Motivic Fundamental Groups and Integral Points

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Contents

Introduction

Siegel's theorem on integral points of affine hyperbolic curves has been one of the corner stones in the theory of Diophantine equations. The original proof by Siegel uses the method of Diophantine approximations. After Faltings proved Mordell's conjecture in [12] by a deep study of semi-simple l-adic Galois representations associated to abelian varieties, a new proof was obtained as a byproduct. More recently, in [24], Kim introduced a new way of proving Siegel's theorem for punctured projective line, which uses instead the unipotent Galois representation associated to the étale realization of the unipotent fundamental group of the projective line minus three points, and comparing it with the de Rham realization. In that proof, as Kim mentions in the introduction of [24], there is no trace of motivic objects, at least in Voevodsky's sense, and it works for the moment, as far as the knowledge of author permits, only for $\mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$ and affine elliptic curves with Mordell-Weil rank at most one, again over $\mathbb{Q}$ (see [25]). Faltings in [17] generalizes ideas in Kim's proof to arbitrary curves over any number field and reduces Siegel's theorem to a difficult problem in estimating dimensions of some global Galois cohomology groups. The problem is that, by then these estimations could not be done neither in case of positive genus, nor for number fields bigger than $\mathbb{Q}$. Hence the only explicit case to which Faltings' generalizations was applied was again $\mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$ in view of difficulties for obtaining such estimates.

In this thesis we use motivic unipotent fundamental groupoids of unirational varieties, as are constructed in [10], in order to generalize these techniques in two direction. Firstly, concerning the case of punctured projective line, we enlarge the base number field. Namely, the first main result of this thesis (see Theorem 4.2.1) is a 'motivic' proof of the fact that for any totally real number field k of degree $d \geq 2$ (or $k = \mathbb{Q}$, but then put $d = 2$), and for any finite set S of finite places of k, if one removes at least $d + 1$ S-integral points from projective line, the resulting curve has only finitely many S-integral points. In particular if we consider the case of $d = 2$, and remove three points 0, 1, and ∞, we obtain Siegel's theorem for totally real quadratic number fields, and for the field $\mathbb{Q}$. To do that, we employ the categories of mixed Tate motives over the base number field and localizations of its ring of integers, to replace the above mentioned global Galois cohomology groups by the algebraic K-groups of the base number field. Then using Borel's explicit calculations, in the case of totally real number fields we obtain good enough estimates for dimensions of these K-groups and derive the promised finiteness result. I would like to mention that what is essential, and probably new in this work, is bringing motivic objects and algebraic K-theory in proving finiteness results for Diophantine equations.

The second direction deals with higher dimensions. Namely by considering the motivic fundamental groupoid of unirational varieties, comparing its different realizations, and using a motivic version of the Lefschetz hyperplane section theorem, we show that integral points of a unirational variety defined over a totally real number field with a highly enough non-abelian fundamental group cannot be dense in the p-adic analytic topology (see Theorem 4.4.3). Here again using motivic, in Voevodsky's sense, fundamental groupoid is very essential in bringing K-theory into the scene and getting the result.

In the first chapter we introduce the notions of motivic unipotent fundamental group and motivic path torsors over them. Finally we state some conditions under which these motivic objects exist. To do that we need two main tools, namely the Tannakian formalism and the abelian categories of mixed Tate motives over number fields and their ring of S-integers, for S a subset of finite places of the number field. In section 1.1 we recall the main notions and results of the Tannakian formalism which are relevant to this work. The main references for this section are [9] and [11]. In section 1.2 we go through Voevodsky's triangulated category of mixed motives and the abelian category of mixed Tate motives over those fields which satisfy the Beilinson-Soulé vanishing conjecture. Then we also recall the variant of the abelian category of mixed Tate motives over the ring of S-integers of number fields, introduced by Deligne and Goncharov. The main references for this section are [1], [4], [8], [10], [26], and [29]. Finally in section 1.3 we use the tools of the previous two sections and give a brief treatment covering the definitions of motivic unipotent fundamental group and path torsors for standard triples over number fields and their ring of S-integers. Here our main reference is [10].

Having studied the existence of motivic unipotent fundamental groupoids for some classes of varieties, we need to have a closer study of the different realizations of these motivic objects in order to extract enough information which is necessary for our future applications. We take that as the subject of the second chapter in which we try to have a closer look at different realizations of unipotent fundamental groups and path torsors of curves. In later chapters we will use the results of this study to obtain similar information in higher dimensional cases. In section 2.1, following Faltings [17], we study different realizations of unipotent fundamental groups and path torsors for any affine carve in a very general situation. Every thing in this section can be applied as well to the projective curves, but since we will be interested only in the affine case, we restrict ourselves to this case. Then in sections 2.2 and 2.3 we study two extra piece of decorations that one can put on the de Rham realization, namely Hodge filtration and Frobenius action. To explain the Frobenius action, we recall in section 2.3 the classical and the logarithmic versions of the crystalline sites and topoi. The main references for this section are [2] and [23]. These extra structures are critical in comparing the de Rham realization and the étale realization which is the subject of another section in chapter four.

In the third chapter we develop some technical tools and constructions which will be crucial in proving our main results. Namely in section 3.1 we fix a profinite group Γ and a complete Hausdorff topological field K and study the representability problem of the functor which assigns to a finitely generated algebra over K the set of finite dimensional Γ-representations which admit a filtration with prescribed subquotients. This will be used in showing the representability of some Galois cohomology groups with values in étale unipotent fundamental groups. Then in section 3.2 we will consider the case that the base field is $\mathbb{Q}_p$ and Γ is the absolute Galois group of a local p-adic field K. In this situation among all $\mathbb{Q}_p$-representations of Γ, there are the very interesting crystalline ones. We will study the locus of these crystalline representations among all of them. The main references for this section are [18] and [19]. Finally in section 3.3 we put all the technical results of the previous two sections together in order to define the period maps. These period maps can be put together and make very important commutative diagrams (see Remark 3.3.2) which make the connection between a variety and different realizations of its unipotent fundamental group. This connection will be the key tool in proving our main results in the last chapter.

In the last chapter we integrate all the tools of the previous chapters to prove our main results. But before that we need one more technical tool. Namely we have to show the commutativity of the diagram appeared in Remark 3.3.2. Note that if we forget about the comparison map, the rest of the diagram is commutative almost by construction, but the fact that the comparison map is compatible with the rest of the diagram is a deep result, sometimes called non-abelian comparison theory. In section 4.1 we review this non-abelian comparison theory and show the commutativity of our main diagram. Using this we are finally able to prove our first main result in section

4.2. Namely in this section we give a new proof of the fact that a sufficiently punctured projective line over a totally real number field has finitely many S-integral points (see Theorem 4.2.1). Note that this is a very special case of Siegel's theorem, but the innovation here is to use mixed Tate motives and K-theory to give a new proof of this well known fact in Diophantine Geometry. Then to launch for the second main result, which generalizes the first one to higher dimensional unirational varieties, we need a motivic version of the Lefschetz hyperplane section theorem for fundamental groups. This is the subject of section 4.3. After that we are ready to state and prove the second main result in section 4.4 (see Theorem 4.4.3). Roughly speaking, this says that if the fundamental group of a unirational variety defined over a totally real number field is highly non-abelian, then the set of S-integral points of that variety cannot be p-adically dense. Although we are only able to prove Theorem 4.4.3 in the stated restricted form, we expect its validity in much more generality (see Conjecture 4.4.4). We finish this thesis by stating some general remarks and questions concerning Theorem 4.4.3 and Conjecture 4.4.4 in section 4.5.

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Chapter 1

Motivic Fundamental Groupoids

The main theme of this thesis is to apply fundamental groups of varieties to study their integral points. Let us for a moment go down to the world of abelian invariants. Then of course the natural replacement for the fundamental group is its abelianization, namely the first (co)homology of the variety. One thing that makes (co)homology a powerful tool in studying varieties is that there are a couple of different (co)homology theories for varieties which admit different kinds of decorations on them, but yet are not completely independent from each other. Roughly speaking, various (co)homology theories of varieties furnished with these different additional structures can be compared to each other via the so called comparison isomorphisms. In the best of the worlds, as Grothendieck has conjectured, one would expect the existence of a motivic (co)homology theory which encompasses all these different (co)homology theories and the comparisons between them. Now let us go back to the fundamental groups. The situation is similar and one has different realizations of fundamental groups associated to varieties and comparison isomorphisms between them. One can ask again if there is a motivic fundamental group which gives all these different realizations. In this chapter we are going to address this question. We will see that for a special class of varieties there is an affirmative answer to this question and this will be one of the most important ingredients in obtaining our main results.

1.1 Tannakian Formalism

In this section we recall the basic notions and results of Tannakian formalism which will be used latter on. Our main references for this theory are [9] and

[11]. Recall that for a category $\mathcal{C}$ a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is called a tensor functor if it satisfies the so called associativity and commutativity constraints, which are compatible with each other. An associativity constraint for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\Phi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z,$$

such that for all objects X, Y, Z, T in $\mathcal{C}$ the diagram

$$\begin{tikzcd}
 & (X \otimes Y) \otimes (Z \otimes T) \arrow[dr] & \\
X \otimes (Y \otimes (Z \otimes T)) \arrow[ur] \arrow[d] & & ((X \otimes Y) \otimes Z) \otimes T \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[rr] & & (X \otimes (Y \otimes Z)) \otimes T \arrow[u]
\end{tikzcd}$$

commutes (the pentagon axiom). A commutativity constraint for $(\mathcal{C}, \otimes)$ is a functorial isomorphism

$$\Psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X,$$

such that for all objects X, Y in $\mathcal{C}$, $\Psi_{Y,X} \circ \Psi_{X,Y} = Id_{X \otimes Y}$. Finally an associativity constraint Φ and a commutativity constraint Ψ are compatible if, for all objects X, Y, Z in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[r] \arrow[dl] & (X \otimes Y) \otimes Z \arrow[dr] & \\
X \otimes (Z \otimes Y) \arrow[dr] & & & Z \otimes (X \otimes Y) \arrow[dl] \\
 & (X \otimes Z) \otimes Y \arrow[r] & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

commutes (the hexagon axiom). The category $\mathcal{C}$ together with the functor $\otimes$ is called a tensor category if it admits an identity object, i.e. a pair $(\underline{1}, e)$ consisting of an object $\underline{1}$ in the category $\mathcal{C}$ and an isomorphism $e : \underline{1} \xrightarrow{\sim} \underline{1} \otimes \underline{1}$ such that the functor $\underline{1} \otimes -$ gives an auto-equivalence of $\mathcal{C}$.

Let $(\mathcal{C}, \otimes)$ be a tensor category and fix two objects X and Y in $\mathcal{C}$. If the contravariant functor

$$T \mapsto Hom_{\mathcal{C}}(T \otimes X, Y)$$

from $\mathcal{C}$ to the category of sets is representable, the representing object is called the internal Hom object from X to Y and is denoted by $\mathcal{H}om(X, Y)$.

Having defined the notion of internal Hom, one can define the dual $X^\vee$ of an object X to be $\mathcal{H}om(X, \underline{1})$, if it exists. Note that by definition the identity map from $\mathcal{H}om(X, Y)$ to itself corresponds to a morphism

$$ev_{X,Y} : \mathcal{H}om(X, Y) \otimes X \to Y,$$

which is called the evaluation morphism. The evaluation map $ev_{X,\underline{1}}$ gives rise to a map $X \otimes X^\vee \to \underline{1}$, hence a map $i_X : X \to (X^\vee)^\vee$. One says that X is a reflexive object when i_X is an isomorphism. Following the same lines, for any finite families of objects $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ one can define a natural morphism from $\otimes_i \mathcal{H}om(X_i, Y_i)$ to $\mathcal{H}om(\otimes_i X_i, \otimes_i Y_i)$. Now we are ready for the following:

Definition 1.1.1. *(Rigid Category) A tensor category* $(\mathcal{C}, \otimes)$ *is rigid if*

- *The internal Hom object* $\mathcal{H}om(X, Y)$ *exists for any pair of objects* X *and* Y*.*
- *The above mentioned maps from* $\otimes_i \mathcal{H}om(X_i, Y_i)$ *to* $\mathcal{H}om(\otimes_i X_i, \otimes_i Y_i)$ *are isomorphisms for all finite families of objects.*
- *Finally, all objects of* $\mathcal{C}$ *are reflexive.*

Having defined the notion of a tensor category we now introduce the notion of functors between tensor categories.

Definition 1.1.2. *(Tensor Functor) Let* $(\mathcal{C}, \otimes)$ *and* $(\mathcal{C}', \otimes')$ *be tensor categories. A tensor functor from* $(\mathcal{C}, \otimes)$ *to* $(\mathcal{C}', \otimes')$ *is a pair* $(\mathcal{F}, c)$ *consisting of a functor* $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ *and a functorial isomorphism*

$$c_{X,Y} : \mathcal{F}(X) \otimes' \mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(X \otimes Y),$$

which commutes with associativity and commutativity constraints and sends the identity element of $(\mathcal{C}, \otimes)$ *to the identity element of* $(\mathcal{C}', \otimes')$*.*

Of course being restricted to tensor categories and tensor functors between them, one must also consider those equivalencies between such categories that respect tensor structures. But it can be checked that as soon as the underlying functor of a tensor functor is an equivalence of categories there exists a tensor inverse such that natural transformations between two possible compositions and the identity functors commute with tensor product as well. We leave it to the reader to make a precise statement of this fact using the evident definition of a morphism of tensor functors. The usual notation for the set of morphisms of tensor functors between two tensor functors $(\mathcal{F}, c)$ and $(\mathcal{G}, d)$ is $\mathcal{H}om^{\otimes}(\mathcal{F}, \mathcal{G})$.

The main feature of the notion of rigidity for tensor categories is that this notion inherits in some sense from tensor categories to the tensor functors between them and even the natural transformations between those tensor functors, and makes such things rigid as well. We mention some phenomena of this kind which are important and whose proofs can be found easily in the literature (see for example [9] or [11]).

Proposition 1.1.3. *Let $\mathcal{F}$ be any tensor functor between two rigid tensor categories $\mathcal{C}$ and $\mathcal{C}'$, then $\mathcal{F}$ commutes with taking internal Hom objects and in particular it commutes with taking dual. Moreover, any morphism of tensor functors between two rigid tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is automatically an isomorphism of functors.*

If one defines an abelian (resp. additive) tensor category to be a tensor category $(\mathcal{C}, \otimes)$ whose underlying category $\mathcal{C}$ is abelian (resp. additive) and tensor product is a bi-additive functor, then another consequence of the rigidity property of rigid tensor categories is that in any abelian rigid tensor category $(\mathcal{C}, \otimes)$, tensor product commutes with direct and inverse limits in each factor and hence in particular is exact in each factor.

Remark 1.1.4. Note that for an additive tensor category $(\mathcal{C}, \otimes)$,

$$R := \mathrm{End}_{\mathcal{C}}(\underline{1})$$

is a ring which acts, via $X \cong \underline{1} \otimes X$, on each object X. It can be shown that the action of R commutes with endomorphisms of X hence in particular R is commutative. Moreover $\mathcal{C}$ is R-linear and $\otimes$ is R-bilinear. ♠

Remark 1.1.5. Let k be a field and R be a k-algebra. Then the categories Vec_k of finite dimensional vector spaces over k and Mod_R of finitely generated modules over R are abelian tensor categories. There is a canonical tensor functor

$$\Phi_R : \mathrm{Vec}_k \to \mathrm{Mod}_R,$$

which sends a vector space V over k to the R-module $V \otimes_k R$. Now for any tensor category $\mathcal{C}$ and any two tensor functors $(\mathcal{F}, c)$ and $(\mathcal{G}, d)$ from $\mathcal{C}$ to Vec_k, we can use the functors Φ_R for all k-algebras in order to build a contravariant functor from k-algebras to sets out of $\mathcal{H}om^{\otimes}(\mathcal{F}, \mathcal{G})$. Namely we define

$$\mathcal{H}om^{\otimes}(\mathcal{F}, \mathcal{G})(R) := \mathcal{H}om^{\otimes}(\Phi_R \circ \mathcal{F}, \Phi_R \circ \mathcal{G}).$$

Finally note that for any tensor functor $\mathcal{F}$ from $\mathcal{C}$ to Vec_k the composition $\Phi_R \circ \mathcal{F}$ takes values in the subcategory Proj_R of finitely generated projective R-modules, which is an additive rigid subcategory of Mod_R. Such a functor will be called a fiber functor on $\mathcal{C}$ with values in the k-algebra R. ♡

Finally we can state the most important result of Tannakian formalism, namely

Theorem 1.1.6. *(Main Theorem of Tannakian Formalism) Let* $(\mathcal{C}, \otimes)$ *be a rigid abelian tensor category such that* $k := \mathrm{End}_{\mathcal{C}}(\underline{1})$ *is a field, and let* $w : \mathcal{C} \to \mathrm{Vec}_k$ *be an exact faithful* k*-linear tensor functor. Then*

1. *The functor* $\mathcal{A}ut^{\otimes}(w)$ *from* k*-algebras to sets (see Remark 1.1.5) is representable by an affine group scheme* G *over* k*.*

2. w *defines an equivalence of tensor categories* $\mathcal{C}$ *and* $\mathrm{Rep}_k(G)$*, where* $\mathrm{Rep}_k(G)$ *is the category of representations of the group scheme* G *over* k*.*

The above Theorem motivates

Definition 1.1.7. *(Neutral Tannakian Category) A neutral Tannakian category over a field* k *is a rigid abelian* k*-linear tensor category* $\mathcal{C}$ *for which there exists an exact faithful* k*-linear tensor functor* $w : \mathcal{C} \to \mathrm{Vec}_k$*. Any such functor* w *is said to be a fiber functor for* $\mathcal{C}$*.*

Theorem 1.1.6 says that for any neutral Tannakian category $\mathcal{C}$ over k equipped with one fiber functor w, the functor $\mathcal{A}ut^{\otimes}(w)$ over k-algebras is representable by an affine group scheme. The next natural question to ask is what if we take two fiber functors and look at natural transformations between them? Is this also representable? There is also a positive answer in this direction whose statement needs some more notations.

Let G be an affine group scheme over the field k and let $U = \mathrm{Spec}(R)$ be an affine k-scheme. A G-torsor over U (for the f.p.q.c. topology) is an affine k-scheme T, faithfully flat over U, together with a morphism $T \times_U G_U \to T$ such that

$$(t, g) \mapsto (t, tg) : T \times_U G_U \to T \times_U T$$

is an isomorphism.

Now let $\mathcal{C}$ be a neutral Tannakian category over a field k with a fixed fiber functor $w : \mathcal{C} \to \mathrm{Vec}_k$. For any fiber functor η with values in a k-algebra R, composition defines a pairing

$$\mathcal{H}om^{\otimes}(\Phi_R \circ w, \eta) \times \mathcal{A}ut^{\otimes}(\Phi_R \circ w) \to \mathcal{H}om^{\otimes}(\Phi_R \circ w, \eta),$$

of functors of R-algebras. By Proposition 1.1.3 we know that

$$\mathcal{H}om^{\otimes}(\Phi_R \circ w, \eta) = \mathcal{I}som^{\otimes}(\Phi_R \circ w, \eta).$$

Moreover one has the following important:

Theorem 1.1.8. *With all the above hypotheses and notations one has*

1. $\mathcal{H}om^{\otimes}(\Phi_R \circ w, \eta)$ *is representable by an affine scheme faithfully flat over* $\mathrm{Spec}(R)$*, which is a* G*-torsor where* $\mathrm{Rep}_k(G) \xrightarrow{\sim} \mathcal{C}$.
2. *The functor* $\eta \mapsto \mathcal{H}om^{\otimes}(\Phi_R \circ w, \eta)$ *gives an equivalence between the category of fiber functors on* $\mathcal{C}$ *with values in* R *and the category of* G*-torsors over* $\mathrm{Spec}(R)$.

Remark 1.1.9. The above line of ideas can be generalized one step further, namely one can replace the target category Vec_k by the category QCoh_S of quasi-coherent sheaves over a k-scheme S, but then one also has to replace the affine group scheme over k, by an affine groupoid over S (see [9]). ♣

1.2 Mixed Tate Motives

In this section we are going to consider the category of mixed motives. The first problem is that the existence of this category is not known yet! So we will talk about Voevodsky's construction which leads to a category $DM(k)$, for any field k of characteristic zero. $DM(k)$ is a candidate for being the derived category of the category of mixed motives over k (note that Voevodsky's construction can be applied to more general setting, namely most of the following constructions and properties are valid over any field and all of them over any perfect field. But since we finally apply these theories to number fields it is harmless to assume from the very beginning that our field has characteristic zero. This would allow us to use the resolution of singularities on which we rely heavily later on). By general facts of homological algebra, having a triangulated category $\mathcal{T}$ which is supposed to be the derived category of an unknown abelian category $\mathcal{A}$, what one needs in order to extract $\mathcal{A}$ from $\mathcal{T}$ is the so called t-structure. Let us briefly recall the notion of a t-structure (see [1] and [26] for more details).

Definition 1.2.1. *A* t*-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ *on a triangulated category* $\mathcal{T}$ *consists of strictly full subcategories* $\mathcal{T}^{\leq 0}$ *and* $\mathcal{T}^{\geq 0}$ *of* $\mathcal{T}$ *such that*

- $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ *and* $\mathcal{T}^{\geq 0}[-1] \subset \mathcal{T}^{\geq 0}$.
- *For* X *in* $\mathcal{T}^{\leq 0}$ *and* Y *in* $\mathcal{T}^{\geq 0}[-1]$*, we have* $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$.
- *For each object* X *of* $\mathcal{T}$*, there are objects* A *in* $\mathcal{T}^{\leq 0}$ *and* B *in* $\mathcal{T}^{\geq 0}[-1]$ *and a distinguished triangle*

$$A \to X \to B \to A[1]$$

Moreover one says that the t-structure is non-degenerate if

- *The intersections $\cap_n \mathcal{T}^{\geq 0}[-n]$ and $\cap_n \mathcal{T}^{\leq 0}[n]$ consist only of the zero object.*

The heart of a t-structure as above is the full subcategory $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

A good example to have in mind is when $\mathcal{T}$ is the derived category of an abelian category $\mathcal{A}$. In that case the subcategory $\mathcal{T}^{\leq 0}$ (resp. $\mathcal{T}^{\geq 0}$) consisting of complexes with trivial homology groups in positive (resp. negative) degrees, form a non-degenerate t-structure on $\mathcal{T}$ whose heart is equivalent to the original abelian category $\mathcal{A}$.

As it has been mentioned above, there is no known t-structure on $DM(k)$ whose heart gives us the category of mixed motives. The existence of such a t-structure depends on the validity of the so called Beilinson-Soulé vanishing conjecture, which is explained below. The important fact for us is that this Beilinson-Soulé vanishing conjecture restricted to a certain subcategory of $DM(k)$ is valid for some fields, like number fields, and hence there is a t-structure on that subcategory which leads to the abelian category of mixed Tate motives over those good fields (see [26, Theorem 4.2., and Corollary 4.3.]). Since our focus is on number fields and in order to avoid extra notations in different settings we start with a brief review of Voevodsky's construction over number fields (see [29]) and the variant over ring of integers of number fields and their localizations introduced by Deligne and Goncharov (see [10]).

Let k be a fixed number field from now on. In [29], Voevodsky has constructed the triangulated category $DM(k)$ of mixed motives over k as follows:

Let $SmCor(k)$ be the category whose objects are smooth separated schemes over k, and for any two objects X and Y, $\mathrm{Hom}(X, Y)$ is the free abelian group generated by closed reduced irreducible subsets Z of $X \times_k Y$, which are finite over X and dominate a connected component of X. We denote by $[X]$ the object in $SmCor(k)$ which corresponds to the smooth separated scheme X, and by $[f]$ the morphism between $[X]$ and $[Y]$ which corresponds to the graph of $f : X \to Y$. Obviously $SmCor(k)$ is an additive category with disjoint union as direct sum. Now the category $DM(k)$ can be obtained from the homotopy category $\mathfrak{H}^b(SmCor(k))$ of bounded complexes over $SmCor(k)$ after the following three steps:

- (Localizing with respect to 'Homotopy Invariance' and 'Mayer-Vietoris') Let T be the thick subcategory of $\mathfrak{H}^b(SmCor(k))$ generated by complexes

$$[X \times_k \mathbb{A}^1] \xrightarrow{[pr_1]} [X]$$

for all X, and all complexes of the form

$$[U \cap V] \to [U] \oplus [V] \to [X]$$

for any Zariski open covering $X = U \cup V$ of X (recall that a thick subcategory of a triangulated category is a triangulated subcategory which is closed under direct summands). Then we localize $\mathfrak{H}^b(SmCor(k))$ with respect to T.

- (Karoubianization) For this, we formally adjoin kernels and cokernels of idempotent endomorphisms. This means that we consider as objects, pairs of the form (X, p) where X is an object in the category resulted from the previous step, and p is an endomorphism of X satisfying $p^2 = p$. Then morphisms between (X, p) and (Y, q) are defined to be the set

$$q \circ \mathrm{Hom}(X, Y) \circ p,$$

where $\mathrm{Hom}(X, Y)$ is the set of homomorphisms from X to Y in the category that come from the previous step. The resulting category in this step will be denoted by $DM^{\mathrm{eff}}(k)$. For any separated smooth scheme X over k, the object corresponding to X in $DM^{\mathrm{eff}}(k)$ will be denoted by $M(X)$. This category can be endowed with a notion of a tensor product in such a way that (see [29, Proposition 2.1.3])

$$M(X \times_k Y) \cong M(X) \otimes M(Y).$$

- (Inverting Tate object) For any separated smooth scheme X over k, the structure morphism $X \to \mathrm{Spec}(k)$ induces a morphism from $M(X)$ to $M(\mathrm{Spec}(k))$ and hence a distinguished triangle

$$\widetilde{M(X)} \to M(X) \to M(\mathrm{Spec}(k)) \to \widetilde{M(X)}[1],$$

where $\widetilde{M(X)}$ is called the reduced motive of X. Then we define the Tate object

$$\mathbb{Z}(1) := \widetilde{M(\mathbb{P}^1)}[-2].$$

For any $n \in \mathbb{N}$, the n-fold tensor product of $\mathbb{Z}(1)$ with itself is denoted by $\mathbb{Z}(n)$, and for any object A in $DM^{\mathrm{eff}}(k)$, we define

$$A(n) := A \otimes \mathbb{Z}(n).$$

Finally the category $DM(k)$ is obtained from $DM^{\mathrm{eff}}(k)$ by inverting $\mathbb{Z}(1)$, namely its objects are pairs (A, n), with A is an object of $DM^{\mathrm{eff}}(k)$ and $n \in \mathbb{Z}$, and the morphisms are defined as follows

$$\mathrm{Hom}_{DM(k)}((A, n), (B, m)) := \varinjlim_{k \geq -n, -m} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(k+n), B(k+m)).$$

> The last point to mention is that the tensor product of $DM^{\text{eff}}(k)$ can be extended to a tensor product on $DM(k)$ (see [29, Corollary 2.1.5]). Moreover there are notions of internal Hom objects and dual objects in $DM(k)$, which all together make $DM(k)$ into a rigid tensor triangulated category (see [29, section 4.3]).

We denote by $DM(k)_{\mathbb{Q}}$ the category obtained from tensorizing $DM(k)$ with $\mathbb{Q}$, and in $DM(k)_{\mathbb{Q}}$ for any $n \in \mathbb{Z}$ we denote by $\mathbb{Q}(n)$ the object which corresponds to $\mathbb{Z}(n)$. $\mathbb{Q}(n)$'s are called Tate objects. The triangulated subcategory $DMT(k)_{\mathbb{Q}}$ of $DM(k)_{\mathbb{Q}}$ generated by these Tate objects is called the triangulated category of mixed Tate motives. One can then extract from $DMT(k)_{\mathbb{Q}}$ the abelian rigid tensor category $MT(k)$ of mixed Tate motives over k as follows.

Writing $\mathrm{Hom}^j(M,N)$ for $\mathrm{Hom}(M,N[j])$, the Beilinson-Soulé vanishing conjecture states that $\mathrm{Hom}^j(\mathbb{Q}(0),\mathbb{Q}(i)) = 0$ whenever $i > 0$ and $j \leq 0$. On the other hand, the groups $\mathrm{Hom}^j(\mathbb{Q}(a),\mathbb{Q}(b))$, for $j,a,b \in \mathbb{Z}$, form a decomposition of rational K-groups of k. Since k is a number field, thanks to Borel's explicit computation of ranks of rational K-groups of number fields (see [4, section 12]), one can show the validity of Beilinson-Soulé vanishing conjecture over k. Moreover one can show the following

$$\mathrm{Hom}^1(\mathbb{Q}(0),\mathbb{Q}(n)) = K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Using validity of Beilinson-Soulé vanishing conjecture for number fields, it is possible to put a non-degenerate t-structure on the triangulated category $DMT(k)_{\mathbb{Q}}$. The heart of this t-structure is the category $MT(k)$ consisting of iterated extensions of objects of the form $\mathbb{Q}(n)$, for $n \in \mathbb{Z}$, which is an abelian rigid tensor category. For any two objects A and B in $MT(k)$, one has a bijection

$$\mathrm{Ext}^1_{MT(k)}(A,B) \xrightarrow{\sim} \mathrm{Hom}^1_{DMT(k)_{\mathbb{Q}}}(A,B).$$

Using the above bijection and the relations between $\mathrm{Hom}^1(\mathbb{Q}(0),\mathbb{Q}(n))$ and rational K-groups of k, one obtains the following important vanishing of Ext^1-groups in $MT(k)$

$$\mathrm{Ext}^1(\mathbb{Q}(n),\mathbb{Q}(m)) = 0, \text{ for all } m \leq n.$$

For any object M in $MT(k)$, using the above mentioned vanishing of Ext^1-groups, one can put a unique finite increasing filtration $W_\bullet$ on M, called filtration by weights, which is indexed by even integers and satisfies the following property:

For any integer $n \in \mathbb{Z}$

$$Gr^W_{-2n}(M) = W_{-2n}(M)/W_{-2(n+1)}(M)$$

is a finite direct sum of copies of $\mathbb{Q}(n)$. Furthermore the filtration $W_\bullet$ is functorial, exact, and compatible with tensor product, and moreover every morphism in $MT(k)$ is strictly compatible with $W_\bullet$. For any $n \in \mathbb{Z}$, define

$$w_n(M) := \mathrm{Hom}(\mathbb{Q}(n), Gr^W_{-2n}(M)),$$

and put all these together to obtain a fiber functor w which sends M to $\bigoplus_n w_n(M)$. This makes $MT(k)$ into a neutral Tannakian category over $\mathbb{Q}$. Whence one can apply the general Theorem 1.1.6 to obtain an affine group scheme G_w, whose category of finite dimensional representations over $\mathbb{Q}$ is equivalent to $MT(k)$. The action of G_w on $w(\mathbb{Q}(1)) = \mathbb{Q}$, and the action of $\mathbb{G}_m$ on the tensor functor w, which is multiplication by λ^n on w_n, give a map

$$\phi : G_w \to \mathbb{G}_m,$$

and a section

$$\tau : \mathbb{G}_m \to G_w$$

for ϕ. Hence if we denote the kernel of ϕ by U_w, we obtain a semi-direct product decomposition

$$G_w \cong \mathbb{G}_m \ltimes U_w.$$

Since the action of U_w respects the weight filtration, and by definition is trivial on $Gr^W_\bullet$, one concludes that U_w is a pro-unipotent affine algebraic group scheme. Now the functor w admits a co-action by

$$\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \cong k^* \otimes_{\mathbb{Z}} \mathbb{Q},$$

and for any subvector space Γ of $k^* \otimes_{\mathbb{Z}} \mathbb{Q}$, one can define the neutral Tannakian category $MT(k)_\Gamma$ to be the full subcategory of $MT(k)$ which contains those objects for which this co-action factors through Γ (see [10, 1.2 and 1.4]). In particular let S be a set of finite places in k, $\mathcal{O}_S$ be the ring of S-integers of k, and put $\Gamma = \mathcal{O}_S^* \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we denote the resulting category $MT(k)_\Gamma$ by $MT(\mathcal{O}_S)$. The analogous definitions of $G_{w,\Gamma}$, $U_{w,\Gamma}$, and the isomorphism $G_{w,\Gamma} \cong \mathbb{G}_m \ltimes U_{w,\Gamma}$ are plain. Now we have

Proposition 1.2.2. [10, Proposition 1.9] *Let Γ be a subvector space of* $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$.

(i) For $r \geq 2$, $\mathrm{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+r))$ is the same in $MT(k)$ and in $MT(k)_\Gamma$.

(ii) The Yoneda Ext^2's are zero in $MT(k)_\Gamma$.

In order to study different realizations of mixed Tate motives over k we need to define the category $\mathcal{R}_k$, or simply $\mathcal{R}$ when k is fixed, of mixed realizations. Any object of $\mathcal{R}$ is given by the following data:

1. A graded vector space V_{dR} over k. The increasing filtration
$$W_{-2n} = \bigoplus_{m \geq n} (V_{dR})_m$$
is called the weight filtration, and the decreasing filtration
$$F^{-n} = \bigoplus_{m \leq n} (V_{dR})_m$$
is called the Hodge filtration.

2. For any embedding σ of k into an algebraic closure C of $\mathbb{R}$, a $\mathbb{Q}$-vector space V_σ equipped with an increasing filtration by weights $W_\bullet$, indexed by even integers. One assumes that V_σ together with its weight filtration is functorial in C.

3. For any σ as above, a comparison isomorphism
$$comp_{\sigma,dR} : V_{dR} \otimes_{k,\sigma} C \xrightarrow{\sim} V_\sigma \otimes_{\mathbb{Q}} C,$$
which respects the weight filtrations and is functorial in C. Its inverse is denoted by $comp_{dR,\sigma}$.

4. For any prime number l and any algebraic closure $\bar{k}$ of k, a $\mathbb{Q}_l$-vector space V_l, equipped with an increasing filtration by weights $W_\bullet$, indexed by even integers. One assumes that V_l together with its weight filtration is functorial in $\bar{k}$. In particular V_l admits a continuous action by $\mathrm{Gal}(\bar{k}/k)$.

5. For any $\sigma : k \hookrightarrow C$ as in (2), and the algebraic closure $\bar{k}$ of k in C, a comparison isomorphism
$$comp_{l,\sigma} : V_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_l \xrightarrow{\sim} V_l$$
which is functorial in C. The inverse is denoted by $comp_{\sigma,l}$.

6. One assumes the existence of a lattice $V_{\sigma,\mathbb{Z}} \subset V_\sigma$, such that
$$comp_{l,\sigma}(V_{\sigma,\mathbb{Z}}) \subset V_l$$
is stable under the Galois action, and is independent of σ. One assumes moreover that Gr^W_{-2n} of (1), (2), and (4) are isomorphic to a direct sum of the corresponding realization of the Tate object $\mathbb{Q}(n)$.

Note that there is an obvious notion of tensor product on $\mathcal{R}$ which makes it into a Tannakian category. Then one can show that for any sub-vector space Γ of $k^* \otimes_{\mathbb{Z}} \mathbb{Q}$, different realizations of Tate objects, with comparison isomorphisms between them, can be extended to a realization functor

$$\mathfrak{real}_\Gamma : MT(k)_\Gamma \to \mathcal{R}$$

and the following holds.

Theorem 1.2.3. *[10, 2.14 and 2.15] The above mentioned realization functor is fully faithful, and the image is essentially stable by sub-objects, i.e. if an object is in the image of* $\mathfrak{real}_\Gamma$*, so are all its sub-objects.*

1.3 Motivic Fundamental Groupoids

Continuing with the conventions of the previous section, assume that k is a fixed number field. In this section our aim is to recall the notions of motivic unipotent fundamental group and motivic path torsors over it for a unirational variety over k which is introduced by Deligne and Goncharov in [10]. We fix the following notations for this section. Let X be a smooth variety over k and let $\overline{X}$ be a smooth projectivization of X such that $D := \overline{X} - X$ is a divisor with normal crossing and smooth irreducible components (Note that since k is a field of characteristic zero, one can apply resolution of singularities and assume these without loss of generality). By studying unipotent vector bundles over $\overline{X}$ (with unipotent integrable logarithmic connection along D), and also by studying unipotent smooth l-adic étale sheaves over X one can construct the de Rham and the étale realizations of the unipotent fundamental group of X. The point is that the categories of these unipotent vector bundles over $\overline{X}$ (with unipotent integrable logarithmic connections along D) and unipotent smooth l-adic étale sheaves over X form rigid tensor categories and any rational point $x \in X(k)$ leads to a fiber functor $\mathcal{F}_x$. Then by applying Theorem 1.1.6 one gets pro-unipotent affine group schemes which are the desired realizations of the unipotent fundamental group of X. These realizations of the unipotent fundamental group together with the Malčev completion of the topological fundamental group of the complex variety associated to X, viewed as the Betti realization, are related to each other via some comparison isomorphisms and putting all these together one gets an affine pro-algebraic group scheme or equivalently a co-commutative Hopf algebra over the category $\mathcal{R}_k$ of mixed realizations over k (see [8] or the next chapter for more details). Let us denote this affine group scheme by $\pi_1^{\mathcal{R}}(X, x)$. On the other hand, one can fix another rational point $y \in X(k)$ and consider the tensor isomorphisms $\mathcal{I}\mathrm{som}^{\otimes}(\mathcal{F}_{\mathrm{y}}, \mathcal{F}_{\mathrm{x}})$ as a functor of k-algebras. Now Theorem 1.1.8 can be applied to give us different realizations of an affine scheme which admit again comparison isomorphisms and hence give rise to an affine scheme over $\mathcal{R}_k$. As one expects from Theorem 1.1.8 this affine scheme is a bi-torsor over affine group schemes $\pi_1^{\mathcal{R}}(X, x)$ and $\pi_1^{\mathcal{R}}(X, y)$, which is called the path torsor from y to x and will be denoted by $\pi_1^{\mathcal{R}}(X; x, y)$. There are obvious concatenation maps

$$\pi_1^{\mathcal{R}}(X; y, z) \times \pi_1^{\mathcal{R}}(X; x, y) \to \pi_1^{\mathcal{R}}(X; x, z)$$

which come from the composition of isomorphisms between corresponding fiber functors. We also mention, although we do not use it in this thesis, that Deligne has extended all these notions and constructions to the case of points at infinity, namely when one (or both) of the base point(s) is (are) represented by a suitable non-zero tangent vector(s) at some point(s) in the support of the divisor D (see [8, Section 15]).

On the other hand, we have seen in the previous section that there are realization functors $\mathfrak{real}_\Gamma$ from $MT(k)_\Gamma$ to $\mathcal{R}_k$ for any subvector space Γ of $k^* \otimes_{\mathbb{Z}} \mathbb{Q}$. So one interesting question to ask is that if the above constructed group schemes and path torsors over them lie in the image of $\mathfrak{real}_\Gamma$ or not. Generally let us say an object or a pro-object in $\mathcal{R}_k$ is motivic if it lies in the image of $\mathfrak{real}$. Then very important for us is

Theorem 1.3.1. *([10, Proposition 4.15]) With above notations and assumptions, assume moreover that X is unirational and the irreducible components of D are absolutely irreducible. Then for any rational point $x \in X(k)$ (resp. any two rational points $x, y \in X(k)$) the pro-unipotent affine group scheme $\pi_1^{\mathcal{R}}(X, x)$ (resp. the path torsor $\pi_1^{\mathcal{R}}(X; x, y)$) is motivic.*

Let us fix some notations concerning the above theorem for future references. With the above hypothesis for any rational point $x \in X(k)$ (resp. any two rational points $x, y \in X(k)$) the above theorem guaranties the existence of the motivic fundamental group (resp. motivic path torsor) which we denote by $\pi_1^{\mathrm{mot}}(X, x)$ (resp. $\pi_1^{\mathrm{mot}}(X; x, y)$). Note that $\pi_1^{\mathrm{mot}}(X, x)$ (resp. $\pi_1^{\mathrm{mot}}(X; x, y)$) is a pro-unipotent affine group scheme (resp. a bi-torsor over $\pi_1^{\mathrm{mot}}(X, x)$ and $\pi_1^{\mathrm{mot}}(X, y)$) over the rigid tensor category $MT(k)$ which is unique up to isomorphism by Theorem 1.2.3 and is sent to $\pi_1^{\mathcal{R}}(X, x)$ (resp. $\pi_1^{\mathcal{R}}(X; x, y)$) by the realization functor $\mathfrak{real}$.

Since we are interested in integral points we are also interested in an integral version of the above results, which fortunately exists. Being concerned with integral points we set up the situation over suitable rings of integers. Let S be a fixed finite set of finite places of the number field k and denote by

$\mathcal{O}_S$ the ring of S-integers in k. Suppose that $\overline{X}_{\mathcal{O}_S}$ is a proper smooth variety over $\mathrm{Spec}(\mathcal{O}_S)$ and $X_{\mathcal{O}_S}$ is the complement of a divisor $D_{\mathcal{O}_S}$ with relative normal crossing and smooth surjective irreducible components over $\mathrm{Spec}(\mathcal{O}_S)$. Then we denote by $\overline{X}_k$ (resp. X_k, resp. D_k) the generic fiber of $\overline{X}_{\mathcal{O}_S}$ (resp. $X_{\mathcal{O}_S}$, resp. $D_{\mathcal{O}_S}$). Note that these generic fibers satisfy the conditions that we fixed in the beginning of this section. Then the promised integral version of Theorem 1.3.1 is

Theorem 1.3.2. *[10, Proposition 4.17] In addition to the above hypotheses assume that the variety X_k is unirational and also that the irreducible components of D_k are absolutely irreducible. Now if the two rational points $x, y \in X_k(k)$ are generic fibers of two integral points $x_{\mathcal{O}_S}, y_{\mathcal{O}_S} \in X_{\mathcal{O}_S}(\mathcal{O}_S)$ then $\pi_1^{\mathrm{mot}}(X, x)$ and $\pi_1^{\mathrm{mot}}(X; x, y)$ belong to the subcategory $MT(\mathcal{O}_S)$ of $MT(k)$.*

The above theorem motivates the following definition of a standard triple which will be used frequently later on. In this definition we keep the above notations unchanged.

Definition 1.3.3. *A standard triple $(\overline{X}, X, D)$ over the ring $\mathcal{O}_S$ of S-integers in a number field k consists of the following data:*

- *A proper smooth variety $\overline{X}$ over* $\mathrm{Spec}(\mathcal{O}_S)$.
- *A relative normal crossing divisor D whose irreducible components are smooth and surjective over* $\mathrm{Spec}(\mathcal{O}_S)$*, and the irreducible components of its generic fiber D_k are absolutely irreducible.*
- *The complement X of D in $\overline{X}$ which is also smooth over* $\mathrm{Spec}(\mathcal{O}_S)$ *and whose generic fiber X_k is unirational.*

Chapter 2

Realizations for Curves

In previous chapter we noted that under some conditions the motivic unipotent fundamental groups and path torsors over them exist. This will be very useful for us, but yet very important is to consider the image of this motivic objects under the realization functor $\mathfrak{real}$, namely to study different realizations of them and the relations between these realizations as much as we can. As long as we only consider different realizations of the unipotent fundamental groups and path torsors over them as objects in the category of mixed realizations, we actually can get rid of many of the restrictions that we had to put to get the motivic ones. For example in this chapter we will see that in a very general context one has different realizations of unipotent fundamental groups and path torsors of (affine) curves. We start by giving a very concrete construction of the de Rham and étale realizations of the unipotent fundamental groups of affine curves and also of the path torsors over them. Very crucial feature of these realizations is that the de Rham realization admits Hodge filtration and Frobenius action, and the étale realization admits Galois action. After giving the explicit constructions, we will discuss the Hodge filtrations and the Frobenius action on the de Rham realization. This additional structures will be used to compare the two realizations which will be very important in our applications.

2.1 Geometric Construction

Let R be a commutative ring with unit, $C \xrightarrow{\pi} \mathrm{Spec}(R)$ a smooth projective curve with geometrically connected fibers, and $X := C - D$ where $D \subset C$ is a divisor which is étale and surjective over $\mathrm{Spec}(R)$. We denote the projection from X to $\mathrm{Spec}(R)$, which is the restriction of $\pi : C \to \mathrm{Spec}(R)$ to X, by the same symbol π. Now consider the following categories:

- Let $\mathcal{C}_{coh}$ be the category of vector bundles $\mathcal{E}$ on C which are iterated extensions of trivial vector bundles. Recall that a vector bundle is called trivial if it has the form $P \otimes_R \mathcal{O}_C$ for a finitely generated projective R-module P. So objects of our category $\mathcal{C}_{coh}$ are vector bundles $\mathcal{E}$ on C which admit a filtration by sub-vector bundles

$$(0) = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}$$

with $\mathcal{E}_i/\mathcal{E}_{i-1}$ being trivial for $1 \leq i \leq n$. Furthermore we denote by T_{coh} the dual of the first cohomology $H^1(C, \mathcal{O}_C)$.

- Let $\mathcal{C}_{dR}$ be the category of vector bundles $\mathcal{E}$ on C, together with a logarithmic connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D),$$

such that $(\mathcal{E}, \nabla)$ is an iterated extension of trivial vector bundles with connection. In this case a vector bundle with connection is called trivial if it is of the form $(P \otimes_R \mathcal{O}_C, Id_P \otimes d)$, where P is again a finitely generated projective R-module, and

$$d : \mathcal{O}_C \to \Omega_{C/\mathrm{Spec}(R)}(D)$$

is the canonical differential. In analogy with the above case, we denote by T_{dR} the dual of the first de Rham cohomology

$$H^1_{dR}(C, \mathcal{O}_C) := \mathbb{H}^1(C, \mathcal{O}_C \xrightarrow{d} \Omega_{C/\mathrm{Spec}(R)}(D)).$$

- For any invertible prime number l in R, one can consider the category $\mathcal{C}_{\text{ét}}$ of smooth l-adic étale sheaves $\mathcal{S}$ on X which are, as in above cases, iterated extensions of trivial ones. A smooth l-adic étale sheaf of the form $\pi^*(\mathcal{G})$ over X, where $\mathcal{G}$ is a smooth l-adic étale sheaf over $\mathrm{Spec}(R)$, is called trivial in this case. We use the notation $T_{\text{ét}}$ for the dual of the first étale cohomology $H^1_{\text{ét}}(X, \mathbb{Q}_l)$.

In the sequel, when we want to refer to any of the above categories, no matter which one, we use the subscript " $-_M$ ". For example $\mathcal{C}_M$ can be replaced by any of $\mathcal{C}_{coh}$, $\mathcal{C}_{dR}$, or $\mathcal{C}_{\text{ét}}$. We follow the same convention for T_M, and so on. We call an object $\mathcal{E}$ in $\mathcal{C}_M$, unipotent of class n, if it admits a filtration of length n by sub-objects with trivial sub-quotients, and moreover we denote by $\mathcal{C}_{M,n}$ the full subcategory of $\mathcal{C}_M$ generated by unipotent objects of class n. Note that by definition any object in $\mathcal{C}_M$ is unipotent of some class, and hence $\mathcal{C}_M = \bigcup_n \mathcal{C}_{M,n}$.

Now fixing a point $x \in X(R)$ gives a functor $\mathcal{F}_M : \mathcal{E} \mapsto \mathcal{E}[x]$ on the category $\mathcal{C}_M$, which in the coherent and de Rham cases takes values in the category of finitely generated projective R-modules, and in the étale case takes values in the category of finite dimensional $\mathbb{Q}_l$-vector spaces. This functor depends on the choice of x, but we omit this dependence in our notation. Following Faltings [17], we are going to show that $\mathcal{F}_M$ is pro-representable in the following sense:

Theorem 2.1.1. *There exists a pro-object $\mathcal{P}_M$ in the category $\mathcal{C}_M$, and an element $p \in \mathcal{P}_M[x]$ such that for any object $\mathcal{E}$ in $\mathcal{C}_M$ and any element $e \in \mathcal{E}[x]$, there exists a unique morphism $\varphi_e : \mathcal{P}_M \to \mathcal{E}$ such that $\varphi_{e,x}(p) = e$. Moreover the pair $(\mathcal{P}_M, p)$ is unique up to a unique isomorphism.*

Remark 2.1.2. Note that when the base ring R is a field, which is always the case in our applications, the category $\mathcal{C}_M$ admits a natural tensor product, any object has a natural dual, and with these notions $\mathcal{C}_M$ is an abelian rigid tensor category. Moreover $\mathcal{F}_M$ is a fiber functor with respect to this natural tensor product, and makes $\mathcal{C}_M$ into a neutral Tannakian category. Then by the general Tannakian formalism (see section 1.1 for details) one knows that tensor automorphisms $\mathcal{A}ut^{\otimes}(\mathcal{F}_M)$, as a functor from algebras over the base field to sets, is representable by an affine group scheme (see Theorem 1.1.6), which will be called G_M. But in what follows, we give an explicit construction of these representing group schemes, and as a byproduct we obtain finer information about them. $\diamondsuit$

To prove Theorem 2.1.1, we need some preliminaries. First we are going to construct the analogue objects in the categories $\mathcal{C}_{M,n}$ for each $n \geq 1$. Note that if we put $\mathcal{P}_{M,1}$ to be $\mathcal{O}_C$ (resp. $(\mathcal{O}_C, d)$, resp. the constant sheaf $\mathbb{Q}_l$) in the coherent (resp. de Rham, resp. étale) case, and take 1 as the distinguished element in $\mathcal{P}_{M,1}[x]$, then it obviously has the required universal property of Theorem 2.1.1 in the subcategory $\mathcal{C}_{M,1}$. Moreover we know that the appropriate zeroth cohomology of $\mathcal{P}_{M,1}$ is R (resp. R, resp. $\mathbb{Q}_l$) in the coherent (resp. de Rham, resp. étale) case, since C is projective and geometrically connected, and the appropriate first cohomology, by our convention, is $T_M^{\otimes -1} = T_M^{\vee}$. Lemma 2.1.3 and Proposition 2.1.6 are stated in the coherent case, but the analogous statements, with trivial modifications, are valid in the de Rham and étale cases, with the same proofs. The important fact to notice is that the second cohomology groups vanish in the de Rham and étale cases, as well as in the coherent case, since D is étale and surjective over Spec(R). Later when studying Hodge filtrations, we will go into more detail in the de Rham case (see section 2.2), and leave the similar étale case to the reader.

Lemma 2.1.3. *For every $n > 1$, there exists an extension $\mathcal{P}_{coh,n}$ of $\mathcal{P}_{coh,n-1}$ by $\pi^*(T_{coh}^{\otimes(n-1)})$ such that $H^0(C, \mathcal{P}_{coh,n}^\vee) = R$ via projection to $\mathcal{P}_{M,1}$,*

$$H^1(C, \mathcal{P}_{coh,n}^\vee) \cong H^1(C, T_{coh}^{\otimes-(n-1)} \otimes_R \mathcal{O}_C) \cong T_{coh}^{\otimes-n},$$

via the inclusion, and there exists an element $p_n \in \mathcal{P}_{coh,n}[x]$ which projects to $1 \in \mathcal{P}_{coh,1}[x]$.

Proof. The proof, as the statement itself, has recursive structure with respect to n. So assume that $\mathcal{P}_{coh,n-1}$ has been constructed with desired properties. Now since $\mathcal{P}_{coh,n}$ is going to be an extension of $\mathcal{P}_{coh,n-1}$ by $T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C$, we consider the corresponding Ext group. We have

$$\mathrm{Ext}^1(\mathcal{P}_{coh,n-1}, T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C) \cong T_{coh}^{\otimes(n-1)} \otimes_R \mathrm{Ext}^1(\mathcal{P}_{coh,n-1}, \mathcal{O}_C),$$

since $T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C$ is a trivial vector bundle. Furthermore

$$T_{coh}^{\otimes(n-1)} \otimes_R \mathrm{Ext}^1(\mathcal{P}_{coh,n-1}, \mathcal{O}_C) \cong T_{coh}^{\otimes(n-1)} \otimes_R H^1(C, \mathcal{P}_{coh,n-1}^\vee),$$

since $\mathcal{P}_{coh,n-1}$ is locally free. Now by applying the induction hypothesis for $\mathcal{P}_{coh,n-1}$, we can continue the above chain of isomorphisms as follows:

$$T_{coh}^{\otimes(n-1)} \otimes_R H^1(C, \mathcal{P}_{coh,n-1}^\vee) \cong T_{coh}^{\otimes(n-1)} \otimes_R T_{coh}^{\otimes-(n-1)},$$

but the right hand side is isomorphic to $\mathrm{Hom}_R(T_{coh}^{\otimes-(n-1)}, T_{coh}^{\otimes-(n-1)})$, and hence has an identity element. Take $\mathcal{P}_{coh,n}$ to be the extension which corresponds to the identity element in the above Ext group. This gives us an exact sequence of vector bundles over C

$$0 \to T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C \to \mathcal{P}_{coh,n} \to \mathcal{P}_{coh,n-1} \to 0.$$

We write the cohomology long exact sequence for the dual of the above exact sequence, use the vanishing of H^2, and apply the induction hypothesis for $\mathcal{P}_{coh,n-1}$, to obtain the following six term exact sequence

$$0 \to R \xrightarrow{\varphi} H^0(C, \mathcal{P}_{coh,n}^\vee) \to T_{coh}^{\otimes-(n-1)} \xrightarrow{Id} T_{coh}^{\otimes-(n-1)} \xrightarrow{f} \dots$$

$$\dots \xrightarrow{f} H^1(C, \mathcal{P}_{coh,n}^\vee) \xrightarrow{\psi} H^1(C, \widetilde{T_{coh}}^{\otimes-(n-1)}) \to 0,$$

where $\widetilde{T_{coh}}$ stands for the trivial bundle $T_{coh} \otimes_R \mathcal{O}_C$ induced by T_{coh}. Because of the identity map in the middle, φ gives the desired isomorphism between $H^0(C, \mathcal{P}_{coh,n}^\vee)$ and R, and ψ also gives

$$H^1(C, \mathcal{P}_{coh,n}^\vee) \cong H^1(C, T_{coh}^{\otimes-(n-1)} \otimes_R \mathcal{O}_C) \cong T_{coh}^{\otimes-(n-1)} \otimes_R H^1(C, \mathcal{O}_C) \cong T_{coh}^{\otimes-n}.$$

Finally take p_n to be any lift of p_{n-1} under the surjection

$$\mathcal{P}_{coh,n}[x] \twoheadrightarrow \mathcal{P}_{coh,n-1}[x]$$

to complete the proof. □

Remark 2.1.4. Note that in the six term exact sequence which appeared in the proof of the above lemma, the map

$$H^1(C, \mathcal{P}^{\vee}_{coh,n-1}) \xrightarrow{f} H^1(C, \mathcal{P}^{\vee}_{coh,n}),$$

is the zero map since it comes after the connecting homomorphism which was the identity map. This means that the pull back of every extension of $\mathcal{P}_{coh,n-1}$ by a trivial vector bundle, via the projection

$$\mathcal{P}_{coh,n} \twoheadrightarrow \mathcal{P}_{coh,n-1},$$

is a split extension. This crucial property will be used in the proof of the next proposition which is almost the proof of Theorem 2.1.1. Moreover note that since $\mathcal{P}_{coh,1}$ is an element of $\mathcal{C}_{coh,1}$ and each $\mathcal{P}_{coh,n}$ is an extension of $\mathcal{P}_{coh,n-1}$ by a trivial bundle, $\mathcal{P}_{coh,n}$ is an element of $\mathcal{C}_{coh,n}$. ♠

Remark 2.1.5. Another important remark about the above construction is that we have constructed the extension

$$0 \to T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C \to \mathcal{P}_{coh,n} \to \mathcal{P}_{coh,n-1} \to 0.$$

But what can one say about the automorphism group $\mathrm{Aut}(\mathcal{P}_{coh,n})$ of this extension? The answer is given by the following computation

$$\mathrm{Aut}(\mathcal{P}_{coh,n}) \cong H^0(C, \mathcal{H}om(\mathcal{P}_{coh,n-1}, T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C)).$$

Factoring out the constant sheaf $T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C$, the above automorphism group becomes isomorphic to

$$T_{coh}^{\otimes(n-1)} \otimes_R H^0(C, \mathcal{P}^{\vee}_{coh,n-1}) \cong T_{coh}^{\otimes(n-1)}.$$

Note in particular that the set of possible lifts p_n of p_{n-1} is a principal homogeneous space over $\mathrm{Aut}(\mathcal{P}_{coh,n})$. ♡

Now we show that $\mathcal{P}_{coh,n}$ has the required universal property of the Theorem 2.1.1 in the subcategory $\mathcal{C}_{coh,n}$. More precisely we have

Proposition 2.1.6. *Let $\mathcal{E} \in \mathcal{C}_{coh}$ be an object of unipotent class m. Then for any $n \geq m$ and any element $e \in \mathcal{E}[x]$, there exists a unique homomorphism $\varphi_e : \mathcal{P}_{coh,n} \to \mathcal{E}$ such that $\varphi_{e,x}(p_n) = e$. In particular for any $n \geq 1$ the pair consisting of the vector bundle $\mathcal{P}_{coh,n}$ in $\mathcal{C}_{coh,n}$ and the element $p_n \in \mathcal{P}_{coh,n}[x]$ is unique up to a unique isomorphism.*

Proof. The proof goes by strong induction on $m \geq 1$. For the induction basis note that any object with unipotent class 1 is of the form $P \otimes_R \mathcal{O}_C$ for a finitely generated projective R-module P. Since $(P \otimes_R \mathcal{O}_C)[x] = P$ we need to show that for any $n \geq 1$ and any $p \in P$ there exists a unique homomorphism $\varphi_p : \mathcal{P}_{coh,n} \to P \otimes_R \mathcal{O}_C$ with $\varphi_{p,x}(p_n) = p$. This can be shown by induction on n. For $n = 1$ we have

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{O}_C, P \otimes_R \mathcal{O}_C) = \mathrm{Hom}_R(R, P),$$

and for any element $p \in P$ there exists a unique homomorphism $\varphi_p : R \to P$ with $\varphi_p(1) = p$, hence we are done. For the induction step, one can use the defining exact sequence

$$0 \to T_{coh}^{\otimes(n-1)} \otimes_R \mathcal{O}_C \to \mathcal{P}_{coh,n} \to \mathcal{P}_{coh,n-1} \to 0,$$

and Remark 2.1.5.

Now suppose that the assertion of the proposition is true for any positive integer smaller than m, and consider an object $\mathcal{E}$ in $\mathcal{C}_{coh}$ of unipotent class m. Also fix an integer $n \geq m$ and a fixed element $e \in \mathcal{E}[x]$. By definition, $\mathcal{E}$ can be written as an extension of a unipotent vector bundle $\mathcal{G}$ of class $m-1$ by a trivial vector bundle, i.e. there exists a finitely generated projective R-module P, and an object $\mathcal{G}$ in $\mathcal{C}_{coh}$ of unipotent class $m-1$ such that we have a short exact sequence

$$0 \to P \otimes_R \mathcal{O}_C \to \mathcal{E} \to \mathcal{G} \to 0.$$

Furthermore assume that $e \in \mathcal{E}[x]$ maps to $g \in \mathcal{G}[x]$. By induction hypothesis, there is a unique morphism $\psi_g : \mathcal{P}_{coh,n-1} \to \mathcal{G}$, which sends p_{n-1} to g. Let us show first that this ψ_g can be extended to a morphism from $\mathcal{P}_{coh,n}$ to $\mathcal{E}$. To do this we pull back the above extension via ψ_g to obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \otimes_R \mathcal{O}_C \arrow[r] \arrow[d, "Id"] & \mathcal{E}' \arrow[r] \arrow[d, "f"] & \mathcal{P}_{coh,n-1} \arrow[r] \arrow[d, "\psi_g"] & 0 \\
0 \arrow[r] & P \otimes_R \mathcal{O}_C \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

We continue pulling back the extension, this time via the surjection

$$\mathcal{P}_{coh,n} \twoheadrightarrow \mathcal{P}_{coh,n-1}.$$

This gives us the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \otimes_R \mathcal{O}_C \arrow[r] \arrow[d, "Id"] & \mathcal{E}'' \arrow[r] \arrow[d, "h"] & \mathcal{P}_{coh,n} \arrow[r] \arrow[d] \arrow[l, dotted, bend right, "s"'] & 0 \\
0 \arrow[r] & P \otimes_R \mathcal{O}_C \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{P}_{coh,n-1} \arrow[r] & 0
\end{tikzcd}$$

But as we have mentioned in Remark 2.1.4 the top row extension is a split extension. Hence there exists a section

$$s : \mathcal{P}_{coh,n} \to \mathcal{E}''.$$

The composite $\alpha := f \circ h \circ s : \mathcal{P}_{coh,n} \to \mathcal{E}$ is the desired extension of ψ_g. This map, by construction, sends p_n to a lift e' of g in $\mathcal{E}[x]$. Now the difference $e - e'$ goes to zero in $\mathcal{G}[x]$ and hence lies in P. By the induction hypothesis again, there exists a unique morphism $\beta : \mathcal{P}_{coh,n-1} \to P \otimes_R \mathcal{O}_C$ which sends p_{n-1} to $e - e' \in P$. If we compose the surjection $\mathcal{P}_{coh,n} \twoheadrightarrow \mathcal{P}_{coh,n-1}$, β, and the injection $P \otimes_R \mathcal{O}_C \hookrightarrow \mathcal{E}$, and finally add the resulting map to α, we obtain a map from $\mathcal{P}_{coh,n}$ to $\mathcal{E}$ which sends p_n to e.

To prove uniqueness, notice that the difference φ of two maps from $\mathcal{P}_{coh,n}$ to $\mathcal{E}$ with same images on p_n, sends p_n to zero. Hence by induction hypothesis the composition of φ with $\mathcal{E} \twoheadrightarrow \mathcal{G}$ is zero. Using this, φ can be factored through $P \otimes_R \mathcal{O}_C$. Now since $P \otimes_R \mathcal{O}_C$ injects into $\mathcal{E}$, image of p_n in P is zero as well, and using induction hypothesis once more, one concludes that φ is zero. □

Recall once more that one can mimic the arguments given above to construct the universal objects $\mathcal{P}_{M,n}$ in the categories $\mathcal{C}_{M,n}$ with properties analogous to those of $\mathcal{P}_{coh,n}$. As an immediate application of this, we can finally prove Theorem 2.1.1.

Proof. (of Theorem 2.1.1) Consider the following pro-object in $\mathcal{C}_M$

$$\mathcal{P}_M := \varprojlim_n \mathcal{P}_{M,n},$$

and consider the element $p := \varprojlim p_n \in \mathcal{P}_M[x]$. For any object $\mathcal{E}$ in $\mathcal{C}_M$, and any element $e \in \mathcal{E}[x]$, $\mathcal{E}$ belongs to $\mathcal{C}_{M,n}$ for sufficiently large n. Fix such a number n, and note that by the above lemma there exists a unique morphism from $\mathcal{P}_{M,n} \to \mathcal{E}$ which sends p_n to e. Any such morphism gives a morphism from $\mathcal{P}_M$ to $\mathcal{E}$, but by uniqueness, these morphisms are all compatible for different n's, and hence give rise to one morphism from $\mathcal{P}_M$ to $\mathcal{E}$ which sends p to e. □

Note that Theorem 2.1.1 is equivalent to saying that for any object $\mathcal{E}$ in $\mathcal{C}_M$, one has

$$\mathrm{Hom}_{\mathcal{C}_M}(\mathcal{P}_M, \mathcal{E}) \cong \mathcal{E}[x].$$

On the other hand, one can naturally define tensor products and duals of objects in $\mathcal{C}_M$, which commute with $\mathcal{F}_M$. We are going to use these facts and put a Hopf-algebra structure on the stalk $\mathcal{P}_M[x]$ of $\mathcal{P}_M$ at x. We concentrate again in the coherent case and leave it to the reader to work out the analogous de Rham and étale cases. Being only concerned with the coherent case, we can and will drop the subscripts " $-_{coh}$ " without causing any confusion.

For any natural number $n \geq 1$, put

$$A_n := \mathcal{P}_n[x]$$

which by Proposition 2.1.6 is isomorphic to $\mathrm{End}(\mathcal{P}_n)$ and hence has a ring structure. By construction, there are ring epimorphisms $A_n \twoheadrightarrow A_{n-1}$, with respect to which we can form the inverse limit

$$A_\infty := \varprojlim A_n \cong \mathcal{P}[x] \cong \mathrm{End}(\mathcal{P}).$$

A_∞ has an obvious ring structure, and in order to make it into a Hopf-algebra we must construct a map from A_∞ to

$$A_\infty \widehat{\otimes} A_\infty \cong \varprojlim_{n,m} (\mathcal{P}_n \otimes \mathcal{P}_m)[x].$$

To do this, note that by Proposition 2.1.6, for any pair of natural numbers n and m, there exists a unique morphism $\mathcal{P}_{n+m} \to \mathcal{P}_n \otimes \mathcal{P}_m$ which sends p_{n+m} in $\mathcal{P}_{n+m}[x]$ to $p_n \otimes p_m$ in $(\mathcal{P}_n \otimes \mathcal{P}_m)[x]$. Taking inverse limit over n and m and fiber over x, we obtain the desired co-product which sends p_∞ to $p_\infty \otimes p_\infty$. Moreover one can consider the co-unit $A_\infty \twoheadrightarrow A_0$ and show

Lemma 2.1.7. *The above co-product and co-unit, make A_∞ into a co-associative co-commutative Hopf-algebra.*

Proof. All the assertions are immediate consequences of appropriate uniqueness assertions in Theorem 2.1.1. □

Having constructed the co-commutative Hopf-algebra A_∞, we can take its dual $A_\infty^\vee$ and obtain a commutative flat Hopf-algebra, in this case over R, and by taking the spectrum of the result, we can construct a flat group scheme

$$G := \mathrm{Spec}(A_\infty^\vee),$$

in this case over R (recall that we could have done the same in the de Rham (resp. étale) case and obtain a commutative flat Hopf-algebra and a flat group scheme over R (resp. $\mathbb{Q}_l$) in that case). Moreover, for each integer $n \geq 1$, we define

$$G_n := G/Z^n(G),$$

where $Z^\bullet$ denotes the descending central series. We will see in Remark 2.1.10 that these algebraic quotients G_n of G are unipotent algebraic group schemes and hence G is a pro-unipotent group scheme.

For any unipotent bundle $\mathcal{E}$, by composing morphisms one has a natural map

$$\mathrm{End}_{\mathcal{C}}(\mathcal{P}) \otimes_R \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}, \mathcal{E}^\vee) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}, \mathcal{E}^\vee).$$

But by Theorem 2.1.1 this is nothing other than a map

$$A_\infty \otimes_R \mathcal{E}^\vee[x] \to \mathcal{E}^\vee[x].$$

Taking the dual of this map gives the map

$$\mathcal{E}[x] \to \mathcal{O}_G \otimes_R \mathcal{E}[x].$$

It is straightforward now to check that the above map puts a co-module structure on $\mathcal{E}[x]$, and hence $\mathcal{F} : \mathcal{E} \mapsto \mathcal{E}[x]$ is a tensor functor from the category of unipotent bundles $\mathcal{C}$ to the category of G-representations on finitely generated projective R-modules. Now we prove that this actually gives an equivalence of categories, but before that we need a preliminary proposition, namely

Proposition 2.1.8. *For any unipotent bundle $\mathcal{E}$ in $\mathcal{C}$, there are positive integers $n, m, r, s \in \mathbb{N}$, such that we have the following exact sequence of unipotent bundles*

$$\mathcal{P}_m^s \to \mathcal{P}_n^r \to \mathcal{E} \to 0.$$

Proof. First of all recall that unipotent bundles, being iterated extensions of trivial bundles, are semi-stable of slope zero. Now fix an object $\mathcal{E}$ in $\mathcal{C}$. Since $\mathcal{E}$ has finite unipotent class, by Proposition 2.1.6 there is an integer n such that for any element $e \in \mathcal{E}[x]$, there is a morphism $\varphi_e \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_n, \mathcal{E})$ whose stalk at x sends p_n to e. Hence the resulting morphism

$$\Phi : \mathcal{P}_n \otimes_R \mathcal{E}[x] \to \mathcal{E}$$

is surjective at x and hence is generically surjective. This implies that $\mathrm{Coker}(\Phi)$ is a torsion coherent module whose length is equal to the slope of $\mathrm{Ker}(\Phi)$ (recall that both $\mathcal{P}_n \otimes_R \mathcal{E}[x]$ and $\mathcal{E}$ have slope zero). On the other hand, $\mathrm{Ker}(\Phi)$ is a sub-vector bundle of $\mathcal{P}_n \otimes_R \mathcal{E}[x]$, which is a semi-stable vector bundle with slope zero, and hence must have slope zero as well. So far we have proved that $\mathrm{Coker}(\Phi)$ is trivial and hence Φ is surjective. Since $\mathcal{E}[x]$ is a finitely generated R-module, for suitable $r \in \mathbb{N}$ there exists a surjection

$$\mathcal{P}_n^r \twoheadrightarrow \mathcal{P}_n \otimes_R \mathcal{E}[x],$$

and hence the composition gives us the exact sequence

$$\mathcal{P}_n^r \to \mathcal{E} \to 0.$$

Obviously the kernel of the surjection $\mathcal{P}_n^r \twoheadrightarrow \mathcal{E}$, being sub-vector bundle of $\mathcal{P}_n^r$, is again semi-stable of slope zero. Using this it is easy to show that this kernel remains unipotent and hence by repeating the above argument we can finish the proof. □

Now we can prove

Theorem 2.1.9. *With above notations and hypotheses, the tensor functor $\mathcal{F}$ gives an equivalence between the category $\mathcal{C}$ of unipotent bundles and the category $\mathrm{Rep}_R(G)$ of G-representations on finitely generated projective R-modules.*

Proof. Let us begin by showing that $\mathcal{F}$ is a fully faithful functor. For this we must show that for any two unipotent bundles $\mathcal{E}$ and $\mathcal{G}$ in $\mathcal{C}$, there is a bijection between $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{E})$ and $\mathrm{Hom}_G(\mathcal{G}[x], \mathcal{E}[x])$. By applying Proposition 2.1.8 to $\mathcal{G}$, we obtain an exact sequence

$$\mathcal{P}_m^s \to \mathcal{P}_n^r \to \mathcal{G} \to 0$$

for suitable $m, n, r, s \in \mathbb{N}$. This leads to the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_n, \mathcal{E})^r \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_m, \mathcal{E})^s \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}_G(\mathcal{G}[x], \mathcal{E}[x]) \arrow[r] & \mathrm{Hom}_G(A_n, \mathcal{E}[x])^r \arrow[r] & \mathrm{Hom}_G(A_m, \mathcal{E}[x])^s
\end{tikzcd}$$

This reduces our problem to showing that for any unipotent bundle $\mathcal{E}$ and any sufficiently large $n \in \mathbb{N}$, the natural map

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}_n, \mathcal{E}) \to \mathrm{Hom}_G(A_n, \mathcal{E}[x])$$

induced by $\mathcal{F}$ is a bijection. This is equivalent, using the language of pro-objects, to show that for any unipotent bundle $\mathcal{E}$ in $\mathcal{C}$ the natural map

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{P}, \mathcal{E}) \to \mathrm{Hom}_G(A_\infty, \mathcal{E}[x])$$

is a bijection. But this is easy to see, since both sides are canonically in bijection with $\mathcal{E}[x]$, left hand side by Theorem 2.1.1 and right hand side by noticing that any G-equivariant map from $A_\infty = \mathcal{O}_G^\vee$ to $\mathcal{E}[x]$ is completely determined by the image of $1 \in A_\infty$, which is an arbitrary element in $\mathcal{E}[x]$.

It remains to show that $\mathcal{F}$ is an essentially surjective functor. To do this fix an arbitrary G-representation M, which is a finitely generated projective R-module. Since for any element $m \in M$, there exists a G-equivariant map from A_∞ to M such that $1 \mapsto m$, and also since M is finitely generated and $A_\infty = \cup_n A_n$, one deduces that for sufficiently large n, there is a natural number $r \in \mathbb{N}$ and a G-equivariant surjection $A_n^r \twoheadrightarrow M$. Repeating this argument for the kernel of this surjection, we obtain an exact sequence

$$A_m^s \xrightarrow{f} A_n^r \to M \to 0$$

for suitable $m, n, r, s \in \mathbb{N}$. A_n and A_m are images under $\mathcal{F}$ of $\mathcal{P}_n$ and $\mathcal{P}_m$ respectively, and since we have shown that $\mathcal{F}$ is a full functor, there exists a morphism $\mathcal{P}_m^s \xrightarrow{\psi} \mathcal{P}_n^r$ between unipotent bundles such that $\mathcal{F}(\psi) = f$. If we denote the cokernel of ψ by $\mathcal{E}$, then it is obvious that $\mathcal{F}(\mathcal{E}) = \mathcal{E}[x]$ is isomorphic to M, and we are done. □

Remark 2.1.10. In addition to all of the above hypotheses, assume that R is a field. Then by above theorem every representation of G on a finite dimensional R-vector space is unipotent. In particular all algebraic quotients G_n of G are unipotent algebraic group schemes over R, and hence G itself is a pro-unipotent group scheme. ♣

Remark 2.1.11. Note that fibers of unipotent bundles are assumed to be finitely generated projective R-modules, hence any exact sequence of these fibers splits. In particular by construction one obtains a (non-canonical) isomorphism

$$A_n \cong \prod_{i=0}^{n-1} T^{\otimes i}.$$

This means that the topological algebra A_∞ is non-canonically isomorphic to the completed free tensor algebra $\prod_n T^{\otimes n}$. Combining this and the previous remark, one obtains that when R is a field of characteristic zero, G is a pro-unipotent group scheme over R which, by Baker-Campbell-Hausdorff formula, is isomorphic to its Lie algebra which in turn is isomorphic to the free Lie algebra in T. ♢

We state the upshot of the above arguments, and analogue ones in the de Rham and étale cases, in the following theorem for future references. Being in general setting now, we go back to our standard subscripts in different cases.

Theorem 2.1.12. *There exists a pro-unipotent group scheme G_M over R (resp. R, resp. $\mathbb{Q}_l$) in the coherent (resp. de Rham, resp. étale) case, such that the category $\mathcal{C}_M$ is equivalent to the category of representations of G_M on finitely generated projective R-modules (resp. R-modules, resp. $\mathbb{Q}_l$-modules). Moreover in the coherent and de Rham cases, provided R is a field of characteristic zero, and without any extra assumption in the étale case, the group G_M is free pro-unipotent isomorphic to its Lie algebra which is the free Lie algebra in T_M.*

Remark 2.1.13. As it has been mentioned briefly in Remark 2.1.2, the group G_M in the above theorem, at least when the base ring R is a field, is the group of tensor automorphisms of the functor $\mathcal{F}_M$, where $\mathcal{F}_M$ is the functor which takes the fiber at the fixed point $x \in X(R)$. One can go further, and for any other point $y \in X(R)$, and even for base points at infinity (see [8, section 15]), define the space of homotopy classes of paths from y to x as follows. In Tannakian formalism it can be described like this. Consider the fiber functor associated to y, which sends an object $\mathcal{E}$ in $\mathcal{C}_M$ to its fiber $\mathcal{E}[y]$ over y, and consider tensor isomorphisms from this functor to $\mathcal{F}_M$. Then one can show that the set of this tensor isomorphisms, as a functor from R-algebras to sets, is representable by an affine scheme over R, which is a right torsor over G_M (see Theorem 1.1.8). This right torsor is called the path torsor from y to x.

One also can give a description of these path torsors parallel to the above explicit construction of G_M. Namely, first of all note that in order to define a co-product on $\mathcal{P}_M[x]$, we took the fiber at x of a morphism $\mathcal{P}_M \to \mathcal{P}_M \otimes \mathcal{P}_M$. Taking the fiber of the same morphism at point y, puts a co-associative, co-commutative co-product on $\mathcal{P}_M[y]$ as well (note that $\mathcal{P}_M[y]$ does not have a ring structure in general). On the other hand, $\mathcal{P}_M[x]$ acts on $\mathcal{P}_M[y]$. In fact, for any $a \in \mathcal{P}_M[x]$ and any $b \in \mathcal{P}_M[y]$, consider the unique endomorphism $\varphi_a \in \mathrm{End}(\mathcal{P}_M)$, which exists by Theorem 2.1.1 and satisfies $\varphi_{a,x}(1) = a$, and define

$$a.b := \varphi_{a,y}(b).$$

It is clear by construction that this action is compatible with co-products on $\mathcal{P}_M[x]$ and $\mathcal{P}_M[y]$, and hence furnishes the affine scheme $\mathrm{Spec}(\mathcal{P}_M[y]^\vee)$ with an action of G_M. We denote the affine scheme $\mathrm{Spec}(\mathcal{P}_M[y]^\vee)$ by $G_M(x,y)$ and call it the path torsor from y to x, referring to its torsor structure over G_M. Finally note that we could have made all these procedures in finite levels, and defined path torsors $G_{M,n}(x,y)$ from y to x over $G_{M,n}$ for all $n \geq 1$, which are actually the push forwards of $G_M(x,y)$ via the projections $G_M \twoheadrightarrow G_{M,n}$. ♠

2.2 The Hodge Filtration

In this section we are going to study an important feature of the de Rham case, namely the Hodge filtration. To start with we look more closely at the de Rham picture. Recall that in the de Rham case, we consider the category $\mathcal{C}_{dR}$ of pairs $(\mathcal{E}, \nabla)$ consisting of a vector bundle $\mathcal{E}$ over the curve C with an automatically flat connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D),$$

which is an iterated extension of trivial vector bundles with connection. In this framework, the proper cohomology theory to work with is the algebraic de Rham cohomology, hence we consider

$$H^i_{dR}(C, \mathcal{E}) := \mathbb{H}^i(C, \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)).$$

One explicit way of computing these algebraic de Rham cohomologies is by using Čeck cohomology. Namely, fix an open affine covering $C = \bigcup_{i \in I} \mathcal{U}_i$ of C, where I is a totally ordered index set. Now consider the following set of n-chains,

$$\underline{\boldsymbol{C}}^n := \bigoplus_{i_0 \leq \dots \leq i_n} \Gamma\left(\bigcap_{s=0}^{n} \mathcal{U}_{i_s}, \mathcal{E}\right) \oplus \bigoplus_{j_0 \leq \dots \leq j_{n-1}} \Gamma\left(\bigcap_{s=0}^{n-1} \mathcal{U}_{j_s}, \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)\right),$$

and the following differentials between them

$$\underline{d} := \begin{pmatrix} d & \nabla \\ 0 & d \end{pmatrix},$$

where d's on the diagonal are the usual differentials appearing in calculating Čech cohomologies of $\mathcal{E}$ and $\mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)$ with respect to the fixed open affine covering. Then the algebraic de Rham cohomologies $H^i_{dR}(C, \mathcal{E})$ are the cohomology groups of the resulting complex. Now it is very easy to prove

Lemma 2.2.1. *One has the following long exact sequence relating coherent and algebraic de Rham cohomologies over C*

$$\dots \to H^i_{dR}(C, \mathcal{E}) \to H^i(C, \mathcal{E}) \xrightarrow{\nabla} H^i(C, \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_C(D)) \to H^{i+1}_{dR}(C, \mathcal{E}) \to \dots$$

Consequently we can prove the following vanishing of H^2_{dR} which we claimed in section 2.1.

Lemma 2.2.2. *With the same notations and hypothesis as in the beginning of previous section about C and D, for any unipotent bundle $\mathcal{E}$ in $\mathcal{C}_{dR}$ one has*

$$H^2_{dR}(C, \mathcal{E}) = 0.$$

Proof. We prove the assertion by induction on the unipotent class of $\mathcal{E}$. For induction basis, using Lemma 2.2.1 and vanishing of coherent H^2 for curves, we obtain the following exact sequence

$$\ldots \to H^1(C, \mathcal{O}_C) \xrightarrow{\nabla} H^1(C, \Omega_{C/\mathrm{Spec}(R)}(D)) \to H^2_{dR}(C, \mathcal{O}_C) \to 0.$$

On the other hand, one knows that $H^1(C, \Omega_{C/\mathrm{Spec}(R)}(D))$ is isomorphic to the dual of $H^0(C, \mathcal{O}(-D))$ by Serre's duality theorem, which is 0 since we assumed that D is surjective over $\mathrm{Spec}(R)$. Whence we obtain that $H^2_{dR}(C, \mathcal{O}_C) = 0$. Now for an arbitrary unipotent bundle $\mathcal{E}$ of unipotent class n, by definition there exists a unipotent bundle $\mathcal{G}$ of class $n-1$ such that $\mathcal{E}$ is an extension of $\mathcal{G}$ by a trivial bundle. By the long exact sequence of algebraic de Rham cohomologies coming from this extension, $H^2_{dR}(C, \mathcal{E})$ sits, in an exact sequence, between $H^2_{dR}(C, \mathcal{G})$ and H^2_{dR} of a trivial bundle, which are both zero by induction hypothesis, and hence we are done. $\square$

This justifies more precisely the de Rham analogues of the objects and theories we developed in section 2.1, but in this case we can go further and enrich our theory. Namely we consider finite decreasing Hodge filtrations $F^\bullet$ by sub-bundles on objects of $\mathcal{C}_{dR}$, and moreover we assume that these Hodge filtrations satisfy Griffiths' transversality property, i.e. for each integer i assume

$$\nabla(F^i(\mathcal{E})) \subset F^{i-1}(\mathcal{E}) \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D).$$

In this context, the trivial bundle is $\mathcal{O}_C$ equipped with the trivial Hodge filtration, i.e. $F^0(\mathcal{O}_C) = \mathcal{O}_C$ and $F^1(\mathcal{O}_C) = 0$. These data induce a filtration on the associated de Rham complex and we can consider algebraic de Rham cohomologies of this filtration on the de Rham complex of a vector bundle $\mathcal{E}$ in different degrees, namely we define

$$F^i(\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)) := F^i(\mathcal{E}) \xrightarrow{\nabla} F^{i-1}(\mathcal{E}) \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D),$$

and

$$F^i H^*_{dR}(C, \mathcal{E}) := \mathbb{H}^*(C, F^i(\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D))).$$

Note that $F^i H^*_{dR}(C, \mathcal{E})$ does not necessarily inject into the $H^*_{dR}(C, \mathcal{E})$. By a morphism between these filtered objects we mean a filtered morphism where a morphism $f : \mathcal{E} \to \mathcal{G}$ between filtered objects is called filtered when for all i, one has $f(F^i(\mathcal{E})) \subset F^i(\mathcal{G})$. Restricting to filtered morphisms, maps and extensions are determined by F^0 of the de Rham complex associated to the bundle $\mathcal{H}om(\mathcal{E}, \mathcal{G})$. Namely, given two filtered objects $\mathcal{E}$ and $\mathcal{G}$, it is obvious that filtered homomorphisms between them are given by $F^0 H^0_{dR}(C, \mathcal{H}om(\mathcal{E}, \mathcal{G}))$, and moreover one has

Proposition 2.2.3. *For two filtered objects $\mathcal{E}$ and $\mathcal{G}$, the set of isomorphism classes of filtered extensions of $\mathcal{G}$ by $\mathcal{E}$ is in bijection with*

$$F^0 H^1_{dR}(C, \mathcal{H}om(\mathcal{G}, \mathcal{E})).$$

Proof. For any filtered extension $0 \to \mathcal{E} \to \mathcal{T} \to \mathcal{G} \to 0$ of $\mathcal{G}$ by $\mathcal{E}$, fix an open affine covering $C = \bigcup_{i \in I} \mathcal{U}_i$ of C, such that for all $i \in I$ one has

$$0 \longrightarrow \mathcal{E}_i \longrightarrow \mathcal{T}_i \underset{\xleftarrow{s_i}}{\longrightarrow} \mathcal{G}_i \longrightarrow 0\ ,$$

where a subscript $-_i$ means restriction to $\mathcal{U}_i$ and s_i are filtered sections. This means that restricting to $\mathcal{U}_i$, one has filtered isomorphism

$$\mathcal{T}_i \cong \mathcal{E}_i \oplus \mathcal{G}_i,$$

and hence the connection of $\mathcal{T}_i$ must have the following form

$$\nabla_{\mathcal{T}_i} = \begin{pmatrix} \nabla_{\mathcal{E}_i} & \lambda_i \\ 0 & \nabla_{\mathcal{G}_i} \end{pmatrix}.$$

Now one can easily check that $\nabla_{\mathcal{T}_i}$ satisfies Griffiths' transversality property if and only if

$$\lambda_i \in \Gamma(\mathcal{U}_i, F^{-1}\mathcal{H} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)),$$

where $\mathcal{H}$ is the sheaf $\mathcal{H}om(\mathcal{G}, \mathcal{E})$. On the other hand, if we consider the differences

$$e_{ij} := s_i\,|_{\mathcal{U}_{ij}} - s_j\,|_{\mathcal{U}_{ij}}$$

where $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$, we get the following collection of sections

$$e_{ij} \in \Gamma(\mathcal{U}_{ij}, F^0\mathcal{H}).$$

Now it is easy, using the Čech approach to compute hyper cohomologies of complexes, which was mentioned at the beginning of this section, to see that the above collection of data, namely

$$(e_{ij}, \lambda_i) \in \bigoplus_{i \leq j} \Gamma(\mathcal{U}_{ij}, F^0\mathcal{H}) \oplus \bigoplus_i \Gamma(\mathcal{U}_i, F^{-1}\mathcal{H} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D))$$

is equivalent to the data of an element in

$$\mathbb{H}^1(C, F^0\mathcal{H} \xrightarrow{\nabla} F^{-1}\mathcal{H} \otimes_{\mathcal{O}_C} \Omega_{C/\mathrm{Spec}(R)}(D)),$$

and hence we are done by definition. $\square$

This proposition allows us to develop a similar filtered de Rham theory. In order to understand this better, we need a closer study of the induced filtration on the de Rham cohomology of the trivial bundle. More precisely we need

Lemma 2.2.4. *The Hodge filtration on $H^1_{dR}(C, \mathcal{O}_C)$ has nonzero grading parts only in degrees 0 and 1. Moreover one has*

$$Gr^0_F(H^1_{dR}(C, \mathcal{O}_C)) \cong H^1(C, \mathcal{O}_C),$$

and

$$F^1(H^1_{dR}(C, \mathcal{O}_C)) \cong H^0(C, \Omega_{C/\mathrm{Spec}(R)}(D)).$$

Hence the induced Hodge filtration on T_{dR} is concentrated in degrees -1 and 0, and moreover $F^0(T_{dR})$ is naturally isomorphic to T_{coh}.

Proof. Simply note that the induced Hodge filtration on the de Rham complex

$$\mathcal{O}_C \to \Omega_{C/\mathrm{Spec}(R)}(D)$$

is as follows. Its F^0 is the complex itself, its F^1 is $0 \to \Omega_{C/\mathrm{Spec}(R)}(D)$, and its F^2 is the zero complex. Hence the nonzero graded parts appear only in degrees 0 and 1, and obviously the 0-th graded part is just the sheaf $\mathcal{O}_C$ in degree 0 and the 1-st graded part is the sheaf $\Omega_{C/\mathrm{Spec}(R)}(D)$ in degree 1. Now everything, except the last claim, is an immediate consequence of [6, (1.4.5)]. But the last assertion is also plain, simply because by definition we have

$$F^0(T_{dR}) = F^0(H^1_{dR}(C, \mathcal{O}_C)^\vee) := \left(\frac{H^1_{dR}(C, \mathcal{O}_C)}{F^1(H^1_{dR}(C, \mathcal{O}_C))} \right)^\vee,$$

which by the previous part is isomorphic to

$$H^1(C, \mathcal{O}_C)^\vee = T_{coh}.$$

Especially, in this case the F^i's are subobjects of the algebraic de Rham cohomology. □

Remark 2.2.5. One could prove the above lemma by employing Lemma 2.2.1 and the above mentioned proof has essentially the same spirit. ♡

Since the Hodge filtration on T_{dR} has nonzero grading parts only in degrees -1 and 0, the induced Hodge filtration on $T_{dR}^{\otimes n}$, and hence the one on $T_{dR}^{\otimes n} \otimes_R \mathcal{O}_C$, has nonzero grading parts only in degrees $\{-n, \ldots, 0\}$. It is easy now to prove the following filtered version of Lemma 2.1.3:

Lemma 2.2.6. *There exists a unique Hodge filtration on $\mathcal{P}_{dR,n}$ which makes the following exact sequence an exact sequence of filtered vector bundles with strictly compatible morphisms*

$$0 \to T_{dR}^{\otimes(n-1)} \otimes_R \mathcal{O}_C \to \mathcal{P}_{dR,n} \to \mathcal{P}_{dR,n-1} \to 0.$$

Moreover, the above exact sequence induces following isomorphisms for each integer i

$$F^i H^0_{dR}(C, \mathcal{P}^\vee_{dR,n}) \cong F^i H^0_{dR}(C, \mathcal{O}_C),$$

$$F^i H^1_{dR}(C, \mathcal{P}^\vee_{dR,n}) \cong F^i H^1_{dR}(C, T_{dR}^{\otimes -(n-1)} \otimes_R \mathcal{O}_C),$$

and finally the element $p_n \in \mathcal{P}_{dR,n}[x]$ in Lemma 2.1.3, lies in $F^0(\mathcal{P}_{dR,n}[x])$.

Proof. Taking into account Proposition 2.2.3, the proof can be done by following exactly the same line of ideas as in the proof of Lemma 2.1.3. Just notice that the class corresponding to $\mathcal{P}_{dR,n}$ in

$$\mathrm{Ext}^1(\mathcal{P}_{dR,n-1}, T_{dR}^{\otimes(n-1)} \otimes_R \mathcal{O}_C) \cong \mathrm{Hom}_R(T_{dR}^{\otimes -(n-1)}, T_{dR}^{\otimes -(n-1)}),$$

was the identity element which lies in F^0. □

Note that, by construction, the resulting Hodge filtration on $\mathcal{P}_{dR,n}$ has nonzero grading parts only in degrees $\{-(n-1), \ldots, 0\}$. Using these and the same ideas as in previous section, one can prove the following filtered analogue of Proposition 2.1.6 in the de Rham case (we omit the similar proof).

Proposition 2.2.7. *For any filtered object $(\mathcal{E}, \nabla, F^\bullet)$ in $\mathcal{C}_{dR,n}$, and any element $e \in F^0(\mathcal{E}[x])$, there exists a unique horizontal, strictly compatible homomorphism $\varphi_e : \mathcal{P}_{dR,n} \to \mathcal{E}$ such that $\varphi_{e,x}(p_n) = e$. Moreover the pair $(\mathcal{P}_{dR,n}, p_n)$ is unique up to a unique isomorphism.*

We finish this section by

Remark 2.2.8. Note that, since by Lemma 2.2.4, $F^0(T_{dR}) \cong T_{coh}$, one also obtains $F^0(T_{dR}^{\otimes n}) \cong T_{coh}^{\otimes n}$. Then simply by the analogy of the constructions, one can see that these relations extend also to the following isomorphism:

$$F^0(\mathcal{P}_{dR,n}) \cong \mathcal{P}_{coh,n}.$$

Furthermore, if one equips the inverse limit $\mathcal{P}_{dR}$ with the induced Hodge filtration, one obtains a Hodge filtration on the stalk $\mathcal{P}_{dR}[x]$ concentrated on negative degrees, and dually a Hodge filtration on the affine coordinate ring $\mathcal{O}(G_{dR})$ of the group scheme G_{dR} concentrated in positive degrees. Now since

$$F^0(\mathcal{P}_{dR}[x]) \cong \mathcal{P}_{coh}[x],$$

one dually obtains the following isomorphism

$$\mathcal{O}(G_{dR})/F^1(\mathcal{O}(G_{dR})) \cong \mathcal{O}(G_{coh}).$$

This gives a closed immersion of the pro-unipotent coherent fundamental group scheme G_{coh} into the pro-unipotent de Rham fundamental group scheme G_{dR}. ♣

2.3 Crystalline Realization and Frobenius

The aim of this section is to introduce a category $\mathcal{C}_{cr}$ which is equivalent to the category $\mathcal{C}_{dR}$ of a curve over a complete discrete valuation ring of unequal characteristic p and absolute ramification index $e < p$. The important property of $\mathcal{C}_{cr}$ is that it only depends on the reduction modulo p of the curve, hence admits a Frobenius action. This leads to a Frobenius action on $\mathcal{C}_{dR}$ and hence a Frobenius action on G_{dR} and all of $G_{dR,n}$'s. This Frobenius action is the second important part of the decoration that one can put on the (pro-)unipotent de Rham fundamental group.

In order to do that, we need to recall some parts of the crystalline and the logarithmic crystalline theories. The main references for this section are [2] for the first or classic part and [23] for the second or logarithmic part. Let A be a commutative ring and $I \subset A$ be an ideal. Recall that a divided power structure on I is a collection of maps $\gamma_i : I \to A$, for all integers $i \geq 0$, such that

1. $\forall x \in I$, $\gamma_0(x) = 1$, $\gamma_1(x) = x$, and $\gamma_i(x) \in I$ if $i \geq 1$.
2. $\forall x, y \in I$, $\gamma_k(x + y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.
3. $\forall \lambda \in A$ and $\forall x \in I$, $\gamma_k(\lambda x) = \lambda^k \gamma_k(x)$.
4. $\forall x \in I$, $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$.
5. $\forall x \in I$, $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$.

In this case we say that (I, γ) is a DP-ideal, (A, I, γ) is a DP-ring, and γ is a DP-structure on I. A DP-morphism $f : (A, I, \gamma) \to (B, J, \delta)$ is a ring homomorphism $f : A \to B$ such that $f(I) \subset J$ and that $\delta_n(f(x)) = f(\gamma_n(x))$ for all $n \in \mathbb{N}$ and $x \in I$. One can easily check that if V is a discrete valuation ring of unequal characteristic p and uniformizing parameter π, then the ideal (π) admits a (unique) DP-structure if and only if $v_\pi(p) < p$. Moreover for any $m \geq 1$, this DP-structure induces a DP-structure on $(\pi) \subset V/(\pi^m)$ which will be called the canonical DP-structure.

Definition 2.3.1. *Let (A, I, γ) be a DP-ring, B an A-algebra, and (J, δ) a DP-ideal in B. One says that γ and δ are compatible if γ extends to a DP-structure $\bar{\gamma}$ on $IB \subset B$ and $\bar{\gamma} = \delta$ on $IB \cap J$.*

If we fix a pair (A, I) consisting of a commutative ring A and an ideal I in A, we can consider the category of pairs over (A, I) whose objects are pairs (B, J) where B is an A-algebra and J is an ideal in B. A morphism from (B, J) to (B', J') is just an A-algebra homomorphism from B to B' which maps J into J'. Moreover if (A, I) admits a DP-structure γ we can consider the category of DP-algebras over (A, I, γ) whose objects are triples (B, J, δ) where (B, J) is a pair over (A, I) and δ is a DP-structure on J which is compatible with γ, and of coarse with DP-morphisms between them. Then we have the obvious forgetful functor from the category of DP-algebras over (A, I, γ) to the category of pairs over (A, I). The following theorem says that this forgetful functor admits a left adjoint which is called the DP-envelope functor and will be denoted by $\mathcal{D}$. More precisely we have

Theorem 2.3.2. *[2, Theorem 3.19] Let (A, I, γ) be a DP-ring and let (B, J) be a pair over (A, I). Then there exists a B-algebra $\mathcal{D}_{B,\gamma}(J)$ with a DP-ideal $(\bar{J}, \theta)$, such that $J\mathcal{D}_{B,\gamma}(J) \subset \bar{J}$, θ is compatible with γ, and with the following universal property: For any triple (C, K, δ) consisting of a B-algebra C, an ideal K in C which contains JC, and a DP-structure δ on K compatible with γ, there is a unique DP-morphism from $(\mathcal{D}_{B,\gamma}(J), \bar{J}, \theta)$ to (C, K, δ) which fits into the following commutative diagram:*

$$\begin{tikzcd}
 & (\mathcal{D}_{B,\gamma}(J), \bar{J}, \theta) \arrow[dr, dotted, "\exists!"] & \\
(B, J) \arrow[ur] \arrow[rr] & & (C, K, \delta) \\
 & (A, I, \gamma) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The following proposition will help us compute some DP-envelopes which will appear later on.

Proposition 2.3.3. *Let $(V, (\pi), \gamma)$ be the above mentioned canonical DP-ring where V is a discrete valuation ring of unequal characteristic p with absolute ramification index smaller than p, and let A be a V-algebra. Then one has*

$$\mathcal{D}_{A,\gamma}(\pi A) \cong (A, \pi A, \bar{\gamma}),$$

in which $\bar{\gamma}$ is the unique extension of γ to πA.

Proof. It suffices to prove that there always exists a (unique) extension $\bar{\gamma}$ of γ to πA, because then DP-envelope by definition does nothing with a pair which already has a compatible DP-structure. Note that $\bar{\gamma}_n(\pi a) = a^n \gamma_n(\pi)$, $\forall a \in A$ defines an extension of γ to πA. It is well defined because if $\pi a = \pi a'$ for $a, a' \in A$, then one has

$$a^n \gamma_n(\pi) - a'^n \gamma_n(\pi) = (a^n - a'^n)\gamma_n(\pi)$$

which is zero since $(a - a') \mid (a^n - a'^n)$ and $\pi \mid \gamma_n(\pi)$ for all $n \in \mathbb{N}$. $\square$

Remark 2.3.4. Note that for any integer $m \geq 1$ the analogue of the above proposition is valid for the DP-ring $(V/(\pi^m), (\pi), \gamma)$, in which γ is the canonical DP-structure again, and it can be proved in exactly the same way as the above proposition was proved. $\diamondsuit$

The next step is to globalize these notions and talk about sheaves of DP-rings, DP-ringed spaces, DP-schemes, and morphisms between them. The exact definitions can be found in [2]. Let us just recall the global version of the notion of DP-envelopes. Let S be a scheme, $\mathcal{I} \subset \mathcal{O}_S$ be a quasi-coherent sheaf of ideals with a DP-structure γ, and suppose $i : X \to Y$ is a closed immersion of S-schemes. Let $\mathcal{J}$ be the defining sheaf of ideals of X in $\mathcal{O}_Y$, then one can show that

$$\mathcal{D}_{X,\gamma}(Y) := \mathcal{D}_{\mathcal{O}_Y,\gamma}(\mathcal{J})$$

is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras and hence we can define the DP-envelope of X in Y to be the DP-scheme

$$D_{X,\gamma}(Y) := \mathrm{Spec}_Y(\mathcal{D}_{X,\gamma}(Y)).$$

Now we have enough tools to define crystalline site and topos. From now on, all schemes under consideration in this section will be killed by a power of a fixed prime number p, unless otherwise specified. Let $\mathcal{S} = (S, \mathcal{I}, \gamma)$ be a DP-scheme, which will play the role of the base in the sequel. For any S-scheme X to which γ extends, the crystalline site of X relative to $\mathcal{S}$, $Crys(X/S)$, consists of the following data: Its objects are pairs $(U \hookrightarrow T, \delta)$, where U is a Zariski open subset of X, $U \hookrightarrow T$ is a closed S-immersion defined by an ideal sheaf $\mathcal{J}$, and δ is a DP-structure on $\mathcal{J}$ which is compatible with γ. Such a pair $(U \hookrightarrow T, \delta)$ will be called an $\mathcal{S}$-DP-thickening of U. A morphism

$$(U \hookrightarrow T, \delta) \xrightarrow{u} (U' \hookrightarrow T', \delta')$$

in $Crys(X/S)$ is just a commutative square

$$\begin{tikzcd} T \arrow[r] & T' \\ U \arrow[u] \arrow[r] & U' \arrow[u] \end{tikzcd}$$

such that $U \to U'$ is a morphism in the Zariski site of X and $T \to T'$ is an $\mathcal{S}$-DP-morphism $(T, \mathcal{J}, \delta) \to (T', \mathcal{J}', \delta')$. A covering family in $Crys(X/S)$ is a collection of morphisms

$$\{u_i : (U_i \hookrightarrow T_i, \delta_i) \to (U \hookrightarrow T, \delta)\}_i$$

such that each $T_i \to T$ is an open immersion and $T = \cup_i T_i$. We denote the topos of sheaves on $Crys(X/S)$ by $(X/S)_{Crys}$ and we call it the crystalline topos of X relative to S.

It can be easily seen from the definition of coverings in the crystalline site that an element in the crystalline topos $(X/S)_{Crys}$ is equivalent to the data of a family of Zariski sheaves on the thickenings T, one for each object in $Crys(X/S)$, which are compatible in an evident way. Using this interpretation, for any crystalline sheaf $\mathcal{F}$ in $(X/S)_{Crys}$ we denote the associated Zariski sheaf on T by $\mathcal{F}_{(U \hookrightarrow T, \delta)}$. The most important element of $(X/S)_{Cris}$ is the structure sheaf $\mathcal{O}_{X/S}$ which associates the Zariski sheaf $\mathcal{O}_T$ on T to the object $(U \hookrightarrow T, \delta)$.

Another important notion which we need is the notion of a crystal. A crystal of $\mathcal{O}_{X/S}$-modules is a sheaf $\mathcal{F}$ of $\mathcal{O}_{X/S}$-modules such that for any morphism

$$u : (U' \hookrightarrow T', \delta') \to (U \hookrightarrow T, \delta)$$

in $Crys(X/S)$, the map

$$u^* \mathcal{F}_{(U \hookrightarrow T, \delta)} \to \mathcal{F}_{(U' \hookrightarrow T', \delta')}$$

is an isomorphism. Note that the sheaf $\mathcal{O}_{X/S}$ itself is an obvious example of a crystal. The following theorem is the first and main step toward our goal.

Theorem 2.3.5. *[2, Theorem 6.6] If $X \to Y$ is a closed immersion of S-schemes, with Y smooth over S, the following categories are naturally equivalent:*

- *The category of crystals of $\mathcal{O}_{X/S}$-modules on $Crys(X/S)$.*
- *The category of $\mathcal{D}_{X,\gamma}(Y)$-modules with an integrable, quasi-nilpotent connection (as an $\mathcal{O}_Y$-module) which is compatible with the canonical connection on $\mathcal{D}_{X,\gamma}(Y)$.*

Recall that a quasi-nilpotent connection ∇ is a connection such that after fixing an open covering $Y = \bigcup_i U_i$ of Y and systems of coordinates $\{y_{i,1}, \ldots, y_{i,t}\}_i$ for all the open subsets U_i of Y over S, there exist natural numbers $n_{i,1}, \ldots, n_{i,t} \in \mathbb{N}$ such that for any i, the differential operator

$$\nabla(\partial/\partial y_{i,1})^{n_{i,1}} \circ \cdots \circ \nabla(\partial/\partial y_{i,t})^{n_{i,t}}$$

kills all the sections over U_i. This notion of quasi-nilpotency can be shown to be independent of the choices of coordinate systems for integrable connections.

If we were only interested in smooth projective cases, we already were equipped enough to endow G_{dR} with a Frobenius action. But since we are mainly interested in the case of open curves we need also the logarithmic crystalline theory. Here we just briefly recall this generalizations and refer the reader to [23] for more precise and general treatment.

Recall that a logarithmic structure on a scheme X is a sheaf of monoids $\mathcal{M}$ on the étale site $X_{\text{ét}}$ of X together with a multiplicative homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$ which satisfies

$$\alpha^{-1}(\mathcal{O}_X^*) \overset{\alpha}{\xrightarrow{\sim}} \mathcal{O}_X^*.$$

Note that for any multiplicative homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$, where $\mathcal{M}$ is a sheaf of monoids on $X_{\text{ét}}$, the push out of the diagram

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r] \arrow[d, "\alpha"] & \mathcal{M} \\ \mathcal{O}_X^* & \end{tikzcd}$$

in the category of sheaves of monoids on $X_{\text{ét}}$ gives rise to a logarithmic structure $\mathcal{M}^a$ on X which is called the associated logarithmic structure to $\mathcal{M}$. Now for any morphism of schemes $f : X \to Y$ and any logarithmic structure $\mathcal{N}$ on Y, the pull back $f^*(\mathcal{N})$ is defined to be the logarithmic structure $(f^{-1}(\mathcal{N}))^a$ on X.

A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ of schemes with logarithmic structures is a pair (f, s) consisting of a morphism of schemes $f : X \to Y$ and a morphism of sheaves $s : f^{-1}(\mathcal{N}) \to \mathcal{M}$ such that

$$\begin{array}{ccc} f^{-1}(\mathcal{N}) & \xrightarrow{s} & \mathcal{M} \\ {\scriptstyle f^{-1}(\beta)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ f^{-1}(\mathcal{O}_Y) & \longrightarrow & \mathcal{O}_X \end{array}$$

commutes.

For example let X be a regular scheme with a fixed reduced divisor D with normal crossing, put $Y = X - D$, and let $i : Y \hookrightarrow X$ be the inclusion. Then the monoid

$$M := i_*\mathcal{O}_Y^* \cap \mathcal{O}_X = \{g \in \mathcal{O}_X : \; g \text{ is invertible outside } D\}$$

and the natural inclusion $M \hookrightarrow \mathcal{O}_X$ puts a fine logarithmic structure on X, where by a fine logarithmic structure we mean a coherent integral one. Recall that a monoid is called integral if it satisfies the cancelation rule and a logarithmic structure $\mathcal{M}$ is called integral if it is formed by a sheaf of integral monoids. Finally a logarithmic structure $\mathcal{M}$ on a scheme X is said to be coherent if it is isomorphic to the logarithmic structure associated with a multiplicative homomorphism

$$P_X \to \mathcal{O}_X,$$

where P_X is the constant sheaf on $X_{\text{ét}}$ associated with a finitely generated monoid P. Now fix a quadruple $(S, L, \mathcal{I}, \gamma)$ as the base, where S is a scheme such that $\mathcal{O}_S$ is killed by a power of a prime number p, L is a fine logarithmic structure on S, $\mathcal{I}$ is a quasi-coherent sheaf of ideals on S, and γ is a DP-structure on $\mathcal{I}$. Assume that (X, M) mentioned above can be considered as a scheme with fine logarithmic structure over (S, L) and finally suppose that γ extends to X. Then one can define the logarithmic crystalline site $Crys(X/S)^{log}$ as follows: Any object in $Crys(X/S)^{log}$ consists of the data $(U \xrightarrow{i} T, M_T, \delta)$ where U is an étale scheme over X, (T, M_T) is a scheme with a fine logarithmic structure over (S, L),

$$i : (U, M) \hookrightarrow (T, M_T)$$

is an exact closed immersion over (S, L), and δ is a DP-structure on the ideal sheaf of $\mathcal{O}_T$ which defines U, compatible with γ. In this context, an exact closed immersion $(i, s) : (X, \mathcal{M}) \to (Y, \mathcal{N})$ between two schemes with logarithmic structures is a morphism of schemes with logarithmic structures such that the underlying morphism of schemes $i : X \to Y$ is a closed immersion and $i^*(\mathcal{N}) \to \mathcal{M}$ is an isomorphism.

Morphisms in $Crys(X/S)^{log}$ are commutative diagrams

$$\begin{tikzcd} T \arrow[r] & T' \\ U \arrow[u, "i"] \arrow[r] & U' \arrow[u, "i'"'] \end{tikzcd}$$

where $U \to U'$ is a morphism in the étale site of X and $T \to T'$ is a DP-morphism between fine logarithmic schemes over (S, L). Coverings are usual coverings in the étale site of X, forgetting the logarithmic structures. The topos of sheaves over $Crys(X/S)^{log}$ is denoted by $(X/S)^{log}_{Crys}$ and is called the logarithmic crystalline topos of X over S. Note that the reason for considering étale schemes over X instead of its Zariski open subsets is that the notion of normal crossing for divisors is well behaved in the étale topology, i.e. it is étale local.

Now, exactly as in the classical case, one can define the structure sheaf $\mathcal{O}_{X/S}$ on $Crys(X/S)^{log}$ to be the sheaf which assigns to an object $(U \xrightarrow{i} T, M_T, \delta)$ the ring of global sections $\Gamma(T, \mathcal{O}_T)$. This obviously belongs to the logarithmic crystalline topos $(X/S)^{log}_{crys}$ of X over S.

Our next aim is to provide ourselves with the logarithmic version of the Theorem 2.3.5. In order to do that, we need the logarithmic version of DP-envelopes, as in Theorem 2.3.2, as well as the logarithmic version of the notion of a crystal. The latter one can be obtained simply by the following:

Definition 2.3.6. *A logarithmic crystal on $Crys(X/S)^{log}$ is a sheaf of $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ in the topos $(X/S)^{log}_{crys}$ such that for any morphism $g : T' \to T$ in $Crys(X/S)^{log}$, the induced map $g^*(\mathcal{F}_T) \to \mathcal{F}_{T'}$ is an isomorphism.*

The former one, namely the logarithmic version of DP-envelope, is given by [23, Proposition 5.3]. Putting all these together one can prove the following theorem which we state in view of its importance for us.

Theorem 2.3.7. *[23, Theorem 6.2] Let (Y, N) be a scheme with fine logarithmic structure which is smooth over (S, L), and let $(X, M) \to (Y, N)$ be a closed immersion. Denote by (D, M_D) the DP-envelope of (X, M) in (Y, N). Then the following two categories are equivalent.*

- *The category of crystals on $Crys(X/S)^{log}$.*
- *The category of $\mathcal{O}_D$-modules with an integrable, quasi-nilpotent connection.*

Having recalled all these generalities, let us go back to the situation of interest to us. Namely let K be a complete non-archimedean field of characteristic zero, V be its ring of integers, and k be the residue field of V which is perfect of characteristic $p > 0$. Moreover fix a uniformizing parameter π for V and assume that $v_\pi(p) < p$. Let X_V be a smooth curve over V which admits an open immersion into a projective smooth curve C_V over V. Assume moreover that the complement D_V of X_V in C_V is étale and surjective over V. Let a subscript " $-_K$ " (resp. " $-_k$ ") denote the generic (resp. special) fiber of the object under study. Now one can apply the general machinery of previous sections to these curves over V, K, and k. In this section we are particularly interested in the de Rham case for X_V. So we consider the category $\mathcal{C}_{dR}$ of unipotent vector bundles on C_V with logarithmic connection along D_V. Then fixing a V-point $x \in X_V(V)$ gives us the functor $\mathcal{F}_{dR}$, and hence a pro-unipotent affine group scheme G_{dR}. Our aim now is to endow G_{dR} with a Frobenius action. To do this, we use the above techniques to obtain a description of G_{dR}, which depends only on the reduction modulo π of objects, i.e. on X_k, C_k, and D_k.

For any integer $m \geq 1$ consider the DP-ring $(V/(\pi^m), (\pi), \gamma)$, where γ is the canonical DP-structure. Equip the resulting DP-scheme with the trivial logarithmic structure and take the resulting DP-scheme with the trivial fine logarithmic structure as our base $\mathcal{S}_m := (S, L, \mathcal{I}, \gamma)_m$. Note that $\mathcal{O}_S$ is killed by a power of p as we always assumed before. Now let C_m, X_m, and D_m be respectively the pull backs of C_V, X_V, and D_V to $V/(\pi^m)$, and note that the new subscript " $-_1$ " does the same as the older one " $-_k$ ". On the other hand, using the normal crossing divisor D_m on the smooth curve C_m, one can put a fine logarithmic structure on C_m and consider it as an smooth scheme with fine logarithmic structure over $\mathcal{S}_m$. Furthermore for any $m \geq 1$, C_1 and its corresponding fine logarithmic structure coming from D_1 is also a scheme with fine logarithmic structure over $\mathcal{S}_m$ which can be embedded by a closed immersion into C_m. Now we can apply Theorem 2.3.7 to this situation. But note that the defining ideal sheaf of C_1 in C_m is the principal ideal sheaf $\pi\mathcal{O}_{C_m}$, and by the evident global versions of Proposition 2.3.3 and Remark 2.3.4 we see that the DP-envelope of C_1 in C_m is C_m itself. Finally it is obvious that under the equivalence given by Theorem 2.3.7 the subcategory of unipotent crystals, which are iterated extensions of the trivial crystal, on $Crys(C_1/S_m)^{log}$ is equivalent to the category of unipotent vector bundles on C_m with logarithmic connections along D_m , which are obviously quasi-nilpotent. Finally since by Grothendieck's algebraicity theorem, the category of unipotent vector bundles on C_V with logarithmic connection along D_V is equivalent with the category of compatible inverse systems of unipotent bundles on the inverse system $\{C_m\}_m$ with logarithmic connections along

$\{D_m\}_m$, one can put everything together and obtain the following

Theorem 2.3.8. *With all the notations and assumptions made above, the category* $\mathcal{C}_{cr}$ *of unipotent crystals on* $Crys(C_1/\mathrm{Spec}(V))^{log}$ *is equivalent to the category* $\mathcal{C}_{dR}$ *of unipotent vector bundles on* C_V *with logarithmic connections along* D_V*.*

Now suppose that k contains the finite field with $q = p^s$ elements $\mathbb{F}_q$, over which C_1 is defined. Consider $F := Fr^s$, where Fr is the absolute Frobenius. Then the curve C_1 admits a Frobenius action induced by F, and since all the above constructions are functorial, F also acts on the category of unipotent crystals on $Crys(C_1/\mathrm{Spec}(V))^{log}$. Now by applying Theorem 2.3.8 one can endow the category $\mathcal{C}_{dR}$ of unipotent vector bundles with logarithmic connection on X_V with a Frobenius action, which is induced by F and furthermore is compatible with tensor product. Moreover, it is easy to see that if we choose the point $x \in X_V(V)$ in such a way that its reduction modulo π, $\bar{x} \in X_k(k)$, belongs to $X_k(\mathbb{F}_q)$, then the above Frobenius action on $\mathcal{C}_{dR}$, induced by F, respects the fiber functor at the point x, and hence induces an action on the pro-unipotent affine group scheme G_{dR}. Let us denote this Frobenius action on G_{dR} by F, and write G_{cr} when we want to refer to the group G_{dR} equipped with this Frobenius action.

Chapter 3

Torsor Spaces and Period Maps

How can one study integral or rational points of a variety using its fundamental group? To explain our general idea in that direction, let us go down once more to the abelianization of the fundamental group and replace it by the Albanese variety. In general, for any projective variety V defined over a number field, there is an Albanese map $V \to A$ from V to its Albanese variety A which is an abelian variety defined over the field of definition of V. More important is that the Albanese map is also defined on the same field as V and A are, and hence maps the rational points of V to the rational points of A. Now since studying rational points of abelian varieties is easier than the general varieties, the naive idea is to study the rational points of A and the Albanese map in order to obtain information about the rational points of V. This principle was applied by Chabauty, long time ago, to attack Mordell's conjecture (see [5]), and has been considered for some time as a powerful tool in studying rational points of projective hyperbolic curves (when the Albanese map has the nice property of being a closed immersion). Here we are trying to develop and use the non-abelian version of these ideas. But first of all we need to build the non-abelian versions of the Albanese variety and the Albanese map, the so called period maps. This is what we are going to do in this chapter.

3.1 Representability

So far we have discussed different notions of pro-unipotent fundamental groups. Some of them have extra structures, namely Galois action, Hodge filtration, Frobenius action, and so on, and all of them, being (pro-)unipotent, admit separated descending central series. On the other hand, we have seen in Remark 2.1.13 that there are natural path spaces which are torsors over these fundamental groups. Since the additional structures mentioned above can be put on the whole unipotent fundamental groupoid, these path torsors can also be furnished with these extra structures. These structures on path torsors are compatible with the corresponding structures on the fundamental groups and the torsor structures. Generally isomorphism classes of torsors are classified by a suitable (not necessarily abelian) cohomology set. As we will see latter, in order to define the period maps we need to have some proper spaces which represent these cohomology sets. To attack this problem, following Faltings [17, section 3], we consider the following general situation.

Let K be a topological field of characteristic zero, which is complete and Hausdorff. Fix a pro-finite group Γ, and finite dimensional continuous representations $\mathbb{L}_1, \ldots, \mathbb{L}_r$ of Γ over K. We are interested in continuous representations $\mathbb{L}$ of Γ over K, which admit a Γ-stable filtration

$$(0) = W_0(\mathbb{L}) \subset W_1(\mathbb{L}) \subset \cdots \subset W_r(\mathbb{L}) = \mathbb{L},$$

such that for each $1 \leq i \leq r$ one has

$$W_i(\mathbb{L})/W_{i-1}(\mathbb{L}) \cong_\Gamma \mathbb{L}_i.$$

Consider the following unipotent group

$$\mathbb{G} := \begin{pmatrix} I & \mathrm{Hom}_K(\mathbb{L}_2, \mathbb{L}_1) & \ldots & \mathrm{Hom}_K(\mathbb{L}_r, \mathbb{L}_1) \\ 0 & I & \ldots & \mathrm{Hom}_K(\mathbb{L}_r, \mathbb{L}_2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \end{pmatrix}$$

Obviously Γ acts on $\mathbb{G}$ in a natural fashion, namely

$$\gamma(g) := \mathrm{Diag}(\rho_i(\gamma)).g.\mathrm{Diag}(\rho_i(\gamma)^{-1}),$$

where

$$\rho_i : \Gamma \to GL(\mathbb{L}_i)$$

is the fixed action of Γ on $\mathbb{L}_i$. We recall very briefly that the set $Z^1(\Gamma, \mathbb{G})$ of continuous 1-cocycles of Γ with values in $\mathbb{G}$ is the set of continuous maps $c : \Gamma \to \mathbb{G},\ \gamma \mapsto c_\gamma$, such that $c_{\gamma_1\gamma_2} = c_{\gamma_1}\gamma_1(c_{\gamma_2})$. The group $\mathbb{G}$ acts on $Z^1(\Gamma, \mathbb{G})$ via the rule

$$(g * c)_\gamma := g^{-1}c_\gamma\gamma(g).$$

When two 1-cocycles lie in the same orbit of this action, we say that they differ by a coboundary. The non-abelian cohomology set $H^1(\Gamma, \mathbb{G})$ is then defined to be the orbit space $Z^1(\Gamma, \mathbb{G})/\mathbb{G}$. Note that $H^1(\Gamma, \mathbb{G})$ is a pointed set, namely we have the orbit of the trivial 1-cocycle, which will be called the trivial class. First of all we have the following:

Lemma 3.1.1. *The isomorphism classes of representations* $\mathbb{L}$ *as above are in bijection with the elements of the non-abelian cohomology set* $H^1(\Gamma, \mathbb{G})$.

Proof. One can use the filtration $W_\bullet$ to obtain an isomorphism

$$\mathbb{L} \cong \bigoplus_{i=1}^{r} \mathbb{L}_i$$

as K-vector spaces. Now since the action by Γ respects $W_\bullet$, if we choose a basis for each $\mathbb{L}_i$ and denote the fixed action of Γ on $\mathbb{L}_i$ by $\rho_i : \Gamma \to GL(\mathbb{L}_i)$, we can write the action of $\gamma \in \Gamma$ on $\mathbb{L}$ as the product of an element g_γ of $\mathbb{G}$ with $\mathrm{Diag}(\rho_i)$. Namely we have

$$\gamma \mapsto \begin{pmatrix} \rho_1(\gamma) & \varphi_\gamma^{2,1} & \dots & \varphi_\gamma^{r,1} \\ 0 & \rho_2(\gamma) & \dots & \varphi_\gamma^{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r(\gamma) \end{pmatrix}$$

which can be factored as

$$\begin{pmatrix} I & \psi_\gamma^{2,1} & \dots & \psi_\gamma^{r,1} \\ 0 & I & \dots & \psi_\gamma^{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{pmatrix} \begin{pmatrix} \rho_1(\gamma) & 0 & \dots & 0 \\ 0 & \rho_2(\gamma) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r(\gamma) \end{pmatrix}$$

where for any $1 \leq i < j \leq r$

$$\varphi_\gamma^{j,i} \text{ and } \psi_\gamma^{j,i} = \varphi_\gamma^{j,i} \rho_j(\gamma)^{-1} \in \mathrm{Hom}_K(\mathbb{L}_j, \mathbb{L}_i).$$

It is an easy and straightforward computation then to check that $\gamma \mapsto g_\gamma$ is a continuous 1-cocycle, and changing the chosen bases for $\mathbb{L}_i$'s will change it by a coboundary. This gives a map from the isomorphism classes of representations like above to $H^1(\Gamma, \mathbb{G})$. The inverse map is also plain. Assume given a continuous 1-cocyle $\gamma \mapsto c_\gamma$, one can easily compute that the map $\gamma \mapsto c_\gamma.\mathrm{Diag}(\rho_i(\gamma))$ gives a continuous representation of Γ on $\bigoplus_{i=1}^r \mathbb{L}_i$, and if we consider the trivial filtration on this direct sum, the resulting object has the property that if we apply the above mentioned procedure on it, we recover the 1-cocycle c. $\square$

The above classical result is nice, but far from being enough for us. We want to consider this non-abelian cohomology set as a functor on the category of finitely generated K-algebras and prove its representability under some conditions. To set up the situation, we fix some conventions. Since K is a topological field, every finite dimensional vector space over K has a natural topology induced from K. On infinite dimensional vector spaces, following Kim [24], we put the inductive limit topology induced by finite dimensional sub-vector spaces, i.e. a subset is open if and only if its intersection with any finite dimensional sub-vector space V is open in V. In particular all K-algebras and all modules over them can be topologized this way. In this topology any compact subset lies in a finite dimensional sub-vector space (see [24, Lemma 4]), and in particular any orbit of any continuous representation of Γ lies in a finite dimensional sub-vector space.

The functor we are interested in, associates to a finitely generated K-algebra R, the set of all continuous representations of Γ on a free R-module M, such that M admits a Γ-stable filtration

$$(0) = W_0(M) \subset W_1(M) \subset \cdots \subset W_r(M) = M,$$

where for each $1 \leq i \leq r$,

$$W_i(M)/W_{i-1}(M) \cong_\Gamma \mathbb{L}_i \otimes_K R.$$

This, by the same argument as in Lemma 3.1.1, is the same as the functor which associates $H^1(\Gamma, \mathbb{G}(R))$ to R, where $\mathbb{G}(R) := \mathbb{G} \otimes_K R$. In order to study this functor, we first study the functor of continuous 1-cocycles, namely $R \mapsto Z^1(\Gamma, \mathbb{G}(R))$. Fixing any finite set of elements $\gamma_1, \ldots, \gamma_s \in \Gamma$ gives us an evaluation map $ev : Z^1(\Gamma, \mathbb{G}) \to \mathbb{G}^s$, which sends a continuous 1-cocycle c to the s-tuple $(c_{\gamma_1}, \ldots, c_{\gamma_s}) \in \mathbb{G}^s$. Our aim is to prove that for sufficiently large s and suitable choice of elements γ_i, the evaluation map ev gives a closed immersion of $Z^1(\Gamma, \mathbb{G})$ into $\mathbb{G}^s$. For this we must assume that $H^1(\Gamma, \mathrm{Hom}_K(\mathbb{L}_j, \mathbb{L}_i))$ is finite dimensional for every $i < j$. In the rest of this section we denote $\mathrm{Hom}_K(\mathbb{L}_j, \mathbb{L}_i)$ by $H_{j,i}$ for short. Now we have

Theorem 3.1.2. *If for all $i < j$*

$$\dim_K(H^1(\Gamma, H_{j,i})) < \infty,$$

then for sufficiently large s, there are elements $\gamma_1, \ldots, \gamma_s \in \Gamma$ such that the above mentioned evaluation map gives a closed immersion of $Z^1(\Gamma, \mathbb{G})$ into $\mathbb{G}^s$. In particular the functor $Z^1(\Gamma, \mathbb{G}(R))$ on finitely generated K-algebras is representable by an affine scheme.

Proof. The proof is by induction on r. For $r = 2$ note that

$$\mathbb{G} \cong H_{2,1}$$

is a finite dimensional vector group over K. Since $H^1(\Gamma, \mathbb{G})$ is finite dimensional as well, we conclude that $Z^1(\Gamma, \mathbb{G})$ is also a finite dimensional vector group over K. The representability assertion is clear now, but let us prove the first claim of the theorem. The evaluation map is K-linear in this case, and since for any non-trivial 1-cocycle c, there exists an element $\gamma \in \Gamma$ such that $c_\gamma \neq 1$, we can reduce the dimension of the kernel of the evaluation map at each step, and hence make the evaluation map injective after finitely many steps.

Now let $\mathbb{G}_1$ (resp. $\mathbb{G}_2$, resp. $\mathbb{G}_{1,2}$) be the group obtained by removing the first row and first column (resp. last row and last column, resp. first and last rows and columns) of $\mathbb{G}$. Note that all these groups are naturally isomorphic to closed sub-groups of $\mathbb{G}$, and on the other hand, $\mathbb{G}_1$ (resp. $\mathbb{G}_2$, resp. $\mathbb{G}_{1,2}$) is the analogue of $\mathbb{G}$ if we had started with representations $\mathbb{L}_2, \ldots, \mathbb{L}_r$ (resp. $\mathbb{L}_1, \ldots, \mathbb{L}_{r-1}$, resp. $\mathbb{L}_2, \ldots, \mathbb{L}_{r-1}$). By the induction hypothesis $Z^1(\Gamma, \mathbb{G}_i(R))$ is representable by a K-algebra R_i and can be embedded by a closed immersion into $\mathbb{G}^{s_i}$ $(i = 1, 2)$. The product scheme, which corresponds to $R_1 \otimes_K R_2$, represents pairs of cocycles $Z^1(\Gamma, \mathbb{G}_1) \times Z^1(\Gamma, \mathbb{G}_2)$, and can be embedded by a closed immersion into $\mathbb{G}^{s_1+s_2}$. Since having the same restriction on $\mathbb{G}_{1,2}$ is a closed condition on this scheme, if we consider the functor $\mathcal{F}$ which sends a K-algebra R to the set of pairs of cocycles in $Z^1(\Gamma, \mathbb{G}_1(R)) \times Z^1(\Gamma, \mathbb{G}_2(R))$ with same restriction on $\mathbb{G}_{1,2}$, we have shown that $\mathcal{F}$ is representable and is a closed subscheme of some finite power of $\mathbb{G}$. Now let $\mathbb{H}$ be defined by the following exact sequence of group schemes over K

$$0 \to H_{r,1} \to \mathbb{G} \to \mathbb{H} \to 0.$$

Clearly $\mathcal{F}(R)$ is nothing other than $Z^1(\Gamma, \mathbb{H}(R))$, which by above argument is representable by a K-algebra S. Recall that for vector groups like $H_{r,1}$, one can define the higher cohomology groups and obtain an exact sequence ending with $H^2(\Gamma, H_{r,1})$ from a short exact sequence like above. Now consider the following commutative diagram of functors

$$\begin{tikzcd}
Z^1(\Gamma, H_{r,1}) \arrow[r, "g"] \arrow[d] & Z^1(\Gamma, \mathbb{G}) \arrow[r, "f"] \arrow[d] & Z^1(\Gamma, \mathbb{H}) \arrow[d] & \\
H^1(\Gamma, H_{r,1}) \arrow[r] & H^1(\Gamma, \mathbb{G}) \arrow[r] & H^1(\Gamma, \mathbb{H}) \arrow[r] & H^2(\Gamma, H_{r,1})
\end{tikzcd}$$

Our goal is to prove the representability of the functor $Z^1(\Gamma, \mathbb{G})$. For that, we first consider the functor $\mathcal{G}$ which is the image of f, i.e. sends a K algebra R to the subset of those continuous 1-cocycles in $Z^1(\Gamma, \mathbb{H})$ which are liftable (not necessarily in a unique way!) to $Z^1(\Gamma, \mathbb{G})$. First note that since $Z^1(\Gamma, \mathbb{H})$

is representable by S, we have a universal 1-cocycle $\xi \in Z^1(\Gamma, \mathbb{H}(S))$, which corresponds to the identity homomorphism in $\mathrm{Hom}_{K-alg}(S,S)$. After fixing a basis for $H^2(\Gamma, H_{r,1})$, the image of ξ in

$$H^2(\Gamma, H_{r,1} \otimes_K S) \cong H^2(\Gamma, H_{r,1}) \otimes_K S$$

can be written in terms of this basis with coefficients in S. Let J be the ideal in S generated by these coefficients, then the image ξ' of ξ in $Z^1(\Gamma, \mathbb{H}(S/J))$ maps to zero in $H^2(\Gamma, H_{r,1}) \otimes_K S/J$, whence can be lifted to $Z^1(\Gamma, \mathbb{G}(S/J))$. Using Yoneda's isomorphism and the universal construction of ξ', one can easily check that ξ' is a universal element and show that $\mathcal{G}$ is representable by the K-algebra S/J. Certainly $\mathcal{G}$ is a closed sub-functor of $\mathcal{F}$ and hence can be embedded by a closed immersion into some power of $\mathbb{G}$.

The final step of the proof is not difficult now. First note that if we could lift ξ' uniquely to $Z^1(\Gamma, \mathbb{G}(S/J))$, it was already clear that $Z^1(\Gamma, \mathbb{G})$ is representable by S/J. In our case, different lifts of ξ' form a principal homogeneous space over the image of $Z^1(\Gamma, H_{r,1})$ under g. This space, by assumption is a finite dimensional vector group over K, and hence we can fix the non-unicity of the lift exactly with the same idea as in the induction basis. □

Note that, having proved that $Z^1(\Gamma, \mathbb{G})$ is representable by an affine scheme over K, we already proved that

$$H^1(\Gamma, \mathbb{G}) = Z^1(\Gamma, \mathbb{G})/\mathbb{G}$$

is an algebraic stack. But we want more, hence we put further assumptions in order to obtain representability by a scheme. The idea is that in order to have a nice quotient in the category of schemes, we need at least that the group $\mathbb{G}$ act freely on $Z^1(\Gamma, \mathbb{G})$, and this can be achieved if we assume that $H^0(\Gamma, H_{j,i})$ vanishes for all $i < j$. This assumption in fact is sufficient for our purpose, more precisely, we have the following

Corollary 3.1.3. *Under the assumptions of Theorem 3.1.2, if one assumes moreover that*

$$H^0(\Gamma, H_{j,i}) = 0$$

for all $i < j$, then $H^1(\Gamma, \mathbb{G})$ is representable by an affine scheme over K.

Proof. Recall that Γ acts in a natural way on $H_{j,i}$'s, and hence every $\gamma \in \Gamma$ induces a K-linear map on these K-vector spaces. Under the above vanishing assumption on Γ-invariants, one can choose finitely many elements $\{\gamma_1, \ldots, \gamma_{s'}\} \subset \Gamma$ such that for all $i < j$ there exists no nonzero element in

$H_{j,i}$ which is fixed by all γ_n , $1 \leq n \leq s'$ (note that there are finitely many indices i and j, and all $H_{j,i}$ are finite dimensional over K). That is, for every $i < j$, one obtains an injection

$$\vartheta^{j,i} : H_{j,i} \hookrightarrow H_{j,i}^{s'}$$

which sends an element $\varphi^{j,i}$ to the s'-tuple $(\gamma_n(\varphi^{j,i}) - \varphi^{j,i})_{n=1}^{s'}$. On the other hand, by Theorem 3.1.2, one can enlarge this set of elements to a bigger, but still finite, set of elements $\{\gamma_1, \dots, \gamma_s\} \subset \Gamma$, to obtain a closed immersion

$$ev : Z^1(\Gamma, \mathbb{G}) \hookrightarrow \mathbb{G}^s$$

at the same time. Now for any $i < j$ fix a complementary sub-vector space $\mathbb{V}^{j,i} \subset H_{j,i}^s$ to the image of $\vartheta^{j,i}$, and let

$$Z^1(\Gamma, \mathbb{G})' \subset Z^1(\Gamma, \mathbb{G})$$

be the set of 1-cocycles c such that all entries of $ev(c)$ lie in $\mathbb{V}^{j,i}$. Obviously $Z^1(\Gamma, \mathbb{G})'$ is a closed sub-functor of $Z^1(\Gamma, \mathbb{G})$, and hence is representable by an affine scheme over K. Our aim is to show that $Z^1(\Gamma, \mathbb{G})'$ forms a complete set of representatives for $\mathbb{G}$ action on $Z^1(\Gamma, \mathbb{G})$, and obtain the representability of $Z^1(\Gamma, \mathbb{G})/\mathbb{G}$.

First we show that for any 1-cocycle $c \in Z^1(\Gamma, \mathbb{G})'$ and any $g \in \mathbb{G}$, $g * c$ lies in $Z^1(\Gamma, \mathbb{G})'$ if and only if $g = 1_{\mathbb{G}}$. To do this assume

$$g = \begin{pmatrix} I & \varphi^{2,1} & \dots & \varphi^{r,1} \\ 0 & I & \dots & \varphi^{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{pmatrix},$$

and

$$c_{\gamma_n} = \begin{pmatrix} I & \psi_{\gamma_n}^{2,1} & \dots & \psi_{\gamma_n}^{r,1} \\ 0 & I & \dots & \psi_{\gamma_n}^{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{pmatrix} \quad \text{for all } 1 \leq n \leq s.$$

By definition and a straightforward computation, one can see that for any $1 \leq i \leq r-1$ the $(i, i+1)$-entry of $(g * c)_{\gamma_n}$ is

$$-\varphi^{i+1,i} + \psi_{\gamma_n}^{i+1,i} + \gamma_n(\varphi^{i+1,i}).$$

If both c and $g * c$ lie in $Z^1(\Gamma, \mathbb{G})'$, we conclude that $\varphi^{i+1,i} = 0$ for all $1 \leq n \leq r-1$. Taking this into account, we can argue the same for $(i, i+2)$-entries and so on to show that $g = 1_{\mathbb{G}}$.

Now in order to finish the prove, we show that for any 1-cocycle c in $Z^1(\Gamma, \mathbb{G})$ there exists an element $g \in \mathbb{G}$, which is unique by the above argument and such that $g * c$ lies in $Z^1(\Gamma, \mathbb{G})'$. Assume that for any $1 \leq n \leq s$, c_{γ_n} has the above matrix form, and consider again the entries immediately above the diagonal. Since $\mathbb{V}^{i+1,i}$ and $\vartheta^{i+1,i}(H_{i+1,i})$ form complementary sub-spaces of $H^s_{i+1,i}$, there are unique elements $\alpha^{i+1,i} \in \mathbb{V}^{i+1,i}$ and $\beta^{i+1,i} \in H_{i+1,i}$ such that for all $1 \leq n \leq s$ and all $1 \leq i \leq r-1$ one has

$$\psi^{i+1,i}_{\gamma_n} = \alpha^{i+1,i} + (\gamma_n(\beta^{i+1,i}) - \beta^{i+1,i}).$$

Now if one considers the element $g_1 \in \mathbb{G}$ with 1 as diagonal entries, $-\beta^{i+1,i}$ as $(i, i+1)$-entries, and 0 elsewhere, it is easy to check that the entries immediately above diagonal in $g_1 * c$ lie in $\mathbb{V}^{i+1,i}$. Similarly we can construct an element $g_2 \in \mathbb{G}$ with nonzero entries only on the diagonal and $(i, i+2)$-entries in such a way that

$$(g_2 * g_1 * c)_{i,j} \in \mathbb{V}^{j,i}\ ,\quad \forall\ 1 \leq i < j \leq i+2 \leq r.$$

Continuing this $r-1$ times and putting $g = g_{r-1}.\dots.g_1$, we are done. □

We have shown so far some representability results, but just for the special unipotent group $\mathbb{G}$. We are interested in its closed Γ-stable sub-group schemes as well. It is not very difficult now to extend the above results to these subgroup schemes, but to do that, we need the following basic lemma:

Lemma 3.1.4. *Let G over K be an algebraic unipotent group scheme, and $H \subset G$ be a closed sub-group scheme. Then the underlying scheme of G/H is an affine space.*

Proof. First let us show that the underlying scheme of an arbitrary algebraic unipotent group scheme over K is an affine space. One can prove this by considering the Lie algebra $\mathfrak{g}$ of G, which is a finite dimensional vector space over K, and use the general fact that $exp : \mathfrak{g} \to G$ is a group isomorphism when we equip $\mathfrak{g}$ with Baker-Campbell-Hausdorff multiplication rule (see [10, Appendix]). But we give the following direct argument. Since G is unipotent, its descending central series is finite, hence we have

$$(1) = Z^t(G) \subset Z^{t-1}(G) \subset \cdots \subset Z^0(G) = G,$$

where t is the unipotent class of G, and for each $0 \leq i \leq t-1$,

$$Z^i(G)/Z^{i+1}(G) \cong \mathbb{G}_a^{n_i}$$

is a finite dimensional vector group over K. Now we prove the claim by induction on t. For $t = 1$, there is nothing to prove. In general, we have the following short exact sequence of groups

$$0 \to Z^{t-1}(G) = \mathbb{G}_a^{n_{t-1}} \to G \to G/Z^{t-1}(G) \to 0.$$

Whence G is a $\mathbb{G}_a^{n_{t-1}}$-torsor over $G/Z^{t-1}(G)$, which has an affine space as its underlying scheme by induction hypothesis. Now since any affine space is cohomologically trivial, we have

$$H^1(G/Z^{t-1}(G), \mathbb{G}_a) = 0.$$

This says that the above short exact sequence splits and we are done. Note that by exactly the same argument, we could prove that any unipotent torsor over an affine space is an affine space. Now to prove the assertion of the lemma, let $N(H)$ denote the normalizer of H in G. Since $N(H)/H$ is a unipotent group, by considering the short exact sequence

$$0 \to N(H)/H \to G/H \to G/N(H) \to 0,$$

it suffices to prove that $G/N(H)$ is an affine space. We can continue this, and since in a unipotent group the normalizer of a subgroup strictly contains the subgroup, we are done after finitely many steps. $\square$

Remark 3.1.5. Note that in proving the above lemma, we used the fact that $\mathrm{char}(K) = 0$ only when we said that the sub-quotients in descending central series are powers of $\mathbb{G}_a$. So one can do the same argument for unipotent group schemes over any perfect field if one assumes in positive characteristic that the unipotent group scheme under study is connected and smooth over the base field. ♠

Using this lemma, we can show that the representability of the first cohomology group for a unipotent group scheme is inherited by its stable closed subgroups. More precisely,

Theorem 3.1.6. *Let G be a unipotent group scheme over K, on which Γ acts and $H \subset G$ be a closed Γ-stable subgroup. If $H^1(\Gamma, G)$ is representable by an affine scheme over K, then so is $H^1(\Gamma, H)$.*

Proof. Assume first that H is a normal Γ-stable subgroup of G. Then from the short exact sequence

$$0 \to H \to G \to G/H \to 0$$

of unipotent group schemes with Γ-action, one obtains the following last terms of the exact sequence of cohomology sets

$$\ldots \to (G/H)^{\Gamma} \to H^1(\Gamma, H) \xrightarrow{\psi} H^1(\Gamma, G) \xrightarrow{\varphi} H^1(\Gamma, G/H).$$

We claim now that cohomology classes in $H^1(\Gamma, H)$ are the same as the pairs consisting of a cohomology class $\xi \in H^1(\Gamma, G)$ plus a trivialization of $\varphi(\xi) \in H^1(\Gamma, G/H)$, i.e. an element $gH \in G/H$ such that for all $\gamma \in \Gamma$ one has

$$\varphi(\xi)_\gamma = g^{-1}\gamma(g)H.$$

To any cohomology class $\zeta \in H^1(\Gamma, H)$, we associate the pair $(\psi(\zeta), 1_G H)$. For the converse correspondence, assume a pair (ξ, gH) is given such that $\xi \in H^1(\Gamma, G)$ and for any $\gamma \in \Gamma$ one has $\xi_\gamma H = g^{-1}\gamma(g)H$. By definition, $\xi = g * \xi = g\xi\gamma(g^{-1})$ in $H^1(\Gamma, G)$. But the map

$$\gamma \mapsto g\xi_\gamma\gamma(g^{-1})$$

takes values in H, and hence gives rise to a cohomology class $\zeta \in H^1(\Gamma, H)$ which we associate to the pair (ξ, gH). It is plain now to check that these correspondences are inverse to each other, whence give the desired bijection. On the other hand, note that given a cohomology class ξ in $H^1(\Gamma, G)$, the set of trivializations of $\varphi(\xi)$ is either empty, which is representable by the empty scheme, or is a homogeneous space over $(G/H)^{\Gamma}$, which is representable by an affine K-scheme (note that the kernel of the map

$$(G/H)^{\Gamma} \to H^1(\Gamma, H)$$

is a closed subgroup of $(G/H)^{\Gamma}$ and hence by Lemma 3.1.4 the homogeneous space of trivializations of $\varphi(\xi)$ is representable by an affine space over K). This gives the representability of the functor $H^1(\Gamma, H)$.

For an arbitrary Γ-stable subgroup H, let $N(H)$ be the normalizer of H in G. Clearly $N(H)$ is Γ-stable as well and hence if we knew that $H^1(\Gamma, N(H))$ is representable, we could apply the above argument to prove representability of $H^1(\Gamma, H)$. This reduces the problem to showing the representability of $H^1(\Gamma, N(H))$. Again we use the fact that in a unipotent group, normalizer of a proper subgroup strictly contains it to finish the proof of theorem. $\square$

Remark 3.1.7. One could prove the above theorem using the more fancy language of topoi. Namely, continuing with the notations and hypotheses of the above theorem, one can consider the category of sets equipped with a continuous action by Γ, and put the canonical Grothendieck topology on it, i.e. the one in which coverings are effective epimorphisms. Since G and H are group objects in this situs, one can consider G-torsors and H-torsors in the associated topos. On the other hand, the space G/H admits a natural action by the group G, and hence to any G-torsor $\mathcal{T}$ is the associated bundle $\mathcal{T} \times^G G/H$ with fiber G/H. Now one can reinterpret the proof of the above theorem in this more general setting and prove that there is a bijection between H-torsors and G-torsors together with a section of the mentioned associated bundle. This bijection can be constructed as follows. For any H-torsor $\mathcal{T}_H$, the associated G-torsor is $\mathcal{T}_G := \mathcal{T}_H \times^H G$, and since $\mathcal{T}_H$ is locally isomorphic to H, the associated bundle of $\mathcal{T}_G$, namely

$$\mathcal{T}_G \times^G G/H \cong \mathcal{T}_H \times^H G/H,$$

locally admits the section $\mathbf{1}_H \times \mathbf{1}_G.H$. Finally one can check easily that this sections can be glued together to form a global section of the associated bundle. Conversely, let $\mathcal{T}_G$ be a G-torsor and s be a section for $\mathcal{T}_G \times^G G/H$. Then the corresponding H-torsor is the inverse image in $\mathcal{T}_G$ of $im(s)$ via the map $t \mapsto t \times \mathbf{1}_G.H$ from $\mathcal{T}_G$ to $\mathcal{T}_G \times^G G/H$. One can conclude the claim now, by noticing that since G/H is an affine space by Lemma 3.1.4, the space of sections of bundles with fiber G/H is representable, and hence H-torsors form a stack or a scheme if G-torsors do so. Let us mention again that the proof given above is essentially the same argument in a more down to earth language, but this proof has the advantage that one doesn't need to deal separately with the cases where H is normal in G or not. ♡

3.2 Crystalline Torsors

In this section we assume that Γ is the absolute Galois group of a finite extension K of $\mathbb{Q}_p$ (more generally one can consider a complete p-adic field of characteristic zero with perfect residue field of characteristic p, but finite extensions of $\mathbb{Q}_p$ are enough for the moment), and we are going to study finite dimensional Γ-representations over $\mathbb{Q}_p$. Among these representations we want to specify the so called crystalline ones. Our aim then is to show that crystalline torsors in $H^1(\Gamma, \mathbb{G})$ can be parametrized by an affine scheme over K.

More precisely let k be a finite field of characteristic p, $V_0 := W(k)$ be the ring of Witt vectors over k, and K_0 be the field of fractions of V_0. Then K_0 is a finite unramified extension of $\mathbb{Q}_p$. Clearly the Frobenius automorphism of k extends to $W(k)$, by functoriality, and hence to a Frobenius automorphism $\mathbf{\Phi}_0$ of K_0 over $\mathbb{Q}_p$. Take K to be a totally ramified extension of K_0, and denote by V the integral closure of V_0 in K. Note that we can assume all these without loss of generality, i.e. every finite extension of $\mathbb{Q}_p$ has this form. Finally let

$$\Gamma := \mathrm{Gal}(\bar{K}/K)$$

be the absolute Galois group of K. In order to study and classify finite dimensional representations of Γ over $\mathbb{Q}_p$, Fontaine has introduced some rings of periods. Among them we mention B_{cr} and B_{dR} (for a brief review of definitions of B_{cr} and B_{dR}, see section 4.1). B_{cr} is a K_0-algebra equipped with an action by Γ, and also a Frobenius semi-linear endomorphism $\mathbf{\Phi}$. B_{dR} is a K-algebra which contains B_{cr} and has a decreasing Hodge-filtration $F^\bullet$, with respect to which it is complete. In general the importance of these rings of periods is that they establish connections between the rigid tensor category $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$ of finite dimensional $\mathbb{Q}_p$-representations of Γ and other categories. For example consider the rigid tensor category $\mathcal{FM}$ whose objects are triples $(E, \mathbf{\Phi}, F^\bullet)$, consisting of a finite dimensional K_0-vector space E, a Frobenius semi-linear automorphism $\mathbf{\Phi}$ of E, and a finite decreasing filtration $F^\bullet$ on $\boldsymbol{E} := E \otimes_{K_0} K$.

There is a common enlargement of $\mathcal{FM}$ and $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$, which contains both as full subcategories. Namely let $\mathcal{FGM}$ be the rigid tensor category whose objects are triples $(\mathbb{E}, \mathbf{\Phi}, F^\bullet)$ which contain the same data as triples in $\mathcal{FM}$ plus a K_0-linear action of Γ on $\mathbb{E}$. Note that this Γ-action is required to be compatible with other structures, i.e. it is induced by a group homomorphism from Γ to $\mathrm{Aut}_{\mathcal{FM}}((\mathbb{E}, \mathbf{\Phi}, F^\bullet))$. The full embeddings from $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$ and $\mathcal{FM}$ into $\mathcal{FGM}$ are given as follows. To any finite dimensional $\mathbb{Q}_p$-representation $\mathbb{L}$ of Γ, associate the triple consisting of $\mathbb{L} \otimes_{\mathbb{Q}_p} K_0$, with induced Γ-action as the first coordinate, $Id_{\mathbb{L}} \otimes \mathbf{\Phi}_0$ as the second, and the trivial filtration on $\boldsymbol{E} = \mathbb{L} \otimes_{\mathbb{Q}_p} K$, i.e. $F^0(\boldsymbol{E}) = \boldsymbol{E}$ and $F^1(\boldsymbol{E}) = 0$, as the third. For any object $(E, \mathbf{\Phi}, F^\bullet)$ in $\mathcal{FM}$, associate the same triple in $\mathcal{FGM}$ where E has been equipped with the trivial Γ-action.

Although B_{cr} has a K_0-linear action by Γ, a Frobenius semi-linear automorphism, and a decreasing Hodge filtration $F^\bullet$ on $B_{cr} \otimes_{K_0} K$ inherited from B_{dR}, it is not an object in $\mathcal{FGM}$, simply because it is not finite dimensional over K_0. But it still helps us to construct important functors between $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$ and $\mathcal{FM}$. For a finite dimensional $\mathbb{Q}_p$-representation $\mathbb{L}$ of Γ, let $\widetilde{\mathbb{L}}$ be its image in $\mathcal{FGM}$. One can then show that

$$\mathfrak{Fil}(\mathbb{L}) := (\widetilde{\mathbb{L}} \otimes B_{cr})^\Gamma$$

has finite dimension over K_0 and hence is an object in $\mathcal{FM}$. Note that $\widetilde{\mathbb{L}} \otimes B_{cr}$ is an abuse of notation, since B_{cr} is not in any of the tensor categories we considered, but the point is that for any object $(\mathbb{E}, \mathbf{\Phi}, F^\bullet)$ in $\mathcal{FGM}$, we can consider the diagonal Γ and $\mathbf{\Phi}$ actions on $\mathbb{E} \otimes_{K_0} B_{cr}$ and the induced Hodge filtration on

$$(\mathbb{E} \otimes_{K_0} K) \otimes_K (B_{cr} \otimes_{K_0} K) \cong (\mathbb{E} \otimes_{K_0} B_{cr}) \otimes_{K_0} K.$$

Similarly for any object $(E, \mathbf{\Phi}, F^\bullet)$ in $\mathcal{FM}$, considered as an object in $\mathcal{FGM}$, one can show that

$$\mathfrak{Gal}((E, \mathbf{\Phi}, F^\bullet)) := (E \otimes_{K_0} B_{cr})^{\mathbf{\Phi}=1} \cap F^0(E \otimes_{K_0} B_{cr} \otimes_{K_0} K)$$

is a finite dimensional $\mathbb{Q}_p$-representation of Γ, and hence is in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$. One says that a finite dimensional Γ-representation $\mathbb{L}$ over $\mathbb{Q}_p$ is crystalline if

$$\dim_{\mathbb{Q}_p}(\mathbb{L}) = \dim_{K_0}(\mathfrak{Fil}(\mathbb{L})),$$

and similarly an object $(E, \mathbf{\Phi}, F^\bullet)$ in $\mathcal{FM}$ is called B_{cr}-admissible if

$$\dim_{K_0}(E) = \dim_{\mathbb{Q}_p}(\mathfrak{Gal}((E, \mathbf{\Phi}, F^\bullet))).$$

An important result of Fontaine's theory is that the functors $\mathfrak{Fil}$ and $\mathfrak{Gal}$ establish an equivalence between the category of crystalline $\mathbb{Q}_p$-representations of Γ and the category of B_{cr}-admissible objects in $\mathcal{FM}$. A crystalline $\mathbb{Q}_p$-representation $\mathbb{L}$ of Γ and a B_{cr}-admissible triple $(E, \mathbf{\Phi}, F^\bullet)$ in $\mathcal{FM}$ are said to be associated to each other, if they correspond to each other under this equivalence. One can show this is the case if and only if there exists a B_{cr}-isomorphism

$$\mathbb{L} \otimes_{\mathbb{Q}_p} B_{cr} \cong_{B_{cr}} E \otimes_{K_0} B_{cr},$$

which respects Γ-action, $\mathbf{\Phi}$, and after the extension of scalars to K, the Hodge filtration (for details and much more about these, see [18] or [19]). There are two crucial properties which make B_{cr} so useful. The first one is that $B_{cr}^{\Gamma} = K_0$, and the other one is the following

Theorem 3.2.1. *[19, Prop. 5.5.] One has $B_{cr}^{\mathbf{\Phi}=1} \cap F^0(B_{dR}) = \mathbb{Q}_p$. One even has a stronger result which says that the following sequence is exact*

$$0 \to \mathbb{Q}_p \to B_{cr} \xrightarrow{(\mathbf{\Phi}-1,i)} B_{cr} \bigoplus \big(B_{dR}/F^0(B_{dR})\big) \to 0,$$

where $i : B_{cr} \hookrightarrow B_{dR} \twoheadrightarrow B_{dR}/F^0(B_{dR})$ is the evident map (see for example [18]).

We also need the notion of a $(K_0, \mathbf{\Phi})$-module. By a $(K_0, \mathbf{\Phi})$-module E, we mean a finite dimensional K_0-vector space E equipped with a Frobenius semi-linear automorphism $\mathbf{\Phi}$. As an immediate corollary of the above theorem, we have

Corollary 3.2.2. *Let*

$$0 \to E_1 \to E \to E_2 \to 0$$

be a short exact sequence of $(K_0, \mathbf{\Phi})$-modules, and $F^\bullet$ a finite decreasing filtration on $E \otimes_{K_0} K$. If the triple consisting of E_i and induced filtration from $F^\bullet$ on $E_i \otimes_{K_0} K$ is B_{cr}-admissible for $i = 1, 2$, then so is the triple consisting of E and $F^\bullet$.

Proof. For $i = 1, 2$, let $\mathbb{L}_i$ be the associated representation to E_i equipped with induced filtration. By Theorem 3.2.1 we have the following short exact sequences for $i = 1, 2$

$$0 \to \mathbb{L}_i \to E_i \otimes_{K_0} B_{cr} \xrightarrow{(\mathbf{\Phi}-1,i)} (E_i \otimes_{K_0} B_{cr}) \bigoplus \left(\frac{E \otimes_{K_0} B_{dR}}{F^0(E \otimes_{K_0} B_{dR})} \right) \to 0.$$

Consider the $\mathbb{Q}_p$-representation $\mathbb{L} := \mathfrak{Gal}((E, \mathbf{\Phi}, F^\bullet))$, i.e. $\mathbb{L}$ is the kernel of the projection

$$E \otimes_{K_0} B_{cr} \xrightarrow{(\mathbf{\Phi}-1,i)} (E \otimes_{K_0} B_{cr}) \oplus (E \otimes_{K_0} B_{dR}/F^0(E \otimes_{K_0} B_{dR})).$$

Now we have the following diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{L}_1 \arrow[r] & E_1 \otimes_{K_0} B_{cr} \arrow[r] \arrow[d] & (E_1 \otimes_{K_0} B_{dR})/F^0(E_1 \otimes_{K_0} B_{dR}) \arrow[r] \arrow[d] & 0 \\
 & & E \otimes_{K_0} B_{cr} \arrow[r] \arrow[d] & (E \otimes_{K_0} B_{dR})/F^0(E \otimes_{K_0} B_{dR}) \arrow[d] & \\
0 \arrow[r] & \mathbb{L}_2 \arrow[r] & E_2 \otimes_{K_0} B_{cr} \arrow[r] \arrow[d] & (E_2 \otimes_{K_0} B_{dR})/F^0(E_2 \otimes_{K_0} B_{dR}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

It is plain now how to apply snake lemma and see that $\mathbb{L}$ fits into an exact sequence

$$0 \to \mathbb{L}_1 \to \mathbb{L} \to \mathbb{L}_2 \to 0.$$

Hence we have

$$\dim_{\mathbb{Q}_p}(\mathbb{L}) = \sum_{i=1}^{2} \dim_{\mathbb{Q}_p}(\mathbb{L}_i) = \sum_{i=1}^{2} \dim_{K_0}(E_i) = \dim_{K_0}(E),$$

and we are done. $\square$

In the previous section, we studied the space of Γ-representations $\mathbb{L}$, which are iterated extensions of fixed ones $\mathbb{L}_i$, and saw that under some conditions they form an affine algebraic space over K. In this section we are going to say some words about the crystalline points on this representing space. Of course we must start by fixing some crystalline representations as well. Hence suppose that we are given finitely many B_{cr}-admissible triples $(E_i, \mathbf{\Phi}_i, F_i^\bullet)$ in $\mathcal{FM}$ with corresponding Γ-representations $\mathbb{L}_i$, for $1 \leq i \leq r$. Moreover assume we have a $(K_0, \mathbf{\Phi})$-module E with an increasing filtration

$$(0) = W_0(E) \subset W_1(E) \subset \cdots \subset W_r(E) = E$$

such that for any $1 \leq i \leq r$, one has

$$W_i(E)/W_{i-1}(E) \cong_{(K_0,\mathbf{\Phi})} E_i.$$

We are interested in the space of possible Hodge filtrations on $\boldsymbol{E} = E \otimes_{K_0} K$ which induce $F_i^\bullet$ on $\boldsymbol{E}_i$. In analogy with the previous section, consider the following unipotent group scheme G over K_0:

$$G := \begin{pmatrix} I & \mathrm{Hom}_{K_0}(E_2, E_1) & \dots & \mathrm{Hom}_{K_0}(E_r, E_1) \\ 0 & I & \dots & \mathrm{Hom}_{K_0}(E_r, E_2) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{pmatrix}$$

Note that G admits a natural Frobenius action $\mathbf{\Phi}$ induced from E_i's, and moreover on $\boldsymbol{G} := G \otimes_{K_0} K$ we have the induced Hodge filtration $F^\bullet$ by subgroups. Moreover, after fixing an isomorphism between E and $\bigoplus_{i=1}^r E_i$ as K_0-vector spaces, in such a way that $W_s(E) = \bigoplus_{i=1}^s E_i$ for all $1 \leq s \leq r$, G acts on E in the natural way, fixing the filtration $W_\bullet$ and inducing trivial action on $Gr_W^i(E)$ for all i. The following lemma gives an answer to our question about possible Hodge filtrations:

Lemma 3.2.3. *Possible Hodge filtrations on $\boldsymbol{E}$, which induce $F_i^\bullet$ on $\boldsymbol{E}_i$ for all $1 \leq i \leq r$, are parametrized by $\boldsymbol{G}/F^0(\boldsymbol{G})$, which by Lemma 3.1.4 is an affine space over K.*

Proof. First note that there is a trivial way of putting a Hodge filtration on $\boldsymbol{E}$ which induces $F_i^\bullet$ on $\boldsymbol{E}_i$. Namely, having fixed the K_0-linear isomorphism

$$E \cong \bigoplus E_i,$$

one can put the sum filtration $F^\bullet = \bigoplus F_i^\bullet$ on $\boldsymbol{E}$, i.e. for any j put

$$F^j(\boldsymbol{E}) = \bigoplus_i F_i^j(\boldsymbol{E}_i).$$

Now any two filtrations on $\boldsymbol{E}$ with same dimensional graded parts can be transformed to each other by suitable elements of $GL(\boldsymbol{E})$, and moreover if these two filtrations induce the given filtrations $F_i^\bullet$ on $\boldsymbol{E}_i$ for all i these transforming elements can be chosen from the group $\boldsymbol{G}$ above. In particular $\boldsymbol{G}$ acts transitively on all possible Hodge filtrations which induce $F_i^\bullet$ on $\boldsymbol{E}_i$ for all i. Now the claim is an immediate consequence of the easy fact that the stabilizer under this action of the above constructed Hodge filtration $F^\bullet$ is $F^0(\boldsymbol{G})$. □

Remark 3.2.4. Note that by Corollary 3.2.2, any such Hodge filtration $F^\bullet$ on $\boldsymbol{E}$ which induces $F_i^\bullet$ on $\boldsymbol{E}_i$ for all i, gives rise to a B_{cr}-admissible triple $(E, \boldsymbol{\Phi}, F^\bullet)$ in $\mathcal{FM}$ with an increasing filtration $W_\bullet$ such that

$$Gr^i_W((E, \boldsymbol{\Phi}, F^\bullet)) = (E_i, \boldsymbol{\Phi}_i, F_i^\bullet).$$

Whence the associated Galois representation $\mathbb{L} = \mathfrak{Gal}(E)$, also admits an increasing filtration $W_\bullet$ such that

$$Gr^i_W(\mathbb{L}) = \mathbb{L}_i.$$

Now by Lemma 3.1.1, the non-abelian cohomology set $H^1(\mathrm{Gal}(\overline{K}/K), \mathbb{G})$ classifies all such Galois representations. A completely similar argument over a finitely generated $\mathbb{Q}_p$-algebra R, implies that for any such algebra R, any element of $(\boldsymbol{G}/F^0(\boldsymbol{G}))(R \otimes_{\mathbb{Q}_p} K)$ gives an element in $H^1(\Gamma, \mathbb{G}(R))$. This means that if we consider $W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G}))(-)$, where $W_{K/\mathbb{Q}_p}$ denotes the Weil restriction functor, and $H^1(\Gamma, \mathbb{G}(-))$, as two functors on the category of finitely generated $\mathbb{Q}_p$-algebras, we have obtained a natural transformation between them. Since these two functors are representable by affine schemes over $\mathbb{Q}_p$, by the Yoneda's lemma we obtain an algebraic comparison map between these two schemes

$$c : W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G})) \to H^1(\Gamma, \mathbb{G}).$$

The points in the image of c are called crystalline points. ♣

We finish this section with following proposition which will be used latter:

Proposition 3.2.5. *With the notations and hypotheses of the above remark, assume moreover that there is no nonzero Frobenius equivariant morphism from E_j to E_i, for any $1 \leq i < j \leq r$. Then the above mentioned comparison map*

$$c : W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G})) \to H^1(\Gamma, \mathbb{G})$$

is injective and identifies $W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G}))$ with a closed sub-stack (or in the situation of Corollary 3.1.3, a closed sub-scheme) of $H^1(\Gamma, \mathbb{G})$.

Proof. Since set of crystalline torsors is the inverse image of the trivial class under the map

$$H^1(\Gamma, \mathbb{G}) \to H^1(\Gamma, \mathbb{G}_{B_{cr}}),$$

being crystalline is a closed condition on elements of $H^1(\Gamma, \mathbb{G})$, and so it suffices to prove the injectivity assertion. Assume by contradiction that c maps two different points in $W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G}))$ to the same point in $H^1(\Gamma, \mathbb{G})$. This means that there are two different Hodge filtrations $F^\bullet$ and $F'^\bullet$ on $\boldsymbol{E}$, such that they both induce $F_i^\bullet$'s on E_i's and

$$\mathfrak{Gal}((E, \boldsymbol{\Phi}, F^\bullet)) \cong \mathfrak{Gal}((E, \boldsymbol{\Phi}, F'^\bullet))$$

as Γ-modules, where this isomorphism respects the increasing filtration $W_\bullet$ and induces the identity on $\mathbb{L}_i$'s. But we know that the category of crystalline Γ-representations is equivalent to the category of B_{cr}-admissible triples, hence there must be an isomorphism

$$\psi : (E, \boldsymbol{\Phi}, F^\bullet) \xrightarrow{\sim} (E, \boldsymbol{\Phi}, F'^\bullet)$$

which respects the increasing filtration $W_\bullet$ and induces identity on E_i's. Any such isomorphism ψ is an element of the unipotent group scheme G, and being an isomorphism in the category $\mathfrak{FM}$, the upper diagonal entries must be Frobenius equivariant. By our assumption there are no nonzero Frobenius equivariant morphisms from E_j to E_i for any $1 \leq i < j \leq r$. This implies that $\psi = Id$, and hence $F^\bullet$ is the same filtration as $F'^\bullet$. This contradiction finishes the proof of our claim. □

Remark 3.2.6. Doing a little more, one can prove by same methods that in general, i.e. without making extra assumption on Frobenius equivariant homomorphisms, the kernel of the comparison map c is the image in $W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G}))$ of Frobenius invariant points in G, and hence it identifies

$$W_{K/\mathbb{Q}_p}(\boldsymbol{G}/F^0(\boldsymbol{G}))/G^{\boldsymbol{\Phi}}$$

with a closed sub-stack (or in nicer situations, closed sub-scheme) of $H^1(\Gamma, \mathbb{G})$. But the above special case will be enough for our applications. ♢

3.3 Period Maps

Now we are going to construct different versions of period maps which in a sense encode the variation of the information coming from different versions of path torsors over different realizations of the unipotent fundamental group. These period maps are going to be maps from the curve under our study to the algebraic spaces which are constructed in previous sections and are parametrizing spaces for the path torsors over different realizations of unipotent fundamental group. In order to do that we fix the following notations for this section.

Let k be a fixed number field, S be a finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S-integers of k. Fix also a smooth, projective, geometrically connected curve C over $\mathcal{O}_S$, a relative divisor D in C which is étale and surjective over $\mathrm{Spec}(\mathcal{O}_S)$, and consider the complement $X := C - D$ which is smooth over $\mathrm{Spec}(\mathcal{O}_S)$.

For any finite place v of k we denote by k_v (resp. $\mathcal{O}_v$, resp. $\mathbb{F}_v$) the completion of k at v (resp. ring of integers of k_v, resp. residue field of k_v). Note that if we take A to be any of the rings k, k_v, $\mathcal{O}_v$, or $\mathbb{F}_v$ then there is a canonical morphism

$$\mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_S)$$

and for any scheme or morphism between schemes defined over $\mathcal{O}_S$, let a subscript $-_A$ denote the pullback via this canonical morphism. Our main interest is to prove that the Diophantine set $X(\mathcal{O}_S)$ is finite (see Theorem 4.2.1). If $X(\mathcal{O}_S)$ is empty then there is nothing to prove, otherwise we fix a point $x \in X(\mathcal{O}_S)$ as the base point to define the following period maps.

Starting with the de Rham period map, let v be a finite place of k outside S and let $x_{k_v} \in X_{k_v}(k_v)$ be the point which is induced by x. Taking x_{k_v} as the base point and using the general Tannakian formalism of section 1.1 or the more explicit constructions of chapter 2, we can construct the pro-unipotent group scheme $G_{dR}(X_{k_v}, x_{k_v})$ and its algebraic quotients $G_{dR,n}(X_{k_v}, x_{k_v})$ for all $n \geq 1$. More precisely, in the Tannakian point of view, $G_{dR}(X_{k_v}, x_{k_v})$ is obtained by applying Theorem 1.1.6 to the category $\mathcal{C}_{dR}$ associated to the triple $(C_{k_v}, X_{k_v}, D_{k_v})$ equipped with the fiber functor coming from the point x_{k_v}. Moreover, for any other point $y \in X_{k_v}(k_v)$ we get path torsors $G_{dR}(X_{k_v}; x_{k_v}, y)$ and $G_{dR,n}(X_{k_v}; x_{k_v}, y)$'s, respectively over the group $G_{dR}(X_{k_v}, x_{k_v})$ and its algebraic quotients $G_{dR,n}(X_{k_v}, x_{k_v})$.

Note that by results of sections 2.2 and 2.3 these group schemes and the path torsors over them admit Hodge filtration and Frobenius action as extra structures. Note also that these extra structures on path torsors are very critical in the sense that they cause the non-triviality of the path torsors. In other words, since a unipotent group scheme over a field of characteristic zero does not admit any non-trivial torsor, without these extra structures all the path torsors mentioned above would simply be trivial torsors and hence pointless. So these extra structures are needed to have non-trivial path torsors. But an important point for us is that even the Frobenius action alone is not sufficient to make these path torsors non-trivial. By this we mean that all these de Rham path torsors admit a system of Frobenius invariant elements compatible with the concatenation maps. More precisely the result [3, Corollary 3.2] says that if both points x_{k_v} and y are invariant under some power of the Frobenius map, then for any $n \geq 1$ there exists a unique element in $G_{dR,n}(X_{k_v}; x_{k_v}, y)$ which is invariant under the same power of the Frobenius map. Obviously these Frobenius invariant elements are compatible with concatenation maps, simply because of their uniqueness. As an upshot, if one considers Frobenius action as the only extra structure then these de Rham path torsors $G_{dR,n}(X_{k_v}; x_{k_v}, y)$ become trivial over unipotent de Rham fundamental groups $G_{dR,n}(X_{k_v}, x_{k_v})$ when both x_{k_v} and y are Frobenius invariant. Of course, by taking limit, the same statement holds for the path torsor $G_{dR}(X_{k_v}; x_{k_v}, y)$ over the pro-unipotent group scheme $G_{dR}(X_{k_v}, x_{k_v})$.

Another way of proving the above fact is that one can prove in general that the pull back of any Frobenius isocrystal on X_{k_v} to the p-adic open unit ball centered at a $W(\mathbb{F}_v)$-point is constant. Very roughly the idea is that one takes the fiber of the Frobenius isocrystal at the center of the open unit ball, takes any section (not necessarily Frobenius equivariant) of this fiber which gives a map (not necessarily Frobenius equivariant) from the constant Frobenius isocrystal to the original one. Then one can check that the conjugates of this map by higher and higher powers of Frobenius converge to a Frobenius equivariant isomorphism from the constant Frobenius isocrystal to the original one (see [14] for more details). Whence all these de Rham path torsors become trivial after forgetting the Hodge filtration, i.e. if v lies above the rational prime p, one has

$$\mathcal{O}_{G_{dR,n}(X_{k_v}; x_{k_v}, y)} \cong \mathcal{O}_{G_{dR,n}(X_{k_v}, x_{k_v})}, \quad \forall n \geq 1,$$

as Frobenius modules over $W(\mathbb{F}_v)[1/p] \subset k_v$. Note that the same also holds in the pro-unipotent case, simply by taking limit. So by identifying the underlying Frobenius modules, if we vary the point $y \in X_{k_v}(k_v)$, we get a varying family of Hodge filtrations on $\mathcal{O}_{G_{dR}}$ and $\mathcal{O}_{G_{dR,n}}$'s for all $n \geq 1$. In exactly the same way that we proved Lemma 3.2.3, one sees in the algebraic quotient case that the set of such Hodge filtrations is parametrized by the affine space $G_{dR,n}/F^0(G_{dR,n})$. Putting all these together, for any $n \geq 1$ we obtain the following period maps

$$p_{dR}^{(n)} : X_{k_v}(k_v) \to G_{dR,n}/F^0(G_{dR,n}).$$

The crucial property of these de Rham period maps is that they are k_v-analytic maps whose images on the set of p-adic integral points in the open p-adic unit ball around x_{k_v} is Zariski dense in $G_{dR,n}/F^0(G_{dR,n})$, for all $n \geq 1$. More precisely we have the following

Theorem 3.3.1. *For any $n \geq 1$, the restriction of the period map $p_{dR}^{(n)}$ to the p-adic integral points in the p-adic open unit ball around x_{k_v} gives a rigid k_v-analytic map with Zariski dense image in $G_{dR,n}/F^0(G_{dR,n})$.*

Proof. Zariski density of the image is very well explained in [17, Section 4], so we only sketch the proof of rigid analyticity.

If one restricts the universal Frobenius crystal G_{dR} to the p-adic open unit ball centered at x_{k_v}, one gets a Frobenius crystal on $W(\mathbb{F}_v)\{\{t\}\}$. This Frobenius crystal is constant as it is mentioned above. So if we denote the Hodge filtration at the base point x_{k_v} by F_0, the filtration at another point y in the open p-adic unit ball is given by $g(y)F_0$ where g is a rigid k_v-analytic map from $W(\mathbb{F}_v)\{\{t\}\}$ to $G_{dR}/F^0(G_{dR})$. Note that the variation of the Hodge filtration is algebraic on the original universal Frobenius crystal, but it becomes only rigid k_v-analytic after making it constant over the p-adic open unit ball because the process of constantification involves rigid k_v-analytic transformations which are not necessarily algebraic. $\square$

There are two other versions of period maps which are important for us, namely the local and the global étale period maps. To define the local étale period map, we work with $X_{\bar{k_v}}$ with the fixed base point $x_{\bar{k_v}}$. This time, as the name suggests, we consider the étale version of the theory developed in section 2.1. Hence we get the pro-unipotent group scheme $G_{\text{ét}}(X_{\bar{k_v}}, x_{\bar{k_v}})$ with its algebraic quotients $G_{\text{ét},n}(X_{\bar{k_v}}, x_{\bar{k_v}})$, on which the absolute Galois group

$$G_v := \mathrm{Gal}(\overline{k_v}/k_v)$$

acts. Moreover for any other point $y \in X_{k_v}(k_v)$, if we denote by $\overline{y}$ the induced point in $X_{\bar{k_v}}$, we obtain the path torsors $G_{\text{ét},n}(X_{\bar{k_v}}; x_{\bar{k_v}}, \overline{y})$ equipped with compatible G_v-action.

Now suppose that we had started with a triple $(\mathbb{P}^1_{\mathcal{O}_S}, X, \{p_1, \ldots, p_{d+1}\})$ over $\mathcal{O}_S$. Then these unipotent groups $G_{\text{ét},n}$, by construction, admit finite increasing filtrations with sub-quotients being isomorphic to tensor powers of the étale realizations of the Tate objects

$$H^1_{\text{ét}}(X_{\bar{k_v}}, \mathbb{Q}_l) \cong \mathbb{Q}_l(1)^d.$$

Hence by putting $\Gamma = G_v$ we can apply Lemma 3.1.1, Corollary 3.1.3, and Theorem 3.1.6 to obtain the following local étale period maps

$$p_{\text{ét}}^{\text{loc},(n)} : X_{k_v}(k_v) \to H^1(G_v, G_{\text{ét},n}).$$

The global étale period map can be constructed in exactly the same way as the local one, but this time we consider the variety $X_{\overline{k}}$, and we note that in this situation the resulting (pro-)unipotent group schemes $G_{\text{ét}}(X_{\overline{k}}, x_{\overline{k}})$ and $G_{\text{ét},n}(X_{\overline{k}}, x_{\overline{k}})$'s are equipped with an action of the global Galois group

$$G_T := \mathrm{Gal}(k_T/k),$$

where k_T is the maximal extension of k unramified outside $T = S \cup \{v\}$. In exactly the same way, when we start with the triple $(\mathbb{P}^1_{\mathcal{O}_S}, X, \{p_1, \ldots, p_{d+1}\})$ over $\mathcal{O}_S$ we get the following global étale period maps

$$p_{\text{ét}}^{\text{glob},(n)} : X(k) \to H^1(G_T, G_{\text{ét},n}).$$

All these maps and the comparison map of Remark 3.2.4 can be put together in the following important diagram.

Remark 3.3.2. For the triple $(\mathbb{P}^1_{\mathcal{O}_S}, X, \{p_1, \ldots, p_{d+1}\})$ over $\mathcal{O}_S$ and any integer $n \geq 1$, one has the following (commutative) diagram

$$\begin{tikzcd}
 & (G_{dR,n}/F^0)(k_v) \arrow[dr, bend left] & \\
X(\mathcal{O}_S) \arrow[r, "i"] \arrow[d, "p_{\text{ét}}^{\text{glob},(n)}"] & X_{k_v}(\mathcal{O}_v) \arrow[u, "p_{dR}^{(n)}"] \arrow[r, dotted] \arrow[d, "p_{\text{ét}}^{\text{loc},(n)}"] & W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)(\mathbb{Q}_p) \arrow[dl, bend left, "c"] \\
H^1(G_T, G_{\text{ét},n})(\mathbb{Q}_p) \arrow[r, "\text{res}"] & H^1(G_v, G_{\text{ét},n})(\mathbb{Q}_p) &
\end{tikzcd}$$

Recall that $p_{dR}^{(n)}$ is a k_v-analytic map with Zariski dense image, c is the comparison map which is injective and whose existence was shown by general constructions of section 3.2, and 'res' is the usual restriction map between group cohomologies induces by the inclusion

$$G_v \subset G_T.$$

Note that the left square in the diagram is evidently commutative, but the commutativity of the lower right triangle is a much deeper claim which will be a consequence of Remark 4.1.2. Moreover note that, the non-abelian cohomology sets of the second row, by general results of section 3.1, are affine algebraic spaces over $\mathbb{Q}_p$, and 'res' is a $\mathbb{Q}_p$-algebraic map with respect to these $\mathbb{Q}_p$-algebraic structures. ♠

Chapter 4

Integral Points and π_1

Now we are almost ready to state and prove the main results of this thesis. I said almost because we still need two more technical tools. First of all, as it has been mentioned in the introduction of the first chapter, one important feature of different cohomology theories for varieties, which makes them very powerful, is that they are not completely independent. Indeed they are actually very closely related via the so called comparison isomorphisms. More precisely, this comparison isomorphisms say that for nice varieties, the étale cohomology groups, as finite dimensional Galois modules, and the crystalline cohomology groups, as finite dimensional filtered Frobenius isocrystals, will become isomorphic over some big ring of periods. Since we are going to employ different realizations of unipotent fundamental group of varieties in our work, we would be very happy to have the non-abelian analogues of these comparison isomorphisms for unipotent fundamental groups. Fortunately such isomorphisms exist and say that the coordinate ring of the unipotent étale fundamental group, as a limit of finite dimensional Galois modules, and the coordinate ring of the unipotent de Rham fundamental group, as a limit of finite dimensional filtered Frobenius isocrystals, will also become isomorphic over some period rings. The same also holds for the coordinate rings of the étale and the de Rham path torsors which will be the topic of section 4.1. Moreover, after being done with the case of curves, in order to generalize things to higher dimensional varieties, we will crucially use a motivic version of the homotopic Lefschetz hyperplane section theorem, which we will develop in section 4.3.

4.1 p-adic Hodge Theory and Comparison

The reader might have noticed that one of the crucial tools for proving our main results in the coming sections is the fundamental diagram of Remark 3.3.2 of the previous section. In that diagram, beyond the period maps which were discussed there, there is an important comparison map

$$c : W_{k_v/\mathbb{Q}_p}(G_{dR,n}/G_{coh,n}) \to H^1(G_v, G_{\text{ét},n}),$$

which plays an important role in the theory. But we still miss a proof of the fact that this map is compatible with different period maps and hence the fundamental diagram is commutative. This is the subject of this section. In order to do that, we need a non-abelian version of the comparison theory between the étale and the crystalline cohomologies of smooth varieties with good reduction over p-adic fields. Fortunately the abelian case has been worked out in [16] (see for example [16, Theorem 9]) in such a natural way that it can be easily generalized to the non-abelian case of our interest. We also recall that the easier case of this theory for curves can be found in [15] (see for example [15, Theorem 3.2]).

Let us first recall the comparison theory over a point. For that we need Fontaine's construction of rings of periods (see [18]). Fix a perfect field k of characteristic $p > 0$, and let $V_0 := W(k)$ be its ring of Witt vectors. Denote by $K_0 := V_0[1/p]$ the field of fractions of V_0, and let K/K_0 be a finite totally ramified extension (note that if k varies through all finite extensions of F_p, we could construct all finite extensions of $\mathbb{Q}_p$ using this procedure). Finally let V and $\bar{V}$ denote integral closures of V_0 in K and $\bar{K}$ respectively, where $\bar{K}$ is an algebraic closure of K. Since $\bar{V}$ is the integral closure in the algebraic closure $\bar{K}$ of K, the equation $x^p = a$ is solvable in $\bar{V}$ for any element $a \in \bar{V}$. This implies that $\bar{V}/p\bar{V}$ has a surjective Frobenius map. Now we consider the inverse limit of this ring with respect to these surjective Frobenius endomorphisms, namely

$$\mathcal{R}(V) := \varprojlim_{\text{Frob}} (\bar{V}/p\bar{V}) = \{(x_0, x_1, x_2, \dots) \in (\bar{V}/p\bar{V})^{\mathbb{N}} \ : \ \forall_{i\in\mathbb{N}},\ x_i = x_{i+1}^p\}.$$

One can represent elements of $\mathcal{R}(V)$ by infinite vectors whose entries lie in $\widehat{\bar{V}}$, where $\widehat{\ }$ stands for p-adic completion. Namely, take arbitrary lifts $y_i \in \bar{V}$ of x_i's, and check easily that for any $i \in \mathbb{N}$ the limit

$$x_i' := \varinjlim_{n} y_{i+n}^{p^n}$$

exists in $\widehat{\bar{V}}$ and is independent of the choices of the lifts y_i's. This way, one can see that there is a bijection between $\mathcal{R}(V)$ and the following set

$$\{(x'_0, x'_1, x'_2, \dots) \in (\widehat{\bar{V}})^{\mathbb{N}} \ : \ \forall_{i\in\mathbb{N}},\ x'_i = (x'_{i+1})^p\}.$$

In what follows we use this latter presentation of elements of $\mathcal{R}(V)$ without writing $'$'s on the entries! Note that in order to make this bijection a ring isomorphism, the component-wise product on the above set works, but the proper addition is more complicated (see [18, 1.2.2]). Now this ring $\mathcal{R}(V)$ is a perfect p-adically complete ring of characteristic p which admits a surjective ring homomorphism

$$\theta : \mathcal{R}(V) \twoheadrightarrow \bar{V}/p\bar{V},$$

where $\theta((x_i)_i) = \overline{x_0}$. Obviously

$$\underline{p} = (p, p^{1/p}, p^{1/p^2}, \dots)$$

lies in the kernel of θ, and conversely if $\underline{x} = (x_i)_i$ belongs to $\mathrm{Ker}(\theta)$ then $p \mid x_0$. This implies that the valuation of x_0 is not less than the valuation of p, and since $x_n^{p^n} = x_0$ for any $n \geq 0$, the valuation of x_n cannot be less that the valuation of p^{1/p^n}. Finally since the multiplication is component-wise, $\underline{p} \mid \underline{x}$, and hence $\mathrm{Ker}(\theta)$ is the ideal generated by $\underline{p}$. Now define

$$A_{\mathrm{inf}}(V) := W(\mathcal{R}(V)),$$

and note that as usual for any element $\underline{x} \in \mathcal{R}(V)$, we denote by

$$[\underline{x}] = (\underline{x}, 0, 0, \dots) \in A_{\mathrm{inf}}(V)$$

its Teichmüller representative. Note also that since $\mathcal{R}(V)$ is perfect, every element in $A_{\mathrm{inf}}(V)$ can be written in the form

$$\sum_{n\geq 0} p^n.[\underline{x_n}]$$

for $\underline{x_n} = (x_{n,i})_i$ in $\mathcal{R}(V)$. Now one can extend the ring homomorphism θ to a ring homomorphism

$$\tilde{\theta} : A_{\mathrm{inf}}(V) \twoheadrightarrow \widehat{\bar{V}},$$

which sends $[\underline{x}]$ to x_0, for any $\underline{x} = (x_i)_i \in \mathcal{R}(V)$. Evidently the element $\xi = [\underline{p}] - p$ belongs to the kernel of $\tilde{\theta}$, and since modulo p, $\tilde{\theta}$ is the same homomorphism as θ from $\mathcal{R}(V)$ to $\widehat{\bar{V}}/p\widehat{\bar{V}} \cong \bar{V}/p\bar{V}$, and $A_{\mathrm{inf}}(V)$ is p-adically complete, it is easy to check that $\mathrm{Ker}(\tilde{\theta})$ is generated by ξ. Consider now

$$A_{cr}(V) := \widehat{A_{\mathrm{inf}}(V)[(\xi^n/n!)_{n\in\mathbb{N}}]}\ ,$$

which is the p-adic completion of the divided power hull of $A_{\inf}(V)$ with respect to $\mathrm{Ker}(\tilde{\theta})$. Note that $A_{cr}(V)$ caries a divided power filtration, which is induced by powers of ξ, which we denote by $F^\bullet$. On the other hand, since all the constructions are functorial, the Frobenius φ of $\mathcal{R}(V)$ extends first to $A_{\inf}(V)$ by the rule $\varphi([\underline{x}]) = [\underline{x}]^p$, and then to $A_{cr}(V)$. Finally since $\mathrm{Gal}(\bar{K}/K)$ acts on $\bar{V}$, again by functoriality of the constructions, it also acts on $A_{cr}(V)$. Now consider the element

$$\underline{1} := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{R}(V),$$

where ζ_p is a primitive p-th root of unity, and define

$$t := log([\underline{1}]) = -\sum_{n=1}^{\infty} \frac{(1-[\underline{1}])^n}{n} = -\sum_{n=1}^{\infty} (n-1)!.(1-[\underline{1}])^{[n]},$$

where $-^{[n]}$ means the n-th divided power. Note that since $1 - [\underline{1}] \in \mathrm{Ker}(\tilde{\theta})$, and p-adic norm of $(n-1)!$ goes to zero when n tends to infinity, t is an element of $A_{cr}(V)$. Obviously t satisfies the identity $\varphi(t) = pt$, and we define

$$B_{cr}(V) := A_{cr}(V)[\frac{1}{pt}].$$

Note that Frobenius and Galois actions immediately extend to B_{cr}, and since the element pt has pure degree 1 in $Gr_{F^\bullet}(A_{cr}(V))$, we can endow $B_{cr}(V)$ with a filtration induced by $F^\bullet$, which by abuse of notation will be denoted by the same symbol $F^\bullet$. As it was mentioned in section 3.2 there are two very important properties of $B_{cr}(V)$ which make it so useful, namely

$$B_{cr}(V)^{\mathrm{Gal}(\bar{K}/K)} = K_0,$$

and

$$B_{cr}(V)^{\varphi - Id} \cap F^0(B_{cr}(V)) = \mathbb{Q}_p.$$

This properties are essential in using $B_{cr}(V)$ to establish comparison functors between finite dimensional $\mathrm{Gal}(\bar{K}/K)$-representations over $\mathbb{Q}_p$ and filtered Frobenius modules over K_0. Recall that we say that a finite dimensional representation V of $\mathrm{Gal}(\bar{K}/K)$ over $\mathbb{Q}_p$ is associated to a finite dimensional filtered Frobenius module E over K_0 if there is a B_{cr}-isomorphism

$$V \otimes_{\mathbb{Q}_p} B_{cr} \cong E \otimes_{K_0} B_{cr},$$

which respects the Galois action, the Frobenius, and the filtration. In this case one can see evidently that

$$\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}((V \otimes_{\mathbb{Q}_p} B_{cr})^{\mathrm{Gal}(\bar{K}/K)}).$$

This motivates us to define a crystalline representation to be a representation for which the above equality between dimensions holds. On the other hand, and in exactly the same manner, one sees that if V and E are associated to each other, then

$$\dim_{K_0}(E) = \dim_{\mathbb{Q}_p}((E \otimes_{K_0} B_{cr})^{\mathbf{\Phi}=1} \cap F^0(E \otimes_{K_0} B_{cr} \otimes_{K_0} K)),$$

and hence we call a filtered Frobenius module over K_0 to be B_{cr}-admissible if the above equality holds for it (see section 3.2).

Finally we recall that $B_{dR}(V)$ can be defined to be the filtered completion of $B_{cr}(V)$ with respect to $F^\bullet$. Then $B_{dR}(V)$ inherits a filtration, again $F^\bullet$ and a Galois action from $B_{cr}(V)$, but it does not admit a Frobenius action anymore. Moreover it satisfies (see [18] for more details and proofs)

$$Gr_{F^\bullet}(B_{dR}(V)) = \widehat{\bar{K}}[t, t^{-1}],$$

and

$$B_{dR}^{\mathrm{Gal}(\bar{K}/K)} = K.$$

The next step is to extend these techniques in order to compare p-adic étale sheaves and de Rham crystals over smooth varieties with good reduction over p-adic fields. In this part we mainly follow Faltings ([13]) to give a brief review of this theory. In sequel R is a smooth V-algebra of relative dimension d. R is said to be small if it is étale over $V[T_1^{\pm 1}, \dots, T_d^{\pm 1}]$.

First of all one replaces $\bar{V}$ in Fontaine's theory with $\bar{R}$, which is defined to be the normal closure of R in the maximal étale extension of $R[1/p]$. Now note that the polynomial $T^{p^2} + pT$ defines an étale extension of $R[1/p]$. In fact, its derivative is

$$p^2 T^{p^2-1} + p = p(1 + pT^{p^2-1})$$

and p lies in the unique maximal ideal of V and hence in Jacobson radical of R as well. This implies that for any element $a \in \bar{R}$ the equation

$$T^{p^2} + pT = a$$

is solvable in $\bar{R}$, hence any element in $\bar{R}$ has a p^2-th root modulo p, and in particular that the Frobenius is surjective on $\bar{R}/p\bar{R}$. This allows us to define analogue period rings for R as follows. First consider

$$\mathcal{R}(R) := \varprojlim_{\text{Frob}} (\bar{R}/p\bar{R}),$$

which obviously admits a projection

$$\theta : \mathcal{R}(R) \twoheadrightarrow (\bar{R}/p\bar{R})$$

whose kernel is generated by $\underline{p}$. Then consider

$$A_{\inf}(R) := W(\mathcal{R}(R)),$$

and extend θ to

$$\tilde{\theta} : A_{\inf} \twoheadrightarrow \widehat{\bar{R}}$$

and note again that $\mathrm{Ker}(\tilde{\theta})$ is generated by $\xi = [\underline{p}] - p$. Finally we can define

$$A_{cr}(R) := \widehat{A_{\inf}(R)[(\xi^n/n!)_{n\in\mathbb{N}}]} ,$$

and so on.

Another tool which Faltings uses to compare p-adic étale sheaves with de Rham crystals is a universal cohomology theory $\mathcal{H}^*$ for a smooth variety X with good reduction over a p-adic field. Consider the site whose objects consist of an open subset $\mathcal{U} \subset X$ plus a finite étale cover $V_K \to \mathcal{U}_K$ over K. A family of objects

$$\{V_{i,K} \to \mathcal{U}_{i,K}\}_{i\in I}$$

in this site forms a covering of the object $V_K \to \mathcal{U}_K$, if the following hold:

- The family $\{\mathcal{U}_i\}_{i\in I}$ forms an open covering of $\mathcal{U}$.
- For any $i \in I$ there is a commutative diagram like

$$\begin{array}{ccc} V_{i,K} & \longrightarrow & V_K \\ \downarrow & & \downarrow \\ \mathcal{U}_{i,K} & \xrightarrow{\varphi_{i,K}} & \mathcal{U}_K \end{array}$$

  where $\varphi_i : \mathcal{U}_i \hookrightarrow \mathcal{U}$ denotes the inclusion of $\mathcal{U}_i$ in $\mathcal{U}$, and $\varphi_{i,K}$ is the base change of φ_i to K.

- Finally the family $\{V_{i,K}\}_{i\in I}$ forms a covering of V_K.

This site gives rise to a topos $\mathcal{T}$, hence a notion of cohomology which will be denoted by $\mathcal{H}^*$.

Now for any p-adic étale local system $\mathbb{L}$ on $X_{\bar{K}}$ and any small open affine subset $\mathrm{Spec}(R)$ of X, one can associate the almost sheaf $\mathbb{L} \otimes_{\mathbb{Q}_p} A_{\inf}(R)$ in $\mathcal{T}$, and prove the following isomorphism

$$H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} A_{\inf}(V) \cong \mathcal{H}^i(\mathcal{T}, \mathbb{L} \otimes_{\mathbb{Q}_p} A_{\inf}(R)).$$

On the other hand, given any filtered Frobenius crystal $(\mathcal{E}, \nabla, F^\bullet)$ on X, since

$$\tilde{\theta} : A_{cr}(R) \twoheadrightarrow \widehat{\bar{R}}$$

is an infinitesimal thickening, one can evaluate $\mathcal{E}$ on the inverse image

$$\tilde{\theta}^{-1}(R) \subset A_{cr}(R)$$

and tensor it with $A_{cr}(R)$. Let us call the result by $\mathcal{E}(A_{cr}(R))$, which is a sheaf in $\mathcal{T}$ equipped with a Frobenius action and filtration. This leads to a map between cohomologies

$$H^i_{cr}(X_K, \mathcal{E}) \otimes_{K_0} A_{cr}(V) \to \mathcal{H}^i(\mathcal{T}, \mathcal{E}(A_{cr}(R))).$$

Finally we say that a p-adic étale local system $\mathbb{L}$ is associated to a filtered Frobenius crystal $\mathcal{E}$, if there exist functorial isomorphisms

$$\mathbb{L} \otimes_{\mathbb{Q}_p} B_{cr}(R) \cong \mathcal{E}(A_{cr}(R)) \otimes_{A_{cr}(R)} B_{cr}(R),$$

respecting the filtration, Frobenius and the Galois action, for all small open affine subsets $\mathrm{Spec}(R)$ of X. Using the above maps between these cohomology groups, one obtains a map between crystalline and étale cohomologies of associated objects as follows

$$H^i_{cr}(X_K, \mathcal{E}) \otimes_{K_0} B_{cr}(V) \to \mathcal{H}^i(\mathcal{T}, \mathcal{E}(A_{cr}(R)) \otimes_{A_{cr}(R)} B_{cr}(R)) =$$

$$= \mathcal{H}^i(\mathcal{T}, \mathbb{L} \otimes_{\mathbb{Q}_p} B_{cr}(R)) \cong H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{cr}(V).$$

Moreover it can be shown that it induces an isomorphism

$$H^i_{cr}(X_K, \mathcal{E}) \otimes_{K_0} B_{cr}(V) \cong H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{cr}(V),$$

i.e. the cohomologies of associated objects in the sense of Faltings' theory, are associated in the sense of Fontaine's theory. In particular one has the following

Theorem 4.1.1. *[16, Theorem 9] With above hypotheses and notations, one has the following isomorphism*

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{cr}(V) \cong H^i_{dR}(X_K, \mathcal{O}_{X_K}) \otimes_{K_0} B_{cr}(V).$$

As an immediate consequence, étale and de Rham cohomologies of X are associated in the sense of Fontaine's theory.

Other important fact for us is that the above isomorphisms between p-adic étale local systems and filtered Frobenius crystals are compatible with connecting homomorphisms. This means that if two pairs of objects are associated to each other so are their extensions provided they have associated classes in proper Ext^1's. Now by Theorem 4.1.1, after passing to duals, one obtains that $G_{\text{ét},1}$ and $G_{dR,1}$ are associated. Moreover, the constant p-adic étale sheaf $\widetilde{T_{\text{ét}}}^{\otimes n}$ with fiber $T_{\text{ét}}^{\otimes n}$ is associated to the constant filtered Frobenius crystal $\widetilde{T_{dR}}^{\otimes n}$ with fiber $T_{dR}^{\otimes n}$, for all $n \geq 1$. Now recall that by Lemma 2.1.3 one has $\mathcal{P}_{\text{ét},n+1}$ (resp. $\mathcal{P}_{dR,n+1}$) is the extension of $\mathcal{P}_{\text{ét},n}$ (resp. $\mathcal{P}_{dR,n}$) by $\widetilde{T_{\text{ét}}}^{\otimes n}$ (resp. $\widetilde{T_{dR}}^{\otimes n}$) associated to the identity element in the proper Ext^1. All these imply, by induction on n, that for any $n \geq 1$ the p-adic étale sheaf $\mathcal{P}_{\text{ét},n}$ is associated in the sense of Faltings' theory to the filtered Frobenius crystal $\mathcal{P}_{dR,n}$. We conclude this section with

Remark 4.1.2. As it has been mentioned above, for any $n \geq 1$ and any small open affine subset $\mathrm{Spec}(R)$ of X one has

$$\mathcal{P}_{\text{ét},n} \otimes_{\mathbb{Q}_p} B_{cr}(R) \cong \mathcal{P}_{dR,n}(A_{cr}(R)) \otimes_{A_{cr}(R)} B_{cr}(R).$$

Now since being associated is preserved by taking pull backs, for any integral point $x \in X(V)$, one can pull back this isomorphisms for a small open affine $\mathrm{Spec}(R)$ containing x and obtain the following isomorphism

$$\mathcal{P}_{\text{ét},n}[x] \otimes_{\mathbb{Q}_p} B_{cr}(V) \cong \mathcal{P}_{dR,n}[x] \otimes_{K_0} B_{cr}(V).$$

In particular for all $n \geq 1$, the coordinate rings of unipotent groups $G_{\text{ét},n}$ (resp. torsors $G_{\text{ét},n}(x,y)$) are associated to the coordinate rings of unipotent groups $G_{dR,n}$ (resp. torsors $G_{dR,n}(x,y)$). $\heartsuit$

4.2 The One Dimensional Case

Finally, we state and prove our first main result, which is a motivic proof of finiteness of integral points on sufficiently punctured projective line over totally real number fields. Our general idea to prove finiteness theorems for integral points on curves is the following: Suppose X is a given curve over $\mathcal{O}_S$ where $\mathcal{O}_S$ is the ring of S integers in a number field k and S is a finite set of finite places of k. The aim is to prove that $X(\mathcal{O}_S)$ is finite. If $X(\mathcal{O}_S)$ is empty, we are done, otherwise fix any element $x \in X(\mathcal{O}_S)$ as base point. Then fixing any finite place v of k, which is not in S, and at which X has good reduction, we can apply above theories and obtain the crucial diagram in Remark 3.3.2. Suppose now that we can estimate the dimensions of the algebraic varieties that appeared in that diagram and prove in particular that for a sufficiently large n one has

$$\mathfrak{D}_n < \dim(G_{dR,n}/G_{coh,n}),$$

where $\mathfrak{D}_n$ is defined to be

$$\mathfrak{D}_n := \dim(\mathrm{Im}(p_{\text{ét}}^{\text{glob},(n)}(X(\mathcal{O}_S)))),$$

where in this section whenever we talk about dimension of a subset of an algebraic variety, we mean the dimension of the Zariski closure of that subset. Then it follows that the closure of the image of $X(\mathcal{O}_S)$ in $G_{dR,n}/G_{coh,n}$ is not Zariski dense. Hence there exists a nonzero algebraic function on $G_{dR,n}/G_{coh,n}$ which vanishes on this image. Since the open p-adic unit disk around $x_v \in X_{k_v}$, $\mathcal{B}_1^\circ(x_v)$, has Zariski dense image in $G_{dR,n}/G_{coh,n}$ (Theorem 3.3.1), the pull back of this nonzero function gives a nonzero p-adic analytic function on $\mathcal{B}_1^\circ(x_v)$, which vanishes on every integral point in $\mathcal{B}_1^\circ(x_v)$. But it is well known that a nonzero p-adic analytic function on $\mathcal{B}_1^\circ(x_v)$ can have only finitely many zeros over any finite extension of $\mathbb{Q}_p$. This says that X has only finitely many integral points in the open p-adic unit disk around any integral point. On the other hand, X_v can be covered by finitely many p-adic unit disks, simply because $X_{F_v}(F_v)$ is a finite set, and we are done.

Let us recall that Kim, in [24], applies this principle to $X = \mathbb{P}^1 - \{0, 1, \infty\}$ over the ring of S-integers in $\mathbb{Q}$ where S is any finite set of rational primes. This proves Siegel's theorem for X over $\mathbb{Q}$. In this proof, in order to estimate the dimension of the related global cohomology group, Kim uses a vanishing theorem of Soulé which says that for any natural number $n \geq 1$, $H^1(G_T, \mathbb{Q}_p(2n)) = 0$, while $H^1(G_T, \mathbb{Q}_p(2n+1))$ is one dimensional. The nonexistence of an analogue of such a vanishing theorem is the main obstruction to the generalization of the result to other number fields. Here, using the so called motivic theory of previous sections, we can prove our main result which generalizes Kim's result, namely we have

Theorem 4.2.1. *Let $k/\mathbb{Q}$ be a totally real number field of degree $d \geq 2$, S be any finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S integers in k. Put*

$$X := \mathbb{P}^1 - \{p_1, p_2, \ldots, p_{d+1}\},$$

where $p_i \in \mathbb{P}^1(\mathcal{O}_S)$ for all $1 \leq i \leq d+1$. Then X has at most finitely many $\mathcal{O}_S$-points.

Remark 4.2.2. Note that the above result does not directly imply Siegel's theorem for all totally real number fields, because we must remove more and more points as degree of the number field grows. But if we consider the particular case of $d = 2$ and $(p_1, p_2, p_3) = (0, 1, \infty)$, we obtain Siegel's theorem for totally real quadratic number fields which of course implies Siegel's theorem for $\mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$. Moreover, the following proof can be directly applied to the case of $\mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$. ♣

Proof. (of Theorem 4.2.1) First of all, note that since we consider X to be a punctured projective line, its first étale and de Rham cohomology groups with values in trivial sheaves are powers of étale and de Rham realizations of the Tate object (this is the essential reason for which we must restrict ourselves to punctured projective line. The point is that for open subcurves of projective curves of higher genus, the first motivic cohomology group is not mixed Tate and hence the following arguments cannot be applied). More precisely, we have

$$T_{\text{ét}} = \mathbb{Q}_p(1)^d \ , \quad T_{dR} = K_0(1)^d,$$

where K_0 is the maximal unramified extension of $\mathbb{Q}_p$ in k_v. Actually one can say much more, and this is the important and main point of this work. Namely, for any rational point $x \in X(k)$, we can put the étale and the de Rham theories discussed in this paper, together with the Malčev unipotent completion of the topological fundamental group of $X(\mathbb{C})$ as Betti realization to obtain a pro-unipotent affine group scheme in the category $\mathcal{R}_k$ of mixed realizations which has been discussed in section 1.2. Moreover for any other rational point $y \in X(k)$, this construction can be also applied to obtain an affine scheme in the category $\mathcal{R}_k$, which is a torsor over the above mentioned $\mathcal{R}_k$-group scheme (see [8, section 13] for more details). Now a very important and crucial fact, that we are going to employ here, is Theorem 1.3.1 which says that all these objects are motivic, at least as long as we restrict ourselves to the punctured projective line. More precisely, for any rational point $x \in X(k)$ (resp. any two rational points $x, y \in X(k)$), there is a pro-unipotent affine group scheme (resp. a torsor over this group scheme) in the category $MT(k)$ whose realization is the above mentioned object in $\mathcal{R}_k$. Finally since a mixed Tate motive $M \in MT(k)$ is unramified at a place v of k, i.e. M belongs to $MT(k)_\Gamma$ where $\Gamma = \mathcal{O}_v^* \otimes_{\mathbb{Z}} \mathbb{Q}$, if and only if its l-adic realization is unramified at v for some prime number l distinct from characteristic of $\mathbb{F}_v$ (see [10, Proposition 1.8] or Theorem 1.3.2), one can deduce that if the points x, y under consideration are T-integral points, then the resulting motivic pro-unipotent fundamental group and path torsor lie in the subcategory $MT(\mathcal{O}_T)$ of $MT(k)$. This implies that the corresponding classes of these path torsors are also motivic, i.e. they come from motivic cohomologies, briefly reviewed in section 1.2, which in turn can be connected to the algebraic K-groups of our base number field k.

The next thing to notice is that since we are working with affine curves, coherent cohomologies vanish in positive degrees. Hence $T_{coh} = 0$ and for any $n \geq 1$, $G_{coh,n}$ is the trivial group. Now by construction we have following exact sequences of unipotent group schemes

$$0 \to k_v(n)^{r_n} \to G_{dR,n+1} \to G_{dR,n} \to 0,$$

where r_n, by Theorem 2.1.12, is the dimension of the n-th graded part of the free Lie algebra over a d dimensional vector space. One can precisely compute these numbers r_n as

$$r_n = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

where μ is the Möbius function (see [28, Part I, Chapter IV, Theorem 4.2]).One obtains then that

$$\dim_{k_v}(G_{dR,n+1}/G_{coh,n+1}) = \dim_{k_v}(G_{dR,n+1}) = r_1 + r_2 + \cdots + r_n.$$

Now in order to estimate the numbers $\mathfrak{D}_n$, we consider the same exact sequences as above for étale unipotent group schemes, which are

$$0 \to \mathbb{Q}_p(n)^{r_n} \to G_{\text{ét},n+1} \to G_{\text{ét},n} \to 0,$$

where these r_n's are the same as before, because we are working with different realizations of the same motivic object. Now we are interested in studying the image of $X(\mathcal{O}_S)$ in $H^1(G_T, \mathbb{Q}_p(n))$ for $n \geq 1$. Since the path torsors coming from $\mathcal{O}_S$-points of X are motivic and they lie in the category $MT(\mathcal{O}_T)$, the global étale period map $p_{\text{ét}}^{glob,(n)}$ factors through motivic cohomology groups $H^1(MT(\mathcal{O}_T), \mathbb{Q}(n))$ whose dimension can be computed as follows. For any $n \geq 2$ we have

$$H^1(MT(\mathcal{O}_T), \mathbb{Q}(n)) \cong \mathrm{Ext}^1_{MT(\mathcal{O}_T)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \mathrm{Ext}^1_{MT(k)}(\mathbb{Q}(0), \mathbb{Q}(n)),$$

where in the last equality we use the first part of Proposition 1.2.2. On the other hand, since $DMT(k)_{\mathbb{Q}}$ is the derived category of the abelian category $MT(k)$, any extension

$$0 \to B \to E \to A \to 0$$

of A by B in $MT(k)$, leads functorially and in a unique way to a distinguished triangle

$$B \to E \to A \to B[1]$$

in $DMT(k)_{\mathbb{Q}}$. This gives a map

$$\mathrm{Ext}^1_{MT(k)}(A,B) \to \mathrm{Hom}^1_{DMT(k)_{\mathbb{Q}}}(A,B),$$

which can be shown to be a bijection. If we take this bijection into account, we can push our calculations one step forward, and see that for all $n \geq 2$

$$H^1(MT(\mathcal{O}_T),\mathbb{Q}(n)) \cong \mathrm{Hom}^1_{DMT(k)_{\mathbb{Q}}}(\mathbb{Q}(0),\mathbb{Q}(n)).$$

But as it has been mentioned in section 1.2, the right hand side of the above isomorphism is isomorphic to $K_{2n-1}(k)\otimes_{\mathbb{Z}}\mathbb{Q}$. This rational K-groups have been explicitly computed by Borel (see [4, section 12]), and in our case, where k is totally real of degree d, one has

$$\dim_{\mathbb{Q}}(K_{2n-1}(k)\otimes_{\mathbb{Z}}\mathbb{Q}) = \begin{cases} 0 & n>1 \text{ is even} \\ d & n>1 \text{ is odd} \end{cases}$$

If we put

$$\alpha := \dim(H^1(MT(\mathcal{O}_T),\mathbb{Q}(1))),$$

and all the above together, we obtain

$$\mathfrak{D}_{n+1} \leq \alpha + d(r_3 + r_5 + \cdots + r_{2\lfloor (n-1)/2\rfloor+1}).$$

Remembering the fact that r_n grows asymptotically like d^n/n, it is straightforward to see that for any sufficiently large even integer n we have

$$\mathfrak{D}_n < \dim(G_{dR,n+1}/G_{coh,n+1}).$$

Now by results of chapter 3 we have the following $\mathbb{Q}_p$-affine schemes and algebraic maps between them

$$H^1(G_T,G_{\text{ét},n}) \xrightarrow{\text{res}} H^1(G_v,G_{\text{ét},n}) \xleftarrow{c} W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)$$

By taking k_v-points, we obtain

$$H^1(G_T,G_{\text{ét},n})(k_v) \xrightarrow{\text{res}} H^1(G_v,G_{\text{ét},n})(k_v) \supset W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)(k_v).$$

But by the definition of the Weil restriction, one has

$$W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)(k_v) = (G_{dR,n}/F^0)(k_v\otimes_{\mathbb{Q}_p}k_v),$$

and the inclusion $k_v \subset k_v\otimes_{\mathbb{Q}_p}k_v$ gives us a projection

$$W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)(k_v) = (G_{dR,n}/F^0)(k_v\otimes_{\mathbb{Q}_p}k_v) \twoheadrightarrow (G_{dR,n}/F^0)(k_v).$$

Note that the image of integral points $X(\mathcal{O}_S)$ in $H^1(G_v, G_{\text{ét},n})(k_v)$ lies in the image of $W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0)(k_v)$ under the comparison map c and hence can be projected to $(G_{dR,n}/F^0)(k_v)$. Now the inequality

$$\mathfrak{D}_n < \dim_{k_v}(G_{dR,n+1}/F^0)$$

for a large n implies that the image of $X(\mathcal{O}_S)$ is not dense in $(G_{dR,n+1}/F^0)(k_v)$ for such a number n. On the other hand, we saw in Theorem 3.3.1 that de Rham period maps from the open unit p-adic balls to $(G_{dR,n+1}/F^0)(k_v)$ have Zariski dense image. Hence the integral points $X(\mathcal{O}_S)$ are not dense in any open unit p-adic ball centered at any S-integral point and we are done. □

4.3 Descent to Lower Dimensions

In this section we are going to briefly explain some observations which lead to an understanding of the structure of unipotent fundamental groups of unirational varieties in higher dimensions. The essential observation is that using different versions of the Lefschetz hyperplane section theorem, many questions concerning the structure of unipotent fundamental groups of varieties in higher dimensions can be reduced to the one dimensional case. Let us start with some generalities on Lefschetz hyperplane section theorem.

It is very well known in Algebraic Topology that low degree (co)homology and homotopy groups of a CW-complex depend only on the low dimensional skeleton of that CW-complex. For example the fundamental group of a CW-complex is completely determined by its 2-skeleton. This is essentially because when we attach a high dimensional cell to a CW-complex both the cell itself and its boundary along which we glue it to the CW-complex are trivial in low degrees and hence cannot alter (co)homology and homotopy groups in low degrees. It is expected that if we take a generic hyperplane section of a (quasi)-projective variety of high dimension, its (co)homology and homotopy groups in low degrees remain unchanged. Lefschetz has made a precise statement in this direction which is called "Lefschetz hyperplane section theorem" or "weak Lefschetz theorem". There are many versions of this theorem which can be easily found in the literature, but for reader's convenience we recall one of them.

Let M be a d-dimensional (quasi)-projective complex manifold embedded into some complex projective space, and let H be a generic hyperplane of the ambient projective space. Then if we denote the hyperplane section $M \cap H$ by M', one can attach some d-cells to M' in such a way that the resulting space becomes homotopically equivalent to M. As an immediate corollary one obtains that the map induced by inclusion from i^{th} homology and homotopy groups of M' to those of M are isomorphisms when $i < d-1$, and are surjections when $i = d-1$. Note that sometimes this corollary is referred to as the Lefschetz hyperplane section theorem.

In this section we are going to put together different versions of Lefschetz hyperplane section theorem, which are applicable to different realizations of unipotent fundamental groups, and obtain in some sense a motivic Lefschetz hyperplane section theorem which can be applied to the unipotent fundamental group of the varieties under our study, as pro-objects in the category of mixed realizations, and hence to the motivic unipotent fundamental groups, when they exist.

Note that the above mentioned Lefschetz hyperplane section theorem provides us with the Betti version. Fortunately the étale version of that theorem exists as well in the literature. In the proper case, we have the following theorem from SGA1.

Theorem 4.3.1. *[21, Lemme 2.10., Exposé X] Let X be a proper scheme over an algebraically closed field k and $g : X \to \mathbb{P}^r_k$ a morphism. Suppose that X is irreducible and normal and* $\dim(g(X)) \geq 2$*. Consider a generic hyperplane H in $\mathbb{P}^r_k$ and put $Y := X \times_{\mathbb{P}^r_k} H$. Then Y is connected and the homomorphism $\pi_1(Y) \to \pi_1(X)$ between corresponding étale fundamental groups is surjective. Moreover this morphism between étale fundamental groups is an isomorphism if* $\dim(g(X)) \geq 3$.

The reader may have noticed that the properness assumption in the above theorem will be annoying for us because we are going to apply that to complements of normal crossing divisors in projective varieties. Actually if we were interested in positive characteristics we would have a serious issue since the Lefschetz hyperplane section theorem is not valid for non-projective varieties over algebraically closed fields of positive characteristics. Fortunately the situation is much better in characteristic zero. Namely if we assume that our base field k is an algebraically closed field of characteristic zero, since everything we are concerned with is defined over a finitely generated subfield of k, such a subfield can be embedded into the field of complex numbers, and the étale fundamental group remains unchanged after all these base changes, we can assume that our base field k is the field of complex numbers. In this case one has the complex analytic space $X(\mathbb{C})$ whose fundamental group completely determines the étale fundamental group of X. Namely if one considers the usual fundamental group of the set of complex points $X(\mathbb{C})$ of X, equipped with complex analytic topology, then its profinite completion will be the étale fundamental group of X. Finally one can drop that annoying properness condition in Theorem 4.3.1 dealing with complex analytic varieties and prove the same result for quasi-projective varieties. More precisely we have the following much more general result which we state carefully for future references.

Theorem 4.3.2. *[20, Theorem, Section 5.1., Part II.] Let X be a purely n-dimensional nonsingular connected algebraic variety. Let $\pi : X \to \mathbb{P}^N_{\mathbb{C}}$ be an algebraic map and let $H \subset \mathbb{P}^N_{\mathbb{C}}$ be a linear subspace of codimension c. Let H_δ be the δ-neighborhood of H with respect to some real analytic Riemannian metric. Define $\varphi(k)$ to be the dimension of the set of points $z \in \mathbb{P}^N_{\mathbb{C}} - H$ such that the fiber $\pi^{-1}(z)$ has dimension k ($-\infty$ if this set is empty). If δ is sufficiently small, then the homomorphism induced by inclusion, $\pi_i(\pi^{-1}(H_\delta)) \to \pi_i(X)$ is an isomorphism for all $i < \hat{n}$ and is a surjection for $i = \hat{n}$, where*

$$\hat{n} = n - \sup_k (2k - (n - \varphi(k)) + \inf(\varphi(k), c-1)) - 1.$$

Furthermore, in this theorem, π is not necessarily proper, and $\pi^{-1}(H_\delta)$ may be replaced by $\pi^{-1}(H)$ if H is generic or if π is proper. The assumption that X is algebraic may be replaced by the assumption that X is the complement of a closed subvariety of a complex analytic variety $\overline{X}$ and the π extends to a proper analytic map $\overline{\pi} : \overline{X} \to \mathbb{P}^N_{\mathbb{C}}$.

Being done with the étale realization, we are going to prove the same statement in the de Rham case as well. To do that once more we use the fact that we are over a field k of characteristic zero and give the following description of the unipotent de Rham fundamental group of the variety X in which we are interested. Let π_1 be the usual fundamental group of the associated complex variety $X(\mathbb{C})$, and consider the group ring $A := k[\pi_1]$. Then the completion

$$\hat{A} := \varprojlim_n A/I^n$$

of A with respect to the augmentation ideal I admits a co-product

$$\Delta : \hat{A} \to \hat{A} \hat{\otimes} \hat{A}$$

which makes it into a completed Hopf algebra over k. Then the Lie algebra of the primitive elements in $\hat{A}$ is a unipotent (not necessarily finite dimensional) Lie algebra over k which, by general Malcev̌ correspondence, is associated to a pro-unipotent affine group scheme over k. One can show that this pro-unipotent affine group scheme is nothing else than the de Rham fundamental group G_{dR}. Going carefully through these constructions one sees easily that for a map $X \to Y$ which induces a surjection (resp. an isomorphism)

$$\pi_1(X(\mathbb{C})) \twoheadrightarrow \pi_1(Y(\mathbb{C}))$$

on the usual fundamental groups of the associated complex varieties, the corresponding map on the above mentioned Lie algebras, and hence the one on the de Rham fundamental groups, is also a surjection (resp. an isomorphism). Putting all these together one obtains the following version of the Lefschetz hyperplane section theorem for the mixed realizations of the unipotent fundamental group.

Theorem 4.3.3. *Let X be a smooth d-dimensional variety over k and let $\overline{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ be a smooth projectivization of X such that $D := \overline{X} - X$ is a divisor with normal crossing and smooth irreducible components. Consider a generic linear $(N-d+l)$-dimensional subspace of the ambient projective space $\mathbb{P}^N_{\mathbb{C}}$, and fix a base point $x \in X \cap H$. Then if $l \geq 1$ (resp. $l \geq 2$) the inclusion $X \cap H \hookrightarrow X$ induces a surjection (resp. an isomorphism)*

$$\pi_1^{\mathcal{R}}(X \cap H, x) \twoheadrightarrow \pi_1^{\mathcal{R}}(X, x)$$

on the corresponding fundamental groups in the category of mixed realizations.

The final step will be the motivic version of the above theorem. So assume that the variety X appeared in the above theorem and the hyperplane H are such that $\pi_1^{\mathcal{R}}(X \cap H, x)$ and $\pi_1^{\mathcal{R}}(X, x)$ are both motivic (for example assume X and $X \cap H$ satisfy the hypothesis of Theorem 1.3.1). Then we have the following diagram.

$$\begin{tikzcd}
\pi_1^{\mathcal{R}}(X \cap H, x) \arrow[r, two heads] \arrow[d, "\mathrm{real}"] & \pi_1^{\mathcal{R}}(X, x) \arrow[d, "\mathrm{real}"'] \\
\pi_1^{\mathrm{mot}}(X \cap H, x) \arrow[r, dotted, two heads, "\phi"] & \pi_1^{\mathrm{mot}}(X, x)
\end{tikzcd}$$

Note that the existence of ϕ in the above diagram is guaranteed by the fullness assertion in Theorem 1.2.3 and moreover ϕ must be surjective (resp. isomorphism) when it becomes so after taking different realizations. This finally gives us the motivic Lefschetz hyperplane section theorem, namely we have

Theorem 4.3.4 (Motivic Lefschetz Hyperplane Section Theorem)**.** *Let D be a normal crossing divisor with smooth irreducible components in a projective smooth variety $\overline{X}$ over a number field k and put $X = \overline{X} - D$. Assume moreover that X is smooth, has dimension at least 2 (resp. at least 3), and that $\pi_1^{\mathcal{R}}(X, x)$ is motivic. Fix a closed immersion $\overline{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ and consider a generic linear subspace H of the ambient projective space through x. If*

$X \cap H$ has positive dimension (resp. dimension at least 2) and $\pi_1^{\mathcal{R}}(X \cap H, x)$ is motivic as well then the map

$$\pi_1^{\mathrm{mot}}(X \cap H, x) \to \pi_1^{\mathrm{mot}}(X, x)$$

induced by inclusion is surjective (resp. isomorphism).

Note that the same result as above is obviously valid for the algebraic quotients $\pi_{1,n}^{\mathrm{mot}}$'s, for any $n \geq 1$. Hence if we denote the kernel of the projection

$$\pi_{1,n+1}^{\mathrm{mot}} \twoheadrightarrow \pi_{1,n}^{\mathrm{mot}}$$

by K_n and we continue with all the assumptions under which the above theorem holds, we get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K_n(X \cap H) \arrow[r] \arrow[d, dotted, "\psi"] & \pi_{1,n+1}^{\mathrm{mot}}(X \cap H, x) \arrow[r] \arrow[d, two heads] & \pi_{1,n}^{\mathrm{mot}}(X \cap H, x) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K_n(X) \arrow[r] & \pi_{1,n+1}^{\mathrm{mot}}(X, x) \arrow[r] & \pi_{1,n}^{\mathrm{mot}}(X, x) \arrow[r] & 0
\end{tikzcd}$$

We have seen in the previous section that when $X \cap H$ is a punctured projective line, then $K_n(X \cap H)$ is isomorphic to some power of the Tate object $\mathbb{Q}(n)$. On the other hand, $\mathbb{Q}(n)$ is a simple object in $MT(k)$. So if we knew for example that the map ψ in the above diagram is a surjection, we could deduce that $K_n(X)$ is also isomorphic to some power of $\mathbb{Q}(n)$. This fact is easy to proof if we take the following interpretation of these K_n's.

For any group G let us denote the descending central series of G by $Z^\bullet G$. Then one has

$$\pi_{1,n}^{\mathrm{mot}}(X, x) \cong \pi_1^{\mathrm{mot}}(X, x)/Z^n(\pi_1^{\mathrm{mot}}(X, x)),$$

and hence one evidently has

$$K_n(X) \cong Z^n(\pi_1^{\mathrm{mot}}(X, x))/Z^{n+1}(\pi_1^{\mathrm{mot}}(X, x)).$$

On the other hand, the epimorphisms in the statement of the Theorem 4.3.4 obviously respect the descending central series of the involving fundamental groups and they remain surjective after being restricted to the n^{th} lower central subgroups Z^n's. Namely under the notations and assumptions of the Theorem 4.3.4 for any $n \geq 1$ we get surjections

$$Z^n(\pi_1^{\mathrm{mot}}(X \cap H, x)) \twoheadrightarrow Z^n(\pi_1^{\mathrm{mot}}(X, x)),$$

and hence the following diagram

$$\begin{tikzcd}
Z^{n+1}(\pi_1^{\mathrm{mot}}(X \cap H, x)) \arrow[r, hook] \arrow[d] & Z^n(\pi_1^{\mathrm{mot}}(X \cap H, x)) \arrow[r, two heads] \arrow[d] & K_n(X \cap H) \arrow[d, "\psi"] \\
Z^{n+1}(\pi_1^{\mathrm{mot}}(X, x)) \arrow[r, hook] & Z^n(\pi_1^{\mathrm{mot}}(X, x)) \arrow[r, two heads] & K_n(X)
\end{tikzcd}$$

Now it is obvious from the above diagram that the map ψ is surjective and hence we get the following important corollary which will be of great use in proving our second main result in the next section.

Corollary 4.3.5. *Under the notations and hypothesis of the Theorem 4.3.4, if one assumes moreover that $(\overline{X}, X, D)$ forms a standard triple over k, then for any $n \geq 1$ one has the following exact sequence*

$$0 \to \mathbb{Q}(n)^{r'_n} \to \pi_{1,n+1}^{\mathrm{mot}}(X, x) \to \pi_{1,n}^{\mathrm{mot}}(X, x) \to 0,$$

where r'_n is the dimension of the vector group

$$Z^n(\pi_1^{\mathrm{mot}}(X, x))/Z^{n+1}(\pi_1^{\mathrm{mot}}(X, x)).$$

Now let k be a fixed number field, S be a finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S-integers of k. Fix also a standard triple $(\overline{X}, X, D)$ over $\mathcal{O}_S$, where $\overline{X}$ is a d-dimensional projective variety over $\mathrm{Spec}(\mathcal{O}_S)$. Finally fix an embedding

$$i : \overline{X} \hookrightarrow \mathbb{P}^N_{\mathcal{O}_S}$$

of $\overline{X}$ into an ambient projective space. Now by enlarging S to a bigger finite set of finite places S' of k, one can find an $(N-d+1)$-dimensional linear subspace $H \cong \mathbb{P}^{N-d+1}_{\mathcal{O}_{S'}}$ of $\mathbb{P}^N_{\mathcal{O}_{S'}}$ in such a way that $(\overline{X} \cap H, X \cap H, D \cap H)$ forms a standard triple over $\mathcal{O}_{S'}$ with $\dim_{\mathcal{O}_{S'}}(X) = 1$. Recall that if we make a base change to k, then by Bertini's theorem for a generic H, $(\overline{X} \cap H)_k$ is a smooth, projective, geometrically connected curve over k and irreducible components of the divisor $(D \cap H)_k$ are étale and absolutely irreducible over k. But if we replace S by a possibly larger finite set of finite places S', we can assume that $\overline{X} \cap H$, $D \cap H$, and hence the complement $C := X \cap H$ are all defined over $\mathcal{O}_{S'}$ and irreducible components of $D \cap H$ are smooth and surjective over $\mathrm{Spec}(\mathcal{O}_{S'})$. Finally note that this enlargement of S to a bigger finite set S' is harmless for us in the sense that the validity of our second main result, namely Theorem 4.4.3, for a smaller set S is a consequence of its validity for a larger set S'. So from now on, we assume without loss of generality that H, $\overline{X} \cap H$, $D \cap H$, and $C = X \cap H$ are all defined over $\mathcal{O}_S$

and $(\overline{X} \cap H, C, D \cap H)$ forms a standard triple over $\mathcal{O}_S$, with $C/\mathrm{Spec}(\mathcal{O}_S)$ a relative curve.

Our main interest is in proving that the Diophantine set $X(\mathcal{O}_S)$ is not locally p-adic analytically dense (see Theorem 4.4.3). If this set is not Zariski dense then there is nothing to prove, otherwise we can choose H in such a way that the intersection C contains an $\mathcal{O}_S$-point. Fix such a point $x \in C(\mathcal{O}_S)$ as the base point in the sequel.

Let us start with the de Rham period map. Let v be a finite place of k outside S and let $x_{k_v} \in X_{k_v}(k_v)$ be the point induced by x. Let $\mathcal{C}_{dR}$ be the category of unipotent vector bundles over $\overline{X}_{k_v}$ equipped with unipotent integrable logarithmic connection along D_{k_v}. Note that here, like in section 2.1, a unipotent object is an object which is an iterated extension of the trivial objects. Taking the fiber at the point x_{k_v} is a fiber functor which makes $\mathcal{C}_{dR}$ a neutral Tannakian category over k_v. Now by applying the general Tannakian formalism of section 1.1, more precisely by applying Theorem 1.1.6, we get the pro-unipotent group scheme $G_{dR}(X_{k_v}, x_{k_v})$. The algebraic quotients $G_{dR,n}(X_{k_v}, x_{k_v})$ can be defined as the quotients of $G_{dR}(X_{k_v}, x_{k_v})$ by its descending central series. Note that here we should impose the integrability condition on the connections in the definition of $\mathcal{C}_{dR}$, while in section 2.1 it was automatic due to one dimensionality. Moreover, for any other point $y \in X_{k_v}(k_v)$ one can apply Theorem 1.1.8 to obtain path torsors $G_{dR}(X; x_{k_v}, y)$ and $G_{dR,n}(X; x_{k_v}, y)$'s, respectively over the groups $G_{dR}(X_{k_v}, x_{k_v})$ and $G_{dR,n}(X_{k_v}, x_{k_v})$'s.

Note that everything in section 2.3 can be applied to arbitrary dimensions and hence one can endow these de Rham fundamental groups with Frobenius action. Moreover since the de Rham fundamental group of C_{k_v} surjects onto the de Rham fundamental group of X_{k_v} (see Theorem 4.3.3), the Hodge filtration on the de Rham fundamental group of C_{k_v} induces a Hodge filtration on the de Rham fundamental group of X_{k_v}. There are also compatible extra structures on path torsors (see [8, sections 11 and 12]). Similar to the one dimensional case, these extra structures make the path torsors non-trivial, and here again the Frobenius action alone is not sufficient for this purpose. More precisely, by similar arguments as in section 3.3, one can show that all the de Rham path torsors become trivial after forgetting the Hodge filtration. That is, if v lies above the rational prime p, one has

$$\mathcal{O}_{G_{dR,n}(X_{k_v}; x_{k_v}, y)} \cong \mathcal{O}_{G_{dR,n}(X_{k_v}, x_{k_v})}, \quad \forall\, n \geq 1,$$

as Frobenius modules over $W(\mathbb{F}_v)[1/p] \subset k_v$. This gives us the following de Rham period maps

$$p_{dR}^{(n)} : X_{k_v}(k_v) \to G_{dR,n}/F^0(G_{dR,n}).$$

Finally we have the analogue of the key Theorem 3.3.1, namely

Theorem 4.3.6. *For any $n \geq 1$, the restriction of the period map $p_{dR}^{(n)}$ to the p-adic integral points in the p-adic open unit ball around x_{k_v} gives a rigid k_v-analytic map with Zariski dense image in $G_{dR,n}/F^0(G_{dR,n})$.*

Proof. Zariski density of the image is a consequence of the same fact in the one dimensional case ([17, Section 4]), and commutativity of the following diagram in which the right vertical map is surjective (see Theorem 4.3.3).

$$\begin{array}{ccc} C_{k_v}(k_v) & \xrightarrow{p_{dR}^{(n)}} & G_{dR,n}(C_{k_v}, x_{k_v})(k_v) \\ \downarrow & & \downarrow \\ X_{k_v}(k_v) & \xrightarrow{p_{dR}^{(n)}} & G_{dR,n}(X_{k_v}, x_{k_v})(k_v) \end{array}$$

The proof of rigid analyticity goes as follows. If one restricts the universal Frobenius crystal G_{dR} to the p-adic open unit ball centered at x_{k_v}, one gets a Frobenius crystal on $W(\mathbb{F}_v)\{\{t_1, \ldots, t_d\}\}$, where d is the dimension of the variety X. This Frobenius crystal is constant as noted above. So if we denote the Hodge filtration at the base point x_{k_v} by F_0, the filtration at another point y in the open p-adic unit ball is given by $g(y)F_0$ where g is a rigid k_v-analytic map from $W(\mathbb{F}_v)\{\{t_1, \ldots, t_d\}\}$ to $G_{dR}/F^0(G_{dR})$. Note that the variation of the Hodge filtration is algebraic on the original universal Frobenius crystal, but it becomes only rigid k_v-analytic after making it constant over the p-adic open unit ball because the process of constantification involves rigid k_v-analytic transformations which are not necessarily algebraic. □

We skip repeating the definitions of the local and global étale period maps due to their similarity with the definitions given in section 3.3. Finally note that one can put these things together to obtain the higher dimensional analogue of the commutative diagram of the Remark 3.3.2. The arguments of section 4.1 are applicable to any dimension if one considers the following observation. By Theorem 4.1.1, after passing to dual, one obtains that $G_{\text{ét},1}$ and $G_{dR,1}$ are associated in the sense of Fontaine's theory. On the other hand, the constant p-adic étale sheaf $\widetilde{\mathbb{Q}_p}(m)^{\otimes n}$ with fiber $\mathbb{Q}_p(m)^{\otimes n}$ is obviously associated to the constant filtered Frobenius crystal $\widetilde{K_0}(m)^{\otimes n}$ with fiber $K_0(m)^{\otimes n}$, for all integers m and all $n \geq 1$. Now we can use Corollary 4.3.5 to see that $\mathcal{P}_{\text{ét},n+1}$ (resp. $\mathcal{P}_{dR,n+1}$) is the extension of $\mathcal{P}_{\text{ét},n}$ (resp. $\mathcal{P}_{dR,n}$) by $\widetilde{\mathbb{Q}_p}(n)^{\otimes r'_n}$ (resp. $\widetilde{K_0}(n)^{\otimes r'_n}$). One can also check that these extensions have associated classes in proper Ext^1's (for example in cases of interest to us this is true because both these extensions are two different realizations of the same extension of Corollary 4.3.5 for motivic fundamental groups). All these imply, by induction on n, that for any $n \geq 1$ the p-adic étale sheaf $\mathcal{P}_{\text{ét},n}$ is associated in the sense of Faltings' theory to the filtered Frobenius crystal $\mathcal{P}_{dR,n}$. Now we get the analogue of the Remark 4.1.2 in higher dimensions and hence the commutativity of the diagram analogue to the fundamental diagram appeared in Remark 3.3.2.

4.4 General Case

Here we are going to prove the second main result of this thesis, which generalizes the first main result to unirational varieties of arbitrary dimension. In order to state that result we need a new notion, which we call $\mathcal{V}$-property ($\mathcal{V}$ for vanishing). Since this notion is very crucial in our result, let us first give an exact definition of it. We continue with the notations and conventions made before. So let k be a number field, S be a finite set of finite places of k, $\mathcal{O}_S$ be the ring of S-integers of k and so on.

Definition 4.4.1 ($\mathcal{V}$-property)**.** *Let X be a variety over $\mathcal{O}_S$ and fix a finite place v of k which lies over a rational prime p. For a given S-integral point $x \in X(\mathcal{O}_S)$, we say that X satisfies the $\mathcal{V}_{S,v}$-property at x, if there exists a nonzero p-adic analytic function on the p-adic open unit ball centered at x_{k_v} which vanishes on the image of any other S-integral point of X in that ball. We say that X satisfies $\mathcal{V}_{S,v}$-property if it satisfies $\mathcal{V}_{S,v}$-property at every S-integral point.*

Remark 4.4.2. Note that there are only finitely many p-adic unit balls in X_{k_v}, because they only depend on the reduction modulo p of the center. On the other hand, when X is a curve, these balls are nothing else than p-adic unit disks, and it is very well known that a nonzero p-adic analytic function has only finitely many zeros over a finite extension of $\mathbb{Q}_p$ in such a disk. As a consequence, a curve X satisfies $\mathcal{V}_{S,v}$-property for at least one v if and only if it has only finitely many S-integral points.

Moreover, notice that if $S \subset S'$ are two finite sets of finite places of k, then $\mathcal{V}_{S',v'}$-property implies $\mathcal{V}_{S,v}$-property for any $v'|v$. $\diamondsuit$

Before stating and proving the main result, let us sketch the general underlying idea, which is the higher dimensional analogue of the idea that we used in section 4.2. Suppose X is a given variety over $\mathcal{O}_S$, where $\mathcal{O}_S$ is the ring of S-integers in a number field k for a finite set of finite places S. We are interested in proving that X satisfies the $\mathcal{V}_{S,v}$-property for some finite place v of k. Suppose that X is nice enough so that we can apply the theories and techniques of previous sections to it. Take a finite place v of k outside of S in such a way that X has good reduction at v. Note that we can choose such a v which lies over an arbitrary large rational prime number p. Then in an analogous manner we obtain a diagram for X similar to that in Remark 3.3.2. Suppose now that one can estimate dimensions of the algebraic varieties appeared in that diagram and prove by any method that for some natural number n one has

$$\mathfrak{D}_n < \dim(G_{dR,n}/G_{coh,n}),$$

where $\mathfrak{D}_n$ is defined to be

$$\mathfrak{D}_n := \dim(\mathrm{Im}(p_{\text{ét}}^{\text{glob},(n)}(X(\mathcal{O}_S)))).$$

In this section again, like in section 4.2, by the dimension of a subset of an algebraic variety, we mean the dimension of its Zariski closure. It follows then that the Zariski closure of the image of $X(\mathcal{O}_S)$ in $G_{dR,n}/F^0(G_{dR,n})$, which has dimension at most $\mathfrak{D}_n$, is not Zariski dense. Hence there exists a nonzero algebraic function on $G_{dR,n}/F^0(G_{dR,n})$ which vanishes on this image. Since the open p-adic unit ball around $x_v \in X_{k_v}$, $\mathcal{B}_1^\circ(x_v)$, has Zariski dense image in $G_{dR,n}/F^0(G_{dR,n})$ (Theorem 3.3.1), the pull back of this nonzero function gives a nonzero p-adic analytic function on $\mathcal{B}_1^\circ(x_v)$, which vanishes on every integral point in $\mathcal{B}_1^\circ(x_v)$ and we are done.

Recall that after fixing a standard triple $(\overline{X}, X, D)$ over $\mathcal{O}_S$, the set

$$\{r'_n = \dim(Z^n(\pi_1^{\mathrm{mot}}(X,x))/Z^{n+1}(\pi_1^{\mathrm{mot}}(X,x))) \ : \ n \geq 1\}$$

is the set of natural numbers appeared in Corollary 4.3.5. Now we can state and prove our second main result, namely we have

Theorem 4.4.3. *Let $k/\mathbb{Q}$ be a totally real number field of degree $d \geq 2$ (or let $k = \mathbb{Q}$ and put $d = 2$), S be any finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S-integers in k. Consider a fixed standard triple $(\overline{X}, X, D)$ over $\mathcal{O}_S$ such that $h^{1,0}(\overline{X}) = 0$. Finally assume that for any constant $c \in \mathbb{N}$ there exists a natural number $n \in \mathbb{N}$ such that*

$$c + d(r'_3 + r'_5 + \cdots + r'_{2\lfloor (n-1)/2 \rfloor + 1}) < r'_1 + r'_2 + \cdots + r'_n$$

Then for almost all finite places v of k, X satisfies the $\mathcal{V}_{S,v}$-property.

Proof. First of all note that the technical condition $h^{1,0}(\overline{X}) = 0$ helps us get rid of the zeroth part of the Hodge filtration on the de Rham (pro)-unipotent fundamental group. By this we mean that when $h^{1,0}(\overline{X}) = 0$, then $F^0(G_{dR})$

and $F^0(G_{dR,n})$'s are all zero as well and hence we can replace all the quotients $G_{dR}/F^0(G_{dR})$ and $G_{dR,n}/F^0(G_{dR,n})$'s, which appear as the target space of the de Rham period maps, by the more simple spaces G_{dR} and $G_{dR,n}$'s (see [22, Remark 1.5]).

Note also that by the last part of Remark 4.4.2 we can replace S by a larger finite set of finite places of k without loose of generality. This allows us to apply the results of section 4.3 and also motivic Lefschetz hyperplane section theorem for the integral versions of the motivic unipotent fundamental groups and path torsors over them.

Now by de Rham realization of the exact sequence in Corollary 4.3.5, for any $n \geq 1$ we have the following exact sequences of unipotent group schemes

$$0 \to K_v(n)^{r'_n} \to G_{dR,n+1} \to G_{dR,n} \to 0.$$

One obtains then that

$$\dim_{k_v}(G_{dR,n+1}/F^0(G_{dR,n+1})) = \dim_{k_v}(G_{dR,n+1}) = r'_1 + r'_2 + \cdots + r'_n.$$

Now in order to estimate the numbers $\mathfrak{D}_n$, firstly we consider the étale realization of the exact sequence of Corollary 4.3.5, which gives the following exact sequence of unipotent group schemes for any $n \geq 1$:

$$0 \to \mathbb{Q}_p(n)^{r'_n} \to G_{\text{ét},n+1} \to G_{\text{ét},n} \to 0.$$

Note that these r'_n's are the same as the ones above that appeared in the de Rham case, because we are working with different realizations of the same motivic exact sequence. Now we are interested in studying the image of $X(\mathcal{O}_S)$ in $H^1(G_T, \mathbb{Q}_p(n))$ for $n \geq 1$. Since the path torsors coming from $\mathcal{O}_S$-points of X are motivic and they lie in the category $MT(\mathcal{O}_T)$ (see Theorem 1.3.2), the map $p_{\text{ét}}^{glob,(n)}$ factors through the motivic cohomology groups

$$H^1(MT(\mathcal{O}_T), \mathbb{Q}(n))$$

whose dimension can be computed as follows. For any $n \geq 2$ we have

$$H^1(MT(\mathcal{O}_T), \mathbb{Q}(n)) \cong \operatorname{Ext}^1_{MT(\mathcal{O}_T)}(\mathbb{Q}(0), \mathbb{Q}(n)) = \operatorname{Ext}^1_{MT(k)}(\mathbb{Q}(0), \mathbb{Q}(n)),$$

where the last equality is a consequence of the first part of Proposition 1.2.2. On the other hand, since $DMT(k)_{\mathbb{Q}}$ is the derived category of the abelian category $MT(k)$, any extension

$$0 \to B \to E \to A \to 0$$

of A by B in $MT(k)$, leads functorially and in a unique way to a distinguished triangle

$$B \to E \to A \to B[1]$$

in $DMT(k)_{\mathbb{Q}}$. This gives a map

$$\mathrm{Ext}^1_{MT(k)}(A, B) \to \mathrm{Hom}^1_{DMT(k)_{\mathbb{Q}}}(A, B),$$

which can be shown to be a bijection. If we take into account this bijection, we can push our calculations one step forward, and see that for all $n \geq 2$

$$H^1(MT(\mathcal{O}_T), \mathbb{Q}(n)) \cong \mathrm{Hom}^1_{DMT(k)_{\mathbb{Q}}}(\mathbb{Q}(0), \mathbb{Q}(n)).$$

But as we discussed in section 1.2, the right hand side of the above isomorphism is isomorphic to

$$K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This rational K-groups have been explicitly computed by Borel (see [4, section 12]), and in our case, where k is totally real of degree d, one has

$$\dim_{\mathbb{Q}}(K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}) = \begin{cases} 0 & n > 1 \text{ is even} \\ d & n > 1 \text{ is odd} \end{cases}$$

Finally note that the rank α of S-units of the number field k is equal to

$$\alpha = \dim(H^1(G_T, \mathbb{Q}_p(1))).$$

By putting all these together, we obtain

$$\mathfrak{D}_n \leq \alpha + d(r'_3 + r'_5 + \cdots + r'_{2\lfloor (n-1)/2 \rfloor + 1}).$$

Now by our hypothesis on the numbers r'_n's one has

$$\mathfrak{D}_n < \dim(G_{dR,n+1}/F^0(G_{dR,n+1}))$$

for some natural number $n \in \mathbb{N}$. The rest of the proof is exactly the same as the final part of the proof of the Theorem 4.2.1 and will be omitted. □

We finish this section by proposing the following conjecture which seems to be very out of reach at the moment. In some sense it is actually true that this conjecture is very far reaching since there are serious obstacles in generalizing the methods of this work to prove it, but all the examples that author managed to check explicitly were supporting it. See the examples and remarks of the following section for more discussion of this.

Conjecture 4.4.4. *Let k be a number field, S be a finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S-integers in k. Let also $\bar{X}$ be a smooth projective variety over* $\mathrm{Spec}(\mathcal{O}_S)$*, D be a relative divisor in $\bar{X}$, and put $X := \bar{X} - D$ which is smooth over* $\mathrm{Spec}(\mathcal{O}_S)$*. Finally assume that the descending central series of the fundamental group of $X(\mathbb{C})$ never stops, i.e. the corresponding numbers r'_n are positive for all $n \in \mathbb{N}$. Then X satisfies $\mathcal{V}_{S,v}$ property for almost all finite places v of k.*

4.5 Remarks and Questions

In this final section, we are going to make some general remarks concerning Theorem 4.4.3 and Conjecture 4.4.4. This, hopefully, not only shows the kind of situations to which we can apply Theorem 4.4.3, but also gives the main obstructions for proving Conjecture 4.4.4. Note that in the sequel we follow the same notations and conventions of the previous sections.

Remark 4.5.1 (Reduction to the 2-dimensional case)**.** Fix a standard triple $(\overline{X}, X, D)$ over $\mathcal{O}_S$ of dimension $d \geq 3$ and assume that $\overline{X}$ has been embedded into an ambient projective space over $\mathrm{Spec}(\mathcal{O}_S)$. Now for a generic hyperplane H in the ambient projective space, the hyperplane section $(\overline{X}, X, D)_H$ of $(\overline{X}, X, D)$ is a standard triple over $\mathcal{O}_{S'}$ for a finite set of finite places S' of k. Then a very important feature of Theorem 4.4.3 is that $(\overline{X}, X, D)$ satisfies its hypotheses if and only if $(\overline{X}, X, D)_H$ does. Simply because by the material of section 4.3, specially by Theorem 4.3.4, when we take a generic hyperplane section the motivic fundamental group and the first cohomology group remain unchanged as long as the dimension of the section is not less than 2. On the other hand, the numbers r'_n which appear in Theorem 4.4.3 are invariants of the fundamental group of X. It is also clear that when H^1 remains unchanged, $h^{1,0}$ does not change as well. The importance of this observation is that it tells us that if we want to prove $\mathcal{V}_{S,v}$-property for a standard triple of dimension $d \geq 3$, in order to check the hypotheses of Theorem 4.4.3 we can take iterated generic hyperplane sections and reduce the problem to the case of dimension 2. This is very remarkable about Theorem 4.4.3 that it reduces an arithmetic problem concerning a family of varieties with arbitrary dimension to a problem which in turn can be reduced to the case of surfaces. Hence in the following, we restrict to surfaces. ♠

Remark 4.5.2 (Divisors with not-necessarily normal crossing and smooth irreducible components)**.** First of all recall that traditionally the number $h^{1,0}$ of a surface is called the irregularity of the surface, hence surfaces for which $h^{1,0} = 0$ are called regular surfaces. Now let $\overline{X}$ be a proper smooth surface over $\mathcal{O}_S$, D be a divisor such that the irreducible components of its generic fiber are absolutely irreducible, and let x be a point in $\overline{X}$ which lies in D. Let $\widetilde{\overline{X}}$ be the blow up of $\overline{X}$ at the point x, and put

$$D' := D + E$$

where E is the exceptional divisor of the blow up. Now $\overline{X} - D$ and $\widetilde{\overline{X}} - D'$ are isomorphic and hence have isomorphic fundamental groups. On the other hand, since blowing up a surface does not change its first cohomology (simply because the blow up $\widetilde{\overline{X}}(\mathbb{C})$ is homeomorphic to the connected sum $\overline{X}(\mathbb{C})\#\overline{\mathbb{CP}}$), $\widetilde{\overline{X}}$ remains regular if $\overline{X}$ is regular.

Now suppose that we start with a proper smooth uni-rational surface $\overline{X}$ over $\mathcal{O}_S$ and a divisor D in $\overline{X}$ such that the irreducible components of the generic fiber of D are absolutely irreducible (note that we do not assume normal crossing or smoothness of the irreducible components). Suppose that after finitely many blow ups the irreducible components of D become smooth and their intersections become normal. Then if we denote the resulting space by $\widetilde{\overline{X}}$, by D' the sum of D with all the resulting exceptional divisors, and by X the complement $\widetilde{\overline{X}} - D'$, we get a standard triple $(\widetilde{\overline{X}}, X, D')$ over $\mathcal{O}_S$ (of course the blow up of a uni-rational surface is uni-rational). Assume we were interested in proving $\mathcal{V}_{S,v}$-property for the complement $\overline{X} - D$ for some finite place v of k. Since X and $\overline{X} - D$ are isomorphic over $\mathrm{Spec}(\mathcal{O}_S)$, $\mathcal{V}_{S,v}$-property is equivalent for them. On the other hand, since they have the same fundamental group, the numbers r'_n which appear in Theorem 4.4.3 are also the same for them. Hence if the numbers r'_n satisfy the hypotheses of Theorem 4.4.3, we can apply Theorem 4.4.3 to the standard triple $(\widetilde{\overline{X}}, X, D')$, conclude the $\mathcal{V}_{S,v}$-property for X and hence for the complement $\overline{X} - D$.

The significance of this observation is that it allows us to work with complements of singular divisors with not necessarily normal crossing in surfaces, in particular in $\mathbb{P}^2$. This is very important for us because of the well known fact that the fundamental group of the complement in $\mathbb{P}^2$ of a divisor with normal crossing whose irreducible components have at worse nodal singularities is abelian (see [7, Theorem 1]). Hence the descending central series of the fundamental group of the complement of such divisors stops after the first step and we cannot apply Theorem 4.4.3 to them. But now we can remove divisors with worse intersections and singularities to get non-abelian fundamental groups to which Theorem 4.4.3 is applicable (see the following remark for instance). ♡

Remark 4.5.3 (Golod-Shafarevich condition)**.** Recall that with the hypothesis of Theorem 4.4.3 we have the motivic fundamental group $\pi_1^{\mathrm{mot}}(X, x)$

which is a pro-unipotent algebraic group scheme over k. Then we defined the numerical invariants r'_n's to be the dimensions of the vector groups

$$Z^n(\pi_1^{\mathrm{mot}})/Z^{n+1}(\pi_1^{\mathrm{mot}}),$$

where $Z^\bullet$ is the lower central series of the group. Then one important assumption in Theorem 4.4.3 is that for any constant number $c \in \mathbb{N}$,

$$\exists n \in \mathbb{N} \quad : \quad c + d(r'_3 + r'_5 + \cdots + r'_{2\lfloor (n-1)/2 \rfloor + 1}) < r'_1 + r'_2 + \cdots + r'_n,$$

where d is the degree of the number field k over $\mathbb{Q}$. In particular, if one can show that the numbers r'_n grow exponentially like $r'_n \sim d^n$, then the above condition is automatic for big enough even n's. In this remark we want to discuss the Golod-Shafarevich condition which guarantees exponential growth of r'_n's.

In general the lower central series $Z^\bullet$ of a pro-unipotent group G satisfies

$$\bigcap_n Z^n(G) = (0).$$

Hence for any nontrivial element $g \in G$ there is a well defined degree d_g which is the unique number n such that

$$g \in Z^n(G) - Z^{n+1}(G).$$

Now suppose that G is finitely presented in the category of pro-unipotent groups over a field of characteristic zero and we are given a presentation

$$G = < g_1, \ldots, g_d | s_1, \ldots, s_r >$$

of G. Assume moreover that all the relations s_i have degree not less than $m \geq 2$ (note that the condition $m \geq 2$ is automatic if we start with a minimal set of generators). Now the Poincaré series $P_G(t)$ of G is defined to be

$$P_G(t) := \sum_{n=0}^{\infty} \mathrm{rank}(Z^n(G)/Z^{n+1}(G)) t^n.$$

With all these notation, one can show

$$\frac{P_G(t).(1 - dt + rt^m)}{1 - t} \geq \frac{1}{1 - t},$$

where the inequality is a term wise inequality for coefficients (see [27, Lemma 3.6.]). One says that G satisfies the Golod-Shafarevich condition if $1-dt+rt^m$

has a root in the open interval $(0,1)$. Now assume that G satisfies the Golod-Shafarevich condition and let $t_0 \in (0,1)$ be a root of $1-dt+rt^m$. Then by the above inequality between power series one sees that $P_G(t)$ cannot converge at t_0. This means that the sequence $\{a_n\}_n$ of ranks of the vector groups $Z^n(G)/Z^{n+1}(G)$ cannot be dominated by the sequence

$$\{\lfloor (1/t_0)^n \rfloor\}_n,$$

and hence a_n's must grow exponentially.

Back to our original situation if one can write a presentation of π_1^{mot} as a pro-unipotent group, which in principle is possible, then it is easy to check the Golod-Shafarevich condition. Then if π_1^{mot} satisfies the Golod-Shafarevich condition, the numerical assumption on numbers r'_n would be automatic for $d < \frac{1}{t_0}$, where t_0 is the above mentioned root in the open interval $(0,1)$. It worth mentioning that it is a reasonable expectation that the unipotent hull of the fundamental groups of the proper varieties whose complex valued points admit a Riemannian metric with strictly negative sectional curvatures (or when it is hyperbolic in the sense of Kobayashi), satisfy the Golod-Shafarevich condition. ♣

Remark 4.5.4 (Main obstructions in proving Conjecture 4.4.4)**.** As we mentioned before, in general the Conjecture 4.4.4 seems to be very difficult to prove. But among all difficulties, there are two main obstacles, when one tries to generalize the methods of this work to prove Conjecture 4.4.4. These obstacles are results of the two main generalizations in Conjecture 4.4.4. Namely, passing from unirational varieties to general ones, and relaxing the totally realness of the ground number field k and growth condition on numbers r'_n. We would like to discuss these two obstacles in this remark.

The first thing to note is that a key step in proving Theorem 4.4.3 was to replace $H^1(G_T, G_{\text{ét},n})$ by algebraic K-groups of the ground number field. We needed this to show that the dimension $\mathfrak{D}_n$ of $\mathrm{Im}(p_{\text{ét}}^{glob,(n)}(X(\mathcal{O}_S)))$ becomes strictly smaller than $\dim(G_{dR,n}/F^0(G_{dR,n})$ and deduce $\mathcal{V}_{S,v}$-property from it. At that point we crucially benefited from the fact that the fundamental group $G_{\text{ét}}$ and all the path torsors over that are the étale realizations of the pro-unipotent motivic fundamental group and motivic path torsors of our unirational variety in the category of mixed Tate motives. This fact helped us reduce the estimation of dimensions of the above global Galois cohomology groups to computing the dimension of motivic cohomology groups with values in some Tate motives, which in turn could be reduced to computing the ranks of rational K-groups of the ground number field, and finish by using Borel's result. In general, when the variety under consideration is not unirational, there are two problems. Firstly, although there are some hopes that the unipotent fundamental group, which has been constructed as a pro-unipotent group scheme in the category of mixed realizations in much more general settings, is always motivic, there is no motivic construction of it for general varieties yet. Secondly, even if one could show that the unipotent fundamental groups are motivic, they are certainly not mixed Tate in general. Simply because even the first homology, which is a tiny quotient of π_1, is not in general mixed Tate and can have odd weights (look for example at the first cohomology of projective curves of positive genus). So even in the case that we put ourselves in the category of mixed motives, in general we will certainly be outside the heaven of mixed Tate ones and hence computing or estimating the dimensions of motivic cohomology groups will remain a difficult problem.

The second obstacle that we are going to mention is about dropping the totally realness of the number field k in Conjecture 4.4.4. Note that in Theorem 4.4.3 we asked for the following condition.

$$\exists n \in \mathbb{N} \quad : \quad c + d(r'_3 + r'_5 + \cdots + r'_{2\lfloor (n-1)/2 \rfloor + 1}) < r'_1 + r'_2 + \cdots + r'_n,$$

where d is the degree of the totally real number field k over $\mathbb{Q}$. This helped us show that the image of integral points of an open p-adic unit ball by the de Rham period map is not Zariski dense in $G_{dR,n}$ and hence the $\mathcal{V}_{S,v}$-property. Note that the above condition is a strong condition in some sense since it essentially says that the numbers r'_n must grow at least as fast as the exponential sequence d^n. This puts a strong constraint on the degree of the totally real number field k to which we can apply Theorem 4.4.3. So why do we feel that one can forget this condition and only asks for non-vanishing of the numbers r'_n and also completely get rid of the total realness of k? We try to explain our intuition here.

First of all recall that for a general number field k of degree d over $\mathbb{Q}$, if we denote by r and s the number of real and conjugate pairs of complex embeddings of k, then by [4, Section 12] we have

$$\dim_{\mathbb{Q}}(K_{2n-1}(k) \otimes_{\mathbb{Z}} \mathbb{Q}) = \begin{cases} s & n > 1 \text{ is even} \\ r + s & n > 1 \text{ is odd} \end{cases}$$

Hence the argument in the proof of Theorem 4.4.3 shows that in this case one has

$$\mathfrak{D}_n = \alpha + s(r'_2 + r'_3 + \cdots + r'_n) + r(r'_3 + r'_5 + \cdots + r'_{2\lfloor (n-1)/2 \rfloor + 1}).$$

Obviously this number cannot be expected to be smaller than

$$\dim(G_{dR,n+1}/F^0(G_{dR,n+1})) = r'_1 + r'_2 + \cdots + r'_n$$

except when $s = 0$. This means that when k has complex places we cannot show that the image of integral points by the de Rham period map is not Zariski dense in $G_{dR,n+1}/F^0(G_{dR,n+1})$. But on the other hand, note that the domain of definition of the comparison map c is the Weil restriction

$$W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0(G_{dR,n}))$$

whose dimension is equal to

$$d(r'_1 + r'_2 + \cdots + r'_n).$$

Since $d = r + 2s$, as long as the numbers r'_n's are positive, for large enough n we always have

$$\mathfrak{D}_n < \dim(W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0(G_{dR,n})))$$

regardless of the number of complex places and the degree of k. This shows that in general the image of integral points by the de Rham period map cannot be Zariski dense in the Weil restriction of $G_{dR,n}/F^0(G_{dR,n})$. But note that by these computations not only the numbers $\mathfrak{D}_n$ become less than the dimension of $W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0(G_{dR,n}))$ but also the difference between these dimensions goes up to infinity as n tends to infinity. This gives the hope that the image of integral points by the de Rham period map must become of higher and higher co-dimension in $W_{k_v/\mathbb{Q}_p}(G_{dR,n}/F^0(G_{dR,n}))$ for larger and larger n. If one could have arranged to prove such high codimensionality, then the analogous results like Theorem 4.4.3, but for not necessarily totally real number fields, would have been a consequence. $\diamondsuit$

Summary

Let k be a totally real number field, S be a finite set of finite places of k, and $\mathcal{O}_S$ be the ring of S-integers in k. Let X be a smooth variety over $\mathcal{O}_S$ which admits a 'nice' projectivization and whose generic fiber is unirational (see Definition 1.3.3). In this thesis we show that if the fundamental group $\pi_1(X(\mathbb{C}), x)$ of the complex points of X is sufficiently non-abelian in a sense that is made precise, then S-integral points $X(\mathcal{O}_S)$ of X, locally in p-adic topology, lie in the zero locus of a non-zero p-adic analytic function for almost all rational prime p (see Theorem 4.2.1 and Theorem 4.4.3). This is done by studying the motivic unipotent fundamental group of X and path torsors over it, which are used to reduce the above assertion to an inequality between an expression involving ranks of rational K-groups of k and an expression involving ranks of subquotients of the descending central series of $\pi_1(X(\mathbb{C}), x)$.

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