every $p > 1/\overline{\rho}$, where $\overline{\rho}$ is the supremum over all $\rho > 0$ for which our induction works. ■

We close this section by indicating how to prove Corollary 3.4. Let the notations be as in the proof of Theorem 3.2. Then Assumption (B3') implies

$$p_n = p(r_n) \geq c\Lambda_n \qquad\qquad \text{where } \Lambda_n = \left(n \prod_{k=1}^{M-1} \log^k(n) \right)^{-1} .$$

We now introduce the function $s(x) = (\log^M(x))^{-1}$ and choose, with a slight abuse of notation, $s_n = s(n)$. Proceeding as in (3.22) and using the mean value theorem to estimate differences of the type $s_k - s_{k-1}$ we obtain

$$\begin{aligned} u(z) - u(y) \leq s_{n+1} \Big[1 - \frac{1}{2} \frac{s_{n-1}}{s_{n+1}} p_n + \frac{-2s'(n-1)}{s_{n+1}} + \frac{1}{s_{n+1}} \sum_{l=1}^{n-3} (-s'(n-l-1)) F_l \\ + \frac{2K}{s_{n+1}} F_n + \frac{s_1}{s_{n+1}} F_{n-2} \Big] . \end{aligned}$$

The second summand on the right hand side is the only negative one and bounded from above by $-c\Lambda_n$. Therefore it suffices to show that the positive summands converge faster to 0 than Λ_n for $n \to \infty$. For the last two terms this is trivial since they are of order $\mathcal{O}(e^{-n\gamma})$ for $n \to \infty$. Moreover,

$$\frac{-2s'(n-1)}{s_{n+1}} = \frac{\log^M(n+1)}{(n-1)(\log^M(n-1))^2 \prod_{k=1}^{M-1} \log^k(n-1)} = \mathcal{O}\left(\frac{\Lambda_n}{\log^M(n)} \right) .$$

Finally, the remaining term of the right hand side can be treated as follows:

$$\begin{aligned}
&\log^M(n+1)\sum_{l=1}^{n-3}\frac{4^{-l\sigma}}{(n-l-1)(\log^M(n-l-1))^2\prod\limits_{k=1}^{M-1}\log^k(n-l-1)}\\
&\qquad= \log^M(n+1)4^{-(n-1)\sigma}\sum_{l=2}^{n-2}\frac{4^{l\sigma}}{l(\log^M(l))^2\prod\limits_{k=1}^{M-1}\log^k(l)}\\
&\qquad= \mathcal{O}\left(\log^M(n)4^{-n\sigma}\int_2^n\frac{4^{x\sigma}\,\mathrm{d}x}{x(\log^M(x))^2\prod_{k=1}^{M-1}\log^k(x)}\right).
\end{aligned}$$

By applying L'Hôspital's rule we end up with

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{\Lambda_n}\log^M(n)4^{-n\sigma}\int_2^n\frac{4^{x\sigma}\,\mathrm{d}x}{x(\log^M(x))^2\prod_{k=1}^{M-1}\log^k(x)}\\
&\qquad= \lim_{n\to\infty} n\Big[\prod_{k=1}^{M}\log^k(n)\Big]4^{-\sigma n}\int_2^n\frac{4^{\sigma x}\,\mathrm{d}x}{x\log^M(x)\prod\limits_{k=1}^{M}\log^k(x)} = 0.
\end{aligned}$$

Thereby we have shown $u(z) - u(y) < s_{n+1}$ for n large.

3.5 The Feller property

In this section we prove and discuss Theorem 3.3. As mentioned in the introduction, one open problem in the area of jump processes is to understand when, given a jump kernel, one can construct a corresponding Feller process and (not less important) when one cannot. In [BBCK09] an example of a jump kernel is given for which the martingale problem fails to be unique. We recall this example. With the help of Theorem 3.3 we then construct an example which is similar to the one in [BBCK09] but results in a Feller process.

Proof of Theorem 3.3:. Let us denote by $(\widetilde{\mathscr{A}}, D(\widetilde{\mathscr{A}}))$ the L^2-generator of $(\mathscr{E}, D(\mathscr{E}))$ and by $(\widetilde{T}_t)$ the corresponding semigroup. Basic calculations imply $\widetilde{\mathscr{A}}u(x) = \mathscr{L}u(x)$ for all $x \in \mathbb{R}^d$ and all functions $u \in C_c^2(\mathbb{R}^d)$, where

$$\begin{aligned}
(\mathscr{L}u)(x) &= \text{ p.v.} \int\limits_{\mathbb{R}^d}\Big(u(x+h)-u(x)\Big)k(x,x+h)\,dh\\
&:= \lim_{\varepsilon\to 0}\int\limits_{|h|>\varepsilon}\Big(u(x+h)-u(x)\Big)k(x,x+h)\,dh
\end{aligned} \tag{3.24}$$

Note that the principal value integral exists for $u \in C_c^2(\mathbb{R}^d)$ because of $k(x,y) = k(y,x)$ and $\alpha < 1$. Hence $(\widetilde{\mathscr{A}}, D(\widetilde{\mathscr{A}}))$ is an extension of $(\mathscr{L}, C_c^2(\mathbb{R}^d))$. Denote by X_t a Hunt process in $\mathbb{R}^d$ corresponding to $(\mathscr{E}, D(\mathscr{E}))$ and by $\mathscr{N}$ the associated properly exceptional set $\mathscr{N}$. Note that any two such processes are equivalent and that the Lebesgue measure of $\mathscr{N}$ is zero, i.e. $|\mathscr{N}| = 0$.

Following Theorem 5.2.2. in [FŌT94] one shows that for any starting point $x_0 \in \mathbb{R}^d \setminus \mathscr{N}$ the process X_t solves the martingale problem for $(\widetilde{\mathscr{A}}, D(\widetilde{\mathscr{A}}))$. For $\lambda > 0$ and $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ denote by $\widetilde{R_\lambda} f$ the resolvent of $(\mathscr{E}, D(\mathscr{E}))$. In the sense of Theorem 4.2.3 in [FŌT94], for any $f \in L^\infty(\mathbb{R}^d)$

$$(\widetilde{R_\lambda} f)(x) = \int_0^\infty e^{-\lambda t} (\widetilde{T}_t f)(x)\, dt = \mathbb{E}^x \Big(\int_0^\infty e^{-\lambda t} f(X_t)\, dt \Big), \quad x \in \mathbb{R}^d \setminus \mathscr{N} .$$

>From here, it needs only three more or less standard steps in order to complete the proof. The proof of Theorem 3.2 can be applied without changes in order to guarantee that bounded functions u which are $\mathscr{L}$-harmonic in $B(x_0, R) \setminus \mathscr{N}$ satisfy

$$\sup_{\substack{x,y \in B(x_0, R/2) \setminus \mathscr{N} \\ |x-y| < t}} |u(x) - u(y)| \leq c_0 \, \|u\|_\infty \, |\ln t|^{-\rho} , \tag{3.25}$$

where c_0 is independent of u but depends on R. Next, let us show that $\widetilde{R_\lambda}$ maps bounded functions into functions uniformly continuous on $\mathbb{R}^d \setminus \mathscr{N}$. We prove

$$\Big| (\widetilde{R_\lambda} f)(x) - (\widetilde{R_\lambda} f)(y) \Big| \leq c_1(\lambda) \, \|f\|_\infty \, \vartheta(|x-y|) , \tag{3.26}$$

with a function $\vartheta \colon (0,1) \to \mathbb{R}_+$ satisfying $\lim\limits_{t \to 0} \vartheta(t) = 0$ and $c_1(\lambda)$ independent of $x, y \in \mathbb{R}^d \setminus \mathscr{N} : |x - y| < 1/2$ and $f \in L^\infty(\mathbb{R}^d)$. (3.26) is only needed for x, y closeby. Choose $x_0 \in \mathbb{R}^d, r > 0$ such that $x, y \in B(x_0, r/2)$. Using the strong Markov property one obtains

$$\begin{aligned} (\widetilde{R_\lambda} f)(x) &= \mathbb{E}^x \left(\int_0^{\tau_{B(x_0,r)}} e^{-\lambda t} f(X_t)\, dt \right) + \mathbb{E}^x \Big((\widetilde{R_\lambda} f)(X_{\tau_{B(x_0,r)}}) \Big) \\ &\quad + \mathbb{E}^x \Big((e^{-\lambda \tau_{B(x_0,r)}} - 1)(\widetilde{R_\lambda} f)(X_{\tau_{B(x_0,r)}}) \Big) , \end{aligned}$$

and a similar expression for $(\widetilde{R_\lambda} f)(y)$. Note that the second term on the right hand side is a $\mathscr{L}$-harmonic function in $B(x_0, r)$ as a function of x. Using the above representation we deduce

$$\begin{aligned} \Big| (\widetilde{R_\lambda} f)(x) - (\widetilde{R_\lambda} f)(y) \Big| &\leq \Big(2\|f\|_\infty + 2\lambda \|\widetilde{R_\lambda} f\|_\infty \Big) \sum_{z \in \{x,y\}} \mathbb{E}^z \tau_{B(x_0,r)} \\ &\quad + c_0 \left\| \widetilde{R_\lambda} f \right\|_\infty |\ln r|^{-\rho} , \end{aligned}$$

where we applied (3.25). Estimate (3.26) follows from an application of Lemma 3.12.

Therefore, there exists a modification R_λ of $\widetilde{R_\lambda}$ such that R_λ satisfies the strong Feller property which means that bounded functions are mapped into $C_b(\mathbb{R}^d)$. Furthermore, for any $v \in C_\infty(\mathbb{R}^d)$ one checks $\lim\limits_{\lambda \to \infty} \|\lambda(\widetilde{R_\lambda} v - v)\|_\infty = 0$. From here, one concludes that there is a modification T_t of $\widetilde{T}_t$ such that (T_t) is a Feller semigroup on $C_\infty(\mathbb{R}^d)$, see Corollary 4.4 in [SU07]. Note that (T_t) might not be a strong Feller semigroup. ■

Let us now review the counter-example of [BBCK09]. We construct a kernel $n_1\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ as follows. Choose $a \in (0,1)$ and $0 < \varepsilon < 1 - a$. For $z \in \mathbb{R}^2$, $z_1 \neq z_2$:

$$m_1(z_1, z_2) = \begin{cases} \min(|z_1|^{-a-2}, |z_2|^{-(a+\varepsilon)-2}) & \text{if } |z_1| \vee |z_2| \leq 1, \\ 0 & \text{if } |z_1| \vee |z_2| > 1. \end{cases} \tag{3.27}$$

Note that there are $0 < c_0 \leq c_1$ such that for $|z_1| \vee |z_2| \leq 1$ one has

$$c_0 |z|^{-a-2} \leq \min(|z_1|^{-a-2}, |z_2|^{-(a+\varepsilon)-2}) \leq c_1 |z|^{-(a+\varepsilon)-2} \,. \tag{3.28}$$

Now for $x, y \in \mathbb{R}^2$, $x \neq y$ define $k_1(x,y)$ as follows. Let $V = \{(x_1, x_2) : |x_1| < |x_2|\}$. Set

$$k_1(x,y) = \begin{cases} m_1\big(|x_1 - y_1|, |x_2 - y_2|\big) \mathbb{1}_{\{x-y \in B_1(0)\}} \,, & x, y \in V \,, \\ m_1\big(|x_2 - y_2|, |x_1 - y_1|\big) \mathbb{1}_{\{x-y \in B_1(0)\}} \,, & x, y \notin V \,, \\ \Big(|x_1 - y_1|^{-2-a} \wedge |x_2 - y_2|^{-2-a}\Big) \mathbb{1}_{\{x-y \in B_1(0)\}} \,, & \text{elsewhere} \,. \end{cases} \tag{3.29}$$

Theorem 3.13 ([BBCK09]) *Let $\mathscr{L}$ be as in (3.24) with k replaced by k_1. Then the martingale problem for $(\mathscr{L}, C_c^2(\mathbb{R}^d))$ is not well-posed for the starting point $0 \in \mathbb{R}^d$.*

The proof of this result is far from being trivial but the main idea can be grasped easily. The above construction has the following effect on the corresponding process X_t. If X_t is started from V it moves in short time intervals rather up or down than left or right. Started in V^c the preferred directions are swapped. One obtains that the transition probability $p(t,x,y)$ is discontinuous for small t at $x = 0$. As a result we find a continuous function v such that $x \mapsto \mathbb{E}^x v(X_t)$ is not continuous at 0 when t is small. On the other hand, if uniqueness to the martingale problem for $\mathscr{L}$ started at 0 were to hold, one would have $\mathbb{P}^x \to \mathbb{P}^0$ as $x \to 0$. This is a contradiction.

Note that $\varepsilon > 0$ can be arbitrarily small in the construction of m_1 and k_1. The following example is a byproduct of Theorem 3.3. It shows that a replacement of an ε-power by a logarithmic term in the construction of k_1 above again leads to a nice Feller semigroup.

Assume $a \in (0,1)$. For $z \in \mathbb{R}^2$, $z_1 \neq z_2$, set

$$m_2(z_1, z_2) = \begin{cases} \min(|z_1|^{-a-2}, |z_2|^{-a-2} \ln(\frac{3}{|z_2|})) & \text{if } |z_1| \vee |z_2| \leq 1, \\ 0 & \text{if } |z_1| \vee |z_2| > 1. \end{cases} \tag{3.30}$$

Define $k_2(x,y)$ with the help of $m_2(z_1,z_2)$ in the same way as $k_1(x,y)$ is defined using $m_1(z_1,z_2)$ above. Further below we show that k_2 satisfies the assumption of Theorem 3.3. Next, let $\big(\mathscr{E}, D(\mathscr{E})\big)$ be as in (3.5) and $\mathscr{L}$ be as in (3.24) with k replaced by k_2. Then Theorem 3.3 applies. If $(\mathscr{A}, D(\mathscr{A}))$ denotes the C_∞-generator of (T_t), then well-posedness of the martingale problem for $(\mathscr{A}, D(\mathscr{A}))$ follows directly from Theorem 4.1, chapter 4 in [EK86] and Dynkin's formula. This statement completes the presentation of the example. Note that the results obtained in [BL02a], [SV04], [BK05b] and [SU07] do not apply to this case.

We close this session with an auxiliary result which we have just used.

Lemma 3.14 *Set $M = \{(x,y) \in \mathbb{R}^2, x \neq y, \max(|x|,|y|) \leq 1\}$. Choose $\beta > 0$. For $(x,y) \in M$ set*

$$m_2(x,y) = \min\Big(|x|^{-\beta}, |y|^{-\beta} \ln\big(\frac{3}{|y|}\big)\Big) \,. \tag{3.31}$$

There are positive constants c_0, c_1 such that for all $(x,y) \in M$

$$\frac{c_0}{\sqrt{x^2+y^2}^{\beta}} \leq m_2(x,y) \leq c_1 \frac{\ln\left(\frac{3}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}^{\beta}} \,. \tag{3.32}$$

Proof: Throughout the proof we assume $x > 0$, $y > 0$. The estimate of $m_2(x,y)$ from below is trivial since

$$m_2(x,y) \geq \frac{1}{x^\beta + y^\beta (\ln\left(\frac{3}{y}\right))^{-1}} \geq \frac{1}{x^\beta + y^\beta} \geq \frac{c_0}{\sqrt{x^2+y^2}^{\beta}} \,.$$

For the estimate from above consider two cases. First we assume $x^{-\beta} \geq y^{-\beta} \ln\left(\frac{3}{y}\right)$. Then $0 < x \leq y \leq 1$ and $y \geq \frac{1}{\sqrt{2}}\sqrt{x^2+y^2}$. Therefore by monotony we get

$$y^{-\beta} \ln\left(\frac{3}{y}\right) \leq 2^{\beta/2} \frac{\ln\left(\frac{3\sqrt{2}}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}^{\beta}} \leq c_1 \frac{\ln\left(\frac{3}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}^{\beta}} \,. \tag{3.33}$$

Now assume $x^{-\beta} \leq y^{-\beta} \ln\left(\frac{3}{y}\right)$. If one has $0 < x \leq y \leq 1$ then we reason as in (3.33). In the case $0 < y \leq x \leq 1$ we have

$$x^{-\beta} \leq y^{-\beta} \ln\left(\frac{3}{x}\right) . \tag{3.34}$$

Again we proceed as in (3.33) finishing the proof. ■

4 Markov chain approximations for symmetric jump processes

4.1 Introduction

Let $Y = (Y_t)_t$ be a continuous-time Markov chain on $\mathbb{Z}^d$. It is a natural question whether the sequence (Y^n) of Markov chains defined by $Y_t^n = n^{-1} Y_{n^\alpha t}$, $\alpha \in (0, 2]$, tends to some reasonable process for $n \to \infty$. The case $\alpha = 2$ is known as diffusive scaling and leads to a diffusion process under certain assumptions on Y, see the classical Donsker's Invariance Principle of [Don51] for the Brownian motion and chapter 11 of [SV06] for diffusion processes in non-divergence form. In the case of symmetric processes Stroock and Zheng derive in [SZ97] a central limit theorem for continuous-time Markov chains of bounded range. In a recent paper, Bass and Kumagai [BK08] remove the restriction of bounded range by replacing it by a second moment condition. In both publications, the generator of the limit object is of the form $Lu(x) = \sum\limits_{i,j=1}^{d} \partial_{x_i}\big(a_{ij}(\cdot)\partial_{x_j}\big)$ and a formula is provided how the diffusion coefficient functions $a_{ij}(\cdot)$ can be computed from the conductivities of the chain (Y_t). The other direction, i.e. constructing a sequence of approximating Markov chains for a given diffusion matrix is not less important and one of the main results of [SZ97].

The aim of this work is to prove results analogous to ones of [SZ97], [BK08] in the case where the limit object is a reversible jump process with corresponding Dirichlet form $\big(\mathcal{E}, D(\mathcal{E})\big)$ given by

$$\begin{aligned} \mathcal{E}(f,g) &= \frac{1}{2} \iint\limits_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus\text{diag}} \big(f(y)-f(x)\big)\big(g(y)-g(x)\big)k(x,y)\,dx\,dy, \quad k(x,y)=k(y,x)\,, \\ D(\mathcal{E}) &= \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}, \quad \text{where } \mathcal{E}_1(f,f) = \mathcal{E}(f,f) + \|f\|_{L^2}\,, \end{aligned} \tag{4.1}$$

and generator L given by

$$Lu(x) = \lim_{\varepsilon\to 0} \int\limits_{|y-x|\ge\varepsilon} \big(u(y)-u(x)\big)k(x,y)\,dy\,. \tag{4.2}$$

Therefore, we study Markov chain approximations for a certain class of reversible jump processes. In [SZ97], [BK08] the generator of the limit object is a uniformly elliptic operator. In our situation, the equivalent concept of uniform ellipticity would be given by $k(x,y) \ge c|x-y|^{-d-\alpha}$ $\forall\, |x-y| \le r_0$ for some $c > 0$, $r_0 > 0$, $\alpha \in (0,2)$. One feature of our approach is that our central limit theorem allows for cases where such an estimate does not hold, i.e. the limit process may be a pure jump process which is anisotropic in some sense. The level of anisotropy is limited since our approach uses a-priori bounds for the modulus of continuity of the heat kernel. As discussed in [BBCK09] these bounds fail for very irregular jump measures.

There are several other contributions to the question how to approximate Hunt processes given by Dirichlet forms, see [MRZ98], [MRS00] and the references therein. However, our results are not covered by these works. We close the introduction by commenting on the differences between this work and [BK08], [SZ97].

1. The limit object in [SZ97] and [BK08] is a diffusion whereas here it is a jump process.

2. The main result of [BK08] is a central limit theorem. In analogy to Theorem 3.9 in [SZ97] we also provide a construction of an approximative Markov chain for a given symmetric jump process, see Theorem 4.3.

3. Our assumption (A5) differs from (A5) of [BK08]. On one hand, we do not assume continuity of the coefficients of the limit process. On the other hand, we assume only L^1_{loc}-convergence of conductivities which is substantially less than uniform convergence on compacts. However, the characterisation and computation of the limit process is much harder in [SZ97], [BK08] because there, the bilinear form corresponding to the limit process contains gradients and not differences.

The chapter is organized as follows. In section 4.2 we present our assumptions and results. We provide a detailed discussion of the assumptions, some definitions, and notation. Furthermore, an auxiliary result on equivalent norms on the Sobolev space $H^{\alpha/2}(\mathbb{R}^d)$ is proved. Sections 4.3 and 4.4 provide the proofs of Theorem 4.13 and Theorem 4.19 both of which are crucial to the proof of our main results. In section 4.5 we prove Theorems 4.1 and 4.2. Theorem 4.3 is proved in section 4.6.

4.2 Assumptions and results

We formulate our assumptions and results in section 4.2.1. In section 4.2.2 we provide a detailed discussion of the assumptions. Section 4.2.3 is devoted to a result on equivalent norms on $H^{\alpha/2}(\mathbb{R}^d)$, $\alpha \in (0,2)$. In section 4.2.4 we define and list various further objects that we deal with in this article.

We denote the counting measure by μ and the Lebesgue measure on $\mathbb{R}^d$ by $\boldsymbol{\lambda}$. In our context we also deal with the function spaces $L^2(\rho^{-1}\mathbb{Z}^d, \rho^{-d}\mu)$ where $\rho > 0$. The scaling factor ρ^{-d} in front of μ is natural from a geometric point of view. Write $B^\rho(x,r) := B(x,r) \cap \rho^{-1}\mathbb{Z}^d$ for the r-ball around x in $\rho^{-1}\mathbb{Z}^d$. We also use the notation $\mu^\rho = \rho^{-d}\mu$. For $x \in \mathbb{R}^d$ we use the abbreviation $|x|_\infty = \max\limits_{i=1,\dots,d} |x_i|$. For $x \in \mathbb{R}$ we write $\lfloor x \rfloor$ instead of $\max\{l \in \mathbb{Z} : l \leq x\}$. For a point $x \in \mathbb{R}^d$ we denote by $[x]_n$ the element of $n^{-1}\mathbb{Z}^d$ satisfying $([x]_n)_i = n^{-1}\lfloor nx_i \rfloor$ for all $i = 1, \dots, d$.

4.2.1 Formulation of assumptions and results

Let $(C^n)_{n\in\mathbb{N}}$ be a sequence of *conductivity functions* $C^n \colon n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d \to [0,\infty)$. Let $\alpha \in (0,2)$. The following assumptions will be important.

(A1) $C^n(x,y) = C^n(y,x)$ and $C^n(x,x) = 0$ for all $n \in \mathbb{N}$, $x, y \in n^{-1}\mathbb{Z}^d$.

(A2) There exists $\kappa_1 > 0$ such that for all $n \in \mathbb{N}$, $x \neq y$

$$C^n(x,y) \leq \kappa_1 |x-y|^{-d-\alpha}.$$

(A3) There exist $N_0 \in \mathbb{N}$ and $\kappa_2 > 0$ with the following property: For any $n \in \mathbb{N}$, $x, y \in n^{-1}\mathbb{Z}^d$, $x \neq y$ there are elements $z_0^{(x,y)}, \ldots, z_l^{(x,y)} \in n^{-1}\mathbb{Z}^d$, $l \leq N_0$, $z_0^{(x,y)} = x$, $z_l^{(x,y)} = y$ satisfying for any $i = 0, \ldots, l-1$

$$C^n\big(z_i^{(x,y)}, z_{i+1}^{(x,y)}\big) \geq \kappa_2 |x-y|^{-d-\alpha} \ , ,$$

and for any $\zeta, \xi \in n^{-1}\mathbb{Z}^d$

$$\#\Big\{(x,y) \in n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d : \zeta = z_k^{(x,y)} \text{ and } \xi = z_{k+1}^{(x,y)} \text{ for some } k\Big\} \leq N_0 .$$

For given $x, y \in n^{-1}\mathbb{Z}^d$, $x \neq y$ we call the ordered set $\{z_0^{(x,y)}, \ldots, z_l^{(x,y)}\}$ above a *chain* and l the length of the chain. The above assumptions are essential for our approach and are discussed in the next section. Note that nearest-neighbor random walks are excluded by (A3). For a mere technical reason discussed below in detail we need an additional assumption:

(A4) There exist $\Theta_1 > 0$ and $\kappa_3 > 0$ such that for all $n \in \mathbb{N}$, $x \in n^{-1}\mathbb{Z}^d$ and $r \geq \Theta_1 n^{-1}$

$$\mu\big(\{y \in B^n(x,r) : C^n(x,y) \geq \kappa_3 |x-y|^{-d-\alpha}\}\big) \geq \big(1 - \tfrac{1}{6 \cdot 2^d}\big)\, \mu\big(B^n(x,r)\big).$$

It is important for our results that the constants $\kappa_1, \kappa_2, \kappa_3, N_0, \Theta_1$ appearing in (A1) through (A4) do not depend on $n \in \mathbb{N}$. We associate to C^n a discrete-time Markov chain $X^n = (X_k^n)_{k \in \mathbb{N}}$ by

$$\mathbb{P}^x(X_1^n = y) = \frac{C^n(x,y)}{\sum\limits_{z \in n^{-1}\mathbb{Z}^d} C^n(x,z)} . \tag{4.3}$$

Let $Y^n = (Y_t^n)_t$ be the continuous-time Markov chain that has the same jumps as X^n while its holding time in the point x is exponentially distributed with parameter $\sum\limits_{z \in n^{-1}\mathbb{Z}^d} C^n(x,z) n^{-d}$. Note that each Y^n starting in $x \in n^{-1}\mathbb{Z}^d$ corresponds to a probability measure on $D([0,\infty); \mathbb{R}^d)$, the space of right-continuous paths in $\mathbb{R}^d$ having left limits, see [EK86], [Bil99] for properties of $D([0,\infty); \mathbb{R}^d)$. Our first result reads as follows:

Theorem 4.1 *Let $(C^n)_n$ be a sequence of conductivity functions satisfying (A1) through (A4). For $x_n \in n^{-1}\mathbb{Z}^d$, $x_n \to x \in \mathbb{R}^d$ the laws of Y^n starting in x_n are tight in $D([0,t_0]; \mathbb{R}^d)$ for any $t_0 > 0$.*

For a precise statement of our results on tightness, see Theorem 4.20.

In order to establish a central limit theorem one needs to prescribe the behavior of C^n for n tending to infinity. For $x \in n^{-1}\mathbb{Z}^d$ set $\mathfrak{Q}_n(x) = \prod_{i=1}^d [x_i, x_i + 1/n)$ and $\mathfrak{Q}_n = \bigcup\limits_{x \in n^{-1}\mathbb{Z}^d} \{\mathfrak{Q}_n(x)\}$.

(A5) There exists a measurable function $k\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ such that for any compact subset K of $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ the functions $(x,y) \mapsto C^n([x]_n, [y]_n)$ converge in $L^1(K)$ to $k(\cdot,\cdot)$ for $n \to \infty$.

(B) There exist $M_0 \in \mathbb{N}$ and $\Lambda_2 > 0$ with the following property: For any $\varepsilon > 0$, $n \in \mathbb{N}$ and $\mathcal{O}, \mathcal{Q} \in \mathfrak{Q}_n$ there are elements $\mathcal{P}_0^{(\mathcal{O},\mathcal{Q})}, \ldots, \mathcal{P}_l^{(\mathcal{O},\mathcal{Q})} \in \mathfrak{Q}_n$, $l \le M_0$, $\mathcal{P}_0^{(\mathcal{O},\mathcal{Q})} = \mathcal{O}$, $\mathcal{P}_l^{(\mathcal{O},\mathcal{Q})} = \mathcal{Q}$ satisfying for any $j = 0, \ldots, l-1$

$$\iint\limits_{\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})} \times \mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}} k(x,y) \mathbb{1}_{\{|x-y| \ge \varepsilon\}} \, dxdy \ge \Lambda_2 \iint\limits_{\mathcal{O} \times \mathcal{Q}} \mathbb{1}_{\{|x-y| \ge \varepsilon\}} |x-y|^{-d-\alpha} \, dxdy \,,$$

and for any $\mathcal{R}, \mathcal{S} \in \mathfrak{Q}_n$

$$\#\Big\{(\mathcal{O},\mathcal{Q}) \in \mathfrak{Q}_n \times \mathfrak{Q}_n : \mathcal{R} = \mathcal{P}_k^{(\mathcal{O},\mathcal{Q})} \text{ and } \mathcal{S} = \mathcal{P}_{k+1}^{(\mathcal{O},\mathcal{Q})} \text{ for some } k\Big\} \le M_0 \,.$$

Again, we call the ordered set $\{\mathcal{P}_0^{(\mathcal{O},\mathcal{Q})}, \ldots, \mathcal{P}_l^{(\mathcal{O},\mathcal{Q})}\}$ above a *chain* and l the length of the chain. Although, to some extent, (B) is a continuous analog of (A3) it does not follow from (A3) and (A5). Such an implication could easily be achieved by adding an additional assumption. In order not to weaken Theorem 4.1 we prefer to work with (B) separately.

Here is our central limit theorem.

Theorem 4.2 *Let $(C^n)_n$ be a sequence of conductivity functions satisfying (A1) through (A5) and (B). Let $\mathscr{X}$ be a Hunt process associated to the regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ given in (4.1) and $\mathcal{N}$ the properly exceptional set. Then, for $x_n \in n^{-1}\mathbb{Z}^d$, $x_n \to x \in \mathbb{R}^d \setminus \mathcal{N}$ the laws of Y^n starting in x_n converge weakly in $D([0,t_0]; \mathbb{R}^d)$ to the law of $\mathscr{X}$ starting in x.*

Let us remark that assumptions (A3) and (B) are technically involved and cover anisotropic situations. In fact, (A3) and (B) are trivially satisfied in the isotropic case, i.e. if C^n satisfies (A1), (A2), (A5) and $C^n(x,y) \ge c\,|x-y|^{-d-\alpha}$ for all $n \in \mathbb{N}$ and $|x-y| > n^{-1}K$ for some $K > 0, c > 0$. Even in this case our theorem is still interesting and new.

It is necessary to allow for some exceptional set in Theorem 4.2. However, due to results in [CK03] the set $\mathcal{N}$ can be removed or assumed to be empty in several situations. There is no need for an exceptional set in our third result, Theorem 4.3. As in Theorem 3.14 of [SZ97] we give an explicit construction of approximating Markov chains.

Theorem 4.3 *Let $k\colon \mathbb{R}^d \times \mathbb{R}^d \to (0,\infty)$ be a measurable function which satisfies $k(x,y) = k(y,x)$ and*

$$\kappa_4 \,|x-y|^{-d-\alpha} \le k(x,y) \le \kappa_5 \,|x-y|^{-d-\alpha} \tag{4.4}$$

for almost all $x, y \in \mathbb{R}^d$, $x \ne y$ with some positive constants $\kappa_4 < \kappa_5$. Define the conductivity functions $C^n\colon n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d \to [0,\infty)$ by

$$C^n(x,y) = \begin{cases} 0 & \text{for } |x-y|_\infty \le n^{-1}\,, \\ n^{2d} \displaystyle\int\limits_{\substack{|x-\xi|_\infty < n^{-1}/2 \\ |y-\zeta|_\infty < n^{-1}/2}} k(\xi,\zeta) d\xi \, d\zeta & \text{for } |x-y|_\infty \ge 2n^{-1}\,. \end{cases}$$

Let $\mathscr{X}$ be the Hunt process corresponding to the Dirichlet form $\big(\mathcal{E}, D(\mathcal{E})\big)$ given by (4.1). Then the sequence of processes corresponding to C^n converges in the sense of Theorem 4.2 to $\mathscr{X}$ for any starting point $x \in \mathbb{R}^d$.

As becomes clear by the discussion below, (A4) allows for quite general cases of sequences C^n. In addition to it, in light of Lemma 4.4 and Lemma 4.10 it is very likely that (A4) can be dropped. This would imply the possibility to weaken the lower bound on k assumed in (4.4) substantially.

4.2.2 Discussion of assumptions

We illustrate assumptions (A1) through (A5) introduced in Section 4.2.1. First, let us look at (A1) through (A4). If, for a fixed scale $n \in \mathbb{N}$, $C^n \colon n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d \to \mathbb{R}^+$ satisfies $(A1)$ through $(A4)$ then the same holds for the conductivity function $C \colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$ with $C(x,y) = n^{-d-\alpha} C^n(n^{-1}x, n^{-1}y)$ for $x, y \in \mathbb{Z}^d$ with the same constants $d, \kappa_1, \kappa_2, N_0$ where the chains in (A3) have to be scaled in an obvious way. Since C is the appropriate conductivity function corresponding to the process Y^n scaled on $\mathbb{Z}^d$ in the obvious ("α-stable") way, it is sufficient to understand (A1) through (A4) for a single conductivity function C on $\mathbb{Z}^d \times \mathbb{Z}^d$. In addition, the main results of Section 3 and 4 are scale-invariant, i.e. the constants appearing are scale-invariant, and depend only on the constants in the assumptions and the dimension d. Therefore it again suffices to prove them for a fixed conductivity function.

Let $C, \widetilde{C} : \mathbb{Z}^d \times \mathbb{Z}^d \to [0,\infty)$ be conductivity functions such that there exists $R > 0$ with $\widetilde{C}(x,y) = C(x,y)$ for $|x-y| \geq R$. If C satisfies one of the assumptions (A3), (A4) then the same assumption also holds for $\widetilde{C}(x,y)$. On the other hand, if $C^n, \widetilde{C}^n \colon n^{-1}\mathbb{Z}^d \times n^{-1}\mathbb{Z}^d \to \mathbb{R}^+$ are two sequences of conductivities with $\widetilde{C}^n(x,y) = C^n(x,y)$ whenever $|x-y| \geq Rn^{-1}$ and if (C^n) satisfies (A5), then (A5) also holds for $(\widetilde{C}^n)$ with the same limit function k.

(A2) bounds $C(x,y)$ from above by the conductivities of a rotationally symmetric α-stable Markov chain on $\mathbb{Z}^d$. (A2) gives in particular

$$\sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} C(x,y) < \infty.$$

(A3) is much more technical. It implies a certain kind of irreducibility of the associated Markov chain. Additionally, it takes into account the highly non-local nature of our objects. Roughly it says that every two points x, y can be connected by chaining together at the utmost N_0 jumps where the probability of each jump is bounded from below by a constant multiple of $|x-y|^{-d-\alpha}$ while at the same time one has enough of these connecting jumps.

(A2) and (A3) imply together $\big|z_i^{(x,y)} - z_{i+1}^{(x,y)}\big| \leq (\kappa_1/\kappa_2)^{1/(d+\alpha)} |x-y|$. This leads us to the following necessary condition for (A2) and (A3).

Lemma 4.4 *Assume (A2) and (A3). Then there exist $\gamma \in (0,1)$, $\Theta_1 > 0$ and $\kappa_3 > 0$ depending only on κ_1, κ_2, N_0, d and α such that for all $x \in \mathbb{Z}^d$, $r > \Theta_1$*

$$\mu\big(\{y \in B^1(x,r) : C(x,y) \geq \kappa_3 |x-y|^{-d-\alpha}\}\big) \geq \gamma \mu\big(B^1(x,r)\big). \tag{4.5}$$

In particular, if the conductivities are stationary, i.e. $C(x,y) = \widetilde{C}(x-y)$ then

$$\mu\big(\{h \in B^1(0,r) : \widetilde{C}(h) \geq \kappa_3 |h|^{-d-\alpha}\}\big) \geq \gamma \mu(B^1(0,r)).$$

Note that Θ_1 from above is not identical to Θ_1 of Assumption (A4) but plays a similar role.
Proof: First notice that (A2) and (A3) imply the existence of $c_1 = c_1(\kappa_1, \kappa_2, d, \alpha, N_0) \geq 1$ such that for all $l \leq N_0$, $\xi, \zeta \in \mathbb{Z}^d$

$$\left|z_l^{(\xi,\zeta)} - \xi\right| \leq c_1 \left|\xi - \zeta\right| .$$

Assume r large enough and $x \in \mathbb{Z}^d$. Then $M = \left\{z_1^{(x,y)} \in \mathbb{Z}^d \colon y \in B^1(x, r/c_1)\right\} \subset B^1(x,r)$. By the second part of (A3)

$$\mu(M) \geq \lfloor \mu\big(B^1(x, r/c_1)\big)/N_0 \rfloor \geq \frac{c_2(d)}{c_1^d N_0} \mu\big(B^1(x,r)\big) . \tag{4.6}$$

Next, set

$$\widetilde{M} = M \setminus B^1\Big(x, \big(\frac{c_2}{2c_1^d N_0}\big)^{1/d} r\Big).$$

Trivially, $\mu(\widetilde{M}) \geq c_3 \mu\big(B^1(x,r)\big)$ where $c_3 = \frac{c_2}{4c_1^d N_0}$ depends on all constants that appeared so far. Assume $r \geq \big(\frac{c_2}{2c_1^d N_0}\big)^{-1/d}$ and $z \in \widetilde{M}$. Then there is $y \in B(x, r/c_1)$ with $z = z_1^{(x,y)} \in \widetilde{M}$ and

$$C\big(x, z_1^{(x,y)}\big) \geq \kappa_2 \left|x-y\right|^{-d-\alpha} \geq \kappa_2 c_1^{d+\alpha} r^{-d-\alpha} \geq \kappa_2 \big(\frac{c_2}{2N_0}\big)^{(d+\alpha)/d} \big|x - z_1^{(x,y)}\big|^{-d-\alpha}.$$

Setting $\kappa_3 = \kappa_2 \big(\frac{c_2}{2N_0}\big)^{(d+\alpha)/d}$ and $\gamma = c_3$ the set

$$\{y \in B^1(x,r) : C(x,y) \geq \kappa_3 \left|x-y\right|^{-d-\alpha}\}$$

contains $\widetilde{M}$ and satisfies (4.5). ■

Lemma 4.4 implies that, under assumptions (A2) and (A3), a second moment condition as in [BK08] cannot hold. Note that (4.5) is not sufficient for (A3) to hold; choose, for instance, $d = 1$ and $C(x,y) = \left|x-y\right|^{-1-\alpha}$ if $x \neq y$, $x - y \in 2\mathbb{Z}$ and $C(x,y) = 0$ elsewhere.

Let us now provide some examples of conductivity functions satisfying our assumptions. If $C(x,y)\left|x-y\right|^{d+\alpha}$ stays bounded between two positive constants then C satisfies (A2), (A3) and (A4). Hence all cases of [BL02b] are covered by our conditions. In addition, our assumptions allow for cases where there are no jumps in the direction of certain cones.

Example 4.5 *Let* $V := \left\{(x_1,x_2) \in \mathbb{Z}^2 : |x_2| \leq \gamma\,|x_1|\right\}$, $\gamma > 0$ *be a double-cone in* $\mathbb{Z}^d$. *Set*

$$C(x,y) = \mathbb{1}_V(x-y) g(x,y) \left|x-y\right|^{-d-\alpha} \tag{4.7}$$

for $x, y \in \mathbb{Z}^2$, $x \neq y$ *where* $g \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to [a,b]$ *for some* $0 < a < b$ *is a measurable, symmetric function. Then these conductivities satisfy (A2) and (A3). If* γ *is large enough (A4) holds, too.*

In fact, if $x = (x_1,x_2)$ *and* $y = (y_1,y_2)$ *set* $z_1^{(x,y)} = (x_1 + \lfloor (2+\gamma^{-1})\left|x-y\right| \rfloor, x_2)$, $z_0^{(x,y)} = x$, *and* $z_2^{(x,y)} = y$. *Then*

$$\begin{aligned} x - z_1^{(x,y)} &= (\lfloor (2+\gamma^{-1})\left|x-y\right| \rfloor, 0) \in V, \\ \big|x - z_1^{(x,y)}\big| &\leq (3+\gamma^{-1})\left|x-y\right| , \end{aligned}$$

$$z_1^{(x,y)} - y = (x_1 - y_1 + \lfloor (2+\gamma^{-1})\,|x-y| \rfloor, x_2 - y_2),$$
$$\left| z_1^{(x,y)} - y \right| \le |x-y| + \sqrt{2} \lfloor (2+\gamma^{-1})\,|x-y| \rfloor \le 2(4+\gamma^{-1})\,|x-y|\,.$$

Finally, $z_1^{(x,y)} - y \in V$ *by*

$$\gamma \left| x_1 - y_1 + \lfloor (2+\gamma^{-1})\,|x-y| \rfloor \right| \ge \gamma \lfloor (1+\gamma^{-1})\,|x-y| \rfloor \ge |x-y| \ge |x_2 - y_2|\,.$$

Example 4.6 *Define C as in Example 4.5 with γ large enough, say $\gamma > 7/8$, and $g \equiv 1$. Set $C^n(x,y) = n^{d+\alpha} C(nx, ny)$, $n \in \mathbb{N}$, $x, y \in n^{-1}\mathbb{Z}^d$. Then $(C^n)_n$ satisfies (A1) through (A5) with*

$$k(x,y) = \mathbb{1}_V(x-y)\,|x-y|^{-d-\alpha}\,, \quad V := \left\{ (x_1,x_2) \in \mathbb{R}^2 : |x_2| \le \gamma\,|x_1| \right\}\,. \tag{4.8}$$

Note also the following counterexample:

Example 4.7 *Set $V := \{0\} \times \mathbb{Z} \cup \mathbb{Z} \times \{0\}$ and $C(x,y) = \mathbb{1}_V(x-y)\,|x-y|^{-d-\alpha}$. Then these conductivities do not satisfy (A3) which follows from Lemma 4.4.*

Finally, let us give the most obvious example of a conductivity function C satisfying (A1) through (A5) .

Example 4.8 *Fix $\alpha \in (0,2)$. Then the conductivity functions*

$$C^n(x,y) = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \pi^{d/2} \Gamma(1-\alpha/2)}\,|x-y|^{-d-\alpha}\,, \quad x, y \in n^{-1}\mathbb{Z}^d$$

satisfy (A1) through (A5). The limit process $\mathscr{X}$ in the sense of Theorem 4.2 is the well-known rotationally invariant α-stable process. The properly exceptional set $\mathcal{N}$ is empty.

4.2.3 Equivalent norms on $H^{\alpha/2}(\mathbb{R}^d)$

So far we have concentrated on a discussion of (A1) through (A5). Let us now look at (B). Let $k\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ be a measurable function satisfying $k(x,y) = k(y,x)$ and $k(x,y) \le \Lambda_1\,|x-y|^{-d-\alpha}$ for almost all (x,y) with $x \ne y$ and some $\Lambda_1 > 0$. In light of (A1), (A2) and (A5) this is the structure of kernels appearing in the limit $n \to \infty$. Under assumptions (A1) through (A5) there still can be large oscillations of C^n in the following sense. Fix two sequences of $x_n^i, y_n^i \in n^{-1}\mathbb{Z}^d$, $i \in \{1,2\}$, with $|x_n^1 - x_n^2| \to 0$ and $|y_n^1 - y_n^2| \to 0$. Then chains connecting x_n^1 and y_n^1 can be very far apart from chains connecting x_n^2 and y_n^2, no matter how large n is. Assumption (B) guarantees that this phenomenon can be avoided by choosing appropriate chains. Having studied (A1) through (A3) it should be clear how to construct examples of kernels k satisfying (B). For instance, the kernel k constructed in example 4.6 satisfies (B).

Let us show that (B) is a natural assumption. Assuming (B) we show that $D(\mathcal{E})$ from (4.1) equals $H^{\alpha/2}(\mathbb{R}^d)$, i.e. (B) determines a class of equivalent norms on $H^{\alpha/2}(\mathbb{R}^d)$, $\alpha \in (0,2)$. One standard definition of this function space is

$$H^{\alpha/2}(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d); \|f\|_{H^{\alpha/2}} := \|f\|_{L^2} + \sqrt{\mathcal{E}_\alpha(f,f)} < \infty \}$$

where

$$\mathcal{E}_\alpha(f,f) = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \big(f(x) - f(y)\big)^2\,|x-y|^{-d-\alpha}\,dx\,dy\,.$$

We show how it is possible to replace $|x-y|^{-d-\alpha}$ in the definition of $\mathcal{E}_\alpha(f,f)$ by some anisotropic kernel $k(x,y)$ without changing the function space. Different versions of such a result are established in the literature on function spaces, see [Tri95, Theorem 2.5.1 (12)]. Our result extends several of them and our assumptions read very different.

Theorem 4.9 *Let $k\colon \mathbb{R}^d\times\mathbb{R}^d \to [0,\infty)$ be a measurable function which satisfies $k(x,y) = k(y,x)$ and $k(x,y) \leq \Lambda_1 |x-y|^{-d-\alpha}$ for almost all (x,y) with $x \neq y$ and some $\Lambda_1 > 0$, $\alpha \in (0,2)$. Assume that k satisfies (B). Then there are two positive constants c_0, c_1 such that*

$$c_0\mathcal{E}_\alpha(f,f) \leq \mathcal{E}(f,f) \leq c_1\mathcal{E}_\alpha(f,f) \quad \forall\, f \in C_c^1(\mathbb{R}^d)\,. \tag{4.9}$$

Hence, the regular Dirichlet form $\big(\mathcal{E}, D(\mathcal{E})\big)$ of (4.1) satisfies under (B)

$$D(\mathcal{E}) = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1} = H^{\alpha/2}(\mathbb{R}^d)\,.$$

Proof: The second estimate in (4.9) follows trivially from the upper bound of k. In order to establish the first one note

$$\mathcal{E}_\alpha(f,f) = \lim_{\varepsilon\to 0} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(f(y)-f(x)\big)^2 \mathbb{1}_{\{|x-y|\geq\varepsilon\}} |x-y|^{-d-\alpha}\, dx\, dy$$

Denote by $z_{\mathcal{Q}}$ the center point of a given cube $\mathcal{Q} \in \mathfrak{Q}_n$. For simplicity we assume that the length of each chain is equal to M_0. Assumption (B) gets involved in the following way:

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(f(y)-f(x)\big)^2 \mathbb{1}_{\{|x-y|\geq\varepsilon\}}\, |x-y|^{-d-\alpha}\, dx\, dy \\
&= \lim_{n\to\infty} \sum_{\mathcal{O},\mathcal{Q}\in\mathfrak{Q}_n} \big(f(z_{\mathcal{O}})-f(z_{\mathcal{Q}})\big)^2 \iint\limits_{\mathcal{O}\times\mathcal{Q}} \mathbb{1}_{\{|x-y|\geq\varepsilon\}}\, |x-y|^{-d-\alpha}\, dx\, dy \\
&\leq M_0 \lim_{n\to\infty} \sum_{\mathcal{O},\mathcal{Q}\in\mathfrak{Q}_n} \sum_{j=0}^{M_0-1} \big(f(z_{\mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}}) - f(z_{\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})}})\big)^2 \iint\limits_{\substack{\mathcal{O}\times\mathcal{Q}\\ |x-y|\geq\varepsilon}} |x-y|^{-d-\alpha}\, dx\, dy \\
&\leq \frac{M_0}{\Lambda_2} \lim_{n\to\infty} \sum_{\mathcal{O},\mathcal{Q}\in\mathfrak{Q}_n} \sum_{j=0}^{M_0-1} \big(f(z_{\mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}}) - f(z_{\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})}})\big)^2 \iint\limits_{\substack{\mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}\times\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})}\\ |x-y|\geq\varepsilon}} k(x,y)\, dx\, dy \\
&\leq \frac{(M_0)^2}{\Lambda_2} \lim_{n\to\infty} \sum_{\mathcal{O},\mathcal{Q}\in\mathfrak{Q}_n} \max_{j=0,\ldots,M_0-1} \big(f(z_{\mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}}) - f(z_{\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})}})\big)^2 \\
&\qquad \times \iint\limits_{\substack{\mathcal{P}_{j+1}^{(\mathcal{O},\mathcal{Q})}\times\mathcal{P}_j^{(\mathcal{O},\mathcal{Q})}\\ |x-y|\geq\varepsilon}} k(x,y)\, dx\, dy \\
&\leq \frac{(M_0)^3}{\Lambda_2} \lim_{n\to\infty} \sum_{\mathcal{O},\mathcal{Q}\in\mathfrak{Q}_n} \big(f(z_{\mathcal{O}})-f(z_{\mathcal{Q}})\big)^2 \iint\limits_{\substack{\mathcal{O}\times\mathcal{Q}\\ |x-y|\geq\varepsilon}} k(x,y)\, dx\, dy
\end{aligned}$$

$$= \frac{(M_0)^3}{\Lambda_2} \iint\limits_{\substack{(\mathbb{R}^d\times\mathbb{R}^d)\setminus\mathrm{diag}\\ |x-y|\geq\varepsilon}} \big(f(y)-f(x)\big)^2 k(x,y)\,dx\,dy\,.$$

■

4.2.4 Further definitions and notation

If $\mathscr{X}$ is a stochastic process and Ω a Borel set write $\tau(\Omega;\mathscr{X})$ resp. $\sigma(\Omega;\mathscr{X})$ for the first time the process exits resp. enters Ω where we omit $\mathscr{X}$ if there is no danger of confusion.

Let $X = (X_k)_k$ be the discrete-time Markov chain associated to C by (4.3). A continuous-time Markov chain $Y = (Y_t)_t$ having the same jumps as X can be constructed as follows: Take a family $(T_{x,j})_{x\in\mathbb{Z}^d, j\in\mathbb{N}}$ of independent random variables, independent also of X, such that $T_{x,j}$ is exponentially distributed with parameter C_x and set $T_{x,0}\equiv 0$. Set $Y_t = X_n$ for $t \in [\sum_{j=0}^{n} T_{X_j,j}, \sum_{j=0}^{n+1} T_{X_j,j})$. Note that in [BL02b] the holding times of the continuous-time process are exponentially distributed with parameter 1 leading to different generators and Dirichlet forms. For more details on Markov chains we refer the reader to [Nor98]. (A2) and (A3) give uniform bounds on the expected holding times of Y. The process Y corresponds to the Dirichlet form

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} \big(f(x)-f(y)\big)^2 C(x,y)$$

with domain $D(\mathcal{E}) = L^2(\mathbb{Z}^d,\mu)$. We derive properties of Y in the next section by comparing $\big(\mathcal{E}, L^2(\mathbb{Z}^d,\mu)\big)$ to the Dirichlet form $\big(\mathcal{E}_\alpha, L^2(\mathbb{Z}^d,\mu)\big)$ of a rotationally invariant α-stable process in $\mathbb{Z}^d$, i.e. $\mathcal{E}_\alpha$ is defined by

$$\mathcal{E}_\alpha(f,f) = \frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} \big(f(x)-f(y)\big)^2 |x-y|^{-d-\alpha}\,, \quad \alpha\in(0,2)\,.$$

By (A2) $\mathcal{E}(f,f)\leq \kappa_1\mathcal{E}_\alpha(f,f)$ for all $f\in L^2(\mathbb{Z}^d,\mu)$.

Define a family $(Y^\rho)_{\rho>0}$ of continuous-time Markov chains on $\rho^{-1}\mathbb{Z}^d$ by $Y^\rho_t = \rho^{-1}Y_{\rho^\alpha t}$. Y^ρ corresponds to the Dirichlet form $\big(\mathcal{E}^\rho, L^2(\rho^{-1}\mathbb{Z}^d, \rho^{-d}\mu)\big)$ defined by

$$\mathcal{E}^\rho(f,f) = \frac{1}{2}\sum_{x,y\in\rho^{-1}\mathbb{Z}^d} \big(f(x)-f(y)\big)^2 C^\rho(x,y)\rho^{-2d} \quad \text{with } C^\rho(x,y) = \rho^{d+\alpha}C(\rho x,\rho y)\,. \tag{4.10}$$

Note that we abuse our own notation here. The above definition of the family $(C^\rho)_{\rho>0}$ does not correspond correctly to our sequence of conductivity functions $(C^n)_{n\in\mathbb{N}}$ defined in the introduction. To be precise: Given an arbitrary sequence $(C^n)_{n\in\mathbb{N}}$ in the sense of the introduction there might be no conductivity function $C\colon \mathbb{Z}^d\times\mathbb{Z}^d\to[0,\infty)$ such that $C^\rho = C^n$ for $n=\rho$. Nevertheless, we use C^ρ in the sense above and C^n in the sense of assumptions (A1) through (A5). This remark carries over to the definition on the family $(Y^n)_{n\in\mathbb{N}}$. We use the symbol Y^n for the continuous-time process corresponding to the conductivity function C^n. That is, the family $(Y^\rho)_{\rho>0}$ is determined by a single conductivity function $C\colon \mathbb{Z}^d\times\mathbb{Z}^d\to[0,\infty)$ whereas the family $(Y^n)_{n\in\mathbb{N}}$ depends on the whole sequence $(C^n)_{n\in\mathbb{N}}$. Y^n corresponds to the Dirichlet form $\big(\mathcal{E}^n, L^2(n^{-1}\mathbb{Z}^d, n^{-d}\mu)\big)$ defined by

$$\mathcal{E}^n(f,f) = \frac{1}{2} \sum_{x,y\in n^{-1}\mathbb{Z}^d} \big(f(x)-f(y)\big)^2 C^n(x,y) n^{-2d} \,. \tag{4.11}$$

Conditions (A1) through (A4) are stable in the following sense. If one fixed conductivity function C satisfies (A1) ((A2), (A3) resp.) then the assumption holds true for the family $(C^\rho)_\rho$ with constants independent of ρ. On the other hand, if (A4) is true for C the conclusion of (A4) holds for any C^ρ with the same γ and κ_3 whenever $r \geq \Theta_1 \rho^{-1}$.

Scaling as above implies a relation between the heat kernel p_{Y^ρ} of Y^ρ with respect to $\rho^{-d}\mu$ and the heat kernel p_Y of Y. Note that, by regarding the heat kernel of the scaled process with respect to $\rho^{-d}\mu$, $p_{Y^\rho}(t,x,y)$ is not anymore the probability that the process starting in x is at time t in y but ρ^{-d} times this probability. One has

$$p_{Y^\rho}(t,x,y) = \rho^d p_Y(\rho^\alpha t, \rho x, \rho y). \tag{4.12}$$

Let $Y^{\rho,\lambda}$ be the process Y^ρ with all jumps bigger than λ removed. $Y^{\rho,\lambda}$ corresponds to the Dirichlet form $\big(\mathcal{E}^{\rho,\lambda}, L^2(\rho^{-1}\mathbb{Z}^d, \rho^{-d}\mu)\big)$ defined by

$$\mathcal{E}^{\rho,\lambda}(f,f) = \frac{1}{2} \sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d \\ |x-y|\leq\lambda}} \big(f(x)-f(y)\big)^2 C^\rho(x,y)\rho^{-2d}.$$

Finally, set

$$\mathcal{E}_\alpha^{\rho,1}(f,f) = \frac{1}{2} \sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d \\ |x-y|\leq 1}} \big(f(x)-f(y)\big)^2 \left|x-y\right|^{-d-\alpha} \rho^{-2d}.$$

Let us finish this section with an overview over all processes which we have introduced so far:

- $X=(X_k)$: discrete-time Markov chain on $\mathbb{Z}^d$ corresponding to the conductivity function C.
- $Y=(Y_t)$: continuous-time Markov chain on $\mathbb{Z}^d$ with the same jumps as (X_n); its Dirichlet form is $\mathcal{E}$.
- $Y^\rho=(Y_t^\rho)$: scaled version of the process (Y_t); it corresponds to C^ρ; its state space is $\rho^{-1}\mathbb{Z}^d$; its Dirichlet form is $\mathcal{E}^\rho$.
- $Y^n=(Y_t^n)$: continuous-time process corresponding to C^n; its state space is $n^{-1}\mathbb{Z}^d$; its Dirichlet form is $\mathcal{E}^n$ defined in (4.11).
- $\mathscr{X}=(\mathscr{X}_t)$: limit of Y^n for $n\to\infty$; its state space is $\mathbb{R}^d$; corresponds to Dirichlet form $\big(\mathcal{E}, D(\mathcal{E})\big)$ defined in (4.1).
- $Y^{\rho,\lambda}=(Y_t^{\rho,\lambda})$: equals the process (Y_t^ρ) but with jumps greater than λ removed; its Dirichlet form is $\mathcal{E}^{\rho,\lambda}$.

4.3 Upper bounds for exit times and the heat kernel

The aim of this section is to establish upper bounds on the heat kernel of the processes $Y^{\rho,\lambda}$ independent of $\rho \geq 1$. These results are applied in order to establish Theorem 4.13 which is the key ingredient needed to show tightness of the family $(Y^n)_{n\in\mathbb{N}}$. Most of the techniques used in this section are borrowed from [CKS87], [BL02b], and [CK03]. The following result, Lemma 4.10, is new and, together with Lemma 4.4, we consider it important for the further development of anisotropic jump processes and Markov chains.

Lemma 4.10 *Assume (A1), (A2) and (A3). Then there exist $c > 0$ and $\Theta_2 \geq 1$ depending on $\kappa_1, \kappa_2, N_0, d, \alpha$ such that for all $f \in L^2(\mathbb{Z}^d, \mu)$, $\rho > 0$, $\lambda > 0$*

$$\mathcal{E}^{\rho,\lambda}_\alpha(f,f) \leq c\mathcal{E}^{\rho,\lambda\Theta_2}(f,f)\,, \qquad \text{and in particular } \mathcal{E}^{\rho}_\alpha(f,f) \leq c\mathcal{E}^{\rho}(f,f)\,.$$

Proof: Let $(z_l^{(x,y)})$ be the chains associated to C by (A3) now scaled on $\rho^{-1}\mathbb{Z}^d$. Note that (A2) and (A3) together imply $|z_{l-1}^{(x,y)} - z_l^{(x,y)}| \leq \Theta_2|x-y|$ with $\Theta_2 = \Theta_2(\kappa_1,\kappa_2,N_0,d,\alpha)$ for any chain in the sense of (A3), any pair (x,y) and any l. For notational convenience we assume the length of all chains to be equal to N_0. Then

$$\begin{aligned}
\mathcal{E}^{\rho,\lambda}_\alpha(f,f) &= \frac{1}{2}\sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d\\|x-y|\leq\lambda}} \big(f(x)-f(y)\big)^2\,|x-y|^{-d-\alpha}\,\rho^{-2d}\\
&\leq N_0 \sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d\\|x-y|\leq\lambda}}\sum_{l=1}^{N_0} \big(f(z_{l-1}^{(x,y)}) - f(z_l^{(x,y)})\big)^2\,|x-y|^{-d-\alpha}\,\rho^{-2d}\\
&\leq N_0(\kappa_2)^{-1} \sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d\\|x-y|\leq\lambda}}\sum_{l=1}^{N_0} \big(f(z_{l-1}^{(x,y)}) - f(z_l^{(x,y)})\big)^2 C^\rho(z_{l-1}^{(x,y)}, z_l^{(x,y)})\rho^{-2d}\\
&\leq N_0^2(\kappa_2)^{-1} \sum_{\substack{x,y\in\rho^{-1}\mathbb{Z}^d\\|x-y|\leq\lambda}} \rho^{-2d} \max_{l=1,\dots,N_0}\Big\{\big(f(z_{l-1}^{(x,y)}) - f(z_l^{(x,y)})\big)^2 C^\rho(z_{l-1}^{(x,y)}, z_l^{(x,y)})\Big\}\\
&\leq (N_0)^3(\kappa_2)^{-1}\mathcal{E}^{\rho,\lambda\Theta_2}(f,f).
\end{aligned}$$

For the last inequality we use the fact that every term of the sum on the left appears at least once in the sum on the right hand side. By the second part of (A3) this happens at most N_0 times. ■

One can use the scaling property (4.12), Lemma 4.10 and the upper bounds on the rotationally symmetric α-stable process on $\mathbb{Z}^d$ (see Proposition 4.2 in [BL02b]) to obtain upper bounds on the heat kernel of (Y^ρ_t):

Lemma 4.11 *Assume (A1), (A2), and (A3). Then there exists a constant $c > 0$ independent of $\rho > 0$ such that*

$$p_{Y^\rho}(t,x,y) \leq ct^{-d/\alpha} \quad \forall\, t > 0, \forall\, x,y \in \rho^{-1}\mathbb{Z}^d\,.$$

In fact, c only depends on d, α and the constants N_0 and κ_2 appearing in (A3).

Applying Lemma 4.10, rough on-diagonal estimates on the heat kernels of the truncated process can be obtained similarly to the corresponding proof in [BL02b]. Then, using Davies' method as in [CKS87], one deduces the following off-diagonal estimates.

Lemma 4.12 *Assume (A1), (A2), and (A3). For any $\lambda \geq \Theta_2$ there exists $c > 0$ such that for all $\rho \geq 1$, $x, y \in \rho^{-1}\mathbb{Z}^d$ and $t \in (0, 1]$*

$$p_{Y^{\rho,\lambda}}(t, x, y) \leq ct^{-d/\alpha} e^{-|x-y|}.$$

These upper bounds imply the following estimates of exit times, cp. [CK03] or [BL02b].

Theorem 4.13 *For any $a > 0$, $b \in (0, 1)$ there exists a constant $\gamma > 0$ depending on $a, b, \kappa_1, \kappa_2, N_0, d$, and α such that for any $R \geq 1$*

$$\mathbb{P}^x(\tau(B^1(x, aR); Y) < \gamma R^\alpha) \leq b \quad \forall\, x \in \mathbb{Z}^d \qquad \textit{and, equivalently,}$$
$$\mathbb{P}^x(\tau(B^1(x, aR); Y^\rho) < \gamma R^\alpha) \leq b \quad \forall\, x \in \rho^{-1}\mathbb{Z}^d, \forall\, \rho \geq 1\,.$$

4.4 Hitting time estimates and the regularity of the heat kernel

In this section we derive an equicontinuity result for the heat kernels of the processes Y^ρ. In our application it is essential that the constants appearing do not depend on the scaling parameter $\rho \geq 1$. Again, our presentation uses results from [BL02b] and [CK03]. Another option would be to adopt methods of [Kom95]. First, observe the following Lévy system identity, cf. [CK03]:

Lemma 4.14 *Let $f\colon \mathbb{R}^+ \times \rho^{-1}\mathbb{Z}^d \times \rho^{-1}\mathbb{Z}^d \to \mathbb{R}^+$ be a bounded measurable function vanishing on the diagonal, i.e. $f(t, x, x) = 0$ for all $x \in \rho^{-1}\mathbb{Z}^d$. Then for all $x \in \rho^{-1}\mathbb{Z}^d$ and predictable stopping times T we have*

$$\mathbb{E}^x\Big[\sum_{s \leq T} f(s, Y^\rho_{s-}, Y^\rho_s)\Big] = \mathbb{E}^x\Big[\int_0^T \Big(\sum_{y \in \rho^{-1}\mathbb{Z}^d} f(s, Y^\rho_s, y) C^\rho(Y^\rho_s, y)\rho^{-d}\Big) ds\Big].$$

Let $W^\rho = (W^\rho_t)_t$ be the space-time process on $\mathbb{R}^+ \times \rho^{-1}\mathbb{Z}^d$ associated to Y^ρ, i.e. $W^\rho_t = (U_t, Y^\rho_t)$ where $U_t = U_0 + t$ is a deterministic process. We call a measurable function $u\colon \mathbb{R}^+ \times \rho^{-1}\mathbb{Z}^d \to \mathbb{R}$ space-time harmonic or caloric on an open set $\Omega \subset \mathbb{R}^+ \times \rho^{-1}\mathbb{Z}^d$ if for all open relative compact sets $\Omega' \subset \Omega$, $(t, x) \in \Omega'$

$$u(t, x) = \mathbb{E}^{(t,x)}\big(u(W^\rho_{\tau(\Omega; W^\rho_s)})\big).$$

Important examples for space-time harmonic functions are given by the heat kernel of Y^ρ; see Lemma 4.5 in [CK03].

Lemma 4.15 *Let $t_0 > 0$, $y \in \rho^{-1}\mathbb{Z}^d$. Then the function $u(t, x) = p_{Y^\rho}(t_0 - t, x, y)$ is space-time harmonic in $[0, t_0) \times \rho^{-1}\mathbb{Z}^d$.*

Next, making use of Theorem 4.13, choose $\tilde{\gamma} = \gamma(1, \frac{1}{2})$, i.e.

$$\mathbb{P}^x(\tau(B^\rho(x, r); Y^\rho) < \tilde{\gamma} r^\alpha) \leq \frac{1}{2}\,. \tag{4.13}$$

Define $Q^\rho(t, x, r) := [t, t + \tilde{\gamma} r^\alpha] \times B^\rho(x, r)$. We have the following estimate on the probability of hitting relatively large sets before exiting $Q^\rho(0, x, r)$:

Lemma 4.16 *Assume (A1), (A2), (A3) and (A4). Set*

$$\omega = \min\{(1-(\tfrac{8}{9})^{\frac{\alpha}{d+\alpha}})^{1/\alpha}, 1-(\tfrac{8}{9})^{\frac{\alpha}{d+\alpha}}\}.$$

Then there exists a constant $c>0$ depending on κ_3, d, and α such that for any $r>\Theta_1\rho^{-1}$, $x\in\rho^{-1}\mathbb{Z}^d$, $(u,y)\in Q^\rho(0,x,r)$ and compact set $A\subset Q^\rho(0,x,r)$ with $\boldsymbol{\lambda}\otimes\mu^\rho(A)\geq \frac{4}{9}\boldsymbol{\lambda}\otimes\mu^\rho\big(Q^\rho(0,x,r)\big)$

$$\mathbb{P}^{(u,y)}\big(\sigma(A;W_t^\rho)<\tau(Q^\rho(0,x,r);W_t^\rho)\big)\geq c.$$

Proof: We prove the assertion of the lemma in the case $(u,y)=(0,x)$ and $A\subset Q^\rho(0,x,r)$ with $\boldsymbol{\lambda}\otimes\mu^\rho(A)\geq\frac{1}{3}\boldsymbol{\lambda}\otimes\mu^\rho\big(Q^\rho(0,x,r)\big)$. For arbitrary (u,y) set $A'=A\cap Q^\rho(u,y,(\frac{8}{9})^{\frac{1}{d+\alpha}}r)$ and note

$$\begin{aligned}\mathbb{P}^{(u,y)}&\big(\sigma(A;W_t^\rho)<\tau(Q^\rho(0,x,r);W_t^\rho)\big)\\ &\geq \mathbb{P}^{(u,y)}\big(\sigma(A';W_t^\rho)<\tau(Q^\rho(u,y,(\tfrac{8}{9})^{\frac{1}{d+\alpha}}r);W_t^\rho)\big)\geq c\end{aligned}$$

because of

$$\boldsymbol{\lambda}\otimes\mu^\rho(A')\geq\tfrac{1}{3}\boldsymbol{\lambda}\otimes\mu^\rho\big(Q^\rho(u,y,(\tfrac{8}{9})^{\frac{1}{d+\alpha}}r)\big)\,.$$

Set $\tau_r=\tau(Q^\rho(0,x,r);W_t^\rho)$, $\sigma_A=\sigma(A;W_t^\rho)$ and $T=\sigma_A\wedge\tau_r$. Without loss of generality we may assume $\mathbb{P}^{(0,x)}(\sigma_A<\tau_r)\leq\frac{1}{4}$. For each $s\in[0,\infty)$ let A_s denote the projection of A on $\{s\}\times\rho^{-1}\mathbb{Z}^d$ and let for $y\in\rho^{-1}\mathbb{Z}^d$

$$N(y)=\Big\{z\in\rho^{-1}\mathbb{Z}^d : C(y,z)<\kappa_3\,|y-z|^{-d-\alpha}\Big\}\,.$$

Choosing $f(s,\xi,\zeta)=\mathbb{1}_{B^\rho(x,r)\times A_s\setminus\{(y,y):y\in\rho^{-1}\mathbb{Z}^d\}}(\xi,\zeta)$ in the Lévy system formula implies

$$\begin{aligned}\mathbb{P}^{(0,x)}\big(\sigma_A<\tau_r\big)&\geq\mathbb{P}^{(0,x)}\big(\sigma_A<\tau_r;Y^\rho_{\sigma_A-}\neq Y^\rho_{\sigma_A}\big)\\
&=\mathbb{E}^{(0,x)}\Big[\sum_{s\leq T}\mathbb{1}_{Q^\rho(0,x,r)\times A_s}(Y^\rho_{s-},Y^\rho_s)\mathbb{1}_{\{Y^\rho_{s-}\neq Y^\rho_s\}}\Big]\\
&=\mathbb{E}^{(0,x)}\Big[\int_0^T\Big(\sum_{z\in A_s}C^\rho(Y^\rho_s,z)\rho^{-d}\Big)\,ds\Big]\\
&\geq\mathbb{E}^{(0,x)}\Big[\int_0^T\Big(\sum_{z\in A_s\setminus N(Y^\rho_s)}C^\rho(Y^\rho_s,z)\rho^{-d}\Big)\,ds\Big]\\
&\geq\kappa_3\mathbb{E}^{(0,x)}\Big[\int_0^T\Big(\sum_{z\in A_s\setminus N(Y^\rho_s)}|Y^\rho_s-z|^{-d-\alpha}\,\rho^{-d}\Big)\,ds\Big]\\
&\geq 2^{-d-\alpha}\kappa_3\mathbb{E}^{(0,x)}\Big[\int_0^T\Big(\sum_{z\in A_s\setminus N(Y^\rho_s)}r^{-d-\alpha}\rho^{-d}\Big)\,ds\Big]\\
&\geq 2^{-d-\alpha}\kappa_3 r^{-d-\alpha}\mathbb{E}^{(0,x)}\Big[\int_0^T\mu^\rho(A_s\setminus N(Y^\rho_s))\,ds\Big]\\
&\geq 2^{-d-\alpha}\kappa_3 r^{-d-\alpha}\mathbb{E}^{(0,x)}\Big[\int_0^{\frac{5}{6}\tilde{\gamma}r^\alpha}\mu^\rho(A_s\setminus N(Y^\rho_s))\,ds\,;\,\sigma_A\wedge\tau_r\geq\tfrac{5}{6}\tilde{\gamma}r^\alpha\Big]\end{aligned}$$

$$\geq \frac{1}{36} \cdot 2^{-d-\alpha} \kappa_3 r^{-d-\alpha} (\boldsymbol{\lambda} \otimes \mu)(Q^\rho(0,x,r)) \mathbb{P}^{(0,x)}\big(\sigma_A \wedge \tau_r \geq \tfrac{5}{6}\tilde{\gamma} r^\alpha\big)$$
$$\geq c(\kappa_3, d, \alpha) \mathbb{P}^{(0,x)}\big(\sigma_A \wedge \tau_r \geq \tfrac{5}{6}\tilde{\gamma} r^\alpha\big).$$

Here we have used $(\boldsymbol{\lambda} \otimes \mu)(Q^\rho(0,x,r)) \asymp r^{d+\alpha}$. For the second last step note that for every path with $\sigma_A \wedge \tau_r \geq \frac{5}{6}\tilde{\gamma} r^\alpha$

$$\frac{1}{3}\boldsymbol{\lambda} \otimes \mu^\rho(Q^\rho(0,x,r)) \leq \boldsymbol{\lambda} \otimes \mu^\rho(A)$$
$$\leq \int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho(A_s \setminus N(Y_s^\rho)) ds + \int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho(A_s \cap N(Y_s^\rho)) ds + \frac{1}{6}\boldsymbol{\lambda} \otimes \mu^\rho(Q^\rho(0,x,r)).$$

Now, we obtain by (A4) and because of $r \geq \Theta_1 \rho^{-1}$

$$\int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho(A_s \cap N(Y_s^\rho)) ds \leq \int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho(B^\rho(Y_s^\rho, 2r) \cap N(Y_s^\rho)) ds$$
$$\leq \frac{1}{6 \cdot 2^d} \int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho\big(B^\rho(Y_s^\rho, 2r)\big) ds = \frac{5}{36}\tilde{\gamma} r^\alpha \mu^\rho\big(B^\rho(x,r)\big) = \frac{5}{36}\boldsymbol{\lambda} \otimes \mu^\rho(Q^\rho(0,x,r)).$$

Hence

$$\int_0^{\frac{5}{6}\tilde{\gamma} r^\alpha} \mu^\rho(A_s \setminus N(Y_s^\rho)) ds \geq \frac{1}{36}\boldsymbol{\lambda} \otimes \mu^\rho(Q^\rho(0,x,r)).$$

By our choice of $\tilde{\gamma}$ we obtain

$$\mathbb{P}^{(0,x)}(\tau_r < \tfrac{5}{6}\tilde{\gamma} r^\alpha) \leq \mathbb{P}^x\big(\tau(B^\rho(x,r); Y^\rho) \leq \tilde{\gamma} r^\alpha\big) \leq \frac{1}{2}.$$

Finally we estimate

$$\mathbb{P}^{(0,x)}(\sigma_A \wedge \tau_r \geq \tfrac{5}{6}\tilde{\gamma} r^\alpha) = 1 - \mathbb{P}^{(0,x)}(\sigma_A \wedge \tau_r < \tfrac{5}{6}\tilde{\gamma} r^\alpha)$$
$$\geq 1 - \mathbb{P}^{(0,x)}(\sigma_A < \tau_r) - \mathbb{P}^{(0,x)}(\tau_r < \tfrac{5}{6}\tilde{\gamma} r^\alpha) \geq \frac{1}{4}.$$

■

We also need the following upper bound on the probability of exiting a ball by large jumps.

Lemma 4.17 *Assume (A1), (A2) and (A3). Let $\Theta_1 \geq 1$ be the constant of Lemma 4.4. Then there exists a constant $c(\kappa_1, \kappa_2, N_0, d, \alpha) > 0$ such that for all $\rho \geq 1$, $s > 2r > \Theta_1 \rho^{-1}$, $(t,x) \in [0,\infty) \times \rho^{-1}\mathbb{Z}^d$ and $(u,z) \in Q^\rho(t,x,r)$*

$$\mathbb{P}^{(u,z)}\big(W^\rho_{\tau(Q^\rho(t,x,r);W^\rho)} \notin Q^\rho(t,x,s)\big) \leq c\frac{r^\alpha}{s^\alpha}.$$

Proof: Due to basic observations it is sufficient to prove the assertion for $(u,y) = (t,x)$. Let $\tau = \tau(B^\rho(x,r); Y^\rho)$. Observe that $|\xi - \zeta| \geq \frac{1}{2}|x - \zeta|$ for $\xi \in B^\rho(x,r)$ and $\zeta \in B^\rho(x,s)^c$. Since the space-time process moves continuously in time, it can only exit

$Q^\rho(t,x,s)$ and $Q^\rho(t,x,r)$ simultaneously by jumping in space. Using this fact together with the Lévy system identity for (Y_t^ρ) and (A2), one obtains

$$\begin{aligned}
&\mathbb{P}^{(t,x)}\big(W^\rho_{\tau(Q^\rho(t,x,r);W^\rho)} \notin Q^\rho(t,x,s)\big) = \mathbb{P}^x\big(Y^\rho_\tau \notin B^\rho(x,s); \tau \le \tilde\gamma r^\alpha\big)\\
&\le \mathbb{P}^x\big(Y^\rho_\tau \notin B^\rho(x,s)\big) = \mathbb{E}^x\Big[\sum_{t\le\tau} \mathbb{1}_{\{Y^\rho_{t-}\in B^\rho(x,r), Y^\rho_t \notin B^\rho(x,s)\}}\Big]\\
&= \mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{|y-x|\ge s} C^\rho(Y^\rho_t,y)\rho^{-d}\Big)dt\Big] \le \mathbb{E}^x\Big[\int_0^\tau \sum_{|y-x|\ge s} \kappa_1 \left|Y^\rho_t - y\right|^{-d-\alpha}\rho^{-d}dt\Big]\\
&\le \kappa_1 2^{d+\alpha}\mathbb{E}^x\Big[\tau \sum_{|y-x|\ge s} |x-y|^{-d-\alpha}\rho^{-d}\Big] \le c_1 s^{-\alpha}\mathbb{E}^x(\tau).
\end{aligned}$$

Therefore it remains to estimate $\mathbb{E}^x(\tau)$. Let κ_3 and γ be the constants of Lemma 4.4. For each $\xi \in \rho^{-1}\mathbb{Z}^d$ define $A(\xi) = \big\{\zeta \in \rho^{-1}\mathbb{Z}^d : C^\rho(\xi,\zeta) \ge \kappa_3 |\xi-\zeta|^{-d-\alpha}\big\}$. We get by applying the Lévy system identity in the above fashion for $r \ge \Theta_1\rho^{-1}$

$$\begin{aligned}
1 &= \mathbb{P}^x\big(Y^\rho_\tau \notin B^\rho(x,r)\big) = \mathbb{E}^x\Big[\sum_{t\le\tau} \mathbb{1}_{\{Y^\rho_{t-}\in B(x,r), Y^\rho_t\notin B(x,r)\}}\Big]\\
&= \mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{|y-x|\ge r} C^\rho(Y^\rho_t,y)\rho^{-d}\Big)dt\Big] \ge \mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{|y-Y^\rho_t|\ge 2r} C^\rho(Y^\rho_t,y)\rho^{-d}\Big)dt\Big]\\
&\ge \mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{\substack{|y-Y^\rho_t|\ge 2r\\ y\in A(Y^\rho_t)}} C^\rho(Y^\rho_t,y)\rho^{-d}\Big)dt\Big]\\
&\ge \kappa_3\mathbb{E}^x\Big[\int_0^\tau \Big(\sum_{\substack{2r\le|y-Y^\rho_t|<6r\gamma^{-1/d}\\ y\in A(Y^\rho_t)}} |Y^\rho_t - y|^{-d-\alpha}\rho^{-d}\Big)dt\Big]\\
&\ge \kappa_3 7^{-d-\alpha}\gamma^{1+d/\alpha}\mathbb{E}^x\Big[\int_0^\tau \rho^{-d}\mu\big(A(Y^\rho_t)\cap B^\rho(Y^\rho_t,6r\gamma^{-1/d})\setminus B^\rho(Y^\rho_t,2r)\big)dt\Big]\\
&\ge c_1(\kappa_3,d,\alpha)\mathbb{E}^x\Big[\int_0^\tau \rho^{-d}\mu\big(B^\rho(Y^\rho_t,2r)\big)dt\Big] \ge c_2(\kappa_3,d,\alpha)r^{-\alpha}\mathbb{E}^x(\tau).
\end{aligned}$$

Here, the second last inequality is due to Lemma 4.4 since there are at least $2\mu\big(B^\rho(Y^\rho_t,2r)\big)$ elements of $A(Y^\rho_t)$ in $B^\rho(Y^\rho_t,6r\gamma^{-1/d})$. ∎

Proposition 4.18 *Assume (A1)-(A4). Then there exist constants $c>0$ and $\beta\in(0,1)$ depending only on the constants appearing in (A1)-(A4) such that for all $R > \Theta_1\rho^{-1}$, $q\colon [0,\tilde\gamma 16R]\times\rho^{-1}\mathbb{Z}^d \to \mathbb{R}$ bounded and space-time harmonic in $Q^\rho(0,x_0,16R)$ the following a-priori continuity estimate holds:*

$$|q(s,x)-q(t,y)| \le c\,\|q\|_\infty R^{-\beta}\big(\,|s-t|^{1/\alpha} + |x-y|\,\big)^\beta$$

for all $(s,x),(t,y)\in Q^\rho(0,x_0,R)$ with $|x-y|\ge\Theta_1\rho^{-1}$.

Moreover, if $|x-y|\le\Theta_1\rho^{-1}$ we have

$$|q(s,x)-q(t,y)| \le c\,\|q\|_\infty R^{-\beta}\big(\,|s-t|^{1/\alpha} + \Theta_1\rho^{-1}\big)^\beta.$$

Proof: The proof given here is similar to the ones in [BK08], [CK03], [BK05b], and [HK09]. We may assume $\|q\|_\infty = \frac{1}{2}$. Fix $(t,x) \in Q^\rho(0,x_0,R)$. Let ω be the constant of Lemma 4.16 and $\xi \in (0,\frac{\omega}{2})$, $\eta > 0$, $Q_k = Q^\rho(t,x,\xi^k R)$, $\tau_{k+1} = \tau(Q^\rho(t,x,\frac{1}{\omega}\xi^{k+1}R);W^\rho)$ and define for $k \in \mathbb{N}$

$$m_k = \inf_{z\in Q_k} q(z), \qquad\qquad M_k = \sup_{z\in Q_k} q(z).$$

We show that it is possible to choose constants ξ,ζ independent of $R>0$, $x,x_0 \in Q^\rho(0,x_0,R)$ and q such that

$$M_k - m_k \leq \zeta^k \tag{4.14}$$

for all $k \geq 0$ with $\xi^k R \geq \Theta_1\rho^{-1}$ where Θ_1 the constant in (A4). The restriction that x and y cannot be arbitrarily close is natural since hitting time estimates in the sense of Lemma 4.16 may not hold for small r. Just consider the conductivities $C(x,y) = |x-y|^{-d-\alpha}\mathbb{1}_{\{|x-y|>R_0\}}$. Then the process W^1 has only jumps with length bigger than R_0 and therefore can't hit sets $A \subset B(x_0,r)$ for $r < R_0$ starting in x_0 unless $x_0 \in A$.

Trivially, (4.14) holds for $k=0$. Now assume that this equation holds already for all $i \leq k$ while still $\xi^{k+1}R \geq \Theta_1\rho^{-1}$. We set

$$A_k = \left\{ z \in Q^\rho(t,x,\tfrac{1}{\omega}\xi^{k+1}R) : q(z) \leq \frac{M_k+m_k}{2} \right\}.$$

Without loss of generality we might assume $\boldsymbol{\lambda}\otimes\mu(A_k)/\boldsymbol{\lambda}\otimes\mu(Q^\rho(t,x,\frac{1}{\omega}\xi^{k+1}R)) \geq \frac{1}{2}$. Else we just look at $\frac{1}{2}-q$ instead of q. We choose a compact set $A_k' \subset A_k$ with $\boldsymbol{\lambda}\otimes\mu(A_k')/\boldsymbol{\lambda}\otimes\mu(Q^\rho(t,x,\frac{1}{\omega}\xi^{k+1}R)) \geq \frac{1}{3}$ and define $T_k = T(A_k';W^\rho)$. Observe that u is harmonic in $Q_k \subset Q^\rho(t,x,R) \subset Q^\rho(0,x_0,16R)$. Therefore we get for arbitrary $z_1,z_2 \in Q_{k+1}$

$$\begin{aligned} q(z_1)-q(z_2) &= \mathbb{E}^{z_1}\big[q(W^\rho_{T_k\wedge\tau_{k+1}})\big] - q(z_2) \\ &= \mathbb{E}^{z_1}\big[q(W^\rho_{T_k}) - q(z_2); T_k < \tau_{k+1}\big] \\ &\quad + \mathbb{E}^{z_1}\big[q(W_{\tau_{k+1}}) - q(z_2); T_k > \tau_{k+1} \text{ and } W^\rho_{\tau_{k+1}} \in Q_k\big] \\ &\quad + \sum_{i=1}^{k} \mathbb{E}^{z_1}\big[q(W_{\tau_{k+1}}) - q(z_2); T_k > \tau_{k+1} \text{ and } W^\rho_{\tau_{k+1}} \in Q_{k-i}\setminus Q_{k-i+1}\big] \\ &\quad + \mathbb{E}^{z_1}\big[q(W_{\tau_{k+1}}) - q(z_2); T_k > \tau_{k+1} \text{ and } W^\rho_{\tau_{k+1}} \notin Q_0\big]. \end{aligned}$$

Here, the first term can be estimated by $\frac{1}{2}(M_k-m_k)\mathbb{P}^{z_1}(T_k > \tau_{k+1})$ while the second term is bounded from above by $(M_k-m_k)\mathbb{P}^{z_1}(T_k > \tau_{k+1}) = (M_k-m_k)(1-\mathbb{P}^{z_1}(T_k < \tau_{k+1}))$. Moreover Lemma 4.16 implies the existence of $c_1>0$ such that $\mathbb{P}^{z_1}(T_k < \tau_{k+1}) \geq c_1$. For the remaining terms note that by Lemma 4.17 there exists $c_2>0$ with

$$\mathbb{P}^{z_1}(W^\rho_{\tau_{k+1}} \notin Q_i) \leq c_2\xi^{(k+1-i)\alpha}, \qquad i = 1,\dots k+1\,.$$

Summing up the terms above we obtain for $\xi^\alpha \leq \zeta/2$

$$\begin{aligned} &q(z_1)-q(z_2) \\ &\quad\leq (M_k-m_k)\big(1-\frac{1}{2}\mathbb{P}^{z_1}(T_k<\tau_{k+1})\big) + \sum_{i=1}^{k+1}(M_{k-i}-m_{k-i})\mathbb{P}^{z_1}(Z^\rho_{\tau_{k+1}} \notin Q_{k-i+1}) \end{aligned}$$

$$\le (1 - \frac{1}{4}c_1)\zeta^k - \frac{1}{4}c_1\zeta^k + c_2 \sum_{i=1}^{k+1} \zeta^{k-i}\xi^{\alpha i}$$
$$\le (1 - \frac{1}{4}c_1)\zeta^k - \frac{1}{4}c_1\zeta^k + c_2\zeta^k \sum_{i=1}^{\infty} (\frac{\xi^\alpha}{\zeta})^i = (1 - \frac{1}{4}c_1)\zeta^k - \frac{1}{4}c_1\zeta^k + 2c_2\xi\zeta^k.$$

Estimate (4.14) follows from the equation above by taking $\zeta = 1 - \frac{c_1}{4}$ and $\xi = \frac{1}{2\omega} \wedge \big(\frac{\zeta}{2}\big)^{1/\alpha} \wedge \frac{c_1}{8c_2}$.

To derive Hölder continuity let $z_i = (s_i, x_i) \in Q^\rho(0, x_0, R)$, $i = 1, 2$ with $z_1 \neq z_2$ and $s_1 \le s_2$. Assume $|x_1 - x_2| \ge \Theta_1\rho^{-1}$ and take k maximal such that $z_2 \in B^\rho(z_1, R\xi^k)$. Then

$$|s_1 - s_2|^{1/\alpha} + |x_1 - x_2| \le (\tilde\gamma^{1/\alpha} + 1)R\xi^k, \qquad \xi^k R \ge \Theta_1\rho^{-1}, \qquad |q(z_1) - q(z_2)| \le \zeta^k.$$

Thus, by optimality, k is the smallest integer such that

$$k \ge (\log\xi)^{-1}\big(\log(|s_1 - s_2|^{1/\alpha} + |x_1 - x_2|) - \log(\tilde\gamma^{1/\alpha} + 1)R\big) - 1,$$

and we get

$$|q(z_1) - q(z_2)| \le \zeta^{-1}\big((\tilde\gamma^{1/\alpha} + 1)R\big)^{-\log\zeta/\log\xi}(|s_1 - s_2|^{1/\alpha} + |x_1 - x_2|)^{\log\zeta/\log\xi},$$

i.e., the proposition holds with $\beta = \log\zeta/\log\xi \in (0,1)$ and $c = \zeta^{-1}(\tilde\gamma^{1/\alpha} + 1)^{-\beta}$. If on the other hand $|x_1 - x_2| \le \Theta_1(\tilde\gamma^{1/\alpha} + 1)\rho^{-1}$ we take k maximal with $R\xi^k \ge \Theta_1\rho^{-1}$ and $z_2 \in B^\rho(z_1, R\xi^k)$. Then in particular $|s_1 - s_2|^{1/\alpha} \le \tilde\gamma^{1/\alpha}R\xi^k$, and we get for k the inequalities

$$k \le (\log\xi)^{-1}\big(\log(\Theta_1\rho^{-1}) - \log R\big),$$
$$k \le (\log\xi)^{-1}\big(\log(|s_1 - s_2|^{1/\alpha}) - \log(\tilde\gamma^{1/\alpha}R)\big)$$

Combining this, we get with β as above

$$|q(z_1) - q(z_2)| \le \zeta^{-1}(\tilde\gamma^{-\beta/\alpha} + 1)R^{-\beta}\big(\,|s_1 - s_2|^{\beta/\alpha} + (\Theta_1\rho^{-1})^\beta\big)$$
$$\le 2\zeta^{-1}(\tilde\gamma^{-\beta/\alpha} + 1)R^{-\beta}\big(\,|s_1 - s_2|^{1/\alpha} + \Theta_1\rho^{-1}\big)^\beta$$

■

In particular, this implies regularity of the heat kernels

Theorem 4.19 *There exist constants $c > 0$ and $\beta \in (0,1)$ such that for all $t_0 \in (0,\infty)$, $x_1, x_2, y \in \rho^{-1}\mathbb{Z}^d$ and $s_1, s_2 \in [t_0, \infty)$ For $y \in \rho^{-1}\mathbb{Z}^d$ we have*

$$|p_{Y^\rho}(s_1, x_1, y) - p_{Y^\rho}(s_2, x_2, y)|$$
$$\le ct_0^{-(d+\beta)/\alpha}\big(\,|s_1 - s_2|^{1/\alpha} + |x_1 - x_2| \vee (\Theta_1\rho^{-1})\big)^\beta.$$

More general, for arbitrary $y_1, y_2 \in \rho^{-1}\mathbb{Z}^d$

$$|p_{Y^\rho}(s_1, x_1, y_1) - p_{Y^\rho}(s_2, x_2, y_2)|$$
$$\le ct_0^{-(d+\beta)/\alpha}\big(\,|s_1 - s_2|^{1/\alpha} + |x_1 - x_2| \vee (\Theta_1\rho^{-1}) + |y_1 - y_2| \vee (\Theta_1\rho^{-1})\big)^\beta.$$

Proof: Fix $t_0 > 0$. For an arbitrary $T_0 \geq t_0$ the function $q(t,x) = p(T_0 - t, x, y)$ is space-time harmonic on $[0, T_0/2) \times \rho^{-1}\mathbb{Z}^d$ by Lemma 4.15 as well as bounded by $c_1 t_0^{-d/\alpha}$ by Lemma 4.11. Now take $R = \big(T_0/(32\tilde{\gamma})\big)^{1/\alpha}$ and $s_1, s_2 \in [0, \tilde{\gamma}R^\alpha)$. Assume first $\Theta_1\rho^{-1} \leq |x_1 - x_2|$. If $|x_1 - x_2| \leq R$ we get by Proposition 4.18

$$\begin{aligned} |p_{Y^\rho}(T-s_1,x_1,y)-p_{Y^\rho}(T-s_2,x_2,y)| &\leq c_2\big(\tfrac{T_0}{32\tilde{\gamma}}\big)^{-\beta/\alpha} t_0^{-d/\alpha}\big(\,|s_1-s_2|^{1/\alpha}+|x_1-x_2|\,\big)^\beta \\ &\leq c_3 t_0^{-(d+\beta)/\alpha}\big(\,|s_1-s_2|^{1/\alpha}+|x_1-x_2|\,\big)^\beta. \end{aligned}$$

In the other case $|x_1 - x_2| > R$ we have $(|s_1-s_2|^{1/\alpha} + |x_1-x_2|)^\beta \geq c_4 t_0^{\beta/\alpha}$ and hence

$$\begin{aligned} |p_{Y^\rho}(T-s_1,x_1,y)-p_{Y^\rho}(T-s_2,x_2,y)| &\leq 2c_1 t_0^{-d/\alpha} \leq 2c_1 t_0^{-(d+\beta)/\alpha} t_0^{\beta/\alpha} \\ &\leq c_5 t_0^{-(d+\beta)/\alpha}\big(\,|s_1-s_2|^{1/\alpha}+|x_1-x_2|\,\big)^\beta. \end{aligned}$$

In the same fashion we can deal with the case $\Theta_1\rho^{-1} \geq |x_1 - x_2|$. The other a-priori estimate asserted in the theorem now follows by symmetry of the heat kernels. ■

4.5 The central limit theorem

The aim of this section is to provide a proof of Theorem 4.1 and Theorem 4.2. Although the limit process $\mathscr{X}$ is a jump process the idea of the proof is very similar to the one in [SZ97] and [BK08]. For $n \in \mathbb{N}$ let $Y^n = (Y^n_t)_t$ be the continuous-time processes defined in the introduction and explained in section 4.2.4. Denote the corresponding semigroup by $(P^{(n)}_t)_t$ and its kernel by $p^{(n)}(t,x,y)$. Recall that the Dirichlet form corresponding to Y^n is given by

$$\mathcal{E}^{(n)}(f,f) = \sum_{x,y\in n^{-1}\mathbb{Z}^d} \big(f(y)-f(x)\big)^2 C^n(x,y) n^{-2d}.$$

We denote the restriction of functions on $\mathbb{R}^d$ to $n^{-1}\mathbb{Z}^d$ by R_n. We also need to extend functions on the grid to continuous functions on $\mathbb{R}^d$ in a (for our purpose) reasonable way: For $x \in n^{-1}\mathbb{Z}^d$ set $\mathfrak{Q}_n(x) := \prod[x_i, x_i + 1/n)$ and $\mathfrak{Q}_n = \bigcup_{x\in n^{-1}\mathbb{Z}^d} \{\mathfrak{Q}_n(x)\}$. The sets of $\mathfrak{Q}_n$ form a partition of $\mathbb{R}^d$. Recall that, for a point $x \in \mathbb{R}^d$ we denote by $[x]_n$ the element of $n^{-1}\mathbb{Z}^d$ satisfying $([x]_n)_i = n^{-1}\lfloor nx_i \rfloor$ for all $i = 1, \dots, d$. For $f\colon n^{-1}\mathbb{Z}^d \to \mathbb{R}$ we denote by $E_n f$ a Lipschitz-continuous function $E_n f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying:

a) $(E_n f)(x) = f(x)$ for any $x \in n^{-1}\mathbb{Z}^d$,

b) $\underline{f}(x) \leq E_n f(x) \leq \overline{f}(x)$ $\forall x \in \mathbb{R}^d$, where $\underline{f}(x) = \min_{\overline{\mathfrak{Q}_n(x)}\cap n^{-1}\mathbb{Z}^d} f$; $\overline{f}(x) = \max_{\overline{\mathfrak{Q}_n(x)}\cap n^{-1}\mathbb{Z}^d} f$,

c) $\big|\nabla E_n f(x)\big| \leq cn \max\big\{|f(\xi) - f(\eta)| : \xi, \eta \in \overline{\mathfrak{Q}_n(x)} \cap n^{-1}\mathbb{Z}^d\big\}$ $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d$.

The precise choice of the function $E_n f$ is not important for our approach as long as E_n is a linear operator.

Let us emphasize that in the following result we adopt the notion $\mathbb{P}^x$ for the probability of a Markov process starting in x. Any stochastic process with càdlàg paths corresponds to a probability measure on $D([0,\infty),\mathbb{R}^d)$. For a Markov process $X=(X_t)$ starting in x we refer to this probability measure as "the law of X under $\mathbb{P}^x$".

The following theorem is an extension of Theorem 4.1. Theorem 4.20 is what we really need in the sequel.

Theorem 4.20 *Let $(C^n)_n$ be a sequence of conductivity functions satisfying (A1) through (A4) and $(x_n)_n$ a sequence of points $x_n \in n^{-1}\mathbb{Z}^d$ with $x_n \to x \in \mathbb{R}^d$ for $n\to\infty$. Then each subsequence (n') of (n) has a subsequence (n'') with the following properties:*

1. *For any $f\in C_c(\mathbb{R}^d)$ the continuous functions $\big(E_{n''}P_t^{(n'')}R_{n''}f\big)$ converge uniformly on compact sets for $n''\to\infty$. The limit defines a family of linear operators $(P_t)_{t>0}$ which extends to the semigroup on $C(\mathbb{R}^d)$ of a strong Markov process $\mathscr{X}$.*
2. *For any $t_0>0$ the laws of $(Y_t^{n''})_{t\in[0,t_0]}$ under $\mathbb{P}^{x_{n''}}$ converge weakly to the law of $(\mathscr{X}_t)_{t\in[0,t_0]}$.*

Once these assertions are proved it remains to show that $\mathscr{X}$ does not depend on the choice of (n'). The proof of Theorem 4.20 makes use of a sufficient condition for tightness provided in [Ald78].

Proof: [of Theorem 4.20 and Theorem 4.1] Let (n') be a subsequence of (n). Fix countable dense subsets (s_i) of $[0,\infty)$ and (f_j) of $C_c(\mathbb{R}^d)$.

By $Q_t^{(n)} := E_nP_t^{(n)}R_n$ we define a positivity-preserving contraction semigroup $(Q_t^{(n)})_t$ on the Banach space $C(\mathbb{R}^d)$. Now, Theorem 4.19 yields that for all i, j the family of functions $(Q_{s_i}^{(n)}f_j)_{n\in\mathbb{N}}$ is equicontinuous. In fact, we have for $x,y\in n^{-1}\mathbb{Z}^d$ with $|x-y|\ge n^{-1}\Theta_1$

$$\begin{aligned}\big|P_{s_i}^{(n)}R_n(f_j)(x)-P_{s_i}^{(n)}R_n(f_j)(y)\big| &\le \sum_{z\in n^{-1}\mathbb{Z}^d}\big|p^{(n)}(s_i,x,z)-p^{(n)}(s_i,y,z)\big|\big|f_j(z)\big|n^{-d}\\ &\le c s_i^{-(d+\beta)/\alpha}\boldsymbol{\lambda}(\operatorname{supp}(f_j))\,\|f_j\|_\infty\,|x-y|^\beta\end{aligned}$$

where $c>0$, $\Theta_1>0$ and $\beta\in(0,1)$ are independent of n and stem from Proposition 4.18. The construction of the extension operators E_n implies for all $x,y\in\mathbb{R}^d$ with $|x-y|\ge n^{-1}\Theta_1$

$$\big|Q_{s_i}^{(n)}f_j(x)-Q_{s_i}^{(n)}f_j(y)\big|\le c s_i^{-(d+\beta)/\alpha}\boldsymbol{\lambda}(\operatorname{supp}(f_j))\,\|f_j\|_\infty\,|x-y|^\beta \tag{4.15}$$

In a similar fashion one establishes for $|x-y|\le n^{-1}\Theta_1$

$$\big|Q_{s_i}^{(n)}f_j(x)-Q_{s_i}^{(n)}f_j(y)\big|\le c\Theta_1 s_i^{-(d+\beta)/\alpha}\boldsymbol{\lambda}(\operatorname{supp}(f_j))\,\|f_j\|_\infty\,n^{-\beta}. \tag{4.16}$$

Furthermore $(Q_{s_i}^{(n)}f_j)$ is equibounded. The Theorem of Arzela-Ascoli and the passage to a diagonal sequence give us therefore a subsequence (n'') of (n') such that for all i, j the sequence $Q_{s_i}^{(n'')}f_j$ converges uniformly on compact sets for $n''\to\infty$. Denote this limit function by $P_{s_i}f_j$. We use an $\varepsilon/3$-argument to extend it for all positive times: Let $t\in(0,\infty)$ and take a subsequence (i') of (i) such that $s_{i'}\to t$ for $i'\to\infty$ and $s_{i'}\ge t/2$. Then

$$\begin{aligned}\big|Q_t^{(n'')}f_j(x)-Q_t^{(m'')}f_j(x)\big| \le \big|Q_t^{(n'')}f_j(x)-Q_{s_{i'}}^{(n'')}f_j(x)\big|+\big|Q_{s_{i'}}^{(n'')}f_j(x)-Q_{s_{i'}}^{(m'')}f_j(x)\big|\\ +\big|Q_t^{(m'')}f_j(x)-Q_{s_{i'}}^{(m'')}f_j(x)\big|.\end{aligned}$$

The second term on the right hand side converges uniformly on compact sets to 0 for $n'', m'' \to \infty$. The other two terms can be handled again by Theorem 4.19:

$$\begin{aligned}\big|P_t^{(n)} R_n(f_j)(x) - P_{s_{i'}}^{(n)} R_n(f_j)(x)\big| &\le \sum_{z \in n^{-1}\mathbb{Z}^d} \big|p^{(n)}(t,x,z) - p^{(n)}(s_i,x,z)\big|\big|f_j(z)\big| n^{-d} \\ &\le c_1 s_i^{-(d+\beta)/\alpha} \boldsymbol{\lambda}(\operatorname{supp}(f_j)) \, \|f_j\|_\infty \left(|s_i - t|^{1/\alpha} + \Theta_1 n^{-1}\right)^\beta .\end{aligned}$$

The right hand side clearly converges to 0 for $n \to \infty$. Hence the limit $P_t f_j$ exists uniformly on compact sets for all $t \in [0,\infty)$. Finally by $\big\|Q_t^{(n'')} f_j\big\| \le \|f_j\|$ and because (f_j) is dense in $C(\mathbb{R}^d)$ in the topology of uniform convergence on compact sets we have established the desired convergence result for all $f \in C(\mathbb{R}^d)$.

It follows from the corresponding properties of the $Q_t^{(n)}$ that P_t is a positivity-preserving contraction semigroup on $C(\mathbb{R}^d)$ which is hence associated to a symmetric strong Markov process $\mathscr{X}$ on $\mathbb{R}^d$.

Fix $t_0 > 0$ and $x \in \mathbb{R}^d$. We apply the tightness criterion of [Ald78]. Take an arbitrary sequence of stopping times $\tau_n \in [0, t_0]$, a sequence (δ_n) of reals converging to 0 and $a > 0$. By Theorem 4.13 for each choice $b \in (0,1)$ there exists a constant $\gamma(a,b)$ with

$$\mathbb{P}^{x_n}\left(\tau\big(B^n(x_n,a); Y^{(n)}\big) \le \gamma(a,b)\right) \le b \tag{4.17}$$

for all $n \in \mathbb{N}$. Therefore, for all n large enough such that $\delta_n \le \gamma(a,b)$,

$$\begin{aligned}\mathbb{P}^{x_n}\left(\big|Y_{\tau_n+\delta_n}^{(n)} - Y_{\tau_n}^{(n)}\big| > a\right) &= \mathbb{P}^{x_n}\left(\big|Y_{\delta_n}^{(n)} - Y_0^{(n)}\big| > a\right) \\ &\le \mathbb{P}^{x_n}\left(\tau\big(B^n(x_n,a); Y^{(n)}\big) \le \delta_n\right) \\ &\le \mathbb{P}^{x_n}\left(\tau\big(B^n(x_n,a); Y^{(n)}\big) \le \gamma(a,b)\right) \le b\end{aligned}$$

where the first equality follows by the strong Markov property. This yields condition (A) in [Ald78]. Moreover, $x_n \to x$ implies the tightness of the starting distributions while (4.17) implies the tightness of $\max_{t\in[0,t_0]} \big|Y_t^{(n)} - Y_{t-}^{(n)}\big|$, both under $\mathbb{P}^{x_n}$. The tightness of the laws of $(Y_t^{(n)})$ under $\mathbb{P}^{x_n}$ follows.

Finally we prove the asserted weak convergence for $n'' \to \infty$ by showing that the finite dimensional distributions of the limit probability $\mathbb{Q}$ on $D([0,t_0],\mathbb{R}^d)$ of a weakly convergent subsequence (n''') are independent of the actual subsequence. For $g \in C_c(\mathbb{R}^d)$, $t \in [0, t_0]$

$$\begin{aligned}\mathbb{E}^x g(\mathscr{X}_t) = \int g(\omega_t) d\mathbb{Q}(\omega) &= \lim_{n'''\to\infty} \mathbb{E}^{x_{n'''}} (R_{n'''} g)\big(Y_t^{(n''')}\big) \\ &= \lim_{n'''\to\infty} P_t^{(n''')} (R_{n'''} g)(x_{n'''}) = P_t g(x) \,,\end{aligned}$$

where the last equality follows from the equicontinuity of the family $P_t^{(n''')} g$. Therefore the one-dimensional distributions are independent of (n'''). More generally let $0 \le s_1 < \ldots < s_k \le t_0$ and $g_1, \ldots, g_k \in C_c(\mathbb{R}^d)$. Then by the time-homogeneity of our Markov chains

$$\begin{aligned}\int g_1(\omega_{s_1}) \cdot \ldots \cdot g_k(\omega_{s_k}) \, d\mathbb{Q}(\omega) &= \lim_{n'''\to\infty} \mathbb{E}^{x_{n'''}} \big(g_1(X_{s_1}^{(n''')}) \cdot \ldots \cdot g_k(X_{s_k})^{(n''')})\big) \\ &= \lim_{n'''\to\infty} \big(P_{s_1}^{(n''')} \big(g_1 P_{s_2-s_1}^{(n''')} (\ldots P_{s_k-s_{k-1}}^{(n''')} g_k)\big)\big)(x_{n'''}) \\ &= \big(P_{s_1}\big(g_1 P_{s_2-s_1}(\ldots P_{s_k-s_{k-1}} g_k)\big)\big)(x) \,.\end{aligned}$$

Here the last equality is again due to the equicontinuity. Hence the k-dimensional distributions of $\mathbb{Q}$ are independent of the choice of the subsequence (n''') and are determined by the semigroup (P_t). Therefore we have weak convergence along (n''). In particular, the stochastic process corresponding to $\mathbb{Q}$ has the same finite-dimensional distributions as $\mathscr{X}$ starting in x. ∎

We proceed to the proof of Theorem 4.2.

Proof of Theorem 4.2: Let (n') be any subsequence of (n). Let $\mathscr{X}$ be a strong Markov process - possibly depending on the choice of (n') - and (n'') be a subsequence of (n') such that the assertions of Theorem 4.20 hold true. We aim to show that $\mathscr{X}$ does not depend on the choice of (n'). It suffices to show that the limiting process $\mathscr{X}$ corresponds to the Dirichlet form (4.1). This is the case if

$$\mathcal{E}(U_\lambda f, g) = (f, g) - \lambda(U_\lambda f, g) \tag{4.18}$$

for any $f, g \in C_c^\infty(\mathbb{R}^d)$ where $U_\lambda f(x) = \int_0^\infty e^{-\lambda t}(P_t f)(x)dt$, $\lambda > 0$. Note that, at this stage, U_λ does depend on the choice of (n'). Equality (4.18) implies

$$\mathcal{E}(U_\lambda f, g) + \lambda(U_\lambda f, g) = \mathcal{E}(G_\lambda f, g) + \lambda(G_\lambda f, g) \quad \text{for any } f, g \in C_c^\infty(\mathbb{R}^d)\,,$$

where G_λ and $(\mathscr{E}, D(\mathscr{E}))$ are independent of (n'). G_λ is then the L^2-resolvent of $\mathscr{X}$ and we are done. Note that Theorem 4.9 implies

$$D(\mathcal{E}) = H^{\alpha/2}(\mathbb{R}^d).$$

We prove (4.18) by approximating each term by its discrete analog. On the discrete level

$$\mathcal{E}^{(n)}(U_\lambda^{(n)} R_n(f), R_n(g)) = (R_n(f), R_n(g)) - \lambda(U_\lambda^{(n)} R_n(f), R_n(g)) \tag{4.19}$$

where $U_\lambda^{(n)} h(x) = \int_0^\infty e^{-\lambda t}\big(P_t^{(n)} h\big)(x)\, dt$, $\lambda > 0$, denotes the resolvent of $\big(Z_t^{(n)}\big)$.

Therefore fix $\lambda > 0$, $f, g \in C_c^\infty(\mathbb{R}^d)$ and abbreviate $f_n = R_n(f)$, $g_n = R_n(g)$. Then $f_n, g_n \in L^2(n^{-1}\mathbb{Z}^d, n^{-1}\mu)$ with $\|f_n\| + \|g_n\| \le c$ for all n. Recalling the definition of section 4.2.1 one sees that $\sum_{x \in n^{-1}\mathbb{Z}^d} f_n(x) \mathbb{1}_{\mathfrak{Q}_n}(x)$ converges in $L^2(\mathbb{R}^d)$ to f and $|(f_n, g_n) - (f, g)|$ converges to zero for $n \to \infty$.

Now by the compactness of the support of f and Theorem 4.19 we get equicontinuity for the family $(E_n U_\lambda^{(n)} R_n f)_{n \in \mathbb{N}}$ analogous to (4.15) resp. (4.16). Together with $\big|[x]_n - x\big| \le \sqrt{d} n^{-1}$ we get

$$\big|E_n U_\lambda^{(n)} f_n(x) - U_\lambda^{(n)} f_n([x]_n)\big| \le c n^{-\beta}$$

for all $x \in \mathbb{R}^d$. In particular, the functions $x \mapsto U_\lambda^{(n')} f_{n'}([x]_{n'})$ on $\mathbb{R}^d$ converge along the subsequence (n'') uniformly on compact sets to $U_\lambda f$. Taking into account that g is compactly supported we get by dominated convergence

$$\big|\big(U_\lambda^{(n'')} f_{n''}, g_{n''}\big) - \big(U_\lambda f, g\big)\big| \to 0 \quad \text{for } n'' \to \infty. \tag{4.20}$$

Therefore the right-hand side of (4.19) converges against the right-hand side of (4.18) for the subsequence $n'' \to \infty$.

It remains to determine the limit of the left-hand side of (4.19) for $n'' \to \infty$. We do this in several steps.

Step 1: $U_\lambda f \in H^{\alpha/2}(\mathbb{R}^d)$.

This result probably follows from standard arguments of approximation theory. For the sake of completeness we give a detailed proof. First note that $U_\lambda^{(n)} f_n$ and $E_n U_\lambda^{(n)} f_n$ form bounded sequences in $L^2(\mathbb{R}^d)$. Set $F_n = E_n U_\lambda^{(n)} f_n$. Then we aim to prove $\|F_n\|_{H^{\alpha/2}(\mathbb{R}^d)} \leq c$ with $c > 0$ independent of n. Define $V_n = \{z \in \mathbb{R}^d : |z|_\infty < 2n^{-1}\}$. Moreover, let $z_1^{(n)}, \dots, z_{2^d}^{(n)}$ be the corners of $\mathfrak{Q}_n(0)$. We write

$$\begin{aligned}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(F_n(\xi)-F_n(\zeta)\big)^2}{|\xi-\zeta|^{d+\alpha}} d\xi\, d\zeta &= \sum_{x\in n^{-1}\mathbb{Z}^d} \iint\limits_{\mathfrak{Q}_n(x)\times V_n} \frac{\big(F_n(\xi+\eta)-F_n(\xi)\big)^2}{|\eta|^{d+\alpha}} d\eta\, d\xi \\ &+ \sum_{x\in n^{-1}\mathbb{Z}^d} \iint\limits_{\mathfrak{Q}_n(x)\times(\mathbb{R}^d\setminus V_n)} \big(F_n(\xi+\eta)-F_n(\xi)\big)^2 |\eta|^{-d-\alpha}\, d\eta\, d\xi =: (I_1) + (I_2)\end{aligned} \tag{4.21}$$

Let us first look at (I_1). For $x \in n^{-1}\mathbb{Z}^d$, $\xi \in \mathfrak{Q}_n(x)$, $\eta \in V_n$, by Taylor's formula

$$\frac{\big(F_n(\xi+\eta)-F_n(\xi)\big)^2}{|\eta|^{d+\alpha}} \leq c_0 n^2 \sum_{\substack{\widetilde{x}\in\overline{\mathfrak{Q}_n(x)}\cap n^{-1}\mathbb{Z}^d \\ \widetilde{y}\in\overline{\mathfrak{Q}_n(x)}+\overline{V}_n\cap n^{-1}\mathbb{Z}^d}} \frac{\big(F_n(\widetilde{x})-F_n(\widetilde{y})\big)^2}{|\eta|^{d+\alpha-2}}$$

Furthermore,

$$\iint\limits_{\mathfrak{Q}_n(x)\times V_n} |\eta|^{2-d-\alpha}\, d\eta\, d\xi \leq n^{-d} \int\limits_{|\eta|\leq 2\sqrt{d}n^{-1}} |\eta|^{2-d-\alpha}\, d\eta \leq c_1 n^{-2+\alpha-d}$$

where $c_1 > 0$ only depends on α and d. Since $\left|\widetilde{y}-\widetilde{x}\right| \leq 4\sqrt{d}n^{-1}$ the first sum in (4.21) can be bounded from above by

$$\begin{aligned} c_2 n^{-2d} &\sum_{x\in n^{-1}\mathbb{Z}^d} \sum_{\substack{\widetilde{x}\in\overline{\mathfrak{Q}_n(x)}\cap n^{-1}\mathbb{Z}^d \\ \widetilde{y}\in\overline{\mathfrak{Q}_n(x)}+\overline{V}_n\cap n^{-1}\mathbb{Z}^d \\ \widetilde{x}\neq\widetilde{y}}} \big(F_n(\widetilde{x})-F_n(\widetilde{y})\big)^2 |\widetilde{x}-\widetilde{y}|^{-d-\alpha} \\ \leq c_3 n^{-2d} &\sum_{\substack{x,h\in n^{-1}\mathbb{Z}^d \\ 0<|h|_\infty\leq 6n^{-1}}} \big(F_n(x+h)-F_n(x)\big)^2 \big|h\big|^{-d-\alpha}. \end{aligned}$$

As expected, (I_1) tends to zero for large n. In order to tackle (I_2) note that, for all $h, x \in n^{-1}\mathbb{Z}^d$, $\eta \in \mathfrak{Q}_n(x+h)$, $\xi \in \mathfrak{Q}_n(x)$

$$\begin{aligned}\big(F_n(\eta)-F_n(\xi)\big)^2 &\leq \max_{i,j=1,\dots,2^d} \big(F_n(x+h+z_i^{(n)}) - F_n(x+z_j^{(n)})\big)^2 \\ &\leq \sum_{i,j}^{2^d} \big(F_n(x+h+z_i^{(n)}) - F_n(x+z_j^{(n)})\big)^2 .\end{aligned}$$

For any $\mathfrak{Q}_n(h) \subset \mathbb{R}^d \setminus V_n$, $h \in \mathbb{Z}^d$, and any $\eta \in \mathfrak{Q}_n(h)$ and $i, j = 1, \dots, 2^d$

$$|\eta| \geq |\eta|_\infty \geq \frac{1}{2}\big|h + z_i^{(n)} - z_j^{(n)}\big|_\infty \geq \frac{1}{2\sqrt{d}}\big|h + z_i^{(n)} - z_j^{(n)}\big| \geq c_4\big|h + z_i^{(n)} - z_j^{(n)}\big|,$$

where c_4 depends only on the dimension. Keeping in mind that the volume of $\mathfrak{Q}_n(x) \times \mathfrak{Q}_n(x+h)$ is n^{-2d} we estimate (I_2) in (4.21) from above by

$$\begin{aligned} c_4^{\alpha+d} n^{-2d} \sum_{i,j=1}^{2^d} \sum_{\substack{x,h \in n^{-1}\mathbb{Z}^d \\ h+z_i^{(n)}-z_j^{(n)} \neq 0}} & \big(F_n(x+h+z_i^{(n)}) - F_n(x+z_j^{(n)})\big)^2 \big|h + z_i^{(n)} - z_j^{(n)}\big|^{-d-\alpha} \\ \leq c_5 n^{-2d} \sum_{\substack{x,h \in n^{-1}\mathbb{Z}^d \\ h \neq 0}} & \big(F_n(x+h) - F_n(x)\big)^2 \big|h\big|^{-d-\alpha}. \end{aligned}$$

Hence we obtain by (A3) and (4.19)

$$\begin{aligned} &\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \big(E_n U_\lambda^{(n)} f_n(\xi) - E_n U_\lambda^{(n)} f_n(\zeta)\big)^2 |\xi - \zeta|^{-d-\alpha} \, d\xi \, d\zeta \\ &\qquad \leq c_5 n^{-2d} \sum_{\substack{x,h \in n^{-1}\mathbb{Z}^d \\ h \neq 0}} \big(U_\lambda^{(n)} f_n(x+h) - U_\lambda^{(n)} f_n(x)\big)^2 |h|^{-d-\alpha} \\ &\qquad \leq c_6 \mathcal{E}^{(n)}\big(U_\lambda^{(n)} f_n, U_\lambda^{(n)} f_n\big) = c_6\big(U_\lambda^{(n)} f_n, f_n\big) - c_6 \lambda \big\|U_\lambda^{(n)} f_n\big\|^2. \end{aligned}$$

The right-hand side is bounded in n. We conclude that $E_n U_\lambda^{(n)} f_n$ is a bounded sequence in the Sobolev space $H^{\alpha/2}(\mathbb{R}^d)$.

In our situation this means that there exists a subsequence (n''') of (n'') such that $E_{n'''} U_\lambda^{(n''')} f_{n'''}$ converges weakly in $H^{\alpha/2}(\mathbb{R}^d)$ and strongly in $L^2(K)$ for $K \subset \mathbb{R}^d$ compact to an element $\widetilde{F} \in H^{\alpha/2}(\mathbb{R}^d)$ for $n''' \to \infty$. Since $E_{n''} U_\lambda^{(n'')} f_{n''} \to U_\lambda f$ pointwise, $U_\lambda f = \widetilde{F}$ almost everywhere. In particular $U_\lambda f \in H^{\alpha/2}(\mathbb{R}^d)$.

Step 2: Setting for $r \in (0,1)$ $\mathcal{E}_r(f,g) = \frac{1}{2} \iint\limits_{|x-y| \geq r} (f(y) - f(x))(g(y) - g(x)) k(x,y) \, dx \, dy$ one observes

$$\big|\mathcal{E}_r(U_\lambda f, g) - \mathcal{E}(U_\lambda f, g)\big| \to 0 \quad \text{for } r \to 0. \tag{4.22}$$

Step 3: In analogy to Step 2 set

$$\mathcal{E}_r^{(n)}(f_n, g_n) = \frac{n^{-2d}}{2} \sum_{\substack{x,y \in n^{-1}\mathbb{Z}^d \\ |x-y| \geq r}} (f_n(y) - f_n(x))(g_n(y) - g_n(x)) C^n(x,y).$$

Then for any $r > 0$

$$\big|\mathcal{E}_r^{(n'')}(U_\lambda^{(n'')} f_{n''}, g_{n''}) - \mathcal{E}_r(U_\lambda f, g)\big| \to 0 \quad \text{for } n'' \to \infty. \tag{4.23}$$

Assertion (4.23) is easily established by estimating the difference for large enough n'' from above by

$$\iint_{|x-y|\geq r/2} \Big| \big(U_\lambda^{(n'')} f([y]_{n''}) - U_\lambda^{(n'')} f([x]_{n''})\big)\big(g([y]_{n''}) - g([x]_{n''})\big) C^{n''}([x]_{n''}, [y]_{n''}) \\ - \big(U_\lambda f(y) - U_\lambda f(x)\big)\big(g(y) - g(x)\big) k(x,y) \Big| \, dx\, dy$$

This term tends to zero for $n'' \to 0$ since $g([x]_{n''}) \to g(x)$ and $U_\lambda^{(n'')} f([x]_{n''}) \to U_\lambda f(x)$ uniformly on compacts for $n'' \to \infty$. Using (A5) we see that the integrand converges uniformly to 0. (4.23) follows by dominated convergence.

Step 4:

$$\sup_{n''} \big|\mathcal{E}^{(n'')}\big(U_\lambda^{(n'')} f_{n''}, g_{n'}\big) - \mathcal{E}_r^{(n'')}\big(U_\lambda^{(n'')} f_{n''}, g_{n''}\big)\big| \to 0 \quad \text{for } r \to 0. \tag{4.24}$$

Recall that both, f and g have compact support, i.e. the number of elements in $\operatorname{supp}(g) \cap n^{-1}\mathbb{Z}^d$ is of order $n^2 d$. Using Cauchy-Schwartz and (A2) we obtain

$$\begin{aligned} &\Big| \sum_{\substack{|x-y|<r \\ x,y\in n^{-1}\mathbb{Z}^d}} \big(U_\lambda^{(n)} f_n(x) - U_\lambda^{(n)} f_n(y)\big)\big(g_n(x) - g_n(y)\big) C^n(x,y) n^{-2d} \Big|^2 \\ &\qquad \leq \Big(\sum_{\substack{|x-y|<r \\ x,y\in n^{-1}\mathbb{Z}^d}} \big(U_\lambda^{(n)} f_n(x) - U_\lambda^{(n)} f_n(y)\big)^2 C^n(x,y) n^{-2d} \Big) \\ &\qquad\qquad \times \sum_{\substack{|x-y|<r \\ x,y\in n^{-1}\mathbb{Z}^d}} \big(g_n(x) - g_n(y)\big)^2 C^n(x,y) n^{-2d} \\ &\qquad \leq c \big|\mathcal{E}^{(n)}\big(U_\lambda^{(n)} f_n, U_\lambda^{(n)} f_n\big)\big| \, \big|\mathcal{E}_r\big(g_n, g_n\big)\big| \,. \end{aligned}$$

Let us look at the above estimate for $n = n''$. Since $\big|\mathcal{E}^{(n'')}\big(U_\lambda^{(n'')} f_{n''}, U_\lambda^{(n'')} f_{n''}\big)\big|$ and $\big|\mathcal{E}\big(g_{n''}, g_{n''}\big)\big|$ are bounded uniformly in n''

$$\sup_{n''} \big|\mathcal{E}^{(n'')}\big(U_\lambda^{(n'')} f_{n''}, U_\lambda^{(n'')} f_{n''}\big)\big| \, \big|\mathcal{E}_r\big(g_{n''}, g_{n''}\big)\big| \to 0 \text{ for } r \to 0 \,.$$

Step 4 is completed.

Finally, combining (4.22), (4.23) and (4.24) by a standard chaining argument proves (4.18). ■

4.6 Approximation of jump processes by Markov chains

Here, we prove Theorem 4.3.

Proof of Theorem 4.3: Obviously, by the symmetry of k (A1) holds. The upper bounds on k imply for $x, y \in n^{-1}\mathbb{Z}^d$, $|x-y|_\infty \geq 2n^{-1}$:

$$\begin{aligned} C^n(x,y) &\leq \kappa_5 n^{2d} \int\limits_{\substack{|x-\xi|_\infty < n^{-1}/2 \\ |y-\zeta|_\infty < n^{-1}/2}} |\xi - \zeta|^{-d-\alpha}\, d\xi\, d\zeta \\ &\leq \kappa_5 (4d)^{(d+\alpha)/2} n^{2d} \int\limits_{\substack{|x-\xi|_\infty < n^{-1}/2 \\ |y-\zeta|_\infty < n^{-1}/2}} |x-y|^{-d-\alpha}\, d\xi\, d\zeta \leq \kappa_5 (4d)^{(d+\alpha)/2} |x-y|^{-d-\alpha}. \end{aligned}$$

since $|\xi-\zeta| \geq |\xi-\zeta|_\infty \geq |x-y|_\infty - n^{-1} \geq \frac{1}{2}|x-y|_\infty \geq \frac{1}{2}d^{-1/2}|x-y|$. In the same way one shows $C^n(x,y) \geq \kappa_4 (4d)^{-(d+\alpha)/2} |x-y|^{-d-\alpha}$ for $|x-y|_\infty \geq 2n^{-1}$. Therefore, (A2), (A3) and (A4) are satisfied.

Let $k_n(x,y) = C^n([x]_n, [y]_n)$. Fix a compact rectangular subset $\Omega = \prod_{i=1}^{2d}[a_i, b_i]$, $a_i < b_i$ of $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ and define $\Omega_\varepsilon = \prod_{i=1}^{2d}[a_i - \varepsilon, b_i + \varepsilon]$ for $\varepsilon > 0$. In particular, there is $c_1 > 0$ with $\boldsymbol{\lambda}(\Omega_\varepsilon \setminus \Omega) < c_1 \varepsilon$ for $\varepsilon < 1$. Then there exists $n_0 > 0$ such that $\text{dist}(\Omega, \text{diag}) > n_0^{-1}$. For all $n > 4n_0$ and $\varepsilon < (4n_0)^{-1}$ the functions k_n are uniformly bounded from above on Ω_ε by $c_2 > 0$. By the Theorem of Lusin, for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that k restricted to K_ε is continuous while $\boldsymbol{\lambda}(\Omega \setminus K_\varepsilon) < \varepsilon$. Furthermore, there exists a continuous function $k^\varepsilon \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ with compact support such that $k^\varepsilon = k$ on K_ε, $\operatorname{supp} k^\varepsilon \subset \Omega_\varepsilon$ and $\|k^\varepsilon\|_\infty < c_2$. We estimate

$$\|k - k^\varepsilon\|_{L^1(\Omega)} = \int_\Omega |k - k^\varepsilon| = \int_{K\varepsilon} |k - k^\varepsilon| + \int_{\Omega \setminus K_\varepsilon} |k - k^\varepsilon| \leq c_2 \varepsilon.$$

Define now $k_n^\varepsilon(x,y) = C_\varepsilon^n([x]_n, [y]_n)$ as above with k replaced by k^ε. Then, since k^ε is Riemann integrable with compact support, k_n^ε converges in $L^1(\mathbb{R}^{2d})$ for $n \to \infty$ to k^ε. Moreover, for $\varepsilon < (4n_0)^{-1}$, $n > \varepsilon^{-1}$ and by the definition of the conductivity functions

$$\|k_n^\varepsilon - k_n\|_{L^1(\Omega)} \leq \|k^\varepsilon - k\|_{L^1(\Omega_\varepsilon)} \leq \|k^\varepsilon - k\|_{L^1(\Omega)} + \|k^\varepsilon - k\|_{L^1(\Omega_\varepsilon - \Omega)} \leq c_2(1 + c_1)\varepsilon$$

Putting all this together we get for n large enough

$$\|k - k_n\|_{L^1(\Omega)} \leq \|k - k^\varepsilon\|_{L^1(\Omega)} + \|k^\varepsilon - k_n^\varepsilon\|_{L^1(\Omega)} + \|k_n^\varepsilon - k_n\|_{L^1(\Omega)} \leq c_3 \varepsilon.$$

This directly yields (A5). Now Theorem 4.1 applies. ∎

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