Cohomology of classes of symbols and classification of traces on corresponding classes of operators with non positive order

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Abstract

This thesis is devoted to the classification issue of traces on classical pseudo-differential operators with fixed non positive order on closed manifolds of dimension $n > 1$. We describe the space of homogeneous functions on a symplectic cone in terms of Poisson brackets of appropriate homogeneous functions, and we use it to find a representation of a pseudo-differential operator as a sum of commutators. We compute the cohomology groups of certain spaces of classical symbols on the n–dimensional Euclidean space with constant coefficients, and we show that any closed linear form on the space of symbols of fixed order can be written either in terms of a leading symbol linear form and the noncommutative residue, or in terms of a leading symbol linear form and the cut-off regularized integral. On the operator level, we infer that any trace on the algebra of classical pseudo-differential operators of order $a \in \mathbb{Z}$ can be written either as a linear combination of a generalized leading symbol trace and the residual trace when $-n+1 \leq 2a \leq 0$, or as a linear combination of a generalized leading symbol trace and any linear map that extends the L^2–trace when $2a \leq -n \leq a$. In contrast, for odd class pseudo-differential operators in odd dimensions, any trace can be written as a linear combination of a generalized leading symbol trace and the canonical trace. We derive from these results the classification of determinants on the Fréchet Lie group associated to the algebras of classical pseudo-differential operators with non positive integer order.

Contents

Introduction

This thesis addresses the classification issue of traces on certain classes of classical pseudo-differential operators on closed manifolds of dimension $n > 1$. The classification was already known for the whole algebra of classical pseudo-differential operators as well as for specific classes such as smoothing operators, non-integer order operators and odd class operators in odd dimensions. Also the case of zero order operators was studied in view of a classification of multiplicative determinants. Interestingly, the above mentioned classes fall into two types, those with traces that vanish on trace-class operators, namely the residual trace and the leading symbol trace, and those equipped with the canonical trace that extends the L^2–trace. This twofold picture extends to classes of operators with fixed non positive order considered here. The residual trace and a generalized leading symbol trace arise when considering operators of integer order a with $-n+1 \leq 2a \leq 0$, whereas the canonical trace arises when restricting to non-integer order, or to odd class operators in odd dimensions.

On the one hand, the noncommutative residue, which falls into the first class of traces, was introduced about 1978 by Adler and Manin in the one-dimensional case; they showed that it defines a trace functional on the algebra generated by one dimensional symbols whose elements are formal Laurent series with a particular composition law. Seven years later Guillemin ([14]) and Wodzicki ([44]) independently extended this definition to all dimensions. This residue yields the only trace (up to a constant) on the whole algebra of classical pseudo-differential operators ([7], [10], [25], [44]), and it has many striking properties, among which its locality, that is very much related with the fact that it vanishes on smoothing operators.
On the other hand, the canonical trace which falls into the second class of traces, was introduced by Kontsevich and Vishik ([23]); they showed that this is actually a trace (even more unique: see [30]) on certain subsets of operators with vanishing residue. In contrast to the noncommutative residue, it is highly non local due to the fact that it extends the L^2–trace.

Fixing the order of the operator as we do throughout this thesis, introduces many technical difficulties, which do not allow a naive and direct implementation of proofs carried out in the case of operators of any order, and one often needs a refined version of previously known results. For the classification of traces it is natural to ask for a representation of a pseudo-differential operator as a sum of commutators of elements in the algebra one considers. Starting from a generalization of a result by Guillemin about the representation of Poisson brackets of homogeneous functions on a symplectic cone ([14]), we generalize and improve a result by Lesch ([25]) concerning the representation of a pseudo-differential operator as a sum of commutators. With this result at hand, in order to classify traces on algebras of classical pseudo-differential operators of fixed non positive order, it remained to solve the issue about the existence of a non-trivial extension of the L^2–trace to a trace functional on the class of operators we consider.

All known traces on algebras of pseudo-differential operators are built using linear forms on symbols which satisfy Stokes' property, i.e., they vanish on partial derivatives of symbols ([35]). Two notable examples are the noncommutative residue, which gives rise to the trace which carries the same name, and the cut-off regularized integral which yields the canonical trace. It is therefore natural to investigate the cohomology groups of spaces of classical symbols on the n–dimensional Euclidean space with constant coefficients, and to look at the dual of those cohomology groups. We compute these cohomology groups, and show that the top cohomology group of certain spaces of symbols is one-dimensional. This implies that in the case of fixed real order a, any closed linear form on the space can be written either in terms of a leading symbol linear form and the noncommutative residue in the case when $a \in \mathbb{Z}$, $a \geq -n+1$, or in terms of a leading symbol linear form and the cut-off regularized integral in the case when $a \notin \mathbb{Z} \cap [-n+1, +\infty)$.

An important consequence of the uniqueness of the noncommutative residue as a linear form which satisfies Stokes' property in the whole space of classical symbols, is that any smoothing symbol is a finite sum of derivatives of symbols; we indeed prove a refined version of this in the case when $a \in \mathbb{Z}$, $a \geq -n+1$. On the operator level, we infer that on the algebra of classical pseudo-differential operators of order $a \in \mathbb{Z}$, when $-n+1 \leq 2a \leq 0$ there is no a non trivial extension of the L^2–trace, and when $2a \leq -n \leq a$ any linear map that extends the L^2–trace is a trace; from this we infer that any trace on this algebra can be written either as a linear combination of a generalized leading symbol trace and the residual trace in the first case, generalizing the result of [28] and [45], or as a linear combination of a generalized leading symbol trace and such a linear map in the second case. In contrast, for odd class pseudo-differential operators in odd dimensions, any trace can be written as a linear combination of a generalized leading symbol trace and the canonical trace.

Finally, we derive from these results the classification of determinants on the Fréchet Lie group associated to those algebras of classical pseudo-differential operators, and we show that any of those determinants can be written either in terms of a generalized leading symbol determinant and the Wodzicki multiplicative determinant, or in terms of a generalized leading symbol determinant and the canonical determinant, generalizing the result of [28].

All these results are organized around five chapters. In the first chapter we study the Poisson bracket representation of homogeneous functions on a symplectic cone, first by using an appropriate differential operator and then by using homogeneous differential forms. We are interested in the case when the symplectic cone is given by the cotangent space of a closed manifold of dimension greater than 1 without the zero section with its standard symplectic form. In Chapter 2 we use integration along the fiber to prove an analogue of the Poincaré Lemma for cohomology with compact support, and we describe the cohomology groups of the space of classical symbols on $\mathbb{R}^n$ with constant coefficients. Using the top cohomology group of some spaces of classical symbols on $\mathbb{R}^n$ with constant coefficients, in Chapter 3 we classify the closed linear forms on those spaces of symbols in terms of a leading symbol linear form, the cut-off regularized integral and the noncommutative residue on symbols.

In Chapter 4 we give a representation of a classical pseudo-differential operator as a sum of commutators. The main fact to give a complete classification of traces on algebras of classical pseudo-differential operators of non positive order is the no existence of a non-trivial extension of the usual trace to the algebra. In Chapter 5, we prove that there does not exist such a non-trivial extension to operators of integer order a when $2a$ is greater than minus the dimension of the manifold, by using the classification of closed linear forms on the space of symbols and by writing a smoothing operator as a sum of commutators of elements in the algebra; then, we consider the case of traces on operators acting on sections of vector bundles over the manifold. In the last part of the chapter we give the classification of multiplicative determinants on the Fréchet Lie group associated to the algebra of non positive integer order classical pseudo-differential operators.

Chapter 1

Poisson Bracket Representation of Homogeneous Functions

In this chapter we study the representation of a homogeneous function in terms of Poisson brackets. In the first section we recall some basic definitions and identities related to homogeneous functions on a symplectic cone. In the second section we recall the definition of the symplectic residue and in the third section we equip the space of homogeneous functions with a pre-Hilbert space structure. In the fourth section we describe the space of homogeneous functions by constructing an operator whose image is either a linear space generated by Poisson brackets or the kernel of the symplectic residue (see Theorem 1.4.1); this construction does not work in the case of Poisson brackets of homogeneous functions of degree zero, but in the last section, we present another proof of that description including this case, using homogeneous differential forms and the homogeneous cohomology of a symplectic cone (see Proposition 1.5.2 – Proposition 1.5.4).

1.1 Homogeneous functions on a symplectic cone

The goal of this section is to provide all the identities needed to define the Poisson bracket of two homogeneous functions and its properties. We will prove some identities of symplectic and differential geometry using references [8] and [32].

Let Y be a symplectic manifold of dimension $2n$, and let ω be the corresponding closed, nondegenerate 2-form on Y, so that ω^n is a volume form on Y. For every $f \in C^\infty(Y)$ there exists a unique Hamiltonian vector field X_f such that $\iota_{X_f}\omega = -df$. The Poisson bracket of two functions $f, g \in C^\infty(Y)$ is defined by

$$\{f, g\} := \omega(X_f, X_g).$$

Thus

$$\{f,g\} = \mathcal{L}_{X_f}(g) = \iota_{X_f}\,dg = dg(X_f) = X_f[g]$$

and $\{f,\cdot\}$ is a derivation (see [4]). Here $\mathcal{L}_{X_f}(g)$ is the Lie derivative of g with respect to the Hamiltonian vector field X_f, ι_{X_f} represents the inner product by X_f, and we use the Cartan identity $\mathcal{L}_{X_f}(g) = d\iota_{X_f}(g) + \iota_{X_f}\,dg$.
For any pair of vector fields X_1, X_2: $[X_1, X_2] = \mathcal{L}_{X_1}X_2$ and we have the identity $\iota_{[X_1,X_2]} = [\mathcal{L}_{X_1}, \iota_{X_2}]$. So, if X_1 and X_2 are Hamiltonian, then $[X_1, X_2]$ is also Hamiltonian with Hamiltonian function $\omega(X_1, X_2)$ (see Def. 18.5 in [8]):

$$\begin{aligned}
\iota_{[X_1,X_2]}\omega &= \mathcal{L}_{X_1}\iota_{X_2}\omega - \iota_{X_2}\mathcal{L}_{X_1}\omega \\
&= d\iota_{X_1}\iota_{X_2}\omega + \iota_{X_1}d\iota_{X_2}\omega - \iota_{X_2}d\iota_{X_1}\omega - \iota_{X_2}\iota_{X_1}d\omega \\
&= d\omega(X_2, X_1) \\
&= -d\omega(X_1, X_2) \\
&= \iota_{X_{\omega(X_1,X_2)}}\omega,
\end{aligned}$$

hence

$$X_{\{f,g\}} = X_{\omega(X_f,X_g)} = [X_f, X_g], \tag{1.1}$$

and $(C^\infty(Y), \{,\})$ is a Poisson algebra (see [4], [8]).

Since $\iota_{X_f}(\omega) = -df$, by induction we prove the following identity:

$$\forall m \in \mathbb{N}, \quad \iota_{X_f}(\omega^m) = -m\,df \wedge \omega^{m-1}. \tag{1.2}$$

Indeed, using the formula $\iota_{X_f}(\alpha \wedge \beta) = \iota_{X_f}(\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \iota_{X_f}(\beta)$, we see that if $\iota_{X_f}(\omega^m) = -m\,df \wedge \omega^{m-1}$, then

$$\begin{aligned}
\iota_{X_f}(\omega^{m+1}) &= \iota_{X_f}(\omega \wedge \omega^m) \\
&= \iota_{X_f}(\omega) \wedge \omega^m + \omega \wedge \iota_{X_f}(\omega^m) \\
&= \iota_{X_f}(\omega) \wedge \omega^m - m\,\omega \wedge df \wedge \omega^{m-1} \\
&= -(m+1)\,df \wedge \omega^m.
\end{aligned}$$

Proposition 1.1.1 (1.2 in [44])**.** *The Poisson bracket of any pair of functions $f, g \in C^\infty(Y)$ satisfies:*

$$\{f,g\}\,\omega^n = n\,df \wedge dg \wedge \omega^{n-1} = d(g\,\iota_{X_f}\omega^n). \tag{1.3}$$

Proof. Since $\omega^{n+1} = 0$, Equation (1.2) implies that

$$\begin{aligned}
0 &= \iota_{X_f}\iota_{X_g}\omega^{n+1} \\
&= -(n+1)\iota_{X_f}(dg \wedge \omega^n) \\
&= -(n+1)\Big(\iota_{X_f}(dg) \wedge \omega^n - dg \wedge \iota_{X_f}(\omega^n)\Big) \\
&= -(n+1)\Big(\iota_{X_f}(dg) \wedge \omega^n + n\,dg \wedge df \wedge \omega^{n-1}\Big) \\
&= (n+1)\Big(n\,df \wedge dg \wedge \omega^{n-1} - \{f,g\}\,\omega^n\Big).
\end{aligned}$$

On the other hand, since $d(\iota_{X_f}\omega) = 0$ and $d\omega = 0$,

$$d(g\,\iota_{X_f}\omega^n) = dg \wedge \iota_{X_f}\omega^n + g\,d(\iota_{X_f}\omega^n) = -n\,dg \wedge df \wedge \omega^{n-1} = n\,df \wedge dg \wedge \omega^{n-1}.$$

$\square$

Let B be a connected smooth manifold of dimension $2n - 1 > 1$.

Definition 1.1.1 ([14])**.** A *cone* over B is a principal bundle $\pi : Y \to B$ with structure group the multiplicative group $\mathbb{R}^+ = (0, +\infty)$. Let $\rho_t : Y \to Y$ be the map of Y onto itself associated with $t \in \mathbb{R}^+$, so that $\pi(\rho_t(y)) = \pi(y)$, for all $y \in Y$, $t \in \mathbb{R}^+$. We say that the cone is smooth if all the above data are smooth. If Y has a symplectic structure with symplectic form ω, Y is a *symplectic cone* if for all $t \in \mathbb{R}^+$: ${\rho_t}^*(\omega) = t\,\omega$, where

$$ {\rho_t}^*(\omega)(X_1, X_2)(y) = \omega(\rho_{t*}X_1, \rho_{t*}X_2)(\rho_t(y)), \tag{1.4}$$

for any $y \in Y$ and for any vector fields X_1, X_2.

Example 1.1.1. The space $Y = \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) = \{(x, \xi) : x \in \mathbb{R}^n, \xi \in (\mathbb{R}^n)^*\}$ is a symplectic cone over S^{2n-1} with symplectic form $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$ and $\rho_t(x, \xi) = (t^{1/2}x, t^{1/2}\xi)$.

Example 1.1.2. Let M be a smooth manifold. The cotangent bundle without the zero section, $Y = T^*M \setminus M$, is a symplectic cone over the cosphere bundle S^*M with its standard symplectic form and $\rho_t(x, \xi) = (x, t\xi)$.

Definition 1.1.2. Let Y be a symplectic cone of dimension $2n$ and assume that Y is connected. Let $\phi_s = \rho_{e^s}$, and let $\mathcal{X}$ be the vector field generating the one-parameter group $\{\phi_s\}$ (see [32]), i.e. for all $y \in Y$,

$$\begin{cases} \phi_0 = id_Y, \\ \left.\dfrac{d(\phi_s)}{ds}\right|_y = \left.\mathcal{X}\right|_{\phi_s(y)}. \end{cases}$$

The map ρ_t preserves $\mathcal{X}$: let $y \in Y$ and $h \in C^\infty(Y)$, then

$$\begin{aligned}
(\rho_t)_*(\mathcal{X}_y)(h) &= \mathcal{X}_y(h \circ \rho_t) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\rho_{e^{-s}}(y)} (h \circ \rho_t) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\rho_t(\rho_{e^{-s}}(y))} (h) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\rho_{e^{-s}}(\rho_t(y))} (h) \\
&= \left.\mathcal{X}\right|_{\rho_t(y)} (h).
\end{aligned}$$

By definition of the Lie derivative with respect to $\mathcal{X}$,

$$\mathcal{L}_{\mathcal{X}}\omega = \lim_{s\to 0} \frac{\phi_s^*\omega - \omega}{s} = \lim_{s\to 0} \frac{e^s\omega - \omega}{s} = \omega.$$

It follows by induction that for all $m \in \mathbb{N}$,

$$\mathcal{L}_{\mathcal{X}}(\omega^m) = m\,\omega^m. \tag{1.5}$$

Indeed, using the formula $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$ (see [32]), we get

$$\mathcal{L}_{\mathcal{X}}(\omega^{m+1}) = \mathcal{L}_{\mathcal{X}}(\omega \wedge \omega^m) = \mathcal{L}_{\mathcal{X}}(\omega) \wedge \omega^m + \omega \wedge \mathcal{L}_{\mathcal{X}}(\omega^m) = (m+1)\,\omega^{m+1}.$$

The vector field $\mathcal{X}$ is not Hamiltonian: if $\mathcal{X}$ were a Hamiltonian vector field corresponding to some $h \in C^\infty(Y)$, it would imply that

$$\omega = \mathcal{L}_{\mathcal{X}}\omega = d(\iota_{\mathcal{X}}\omega) = -d^2h = 0,$$

which is a contradiction.

By definition, $\mathcal{X}$ is a vertical vector field (see [5]) since for any $h \in C^\infty(B)$,

$$\mathcal{X}[\pi^*h] = \mathcal{X}[h \circ \pi] = 0.$$

Let α be the one-form

$$\alpha := \iota_{\mathcal{X}}\omega. \tag{1.6}$$

By Cartan's identity, since $d\omega = 0$, we have

$$d\alpha = d(\iota_{\mathcal{X}}\omega) = \mathcal{L}_{\mathcal{X}}\omega = \omega. \tag{1.7}$$

Since ${\rho_t}^*(\omega) = t\omega$ and ρ_t preserves $\mathcal{X}$ we get from (1.4) and (1.6)

$${\rho_t}^*(\alpha) = t\,\alpha. \tag{1.8}$$

Consider the $(2n-1)$–form on Y defined by

$$\mu := \alpha \wedge \omega^{n-1}. \tag{1.9}$$

From (1.8) we have

$${\rho_t}^*(\mu) = {\rho_t}^*(\alpha \wedge \omega^{n-1}) = {\rho_t}^*(\alpha) \wedge {\rho_t}^*(\omega^{n-1}) = t\,\alpha \wedge t^{n-1}\omega^{n-1} = t^n\mu. \tag{1.10}$$

By (1.6), μ is horizontal (see [5]) with respect to the fibration $\pi : Y \to B$:

$$\iota_{\mathcal{X}}\mu = \iota_{\mathcal{X}}(\alpha \wedge \omega^{n-1}) = \iota_{\mathcal{X}}(\alpha) \wedge \omega^{n-1} + \iota_{\mathcal{X}}\omega \wedge (\iota_{\mathcal{X}}\omega^{n-1}) = 0. \tag{1.11}$$

Equation (1.7) implies that μ also satisfies

$$d\mu = d(\alpha \wedge \omega^{n-1}) = d(\alpha) \wedge \omega^{n-1} - \alpha \wedge d(\omega^{n-1}) = \omega^n. \tag{1.12}$$

Since ω is a 2-form, $\iota_{\mathcal{X}}\omega^2 = \iota_{\mathcal{X}}\omega \wedge \omega + \omega \wedge \iota_{\mathcal{X}}\omega = 2\,\iota_{\mathcal{X}}\omega \wedge \omega$, and if we assume that $\iota_{\mathcal{X}}\omega^m = m\,\iota_{\mathcal{X}}\omega \wedge \omega^{m-1}$, then

$$\iota_{\mathcal{X}}\omega^{m+1} = \iota_{\mathcal{X}}\omega \wedge \omega^m + \omega \wedge \iota_{\mathcal{X}}\omega^m = (m+1)\,\iota_{\mathcal{X}}\omega \wedge \omega^m.$$

Hence, we have proved by induction

$$\forall m \in \mathbb{N}, \quad \iota_{\mathcal{X}}\omega^m = m\,\iota_{\mathcal{X}}\omega \wedge \omega^{m-1}. \tag{1.13}$$

In particular, if $m = n$,

$$\iota_{\mathcal{X}}\omega^n = n\,\mu, \tag{1.14}$$

then (1.11) and (1.12) imply that

$$\mathcal{L}_{\mathcal{X}}\mu = n\,\mu. \tag{1.15}$$

Definition 1.1.3. A function $f \in C^\infty(Y)$ is homogeneous of degree $a \in \mathbb{R}$ if $\rho_t^*(f)(y) = t^a f(y)$ for all $y \in Y$, $t \in \mathbb{R}^+$. Let $\mathcal{P}_a$ be the space of smooth functions on Y which are homogeneous of degree a. We will use Euler's identity:

$$f \in \mathcal{P}_a \Rightarrow \mathcal{L}_{\mathcal{X}} f = \iota_{\mathcal{X}} df = \mathcal{X}[f] = af, \tag{1.16}$$

which can be proved using the definition of the Lie derivative of a function with respect to $\mathcal{X}$: for any $y \in Y$

$$\mathcal{L}_{\mathcal{X}} f(y) = \lim_{s\to 0} \frac{\phi_s^* f(y) - f(y)}{s} = \lim_{s\to 0} \frac{e^{as} f(y) - f(y)}{s} = af(y).$$

If $f \in \mathcal{P}_a$, then $\rho_t^*(df) = d(\rho_t^* f) = d(t^a f) = t^a df$. For $t > 0$, applying ρ_t^* to both sides of identity (1.3), yields that $\{f, g\}$ lies in $\mathcal{P}_{a+b-1}$, for all $f \in \mathcal{P}_a$, $g \in \mathcal{P}_b$. Let $\{\mathcal{P}_a, \mathcal{P}_b\}$ be the linear subspace of $\mathcal{P}_{a+b-1}$ spanned by all functions of the form $\{f, g\}$ with $f \in \mathcal{P}_a, g \in \mathcal{P}_b$.

Lemma 1.1.1. *If $f \in \mathcal{P}_a$, then $d(f\iota_{\mathcal{X}}\omega^n) = (n+a)f\omega^n$.*

Proof. Let $f \in \mathcal{P}_a$. From (1.5) and (1.16) we get

$$\begin{aligned} d(f\iota_{\mathcal{X}}\omega^n) &= df \wedge \iota_{\mathcal{X}}(\omega^n) + nf\omega^n \\ &= \iota_{\mathcal{X}}(df) \wedge \omega^n + nf\omega^n \\ &= (n+a)f\omega^n. \end{aligned}$$

□

Lemma 1.1.2 (1.7 in [44])**.** *If $g \in \mathcal{P}_l$, $[\mathcal{X}, X_g] = (l-1)X_g$.*

Proof. For any pair of vector fields X_1, X_2: $\iota_{[X_1,X_2]} = [\mathcal{L}_{X_1}, \iota_{X_2}]$ so we have

$$\begin{aligned} \iota_{[\mathcal{X},X_g]}\omega &= [\mathcal{L}_{\mathcal{X}}, \iota_{X_g}]\omega \\ &= \mathcal{L}_{\mathcal{X}}(\iota_{X_g}\omega) - \iota_{X_g}(\mathcal{L}_{\mathcal{X}}\omega) \\ &= \mathcal{L}_{\mathcal{X}}(-dg) - \iota_{X_g}\omega \\ &= -l\,dg + dg \\ &= -d((l-1)g) \\ &= \iota_{(l-1)X_g}\omega; \end{aligned}$$

and since ω is non-degenerate we reach the conclusion. □

If $\{\psi_t\}$ denotes the one-parameter group corresponding to X_g, for all $y \in Y$,

$$\begin{cases} \psi_0 = id_Y, \\ \left.\dfrac{d(\psi_t)}{dt}\right|_y = X_g|_{\psi_t(y)}. \end{cases}$$

Every diffeomorphism ψ_t is a symplectomorphism, i.e. it preserves the symplectic form ω: for any $t \in \mathbb{R}^+$, $\psi_t^*\omega = \omega$; the reason is that $\psi_0^*\omega = \omega$ and

$$\left.\frac{d}{dt}\right|_{t=0} \psi_t^*\omega = \mathcal{L}_{X_g}\omega = d\iota_{X_g}\omega + \iota_{X_g}d\omega = -d^2 g = 0. \tag{1.17}$$

Remark 1.1.1. From Lemma 1.1.2 above and Prop. 2.18 in [32] we see that when $l = 1$ the groups of diffeomorphisms $\{\psi_t\}$ and $\{\phi_s\}$ are mutually commutative.

Lemma 1.1.3 (1.11 in [44])**.** *If $f \in \mathcal{P}_l$, $g \in \mathcal{P}_m$, then*

$$\{f, g\}\,\mu = d(g\iota_{X_f}\mu) - (l + m - 1 + n)\, g\, df \wedge \omega^{n-1}.$$

Proof. By the previous identities and some properties of the Lie derivative, we have

$$\begin{aligned}
\{f, g\}\,\mu &\overset{(1.14)}{=} \frac{1}{n}\{f, g\}\,\iota_{\mathcal{X}}\omega^n \overset{(1.3)}{=} \frac{1}{n}\iota_{\mathcal{X}} d(g\,\iota_{X_f}\omega^n) \\
&= \frac{1}{n}\mathcal{L}_{\mathcal{X}}(g\,\iota_{X_f}\omega^n) - \frac{1}{n}d(g\,\iota_{\mathcal{X}}\iota_{X_f}\omega^n) \\
&\overset{(1.2),(1.13)}{=} d(g\,\iota_{X_f}\mu) - \mathcal{L}_{\mathcal{X}}(g\,df \wedge \omega^{n-1}) \\
&= d(g\,\iota_{X_f}\mu) - g\,df \wedge \mathcal{L}_{\mathcal{X}}(\omega^{n-1}) - \mathcal{L}_{\mathcal{X}}(g\,df) \wedge \omega^{n-1} \\
&\overset{(1.5)}{=} d(g\,\iota_{X_f}\mu) - (n-1)\,g\,df \wedge \omega^{n-1} - \mathcal{L}_{\mathcal{X}}(g\,df) \wedge \omega^{n-1} \\
&\overset{(1.16)}{=} d(g\,\iota_{X_f}\mu) - (n-1)\,g\,df \wedge \omega^{n-1} - (m+l)\,g\,df \wedge \omega^{n-1} \\
&= d(g\,\iota_{X_f}\mu) - (l + m - 1 + n)\,g\,df \wedge \omega^{n-1}.
\end{aligned}$$

□

Remark 1.1.2. Fix a domain $\mathcal{U} \subseteq \mathbb{R}^n$ and let $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the corresponding coordinates in the cotangent space $T^*\mathcal{U} = \mathcal{U} \times \mathbb{R}^n$. For the symplectic cone $T^*\mathcal{U} \setminus \mathcal{U} \to S^*\mathcal{U}$ we explicitly have (see 2.1 in [44])

$$\begin{aligned}
&\omega = \sum_{i=1}^n d\xi_i \wedge dx_i, \qquad \mathcal{X} = \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}, \qquad \alpha = \sum_{i=1}^n \xi_i\, dx_i, \\
&\omega^n = (-1)^{\frac{n(n-1)}{2}} n! d\xi_1 \wedge \cdots \wedge d\xi_n \wedge dx_1 \wedge \cdots \wedge dx_n = (-1)^{\frac{n(n-1)}{2}} n!\, d\xi\, dx, \\
&\mu = (-1)^{\frac{n(n-1)}{2}} (n-1)! \left(\sum_{i=1}^n (-1)^{i-1} \xi_i d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n \right) \wedge dx \\
&\quad =: \overline{\mu}(\xi) \wedge dx,
\end{aligned}$$

where $\overline{\mu}(\xi)$ is a volume form on the sphere S^{n-1}.

We will fix once and for all, a function $p \in \mathcal{P}_1$ such that p is everywhere positive. By Euler's identity this implies that $dp \neq 0$ everywhere: for $y \in Y$,

$$p(y) = \mathcal{L}_{\mathcal{X}} p(y) = dp(\mathcal{X})(y).$$

Therefore 1 is a regular value for p, and if $Z = \{y \in Y : p(y) = 1\} = p^{-1}(1)$, by the Preimage Theorem (see [16]), Z is a submanifold of Y of dimension $2n - 1$. Moreover, $\pi \restriction_Z : Z \to B$ is a diffeomorphism, since Z is identified with the quotient space $Y/\mathbb{R}^+$, and this space is identified with the base space B (because the fibers of the bundle Y are diffeomorphic to $\mathbb{R}^+$).

Proposition 1.1.2 (Sect. 4 in [16])**.** *Let Z be the preimage of a regular value $y \in Y$ under the smooth map $p : Y \to \mathbb{R}$. Then the kernel of the derivative $dp_z : T_zY \to T_y\mathbb{R}$ at any point $z \in Z$ is precisely the tangent space to Z, T_zZ.*

Proof. Since p is constant on Z, dp_z is zero on T_zZ. But $dp_z : T_zY \to T_y\mathbb{R}$ is surjective, so the dimension of the kernel of dp_z must be

$$\dim(T_zY) - \dim(T_y\mathbb{R}) = \dim(Y) - \dim(\mathbb{R}) = \dim(Z).$$

Thus T_zZ is a subspace of the kernel that has the same dimension as the complete kernel, hence $\ker(dp_z) = T_zZ$. □

The previous proposition leads us to consider, for all $z \in Z$, the short exact sequence

$$0 \to T_zZ \to T_zY \overset{dp_z}{\to} T_1\mathbb{R} \to 0.$$

Let z be any point in Z. Then,

1. $\alpha(\mathcal{X}) = 0$, since $\alpha := \iota_{\mathcal{X}}\omega$.
2. $dp(X_p) = -\iota_{X_p}\iota_{X_p}\omega = 0$, then $X_p \in \ker(dp)$, and hence $X_p|_z \in T_zZ$.
3. $\alpha(X_p) = \omega(\mathcal{X}, X_p) = \mathcal{X}[p] = p \neq 0$, then $\alpha(X_p)|_z = 1$.
4. $dp(\mathcal{X}) = \mathcal{X}[p] = p \neq 0$, then $dp(\mathcal{X})|_z = 1$.

Hence for every point $z \in Z$, by item 4 and Proposition 1.1.2, $\mathcal{X}|_z \notin T_zZ$ and since

$$\dim(T_zY) - \dim(T_zZ) = 1, \tag{1.18}$$

we have the following decomposition:

$$T_zY = T_zZ \oplus \mathbb{C} \cdot \mathcal{X}|_z . \tag{1.19}$$

In general, for any point $y \in Y$ we have

$$T_yY = \ker(dp_y) \oplus \mathbb{C} \cdot \mathcal{X}|_y . \tag{1.20}$$

1.2 The symplectic residue

In this section we recall the definition and the properties of the symplectic residue of a homogeneous function of degree $-n$ defined on a symplectic cone of dimension $2n$, following [14].

Let $Y \xrightarrow{\pi} B$ be a symplectic cone, and we will assume that B is closed unless we indicate something else. Let f be a homogeneous function of degree $-n$ on Y, and consider the $(2n-1)$–form $f\mu$, where μ is the $(2n-1)$–form defined in (1.9). The form $f\mu$ satisfies the following

1. It is closed: by (1.12), (1.13), (1.16) and since $\iota_{\mathcal{X}}(df) \wedge \omega^n = df \wedge \iota_{\mathcal{X}}\omega^n$,
$$\begin{aligned} d(f\mu) &= df \wedge \mu + f\, d\mu = \frac{1}{n}\, df \wedge \iota_{\mathcal{X}}\omega^n + f\,\omega^n \\ &= \frac{1}{n}\iota_{\mathcal{X}}(df) \wedge \omega^n + f\,\omega^n = -f\,\omega^n + f\,\omega^n = 0. \end{aligned} \tag{1.21}$$

2. It is horizontal: by (1.11),
$$\iota_{\mathcal{X}}(f\mu) = f\iota_{\mathcal{X}}\mu = 0.$$

3. It is invariant under the action of $\mathbb{R}^+$: by (1.10),
$$\rho_t{}^*(f\mu) = \rho_t{}^*(f)\rho_t{}^*(\mu) = t^{-n} f\, t^n \mu = f\mu. \tag{1.22}$$

So there exists a unique $(2n-1)$–form μ_f on B such that
$$f\mu = \pi^*\mu_f. \tag{1.23}$$
In fact, given a point $x \in B$, take $y \in Y$ such that $\pi(y) = x$; because of the invariance of $f\mu$ it does not depend on the choice of the point y, so we can choose $y \in Z$. Since $f\mu$ is horizontal, by (1.19) we can also choose a basis $V_1, \ldots, V_{2n-1}$ of T_yZ, and define the form μ_f by
$$\mu_f|_x(\pi_* V_1, \ldots, \pi_* V_{2n-1}) = f(y)\mu|_y(V_1, \ldots, V_{2n-1}).$$
If f is non-vanishing then by (1.23), μ_f is also non-vanishing, so we will orient B by requiring the form μ_f to be positively oriented when f is everywhere positive.

Definition 1.2.1 ([14])**.** Let f be a homogeneous function of degree $-n$ on Y. We define the *symplectic residue* of f to be the integral
$$\operatorname{res}(f) := \int_B \mu_f. \tag{1.24}$$
Since $\pi \restriction_Z : Z \to B$ is a diffeomorphism, we can orient B such that the degree of π is 1 and by the Degree Formula (see [16])
$$\int_B \mu_f = \int_Z (f\mu) \restriction_Z .$$
The symplectic residue has the following properties (see Prop. 6.1 in [14]):

Proposition 1.2.1. *1. The map* res *is linear and continuous (as a distribution) on the space $\mathcal{P}_{-n}$.*

2. Let $\Phi : Y \to Y$ be a symplectomorphism commuting with the action of $\mathbb{R}^+$. Then for $f \in \mathcal{P}_{-n}$, $\mathrm{res}(\Phi^* f) = \mathrm{res}(f)$.

3. If $g \in \mathcal{P}_l$, $f \in \mathcal{P}_m$ and $l + m = -n + 1$, then $\mathrm{res}(\{f, g\}) = 0$.

Proof. 1. If $f, g \in \mathcal{P}_{-n}$, and $c \in \mathbb{C}$,

$$\pi^* \mu_{f+c\,g} = (f + c\,g)\mu = f\mu + c\,g\,\mu = \pi^* \mu_f + c\,\pi^* \mu_g,$$

so the linearity holds as a consequence of this and the linearity of the integral over B. If $\{f_k\}$ is a sequence of functions in $\mathcal{P}_{-n}$ such that $f_k \to 0$ as $k \to \infty$ uniformly in the C^∞–topology of Y, then $\mathrm{res}(f_k) \to 0$ as $k \to \infty$.

2. If $\Phi : Y \to Y$ is a diffeomorphism which commutes with the action of $\mathbb{R}^+$, then there exists a diffeomorphism of the base $\Psi : B \to B$, such that $\pi \circ \Phi = \Psi \circ \pi$: For $x \in B$, let $\widehat{x} \in Y$ be such that $\pi(\widehat{x}) = x$, define $\Psi(x) := \pi(\Phi(\widehat{x}))$. Ψ is well-defined: another element in the fiber of x is of the form $\rho_t(\widehat{x})$ for some $t \in \mathbb{R}^+$, so

$$\pi(\Phi(\rho_t(\widehat{x}))) = \pi(\rho_t(\Phi(\widehat{x}))) = \pi(\Phi(\widehat{x})) = \Psi(x).$$

If $f \in \mathcal{P}_{-n}$, then, since Φ commutes with the action of $\mathbb{R}^+$, for any $t \in \mathbb{R}^+$, $\rho_t^*(\Phi^* f) = \Phi^*(\rho_t^* f) = t^{-n}(\Phi^* f)$, so $\Phi^* f \in \mathcal{P}_{-n}$.
Φ preserves $\mathcal{X}$: For $h \in C^\infty(Y)$ and for any $y \in Y$,

$$\begin{aligned}
(\Phi)_*(\mathcal{X}_y)(h) &= \mathcal{X}_y(h \circ \Phi) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\rho_{e^{-s}}(y)} (h \circ \Phi) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\Phi(\rho_{e^{-s}}(y))} (h) \\
&= \left.\frac{d(\rho_{e^s})}{ds}\right|_{\rho_{e^{-s}}(\Phi(y))} (h) \\
&= \left.\mathcal{X}\right|_{\Phi(y)} (h).
\end{aligned}$$

From this, and since $\Phi^*\omega = \omega$, we get $\Phi^*\mu = \mu$. Therefore, the equality $\pi\Phi = \Psi\pi$ implies that

$$\pi^*\Psi^*\mu_f = \Phi^*\pi^*\mu_f \; = \Phi^*(f\mu) \; = \Phi^*(f)\mu \; = \pi^*\mu_{\Phi^* f}$$

and hence $\Psi^*\mu_f = \mu_{\Phi^* f}$.
Hence, since Ψ is a diffeomorphism on B and its degree is equal to 1 (that is the degree of Φ since Φ is a symplectomorphism), we use again the Degree Formula to conclude

$$\mathrm{res}(f) = \int_B \mu_f = \int_B \Psi^*\mu_f = \int_B \mu_{\Phi^* f} = \mathrm{res}(\Phi^* f).$$

3. First consider the case $l = 1$ and $m = -n$. If $\{\Phi_t\}$ is the one-parameter group of symplectomorphisms generated by the Hamiltonian vector field X_g, by Remark 1.1.1 every Φ_t commutes with the action of $\mathbb{R}^+$, and therefore by the previous item, $\operatorname{res}(\Phi_t^* f) = \operatorname{res}(f)$. So by continuity of res, we differentiate and evaluate at zero to get:

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} \operatorname{res}(f) &= \left.\frac{d}{dt}\right|_{t=0} \operatorname{res}(\Phi_t^* f) \\ &= \operatorname{res}\left(\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* f \right) \\ &= \operatorname{res}(\mathcal{L}_{X_g}(f)) \\ &= -\operatorname{res}(\{f, g\}). \end{aligned}$$

The general case follows from Lemma 1.1.3:

$$\begin{aligned} \operatorname{res}(\{f, g\}) &= \int_B \mu_{\{f,g\}} = \int_Z \{f, g\}\, \mu \\ &= \int_Z \left(d(g\, \iota_{X_f} \mu) - (l + m - 1 + n)\, g\, df \wedge \omega^{n-1} \right). \end{aligned}$$

Hence by Stokes' Theorem, $\operatorname{res}(\{f, g\}) = 0$ when $l + m = -n + 1$.

□

1.3 L^2–structure on $\mathcal{P}_s$

Let us now introduce a pre-Hilbert space structure on $\mathcal{P}_s$. Following the notation of Section 1.1, since B and Z are diffeomorphic, we will consider the symplectic cone $\pi : Y \to Z$; we will also assume that Y is connected and Z is compact. We construct the following maps:

1. The restriction mapping, i.e. the map:

$$\begin{aligned} R_s : \mathcal{P}_s &\to C^\infty(Z) \\ f &\mapsto R_s(f) = f \restriction_Z . \end{aligned}$$

2. The extension mapping, i.e. the map:

$$\begin{aligned} &T_s : C^\infty(Z) \to \mathcal{P}_s \\ &\qquad g \mapsto T_s(g) \text{ such that } \forall y \in Y,\ T_s(g)(y) := g\left(\rho_{\frac{1}{p(y)}}(y)\right)(p(y))^s. \end{aligned}$$

These maps provide an identification

$$\mathcal{P}_s \cong C^\infty(Z), \tag{1.25}$$

since $R_s \circ T_s = id_{C^\infty(Z)}$ and $T_s \circ R_s = id_{\mathcal{P}_s}$.

Let ν be the restriction of μ to Z. Then ν is a volume form on Z: let $z \in Z$; since $T_z Z$ is of dimension $2n-1$, if $V_1, \ldots, V_{2n-1}$ form a basis of $T_z Z$, by (1.19), $\mathcal{X}, V_1, \ldots, V_{2n-1}$ form a basis of $T_z Y$, and hence

$$\nu_z(V_1, \ldots, V_{2n-1}) = \omega_z^n(\mathcal{X}, V_1, \ldots, V_{2n-1})$$

does not vanish since ω^n is a volume form on Y. Hence, ν defines an L^2-structure on $C^\infty(Z)$: for $f, g \in C^\infty(Z)$

$$\langle f, g \rangle_Z := \int_Z \bar{f} \cdot g \, \nu,$$

where $\bar{f}$ denotes the complex conjugate of the function $f : Z \to \mathbb{C}$. The associated norm is

$$\|f\| = \left(\langle f, f \rangle_Z\right)^{1/2},$$

and by completion we obtain a Hilbert space $L^2(Z)$. By means of (1.25), we have an L^2-structure on $\mathcal{P}_s$ with the following inner product: for $f, g \in \mathcal{P}_s$ we define

$$\langle f, g \rangle_s := \langle R_s(f), R_s(g) \rangle_Z = \int_Z \overline{f \restriction_Z} \cdot g \restriction_Z \ \nu = \int_Z \left(\bar{f} \cdot g\right) \restriction_Z \ \nu,$$

and the associated norm is

$$\|f\|_s := \|R_s(f)\|.$$

1.4 A differential operator on $\mathcal{P}_s$

In this section we study the representation of a homogeneous function on a symplectic cone in terms of Poisson brackets, by using a generalization of Sect. 6 of [14] (see also Sect. 2 of [25]) in which the case when $l = 1$ is investigated. We consider homogeneous functions on a symplectic cone $Y \to B$, and as before, we assume that Y is connected and B is compact. In the following result we study the case $l \neq 0$; the case $l = 0$ will be treated separately right after the proof of this theorem.

Theorem 1.4.1. *Let l, m be real numbers with $l \neq 0$.*

1. *If $l + m - 1 \neq -n$, then $\{\mathcal{P}_l, \mathcal{P}_m\} = \mathcal{P}_{l+m-1}$.*
2. *If $l + m - 1 = -n$, then $\{\mathcal{P}_l, \mathcal{P}_m\} = \ker(\mathrm{res}) \subseteq \mathcal{P}_{-n}$ and $\{\mathcal{P}_l, \mathcal{P}_m\}$ is of codimension 1 on $\mathcal{P}_{-n}$.*

Proof. We will prove the theorem in several steps:

1. Definition of the operators D_q.
   Given $q \in \mathcal{P}_l$, let X_q be the Hamiltonian vector field associated with q. Let $D_q : \mathcal{P}_m \to \mathcal{P}_{l+m-1}$ be the operator "differentiation by X_q": $D_q(h) = \mathcal{L}_{X_q}(h) = \{q, h\}$.

By (1.20), every vector field on Y can be expressed as a linear combination of some vector field in $\ker(dp)$ and $\mathcal{X}$. In particular, the vector X_q decomposes uniquely as a sum

$$X_q = W_q + \widetilde{q}\,\mathcal{X}, \tag{1.26}$$

where W_q is a vector field on Y such that $W_q[p] = 0$ and $\widetilde{q} \in C^\infty(Y)$.

From (1.26) and using (1.16) we get $X_q[p] = W_q[p] + \widetilde{q}\,\mathcal{X}[p] = p\,\widetilde{q}$, therefore

$$\widetilde{q} = p^{-1} X_q[p] = p^{-1}\{q, p\} \in \mathcal{P}_{l-1}. \tag{1.27}$$

By (1.16), if $f \in \mathcal{P}_m$, $\mathcal{X}[f] = mf$ and hence,

$$\begin{aligned} X_q[f] &= W_q[f] + \widetilde{q}\,\mathcal{X}[f] \\ D_q(f) &= W_q[f] + m\,\widetilde{q} f. \end{aligned}$$

2. Construction of the transpose of D_q.
First of all, by Prop. 6.5.17 in [1], since ν is a volume form on Z, for any $h \in C^\infty(Z)$ and for any vector field X on Z we have the identity

$$d(\iota_X(h\nu)) = X[h]\nu + h\,\mathrm{Div}(X)\nu, \tag{1.28}$$

where $\mathrm{Div}(X)$ represents the divergence of X. Let $f \in \mathcal{P}_m,\ g \in \mathcal{P}_{l+m-1}$. Taking $h = \overline{f} \cdot g$ and $X = W_q$, we have

$$\begin{aligned}
\langle D_q f, g\rangle_{l+m-1} &= \langle R_{l+m-1}(D_q f), R_{l+m-1}(g)\rangle_Z = \int_Z \left(\overline{D_q f} \cdot g\right) \restriction_Z \nu \\
&= \int_Z \left(\overline{(W_q[f] + m\,\widetilde{q}\,f)} \cdot g\right) \restriction_Z \nu \\
&= \int_Z \left(\overline{W_q[f]} \cdot g\right) \restriction_Z \nu + \int_Z \left(\overline{m\,\widetilde{q}\,f} \cdot g\right) \restriction_Z \nu \\
&= \int_Z \left(\overline{W_q[f]} \cdot g\right) \restriction_Z \nu + \int_Z \left(\overline{f} \cdot m\,\widetilde{q}\,g\right) \restriction_Z \nu \\
&\overset{(1.28)}{=} - \int_Z \left(\overline{f} \cdot W_q[g]\right) \restriction_Z \nu - \int_Z \left(\overline{f} \cdot g\,\mathrm{Div}(W_q)\right) \restriction_Z \nu + \int_Z \left(\overline{f} \cdot m\,\widetilde{q}\,g\right) \restriction_Z \nu \\
&= - \int_Z \left(\overline{f} \cdot (W_q[g] + (\mathrm{Div}(W_q))g - m\,\widetilde{q}\,g)\right) \restriction_Z \nu \\
&= \langle T_m(f \restriction_Z), T_m((-W_q - \mathrm{Div}(W_q) + m\,\widetilde{q})(g) \restriction_Z)\rangle_m = \langle f, D_q^t(g)\rangle_m.
\end{aligned}$$

Hence the transpose of D_q is the operator $D_q^t : \mathcal{P}_{l+m-1} \to \mathcal{P}_m$ given by

$$D_q^t(g) = T_m((-W_q - \mathrm{Div}(W_q) + m\,\widetilde{q})(g) \restriction_Z). \tag{1.29}$$

So we must compute $\mathrm{Div}(W_q)$. We will show that

$$\mathrm{Div}(W_q) = (-n + 1 - l)\,\widetilde{q}. \tag{1.30}$$

By Equation (1.21) $d(p^{-n}\mu) = 0$, and hence by (1.12)

$$0 = d(p^{-n}\mu) = d(p^{-n}) \wedge \mu + p^{-n} d\mu = -n\, p^{-n-1} dp \wedge \mu + p^{-n}\omega^n$$

and on Z, we get

$$w^n = n\, dp \wedge \nu. \tag{1.31}$$

By definition $\mathcal{L}_{W_q}\nu = \mathrm{Div}(W_q)\nu$ (see Def. 6.5.16 in [1]). Therefore, in view of the fact that $W_q[p] = 0$, we get from (1.31)

$$\begin{aligned} \mathcal{L}_{W_q}\omega^n &= \mathcal{L}_{W_q}(n\, dp \wedge \nu) \\ &= n\, d(\mathcal{L}_{W_q}(p)) \wedge \nu + n\, dp \wedge \mathcal{L}_{W_q}(\nu) \\ &= n\, dp \wedge \mathrm{Div}(W_q)\nu \\ &= \mathrm{Div}(W_q)\, \omega^n. \end{aligned} \tag{1.32}$$

For any function $h \in C^\infty(Y)$ and any vector field W on Y, we have $\mathrm{Div}(hW) = h\, \mathrm{Div}(W) + W[h]$ (see Prop. 6.5.17 in [1]); therefore,

$$\begin{aligned} \mathcal{L}_{W_q}\omega^n &\overset{(1.26)}{=} \mathcal{L}_{X_q}\omega^n - \mathcal{L}_{(\widetilde{q}\mathcal{X})}\omega^n \\ &\overset{(1.17)}{=} -\mathrm{Div}(\widetilde{q}\mathcal{X})\, \omega^n \\ &= -\mathcal{X}[\widetilde{q}]\, \omega^n - \widetilde{q}\, \mathcal{L}_{\mathcal{X}}\omega^n \\ &\overset{(1.15),(1.16)}{=} (1-l-n)\, \widetilde{q}\, \omega^n. \end{aligned}$$

Hence, by (1.32) on Z we have $\mathrm{Div}(W_q) = (1-l-n)\, \widetilde{q}$.

Going back to (1.29) we get for the transpose of D_q, for $g \in \mathcal{P}_{l+m-1}$

$$\begin{aligned} D_q^t(g) &= T_m((-W_q - \mathrm{Div}(W_q) + m\, \widetilde{q})(g) \restriction_Z) \\ &= T_m((-W_q - (1-l-n)\, \widetilde{q} + m\, \widetilde{q})(g) \restriction_Z) \\ &= p^{2-2l}\left(-D_q + (l+m-1)\, \widetilde{q} - (1-l-n)\, \widetilde{q} + m\, \widetilde{q}\right)(g), \end{aligned}$$

$$D_q^t = -p^{2-2l}\left(D_q - (2m+2l+n-2)\, \widetilde{q}\right). \tag{1.33}$$

Now, since $\{q, p^r g\} = p^r(\{q,g\} + r\, p^{-1}\{q,p\}\, g)$, if $r = -(2m+2l+n-2)$, by (1.27), Equation (1.33) implies that

$$\begin{aligned} D_q^t(g) &= -p^{2-2l}\left(D_q(g) + r\, \widetilde{q}\, g\right) \\ &= -p^{2-2l}\left(\{q,g\} + r\, p^{-1}\{q,p\}\, g\right) \\ &= -p^{2-2l} p^{-r}\{q, p^r g\} \\ &= -p^{2m+n}\{q, p^r g\}. \end{aligned} \tag{1.34}$$

3. Construction of the operator Δ_{l+m-1}.
   Choose a system of functions $g_1, \ldots, g_N$ in $\mathcal{P}_l$ whose differentials span the

cotangent space at every point of Y. By non-degeneracy of ω, this implies that the Hamiltonian vector fields X_{g_i} generate the tangent space T_yY at every point y of Y. We denote by D_i the operator D_{g_i}.
Consider the linear operator

$$\Delta_{l+m-1} = \sum_{i=1}^{N} D_i D_i^t. \tag{1.35}$$

The theorem is an immediate consequence of the following lemma

Lemma 1.4.1. *When $l+m-1 \neq -n$, $\Delta_{l+m-1} : \mathcal{P}_{l+m-1} \to \mathcal{P}_{l+m-1}$ is bijective, and when $l+m-1 = -n$,* $\mathrm{Image}(\Delta_{l+m-1}) = \ker(\mathrm{res})$.

Proof. Clearly, Δ_{l+m-1} is a self-adjoint operator on $\mathcal{P}_{l+m-1}$. To compute the leading symbol (see [43]) of Δ_{l+m-1}, take $y \in Y, \xi \in T_y^*Y$, and choose $f \in \mathcal{P}_{l+m-1}$ such that $f(y) = 0$ and $df_y = \xi$, then

$$\sigma_{D_i}(\xi) = D_i f_y = \{g_i, f\}_y = (df)_y(X_{g_i}) = \xi(X_{g_i}),$$

which implies that Δ_{l+m-1} has leading symbol $\sigma_{D_i^2}(\xi) = |\xi(X_{g_i})|^2$.
Therefore Δ_{l+m-1} is elliptic and following Thm. 5.5 of Chap. III in [24], we can conclude that $\mathrm{Image}(\Delta_{l+m-1})$ is closed and hence

$$\mathcal{P}_{l+m-1} = \ker(\Delta_{l+m-1}) \oplus \mathrm{Image}(\Delta_{l+m-1}). \tag{1.36}$$

Let us compute $\ker(\Delta_{l+m-1})$: for all $f \in \mathcal{P}_{l+m-1}$,

$$\langle \Delta_{l+m-1} f, f\rangle_{l+m-1} = \sum_{i=1}^{N} \langle D_i D_i^t f, f\rangle_{l+m-1} = \sum_{i=1}^{N} \| D_i^t f \|_m^2,$$

so $f \in \ker(\Delta_{l+m-1})$ if and only if $f \in \ker(D_i^t)$ for all i.

In view of (1.34), for all $i = 1, \dots, N$, $f \in \ker(D_i^t)$ if and only if on Y we have

$$X_{g_i}[p^r f] = \{g_i, p^r f\} = 0. \tag{1.37}$$

Since the X_{g_i}'s span the tangent space to Y at every point and Y is connected, $p^r f$ must be a constant, $p^r f = c \;\Rightarrow\; f = c\,p^{-r}$.

Remember that $f \in \mathcal{P}_{l+m-1}$ and that $-r = l+m-1 \Rightarrow l+m-1 = -n$. Hence

- If $l+m-1 \neq -n$, the constant c must be 0 and by (1.36), Δ_{l+m-1} is bijective, i.e. $\mathrm{Image}(\Delta_{l+m-1}) = \mathcal{P}_{l+m-1}$, and from

  $$\mathrm{Image}(\Delta_{l+m-1}) \subseteq \{\mathcal{P}_l, \mathcal{P}_m\} \subseteq \mathcal{P}_{l+m-1},$$

  we get

  $$\mathrm{Image}(\Delta_{l+m-1}) = \{\mathcal{P}_l, \mathcal{P}_m\} = \mathcal{P}_{l+m-1}. \tag{1.38}$$

- If $l+m-1=-n$, $f=c\,p^{-n}$; in this case, the space generated by p^{-n} coincides with $\ker(D_i^t)$ for all $i=1,\ldots,N$, i.e. $\ker(\Delta_{l+m-1})$ has dimension 1.
  If $f=p^{-n}$ then for all $g\in\mathcal{P}_{-n}$, the L^2-inner product of f and g is
  $$\langle f,g\rangle_{-n}=\int_Z g\,\mu=\int_B \mu_g=\mathrm{res}(g).$$
  By Proposition 1.2.1, we have
  $$\mathrm{Image}(\Delta_{l+m-1})\subseteq\{\mathcal{P}_l,\mathcal{P}_m\}\subseteq\ker(\mathrm{res}), \tag{1.39}$$
  so, with the short exact sequence
  $$0\to\ker(\mathrm{res})\to\mathcal{P}_{-n}\to\mathbb{C}\to 0,$$
  where the first map is the inclusion and the second map is res, the space ker(res) has codimension 1 on $\mathcal{P}_{-n}$, and therefore
  $$\mathrm{Image}(\Delta_{l+m-1})=\ker(\mathrm{res}). \tag{1.40}$$

□

When $l+m-1=-n$, the theorem follows from (1.39) and (1.40), otherwise, it follows from (1.38).

□

The differential operator in degree zero

Let us consider the construction of the operator Δ_{l+m-1}, when $l=0$:

In the proof of Theorem 1.4.1, we needed a system of functions $g_1,\ldots,g_N$ in $\mathcal{P}_l$ whose differentials span the cotangent space at every point of Y. In the case $l=0$ it is not possible to choose such a system: the dimension of a set generated by differentials of homogeneous functions of degree zero is at most $2n-1$, because they will not generate the part in the cotangent space corresponding to the fiber variable[1]. Instead, in the proof of Proposition 1.5.4 below, we choose a system of functions $g_1,\ldots,g_N$ in $\mathcal{P}_0\cong C^\infty(Z)$, whose differentials span the cotangent space of Z at every point, i.e. for all $z\in Z$, $\mathrm{span}_{1\le i\le N}\{\,dg_i|_z\}=T_z^*Z$.

For any integer $m\le 0$, we denote by D_i the operator $\{g_i,\cdot\}$ from $\mathcal{P}_m$ to $\mathcal{P}_{m-1}$. If $r=-2m-n+2=-2(m-1)-n$, by (1.34) the transpose D_i^t corresponds to the operator $-p^{2m+n}\{g_i,p^r\cdot\}$. Define the linear operator

$$\begin{aligned}\Delta_{m-1}:\mathcal{P}_{m-1}&\to\mathcal{P}_{m-1}\\ f&\mapsto\Delta_{m-1}(f):=\sum_{i=1}^N D_iD_i^t(f).\end{aligned}$$

[1] I thank Prof. Jean-Marie Lescure for pointing this out to me.

We are going to compute $\ker(\Delta_{m-1})$, i.e. the functions $f \in \mathcal{P}_{m-1}$ such that $\Delta_{m-1}(f) = 0$:

$$\begin{aligned}\Delta_{m-1}(f) = 0 &\Leftrightarrow D_i^t(f) = 0 \ \ \forall i = 1, \ldots, N\\ &\Leftrightarrow \{g_i, p^r f\} = 0 \ \ \forall i = 1, \ldots, N\\ &\Leftrightarrow \omega(X_{g_i}, X_{p^r f}) = 0 \ \ \forall i = 1, \ldots, N\\ &\Leftrightarrow dg_i(X_{p^r f}) = 0 \ \ \forall i = 1, \ldots, N.\end{aligned}$$

Since the differentials of $g_1, \ldots, g_N$ span the cotangent space of Z at every point z, $X_{p^r f}|_z$ does not belong to $T_z Z$. Moreover, since $g_i \in \mathcal{P}_0$, $\forall i = 1, \ldots, N$, by (1.16) we also have

$$dg_i(\mathcal{X}) = \mathcal{X}[g_i] = 0.$$

Therefore, by (1.20) we can pick a function $h \in C^\infty(Y)$ such that

$$X_{p^r f} = h\,\mathcal{X}.$$

Let $A := \{g \in C^\infty(Y) : \exists\, h \in C^\infty(Y) : \ X_g = h\,\mathcal{X}\}$. Then

$$\ker(\Delta_{m-1}) = p^{-r}\left(A \cap \mathcal{P}_{-m+1-n}\right). \tag{1.41}$$

Proposition 1.4.1. *Let a be a real number $a \leq 0$.*

1. *If $a \neq 0$, then $A \cap \mathcal{P}_a = \{0\}$.*
2. *For $g \in \mathcal{P}_0$ and $f \in A \cap \mathcal{P}_a$, $\{f, g\} = 0$.*
3. *If $n = 1$, then $\{\mathcal{P}_0, \mathcal{P}_0\} = 0$.*

Proof.

1. If $f \in \mathcal{P}_a$, then $\mathcal{X}[f] = af$, and if $f \in A$, there exists a function $h \in C^\infty(Y)$ such that $X_f = h\,\mathcal{X}$. Then

$$af = \mathcal{X}[f] = \alpha(X_f) = -\omega(\mathcal{X}, X_f) = -\omega(\mathcal{X}, h\,\mathcal{X}) = -h\,\omega(\mathcal{X}, \mathcal{X}) = 0,$$

and since $a \neq 0$, we conclude that $f = 0$.

2. If $f \in A$, there exists a function $h \in C^\infty(Y)$ such that $X_f = h\,\mathcal{X}$, then by (1.16), for any $g \in \mathcal{P}_0$ we have

$$\{f, g\} = X_f[g] = h\,\mathcal{X}[g] = h \cdot 0 = 0.$$

In particular, $\{f, g\} = 0$ whenever $g \in \mathcal{P}_0$ and $f \in A \cap \mathcal{P}_0$; when $f \in A \cap \mathcal{P}_a$ and $a \neq 0$ the statement follows immediately from the first part of the proposition.

3. By Equation (1.3) with $n = 1$, $\{f, g\}\,\omega = df \wedge dg$, so if $f, g \in \mathcal{P}_0$ we have $\{f, g\} = 0$.

$\square$

Lemma 1.4.2. *If $n \geq 2$, there exists a system of functions $g_i \in \mathcal{P}_0$, $1 \leq i \leq N$, such that for some $i \neq j$, $\{g_i, g_j\} \in \mathcal{P}_{-1}$.*

Proof. We prove this locally: consider that the manifold structure on the symplectic cone Y is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and the corresponding symplectic form is the 2–form $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$. For $i = 1, \ldots, n$, define functions $g_i(x, \xi) := x_i$, and $g_{n+i}(x, \xi) := \xi_i |\xi|^{-1}$, which are homogeneous of degree 0 on Y. Then for $i, j = 1, \ldots, n$,

$$\{g_i, g_j\} = 0 = \{g_{n+i}, g_{n+j}\},$$

but

$$\{g_i, g_{n+i}\} = |\xi|^{-1} - \xi_i^2 |\xi|^{-3},$$

which is not identically zero when $n \geq 2$. □

Definition 1.4.1 (See e.g. [20])**.** Let M be a smooth manifold of dimension n. A linear partial differential operator P on $C^\infty(M)$ is called hypoelliptic if for every distribution u on M, the condition $Pu \in C^\infty(M)$ implies that $u \in C^\infty(M)$.

Theorem 1.4.2 (Thm. 1.1 in [18])**.** *If $X_1, \ldots, X_r$ denote first order homogeneous differential operators in an open set $\Omega \subset \mathbb{R}^n$ with smooth coefficients, such that among the operators*

$$X_{j_1},\ [X_{j_1}, X_{j_2}],\ [X_{j_1}, [X_{j_2}, X_{j_3}]], \ldots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \ldots, X_{j_k}]]], \ldots$$

where $j_i = 1, \ldots, r$, there exist n which are linearly independent at any given point in Ω, then the operator $\sum_{i=1}^{r} X_i^2$ is hypoelliptic.

This theorem also holds when considering the composition of X_i with its adjoint $X_i^* X_i$ instead of X_i^2, as we can see in [20] (see also [12], [17]). In our case, the second order differential operator Δ_{m-1} is constructed from N first order homogeneous differential operators $X_1, \ldots, X_N$, which generate a subspace of TY of dimension $2n-1$. These operators correspond to homogeneous functions $g_1, \ldots, g_N$ of degree zero.
By Lemma 1.4.2, there exist i, j such that the function $\{g_i, g_j\}$ is homogeneous of degree -1, in which case the differentials of the functions $g_i, \{g_i, g_j\}$ generate the cotangent space of Y at every point. Hence, by non-degeneracy of ω and by (1.1), the vector fields $X_{g_i},\ X_{\{g_i, g_j\}} = [X_{g_i}, X_{g_j}]$ generate the tangent space of Y at every point and hence they satisfy the hypothesis in Theorem 1.4.2. So Δ_{m-1} is a hypoelliptic operator.

Remark 1.4.1. For any $m \neq 0$, the space $\{\mathcal{P}_0, \mathcal{P}_m\}$ coincides with the space $\{\mathcal{P}_m, \mathcal{P}_0\}$, so when $n > 1$ and $m - 1 = -n$, by Theorem 1.4.1,

$$\{\mathcal{P}_0, \mathcal{P}_{-n+1}\} = \ker(\text{res}).$$

Therefore, the remaining case is when $n > 1$ and $m = 0$, but then $-m+1-n \neq 0$, so by Equation (1.41) and Proposition 1.4.1, we get

$$\ker(\Delta_{-1}) = \{0\}. \tag{1.42}$$

By construction of the operator $\Delta_{-1} : \mathcal{P}_{-1} \to \mathcal{P}_{-1}$, we have

$$\mathrm{Image}(\Delta_{-1}) \subseteq \{\mathcal{P}_0, \mathcal{P}_0\} \subseteq \mathcal{P}_{-1}. \tag{1.43}$$

Remark 1.4.2. We do not know if it is possible to use the fact that Δ_{-1} is hypoelliptic, to argue as in (1.36) and say that

$$\mathcal{P}_{-1} = \ker(\Delta_{-1}) \oplus \mathrm{Image}(\Delta_{-1}), \tag{1.44}$$

hence, by (1.42),

$$\mathcal{P}_{-1} = \mathrm{Image}(\Delta_{-1}), \tag{1.45}$$

in which case by (1.43) we could conclude that

$$\mathcal{P}_{-1} = \{\mathcal{P}_0, \mathcal{P}_0\}. \tag{1.46}$$

An explicit description of Δ in degree 0

Let M be a closed manifold of dimension $n > 1$. If the manifold structure on M is described by coordinate charts $(\mathcal{U}, x_1, \ldots, x_n)$ with $x_i : \mathcal{U} \to \mathbb{R}$, then at any $x \in \mathcal{U}$ the differentials $(dx_i)_x$ form a basis of T_x^*M. Namely if $\xi \in T_x^*M$, then $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some real coefficients $\xi_1, \ldots, \xi_n$ (see [8]). The chart $(T^*\mathcal{U}, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ is a coordinate chart for T^*M, and the coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ are the cotangent coordinates associated to $(x_1, \ldots, x_n)$ on $\mathcal{U}$. The 2–form $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$ is the canonical symplectic form and we have the symplectic cone $T^*M \setminus M \to S^*M$ (see Example 1.1.2).

The function $p(x, \xi) := |\xi|$ is a homogeneous function on $Y := T^*M \setminus M$ of degree 1, everywhere positive. For $i = 1, \ldots, n$, define functions $g_i(x, \xi) := x_i$, and $g_{n+i}(x, \xi) := \xi_i |\xi|^{-1}$, which are homogeneous of degree 0 on Y. They satisfy:

$$dg_i = dx_i \text{ and } dg_{n+i} = |\xi|^{-1} \left(-\sum_{k \neq i} \xi_k \xi_i |\xi|^{-2} d\xi_k + (1 - \xi_i^2 |\xi|^{-2}) d\xi_i \right).$$

The Hamiltonian vector field corresponding to a function g is

$$X_g = \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i} \frac{\partial}{\partial \xi_i} - \frac{\partial g}{\partial \xi_i} \frac{\partial}{\partial x_i} \right).$$

Therefore, for $i = 1, \ldots, n$,

$$X_{g_i} = \frac{\partial}{\partial \xi_i} \text{ and } X_{g_{n+i}} = -|\xi|^{-1} \left(-\sum_{k \neq i} \xi_k \xi_i |\xi|^{-2} \frac{\partial}{\partial x_k} + (1 - \xi_i^2 |\xi|^{-2}) \frac{\partial}{\partial x_i} \right).$$

By (1.34), the operators $D_i^t : \mathcal{P}_{m-1} \to \mathcal{P}_m$ are given by

$$D_i^t(h) = -|\xi|^{2m+n} \{g_i, |\xi|^{-2m-n+2} h\} \text{ for } h \in \mathcal{P}_{m-1}.$$

Therefore, for $i = 1, \dots, n$,

$$\begin{aligned} D_i^t(h) &= -|\xi|^{2m+n}\partial_{\xi_i}(|\xi|^{-2m-n+2}h) \\ &= (2m+n-2)\xi_i h - |\xi|^2\partial_{\xi_i}h; \end{aligned}$$

$$\begin{aligned} D_iD_i^t(h) &= \partial_{\xi_i}((2m+n-2)\xi_i h - |\xi|^2\partial_{\xi_i}h) \\ &= (2m+n-2)h + (2m+n-4)\xi_i\partial_{\xi_i}h - |\xi|^2\partial_{\xi_i}^2 h; \end{aligned} \tag{1.47}$$

$$\begin{aligned} &D_{n+i}^t(h) \\ &= |\xi|^{2m+n-1}\left(-\sum_{k\neq i}\xi_k\xi_i|\xi|^{-2}\partial_{x_k}(|\xi|^{-2m-n+2}h) + (1-\xi_i^2|\xi|^{-2})\partial_{x_i}(|\xi|^{-2m-n+2}h)\right) \\ &= |\xi|\left(-\sum_{k\neq i}\xi_k\xi_i|\xi|^{-2}\partial_{x_k}h + (1-\xi_i^2|\xi|^{-2})\partial_{x_i}h\right); \end{aligned}$$

$$\begin{aligned} &D_{n+i}D_{n+i}^t(h) \\ &= -|\xi|^{-1}\left(-\sum_{k\neq i}\xi_k\xi_i|\xi|^{-2}\partial_{x_k}(D_{n+i}^t(h)) + (1-\xi_i^2|\xi|^{-2})\partial_{x_i}(D_{n+i}^t(h))\right) \\ &= \sum_{j\neq i}\xi_j\xi_i|\xi|^{-2}\left(-\sum_{k\neq i}\xi_k\xi_i|\xi|^{-2}\partial_{x_j}\partial_{x_k}h + (1-\xi_i^2|\xi|^{-2})\partial_{x_j}\partial_{x_i}h\right) \\ &\quad - (1-\xi_i^2|\xi|^{-2})\left(-\sum_{k\neq i}\xi_k\xi_i|\xi|^{-2}\partial_{x_i}\partial_{x_k}h + (1-\xi_i^2|\xi|^{-2})\partial_{x_i}^2 h\right). \end{aligned} \tag{1.48}$$

Adding up the terms in (1.47) and (1.48) we obtain

$$\begin{aligned} \Delta(h) &= n(2m+n-2)h + (2m+n-4)(m-1)h - |\xi|^2\Delta_\xi h \\ &\quad - (A_1, \dots, A_n)\operatorname{Hess}_x(h)(A_1, \dots, A_n)^t \\ &= (2m^2+n^2+3mn-3n-6m+4)h - |\xi|^2\Delta_\xi h \\ &\quad - (A_1, \dots, A_n)\operatorname{Hess}_x(h)(A_1, \dots, A_n)^t, \end{aligned}$$

where we have used the following:

- since h is homogeneous of degree $m-1$, by (1.16) we have

$$\sum_{i=1}^n \xi_i\partial_{\xi_i}h = (m-1)h.$$

- $\Delta_\xi h = \sum_{i=1}^{n} \partial^2_{\xi_i} h.$

- $(A_1, \ldots, A_n) = (1, \ldots, 1) - |\xi|^{-2} \begin{pmatrix} \xi_1 \, \xi_1 \cdots \xi_1 \\ \xi_2 \, \xi_2 \cdots \xi_2 \\ \cdots\cdots \\ \xi_n \xi_n \cdots \xi_n \end{pmatrix} \cdot (\xi_1, \ldots, \xi_n)^t.$

- $\mathrm{Hess}_x(h)$ denotes the Hessian matrix in x of h:

$$\mathrm{Hess}_x = \begin{pmatrix} \partial^2_{x_1} & \partial_{x_1}\partial_{x_2} \cdots \partial_{x_1}\partial_{x_n} \\ \partial_{x_2}\partial_{x_1} & \partial^2_{x_2} \quad \cdots \partial_{x_2}\partial_{x_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_{x_n}\partial_{x_1} & \partial_{x_n}\partial_{x_2} \cdots \quad \partial^2_{x_n} \end{pmatrix}.$$

In the case when $m = 0$, the operator reads

$$\Delta(h) = (n^2 - 3n + 4)h - |\xi|^2 \Delta_\xi h - (A_1, \ldots, A_n)\, \mathrm{Hess}_x(h)(A_1, \ldots, A_n)^t,$$

but we cannot see from this that the operator Δ is hypoelliptic and then use it to conclude (1.46). In Proposition 1.5.4 below, we present the proof of (1.46) by using a different argument.

1.5 Homogeneous differential forms

In this section we study homogeneous differential forms on the symplectic cone $\pi : Y \to Z$, in order to give a more explicit representation of a homogeneous function in terms of Poisson brackets, adding the case $l = 0$ to Theorem 1.4.1.

Let us consider the set $\Omega^{k,a}(Y)$ of a–homogeneous k–differential forms on Y, that is $\eta \in \Omega^{k,a}(Y)$ if and only if for all $t \in \mathbb{R}^+$,

$$\rho_t^* \eta = t^a \eta,$$

where $\rho_t^* \eta$ is defined as in (1.4). The usual exterior derivation maps

$$d : \Omega^{k,a}(Y) \to \Omega^{k+1,a}(Y),$$

and $(\Omega^{k,a}(Y),\, d)$ is a subcomplex of the usual de Rham complex $(\Omega^k(Y), d)$. We can compute the cohomology of this complex:

$$H^{k,a}(Y) = \frac{\ker(d : \Omega^{k,a}(Y) \to \Omega^{k+1,a}(Y))}{\mathrm{Image}(d : \Omega^{k-1,a}(Y) \to \Omega^{k,a}(Y))}.$$

Proposition 1.5.1. *The homogeneous cohomology groups of the symplectic cone $\pi : Y \to Z$ are given by:*

$$H^{k,a}(Y) \cong \begin{cases} H^k(Z) \oplus H^{k-1}(Z), & \text{if } a = 0, \\ 0, & \text{if } a \neq 0. \end{cases}$$

In particular, $H^{2n,0}(Y) \cong \mathbb{C}$.

Proof. By (1.25), locally any a–homogeneous k–differential form $\omega \in \Omega^{k,a}(Y)$ can be expressed as

$$\omega = p^a \pi^*(\omega_1) + p^{a-1}\, dp \wedge \pi^*(\omega_2), \tag{1.49}$$

for some $\omega_1 \in \Omega^k(Z)$, $\omega_2 \in \Omega^{k-1}(Z)$, and therefore

$$d\omega = p^a \pi^*(d\omega_1) + p^{a-1}\, dp \wedge (-\pi^*(d\omega_2 - a\,\omega_1)). \tag{1.50}$$

If $d\omega = 0$ then

$$d\omega_1 = 0 \text{ and } d\omega_2 = a\,\omega_1.$$

If $a \neq 0$, $\omega_1 = \frac{1}{a} d\omega_2$ and hence

$$\omega = \frac{1}{a} p^a \pi^*(d\omega_2) + p^{a-1} dp \wedge \pi^*(\omega_2) = d\left(\frac{1}{a} p^a \pi^*(\omega_2)\right).$$

Since $\frac{1}{a} p^a \pi^*(\omega_2) \in \Omega^{k-1,a}(Y)$, we conclude that if $a \neq 0$, then $H^{k,a}(Y) = 0$.

If $a = 0$ we have $\omega = \pi^*(\omega_1) + p^{-1}\, dp \wedge \pi^*(\omega_2) \in \Omega^{k,0}(Y)$, and by (1.50), $d\omega = 0$ implies that $d\omega_1 = 0$ and $d\omega_2 = 0$, so $[\omega_1] \in H^k(Z)$, $[\omega_2] \in H^{k-1}(Z)$. If $\tau = \pi^*(\tau_1) + p^{-1}\, dp \wedge \pi^*(\tau_2) \in \Omega^{k-1,0}(Y)$ is such that $\omega = d\tau$, then $\omega_1 = d\tau_1$ and $\omega_2 = d\tau_2$. This gives rise to a map

$$\begin{aligned} H^{k,0}(Y) &\to H^k(Z) \oplus H^{k-1}(Z) \\ [\omega] &\mapsto [\omega_1] \oplus [\omega_2], \end{aligned}$$

which is an isomorphism. Indeed, if $\omega_1 = d\alpha_1$, $\omega_2 = d\alpha_2$, then

$$\omega = \pi^*(d\alpha_1) + p^{-1} dp \wedge \pi^*(d\alpha_2) = d\left(\pi^*(\alpha_1) - p^{-1} dp \wedge \pi^*(\alpha_2)\right).$$

In particular, $H^{2n,0}(Y) \cong H^{2n-1}(Z)$, and since Z is an oriented $(2n-1)$–dimensional manifold, we have $H^{2n-1}(Z) \cong \mathbb{C}$ where the isomorphism is given by integration over Z. □

By definition of a symplectic cone (Definition 1.1.1), the symplectic form $\omega \in \Omega^{2,1}(Y)$ is a 1–homogeneous 2–form on Y, and $\omega^n \in \Omega^{2n,n}(Y)$ is a volume form on Y. By (1.6), $\alpha = \iota_{\mathcal{X}}\omega \in \Omega^{1,1}(Y)$ satisfies $\omega = d\alpha$.

Proposition 1.5.2. *For any $m \in \mathbb{R}$ and any $l \neq 0$, such that $l + m - 1 \neq -n$, $\{\mathcal{P}_l, \mathcal{P}_m\} = \mathcal{P}_{l+m-1}$.*

Proof. Let $f \in \mathcal{P}_{l+m-1}$. The differential form $f\omega^n \in \Omega^{2n,n+l+m-1}(Y)$ is closed and by Proposition 1.5.1 it is exact; indeed, by Lemma 1.1.1, the differential form $f\iota_{\mathcal{X}}\omega^n \in \Omega^{2n-1,n+l+m-1}(Y)$ satisfies

$$d(f\iota_{\mathcal{X}}\omega^n) = (n + l + m - 1) f\omega^n. \tag{1.51}$$

From (1.14)

$$\iota_{\mathcal{X}}\omega^n = n\,\alpha \wedge \omega^{n-1}, \tag{1.52}$$

and by (1.7)

$$\omega^{n-1} = d(\alpha \wedge \omega^{n-2}). \tag{1.53}$$

Choose a system of functions $g_1, \dots, g_N$ in $\mathcal{P}_l$ such that at every point y of Y, their differentials $dg_1|_y, \dots, dg_N|_y$ are linearly independent and span the cotangent space $T^*_y Y$. Since the linear form $f\alpha$ belongs to $\Omega^{1,l+m}(Y)$, there exist functions $f_1, \dots, f_N \in C^\infty(Y)$ such that

$$f\alpha = \sum_{i=1}^{N} f_i \, dg_i. \tag{1.54}$$

For all $i = 1, \dots, N$, $f_i \in \mathcal{P}_m$: from the homogeneity of $f\alpha$ and g_i, for all $t \in \mathbb{R}^+$, it follows that

$$\rho_t^*(f\alpha) = t^{l+m} f\alpha = t^l \sum_{i=1}^{N} \rho_t^*(f_i) \, dg_i.$$

From this,

$$\sum_{i=1}^{N} (\rho_t^*(f_i) - t^m f_i) \, dg_i = 0$$

and from the linear independence of $dg_1, \dots, dg_N$ at every point of Y we have $\rho_t^*(f_i) = t^m f_i$, so $f_i \in \mathcal{P}_m$ for all $i = 1, \dots, N$. Moreover,

$$\begin{aligned}
f\,\omega^n &\overset{(1.51)}{=} \frac{1}{n+l+m-1} d(f \iota_{\mathcal{X}}(\omega^n)) \\
&\overset{(1.52)}{=} \frac{n}{n+l+m-1} d(f\alpha \wedge \omega^{n-1}) \\
&= \frac{n}{n+l+m-1} d(d(f\alpha) \wedge \alpha \wedge \omega^{n-2}) \\
&= \frac{n}{n+l+m-1} d(f\alpha) \wedge d(\alpha \wedge \omega^{n-2}) \\
&\overset{(1.53)}{=} \frac{n}{n+l+m-1} d(f\alpha) \wedge \omega^{n-1} \\
&= \frac{n}{n+l+m-1} d\left(\sum_{i=1}^{N} f_i \, dg_i\right) \wedge \omega^{n-1} \\
&= \frac{n}{n+l+m-1} \sum_{i=1}^{N} df_i \wedge dg_i \wedge \omega^{n-1} \\
&\overset{(1.3)}{=} \frac{1}{n+l+m-1} \sum_{i=1}^{N} \{f_i, g_i\} \, \omega^n.
\end{aligned}$$

Therefore

$$f = \frac{-1}{n+l+m-1} \sum_{i=1}^{N} \{g_i, f_i\} \in \{\mathcal{P}_l, \mathcal{P}_m\}.$$

$\square$

Proposition 1.5.3. *For any $l, m \in \mathbb{R}$ such that $l + m - 1 = -n$,* $\{\mathcal{P}_l, \mathcal{P}_m\} = \ker(\mathrm{res})$.

Proof. By Proposition 1.2.1 we already have the inclusion $\{\mathcal{P}_l, \mathcal{P}_m\} \subseteq \ker(\mathrm{res})$. Let $f \in \ker(\mathrm{res}) \subset \mathcal{P}_{-n}$. Then by (1.21) and (1.22), $f\mu \in \Omega^{2n-1,0}(Y)$, and $[(f\mu)\restriction_Z] \in H^{2n-1}(Z) \cong \mathbb{C}$. By Definition 1.2.1,

$$\mathrm{res}(f) = \int_Z (f\mu)\restriction_Z,$$

so that $\mathrm{res}(f) = 0$ if and only if $[(f\mu)\restriction_Z] = 0$ in $H^{2n-1}(Z)$. Then, there exists $\beta \in \Omega^{2n-2}(Z)$ such that $(f\mu)\restriction_Z = d\beta$.

The condition $l + m - 1 = -n$ implies that either $l \neq 0$ or $m \neq 0$, so we can assume that $l \neq 0$. Choose a system of functions $g_1, \ldots, g_N$ in $\mathcal{P}_l$ such that at every point y of Y, their differentials $dg_1|_y, \ldots, dg_N|_y$ span the cotangent space T_y^*Y, and such that the differential forms $\iota_{X_{g_1}}\mu, \ldots, \iota_{X_{g_N}}\mu$ are linearly independent and span the space $\Omega^{2n-2}T^*Y$ at every point. Then there exist functions $f_1, \ldots. f_N \in C^\infty(Z)$ such that

$$\pi^*(\beta) = \sum_{i=1}^{N} \pi^*(f_i)\iota_{X_{g_i}}\mu. \tag{1.55}$$

For all $t \in \mathbb{R}^+$, by (1.22) we have $\rho_t^*(f\mu) = f\mu$ and

$$\rho_t^*(\iota_{X_{g_i}}\mu) = t^{l+n-1}\iota_{X_{g_i}}\mu = t^{-m}\iota_{X_{g_i}}\mu.$$

Hence, from (1.55), the linear independence of $\iota_{X_{g_1}}\mu, \ldots, \iota_{X_{g_N}}\mu$ implies that $\rho_t^*(\pi^*(f_i)) = t^m\pi^*(f_i)$, so $\pi^*(f_i) \in \mathcal{P}_m$ for all $i = 1, \ldots, N$. Moreover, by Lemma 1.1.3 we get

$$f\mu = d(\pi^*(\beta)) = \sum_{i=1}^{N} d(\pi^*(f_i)\iota_{X_{g_i}}\mu) = \sum_{i=1}^{N}\{g_i, \pi^*(f_i)\}\mu, \tag{1.56}$$

since $\mu\restriction_Z$ is a volume form on Z, this implies that

$$f = \sum_{i=1}^{N}\{g_i, \pi^*(f_i)\} \in \{\mathcal{P}_l, \mathcal{P}_m\},$$

so we have

$$\ker(\mathrm{res}) \subseteq \{\mathcal{P}_l, \mathcal{P}_m\}.$$

□

Now we present a proof of (1.46) by using homogeneous differential forms:

Proposition 1.5.4. $\{\mathcal{P}_0, \mathcal{P}_0\} = \mathcal{P}_{-1}$.

Proof. The proof is very similar to the one of Proposition 1.5.2. However, since we cannot choose a system of homogeneous functions of degree zero whose differentials generate the cotangent space of Y at every point (see paragraph after the proof of Theorem 1.4.1), we choose a system of functions $g_1, \ldots, g_N$ in $\mathcal{P}_0$ whose differentials $d(R_0(g_1)), \ldots, d(R_0(g_N))$ are linearly independent and span the cotangent space at every point of Z. From (1.25), for any $g \in \mathcal{P}_0$ we have that $g = \pi^*(R_0(g))$. Let $f \in \mathcal{P}_{-1}$; the differential form $f\alpha \in \Omega^{1,0}(Y)$ and $f\alpha = \pi^*((f\alpha) \restriction_Z)$. Therefore, there exist $f_1^0, \ldots, f_N^0 \in C^\infty(Z)$ such that

$$(f\alpha) \restriction_Z = \sum_{i=1}^{N} f_i^0 \, d(R_0(g_i)), \tag{1.57}$$

which implies that

$$f\alpha = \sum_{i=1}^{N} \pi^*(f_i^0) \, dg_i.$$

As before, we can prove that the functions $\pi^*(f_i^0)$ belong to $\mathcal{P}_0$. Moreover,

$$\begin{aligned} f\,\omega^n &= \frac{1}{n-1} d(f \iota_{\mathcal{X}}(\omega^n)) \\ &= \frac{n}{n-1} d(f\alpha) \wedge \omega^{n-1} \\ &= \frac{n}{n-1} d\left(\sum_{i=1}^{N} \pi^*(f_i^0) \, dg_i \right) \wedge \omega^{n-1} \\ &= \frac{1}{n-1} \sum_{i=1}^{N} \{\pi^*(f_i^0), g_i\} \, \omega^n. \end{aligned}$$

We conclude that

$$f = \frac{1}{n-1} \sum_{i=1}^{N} \{\pi^*(f_i^0), g_i\} \in \{\mathcal{P}_0, \mathcal{P}_0\}.$$

□

Thus, we have another proof of Theorem 1.4.1 including the case $l = 0$. We see that the method described in this section gives us an explicit expression of a homogeneous function in terms of Poisson brackets.
To conclude, for any real numbers l, m,

$$\{\mathcal{P}_l, \mathcal{P}_m\} = \begin{cases} \mathcal{P}_{l+m-1}, & \text{if } l + m \neq -n + 1, \\ \ker(\mathrm{res}), & \text{if } l + m = -n + 1. \end{cases} \tag{1.58}$$

Chapter 2

Cohomology Groups of the Space of Symbols

We are interested in the classification of closed linear forms on certain spaces of classical symbols; this classification comes from the dual of the top cohomology groups of those spaces. For that reason, we want to compute the cohomology groups of some spaces of classical symbols on $\mathbb{R}^n$. An analogue of the Poincaré Lemma in the classical case of cohomology with compact support ([6]) can be used to compute the cohomology groups for smoothing symbols as it is usually done for smooth functions with compact support, via the usual integration map on $\mathbb{R}^n$. For more general classes of symbols, we still use an analogue of the Poincaré Lemma, but substituting to the ordinary integration map, an integration map along the fiber as described in the first part of this chapter. We investigate examples related directly with classical symbols, where the linear map that gives us integration along the fiber will produce either the noncommutative residue or the cut-off regularized integral. At the end of the chapter we use a Mayer-Vietoris sequence argument to conclude the computation of those cohomology groups (see Theorem 2.4.1). This chapter is based on Section 7 of [26].

2.1 Integration along the fiber

In this section we introduce a map, called integration along the fiber, which will allow us to compute the cohomology groups of spaces of classical symbols.
Let M be a connected compact manifold of dimension $n-1>0$ and let

$$\mathcal{C}(\mathbb{R}^+ \times M) \subseteq C^\infty(\mathbb{R}^+ \times M),$$

be a set of smooth functions, closed under partial derivatives and such that if $r \in \mathbb{R}^+$ denotes the radial coordinate,

$$\forall f \in \mathcal{C}(\mathbb{R}^+ \times M),\ \exists r_0 > 0 : f(r, \cdot) = 0,\, \forall r \leq r_0. \tag{2.1}$$

We will also assume that the set $\mathcal{C}(\mathbb{R}^+ \times M)$ satisfies the following:

Assumption 2.1.1. *There exists a map,*

$$\mathcal{J} : \mathcal{C}(\mathbb{R}^+ \times M) \to C^\infty(M),$$

such that

1. *$\mathcal{J}$ is linear in the sense that for all $a, b \in \mathbb{C}$, whenever f, g and $af + b\,g$ belong to $\mathcal{C}(\mathbb{R}^+ \times M)$ we have*
$$\mathcal{J}(af + b\,g) = a\,\mathcal{J}(f) + b\,\mathcal{J}(g).$$
2. *$\mathcal{J} \circ \partial_r = 0$.*
3. *If $f \in \mathcal{C}(\mathbb{R}^+ \times M)$ is such that $\mathcal{J}(f) = 0$, then for any $r \in \mathbb{R}^+$ the function $(r, \cdot) \mapsto F(r, \cdot) := \int_0^r f(t, \cdot)\, dt$ belongs to $\mathcal{C}(\mathbb{R}^+ \times M)$.*
4. *$\mathcal{J}$ commutes with partial derivatives on M.*
5. *$\mathcal{J}$ is non-trivial, so there exists $e \in \mathcal{C}(\mathbb{R}^+ \times M)$ such that $e(r, \cdot)$ is constant on M, and such that $\mathcal{J}(e) = 1$.*

Consider the projection map

$$\begin{aligned} p :\ \mathbb{R}^+ \times M &\to M \\ (r, \eta) &\mapsto \eta. \end{aligned}$$

In the following we denote by $\Omega^k(\mathcal{C}(A))$, the set of k–differential forms on A with coefficients in $\mathcal{C}(A)$. Every k–differential form $\omega \in \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M))$ is locally a sum of differential forms:

$$\omega = f_1(r, \eta) p^* \tau_1 + f_2(r, \eta) p^* \tau_2 \wedge\, dr,$$

with $f_i \in \mathcal{C}(\mathbb{R}^+ \times M),\ \tau_1 \in \Omega^k(M),\ \tau_2 \in \Omega^{k-1}(M)$.

Consider local coordinates $(\eta_1, \ldots, \eta_{n-1})$ on M. The differential operator

$$d_{\mathbb{R}^+ \times M} : \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)) \to \Omega^{k+1}(\mathcal{C}(\mathbb{R}^+ \times M))$$

is defined as follows:

1. If $f \in \Omega^0(\mathcal{C}(\mathbb{R}^+ \times M)) = \mathcal{C}(\mathbb{R}^+ \times M)$, then
$$d_{\mathbb{R}^+ \times M}(f) = \partial_r f\, dr + d_M(f) := \partial_r f\, dr + \sum_{i=1}^{n-1} \partial_{\eta_i} f\, d\eta_i \in \Omega^1(\mathcal{C}(\mathbb{R}^+ \times M)).$$
2. If $\omega = f_1 p^* \tau_1 + f_2 p^* \tau_2 \wedge\, dr \in \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M))$, then
$$d_{\mathbb{R}^+ \times M}(\omega) = \partial_r f_1\, dr \wedge p^* \tau_1 + d_M(f_1 p^* \tau_1) + d_M(f_2 p^* \tau_2) \wedge dr \in \Omega^{k+1}(\mathcal{C}(\mathbb{R}^+ \times M)).$$

The operator $d_{\mathbb{R}^+\times M}$ satisfies $d^2_{\mathbb{R}^+\times M} = 0$, and we can see that the complex $\{\Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)), d_{\mathbb{R}^+\times M}\}$ is a subcomplex of the usual de Rham complex $\{\Omega^k(\mathbb{R}^+ \times M), d_{\mathbb{R}^+\times M}\}$.

Consider the set of closed k–forms on $\mathbb{R}^+ \times M$ with coefficients in $\mathcal{C}(\mathbb{R}^+ \times M)$

$$Z^k(\mathcal{C}(\mathbb{R}^+ \times M)) = \ker\big(d_{\mathbb{R}^+\times M} : \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)) \to \Omega^{k+1}(\mathcal{C}(\mathbb{R}^+ \times M))\big),$$

and the set of exact k–forms on $\mathbb{R}^+ \times M$ with coefficients in $\mathcal{C}(\mathbb{R}^+ \times M)$

$$B^k(\mathcal{C}(\mathbb{R}^+ \times M)) = \mathrm{Image}\big(d_{\mathbb{R}^+\times M} : \Omega^{k-1}(\mathcal{C}(\mathbb{R}^+ \times M)) \to \Omega^{k}(\mathcal{C}(\mathbb{R}^+ \times M))\big).$$

The k–th cohomology group of the complex $\{\Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)), d_{\mathbb{R}^+\times M}\}$ is given by the quotient space

$$H^k(\mathcal{C}(\mathbb{R}^+ \times M)) := Z^k(\mathcal{C}(\mathbb{R}^+ \times M))/B^k(\mathcal{C}(\mathbb{R}^+ \times M)).$$

Inspired in [6], we define the following map, called *integration along the fiber*

$$\begin{aligned} p_* : \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)) &\to \Omega^{k-1}(M) \\ \omega = f_1 p^*\tau_1 + f_2 p^*\tau_2 \wedge\, dr \ &\mapsto \mathcal{J}(f_2)\tau_2. \end{aligned}$$

Remark 2.1.1. Items (3) and (5) in Assumption 2.1.1 will imply, respectively, the injectivity and the surjectivity of the map p_* in cohomology.

Lemma 2.1.1. $d_M \circ p_* = p_* \circ d_{\mathbb{R}^+\times M}$.

Proof. For $\omega = f_1 p^*\tau_1 + f_2 p^*\tau_2 \wedge\, dr \in \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M))$ we have (we write in parentheses the number corresponding to the item in Assumption 2.1.1 that we use):

$$\begin{aligned} p_*(d_{\mathbb{R}^+\times M}(\omega)) &= p_*\,(\partial_r(f_1)dr \wedge p^*\tau_1 + d_M(f_1) \wedge p^*\tau_1 + f_1 p^*(d_M(\tau_1)) \\ &\qquad + d_M(f_2) \wedge p^*\tau_2 \wedge\, dr + f_2 p^*(d_M(\tau_2)) \wedge\, dr) \\ &= (-1)^k \mathcal{J}(\partial_r(f_1))\tau_1 + \sum_{i=1}^{n-1} \mathcal{J}(\partial_{\eta_i}(f_2))d\eta_i \wedge \tau_2 + \mathcal{J}(f_2)d_M(\tau_2) \\ &\overset{(2),(4)}{=} \sum_{i=1}^{n-1} \partial_{\eta_i}(\mathcal{J}(f_2))d\eta_i \wedge \tau_2 + \mathcal{J}(f_2)d_M(\tau_2) \\ &= d_M(\mathcal{J}(f_2)) \wedge \tau_2 + \mathcal{J}(f_2)d_M(\tau_2) \\ &= d_M(\mathcal{J}(f_2)\tau_2) \\ &= d_M(p_*(\omega)). \end{aligned}$$

$\square$

By Assumption 2.1.1 (5), there exists $e \in \mathcal{C}(\mathbb{R}^+ \times M)$, constant on M, and such that $\mathcal{J}(e) = 1$. We define the map

$$\begin{aligned} e_* : \Omega^{k-1}(M) &\to \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)) \\ \tau &\mapsto e\, p^*\tau \wedge dr. \end{aligned}$$

Lemma 2.1.2. $d_{\mathbb{R}^+ \times M} \circ e_* = e_* \circ d_M$.

Proof. Let $\tau \in \Omega^{k-1}(M)$. Then

$$\begin{aligned} d_{\mathbb{R}^+ \times M}(e_*(\tau)) &= d_{\mathbb{R}^+ \times M}(e\, p^*\tau \wedge dr) \\ &= d_M(e\, p^*\tau) \wedge dr \\ &= e\, p^*(d_M\tau) \wedge dr \\ &= e_*(d_M(\tau)). \end{aligned}$$

□

Lemma 2.1.3. $p_* \circ e_* = 1$ *on* $\Omega^{k-1}(M)$.

Proof. Let $\tau \in \Omega^{k-1}(M)$. Then

$$p_*(e_*(\tau)) = p_*(e\, p^*\tau \wedge dr) = \mathcal{J}(e)\tau = \tau.$$

□

Lemma 2.1.4. $e_* \circ p_* = 1$ *on* $H^k(\mathcal{C}(\mathbb{R}^+ \times M))$.

Proof. Let $\omega = f_1 p^*\tau_1 + f_2 p^*\tau_2 \wedge dr \in \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M))$. The form

$$\omega - e_*(p_*(\omega)) = f_1 p^*\tau_1 + f_2 p^*\tau_2 \wedge dr - e\, p^*(\mathcal{J}(f_2)\tau_2) \wedge dr \tag{2.2}$$

satisfies

$$p_*(\omega - e_*(p_*(\omega))) = \mathcal{J}(f_2)\tau_2 - \mathcal{J}(e)\mathcal{J}(f_2)\tau_2 = 0.$$

Since $\mathcal{J}(f_2 - e\, p^*(\mathcal{J}(f_2))) = 0$, by Assumption 2.1.1 (3) we have

$$\int_0^r \Big(f_2(t,\eta) - e(t,\eta) p^*(\mathcal{J}(f_2))(t,\eta) \Big)\, dt \in \mathcal{C}(\mathbb{R}^+ \times M).$$

We define an operator $K : \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M)) \to \Omega^{k-1}(\mathcal{C}(\mathbb{R}^+ \times M))$ in the following way: let $A(r,\eta) := \int_0^r e(t,\eta)\, dt$. For $\omega = f_1 p^*\tau_1 + f_2 p^*\tau_2 \wedge dr \in \Omega^k(\mathcal{C}(\mathbb{R}^+ \times M))$ we set

$$\begin{aligned} K(\omega) &:= \left(\int_0^r f_2(t,\eta)\, dt - A(r,\eta) p^* \left(\mathcal{J}(f_2)\right)(r,\eta) \right) p^*\tau_2 && (2.3) \\ &= \left(\int_0^r \Big(f_2(t,\eta) - e(t,\eta) p^*(\mathcal{J}(f_2))(t,\eta) \Big)\, dt \right) p^*\tau_2. && (2.4) \end{aligned}$$

Thus we have

$$\begin{aligned}
dK(\omega) &= d\left(\left(\int_0^r f_2(t,\eta)\,dt - A(r,\eta)p^*\left(\mathcal{J}(f_2)\right)(r,\eta)\right)p^*\tau_2\right)\\
&= \Big(f_2(r,\eta)\,dr + d_M\left(\int_0^r f_2(t,\eta)\,dt\right) - e\,dr\,p^*\left(\mathcal{J}(f_2)\right)(r,\eta)\\
&\quad - d_M\left(A(r,\eta)p^*\left(\mathcal{J}(f_2)\right)(r,\eta)\right)\Big)p^*\tau_2\\
&\quad + \left(\int_0^r \Big(f_2(t,\eta) - e(t,\eta)p^*(\mathcal{J}(f_2))(t,\eta)\Big)\,dt\right)p^*(d_M\tau_2)\\
&= \Big(f_2(r,\eta)\,dr + d_M\left(\int_0^r f_2(t,\eta)\,dt\right) - e\,dr\,p^*\left(\mathcal{J}(f_2)\right)(r,\eta)\\
&\quad - A(r,\eta)p^*\left(d_M\left(\mathcal{J}(f_2)\right)\right)(r,\eta)\Big)p^*\tau_2\\
&\quad + \left(\int_0^r \Big(f_2(t,\eta) - e(t,\eta)p^*(\mathcal{J}(f_2))(t,\eta)\Big)\,dt\right)p^*(d_M\tau_2).
\end{aligned}$$

$$\begin{aligned}
Kd(\omega) &= (-1)^k\left(\int_0^r \partial_t f_1(t,\eta)\,dt - A(r,\eta)p^*\left(\mathcal{J}(\partial_r f_1)\right)(r,\eta)\right)p^*\tau_1\\
&\quad + \left(\sum_{i=1}^{n-1}\int_0^r \partial_{\eta_i} f_2(t,\eta)\,dt - \sum_{i=1}^{n-1} A(r,\eta)p^*\left(\mathcal{J}(\partial_{\eta_i} f_2)\right)(r,\eta)\right)d\eta_i\wedge p^*\tau_2\\
&\quad + \left(\int_0^r f_2(t,\eta)\,dt - A(r,\eta)p^*\left(\mathcal{J}(f_2)\right)(r,\eta)\right)p^*(d_M\tau_2)\\
&\overset{(2),(2.1)}{=} (-1)^k f_1(r,\eta)p^*\tau_1 + \left(\int_0^r \Big(f_2(t,\eta) - e(t,\eta)p^*(\mathcal{J}(f_2))(t,\eta)\Big)\,dt\right)p^*(d_M\tau_2)\\
&\quad + \left(\sum_{i=1}^{n-1}\int_0^r \partial_{\eta_i} f_2(t,\eta)\,dt - \sum_{i=1}^{n-1} A(r,\eta)p^*\left(\mathcal{J}(\partial_{\eta_i} f_2)\right)(r,\eta)\right)d\eta_i\wedge p^*\tau_2\\
&\overset{(4)}{=} (-1)^k f_1(r,\eta)p^*\tau_1 + \left(\int_0^r \Big(f_2(t,\eta) - e(t,\eta)p^*(\mathcal{J}(f_2))(t,\eta)\Big)\,dt\right)p^*(d_M\tau_2)\\
&\quad + d_M\left(\int_0^r f_2(t,\eta)\,dt\right) - A(r,\eta)p^*\left(d_M\left(\mathcal{J}(f_2)\right)\right)(r,\eta)p^*\tau_2.
\end{aligned}$$

Therefore,

$$(Kd - dK)(\omega) = (-1)^k\Big(f_1(r,\eta)p^*\tau_1 + f_2(r,\eta)p^*\tau_2\wedge\,dr - e\,p^*(\mathcal{J}(f_2)\tau_2)\wedge\,dr\Big),$$

and by (2.2) we get $1 - e_*\circ p_* = (-1)^k(Kd - dK)$ on $\Omega^k(\mathcal{C}(\mathbb{R}^+\times M))$. □

Proposition 2.1.1. *For all $k\geq 1$, $H^k(\mathcal{C}(\mathbb{R}^+\times M))\cong H^{k-1}(M)$.*

Proof. By the previous lemmas, the map $p_*:\Omega^k(\mathcal{C}(\mathbb{R}^+\times M))\to\Omega^{k-1}(M)$ yields the isomorphism and $e_*:\Omega^{k-1}(M)\to\Omega^k(\mathcal{C}(\mathbb{R}^+\times M))$ yields the corresponding inverse. □

Corollary 2.1.1. $H^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})) \cong \begin{cases} \mathbb{C}, & \textit{if } k = 1, n, \\ 0, & \textit{otherwise.} \end{cases}$

Proof. For $k \geq 1$, this follows from previous proposition and the cohomology groups of the $(n-1)$–dimensional sphere. By (2.1), we also have that the zero–th cohomology group $H^0(\mathcal{C}(\mathbb{R}^+ \times S^{n-1}))$ vanishes. □

Consider the polar coordinate diffeomorphism

$$\begin{aligned} q : \ \mathbb{R}^n \setminus 0 &\to \mathbb{R}^+ \times S^{n-1} \\ \xi &\mapsto \left(|\xi|, \frac{\xi}{|\xi|} \right). \end{aligned} \tag{2.5}$$

Then, if we set

$$\mathcal{C}(\mathbb{R}^n \setminus 0) := q^*(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})), \tag{2.6}$$

q^* induces the following isomorphism in cohomology:

$$H^k(\mathcal{C}(\mathbb{R}^n \setminus 0)) \cong H^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})). \tag{2.7}$$

2.2 Examples

In this section we give examples of sets related directly with classical symbols on $\mathbb{R}^n$, which satisfy Assumption 2.1.1, i.e. they admit a map that produces an integration along the fiber. This map is related to the usual integral on $\mathbb{R}^n$, the cut-off regularized integral or the noncommutative residue, as we explain later in Chapter 3.

2.2.1 The usual integral

Let $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ be the set of smooth functions $\sigma \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ that satisfy the following conditions

1. As $r \to \infty$, for all $s, m \in \mathbb{N}$, $|\partial_r^s \sigma(r, \cdot)| = \mathcal{O}((1+r)^{-m-s})$.
2. If $\sigma \in \mathcal{C}(\mathbb{R}^+ \times S^{n-1})$, then there exists $r_0 > 0$ such that $\sigma(r, \cdot) = 0$ for all $r \leq r_0$.

The set $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ satisfies Assumption 2.1.1: for $\eta \in S^{n-1}$ consider the map

$$\mathcal{J}(\sigma)(\eta) := \int_0^\infty \sigma(r, \eta)\, dr. \tag{2.8}$$

1. The usual integral on $\mathbb{R}^+$, $\mathcal{J}$, is linear in the set $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$.
2. We want to show that $\mathcal{J} \circ \partial_r = 0$. Let $\sigma \in \mathcal{C}(\mathbb{R}^+ \times S^{n-1})$; then,

$$\int_0^\infty \partial_r \sigma(r, \eta)\, dr = \lim_{R \to \infty} \sigma(R, \eta) - \lim_{r \to 0^+} \sigma(r, \eta) = 0. \tag{2.9}$$

3. Let $f \in \mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ be such that $\mathcal{J}(f) = 0$, i.e. if $F(r, \cdot) := \int_0^r f(t, \cdot)\, dt$, then
$$\lim_{r \to +\infty} F(r, \cdot) = 0.$$
For $s = 0$ we can use L'Hôpital's rule, and for $s \geq 1$ we can use that $\partial_r^s F = \partial_r^{s-1} f$ to show that for any $m \in \mathbb{N}$, as $r \to \infty$
$$|\partial_r^s F(r, \cdot)| = \mathcal{O}((1 + r)^{-m-s}),$$
so $F \in \mathcal{C}(\mathbb{R}^+ \times S^{n-1})$.

4. For all $i = 1, \ldots, n-1$, $\partial_{\eta_i} \circ \mathcal{J} = \mathcal{J} \circ \partial_{\eta_i}$.
By the properties of σ one can interchange ∂_{η_i} with $\int$ and for all $\eta \in S^{n-1}$ we get
$$\partial_{\eta_i}(\mathcal{J}(\sigma))(\eta) = \partial_{\eta_i}\left(\int_{\mathbb{R}^+} \sigma(r, \eta) dr\right) = \int_{\mathbb{R}^+} \partial_{\eta_i}(\sigma)(r, \eta) dr = \mathcal{J}(\partial_{\eta_i}(\sigma))(\eta). \tag{2.10}$$

5. Let $e \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ be any smooth function with compact support on $\mathbb{R}^+$, constant on S^{n-1} and with total integral 1 on $\mathbb{R}^+$. The function e belongs to $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ and $\int_0^\infty e(r, \eta)\, dr = 1$.

2.2.2 Towards the residue map and the cut-off integral

In view of the examples to come, we introduce the following notations:

Let $\psi \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ be a smooth cut-off function which vanishes in a neighborhood of $r = 0$ and is identically one for $r \geq 1$. For any $a \in \mathbb{R}$ and for all $j \geq 0$, let $\sigma_{a-j} \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ be a smooth function such that for all $(r, \eta) \in \mathbb{R}^+ \times S^{n-1}$,
$$\sigma_{a-j}(r, \eta) = r^{a-j} \sigma_{a-j}(1, \eta).$$
For all $N \in \mathbb{N}$, let $g_{<-N} \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ be a smooth function that satisfies the following:
$$\exists\, r_0 > 0, \text{ such that } g_{<-N}(r, \cdot) = 0, \ \forall r \leq r_0, \text{ and} \tag{2.11}$$
$$\exists\, m < -N : \text{ as } r \to \infty, \forall\, s \in \mathbb{N}, \ |\partial_r^s(g_{<-N}(r, \cdot))| = \mathcal{O}((1 + r)^{m-s}). \tag{2.12}$$
For any $a \in \mathbb{R}$, let us define $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ as the set of smooth functions $\sigma \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ such that for all $N \in \mathbb{N}$, there exist k_N, ψ, σ_{a-j}, $g_{<-N}$ as above, such that
$$\sigma = \sum_{j=0}^{k_N} \psi\, \sigma_{a-j} + g_{<-N}. \tag{2.13}$$
We assume that l is the smallest integer such that $a - l < 0$, i.e. $l = \lfloor a \rfloor + 1$. Setting
$$\pi_{<-1}(\sigma)(r, \eta) := \sum_{j=l+1}^{k_N} \psi(r, \eta) \sigma_{a-j}(r, \eta) + g_{<-N}(r, \eta),$$

then $\pi_{<-1}(\sigma)$ satisfies (2.11) and (2.12) with -1 instead of $-N$. Therefore (2.13) reads

$$\sigma = \sum_{j=0}^{l} \psi\, \sigma_{a-j} + \pi_{<-1}(\sigma). \tag{2.14}$$

The set $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ is closed under partial derivatives and satisfies (2.1). In particular, for any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, the partial derivative with respect to r, $\partial_r(\sigma)$ belongs to $\mathcal{C}^{a-1}(\mathbb{R}^+ \times S^{n-1}) \subset \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, and from (2.14) it reads

$$\begin{aligned}
\partial_r \sigma &= \sum_{j=0}^{l} \psi\, \partial_r(\sigma_{a-j}) + \sum_{j=0}^{l} \partial_r(\psi)\sigma_{a-j} + \partial_r(\pi_{<-1}(\sigma)) \\
&= \sum_{j=0}^{l} (a-j)\psi\, \sigma_{a-j} r^{-1} + \sum_{j=0}^{l} \partial_r(\psi)\sigma_{a-j} + \partial_r(\pi_{<-1}(\sigma)).
\end{aligned} \tag{2.15}$$

Remark 2.2.1. Since $\partial_r(\psi)$ has compact support in the open interval $(0,1)$, the expression with this term satisfies (2.12) for all $m \in \mathbb{N}$.

For any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ and for all $r \in \mathbb{R}^+$, the integral $\int_0^r \sigma(t,\cdot)\, dt$ belongs to $\mathcal{C}^{a+1}(\mathbb{R}^+ \times S^{n-1})$. Using integration by parts and the fact that

$$\text{if } a-j \neq -1 \text{ then } \; r^{a-j} = \partial_r\left(\frac{r^{a-j+1}}{a-j+1}\right), \quad r^{-1} = \partial_r(\ln(r)),$$

from (2.13) we get for any $r \in \mathbb{R}^+$

$$\begin{aligned}
\int_0^r \sigma &= \sum_{j=0}^{l+1} \int_0^r \psi\, \sigma_{a-j} + \int_0^r \pi_{<-2}(\sigma) \\
&= \sum_{j=0}^{l+1} \sigma_{a-j}(1,\cdot)\left(\int_0^r \psi(t,\eta)\, t^{a-j}\, dt\right) + \int_0^r \pi_{<-2}(\sigma) \\
&= \sum_{\substack{j=0 \\ a-j\neq -1}}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1}\left(\psi\, r^{a-j+1} - \int_0^r \partial_t(\psi)\, t^{a-j+1}\, dt\right) \\
&\quad + \sigma_{-1}(1,\cdot)\left(\psi \ln(r) - \int_0^r \partial_t(\psi)\ln(t)\, dt\right) + \int_0^r \pi_{<-2}(\sigma).
\end{aligned} \tag{2.16}$$

We also define the following sets

(a) $\mathcal{C}^{\infty}(\mathbb{R}^+ \times S^{n-1}) := \left\langle \bigcup_{a\in\mathbb{R}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}) \right\rangle$, the linear space spanned by all the spaces $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$.

(b) $\mathcal{C}^{-\infty}(\mathbb{R}^+ \times S^{n-1}) := \bigcap_{a\in\mathbb{R}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$.

Any k–differential form $\omega \in \Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}))$ is locally a sum of differential forms:

$$\omega = f\, p^*\tau_1 + g\, p^*\tau_2 \wedge\, dr,$$

where $f \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, $g \in \mathcal{C}^{a-1}(\mathbb{R}^+ \times S^{n-1}) \subset \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, and the differential forms $\tau_1 \in \Omega^k(S^{n-1})$ and $\tau_2 \in \Omega^{k-1}(S^{n-1})$. In particular, for any element $f \in \mathcal{C}^{a-k}(\mathbb{R}^+ \times S^{n-1})$ we have $f\, r^k \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, and $f\, r^{k-1} \in \mathcal{C}^{a-1}(\mathbb{R}^+ \times S^{n-1})$, so we want to study the following sets of differential forms:

$$\begin{aligned}\Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})) = \{f\, r^k p^*\tau_1 + g\, r^{k-1} p^*\tau_2 \wedge dr : f, g \in \mathcal{C}^{a-k}(\mathbb{R}^+ \times S^{n-1}),\\ \tau_1 \in \Omega^k(S^{n-1}),\ \tau_2 \in \Omega^{k-1}(S^{n-1})\}.\end{aligned} \tag{2.17}$$

Remark 2.2.2. We can verify that Proposition 2.1.1 and Corollary 2.1.1 also hold for these sets of differential forms.
Note that the map $p_* : \Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})) \to \Omega^{k-1}(S^{n-1})$ acting on the form $\omega = f\, r^k p^*\tau_1 + g\, r^{k-1} p^*\tau_2 \wedge dr$ is given by $p_*(\omega) = \mathcal{J}(g\, r^{k-1})\tau_2$.

The sets of 0–th and top degree differential forms are given by:

- $\Omega^0(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})) = \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$.
- $\Omega^n(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})) = \{f\, r^{n-1} p^*(dvol_{S^{n-1}}) \wedge dr : f \in \mathcal{C}^{a-n}(\mathbb{R}^+ \times S^{n-1})\}$.

If $\omega = f\, r^k p^*\tau_1 + g\, r^{k-1} p^*\tau_2 \wedge dr \in \Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}))$, then if d_S denotes the differential on S^{n-1},

$$d_{\mathbb{R}^+\times S^{n-1}}(\omega) = d_S(f\, r^k p^*\tau_1) + (-1)^k \partial_r(f\, r^k) p^*\tau_1 \wedge\, dr + d_S(g\, r^{k-1} p^*\tau_2) \wedge dr,$$

belongs to $\Omega^{k+1}(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}))$, so we see that the exterior derivative maps

$$d_{\mathbb{R}^+\times S^{n-1}} : \Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})) \to \Omega^{k+1}(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})).$$

Thus, in the following we consider the complexes $\{\Omega^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})), d_{\mathbb{R}^+\times S^{n-1}}\}$, where $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ is either $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ for any $a \in \mathbb{R}$, $\mathcal{C}^{-\infty}(\mathbb{R}^+ \times S^{n-1})$, $\mathcal{C}^{\infty}(\mathbb{R}^+ \times S^{n-1})$, $\bigcup_{a\in\mathbb{Z}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-1,+\infty)} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})\right\rangle$.

Remark 2.2.3. By Remark 2.2.2, in the space $\Omega^k(\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}))$ we compute the map $\mathcal{J}$ in elements of $\mathcal{C}^{a-1}(\mathbb{R}^+ \times S^{n-1})$. Therefore, condition (3) in Assumption 2.1.1 reads: If $f \in \mathcal{C}^{a-1}(\mathbb{R}^+ \times M)$ is such that $\mathcal{J}(f) = 0$, then for any $r \in \mathbb{R}^+$, $\int_0^r f(t,\cdot)\, dt$ belongs to $\mathcal{C}^a(\mathbb{R}^+ \times M)$.

Corollary 2.2.1. $H^k(\mathcal{C}^{-\infty}(\mathbb{R}^+ \times S^{n-1})) \cong \begin{cases}\mathbb{C}, & \textit{if } k = 1, n,\\ 0, & \textit{otherwise.}\end{cases}$

Proof. The space $\mathcal{C}^{-\infty}(\mathbb{R}^+ \times S^{n-1})$ coincides with the space $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ defined in Subsection 2.2.1. Then, the linear map $\mathcal{J} = \int_{\mathbb{R}^+}$ satisfies Assumption 2.1.1, and the statement follows by Corollary 2.1.1. $\square$

Towards the cut-off regularized integral

For the next example, let us define the map $\mathcal{J}$ as follows: for any $a \in \mathbb{R}$, for any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, we consider the finite part of (2.16) when $r \to +\infty$:

$$\begin{aligned}
\mathcal{J}(\sigma)(\cdot) &:= \mathrm{cf}(\sigma)(\cdot) := \operatorname*{fp}_{R\to\infty} \int_0^R \sigma(r,\cdot)dr \\
&= \int_{\mathbb{R}^+} \left(\pi_{<-2}(\sigma) - \sum_{\substack{j=0 \\ a-j\neq -1}}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \partial_r(\psi) r^{a-j+1} - \sigma_{-1}(1,\cdot)\partial_r(\psi)\ln(r) \right) dr.
\end{aligned} \tag{2.18}$$

Let us assume that $a \notin \mathbb{Z} \cap [-1, +\infty)$; then $\sigma_{-1} \equiv 0$ and

$$\mathrm{cf}(\sigma)(\cdot) = \int_{\mathbb{R}^+} \left(\pi_{<-2}(\sigma)(r,\cdot) - \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \partial_r(\psi) r^{a-j+1} \right) dr. \tag{2.19}$$

The map $\mathcal{J}$ satisfies Assumption 2.1.1:

1. The linearity of $\mathcal{J}$ follows from the vector space structure of the spaces $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ and the definition of $\mathcal{J}$.

2. $\mathcal{J} \circ \partial_r = 0$: By condition (2.11) on the functions in $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, $\int_0^R \partial_r \sigma(r,\eta)\, dr = \sigma(R,\eta)$ and since

$$\lim_{R\to\infty} \pi_{<-1}(\sigma)(R,\eta) = 0,$$

we obtain from (2.14)

$$\begin{aligned}
\operatorname*{fp}_{R\to\infty} \int_0^R \partial_r \sigma(r,\eta)\, dr &= \operatorname*{fp}_{R\to\infty} \sigma(R,\eta) \\
&= \operatorname*{fp}_{R\to\infty} \left(\sum_{j=0}^{l} \psi(R,\eta)\sigma_{a-j}(R,\eta) \right) \\
&= \sigma_0(1,\eta),
\end{aligned} \tag{2.20}$$

which vanishes whenever $a \notin \mathbb{Z} \cap [-1, +\infty)$.

Remark 2.2.4. Even more, whenever $a \notin \mathbb{Z}$ or $a < 0$, the set $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ does not admit functions that are constant in r for $r \geq 1$, and hence, for any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ we have

$$\mathrm{cf}(\partial_r \sigma)(\eta) = \operatorname*{fp}_{r\to\infty} \sigma(r,\eta) = \sigma_0(1,\eta) = 0.$$

Remark 2.2.5. From this computation we can also infer that for all $k = 1, \dots, n$, and for any $\sigma \in \mathcal{C}^{a-k}(\mathbb{R}^+ \times S^{n-1})$,

$$\mathrm{cf}(\partial_r(\sigma r^{k-1}))(\cdot) = \sigma_{-k+1}(1, \cdot). \tag{2.21}$$

3. Let $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ be as in (2.13) with $\mathcal{J}(\sigma) = 0$, i.e. $\mathrm{cf}(\sigma) \equiv 0$. Since $a \notin \mathbb{Z} \cap [-1, +\infty)$, from (2.16) we get

$$\int_0^r \sigma = \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \left(\psi\, r^{a-j+1} - \int_0^r \partial_t(\psi)\, t^{a-j+1}\, dt \right) + \int_0^r \pi_{<-2}(\sigma)$$
$$= \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \psi r^{a-j+1} + \int_0^r \left(\pi_{<-2}(\sigma) - \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \partial_t(\psi) t^{a-j+1} \right) dt.$$

For $s = 0$ we can use L'Hôpital's rule, and for $s \geq 1$ we can use that for some $p < -2$, $\partial_r^s(\pi_{<-2}(\sigma)) = \mathcal{O}((1+r)^{p-s})$, to show that as $r \to \infty$, by Remark 2.2.1, there exists $m < -1$ such that

$$\partial_r^s \left(\int_0^r \left(\pi_{<-2}(\sigma) - \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \partial_t(\psi)\, t^{a-j+1} \right) dt \right) = \mathcal{O}((1+r)^{m-s}). \tag{2.22}$$

Therefore $\int_0^r \sigma \in \mathcal{C}^{a+1}(\mathbb{R}^+ \times S^{n-1})$.

4. For all $i = 1, \dots, n-1$, $\partial_{\eta_i} \circ \mathcal{J} = \mathcal{J} \circ \partial_{\eta_i}$.
By the properties of σ we can interchange ∂_{η_i} with $\int$ and we get

$$\partial_{\eta_i}(\mathcal{J}(\sigma))(\eta) = \partial_{\eta_i}\left(\mathrm{cf}(\sigma(r,\eta))\right)$$
$$= \partial_{\eta_i} \left(\int_{\mathbb{R}^+} \left(\pi_{<-2}(\sigma) - \sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\eta)}{a-j+1} \partial_r(\psi(r,\eta)) r^{a-j+1} \right) dr \right)$$
$$= \int_{\mathbb{R}^+} \left(\partial_{\eta_i} \pi_{<-2}(\sigma) - \partial_{\eta_i} \left(\sum_{j=0}^{l+1} \frac{\sigma_{a-j}(1,\eta)}{a-j+1} \partial_r(\psi(r,\eta)) r^{a-j+1} \right) \right) dr$$
$$= \mathrm{cf}(\partial_{\eta_i}(\sigma)(r,\eta)).$$

5. If $e \in C^\infty(\mathbb{R}^+ \times S^{n-1})$ is any smooth function with compact support on $\mathbb{R}^+$, constant on S^{n-1} and with total integral 1 on $\mathbb{R}^+$, then all the derivatives of e are of order $\mathcal{O}((1+r)^{-m})$ for all $m \in \mathbb{N}$. Hence, the function e belongs to $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ for all $a \notin \mathbb{Z} \cap [-1, +\infty)$, and $\mathcal{J}(e) = 1$.

Corollary 2.2.2. *If the space* $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ *is either* $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ *where* $a \notin \mathbb{Z} \cap [-1, +\infty)$, *or* $\left\langle \bigcup_{a \notin \mathbb{Z} \cap [-1,+\infty)} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}) \right\rangle$, *then*

$$H^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})) \cong \begin{cases} \mathbb{C}, & \text{if } k = 1, n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.23}$$

Proof. The set $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ satisfies Assumption 2.1.1 with the linear map $\mathcal{J} = \mathrm{cf}$, and the statement follows by Corollary 2.1.1. $\square$

Remark 2.2.6. The isomorphism in (2.23) when $k = n$ is produced by integration over S^{n-1} composed with integration along the fiber. If $\overline{\mu}(\eta)$ denotes a volume form on S^{n-1} as in Remark 1.1.2, then to the n–form $\sigma r^{n-1}\overline{\mu}(\eta) \wedge dr$ it corresponds the complex number

$$\int_{S^{n-1}} \left(\mathrm{cf}(\sigma(r,\eta)r^{n-1})\right) \overline{\mu}(\eta),$$

which corresponds to the cut-off regularized integral of a symbol with constant coefficients (see Subsection 3.1.2).

Towards the noncommutative residue

For the next example, let us define the map $\mathcal{J}$ as follows: for any $a \in \mathbb{R}$, for any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, for all $\eta \in S^{n-1}$,

$$\mathcal{J}(\sigma)(\eta) := \mathrm{rs}(\sigma)(\eta) := \sigma_{-1}(1, \eta). \tag{2.24}$$

From this we can infer that for all $k = 1, \ldots, n$, and for any $\sigma \in \mathcal{C}^{a-k}(\mathbb{R}^+ \times S^{n-1})$

$$\mathrm{rs}(\sigma r^{k-1}) = \sigma_{-k}(1, \cdot). \tag{2.25}$$

The map $\mathcal{J}$ satisfies Assumption 2.1.1:

1. The linearity of $\mathcal{J}$ follows from
$$(\sigma + \tau)_{-1} = \sigma_{-1} + \tau_{-1},\ \forall\, \sigma, \tau \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}).$$

2. $\mathcal{J} \circ \partial_r = 0$: for any $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, the coefficient of r^{-1} in (2.15) vanishes, so $\mathrm{rs}(\partial_r \sigma) = 0$.

3. Let $\sigma \in \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ be as in (2.13) with $\mathcal{J}(\sigma) = 0$, i.e. $\sigma_{-1} \equiv 0$; in particular, this holds whenever $a \notin \mathbb{Z} \cap [-1, +\infty)$. Then, from (2.16) we get

$$\begin{aligned}
\int_0^r \sigma &= \sum_{\substack{j=0 \\ a-j\neq -1}}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \left(\psi r^{a-j+1} - \int_0^r \partial_t(\psi) t^{a-j+1}\, dt\right) + \int_0^r \pi_{<-2}(\sigma) \\
&= \sum_{\substack{j=0 \\ a-j\neq -1}}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \psi r^{a-j+1} \\
&\quad + \int_0^r \left(\pi_{<-2}(\sigma) - \sum_{\substack{j=0 \\ a-j\neq -1}}^{l+1} \frac{\sigma_{a-j}(1,\cdot)}{a-j+1} \partial_t(\psi) t^{a-j+1}\right) dt.
\end{aligned} \tag{2.26}$$

Using Remark 2.2.1, the expression in parentheses in the integral is of order less than -2. Since the function $\partial_t\psi$ has compact support, all the terms $\int_0^r \partial_t(\psi)t^{a-j+1}\,dt$ satisfy (2.11) and (2.12). On the other hand, if $r_0>0$ is such that $\pi_{<-2}(\sigma)(r,\cdot)=0$ for all $r\le r_0$, then $\int_0^r \pi_{<-2}(\sigma)(t,\cdot)\,dt=0$ for all $r\le r_0$, satisfying (2.11).
Now, from (2.18)

$$\mathrm{cf}(\sigma)(\cdot)=\int_{\mathbb{R}^+}\left(\pi_{<-2}(\sigma)-\sum_{\substack{j=0\\a-j\neq-1}}^{l+1}\frac{\sigma_{a-j}(1,\cdot)}{a-j+1}\partial_r(\psi)r^{a-j+1}\right)dr. \tag{2.27}$$

Since $\mathrm{supp}(\partial_t\psi)\subset(0,1)$, when $r\ge 1$ Equation (2.26) becomes

$$\begin{aligned}\int_0^r\sigma &= \sum_{\substack{j=0\\a-j\neq-1}}^{l+1}\frac{\sigma_{a-j}(1,\cdot)}{a-j+1}r^{a-j+1}\\ &\quad+\int_0^r\pi_{<-2}(\sigma)-\sum_{\substack{j=0\\a-j\neq-1}}^{l+1}\frac{\sigma_{a-j}(1,\cdot)}{a-j+1}\int_{\mathbb{R}^+}\partial_r(\psi)r^{a-j+1}\,dr\\ &=\sum_{\substack{j=0\\a-j\neq-1}}^{k+1}\frac{\sigma_{a-j}(1,\cdot)}{a-j+1}r^{a-j+1}+\mathrm{cf}(\sigma)-\int_r^\infty\pi_{<-2}(\sigma).\end{aligned}\tag{2.28}$$

From the properties of $\pi_{<-2}(\sigma)$, namely, (2.11) and (2.12) with -2 instead of $-N$, we find that for r sufficiently large, the integral $\int_r^\infty\pi_{<-2}(\sigma)$ is of order less than -1 up to a constant. So we can conclude that if the set $\mathcal{C}^{a+1}(\mathbb{R}^+\times S^{n-1})$ admits functions which are constant in r for $r\ge1$ (which is the case when $a\in\mathbb{Z}\cap[-1,+\infty)$), the term

$$\int_0^r\left(\pi_{<-2}(\sigma)-\sum_{\substack{j=0\\a-j\neq-1}}^{l+1}\frac{\sigma_{a-j}(1,\cdot)}{a-j+1}\partial_t(\psi)t^{a-j+1}\,dt\right) \tag{2.29}$$

is of order -1 up to a constant, and therefore from (2.28) we conclude that whenever $a\in\mathbb{Z}\cap[-1,+\infty)$, for any $\sigma\in\mathcal{C}^a(\mathbb{R}^+\times S^{n-1})$ such that $\mathrm{rs}(\sigma)=0$, $\int_0^r\sigma\in\mathcal{C}^{a+1}(\mathbb{R}^+\times S^{n-1})$.

4. For all $i=1,\dots,n-1$, $\partial_{\eta_i}\circ\mathcal{J}=\mathcal{J}\circ\partial_{\eta_i}$.

$$\mathcal{J}(\partial_{\eta_i}\sigma)(\eta)=(\partial_{\eta_i}\sigma)_{-1}(1,\eta)=\partial_{\eta_i}(\sigma_{-1})(1,\eta)=\partial_{\eta_i}(\mathcal{J}(\sigma))(\eta).$$

5. Let $\psi\in C^\infty(\mathbb{R}^+)$ be any smooth cut-off function which vanishes in a neighborhood of $r=0$ and is identically one for $r\ge1$. Let us consider the function $e(r,\eta)=\psi(r)r^{-1}\in\mathcal{C}^a(\mathbb{R}^+\times S^{n-1})$ for all $a\in\mathbb{Z}\cap[-1,+\infty)$. Then e is constant on S^{n-1} and $\mathcal{J}(e)=1$.

Corollary 2.2.3. *If the space $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ is either $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ where $a \in \mathbb{Z} \cap [-1, +\infty)$, $\mathcal{C}^\infty(\mathbb{R}^+ \times S^{n-1})$ or $\bigcup_{a\in\mathbb{Z}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$, then*

$$H^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1})) \cong \begin{cases} \mathbb{C}, & \textit{if } k = 1, n, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.30}$$

Proof. The sets $\mathcal{C}^{a+1}(\mathbb{R}^+ \times S^{n-1})$ where $a \in \mathbb{Z} \cap [-1, +\infty)$, $\mathcal{C}^\infty(\mathbb{R}^+ \times S^{n-1})$ and $\bigcup_{a\in\mathbb{Z}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ admit functions which are constant in r for $r \geq 1$, so they satisfy Assumption 2.1.1 with the linear map $\mathcal{J} = \mathrm{rs}$, and the statement follows by Corollary 2.1.1. ☐

Remark 2.2.7. The isomorphism in (2.30) when $k = n$ is produced by integration over S^{n-1} composed with integration along the fiber. If $\overline{\mu}(\eta)$ denotes a volume form on S^{n-1} as in Remark 1.1.2, then to the n–form $\sigma r^{n-1}\overline{\mu}(\eta) \wedge dr$ it corresponds the complex number

$$\int_{S^{n-1}} \sigma_{-n}(1, \eta)\overline{\mu}(\eta),$$

which corresponds to the noncommutative residue of a symbol with constant coefficients (see Subsection 3.1.1).

2.3 Classes of symbols

Let us recall the definition of symbols on an open subset of $\mathbb{R}^n$ following [20] and [39]. We assume that the dimension n is such that $n > 1$, unless we indicate something else.

Definition 2.3.1. Let $U \subseteq \mathbb{R}^n$ be an open set and let $a \in \mathbb{R}$. We say that $\sigma(x, \xi)$ is a *symbol* of order $\mathrm{ord}(\sigma) = a$ on U, and we write $\sigma \in S^a(U)$, if:

(a) $\sigma(x, \xi)$ is smooth in $(x, \xi) \in U \times \mathbb{R}^n$.

(b) For every compact set $K \subset U$ and for all $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ there are constants $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{a-|\beta|}, \quad \forall (x, \xi) \in K \times \mathbb{R}^n. \tag{2.31}$$

Remark 2.3.1. $S^a(U)$ is a Fréchet space with semi-norms given by the smallest constants which can be used in (2.31) (see Sect. 18.1 in [20]).

The symbol σ is *smoothing* if $\sigma \in S^{-\infty}(U) := \bigcap_{a\in\mathbb{R}} S^a(U)$. Thus, moding out by smoothing symbols, given $\sigma_j \in S^{m_j}(U)$ where $m_j \to -\infty$ as $j \to \infty$, we write $\sigma \sim \sum_{j=0}^\infty \sigma_j$ if for every $N \in \mathbb{N}$ there is an integer K_N such that $\sigma - \sum_{j=0}^{K_N} \sigma_j \in S^{-N}(U)$ and we say that $\sum_{j=0}^\infty \sigma_j$ is an asymptotic expansion of the symbol σ.

The symbol $\sigma \in S^a(U)$ is called a *classical symbol of order a* if for any $N \in \mathbb{N}$, there is an integer K_N and there are functions σ_{a-j}, ψ such that

$$\sigma(x,\xi) - \sum_{j=0}^{K_N} \psi(\xi)\sigma_{a-j}(x,\xi) \in S^{-N}(U), \quad \forall (x,\xi) \in U \times \mathbb{R}^n, \tag{2.32}$$

where σ_{a-j} is positively homogeneous in $\xi \in \mathbb{R}^n \setminus 0$ of degree $a-j$, i.e. for all $t > 0$, $\sigma_{a-j}(x, t\xi) = t^{a-j}\sigma_{a-j}(x,\xi)$, and ψ is a smooth cut-off function on $\mathbb{R}^n$ such that $\psi(\xi) = 0$ for all $|\xi| \leq \frac{1}{4}$, $\psi(\xi) = 1$ for all $|\xi| \geq \frac{1}{2}$. In short we write

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \psi(\xi)\sigma_{a-j}(x,\xi). \tag{2.33}$$

We can compare this definition with the one given in Subsection 2.2.2 of a classical symbol in polar coordinates. We denote by $CS^a(U)$ the set of classical symbols of order a on U. Given $\sigma(x,\xi) \in CS^a(U)$ with asymptotic expansion as in (2.33), we call σ_a the *leading symbol of σ*.
By $CS^a_{cc}(\mathbb{R}^n)$ we denote the set of classical symbols of order a with constant coefficients, i.e. its elements do not depend on x but on $\xi \in T^*_x U \cong \mathbb{R}^n$.
By $CS^a_{\text{comp}}(U)$ we denote the set of symbols of order a with x–compact support on U.
If $\sigma \in CS^a(U)$, $\tau \in CS^b(U)$ and $a - b \notin \mathbb{Z}$, then $\sigma + \tau$ is not a classical symbol anymore, but we can consider the whole space of classical symbols on U, $CS(U) := \left\langle \bigcup_{a\in\mathbb{R}} CS^a(U) \right\rangle$, as the linear space generated by $CS^a(U)$ for all $a \in \mathbb{R}$.
The space of symbols with non integer order or with order less than $-n$: $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a(U) \right\rangle$.
The space of symbols of order less than $-n$: $CS^{<-n}(U) := \bigcup_{a<-n} CS^a(U)$.

The set $CS^a(U,W) \cong CS^a(U) \otimes \text{End}(W)$ of symbols of order a on an open subset U of $\mathbb{R}^n$ with values in a Euclidean space W (with norm $\|\cdot\|$) can be equipped with a Fréchet structure. The following semi-norms labelled by multiindices α, β and integers $j \geq 0, N$, give rise to a Fréchet topology on $CS^a(U,W)$:

$$\sup_{x\in K, \xi\in\mathbb{R}^n} (1+|\xi|)^{-a+|\beta|} \left\| \partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi) \right\|, \tag{2.34}$$

$$\sup_{x\in K, \xi\in\mathbb{R}^n} (1+|\xi|)^{-a+N+|\beta|} \left\| \partial_x^\alpha \partial_\xi^\beta \left(\sigma(x,\xi) - \sum_{j=0}^{N-1} \sigma_{a-j}(x,\xi) \right) \right\|, \tag{2.35}$$

$$\sup_{x\in K, |\xi|=1} \left\| \partial_x^\alpha \partial_\xi^\beta \sigma_{a-j}(x,\xi) \right\|. \tag{2.36}$$

where K is a compact set in U. In this topology $\left(C^\infty(U) \widehat{\otimes} CS^a_{cc}(\mathbb{R}^n)\right) \otimes \text{End}(W)$ is dense in $CS^a(U,W)$ (see [35]), where $\widehat{\otimes}$ denotes the completion of the tensor product in this topology (see Chap. 43 in [42]).

2.4 A Mayer-Vietoris sequence

In this section we consider a Mayer-Vietoris sequence argument following [6], to give a description of the cohomology groups of spaces of classical symbols on $\mathbb{R}^n$ with constant coefficients. Let us consider the two open subsets of $\mathbb{R}^n$, $U := B(0,1)$ the open unit ball, and $V := \mathbb{R}^n \setminus 0$. Then $\mathbb{R}^n = U \cup V$ and $U \cap V = B(0,1) \setminus 0$.

Here we use the notation of Section 2.1 and Subsection 2.2.2. Any function in $\mathcal{C}(\mathbb{R}^n \setminus 0)$, defined in (2.6), can be considered as a function on $\mathbb{R}^n$ vanishing at zero, and any function in $C_c^\infty(U)$, the set of smooth functions with compact support in U, can be considered as a function on $\mathbb{R}^n$ vanishing outside U.

By $\mathcal{C}(\mathbb{R}^n)$ we denote the subset of $CS_{cc}(\mathbb{R}^n)$ that corresponds to $\mathcal{C}(\mathbb{R}^n \setminus 0)$ according to the behavior of its elements outside any neighborhood of 0 and (2.6). Hence, if $\mathcal{C}(\mathbb{R}^+ \times S^{n-1})$ is either $\mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ with $a \in \mathbb{R}$, $\mathcal{C}^{-\infty}(\mathbb{R}^+ \times S^{n-1})$, $\mathcal{C}^\infty(\mathbb{R}^+ \times S^{n-1})$, $\bigcup_{a\in\mathbb{Z}} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1})$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-1,+\infty)} \mathcal{C}^a(\mathbb{R}^+ \times S^{n-1}) \right\rangle$, then $\mathcal{C}(\mathbb{R}^n)$ is respectively either $CS^a_{cc}(\mathbb{R}^n)$ with $a \in \mathbb{R}$, $CS^{-\infty}_{cc}(\mathbb{R}^n)$, $CS_{cc}(\mathbb{R}^n)$, $\bigcup_{a\in\mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-1,+\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$.

If we consider the space $\mathbb{R}^n$ with coordinates $(\xi_1, \dots, \xi_n)$, any of the sets $\mathcal{C}(\mathbb{R}^n)$ satisfies $CS^{-\infty}_{cc}(\mathbb{R}^n) \subseteq \mathcal{C}(\mathbb{R}^n) \subseteq CS_{cc}(\mathbb{R}^n)$ and is stable under partial derivatives, i.e. for all $i = 1, \dots, n$,

$$\sigma \in \mathcal{C}(\mathbb{R}^n) \Rightarrow \partial_{\xi_i}\sigma \in \mathcal{C}(\mathbb{R}^n). \tag{2.37}$$

For the cases when $\mathcal{C}(\mathbb{R}^n)$ is either $CS^{-\infty}_{cc}(\mathbb{R}^n)$, $CS_{cc}(\mathbb{R}^n)$, $\bigcup_{a\in\mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-1,+\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$, any k–differential form ω on $\mathbb{R}^n$ with coefficients in $\mathcal{C}(\mathbb{R}^n)$ can be written as

$$\omega = \sum_I \sigma_I \, d\xi_I \in \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)),$$

where $\sigma_I \in \mathcal{C}(\mathbb{R}^n)$, $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and $d\xi_I = d\xi_{i_1} \wedge \dots \wedge d\xi_{i_k}$. We have the differential operator

$$d : \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \Omega^{k+1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$$

defined as follows:

1. If $\sigma \in \Omega^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) = \mathcal{C}(\mathbb{R}^n)$, then $d\sigma = \sum_{i=1}^n \partial_{\xi_i}\sigma \, d\xi_i \in \Omega^1(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$.
2. If $\omega = \sum_I \sigma_I \, d\xi_I \in \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$, then $d\omega = \sum_I d\sigma_I \, d\xi_I \in \Omega^{k+1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$.

The operator d satisfies $d^2 = 0$, and the complex $\{\Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)),\, d\}$ is a subcomplex of the usual de Rham complex $\{\Omega^k(\mathbb{R}^n),\, d\}$. Consider the set of closed

k–forms on $\mathbb{R}^n$ with coefficients in $\mathcal{C}(\mathbb{R}^n)$

$$Z^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) = \ker(d : \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \Omega^{k+1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))),$$

and the set of exact k–forms on $\mathbb{R}^n$ with coefficients in $\mathcal{C}(\mathbb{R}^n)$

$$B^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) = \mathrm{Image}(d : \Omega^{k-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))).$$

The k–th cohomology of the complex $\{\Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)),\, d\}$ is given by

$$H^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) := Z^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))/B^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)).$$

Now, for the case $\mathcal{C}(\mathbb{R}^n) = CS^a_{cc}(\mathbb{R}^n)$,

$$\sigma \in CS^a_{cc}(\mathbb{R}^n) \Rightarrow \partial_{\xi_i}\sigma \in CS^{a-1}_{cc}(\mathbb{R}^n) \subset CS^a_{cc}(\mathbb{R}^n).$$

Moreover, like the sets of differential forms in (2.17), for any $0 \le k \le n$

$$\Omega^k(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)) = \operatorname*{span}_{|I|=k} \left\{\sigma_I(\xi)\, d\xi_I : \sigma_I \in CS^{a-k}_{cc}(\mathbb{R}^n)\right\}, \tag{2.38}$$

and the set of exact k–differential forms is given by

$$B^k(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)) = \operatorname*{span}_{|J|=k-1} \left\{\sum_{i=1}^n \partial_{\xi_i}(\sigma_J)\, d\xi_i \wedge\, d\xi_J : \sigma_J \in CS^{a-k+1}_{cc}(\mathbb{R}^n)\right\}. \tag{2.39}$$

As in Equation (2.7), we have the complex

$$\Omega^k(\mathbb{R}^n \setminus 0,\, \mathcal{C}(\mathbb{R}^n \setminus 0)) = q^*(\Omega^k(\mathcal{C}(\mathbb{R}^+ \times S^{n-1}))).$$

Hence, $\Omega^*(\mathbb{R}^n \setminus 0,\, \mathcal{C}(\mathbb{R}^n \setminus 0))$ is the subcomplex of $\Omega^*(\mathbb{R}^n,\, \mathcal{C}(\mathbb{R}^n))$ consisting of differential forms whose coefficients vanish in a neighborhood of 0. By $\Omega^*_c(U)$ we denote the subcomplex of $\Omega^*(\mathbb{R}^n,\, \mathcal{C}(\mathbb{R}^n))$ consisting of differential forms whose coefficients have compact support in U.

If we have the inclusion $A \to B$, we denote by $j_{A,B} : \Omega^k(A) \to \Omega^k(B)$ the map such that for all $\omega \in \Omega^k(A)$, $j_{A,B}(\omega) \in \Omega^k(B)$ is the form which vanishes outside A and coincides with ω in A.
In view of the inclusions

$$U \cap V \rightrightarrows U \bigsqcup V \to \mathbb{R}^n,$$

we use a Mayer-Vietoris sequence argument ([6]). Indeed we have the following maps

$$\begin{aligned} \Omega^k_c(U \cap V) &\xrightarrow{F} \Omega^k_c(U) \oplus \Omega^k(V, \mathcal{C}(V)) \\ \omega &\mapsto j_{U\cap V,U}(\omega) \oplus j_{U\cap V,V}(\omega); \end{aligned} \tag{2.40}$$

$$\begin{aligned} \Omega^k_c(U) \oplus \Omega^k(V, \mathcal{C}(V)) &\xrightarrow{G} \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \\ \tau \oplus \eta &\mapsto j_{U,\mathbb{R}^n}(\tau) - j_{V,\mathbb{R}^n}(\eta). \end{aligned} \tag{2.41}$$

Lemma 2.4.1. *For all $k = 0, \ldots, n$, the sequence*

$$0 \to \Omega_c^k(U \cap V) \xrightarrow{F} \Omega_c^k(U) \oplus \Omega^k(V, \mathcal{C}(V)) \xrightarrow{G} \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to 0 \tag{2.42}$$

is exact.

Proof. By definition of the maps $j_{U\cap V,U}, j_{U\cap V,V}$, the map F is injective:

$$\ker(F) = \{\omega \in \Omega_c^k(U \cap V) : \; j_{U\cap V,U}(\omega) = 0 = j_{U\cap V,V}(\omega)\} = \{0\}.$$

Now, the null space of G is given by

$$\ker(G) = \{\tau \oplus \eta \in \Omega_c^k(U) \oplus \Omega^k(V, \mathcal{C}(V)) : \; j_{U,\mathbb{R}^n}(\tau) = j_{V,\mathbb{R}^n}(\eta)\},$$

so $\tau = \eta$ on $U \cap V$, and $j_{U,\mathbb{R}^n}(\tau) = 0 = j_{V,\mathbb{R}^n}(\eta)$ on $(U \cap V)^c$. Hence, this set coincides with the image of F.
In order to verify that G is surjective we consider a partition of unity $\{\rho_U, \rho_V\}$ subordinate to the open cover $\{U, V\}$. Let $\omega \in \Omega^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$. Since

$$\operatorname{supp}(\rho_U \omega) \subseteq \operatorname{supp}(\rho_U) \subseteq U,$$

we get $\rho_U \omega \in \Omega_c^k(U)$. By definition of $\mathcal{C}(\mathbb{R}^n)$ and since $\rho_V \equiv 1$ outside U we have $\rho_V \omega \in \Omega^k(V, \mathcal{C}(V))$. Then

$$\begin{aligned} G(\rho_U \omega \oplus (-\rho_V \omega)) &= j_{U,\mathbb{R}^n}(\rho_U \omega) + j_{V,\mathbb{R}^n}(\rho_V \omega) \\ &= \rho_U \omega + \rho_V \omega \\ &= \omega. \end{aligned}$$

□

As a consequence, like in ordinary compact support cohomology (see [6]), the sequence in (2.42) gives rise to a long exact sequence in cohomology:

$$\begin{aligned} &H_c^0(U \cap V) \hookrightarrow H_c^0(U) \oplus H^0(V, \mathcal{C}(V)) \to H^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to H_c^1(U \cap V) \to \cdots \\ &\quad \cdots \to H^{k-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to H_c^k(U \cap V) \to H_c^k(U) \oplus H^k(V, \mathcal{C}(V)) \to \\ &\quad \to H^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to H_c^{k+1}(U \cap V) \to \cdots \to H^{n-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \\ &\quad \to H_c^n(U \cap V) \to H_c^n(U) \oplus H^n(V, \mathcal{C}(V)) \to H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to 0. \end{aligned}$$

The cohomology groups of $\Omega_c^*(U)$ and $\Omega_c^*(U \cap V)$ are given by

$$H_c^k(U) \cong \begin{cases} \mathbb{C}, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \qquad H_c^k(U \cap V) \cong \begin{cases} \mathbb{C}, & \text{if } k = 1, n, \\ 0, & \text{otherwise.} \end{cases}$$

For our examples, by Equations (2.7), (2.23), (2.30) and Corollary 2.2.1, we have

$$H^k(V,\mathcal{C}(V)) \cong \begin{cases} \mathbb{C}, & \text{if } k = 1, n, \\ 0, & \text{otherwise.} \end{cases}$$

From the long exact sequence in cohomology we infer that:

For $n > 2$:

1. For all $k = 2, \ldots, n-2$, $H^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong 0$.

2. From

$$0 \oplus 0 \to H^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C} \to 0 \oplus \mathbb{C} \to H^1(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to 0,$$

   we get

$$H^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong H^1(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)). \tag{2.43}$$

3. From

$$0 \oplus 0 \to H^{n-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C} \xrightarrow{f} \mathbb{C} \oplus \mathbb{C} \to H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to 0,$$

   we get

$$\dim(H^{n-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))) \leq 1 \text{ and}$$

$$\dim(H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))) = \dim(H^{n-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))) + 1. \tag{2.44}$$

For $n = 2$:

$$0 \oplus 0 \to H^0(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \to \mathbb{C} \to 0 \oplus \mathbb{C} \to H^1(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \to \mathbb{C} \xrightarrow{f} \mathbb{C} \oplus \mathbb{C} \to H^2(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \to 0,$$

hence

$$\dim(H^2(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2))) \leq 2 \text{ and}$$

$$\dim(H^2(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2))) = \dim(H^1(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2))) - \dim(H^0(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2))) + 1. \tag{2.45}$$

For any $n \geq 2$ we distinguish the following cases:

(I) $\mathcal{C}(\mathbb{R}^n)$ is either $CS^a_{cc}(\mathbb{R}^n)$ with $a \in \mathbb{Z} \cap [0, +\infty)$, $CS_{cc}(\mathbb{R}^n)$ or $\bigcup\limits_{a \in \mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$, this set contains the constant functions and therefore

$$H^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \mathbb{C}.$$

(II) $\mathcal{C}(\mathbb{R}^n)$ is either $CS^a_{cc}(\mathbb{R}^n)$ with $a \notin \mathbb{Z} \cap [0, +\infty)$, $CS^{-\infty}_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a \notin \mathbb{Z} \cap [-1, +\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$, this set does not contain the constant functions and therefore

$$H^0(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong 0.$$

On the other hand, the linear map $f : \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$, which comes from the map F defined in (2.40) is always injective, hence

$$H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \mathbb{C}. \tag{2.46}$$

Therefore, for $n > 2$ by (2.44) we have $H^{n-1}(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong 0$.
By (2.43) we also obtain

(I) $H^1(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \mathbb{C}$.

(II) $H^1(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong 0$.

For $n = 2$ by (2.45) we obtain

(I) $H^0(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong H^1(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong H^2(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong \mathbb{C}$.

(II) $H^0(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong H^1(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong 0$ and $H^2(\mathbb{R}^2, \mathcal{C}(\mathbb{R}^2)) \cong \mathbb{C}$.

Therefore, we have proved the following:

Theorem 2.4.1. *For $n \geq 2$ the cohomology groups of $\mathbb{R}^n$ with coefficients in a space of classical symbols with constant coefficients are given by:*

(I) If $\mathcal{C}(\mathbb{R}^n)$ is either $CS^a_{cc}(\mathbb{R}^n)$ with $a \in \mathbb{Z} \cap [0, +\infty)$, $CS_{cc}(\mathbb{R}^n)$ or $\bigcup\limits_{a \in \mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$:

$$H^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \begin{cases} 0, & \text{if } k = 2, \ldots, n-1, \\ \mathbb{C}, & \text{if } k = 0, 1, n. \end{cases}$$

(II) If $\mathcal{C}(\mathbb{R}^n)$ is either $CS^a_{cc}(\mathbb{R}^n)$ with $a \notin \mathbb{Z} \cap [0, +\infty)$, $CS^{-\infty}_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a \notin \mathbb{Z} \cap [-1, +\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$:

$$H^k(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \begin{cases} 0, & \text{if } k = 0, \ldots, n-1, \\ \mathbb{C}, & \text{if } k = n. \end{cases}$$

Chapter 3

Closed Linear Forms on Symbols

In this chapter we classify closed linear forms on certain subsets of the space of symbols by using results of Chapter 2 relative to the top cohomology group of the corresponding space. In the first section we recall the definition and properties of the noncommutative residue and the cut-off regularized integral, as closed linear forms in certain spaces of symbols. In the second section we classify closed linear forms on spaces of symbols with constant coefficients (see Corollary 3.2.1, Corollary 3.2.2, Corollary 3.2.3 and Theorem 3.2.1). In the third section, we use this description to classify closed linear forms on the corresponding subsets of the whole space of symbols on a connected subset of $\mathbb{R}^n$, in terms of a leading symbol linear form, the noncommutative residue and the cut-off regularized integral (see Corollary 3.3.1). In the last section, we classify closed linear forms on the space of odd class symbols on $\mathbb{R}^n$ in odd dimensions (see Corollary 3.4.1). We also assume that $n \geq 2$.

3.1 Closed linear forms

In this section we give the definition of a closed linear form in a space of symbols and its relation with the dual of the top cohomology group of the space.

We consider the space $\mathbb{R}^n$ with coordinates $(\xi_1, \ldots, \xi_n)$. Let $\mathcal{A}_n$ be a set $CS_{cc}^{-\infty}(\mathbb{R}^n) \subseteq \mathcal{A}_n \subseteq CS_{cc}(\mathbb{R}^n)$ which is stable under partial derivatives as in (2.37). We say that a functional $\lambda : \mathcal{A}_n \to \mathbb{C}$ is

1. *Linear* if for any $a, b \in \mathbb{C}$, whenever σ, τ and $a\,\sigma + b\,\tau$ belong to $\mathcal{A}_n$ we have
$$\lambda(a\,\sigma + b\,\tau) = a\,\lambda(\sigma) + b\,\lambda(\tau).$$

2. *Closed*, equivalently, it satisfies Stokes' property on $\mathcal{A}_n$, if λ vanishes on

partial derivatives of elements of $\mathcal{A}_n$:

$$\forall \sigma \in \mathcal{A}_n,\ \forall i = 1, \ldots, n: \quad \lambda \circ \partial_{\xi_i}(\sigma) = 0. \tag{3.1}$$

Remark 3.1.1. For an open subset $U \subseteq \mathbb{R}^n$, if $CS^{-\infty}(U) \subseteq \mathcal{B}_n \subseteq CS(U)$, we say that $\mathcal{B}_n$ is stable under partial derivatives, if for all $i = 1, \ldots, n$,

$$\sigma \in \mathcal{B}_n \Rightarrow \partial_{x_i}\sigma,\ \partial_{\xi_i}\sigma \in \mathcal{B}_n,$$

and a functional $\lambda : \mathcal{B}_n \to \mathbb{C}$ is *closed* on $\mathcal{B}_n$, if λ vanishes on partial derivatives of elements of $\mathcal{B}_n$:

$$\forall \sigma \in \mathcal{B}_n, \forall i = 1, \ldots, n: \quad \lambda \circ \partial_{x_i}(\sigma) = 0 \text{ and } \lambda \circ \partial_{\xi_i}(\sigma) = 0. \tag{3.2}$$

As in (2.38), in the case when the space $\mathcal{C}(\mathbb{R}^n)$ defined in Section 2.4 is $\mathcal{C}(\mathbb{R}^n) = CS^a_{cc}(\mathbb{R}^n)$, the coefficients of the corresponding sets of n–differential forms belong to the space $\mathcal{A}_n = CS^{a-n}_{cc}(\mathbb{R}^n)$. By (2.38), the set of n–differential forms on $\mathbb{R}^n$ with coefficients in $\mathcal{A}_n$ is given by

$$\Omega^n(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)) = \left\{ \sum_{|I|<\infty} \sigma_I(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n : \sigma_I \in CS^{a-n}_{cc}(\mathbb{R}^n) \right\}. \tag{3.3}$$

By (2.39), the set of exact n–differential forms on $\mathbb{R}^n$ with coefficients in $\mathcal{A}_n$:

$$B^n(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)) = \left\{ \sum_{i=1}^n \partial_{\xi_i}\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n :\ \sigma \in CS^{a-n+1}_{cc}(\mathbb{R}^n) \right\}. \tag{3.4}$$

Thus, the corresponding n–th cohomology group is given by

$$H^n(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)) := \Omega^n(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n))/B^n(\mathbb{R}^n, CS^a_{cc}(\mathbb{R}^n)).$$

Remark 3.1.2. With the notation of Chapter 2, if q represents the polar coordinate diffeomorphism of (2.5),

$$\sigma \in CS^a_{cc}(\mathbb{R}^n) \ \Rightarrow\ q_*(\sigma \restriction_{\mathbb{R}^n\setminus 0}) r^{n-1} \in \mathcal{C}^{a+n-1}(\mathbb{R}^+ \times S^{n-1}).$$

Hence closed linear forms on $CS^a_{cc}(\mathbb{R}^n)$ come from the top cohomology group $H^n(\mathbb{R}^n, CS^{a+n}_{cc}(\mathbb{R}^n))$, and they are related to linear forms on $\mathcal{C}^{a+n-1}(\mathbb{R}^+ \times S^{n-1})$ which satisfy Assumption 2.1.1. Therefore the condition given for the choice of the map $\mathcal{J}$ in view of the order of the symbols in Corollary 2.2.2 and Corollary 2.2.3 amounts to choosing $a \notin \mathbb{Z} \cap [-n, +\infty)$ for $\mathcal{J} = \text{cf}$, and $a \in \mathbb{Z} \cap [-n, +\infty)$ for $\mathcal{J} = \text{rs}$.

In the case when $\mathcal{C}(\mathbb{R}^n)$ is either $CS^{-\infty}_{cc}(\mathbb{R}^n)$, $CS_{cc}(\mathbb{R}^n)$, $\bigcup_{a\in\mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$, the coefficients of the corresponding sets of differential forms belong to the space $\mathcal{A}_n := \mathcal{C}(\mathbb{R}^n)$. The set of n–differential forms on $\mathbb{R}^n$ with coefficients in $\mathcal{A}_n$:

$$\Omega^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) := \left\{ \sum_{|I|<\infty} \sigma_I(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n :\ \sigma_I \in \mathcal{A}_n \right\},$$

as well as the set of exact n–differential forms on $\mathbb{R}^n$ with coefficients in $\mathcal{A}_n$:

$$B^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) = \left\{ \sum_{i=1}^{n} \partial_{\xi_i}\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n : \ \sigma \in \mathcal{A}_n \right\}.$$

Thus, the corresponding n–th cohomology group is given by

$$H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) := \Omega^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))/B^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)).$$

With this notation of the sets $\mathcal{C}(\mathbb{R}^n)$ and $\mathcal{A}_n$, we prove the following lemma, which explains the use of the term "closed" in (3.1):

Lemma 3.1.1 (See Lemma 1 in [35])**.** *If $\lambda : \mathcal{A}_n \to \mathbb{C}$ is a closed linear functional on $\mathcal{A}_n$, then it induces a well-defined linear form $\widetilde{\lambda} : H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C}$. Moreover, any linear form $\widetilde{\lambda} : H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C}$ induces a closed linear form $\lambda : \mathcal{A}_n \to \mathbb{C}$.*

Proof. Any functional $\lambda : \mathcal{A}_n \to \mathbb{C}$ induces a linear form $\widetilde{\lambda} : H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C}$ in the following way: If $\omega \in H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$ is represented by the top degree differential form $\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n \in \Omega^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$, then one can define

$$\widetilde{\lambda}(\omega) := \lambda(\sigma).$$

Similarly, any linear form $\widetilde{\lambda} : H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C}$ induces a linear functional $\lambda : \mathcal{A}_n \to \mathbb{C}$ as follows: If $\sigma \in \mathcal{A}_n$, consider the cohomology class of the top degree differential form $\omega = \sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n \in \Omega^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$, and define

$$\lambda(\sigma) := \widetilde{\lambda}([\omega]).$$

For any $\sigma \in \mathcal{A}_n$ and for all $i = 1, \dots, n$

$$\begin{aligned} d(\sigma\, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge\, d\xi_n) &= (-1)^i \partial_{\xi_i}\sigma\, d\xi_1 \wedge \cdots \wedge d\xi_n \\ \Longrightarrow \widetilde{\lambda}([d(\sigma\, d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge\, d\xi_n)]) &= (-1)^i \lambda(\partial_{\xi_i}\sigma). \end{aligned}$$

Then λ is closed on $\mathcal{A}_n$ if and only if $\widetilde{\lambda}$ is well defined on $H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$. □

3.1.1 The noncommutative residue

Let U be an open subset of $\mathbb{R}^n$. In Definition 1.2.1 we defined the symplectic residue of homogeneous functions of degree $-n$, then for a symbol $\sigma \in CS(U)$ with asymptotic expansion $\sigma \sim \sum_{j=0}^{\infty} \sigma_{a-j}$ the *noncommutative residue* of σ is the symplectic residue of its homogeneous component of order $-n$:

$$\operatorname{res}(\sigma) := \operatorname{res}(\sigma_{-n}).$$

Consider the symplectic cone $T^*U \setminus U \to S^*U$ as in Remark 1.1.2, so if $\overline{\mu}(\xi)$ is a volume form on S^*_xU, the noncommutative residue of σ is (see [44]):

$$\operatorname{res}(\sigma) = \int_U \int_{S^*_xU} \sigma_{-n}(x, \xi)\, \overline{\mu}(\xi) \wedge dx. \tag{3.5}$$

Clearly, $\operatorname{res}(\sigma) = 0$ whenever $\operatorname{ord}(\sigma) \notin \mathbb{Z} \cap [-n, +\infty)$.

Proposition 3.1.1. *For any $\sigma \in CS(U)$, for all $i = 1, \dots, n$,* $\mathrm{res}(\partial_{x_i}\sigma) = 0$ *and* $\mathrm{res}(\partial_{\xi_i}\sigma) = 0$.

Proof. See Sect. 1 in [10], Lemma 2.6 in [25] and Sect. 1.3 in [35]. $\square$

Remark 3.1.3. If $\sigma \in CS_{cc}(\mathbb{R}^n)$, its noncommutative residue reads

$$\mathrm{res}(\sigma) = \int_{S^{n-1}} \sigma_{-n}(\xi)\,\overline{\mu}(\xi).$$

Using the notation of (2.24) we have the following:

Lemma 3.1.2. *For all $\sigma \in CS_{cc}(\mathbb{R}^n)$,* $\mathrm{res}(\sigma) = \int_{S^{n-1}} \mathrm{rs}(q_*(\sigma \restriction_{\mathbb{R}^n\setminus 0})r^{n-1})(\eta)\,\overline{\mu}(\eta)$.

Proof. Using polar coordinates, by (2.25) and Remark 2.2.7 we have

$$\begin{aligned}\mathrm{res}(\sigma) &= \int_{S^{n-1}} \sigma_{-n}(\eta)\,\overline{\mu}(\eta)\\ &= \int_{S^{n-1}} \mathrm{rs}(q_*(\sigma \restriction_{\mathbb{R}^n\setminus 0})r^{n-1})(\eta)\,\overline{\mu}(\eta).\end{aligned}$$

$\square$

3.1.2 The cut-off regularized integral

Let $k_n \in \mathbb{N}$ be the smallest integer such that $a - k_n < -n+1$, i.e. $k_n = \lfloor a \rfloor + n$. As in (2.32), a symbol σ is classical of order a ($\sigma \in CS^a_{cc}(\mathbb{R}^n)$) if there exist a cut-off function ψ which vanishes in a neighborhood of 0 and is identically 1 outside the unit ball, and functions σ_{a-j} positively homogeneous of degree $a-j$, such that

$$\sigma = \sum_{j=0}^{k_n} \psi\sigma_{a-j} + \pi^{\psi}_{<-n}(\sigma). \tag{3.6}$$

For any $R > 0$, $B(0,R)$ denotes the ball of radius R centered at 0 in $\mathbb{R}^n$. The map $R \mapsto \int_{B(0,R)} \sigma$ has an asymptotic expansion as $R \to \infty$ of the form

$$\int_{B(0,R)} \sigma \underset{R\to\infty}{\sim} \alpha_0(\sigma) + \sum_{\substack{j=0\\ a-j\neq -n}}^{k_n} \sigma_{a-j} R^{a-j+n} + \mathrm{res}(\sigma)\log R, \tag{3.7}$$

Definition 3.1.1 ((5.9) in [25], Sect. 1.4 in [35])**.** The *cut-off regularized integral* (or Hadamard partie finie) of σ is the constant term $\alpha_0(\sigma)$ in (3.7), given by

$$\begin{aligned}&⨍_{\mathbb{R}^n} \sigma(\xi)d\xi := \underset{R\to\infty}{\mathrm{fp}} \int_{B(0,R)} \sigma =\\ &= \int_{\mathbb{R}^n} \pi^{\psi}_{<-n}(\sigma) + \sum_{j=0}^{k_n} \int_{B(0,1)} \psi\sigma_{a-j} - \sum_{\substack{j=0\\ a-j\neq -n}}^{k_n} \frac{1}{a-j+n} \int_{S(0,1)} \sigma_{a-j}. \end{aligned}\tag{3.8}$$

Lemma 3.1.3. *The cut-off integral* $-\!\!\!\!\!\!\int_{\mathbb{R}^n}\sigma$ *does not depend on the choice of the cut-off function in the asymptotic expansion of* σ.

Proof. Let us consider ψ, χ smooth cut-off functions, which vanish in a neighborhood of $\xi = 0$ and which are 1 for $|\xi| \geq 1$. The support of the difference $\psi - \chi$ is a compact set contained in $B(0,1)$. If we consider the difference of the terms in the right hand side of (3.8) with the cut-off functions ψ and χ we get the expression

$$\int_{\mathbb{R}^n} (\pi^{\psi}_{<-n}(\sigma) - \pi^{\chi}_{<-n}(\sigma)) + \sum_{j=0}^{k_n} \int_{B(0,1)} (\psi - \chi)\sigma_{a-j}. \tag{3.9}$$

By (3.6),

$$\pi^{\psi}_{<-n}(\sigma) - \pi^{\chi}_{<-n}(\sigma) = -\sum_{j=0}^{k_n} (\psi - \chi)\sigma_{a-j},$$

and since the support of $\psi - \chi$ is contained in $B(0,1)$, the expression in (3.9) vanishes. □

Let us recall the following equivalent version of Stokes' theorem on Riemannian manifolds:

Lemma 3.1.4 (Cor. 8.2.10 in [1])**.** *Let M be a Riemannian manifold, such that M and ∂M carry correspond uniquely determined volume forms μ_M and $\mu_{\partial M}$. If X is a smooth vector field with compact support on M, and if $\vec{n}$ is the unit outer normal vector field on ∂M, then*

$$\int_M \mathrm{Div}(X)\mu_M = \int_{\partial M} \langle X, \vec{n}\rangle \mu_{\partial M}.$$

We use this to describe the action of the cut-off regularized integral on partial derivatives. The following lemma is a particular case of Lemma 5.5 in [27]; we denote by $\mu_{S(0,R)}(\xi)$ a volume form on $S(0,R)$:

Proposition 3.1.2. *Let* $\sigma = \sum_{j=0}^{k_n} \psi\, \sigma_{a-j} + \pi^{\psi}_{<-n}(\sigma) \in CS^a_{cc}(\mathbb{R}^n)$ *be as in* (3.6)*. Then, for all* $i = 1, \dots, n$

$$-\!\!\!\!\!\!\int_{\mathbb{R}^n} \partial_{\xi_i}\sigma(\xi) d\xi = \int_{S(0,1)} \sigma_{-n+1}(\xi)\, \xi_i\, \mu_{S(0,1)}(\xi).$$

Proof. We apply Lemma 3.1.4 to $M = B(0,R) := \{\xi \in \mathbb{R}^n : |\xi| \leq R\}$ for some positive number R, with boundary $S(0,R) = \{\xi \in \mathbb{R}^n : |\xi| = R\}$, and to the vector field $X = \sigma e_i$, where $(e_1, \dots, e_n)$ is the canonical orthonormal basis of

$\mathbb{R}^n$. This yields, for $\vec{n} = \dfrac{\xi}{|\xi|}$:

$$
\begin{aligned}
\int_{B(0,R)} \partial_{\xi_i}\sigma(\xi)\, d\xi &= \int_{S(0,R)} \sigma(\xi)\, \langle e_i, \vec{n}\rangle\, \mu_{S(0,R)}(\xi) \\
&= \int_{S(0,R)} \sigma(\xi)\, \frac{\xi_i}{|\xi|}\, \mu_{S(0,R)}(\xi) \\
&= \int_{S(0,1)} \sigma(R\eta)\, \eta_i\, R^{n-1} \mu_{S(0,1)}(\eta) \\
&= \sum_{j=0}^{k_n} \int_{S(0,1)} \psi(R\eta)\sigma_{a-j}(R\eta)\, \eta_i\, R^{n-1} \mu_{S(0,1)}(\eta) \\
&\qquad + \int_{S(0,1)} \pi^{\psi}_{<-n}(\sigma)(R\eta)\, \eta_i\, R^{n-1} \mu_{S(0,1)}(\eta) \\
&= \sum_{j=0}^{k_n} \int_{S(0,1)} \psi(R\eta)\sigma_{a-j}(\eta)\, \eta_i\, R^{a-j+n-1} \mu_{S(0,1)}(\eta) \\
&\qquad + R^{n-1} \int_{S(0,1)} \pi^{\psi}_{<-n}(\sigma)(R\eta)\, \eta_i\, \mu_{S(0,1)}(\eta).
\end{aligned}
$$

We reach the statement from

$$
\lim_{R\to\infty} R^{n-1} \int_{S(0,1)} \pi^{\psi}_{<-n}(\sigma)(R\eta)\, \eta_i\, \mu_{S(0,1)}(\eta) = 0,
$$

and

$$
\begin{aligned}
\operatorname*{fp}_{R\to\infty} \int_{S(0,1)} \psi(R\eta)\sigma_{a-j}(\eta)\, \eta_i\, R^{a-j+n-1} \mu_{S(0,1)}(\eta) &= \\
&= \begin{cases} 0, & \text{if } a-j+n-1 \neq 0, \\ \int_{S(0,1)} \sigma_{a-j}(\eta)\, \eta_i\, \mu_{S(0,1)}(\eta), & \text{if } a-j+n-1 = 0. \end{cases}
\end{aligned}
$$

□

Corollary 3.1.1. *If $a \notin \mathbb{Z} \cap [-n+1, +\infty)$, the cut-off regularized integral is closed on $CS^a_{cc}(\mathbb{R}^n)$.*

Proof. It follows from Proposition 3.1.2, since in this case, given $\sigma \in CS^a_{cc}(\mathbb{R}^n)$, the homogeneous component σ_{-n+1} vanishes. Compare with (2.21) □

Using the notation of (2.19) we have the following, where $\overline{\mu}(\eta)$ is a volume form on S^{n-1}:

Lemma 3.1.5. *For any $\sigma \in \left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$, the cut-off regularized integral of σ satisfies $⨍_{\mathbb{R}^n} \sigma(\xi) d\xi = \int_{S^{n-1}} \mathrm{cf}(q_*(\sigma \restriction_{\mathbb{R}^n\setminus 0}) r^{n-1})(\eta) \overline{\mu}(\eta)$.*

Proof. Using polar coordinates, by Remark 2.2.6 we have

$$\begin{aligned}
-\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(\xi)d\xi &= \operatorname*{fp}_{R\to\infty} \int_{B(0,R)} \sigma(\xi)d\xi \\
&= \int_{S^{n-1}} \mathrm{cf}(q_*(\sigma \restriction_{\mathbb{R}^n\setminus 0}) r^{n-1})(\eta)\overline{\mu}(\eta).
\end{aligned}$$

□

Remark 3.1.4. Given an open subset $U \subseteq \mathbb{R}^n$, the cut-off regularized integral of a symbol $\sigma \in CS(U)$ is given by

$$\int_U -\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(x,\xi)d\xi\, dx.$$

Proposition 3.1.3 (Prop. 5.2 in [25])**.** *Let $\sigma \in CS(\mathbb{R}^n)$ be a symbol with asymptotic expansion as in* (3.6) *and let $C \in GL(n,\mathbb{R})$. We have the transformation rule (we omit the integration on U and the variable x)*

$$-\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(C\xi)d\xi = |\det(C)|^{-1}\left(-\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(\xi)d\xi - \int_{S^{n-1}} \sigma_{-n}(\xi)\log(C^{-1}\xi)\,d\xi\right). \tag{3.10}$$

Remark 3.1.5. This proposition implies that in $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a(U)\right\rangle$, the cut-off regularized integral may be used to define a global density on a manifold.

3.2 Closed linear forms on classes of symbols with constant coefficients

In this section we classify closed linear forms on classes of symbols with constant coefficients on $\mathbb{R}^n$ in terms of a leading symbol linear form, the noncommutative residue and the cut-off regularized integral.

First of all, consider the case when $\mathcal{C}(\mathbb{R}^n)$ is either $CS_{cc}^{-\infty}(\mathbb{R}^n)$, $CS_{cc}(\mathbb{R}^n)$, $\bigcup_{a\in\mathbb{Z}} CS^a_{cc}(\mathbb{R}^n)$ or $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{cc}(\mathbb{R}^n)\right\rangle$ (the coefficients of the corresponding sets of differential forms belong to the space $\mathcal{A}_n := \mathcal{C}(\mathbb{R}^n)$). The top cohomology group of $\mathcal{C}(\mathbb{R}^n)$ is one-dimensional. Consider the functional $\mathcal{I} : \mathcal{A}_n \to \mathbb{C}$ that induces the isomorphism $H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \cong \mathbb{C}$ in (2.46), as in Lemma 3.1.1.

Results in [30], [35] provide conditions under which one can express a classical symbol as a sum of derivatives, however, this result is up to smoothing symbols (see Proposition 3.4.2 below). Using the fact that the top cohomology group is one-dimensional, we can express a complete classical symbol with constant coefficients as a sum of derivatives under an appropriate condition:

Lemma 3.2.1. *Let $\sigma \in \mathcal{A}_n$, and suppose that $\mathcal{I}(\sigma) = 0$. Then there exist symbols $\tau_i \in \mathcal{A}_n$, such that $\sigma = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i$.*

Proof. Let $\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n \in \Omega^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$. If $\mathcal{I}(\sigma) = 0$, by Lemma 3.1.1, the isomorphism $\widetilde{I} : H^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n)) \to \mathbb{C}$ satisfies $\widetilde{I}([\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n]) = 0$, which implies that $\sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_n \in B^n(\mathbb{R}^n, \mathcal{C}(\mathbb{R}^n))$, so there exist symbols $\tau_i \in \mathcal{A}_n$ such that $\sigma(\xi) = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i(\xi)$. □

Proposition 3.2.1. *Any closed linear form on $\mathcal{A}_n$ is a multiple of $\mathcal{I}$.*

Proof. Consider a function $e \in \mathcal{A}_n$ such that $\mathcal{I}(e) = 1$. Then any symbol $\sigma \in \mathcal{A}_n$ can be expressed as a linear combination of a symbol on which $\mathcal{I}$ vanishes and the term $\mathcal{I}(\sigma)\, e$:

$$\sigma = (\sigma - \mathcal{I}(\sigma)\, e) + \mathcal{I}(\sigma)\, e.$$

Hence, by Lemma 3.2.1 there exist functions $\tau_i \in \mathcal{A}_n$ such that

$$\sigma = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i + \mathcal{I}(\sigma)\, e. \tag{3.11}$$

Let $\lambda : \mathcal{A}_n \to \mathbb{C}$ be a closed linear form on $\mathcal{A}_n$. Applying λ to both sides of (3.11), we get

$$\lambda(\sigma) = \mathcal{I}(\sigma)\lambda(e). \tag{3.12}$$

□

In the following we provide examples of sets $\mathcal{A}_n$ which admit a unique (up to a constant) closed linear functional λ, as a consequence of (2.46).

Corollary 3.2.1. *If $\mathcal{A}_n = CS_{cc}^{-\infty}(\mathbb{R}^n)$ then for any $\sigma \in \mathcal{A}_n$,*

$$\mathcal{I}(\sigma) = \int_{\mathbb{R}^n} \sigma(\xi) d\xi.$$

Thus, any closed linear form λ on $\mathcal{A}_n$ is proportional to the usual integral on $\mathbb{R}^n$, that is, there exists a constant $c \in \mathbb{C}$ such that for all $\sigma \in CS_{cc}^{-\infty}(\mathbb{R}^n)$, $\lambda(\sigma) = c \int_{\mathbb{R}^n} \sigma$.

Proof. The linear form $\int_{\mathbb{R}^n}$ is closed on $\mathcal{A}_n$: if $\sigma \in \mathcal{A}_n$, then for all $i = 1, \ldots, n$, by the decreasing condition of the functions in $\mathcal{A}_n$ and by Fubini's theorem, with the notation $\overline{d\xi} := d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \partial_{\xi_i} \sigma(\xi)\, d\xi_1 \wedge \cdots \wedge d\xi_i \wedge \cdots \wedge d\xi_n = \\
&= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \partial_{\xi_i} \sigma(\xi)\, d\xi_i \right) d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n \\
&= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\lim_{R \to \infty} \sigma(\xi_1, \ldots, R, \ldots, \xi_n) - \lim_{r \to -\infty} \sigma(\xi_1, \ldots, r, \ldots, \xi_n) \right) \overline{d\xi} \\
&= 0.
\end{aligned}$$

We conclude the statement from Proposition 3.2.1, where the symbol e can be any smooth function on $\mathbb{R}^n$ with compact support and $\int_{\mathbb{R}^n} e = 1$. $\square$

Corollary 3.2.2. *If* $\mathcal{A}_n = \left\langle \bigcup_{a \notin \mathbb{Z} \cap [-n, +\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$ *for any* $\sigma \in \mathcal{A}_n$,

$$\mathcal{I}(\sigma) = -\!\!\!\!\!\!\int_{\mathbb{R}^n} \sigma(\xi) d\xi.$$

Thus, any closed linear form on $\mathcal{A}_n$ *is proportional to the cut-off regularized integral on* $\mathbb{R}^n$.

Proof. By Corollary 3.1.1, the cut-off regularized integral is a closed linear form on $\mathcal{A}_n$. We conclude the statement from Proposition 3.2.1, where the symbol e can be any smooth function on $\mathbb{R}^n$ with compact support and $\int_{\mathbb{R}^n} e = 1$. Notice that in this case $\mathcal{A}_n \subseteq \ker(\text{res})$. If we restrict to $CS^{-\infty}_{cc}(\mathbb{R}^n)$ the constant in Corollary 3.2.1 is $c = 1$. $\square$

Corollary 3.2.3. *If* $\mathcal{A}_n \in \left\{ CS_{cc}(\mathbb{R}^n), \bigcup_{a \in \mathbb{Z}} CS^a_{cc}(\mathbb{R}^n) \right\}$, *for any* $\sigma \in \mathcal{A}_n$,

$$\mathcal{I}(\sigma) = \text{res}(\sigma).$$

Thus, any closed linear form on $\mathcal{A}_n$ *is proportional to the noncommutative residue.*

Proof. By Proposition 3.1.1 and Remark 3.1.3, the noncommutative residue is a closed linear form on $\mathcal{A}_n$. We conclude the statement from Proposition 3.2.1, where the symbol e can be $e(\xi) = \frac{1}{\text{Vol}(S^{n-1})} \psi(\xi) |\xi|^{-n}$, for any cut-off function ψ which vanishes in a small neighborhood of $|\xi| = 0$ and is 1 outside the unit ball. If we restrict to $CS^{-\infty}_{cc}(\mathbb{R}^n)$ the constant in Corollary 3.2.1 is $c = 0$. $\square$

Let us now fix a real number a. The leading symbol map on S^{n-1}:

$$\begin{aligned} L_a : CS^a_{cc}(\mathbb{R}^n) &\to C^\infty(S^{n-1}) \\ \sigma \sim \sum_{j=0}^\infty \psi \, \sigma_{a-j} &\mapsto \sigma_a \restriction_{S^{n-1}}, \end{aligned}$$

leads us to consider a linear form, which we call a *leading symbol linear form* in the following way: If ρ is any linear functional on S^{n-1}, then

$$\begin{aligned} \rho \circ L_a : CS^a_{cc}(\mathbb{R}^n) &\to \mathbb{C} \\ \sigma &\mapsto \rho(\sigma_a \restriction_{S^{n-1}}), \end{aligned}$$

is a linear form on $CS^a_{cc}(\mathbb{R}^n)$ since the leading symbol map is linear, and it is closed since for any $\sigma \in CS^a_{cc}(\mathbb{R}^n)$, for all $i = 1, \ldots, n$, $\partial_{\xi_i} \sigma$ is of order $a - 1$ and hence $L_a(\partial_{\xi_i} \sigma) = 0$.
In (2.46) we found that the top cohomology group of $CS^{a+n}_{cc}(\mathbb{R}^n)$ is one dimensional. Consider the functional $\mathcal{I} : CS^a_{cc}(\mathbb{R}^n) \to \mathbb{C}$ constructed as in Lemma 3.1.1, that induces the isomorphism $H^n(\mathbb{R}^n, CS^{a+n}_{cc}(\mathbb{R}^n)) \cong \mathbb{C}$. Then we get:

Theorem 3.2.1. *Any closed linear form on $CS^a_{cc}(\mathbb{R}^n)$ can be written either as a linear combination of a leading symbol linear form and the cut-off regularized integral when $a \notin \mathbb{Z} \cap [-n+1, +\infty)$, or as a linear combination of a leading symbol linear form and the noncommutative residue when $a \in \mathbb{Z} \cap [-n+1, +\infty)$.*

Proof. From (3.4) and Lemma 3.2.1, for any $\sigma \in CS^a_{cc}(\mathbb{R}^n)$ such that $\mathcal{I}(\sigma) = 0$, there exist symbols $\tau_i \in CS^{a+1}_{cc}(\mathbb{R}^n)$ such that $\sigma = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i$.
For any $\sigma \sim \sum_{j=0}^{\infty} \psi\, \sigma_{a-j} \in CS^a_{cc}(\mathbb{R}^n)$ we have

$$\sigma - \psi\sigma_a \in CS^{a-1}_{cc}(\mathbb{R}^n).$$

If as in Proposition 3.2.1, the symbol $e \in CS^a_{cc}(\mathbb{R}^n)$ is such that $\mathcal{I}(e) = 1$, then

$$\sigma - \psi\sigma_a = (\sigma - \psi\sigma_a - \mathcal{I}(\sigma - \psi\sigma_a)\, e) + \mathcal{I}(\sigma - \psi\sigma_a)\, e.$$

Hence, there exist symbols $\tau_i \in CS^a_{cc}(\mathbb{R}^n)$ such that

$$\sigma - \psi\sigma_a = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i + \mathcal{I}(\sigma - \psi\sigma_a)\, e. \tag{3.13}$$

Let $\lambda : CS^a_{cc}(\mathbb{R}^n) \to \mathbb{C}$ be a closed linear form on $CS^a_{cc}(\mathbb{R}^n)$. Applying λ to both sides of (3.13) we get

$$\lambda(\sigma - \psi\sigma_a) = \mathcal{I}(\sigma - \psi\sigma_a)\lambda(e), \tag{3.14}$$

which implies that

$$\lambda(\sigma) = \lambda(\psi\sigma_a) - \mathcal{I}(\psi\sigma_a)\lambda(e) + \mathcal{I}(\sigma)\lambda(e). \tag{3.15}$$

Then there exists a linear functional ρ on S^{n-1} and a constant $c \in \mathbb{C}$ such that

$$\lambda(\sigma) = \rho(L_a(\sigma)) + c\,\mathcal{I}(\sigma). \tag{3.16}$$

1. If $a \notin \mathbb{Z} \cap [-n+1, +\infty)$, $\mathcal{I}$ is the map in Corollary 3.2.2; the symbol $e \in CS^a_{cc}(\mathbb{R}^n)$ can be any smooth function on $\mathbb{R}^n$ with compact support and $\int_{\mathbb{R}^n} e = 1$. Thus, any closed linear form on $CS^a_{cc}(\mathbb{R}^n)$ is a linear combination of a leading symbol linear form and the cut-off regularized integral.

2. If $a \in \mathbb{Z} \cap [-n+1, +\infty)$, $\mathcal{I}$ is the map in Corollary 3.2.3; for any cut-off function ψ which vanishes in a small neighborhood of $|\xi| = 0$ and is 1 outside the unit ball, the symbol $e(\xi) = \frac{1}{\mathrm{Vol}(S^{n-1})}\psi(\xi)|\xi|^{-n} \in CS^a_{cc}(\mathbb{R}^n)$. Thus, any closed linear form on $CS^a_{cc}(\mathbb{R}^n)$ is a linear combination of a leading symbol linear form and the noncommutative residue.

□

Proposition 3.2.2. *If $a \in \mathbb{Z} \cap [-n+1, +\infty)$, any smoothing symbol can be written as a sum of derivatives of elements in $CS^a_{cc}(\mathbb{R}^n)$.*

Proof. If $a \in \mathbb{Z} \cap [-n, +\infty)$, the functional res : $CS^a_{cc}(\mathbb{R}^n) \to \mathbb{C}$ induces the isomorphism $H^n(\mathbb{R}^n, CS^{a+n}_{cc}(\mathbb{R}^n)) \cong \mathbb{C}$. Since the noncommutative residue vanishes on smoothing symbols, from Equation (3.4) and Lemma 3.2.1, for any symbol $\sigma \in CS^{-\infty}_{cc}(\mathbb{R}^n)$, there exist symbols $\tau_i \in CS^{a+1}_{cc}(\mathbb{R}^n)$ such that $\sigma = \sum_{i=1}^{n} \partial_{\xi_i} \tau_i$. $\square$

3.3 Closed linear forms on classes of symbols on $\mathbb{R}^n$

In this section we study closed linear forms on some subsets of the space $CS_{\text{comp}}(U)$ (see Section 2.3) of classical symbols with x–compact support on a subset $U \subseteq \mathbb{R}^n$, using the classification of closed linear forms on the corresponding subsets of $CS_{cc}(\mathbb{R}^n)$.

Let U be a connected subset of $\mathbb{R}^n$ with $n \geq 2$. Then, by the Poincaré Lemma (see [6]), the top cohomology group with compact support of U is one–dimensional, $H^n_c(U) \cong \mathbb{C}$ and the isomorphism is given by integration over U:

$$\begin{aligned} C^\infty_c(U) &\to \mathbb{C} \\ f &\mapsto \int_U f(x)\, dx, \end{aligned}$$

where dx denotes a volume form on $\mathbb{R}^n$. So any closed linear form on $C^\infty_c(U)$ is proportional to the integral over U.

Let $\mathcal{D}(U)$ be a subset of $CS_{\text{comp}}(U)$ stable under partial derivatives $\partial_{x_k}, \partial_{\xi_k}$, $k = 1, \ldots, n$, as in Remark 3.1.1, and under multiplication by functions in $C^\infty_c(U)$. Any element in $\mathcal{D}(U)$ can be considered as a function on $T^*U = U \times \mathbb{R}^n$. We define

$$\mathcal{S}(U) := \{\tau \in CS_{cc}(\mathbb{R}^n) : f \otimes \tau \in \mathcal{D}(U) \;\; \forall f \in C^\infty_c(U)\}.$$

The set $\mathcal{S}(U)$ is stable under partial differentiation ∂_{ξ_k}, as in (3.1).

For the rest of this section we assume that $\mathcal{S}(U)$, $\mathcal{D}(U)$ correspond to one of the following examples, here $\lambda : \mathcal{S}(U) \to \mathbb{C}$ is a closed linear functional as in Section 3.2.

Example 3.3.1.
1. $\mathcal{D}(U) = CS_{\text{comp}}(U)$, $\mathcal{S}(U) = CS_{cc}(\mathbb{R}^n)$, $\lambda = \text{res}$.
2. $\mathcal{D}(U) = CS^{-\infty}_{\text{comp}}(U)$, $\mathcal{S}(U) = CS^{-\infty}_{cc}(\mathbb{R}^n)$, $\lambda = \int_{\mathbb{R}^n}$.

3. $\mathcal{D}(U) = \left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{\mathrm{comp}}(U) \right\rangle$, $\mathcal{S}(U) = \left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{cc}(\mathbb{R}^n) \right\rangle$, $\lambda = -\!\!\!\!\!\!\int_{\mathbb{R}^n}$.

4. If $a \notin \mathbb{Z} \cap [-n+1, +\infty)$ and $\mathcal{D}(U) = CS^a_{\mathrm{comp}}(U)$, $\mathcal{S}(U) = CS^a_{cc}(\mathbb{R}^n)$, $\lambda = \rho \circ L_a + c -\!\!\!\!\!\!\int_{\mathbb{R}^n}$.

5. If $a \in \mathbb{Z} \cap [-n+1, +\infty)$ and $\mathcal{D}(U) = CS^a_{\mathrm{comp}}(U)$, $\mathcal{S}(U) = CS^a_{cc}(\mathbb{R}^n)$, $\lambda = \rho \circ L_a + c\,\mathrm{res}$.

Let us assume that U is compact. The space $C^\infty_c(U)$ has a Fréchet space structure (see e.g. Sect. 10 in [42]), and in the previous examples the space $C^\infty_c(U)\widehat{\otimes}\mathcal{S}(U)$ is dense in $\mathcal{D}(U)$ for the Fréchet topology of symbols of constant order defined in Section 2.3. From the density of $C^\infty_c(U)\widehat{\otimes}CS^a_{cc}(\mathbb{R}^n)$ in $CS^a_{\mathrm{comp}}(U)$ we infer that for any $\sigma \in CS^a_{\mathrm{comp}}(U)$, for all $N \in \mathbb{N}$ there exists $K_N \in \mathbb{N}$, such that for all $i = 1, \ldots, K_N$, there exist $f_i \in C^\infty_c(U)$, $\tau_i \in CS^a_{cc}(\mathbb{R}^n)$ which in the Fréchet topology of $CS^a_{cc}(\mathbb{R}^n)$ satisfy

$$\sum_{i=1}^{K_N} f_i \otimes \tau_i \underset{N\to\infty}{\longrightarrow} \sigma, \tag{3.17}$$

this means that for all $\alpha, \beta \in \mathbb{N}^n$,

$$\sum_{i=1}^{K_N} \partial^\alpha_x f_i \otimes \partial^\beta_\xi \tau_i \underset{N\to\infty}{\longrightarrow} \partial^\alpha_x \partial^\beta_\xi \sigma. \tag{3.18}$$

From this density it is also possible to define a linear form on $CS^a_{\mathrm{comp}}(U)$ by defining it on the tensor product space $C^\infty_c(U) \otimes CS^a_{cc}(\mathbb{R}^n)$ and then extending it by continuity.

Proposition 3.3.1. *Let $\lambda : \mathcal{S}(U) \to \mathbb{C}$ be a closed linear functional on $\mathcal{S}(U)$. The linear form*

$$\begin{aligned} \Lambda : C^\infty_c(U) \otimes \mathcal{S}(U) &\to \mathbb{C} \\ f \otimes \tau &\mapsto \left(\int_U f \right) \cdot \lambda(\tau), \end{aligned}$$

defines a closed linear form on $\mathcal{D}(U)$.

Proof. We have to prove that for any $f \otimes \tau \in C^\infty_c(U) \otimes \mathcal{S}(U)$ and for all $k = 1, \ldots, n$, $\Lambda(\partial_{x_k}(f \otimes \tau)) = 0$ and $\Lambda(\partial_{\xi_k}(f \otimes \tau)) = 0$. Indeed,

$$\begin{aligned} \Lambda(\partial_{x_k}(f \otimes \tau)) &= \left(\int_U \partial_{x_k} f \right) \cdot \lambda(\tau) \\ &= 0 \cdot \lambda(\tau) \\ &= 0, \end{aligned}$$

and similarly

$$\begin{aligned}\Lambda(\partial_{\xi_k}(f \otimes \tau)) &= \left(\int_U f\right) \cdot \lambda(\partial_{\xi_k}\tau) \\ &= \left(\int_U f\right) \cdot 0 \\ &= 0.\end{aligned}$$

□

Proposition 3.3.2. *Let $\Lambda : \mathcal{D}(U) \to \mathbb{C}$ be a continuous closed linear form on $\mathcal{D}(U)$. Then, there exists a closed linear form $\lambda : \mathcal{S}(U) \to \mathbb{C}$ such that*

$$\Lambda(\sigma) = \left(\int_U \widehat{\otimes}\lambda\right)(\sigma) := \int_U \lambda(\sigma(x, \cdot))dx.$$

Proof. For fixed $f \in C_c^\infty(U)$, we define the linear form

$$\begin{aligned}\Lambda_f : \mathcal{S}(U) &\to \mathbb{C} \\ \tau &\mapsto \Lambda(f \otimes \tau).\end{aligned}$$

Since Λ is closed, for all $k = 1, \ldots, n$,

$$\Lambda_f(\partial_{\xi_k}\tau) = \Lambda(f \otimes \partial_{\xi_k}\tau) = \Lambda(\partial_{\xi_k}(f \otimes \tau)) = 0.$$

Thus Λ_f satisfies Stokes' property so that there is a constant b_f such that

$$\Lambda_f = b_f \lambda,$$

where λ is the closed linear form corresponding to $\mathcal{S}(U)$ as in Example 3.3.1. Similarly, for fixed $\tau \in \mathcal{S}(U)$, we can define the linear form

$$\begin{aligned}\Lambda_\tau : C_c^\infty(U) &\to \mathbb{C} \\ f &\mapsto \Lambda(f \otimes \tau).\end{aligned}$$

Since Λ is closed, for all $k = 1, \ldots, n$,

$$\Lambda_\tau(\partial_{x_k} f) = \Lambda((\partial_{x_k} f) \otimes \tau) = \Lambda(\partial_{x_k}(f \otimes \tau)) = 0.$$

Thus Λ_τ satisfies Stokes' property so that there is a constant c_τ such that

$$\Lambda_\tau = c_\tau \int_U .$$

Since the map

$$(f, \tau) \mapsto \Lambda(f \otimes \tau)$$

is bilinear, by Prop. 50.7 in [42], it follows that there exists a constant C such that

$$\Lambda(f \otimes \tau) = C \left(\int_U f\right) \cdot \lambda(\tau).$$

From the continuity of Λ combined with the density of $C_c^\infty(U)\widehat{\otimes}\mathcal{S}(U)$ in $\mathcal{D}(U)$ we can infer that for any $\sigma \in \mathcal{D}(U)$,

$$\Lambda(\sigma) = C\int_U \lambda(\sigma(x,\cdot))dx.$$

□

Let us conclude the classification of closed linear forms on $\mathcal{D}(U)$ for the sets $\mathcal{D}(U)$ in Example 3.3.1:

Corollary 3.3.1. *Any closed linear form on $\mathcal{D}(U)$ is a multiple of the map indicated below in each of the following cases. Let $\sigma \in \mathcal{D}(U)$ be a symbol with x–compact support on U and with asymptotic expansion $\sigma \sim \sum_{j=0}^\infty \psi\sigma_{a-j}$.*

1. *$\mathcal{D}(U) = CS_{\rm comp}(U)$, $\Lambda(\sigma) = \mathrm{res}(\sigma) = \int_U \int_{S_x^*U} \sigma_{-n}(x,\xi)\overline{\mu}(\xi) \wedge dx$.*
2. *$\mathcal{D}(U) = CS^{-\infty}_{\rm comp}(U)$, $\Lambda(\sigma) = \int_U \int_{\mathbb{R}^n} \sigma(x,\xi)d\xi\, dx$.*
3. *$\mathcal{D}(U) = \left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} CS^a_{\rm comp}(U)\right\rangle$, $\Lambda(\sigma) = \int_U ⨍_{\mathbb{R}^n} \sigma(x,\xi)d\xi\, dx$.*
4. *If $a \notin \mathbb{Z}\cap[-n+1,+\infty)$ and $\mathcal{D}(U) = CS^a_{\rm comp}(U)$,
$\Lambda(\sigma) = \rho\circ L_a(\sigma) + c\int_U ⨍_{\mathbb{R}^n} \sigma(x,\xi)d\xi\, dx$.*
5. *If $a \in \mathbb{Z}\cap[-n+1,+\infty)$ and $\mathcal{D}(U) = CS^a_{\rm comp}(U)$,
$\Lambda(\sigma) = \rho\circ L_a(\sigma) + c\,\mathrm{res}(\sigma)$.*

3.4 Closed linear forms on odd-class symbols

Let U be an open subset of $\mathbb{R}^n$. The space of odd-class symbols $CS^{(\rm odd)}(U)$ is the set of symbols $\sigma \in CS(U)$ of integer order and with asymptotic expansion as in (2.33), $\sigma \sim \sum_{j=0}^{\infty} \psi\,\sigma_{a-j}$ such that additionally, for all $j \geq 0$, (see Sect. 4 in [23], [30] and Sect. 1 in [35]):

$$\sigma_{a-j}(x,-\xi) = (-1)^{a-j}\sigma_{a-j}(x,\xi),\ \forall\,\xi : |\xi| \geq 1. \tag{3.19}$$

Lemma 3.4.1. *$CS^{(\rm odd)}(U)$ is closed under partial derivatives.*

Proof. Let $\sigma \sim \sum_{j=0}^\infty \psi\,\sigma_{a-j} \in CS^{(\rm odd)}(U)$ be an odd-class symbol of order a, where

$$\sigma_{a-j}(x,t\xi) = t^{a-j}\sigma_{a-j}(x,\xi), \quad \forall\, t>0,$$
$$\sigma_{a-j}(x,-\xi) = (-1)^{a-j}\sigma_{a-j}(x,\xi).$$

Hence, since $\partial_{x_i}\psi = 0$ and $\partial_{\xi_i}\psi$ has compact support, $\partial_{x_i}(\sigma) \sim \sum_{j=0}^\infty \psi\,\partial_{x_i}(\sigma_{a-j})$ is a symbol of order a, $\partial_{\xi_i}(\sigma) \sim \sum_{j=0}^\infty \psi\,\partial_{\xi_i}(\sigma_{a-j})$ is a symbol of order $a-1$;

for all $j \geq 0$,

$$\begin{aligned}
\partial_{x_i}(\sigma_{a-j})(x, t\xi) &= t^{a-j}\partial_{x_i}(\sigma_{a-j})(x,\xi), \quad \forall t > 0,\\
\partial_{x_i}(\sigma_{a-j})(x, -\xi) &= (-1)^{a-j}\partial_{x_i}(\sigma_{a-j})(x,\xi),\\
\partial_{\xi_i}(\sigma_{a-j})(x, t\xi) &= t^{a-j-1}\partial_{\xi_i}(\sigma_{a-j})(x,\xi), \quad \forall t > 0,\\
\partial_{\xi_i}(\sigma_{a-j})(x, -\xi) &= (-1)^{a-j-1}\partial_{\xi_i}(\sigma_{a-j})(x,\xi).
\end{aligned}$$

□

Let us denote by $CS^{(\mathrm{odd}),a}(U)$ the set of odd-class symbols of order $a \in \mathbb{Z}$ on U.

Lemma 3.4.2. *In odd dimensions, the noncommutative residue of any odd-class symbol vanishes.*

Proof. Let $\sigma \in CS^{(\mathrm{odd}),a}(U)$ be with asymptotic expansion $\sigma \sim \sum_{j=0}^{\infty} \psi\, \sigma_{a-j}$ as in (2.33). Since n is odd, we have $\sigma_{-n}(x,-\xi) = (-1)^n\sigma_{-n}(x,\xi) = -\sigma_{-n}(x,\xi)$. Therefore, by (3.5) we obtain

$$\begin{aligned}
\mathrm{res}(\sigma) &= \int_U \int_{S_x^*U} \sigma_{-n}(x,\xi)\overline{\mu}(\xi) \wedge dx\\
&= -\int_U \int_{S_x^*U} \sigma_{-n}(x,-\xi)\overline{\mu}(\xi) \wedge dx\\
&= -\int_U \int_{S_x^*U} \sigma_{-n}(x,\xi)\overline{\mu}(\xi) \wedge dx\\
&= -\,\mathrm{res}(\sigma).
\end{aligned}$$

Therefore $\mathrm{res}(\sigma) = 0$. □

Proposition 3.4.1. *In odd dimensions, the cut-off regularized integral is closed on $CS^{(\mathrm{odd})}(U)$.*

Proof. See Cor. 2 in [35]. □

Proposition 3.4.2. *Let $n \in \mathbb{Z}$ be odd. For any $\sigma \in CS^{(\mathrm{odd}),a}(U)$, there exist τ_i in $CS^{(\mathrm{odd}),a+1}(U)$ such that*

$$\sigma \sim \sum_{i=1}^{n} \partial_{\xi_i}\tau_i. \tag{3.20}$$

Proof. (See Lemma 1.3 in [10], and [30]). For a cut-off function ψ as in Section 2.3 consider

$$\sigma \sim \sum_{j=0}^{\infty} \psi\, \sigma_{a-j},$$

with σ_{a-j} a positively homogeneous function of degree $a-j$ in ξ which satisfies (3.19).

- If $a-j\neq -n$, consider the homogeneous function $\tau_{i,a-j+1} := \frac{\xi_i \sigma_{a-j}(x,\xi)}{a-j+n}$. By Euler's identity we have

$$\sum_{i=1}^{n} \partial_{\xi_i}(\tau_{i,a-j+1})(x,\xi) = \sigma_{a-j}(x,\xi).$$

From the definition it is immediate to see that the $\tau_{i,a-j+1}$ satisfy (3.19):

$$\begin{aligned}\tau_{i,a-j+1}(x,t\xi) &= t^{a-j+1}\tau_{i,a-j+1}(x,\xi), \quad \forall t>0,\\ \tau_{i,a-j+1}(x,-\xi) &= (-1)^{a-j+1}\tau_{i,a-j+1}(x,\xi).\end{aligned}$$

- Let $a-j=-n$. In polar coordinates $(r,\omega)\in\mathbb{R}^+\times S^{n-1}$, the Laplacian in ξ reads

$$\Delta = -\sum_{i=1}^{n}\partial_{\xi_i}^2 = -r^{1-n}\partial_r(r^{n-1}\partial_r) - r^{-2}\Delta_{S^{n-1}}.$$

Therefore, for any function $f\in C^\infty(S^{n-1})$,

$$\Delta(f(\omega)r^{2-n}) = r^{-n}\Delta_{S^{n-1}}f(\omega).$$

Since n is odd and $\sigma\in CS^{(\mathrm{odd}),a}(U)$, by Lemma 3.4.2, $\mathrm{res}(\sigma)=0$. Therefore $\sigma_{-n}(x,\cdot)\restriction_{S^{n-1}}$ is orthogonal to the constants which form the kernel $\ker(\Delta_{S^{n-1}})$. Hence there exists a unique function $h(x,\cdot)\in C^\infty(S^{n-1})$, orthogonal to the constants, such that $\Delta_{S^{n-1}}(h(x,\cdot)) = \sigma_{-n}(x,\cdot)\restriction_{S^{n-1}}$. The function $h(x,\cdot)$ is an odd function on S^{n-1}: $h(x,-\xi)=-h(x,\xi)$.

We choose a smooth function χ on $\mathbb{R}$ which vanishes for small r and is equal to 1 for $r\geq 1/2$. For $r=|\xi|$, we set

$$b_{-n}(x,\xi) := \chi(|\xi|)|\xi|^{2-n}h\left(x,\frac{\xi}{|\xi|}\right).$$

The function b_{-n} is smooth on $U\times\mathbb{R}^n$ and is homogeneous of degree $-n+2$ in ξ for $|\xi|\geq 1$. As $\sigma_{-n}(x,\xi)$ vanishes for x outside a compact set, so does $b_{-n}(x,\xi)$. In particular, b_{-n} is a symbol of order $2-n$ on U. Let us define $\tau_{i,-n+1} := -\partial_{\xi_i}b_{-n}$. Since h is odd so is b_{-n} and therefore,

$$\begin{aligned}\tau_{i,-n+1}(x,-\xi) &= -(\partial_{\xi_i}b_{-n})(x,-\xi)\\ &= -\partial_{\xi_i}b_{-n}(x,\xi)\\ &= (-1)^{-n+1}\tau_{i,-n+1}(x,\xi).\end{aligned}$$

Moreover, we have for $|\xi|\geq 1$

$$\Delta b_{-n} = \Delta(r^{2-n}h(x,\cdot)) = r^{-n}\sigma_{-n}(x,\cdot)\restriction_{S^{n-1}} = \sigma_{-n}.$$

Let $\tau_i \sim \sum\limits_{j=0}^{\infty} \psi\, \tau_{i,a-j+1}$, then since $\partial_{\xi_i}\psi$ has compact support, the difference $\sigma - \sum\limits_{i=1}^{n} \partial_{\xi_i}\tau_i$ is smoothing and

$$\sigma \sim \sum_{i=1}^{n}\sum_{j=0}^{\infty} \psi\, \partial_{\xi_i}\tau_{i,a-j+1} \sim \sum_{i=1}^{n} \partial_{\xi_i}\tau_i.$$

□

Corollary 3.4.1. *In odd dimensions, the cut-off regularized integral induces an isomorphism*

$$H^n(\mathbb{R}^n, CS_{cc}^{(\mathrm{odd})}(\mathbb{R}^n)) \cong \mathbb{C}.$$

Therefore, any closed linear form on $CS_{cc}^{(\mathrm{odd})}(\mathbb{R}^n)$ is proportional to the cut-off regularized integral.

Proof. By Lemma 3.4.1 and Proposition 3.4.1, $⨍_{\mathbb{R}^n}$ yields a well defined function

$$\widetilde{⨍_{\mathbb{R}^n}} : H^n(\mathbb{R}^n, CS_{cc}^{(\mathrm{odd})}(\mathbb{R}^n)) \to \mathbb{C}.$$

Since the cut-off regularized integral coincides with usual integration when restricting to smoothing symbols, we can take any smooth function with compact support in $\mathbb{R}^n$ and with total integral 1 to prove that the map $\widetilde{⨍_{\mathbb{R}^n}}$ is surjective. To prove that this map is injective, let $\sigma \in CS_{cc}^{(\mathrm{odd})}(\mathbb{R}^n)$ be such that $⨍_{\mathbb{R}^n}\sigma = 0$. By Lemma 3.4.2, $\mathrm{res}(\sigma) = 0$ and therefore by Proposition 3.4.2, there exist $\tau_i \in CS_{cc}^{(\mathrm{odd})}(\mathbb{R}^n)$ and $s \in CS_{cc}^{-\infty}(\mathbb{R}^n)$ such that

$$\sigma = \sum_{i=1}^{n} \partial_{\xi_i}\tau_i + s. \tag{3.21}$$

Then, by Proposition 3.4.1 we have the following

$$0 = ⨍_{\mathbb{R}^n} \sigma = \sum_{i=1}^{n} ⨍_{\mathbb{R}^n} \partial_{\xi_i}\tau_i + ⨍_{\mathbb{R}^n} s = 0 + \int_{\mathbb{R}^n} s = \int_{\mathbb{R}^n} s.$$

Hence, by Corollary 3.2.1, there exist smoothing symbols $s_i \in CS_{cc}^{-\infty}(\mathbb{R}^n)$ such that

$$s = \sum_{i=1}^{n} \partial_{\xi_i} s_i. \tag{3.22}$$

Equations (3.21) and (3.22) imply that $\sigma = \sum_{i=1}^{n} \partial_{\xi_i}(\tau_i + s_i)$. The rest of the statement follows from Proposition 3.2.1. □

As a consequence of this, as in Section 3.3 we conclude that if U is a compact subset of $\mathbb{R}^n$, any closed linear form on $CS^{(\mathrm{odd})}(U)$ is a multiple of the cut-off regularized integral; and if we fix the order a, any closed linear form on $CS^{(\mathrm{odd}),a}(U)$ is a linear combination of a leading symbol closed linear form and the cut-off regularized integral.

Chapter 4

Commutators and Traces

Our main interest is the classification of traces on algebras of the form $Cl^a(M)$ with $a \leq 0$. Since traces vanish on commutators, we devote this chapter to the study of the representation of a pseudo-differential operator as a sum of commutators. For this, we use the results of Chapter 1 relative to the representation of a homogeneous function in terms of Poisson brackets of homogeneous functions of appropriate degree and the fact that the leading symbol of a commutator of pseudo-differential operators is, up to a constant, the Poisson bracket of the corresponding leading symbols. This enables us to extend and refine in the context of operators with order bounded from above, known results by Lesch [25] (see our Theorem 4.3.1) on the one hand, and by Ponge [37] (see our Proposition 4.4.1) on the other hand, concerning the representation of a pseudo-differential operator as a sum of commutators. Throughout the chapter we denote by M a closed connected smooth manifold of dimension $n > 1$, unless we indicate something else.

4.1 Classical pseudo-differential operators

Let us recall the definition and some properties of classical pseudo-differential operators on a manifold M following [20] and [39]. We use the notation of Section 2.3 for spaces of symbols on an open subset $U \subseteq \mathbb{R}^n$.

Definition 4.1.1. For a symbol $\sigma \in S^a(U)$, the canonical operator $P := \mathrm{Op}(\sigma)$ associated to σ is defined by:

$$P(f)(x) = \frac{1}{(2\pi)^n} \int_{U\times\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)f(y)d\xi dy$$

as a linear operator mapping the space of smooth functions with compact support on U, $C_c^\infty(U)$, to $C^\infty(U)$, and P is called a *pseudo-differential operator (ψDO) of order* $\mathrm{ord}(P) = a$. If $\sigma \in CS^a(U)$, P is a *classical pseudo-differential operator* of order a over U and we write $P \in Cl^a(U)$; if $\sigma \in S^{-\infty}(U)$, P is a *smoothing operator* over U and we write $P \in Cl^{-\infty}(U)$.

The ψDO P can also be expressed as an integral operator ([39]), namely:

$$P(f)(x) = \int_U K(x,y) f(y) dy,$$

where

$$K(x,y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma(x,\xi) d\xi,$$

is the *Schwartz kernel* of P, which is a continuous function on $U \times U$ whenever $a < -n$, and which can be interpreted as a distribution whenever $a \geq -n$ (see [9], [41]). By $Cl^a_{\text{comp}}(U)$ we denote the space of classical ψDOs of order a on U whose Schwartz kernel has compact support on $U \times U$.

Using the representation in local charts, one defines classical pseudo-differential operators on manifolds which can be generalized to operators acting on sections of a vector bundle over a manifold (see [20], [39]): Let M be a smooth n–dimensional manifold. A linear operator $A : C^\infty_c(M) \to C^\infty(M)$ is a pseudo-differential operator of order a on M if for every U local coordinate chart of M, with diffeomorphism $\varphi : U \to V$, from U to an open set $V \subseteq \mathbb{R}^n$, the operator $\varphi^\# A : C^\infty_c(V) \to C^\infty(V)$ defined by the following diagram is a pseudo-differential operator of order a on V:

$$\begin{array}{ccc} C^\infty_c(V) & \longrightarrow & C^\infty(V) \\ \Big\downarrow \varphi^* & & \Big\downarrow \varphi^* \\ C^\infty_c(U) & \longrightarrow & C^\infty(U) \end{array} \tag{4.1}$$

in the lower row we have the operator $r_U \circ A \circ i_U : C^\infty_c(U) \to C^\infty(U)$, where $i_U : C^\infty_c(U) \to C^\infty_c(M)$ is the natural embedding, and $r_U : C^\infty(M) \to C^\infty(U)$ is the natural restriction.

Let E and F be smooth vector bundles over M of rank k and l respectively. By Def. 18.1.32 in [20], a classical ψDO of order a from sections of E to sections of F is a continuous linear map $A : C^\infty_c(M,E) \to C^\infty(M,F)$ such that for every open set $U \subseteq M$ where E and F are trivialized by

$$\phi_E : E \restriction_U \to \varphi(U) \times \mathbb{C}^k, \quad \phi_F : F \restriction_U \to \varphi(U) \times \mathbb{C}^l,$$

there is an $l \times k$ matrix of ψDOs $A_{ij} \in Cl^a(\varphi(U))$ such that

$$(\phi_F((Ag)\restriction_U))_i = \sum_j A_{ij}(\phi_E(g))_j, \quad g \in C^\infty_c(U,E).$$

We then write $A \in Cl^a(M,E,F)$, and $A \in Cl^a(M,E)$ in the case when $E = F$.

We now equip these infinite dimensional sets of operators by using the Fréchet topology on constant order symbols defined in Section 2.3. For any closed manifold M and for any $a \in \mathbb{R}$, the linear space $Cl^a(M,E)$ of classical ψDOs of order a can be equipped with a Fréchet topology. The Fréchet structure on $CS^a(U,W)$ induces one on $Cl^a(M,E)$. Indeed, let $(U_i, \varphi_i)_{i\in I}$ be an atlas on M for some finite set I, with local trivializations $\phi_i : E\restriction_{U_i} \xrightarrow{\cong} \varphi_i(U_i)\times W$ compatible with the charts, a partition of unity $(\chi_i)_{i\in I}$ subordinate to the chosen atlas and smooth functions $(\widetilde{\chi_i})_{i\in I}$ on M such that $\mathrm{supp}(\widetilde{\chi_i}) \subset U_i$ and $\widetilde{\chi_i} = 1$ near the support of χ_i. We write an operator $A \in Cl^a(M,E)$ as follows:

$$A = \sum_{i\in I_0 \subseteq I} A_i + R(A),$$

where $A_i := \chi_i \cdot \left(\phi_i^{-1} \circ \mathrm{Op}(\sigma_i) \circ \phi_i\right) \cdot \widetilde{\chi_i} \in Cl^a(M,E)$ are ψDOs with compactly supported symbols in $CS^a(\varphi_i(U_i), W)$ and $R(A) \in Cl^{-\infty}(M,E)$.
The Fréchet topology on $Cl^a(M,E)$ is provided by the countable family of semi-norms built from:

1. A countable family of semi-norms given by the supremum norm of the kernel of $R(A)$ and its derivatives on a countable family of compact subsets.
2. The countable family of semi-norms on $\mathrm{Op}(\sigma_i)$ induced by the ones on the symbols σ_i described in Equations (2.34)–(2.36).

Let us recall a result about the existence of a symbol for a given sequence of symbols of decreasing order:

Proposition 4.1.1 (Prop. 1.8 in [13])**.** *Let U be an open subset of $\mathbb{R}^n$ and let $\sigma_{a-j} \in S^{a-j}(U)$ for all $j \in \mathbb{N}$. Then there exists $\sigma \in S^a(U)$, unique modulo (i.e. up to some element in) $S^{-\infty}(U)$, such that $\sigma - \sum_{0\le j<k} \sigma_{a-j} \in S^{a-k}(U)$ for all $k \in \mathbb{N}$.*

Remark 4.1.1. As a consequence of this proposition we find, e.g. in Sect. 18.1 of [20], that if $A_j \in Cl^{m_j}(U)$, and $m_j \downarrow -\infty$, there exists $A \in Cl^{m_0}(U)$ such that for all $k \in \mathbb{N}$,

$$A - \sum_{j<k} A_j \in Cl^{m_k}(U).$$

Let us recall a fact about splittings of short exact sequences in certain graded algebras that we will use in the case of classical ψDOs of integer oder.

Proposition 4.1.2. *Consider a graded algebra $\{\mathcal{A}_k\}_{k\in\mathbb{Z}}$. Suppose that for each fixed k, the sequence*

$$0 \to \mathcal{A}_{k-1} \to \mathcal{A}_k \to \mathcal{A}_k/\mathcal{A}_{k-1} \to 0$$

splits. Then, for all $j < k$ there is a canonical way to construct splittings of the sequences

$$0 \to \mathcal{A}_j \to \mathcal{A}_k \to \mathcal{A}_k/\mathcal{A}_j \to 0.$$

Proof. Let $q_k : \mathcal{A}_k/\mathcal{A}_{k-1} \to \mathcal{A}_k$ be a splitting of the sequence

$$0 \to \mathcal{A}_{k-1} \to \mathcal{A}_k \xrightarrow{p_k} \mathcal{A}_k/\mathcal{A}_{k-1} \to 0, \tag{4.2}$$

i.e. $p_k \circ q_k = id_{\mathcal{A}_k/\mathcal{A}_{k-1}}$, or equivalently,

$$\begin{aligned}\mathcal{A}_k &= \ker(p_k) \oplus \mathrm{Image}(q_k)\\ &\cong \mathcal{A}_{k-1} \oplus q_k(p_k(\mathcal{A}_k)).\end{aligned}$$

Now, since $\mathcal{A}_{k-1} \cong \mathcal{A}_{k-2} \oplus q_{k-1}(p_{k-1}(\mathcal{A}_{k-1}))$, we have

$$\mathcal{A}_k \cong \mathcal{A}_{k-2} \oplus q_{k-1}(p_{k-1}(\mathcal{A}_{k-1})) \oplus q_k(p_k(\mathcal{A}_k)). \tag{4.3}$$

The sequence (4.2), produces the following split exact sequence:

$$0 \to \mathcal{A}_{k-1}/\mathcal{A}_{k-2} \to \mathcal{A}_k/\mathcal{A}_{k-2} \to \mathcal{A}_k/\mathcal{A}_{k-1} \to 0. \tag{4.4}$$

From this, we consider the exact sequence

$$0 \to \mathcal{A}_{k-2} \to \mathcal{A}_k \xrightarrow{\pi_k} \mathcal{A}_k/\mathcal{A}_{k-2} \to 0,$$

where for $\xi \in \mathcal{A}_k$, $\xi - q_k(p_k(\xi)) \in \mathcal{A}_{k-1}$

$$\pi_k(\xi) = p_{k-1}(\xi - q_k(p_k(\xi))) + p_k(\xi).$$

For $\xi \in \mathcal{A}_k$, by (4.3) we can define a map

$$\theta_k : \mathcal{A}_k/\mathcal{A}_{k-2} \to \mathcal{A}_k$$

by $\theta_k(\pi_k(\xi)) := q_{k-1}(p_{k-1}(\xi - q_k(p_k(\xi)))) + q_k(p_k(\xi))$. Then

$$\pi_k \circ \theta_k \circ \pi_k = \pi_k,$$

i.e. θ_k is a well defined right inverse of the projection π_k.
The proposition follows by induction on $k - j$ for all $j < k$ with the sequence

$$0 \to \mathcal{A}_j \to \mathcal{A}_k \to \mathcal{A}_k/\mathcal{A}_j \to 0.$$

□

4.2 Known traces on pseudo-differential operators

Let M be a closed connected manifold of dimension $n > 1$, and let $\mathcal{A} \subseteq Cl(M)$ be a subset of the whole algebra of ψDOs on M. A *trace* on $\mathcal{A}$ is a map

$$\tau : \mathcal{A} \to \mathbb{C},$$

linear in the sense that for all a, $b \in \mathbb{C}$, whenever A, B and $aA + bB$ belong to $\mathcal{A}$ we have

$$\tau(aA + bB) = a\,\tau(A) + b\,\tau(B),$$

and such that for any $A, B \in \mathcal{A}$, whenever $AB, BA \in \mathcal{A}$ it satisfies

$$\tau([A, B]) = 0, \text{ or equivalently, } \tau(AB) = \tau(BA).$$

4.2.1 The L^2–trace

A pseudo-differential operator A of order $\mathrm{ord}(A) < -n$ is a trace-class operator. The L^2–trace or usual trace is the functional

$$\begin{aligned}\mathrm{Tr}_{L^2} : Cl^{<-n}(M) &\to \mathbb{C}\\ A \mapsto \mathrm{Tr}_{L^2}(A) &:= \int_M K_A(x,x)\,dx,\end{aligned} \tag{4.5}$$

where K_A is the Schwartz kernel of the operator A. If $\sigma(A)$ is the symbol of A, we can also write

$$\mathrm{Tr}_{L^2}(A) = \frac{1}{(2\pi)^n}\int_{T^*M} \sigma(A); \tag{4.6}$$

the last integral is defined using a finite covering of M, a partition of unity subordinate to it and the local representation of the symbol, but this definition is independent of such choices. This trace is continuous for the Fréchet topology on the space of ψDOs of constant order less than $-n$.
This is the unique trace on the algebra of smoothing operators $Cl^{-\infty}(M)$, since we have the exact sequence (see [15])

$$0 \to [Cl^{-\infty}(M), Cl^{-\infty}(M)] \to Cl^{-\infty}(M) \xrightarrow{\mathrm{Tr}_{L^2}} \mathbb{C} \to 0.$$

Equivalently we can say the following

Theorem 4.2.1 (Thm. A.1 in [15])**.** *If R is a smoothing operator then, for any pseudo-differential idempotent of rank 1, J, there exist smoothing operators $S_1, \ldots, S_N$, $T_1, \ldots, T_N$, such that*

$$R = \mathrm{Tr}_{L^2}(R)J + \sum_{j=1}^{N}[S_j, T_j].$$

Therefore, any smoothing operator with vanishing L^2–trace is a sum of commutators in the space $[Cl^{-\infty}(M), Cl^{-\infty}(M)]$.

Lemma 4.2.1. *Given a real number a with $2a < -n$, let $\alpha : Cl^a(M) \to \mathbb{C}$ be any linear functional such that $\alpha \restriction_{Cl^{<-n}(M)} = \mathrm{Tr}_{L^2}$. Then α is a trace.*

Proof. If $\mathfrak{L}^p$ denotes the class of operators A such that $\mathrm{Tr}_{L^2}(|A|^p) < \infty$, then for any $T \in \mathfrak{L}^p$, $S \in \mathfrak{L}^q$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have $\mathrm{Tr}_{L^2}([T,S]) = 0$ (see e.g. Cor. 3.8 in [40]). Let $T, S \in Cl^a(M)$. Since $2a < -n$, the operators T, S belong to $\mathfrak{L}^2$, and since $\mathrm{ord}([T,S]) \le 2a - 1 < -n - 1$, we get

$$\alpha([T,S]) = \mathrm{Tr}_{L^2}([T,S]) = 0.$$

$\square$

Proposition 4.2.1 (See e.g. [25] and Prop. 4.4 in [26])**.** *The trace* Tr_{L^2} *does not extend to a trace functional neither on the whole algebra $Cl(M)$, nor on the algebra $Cl^0(M)$.*

Remark 4.2.1. We postpone one proof of this proposition to Subsection 5.2.1, where we study traces defined on operators acting on sections of vector bundles over the manifold. In Corollary 5.1.1 we refine this result to the algebra $Cl^a(M)$ when $a \in \mathbb{Z}$ is such that $2a \in [-n+1, 0]$.

4.2.2 The Wodzicki residue

Considering the symplectic cone $T^*M \setminus M \to S^*M$, in Subsection 3.1.1 we gave the definition of the noncommutative residue at the level of symbols. The *Wodzicki residue* (also known as the noncommutative residue or the residual trace) of an operator $A \in Cl(M)$ is defined from the noncommutative residue of the symbol of A (see [44]):

$$\mathrm{Res}(A) := \frac{1}{(2\pi)^n}\,\mathrm{res}(\sigma(A)) = \frac{1}{(2\pi)^n}\int_M \int_{S_x^*M} \sigma_{-n}(A)(x,\xi)\,\overline{\mu}(\xi)\wedge dx,$$

where $\overline{\mu}(\xi)$ is a volume form on S_x^*M. This is the unique trace on the whole algebra of pseudo-differential operators $Cl(M)$ as we can see in [7], [10], [25], [44]. By definition, this trace vanishes on trace-class ψDOs, non-integer order ψDOs and differential operators. The continuity of the residual trace for the Fréchet topology on the space of constant order ψDOs, follows from the fact that it is defined in terms of a finite number of homogeneous parts of the symbols of the operators.

4.2.3 The canonical trace

By Corollary 3.1.1 and Proposition 3.1.3, the cut-off regularized integral (see Subsection 3.1.2) is closed and covariant on $CS^a_{cc}(\mathbb{R}^n)$ for all $a \notin \mathbb{Z}\cap[-n,+\infty)$. As we said in Remark 3.1.5, this allows us to construct the canonical trace on the space $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} Cl^a(M)\right\rangle$.

Proposition 4.2.1 shows that there is no a non-trivial trace on $Cl(M)$ which extends the L^2–trace. However, the L^2–trace does extend to non-integer order operators. Indeed, Kontsevich and Vishik ([23]) constructed a functional, the *canonical trace*

$$\begin{aligned}\mathrm{TR} : \left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} Cl^a(M)\right\rangle &\to \mathbb{C}\\ A &\mapsto \mathrm{TR}(A) := \frac{1}{(2\pi)^n}\int_M dx \,⨍_{T_x^*M} \sigma(A)(x,\xi)d\xi,\end{aligned}$$

where the right hand side is interpreted in the same way as (4.6).

If $A \in Cl^a(M)$, $B \in Cl^b(M)$ and if $a, b \notin \mathbb{Z}$, then $\mathrm{ord}(AB) = a+b$ may be an integer, so the linear space $\left\langle \bigcup_{a\notin\mathbb{Z}\cap[-n,+\infty)} Cl^a(M)\right\rangle$ is not an algebra. In spite of this, the canonical trace has the following properties (see [23] and Sect. 5 in [25]):

1. For any $A, B \in \left\langle \bigcup_{a \notin \mathbb{Z} \cap [-n,+\infty)} Cl^a(M) \right\rangle$ and for any $c \in \mathbb{C}$, such that $\mathrm{ord}(cA+B) \notin \mathbb{Z} \cap [-n,+\infty)$, $\mathrm{TR}(cA+B) = c\,\mathrm{TR}(A) + \mathrm{TR}(B)$.

2. For any $A \in Cl(M)$ such that $\mathrm{ord}(A) < -n$, $\mathrm{TR}(A) = \mathrm{Tr}_{L^2}(A)$.

For any elliptic ψDO $P \in Cl^1(M)$ with positive leading symbol, the canonical trace satisfies (see (5.2) in [25])

$$\mathrm{TR}(A) = \operatorname*{LIM}_{t \to 0^+} \mathrm{Tr}_{L^2}(Ae^{-tP}),$$

where $\operatorname*{LIM}_{t \to 0^+} f(t)$ denotes the constant term in $f(t)$ when $t \to 0^+$.

Lemma 4.2.2. *If* $A, B \in Cl(M)$ *are such that* $\mathrm{ord}(A)+\mathrm{ord}(B) \notin \mathbb{Z} \cap [-n,+\infty)$, *then* $\mathrm{TR}(AB) = \mathrm{TR}(BA)$.

Proof. (see Sect. 4 in [25]) Let $P \in Cl^1(M)$ be an elliptic ψDO whose leading symbol is positive and let $A \in Cl^a(M)$, $B \in Cl^b(M)$. We put

$$\nabla_P^0(B) := B, \quad \nabla_P^{j+1} B := [P, \nabla_P^j B],$$

and by induction, for all $j \in \mathbb{N}$ we have

$$\nabla_P^j B \in Cl^b(M).$$

Then, for N large enough we have the formula

$$e^{-tP}B = \sum_{j=0}^{N-1} \frac{(-t)^j}{j!} (\nabla_P^j B) e^{-tP} + R_N(t),$$

where $\mathrm{Tr}_{L^2}\left((R_N(t)^* R_N(t))^{1/2}\right) = O(t)$ as $t \to 0$; therefore

$$Ae^{-tP}B = ABe^{-tP} + \sum_{j=1}^{N-1} \frac{(-t)^j}{j!} A(\nabla_P^j B) e^{-tP} + R'_N(t).$$

Thus

$$\mathrm{Tr}_{L^2}([A,B]e^{-tP}) = -\sum_{j=1}^{N-1} \frac{(-t)^j}{j!} \mathrm{Tr}_{L^2}(A(\nabla_P^j B)e^{-tP}) + O(t), \quad t \to 0.$$

Since

$$\mathrm{Tr}_{L^2}(A(\nabla_P^j B)e^{-tP}) \underset{t \to 0^+}{\sim} \sum_{k=0}^{\infty} (c_k + d_k \log t) t^{k-(a+b)-n} + \sum_{k=0}^{\infty} e_k t^k,$$

and

$$\begin{aligned}\mathrm{TR}([A,B]) &= \operatorname*{LIM}_{t\to 0^+} \mathrm{Tr}_{L^2}([A,B]e^{-tP}) \\ &= -\operatorname*{LIM}_{t\to 0^+} \sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \mathrm{Tr}_{L^2}(A(\nabla_P^j B)e^{-tP}), \end{aligned} \tag{4.7}$$

we obtain

$$\mathrm{TR}([A,B]) = 0 \text{ whenever } a+b \notin \{-n+1, -n+2, -n+3, \ldots\}.$$

In the case that $a+b=-n$, the canonical trace is not well-defined on AB. $\square$

From the proof of previous lemma we can improve Lemma 4.2.1 in the following way:

Lemma 4.2.3. *Given a real number a with $2a-1<-n$, let $\alpha : Cl^a(M) \to \mathbb{C}$ be any linear functional such that $\alpha \restriction_{Cl^{<-n}(M)} = \mathrm{Tr}_{L^2}$. Then α is a trace.*

Proof. Let $A, B \in Cl^a(M)$. Then $[A,B] \in Cl^{2a-1}(M) \subset Cl^{<-n}(M)$, so $[A,B]$ is a trace-class operator and $\mathrm{TR}([A,B]) = \mathrm{Tr}_{L^2}([A,B])$. As in the proof of Lemma 4.2.2, from (4.7) we get

$$\alpha([A,B]) = \mathrm{Tr}_{L^2}([A,B]) = -\operatorname*{LIM}_{t\to 0^+} \sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \mathrm{Tr}_{L^2}(A(\nabla_P^j B)e^{-tP}) = 0. \tag{4.8}$$

$\square$

Remark 4.2.2. From the previous result we conclude that the functional

$$\begin{aligned}\mathrm{Tr}^P : Cl^a(M) &\to \mathbb{C} \\ A &\mapsto \mathrm{Tr}^P(A) := \operatorname*{LIM}_{t\to 0^+} \mathrm{Tr}_{L^2}(Ae^{-tP}),\end{aligned}$$

defines a trace on $Cl^a(M)$ when $2a \le -n \le a$, since $\mathrm{Tr}^P \restriction_{Cl^{<-n}(M)} = \mathrm{Tr}_{L^2}$. If we consider another elliptic ψDO $Q \in Cl^1(M)$ with positive leading symbol, then (see Prop. 2.2 in [23])

$$\mathrm{Tr}^P(A) - \mathrm{Tr}^Q(A) = \mathrm{res}(A(\log Q - \log P)).$$

From this we can deduce that Tr^P is independent of P whenever A has order $\mathrm{ord}(A) \notin \mathbb{Z} \cap [-n, +\infty)$.

The uniqueness of the canonical trace on its domain is proved in [30] and [35]. The canonical trace is continuous for the Fréchet topology on the space of ψDOs of constant order where is well defined.

4.2.4 Leading symbol traces

In [36] the authors describe some traces on certain spaces of ψDOs in order to construct characteristic classes of infinite dimensional vector bundles over a closed manifold, these are the *leading symbol traces*, defined for $a \leq 0$ by using the leading symbol map:

$$\begin{array}{rcl} \mathrm{tr}_a : Cl^a(M) & \to & C^\infty(S^*M) \\ A & \mapsto & \sigma_a(A) \restriction_{S^*M}, \end{array}$$

where $\sigma_a(A)$ denotes the leading symbol of the operator A, whose symbol belongs to $CS^a(U)$ for any local chart U of M.

Lemma 4.2.4 (Lemma 3.1 in [36])**.** *The map* tr_a *is linear and for any distribution* $\Lambda \in \mathcal{D}'(S^*M)$*, the map* $\mathrm{tr}_a^\Lambda : Cl^a(M) \to \mathbb{R}$ *given by* $\mathrm{tr}_a^\Lambda(A) = \Lambda(\mathrm{tr}_a(A))$ *is a trace.*

Proof. Taking the $a-$th component of the symbol is a linear application. If $a = 0$, since the leading symbol is multiplicative, we get

$$\sigma_a(AB) = \sigma_a(A)\sigma_a(B) = \sigma_a(B)\sigma_a(A) = \sigma_a(BA). \tag{4.9}$$

When $a < 0$, for $A, B \in Cl^a(M)$, the products AB and BA lie in $Cl^{2a}(M)$, so we have

$$\sigma_a(AB) = 0 = \sigma_a(BA).$$

□

Remark 4.2.3. If $a < 0$, for $r \in [2a, a)$, $\mathrm{tr}_r(A) := \sigma_r(A)$ defines also a trace on $Cl^a(M)$, as $\mathrm{tr}_r(AB)$ trivially vanishes for $r > 2a$, and the proof of the lemma covers the case $r = 2a$. In this case if $\Lambda(\phi) = \int_{S^*M} \phi(x,\xi)$, these traces are defined only after a choice of coordinates and a partition of unity on M, since integrals of non-leading symbols depend on such choices.

Leading symbol traces are continuous for the Fréchet topology on the space of constant order ψDOs, since they are defined in terms of a finite number of homogeneous parts of the symbols of the operators.

4.3 Pseudo-differential operators in terms of commutators

In Example 1.1.2 we saw that the cotangent bundle of M without the zero section $Y := T^*M \setminus M$, is a symplectic cone over the cosphere bundle S^*M, and we can consider that the symbol of a classical ψDO on M has an asymptotic expansion whose components are homogeneous functions on Y. In this section we use the representation of a homogeneous function in terms of Poisson brackets given in (1.58) in order to write a ψDO as a sum of commutators.

By means of the local representation of the symbol of an operator and the symplectic form, we can see that the leading symbol of the product of two ψDOs is the product of the corresponding leading symbols as in (4.9), and that the leading symbol of a commutator is proportional to the Poisson bracket of the leading symbols of the corresponding operators (see e.g. Sect. 1.5 in [2]). This leads to the following theorem, inspired by the proof carried out in [25] in the case $Cl(M)$, we actually improve Prop. 4.7 in [25] for the case $Cl(M)$, in which the case when $m = 1$ is investigated.

Theorem 4.3.1. *Let $Q \in Cl^{-n}(M)$ be an operator with $\mathrm{Res}(Q) = 1$. For any real numbers m, s there exist $P_1, \dots, P_N \in Cl^m(M)$, such that for any $A \in Cl^s(M)$ there exist $Q_1, \dots, Q_N \in Cl^{s-m+1}(M)$ and $R \in Cl^{-\infty}(M)$ such that*

$$A = \sum_{i=1}^{N} [P_i, Q_i] + \mathrm{Res}(A)Q + R. \tag{4.10}$$

Proof. We choose $P_1, \dots, P_N \in Cl^m(M)$ such that either the differentials of their leading symbols span the cotangent bundle of Y at every point if $m \neq 0$, or such that the differentials of their leading symbols restricted to Z span the cotangent bundle of Z at every point if $m = 0$. We consider the leading symbol $\sigma_s(A) \in \mathcal{P}_s(Y)$ of A. If $s \neq -n$, then by the first part of (1.58), there exist $Q_1^{(1)}, \dots, Q_N^{(1)} \in Cl^{s-m+1}(M)$ such that

$$A - \sum_{i=1}^{N} [P_i, Q_i^{(1)}] \in Cl^{s-1}(M).$$

If $s - 1 \neq -n$, we iterate the procedure: by the first part of (1.58) there exist $B_1^{(1)}, \dots, B_N^{(1)} \in Cl^{(s-1)-m+1}(M)$ such that, if $Q_i^{(2)} := Q_i^{(1)} + B_i^{(1)}$ for all $i = 1, \dots, N$, then

$$A - \sum_{i=1}^{N} [P_i, Q_i^{(2)}] \in Cl^{s-2}(M).$$

- If $s \notin \{l \in \mathbb{Z} : l \geq -n\}$, then by induction we find operators $Q_i^{(l)} \in Cl^{s-m+1}(M)$ such that

$$A^{(l)} := A - \sum_{i=1}^{N} [P_i, Q_i^{(l)}] \in Cl^{s-l}(M). \tag{4.11}$$

- If $s \in \{l \in \mathbb{Z} : l \geq -n\}$, then (4.11) holds for $l \leq s + n$. After that, since $\mathrm{Res}([P_i, Q_i^{(l)}]) = 0$ for all $i = 1, \dots, N$ and $A^{(s+n)} \in Cl^{-n}(M)$, we have

$$\mathrm{Res}\left(A^{(s+n)} - \mathrm{Res}(A)Q\right) = 0. \tag{4.12}$$

By the second part of (1.58) there exist $B_1^{(s+n)}, \ldots, B_N^{(s+n)} \in Cl^{-n-m+1}(M)$ such that

$$A^{(s+n)} - \operatorname{Res}(A)Q - \sum_{i=1}^{N}[P_i, B_i^{(s+n)}] \in Cl^{-n-1}(M).$$

Using the first part of (1.58) once again, and by induction we find operators $Q_i^{(l)} \in Cl^{s-m+1}(M)$ such that

$$A^{(l)} := A - \sum_{i=1}^{N}[P_i, Q_i^{(l)}] - \operatorname{Res}(A)Q \in Cl^{s-l}(M). \tag{4.13}$$

Now the operators $Q_i^{(l)}$ are constructed as: $Q_i^{(l)} = \sum_{j=0}^{l-1} B_i^{(j)}$ where $B_i^{(0)} := Q_i^{(1)}$ and for all $j \in \mathbb{N}$, $B_i^{(j)} \in Cl^{s-j-m+1}(M)$, so by Remark 4.1.1, we can choose $Q_i \in Cl^{s-m+1}(M)$ with $Q_i - Q_i^{(l)} \in Cl^{s-m+1-l}(M)$ for all $l \in \mathbb{N}$. Then we reach the conclusion. □

Remark 4.3.1. Consider an algebra $Cl^a(M)$ with $a \leq 0$. In order to apply Theorem 4.3.1 to express any element in the algebra as a sum of commutators of operators in this space, we need that the conditions

$$s \leq a, \ m \leq a, \ s - m + 1 \leq a$$

be fulfilled, which only holds when $s \leq 2a - 1$. The composition of operators on the space $Cl(M)$ is given by a product on classical symbols (see [20]), and therefore, the space $Cl^a(M)$ is an algebra only when a is an integer and $a \leq 0$.

Remark 4.3.2. A particular case of Proposition 4.1.2 is the graded algebra $\{\mathcal{A}_k\}_{k \in \mathbb{Z}, k \leq 0}$ with $\mathcal{A}_k := Cl^k(M)$. In this case, the quotient map p_k represents the leading symbol map, so by (1.25), the quotient $\mathcal{A}_k/\mathcal{A}_{k-1}$ can be identified with $\mathcal{P}_k(Y) \cong C^\infty(S^*M)$, the set of homogeneous functions on Y of degree k. Given an operator in $Cl^k(M)$, its leading symbol is a well-defined function in $\mathcal{P}_k(Y)$, and the splitting q_k can be constructed after introducing a Riemannian metric on M, e.g. as in Thm. 3.19 in Chapt. III of [24]. Therefore, given a non positive integer number a, we can consider the projection map π_a:

$$0 \to Cl^{2a-1}(M) \to Cl^a(M) \overset{\pi_a}{\to} Cl^a(M)/Cl^{2a-1}(M) \to 0, \tag{4.14}$$

with corresponding splitting $\theta_a : Cl^a(M)/Cl^{2a-1}(M) \to Cl^a(M)$ constructed as in Proposition 4.1.2.

For any $A \in Cl^a(M)$, $A - \theta_a(\pi_a(A)) \in Cl^{2a-1}(M)$ and by Theorem 4.3.1, given an operator $Q \in Cl^{-n}(M)$ with $\operatorname{Res}(Q) = 1$, there exist operators $P_1, \ldots, P_N, Q_1, \ldots, Q_N \in Cl^a(M)$ and $R \in Cl^{-\infty}(M)$ such that

$$A - \theta_a(\pi_a(A)) = \sum_{i=1}^{N}[P_i, Q_i] + \operatorname{Res}(A - \theta_a(\pi_a(A)))Q + R. \tag{4.15}$$

4.4 Smoothing operators as sums of commutators

Theorem 4.2.1 states that if $R \in Cl^{-\infty}(M)$ is a smoothing operator then, for any J pseudo-differential idempotent of rank 1, there exist smoothing operators $S_1, \ldots, S_{N'}$, $T_1, \ldots, T_{N'}$, such that

$$R = \mathrm{Tr}_{L^2}(R)J + \sum_{j=1}^{N'} [S_j, T_j]. \tag{4.16}$$

From the Schwartz kernel representation of an operator, it is possible to go further and prove that if $b \in \mathbb{Z} \cap [-n+1, +\infty)$, then any smoothing operator can be written as a sum of commutators of $[Cl^0(M), Cl^b(M)]$.

In Sect. 4 of [37] there is a proof of the representation of a smoothing ψDO as a sum of commutators. Here we present a similar proof, but unlike the proof in loc. cit., we base our proof in the following lemma:

Lemma 4.4.1. *Let $g \in C^\infty(\mathbb{R}^n)$ be a smooth function on $\mathbb{R}^n$ and for all $j = 1, \ldots, n$, let Q_j be the operator defined by the kernel*

$$(x, y) \mapsto K_{Q_j}(x, y) := \frac{x_j - y_j}{|x - y|^2} g(x).$$

Then Q_j is a classical pseudo-differential operator on $\mathbb{R}^n$ of order $-n+1$.

Proof. Consider the function

$$f(y) := \frac{y_j}{|y|^2} = \partial_{y_j}(\log |y|),$$

that belongs to $C^\infty(\mathbb{R}^n \setminus 0)$ and is positively homogeneous of degree -1. Since f is locally integrable in $\mathbb{R}^n \setminus 0$, it defines a distribution homogeneous of degree -1 (see Def. 3.2.2 in [19]). Then by Thms. 7.1.18 and 7.1.16 in [19], its Fourier transform $\hat{f}$ is a homogeneous distribution of degree $-n+1$ in $\mathbb{R}^n$ which is smooth in $\mathbb{R}^n \setminus 0$.
For all $u \in C_c^\infty(\mathbb{R}^n)$, we understand the following integral as a distribution:

$$\begin{aligned} Q_j u(x) &= \int_{\mathbb{R}^n} K_{Q_j}(x, y) u(y) dy \\ &= \int_{\mathbb{R}^n} g(x) f(x - y) u(y) dy \\ &= g(x) \int_{\mathbb{R}^n} f(x - y) u(y) dy \\ &= g(x)(f * u)(x), \end{aligned}$$

where $f * u$ denotes the convolution product between f and u. Therefore, Q_j is a linear operator from $C_c^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. Let ψ be a cut-off function which is 0 for $|\xi| \leq \frac{1}{4}$ and which is 1 for $|\xi| \geq \frac{1}{2}$. Then Q_j is a pseudo-differential operator with symbol $g(x)\psi(\xi)\hat{f}(\xi)$, and since this function is positively homogeneous of degree $-n+1$ outside a neighborhood of 0, Q_j is a classical ψDO of order $-n+1$ as we claimed. □

In the following, for all $j = 1, \dots, n$, $\mathrm{Op}(x_j)$ denotes the operator multiplication by x_j (see Definition 4.1.1).

Lemma 4.4.2 (See Lemma 4.1 in [37]). *Any smoothing operator $R \in Cl^{-\infty}(\mathbb{R}^n)$ can be written as a finite sum of commutators*

$$R = \sum_{j=1}^{n} [\mathrm{Op}(x_j), B_j]$$

with $B_j \in Cl^{-n+1}(\mathbb{R}^n)$.

Proof. A smoothing operator R has smooth kernel $K_R(x, y)$, and therefore, $K_R(x, y) - K_R(x, x)$ is smooth and vanishes on the diagonal. It follows that there are smooth functions $K_1, \dots, K_n$ such that (see Thm. 1.1.9 in [19])

$$K_R(x, y) = K_R(x, x) + \sum_{j=1}^{n} (x_j - y_j) K_j(x, y).$$

Let Q be the operator defined by the kernel

$$K_Q(x, y) = K_R(x, x),$$

and let R_j be the smoothing operators defined by the kernels $K_j(x, y)$, then

$$R = Q + \sum_{j=1}^{n} [\mathrm{Op}(x_j), R_j].$$

Set $H_j(x, y) := \dfrac{y_j}{|y|^2} K_R(x, x)$ and let Q_j be the operator with kernel

$$(x, y) \mapsto H_j(x, x - y).$$

By Lemma 4.4.1, Q_j is a classical pseudo-differential operator of order $-n + 1$. Since

$$\begin{aligned} \sum_{j=1}^{n} (x_j - y_j) H_j(x, x - y) &= \sum_{j=1}^{n} \frac{(x_j - y_j)^2}{|x - y|^2} K_R(x, x) \\ &= K_R(x, x) \\ &= K_Q(x, y), \end{aligned}$$

it follows that

$$Q = \sum_{j=1}^{n} [\mathrm{Op}(x_j), Q_j].$$

Since the operators R_j are smoothing and Q_j are of order $-n + 1$, the result of the lemma follows with $B_j := R_j + Q_j \in Cl^{-n+1}(\mathbb{R}^n)$. □

Proposition 4.4.1 (See Prop. 4.2 in [37])**.** *Let $b \in \mathbb{Z} \cap [-n+1, +\infty)$. Any $R \in Cl^{-\infty}(M)$ belongs to $[Cl^0(M), Cl^b(M)]$.*

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set and let $R \in Cl^{-\infty}_{\mathrm{comp}}(U)$ be a smoothing operator with compactly supported Schwartz kernel $K_R \in C_c^\infty(U \times U)$. Let $\psi \in C_c^\infty(U)$ be such that $\psi(x)\psi(y) = 1$ near the support of the kernel of R, then $\psi R \psi = R$; in fact, for any $u \in C_c^\infty(U)$ we have

$$\begin{aligned}
\psi R \psi u(x) &= \psi(x) \int_U K_R(x,y)\psi(y)u(y)dy \\
&= \int_U K_R(x,y)\psi(x)\psi(y)u(y)dy \\
&= \int_U K_R(x,y)u(y)dy \\
&= Ru(x).
\end{aligned}$$

By Lemma 4.4.2 there exist $P_i \in Cl^b(U)$ such that $R = \sum_{i=1}^n [\mathrm{Op}(x_i), P_i]$. Let $\chi \in C_c^\infty(U)$ be such that $\chi = 1$ near $\mathrm{supp}(\psi)$. Then we have

$$\psi[\mathrm{Op}(x_i), P_i]\psi = \mathrm{Op}(x_i)\chi\psi P_i \psi - \psi P_i \psi \, \mathrm{Op}(x_i)\chi = [\mathrm{Op}(x_i)\chi, \psi P_i \psi],$$

so we get

$$R = \sum_{i=1}^n [\mathrm{Op}(x_i\chi), \psi P_i \psi], \tag{4.17}$$

where $\mathrm{Op}(x_i\chi)$ denotes the multiplication by the function $x_i\chi \in C_c^\infty(U)$, and the operator $\psi P_i \psi$ belongs to $Cl^b_{\mathrm{comp}}(U)$.
Now let $(\varphi_j) \subset C^\infty(M)$ be a partition of unity subordinate to an open covering (U_j) of M by local coordinate charts. For each index j let $\psi_j \in C_c^\infty(U_j)$ be such that $\psi_j = 1$ near $\mathrm{supp}(\varphi_j)$. Then for any $R \in Cl^{-\infty}(M)$ we have

$$R = \sum_{j=1}^N \varphi_j R \psi_j + \sum_{j=1}^N \varphi_j R(1-\psi_j). \tag{4.18}$$

For each index j the operator $\varphi_j R \psi_j$ belongs to $Cl^{-\infty}_{\mathrm{comp}}(U_j)$, so by the previous argument it can be written as a sum of commutators of the form (4.17). Moreover, the operator $S := \sum_{j=1}^N \varphi_j R(1-\psi_j)$ is smoothing and has Schwartz kernel that vanishes on the diagonal, so its L^2–trace vanishes and by Theorem 4.2.1 it can be written as a sum of commutators in $[Cl^{-\infty}(M), Cl^{-\infty}(M)]$. Hence R belongs to the space $[Cl^0(M), Cl^b(M)]$. □

Chapter 5

Classification of Traces and Associated Determinants

In this chapter we use the representation of a pseudo-differential operator as a sum of commutators given in Chapter 4, to classify the traces on algebras of non positive order classical pseudo-differential operators on a closed manifold M of dimension $n > 1$ (Theorem 5.1.1 and Corollary 5.1.2). From this we deduce a classification of traces on operators acting on sections of a vector bundle over the manifold. We also show that any trace on the algebra of odd-class operators of non positive even order in odd dimensions is a linear combination of a generalized leading symbol trace and the canonical trace (Theorem 5.1.2). The classification of traces on algebras of non positive order classical pseudo-differential operators induces a related classification of multiplicative determinants on the Fréchet-Lie group of invertible operators corresponding to those algebras, that we describe at the end of the chapter (Proposition 5.3.3). Along the chapter, M denotes a closed smooth manifold of dimension $n > 1$.

5.1 Traces on $Cl^a(M)$ for $a \leq 0$

In [28] (see also [45]), there is a homological proof of the classification of the traces on the algebra $Cl^0(M)$, which shows that any trace on this algebra can be written as a linear combination of a leading symbol trace and the residual trace. We address the issue of the classification of traces on $Cl^a(M)$ for a negative integer a. In Proposition 5.1.1 we prove the key fact that allows us to conclude that the L^2–trace does not extend to a trace functional on $Cl^a(M)$ for an integer a such that $2a \in [-n+1, 0]$. We show that traces on $Cl^a(M)$ can be written either as a linear combination of a generalized leading symbol trace and the residual trace, or as a linear combination of a generalized leading symbol trace and a linear extension of the L^2–trace, depending on the value of a.

5.1.1 No non-trivial extension of the L^2–trace to $Cl^a(M)$

As was shown in Subsection 4.2.1, there is a unique trace on the algebra of smoothing operators: the L^2–trace Tr_{L^2}. Hence, for any $a \leq 0$, the restriction $\Lambda\restriction_{Cl^{-\infty}(M)}$ of any trace $\Lambda : Cl^a(M) \to \mathbb{C}$, is proportional to Tr_{L^2}, i.e.

$$\exists\, c \in \mathbb{C}: \quad \Lambda(R) = c\ \mathrm{Tr}_{L^2}(R), \quad \forall R \in Cl^{-\infty}(M). \tag{5.1}$$

We now address the following problem: *for which values of $a \leq 0$, does c vanish?.*

Certainly this is not the case for $a < -n$, since an operator of order $a < -n$ is trace-class, or for $a \notin \mathbb{Z}$, since the canonical trace satisfies (5.1) with $c = 1$. Using the results in Chapter 2, we prove that c vanishes when $a \in \mathbb{Z}$ is such that $2a \in [-n+1, 0]$.

Lemma 5.1.1. *For any $a \in \mathbb{R}$ there exists an inclusion map*

$$[Cl^0(M), Cl^{2a}(M)] \hookrightarrow [Cl^a(M), Cl^a(M)],$$

meaning that any commutator in $[Cl^0(M), Cl^{2a}(M)]$ can be written as a sum of commutators in $[Cl^a(M), Cl^a(M)]$.

Proof. Let $A \in Cl^0(M)$, $B \in Cl^{2a}(M)$. Consider a Laplacian operator Δ and the second order elliptic operator $(1+\Delta)$. For any $a \in \mathbb{R}$, $(1+\Delta)^{a/2}$ and $(1+\Delta)^{-a/2}$ are operators of order a and $-a$, respectively, and therefore $A(1+\Delta)^{a/2}$, $(1+\Delta)^{a/2}A$, $(1+\Delta)^{a/2}$, $B(1+\Delta)^{-a/2}$, $(1+\Delta)^{-a/2}B$, $AB(1+\Delta)^{-a/2}$, $(1+\Delta)^{-a/2}BA$ are operators in $Cl^a(M)$. Moreover,

$$[A(1+\Delta)^{a/2}, (1+\Delta)^{-a/2}B] = AB - (1+\Delta)^{-a/2}BA(1+\Delta)^{a/2}, \tag{5.2}$$
$$[(1+\Delta)^{a/2}A, B(1+\Delta)^{-a/2}] = (1+\Delta)^{a/2}AB(1+\Delta)^{-a/2} - BA, \tag{5.3}$$
$$[AB(1+\Delta)^{-a/2}, (1+\Delta)^{a/2}] = AB - (1+\Delta)^{a/2}AB(1+\Delta)^{-a/2}, \tag{5.4}$$
$$[(1+\Delta)^{-a/2}BA, (1+\Delta)^{a/2}] = (1+\Delta)^{-a/2}BA(1+\Delta)^{a/2} - BA. \tag{5.5}$$

Adding up the expressions in (5.2), (5.3), (5.4) and (5.5) yields twice the commutator $[A, B]$, so that the resulting expression belongs to the space of commutators $[Cl^a(M), Cl^a(M)]$. □

We will need the following (we use the notation of Definition 4.1.1):

Lemma 5.1.2 (Thm. 18.1.6 in [20]). *For any $\sigma \in CS(\mathbb{R}^n)$ and for all $k = 1, \ldots, n$,*

$$\mathrm{Op}(\partial_{\xi_k}\sigma) = -i[\mathrm{Op}(x_k), \mathrm{Op}(\sigma)], \tag{5.6}$$

where as before, $\mathrm{Op}(x_k)$ denotes the operator multiplication by x_k.

Proof. For any $f \in \mathcal{S}(\mathbb{R}^n)$ and for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\mathrm{Op}(\partial_{\xi_k}\sigma)(f)(x) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}\partial_{\xi_k}\sigma(x,\xi)\hat{f}(\xi)d\xi \\
&= \int_{\mathbb{R}^n} \partial_{\xi_k}\left(e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\right)d\xi - \int_{\mathbb{R}^n}\sigma(x,\xi)\partial_{\xi_k}\left(e^{ix\cdot\xi}\hat{f}(\xi)\right)d\xi \\
&= A(x) - i\int_{\mathbb{R}^n}\sigma(x,\xi)e^{ix\cdot\xi}\Big(x_k\hat{f}(\xi)d\xi - \widehat{x_k f}(\xi)\Big)d\xi \\
&= A(x) - ix_k\,\mathrm{Op}(\sigma)(f)(x) + i\,\mathrm{Op}(\sigma)(x_k f)(x) \\
&= A(x) - i\,\mathrm{Op}(x_k)\,\mathrm{Op}(\sigma)(f)(x) + i\,\mathrm{Op}(\sigma)\,\mathrm{Op}(x_k)(f)(x) \\
&= A(x) - i[\mathrm{Op}(x_k),\mathrm{Op}(\sigma)](f)(x).
\end{aligned}$$

The term

$$A(x) := \int_{\mathbb{R}^n}\partial_{\xi_k}\left(e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\right)d\xi$$

vanishes for all $x \in \mathbb{R}^n$: if we denote by $\overline{d\xi} := d\xi_1 \wedge \ldots \wedge \widehat{d\xi_k} \wedge \ldots \wedge d\xi_n$, then

$$\begin{aligned}
|A(x)| &\le \int_{\mathbb{R}^{n-1}}\left|\int_{\mathbb{R}}\partial_{\xi_k}\left(e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\right)d\xi_k\right|\overline{d\xi} \\
&= \int_{\mathbb{R}^{n-1}}\left(\lim_{R\to\infty}\left|\sigma(x,\xi)\hat{f}(\xi)\Big|_{\xi_k=R}\right| - \lim_{R\to\infty}\left|\sigma(x,\xi)\hat{f}(\xi)\Big|_{\xi_k=-R}\right|\right)\overline{d\xi} \\
&= 0,
\end{aligned}$$

since $\hat{f}$ lies in $\mathcal{S}(\mathbb{R}^n)$. Then for all $k = 1,\ldots,n$, $\mathrm{Op}(\partial_{\xi_k}\sigma) = -i[\mathrm{Op}(x_k),\mathrm{Op}(\sigma)]$. □

In the following we use the notation of Section 2.3 and Section 4.1. Remember that by $CS^a_{\mathrm{comp}}(U)$ we denote the set of classical symbols of order a on U with x–compact support, and $Cl^a_{\mathrm{comp}}(U)$ denotes the space of classical ψDOs of order a on U whose Schwartz kernel has compact support on $U \times U$.

Proposition 5.1.1. *Let $a \in \mathbb{Z}$ be such that $2a \in [-n+1, 0]$. For any trace Λ on $Cl^a(M)$ the constant c in (5.1) vanishes.*

Proof. Let $(U, x_1, \ldots, x_n)$ be a local coordinate chart of M. For any symbol $\sigma \in CS^a_{\mathrm{comp}}(U)$, the canonical operator $\mathrm{Op}(\sigma)$ associated to σ is a linear operator that maps the space $C^\infty_c(U)$ to $C^\infty_c(U)$. However, we cannot say that $\mathrm{Op}(\sigma) \in Cl^a_{\mathrm{comp}}(U)$, since its Schwartz kernel has x–compact support but not necessarily y–compact support.

Remark 5.1.1. Any operator in $Cl^a_{\mathrm{comp}}(U)$ can be extended by zero to an operator in $Cl^a(M)$ (the new operator vanishes outside U), and we have the natural inclusion $Cl^a_{\mathrm{comp}}(U) \subset Cl^a(M)$.

Let $\tau \in CS^{-\infty}_{cc}(\mathbb{R}^n)$ be a smoothing symbol such that $\int_{\mathbb{R}^n}\tau(\xi)d\xi \neq 0$. By Proposition 3.2.2 there exist $\tau_1, \ldots, \tau_n \in CS^{2a}_{cc}(\mathbb{R}^n)$ such that

$$\tau = \sum_{k=1}^n \partial_{\xi_k}\tau_k. \tag{5.7}$$

Remark 5.1.2. Since the symbol τ has non-vanishing integral, one at least of the symbols τ_k does not lie in $CS_{cc}^{-\infty}(\mathbb{R}^n)$.

Choose a smooth function $f \in C_c^\infty(U)$ with compact support on U such that $\int_U f(x)dx \neq 0$. Then $\sigma := f \otimes \tau \in CS_{\text{comp}}^{-\infty}(U)$, defined by $\sigma(x,\xi) := f(x)\tau(\xi)$, is a smoothing symbol with x–compact support on U, and

$$\sigma = f \otimes \tau = f \otimes \sum_{k=1}^n \partial_{\xi_k} \tau_k = \sum_{k=1}^n \partial_{\xi_k}(f \otimes \tau_k)$$

is such that

$$\int_{U\times\mathbb{R}^n} \sigma(x,\xi) d\xi\, dx \neq 0. \tag{5.8}$$

By Lemma 5.1.2,

$$\operatorname{Op}(\sigma) = \sum_{k=1}^n \operatorname{Op}(\partial_{\xi_k}(f \otimes \tau_k)) = -i \sum_{k=1}^n [\operatorname{Op}(x_k), \operatorname{Op}(f \otimes \tau_k)]. \tag{5.9}$$

Let $\psi \in C_c^\infty(U)$ be a function such that $\psi = 1$ near the support of f; then $\psi f = f$ on U. Moreover, for all $k = 1, \dots, n$,

$$\begin{aligned}
&[\operatorname{Op}(x_k), \operatorname{Op}(f \otimes \tau_k)] \operatorname{Op}(\psi) = [\operatorname{Op}(x_k), \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi)] \\
&\quad = [\operatorname{Op}(x_k), \operatorname{Op}(\psi f \otimes \tau_k) \operatorname{Op}(\psi)] \\
&\quad = [\operatorname{Op}(x_k), \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi)] \\
&\quad = \operatorname{Op}(x_k) \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) - \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \\
&\quad = \operatorname{Op}(x_k) \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) - \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \operatorname{Op}(\psi) \\
&\qquad + \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \operatorname{Op}(\psi) - \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \\
&\quad = [\operatorname{Op}(x_k) \operatorname{Op}(\psi), \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi)] + A_k \\
&\quad = [\operatorname{Op}(\psi x_k), \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi)] + A_k,
\end{aligned} \tag{5.10}$$

where we use that the operator $\operatorname{Op}(x_k)$ commutes with the operator multiplication by ψ, $\operatorname{Op}(\psi)$, and where the operator A_k is defined by

$$\begin{aligned}
A_k &:= \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \operatorname{Op}(\psi) - \operatorname{Op}(\psi) \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k) \\
&= \operatorname{Op}(f \otimes \tau_k) \operatorname{Op}(\psi) \operatorname{Op}(x_k)(\operatorname{Op}(\psi) - 1).
\end{aligned} \tag{5.11}$$

Remark. If $\sigma \in CS_{\text{comp}}^a(U)$ the Schwartz kernel of the operator $\operatorname{Op}(\sigma)\operatorname{Op}(\psi)$ is given by

$$K_{\operatorname{Op}(\sigma)\operatorname{Op}(\psi)}(x,y) = K_{\operatorname{Op}(\sigma)}(x,y)\psi(y);$$

so it has compact support on $U \times U$ and hence $\operatorname{Op}(\sigma)\operatorname{Op}(\psi)$ lies in $Cl_{\text{comp}}^a(U)$.

Since $f \otimes \tau_k \in CS_{\text{comp}}^{2a}(U)$, the operator $\operatorname{Op}(f \otimes \tau_k)\operatorname{Op}(\psi)$ lies in $Cl_{\text{comp}}^{2a}(U)$; similarly, $\psi x_k \in CS_{\text{comp}}^0(U)$ and the operator multiplication by ψx_k: $\operatorname{Op}(\psi x_k)$ lies in $Cl_{\text{comp}}^0(U)$.

Let Λ be a trace on $Cl^a(M)$. By Lemma 5.1.1, Λ vanishes on $[Cl^0(M), Cl^{2a}(M)]$, and by Remark 5.1.1, Λ vanishes on $[Cl^0_{\mathrm{comp}}(U), Cl^{2a}_{\mathrm{comp}}(U)]$. In particular, for all $k = 1, \dots, n$,

$$\Lambda([\mathrm{Op}(\psi x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)]) = 0.$$

Now, since $\psi = 1$ near the support of f, by (5.11) the operator A_k is smoothing and its Schwartz kernel vanishes on the diagonal. Hence, by (4.5) its L^2–trace vanishes and by Theorem 4.2.1 A_k can be written as a sum of commutators in $[Cl^{-\infty}(M), Cl^{-\infty}(M)]$, and therefore, for all $k = 1, \dots, n$, $\Lambda(A_k) = 0$.

Thus, for $\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi) \in Cl^{-\infty}_{\mathrm{comp}}(U)$, from (5.9) and (5.10) we conclude that,

$$\begin{aligned}\Lambda(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) &= -i\sum_{k=1}^{n} \Lambda([\mathrm{Op}(x_k), \mathrm{Op}(f \otimes \tau_k)]\,\mathrm{Op}(\psi)) \\ &= -i\sum_{k=1}^{n} \left(\Lambda([\mathrm{Op}(\psi x_k), \mathrm{Op}(f \otimes \tau_k)\,\mathrm{Op}(\psi)]) + \Lambda(A_k)\right) \\ &= 0. \end{aligned} \tag{5.12}$$

On the other hand, by (4.6) and (5.1),

$$\Lambda(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = c\ \mathrm{Tr}_{L^2}(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = c\int_{U\times\mathbb{R}^n} \sigma(x,\xi)d\xi\,dx. \tag{5.13}$$

Therefore, by (5.12) we obtain

$$c\ \mathrm{Tr}_{L^2}(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = \Lambda(\mathrm{Op}(\sigma)\,\mathrm{Op}(\psi)) = 0,$$

which, by (5.8) implies that $c = 0$. □

Remark 5.1.3. As a consequence of this proposition, by (5.1) whenever $a \in \mathbb{Z}$ is such that $2a \in [-n+1, 0]$, any trace on $Cl^a(M)$ vanishes on smoothing operators on M.

Corollary 5.1.1. *If $a \in \mathbb{Z}$ is such that $2a \in [-n+1, 0]$, the trace Tr_{L^2} does not extend to a trace functional on the algebra $Cl^a(M)$.*

5.1.2 Generalized leading symbol traces

In Subsection 4.2.4 we studied the leading symbol traces defined on an algebra of operators $Cl^a(M)$ for $a \leq 0$; in this section we consider a more general definition which actually coincides with a leading symbol trace for $a = 0$. As in Section 4.3, for a non positive integer order a we consider the projection map π_a:

$$0 \to Cl^{2a-1}(M) \to Cl^a(M) \xrightarrow{\pi_a} Cl^a(M)/Cl^{2a-1}(M) \to 0. \tag{5.14}$$

Lemma 5.1.3. *Any continuous linear map λ on $Cl^a(M)/Cl^{2a-1}(M)$ defines a trace on $Cl^a(M)$ by $\lambda \circ \pi_a$ called* generalized leading symbol trace.

Proof. If $A, B \in Cl^a(M)$, their commutator $[A, B]$ belongs to $Cl^{2a-1}(M)$, and since $\lambda \circ \pi_a$ vanishes on $Cl^{2a-1}(M)$, this is a trace on $Cl^a(M)$. □

By Proposition 4.1.2 and Remark 4.3.2, λ is a map on

$$Cl^a(M)/Cl^{2a-1}(M) \cong \mathcal{P}_a(Y) \oplus \ldots \oplus \mathcal{P}_{2a}(Y). \tag{5.15}$$

By (1.25) it follows that $\mathcal{P}_k(Y) \cong C^\infty(S^*M)$ for any $k = a, a-1, \ldots, 2a$.

For $A \in Cl^a(M)$, $\lambda(\pi_a(A))$ depends on $\sigma_a(A), \ldots, \sigma_{2a}(A)$, where $\sigma_{a-i}(A)$ represents the homogeneous component of degree $a-i$ in the asymptotic expansion of the symbol of A. Since $\lambda \circ \pi_a$ is linear in A, it is a linear combination of linear maps λ_{a-i} on S^*M, in the terms $\sigma_{a-i}(A)$, hence it reads,

$$\lambda(\pi_a(A)) = \sum_{i=0}^{|a|} \lambda_{a-i}(\sigma_{a-i}(A)).$$

Generalized leading symbol traces are continuous for the Fréchet topology on the space of constant order ψDOs, since they are defined in terms of a finite number of homogeneous parts of the symbols of the operators.

5.1.3 Classification of traces on $Cl^a(M)$

Now we can give the classification of traces on the algebra of operators $Cl^a(M)$ for a non positive integer a, using the representation of a classical ψDO given in Section 4.3. We choose a linear functional $\widetilde{\mathrm{Tr}}_a : Cl^a(M) \to \mathbb{C}$ as follows:

$$\widetilde{\mathrm{Tr}}_a = \begin{cases} \text{trace on } Cl^a(M) \text{ that extends } \mathrm{Tr}_{L^2}, \text{ if it exists,} \\ 0, \text{ if such an extension does not exist.} \end{cases}$$

Theorem 5.1.1. *Let $a \in \mathbb{Z}$ be such that $a \le 0$ and let τ be a trace on $Cl^a(M)$. There exist constants $c_1, c_2 \in \mathbb{C}$ and a linear map λ on $Cl^a(M)/Cl^{2a-1}(M)$ such that τ can be expressed in the form*

$$\tau = \lambda \circ \pi_a + c_1 \operatorname{Res} + c_2 \widetilde{\mathrm{Tr}}_a. \tag{5.16}$$

Proof. For any $A \in Cl^a(M)$, Equation (4.15) reads

$$A - \theta_a(\pi_a(A)) = \sum_{i=1}^{N} [P_i, Q_i] + \operatorname{Res}(A - \theta_a(\pi_a(A)))Q + R, \tag{5.17}$$

where $P_i, Q_i \in Cl^a(M)$, $Q \in Cl^{-n}(M)$ has $\operatorname{Res}(Q) = 1$ and $R \in Cl^{-\infty}(M)$. Applying $\widetilde{\mathrm{Tr}}_a$ to both sides of (5.17), for all $i = 1, \ldots, N$, $\widetilde{\mathrm{Tr}}_a([P_i, Q_i]) = 0$ and hence

$$\widetilde{\mathrm{Tr}}_a(A - \theta_a(\pi_a(A))) - \operatorname{Res}(A - \theta_a(\pi_a(A)))\widetilde{\mathrm{Tr}}_a(Q) = \widetilde{\mathrm{Tr}}_a(R) = \mathrm{Tr}_{L^2}(R). \tag{5.18}$$

Therefore, by Theorem 4.2.1 and (5.18), for any pseudo-differential idempotent of rank 1, J, there exist smoothing operators $S_1, \ldots, S_{N'}$, $T_1, \ldots, T_{N'}$, such that (5.17) becomes

$$A - \theta_a(\pi_a(A)) = \sum_{i=1}^{N} [P_i, Q_i] + \operatorname{Res}(A - \theta_a(\pi_a(A)))Q + \operatorname{Tr}_{L^2}(R)J + \sum_{j=1}^{N'} [S_j, T_j], \tag{5.19}$$

where $\operatorname{Tr}_{L^2}(R) = \widetilde{\operatorname{Tr}_a}(A - \theta_a(\pi_a(A))) - \operatorname{Res}(A - \theta_a(\pi_a(A)))\widetilde{\operatorname{Tr}_a}(Q)$.
Applying τ to both sides of (5.19) we have

$$\begin{aligned}
\tau(A) &= \tau(\theta_a(\pi_a(A))) + \operatorname{Res}(A - \theta_a(\pi_a(A)))\tau(Q) + \operatorname{Tr}_{L^2}(R)\tau(J) \\
&= \tau(\theta_a(\pi_a(A))) - \operatorname{Res}(\theta_a(\pi_a(A)))\tau(Q) - \widetilde{\operatorname{Tr}_a}(\theta_a(\pi_a(A)))\tau(J) \\
&\quad + \operatorname{Res}(\theta_a(\pi_a(A)))\widetilde{\operatorname{Tr}_a}(Q)\tau(J) + \operatorname{Res}(A)\left(\tau(Q) - \widetilde{\operatorname{Tr}_a}(Q)\tau(J)\right) \\
&\quad + \widetilde{\operatorname{Tr}_a}(A)\tau(J).
\end{aligned}$$

If $c_1 := \tau(Q) - \widetilde{\operatorname{Tr}_a}(Q)\tau(J)$, and $c_2 := \tau(J)$, denoting by λ the linear map on $Cl^a(M)/Cl^{2a-1}(M)$:

$$\lambda := \tau \circ \theta_a - c_1 \operatorname{Res} \circ \theta_a - c_2 \widetilde{\operatorname{Tr}_a} \circ \theta_a, \tag{5.20}$$

we obtain

$$\tau(A) = \lambda(\pi_a(A)) + c_1 \operatorname{Res}(A) + c_2 \widetilde{\operatorname{Tr}_a}(A). \tag{5.21}$$

□

Remark 5.1.4. Notice that we can fix the operators $P_1, \ldots, P_N$, Q and J from the beginning. The constants c_1 and c_2 depend on the choice of Q and J. A priori λ depends on the choice of splitting θ_a, however, if we choose another splitting θ', the difference between the expressions in (5.21) for θ_a and for θ'_a yields

$$\lambda(\pi_a(A)) = \lambda'(\pi_a(A)),$$

so the expression in (5.16) is independent of the choice of splitting θ_a.

Remark 5.1.5. If the trace τ is continuous for the Fréchet topology of $Cl^a(M)$, we can choose $\widetilde{\operatorname{Tr}_a}$ continuous for the same topology so that $\lambda \circ \pi_a$ is also continuous as the linear combination in (5.20).

Corollary 5.1.2. *Let $a \in \mathbb{Z}$ be such that $a \leq 0$ and let $\tau : Cl^a(M) \to \mathbb{C}$ be a trace. Then*

1. *If $-n + 1 \leq 2a \leq 0$, τ is a linear combination of a generalized leading symbol trace and the residual trace.*

2. *If $a < -n$, τ is a linear combination of a generalized leading symbol trace and the L^2–trace.*

3. *If $2a \leq -n \leq a$, τ is a linear combination of a generalized leading symbol trace and a linear extension of the L^2–trace.*

Proof. 1. If $-n+1 \leq 2a \leq 0$, we can use Proposition 5.1.1 or Proposition 4.4.1 to show that $\widetilde{\mathrm{Tr}_a} \equiv 0$. Note that Proposition 4.2.1 yields this result for $a = 0$. Therefore, there exists a constant $c \in \mathbb{C}$ such that (5.16) reads

$$\tau = \lambda \circ \pi_a + c\,\mathrm{Res}\,.$$

For $a = 0$, this confirms the corresponding result in [28] (see also [45]).

2. If $a < -n$, the residual trace vanishes on $Cl^a(M)$ and $\widetilde{\mathrm{Tr}_a} = \mathrm{Tr}_{L^2}$. Therefore, there exists a constant $c \in \mathbb{C}$ such that (5.16) reads

$$\tau = \lambda \circ \pi_a + c\,\mathrm{Tr}_{L^2}\,.$$

3. If $2a \leq -n \leq a$, we consider $Cl^{2a-1}(M)$ as a linear subspace of $Cl^a(M)$. Let $\{p_j\}_{j\in\mathbb{N}}$ be a countable family of semi-norms on $Cl^a(M)$. Since Tr_{L^2} is a linear form on $Cl^{2a-1}(M)$, continuous for the Fréchet topology of $Cl^{2a-1}(M)$, Tr_{L^2} is also continuous for the Fréchet topology of $Cl^a(M)$ (see Section 4.1, or [13]), that is, for any $A \in Cl^{2a-1}(M)$, for all $j \in \mathbb{N}$,

$$|\,\mathrm{Tr}_{L^2}(A)| \leq p_j(A).$$

By the Hahn-Banach Theorem (see e.g. Thm. 18.1 in [42]), there exists a linear form α on $Cl^a(M)$, extending Tr_{L^2}, i.e. such that

$$\forall A \in Cl^{2a-1}(M),\ \alpha(A) = \mathrm{Tr}_{L^2}(A),\ \text{and furthermore, } |\alpha(A)| \leq p_j(A).$$

Lemma 4.2.3 states that $\widetilde{\mathrm{Tr}_a}$ is any such a linear form α. Therefore, there exists a constant $c \in \mathbb{C}$ such that (5.16) reads

$$\tau = \lambda \circ \pi_a + c\,\alpha.$$

□

Remark 5.1.6. If $2a \leq -n \leq a$, Res is a non-trivial trace on $Cl^a(M)$; for any $A \in Cl^a(M)$, $\mathrm{Res}(A - \theta_a(\pi_a(A))) = 0$, but $\mathrm{Res}(A)$ is not necessarily 0. Considering $\tau = \mathrm{Res}$ in the previous corollary, we get $c = \mathrm{Res}(J) = 0$, and $\mathrm{Res} = (\mathrm{Res} \circ \theta_a) \circ \pi_a$.

5.1.4 Classification of traces on $Cl^{(\mathrm{odd}),a}(M)$

Associated to the odd-class symbols defined in Section 3.4, we consider the set $Cl^{(\mathrm{odd})}(M)$ of odd-class operators on the manifold M. The canonical trace defined in Subsection 4.2.3 is the unique trace on $Cl^{(\mathrm{odd})}(M)$ when the dimension of the manifold is odd (see [30], [35]). In this section we assume that the dimension n is odd, and we prove that any trace on the algebra of odd-class operators of negative even order is a linear combination of a generalized leading symbol trace and the canonical trace.

The following lemma implies that $Cl^{(\mathrm{odd})}(M)$ is an algebra (see Sect. 4 in [23]):

Lemma 5.1.4. *Let $A \in Cl^{(\mathrm{odd}),a}(M)$ and $B \in Cl^{(\mathrm{odd}),b}(M)$, $a, b \in \mathbb{Z}$. Then $AB \in Cl^{(\mathrm{odd}),a+b}(M)$. If besides B is an invertible elliptic operator, then $B^{-1} \in Cl^{(\mathrm{odd}),-b}(M)$ and $AB^{-1} \in Cl^{(\mathrm{odd}),a-b}(M)$.*

We also have Lemma 5.1.1 in the case of odd-class operators:

Lemma 5.1.5. *If $a \in \mathbb{Z}$ is even, then there exists an inclusion map*

$$[Cl^{(\mathrm{odd}),0}(M), Cl^{(\mathrm{odd}),2a}(M)] \hookrightarrow [Cl^{(\mathrm{odd}),a}(M), Cl^{(\mathrm{odd}),a}(M)],$$

meaning that any commutator in $[Cl^{(\mathrm{odd}),0}(M), Cl^{(\mathrm{odd}),2a}(M)]$ can be written as a sum of commutators in $[Cl^{(\mathrm{odd}),a}(M), Cl^{(\mathrm{odd}),a}(M)]$.

Proof. Differential operators are examples of odd-class operators, then Lemma 5.1.4 implies that integer powers of an invertible Laplacian operator (as the one used in the proof of Lemma 5.1.1) are odd-class operators. Hence we proceed as in the proof of Lemma 5.1.1. □

As in Chapter 4, for a non positive integer a we consider the projection map π_a:

$$Cl^{(\mathrm{odd}),a}(M) \overset{\pi_a}{\to} Cl^{(\mathrm{odd}),a}(M)/Cl^{(\mathrm{odd}),2a-1}(M), \tag{5.22}$$

with corresponding splitting $\theta_a : Cl^{(\mathrm{odd}),a}(M)/Cl^{(\mathrm{odd}),2a-1}(M) \to Cl^{(\mathrm{odd}),a}(M)$, and hence for any $A \in Cl^{(\mathrm{odd}),a}(M)$, $A - \theta_a(\pi_a(A)) \in Cl^{(\mathrm{odd}),2a-1}(M)$.

We would like to classify traces on $Cl^{(\mathrm{odd}),a}(M)$ as we did in the previous section for the algebra $Cl^a(M)$; however we cannot apply Theorem 4.3.1 directly to the operator $A - \theta_a(\pi_a(A))$ as in the proof of Theorem 5.1.1, since we do not know if (1.58) holds in the set of odd functions, but we still can conclude a classification of traces on $Cl^{(\mathrm{odd}),a}(M)$ for $a \in \mathbb{Z}$ even and such that $a \leq 0$.

Theorem 5.1.2. *If $a \in \mathbb{Z}$ is even and $a \leq 0$, any trace on $Cl^{(\mathrm{odd}),a}(M)$ can be written as a linear combination of a generalized leading symbol trace and the canonical trace.*

Proof. Let $A \in Cl^{(\mathrm{odd}),a}(M)$. Locally, for a chart U of M, the symbol of $A - \theta_a(\pi_a(A))$ belongs to $CS^{(\mathrm{odd}),2a-1}(U)$, then by Proposition 3.4.2 there exist τ_i in $CS^{(\mathrm{odd}),2a}(U)$ such that

$$\sigma(A - \theta_a(\pi_a(A))) \sim \sum_{i=1}^{n} \partial_{\xi_i} \tau_i.$$

By Lemma 5.1.2, on U we get

$$A - \theta_a(\pi_a(A)) \sim \sum_{i=1}^{n} [\mathrm{Op}(x_i), \mathrm{Op}(\tau_i)],$$

where $[\mathrm{Op}(x_i), \mathrm{Op}(\tau_i)] \in [Cl^{(\mathrm{odd}),0}(U), Cl^{(\mathrm{odd}),2a}(U)]$. Using a partition of unity subordinate to an open cover of M and multiplying by appropriate cut-off functions as in the proof of Proposition 4.4.1 and Proposition 5.1.1, one proves the existence of operators $B_i \in Cl^{(\mathrm{odd}),0}(M)$, $C_i \in Cl^{(\mathrm{odd}),2a}(M)$, and of a smoothing operator R such that

$$A - \theta_a(\pi_a(A)) = \sum_{i=1}^{n} [B_i, C_i] + R. \tag{5.23}$$

By Lemma 5.1.5, there exist operators $D_1, \ldots, D_N, E_1, \ldots, E_N \in Cl^{(\mathrm{odd}),a}(M)$, such that

$$A - \theta_a(\pi_a(A)) = \sum_{k=1}^{N} [D_k, E_k] + R. \tag{5.24}$$

Applying TR to both sides of (5.24) yields

$$\mathrm{TR}(A - \theta_a(\pi_a(A))) = \sum_{k=1}^{N} \mathrm{TR}([D_k, E_k]) + \mathrm{TR}(R) = \mathrm{Tr}_{L^2}(R). \tag{5.25}$$

Hence, as in the proof of Theorem 5.1.1, for any J pseudo-differential idempotent of rank 1, there exist smoothing operators $S_1, \ldots, S_{N'}$, $T_1, \ldots, T_{N'}$, such that (5.24) becomes

$$A - \theta_a(\pi_a(A)) = \sum_{k=1}^{N} [D_k, E_k] + \mathrm{TR}(A - \theta_a(\pi_a(A)))J + \sum_{j=1}^{N'} [S_j, T_j]. \tag{5.26}$$

Let $\tau : Cl^{(\mathrm{odd}),a}(M) \to \mathbb{C}$ be a trace on $Cl^{(\mathrm{odd}),a}(M)$. If we apply τ to both sides of (5.26) we get

$$\begin{aligned} \tau(A) &= \tau(\theta_a(\pi_a(A))) + \mathrm{TR}(A - \theta_a(\pi_a(A)))\tau(J) && (5.27)\\ &= \tau(\theta_a(\pi_a(A))) - \mathrm{TR}(\theta_a(\pi_a(A)))\tau(J) + \mathrm{TR}(A)\tau(J). && (5.28) \end{aligned}$$

So, as in the proof of Theorem 5.1.1, we conclude that τ is a linear combination of a generalized leading symbol trace and the canonical trace. $\square$

5.2 Traces on operators acting on sections of vector bundles

An operator acting on sections of a vector bundle over a closed manifold M of dimension $n > 1$, can be seen as a matrix of operators on the manifold. In this section we study traces defined on matrices with coefficients in an algebra $\mathcal{A}$ over $\mathbb{C}$. When the algebra is unital, we obtain a characterization of any trace on that space of matrices; however, when the algebra is non-unital we cannot conclude a similar result. We apply this to the case of pseudo-differential operators acting on sections of a vector bundle, first in the case that the vector bundle is trivial and then in the general case.

5.2.1 Trivial vector bundles

Given a closed manifold M, any classical ψDO of order a acting on the sections of a trivial vector bundle $M \times \mathbb{C}^N$ over M, can be seen as an $N \times N$ matrix whose entries are classical ψDOs of order a on M (see Section 4.1). To give a classification of the traces on those operators, we study a more general case of traces on the space $M_N(\mathcal{A})$ of $N \times N$ matrices, whose entries belong to an algebra $\mathcal{A}$ over $\mathbb{C}$.

Consider the space $M_N(\mathbb{C})$ of $N \times N$ matrices with coefficients in $\mathbb{C}$. For all $i, j = 1, \ldots, N$, we denote by E_{ij} the elementary matrix in $M_N(\mathbb{C})$ with 1 in the (i, j)–position and 0 everywhere else. The matrices E_{ij} form a basis of $M_N(\mathbb{C})$ and we have

1. $E_{ii} - E_{jj} = [E_{ij}, E_{ji}]$.

2. If $i \neq j$ then $E_{ij} = [E_{ij}, E_{jj}]$.

Let us denote by tr the unique trace on the algebra $M_N(\mathbb{C})$ such that for all $i = 1, \ldots, N$, $\mathrm{tr}(E_{ii}) = 1$.

Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. We will use the following isomorphism that gives an identification of $M_N(\mathcal{A})$ with $\mathcal{A} \otimes M_N(\mathbb{C})$:

$$\begin{aligned}\phi : M_N(\mathcal{A}) &\to \mathcal{A} \otimes M_N(\mathbb{C})\\ A := (A_{ij})_{i,j} &\mapsto \phi(A) := \sum_{i,j=1}^{N} A_{ij} \otimes E_{ij},\end{aligned}$$

Definition 5.2.1. A trace on $M_N(\mathcal{A})$ is a linear map $\varphi : M_N(\mathcal{A}) \to \mathbb{C}$ such that for any $X, Y \in M_N(\mathcal{A})$ it satisfies

$$\varphi([X, Y]) = 0.$$

Lemma 5.2.1. *Let $\mathcal{A}$ be a unital algebra and let $\overline{\varphi} : \mathcal{A} \otimes M_N(\mathbb{C}) \to \mathbb{C}$ be a bilinear map. The following are equivalent*

1. *For all $x, y \in \mathcal{A}$, $P, Q \in M_N(\mathbb{C})$, $\overline{\varphi}([x \otimes P, y \otimes Q]) = 0$.*

2. *For all $x, y \in \mathcal{A}$, $P, Q \in M_N(\mathbb{C})$, $\overline{\varphi}([x, y] \otimes P) = 0$ and $\overline{\varphi}(x \otimes [P, Q]) = 0$.*

Remark 5.2.1. This implies that when the algebra $\mathcal{A}$ is unital, any bilinear map on $\mathcal{A} \otimes M_N(\mathbb{C})$ that satisfies the second condition, yields a trace on $M_N(\mathcal{A})$.

Proof. It is enough to express the elements on which $\overline{\varphi}$ vanishes in the first item in terms of the elements on which $\overline{\varphi}$ vanishes in the second item and vice versa.

1. $\Rightarrow$ *2.* Any element of the form $[x,y]\otimes P$ or $x\otimes[P,Q]$ can be written as commutators $[a\otimes A, b\otimes B]$ for some $a,b\in\mathcal{A}$, and $A,B\in M_N(\mathbb{C})$:
Since $M_N(\mathbb{C})$ is a unital algebra with unit 1_N, we have

$$\begin{aligned}[x,y]\otimes P &= xy\otimes P - yx\otimes P\\ &= xy\otimes P1_N - yx\otimes 1_N P\\ &= (x\otimes P)(y\otimes 1_N) - (y\otimes 1_N)(x\otimes P)\\ &= [x\otimes P, y\otimes 1_N]. \end{aligned} \tag{5.29}$$

Note that for this we do not need $\mathcal{A}$ to be unital.
Similarly, if $\mathcal{A}$ is a unital algebra with unit $1_{\mathcal{A}}$, we have

$$\begin{aligned}x\otimes[P,Q] &= x\otimes PQ - x\otimes QP\\ &= x1_{\mathcal{A}}\otimes PQ - 1_{\mathcal{A}}x\otimes QP\\ &= (x\otimes P)(1_{\mathcal{A}}\otimes Q) - (1_{\mathcal{A}}\otimes Q)(x\otimes P)\\ &= [x\otimes P, 1_{\mathcal{A}}\otimes Q]. \end{aligned} \tag{5.30}$$

2. $\Rightarrow$ *1.* Any commutator $[x\otimes P, y\otimes Q]\in\mathcal{A}\otimes M_N(\mathbb{C})$ can be written as a linear combination of elements of the form $[a,b]\otimes A$ and $a\otimes[A,B]$ for $a,b\in\mathcal{A}$, and $A,B\in M_N(\mathbb{C})$: If $P=(P_{ij})_{i,j}, Q=(Q_{ij})_{i,j}\in M_N(\mathbb{C})$

$$\begin{aligned}[x\otimes P, y\otimes Q] &= (x\otimes P)(y\otimes Q) - (y\otimes Q)(x\otimes P)\\ &= xy\otimes PQ - yx\otimes QP\\ &= \sum_{i,k=1}^{N} P_{ik}Q_{ki}(xy\otimes E_{ii}) + \sum_{\substack{i,j,k=1\\ i\neq j}}^{N} P_{ik}Q_{kj}(xy\otimes[E_{ij},E_{jj}])\\ &\quad - \sum_{i,k=1}^{N} Q_{ik}P_{ki}(yx\otimes E_{ii}) - \sum_{\substack{i,j,k=1\\ i\neq j}}^{N} Q_{ik}P_{kj}(yx\otimes[E_{ij},E_{jj}])\\ &= \sum_{i,k=1}^{N} P_{ik}Q_{ki}([x,y]\otimes E_{ii}) + \sum_{\substack{i,j,k=1\\ i\neq j}}^{N} P_{ik}Q_{kj}(xy\otimes[E_{ij},E_{jj}])\\ &\quad - \sum_{\substack{i,j,k=1\\ i\neq j}}^{N} Q_{ik}P_{kj}(yx\otimes[E_{ij},E_{jj}]). \end{aligned}$$ □

For any $f\in\mathcal{A}^*$, $g\in(M_N(\mathbb{C}))^*$, consider the linear map $f\otimes g\in(\mathcal{A}\otimes M_N(\mathbb{C}))^*$ defined by

$$\begin{aligned}\mathcal{A}\otimes M_N(\mathbb{C}) &\to \mathbb{C}\\ x\otimes P &\mapsto (f\otimes g)(x\otimes P) := f(x)g(P).\end{aligned}$$

Lemma 5.2.2. *Given any trace τ on $\mathcal{A}$, the linear map $\varphi := (\tau \otimes \mathrm{tr}) \circ \phi$ is a trace on $M_N(\mathcal{A})$.*

Proof. The linearity of φ holds from its definition and the linearity of τ and tr. If $A = (A_{ij})_{i,j}, B = (B_{ij})_{i,j} \in M_N(\mathcal{A})$, then their commutator $C = [A, B] = (C_{ij})_{i,j} \in M_N(\mathcal{A})$ is given by the following:

$$C_{ij} = \sum_{k=1}^{N}(A_{ik}B_{kj} - B_{ik}A_{kj}), \quad \forall\, i, j = 1, \ldots, N$$

and it satisfies

$$\sum_{i=1}^{N} C_{ii} = \sum_{i=1}^{N}\sum_{k=1}^{N}[A_{ik}, B_{ki}].$$

Then

$$\begin{aligned}\phi([A,B]) &= \sum_{i,j=1}^{N} C_{ij} \otimes E_{ij} \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N}[A_{ik}, B_{ki}] \otimes E_{ii} + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} C_{ij} \otimes [E_{ij}, E_{jj}].\end{aligned}$$

For any $A, B \in M_N(\mathcal{A})$ we have

$$\begin{aligned}\varphi([A,B]) &= (\tau \otimes \mathrm{tr}) \circ \phi([A,B]) \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} \tau([A_{ik}, B_{ki}])\,\mathrm{tr}(E_{ii}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \tau(C_{ij})\,\mathrm{tr}([E_{ij}, E_{jj}]) \\ &= 0.\end{aligned}$$

□

Let us consider the case when $\mathcal{A}$ is unital.

Lemma 5.2.3. *Given any trace φ on $\mathcal{A} \otimes M_N(\mathbb{C})$, there exists a trace τ on $\mathcal{A}$ such that $\varphi = \tau \otimes \mathrm{tr}$.*

Proof. Let $\varphi : \mathcal{A} \otimes M_N(\mathbb{C}) \to \mathbb{C}$ be a trace on $\mathcal{A} \otimes M_N(\mathbb{C})$. Let us define the linear maps

$$\begin{aligned}\varphi_{ij} &: \mathcal{A} \to \mathbb{C} \\ &x \mapsto \varphi_{ij}(x) := \varphi(x \otimes E_{ij}).\end{aligned}$$

These linear maps satisfy the following properties:

1. For all $i,j=1,\dots,N$, φ_{ij} is a trace on $\mathcal{A}$:
   By (5.29), for any x,y in $\mathcal{A}$ we have
   $$\varphi_{ij}([x,y]) = \varphi([x,y]\otimes E_{ij}) = 0.$$
   Remember that for this we do not use the assumption that $\mathcal{A}$ is unital.

2. For all $i,j=1,\dots,N$, if $i\neq j$ then $\varphi_{ij}=0$:
   By (5.30), for any element x in $\mathcal{A}$ we have
   $$\varphi_{ij}(x) = \varphi(x\otimes E_{ij}) = \varphi(x\otimes [E_{ij},E_{jj}]) = 0.$$

3. For all $i,j=1,\dots,N$, $\varphi_{ii}=\varphi_{jj}$:
   By (5.30), for any element x in $\mathcal{A}$ we have
   $$\begin{aligned}\varphi_{ii}(x)-\varphi_{jj}(x) &= \varphi(x\otimes E_{ii})-\varphi(x\otimes E_{jj})\\ &= \varphi(x\otimes (E_{ii}-E_{jj}))\\ &= \varphi(x\otimes [E_{ij},E_{ji}])\\ &= 0.\end{aligned}$$

Using the last relation we can denote by τ the common linear map φ_{ii} for all $i=1,\dots,N$. Then

$$\begin{aligned}\varphi\left(\sum_{i,j=1}^{N} A_{ij}\otimes E_{ij}\right) &= \sum_{i,j=1}^{N}\varphi_{ij}\left(A_{ij}\right)\\ &= \sum_{i=1}^{N}\varphi_{ii}\left(A_{ii}\right)\\ &= \sum_{i=1}^{N}\tau\left(A_{ii}\right)\\ &= (\tau\otimes \mathrm{tr})\left(\sum_{i,j=1}^{N} A_{ij}\otimes E_{ij}\right).\end{aligned}$$

□

Remark 5.2.2. If $\mathcal{A}=Cl(M)$ or $\mathcal{A}=\bigcup_{a\in\mathbb{Z}} Cl^a(M)$, then $\mathcal{A}$ is unital and any trace on $M_N(\mathcal{A})\cong Cl(M, M\times\mathbb{C}^N)$ or $M_N(\mathcal{A})\cong\bigcup_{a\in\mathbb{Z}} Cl^a(M,M\times\mathbb{C}^N)$ resp., reads $(\mathrm{Res}\otimes\mathrm{tr})\circ\phi$, and if $\mathcal{A}=Cl^0(M)$, then $\mathcal{A}$ is also unital and with the notation of Corollary 5.1.2, any trace on $M_N(\mathcal{A})$ reads $((\lambda\circ\pi_0+c\,\mathrm{Res})\otimes\mathrm{tr})\circ\phi$.

Using Lemma 5.2.2, we can write the proof of Proposition 4.2.1 for the unital algebra $\mathcal{A}=Cl(M)$ or for the unital algebra $\mathcal{A}=Cl^0(M)$ as in [26]:

Proposition 4.2.1. *The trace* Tr_{L^2} *does not have a continuation as a trace functional on the whole algebra* $Cl(M)$.

Proof. Assume τ is a trace on $Cl(M)$ such that for any $P \in Cl^a(M)$, if $a < -n$, then $\tau(P) = \mathrm{Tr}_{L^2}(P)$. We may choose N big enough and an elliptic operator $A \in Cl(M, M \times \mathbb{C}^N)$ of non-vanishing Fredholm index (In dimensions greater than 2, the index of any scalar elliptic ψDO vanishes ([9], [33]), however this assumption is valid considering N big enough and A an elliptic ψDO on $Cl(M, M \times \mathbb{C}^N)$). Let $B \in Cl(M, M \times \mathbb{C}^N)$ be a parametrix of A. Then the L^2–trace on $Cl^{-\infty}(M, M \times \mathbb{C}^N)$ reads

$$\mathrm{Tr}_{L^2}^{(M\times\mathbb{C}^N)} = (\mathrm{Tr}_{L^2} \otimes \mathrm{tr}) \circ \phi.$$

Hence (see Thm. 18.1.24 in [20])

$$I - BA,\ I - AB \in Cl^{-\infty}(M, M \times \mathbb{C}^N),$$

and we get the contradiction (see Prop. 19.1.14 in [20])

$$\begin{aligned} 0 &\neq \mathrm{ind}(A) \\ &= \mathrm{Tr}_{L^2}^{(M\times\mathbb{C}^N)}(I - BA) - \mathrm{Tr}_{L^2}^{(M\times\mathbb{C}^N)}(I - AB) \\ &= (\mathrm{Tr}_{L^2} \otimes \mathrm{tr}) \circ \phi(I - BA) - (\mathrm{Tr}_{L^2} \otimes \mathrm{tr}) \circ \phi(I - AB) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi(I - BA) - (\tau \otimes \mathrm{tr}) \circ \phi(I - AB) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi([A, B]) \\ &= 0. \end{aligned}$$

□

In the case when $\mathcal{A}$ is non-unital, we can consider its unitization (see e.g. [3]) $\widetilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$, with the product

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu),$$

and with unit element $(0, 1)$. We denote by inc the inclusion

$$\begin{aligned} \mathrm{inc} : \mathcal{A} &\to \widetilde{\mathcal{A}} \\ a &\mapsto (a, 0). \end{aligned}$$

One can apply Lemma 5.2.3 to $\widetilde{\mathcal{A}}$ to obtain:
Given any trace φ on $\widetilde{\mathcal{A}} \otimes M_N(\mathbb{C})$, there exists a trace τ on $\widetilde{\mathcal{A}}$ such that

$$\varphi = \tau \otimes \mathrm{tr}\,.$$

In particular, given any $A, B \in \mathcal{A}$, $P, Q \in M_N(\mathbb{C})$

$$\varphi \circ (\mathrm{inc} \otimes id_{M_N(\mathbb{C})})([A \otimes P, B \otimes Q]) = \varphi([\mathrm{inc}(A) \otimes P, \mathrm{inc}(B) \otimes Q]) = 0,$$

so $\varphi \circ (\mathrm{inc} \otimes id_{M_N(\mathbb{C})})$ is a trace on $\mathcal{A} \otimes M_N(\mathbb{C})$.
Similarly,

$$\tau \circ \mathrm{inc}([A, B]) = \tau([\mathrm{inc}(A), \mathrm{inc}(B)]) = 0,$$

so $\tau \circ \mathrm{inc}$ is a trace on $\mathcal{A}$, and it is such that

$$\varphi \circ (\mathrm{inc} \otimes id_{M_N(\mathbb{C})}) = (\tau \circ \mathrm{inc}) \otimes \mathrm{tr}\,.$$

However, we do not know if Lemma 5.2.3 holds in the case that $\mathcal{A}$ is a non-unital algebra, for example $\mathcal{A} = Cl^a(M)$ with $a \in \mathbb{Z}$, $a < 0$.

5.2.2 General vector bundles

In this section we give a classification of the traces on classical ψDOs acting on the sections of a vector bundle E over M, by using the results of the previous section and the fact that the space of sections $\Gamma(E)$ of any vector bundle E over M is isomorphic to the image of some power of the space of smooth functions on M, $C^\infty(M)$, by an idempotent.

We start following the argument given in Lemma 3 of [28]. Let E be any vector bundle over M. There is a positive integer N, such that E is a direct summand of $M \times \mathbb{C}^N$; let $e \in M_N(C^\infty(M))$ be a smooth projection onto E. There exists an idempotent $\varepsilon \in \mathrm{End}_{C^\infty(M)}(C^\infty(M)^N)$ so that the $C^\infty(M)$–module of sections of E satisfies

$$\Gamma(E) \cong \varepsilon(C^\infty(M)^N).$$

Since the endomorphism algebra $\mathrm{End}_{C^\infty(M)}(C^\infty(M)^N)$ can be identified with the matrix algebra $M_N(C^\infty(M))$, the matrix $e \in M_N(C^\infty(M))$ is such that $e = e^2$ and

$$\Gamma(E) \cong e(C^\infty(M)^N)$$

as right $C^\infty(M)$–modules (see Prop. 2.9 and Prop. 2.22 in [11]). Let k be the rank of E, then E has a constant pointwise trace equal to k:

$$\forall x \in M, \quad \mathrm{tr}(e(x)) = \sum_{i=1}^{N} e_{ii}(x) = k \in \mathbb{N}^*.$$

In the following, we take $\mathcal{A}$ a unital algebra on $Cl(M)$ and denote by $\mathcal{A}(M,E)$ the corresponding subset of operators in $Cl(M,E)$.
Let $K : \Gamma(E) \to e(C^\infty(M)^N)$ be a $C^\infty(M)$–module isomorphism and let us denote also by e the 0th–order operator in $\mathcal{A}(M, M \times \mathbb{C}^N)$ that is multiplication by the matrix e. Consider the following maps:

$$\begin{aligned} \Phi : \mathcal{A}(M,E) &\to \mathcal{A}(M, M \times \mathbb{C}^N) \\ T &\mapsto eKTK^{-1}e; \end{aligned} \qquad \begin{aligned} \Psi : \mathcal{A}(M, M \times \mathbb{C}^N) &\to \mathcal{A}(M,E) \\ S &\mapsto K^{-1}eSeK. \end{aligned}$$

Observe that for all $T \in \mathcal{A}(M,E)$, $\Psi \circ \Phi(T) = T$ and for all $S \in \mathcal{A}(M, M \times \mathbb{C}^N)$, $\Phi \circ \Psi(S) = eSe$, which implies that we have an isomorphism:

$$\begin{aligned} \mathcal{A}(M,E) &\to e\,\mathcal{A}(M, M \times \mathbb{C}^N)\,e \\ K^{-1}eSeK &\mapsto eSe. \end{aligned}$$

We can also prove that

$$M_N(\mathcal{A}) = M_N(\mathcal{A})\, e\, M_N(\mathcal{A}),$$

where the left hand side denotes the subset of $M_N(\mathcal{A})$ consisting of finite sums: $\sum_i X_i e Y_i$, where X_i, Y_i are arbitrary elements in $M_N(\mathcal{A})$. In fact, it is sufficient to check that the identity matrix I_N belongs to $M_N(\mathcal{A})\, e\, M_N(\mathcal{A})$. As before, for all $i, j = 1, \ldots, N$, we denote by E_{ij} the elementary matrix in $M_N(\mathbb{C})$ with 1 in the (i, j)–position and 0 everywhere else. We have

$$\sum_{i,j=1}^{N} E_{ij}\, e\, E_{ji} = \sum_{i=1}^{N} e_{ii} \cdot I_N = k \cdot I_N.$$

From the isomorphism $\mathcal{A}(M, M \times \mathbb{C}^N) \cong M_N(\mathcal{A})$, we can conclude that the algebras $\mathcal{A}(M, E)$ and $\mathcal{A}(M, M \times \mathbb{C}^N)$ are Morita equivalent and therefore there are natural isomorphisms between their Hochschild homology groups (see Sect. 1.2 in [29]). Since the space of traces on $\mathcal{A}(M, E)$ is isomorphic to the dual of its zeroth Hochschild homology group, it implies that there is an isomorphism between the space of traces of the algebra $\mathcal{A}(M, E)$ with the space of traces of the algebra $\mathcal{A}(M, M \times \mathbb{C}^N)$.

Lemma 5.2.4. *Given a trace τ on $\mathcal{A}$, the map*

$$\begin{aligned} \tau_E : \mathcal{A}(M, E) &\to \mathbb{C} \\ T &\mapsto \tau_E(T) := (\tau \otimes \mathrm{tr}) \circ \phi(eKTK^{-1}e) \end{aligned}$$

is a trace on $\mathcal{A}(M, E)$.

Proof. If $T \in \mathcal{A}(M, E)$, the operator $KTK^{-1}e$ takes values in $e(C^\infty(M)^N)$ and since $e^2 = e$, $KTK^{-1}e = eKTK^{-1}e$. As already observed in Lemma 5.2.2, if τ is any trace on $\mathcal{A}$, the linear map $(\tau\otimes\mathrm{tr})\circ\phi$ is a trace on $M_N(\mathcal{A}) \cong \mathcal{A}(M, M\times\mathbb{C}^N)$. Therefore by Lemma 5.2.2, for any $T_1, T_2 \in \mathcal{A}(M, E)$

$$\begin{aligned} \tau_E(T_1T_2) &= (\tau \otimes \mathrm{tr}) \circ \phi(eKT_1T_2K^{-1}e) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi(eKT_1K^{-1}KT_2K^{-1}e) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi(eKT_1K^{-1}eKT_2K^{-1}e) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi((eKT_1K^{-1}e)(eKT_2K^{-1}e)) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi((eKT_2K^{-1}e)(eKT_1K^{-1}e)) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi(eKT_2K^{-1}eKT_1K^{-1}e) \\ &= (\tau \otimes \mathrm{tr}) \circ \phi(eKT_2T_1K^{-1}e) \\ &= \tau_E(T_2T_1). \end{aligned}$$

□

Thus, if tr_E denotes the trace on E, the following are traces on $\mathcal{A}(M, E)$ (as in Section 4.2):

1. The Wodzicki residue: For $A \in \{Cl(M,E),\ \bigcup_{a\in\mathbb{Z}} Cl^a(M,E)\}$,

$$\operatorname{Res}(A) = \frac{1}{(2\pi)^n}\int_M \int_{S_x^*M} \operatorname{tr}_{E_x}(\sigma_{-n}(A)(x,\xi))\overline{\mu}(\xi)\wedge dx.$$

2. The leading symbol trace: For $A \in Cl^0(M,E)$, $\lambda(\operatorname{tr}_E(\pi_0(A)))$.

Therefore when $\mathcal{A}$ is a unital algebra, the traces given in Remark 5.2.2 define in this way the only traces on $\mathcal{A}(M,E)$.

5.3 Classification of determinants on the group $(Id + Cl^a(M))^*$

In [28] the authors give a description of the determinants on the space of invertible operators $(Id+Cl^0(M))^*$, namely, every determinant on this space can be written in a unique way in terms of the residue determinant and a leading symbol determinant. In this section we consider the case $a<0$, and we use the classification of traces on the Fréchet-Lie algebra $Cl^a(M)$ given in Subsection 5.1.3 to describe the determinants in the Fréchet-Lie group $(Id+Cl^a(M))^*$.

Definition 5.3.1. Let $\mathcal{G}$ be a Fréchet-Lie group and $\widetilde{\mathcal{G}}$ its subgroup of elements pathwise connected to the identity 1. A *determinant map* or *multiplicative map* on $\mathcal{G}$ is a group morphism $\operatorname{Det} : \widetilde{\mathcal{G}} \to \mathbb{C}$, that is

$$\forall\, g,h \in \widetilde{\mathcal{G}},\ \operatorname{Det}(g\cdot h) = \operatorname{Det}(g)\operatorname{Det}(h).$$

Proposition 5.3.1 (Cor. 5.12 in [21], and [22])**.** *If $a<0$, the space of invertible operators*

$$\mathcal{G} := (Id + Cl^a(M))^* = (\{1 + A : A \in Cl^a(M)\})^*$$

is a Fréchet-Lie group with Fréchet-Lie algebra $Cl^a(M)$, which admits an exponential mapping from $Cl^a(M)$ to $(Id+Cl^a(M))^*$.

Explicitly this exponential mapping is given by:

$$\begin{aligned}\operatorname{Exp} : Cl^a(M) &\to \widetilde{\mathcal{G}}\\ A &\mapsto \operatorname{Exp}(A) = \sum_{k=0}^{\infty}\frac{1}{k!}A^k.\end{aligned}$$

This map restricts to a diffeomorphism from some neighborhood of the identity in $Cl^a(M)$ to a neighborhood of the identity in $\widetilde{\mathcal{G}}$. The inverse is given by

$$\begin{aligned}\operatorname{Log} : \widetilde{\mathcal{G}} &\to Cl^a(M)\\ 1 + A &\mapsto \operatorname{Log}(1+A) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}A^k.\end{aligned}$$

Remark 5.3.1. Prop. 3 in [28] shows that Proposition 5.3.1 also holds in the case $a = 0$.

Definition 5.3.2. If T is a continuous trace on $Cl^a(M)$ and exp denotes the exponential function on $\mathbb{C}$, we define a map

$$\begin{aligned} \mathrm{Det} : \mathrm{Exp}(Cl^a(M)) &\to \mathbb{C}^* \\ 1 + A &\mapsto \mathrm{Det}(1 + A) := \exp(T(\mathrm{Log}(1 + A))). \end{aligned}$$

Proposition 5.3.2. *The map* Det *is multiplicative.*

Proof. Consider two elements $g_1, g_2 \in \mathrm{Exp}(Cl^a(M))$. By the Campbell-Hausdorff formula on $\mathrm{Exp}(Cl^a(M))$ (see [34]), we have

$$\begin{aligned} \mathrm{Det}(g_1 \cdot g_2) &= \exp(T(\mathrm{Log}(g_1 \cdot g_2))) \\ &= \exp\left(T\left(\mathrm{Log}(g_1) + \mathrm{Log}(g_2) + \sum_{k=1}^{\infty} C^{(k)}(\mathrm{Log}(g_1), \mathrm{Log}(g_2))\right)\right) \\ &= \exp\left(T(\mathrm{Log}(g_1)) + T(\mathrm{Log}(g_2))\right) \\ &= \exp\left(T(\mathrm{Log}(g_1))\right) \exp\left(T(\mathrm{Log}(g_2))\right) \\ &= \mathrm{Det}(g_1)\,\mathrm{Det}(g_2). \end{aligned}$$

Here we have used that $T(C^{(k)}(\mathrm{Log}(g_1), \mathrm{Log}(g_2))) = 0$ for all k since T is a trace and $C^{(k)}(\mathrm{Log}(g_1), \mathrm{Log}(g_2))$ are commutators on $\mathrm{Log}(g_1), \mathrm{Log}(g_2) \in Cl^a(M)$. □

In Corollary 5.1.2, we gave an explicit description of the traces on the algebra $Cl^a(M)$, and in Section 4.2 (see also Remark 5.1.5), we noticed that those traces are indeed continuous for the Fréchet topology of $Cl^a(M)$. So together with Proposition 5.3.2, we have an explicit description of the determinant maps on the Fréchet-Lie group $(Id + Cl^a(M))^*$, namely:

Proposition 5.3.3. *Let $a \in \mathbb{Z}$ be such that $a \leq 0$. Determinant maps on $(Id + Cl^a(M))^*$ are given by a two parameter family: for $c_1, c_2 \in \mathbb{C}$, and for any linear map $\lambda : Cl^a(M)/Cl^{2a-1}(M) \to \mathbb{C}$,*

1. *If $-n + 1 \leq 2a \leq 0$,*

$$\mathrm{Det}_{c_1,c_2}(\cdot) = \exp\Big(c_1 \lambda \circ \pi_a(\mathrm{Log}(\cdot)) + c_2\,\mathrm{Res}(\mathrm{Log}(\cdot))\Big).$$

2. *If $a < -n$,*

$$\mathrm{Det}_{c_1,c_2}(\cdot) = \exp\Big(c_1 \lambda \circ \pi_a(\mathrm{Log}(\cdot)) + c_2\,\mathrm{Tr}_{L^2}(\mathrm{Log}(\cdot))\Big).$$

3. *If $2a - 1 < -n \leq a$, for a continuous linear extension α of Tr_{L^2} as in Corollary 5.1.2,*

$$\mathrm{Det}_{c_1,c_2}(\cdot) = \exp\Big(c_1 \lambda \circ \pi_a(\mathrm{Log}(\cdot)) + c_2 \alpha(\mathrm{Log}(\cdot))\Big).$$

These determinants differ from the ones sometimes used by physicists for operators of the type $1+$ *Schatten class operator* ([31], [40]) which in contrast to these are not multiplicative but do extend the ordinary determinant for determinant class operators.

Here are some relevant specific cases

- $\mathrm{Det}_{1,0}(\cdot) = \exp\Big(\lambda \circ \pi_a(\mathrm{Log}(\cdot))\Big)$ are related to the leading symbol determinants (see [36] for the case $a = 0$).
- If $a \in \mathbb{Z}$ and $2a \in [-n+1, 0]$, $\mathrm{Det}_{0,1}(\cdot) = \exp\Big(\mathrm{Res}(\mathrm{Log}(\cdot))\Big)$ is the Wodzicki multiplicative determinant (also called the exotic logarithmic determinant) (see [44]).
- If $a < -n$, $\mathrm{Det}_{0,1}(\cdot) = \exp\Big(\mathrm{Tr}_{L^2}(\mathrm{Log}(\cdot))\Big)$ is the Fredholm determinant (see Lemma 2.1 in [38]).
- If $2a \leq -n$, $\mathrm{Det}_{c_1,c_2}(\cdot) = \exp\Big(c_1\lambda \circ \pi_a(\mathrm{Log}(\cdot)) + c_2\alpha(\mathrm{Log}(\cdot))\Big)$ is an extension of the Fredholm determinant (see [38], [40]).

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