Fourier–Mukai transform for twisted sheaves

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Hermes Jackson Martinez Navas

aus Bogota, Kolumbien

Bonn 2010

Angefertigt mit Genehmigung
der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich–Wilhelms–Universität Bonn

Erster Referent: Prof. Dr. Daniel Huybrechts
Zweiter Referent: Prof. Dr. Günter Harder

Tag der mündlichen Prüfung: 06.07.2010

Erscheinungsjahr 2010

For Jesus, my best friend!

Acknowledgements

I would like to thank my supervisor Daniel Huybrechts, for he has been of much help during my PhD. I really thank him for all the time he spent with me in his office and for all the suggestions and corrections he usually made. Another special word of thanks is given to Dr. Christian Kaiser, the coordinator of the IMPRS graduate programme, who helped me a lot with the IMPRS seminars at the Max–Planck–Institut für Mathematik and with all the discussions we had together.

I would like to thank Max–Planck–Institut für Mathematik, for the scholarship and all the travel grants to attend conferences.

I would like to thank Carolina and all my friends with whom I have shared great time in Bonn; I especially thank my wife Tatiana, for all the love and support she gave me during my studies. Finally, I would like to thank my best friend Jesus the Messiah. All his love and support was invaluable during all this time.

Contents

Introduction

Let $\mathrm{D}^b(X)$ denote the bounded derived category of coherent sheaves on a variety X. This category is obtained by adding morphisms to the homotopic category of bounded complexes of coherent sheaves on X, in order to ensure that any morphism that induces an isomorphism in cohomology (i.e. quasi-isomorphism) becomes an isomorphism.

Let α be an element in the cohomological Brauer group of X, i.e. $\alpha \in \mathrm{Br}'(X) := H^2(X, \mathcal{O}_X^*)_{tors}$ and $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)$ be a 2-cocycle on an open covering $\{U_i\}_{i \in I}$ of X, that satisfy the boundary conditions and whose image in $H^2(X, \mathcal{O}_X^*)$ is α. An α-twisted sheaf is a collection

$$(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j \in I})$$

of sheaves $\mathcal{F}_i$ on U_i, and isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ satisfying the following conditions:

(i) $\varphi_{ii} = \mathrm{id}$,

(ii) $\varphi_{ij} = \varphi_{ji}^{-1}$,

(iii) $\varphi_{jk} \circ \varphi_{ij} \circ \varphi_{ki} = \alpha_{ijk}.\,\mathrm{id}$.

Similarly to the definition of $\mathrm{D}^b(X)$, we define $\mathrm{D}^b(X, \alpha)$ to be the bounded derived category of α-twisted coherent sheaves on X obtained by adding morphisms to the homotopic category of bounded complexes of α-twisted coherent sheaves on X in order to ensure that any morphism that induces an isomorphism in cohomology becomes an isomorphism.

In [31], Mukai realized the importance of Fourier–Mukai transforms when he proved that the Poincaré bundle over the product of an abelian variety with its dual, $A \times \hat{A}$, defines an equivalence of categories between the derived categories of coherent sheaves on A and $\hat{A}$.

More generally, it has been observed that the universal sheaf on the product of a variety and a fine moduli space on this variety leads to an interesting interplay between the two derived categories. Sometimes the variety and its moduli space are found to even have equivalent derived categories.

This can be extended to coarse moduli spaces, as has been observed by Căldăraru. More precisely, let $X/\mathbb{C}$ be a smooth projective variety and let M^s denote a moduli space of stable sheaves (with respect to a given polarization and with fixed Hilbert polynomial). Then one can find an étale or an analytic covering $\{U_i\}$ of M^s with a local universal sheaf $\mathcal{F}_i$ over $X \times_{\mathbb{C}} U_i$ together with isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ which makes $(\mathcal{F}_i, \varphi_{ij})$ an $\pi_{M^s}^* \alpha$-twisted sheaf for $\alpha \in \mathrm{Br}'(M^s)$. Thus, the obstruction to get a universal bundle is given by an element in $H^2(M^s, \mathcal{O}_{M^s}^*)$, which motivates the study of α-twisted sheaves. The twisted universal sheaf can be used to compare the untwisted derived category $\mathrm{D}^b(X)$ with the twisted category $D^b(M^s, \alpha)$. This motivates to study, more generally, Fourier-Mukai transforms between arbitrary twisted derived categories.

Bridgeland in his thesis, showed a classification of surfaces under derived categories. Analogously, we show in Chapter 1 that some of his and other well known results extend naturally to the derived category of twisted sheaves. First, we show that the following result proven by Kawamata in the untwisted case also holds in the derived category of twisted coherent sheaves. This theorem plays an important role in the classification of varieties under derived categories of coherent sheaves and derived categories of twisted coherent sheaves.

Theorem (Kawamata). *Let X be a smooth projective surface containing a (-1)-curve and Y a smooth projective variety and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence. Then one of the following holds*

(i) $X \cong Y$.

(ii) X is a relatively minimal elliptic rational surface.

In the case of surfaces of general type, i.e. of Kodaira dimension 2, we get the following result:

Proposition. *Let X be a surface of general type and Y a smooth projective variety. If $\mathrm{D}^b(X, \alpha) \cong \mathrm{D}^b(Y, \beta)$, then $X \cong Y$.*

In the case of surfaces of Kodaira dimension 1, we get the following generalization of a result obtained by Bridgeland for the derived category of coherent sheaves, where we denote by $M(v)$ the moduli space of stable sheaves E on Y with Mukai vector $v(E) = (\mathrm{rk}(E), c_1(E), c_1(E)^2/2 - c_2(E) + \mathrm{rk}(E)) = v$.

Proposition. *Let $\pi : Y \to C$ be a relatively minimal elliptic surface with $\mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. Then there exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and $X \cong M(v)$.*

For surfaces of Kodaira dimension $\mathrm{kod}(X) = -\infty$, the cohomological Brauer group $\mathrm{Br}'(X)$ is trivial. Thus, the derived category of twisted coherent sheaves does not provide anything new in this case.

In Chapter 2, we study the injectivity of the induced morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ given by the K3 cover $\pi : X \to Y$ of an Enriques surface Y. In order to do that, we use the Hochschild–Serre spectral sequence and we find an explicit projective bundle (if possible) that represents a nontrivial class of the Brauer group of the K3 surface X such that this projective bundle descends on the Enriques surface to a projective bundle that does not come from a vector bundle (i.e. it can not be written as $\mathbb{P}(E)$ for some rank 2 vector bundle E on Y).

Besides, by using the results of this chapter we also describe the moduli space of marked Enriques surfaces. Some of the results in this chapter were also obtained independently by Beauville who also pointed out a mistake in an earlier version. I will say more about his results in Chapter 2.

For K3 surfaces of Picard number 11 covering Enriques surfaces, Ohashi, ([36], Prop. 3.5), proved that the Néron–Severi lattice is either

(1) $U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, where $N \geq 2$, or

(2) $U \oplus E_8(2) \oplus \langle -4M \rangle$, where $M \geq 1$.

For the first possibility we show that the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective if and only if N is an even number. Unfortunately, we could not settle the second case.

In the last chapter we study derived equivalences of K3 surfaces of Picard number 11 that cover Enriques surfaces and derived equivalences of supersingular surfaces. For example, in the first case, we provide an example of a twisted K3 surface that covers an Enriques surface with no twisted FM partners, i.e. if (Z, α) is a FM partner such that Z covers an Enriques surface, then $Z \cong X$ and $\alpha = 1$. In the second case, we recall that Sertöz found explicit conditions on the entries of the intersection matrix of the transcendental lattice of a supersingular K3 surface ensuring that the K3 surface covers an Enriques surface. We study some of these cases and impose some additional conditions on the entries of two intersection matrices (of the transcendental lattices) of two supersingular surfaces related by an equivalence of categories $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Z)$ with $\mathrm{ord}(\alpha) \leq 2$ and we show that this implies an isomorphism of the two K3 surfaces X and Z.

Chapter 1

General Results

1.1 Brauer groups

Let X be a smooth projective variety. We define the *cohomological Brauer group* of X to be the torsion part of the cohomology group $H^2(X, \mathcal{O}_X^*)$ in the analytic topology (or, in $H^2_{et}(X, \mathcal{O}_X^*)$ for the étale topology). We denote it by $\mathrm{Br}'(X)$. Căldăraru gave in [8] the following characterization for the Brauer group $\mathrm{Br}'(X)$:

Lemma 1.1.1. *Let X be a smooth projective variety. Then there exists the following exact sequence:*

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(X, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Br}'(X) \to 0.$$

Example 1.1.2. Let X be a smooth projective curve. The long exact sequence obtained from the short exponential exact sequence yields

$$H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \to H^3(X, \mathbb{Z}).$$

Hence the cohomological Brauer group $\mathrm{Br}'(X)$ is trivial because $H^2(X, \mathcal{O}_X) = H^3(X, \mathbb{Z}) = 0$.

For any positive integer n, consider the short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 0.$$

The long exact sequence associated to this short exact sequence yields

$$H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathrm{GL}(n)) \to H^1(X, \mathrm{PGL}(n)) \xrightarrow{\delta_n} H^2(X, \mathcal{O}_X^*).$$

An element in $H^1(X, \mathrm{PGL}(n))$ corresponds to a projective bundle $Z \to X$, which is $Z = \mathbb{P}(E)$ for some vector bundle E of rank n if and only if $\delta_n([Z]) = 0$. Moreover, it is well known (cf. [15] Prop. 1.4) that $\mathrm{im}(\delta_n)$ consists of torsion elements, i.e. $\mathrm{im}(\delta_n) \subseteq \mathrm{Br}'(X)$, because from the long exact sequences associated to the following short exact sequences

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathrm{SL}(n) \to \mathrm{PGL}(n) \to 0$$

and

$$0 \to \mathcal{O}_X^* \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 0$$

one concludes that the map δ_n factors through $H^2_{et}(X, \mathbb{Z}/n\mathbb{Z})$, which is killed by n.

Definition 1.1.3. *The subgroup* $\mathrm{Br}(X) := \bigcup_{n\in\mathbb{N}} \mathrm{im}(\delta_n) \subseteq \mathrm{Br}'(X)$ *is the Brauer group of* X.

Grothendieck conjectured that the inclusion $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X)$ is an isomorphism for all smooth quasi-projective varieties. He showed the conjecture in the case when X is an arbitrary algebraic curve or a smooth projective surface (cf. [15]). It is also known for abelian varieties (cf. [19]), for normal separated algebraic surfaces (cf. [39]), for smooth toric varieties (cf. [9]). Gabber proved the conjecture for separated unions of two affine varieties (cf. [13]), and also for schemes with an ample invertible sheaf. An alternative proof for the last is due to De Jong who uses techniques of twisted sheaves. Now, we describe the Brauer group in a different way. Let R be a commutative ring.

Definition 1.1.4. *An Azumaya algebra* A *is an* R*-algebra which is a finitely generated projective* R*-module and such that the natural homomorphism*

$$\begin{aligned} A \otimes_R A^\circ &\longrightarrow \mathrm{End}_R(A) \\ a \otimes a' &\longmapsto (x \mapsto axa') \end{aligned}$$

is an isomorphism, where A° *denotes the opposite algebra, i.e. the algebra with the multiplication reversed.*

The sheafification of A, which is a sheaf of algebras $\mathcal{A}$ on $\mathrm{Spec}(R)$, is called the *sheaf of Azumaya algebras*. A sheaf of algebras $\mathcal{A}$ on a scheme X is a sheaf of Azumaya algebras if it is a sheaf of Azumaya algebras over each open subset $\mathrm{Spec}(R)$ for some commutative ring R in an open affine cover of X. We say that two sheaves of Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ on X are Morita equivalent if there exist two vector bundles E and E' on X such that

$$\mathcal{E}\mathrm{nd}(E) \otimes \mathcal{A} \cong \mathcal{E}\mathrm{nd}(E') \otimes \mathcal{A}'.$$

Remark 1.1.5. Let X be a complex variety and let $\mathrm{Proj}_r(X)$ denote the set of isomorphism classes of holomorphic fibre bundles with fibre $\mathbb{P}^r$. The composition law on $\mathrm{Proj}(X) := \bigcup_{r\in\mathbb{N}} \mathrm{Proj}_r(X)$ is given by $\otimes$ and if we define the equivalence relation on $\mathrm{Proj}(X)$ by

$$P \sim Q \text{ if and only if } P \otimes \mathbb{P}(E) \cong Q \otimes \mathbb{P}(F), \text{ with } E, F \text{ vector bundles,}$$

we can also define the Brauer group as the quotient $\mathrm{Br}(X) = \mathrm{Proj}(X)/\sim$. We also call the projective bundles in $\mathrm{Proj}(X)$ to be *Brauer–Severi varieties*.

1.2 Twisted derived categories

Definition 1.2.1. *A twisted variety* (X, α) *consists of a variety* X *together with a Brauer class* $\alpha \in \mathrm{Br}'(X)$.

If (X, α) is a twisted variety, $\alpha \in \mathrm{Br}'(X)$ can be represented as a Čech 2-cocycle on an open analytic cover $\{U_i\}_{i \in I}$ of X by sections

$$\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*).$$

We say that $\mathcal{F}$ is an *α-twisted quasi-coherent (coherent) sheaf* if this consists of a pair $(\mathcal{F}_i, \{\varphi_{ij}\}_{i,j \in I})$ where $\mathcal{F}_i$ is a quasi-coherent (coherent) sheaf on U_i and

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$$

is an isomorphism satisfying the following conditions (i.e. the α-twisted cocycle conditions):

(i) $\varphi_{ii} = \mathrm{id}$,

(ii) $\varphi_{ij} = \varphi_{ji}^{-1}$,

(iii) $\varphi_{jk} \circ \varphi_{ij} \circ \varphi_{ki} = \alpha_{ijk}.\,\mathrm{id}$.

If for every $i \in I$, $\mathcal{F}_i$ is only a sheaf of $\mathcal{O}_X$-modules on U_i, we say that $\mathcal{F}$ is an α-twisted sheaf and we denote by $\mathrm{Mod}(X, \alpha)$ the abelian category of α-twisted sheaves.

Lemma 1.2.2 ([8], Lemma 2.1.1). $\mathrm{Mod}(X, \alpha)$ *has enough injectives for all* $\alpha \in H^2(X, \mathcal{O}_X^*)$.

Definition 1.2.3. *The category of the α-twisted quasi-coherent (respectively α-twisted coherent) sheaves on* X *will be denoted by* $\mathrm{QCoh}(X, \alpha)$ *(respectively* $\mathrm{Coh}(X, \alpha)$*).*

Remark 1.2.4. If X is a smooth projective variety (defined over an arbitrary field) and $\alpha \in H^2_{et}(X, \mathcal{O}_X^*)$, the abelian category $\mathrm{Coh}(X, \alpha)$ contains a locally free α-twisted coherent sheaf.

Now, we recall the definition of the derived category of twisted sheaves on a variety X. Let (X, α) be a twisted variety and let $C(X, \alpha)$ denote the abelian category whose objects are complexes of sheaves in $\mathrm{Coh}(X, \alpha)$

$$\mathcal{E}^\bullet := (\ldots \xrightarrow{d^{i-2}} \mathcal{E}^{i-1} \xrightarrow{d^{i-1}} \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \xrightarrow{d^{i+1}} \ldots)$$

and morphisms are given by morphisms of complexes:

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{d_1^{i-2}} & \mathcal{E}^{i-1} & \xrightarrow{d_1^{i-1}} & \mathcal{E}^i & \xrightarrow{d_1^{i}} & \mathcal{E}^{i+1} & \xrightarrow{d_1^{i+1}} & \ldots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^{i}} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\ldots & \xrightarrow{d_2^{i-2}} & \mathcal{F}^{i-1} & \xrightarrow{d_2^{i-1}} & \mathcal{F}^i & \xrightarrow{d_2^{i}} & \mathcal{F}^{i+1} & \xrightarrow{d_2^{i+1}} & \ldots
\end{array}$$

i.e. for any $i \in \mathbb{Z}$, $f^i \circ d_1^{i-1} = d_2^{i-1} \circ f^{i-1}$.

We define the *i-th cohomology sheaf* of a complex $\mathcal{E}^\bullet$ to be

$$\mathcal{H}^i(\mathcal{E}^\bullet) := \frac{\ker(d^i)}{\operatorname{im}(d^{i-1})}.$$

This induces for a morphism of complexes $f^\bullet : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ a morphism of twisted sheaves

$$\mathcal{H}^i(f^\bullet) : \mathcal{H}^i(\mathcal{E}^\bullet) \to \mathcal{H}^i(\mathcal{F}^\bullet).$$

The *homotopy category*, $\mathrm{Kom}(X, \alpha)$, is the category whose objects are complexes of $C(X, \alpha)$ and morphisms are

$$\mathrm{Mor}_{\mathrm{Kom}(X,\alpha)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \mathrm{Mor}_{C(X,\alpha)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)/\sim$$

where $f^\bullet \sim g^\bullet$ if there exists morphisms $\{\delta_i : \mathcal{E}^i \to \mathcal{F}^{i-1}\}_{i \in \mathbb{Z}}$ such that

$$f^i - g^i = \delta^{i+1} \circ d^i_{\mathcal{E}} + d^{i-1}_{\mathcal{F}} \circ \delta^i$$

and we say in this case that $f^\bullet$ and $g^\bullet$ are *homotopically equivalent*. By localizing $\mathrm{Kom}(X, \alpha)$ with respect to the class Qis whose elements are the *quasi-isomorphisms* (i.e. morphisms of complexes $f^\bullet$ such that, for any i, $\mathcal{H}^i(f^\bullet)$ is an isomorphism) we obtain the derived category of twisted coherent sheaves $\mathrm{D}(X, \alpha)$. There exists a functor

$$Q_{(X,\alpha)} : C(X, \alpha) \to \mathrm{D}(X, \alpha)$$

such that

(i) $Q_{(X,\alpha)}$(quasi-isom)=isom,

(ii) for any category T and a functor $F : C(X, \alpha) \to T$ such that F(quasi-isom)=isom, there exists a functor $R : \mathrm{D}(X, \alpha) \to T$ such that $F = R \circ Q_{(X,\alpha)}$.

The subcategory of $\mathrm{D}(X, \alpha)$ whose objects are complexes with finitely many sheaves different from 0 will be called the *bounded derived category* of α-twisted coherent sheaves on X and denoted by $\mathrm{D}^b(X, \alpha)$.

Example 1.2.5. Let (X, α) be a twisted variety. For any closed point $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ is in $\mathrm{D}^b(X, \alpha)$.

1.3 Derived functors

Let (X, α) and (Y, β) be twisted varieties. Suppose that a functor

$$F : \mathrm{Coh}(X, \alpha) \to \mathrm{Coh}(Y, \beta)$$

is left exact (i.e. it preserves the exactness of short exact sequences in $\mathrm{Coh}(X,\alpha)$ on the left hand side). We define the *right derived functor* of F if it exists to be the functor

$$RF : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$$

which is uniquely determined (up to a unique isomorphism) by the properties:

(i) RF is exact (as a functor between triangulated categories),

(ii) there exists a morphism $Q_{(Y,\beta)} \circ \mathrm{Kom}(F) \to RF \circ Q_{(X,\alpha)}$, where $\mathrm{Kom}(F)$ is the functor $\mathrm{Kom}(F) : \mathrm{Kom}(X,\alpha) \to \mathrm{Kom}(Y,\beta)$ that extends F,

(iii) if $G : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is an exact functor. Then any morphism of functors $Q_{(Y,\beta)} \circ \mathrm{Kom}(F) \to G \circ Q_{(X,\alpha)}$ factorizes over a morphism $RF \to G$.

Similarly, we can define the *left derived functor* $LG : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ of a right exact functor $G : \mathrm{Coh}(X,\alpha) \to \mathrm{Coh}(Y,\beta)$.

Proposition 1.3.1 ([8], Theorem 2.2.6). *Assume that X and Y are smooth schemes or analytic spaces of finite dimension. Suppose moreover that $f : X \to Y$ is a proper morphism. If $\alpha, \alpha' \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$, then the following functors are defined:*

$$R\mathcal{H}om^\bullet : \mathrm{D}^b(X,\alpha) \times \mathrm{D}^b(X,\alpha') \to \mathrm{D}^b(X,\alpha^{-1}.\alpha'),$$

$$- \overset{L}{\otimes} - : \mathrm{D}^b(X,\alpha) \times \mathrm{D}^b(X,\alpha') \to \mathrm{D}^b(X,\alpha.\alpha'),$$

$$Lf^* : \mathrm{D}^b(Y,\beta) \to \mathrm{D}^b(X,f^*\beta),$$

$$Rf_* : \mathrm{D}^b(X,f^*\beta) \to \mathrm{D}^b(Y,\beta)$$

Furthermore, if X is a scheme or a compact complex analytic space, then

$$\mathrm{RHom}^\bullet : \mathrm{D}^b(X,\alpha)^{op} \times \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Ab)$$

is also defined where Ab is the abelian category of abelian groups.

We proceed to name a few properties from ([8], Sect. 2.3). If $(X,\alpha), (Y,\beta), (Z,\gamma)$ are twisted varieties and $f : X \to Y$, $g : Y \to Z$ are proper morphisms. Then there exists the following (natural) isomorphisms of functors:

- $R(g_* \circ f_*) \cong Rg_* \circ Rf_*$ as functors from $\mathrm{D}(X, f^*(g^*(\gamma)))$ to $\mathrm{D}(Z,\gamma)$,
- $L(f^* \circ g^*) \cong Lf^* \circ Lg^*$ as functors from $\mathrm{D}^-(Z,\gamma)$ to $\mathrm{D}^-(X, f^*g^*\gamma)$,
- $\mathrm{RHom}^\bullet(\mathcal{F},\mathcal{G}) \cong R\Gamma(X, R\mathcal{H}om^\bullet(\mathcal{F},\mathcal{G}))$, $\mathcal{F},\mathcal{G} \in \mathrm{D}^b(X,\alpha)$,
- $Rf_*R\mathcal{H}om^\bullet(\mathcal{F},\mathcal{G}) \cong R\mathcal{H}om^\bullet(Rf_*(\mathcal{F}), Rf_*(\mathcal{G}))$, for $\mathcal{F} \in \mathrm{D}^-(X, f^*\alpha)$ and $\mathcal{G} \in \mathrm{D}^-(X, f^*\alpha')$,

- (Projection Formula) $Rf_*(\mathcal{F}) \overset{L}{\otimes} \mathcal{G} \cong Rf_*(\mathcal{F} \overset{L}{\otimes} Lf^*(\mathcal{G}))$, for any $\mathcal{F} \in \mathrm{D}^-(X, f^*\alpha)$ and $\mathcal{G} \in \mathrm{D}^-(Y, \beta)$,

- $R\mathcal{H}om^\bullet(Lf^*(\mathcal{F}), \mathcal{G}) \cong R\mathcal{H}om^\bullet(\mathcal{F}, Rf_*(\mathcal{G}))$, for $\mathcal{F} \in D^-(Y, \beta)$ and $\mathcal{G} \in D^-(Y, f^*\beta')$,

- $Lf^*(\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong Lf^*(\mathcal{F}) \overset{L}{\otimes} Lf^*(\mathcal{G})$, for any $\mathcal{F} \in \mathrm{D}^-(X, \alpha)$ and $\mathcal{G} \in \mathrm{D}^-(X, \alpha')$,

- $\mathcal{F} \overset{L}{\otimes} \mathcal{G} \cong \mathcal{G} \overset{L}{\otimes} \mathcal{F}$ and $\mathcal{F} \overset{L}{\otimes} (\mathcal{G} \overset{L}{\otimes} \mathcal{H}) \cong (\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \overset{L}{\otimes} \mathcal{H}$, for any $\mathcal{F} \in \mathrm{D}^-(X, \alpha)$, $\mathcal{G} \in \mathrm{D}^-(X, \alpha')$ and $\mathcal{H} \in \mathrm{D}^-(X, \alpha'')$,

- $R\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G}) \overset{L}{\otimes} \mathcal{H} \cong R\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G} \overset{L}{\otimes} \mathcal{H})$, for any $\mathcal{F} \in \mathrm{D}^-(X, \alpha)$, $\mathcal{G} \in \mathrm{D}^+(X, \alpha')$ and $\mathcal{H} \in \mathrm{D}(X, \alpha'')$,

- $R\mathcal{H}om^\bullet(\mathcal{F}, R\mathcal{H}om^\bullet(\mathcal{G}, \mathcal{H})) \cong R\mathcal{H}om^\bullet(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, \mathcal{H})$, for $\mathcal{F} \in \mathrm{D}^-(X, \alpha), \mathcal{G} \in \mathrm{D}^-(X, \alpha')$ and $\mathcal{H} \in \mathrm{D}^+(X, \alpha'')$,

- $R\mathcal{H}om^\bullet(\mathcal{F}, \mathcal{G} \overset{L}{\otimes} \mathcal{H}) \cong R\mathcal{H}om^\bullet(\mathcal{F} \overset{L}{\otimes} \mathcal{H}^\vee, \mathcal{G})$, for $\mathcal{F} \in \mathrm{D}^-(X, \alpha')$ and $\mathcal{G} \in \mathrm{D}^+(X, \alpha')$ and for a bounded α-complex $\mathcal{H}$ where $\mathcal{H}^\vee := R\mathcal{H}om(\mathcal{H}, \mathcal{O}_X)$.

And, finally we recall the Flat Base Change Theorem for the derived category of twisted sheaves, i.e if $u : Y' \to Y$ is a flat morphism in the following commutative diagram

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y,
\end{tikzcd}$$

then there exists a functorial isomorphism

$$u^* Rf_*(\mathcal{F}) \cong Rg_* v^*(\mathcal{F}),$$

for any $\mathcal{F} \in \mathrm{D}(X, f^*\beta)$.

Theorem 1.3.2 ([8], Theorem 2.4.1). *Let $f : X \to Y$ be a proper smooth morphism of relative dimension n between smooth schemes or between smooth analytic spaces, and let $\alpha \in \mathrm{Br}(Y)$. Define $f^! : \mathrm{D}^b(Y, \alpha) \to \mathrm{D}^b(X, f^*\alpha)$ by*

$$f^!(-) = Lf^*(-) \otimes_{\mathcal{O}_X} \omega_{X/Y}[n]$$

where $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ and $\Omega_{X/Y}$ is the locally free (cf. [18], III. 10.0.2) sheaf of relative differentials. Then for any $G^\bullet \in \mathrm{D}^b(Y, \alpha)$ there is a natural homomorphism

$$Rf_* f^! G^\bullet \to G^\bullet,$$

which induces a natural homomorphism

$$Rf_* R\mathcal{H}om^\bullet(F^\bullet, f^! G^\bullet) \to R\mathcal{H}om^\bullet(Rf_* F^\bullet, G^\bullet)$$

for every $F^\bullet \in \mathrm{D}^b(X, f^*\alpha)$*, which is an isomorphism.*

Corollary 1.3.3. *Under the conditions of the previous theorem,* $f^!$ *is a right adjoint to* Rf_* *as functors between* $\mathrm{D}^b(X, f^*\alpha)$ *and* $\mathrm{D}^b(Y, \alpha)$*.*

1.4 Spectral sequences

Theorem 1.4.1 ([28], App. C). *Let* $\mathcal{A}, \mathcal{B}$ *be abelian categories, and let* $F : \mathcal{A} \to \mathcal{B}$ *be an additive, left exact functor. Assume that* $\mathcal{A}$ *has enough injectives, so that the derived functor*

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

exists. Let $X^\bullet$ *be a complex in* $D^+(\mathcal{A})$*. Then there exists a spectral sequence* $E_k^{i,j}$ *such that*

$$E_2^{i,j} = R^i F(H^j(X^\bullet)) \Rightarrow H^{i+j}(RF(X^\bullet))$$

We recall some spectral sequences defined in the derived category $\mathrm{D}^b(X)$ on a smooth variety X (cf. [21], Ch. II and III).

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{F}^\bullet, \mathcal{H}^q(\mathcal{E}^\bullet)) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet). \tag{1.1}$$

$$E_2^{p,q} = \mathcal{E}xt^p(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{E}^\bullet) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet). \tag{1.2}$$

$$E_2^{p,q} = \mathcal{T}or^{-p}(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{E}^\bullet) \Rightarrow \mathcal{T}or^{-(p+q)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet). \tag{1.3}$$

for any $\mathcal{E}^\bullet, \mathcal{F}^\bullet$ in $\mathrm{D}^b(X)$.

We see now some applications of this spectral sequences in twisted derived categories. Let (X, α) be a smooth variety and $\mathcal{P} \in \mathrm{D}^b(X, \alpha)$. We use the spectral sequence (1.2) to show that the support of the object $\mathcal{P}$ remains the same under taking its dual. Take a locally free α^{-1}-twisted sheaf L on X and consider the spectral sequence:

$$\mathcal{E}xt^p(\mathcal{H}^{-q}(\mathcal{P} \otimes L), \mathcal{O}_X) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{P} \otimes L, \mathcal{O}_X) = \mathcal{H}^{p+q}(\mathcal{P}^\vee \otimes L^\vee).$$

Hence

$$\mathrm{supp}(\mathcal{P}^\vee \otimes L^\vee) = \bigcup \mathrm{supp}(\mathcal{H}^i(\mathcal{P}^\vee \otimes L^\vee)) \subseteq \bigcup \mathrm{supp}(\mathcal{H}^i(\mathcal{P} \otimes L)) = \mathrm{supp}(\mathcal{P} \otimes L).$$

Since L is a locally free α^{-1}-twisted sheaf,

$$\mathrm{supp}(\mathcal{P}^\vee) = \mathrm{supp}(\mathcal{P}^\vee \otimes L^\vee) \subseteq \mathrm{supp}(\mathcal{P} \otimes L) = \mathrm{supp}(\mathcal{P})$$

and from $(\mathcal{P}^\vee)^\vee \cong \mathcal{P}$, we get the other inclusion. Thus

$$\mathrm{supp}(\mathcal{P}) = \mathrm{supp}(\mathcal{P}^\vee).$$

Let $\mathcal{A}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism

$$\eta_{A,B} : \mathrm{Hom}(A, B) \xrightarrow{\sim} \mathrm{Hom}(B, S(A))^\vee$$

of k-vector spaces which is functorial in A and B.

Example 1.4.2. Let X be a smooth projective variety. The functor

$$\begin{aligned} S : \mathrm{D}^b(X) &\to \mathrm{D}^b(X) \\ \mathcal{E} &\mapsto \mathcal{E} \otimes \omega_X[\dim(X)] \end{aligned}$$

where ω_X is the dualizing sheaf of X, is a Serre functor.

Example 1.4.3. If (X, α) is a twisted smooth projective variety, the functor

$$\begin{aligned} S_{(X,\alpha)} : \mathrm{D}^b(X, \alpha) &\to \mathrm{D}^b(X, \alpha) \\ \mathcal{E} &\mapsto \mathcal{E} \otimes \omega_X[\dim(X)], \end{aligned}$$

is a Serre functor. Indeed, if $\mathcal{F}, \mathcal{G} \in \mathrm{D}^b(X, \alpha)$,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}^b(X,\alpha)}(\mathcal{F}, S_{(X,\alpha)}\mathcal{G}) &= \mathrm{Hom}_{\mathrm{D}^b(x,\alpha)}(\mathcal{F}, \mathcal{G} \otimes \omega_X[\dim(X)]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{G}^\vee \otimes \mathcal{F}, \omega_X[\dim(X)]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X)}(\omega_X[\dim(X)], S(\mathcal{G}^\vee \otimes \mathcal{F}))^\vee \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X)}(\omega_X[\dim(X)], \mathcal{G}^\vee \otimes \mathcal{F} \otimes \omega_X[\dim(X)])^\vee \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X,\alpha)}(\mathcal{G}, \mathcal{F})^\vee \end{aligned}$$

because $\mathcal{G}^\vee \otimes \mathcal{F} \in \mathrm{D}^b(X)$ and by the previous example, S is a Serre functor.

Definition 1.4.4. *A collection of objects* Ω *in the category* $\mathrm{D}^b(X, \alpha)$ *is a spanning class of (or spans)* $\mathrm{D}^b(X, \alpha)$ *if for all* $G \in \mathrm{D}^b(X, \alpha)$ *the following equivalent conditions hold:*

(i) If $\mathrm{Hom}(F, G[i]) = 0$ *for all* $F \in \Omega$ *and all* $i \in \mathbb{Z}$ *then* $G \cong 0$.

(ii) If $\mathrm{Hom}(G[i], F) = 0$ *for all* $F \in \Omega$ *and all* $i \in \mathbb{Z}$ *then* $G \cong 0$.

The equivalence in the last definition follows immediately by using the Serre functor $S_{(X,\alpha)}$. The proof of the following proposition is identical as in the untwisted case (cf. [21], Prop. 3.16).

Proposition 1.4.5. *Let* (X, α) *be a twisted smooth projective variety. The objects of the form* $k(x)$ *with* $x \in X$ *a closed point span the derived category* $\mathrm{D}^b(X, \alpha)$.

Proof. We show that for a given $\mathcal{E}^\bullet \in \mathrm{D}^b(X,\alpha)$ there exists a point $x \in X$ and an integer n such that $\mathrm{Hom}(\mathcal{E}^\bullet, k(x)[n]) \neq 0$. Consider the spectral sequence

$$E_2^{p,q} := \mathrm{Hom}(\mathcal{H}^{-q}(\mathcal{E}^\bullet \otimes L), k(x)[p]) \Rightarrow \mathrm{Hom}(\mathcal{E}^\bullet \otimes L, k(x)[p+q]),$$

where L is a locally free α^{-1}-twisted sheaf. Let m be the maximal integer with $\mathcal{H}^m(\mathcal{E}^\bullet \otimes L) \neq 0$. This implies that the differentials with source $E_2^{0,-m}$ are trivial. On the other hand, from the triviality of the negative Ext-groups between coherent sheaves we obtain the triviality of all the differentials with target $E_r^{0,-m}$. Thus, $E_\infty^{0,-m} = E_2^{0,-m}$. Hence if $x \in \mathrm{supp}(\mathcal{E}^\bullet \otimes L) = \mathrm{supp}(\mathcal{E}^\bullet)$, then $E_\infty^{0,-m} = E_2^{0,-m} = \mathrm{Hom}(\mathcal{H}^m(\mathcal{E}^\bullet \otimes L), k(x)) \neq 0$ and hence

$$\mathrm{Hom}(\mathcal{E}^\bullet, k(x)^{\oplus n}[-m]) = \mathrm{Hom}(\mathcal{E}^\bullet, k(x) \otimes L^\vee[-m]) = \mathrm{Hom}(\mathcal{E}^\bullet \otimes L, k(x)[-m]) \neq 0,$$

where $n := \mathrm{rk}(L)$. Thus $\mathrm{Hom}(\mathcal{E}^\bullet, k(x)) \neq 0$. □

Lemma 1.4.6. *Let $\pi : S \to T$ be a morphism of schemes, and for each point $t \in T$, let $i_t : S_t \to S$ denote the inclusion of the fibre $\pi^{-1}(t)$ in S. Let E be an object of $\mathrm{D}^b(S,\alpha)$ such that for all $t \in T$, $Li_t^*(E)$ is a twisted sheaf on S_t. Then E is a twisted sheaf on S, flat over T. (See [21], Lemma 3.31)*

Proof. Let $t \in T$ and consider the spectral sequence

$$E_2^{p,q} = L_{-p} i_t^*(\mathcal{H}^q(E)) \Rightarrow L_{-(p+q)} i_t^*(E).$$

Since $Li_t^*(E)$ is a twisted sheaf, the right-hand side is zero unless $p + q = 0$. Take q_0 the largest q such that $\mathcal{H}^q(E) \neq 0$, then since E_2^{0,q_0} does not vanish in the spectral sequence we get $q_0 = 0$. We get also the same if we replace E by $E \otimes F$ where F is a locally free α^{-1}-twisted sheaf. Hence since $E_2^{-1,0}$ survives in the spectral sequence then the sheaf $\mathcal{H}^0(E \otimes F)$ is a flat sheaf over T. From the flatness we also deduce that $E_2^{p,0}$ are trivial for $p < 0$. Then $E \otimes F$ is a sheaf and flat over T. Hence E is also a sheaf. If $E = (E_i, \varphi_{ij})$ then E_i is flat because the term $E_2^{-1,0}$ vanishes for the spectral sequence applied to the sheaf E_i. Hence E is flat over T. □

The last lemma has a useful application. Suppose $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is a FM equivalence (See Def. 1.6.5) such that for all $x \in X$ there exists $f(x) \in Y$ with $\Phi_{\mathcal{P}}(k(x)) = k(f(x))$. Hence

$$\mathcal{P}|_{\{x\}\times Y} \cong k(f(x)) \tag{1.4}$$

for all $x \in X$ and then by the previous lemma $\mathcal{P}$ is a twisted sheaf (which is X-flat). By taking local sections of $\mathcal{P}$ we define a morphism $X \to Y$ and by the isomorphism 1.4, we get that this induces f on closed points. We call this morphism again f. By following the same argument given in ([21], Cor. 5.23), we obtain

$$\Phi_{\mathcal{P}}(-) = (L \otimes (-)) \circ f_* \tag{1.5}$$

where L is a line bundle and that f is an isomorphism because $\Phi_{\mathcal{P}}$ is an equivalence.

1.5 Moduli spaces of sheaves

In this section we recall basic facts about moduli spaces of sheaves and Mukai's theory of fine moduli spaces of sheaves on K3 surfaces. A good reference concerning moduli spaces is [22].

1.5.1 Basic facts about moduli spaces

Let X be a projective scheme. If $\mathcal{O}(1)$ is an ample line bundle and $\mathcal{E}$ is a coherent sheaf on X, then the *Hilbert polynomial* of $\mathcal{E}$ is defined by

$$P(\mathcal{E}, m) := \chi(\mathcal{E} \otimes \mathcal{O}(m)).$$

We recall that the *dimension* of a coherent sheaf $\mathcal{E}$ on a projective scheme X is the dimension of the support of $\mathcal{E}$, and we denoted it by $\dim(\mathcal{E})$. We say that a coherent sheaf $\mathcal{E}$ is *pure* of dimension d if for every subsheaf $\mathcal{F}$, $d = \dim(\mathcal{E}) = \dim(\mathcal{F})$. It is well known that the Hilbert polynomial can be written as

$$P(\mathcal{E}, m) = \sum_{i=0}^{\dim(\mathcal{E})} \alpha_i(\mathcal{E}) \frac{m^i}{i!}$$

where $\alpha_i(\mathcal{E})$ is an integer, for any $i \in \{0, ..., \dim(\mathcal{E})\}$. We define the *rank* of a coherent sheaf $\mathcal{E}$ of dimension $d = \dim X$ to be the number

$$\operatorname{rk}(\mathcal{E}) := \frac{\alpha_d(\mathcal{E})}{\alpha_d(\mathcal{O}_X)}.$$

The *reduced Hilbert polynomial* of a coherent sheaf $\mathcal{E}$ of dimension d is defined by

$$p(\mathcal{E}, m) := \frac{P(\mathcal{E}, m)}{\alpha_d(\mathcal{E})}.$$

We consider the natural order on the ring $\mathbb{Q}[x]$ of polynomials with rational coefficients given by: if f and g are polynomials, we write $f \leq g$ if $f(m) \leq g(m)$, for $m \gg 0$ and $f < g$ if $f(m) < g(m)$, for $m \gg 0$. Under this order we introduce the concept of stability.

Definition 1.5.1. *A coherent sheaf $\mathcal{E}$ of dimension d is semistable if it is pure, and for any subsheaf $\mathcal{F} \subset \mathcal{E}$, $p(\mathcal{F}) \leq p(E)$. We say that the sheaf $\mathcal{E}$ is stable if the strict inequality holds.*

We proceed now to introduce the moduli functor. Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme and P be a fixed polynomial in $\mathbb{Q}[x]$. We define the category $(\mathrm{Sch}\,/k)^{op}$ as the opposite category of the category of schemes over a field k and Sets the category of sets. Take the functor

$$\mathcal{M}' : (\mathrm{Sch}\,/k)^{op} \to \mathrm{Sets}$$

such that for any k-scheme S, $\mathcal{M}'(S)$ is the set of isomorphism classes of coherent sheaves on $X \times S$ so that

(1) they have Hilbert polynomial P,

(2) they are semistable on the fibres of $X \times S \to S$,

(3) they are flat over S with respect to the projection $p : X \times S \to S$,

and if $f : S' \to S$ is morphism in (Sch/k), $\mathcal{M}'(f)$ is defined to be the map

$$\mathcal{M}'(f) : \mathcal{M}'(S) \to \mathcal{M}'(S') \tag{1.6}$$
$$\mathcal{F} \mapsto [f_X^* \mathcal{F}] \tag{1.7}$$

where $f_X := \mathrm{id}_X \times f$. For any line bundle L on S and family $\mathcal{F} \in \mathcal{M}'(S)$, the family $\mathcal{F} \otimes p^* L$ is also a family in $\mathcal{M}'(S)$ with fibres

$$(\mathcal{F} \otimes p^* L)_s = \mathcal{F}_s \otimes_{k(s)} L(s) \cong \mathcal{F}_s$$

for any $s \in S$ and $p : X \times S \to S$ the projection. Thus, we can define an equivalence relation by

$$\mathcal{F} \sim \mathcal{G} \text{ for } \mathcal{F}, \mathcal{G} \in \mathcal{M}'(S) \text{ if and only if } \mathcal{F} \cong \mathcal{G} \otimes p^* L \text{ for some } L \in \mathrm{Pic}(S).$$

Now, we define the quotient functor by

$$\mathcal{M} = \mathcal{M}' / \sim .$$

Definition 1.5.2. *A functor $F : (\mathrm{Sch}/k)^{op} \to \mathrm{Sets}$ is representable if there exists a scheme M and a natural isomorphism of functors $F \cong \mathrm{Hom}(-, M)$. We say that a scheme M corepresents the functor F if there exists a morphism of functors $\psi : F \to \mathrm{Hom}(-, M)$ such that for every morphism $\varphi : F \to \mathrm{Hom}(-, N)$ uniquely factorizes over a morphism $\alpha : \mathrm{Hom}(-, M) \to \mathrm{Hom}(-, N)$ induced by a morphism of schemes $M \to N$.*

Definition 1.5.3. *(i) A scheme M is called a (coarse) moduli space of semistable sheaves if it corepresents the functor $\mathcal{M}$,*

(ii) If the functor $\mathcal{M}$ is representable by a scheme M, we say that M is the fine moduli space associated to $\mathcal{M}$.

Analogously, one defines the moduli functor of semistable sheaves for a family $X \to S$.

Proposition 1.5.4 ([8], Prop. 3.3.2). *Let X/S be a flat, projective morphism, and let $\mathcal{O}(1)$ be a relatively ample sheaf on X/S. For a polynomial P, consider the relative moduli space M^s/S of stable sheaves with Hilbert polynomial P on the fibres of X/S. Then there exists a covering $\{U_i\}$ of M^s (by analytic open sets in the analytic setting, and by étale open sets in the algebraic setting) such that on each $X \times_S U_i$ there exists a local universal sheaf $\mathcal{U}_i$. Furthermore, there exists an $\alpha \in H^2(M^s, \mathcal{O}^*_{M^s})$ (that only depends on X/S, $\mathcal{O}(1)$ and P) and isomorphisms $\varphi_{ij} : \mathcal{U}_i|_{U_i \cap U_j} \to \mathcal{U}_j|_{U_i \cap U_j}$ that make $(\{\mathcal{U}_i, \{\varphi_{ij}\}\})$ into a α-sheaf called the universal α-sheaf.*

Definition 1.5.5. *The element $\alpha \in H^2(M^s, \mathcal{O}^*_{M^s})$ described above is called the obstruction to the existence of a universal sheaf on $X \times_S M$, and is denoted by* $\mathrm{Obs}(X/S, P)$.

1.5.2 Moduli spaces on K3 surfaces

The results presented in this section are due to Mukai (cf. [32]). Let X be a K3 surface. The weight two Hodge structure of $H^*(X,\mathbb{Z})$ is defined by

$$\begin{aligned}\tilde{H}^{2,0}(X) &:= H^{2,0}(X),\\ \tilde{H}^{0,2}(X) &:= H^{0,2}(X),\\ \tilde{H}^{1,1}(X) &:= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}).\end{aligned}$$

If $\mathcal{E}$ is a locally free sheaf on X with rank $r := \mathrm{rk}(\mathcal{E})$, first Chern class $c_1 := c_1(\mathcal{E})$ and second Chern class $c_2 := c_2(\mathcal{E})$, the *Mukai vector* is defined by

$$v(\mathcal{E}) := ch(\mathcal{E}).\sqrt{td_X} = (r, c_1, c_1^2/2 - c_2 + r)$$

and the *Euler characteristic* of a pair $(\mathcal{E},\mathcal{F})$ of coherent sheaves by

$$\chi(\mathcal{E},\mathcal{F}) := \sum_i (-1)^i \dim \mathrm{Ext}^i(\mathcal{E},\mathcal{F}).$$

If (.) is the cup product, we define the *Mukai pairing* on $H^*(X,\mathbb{Z})$ to be the bilinear form

$$\langle \alpha, \beta \rangle := -(\alpha_1.\beta_3) + (\alpha_2.\beta_2) - (\alpha_3.\beta_1),$$

for any element $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ in $H^*(X,\mathbb{Z})$.

Proposition 1.5.6. *Let $\mathcal{E}$ and $\mathcal{F}$ be two locally free sheaves on a K3 surface. Then $\chi(\mathcal{E},\mathcal{F}) = -\langle v(E), v(F)\rangle$.*

Definition 1.5.7. *Let X be a K3 surface and let $v = (r,h,s) \in \tilde{H}(X,\mathbb{Z})$ be a fixed Mukai vector.*

(i) We denote by $M(v)$ the moduli space of semistable sheaves $\mathcal{E}$ on X such that $\mathrm{rk}(E) = r, c_1(\mathcal{E}) = h$ and $c_1(\mathcal{E})^2/2 - c_2(\mathcal{E}) + \mathrm{rk}(\mathcal{E}) = s$.

(ii) We denote by $M(v)^s$ the subscheme corresponding to stable sheaves.

Definition 1.5.8. *We say that a Mukai vector v on an K3 surface X is isotropic if $\langle v, v\rangle = 0$.*

Theorem 1.5.9. *Let X be a K3 surface with an ample line bundle H and let $v = (r,h,s)$ be a primitive, isotropic Mukai vector such that $gcd(r, h.H, s) = 1$. Then $M(v)^s$ is a fine moduli space of sheaves which are stable sheaves with respect to H. Furthermore $M(v)^s$ is a K3 surface.*

Let $v = (r,h,s)$ be an isotropic Mukai vector and assume $M(v) = M(v)^s$ non-empty, i.e. $M(v)^s$ is smooth and irreducible. Let $\mathcal{E}$ be a quasi-universal family on $X \times M(v)$ and p, q the natural projections

$$\begin{tikzcd} X \times M(v) \arrow[r, "p"] \arrow[d, "q"] & M(v). \\ X & \end{tikzcd}$$

The sheaf $\mathcal{E}$ induces a morphism $f_{\mathcal{E}} : \tilde{H}(M(v), \mathbb{Q}) \to \tilde{H}(X, \mathbb{Q})$ defined by

$$c \longmapsto q_*(v(\mathcal{E}).p^*(c)).$$

Theorem 1.5.10 ([32], Theorem 4.9.). *Let X be a K3 surface and let $v = (r, h, s)$ be an isotropic Mukai vector. Assume that $M(v) = M(v)^s$ is non-empty and that $M(v)$ is fine. Let $\mathcal{E}$ be a universal family on $X \times M(v)$. Then $f_{\mathcal{E}}$ induces a Hodge isometry of the Mukai lattices $\tilde{H}(X, \mathbb{Z})$ and $\tilde{H}(M(v), \mathbb{Z})$.*

We can see that $f_{\mathcal{E}}^{-1}$ induces a morphism

$$\varphi_{\mathbb{Q}} : (v^{\perp} \otimes \mathbb{Q})/\mathbb{Q}v \to H^2(M(v), \mathbb{Q}). \tag{1.8}$$

Theorem 1.5.11 ([32], Theorem 1.5.). *Let X be a K3 surface and $v = (r, h, s)$ an isotropic Mukai vector such that $M(v) = M(v)^s$ is non-empty and $M(v)$ is fine. Let $\mathcal{E}$ be a universal family on $X \times M(v)$ and $\varphi_{\mathbb{Q}}$ as in (1.8). Then*

(i) $\varphi_{\mathbb{Q}}$ does not depend on the choice of the universal family $\mathcal{E}$,

(ii) $\varphi_{\mathbb{Q}}$ is an isomorphism of Hodge structures compatible with the pairing

(iii) $\varphi_{\mathbb{Q}}$ is defined over $\mathbb{Z}$, i.e. $\varphi_{\mathbb{Q}} : v^{\perp}/\mathbb{Z}v \to H^2(M(v), \mathbb{Z})$ is an isometry.

1.6 Ample (antiample) canonical bundle

For the rest of the chapter we consider all the varieties to be smooth and projective.

Definition 1.6.1. *An object $P \in \mathrm{D}^b(X, \alpha)$ is called a point of codimension d if*

(i) $S_{(X,\alpha)}(P) \cong P[d]$, (where $S_{(X,\alpha)}$ is the Serre functor).

(ii) $\mathrm{Hom}(P, P[i]) = 0$ for $i < 0$.

(iii) The object P is simple, i.e. $k := \mathrm{Hom}(P, P)$.

We follow the untwisted proofs of the next two lemmas in order to get a twisted version of them (cf. [21], Lemma 4.5 and Prop. 4.6, and the original proof in [3]).

Lemma 1.6.2. *Let $\mathcal{F}^\bullet \in \mathrm{D}^b(X, \alpha)$ be a simple complex concentrated in dimension 0 such that $\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i]) = 0$ for $i < 0$. Then $\mathcal{F}^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer m.*

Lemma 1.6.3. *Let X be a smooth projective variety of dimension n. If ω_X is ample or antiample, then the point like objects in $\mathrm{D}^b(X, \alpha)$ are the objects P isomorphic to $k(x)[m]$, where $x \in X$ is a closed point and $m \in \mathbb{Z}$.*

Proof. It can be easily seen that the objects of the form $k(x)[m]$ are point like objects in $\mathrm{D}^b(X,\alpha)$. Now, we show that all point like objects are of this form. Take $P \in \mathrm{D}^b(X,\alpha)$ a point like object. By (i) in Definition 1.6.1

$$\mathcal{H}^i(P \otimes \omega_X[n-d]) \cong \mathcal{H}^i(P).$$

Thus

$$\mathcal{H}^{i+n-d}(P \otimes \omega_X) \cong \mathcal{H}^i(P),$$

i.e.

$$\mathcal{H}^{i+n-d}(P) \otimes \omega_X \cong \mathcal{H}^i(P). \tag{1.9}$$

If $n > d$, then we take the maximal integer i between all the indices of the non-vanishing cohomologies $\mathcal{H}^i$. This yields to a contradiction by using (1.9). On the other hand, if $n < d$, we take i to be minimal, and (1.9) also yields to a contradiction. Thus, $n = d$ and hence

$$\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P) \tag{1.10}$$

Now, we show that this isomorphism implies that $\mathcal{H}^i(P)$ is supported in dimension 0. Recall that the Hilbert polynomial

$$P_{\mathcal{F}}(k) = \chi(\mathcal{F} \otimes \omega_X^k)$$

has degree

$$\deg(P_{\mathcal{F}}) = \dim(\operatorname{supp} \mathcal{F})$$

when ω_X (or $\omega_X^\vee$) is ample and $\mathcal{F}$ is any coherent sheaf. Let $\mathcal{E} \in \mathrm{Coh}(X,\alpha^{-1})$ be a locally free α^{-1}-twisted sheaf and denote by $\mathcal{F}^i := \mathcal{H}^i(\mathcal{P}) \otimes \mathcal{E}$. Hence

$$\mathcal{F}^i \otimes \omega_X \cong \mathcal{F}^i \tag{1.11}$$

and $\mathcal{F}^i$ is a coherent sheaf on X. If $n = \dim(\operatorname{supp}(\mathcal{F}^i)) > 0$, we deduce from the isomorphism (1.11) that for all k, $P_{\mathcal{F}^i}(k)$ is a fixed number, i.e. the polynomial $P_{\mathcal{F}^i}$ is a constant polynomial, a contradiction. Then $\mathcal{F}^i$ is supported in dimension 0, and since $\mathcal{E}$ is locally free, $\mathcal{H}^i(P)$ has also support of dimension 0. Thus, P is a complex concentrated in dimension 0 and by Lemma 1.6.2, $P \cong k(x)[m]$ for some closed point x and integer m. □

Definition 1.6.4. *Let $\mathcal{D}$ be a triangulated category with a Serre functor S. An object $L \in \mathcal{D}$ is invertible if for any point like object $P \in \mathcal{D}$ there exists $n_P \in \mathbb{Z}$ such that*

$$Hom(L, P[i]) = \begin{cases} k(P), & \text{if } i = n_P \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.6.5. *Let (X,α) and (Y,β) be two twisted varieties. A functor $F : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is of Fourier–Mukai type (or a Fourier–Mukai functor) if there exists $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{P}}$,* *where we denote by* $p : X \times Y \to Y$ *and* $q : X \times Y \to X$ *the natural projections,* $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ *is the exact functor defined by*

$$\Phi_{\mathcal{P}} := Rp_*(\mathcal{P} \overset{L}{\otimes} q^*(-)).$$

If the Fourier–Mukai functor is an equivalence we will call it a Fourier–Mukai transform.

From now, we will often write a functor and its derived functor in the same way.

In the category of twisted coherent sheaves Canonaco and Stellari proved in [6] that every equivalence can be seen as a Fourier–Mukai transform. In fact, they showed the following more general statement:

Theorem 1.6.6. *Let* (X, α) *and* (Y, β) *be twisted varieties and let* $F : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ *be an exact functor such that, for any* $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X, \alpha)$,

$$Hom_{\mathrm{D}^b(Y,\beta)}(F(\mathcal{F}), F(\mathcal{G})[j]) = 0 \text{ if } j < 0.$$

Then there exist $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$ *and an isomorphism of functors* $F \cong \Phi_{\mathcal{P}}$. *Moreover,* $\mathcal{P}$ *is uniquely determined up to isomorphism.*

By this theorem, we focus only on Fourier–Mukai transforms. If we take any exact functor

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta),$$

then by an application of the Grothendieck–Verdier duality (cf. Theorem 1.3.2) as was given by Mukai (a good exposition by Orlov is found in [37]) we can prove that the functor $\Phi_{\mathcal{P}}$ has a left and a right adjoint functor with kernels

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)]$$

and

$$\mathcal{P}_R := \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)]$$

respectively. In particular, if $\Phi_{\mathcal{P}}$ is an equivalence, these adjoints must be quasi-inverses to $\Phi_{\mathcal{P}}$. However, from the uniqueness of the kernel of a twisted Fourier–Mukai transform we conclude that $\mathcal{P}_L$ is isomorphic to $\mathcal{P}_R$ and then

$$\mathcal{P}^\vee \cong \mathcal{P}^\vee \otimes (p^*\omega_Y \otimes q^*\omega_X^\vee[\dim(X) - \dim(Y)]).$$

This isomorphism implies: $\dim(X) = \dim(Y)$.

Remark 1.6.7. If $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is an equivalence, the isomorphism $\mathcal{P}_L \cong \mathcal{P}_R$ and projection formula imply that for any point $x \in X$,

$$\begin{aligned}
\Phi_{\mathcal{P}}(k(x)) &= p_*(\mathcal{P} \otimes q^*k(x)) \\
&= p_*(\mathcal{P} \otimes q^*k(x) \otimes q^*\omega_X) \\
&= p_*(\mathcal{P} \otimes p^*\omega_Y \otimes q^*k(x)) \\
&= \omega_Y \otimes p_*(\mathcal{P} \otimes q^*k(x)) \\
&= \omega_Y \otimes \Phi_{\mathcal{P}}(k(x)).
\end{aligned}$$

Let X, Y and Z be three smooth varieties. Define the projections π_{XZ}, π_{XY} and π_{YZ} from $X \times Y \times Z$ to $X \times Z$, $X \times Y$ and $Y \times Z$ respectively. Let $\mathcal{P} \in \mathrm{D}^b(X \times Y, q^*(\alpha)^{-1}.p^*(\beta))$ and $\mathcal{Q} \in \mathrm{D}^b(Y \times Z, u^*(\beta)^{-1}.t^*(\gamma))$ where q, p and u, t are the natural projections:

$$\begin{array}{ccc} X \times Y & \xrightarrow{p} & Y, \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} Y \times Z & \xrightarrow{t} & Z. \\ {\scriptstyle u}\downarrow & & \\ Y & & \end{array}$$

We define the object

$$\mathcal{R} := \pi_{XZ*}(\pi_{XY}^* \mathcal{P} \otimes \pi_{YZ}^* \mathcal{Q}),$$

and let us show that this element is in $\mathrm{D}^b(X \times Z, s^*(\alpha)^{-1}.r^*(\gamma))$ where r and s denote the projections from $X \times Z$ to Z and X respectively. Let π_X, π_Y and π_Z denote the projections from $X \times Y \times Z$ to X, Y and Z respectively. The object $\pi_{XY}^*(\mathcal{P}) \otimes \pi_{YZ}^*(\mathcal{Q})$ is in

$$\begin{aligned} &\mathrm{D}^b(X \times Y \times Z, \pi_{XY}^*(q^*(\alpha)^{-1}.p^*(\beta)).\pi_{YZ}^*(u^*(\beta)^{-1}.t^*(\gamma))) \\ &\cong \mathrm{D}^b(X \times Y \times Z, \pi_X^*(\alpha)^{-1}.\pi_Y^*(\beta).\pi_Y^*(\beta)^{-1}.\pi_Z^*(\gamma)) \\ &\cong \mathrm{D}^b(X \times Y \times Z, \pi_X^*(\alpha)^{-1}.\pi_Z^*(\gamma)) \\ &\cong \mathrm{D}^b(X \times Y \times Z, \pi_{XZ}^*(s^*(\alpha)^{-1}).\pi_{XZ}^*(r^*(\gamma))) \\ &\cong \mathrm{D}^b(X \times Y \times Z, \pi_{XZ}^*(s^*(\alpha)^{-1}.r^*(\gamma))). \end{aligned}$$

Hence

$$\mathcal{R} = \pi_{XZ*}(\pi_{XY}^*(\mathcal{P}) \otimes \pi_{YZ}^*(\mathcal{Q})) \in \mathrm{D}^b(X \times Z, s^*(\alpha)^{-1}.r^*(\gamma)).$$

We note that the following twisted version of a result of Mukai holds by just following his proof.

Proposition 1.6.8 (Mukai, [31]). *The composition of two Fourier–Mukai transforms*

$$\mathrm{D}^b(X, \alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y, \beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(Z, \gamma)$$

is isomorphic to the Fourier–Mukai transform

$$\Phi_{\mathcal{R}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Z, \gamma).$$

We follow only a part of the proof given in ([21], Prop. 4.11) of the untwisted version of the next proposition originally proved by Bondal and Orlov in [3].

Proposition 1.6.9. *Let X be a smooth projective variety with ample (or antiample) canonical bundle. If there exists an exact equivalence $F : \mathrm{D}^b(X, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y, \beta)$ with Y a smooth projective variety, then there exists an isomorphism $f : X \xrightarrow{\sim} Y$ with $f^*(\beta) = \alpha$.*

Proof. First, note that from the definition of point like objects there exists a bijection between the set of point like objects in $\mathrm{D}^b(X,\alpha)$ and the point like objects in $\mathrm{D}^b(Y,\beta)$. Since we have

$$\{\text{points like objects in } \mathrm{D}^b(X,\alpha)\} = \{k(x)[m] | x \in X \text{ closed and } m \in \mathbb{Z}\}$$

and

$$\{k(y)[m] \,|\, y \in Y \text{ closed and } m \in \mathbb{Z}\} \hookrightarrow \{\text{point like objects in } \mathrm{D}^b(Y,\beta)\}$$

we conclude that $F(k(x)[n])$ is a point like object but we still do not know whether it is of the form $k(y)[m]$ for some closed point $y \in Y$ and $m \in \mathbb{Z}$.

Claim Every point like object in $\mathrm{D}^b(Y,\beta)$ is of the form $k(y)[m]$ for some closed point $y \in Y$ and $m \in \mathbb{Z}$.

Proof. Suppose not and let P be a point like object not isomorphic to any $k(y)[m]$. We know that for every $y \in Y$ there exists $x_y \in X$ and $m_y \in \mathbb{Z}$ such that

$$F(k(x_y)[m_y]) = k(y).$$

From the bijection between point like objects in $\mathrm{D}^b(X,\alpha)$ and in $\mathrm{D}^b(Y,\beta)$, we find $x_P \in X, m_P \in \mathbb{Z}$ such that $x_P \neq x_y$ for all $y \in Y$ and

$$F(k(x_P)[m_P]) = P.$$

Then,

$$\begin{aligned}
\mathrm{Hom}(k(y)[n], P) &= \mathrm{Hom}(F(k(x_y)[m_y])[n], F(k(x_P)[m_P])) \\
&= \mathrm{Hom}(k(x_y)[m_y+n], k(x_P)[m_P]) \\
&= \mathrm{Hom}(k(x_y), k(x_P)[m_P - m_y - n]) \\
&= 0
\end{aligned}$$

for all y. Hence, since by Prop. 1.4.5 the set

$$\{k(y)[n] \,|\, y \in Y \text{ closed}, \, n \in \mathbb{Z}\}$$

span the category $\mathrm{D}^b(Y,\beta)$, we conclude $P = 0$. $\square$

Thus, for every $x \in X$ there exists $y_x \in Y$ and $m_x \in \mathbb{Z}$ such that $F(k(x)) = k(y_x)[m_x]$. Besides, for every $x \in X$ there exists V_x a neighborhood of x such that for every $z \in V_x$, $F(k(z)) = k(y_z)[m_x]$ and we can conclude that $m_x = m_z$ for all $z \in X$. Therefore we can assume that $F(k(x)) = k(y_x)$ for all x in X and so F defines a bijection $f : X \to Y$ by $x \mapsto y_x$. Since $F \cong \Phi_{\mathcal{P}}$, we have $\mathcal{P}|_{\{x\}\times Y} \cong k(y_x)$ and from this we can assume that f is a morphism (cf. commentary after Lemma 1.4.6). Since F is an equivalence, we conclude that f is injective. The surjectivity of the map was shown above. By using F^{-1} we also show that f^{-1} is a morphism. On the other hand, $\mathcal{P}$ is a sheaf supported on the graph of f and the second projection gives an isomorphism $\mathrm{supp}(\mathcal{P}) \cong Y$. Then if we consider $\mathcal{P}$ as a sheaf over its support, we can consider it as a twisted sheaf over Y. Besides, we also know that it is a twisted sheaf of constant fibre dimension 1, i.e. an untwisted line bundle L over Y. Then $F \cong L \otimes f_*(-)$ (up to shift). Therefore, f is an isomorphism with $f^*(\beta) = \alpha$. $\square$

Let (X, α) be a twisted variety and let $\mathcal{F}$ be an α-twisted coherent sheaf. We proceed to define the exterior algebra $\bigwedge \mathcal{F}$. By definition $\mathcal{F} = (\mathcal{F}_i, \varphi_{ij})_{i,j \in I}$ where $\mathcal{F}_i$ is a coherent sheaf on an element U_i of an open covering $\{U_i\}_{i \in I}$ of X and

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$$

are morphisms that satisfies the α-twisted cocycle conditions. We define the exterior algebras as usual for any coherent sheaf $\mathcal{F}_i$ and we need only to check that the resulting transition maps satisfies the cocycle conditions. But this follows inmediately and it shows that for any $r \in \mathbb{N}$, $\bigwedge^r \mathcal{F}$ is a α^r-twisted sheaf. In particular, if $\mathcal{F}$ is a locally free α-twisted sheaf of rank r, the maximal exterior power of $\mathcal{F}$, $\bigwedge^r \mathcal{F}$, is a line bundle called the *determinant bundle of* $\mathcal{F}$ and we denote it by $\det(\mathcal{F})$. Now, we follow the proofs of the untwisted version of the following three lemmas and the corresponding corollary to get a twisted version of them (cf. [21]).

Lemma 1.6.10. *Let Z be a normal variety and $\mathcal{F} \in \mathrm{Coh}(Z, \alpha)$. If L_1 and L_2 are two line bundles with $\mathcal{F} \otimes L_1 \cong \mathcal{F} \otimes L_2$, then $L_1^r \cong L_2^r$ where r is the generic rank of $\mathcal{F}$.*

Proof. By definition $\mathcal{F} = (\mathcal{F}_i, \varphi_{ij})_{i,j \in I}$, where $\mathcal{F}_i$ is coherent sheaf on an open set U_i of an open covering $\{U_i\}_{i \in I}$ of X. Let $f = \{f_i\}_{i \in I}$ be the isomorphism $f : \mathcal{F} \otimes L_1 \cong \mathcal{F} \otimes L_2$ given in the statement, i.e.

$$f_i : \mathcal{F}_i \otimes L_1 \cong \mathcal{F}_i \otimes L_2$$

is an isomorphism for every $i \in I$ such that the following diagram commutes

$$\begin{tikzcd}
(\mathcal{F}_j \otimes L_1)|_{U_i \cap U_j} \arrow[r, "f_j|_{U_i \cap U_j}"] & (\mathcal{F}_j \otimes L_2)|_{U_i \cap U_j} \\
(\mathcal{F}_i \otimes L_1)|_{U_i \cap U_j} \arrow[r, "f_i|_{U_i \cap U_j}"'] \arrow[u, "\varphi_{ij}^1"] & (\mathcal{F}_i \otimes L_2)|_{U_i \cap U_j} \arrow[u, "\varphi_{ij}^2"']
\end{tikzcd}$$

where φ_{ij}^k are defined by $\varphi_{ij} \otimes \mathrm{id}$, $k = 1, 2$. First, let us suppose that $\mathcal{F}$ is a locally free α-twisted sheaf of rank r. The last diagram induces the following commutative diagram

$$\begin{tikzcd}
(\det(\mathcal{F}_j) \otimes L_1^r)|_{U_i \cap U_j} \arrow[r, "\tilde{f}_j"] & (\det(\mathcal{F}_j) \otimes L_2^r)|_{U_i \cap U_j} \\
(\det(\mathcal{F}_i) \otimes L_1^r)|_{U_i \cap U_j} \arrow[u, "\tilde{\varphi}_{ij}^1"] \arrow[r, "\tilde{f}_i"'] & (\det(\mathcal{F}_i) \otimes L_2^r)|_{U_i \cap U_j} \arrow[u, "\tilde{\varphi}_{ij}^2"']
\end{tikzcd}$$

Hence $\det(\mathcal{F}) \otimes L_1^r \cong \det(\mathcal{F}) \otimes L_2^r$ and so $L_1^r \cong L_2^r$.

In general, let $\mathcal{F}$ be an α-twisted coherent sheaf. Dividing by the torsion part, we can assume that $\mathcal{F}$ is torsion free. Since Z is normal, $\mathcal{F}$ is a locally free α-twisted sheaf on an open set U with $\operatorname{codim}(Z - U) \geq 2$. Therefore by the argument given above we have that $L_1^r|_U \cong L_2^r|_U$. Then it defines a trivializing section $s \in H^0(U, L_1^r \otimes L_2^{-r})$ which can be extended to another trivializing section $\tilde{s} \in H^0(Z, L_1^r \otimes L_2^{-r})$ and it defines an isomorphism $L_1^r \cong L_2^r$. $\square$

Lemma 1.6.11. *If $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is an equivalence, then the projection $q : \operatorname{supp}(\mathcal{P}) \twoheadrightarrow X$ is surjective.*

Proof. Let L be a locally free $q^*(\alpha).p^*(\beta)^{-1}$-twisted sheaf. Suppose that q is not surjective, i.e. there exists a point $x \in X \setminus q(\operatorname{supp}(\mathcal{P}))$. Consider the spectral sequence

$$E_2^{p,q} = \mathcal{T}or^{-p}(\mathcal{H}^q(\mathcal{P} \otimes L), q^*k(x)) \Rightarrow \mathcal{T}or^{-(p+q)}(\mathcal{P} \otimes L, q^*k(x)).$$

Since $\mathcal{H}^q(\mathcal{P} \otimes L)$ is a (untwisted) coherent sheaf, $\mathcal{T}or^{-p}(\mathcal{P} \otimes L, q^*k(x)) = 0$ because $\mathcal{H}^q(\mathcal{P} \otimes L)$ and $q^*k(x)$ have disjoint support. Hence from the spectral sequence $\mathcal{P} \otimes q^*k(x)$ is trivial. This implies that $\Phi_{\mathcal{P}}(k(x)) \cong 0$. This contradicts the fact that $\Phi_{\mathcal{P}}$ is an equivalence. $\square$

Remark 1.6.12. Since the support of a complex does not change when we take tensor product with a line bundle, one has

$$\operatorname{supp}(\mathcal{P}) = \operatorname{supp}(\mathcal{P}^\vee) = \operatorname{supp}(\mathcal{P}_R) = \operatorname{supp}(\mathcal{P}_L).$$

Thus, we also deduce from the equivalence that $p : \operatorname{supp}(P) \twoheadrightarrow Y$ is surjective. Hence, there exist two irreducible components $Z_1 \subset \operatorname{supp}(\mathcal{H}^i(\mathcal{P}))$ and $Z_2 \subset \operatorname{supp}(\mathcal{H}^j(\mathcal{P}))$ that project onto X and Y respectively. Note that the components could be different.

Lemma 1.6.13. *Let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence and let $Z \subseteq \operatorname{supp}(\mathcal{P})$ be a closed irreducible subvariety with normalization $\nu : \tilde{Z} \to Z$. Then there exists an integer $r > 0$ such that*

$$\pi_X^* \omega_X^r \cong \pi_Y^* \omega_Y^r$$

where $\pi_X := q \circ \nu$ and $\pi_Y := p \circ \nu$.

Proof. Let $Z \subseteq \mathrm{supp}(\mathcal{P})$ be a closed irreducible subvariety and let $\nu : \tilde{Z} \to Z$ be its normalization. Then there exists an integer i such that $Z \subseteq \mathrm{supp}(\mathcal{H}^i)$ where $\mathcal{H}^i := \mathcal{H}^i(\mathcal{P})$ is the i-th cohomology of $\mathcal{P}$. We will apply Lemma 1.6.10 to the coherent sheaf $\nu^*\mathcal{H}^i$ on $\tilde{Z}$. Since $\Phi_{\mathcal{P}}$ is an equivalence,

$$\mathcal{P} \otimes q^*\omega_X \cong \mathcal{P} \otimes p^*\omega_Y,$$

and by taking cohomology on both sides yields

$$\mathcal{H}^i \otimes q^*\omega_X \cong \mathcal{H}^i \otimes p^*\omega_Y$$

and by taking the pullback of ν we get

$$\nu^*(\mathcal{H}^i) \otimes \pi_X^*\omega_X \cong \nu^*(\mathcal{H}^i) \otimes \pi_Y^*\omega_Y.$$

Thus we can conclude that there exists $r > 0$ such that $\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$. □

Corollary 1.6.14. *Let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence and let $Z \subset \mathrm{supp}(\mathcal{P})$ be a closed subvariety such that ω_X (or $\omega_X^\vee$) restricted to the image of $q : Z \to X$ is ample. Then $p : Z \to Y$ is a finite morphism.*

Proof. Suppose that $p : Z \to Y$ is not finite, i.e. there exists an irreducible curve $i : C \hookrightarrow Z$ such that $p \circ i : C \to Y$ is constant. Thus, $i^*p^*\omega_Y$ is a numerically trivial line bundle on C. By Lemma 1.6.13, $i^*q^*\omega_X$ is also numerically trivial. On the other hand, ω_X (or $\omega_X^\vee$) is ample on $q(Z)$ and so on $q(i(C))$ because $q \circ i$ is non-trivial. □

The following result is the twisted version of a result of Orlov (cf. [37]). We follow the proof given in ([21], Prop. 6.1).

Theorem 1.6.15. *Let X and Y be two projective varieties with $\alpha \in \mathrm{Br}'(X)$ and $\beta \in \mathrm{Br}'(Y)$. Any equivalence of categories $F : \mathrm{D}^b(X, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y, \beta)$ implies an isomorphism of the canonical rings $R(X) \cong R(Y)$.*

Proof. Let d be the diagonal morphism $d : X \hookrightarrow X \times X$. Then $d_*\mathcal{O}_X$ can be regarded as a $\alpha \boxtimes \alpha^{-1}$-twisted sheaf. Denote $\mathcal{O}_\Delta := d_*\mathcal{O}_X$, which viewed as a Fourier–Mukai kernel induces the identity $\mathrm{id} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(X, \alpha)$.

The equivalence F is given by a Fourier–Mukai transform

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$$

with $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$. Then the Fourier–Mukai transform

$$\Phi_{\mathcal{Q}} : \mathrm{D}^b(X, \alpha^{-1}) \to \mathrm{D}^b(Y, \beta^{-1})$$

with

$$\mathcal{Q} := \mathcal{P}^\vee \otimes q^*\omega_X[n] \cong \mathcal{P}^\vee \otimes p^*\omega_Y[n] \in \mathrm{D}^b(Y \times X, \beta^{-1} \boxtimes \alpha)$$

is also an equivalence. Indeed, since the composition

$$\mathrm{D}^b(X,\alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y,\beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(X,\alpha)$$

is isomorphic to the identity, and the kernel of this composition is given by $\mathcal{R} = \pi_{13*}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{Q})$, one has $\mathcal{R} \cong \mathcal{O}_\Delta \in \mathrm{D}^b(X \times X, \alpha^{-1} \boxtimes \alpha)$. Consider the automorphism $\tau_{12} : X \times X \to X \times X$ that interchanges the two factors,

$$\mathcal{O}_\Delta \cong \tau_{12}^*\mathcal{O}_\Delta \cong \tau_{12}^*\mathcal{R} \cong \pi_{13*}\tau_{13}^*(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{Q}) \cong \pi_{13*}(\pi_{12}^*\mathcal{Q} \otimes \pi_{23}^*\mathcal{P}).$$

Thus the composition of

$$\mathrm{D}^b(X,\alpha^{-1}) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(Y,\beta^{-1}) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(X,\alpha^{-1})$$

is isomorphic to the identity.

In the same way we can prove that

$$\mathrm{D}^b(Y,\beta^{-1}) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(X,\alpha^{-1}) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(X,\beta^{-1})$$

is isomorphic to the identity.

Moreover $\mathcal{P} \boxtimes \mathcal{Q} \in \mathrm{D}^b((X \times X) \times (Y \times Y), \alpha^{-1} \boxtimes \alpha \boxtimes \beta \boxtimes \beta^{-1})$ defines the Fourier–Mukai equivalence

$$\Phi_{\mathcal{P}\boxtimes\mathcal{Q}} : \mathrm{D}^b(X \times X, \alpha^{-1} \boxtimes \alpha) \longrightarrow \mathrm{D}^b(Y \times Y, \beta^{-1} \boxtimes \beta)$$

Now, we show that this equivalence implies an isomorphism between the canonical rings. Since $d_*(\omega_X^m)$ can be considered as an element in $\mathrm{D}^b(X \times X, \alpha^{-1} \boxtimes \alpha)$, by defining $S := \Phi_{\mathcal{P}\boxtimes\mathcal{Q}}(d_*\omega_X^m)$ we have that

$$\Phi_S : \mathrm{D}^b(Y,\beta) \to \mathrm{D}^b(Y,\beta)$$

is an equivalence that can be obtained as the composition

$$\mathrm{D}^b(Y,\beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(X,\alpha) \xrightarrow{\Phi_{d_*\omega_X^m}} \mathrm{D}^b(X,\alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y,\beta).$$

That is,

$$\Phi_S \cong \Phi_{\mathcal{P}} \circ \Phi_{d_*\omega_X^m} \circ \Phi_{\mathcal{Q}}.$$

Note that $\Phi_{d_*\omega_X^m} = S_{(X,\alpha)}^m[-mn]$ where $S_{(X,\alpha)}$ denotes the Serre functor defined on the category $\mathrm{D}^b(X,\alpha)$. From the fact that equivalences commutes with Serre functor, we conclude that

$$\Phi_S \cong S_{(Y,\beta)}^m[-mn].$$

Then, the uniqueness of the kernel of a Fourier–Mukai transform yields

$$S \cong d_*\omega_Y^m,$$

i.e. $\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}(d_*\omega_X^m) \cong d_*(\omega_Y^m)$. Thus

$$\begin{aligned} H^0(X,\omega_X^m) &= Hom_{D^b(X\times X,\alpha\boxtimes\alpha^{-1})}(d_*\mathcal{O}_X, d_*\omega_X^m) \\ &\cong Hom_{\mathrm{D}^b(Y\times Y,\beta\boxtimes\beta^{-1})}(d_*\mathcal{O}_Y, d_*\omega_Y^m) \\ &= H^0(Y,\omega_Y^m). \end{aligned}$$

Since the algebra structure is given by composition of Ext's just by using

$$\mathrm{Ext}^i_{\mathrm{D}^b(X\times X,\alpha^{-1}\boxtimes\alpha)}(d_*\mathcal{O}_X, d_*(\omega_X^k)) \cong \mathrm{Ext}^i_{\mathrm{D}^b(X\times X,\alpha^{-1}\boxtimes\alpha)}(d_*\omega_X^m, d_*(\omega_X^{m+k})).$$

Hence $R(X) \cong R(Y)$. □

The following result in the untwisted case is due to Kawamata (cf. [25]) but copying his proof yields a proof in the twisted case.

Theorem 1.6.16 (Kawamata). *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \longrightarrow \mathrm{D}^b(Y,\beta)$ be an equivalence such that the canonical bundle ω_X is big or anti-big (i.e. $\omega_X^\vee$ is big). Then there exists a birational morphism $f : X \dashrightarrow Y$ with $f^*(\beta) = \alpha$ where it is defined.*

Proof. Assume ω_X is big, i.e. there exists $m > 0$ such that $\omega_X^m \equiv_{lin} H + D$ with H ample and D effective. Let Z be an irreducible component of $\mathrm{supp}(\mathcal{P})$ that surjects onto X. Let us show that

$$\pi_Y : \tilde{Z}\backslash\pi_X^{-1}(D) \to Y$$

is quasifinite where $\pi_X := q\circ\nu$, $\pi_Y := p\circ\nu$ and $\nu : \tilde{Z} \to Z$ is the normalization map. Suppose that our map π_Y is not quasifinite, i.e. there exists an irreducible curve $C \nsubseteq \pi_X^{-1}(D)$ such that $\pi_Y(C)$ is a point. Thus $(\pi_Y^*(K_Y).C) = 0$. On the other hand,

$$m(\pi_X^* K_X.C) = (\pi_X^* H.C) + (\pi_X^* D.C).$$

Then,

$$(\pi_X^* K_X.C) \geq \frac{1}{m}(\pi_X^* H.C) > 0 \tag{1.12}$$

because H is ample. By Lemma 1.6.13 there exists an integer r such that

$$(\pi_X^* rK_X.C) = (\pi_Y^* rK_Y.C).$$

Thus, by inequality (1.12), $(\pi_Y^* rK_Y.C) > 0$. This contradicts the fact

$$(\pi_Y^*(K_Y).C) = 0.$$

Therefore, the morphism $q : Z \to Y$ is generically finite which implies $\dim(Z) \leq \dim(Y)$. Since Z dominates X, $\dim(Z) \geq \dim(X)$. Thus, from the equality $\dim(X) = \dim(Y)$ we get

$$\dim(X) = \dim(Z) = \dim(Y).$$

Now, we show for $x \in X$ and $y \in Y$ generic that both $Z \cap (X \times \{y\})$ and $Z \cap (\{x\} \times Y)$ consist of just one point. For $x \in X$ generic, the intersection $Z \cap (\{x\} \times Y)$ is a finite set of reduced points $\{y_1, ..., y_m\}$ and disjoint from any other irreducible component of $\bigcup \mathrm{supp}(\mathcal{H}^i(\mathcal{P}))$. Then, locally around y_i the image $\Phi_{\mathcal{P}}(k(x))$ has support in y_i. Thus, $\mathrm{Hom}(\Phi_{\mathcal{P}}(k(x)), \Phi_{\mathcal{P}}(k(x)))$ is m-dimensional and hence by faithfulness of the functor $\Phi_{\mathcal{P}}$, we deduce that $m = 1$. If y is generic, we use a similar argument by using $\Phi_{\mathcal{P}_R}$ instead of $\Phi_{\mathcal{P}}$ itself. The only thing we need to check is that Z is also a component of $\mathrm{supp}(\mathcal{P}^\vee)$, but this follows from the equality (see section 1.4)

$$\mathrm{supp}(\mathcal{P}) = \mathrm{supp}(\mathcal{P}^\vee).$$

Since $Z \cap (X \times \{y\})$, $Z \cap (\{x\} \times Y)$ consist of only one reduced point, it defines a birational morphism $f : X \to Y$ with $f^*(\beta) = \alpha$ because the sheaf $\mathcal{P}$ is a line bundle considered as a sheaf over the intersection between Z and the open set where f is defined (since for general x in X there exists y in Y with $\Phi(k(x)) = \mathcal{P}|_{\{x\}\times Y} = k(y)$). □

Remark 1.6.17. If X and Y are two smooth projective varieties with a birational correspondence

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \\ X & & \end{array}$$

where Z is a normal smooth variety. If $\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$, then $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$.

Remark 1.6.18. Let X and Y be two K-equivalent surfaces, i.e. there exists a birational correspondence

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \\ X & & \end{array}$$

such that $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$. Then $X \cong Y$.

1.7 Classification of surfaces under twisted derived categories.

In this section we show that a theorem of Kawamata remains true when we consider twisted derived categories.

Definition 1.7.1. *If L is a line bundle on a projective scheme X, we define the numerical Kodaira dimension $\nu(X, L)$ to be the maximal integer m such that there exists a proper morphism $\phi : W \to X$ with W of dimension m and $([\phi^*(L)]^m.W) \neq 0$. In particular, if $L = \omega_X$, we denote $\nu(X) := \nu(X, \omega_X)$.*

Lemma 1.7.2 ([21], Lemma 6.26). *Let $\pi : Z \to X$ be a projective morphism of proper schemes and $L \in \operatorname{Pic}(X)$.*

(i) If L is a nef line bundle on X then $\pi^*(L)$ is nef.

(ii) If π is surjective, then L is nef if and only if $\pi^*(L)$ is nef.

Lemma 1.7.3 ([21], Lemma 6.28). *Let $\pi : Z \to X$ be a projetive morphism of projective schemes and $L \in \operatorname{Pic}(X)$.*

(i) Then $\nu(X, L) \geq \nu(Z, \pi^* L)$.

(ii) If $\pi : Z \to X$ is surjective, then $\nu(X, L) = \nu(Z, \pi^* L)$.

Proposition 1.7.4 ([21], Prop. 6.17). *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence. Then $\nu(X) = \nu(Y)$.*

Proof. Since $\Phi_{\mathcal{P}}$ is an equivalence, there exists a component Z of $\operatorname{supp}(\mathcal{P})$ such that $p : Z \to Y$ is surjective. If $\nu : \tilde{Z} \to Z$ is the normalization, then by Lemma 1.6.13, there exists an integer m such that $\pi_X^* \omega_X^r \cong \pi_Y^* \omega_Y^r$ where $\pi_X = q \circ \nu$ and $\pi_Y = p \circ \nu$. Hence

$$\nu(\tilde{Z}, \pi_X^* \omega_X^r) = \nu(\tilde{Z}, \pi_Y^* \omega_Y^r)$$

and then

$$\nu(X, \omega_X) \geq \nu(\tilde{Z}, \pi_X^* \omega_X) = \nu(\tilde{Z}, \pi_X^* \omega_X^r) = \nu(\tilde{Z}, \pi_Y^* \omega_Y^r) = \nu(Y, \omega_Y).$$

The other inequality holds by considering $\Phi_{\mathcal{P}_R}$ instead of $\Phi_{\mathcal{P}}$. $\square$

Definition 1.7.5. *A rational surface is a surface that is birationally equivalent to $\mathbb{P}^2$.*

Definition 1.7.6. *A ruled surface, is a smooth projective surface X, together with a surjective morphism $\pi : X \to C$ to a nonsingular curve C, such that the fibre X_y is isomorphic to $\mathbb{P}^1$ for every point $y \in C$.*

Theorem 1.7.7 (Castelnuovo). *A surface is rational if and only if the irregularity and second geometric genus are trivial, i.e. $h^1(X, \mathcal{O}_X) = h^0(X, \omega_X^2) = 0$.*

Definition 1.7.8. *A smooth surface X is an elliptic surface if there exists a curve C and a morphism $\pi : X \to C$ whose general fibre is an elliptic curve.*

The proof of the following result is identical to the proof of its untwisted version given in ([21], Prop. 6.18), which was originally proved by Kawamata in [25].

Theorem 1.7.9 (Kawamata). *Let X be a smooth projective surface containing a (-1)-curve and Y a smooth projective variety and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence. Then one of the following holds* *(i)* $X \cong Y$.

(ii) *X is a relatively minimal elliptic rational surface.*

Proof. Let Z be a component of $\Gamma := \operatorname{supp}(\mathcal{P})$ that dominates X.

Case 1 $\dim(Z) = 2$

Since $\dim(Z) = 2$, the map $q : Z \to X$ is generically finite. It can be seen as in the proof of Theorem 1.6.16 that q is a birational morphism. We show that Z dominates Y. For a general point x in X the birationality of q implies that $\Phi_{\mathcal{P}}(k(x))$ is concentrated in only one point, say y. Thus, by Lemma 1.6.2

$$\Phi(k(x)) = k(y)[m]$$

for some integer m. Then there exists an open dense subset $U \subseteq X$ such that $U \cong Z_U = \Gamma_U$. If we suppose that p is not dominant, there exist distinct x_1, x_2 in U with $p(x_1) = p(x_2) =: y$ (because $\dim(Z) = 2$ and if p is not dominant then $\overline{p(Z)}$ is of dimension at most 1). Since $\Phi_{\mathcal{P}}(k(x_1))$ and $\Phi_{\mathcal{P}}(k(x_2))$ are concentrated in the single point y, we find a non-trivial morphism

$$\mathcal{H}^{m_1}(\Phi(k(x_1))) \to \mathcal{H}^{m_2}(\Phi(k(x_2)))$$

where m_1 and m_2 are the maximal and the minimal integers where the cohomologies for $\Phi(k(x_1))$ and $\Phi(k(x_2))$ are nonzero. This morphism defines a non-trivial morphism

$$\Phi(k(x_1))[m_1] \to \Phi(k(x_2))[m_2].$$

This contradicts the fact that

$$\operatorname{Ext}^j(k(x_1), k(x_2)) = 0 \text{ for all } j.$$

Hence, the morphism $p : Z \to Y$ is dominant and thus it is a birational map which defines a K-equivalence $X \xleftarrow{q} Z \xrightarrow{p} Y$. This implies that the surfaces X and Y are isomorphic.

Case 2 Any irreducible component $Z \subset \Gamma$ that dominates X or Y has dimension at least three.

Since X is not minimal, there exists a (-1)-curve $E(\cong \mathbb{P}^1)$ in X. Denote $\Gamma_E := \Gamma \times_X E$. Since the canonical bundle ω_X over E is an antiample line bundle ($\omega_X|_E \cong \mathcal{O}(-1)$), by Corollary 1.6.14 the projection $p_E : \Gamma_E \to Y$ is a finite morphism. Hence $\dim(\Gamma_E) = 2$ (the dimension of the fibres of $q_E : \Gamma_E \to X$ are at least one dimensional). Then, as we did in the first case we can show that the morphism $p_E : \Gamma_E \to Y$ is dominant. If $\nu : \tilde{Z}_E \to Z_E$ is the normalization of a component Z_E of Γ_E, then by Lemma 1.6.13, there exists an integer r such that $\nu^* q_E^* \omega_X^r \cong \nu^* p_E^* \omega_Y^r$ where p_E and q_E are considered defined on the component Z_E. Thus,

$$\nu^* q_E^* \omega_X \equiv_{num} \nu^* p_E^* \omega_Y. \tag{1.13}$$

Since $\omega_X^\vee$ is ample on E, $\omega_Y^\vee$ is nef (because $q_E \circ \nu$ is surjective and Lemma 1.7.2). By Lemma 1.7.3 and (1.13)

$$\nu(Y, \omega_Y^\vee) = \nu(\tilde{Z}_E, \nu^* p_E^*(\omega_Y^\vee)) = \nu(\tilde{Z}_E, \nu^* q_E^* \omega_X^\vee) = 1.$$

Since $\omega_Y^\vee$ is nef, $\omega_X^\vee$ is nef because $q : \Gamma \to X$ is a surjective (cf. Lemma 1.7.2) morphism and (1.13). Since $\Phi_{\mathcal{P}}$ is an equivalence, $\nu(X, \omega_X^\vee) = \nu(Y, \omega_Y^\vee) = 1$. Let us see that $\mathrm{kod}(X) = -\infty$. Suppose that $H^0(X, \omega_X^k) \neq 0$ for some $k > 0$. Let $0 \neq s$ be a section in $H^0(X, \omega_X^k)$. Then $Z(s)$ is either empty or a curve. The first case can not happen because $\nu(X, \omega_X^{-k}) = 1$ and neither the second because otherwise $Z(s)$ intersects non-trivially with an ample divisor and this contradicts that $\omega_X^\vee$ is nef. This shows that $\mathrm{kod}(X) = -\infty$.

By the classification of surfaces we know that the minimal model for X is either a rational or a ruled surface over a curve of genus ≥ 1. If the minimal model of X is a ruled surface, then it satisfies $c_1^2 = 8(1-g)$. Then $g = 1$ because $\omega_X^\vee$ is nef and since c_1^2 decreases under blow-ups, X is a minimal ruled surface over an elliptic curve (again because $\omega_X^\vee$ is nef). This contradicts our assumption that X contains a (-1)-curve. Thus X is a rational surface. By Theorem 1.6.15, $h^0(X, \omega_X^2) = h^0(Y, \omega_Y^2)$ and $h^1(X, \mathcal{O}_X) = h^1(X, \mathcal{O}_Y)$. Hence, from the Castelnuovo criterion (cf. Theorem 1.7.7), Y is a rational surface. Since Y is a rational surface, we can pick a smooth rational curve E' in Y such that $\omega_Y^\vee$ is ample on it. By the same discusion as before, we obtain a finite dominating morphism $\Gamma'_E \to X$. Since the pullbacks of ω_X and ω_Y are numerically equivalent, the restriction of ω_X to $D := q(F)$ where F is a fibre of $p : \Gamma'_E \to X$ is numerically trivial. By the Hodge index theorem, either $c_1^2(X) < 0$ or $c_1(X) = 0$ because D moves in a family (because from $F.\omega_Y = 0$ we conclude that $D.\omega_X = 0$). The first contradicts the fact that $\omega_X^\vee$ is nef and the second the fact that $H^0(X, \omega_X^k) = 0$ for any k (because $\mathrm{kod}(X) = -\infty$ and the fact that $\omega_X \equiv 0$ implies $\omega^k \cong \mathcal{O}_X$ for some k). Thus $D^2 = 0$ and this defines the desired covering of X by elliptic curves. $\square$

1.7.1 Surfaces with kod$=-\infty$,2

We also have the following twisted version of a proposition due to Bridgeland and Maciocia. The proof is identical to ([21], Prop. 12.16)

Proposition 1.7.10. *Let X be a surface of general type and Y a smooth projective variety. If $\mathrm{D}^b(X, \alpha) \cong \mathrm{D}^b(Y, \beta)$, then $X \cong Y$.*

Proof. Since X is of general type, Y is also of general type by Theorem 1.6.15. Moreover, by Theorem 1.6.16, X and Y are birational. If X is not minimal, by Theorem 1.7.9, $X \cong Y$. Thus we can assume that X and Y are minimal surfaces. Since the minimal model of a surface of general type is unique, the birational morphism between X and Y yields an isomorphism $X \cong Y$. $\square$

Let X be a rational surface. Thus, $H^i(X, \mathcal{O}_X) = 0$ for any $i > 0$. From the exponential short exact sequence we obtain the isomorphism

$$\mathrm{Br}'(X) \cong H^3(X, \mathbb{Z}).$$

Since the cohomological Brauer group is a birational invariant,

$$\mathrm{Br}'(X) \cong \mathrm{Br}'(\mathbb{P}^2) = H^3(\mathbb{P}^2, \mathbb{Z}) = 0.$$

Now, let $\pi : X \to C$ be a ruled surface. Consider the Leray spectral sequence associated to π:

$$E_2^{p,q} = H^p(C, R^q\pi_*\mathcal{O}_X^*) \Rightarrow H^{p+q}(X, \mathcal{O}_X^*).$$

Since $H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $q \geq 1$, we obtain

$$R^q\pi_*\mathcal{O}_X = 0, \text{ for } q \geq 1. \tag{1.14}$$

The exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ yields a long exact sequence

$$\ldots \to R^q\pi_*\mathcal{O}_X \to R^q\pi_*\mathcal{O}_X^* \to R^{q+1}\pi_*\mathbb{Z} \to R^{q+1}\pi_*\mathcal{O}_X \to \ldots$$

so that by equation (1.14),

$$R^q\pi_*\mathcal{O}_X^* \cong R^{q+1}\pi_*\mathbb{Z}, \text{ for any } q \geq 1. \tag{1.15}$$

Clearly $R^q\pi_*\mathcal{O}_X^* = 0$ for $q \geq 2$ and $R^0\pi_*\mathcal{O}_X^* = O_C^*$. On the other hand, the sheaf $R^2\pi_*\mathbb{Z}$ is a local system of coefficients with stalk $\mathbb{Z}$ and the complex structure of the morphism π gives a canonical generator for each stalk on this local system. Thus $R^2\pi_*\mathbb{Z}$ is trivial, i.e. $R^2\pi_*\mathbb{Z} = \mathbb{Z}$. Hence by the isomorphism (1.15)

$$R^1\pi_*\mathcal{O}_X^* = \mathbb{Z}. \tag{1.16}$$

The Leray spectral sequence yields a long exact sequence

$$H^0(C, R^1\pi_*\mathcal{O}_X^*) \to H^2(C, \mathcal{O}_C^*) \to H^2(X, \mathcal{O}_X^*) \to H^1(C, R^1\pi_*\mathcal{O}_X^*). \tag{1.17}$$

By equation 1.16, $H^1(C, R^1\pi_*\mathcal{O}_X^*) = H^1(C, \mathbb{Z}) = \mathbb{Z}^{2g(C)}$. Since X is smooth, $H^2(X, \mathcal{O}_X^*)$ is a torsion group. Thus the last map in the sequence (1.17) is trivial and since $H^2(C, \mathcal{O}_C^*) = 0$, we obtain $\mathrm{Br}'(X) = H^2(X, \mathcal{O}_X^*) = 0$ (if X is not smooth we also obtain that $\mathrm{Br}'(X) = H^2(X, \mathcal{O}_X^*)_{tors} = 0$). Therefore, we have shown the following proposition:

Proposition 1.7.11. *If X is a smooth projective surface of* $\mathrm{kod}(X) = -\infty$*, then* $\mathrm{Br}'(X) = 0$*.*

Proposition 1.7.12. *Let X be a smooth projective surface containing a* (-1)*-curve and Y a smooth projective variety. If* $\mathrm{Br}'(X) \neq 0$ *and* $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ *is an equivalence. Then $X \cong Y$.*

Proof. By Theorem 1.7.9, either $X \cong Y$ or X is a rational surface that is elliptically fibred. Thus, if X is rational, Proposition 1.7.11 implies $\mathrm{Br}'(X) = 0$, a contradiction. □

1.7.2 Surfaces with $\mathrm{kod} = 1$

Definition 1.7.13. *A vector bundle $\mathcal{F}$ on a curve C is decomposable if it is isomorphic to a direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ of two non-zero vector bundles. Otherwise, we say that $\mathcal{F}$ is indecomposable.*

Lemma 1.7.14 ([38], Cor 14.8). *Let $\mathcal{F}$ be an indecomposable vector bundle of rank r and degree d on an elliptic curve E. The following conditions are equivalent.*

(i) $\mathcal{F}$ is stable;

(ii) $\mathcal{F}$ is simple;

(iii) d and r are relatively prime.

Theorem 1.7.15 ([12], Prop. I. 3.24). *Let X be a minimal projective surface of Kodaira dimension 1. Then there is a unique curve C and a unique morphism $\pi : X \to C$ making X an elliptic surface.*

Definition 1.7.16. *Let $\pi : X \to C$ be an elliptic surface and $c \in C$. The fibre $\pi^{-1}(c)$ is called a multiple fibre if there is a divisor D on X with $\pi^{-1}(c) = mD$ for some integer $m > 1$.*

Let $\pi : X \to C$ be a relatively minimal elliptic surface with $\mathrm{kod}(X) = 1$. The cohomology class of the fibre $F_x := \pi^{-1}(x)$ is denoted by $f \in H^2(X, \mathbb{Z})$. Note that F_x is a smooth elliptic curve for generic $x \in C$. The canonical bundle formula (cf. [1], V.12) states that

$$\omega_X \cong \pi^*\mathcal{L} \otimes \mathcal{O}(\sum (m_i - 1)F_i) \tag{1.18}$$

where $\mathcal{L} \in \mathrm{Pic}(C)$ and F_i are the multiple fibres. Hence $c_1(X) = \lambda f$ (in $H^2(X, \mathbb{Q})$) for some $\lambda \neq 0$ (because $\mathrm{kod}(X) = 1$). We also define the moduli space $M_H(v)$ similarly as for K3 surfaces to be the moduli space of semi-stable (with respect to H) sheaves E with $v(E) = v$.

Remark 1.7.17. Suppose $v = (0, rf, d)$ and E a stable sheaf of rank r and degree d. By definition one has $\chi(E) = d$ by the Hirzebruch–Riemann–Roch formula and $f.c_1(X) = 0$. On the other hand, if $[E] \in M_H(v)$ corresponds to a stable sheaf E, $\mathrm{supp}(E)$ is connected, so that $\mathrm{supp}(E) \subseteq F_x$ for some fibre F_x because $v(E) = (0, rf, d)$ (if $\mathrm{supp}(E)$ has an horizontal component it would intersect non-trivially the fibre class f).

Definition 1.7.18. *Let $\pi : X \to C$ be an elliptic surface with* $\mathrm{kod}(X) = 1$ *and let $\lambda_{X/C}$ denote the smallest positive number such that there exists a divisor σ on X with $\sigma.f = \lambda_{X/C}$. We also denote it sometimes by only λ_X (recall that from Theorem* 1.7.15 *there is only one C and morphism making X an elliptic fibration).*

Theorem 1.7.19 ([8], Theorem 3.2.1). *The functor* $F = \Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ *is fully faithful, if and only if, for each* $x \in X$,

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}(F(k(x)), F(k(x))) = \mathbb{C},$$

and for each pair of points $x_1, x_2 \in X$, *and each integer* i,

$$\mathrm{Ext}^i_{\mathrm{D}^b(Y)}(F(k(x_1)), F(k(x_2))) = 0$$

unless $x_1 = x_2$ *and* $0 \leq i \leq \dim X$. *Assuming the above conditions satisfied,* F *is an equivalence if and only if, for every point* $x \in X$,

$$F(k(x)) \overset{L}{\otimes} \omega_Y \cong F(k(x)).$$

Căldăraru proved in [8] a version of the following proposition in the case of K3 surfaces. In that case the proof followed inmediately from the last theorem because of the triviality of the canonical bundle for K3 surfaces. This is not the case for properly elliptic surfaces.

Proposition 1.7.20. *Let* X *be a properly elliptic surface, i.e.* $\mathrm{kod}(X) = 1$ *that is relatively minimal, and let* $v = (0, rf, d)$ *be a Mukai vector with* $gcd(r, d) = 1$. *Let* M *be a connected component of the moduli space of stable sheaves with Mukai vector* v *and let* $\alpha = \mathrm{Obs}(X, v)$ *(see Definition 1.5.5). Then we have*

$$\mathrm{D}^b(X) \cong \mathrm{D}^b(M, \alpha^{-1}).$$

Proof. The $\pi_M^*\alpha$-universal sheaf $\mathcal{E}$ on $X \times M$ defines a functor

$$\Phi_{\mathcal{E}} : \mathrm{D}^b(M, \alpha^{-1}) \to \mathrm{D}^b(X).$$

Let $[\mathcal{F}] \in M$ be a point corresponding to a stable sheaf $\mathcal{F}$ on X and Mukai vector $v = (0, rf, d)$. Then, by definition of the universal sheaf, $\Phi_{\mathcal{E}}(k([\mathcal{F}])) = \mathcal{F}$. We check the conditions of Theorem 1.7.19. Let $[\mathcal{F}]$ and $[\mathcal{G}]$ be two distinct points in M corresponding to two nonisomorphic stable sheaves $\mathcal{F}$ and $\mathcal{G}$ on X respectively. Since $\mathcal{F}$ is a stable sheaf,

$$\mathrm{Hom}(\Phi_{\mathcal{E}}(k([\mathcal{F}])), \Phi_{\mathcal{E}}(k([\mathcal{F}]))) = \mathrm{Hom}(\mathcal{F}, \mathcal{F}) = \mathbb{C}.$$

If $i < 0$ or $i > 2$, trivially $\mathrm{Ext}^i(\Phi_{\mathcal{E}} k([\mathcal{F}]), \Phi_{\mathcal{E}} k([\mathcal{G}])) = 0$. Since $\mathcal{F}$ and $\mathcal{G}$ are stables,

$$\mathrm{Hom}(\Phi_{\mathcal{E}}(k([\mathcal{F}])), \Phi_{\mathcal{E}}(k([\mathcal{G}]))) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0.$$

By Serre duality,

$$\mathrm{Ext}^2(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^{\vee}. \tag{1.19}$$

Let us show that $\mathcal{F} \cong \mathcal{F} \otimes \omega_X$. If $\mathcal{F}$ is supported on a non-singular fibre, by the canonical bundle formula (cf. (1.18)), the restriction of ω_X to the non-singular fibre is trivial. Hence $\mathcal{F} \cong \mathcal{F} \otimes \omega_X$. Since the dimension of $\mathrm{Hom}(\mathcal{E}_{[\mathcal{F}]}, \mathcal{E}_{[\mathcal{F}]} \otimes \omega_X)$ is upper semi-continuous on M (cf. [14], III. 7.7.8), for all $[\mathcal{F}] \in M$ there is a non-zero morphism $\mathcal{E}_{[\mathcal{F}]} \to \mathcal{E}_{[\mathcal{F}]} \otimes \omega_X$ (i.e. $\mathcal{F} \to \mathcal{F} \otimes \omega_X$ is non-zero). Since $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F} \otimes \omega_X)$ and

$$c_1(\mathcal{F}).f = c_1(\mathcal{F}).f + c_1(X).f = c_1(\mathcal{F} \otimes \omega_X).f$$

and both sheaves $\mathcal{F}$ and $\mathcal{F} \otimes \omega_X$ are stable, we obtain an isomorphism

$$\mathcal{F} \cong \mathcal{F} \otimes \omega_X$$

for all $\mathcal{F}$ stable. Thus, by isomorphism 1.19

$$\mathrm{Ext}^2(\Phi_{\mathcal{E}}(k([\mathcal{F}])), \Phi_{\mathcal{E}}(k([\mathcal{G}]))) = \mathrm{Ext}^2(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^\vee = \mathrm{Hom}(\mathcal{G}, \mathcal{F})^\vee = 0$$

for any two points $[\mathcal{F}] \neq [\mathcal{G}]$ in M (corresponding to two stable sheaves on X). Since

$$\chi(\mathcal{F}, \mathcal{G}) = -\langle v(\mathcal{F}), v(\mathcal{G}) \rangle = -\langle v, v \rangle = 0,$$

we obtain $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$. Thus, since we have verified all the conditions of Theorem 1.7.19, $\Phi_{\mathcal{E}}$ is an equivalence of categories. □

The following result is a generalization of a result of Bridgeland and Maciocia (cf. [5]). We follow the proof given in [21] with some little modifications.

Proposition 1.7.21. *Let $\pi : Y \to C$ be a relatively minimal elliptic surface with $\mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. Then there exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and $X \cong M(v)$.*

Proof. If either X or Y is not minimal, then they are isomorphic (see Theorem 1.7.9) and we pick $v = (0, f, 1)$. Hence, we may assume that X and Y are minimal surfaces. For any closed point x in X, $E := \Phi(k(x))$ satisfies

$$E \otimes \omega_Y \cong E,$$

because of Remark 1.6.7. Since $\mathrm{Hom}(k(x), k(x)) = \mathrm{Hom}(E, E)$, E is simple and thus $\mathrm{supp}(E)$ is connected. Since $E \cong E \otimes \omega_Y$, $\mathrm{supp}(E) \subset F_y$ for some fibre $F_y \subset Y$ because $\mathrm{kod}(Y) = 1$ and the isomorphism (1.18). For general x, we may assume that F_y is a smooth fibre. Thus, since $\mathrm{supp}(E)$ is connected, either $\mathrm{supp}(E) = F_y$ or $\mathrm{supp}(E)$ consists of only a closed point in F_y.

Claim We can assume that E is a shifted sheaf, i.e. $\mathcal{H}^i(E) = 0$ for all but one $i \in \mathbb{Z}$.

Proof. Consider the spectral sequence

$$E_2^{p,q} = \bigoplus_i \mathrm{Ext}^p(\mathcal{H}^i(E), \mathcal{H}^{i+q}(E)) \Rightarrow Ext^{p+q}(E, E).$$

Since Y is a surface, $E_2^{p,q}$ are trivial for $p \notin [0, 2]$ In particular

$$\bigoplus_i \mathrm{Ext}^1(\mathcal{H}^i(E), \mathcal{H}^i(E)) \subset \mathrm{Ext}^1(E, E). \tag{1.20}$$

Since E is supported on a smooth elliptic curve F_x, all its cohomologies are. This means that if $\mathcal{H}^i(E) \neq 0$, $\mathrm{Ext}^1(\mathcal{H}^i(E), \mathcal{H}^i(E)) \neq 0$ (because $\mathrm{Ext}^1_{F_x}(\mathcal{H}^i(E), \mathcal{H}^i(E)) \hookrightarrow \mathrm{Ext}^1_Y(\mathcal{H}^i(E), \mathcal{H}^i(E))$). Moreover, since $\mathcal{H}^i(E)$ is supported on a smooth elliptic curve, $\chi(\mathcal{H}^i(E), \mathcal{H}^i(E)) = -\langle v(\mathcal{H}^i(E)), v(\mathcal{H}^i(E))\rangle = 0$ and $\mathcal{H}^i(E) = \mathcal{H}^i(E) \otimes \omega_X$ (cf. the proof of Proposition 1.7.20). Thus, by Serre duality $\dim \mathrm{Ext}^1_Y(\mathcal{H}^i(E), \mathcal{H}^i(E))$ is even (≥ 2) for any $\mathcal{H}^i(E) \neq 0$. Hence by (1.20), $2n \leq \dim \mathrm{Ext}^1_Y(E, E) = 2$, where n is the number of non-trivial cohomologies $\mathcal{H}^i$. Thus, E is a shifted sheaf. □

By composing the original equivalence with a shift, we can assume that E is a sheaf. If E is concentrated in one point y, from $\Phi(k(x)) = k(y)$ we get that X and Y are birational. Hence they are isomorphic because they are minimal surfaces (the minimal model for surfaces of Kodaira dimension 1 is unique). Thus, we can assume that E is a vector bundle on F_y. Since E is simple, by Lemma 1.7.14, E is stable (with respect to some polarization H) and $(\mathrm{rk}(E), \deg(E)) = 1$. Set $v = (0, rf, d)$ where $r := \mathrm{rk}(E), d := \deg(E)$. Then v is isotropic, i.e. $\langle v, v\rangle = 0$. Hence the moduli space $M = M_H(v)$ of stable sheaves with Mukai vector v is 2-dimensional.

By Proposition 1.7.20, for $\gamma = \mathrm{Obs}(Y, v)$, the $\pi_M^*\gamma$-universal sheaf yields an equivalence

$$\Phi_{\mathcal{E}} : \mathrm{D}^b(M, \gamma^{-1}) \to \mathrm{D}^b(Y).$$

Thus, the composition

$$\Psi := \Phi_{\mathcal{E}}^{-1} \circ \Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(M, \gamma^{-1}).$$

satisfies $\Psi(k(x)) = k(e)$ where $e \in M$ is the point that corresponds to E. Hence, M is birational to X. Since X is minimal and $\mathrm{kod}(X) = 1$, $M \cong X$. Moreover Ψ defines an isomorphism $f : X \to M$ such that $\Psi|_U \cong L \otimes f_*(-)|_U$, hence $f^*\gamma^{-1} = \alpha$ (the restriction morphism $\mathrm{Br}'(X) \to \mathrm{Br}'(U)$ is injective). □

Corollary 1.7.22. *Let X and Y be relatively minimal elliptic surfaces with $\mathrm{kod}(X) = \mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. Then one of the following holds*

(1) $X \cong Y$ and $\alpha = 1$ in $\mathrm{Br}'(X)$,

(2) There exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and an isomorphism $f : X \cong M(v)$ with $f^*(\gamma^{-1}) = \alpha$, where $\gamma = \mathrm{Obs}(Y, v)$.

Remark 1.7.23. In general, the moduli space $M(v)$ obtained in the previous Proposition is coarse.

Corollary 1.7.24. *Let X and Y be relatively minimal elliptic surfaces with $\mathrm{kod}(X) = \mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. If Y is elliptically fibred with a section, then $\alpha = 1$ in $\mathrm{Br}(X)$.*

Proof. By the last corollary there exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and an isomorphism $f : X \cong M(v)$ with $f^*(\gamma^{-1}) = \alpha$, where $\gamma = \mathrm{Obs}(Y, v)$. Since $\lambda_Y = 1$, there exists H such that $gcd(d, r(f.H)) = 1$ with H ample. Thus $M(v)$ is a fine moduli space, i.e. $\gamma = 1$ in $\mathrm{Br}'(Y)$ and hence $\alpha = 1$ in $\mathrm{Br}'(X)$. $\square$

The previous corollary provides a very interesting application. First we introduce the notion of the Tate–Shafarevich group. For an elliptic surface $\pi : X \to C$ with a section σ and integral fibres, we define the Tate–Shafarevich group by

$$\mathrm{Sh}(X) := H^1(C, X^\#)$$

where $X^\#$ is the sheaf of abelian groups on C such that

$$X^\#(U) = \text{the group of sections of } X_U \to U$$

and the natural group structure on $X^\#$ is the one given by the section $\sigma : C \to X$. This group is in 1-1 correspondence with the set of elliptic fibrations $Y \to C$ whose Jacobian is $\pi : X \to C$ (Note that we are in the analytic or étale setup).

Notation 1. Let $\pi : X \to C$ be an elliptic surface with a section. For any $\alpha \in \mathrm{Sh}(X)$, let $\pi_\alpha : X_\alpha \to C$ denote the elliptic fibration corresponding to the element α.

Let $\pi : X \to C$ be an elliptic fibration with a section and integral fibres and let $\pi_\alpha : X_\alpha \to C$ be an elliptic fibration in $\mathrm{Sh}(X)$. We proceed to define a morphism $T_\alpha : \mathrm{Sh}(X) \to \mathrm{Br}'(X_\alpha)$. First, for a given $\alpha \in \mathrm{Sh}(X)$ we can define a homomorphism

$$T_\alpha : H^1(C, X^\#) \to H^1(C, \mathcal{P}ic(X_\alpha/C)) \tag{1.21}$$

by considering the long exact sequence obtained from the exact sequence

$$0 \longrightarrow X^\# \longrightarrow \mathcal{P}ic(X_\alpha/C) \xrightarrow{\deg_\alpha} \mathbb{Z} \longrightarrow 0$$

where $\mathcal{P}ic(X_\alpha/C)$ is the relative Picard sheaf of π_α (note that the relative Picard functor for an elliptic fibration with integral fibres is representable. If the elliptic fibration allows non-integral fibres the functor is non-representable, but it has a maximal representable quotient (cf. [10])) and $\deg_\alpha$ is the map that sends any $L \in \mathrm{Pic}(\pi_\alpha^{-1}(U))/\pi_\alpha^* \mathrm{Pic}(U)$ to its degree along a smooth fibre. From the Leray spectral sequence associated to $\pi_\alpha : X_\alpha \to C$ and $\mathcal{O}^*_{X_\alpha}$, we get the following exact sequence

$$\mathrm{Br}'(C) \to \mathrm{Br}'(X_\alpha) \to H^1(C, \mathcal{P}ic(X_\alpha/C)) \to H^3(C, \mathcal{O}^*_C),$$

where all cohomologies are taken either in the analytic topology or in the étale topology (note that $R^1\pi_{\alpha,*}\mathcal{O}^*_{X_\alpha} = \mathcal{P}ic(X_\alpha/C)$). Hence, since $H^3(C, \mathcal{O}^*_C) = H^2(C, \mathcal{O}^*_C) = 0$,

$$H^1(C, \mathcal{P}ic(X_\alpha/C)) \cong \mathrm{Br}'(X_\alpha). \tag{1.22}$$

Since $\mathrm{Sh}(X) = H^1(C, X^{\#})$, from 1.21 and 1.22 we get the morphism

$$T_\alpha : \mathrm{Sh}(X) \to \mathrm{Br}'(X_\alpha).$$

In particular, for the elliptic fibration $\pi : X \to C$ we get the exact sequence

$$0 \longrightarrow \mathrm{Sh}(X) \xrightarrow{T_0} \mathrm{Br}'(X) \longrightarrow H^1(C, \mathbb{Z}).$$

Thus T_0 is an isomorphism because $\mathrm{Br}'(X)$ is a torsion group and $H^1(C, \mathbb{Z})$ is torsion free.

Theorem 1.7.25 (Donagi–Pantev, [11]). *Let $\pi : X \to C$ be an elliptic fibration with a section. Fix a positive integer m and let $\alpha, \beta \in \mathrm{Sh}(X)$ be two elements such that α is m-divisible and β is m-torsion. Then there is an equivalence*

$$\Phi : \mathrm{D}^b(X_\alpha, T_\alpha(\beta)) \cong \mathrm{D}^b(X_\beta, T_\beta(\alpha)^{-1}).$$

Remark 1.7.26. Let X be a relatively elliptic surface with a section and $\alpha \in \mathrm{Sh}(X)$. Due to Theorem 1.7.25, there exists an equivalence

$$\mathrm{D}^b(X_\alpha) = \mathrm{D}^b(X_\alpha, T_\alpha(0)) \cong \mathrm{D}^b(X, T_0(\alpha)^{-1}).$$

Since T_0 is an isomorphism, we denote the element α and $T_0(\alpha)$ by the same letter α when there is no confusion. For example, if α is of order 2 we get an equivalence $\mathrm{D}^b(X_\alpha) \cong \mathrm{D}^b(X, \alpha^{-1}) \cong \mathrm{D}^b(X, \alpha)$.

Proposition 1.7.27. *Let X be a relatively minimal elliptic surface with a section and* $\mathrm{kod}(X) = 1$. *If* $Y \in \mathrm{Sh}(X)$ *and* $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ *is an equivalence. Then $X \cong Y$ as elliptic surfaces.*

Proof. Since $Y \in \mathrm{Sh}(X)$, there exists $\alpha \in \mathrm{Sh}(X)$ such that $X_\alpha \cong Y$. By Theorem 1.7.25

$$\mathrm{D}^b(X, T_0(\alpha)^{-1}) \cong \mathrm{D}^b(X_\alpha) \cong \mathrm{D}^b(Y) \cong \mathrm{D}^b(X)$$

and by Corollary 1.7.24, $T_0(\alpha)^{-1} = 1$ in $\mathrm{Br}'(X)$. Thus X and Y are isomorphic as elliptic surfaces. $\square$

Chapter 2

Enriques Surfaces

Let Y be an Enriques surface and $\pi : X \to Y$ its K3 cover with the fixed point free involution τ compatible with π. Since the cohomological Brauer group $\mathrm{Br}'(Y)$ is $\mathbb{Z}/2\mathbb{Z}$, it is natural to ask about the triviality of the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$. Indeed, we show that π^* is trivial if and only if there exists a holomorphic line bundle $\mathcal{L}$ on X such that $\tau^*\mathcal{L} \otimes \mathcal{L} \cong \mathcal{O}_X$ and there is no holomorphic line bundle $\mathcal{M}$ with $\mathcal{L} = \tau^*\mathcal{M} \otimes \mathcal{M}^\vee$ satisfying $N_{X/Y}(\mathcal{L}) = 0$. As as consequence, we show that for any Enriques surface whose K3 cover has Picard number 10, the homomorphism π^* is injective.

2.1 Basic facts about Enriques surfaces

We briefly recall some fundamental facts about Enriques and K3 surfaces and lattice theory.

Definition 2.1.1. *A K3 surface is a compact complex surface X with trivial canonical bundle, i.e. $\omega_X \cong \mathcal{O}_X$, and $H^1(X, \mathcal{O}_X) = 0$.*

Definition 2.1.2. *An Enriques surface is a compact complex surface X with $\omega_X^2 \cong \mathcal{O}_X$, $\omega_X \neq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.*

Example 2.1.3. Let X be a smooth complete intersection surface in $\mathbb{P}^n$, $n \geq 3$ of $n-2$ hypersurfaces of degrees $d_1, ..., d_{n-2}$. Then X is a K3 surface if and only if $\sum_{i=1}^{n-2} d_i = n+1$, because by adjunction formula the dualizing sheaf ω_X of X is $\mathcal{O}_X(\sum_{i=1}^{n-2} d_i - n - 1)$.

Example 2.1.4 *(Kummer surfaces).* Let A be an abelian surface and $\iota : A \to A$ the automorphism given by $\iota(a) = -a$ for any $a \in A$. The quotient $Y := A/\langle\iota\rangle$ has 16 singular points. Let $\pi : \tilde{A} \to A$ be the blow-up of A along those 16 points. Thus, there exists a unique automorphism $\tilde{\iota}$ such that $\tilde{\iota} \circ \pi = \pi \circ \tilde{\iota}$. Then $\tilde{\iota}^2 = \mathrm{id}$, so we can consider the quotient

$$\mathrm{Km}(A) := \tilde{A}/\langle\tilde{\iota}\rangle.$$

This is called the Kummer surface associated to the abelian surface A and is also an example of a K3 surface

A *lattice* is a free abelian group L of finite rank with a non-degenerate symmetric bilinear form $b : L \times L \to \mathbb{Z}$. A lattice (L, b) is *even* if, for all $x \in L$, $b(x, x) \in 2\mathbb{Z}$, and it is *odd* if there exists $x \in L$ such that $b(x, x) \notin 2\mathbb{Z}$. Given $\{e_1, ..., e_{\mathrm{rk}\, L}\}$ a basis for L, the determinant of the matrix associated to the bilinear form $(b(e_1, e_j))$ is determined uniquely independent of the choice of the basis. This number $\mathrm{disc}(L) := \det(b(e_i, e_j))$ is called the discriminant of L.

The lattice (L, b) is *unimodular* if $\mathrm{disc}(L) = \pm 1$ while it is *non-degenerate* if $\mathrm{disc}(L) \neq 0$. Moreover, a lattice (L, b) is positive definite if $b(x, x) > 0$, for any $x \in L \setminus \{0\}$ and *negative definite* if $b(x, x) < 0$, for any $x \in L \setminus \{0\}$.

Example 2.1.5. (i) The *hyperbolic plane* U is the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ with the quadratic form represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with respect to the standard basis. This is an even, unimodular and indefinite lattice.

(ii) The *root-lattice* E_8 is the free abelian group $\mathbb{Z}^8$ endowed with the quadratic form represented by the matrix

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

with respect to the standard basis. This is an even, unimodular and negative definite lattice.

We recall that the *dual lattice* of a lattice L is defined by

$$L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \cong \{l \in L \otimes \mathbb{Q} \,|\, b(l, p) \in \mathbb{Z}, \text{ for any } p \in L\}.$$

The quotient of the natural inclusion $L \hookrightarrow L^\vee$

$$A_L := L^\vee / L$$

is called the *discriminant group* of L. The order of A_L is $|\,\mathrm{disc}\, L|$ (cf. [1], Lemma 2.1) and is denoted by $l(A_L)$. The bilinear form b of L induces a symmetric bilinear form $b_L : A_l \times A_L \to \mathbb{Q}/\mathbb{Z}$ and hence a quadratic form

$$q_L : A_L \to \mathbb{Q}/\mathbb{Z}.$$

The second cohomoloy of a K3 surface $H^2(X,\mathbb{Z})$ endowed with the cup-product is an even unimodular lattice of rank 22 and signature (3,19). Thus,

$$H^2(X,\mathbb{Z}) \cong E_8^{\oplus 2} \oplus U^{\oplus 3}$$

where E_8, U are the root and hyperbolic lattices respectively.

Theorem 2.1.6 (Global Torelli). *Two K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry $\varphi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$. If φ maps at least one Kähler class on X to a Kähler class on Y, then there exists an isomorphism $f : X \xrightarrow{\sim} Y$ with $f_* = \varphi$.*

Let Y be a smooth Enriques surface, $\pi : X \to Y$ its K3 cover and $\tau : X \to X$ the corresponding fixed point free involution such that $X/\tau \cong Y$. Thus we obtain the following lemma

Lemma 2.1.7. $0 \to \langle \omega_Y \rangle \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X)^\tau \to 0$ *is an exact sequence.*

Proof. Let $\mathcal{L}$ be a sheaf with $\pi^*(\mathcal{L}) = \mathcal{O}_X$. Then $\mathcal{L} \otimes (\mathcal{O}_Y \oplus \omega_Y) = \pi_*(\pi^*(\mathcal{L})) = \pi_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus \omega_Y$. Therefore $\mathcal{L}$ is either $\mathcal{O}_Y$ or ω_Y. On the other hand, if $\lambda_\tau : \mathcal{M} \to \tau^*(\mathcal{M})$ is an isomorphism for some line bundle $\mathcal{M} \in \mathrm{Pic}(X)$. Then, since $\mathcal{M}$ is simple (because it is a line bundle) $\tau^*\lambda_\tau \circ \lambda_\tau = c.\,\mathrm{id}$ for some $c \in \mathbb{C}$. Thus, we can replace λ_τ by $\frac{1}{\sqrt{c}}\lambda_\tau$ to obtain a linearization on $\mathcal{M}$ (see Definition 2.2.2 below). Hence, there exists a line bundle $\mathcal{L}$ on Y such that $\pi^*\mathcal{L} = \mathcal{M}$. □

Lemma 2.1.8. *(1) If X is a K3 surface, then $H_1(X,\mathbb{Z}) = H^2(X,\mathbb{Z})_{tors} = 0$ (see [1], Prop. 3.3)*

(2) If Y is an Enriques surface, then $H_1(Y,\mathbb{Z}) = H^2(Y,\mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$

Lemma 2.1.9. *If Y is an Enriques surface, then* $\mathrm{Br}'(Y) = H^3(Y,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. By Serre duality and Lemma 2.1.8 (2), it follows $0 = b_1(Y) = b_3(Y)$ and $H^3(Y,\mathbb{Z})_{tors} = H^2(Y,\mathbb{Z})_{tors} = \mathbb{Z}/2\mathbb{Z}$ (see [1], page 15). Since $p_g(Y) = 0$, the exponential sequence induces the following exact sequence

$$0 \to H^2(Y,\mathcal{O}_Y^*) \to H^3(Y,\mathbb{Z}) \to H^3(Y,\mathcal{O}_X)$$

Then, from the vanishing of $H^3(Y,\mathcal{O}_X)$, we conclude the isomorphism $\mathrm{Br}'(Y) = H^3(Y,\mathbb{Z})$ and from the vanishing $b_3(Y) = 0$, we deduce $H^3(Y,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. □

2.2 The kernel of $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$

First of all, I want to remark that the Lemmas 2.2.4 and 2.2.6 were independently obtained by Beauville in [2]. He also let me knew a little mistake that I have made in a previous version of Lemma 2.2.6.

Now, we will study the kernel of the map $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ induced by the universal cover, $\pi : X \to Y$, of the Enriques surface Y. In a particular case we will be able to describe the non trivial element of $\mathrm{Br}'(Y)$ as a Brauer–Severi variety over Y. For the basic facts about group cohomology we refer to [43]. In order to describe $\ker(\pi^*)$, we use the Hochschild–Serre spectral sequence (see [29], Th. 14.9)

$$E_2^{p,q} := H^p(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathcal{O}_X^*)) \Rightarrow H^{p+q}(Y, \mathcal{O}_Y^*). \tag{2.1}$$

and the following theorem (cf. [43], Thm 6.2.2). First, we recall that for a cyclic group G of order m with a generator τ, the norm in $\mathbb{Z}G$ is the element $N = 1 + \tau + ... + \tau^{m-1}$.

Theorem 2.2.1. *If A is a G-module with G a cyclic group generated by τ, then*

$$H^n(G, A) = \begin{cases} A^G, & \textit{if } n = 0 \\ \{a \in A : Na = 0\}/(\tau - 1)A, & \textit{if } n \textit{ is odd} \\ A^G/NA, & \textit{otherwise} \end{cases}$$

The last theorem can be used to compute $E_2^{n,0}$ for all n. First, since the action of $\langle\tau\rangle = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^* = H^0(X, \mathcal{O}_X^*)$ is trivial, one has

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = 0 \tag{2.2}$$

for all even integers $n \neq 0$. On the other hand, if n is an odd integer and $a \in \mathbb{C}^*$ with $N(a) = 1$, it follows from the definition of the norm map that $1 = a\tau(a) = a^2$. Thus

$$E_2^{n,0} = H^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}. \tag{2.3}$$

Since $E_2^{2,0} = 0$, also $E_\infty^{2,0} = 0$ and the following exact sequence follows

$$0 \to E_\infty^{1,1} \to H^2(Y, \mathcal{O}_Y^*) \to H^2(X, \mathcal{O}_X^*)^\tau. \tag{2.4}$$

Let us recall first a few facts about linearization for finite group actions. Let Z be a smooth projective variety with an action by a finite group G. Let $\sigma : G \times Z \to Z$ be the action on Z, $\mu : G \times G \to G$ be the multiplication map of G and $p_2 : G \times Z \to Z$, $p_{23} : G \times G \times Z \to G \times Z$ be the projections.

Definition 2.2.2. *A G-linearization of a coherent sheaf F is an isomorphism $\lambda : \sigma^* F \tilde{\to} p_2^* F$ of $\mathcal{O}_{G\times Z}$-modules that satisfies the cocycle condition $(\mu \times \mathrm{id}_Z)^*\lambda = p_{23}^*\lambda \circ (\sigma \times \mathrm{id}_G)^*\lambda$.*

In the particular case that G is a finite group, the last definition can be reformulated as: A G-linearization of F is given by isomorphisms $\lambda_g : F \tilde{\to} g^* F$ for all $g \in G$ satisfying $\lambda_1 = \mathrm{id}_F$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$. If (F, λ) and (F', λ') are two G-linearised sheaves, then $\mathrm{Hom}(F, F')$ becomes a G-representation defined by the right action $g.f = (\lambda'_g)^{-1} \circ g^* f \circ \lambda_g$ for $f : F \to F'$.

Example 2.2.3. There is a canonical G-linearization of $\mathcal{O}_Z$ given by $\lambda_g = 1$ for all $g \in G$. However, every group homomorphism $\chi : G \to \mathbb{C}^*$ gives rise to a different linearization and two different homomorphisms $G \to \mathbb{C}^*$ endow $\mathcal{O}_Z$ with different G-linearizations.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal cover map. We proceed to define the *relative norm map* $N_{X/Y}$. Let U_i be an open covering of Y such that $\hat{U}_i := \pi^{-1}(U_i)$ consists of two copies of U_i. Take $f = (f_0, f_1) \in \mathcal{O}^*(\hat{U}_i)$ and define the *sheaf relative norm map* by $f_0 f_1$. Thus, the relative norm map induced in the Picard groups can be defined as follows. Take a 1-cocycle $\{\hat{\varphi}_i = (\varphi_0^i, \varphi_1^i)\}_i$ over X that represents a line bundle $\mathcal{L}$, and define our desired morphism by $N_{X/Y}(\{(\varphi_0^i, \varphi_1^i)\}_i) = \{\varphi_0^i . \varphi_1^i\}_i$. This is also the cocycle defining the line bundle $\det(\pi_*(\mathcal{L}))$. Hence, we obtain $N_{X/Y}(-) = \det(\pi_*(-))$.

Lemma 2.2.4. *The kernel of* $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is*

$$(\ker N_{X/Y})/((1-\tau)\,\mathrm{Pic}(X)).$$

Proof. First, we recall that the kernel of the map $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is $E_\infty^{1,1}$. We take an open covering $\{U_i\}$ of Y such that $\hat{U}_i := \pi^{-1}(U_i)$ consists of two copies of U_i. We consider the following commutative diagram given by differentials of Cech cohomology and group cohomology differentials in the vertical and horizontal arrows respectively.

$$\begin{tikzcd}
\bigoplus \Gamma(\hat{U}_i \cap \hat{U}_j \cap \hat{U}_k) \arrow[r, "1+\tau"] & \bigoplus \Gamma(\hat{U}_i \cap \hat{U}_j \cap \hat{U}_k) & \\
\bigoplus \Gamma(\hat{U}_i \cap \hat{U}_j) \arrow[r, "1+\tau"] \arrow[u, "d_1"] & \bigoplus \Gamma(\hat{U}_i \cap \hat{U}_j) \arrow[u, "d_1"] & \\
\bigoplus \Gamma(\hat{U}_i) \arrow[r, "1+\tau"] \arrow[u, "d_0"] & \bigoplus \Gamma(\hat{U}_i) \arrow[r, "1-\tau"] \arrow[u, "d_0"] & \bigoplus \Gamma(\hat{U}_i)
\end{tikzcd}$$

If we define the filtration F^p of the above double complex to be the subcomplex where the entries in the first $p-1$ vertical columns are zero, then we can define the maps $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ for the E_2 terms in the usual way. Let $(f'_{ij})_{i,j}$ be an element in $\ker(d_2^{1,1})$. Since $(f'_{ij})_{i,j} \in E_2^{1,1}$, $f_{ij} := (1+\tau)f'_{ij} = f_i - f_j$ (i.e. $(f_{ij})_{i,j}$ is the cocycle corresponding to $\mathcal{O}_X$) and since $d_2^{1,1}(f'_{ij}) = 1$, $(1-\tau)f_i = 1$, i.e. $\tau f_i = f_i$. Thus, τ defines the trivial character $\chi : G \to \mathbb{C}^*$. This implies that $(f_i)_i$ has a descend data over Y which gives us the cocycle that represents the trivial sheaf $\mathcal{O}_Y$. On the other hand if $(1-\tau)f_i = -1$ then $\tau f_i = -f_i$ and this defines a nontrivial character which corresponds to another linearization of $\mathcal{O}_X$. Hence in this case $\mathcal{O}_X$ descends to K_Y. $\square$

Definition 2.2.5. *Let X be a surface and $\mathcal{P}$ a $\mathbb{P}^1$-bundle on X. We say that $\mathcal{P}$ comes from a vector bundle if there exists a vector bundle E on X such that $\mathcal{P} \cong \mathbb{P}(E)$.*

Lemma 2.2.6. *Let Y be an Enriques surface and $\pi : X \to Y$ its universal cover map. Let $\mathcal{L}$ be a line bundle satisfying $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, $N_{X/Y}(\mathcal{L}) = 0$, and such that $[\mathcal{L}]$ is nontrivial in $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$. Then $\mathbb{P}(\mathcal{O} \oplus \mathcal{L})$ descends to a projective bundle that does not come from a vector bundle.*

Proof. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle with $N_{X/Y}(\mathcal{L}) = 0$ representing a nontrivial element in

$$\begin{aligned} E_2^{1,1} &= H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) \\ &= \frac{\{L \in \mathrm{Pic}(X) : \tau^*L \otimes L = \mathcal{O}_X\}}{\{\tau^*M \otimes M^\vee : M \in \mathrm{Pic}(X)\}}. \end{aligned}$$

We proceed to give a G-linearization on $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$:

$$\lambda_\tau : \mathbb{P}(\tau^*(\mathcal{O}_X \oplus \mathcal{L})) \longrightarrow \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}).$$

Since $N_{X/Y}(\mathcal{L}) = 0$ we can find a G-linearised isomorphism $i : \mathcal{L} \otimes \tau^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$ where we consider $\mathcal{O}_X$ endowed with the canonical G-linearization. We define λ_τ as the composition of morphisms

$$\begin{gathered} \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) \to \mathbb{P}(\tau^*\mathcal{L} \oplus (\mathcal{L} \otimes \tau^*\mathcal{L})) \to \mathbb{P}(\tau^*\mathcal{L} \oplus \mathcal{O}_X) \to \mathbb{P}(\mathcal{O}_X \oplus \tau^*\mathcal{L}) \\ [a : b] \mapsto [a\tau^*b : b\tau^*b] \mapsto [a\tau^*b : i(b\tau^*b)] \mapsto [i(b\tau^*b) : a\tau^*b] \end{gathered}$$

where a and b are sections of $\mathcal{O}_X$ and $\mathcal{L}$ respectively. Note that $\mathbb{P}(\mathcal{O}_X \oplus \tau^*\mathcal{L}) = \mathbb{P}(\tau^*\mathcal{O}_X \oplus \tau^*\mathcal{L})$ because we consider the canonical linearization on $\mathcal{O}_X$, i.e. $\tau^*\mathcal{O}_X = \mathcal{O}_X$. Since i is a G-linearised isomorphism, it commutes with τ and from this we can check that $\lambda_\tau^2 = \mathrm{id}$ as follows:

$$\begin{aligned} \lambda_\tau^2([a : b]) &= \lambda_\tau([i(b\tau^*b) : a\tau^*b]) \\ &= [i((a\tau^*b)\tau^*(a\tau^*b)) : i(b\tau^*b)\tau^*(a\tau^*b)] \\ &= [a\tau^*a.i(b\tau^*b) : i(b\tau^*b)\tau^*(a\tau^*b)] \\ &= [a\tau^*a : \tau^*(a\tau^*b)] \\ &= [a\tau^*a : b\tau^*a] \\ &= [a : b] \end{aligned}$$

Hence, the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ descends to a projective bundle $\mathcal{P}$ over Y. Now, we show that $\mathcal{P}$ does not come from a vector bundle on Y. Suppose $\mathcal{P} = \mathbb{P}(E)$ for some vector bundle E over Y and so $\mathbb{P}(\pi^*(E)) = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$. Thus, it follows that $\pi^*(E) = M \otimes (\mathcal{O}_X \oplus \mathcal{L})$, for some $M \in \mathrm{Pic}(X)$. By taking determinants on both sides of this isomorphism we get $\det(\pi^*(E)) = M^{\otimes 2} \otimes \mathcal{L}$.

In particular, this implies that M is not invariant. Indeed, if M is an invariant line bundle, $\mathcal{L} = \det(\pi^*(E)) \otimes (M^\vee)^{\otimes 2}$ is an invariant bundle. Hence $\mathcal{L} \cong \mathcal{O}_X$ because $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, a contradiction. Since $M^{\otimes 2} \otimes \mathcal{L}$ is invariant and $\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X$, one has

$$M^{\otimes 2} \otimes \mathcal{L} = \tau^*(M^{\otimes 2} \otimes \mathcal{L}) = \tau^* M^{\otimes 2} \otimes \mathcal{L}^\vee$$

and so, $\tau^* M^{\otimes 2} = M^{\otimes 2} \otimes \mathcal{L}^{\otimes 2}$. Hence, from the torsion freeness of $\mathrm{Pic}(X)$ we obtain $\tau^* M = M \otimes \mathcal{L}$, i.e., $\mathcal{L} = \tau^* M \otimes M^\vee$, but this contradicts the assumption that $\mathcal{L}$ defines a non trivial element in $E_2^{1,1}$. □

Lemma 2.2.7. *Let $\pi : X \to Y$ be the universal cover of an Enriques surface Y with $\rho(X) = 10$, then $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is a nontrivial homomorphism.*

Proof. We show that $\rho(X) = 10$, implies $\mathrm{Pic}(X)^\tau = \mathrm{Pic}(X)$, i.e. all the line bundles on X are invariant. Since $\rho(X) = 10$, $\mathrm{Pic}(X)^\tau \subseteq \mathrm{Pic}(X)$ is a sublattice of finite index. Thus, if $\mathcal{L}$ is a line bundle, there exists a positive integer r with $\mathcal{L}^{\otimes r} \in \mathrm{Pic}(X)^\tau$, i.e.

$$\tau^* \mathcal{L}^{\otimes r} = \mathcal{L}^{\otimes r}.$$

Hence

$$(\tau^* \mathcal{L} \otimes \mathcal{L}^\vee)^{\otimes r} = \mathcal{O}_X,$$

Since $\mathrm{Pic}(X)$ is torsion free, we obtain

$$\tau^* \mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}_X.$$

i.e. $\mathcal{L}$ is an invariant line bundle. Thus, the group $H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$ vanishes and the lemma follows by using the exact sequence (2.4). □

Example 2.2.8. In this example we show the existence of a K3 surface X with $\rho(X) = 10$ that covers an Enriques surface. First, we find a K3 surface with Picard number 10. Let us define $\Lambda := E_8 \oplus E_8 \oplus U \oplus U \oplus U$ and an involution ρ of L by

$$\rho : \Lambda \to \Lambda,\ (e_1, e_2, h_1, h_2, h_3) \mapsto (e_2, e_1, -h_1, h_3, h_2).$$

Note that this involution is the universal action (cf. [1], Ch. VIII, Lemma 19.1), i.e. whenever $\pi : X \to Y$ is the universal covering of an Enriques surface Y with $\tau : X \to X$ the covering involution, then there exists an isometry $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. The ρ-invariant sublattice of Λ is

$$\Lambda^+ = \{x \in \Lambda \,|\, \rho(x) = x\} = \{(e, e, 0, h, h) \,|\, e \in E_8, h \in U\},$$

which is isometric to $E_8(2) \oplus U(2)$, where the isometry is given as follows

$$\rho^+ : \Lambda^+ \to E_8(2) \oplus U(2), \qquad (e, e, 0, h, h) \mapsto (e, h)$$

Hence, $E_8(2) \oplus U(2) \hookrightarrow E_8^{\oplus 2} \oplus U^{\oplus 3}$ is a primitive embedding. Since this lattice has Picard number 10 and signature (1,9), by ([30], Cor. 2.9) we can find an algebraic K3 surface X with $NS(X) = E_8(2) \oplus U(2)$. Now, we show that X has a fixed point free involution. The isometry ρ^+ also yields an isomorphism

$$(\Lambda^+)^\vee/\Lambda^+ \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

It means that Λ^+ is a 2-elementary lattice with $l(A_{\Lambda^+}) = 10$. This gives us an involution

$$\tau^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

which is the identity on Λ^+ and acts like multiplication by (-1) on $T_X = (\Lambda^+)^\perp = (\mathrm{NS}(X))^\perp$ where the orthogonal complement is taken in $H^2(X,\mathbb{Z})$. Since τ^* is the identity on Λ^+ (=NS(X) through the isometry ρ^+), it is effective and so it maps a Kähler class to a Kähler class. By the global Torelli theorem for K3 surfaces, there exists a unique involution $\tau : X \to X$ which induces τ^* on $H^2(X,\mathbb{Z})$. Then it follows from ([34], Thm. 4.2.2) that the set of fixed points X^τ is empty. It means that the involution τ is fixed point free, hence X/τ is an Enriques surface.

2.3 The Brauer group $\mathrm{Br}_{top}(Y)$

For any smooth projective variety X we can define the cohomogical topological Brauer group as $\mathrm{Br}'_{top}(X) := H^2(X, \mathcal{C}^*_X)$. If Y is an Enriques surface then since $H^3(X,\mathbb{Z}) = 0$, we get that the homomorphism $\pi^* : \mathrm{Br}'_{top}(Y) \to \mathrm{Br}'(X)$ is trivial. In this section we will study the Brauer group $\mathrm{Br}_{top}(Y)$ which is defined later and we will give a topological view of the results obtained in Section 2.2. First we recall the following well known Theorem:

Theorem 2.3.1 (Schwarzenberger, [40]). *Let X be a projective surface. A topological complex vector bundle admits a holomorphic structure if and only if its first Chern class belongs to the Neron–Severi group of the surface.*

Lemma 2.3.2. *Let Y be an Enriques surface. Then every topological vector bundle on Y has a holomorphic structure.*

Proof. Let E be a $\mathcal{C}_X$–bundle on Y. Since Y is an Enriques surface then $NS(Y) \cong H^2(Y,\mathbb{Z})$. Hence $c_1(E) \in \mathrm{NS}(Y)$ and by Theorem 2.3.1, E has a holomorphic structure. □

Lemma 2.3.3. *On any K3 surface, every topological projective bundle comes from a topological vector bundle.*

Proof. Let X be a K3 surface. Since $H^i(X, \mathcal{C}_X) = 0$ for all $i > 0$, then by the long exact sequence obtained from the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{C}_X \to \mathcal{C}^*_X \to 0$$

we get $H^2(X, \mathcal{C}^*_X) \cong H^3(X,\mathbb{Z})$. Hence $H^2(X, \mathcal{C}^*_X) = 0$, because X is a K3 surface. Now from the exact sequence

$$0 \to \mathcal{C}^*_X \to GL_n(\mathcal{C}_X) \to PGL_n(\mathcal{C}_X) \to 0 \tag{2.5}$$

we get
$$H^1(X, GL_n(\mathcal{C}_X)) \to H^1(X, PGL_n(\mathcal{C}_X)) \to 0,$$
and the statement follows. □

Lemma 2.3.4. *If Y is an Enriques surface, then the inclusion $\mathcal{O}_Y \subset \mathcal{C}_Y$ induces a surjective map*
$$H^1(Y, PGL_2(\mathcal{O}_Y)) \twoheadrightarrow H^2(Y, \mathcal{C}_Y^*)$$

Proof. From the exponential exact sequence
$$0 \to \mathbb{Z} \to \mathcal{C}_Y \to \mathcal{C}_Y^* \to 0$$
and Lemma 2.1.9 we get
$$H^2(Y, \mathcal{C}_Y^*) \cong H^3(Y, \mathbb{Z}) \cong H^2(Y, \mathcal{O}_Y^*) = \mathbb{Z}/2\mathbb{Z}. \tag{2.6}$$
From (2.5) we get a surjective morphism
$$f : H^1(Y, PGL_2(\mathcal{O}_Y)) \twoheadrightarrow H^2(Y\mathcal{O}_Y^*)$$
and an isomorphism $H^2(Y, \mathcal{O}_Y^*) \cong H^2(Y, \mathcal{C}^*)$, and these define a surjective morphism $H^2(Y, PGL_2(\mathcal{C}_Y)) \twoheadrightarrow H^2(Y, \mathcal{C}_Y^*)$.

□

We define the topological Brauer group of X by
$$\mathrm{Br}_{top}(X) := \mathrm{BS}(X)/\sim$$
where BS(X)={isomorphism classes of topological Brauer–Severi varieties over X} and $[\mathcal{P}] \sim [\mathcal{Q}]$ if and only if there exists topological vector bundles E and F of positive rank such that $\mathcal{P} \otimes \mathbb{P}(E) \cong \mathcal{Q} \otimes \mathbb{P}(F)$. If Y is an Enriques surface then $\mathrm{Br}_{top}(Y) = \mathrm{Br}'_{top}(Y) = \mathbb{Z}/2\mathbb{Z}$ because of (2.6).

Now, we introduce the following spectral sequence
$$E_{2,\mathbb{Z}}^{p,q} := H^p(\mathbb{Z}/2\mathbb{Z}, H^q(X, \mathbb{Z})) \Rightarrow H^{p+q}(Y, \mathbb{Z}) \tag{2.7}$$
associated to the covering map $\pi : X \to Y$ of an Enriques surface Y and we compute some of its terms. Since X is a K3 surface, the vanishing $H^1(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = 0$ implies
$$E_{2,\mathbb{Z}}^{n,1} = E_{2,\mathbb{Z}}^{n,3} = 0 \tag{2.8}$$
for all integers n. Now, we compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for all integers n. First, we note that the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $\mathbb{Z}$. Since the term $E_{2,\mathbb{Z}}^{0,0}$ corresponds to the invariant elements of $\mathbb{Z}$ under the action of $\mathbb{Z}/2\mathbb{Z}$ we obtain that $E_{2,\mathbb{Z}}^{0,0} = \mathbb{Z}$. Now, let us compute the terms $E_{2,\mathbb{Z}}^{n,0}$ for odd n. Since the action is trivial, we deduce that
$$0 = N(m) = \tau^*(m) + m = 2m.$$

Then it follows that $m = 0$ and hence by Theorem 2.2.1 that $E_{2,\mathbb{Z}}^{n,0} = 0$. On the other hand, if n is an even number we can see that $E_{2,\mathbb{Z}}^{n,0} = \mathbb{Z}/2\mathbb{Z}$. Summarizing, we have proved that

$$E_{2,\mathbb{Z}}^{n,0} = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{if } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is even, } n \neq 0. \end{cases} \tag{2.9}$$

From (2.8) and (2.9) we deduce

$$E_{\infty,\mathbb{Z}}^{0,3} = E_{\infty,\mathbb{Z}}^{2,1} = E_{\infty,\mathbb{Z}}^{3,0} = 0$$

and this implies

$$E_{\infty,\mathbb{Z}}^{1,2} = \mathbb{Z}/2\mathbb{Z}. \tag{2.10}$$

The homomorphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ induces a homomorphism $C : E_2^{1,1} \to E_{2,\mathbb{Z}}^{1,2}$ which can be easily described by using Theorem 2.2.1 as

$$C : \frac{\{L \in \mathrm{Pic}(X) | \tau^* L \otimes L \cong \mathcal{O}_X\}}{\{\tau^* M \otimes M^\vee | M \in \mathrm{Pic}(X)\}} \to \frac{\{\ell \in H^2(X, \mathbb{Z}) | \tau^* \ell + \ell = 0\}}{\{\tau^* m - m | m \in H^2(X, \mathbb{Z})\}}, \tag{2.11}$$

sending $[L]$ to $[c_1(L)]$. The following lemma was also independently proved by Beauville in [2].

Lemma 2.3.5. *The homomorphism C is injective.*

Proof. Let $[L]$ be the class of a line bundle L such that $\tau^* L \otimes L = \mathcal{O}_X$. Suppose that $C(L) = 0$. Thus, there exists a topological line bundle M such that $L = M^\vee \otimes \tau^* M$ and so

$$-c_1(M) + c_1(\tau^* M) = c_1(M^\vee \otimes \tau^* M) = c_1(L) \in \mathrm{NS}(X). \tag{2.12}$$

On the other hand, since the topological rank 2 vector bundle $\tau^* M \oplus M$ has a linearization (i.e. the trivial linearization), there exists a topological vector bundle E on Y such that $\pi^* E = \tau^* M \oplus M$. By Lemma 2.3.2, E has a holomorphic structure and induces one on $\tau^* M \oplus M$. Thus, by Theorem 2.3.1,

$$c_1(\tau^* M \oplus M) \in \mathrm{NS}(X). \tag{2.13}$$

Therefore, by equations 2.12 and 2.13, $2c_1(\tau^* M) = (c_1(\tau^* M) - c_1(M)) + c_1(\tau^* M \otimes M) \in \mathrm{NS}(X)$. Since X is a K3 surface, $c_1 : \mathrm{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z})$ is injective and so

$$\frac{H^2(X, \mathbb{Z})}{\mathrm{NS}(X)} \hookrightarrow H^2(X, \mathcal{O}_X).$$

Thus $c_1(\tau^* M) \in \mathrm{NS}(X)$ because $2c_1(\tau^* M) \in \mathrm{NS}(X)$ and $H^2(X, \mathcal{O}_X)$ is torsion free, and so we conclude $[L] = 0$ in $E_2^{1,1}$. $\square$

In Example 2.2.8 we have introduced the involution ρ on the K3 lattice $\Lambda := (E_8)^{\oplus 2} \oplus U^{\oplus 3}$ and also defined the invariant lattice Λ^+. We define similarly the ρ-anti-invariant sublattice of Λ by

$$\Lambda^- := \{\ell \in \Lambda \mid \rho(\ell) = -\ell\}.$$

Given $\ell = (x, y, z_1, z_2, z_3) \in \Lambda$, we get $\rho(\ell) = -\ell$ if and only if

$$\ell = (x, -x, z_1, z_2, -z_2).$$

Let $m = (m_1, m_2, n_1, n_2, n_3) \in \Lambda$, then

$$\rho(m) - m = (m_2 - m_1, -(m_2 - m_1), -2n_1, n_3 - n_2, -(n_3 - n_2)).$$

It yields that

$$\ell = (x, -x, z, y, -y) \in \Lambda^-$$

can be written as $\rho(m) - m$ for some $m \in \Lambda$ if and only if $z = -2n$ for some $n \in U$.

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Consider the spectral sequence $E^{1,2}_{2,\mathbb{Z}}$ associated to this (see (2.7)). Let $\ell \in H^2(X, \mathbb{Z})$ such that $\tau^*\ell = -\ell$. Thus, $2\ell = \ell - \tau^*\ell$, i.e. $[2\ell] = 0$ in $E^{1,2}_{2,\mathbb{Z}} = H^1(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathbb{Z}))$. Therefore, any element in $E^{1,2}_{2,\mathbb{Z}} = H^1(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathbb{Z}))$ is 2-torsion.

By definition, $E^{1,2}_{3,\mathbb{Z}} = \ker(d_2^{1,2} : E^{1,2}_{2,\mathbb{Z}} \to E^{3,1}_{2,\mathbb{Z}})$. Thus

$$E^{1,2}_{3,\mathbb{Z}} = E^{1,2}_{2,\mathbb{Z}}$$

because $E^{3,1}_{2,\mathbb{Z}} = H^3(\mathbb{Z}/2\mathbb{Z}, H^1(X, \mathbb{Z})) = 0$. Since

$$\mathbb{Z}/2\mathbb{Z} = E^{1,2}_{\infty,\mathbb{Z}} = \ker(d_3^{1,2} : E^{1,2}_{3,\mathbb{Z}} \to E^{4,0}_{3,\mathbb{Z}}),$$

we have only the following two options:

(1) $E^{1,2}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} \neq 0$,

(2) $E^{1,2}_{2,\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$ and $d_3^{1,2} = 0$.

Now, we show that (2) can not occur.

Lemma 2.3.6. *Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. Then the map $d_3^{1,2} \neq 0$ of the spectral sequence $E^{p,q}_{2,\mathbb{Z}}$ associated to the morphism $\pi : X \to Y$.*

Proof. First, we compute the term $E^{0,4}_{\infty,\mathbb{Z}}$. Since

$$E^{1,3}_{\infty,\mathbb{Z}} = E^{3,1}_{\infty,\mathbb{Z}} = 0,$$

$E_{2,\mathbb{Z}}^{4,0} = \mathbb{Z}/2\mathbb{Z}$ and $E_{2,\mathbb{Z}}^{2,2}$ is a torsion group, one finds

$$E_{\infty,\mathbb{Z}}^{0,4} = \mathbb{Z}.$$

Suppose that $d_3^{1,2} = 0$. Since X is a K3 surface

$$E_{2,\mathbb{Z}}^{0,3} = H^0(\mathbb{Z}/2\mathbb{Z}, H^3(X,\mathbb{Z})) = 0 \tag{2.14}$$

$$E_{2,\mathbb{Z}}^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, H^1(X,\mathbb{Z})) = 0. \tag{2.15}$$

By definition of the terms of the spectral sequence

$$E_{3,\mathbb{Z}}^{4,0} = \frac{E_{2,\mathbb{Z}}^{4,0}}{\operatorname{im}(d_2^{2,1} : E_{2,\mathbb{Z}}^{2,1} \to E_{2,\mathbb{Z}}^{4,0})}$$

and by equation (2.15), $E_{3,\mathbb{Z}}^{4,0} = E_{2,\mathbb{Z}}^{4,0}$. Since $d_3^{1,2} = 0$,

$$E_{4,\mathbb{Z}}^{4,0} = \frac{E_{3,\mathbb{Z}}^{4,0}}{\operatorname{im}(d_3^{1,2} : E_{3,\mathbb{Z}}^{1,2} \to E_{3,\mathbb{Z}}^{4,0})} = E_{3,\mathbb{Z}}^{4,0},$$

and finally by equation (2.14)

$$E_{\infty,\mathbb{Z}}^{4,0} = E_{5,\mathbb{Z}}^{4,0} = \frac{E_{4,\mathbb{Z}}^{4,0}}{\operatorname{im}(d_4^{0,3} : E_{4,\mathbb{Z}}^{0,3} \to E_{4,\mathbb{Z}}^{4,0})} = E_{4,\mathbb{Z}}^{4,0}.$$

Hence we conclude $E_{\infty,\mathbb{Z}}^{4,0} = E_{2,\mathbb{Z}}^{4,0} = \mathbb{Z}/2\mathbb{Z}$, a contradiction. □

2.4 The family of marked Enriques surfaces

Let $\mathfrak{M}$ be the (fine) moduli space of marked Enriques surfaces and let $f : \mathcal{Y} \to \mathfrak{M}$ be the universal family of Enriques surfaces parametrized by $\mathfrak{M}$ with an f-ample line bundle $\mathcal{H}$. Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathfrak{M} \\ {\scriptstyle\pi}\downarrow & & \| \\ \mathcal{Y} & \xrightarrow{f} & \mathfrak{M} \end{array}$$

where $\mathcal{X}$ is a family of K3 surfaces such that for all $t \in \mathfrak{M}$, $\pi_t : \mathcal{X}_t \to \mathcal{Y}_t$ is the covering map. Denote by $\tau_t : \mathcal{X}_t \to \mathcal{X}_t$ the covering involution.

Remark 2.4.1. Let $\mathcal{M}$ be the moduli space of unmarked Enriques surfaces and let $f : \mathcal{Y} \to \mathcal{M}$ be a local universal family (i.e. Kuranishi family). Thus, for any small neighbourhood V (contractible) around a point $0 \in \mathcal{M}$ corresponding to an Enriques surface $\mathcal{Y}_0$ we may form the restriction covering $g|V : \mathcal{X}|V \to V$ and give an isometry $\phi_0 : H^2(\mathcal{X}_0, \mathbb{Z}) \to \Lambda$ with $\phi_0 \circ \tau_0^* = \rho \circ \phi_0$ (where ρ

is the involution defined in Example 2.2.8). This can be extended to a unique marking $\phi : R^2(g|V)_*\mathbb{Z} \cong \underline{\Lambda}$ and $\phi_t \circ \tau_t^* = \rho \circ \phi_t$ holds for any $t \in V$ because V is contractible. Since the following arguments are local, we may assume that there exists a universal involution $\tau : \mathcal{X} \to \mathcal{X}$ and we can show a similar statement for the moduli space $\mathcal{M}$ instead of $\mathfrak{M}$.

Let $\tau : \mathcal{X} \to \mathcal{X}$ be a universal involution and a marking $\phi : R^2 g_*\mathbb{Z} \cong \underline{\Lambda}$ such that for all $t \in \mathfrak{M}$ the restricted morphisms $\phi_t : H^2(\mathcal{X}_t, \mathbb{Z}) \cong \Lambda$ satisfies $\phi_t \circ \tau_t^* = \rho \circ \phi_t$. Pick $\ell \in \Lambda$ with $\rho(\ell) = -\ell$, i.e. $\ell = (x, -x, z, y, -y)$ for some $(x, y, z) \in E_8 \oplus U \oplus U$ such that $z \neq 2n$ for all $n \in U$. This is equivalent to say that ℓ can not be written as $\rho(m) - m$ for some $m \in \Lambda$. In other words for all $t \in \mathfrak{M}$, $\tau_t^*(\phi^{-1}(\ell)) = -\phi^{-1}(\ell)$ and $\phi^{-1}(\ell)$ can not be written as $\tau_t^* m - m$ for some $m \in H^2(\mathcal{X}_t, \mathbb{Z})$.

Let $\mathfrak{M}_\ell \subset \mathfrak{M}$ be defined by

$$\mathfrak{M}_\ell := \{t \in \mathfrak{M} | \exists p_t \in \mathrm{NS}(\mathcal{X}_t)\, \exists m \in H^2(\mathcal{X}_t, \mathbb{Z}), p_t - \phi_t^{-1}(\ell) = \tau^* m - m, d_{3,\mathbb{Z}}^{1,2}([\phi_t^{-1}(\ell)]) = 0\}.$$

Thus, for every $t \in \mathfrak{M}_\ell$ there exists $\mathcal{L}_t \in \mathrm{Pic}(\mathcal{X}_t)$ with $c_1(\mathcal{L}_t) = p_t$ and so $[\mathcal{L}_t] \in E_2^{1,1}$ is a nontrivial class. Hence by Lemma 2.2.6, $\mathbb{P}(\mathcal{O}_{\mathcal{X}_t} \oplus \mathcal{L}_t)$ descends to a projective bundle $\mathcal{P}_t$ that does not come from a vector bundle on $\mathcal{Y}_t$, i.e. the class of $\mathcal{P}_t$ is the nontrivial class of $\mathrm{Br}'(\mathcal{Y}_t)$. In particular $\pi_t^* : \mathrm{Br}'(\mathcal{Y}_t) \to \mathrm{Br}'(\mathcal{X}_t)$ is trivial. Define I to be the set

$$I := \{\ell \in \Lambda \,|\, \rho(\ell) = -\ell\, \forall m \in \Lambda, \ell \neq \rho(m) - m\}$$

We claim that $\bigcup_{\ell \in I} \mathfrak{M}_\ell$ parametrizes all the Enriques surfaces Y with trivial morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$. Let Y be an Enriques surface such that the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial. Take $[Y] =: t \in \mathfrak{M}$ the corresponding point in the moduli space $\mathfrak{M}$. Since $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial, $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$ and by Lemma 2.2.4, there exists a line bundle (holomorphic) $\mathcal{L}_t$ such that $d_2^{1,1}([\mathcal{L}_t]) = 0$. This shows that $d_{3,\mathbb{Z}}^{1,2}(\mathcal{L}_t) = 0$ and so $t \in \mathfrak{M}_\ell$ where $\ell = \phi_t(c_1(\mathcal{L}_t))$. For every $t \in \mathfrak{M} \backslash \bigcup_{\ell \in I} \mathfrak{M}_\ell$ there is a topological line bundle $\mathcal{L}_t$ such that $\mathbb{P}(\mathcal{O}_{\mathcal{X}_t} \oplus \mathcal{L}_t)$ (this represents a nontrivial class in $\mathrm{Br}_{top}(\mathcal{X}_t)$) descends to the nontrivial class of $\mathrm{Br}_{top}(\mathcal{Y}_t)$ (a similar proof holds for it as in the holomorphic case). However there also exists a holomorphic projective bundle $\mathcal{P}_t$ that represents the nontrivial class of $\mathrm{Br}(\mathcal{Y}_t) (= \mathrm{Br}_{top}(\mathcal{Y}_t))$. When we consider the projective bundle as a topological bundle, this also represents the nontrivial class of $\mathrm{Br}_{top}(\mathcal{Y}_t)$. Hence $\pi^*\mathcal{P}_t$ is a holomorphic projective bundle and represents the nontrivial class $[\pi^*\mathcal{P}_t] = [\mathbb{P}(\mathcal{O}_{\mathcal{X}_t} \oplus \mathcal{L}_t)]$ (the holomorphic projective bundle $\mathcal{P}_t$ is considered as a topological bundle) class of $\mathrm{Br}_{top}(\mathcal{X}_t)$. Thus, $\mathfrak{M} \setminus \bigcup_{\ell \in I} \mathfrak{M}_\ell$ parametrizes all the Enriques surfaces Y with $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ nontrivial.

The results described so far were obtained independently. Only at this point Beauville's article appeared on the arxiv. He was able to deal with the differential $d_{3,\mathbb{Z}}^{1,2}$ more effectively and proved:

Proposition 2.4.2 ([2], Cor. 5.6 and its proof). *Let $\lambda = (\alpha, \alpha', \beta) \in H^2(X, \mathbb{Z})$ such that $\alpha, \alpha' \in E_8 \oplus U$ and $\beta \in U$ and ε the class of $e + f$ in $U_2 := U/2U$ where $\{e, f\}$ is the basis of the hyperbolic lattice U. Then the following conditions are equivalent:*

(i) $\pi_*\lambda = 0$ *and* $\lambda \notin (1-\tau^*)(H^2(X,\mathbb{Z}))$*;*

(ii) $\tau^*\lambda = -\lambda$ *and* $\lambda^2 \equiv 2(mod.\ 4)$.

(iii) the class $\bar{\beta} = \varepsilon$ *and* $\alpha' = -\alpha$.

Corollary 2.4.3. $\pi : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ *is trivial if and only if there exists a line bundle* L *on* X *with* $\tau^*L = L^\vee$ *and* $c_1(L)^2 \equiv 2\,(mod 4)$

We will say more about Beauville's work in Section 2.6. Now, we quickly recall a kind of divisors in the period domain Ω of $E_8(2)\oplus U(2)$-polarized marked K3 surfaces. If we fix the unique primitive embedding of $E_8(2)\oplus U(2)$ in the K3 lattice Λ, then Ω is by definition

$$\Omega := \{[\omega] \in \mathbb{P}((E_8(2)\oplus U(2))^{\perp}_{\mathbb{C}}) \mid \omega^2 = 0,\ \omega\bar{\omega} > 0\}$$

Let $S \subset \Lambda$ be a primitive sublattice of rank 11 containing the lattice $E_8(2)\oplus U(2)$. Then the subset

$$\Omega(S) := \{[\omega] \in \mathbb{P}(S^{\perp}_{\mathbb{C}}) \mid \omega^2 = 0,\ \omega\bar{\omega} > 0\}$$

is called the Heegner divisor of type S in Ω.

Proposition 2.4.4 ([36], Prop. 3.1). *If* X *corresponds to a very general point of* $\Omega(S)$*, i.e. in the complement of a union of countably many proper closed analytic subset of* $\Omega(S)$*, then we have* $\mathrm{NS}(X) = S$.

Remark 2.4.5. Ohashi proved in ([36], Thm. 3.4) that for a lattice $S = U(2)\oplus E_8(2)\oplus\langle -2N\rangle$ with $N \equiv 0(mod\,4)$, there exists a K3 surface X with an Enriques quotient and such that $NS(X) = S$.

Example 2.4.6. Now, we will show the existence of a K3 surface X covering an Enriques surface Y with $\rho(X) = 11$ and $E_2^{1,1} = 0$ which from (2.4) implies that $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. Let $\alpha \in \Lambda$, defined by

$$\alpha = (\sum_{i\,\mathrm{odd}} a_i e_i, -\sum_{i\,\mathrm{odd}} a_i e_i, 0, f_1 - f_2, -f_1 + f_2),$$

where a_i are integers. This is a primitive element, $\alpha = \beta - \rho(\beta)$ where

$$\beta = (a_1e_1 + a_3e_3, -a_5e_5 - a_7e_7, 0, f_1, f_2)$$

and

$$\alpha^2 = -4\sum_{i\,\mathrm{odd}} a_i^2 = -4m.$$

Thus, $E_8(2)\oplus U(2)\oplus\alpha\mathbb{Z} \hookrightarrow E_8^{\oplus 2}\oplus U^{\oplus 3}$ is a primitive embedding (note that $E_8(2)\oplus U(2)$ diagonally embedds in $E_8^{\oplus 2}\oplus U^{\oplus 3}$). Note that by the Lagrange's four-square theorem ([24], Prop. 17.7.1), m can take any positive integer value. By Proposition 2.4.4 and Remark 2.4.5, there exists a K3 surface X with an Enriques quotient Y and such that

$$NS(X) = E_8(2)\oplus U(2)\oplus\alpha\mathbb{Z}$$

and by ([1], Lemma 19.1) there exists an isometry $\phi : H^2(X,\mathbb{Z}) \to \Lambda$ such that $\phi \circ \tau^* = \rho \circ \phi$. Now, we take a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \phi^{-1}(\alpha)$. Then,

$$\begin{aligned}\alpha &= -\rho(\alpha)\\ &= -\rho(\phi(\phi^{-1}(\alpha)))\\ &= -\phi(\tau^*(\phi^{-1}(\alpha)))\\ &= -\phi(\tau^*(c_1(\mathcal{L})))\\ &= -\phi(c_1(\tau^*\mathcal{L}))\\ &= \phi(c_1(\tau^*\mathcal{L}^\vee)).\end{aligned}$$

Then, from the injectivity of ϕ, it follows that

$$c_1(\tau^*\mathcal{L} \otimes \mathcal{L}) = 0,$$

and since X is a K3 surface we deduce

$$\tau^*\mathcal{L} \otimes \mathcal{L} = \mathcal{O}_X,$$

i.e. $[\mathcal{L}] \in E_2^{1,1}$. Now, since $\alpha = \beta - \rho(\beta)$ and $E_2^{1,1} \subseteq E_{2,\mathbb{Z}}^{1,2}$ (Lemma 2.3.5), then $[\mathcal{L}] = 0$ in $E_2^{1,1}$.

Now, we show that for any line bundle $\mathcal{M}$ such that $\tau^*\mathcal{M} \otimes \mathcal{M} = \mathcal{O}_X$, there exists an integer n such that $\mathcal{M} = \mathcal{L}^{\otimes n}$. By construction of the above primitive embedding, we have that the action of τ^* on $E_8(2) \oplus U(2)$ is the identity. Thus, if $\mathcal{M}$ is a line bundle, it can be written as $\mathcal{M} = \mathcal{L}^{\otimes n} \otimes \mathcal{F}$ for some invariant line bundle $\mathcal{F}$. Hence

$$\mathcal{O}_X = \tau^*\mathcal{M} \otimes \mathcal{M} = \tau^*\mathcal{L}^{\otimes n} \otimes \tau^*\mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{F} = \mathcal{F}^{\otimes 2}.$$

Hence $\mathcal{F} = \mathcal{O}_X$ because $\mathrm{Pic}(X)$ is torsion free and thus $\mathcal{M} = \mathcal{L}^{\otimes n}$. Thus, we have showed that $E_2^{1,1} = 0$.

Remark 2.4.7. We have given until here only examples of K3 surfaces X covering Enriques surfaces Y such that $\pi* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. We will show in the next chapter that this is not the case for any K3 surface that has an Enriques quotient.

Example 2.4.8. Let E_1, E_2 be elliptic curves over k (a field of characteristic 0) which are not isogeneous over $\bar{k}$ and such that their points of order 2 are defined over k. For $i = 1, 2$, let D_i be a principal homogeneous space of E_i whose class in $H^1(Gal(\bar{k}/k), E_i)$ has order 2. The antipodal involution $P \mapsto -P$ defines an involution on D_1 and on D_2, and defines a Kummer surface X by considering the minimal desingularization of the quotient of $D_1 \times D_2$ by the simultaneous antipodal involution. Since X is a Kummer surface, it covers an Enriques surface Y. Harari and Skorobogatov were able to prove that for this example the morphism $\pi^* : \mathrm{Br}'(\bar{Y}) \to \mathrm{Br}'(\bar{X})$ is injective (See Cor 2.8, [16]) where $\bar{X}$ and $\bar{Y}$ are the surfaces over $\bar{k}$ obtained from X and Y respectively by extending the ground field from k to $\bar{k}$. We also know from Corollary 4.4 in [30] that $\rho(\bar{X}) \geq 17$ because X is a Kummer surface.

2.5 More about the morphism $\mathrm{Br}'(Y) \to \mathrm{Br}'(X)$

Let $\pi : X \to Y$ be the universal covering map of an Enriques surface Y and let τ be the fixed point free involution of X associated to π. We proceed to study how τ acts on $H^2(X, \mathcal{O}_X^*)$ and on $H^3(X, \mathcal{O}_X^*)$.

Lemma 2.5.1. *Let X be a K3 surface with a fixed point free involution τ. The involution τ acts on $H^2(X, \mathcal{O}_X^*)$ as $\tau^*\alpha = \alpha^{-1}$.*

Proof. The involution τ acts on $H^2(X, \mathcal{O}_X)$ as $-\mathrm{id}$. Indeed, since $H^2(X, \mathcal{O}_X)$ is one dimensional then the action τ on this is $\pm\,\mathrm{id}$. If θ is a 2-form and $\tau^*\theta = \theta$, the form descends to a 2-form on $Y := X/\tau$. This is a contradiction because for any Enriques surface $h^{0,2}(Y) = 0$. From the exponential sequence we get

$$\begin{tikzcd} H^2(X, \mathcal{O}_X) \arrow[r, two heads] \arrow[d, "\text{-id}"'] & H^2(X, \mathcal{O}_X^*) \arrow[d, "\tau^*"'] \\ H^2(X, \mathcal{O}_X) \arrow[r, two heads] & H^2(X, \mathcal{O}_X^*) \end{tikzcd}$$

Hence for every $\alpha \in H^2(X, \mathcal{O}_X^*)$, $\tau^*\alpha = \alpha^{-1}$. □

Lemma 2.5.2. *Let X be a K3 surface. Any element in the Brauer group $\mathrm{Br}'(X)$ is 2-divisible.*

Proof. From the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{O}_X^* \to \mathcal{O}_X^* \to 0$$

we get

$$0 \to \mathrm{Br}'(X)_2 \to \mathrm{Br}'(X) \to \mathrm{Br}'(X) \to 0$$

because $H^3(X, \mathbb{Z}/2\mathbb{Z}) = 0$. □

Notation 2. Let $\rho := \rho(X)$ denote the Picard number of a surface X.

Remark 2.5.3. Let X be a K3 surface with an involution τ that has no fixed points. For any invariant line bundle L under τ, there is a line bundle M on the Enriques surface $Y := X/\tau$ such that $\pi^*M = L$. This is no longer true for Brauer classes. Indeed, by Lemma 2.5.1, the invariant elements of $\mathrm{Br}'(X)$ under τ consist of all the 2-torsion elements of $\mathrm{Br}'(X)$. Since X is a K3 surface, $\mathrm{Br}'(X)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Hence, since $\rho \le 20$, there exists an element $\alpha \in \mathrm{Br}'(X)$ such $\tau^*\alpha = \alpha$ which is not in the image $\mathrm{im}(\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X))$. In conclusion, you may have picked α that happens to be in the image, but since $22 - \rho \ge 2$, there is always another one.

Now, let us compute some elements of the spectral sequence $E_2^{p,q}$ introduced in (2.1), associated to the universal covering map $\pi : X \to Y$ of an Enriques surface Y. First, we know from the exponential sequence that

$$H^3(Y, \mathcal{O}_Y^*) \cong H^4(Y, \mathbb{Z}) = \mathbb{Z}. \tag{2.16}$$

Remark 2.5.4. By Theorem 2.2.1,

$$E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = \frac{\{L \in \mathrm{Pic}(X) \,|\, \tau^*L \otimes L^\vee = \mathcal{O}_X\}}{\{\tau^*M \otimes M \,|\, M \in \mathrm{Pic}(X)\}}$$

and

$$E_2^{1,2} = H^1(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathcal{O}_X^*)) = \frac{\{\alpha \in H^2(X, \mathcal{O}_X^*) \,|\, \tau^*(\alpha).\alpha = 1\}}{\{\tau^*(\beta).\beta^{-1} \,|\, \beta \in H^2(X, \mathcal{O}_X^*)\}}.$$

By Lemmas 2.5.1 and 2.5.2, $E_2^{1,2} = 0$. Now, if $L \in \mathrm{Pic}(X)$ with $\tau^*L \otimes L^\vee = \mathcal{O}_X$. Then $[L^{\otimes 2}] = [\tau^*(L) \otimes L]$, i.e. $[L]$ is a 2-torsion element in $E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$.

Thus $E_2^{1,2} = 0$, $E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ (cf. equation 2.3) and $E_2^{2,1}$ is a torsion group (by the last remark). In conclusion, we get from the equation 2.16 which says that $E^3 = \mathbb{Z}$, that

$$E_\infty^{0,3} = \mathbb{Z}, \tag{2.17}$$

$$E_\infty^{1,2} = E_\infty^{2,1} = E_\infty^{3,0} = 0. \tag{2.18}$$

The action τ on $H^3(X, \mathcal{O}_X^*) = H^4(X, \mathbb{Z}) = \mathbb{Z}$ is $\pm\,\mathrm{id}$. If $\tau^* = -\,\mathrm{id}$, then $E_2^{0,3} = H^0(\mathbb{Z}/2\mathbb{Z}, H^3(X, \mathcal{O}_X^*)) = H^3(X, \mathcal{O}_X^*)^\tau = 0$, but this contradicts the fact $E_\infty^{0,3} = \mathbb{Z}$. Thus, we have shown the following lemma. (Note that this lemma trivially follows only from the fact that $H^3(X, \mathcal{O}_X^*) = H^4(X, \mathbb{Z}) = \mathbb{Z}$ and the action on the last cohohomogy group is id, but the computations above are needed).

Lemma 2.5.5. *Let X be a K3 surface with a fixed point free involution τ. Then the action of τ on $H^3(X, \mathcal{O}_X^*)$ is trivial.* □

Remark 2.5.6. Let L be a line bundle such that $\tau^*L \otimes L = \mathcal{O}_X$. Thus, $L^{\otimes 2} = L \otimes (\tau^*L)^\vee$, i.e. $[L] \otimes [L] = [L^{\otimes 2}] = 0$ in $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X))$. Since

$$E_2^{0,2} = H^0(\mathbb{Z}/2\mathbb{Z}, H^2(X, \mathcal{O}_X^*)) = H^2(X, \mathcal{O}_X^*)^\tau,$$

by Lemma 2.5.1, $E_2^{0,2} = \mathrm{Br}'(X)_2$. Indeed, if $\alpha \in \mathrm{Br}'(X)$ with $\tau^*\alpha = \alpha$, then by Lemma 2.5.1, $\alpha = \tau^*\alpha = \alpha^{-1}$, i.e. α is a 2-torsion element of $\mathrm{Br}'(X)$. On the other hand, if $\alpha \in \mathrm{Br}'(X)_2$, then by Lemma 2.5.1, $\alpha = \alpha^{-1} = \tau^*\alpha$. Finally, by Remark 2.5.3, $E_2^{0,2} = \mathrm{Br}'(X)_2 = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Since any element in $E_2^{1,1}$ is a 2-torsion element, we have only the following four cases:

(1) $E_2^{1,1} = 0$ or

(2) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}$, $d_2^{1,1} = \mathrm{id}$, i.e. $E_\infty^{1,1} = 0$ or

(3) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}$, $d_2^{1,1} = 0$, i.e. $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$ or

(4) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $d_2^{1,1} \neq 0$, i.e. $0 \to \mathbb{Z}/2\mathbb{Z} \to E_2^{1,1} \xrightarrow{d_2^{1,1}} E_2^{3,0} \to 0$.

Lemma 2.5.7. *Let Y be an Enriques surface, $\pi : X \to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = 0$, then $E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$.*

Proof. Since $E_2^{1,1} = 0$,

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$$

and by (2.18)

$$0 = E_\infty^{3,0} = E_4^{3,0} = \frac{E_3^{3,0}}{\mathrm{im}(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0})}.$$

Thus $d_3^{0,2}$ is surjective. Since $E_2^{1,1} = 0$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}), \tag{2.19}$$

and since $E_3^{3,0} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ and all elements in $E_2^{0,2}$ are 2-torsion,

$$E_3^{0,2} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}. \tag{2.20}$$

By equation (2.18)

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})},$$

and thus the morphism $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is surjective. Hence, by (2.19) and the fact that any element in $E_2^{0,2}$ is a 2-torsion element (cf. Remark 2.5.6)

$$E_2^{0,2} = E_3^{0,2} \times E_2^{2,1}.$$

From $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 2.5.6) and (2.20), $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. $\square$

Lemma 2.5.8. *Let Y be an Enriques surface, $\pi : X \to Y$ the universal covering map of Y and τ the fixed point free involution given by π. If $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = 0$. Then $E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. Since $E_2^{1,1} \neq 0$ and $E_\infty^{1,1} = 0$, $\mathrm{im}(d_2^{1,1}) = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$ (cf. equation 2.3). Thus

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})} = 0. \tag{2.21}$$

By Remark 2.5.4, any element in $E_2^{2,1}$ is 2-torsion. Then there is an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By equation 2.18

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})}$$

and thus $\mathrm{im}(d_2^{0,2}) = (\mathbb{Z}/2\mathbb{Z})^m$. Hence

$$\ker(d_2^{0,2}) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

because $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Since $E_\infty^{0,2} = \mathbb{Z}/2\mathbb{Z}$,

$$\mathbb{Z}/2\mathbb{Z} = E_\infty^{0,2} = E_4^{0,2} = \ker(d_3^{0,2} : \ker(d_2^{0,2}) \to E_3^{3,0})$$

and from equation 2.21

$$\mathbb{Z}/2\mathbb{Z} = \ker(d_2^{0,2}) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}$$

and so $m = 21 - \rho$. □

Lemma 2.5.9. *Let X be a K3 surface that covers an Enriques surface Y and such that its spectral sequence satisfies $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Then $E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$.*

Proof. By assumptions $d_2^{1,1}$ is trivial and so

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})} = E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$$

and by definition

$$E_4^{3,0} = \frac{E_3^{3,0}}{\mathrm{im}(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0})}. \tag{2.22}$$

On the other hand,

$$0 = E_\infty^{0,2} = \ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0})$$

because $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. Hence $d_3^{0,2} : E_3^{0,2} \to E_3^{3,0} = \mathbb{Z}/2\mathbb{Z}$ is injective and this and (2.22) imply the following equivalence

(1) $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$ if and only if $E_\infty^{3,0} = E_4^{3,0} = 0$.

By (2.18), $E_\infty^{3,0} = 0$. Thus, the equivalence (1) implies $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Since by Remark 2.5.4, any element in $E_2^{2,1}$ is a 2-torsion element, there exists an integer m such that $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^m$. By (2.18)

$$0 = E_\infty^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})},$$

and thus

$$\mathrm{im}(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}) = (\mathbb{Z}/2\mathbb{Z})^m,$$

i.e. the map $d_2^{0,2}$ is surjective. Since $E_2^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$ (cf. Remark 2.5.6), $E_3^{0,2} = \ker(d_2^{0,2})$ it yields from the surjectivity of $d_2^{0,2}$ that

$$E_3^{0,2} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho-m}.$$

Thus, $m = 21 - \rho$ because $E_3^{0,2} = \mathbb{Z}/2\mathbb{Z}$. Hence

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}.$$

□

Lemma 2.5.10. *Let Y be an Enriques surface and $\pi : X \to Y$ the universal covering map of Y such that $E_2^{1,1} = H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^2$. Then $E_2^{2,1} = H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Moreover $\rho(X) \geq 12$.*

Proof. Since $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2$ and $E_2^{3,0} = \mathbb{Z}/2\mathbb{Z}$, the map $d_2^{1,1} \neq 0$. Hence $E_\infty^{1,1} = E_3^{1,1} = \ker(d_2^{1,1})$ is nontrivial, and thus it must be $\mathbb{Z}/2\mathbb{Z}$. By definition

$$E_3^{3,0} = \frac{E_2^{3,0}}{\mathrm{im}(d_2^{1,1} : E_2^{1,1} \to E_2^{3,0})} = 0 \tag{2.23}$$

and by (2.18)

$$E_\infty^{2,1} = E_3^{2,1} = \frac{E_2^{2,1}}{\mathrm{im}(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})} = 0. \tag{2.24}$$

Since $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$, then

$$0 = E_\infty^{0,2} = E_4^{0,2} = \ker(d_3^{0,2} : E_3^{0,2} \to E_3^{3,0}).$$

Thus, by equation 2.23, $E_3^{0,2} = 0$. By definition

$$E_3^{0,2} = \ker(d_2^{0,2} : E_2^{0,2} \to E_2^{2,1})$$

and then $d_2^{0,2} : E_2^{0,2} \to E_2^{2,1}$ is injective. Hence by (2.24),

$$E_2^{2,1} = E_2^{0,2}.$$

Since

$$E_2^{0,2} = \mathrm{Br}'(X)_2 = (\mathbb{Z}/2\mathbb{Z})^{22-\rho},$$

one finds

$$E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}.$$

Since $E_2^{2,1}$ is a quotient of $\mathrm{Pic}(X)^\tau$ and thus of $\mathrm{Pic}(Y) = \mathbb{Z}^{10} \times \mathbb{Z}/2\mathbb{Z}$, one finds $22 - \rho \leq 10$ (note that $\mathbb{Z}/2\mathbb{Z} \subset \mathrm{Pic}(Y)$ goes to zero in $E_2^{2,1}$). Thus $\rho \geq 12$ ☐

In conclusion, by Lemmas 2.5.7, 2.5.8, 2.5.9, 2.5.10 and the statement before Lemma 2.5.7, we obtain that we only have the following four cases:

(1) $E_2^{1,1} = 0, E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$ or

(2) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, E_\infty^{1,1} = 0, E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

(3) $E_2^{1,1} = \mathbb{Z}/2\mathbb{Z}, E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{21-\rho}$ or

(4) $E_2^{1,1} = (\mathbb{Z}/2\mathbb{Z})^2, E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}, E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$.

Note that in the cases (2) and (3) we have that $\rho \geq 11$.

Proposition 2.5.11. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$. Then the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial, where $Y := X/\langle\tau\rangle$.*

Proof. Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{22-\rho}$, we are in case (4). Hence $E_\infty^{1,1} = \mathbb{Z}/2\mathbb{Z}$. By (2.4), the morphism $\pi : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is trivial. □

Proposition 2.5.12. *Let X be a K3 surface with a fixed point free involution τ and Picard number ρ such that $H^2(\mathbb{Z}/2\mathbb{Z}, \mathrm{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$. Then the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is nontrivial, where $Y := X/\langle\tau\rangle$.*

Proof. Since $E_2^{2,1} = (\mathbb{Z}/2\mathbb{Z})^{20-\rho}$, we are in case (1). Hence $E_\infty^{1,1} = 0$. By (2.4), the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. □

Let Y be an Enriques surface and $\pi : X \to Y$ its universal covering map. We know that if X is as in the first case above, then $\rho(X) \geq 10$, and if X is one of the cases (2) or (3), then $\rho(X) \geq 11$ and if X is as in the case (4), then $\rho(X) \geq 12$. Thus, if $\rho(X) = 10$, the K3 surface X satisfies the conditions of the first case and we obtain $E_2^{1,1} = 0$. Hence, by equation 2.4, the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective. This is another proof for the same result obtained before in Lemma 2.2.7.

Proposition 2.5.13. *Let X be a K3 cover of an Enriques surface Y such that $\rho(X) = 11$ and $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle$, where $N \geq 2$. Then $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is injective if and only if N is an even number.*

Proof. Note that $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N\rangle = \pi^* \mathrm{NS}(Y) \oplus \langle -2N\rangle$ (because as in Example 2.2.8, $\Lambda^+ \cong U(2) \oplus E_8(2)$ and this is diagonally embedded in the K3 lattice), i.e. τ^* acts trivially on $U(2) \oplus E_8(2)$. Now, we show that τ acts as $-\mathrm{id}$ on $\langle -2N\rangle$. Let $L \in \mathrm{NS}(X)$ denote the generator of $\langle -2N\rangle$, i.e. $c_1^2(L) = -2N$. Thus,

$$\tau^* L = I \otimes L^{\otimes k} \tag{2.25}$$

for some integer k and invariant line bundle I and since τ is an involution:

$$\begin{aligned} L = \tau^*\tau^* L &= \tau^* I \otimes \tau^* L^{\otimes k} \\ &= I \otimes \tau^* L^{\otimes k} \\ &= I \otimes (I \otimes L^{\otimes k})^{\otimes k} \\ &= I^{\otimes (k+1)} \otimes L^{\otimes k^2}. \end{aligned}$$

Hence

$$L^{\otimes (k^2-1)} \otimes I^{\otimes (k+1)} = \mathcal{O}_X \tag{2.26}$$

and we find that $L^{\otimes (k^2-1)}$ is an invariant line bundle. Thus,

$$\mathcal{O}_X = L^{\otimes (-k^2+1)} \otimes \tau^* L^{\otimes (k^2-1)} = (L^\vee \otimes \tau^* L)^{\otimes (k^2-1)}$$

and if $k \neq 1, -1$, then $\mathcal{O}_X = L^\vee \otimes \tau^* L$ (because $\mathrm{Pic}(X)$ is a free torsion group), i.e. L is an invariant line bundle which contradicts our assumption about L. If $k = 1$, then from (2.26) we get $I = \mathcal{O}_X$ and then by (2.25), L is an invariant bundle which contradicts our assumption on L. Thus $k=-1$ and from (2.26), $I=\mathcal{O}_X$ and from (2.25) we obtain $\tau^*L\otimes L=\mathcal{O}_X$, i.e. τ acts as $-\operatorname{id}$ in $\langle -2N\rangle$.

Now, we show that if M is a line bundle such that $\tau^*M\otimes M=\mathcal{O}_X$, then $M=L^{\otimes m}$ for some integer m. Indeed, if $M=L^{\otimes m}\otimes F$ where F is an invariant line bundle, then

$$\mathcal{O}_X=\tau^*M\otimes M=\tau^*L^{\otimes m}\otimes\tau^*F\otimes L^{\otimes m}\otimes F=F^{\otimes 2}.$$

Hence $F=\mathcal{O}_X$ because $\operatorname{Pic}(X)$ is torsion free and thus $M=L^{\otimes m}$.

Suppose that N is an even number and that $\pi^*:\operatorname{Br}'(Y)\to\operatorname{Br}'(X)$ is trivial. By Corollary 2.4.3 there exists a line bundle $M=L^{\otimes m}$ for some integer m such that $c_1(M)^2\equiv 2(mod\,4)$. Thus $-2m^2N\equiv 2(mod\,4)$, which implies that m^2N is an odd number and thus N is an odd number, a contradiction. On the other hand, let us suppose that $\pi^*:\operatorname{Br}'(Y)\to\operatorname{Br}'(X)$ is injective. By Corollary 2.4.3, $c_1^2(L)\not\equiv 2(mod\,4)$. Hence, $(1-N)\not\equiv 0(mod\,2)$ and thus N is an even number.

$\square$

2.6 Overview of the paper of Beauville

In this section we will coment briefly the paper [2] of Beauville. Let Y be an Enriques surface over $\mathbb{C}$. We denote by k_Y the image of K_Y in $H^2(Y,\mathbb{Z}/2\mathbb{Z})$ and by b_Y the nonzero element of $\operatorname{Br}'(Y)$.

Proposition 2.6.1 (Prop. 3.5, [2]). *(1) The kernel of $\pi^*:H^2(Y,\mathbb{Z}/2\mathbb{Z})\to H^2(X,\mathbb{Z}/2\mathbb{Z})$ is $\{0,k_Y\}$.*

(ii) The Gysin map $\pi_*:H^2(X,\mathbb{Z})\to H^2(Y,\mathbb{Z})$ is surjective.

Beauville also introduced the spectral sequence 2.1 as we did in this chapter and proved the following proposition.

Proposition 2.6.2 (Prop. 4.1, [2]). *Let $\pi:X\to Y$ be an étale, cyclic covering of smooth projective varieties over an algebraically closed field k. Let σ be a generator of the Galois group G of π, and let $Nm:\operatorname{Pic}(X)\to\operatorname{Pic}(Y)$ be the norm homomorphism. Then the kernel of $\pi^*:\operatorname{Br}'(Y)\to\operatorname{Br}'(X)$ is canonically isomorphic to* $\operatorname{Ker}\operatorname{Nm}/(1-\sigma^*)(\operatorname{Pic}(X))$.

This Proposition yields to the following corollary similar to our Lemma 2.2.6.

Corollary 2.6.3 (Cor. 4.3, [2]). *Assume $k=\mathbb{C}$, and $H^1(X,\mathcal{O}_X)=H^2(Y,\mathcal{O}_Y)=0$. Then the following conditions are equivalent*

(i) The map $\pi^*:\operatorname{Br}'(Y)\to\operatorname{Br}'(X)$ is not injective;

(ii) there exists $L\in\operatorname{Pic}(X)$ whose class $\lambda=c_1(L)$ in $H^2(X,\mathbb{Z})$ satisfies $\pi_*\lambda=0$ and $\lambda\notin(1-\sigma^*)(H^2(X,\mathbb{Z}))$.

It is worth to mention that the difference between this result and ours is that in the Lemma 2.2.6 we found explicitily the Brauer–Severi variety on X that corresponds to the pullback of the nontrivial element b_Y.

Let E be the lattice $E_8 \oplus U$ and let $H^2(Y,\mathbb{Z})_{tf}$ be the quotient of $H^2(Y,\mathbb{Z})$ by its torsion subgroup $\{0, k_Y\}$. We have isomorphisms

$$H^2(Y,\mathbb{Z})_{tf} \cong E, \;\; H^2(X,\mathbb{Z}) \cong E \oplus E \oplus U$$

such that $\pi^* : H^2(Y,\mathbb{Z})_{tf} \to H^2(X,\mathbb{Z})$ is identified with the diagonal embedding $\delta : E \hookrightarrow E \oplus E$ and σ^* with the involution ρ introduced in Example 2.2.8.

For any lattice M, its scalar product induces a product $M_2 \otimes M_2 \to \mathbb{Z}/2\mathbb{Z}$ where $M_2 := M/2M$. Moreover, if M is even, there exists a quadratic form $q : M_2 \to \mathbb{Z}/2\mathbb{Z}$ defined by $q(m) = \frac{1}{2}\tilde{m}^2$, where $\tilde{m} \in M$ is any lift of $m \in M_2$. Let ε denote the unique element with $q(\varepsilon) = 1$, i.e. the class of $e + f$ where $\{e, f\}$ is a hyperbolic basis of U. Hence, Beauville showed:

Proposition 2.6.4 (Prop. 5.3, [2]). *The image of* $\pi^* : H^2(Y,\mathbb{Z}/2\mathbb{Z}) \to H^2(X,\mathbb{Z}/2\mathbb{Z})$ *is* $\delta(E_2) \oplus (\mathbb{Z}/2\mathbb{Z})\varepsilon$.

Corollary 2.6.5 (Cor. 5.5, [2]). *The kernel of* $\pi_* : U_2 \to \{0, k_Y\}$ *is* $\{0, \varepsilon\}$.

From this result Beauville obtained the Proposition 2.4.2, which is very useful to describe the moduli space of marked Enriques surfaces. I want to remark that since I was not aware about a mistake that I had made in the proof of Lemma 2.2.6, the moduli space that I have explained in Section 2.4. in a previous version was still not well described.

For $\lambda \in \Lambda^-$, Beauville associated the hypersurface H_λ of Ω defined by $\lambda.\omega = 0$ where $\Omega \subset \mathbb{P}(\Lambda^-_{\mathbb{C}})$ is the domain defined by the equations

$$\omega.\omega = 0, \; \omega.\bar{\omega} > 0, \; \omega.\lambda \neq 0 \text{ for all } \lambda \in \Lambda^- \text{ with } \lambda^2 = -2.$$

Proposition 2.6.6 (Prop. 6.2, [2]). *We have* $\pi^* b_Y = 0$ *if and only if the period map* $\wp(Y,\varphi)$ *belongs to one of the hypersurfaces* H_λ *for some vector* $\lambda \in \Lambda^-$ *with* $\lambda^2 \cong 2(mod.\,4)$

Finally, Beauville completed the picture by proving the following lemma.

Lemma 2.6.7 (Lem. 6.3, [2]). *Let* λ *be a primitive element of* Λ^-.

(i) The hypersurface H_λ *is non-empty if and only if* $\lambda^2 < -2$.

(ii) If μ *is another primitive element of* Λ^- *with* $H_\mu = H_\lambda \neq \emptyset$, *then* $\mu = \pm\lambda$.

Chapter 3

Quotient Varieties

In this note we will show that some statements proved by Bridgeland in [4] hold also in the twisted case.

3.1 Quotient varieties

It is well known that if G is a finite cyclic group that acts freely on a smooth projective variety $\tilde{X}$ then the quotient $X = \tilde{X}/G$ is also a smooth projective variety. As an example of this we can consider X to be an Enriques surface, and then there is a K3 surface $\tilde{X}$ and a fixed point free involution, τ, on $\tilde{X}$ such that $X = \tilde{X}/\tau$. Now we recall the following proposition (Prop. 3.2 [4])

Proposition 3.1.1. *Let X be a smooth projective variety whose canonical bundle has finite order n. Then there is a smooth projective variety $\tilde{X}$ with trivial canonical bundle and an unbranched cover $\pi_X : \tilde{X} \to X$ of degree n, such that*

$$\pi_{X,*}\mathcal{O}_{\tilde{X}} \cong \bigoplus_{i=0}^{n-1} \omega_X^i$$

Furthermore, $\tilde{X}$ is uniquely determined up to isomorphism and there is a free action of the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ on $\tilde{X}$ such that $\pi_X : \tilde{X} \to X = \tilde{X}/G$ is the quotient morphism.

Definition 3.1.2. *Let X be a smooth projective variety with canonical bundle of order n. The unique smooth projective variety $\tilde{X}$ of Proposition 3.1.1 together with the quotient morphism $\pi_X : \tilde{X} \to X$ is called the canonical cover of X.*

We assume that all varieties are smooth varieties with canonical bundle of finite order and we also assume that for any smooth variety X, the Picard group of its canonical cover, $\mathrm{Pic}(\tilde{X})$, is torsion free. Note that Enriques surfaces X satisfy this condition. In this section we fix two twisted varieties (X, α) and (Y, β) and their respective canonical covers $\pi_X : \tilde{X} \to X$ and $\pi_Y : \tilde{Y} \to Y$.

Definition 3.1.3. *A functor* $\tilde{\Phi} : \mathrm{D}^b(\tilde{Y}, \pi_Y^*\beta) \to \mathrm{D}^b(\tilde{X}, \pi_X^*\alpha)$ *is called equivariant if there is an automorphism* $\mu : G \to G$*, and an isomorphism of functors*

$$g^* \circ \tilde{\Phi} \cong \tilde{\Phi} \circ \mu(g)^*$$

for each $g \in G$*.*

Definition 3.1.4. *A lift of a FM transform* $\Phi : \mathrm{D}^b(Y, \beta) \to \mathrm{D}^b(X, \alpha)$ *is a FM transform* $\tilde{\Phi} : \mathrm{D}^b(\tilde{Y}, \pi_Y^*\beta) \to \mathrm{D}^b(\tilde{X}, \pi_X^*\alpha)$ *such that the following diagrams*

$$\begin{tikzcd}
\mathrm{D}^b(\tilde{Y}, \pi_Y^*\beta) \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi_{Y*}"'] & \mathrm{D}^b(\tilde{X}, \pi_X^*\alpha) \arrow[d, "\pi_{X*}"] \\
\mathrm{D}^b(Y, \beta) \arrow[r, "\Phi"'] & \mathrm{D}^b(X, \alpha)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{D}^b(\tilde{Y}, \pi_Y^*\beta) \arrow[r, "\tilde{\Phi}"] & \mathrm{D}^b(\tilde{X}, \pi_X^*\alpha) \\
\mathrm{D}^b(Y, \beta) \arrow[r, "\Phi"'] \arrow[u, "\pi_Y^*"] & \mathrm{D}^b(X, \alpha) \arrow[u, "\pi_X^*"']
\end{tikzcd}$$

commute.

Let us suppose that the canonical bundle of the variety X has order n, i.e., X is the quotient of $\tilde{X}$ by a free action of $G = \mathbb{Z}/n\mathbb{Z}$ on $\tilde{X}$. Let g be a generator of G and $\tilde{\alpha} := \pi_X^*\alpha$. We take this cocycle to be defined over an $g-$invariant covering, i.e., defined over a covering $\pi^{-1}U_i$ where U_i covers X and $\pi^{-1}(U_i)$ consists of n copies of U_i. Let G-Coh$(\tilde{X}, \tilde{\alpha})$ be the category of G-linearized $\tilde{\alpha}$-twisted sheaves on $\tilde{X}$ and Sp-Coh(X, α) be the category of α-twisted coherent sheaves E such that $E \otimes \omega_X \cong E$. The following lemma follows from the untwisted version

Remark 3.1.5. If G is a cyclic group, a twisted sheaf $F \in$ G-Coh$(\tilde{X}, \tilde{\alpha})$ if and only if $g^*F \cong F$, where $g \in$ G is a generator.

Lemma 3.1.6. *Let* F *be an element in* $\mathrm{Coh}(\tilde{X}, \pi^*\alpha)$*. Then there is an element* $E \in \mathrm{Coh}(X, \alpha)$ *such that* $\pi^*E = F$ *if and only if there is an isomorphism* $g^*F \cong F$*.*

Proof. Clearly, if $F = \pi^*E$, then $g^*F = F$. On the other hand, let us suppose that $g^*F \cong F$. Let $F = (F_i, \varphi_{ij})$ be an element in G-Coh$(\tilde{X}, \pi^*\alpha)$. Since the open covering $\{U_i\}$ of X that defines α satisfies that $\pi^{-1}(U_i)$ consists of $n := |G|$ copies of U_i, the coherent sheaf F_i on $\pi^{-1}(U_i)$ descends to a coherent sheaf E_i defined on U_i. Since $\{\varphi_{ij}\}$ is g-invariant (on disjoint copies of U_i) it descends to maps $\psi_{ij} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$ which defines (E_i, ψ_{ij}) as an element in $\mathrm{Coh}(X, \alpha)$. ☐

Remark 3.1.7. Suppose that $E \in \mathrm{Coh}(X, \alpha)$ and $F \in \mathrm{Coh}(\tilde{X}, \pi^*\alpha)$ such that $\pi_*F = E$. Thus, by projection formula

$$E \otimes \omega_X \cong \pi_*(F \otimes \pi^*(\omega_X)) = \pi_*F = E.$$

Thus, it induces on E the structure of a module over $\pi_*(\mathcal{O}_{\tilde{X}})$. Unfortunately, this does not imply that there exists a $\pi^*\alpha$-twisted coherent sheaf F such that $\pi_* F = E$. Indeed, suppose α has order 2 and $\pi^* : \mathrm{Br}'(X) \to \mathrm{Br}'(\tilde{X})$ injective. Let E be a locally free α-twisted sheaf of rank 2 on X (this exists by Theorem 3.13 in [7]). Thus, if there exists a locally free $\pi^*\alpha$-twisted sheaf F such that $\pi_* F = E$, then F is a line bundle and hence $\pi^*\alpha$ is trivial, a contradiction.

Lemma 3.1.8 (Cor. 5.3, [6]). *Let X be an Enriques surface. If $\Phi : \mathrm{Coh}(X, \alpha) \to \mathrm{Coh}(Y, \beta)$ is an equivalence, then there exists an isomorphism $f : X \xrightarrow{\sim} Y$, such that $f^*\beta = \alpha$.*

Corollary 3.1.9. *Let X be an Enriques surface. If $\Phi : \mathrm{Coh}(X, \alpha) \to \mathrm{Coh}(X, \beta)$ is an equivalence, then $[\alpha] = [\beta] \in \mathrm{Br}'(X)$.*

Remark 3.1.10. By Lemma 4.6 in ([17], Ch.I), any complex $E^\bullet$ in $\mathrm{D}^b(X, \alpha)$ has an injective resolution $I^\bullet$.

Definition 3.1.11. *A G-object of $\mathrm{D}^b(\tilde{X}, \pi_X^*\alpha)$ is an object $\tilde{E}$ together with an isomorphism $\lambda_h : \tilde{E} \to h^*(\tilde{E})$ in $\mathrm{D}^b(\tilde{X}, \pi_X^*\alpha)$ for each $h \in G$ such that for any pair $g, h \in G$,*

$$\lambda_{hg} = g^*(\lambda_h) \circ \lambda_g.$$

Proposition 3.1.12. *Let $\tilde{E}$ be a G-object of $\mathrm{D}^b(\tilde{X}, \pi^*\alpha)$. Then there is an object E of $\mathrm{D}^b(X, \alpha)$ such that $\pi^* E \cong \tilde{E}$.*

Proof. The proof is completely analogous of the untwisted version of the proposition proved in ([4], Prop. 3.1). □

Notation 3. Given $\pi_X : \tilde{X} \to X$ the canonical cover of a variety and $\alpha \in \mathrm{Br}'(X)$ we denote by $\tilde{\alpha} := \pi_X^*(\alpha) \in \mathrm{Br}'(\tilde{X})$ if there is no confusion.

Lemma 3.1.13. *Let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(X, \alpha)$ and $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{X}, \tilde{\alpha})$ be FM transforms, such that $\tilde{\Phi}$ lifts Φ.*

(a) If $\Phi \cong \mathrm{id}_{\mathrm{D}^b(X,\alpha)}$, then $\tilde{\Phi} \cong g_*$ for some $g \in G$ with $g^*\tilde{\alpha} = \tilde{\alpha}$.

(b) If $\tilde{\Phi} \cong \mathrm{id}_{\mathrm{D}^b(\tilde{X},\tilde{\alpha})}$, then Φ is an equivalence and $\Phi(-) = (\omega_X^{\otimes d} \otimes -)$ for some integer d.

Proof. To prove (a), let us take a point $\tilde{x} \in \tilde{X}$ and $x := \pi_X(\tilde{x})$. Then for $E := \tilde{\Phi}(\mathcal{O}_{\tilde{x}})$ we have $\pi_{X,*}(E) = \mathcal{O}_x$. Indeed, since $\tilde{\Phi}$ lifts $\Phi \cong \mathrm{id}_{\mathrm{D}^b(X,\alpha)}$ we get

$$\pi_{X,*}(E) = \Phi(\pi_{X,*}(\mathcal{O}_{\tilde{x}})) = \mathcal{O}_x.$$

Hence $E = \mathcal{O}_{f(\tilde{x})}$ for some point $f(\tilde{x}) \in \pi_X^{-1}(x)$, and thus $\tilde{\Phi}$ defines a morphism $f : \tilde{X} \to \tilde{X}$ such that

$$\tilde{\Phi}(-) \cong f_*(L \otimes -)$$

for some line bundle $L \in \mathrm{Pic}(\tilde{X})$. Besides, since $f(\tilde{x}) \in \pi_X^{-1}(x)$, $f = g \in G$. Take an arbitrary α-twisted locally free sheaf F. Since $g_*^{-1} \circ \tilde{\Phi}$ also lifts the identity, we have

$$(g_*^{-1} \circ \tilde{\Phi})(\pi_X^* F) = \pi_X^* F$$

and so

$$L \otimes \pi_X^* F \cong \pi_X^* F.$$

Hence $L^r \otimes \det(\pi_X^* F) = \det(\pi_X^* F)$, where $r := \mathrm{rk}(F)$. Thus $L^{\otimes r} \cong \mathcal{O}_{\tilde{X}}$ and then $L \cong \mathcal{O}_{\tilde{X}}$ because $\mathrm{Pic}(\tilde{X})$ is torsion free by our general assumption.

To prove (b), take points x in X and $\tilde{x}$ in $\tilde{X}$ such that $\pi_X(\tilde{x}) = x$. Since $\tilde{\Phi}$ lifts Φ, one has

$$\begin{gathered} \pi_{X,*}(\tilde{\Phi}(\mathcal{O}_{\tilde{x}})) = \Phi(\pi_{X,*}(\mathcal{O}_{\tilde{x}})) \\ \pi_{X,*}(\mathcal{O}_{\tilde{x}}) = \Phi(\mathcal{O}_x) \\ \mathcal{O}_x = \Phi(\mathcal{O}_x). \end{gathered}$$

Thus Φ is an equivalence of the form $(L \otimes -)$ for some line bundle L on X. As $\tilde{\Phi}$ lifts Φ, we get $\tilde{\Phi}(\pi_X^*(E)) = \pi_X^*(\Phi(E))$ for any locally free α-twisted sheaf E. Since $\tilde{\Phi} = \mathrm{id}_{\tilde{X}}$,

$$\pi_X^*(E) = \pi_X^*(L) \otimes \pi_X^* E.$$

Hence $\pi_X^* \mathcal{O}_{\tilde{X}} = \pi_X^* L^{\otimes r}$ where $r := \mathrm{rk}(E)$ and so $\pi_X^* L = \mathcal{O}_{\tilde{X}}$ because $\mathrm{Pic}(\tilde{X})$ has no torsion. Now, we use projection formula to compute L in terms of the canonical bundle

$$L \otimes (\bigoplus_{i=0}^{n-1} \omega_X^{\otimes i}) = L \otimes \pi_{X,*}(\pi_X^* \mathcal{O}_X) = \pi_{X,*}(\pi_X^* L) = \bigoplus_{i=0}^{n-1} \omega_X^{\otimes i}$$

Hence L is some power of ω_X. □

In the following lemma we explain one way to find liftings and the proof is exactly the same as in [4], which uses projection formula and base change theorem for twisted sheaves.

Lemma 3.1.14. *Let $\tilde{\mathcal{P}}$ and $\mathcal{P}$ be objects of $\mathrm{D}^b(\tilde{Y} \times \tilde{X}, \tilde{\beta}^{-1} \boxtimes \tilde{\alpha})$ and $\mathrm{D}^b(Y \times X, \beta^{-1} \boxtimes \alpha)$ respectively, such that*

$$(\pi_Y \times \mathrm{id}_X)^*(\mathcal{P}) \cong (\mathrm{id}_{\tilde{Y}} \times \pi_X)_*(\tilde{\mathcal{P}}) \tag{3.1}$$

Then $\tilde{\Phi} = \Phi^{\tilde{\mathcal{P}}}_{\tilde{Y} \to \tilde{X}}$ is a lift of $\Phi = \Phi^{\mathcal{P}}_{Y \to X}$.

Remark 3.1.15. The object $(\pi_Y \times \mathrm{id}_X)^*(\mathcal{P})$ is in $\mathrm{D}^b(\tilde{Y} \times X, (\pi_Y \times \mathrm{id}_X)^*(\beta^{-1} \boxtimes \alpha))$ and since

$$\begin{aligned} ((\pi_Y \times \mathrm{id}_X)^*(p_Y^* \beta^{-1} \otimes p_X^* \alpha))(\tilde{y}, x) &= (p_Y^* \beta^{-1})(y, x) \otimes (p_X^* \alpha)(y, x) \\ &= \beta^{-1}(y) \otimes \alpha(x) \\ &= (\pi_Y^* \beta^{-1})(\tilde{y}) \otimes \alpha(x) \\ &= (p_{\tilde{Y}}^*(\pi_Y^* \beta^{-1}) \otimes p_X^* \alpha)(\tilde{y}, x) \\ &= (\tilde{\beta}^{-1} \boxtimes \alpha)(\tilde{y}, x) \end{aligned}$$

then $(\pi_Y \times \mathrm{id}_X)^*(\mathcal{P}) \in \mathrm{D}^b(\tilde{Y} \times X, \tilde{\beta}^{-1} \boxtimes \alpha)$.

Therefore by following Theorem 4.5 in [4], we obtain the following theorem.

Theorem 3.1.16. *Let X and Y be smooth projective varieties with canonical bundles of order n, and $\pi_X : \tilde{X} \to X$, $\pi_Y : \tilde{Y} \to Y$ their canonical covers. Let $\Phi_{\mathcal{P}}$ be a FM transform*

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(Y, \beta) \to \mathrm{D}^b(X, \alpha)$$

such that there exists $\tilde{\mathcal{P}} \in \mathrm{D}^b(\tilde{Y} \times \tilde{X}, \tilde{\beta}^{-1}.\tilde{\alpha})$ that satisfies $(\pi_Y \times \mathrm{id}_X)^*(\mathcal{P}) \cong (\mathrm{id}_{\tilde{Y}} \times \pi_X)_*(\tilde{P})$. Then $\Phi_{\mathcal{P}}$ lifts to an equivariant FM transform

$$\tilde{\Phi} : \mathrm{D}^b(\tilde{Y}, \pi_Y^* \beta) \to \mathrm{D}^b(\tilde{X}, \pi_X^* \alpha).$$

Conversely, any equivariant FM transform $\tilde{\Phi} : \mathrm{D}^b(\tilde{Y}, \pi_Y^* \beta) \to \mathrm{D}^b(\tilde{X}, \pi_X^* \alpha)$ is the lift of a FM transform $\Phi : \mathrm{D}^b(Y, \beta) \to \mathrm{D}^b(X, \alpha)$.

3.2 Derived categories of Enriques surfaces

In this section we use X and Y to represent smooth Enriques surfaces. In particular, $\mathrm{Br}'_{et}(X) = \mathrm{Br}'_{an}(X)_{tors} = \mathrm{Br}'_{an}(X) = \mathbb{Z}/2\mathbb{Z}$ and we simply write $\mathrm{Br}'(X)$ for any of the Brauer groups.

Definition 3.2.1. *An Enriques surface is called special, if it carries an elliptic pencil together with a 2-section which is a (-2)-curve.*

If X is a special Enriques surface, the K3 covering $\tilde{X}$ admits an elliptic fibration with two sections. Let $\mathrm{Sh}(\tilde{X})$ be the Tate–Shafarevich group consisting of all algebraic elliptic fibrations whose Jacobian is $\tilde{X}$ (see page 35). Shafarevich defined an identification between the Tate–Shafarevich group of $\tilde{X}$ and the étale cohomological Brauer group $H^2(\tilde{X}, \mathbb{G}_m)$ (see page 36) i.e, $\mathrm{Sh}(\tilde{X}) = \mathrm{Br}'(\tilde{X})$.

Notation 4. Let $\pi : X \to \mathbb{P}^1$ be an algebraic elliptic fibration with a section. To any $\alpha \in \mathrm{Br}'(X)$, let $\pi_\alpha : X_\alpha \to \mathbb{P}^1$ denote the corresponding genus one fibration in $\mathrm{Sh}_{et}(X)$ (see page 31).

In this chapter, if we consider a Fourier–Mukai transform $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ between Enriques surfaces X and Y, we suppose that this has a lifting to their K3 covers.

We recall some facts about K3 surfaces. Let X be a K3 surface. Since $H^2(X, \mathbb{Z})$ is unimodular, $H_1(X, \mathbb{Z})$ is torsion free and Lemma 1.1.1,

$$\mathrm{Br}'(X) \cong T(X)^\vee \otimes \mathbb{Q}/\mathbb{Z} \cong \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

Thus, an n-torsion element $\alpha \in \mathrm{Br}'(X)$ is identified with a surjective morphism $\alpha : T(X) \to \mathbb{Z}/n\mathbb{Z}$ and we define the non-primitive sublattice of $T(X)$:

$$T(X, \alpha) := \ker \alpha$$

Definition 3.2.2. *Let X be a K3 surface with a B-field $B \in H^2(X,\mathbb{Q})$. Let $\tilde{H}(X,B,\mathbb{Z})$ denote the weight-two Hodge structure on $H^*(X,\mathbb{Z})$ with*

$$\tilde{H}^{2,0}(X,B) := \exp(B)(H^{2,0}(X))$$

and $\tilde{H}^{1,1}(X,B)$ its orthogonal complement with respect to the Mukai pairing.

Theorem 3.2.3 (Cor. 7.8, [23]). *Let (X,α) and (Y,β) be twisted K3 surfaces. Choose $B \in H^2(X,\mathbb{Q})$ and $B' \in H^2(X',\mathbb{Q})$ such that $\alpha = \alpha_B$ and $\alpha' = \alpha_{B'}$. If*

$$\Phi : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(X',\alpha')$$

is an equivalence, then there exists a naturally defined Hodge isometry

$$\Phi_*^{B,B'} : \tilde{H}(X,B,\mathbb{Z}) \to \tilde{H}(X',B',\mathbb{Z})$$

The same statement also holds for abelian surfaces. This theorem implies the following corollary

Corollary 3.2.4 (Cor. 3.1.10, [42]). *Let (X,α) be a twisted K3 surface and (Y,β) be a twisted variety. If there exists an equivalence $\Phi : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$, then Y is a K3 surface.*

Theorem 3.2.5 (Mukai). *Suppose that X_1 and X_2 are two K3 surfaces with $\rho(X_i) \geq 12$. Then up to a sign any Hodge isometry $T(X_1) \cong T(X_2)$ is induced by an isomorphism $X_1 \cong X_2$.*

Lemma 3.2.6. *Let (X,α) be an Enriques surface and (Y,β) a twisted variety. If there exists an equivalence $\Phi : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ then Y is an Enriques surface.*

Proof. Due to Theorem 3.1.16 (and our general), Φ extends to an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X},\tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y},\tilde{\beta})$. From the last corollary, $\tilde{Y}$ is a K3 surface and thus Y is an Enriques surface. □

Remark 3.2.7. In the particular case that X of the last lemma is a special Enriques surface we get another proof for it. We comment quickly here. From the equivalence Φ, we get that ω_Y is also of order 2 and that $\mathrm{kod}(Y) = 0$ (by Theorem 1.6.15). Then either Y is a bielliptic surface or an Enriques surface. Suppose that Y is a bielliptic surface. Hence its canonical cover $\tilde{Y}$ is an abelian surface given as a quotient of a product of two elliptic curves by a finite group or a K3 surface. Since the equivalence Φ extends to an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X},\tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y},\tilde{\beta})$, we get by Theorem 1.7.25 an equivalence $\Psi : \mathrm{D}^b(\tilde{X}_{\tilde{\alpha}}) \to \mathrm{D}^b(\tilde{Y}_{\tilde{\beta}})$. Hence $\tilde{Y}_{\tilde{\beta}}$ is a K3 surface and so $\tilde{Y}$ also is. Therefore Y is an Enriques surface.

Lemma 3.2.8. *Let X and Y be special Enriques surfaces and suppose that either $\rho(\tilde{X}) \geq 12$ or $\rho(\tilde{Y}) \geq 12$. Then any equivalence $\Phi : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ implies an isomorphism $\tilde{X}_{\tilde{\alpha}} \cong \tilde{Y}_{\tilde{\beta}}$.*

Proof. The equivalence Φ can be lifted to an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y}, \tilde{\beta})$ and this implies an equivalence $\Psi : \mathrm{D}^b(\tilde{X}_{\tilde{\alpha}}) \to \mathrm{D}^b(\tilde{Y}_{\tilde{\beta}})$ by Theorem 1.7.25. Suppose that $\rho(\tilde{X}) \geq 12$. Since $\mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \cong \mathrm{D}^b(\tilde{X}_\alpha)$, we get an isometry $T(\tilde{X}_\alpha) \cong T(\tilde{X}, \tilde{\alpha}) \subseteq T(\tilde{X})$. Since $\mathrm{rk}(T(X)) = \mathrm{rk}(T(X, \alpha))$, also $\rho(\tilde{X}_{\tilde{\alpha}}) \geq 12$. From the equivalence Ψ we get $\rho(\tilde{Y}_\beta) = \rho(\tilde{X}_\alpha) \geq 12$. Hence the equivalence Ψ implies an isomorphism $\tilde{X}_{\tilde{\alpha}} \cong \tilde{Y}_{\tilde{\beta}}$ by Theorem 3.2.5. □

Remark 3.2.9. Let X, Y be Enriques surfaces and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ an equivalence. Consider $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y}, \tilde{\beta})$ a lifting of Φ. It induces an isometry $T_{\tilde{\Phi}} : T(\tilde{X}, \tilde{\alpha}) \cong T(\tilde{Y}, \tilde{\beta})$. On the other hand we have the standard formula on lattices

$$|\operatorname{disc}(T(\tilde{X}, \tilde{\alpha}))| = \operatorname{ord}(\tilde{\alpha})^2.|\operatorname{disc}(T(\tilde{X}))|,$$

and

$$|\operatorname{disc}(T(\tilde{Y}, \tilde{\beta}))| = \operatorname{ord}(\tilde{\beta})^2.|\operatorname{disc}(T(\tilde{Y}))|.$$

Hence it follows that if $\operatorname{ord}(\tilde{\alpha}) \neq \operatorname{ord}(\tilde{\beta})$ then $X \ncong Y$.

Lemma 3.2.10. *Let X be an Enriques surface such that $\rho(\tilde{X}) = 10$. If $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(X)$ is an equivalence then $\alpha = 1$ in $\mathrm{Br}'(X)$.*

Proof. The equivalence Φ lifts to an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{X})$. This induces an isometry $T(\tilde{X}, \tilde{\alpha}) \cong T(\tilde{X})$, and then $\tilde{\alpha} = 1$. Since $\rho(\tilde{X}) = 10$, the morphism $\pi^* : \mathrm{Br}'(X) \to \mathrm{Br}'(\tilde{X})$ is injective by Lemma 2.2.7. Hence $\alpha = 1$. □

Lemma 3.2.11. *Let X, Y be Enriques surfaces. If $\pi : \tilde{X} \to X$ is the universal covering map and $\rho(\tilde{X}) = 10$ and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is an equivalence. Then α is trivial in $\mathrm{Br}'(X)$ if and only if β is trivial in $\mathrm{Br}'(Y)$.*

Proof. Since $\rho(\tilde{X}) = 10$, then $\rho(\tilde{Y}) = 10$. Hence both of them have Picard lattice $E_8(2) \oplus U(2)$. Suppose $\pi^*(\alpha) = 1$ in $\mathrm{Br}'(\tilde{X})$. The equivalence Φ lifts to an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y}, \tilde{\beta})$. Thus, if $\pi^*(\beta)$ is nontrivial in $\mathrm{Br}'(\tilde{Y})$,

$$\operatorname{disc}(T(\tilde{X})) = \operatorname{disc}(T(\tilde{Y}, \tilde{\beta})) = 4 \operatorname{disc}(T(\tilde{Y})). \tag{3.2}$$

But, since $NS(\tilde{X}) = NS(\tilde{Y}) = E_8(2) \oplus U(2)$, $\operatorname{disc}(T(\tilde{X})) = \operatorname{disc}(T(\tilde{Y}))$. Thus by equation (3.2), $\operatorname{disc}(T(\tilde{X})) = 0$, a contradiction.

3.3 K3 cover of Picard number 11

The following lemma can be obtained by following the argument in (Prop. 7.3, [23]). However we will give the proof.

Lemma 3.3.1. *Let $(X, \alpha = \alpha_B)$ be a twisted K3 surface with $\rho(X) \geq 11$ and such that X is elliptically fibred with a section. Then there exists a K3 surface Z and a Fourier–Mukai equivalence $\Phi : \mathrm{D}^b(Z) \to \mathrm{D}^b(X, \alpha_B)$.*

Proof. Since $T(X,B) \subset T(X)$ is a sublattice, Theorem 1.12.4 in [33] (see Remark 3.3.2) yields a primitive embedding $T(X,B) \hookrightarrow \Lambda$. By the surjectivity of the period map, there is a K3 surface Z and a Hodge isometry $T(Z) \cong T(X,B)$. This induces an embedding $i : T(Z) \hookrightarrow T(X)$ such that it is the kernel of $\alpha : T(X) \to \mathbb{Z}/n\mathbb{Z}$, where n is the order of α in $\mathrm{Br}'(X)$. Let $\ell \in T(X)$ with $\alpha(\ell) = 1$ and $t \in T(Z)$ the element such that $i(t) = n.\ell$. By ([32], Prop. 6.6), there exists a compact, smooth, two-dimensional moduli space M of stable sheaves on Z such that $\phi : T(Z) \to T(M)$ maps t to an element divisible by n and such that $(1/n)\phi(t)$ generates the quotient $\mathrm{Coker}(\phi) = \mathbb{Z}/n\mathbb{Z}$. Căldăraru showed that the Brauer class defined by $\mathrm{Coker}(\phi) \cong \mathbb{Z}/n\mathbb{Z}$ is the obstruction for the existence of a universal sheaf. Moreover he showed the existence of a FM equivalence $\Phi_{\mathcal{P}} : \mathrm{D}^b(Z) \cong D^b(M, \beta^{-1})$ defined by the the $1 \boxtimes \beta^{-1}$-twisted universal sheaf. There is an isomorphism $\psi : T(X) \cong T(M)$ that sends $(1/n)i(t)$ to $(1/n)\phi(t)$ and yields the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T(Z) \arrow[r, "i"] \arrow[d, equal] & T(X) \arrow[r, "\alpha"] \arrow[d, "\psi"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T(Z) \arrow[r] & T(M) \arrow[r, "\beta^{-1}"] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

Since $T(X) \cong T(M)$, there exists an isomorphism $f : M \cong X$ such that $f^*|_{T(X)} = \pm\psi$, because the Néron–Severi group of every elliptic fibred K3 surface with a section contains the hyperbolic lattice. Besides from the commutativity of the diagram we get $f^*\alpha = \beta^{-1}$ and this gives the equivalence

$$\mathrm{D}^b(Z) \xrightarrow{\Phi} \mathrm{D}^b(M, \beta^{-1}) \xrightarrow{f_*} \mathrm{D}^b(X, \alpha)$$

□

Remark 3.3.2. Let us recall the result of Nikulin ([33], Thm. 1.12.4) used in the last lemma: Every even lattice T of signature (t_+, t_-) admits a primitive embedding into some even unimodular lattice Γ of signature (l_+, l_-) if $l_+ - l_- \equiv 0 \pmod 8$, $t_+ \leq l_+$, $t_- \leq l_-$ and

$$t_+ + t_- \leq \frac{1}{2}(l_+ + l_-).$$

If we take $(l_+, l_-) = (3, 19)$ then $\Gamma = E_8^{\oplus 2} \oplus U^{\oplus 3}$ and the conditions are immediately satisfied for $T = T(X,B)$.

The following proposition is due to Hisanori Ohashi ([36], Thm. 3.5).

Proposition 3.3.3. *Let X be a K3 surface with Picard number 11 covering an Enriques surface. Then the Néron–Severi lattice of X is one of the followings*

(1) $U(2) \oplus E_8(2) \oplus \langle -2N \rangle$*, where* $N \geq 2$*,*

(2) $U \oplus E_8(2) \oplus \langle -4M \rangle$ *where* $M \geq 1$*.*

Notation 5. A K3 surface X with Picard number 11 covering an Enriques surface is said to be of type T_1 if the Néron-Severi lattice of X is $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, where $N \geq 2$ and the K3 surface is said to be of type T_2 if the Néron–Severi lattice of X is $\mathrm{NS}(X) = U \oplus E_8(2) \oplus \langle -4M \rangle$ where $M \geq 1$.

Remark 3.3.4. If X is a K3 surface with $NS(X) = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, then we can assume that the involution acts as -1 on $\langle -2N \rangle$.

Note that Proposition 3.3.3 implies that $T(X)$ is one of the following

(1) $T(X) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $\mathrm{NS}(X) = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, where $N \geq 2$;

(2) $T(X) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $\mathrm{NS}(X) = U \oplus E_8(2) \oplus \langle -4M \rangle$, where $M \geq 1$.

Lemma 3.3.5. *Let X, Y be K3 surfaces covering Enriques surfaces with Picard number $\rho(X) = \rho(Y) = 11$ and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ an equivalence with $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$ in their respective Brauer groups. Then one of the following holds*

(1) $T(X) \cong T(Y)$ as lattices,

(2) $T(X) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$ and $T(Y) = U \oplus E_8(2) \oplus \langle 8N \rangle$, $N \geq 2$,

(3) $T(X) = U \oplus E_8(2) \oplus \langle 8N \rangle$ and $T(Y) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $N \geq 2$.

Proof. By Proposition 3.3.3, one of the following holds

(1) $T(X) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $T(Y) = U \oplus E_8(2) \oplus \langle 4N \rangle$, $M, N \geq 1$,

(2) $T(X) = U(2) \oplus E_8(2) \oplus \langle 2M \rangle$ and $T(Y) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $M, N \geq 2$,

(3) $T(X) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$ and $T(Y) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $N \geq 2$, $M \geq 1$,

(4) $T(X) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $T(Y) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $M \geq 1$, $N \geq 2$.

Since E_8 is unimodular and negative definite,

$$\mathrm{disc}(U(2) \oplus E_8(2) \oplus \langle 2m \rangle) = 2^{11} m,$$

$$\mathrm{disc}(U \oplus E_8(2) \oplus \langle 4m \rangle) = 2^{10} m.$$

Since Φ is an equivalence, there exists a Hodge isometry $T(X, \alpha) \cong T(Y, \beta)$ and this implies that $\mathrm{disc}\, T(X) = \mathrm{disc}\, T(Y)$ (by Remark 3.2.9). Thus, if either (1) or (2) holds, we have $M = N$, i.e, $T(X) \cong T(Y)$ as lattices. On the other hand, if (3) holds, one has

$$2^{11} N = \mathrm{disc}\, T(X) = \mathrm{disc}\, T(Y) = 2^{10} M,$$

i.e. $M = 2N$. Similarly if (4) holds, we also get that $M = 2N$. $\square$

Corollary 3.3.6. *Let X be a K3 surface that covers an Enriques surface. If $T(X) = U \oplus E_8(2) \oplus \langle 4M \rangle$ such that either $M \not\equiv 0 \mod 2$ or $M = 2$ and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is an equivalence of categories such that $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$. Then $T(X) \cong T(Y)$ as lattices.*

Proof. Suppose that $T(Y) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, for some $N \geq 2$. Thus, by the last lemma $M = 2N$. This contradicts the assumption $M \not\equiv 0 \mod 2$. □

Corollary 3.3.7. *Let X, Y be K3 surfaces that cover Enriques surfaces and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence with α of order n. If $T(Y) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $M \geq 1$. Then $M \equiv 0 \mod n^2$.*

Proof. It follows from the equivalence Φ that there is an isometry

$$T(X, \alpha) \cong T(Y) = U \oplus E_8(2) \oplus \langle 4M \rangle$$

for some $M \geq 1$. Suppose that α is non-trivial. Since X covers an Enriques surface, one of the following holds

(1) $T(X) = U \oplus E_8(2) \oplus \langle 4N \rangle$, $N \geq 1$,

(2) $T(X) = U(2) \oplus E_8(2) \oplus \langle 2N \rangle$, $N \geq 2$.

If we are in the second case,

$$2^{10}M = \operatorname{disc} T(X, \alpha) = \mathrm{ord}(\alpha)^2 \operatorname{disc} T(X) = \mathrm{ord}(\alpha)^2 2^{11} N.$$

Hence $M \equiv 0 \mod n^2$. On the other hand, in the first case we get

$$2^{10}M = \operatorname{disc} T(X, \alpha) = \mathrm{ord}(\alpha)^2 \operatorname{disc} T(X) = \mathrm{ord}(\alpha)^2 2^{10} N$$

and also $M \equiv 0 \mod n^2$. □

Remark 3.3.8. Let X be a K3 surface such that hyperbolic plane $U \hookrightarrow NS(X)$. Kondo proved in [27] that if the orthogonal of U in $NS(X)$ is a negative definite even lattice, there exists an elliptic fibration for X with a section. In particular, if the K3 surface X covers an Enriques surface and $\rho(X) = 11$ such that $NS(X) = U \oplus E_8(2) \oplus \langle -4M \rangle$, $M \geq 1$ (See Example 2.1.5). Then X has an elliptic fibration with a section.

Lemma 3.3.9. *Let X be an Enriques surface. Let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an equivalence such that $T(\tilde{Y}) = U \oplus E_8(2) \oplus \langle 4M \rangle$, $2 \nmid M$ and $\pi_Y^* \beta = 1$. Then $\tilde{X} \cong \tilde{Y}$*

Proof. Since $\mathrm{ord}(\pi_Y^* \alpha)$ is either 1 or 2, then by Corollary 3.3.7, $\pi_X^* \alpha = 1$ in $\mathrm{Br}(\tilde{X})$. Thus, there exists an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}) \to \mathrm{D}^b(\tilde{Y})$. Since $\tilde{Y}$ is elliptically fibred with a section (by Remark 3.3.8), the number of Fourier–Mukai partners of $\tilde{Y}$ is 1 (by Cor. 2.7 in [20]), i.e. $\tilde{X} \cong \tilde{Y}$. □

Let us recall a proposition proved by Ohashi in [36] under the notation $2N = 2^e p_1^{e_1} ... p_l^{e_l}$ and $4M = 2^e p_1^{e_1} ... p_l^{e_l}$ in Proposition 3.3.3.

Proposition 3.3.10 (Ohashi). *The number of Enriques quotient for a K3 surface X, i.e. $\{Y \mid Y$ Enriques surface, $\exists X \twoheadrightarrow Y\}/\simeq$, does not exceed the number*

$$B_0 := \begin{cases} 2^{l-1} & \text{if } X \text{ is of type } T_1 \text{ and } e = 1 \\ (2^5+1).2^{l+4} & \text{if } X \text{ is of type } T_1 \text{ and } e = 2 \\ 2^{l+10} & \text{if } X \text{ is of type } T_1 \text{ and } e \geq 3 \\ 1 & \text{if } X \text{ is of type } T_2 \text{ and } e = 2, l = 0 \\ 2^{l-1} & \text{if } X \text{ is of type } T_2 \text{ and } e = 2, l > 0 \\ 2^{2l+5} & \text{if } X \text{ is of type } T_2 \text{ and } e \geq 3 \end{cases}$$

Proposition 3.3.11. *Let X and Y be Enriques surfaces such that the K3 covering $\tilde{Y}$ of Y has transcendental lattice $T(\tilde{Y}) = U(2) \oplus E_8(2) \oplus \langle 4M \rangle$ and such that $M \not\equiv 0(mod\, 4)$. If $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ is an equivalence. Then one of the following holds*

(1) $T(\tilde{X}) = U \oplus E_8(2) \oplus \langle 4M \rangle$;

(2) $X \cong Y$.

Proof. The equivalence Φ induces a Hodge isometry $T(\tilde{X}, \tilde{\alpha}) \cong T(\tilde{Y})$. If $\pi_X^*(\alpha) = 1$ in $\mathrm{Br}'(\tilde{X})$, then $T(\tilde{X}) \cong T(\tilde{Y})$. By Proposition 2.5.13, $\alpha = 1$ in $\mathrm{Br}'(X)$. Hence $X \cong Y$ (by Prop. 6.1. in [5]). Suppose that $\pi_X^*(\alpha)$ has order 2. Since $T(\tilde{X}, \tilde{\alpha}) \cong T(\tilde{Y})$, one of the following holds

(1) $T(\tilde{X}) = U(2) \oplus E_8(2) \oplus \langle M \rangle$ and $M \equiv 0(mod\, 2)$;

(2) $T(\tilde{X}) = U \oplus E_8(2) \oplus \langle 4M \rangle$.

Since $M \not\equiv 0(mod\, 4)$, by Proposition 2.5.13, the first one option above does not hold, i.e. $T(\tilde{X}) = U \oplus E_8(2) \oplus \langle 4M \rangle$. □

Example 3.3.12. Ohashi gave an explicit example in [35] of a K3 surface with only one Enriques quotient. Let us recall it here. Let $(x_0 : x_1, y_0 : y_1)$ be the homogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$ and $i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ the involution defined by $i(x_0 : x_1, y_0 : y_1) = (x_1 : x_0, y_1 : y_0)$. Consider the linear system L consisting of divisors D of bidegree $(4,4)$ such that:

(a) the bihomogeneous equation of D is invariant under i,

(b) D has multiplicities at least 2 at both $(0:1,1:0)$ and $(1:0,0:1)$.

The linear system L is given by the divisors of bidegree $(4,4)$:

$$a_0x_0^2x_1^2y_0^2y_1^2 + a_1(x_0^2x_1^3y_0^2y_1^2 + x_0^3x_1y_0^2y_1^2) + a_2(x_1^4y_0^2y_1^2 + x_0^4y_0^2y_1^2) + a_3(x_0^3x_1y_0y_1^3 + x_0x_1^3y_0^3y_1) + a_4(x_0^2x_1^2y_0y_1^3 + x_0^2x_1^2y_0^3y_1) + a_5(x_0x_1^3y_0y_1^3 + x_0^3x_1y_0^3y_1) + a_6(x_1^4y_0y_1^3 + x_0^4y_0^3y_1) + a_7(x_0^2x_1^2y_1^4 + x_0^2x_1^2y_0^4) + a_8(x_0x_1^3y_1^4 + x_0^3x_1y_0^4) + a_9(x_1^4y_1^4 + x_0^4y_0^4).$$

The general member of L has exactly two ordinary nodes at $(0:1,1:0)$

and $(1:0,0:1)$ as singularities and does not contain the four fixed points $(1:\pm 1, 1:\pm 1)$. The double covering $\hat{X}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ along D is a K3 surface with two nodes and one of the liftings of i, say τ, is a free involution of $\hat{X}$. Thus $\hat{Y} = \hat{X}/\tau$ defines a family of Enriques surfaces with one node. Denote by X and Y the minimal desingularizations of $\hat{X}$ and $\hat{Y}$ respectively. The K3 surface is such that $\mathrm{Pic}(X) \cong U \oplus E_8(2) \oplus \langle -4 \rangle$ and only covers one Enriques surface. We sketch this fact now. First, we note that if X and X' are isomorphic as quasi-polarized varieties (a quasi polarization is a nef line bundle), then the isomorphism is induced by an element $\phi \in \mathrm{Aut}(\mathbb{P}^3)$ that preserves the defining quadratic equation of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and takes D to D' (i.e. ϕ stabilizes L). It can be showed that the stabilizer is $G = \langle i, \sigma \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$, where $\sigma : (x_0 : x_1, y_0 : y_1) \mapsto (y_0 : y_1, x_0 : x_1)$. Then the family has dimension $10 - (2-1) = 9$, and then a general member X has Picard number 11. Hence, either $\mathrm{Pic}(X) = U(2) \oplus E_8(2) \oplus \langle -2N \rangle$, $N \geq 2$ or $\mathrm{Pic}(X) = U \oplus E_8(2) \oplus \langle -4M \rangle$, $M \geq 1$. Let M (resp. K) be the invariant (resp. antinvariant) part of the action of τ on $\mathrm{Pic}(X)$ and let E_1, E_2 the two (-2)-curves on X arising from two nodes on $\hat{X}$. Since the involution τ interchanges E_1 and E_2, then $E_1 + E_2 \in M$ and $E_1 - E_2 \in K$. Thus, $K \cong \langle -4 \rangle$. Since $\mathcal{O}_X(E_1)$ is a line bundle, $[\mathrm{Pic}(X) : M \oplus K] = 2$. This implies that $\mathrm{Pic}(X) = U \oplus E_8(2) \oplus \langle -4 \rangle$.

Now, let us see that this K3 surface $(X, 1)$ has only one FM–partner. Let (Y, α) be a K3 surface such that $\Phi : \mathrm{D}^b(Y, \alpha) \to \mathrm{D}^b(X)$ is an equivalence. By Corollary 3.3.7, $\alpha = 1$ in $\mathrm{Br}'(Y)$. Hence we obtain an untwisted equivalence $\Phi : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ and then $Y \cong X$ because X is elliptically fibred with a section by Remark 3.3.8, i.e. we have the following proposition.

Proposition 3.3.13. *Let $(X, 1)$ be as in the example. If (Y, α) is a twisted FM–partner of $(X, 1)$ such that Y covers an Enriques surface, then $Y \cong X$ and $\alpha = 1$ in $\mathrm{Br}'(Y)$.*

Corollary 3.3.14. *Let $(Y, 1)$ be the twisted Enriques surface given in the example and (X, α) a twisted variety. If $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ is an equivalence, then $X \cong Y$.*

Proof. Since Φ is an equivalence, there exists an equivalence $\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \tilde{\alpha}) \to \mathrm{D}^b(\tilde{Y})$. By Proposition 3.3.13, $\tilde{\alpha} = 1$ and $\tilde{X} \cong \tilde{Y}$. Since the number of Enriques quotients of $\tilde{Y}$ is 1 (Proposition 3.3.10), $X \cong Y$. □

Lemma 3.3.15. *Let X and Y be Enriques surfaces. Suppose $\rho(\tilde{Y}) \geq 12$ or that $\tilde{Y}$ is elliptically fibred with a section and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ is an equivalence. Then $\tilde{X} \cong \tilde{Y}$ if and only if $\pi_X^* \alpha = 0$.*

Proof. Assume $\pi_X^* \alpha = 0$. The equivalence Φ lifts to an equivalence

$$\tilde{\Phi} : \mathrm{D}^b(\tilde{X}, \pi_X^* \alpha) \cong \mathrm{D}^b(\tilde{Y}).$$

Hence there is an equivalence $\Psi : \mathrm{D}^b(\tilde{X}) \cong \mathrm{D}^b(\tilde{Y})$ and so $\tilde{Y} \cong \tilde{X}$ because either $\rho(\tilde{Y}) \geq 12$ (and Theorem 3.2.5) or $\tilde{Y}$ is elliptically fibred with a section (and

Cor. 2.7 in [20]). On the other hand, if $f : \tilde{Y} \to \tilde{X}$ is an isomorphism, we get an equivalence $\Psi := \tilde{\Phi} \circ f_* : \mathrm{D}^b(\tilde{Y}, f^*\pi_X^*\alpha) \to \mathrm{D}^b(\tilde{Y})$ where $\tilde{\Phi}$ is the lift of Φ and f_* is the equivalence induced by the isomorphism f. Hence from the equivalence Ψ, we get by Remark 3.2.9 $f^*\pi_X^*\alpha = 0$ in $\mathrm{Br}'(\tilde{Y})$ and then $\pi_X^*\alpha = 0$. □

3.4 Supersingular K3 surfaces

Let X be a supersingular K3 surface, i.e. X is a K3 surface with Picard number $\rho(X) = 20$. Thus, the transcendental lattice of X given by its intersection matrix is

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix} \tag{3.4}$$

with respect to some basis $\{e_1, e_2\}$, where $a, b > 0$ and $4ab - c^2 > 0$. Keum gave a criterion to know when a K3 surface covers an Enriques surface, which later was improved by Ohashi in ([35], Theorem 1.2) where he proved:

Theorem 3.4.1 (Ohashi). *Let X be an algebraic K3 surface. Then the following are equivalent*

(1) X admits a fixed-point-free involution.

(2) There exists a primitive embedding of $T(X)$ into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$such that the orthogonal complement of $T(X)$ in Λ^- contains no vectors of self-intersection -2.

By using this criterion, Sertöz found in [41] explicit conditions to know when a supersingular K3 surface covers an Enriques surface in terms of the entries of the intersection matrix of its transcendental lattice.

Theorem 3.4.2. *If X is a supersingular K3 surface with transcendental lattice given as in (3.4), then X covers an Enriques surface if and only if one of the following conditions holds:*

(1) a, b, and c are even.

(2) c is odd and ab is even.

(3) c is even, a or b is odd. The form $ax^2 + cxy + by^2$does not represent 1.

(4) c is even, a or b is odd. The form $ax^2 + cxy + by^2$represents 1, and $4ab - c^2 \neq 4, 8, 16$.

Let X be a K3 surface with its transcendental $T(X)$ generated by e_1, e_2 and its corresponding matrix given by

$$\begin{pmatrix} (e_1.e_1) & (e_1.e_2) \\ (e_1.e_2) & (e_2.e_2) \end{pmatrix}$$

such that $e_1^2 > 0, e_2^2 > 0$. We show that there are only three cases of sublattices of degree 2 for the lattice $T(X)$. Before going on, we write $(x, y) \in T(X)$ to denote $xe_1 + ye_2$ and $x, y \in \mathbb{Z}$. The possible sublattices of index two are

(1) $T_1 := \bigcup_{n\in\mathbb{Z}}\{(x,y)\in T(X)|y=-x+2n\}$,

(2) $T_2 := \bigcup_{n\in\mathbb{Z}}\{(x,y)\in T(X)|2y=-x+2n\}, (2\mathbb{Z}\times\mathbb{Z}\hookrightarrow\mathbb{Z}\times\mathbb{Z})$,

(3) $T_3 := \bigcup_{n\in\mathbb{Z}}\{(x,y)\in T(X)|y=-2x+2n\}, (\mathbb{Z}\times 2\mathbb{Z}\hookrightarrow\mathbb{Z}\times\mathbb{Z})$.

Let us see now that all the sublattices of degree 2 of $T(X)$ are the described above. All the points of the form $2ne_1+2me_2$ belong to all the sublattices of degree 2. Now if there is a point in the sublattice T of the form $(2k+1)e_1+2ne_2$ in the lattice, then the element $(1,0)$ also is in the lattice T, and so T consist of all the elements of the form $ne_1+(2m)e_2$. Hence $T=T_3$.

If there is an element of the form $2me_1+(2k+1)e_2$ in the sublattice, the element $(0,1)$ is also in the lattice, and then the sublattice consists of all the points of the form $2ne_1+me_2$ for all integers m,n. Hence $T=T_2$.

Finally, the last possibility is that there is an element $(2k_1+1)e_1+(2k_2+1)e_2$ in the sublattice. Thus the element $(1,1)$ is in the sublattice, and this gives the lattice T_1 that consists of all the points in

$$\bigcup_{k\in\mathbb{Z}}\{me_1+ne_2\in T(X)|n+m=2k, m,n\in\mathbb{Z}\}.$$

Now, we will find the set of generators for all the lattices. We treat first with the lattice T_3. Suppose that $\{(a,2b),\,(c,2d)\}$ is a basis for the lattice T_3. Since $(1,0)\in T_3$, there exist integers m,n such that

$$ma+nc=1, \tag{3.5}$$

$$2mb+2nd=0.$$

Thus

$$m=\frac{-d}{bc-ad},\ n=\frac{b}{bc-ad}.$$

Since $(0,2)\in T_3$, there exists integers s,t such that

$$sa+tc=0, \tag{3.6}$$

$$2sb+2td=2. \tag{3.7}$$

Notation 6. Let $k,l\in\mathbb{Z}$. We define $gcd(k,l)$ to be the greater commun divisor between k and l in the case that both k and l are nonzero and $k+l$ in other case.

From the equation (3.5), $gcd(a,c)=1$. Hence from the equations (3.6) and (3.7), $s=\pm c, t=\mp a$. Then, $bc-ad=\pm1$. And so, $m=\mp d$, $n=\pm b$.

Thus, we have showed that $\{(a,2b),\,(c,2d)\}$ is a basis for T_3 if and only if $bc-ad=\pm1$. Similarly, $\{(2a,b),(2c,d)\}$ is a basis for T_2 if and only if

$ad - bc = \pm 1$. Now, suppose that $\{(a,b),(c,d)\}$ is a basis for the lattice T_1. Since $(1,1) \in T_1$, there exists integers m, n such that

$$\begin{aligned} ma + nc &= 1, \\ mb + nd &= 1. \end{aligned} \tag{3.8}$$

Thus,

$$m = \frac{d-c}{ad-bc}, n = \frac{a-b}{ad-bc}.$$

Since $(2,0) \in T_1$, there exists integers s, t such that

$$sa + tc = 2, \tag{3.9}$$

$$sb + td = 0. \tag{3.10}$$

By equation (3.8), $gcd(b,d) = 1$. Thus by equation (3.10), $s = \pm \ell d, t = \mp \ell b$ for some integer $\ell > 0$. Hence $\ell(ad - bc) = \pm 2$ and so $\ell = 1$ or $\ell = 2$. If $\ell = 2$, then without loss of generality ad is even and bc is odd (because $ad - bc = \pm 1$). Thus, since $(a,b),(c,d)$ are in T_1 and b, c are odd numbers, we conclude that a and d are also odd numbers, a contradiction. Thus $ad - bc = \pm 2$ and

$$m = \frac{d-c}{\pm 2}, n = \frac{a-b}{\pm 2}.$$

This implies that both d and c are either odd or even and that both a and b are either odd or even.

Thus we have showed that $\{(a,b),(c,d)\}$ is a basis for T_1 if and only if $gcd(a,c) = gcd(b,d) = 1$, $ad - bc = \pm 2$ and both of a, b are either odd or even and both of c, d are either odd or even (Note that we have explained only one direction of the implication but the other is completely clear). Summarizing, we have the following result:

Lemma 3.4.3. *Let X be a K3 surface with transcendental lattice $T(X)$ and basis $\{e_1, e_2\}$. Let $S \subset T(X)$ be a sublattice of index 2. Then*

(1) If S is of type T_1, $\{ae_1 + be_2, ce_1 + de_2\}$ is a basis of S if and only if $gcd(a,c) = gcd(b,d) = 1$, $ad - bc = \pm 2$ and $2|(a-b)$, $2|(c-d)$.

(2) If S is of type T_2, $\{2ae_1 + be_2, 2ce_1 + de_2\}$ is a basis of S if and only if $ad - bc = \pm 1$,

(3) If S is of type T_3, $\{ae_1 + 2be_2, ce_1 + 2de_2\}$ is a basis of S is and only if $ad - bc = \pm 1$.

Proposition 3.4.4. *Let X and Y be K3 covers of Enriques surfaces such that the intersection matrices of $T(X)$ and $T(Y)$ are given by*

$$\begin{pmatrix} 2m & k \\ k & 2n \end{pmatrix}, \begin{pmatrix} 2s & r \\ r & 2t \end{pmatrix}$$

respectively and such that s, t, m, n *are positive numbers,* k, r *are even numbers and* $2|(s-m)$, $2|(t-n)$ *and* $4 \nmid s$, $4 \nmid t$ *and one and only one of* s, t *is an odd number. Let* $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ *be an equivalence such that* $\mathrm{ord}(\alpha) \leq 2$. *Then* $\alpha = 1$ *in* $\mathrm{Br}'(X)$ *and* $X \cong Y$.

Proof. Let $\{e_1, e_2\}$ and $\{f_1, f_2\}$ be basis of $T(X)$ and $T(Y)$ respectively such that $e_1^2 = 2m, e_2^2 = 2n, (e_1.e_2) = k$ and $f_1^2 = 2s, f_2^2 = 2t, (f_1.f_2) = r$. From the eqivalence Φ we get an isometry $T_\Phi : T(X, \alpha) \to T(Y)$. If α is non-trivial, $T(X, \alpha)$ is a sublattice of index 2 in $T(X)$. Suppose $T(X, \alpha)$ is of type T_1. Let a, b, c, d be integers such that

$$T^{-1}(f_1) = ae_1 + be_2,$$

$$T^{-1}(f_2) = ce_1 + de_2.$$

Thus,

$$\begin{aligned}(T^{-1}(f_1).T^{-1}(f_1)) &= a^2e_1^2 + b^2e_2^2 + 2ab(e_1.e_2)\\ &= 2ma^2 + 2nb^2 + 2abk\\ &= 2s,\end{aligned}$$

and

$$\begin{aligned}(T^{-1}(f_2).T^{-1}(f_2)) &= c^2e_1^2 + d^2e_2^2 + 2cd(e_1.e_2)\\ &= 2mc^2 + 2nd^2 + 2cdk\\ &= 2t.\end{aligned}$$

Since $2|(s-m)$ and $2|(t-n)$, s is even if and only if m is even and t is even if and only if t is even. Since $T(X, \alpha)$ is of type T_1, by Lemma 3.4.3(1) we obtain that $2|(a-b)$, $2|(c-d)$ and $gcd(a, c) = gcd(b, d) = 1$, i.e. we have that either a, b, c, d are all odd integers or a, b are even and c, d are odd or a, b are odd and c, d are even.

Case 1 s, m are even numbers and t, n are odd numbers.

If a, b, c, d are odd numbers, $ma^2 + nb^2 + abk (= s)$ is an odd number. This contradicts our assumption that s is an even number. Now, if a, b are even numbers and c, d are odd numbers, $ma^2 + nb^2 + abk (= s)$ is an even number and $4|s$, a contradiction. Finally, if a, b are odd numbers and c, d are even numbers, $mc^2 + nd^2 + cdk (= t)$ is an even number. This contradicts our assumption that t is an odd number.

Case 2 s, m are odd numbers and t, n are even numbers.

If a, b, c, d are odd numbers, $mc^2 + nd^2 + cdk (= t)$ is an odd number. This contradicts our assumption that t is even number. Now, if a, b are even numbers and c, d are odd numbers, $mc^2 + nd^2 + cdk (= t)$ is an odd number. This contradicts our assumption that t is an even number. Finally, if a, b are odd numbers and c, d are even numbers, $mc^2 + nd^2 + cdk(= t)$ is an even number and $4|t$. A contradiction.

Now, let us suppose that the sublattice $T(X, \alpha)$ is of type T_3. Let a, b, c, d be integers such that

$$T^{-1}(f_1) = ae_1 + 2be_2,$$
$$T^{-1}(f_2) = ce_1 + 2de_2.$$

Then

$$\begin{aligned}(T^{-1}(f_1).T^{-1}(f_1)) &= a^2e_1^2 + 4b^2e_2^2 + 4ab(e_1.e_2)\\ &= 2ma^2 + 8nb^2 + 4abk\\ &= 2s,\end{aligned}$$

and

$$\begin{aligned}(T^{-1}(f_2).T^{-1}(f_2)) &= c^2e_1^2 + 4d^2e_2^2 + 4cd(e_1.e_2)\\ &= 2mc^2 + 8nd^2 + 4cdk\\ &= 2t,\end{aligned}$$

If a is even, $4|s$ and if c is even, $4|t$. In both cases we get a a contradiction. Thus, we can assume that a, c are odd numbers. If m is an odd (even) number, $ma^2+4nb^2+2abk(= s)$ and $mc^2+4nd^2+2cdk(= t)$ are odd (even) numbers. This contradicts our assumption that one and only one of s and t is an odd number. Finally, if $T(X, \alpha)$ is a sublattice of type T_2, we get also a contradiction by following a similar argument as in the case that $T(X, \alpha)$ was of type T_3. $\square$

Proposition 3.4.5. *Let X and Y be K3 covers of Enriques surfaces such that the intersection matrixes of $T(X)$ and $T(Y)$ are given by*

$$\begin{pmatrix} 2m & k \\ k & 2n \end{pmatrix}, \begin{pmatrix} 2s & r \\ r & 2t \end{pmatrix}$$

respectively and such that s, t, m, n are positive numbers, k, r are odd numbers and $2|(s - m)$, $2|(t - n)$ and $4 \nmid s$, $4 \nmid t$ and one and only one of s, t is an odd number. Let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence such that $\mathrm{ord}(\alpha) \leq 2$. *Then $\alpha = 1$ in $\mathrm{Br}'(X)$ and $X \cong Y$.*

Proof. Let $\{e_1, e_2\}$ and $\{f_1, f_2\}$ be basis of $T(X)$ and $T(Y)$ respectively such that $e_1^2 = 2m, e_2^2 = 2n, (e_1.e_2) = k$ and $f_1^2 = 2s, f_2^2 = 2t, (f_1.f_2) = r$. From the eqivalence Φ we get an isometry $T_\Phi : T(X, \alpha) \to T(Y)$. If α is non-trivial, $T(X, \alpha)$ is a sublattice of index 2 in $T(X)$. Suppose $T(X, \alpha)$ is of type T_1. Let a, b, c, d be integers such that

$$T^{-1}(f_1) = ae_1 + be_2,$$

$$T^{-1}(f_2) = ce_1 + de_2.$$

Thus,

$$\begin{aligned}(T^{-1}(f_1).T^{-1}(f_1)) &= a^2e_1^2 + b^2e_2^2 + 2ab(e_1.e_2)\\ &= 2ma^2 + 2nb^2 + 2abk\\ &= 2s,\end{aligned}$$

and

$$\begin{aligned}(T^{-1}(f_2).T^{-1}(f_2)) &= c^2e_1^2 + d^2e_2^2 + 2cd(e_1.e_2)\\ &= 2mc^2 + 2nd^2 + 2cdk\\ &= 2t.\end{aligned}$$

Since $2|(s-m)$ and $2|(t-n)$, s is even if and only if m is even and t is even if and only if n is even. Since $T(X,\alpha)$ is of type T_1, by Lemma 3.4.3(1) we obtain that $2|(a-b)$, $2|(c-d)$ and $gcd(a,c) = gcd(b,d) = 1$, i.e. we have that either a,b,c,d are all odd integers or a,b are even and c,d are odd or a,b are odd and c,d are even.

Case 1 s,m are even and t,n are odd.

If a,b,c,d are odd numbers, $mc^2 + nd^2 + cdk(= t)$ is an even number. This contradicts our assumption that t is odd. If a,b are even and c,d are odd numbers, $mc^2 + nd^2 + cdk(= t)$ is an even number. This contradicts our assumption that t is an odd number. If a,b are odd and c,d are even numbers, $mc^2 + nd^2 + cdk(= t)$ is an even number. This contradicts our assumption that t is odd.

Case 2 s,m are odd numbers and t,n are even numbers.

If a,b,c,d are odd numbers, $ma^2 + nb^2 + bak(= s)$ is an even number. This contradicts our assumption that s is an odd number. If a,b are even numbers and c,d are odd numbers, $ma^2 + nb^2 + bak(= s)$ is an even number. This contradicts our assumption that s is an odd number. If a,b are odd numbers and c,d are even numbers, $ma^2 + nb^2 + bak(= s)$ is an even number. This also contradicts our assumption that s is an odd number.
Now, we suppose that the sublattice $T(X,\alpha)$ is of type T_3. Let a,b,c,d be integers such that

$$\begin{aligned}T^{-1}(f_1) &= ae_1 + 2be_2,\\ T^{-1}(f_2) &= ce_1 + 2de_2.\end{aligned}$$

Then

$$\begin{aligned}(T^{-1}(f_1).T^{-1}(f_1)) &= a^2e_1^2 + 4b^2e_2^2 + 4ab(e_1.e_2)\\ &= 2ma^2 + 8nb^2 + 4abk\\ &= 2s,\end{aligned}$$

and

$$\begin{aligned}(T^{-1}(f_2).T^{-1}(f_2)) &= c^2e_1^2 + 4d^2e_2^2 + 4cd(e_1.e_2)\\ &= 2mc^2 + 8nd^2 + 4cdk\\ &= 2t.\end{aligned}$$

If s, m are even and t, n are odd numbers, $mc^2 + 4nd^2 + 2cdk (= t)$ is an even number. This contradicts our assumption that t is an odd number. Now, we suppose that s, m are odd and t, n are even numbers. Since $ma^2+4nb^2+2abk = s$ and $mc^2+4nd^2+2cdk = t$, a is odd and c is even. Hence $4|t$, a contradiction. $\square$

Lemma 3.4.6. *Let X and Y be K3 covers of Enriques surfaces such that the intersection matrixes of $T(X)$ and $T(Y)$ are given by*

$$\begin{pmatrix} 2m & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2s & r \\ r & 2t \end{pmatrix}$$

such that $r > 0$, $4 \nmid r$, $2|(s-m)$ and $2|(t-n)$. If $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ and $\mathrm{ord}(\alpha) \leq 2$ and one and only one of s, t is an odd number. Then $\alpha = 1$ in $\mathrm{Br}'(X)$ and $X \cong Y$.

Proof. Let $\{e_1, e_2\}$ and $\{f_1, f_2\}$ be basis of $T(X)$ and $T(Y)$ respectively such that $e_1^2 = 2m, e_2^2 = 2n, (e_1.e_2) = 0$ and $f_1^2 = 2s, f_2^2 = 2t, (f_1.f_2) = r$. From the equivalence Φ we get an isometry $T_\Phi : T(X, \alpha) \to T(Y)$. If α is non-trivial, $T(X, \alpha)$ is a sublattice of index 2 in $T(X)$. Suppose $T(X, \alpha)$ is of type T_1. Let a, b, c, d be integers such that

$$T^{-1}(f_1) = ae_1 + be_2,$$

$$T^{-1}(f_2) = ce_1 + de_2.$$

Thus,

$$\begin{aligned}(T^{-1}(f_1).T^{-1}(f_1)) &= a^2e_1^2 + b^2e_2^2 + 2ab(e_1, e_2)\\ &= 2ma^2 + 2nb^2 + 2abk\\ &= 2s,\end{aligned}$$

and

$$\begin{aligned}(T^{-1}(f_2).T^{-1}(f_2)) &= c^2e_1^2 + d^2e_2^2 + 2cd(e_1, e_2)\\ &= 2mc^2 + 2nd^2 + 2cdk\\ &= 2t.\end{aligned}$$

Case 1 s, m are even and t, n are odd numbers.

Since $ma^2 + nb^2 = s$, b is an even number. Hence by 3.4.3(1), a, b are even and c, d are odd numbers. Since $r > 0$ and

$$r = (T^{-1}(f_1).T^{-1}(f_2)) = ace_1^2 + bde_2^2 = 2mac + 2nbd,$$

then $4|r$, a contradiction.

Case 2 s, m are odd and t, n are even numbers.

Since $mc^2 + nd^2 = t$, c is an even number. Thus, by 3.4.3(1), d is also an even number. Hence, $4|r$, a contradiction.

Now, we suppose that the lattice $T(X, \alpha)$ is of type T_3 and we define the integers a, b, c, d as we did in the last proposition when we considered the lattice T_3. If m is an even number, $4|r$ because $2acm + 8bdn = r$, a contradiction. Thus m is an odd number, and so s is odd and t, n are even numbers, because $2|(m - s)$ and the assumption on s, t. Since $mc^2 + 4nd^2 = t$, c is an even number and then $4|r$ because $2acm + 8bdn = r$. □

Lemma 3.4.7. *Let X and Y be K3 covers of Enriques surfaces such that the intersection matrixes of $T(X)$ and $T(Y)$ are given by*

$$\begin{pmatrix} 2m & 0 \\ 0 & 2n \end{pmatrix}, \begin{pmatrix} 2s & r \\ r & 2s \end{pmatrix}$$

If $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ and $\mathrm{ord}(\alpha) = 2$. Then either $s = m + n$ or $s = 4n, m = 3n$ or $n = 3m, s = 4m$ or $m = s = 4n$ or $n = s = 4m$.

Proof. Let $\{e_1, e_2\}$ and $\{f_1.f_2\}$ be basis of $T(X)$ and $T(Y)$ respectively such that $e_1^2 = 2m, e_2^2 = 2n, (e_1.e_2) = 0$ and $f_1^2 = 2s, f_2^2 = 2s, (f_1.f_2) = r$. From the eqivalence Φ we get an isometry $T_\Phi : T(X, \alpha) \to T(Y)$. Since α is non-trivial, $T(X, \alpha)$ is a sublattice of index 2 in $T(X)$. Suppose $T(X, \alpha)$ is of type T_1. Let a, b, c, d be integers such that

$$T_\Phi^{-1}(f_1) = ae_1 + be_2,$$

$$T_\Phi^{-1}(f_2) = ce_1 + de_2.$$

Thus,

$$\begin{aligned} 2s &= (T_\Phi^{-1}(f_1).T_\Phi^{-1}(f_1)) \\ &= a^2e_1^2 + b^2e_2^2 + 2ab(e_1.e_2) \\ &= 2ma^2 + 2nb^2, \end{aligned}$$

and

$$\begin{aligned} 2s &= (T_\Phi^{-1}(f_2).T_\Phi^{-1}(f_2)) \\ &= c^2e_1^2 + d^2e_2^2 + 2cd(e_1.e_2) \\ &= 2mc^2 + 2nd^2. \end{aligned}$$

Case 1 $abcd = 0$.

Suppose $a = 0$. By Lemma 3.4.3(1), $ad - bc = \pm 2$, $2|(a-b)$ and $gcd(a,c) = 1$. Thus, $b = \pm 2$ and $c = \pm 1$ because $gcd(a,c) = 1$. By replacing this values above, we obtain $8n = 2s = 2m + 2nd^2$ and since $m > 0, n > 0$, then $d = \pm 1$ (because $2n(4-d^2) = 2m$ implies $d \in (-2,2)$ and by Lemma 3.4.3(1), $gcd(b,d) = 1$ which implies that $d \neq 0$ because $b = \pm 2$). Hence $m = 3n$, $s = 4n$. Similarly, if any of b, c, d is 0, then either $m = 3n$, $s = 4n$ or $n = 3m$, $s = 4m$.

Case 2 $abcd \neq 0$.

Let us first study the case $a = \pm c$. By Lemma 3.4.3(1), $gcd(a,c) = 1$ and $ad - bc = \pm 2$. Thus, $|a| = |c| = 1$. By replacing these values in the equations above we get $m + nb^2 = s = m + nd^2$ and so $n(b^2 - d^2) = 0$. Thus $b = \pm d$ and from $ad - bc = \pm 2$ we obtain $|a| = |b| = |c| = |d| = 1$. Hence $s = ma^2 + nb^2 = m + n$. Now, we show that if $|a| > |c|$, then $b^2 > d^2$. If exactly three elements in $\{a,b,c,d\}$ have the same sign, from $ad - bc = \pm 2$ we obtain that $|a| = |b| = |c| = |d| = 1$. This case was already studied and we obtained $s = m + n$. Thus, we may assume for the terms in $\{a,b,c,d\}$ that either all of them are positive or all of them are negative or only two of them are positive. We may assume that all a,b,c,d are positive because if $a < 0, d < 0$, then $(-a)(-d) - bc = \pm 2$, or if $a < 0, b < 0$, $(-a)d - (-b)c = \pm 2$ (the other cases are similar). Without loss of generality, $a > c$. If $b < d$, $ad \geq (c+1)(b+1) = bc + b + c + 1$ and so $ad - bc \geq b + c + 1 \geq 3$, a contradiction. Hence $b \geq d$ and we have proved our statement. Now, since $a^2m + b^2n = s = c^2m + d^2n$, then $(a^2 - c^2)m + (b^2 - d^2)n = 0$ which is a contradiction because $m > 0, n > 0, a > b, b \geq d$.

Now, we suppose that the sublattice $T(X,\alpha)$ is of type T_3. Let a,b,c,d be integers such that

$$T_\Phi^{-1}(f_1) = ae_1 + 2be_2,$$
$$T_\Phi^{-1}(f_2) = ce_1 + 2de_2.$$

Hence

$$\begin{aligned} 2s &= (T_\Phi^{-1}(f_1).T_\Phi^{-1}(f_1)) \\ &= a^2e_1^2 + 4b^2e_2^2 + 2ab(e_1,e_2) \\ &= 2ma^2 + 8nb^2, \end{aligned}$$

and

$$\begin{aligned} 2s &= (T_\Phi^{-1}(f_2).T_\Phi^{-1}(f_2)) \\ &= c^2e_1^2 + 4d^2e_2^2 + 4cd(e_1,e_2) \\ &= 2mc^2 + 8nd^2. \end{aligned}$$

Thus,

$$ma^2 + 4nb^2 = s = mc^2 + 4nd^2 \tag{3.11}$$

and by Lemma 3.4.3(2), $ad - bc = \pm 1$.

Case $ad = 0$ or $bc = 0$.

Assume $ad = 0$. Since $ad - bc = \pm 1$, then $b = \pm 1$, $c = \pm 1$. If $d \neq 0$ $(a = 0)$, we obtain from equation (3.11) that $m + 4n(d^2 - 1) = 0$. This is a contradiction because $m, n > 0$ and $d \neq 0$. On the other hand, if $a \neq 0$ $(d = 0)$, we obtain $m(a^2 - 1) + 4nb^2 = 0$. This is also a contradiction because $m, n > 0$ and $a \neq 0$. Thus $a = d = 0$ and we obtain from equation (3.11) that $4n = s = m$. Similarly, we also get a contradiction if $bc = 0$ and one of b, c is nonzero. If $b = c = 0$ we get $m = s = 4n$.

Now, we may assume $abcd \neq 0$. Let us see that $a \neq \pm c$. Otherwise, if $a = \pm c$, then $c(\pm d - b) = ad - bc = \pm 1$ and so $\pm d - b = \pm 1$ (the signs are not necessarily in the respective order). By equation (3.11), $4n(d^2 - b^2) = 0$, a contradiction. Similarly, we can also check that $c \neq \pm d$. As in Case 2, we can suppose that all a, b, c, d are positive and we may assume that $a > c$ to prove $b \geq d$ (the stament is similar as in Case 2). Let us suppose $b < d$. Hence, $ad \geq (c+1)(b+1) = cb + c + b + 1$ and so $ad - bc \geq c + b + 1 \geq 3$, a contradiction. This shows that if a, b, c, d are all nonzero and $|a| > |c| > 0$, then $|b| > |d|$ (similarly if $|c| > |a| > 0$, then $|d| > |b| > 0$). But, in this case we get a contradiction because $m(a^2 - c^2) + 4n(b^2 - d^2) = 0$, $m > 0$ and $n > 0$.

Now, since $T(X, \alpha) \cong T(Y)$, then $\operatorname{disc} T(X, \alpha) = \operatorname{disc} T(Y)$ and since $[T(X) : T(X, \alpha)] = 2$, $4 \operatorname{disc} T(X) = \operatorname{disc} T(X, \alpha)$.

Caso 1 $s = m + n$

$$\begin{aligned} 16mn = 4 \operatorname{disc} T(X) &= \operatorname{disc} T(X, \alpha) \\ &= \operatorname{disc} T(Y) \\ &= 4s^2 - r^2 \\ &= 4(m+n)^2 - r^2. \end{aligned}$$

Hence, $0 = 4(m+n)^2 - 16mn - r^2 = 4(m-n)^2 - r^2 = (2(m-n) - r)(2(m-n) + r)$ and then $r = \pm 2(m - n)$.

Case 2 $s = 4n, m = 3n$ or $s = 4m$, $n = 3m$.
If $s = 4n, m = 3n$, $48n^2 = 4 \operatorname{disc} T(X) = \operatorname{disc} T(X, \alpha) = \operatorname{disc} T(Y) = 64n^2 - r^2$. Thus $r = \pm 4n$. On the other hand, if $n = 3m, s = 4m$, then $r = \pm 4m$.

Case 3 $m = s = 4n$ or $n = s = 4m$.
If $m = s = 4n$, $64n^2 = 4 \operatorname{disc} T(X) = \operatorname{disc} T(X, \alpha) = \operatorname{disc} T(Y) = 64n^2 - r^2$. Thus $r = 0$. On the other hand, if $n = s = 4m$ we also obtain $r = 0$. $\square$

Example 3.4.8. Let $(Y, 1), (X, \alpha)$ be K3 covers of Enriques surfaces, such that

$\mathrm{ord}(\alpha) = 2$ and such that their transcendental lattices are defined by the corresponding matrices

$$\begin{pmatrix} 4k & 0 \\ 0 & 4k \end{pmatrix}, \begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix},$$

where k is odd. We show in this example that if there exists an equivalence of categories $\Phi : \mathrm{D}^b(X, \alpha) \xrightarrow{\sim} \mathrm{D}^b(Y)$, then c is even, a or b is odd and the form $ax^2 + cxy + by^2$ does not represent 1.

From the equivalence Φ, we get an isometry $T(X, \alpha) \cong T(Y)$, and this implies that $4(4ab - c^2) = \mathrm{disc}(T(X, \alpha)) = \mathrm{disc}(T(Y)) = 16k^2$. Thus, c is an even number and by Theorem 3.4.2, one of the following holds

(i) a, b, c are even,

(ii) c is even, a or b is odd and the form $ax^2 + cxy + by^2$ does not represent 1,

(iii) c is even, a or b is odd, the form $ax^2 + cxy + by^2$ represents 1 and $4ab - c^2 \neq 4, 8, 16$.

Assume that (i) holds, i.e. $a = 2a_1, b = 2b_1, c = 2c_1$ for some integers a_1, b_1, c_1. Thus,

$$k^2 = 4a_1b_1 - c_1^2$$

and then c_1 is odd. Hence $k = 2p + 1$ and $c_1 = 2q + 1$ for some integers p, q, so

$$\begin{aligned} 4a_1b_1 &= k^2 + c_1^2 \\ &= (2p + 1)^2 + (2q + 1)^2 \\ &= 4(p^2 + p + q^2 + q) + 2, \end{aligned}$$

which is a contradiction. Now, assume that (iii) holds and hence there exists a basis $\{u, v\}$ such that the matrix associated to the transcendental lattice with respect to this basis is (see [41], pág. 5):

$$\begin{pmatrix} 2 & 0 \\ 0 & 2(\frac{4ab-c^2}{4}) \end{pmatrix}$$

and by Lemma 3.4.7, one of the following holds

(1) $s := 2k = \pm 2(1 + \frac{1}{4}(4ab - c^2))$, $r = \pm 2(1 - \frac{1}{4}(4ab - c^2)) = 0$;

(2) $s := 2k = 4n$, $1 = \frac{3}{4}(4ab - c^2)$, $4ab - c^2 = 0$;

(3) $s := 2k = 4$, $\frac{1}{4}(4ab - c^2) = 3$, $4m = 0$;

(4) $1 = s := 2k = 4ab - c^2$, $r = 0$;

(5) $\frac{1}{4}(4ab - c^2) = s := 2k = 4$, $r = 0$.

We can check that all cases lead to contradictions and this shows our statement.

Lemma 3.4.9. *Let X and Y be K3 covering of Enriques surfaces with their transcendental lattices given by*

$$\begin{pmatrix} 2s & r \\ r & 2t \end{pmatrix}, \begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$$

respectively. If $\Phi : \mathrm{D}^b(Y, \beta) \to \mathrm{D}^b(X)$ is an equivalence and r is an odd number. Then c is an odd number and ab is even.

Proof. Let us suppose that c is an even number. From the equivalence Φ we obtain an isometry $T_\Phi : T(Y, \beta) \cong T(X)$. If $\{f_1, f_2\} \subseteq \langle e_1, e_2 \rangle$ is a basis for $T(Y, \beta)$ where $\{e_1, e_2\}$ is a basis of $T(Y)$ such that

$$e_1^2 = 2a,\ e_2^2 = 2b,\ (e_1.e_2) = c,$$

and

$$T_\Phi(f_1)^2 = 2s,\ T_\Phi^2(f_2) = 2t,\ (T_\Phi(f_1).T_\Phi(f_2)) = r.$$

Since $f_1 = le_1 + me_2$, $f_2 = ne_1 + ke_2$ then $(f_1.f_2)$ is an even number because e_1^2, e_2^2 and c are even numbers. On the other hand, $(f_1.f_2) = (T_\Phi(f_1).T_\Phi(f_2)) = r$ which is an odd number. A contradiction. □

3.5 Kummer surfaces

Proposition 3.5.1 (Morrison, Cor. 4.4, [30]). *Let X be an algebraic K3 surface.*

(1) If $\rho(X) = 19$ or 20, then X is a Kummer surface if and only if there is an even lattice T' with $T(X) \cong T'(2)$.

(2) If $\rho(X) = 18$, then X is a Kummer surface if and only if there is an even lattice T' with $T(X) \cong U(2) \oplus T'(2)$.

(3) If $\rho(X) = 17$, then X is a Kummer surface if and only if there is an even lattice T' with $T(X) \cong U(2)^2 \oplus T'(2)$.

(4) If $\rho(X) < 17$, then X is not a Kummer surface.

Corollary 3.5.2. *Let X be an algebraic Kummer surface.*

(1) If $\rho(X) = 20$, then $|\operatorname{disc} \operatorname{Pic}(X)| \geq 12$.

(2) If $\rho(X) = 19$, then $|\operatorname{disc} \operatorname{Pic}(X)| \geq 16$.

(3) If $\rho(X) = 18$, then $|\operatorname{disc} \operatorname{Pic}(X)| \geq 16$.

(4) If $\rho(X) = 17$, then $|\operatorname{disc} \operatorname{Pic}(X)| \geq 64$.

Lemma 3.5.3. *Let X and Y be K3 covering of Enriques surfaces such that $\rho(X) = \rho(Y) = 20$, $\beta \in \operatorname{Br}(Y)$ of order 2 and let $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y, \beta)$ be an equivalence. If Y is a Kummer surface, then X is also a Kummer surface.*

Proof. Since Y is a Kummer surface, its transcendental lattice is defined by the matrix

$$\begin{pmatrix} 4a & 2c \\ 2c & 4b \end{pmatrix},$$

i.e. $T(Y)$ is generated by $\{e_1, e_2\}$ such that $e_1^2 = 4a, e_2^2 = 4b, (e_1.e_2) = 2c$. From the equivalence Φ, we get an isometry $T_\Phi : T(Y, \beta) \cong T(X)$. If $\{f_1, f_2\}$ generates $T(Y, \beta)$, then there exist integers m, n, k, l such that $f_1 = me_1 + ne_2, f_2 = ke_1 + le_2$. Thus, $4|f_1^2, 4|f_2^2, 2|(f_1.f_2)$. Hence the lattice T' defined by the matrix

$$\begin{pmatrix} (f_1.f_1)/2 & (f_1.f_2)/2 \\ (f_1.f_2)/2 & (f_2.f_2)/2 \end{pmatrix}$$

is an even lattice such that $T'(2) = T(Y, \beta)$. Hence, $T(X) \cong T'(2)$ and this shows that X is a Kummer surface. □

Proposition 3.5.4 ([26], Prop. 2.5). *Let S be a Kummer surface $Km(E \times E')$ of the product of non-isogeneous elliptic curves E and E'. Then there exists an elliptic fibration on S whose Jacobian surface is not a Kummer surface.*

Remark 3.5.5. If $S = Km(E \times E')$ is the Kummer surface of the product of non-isogeneous elliptic curves E and E', the transcendental lattice $T(S) = U(2) \oplus U(2)$, $\mathrm{Pic}(S) = D_8 \oplus D_8 \oplus U$. Thus $\rho(S) = 18$ and $\mathrm{disc}(\mathrm{Pic}(S)) = 16$.

Lemma 3.5.6. *There exist K3 surfaces X, Y and an equivalence of categories $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y, \alpha)$ such that only X is a Kummer surface and $\mathrm{ord}(\alpha) = 2$.*

Proof. Let X be the Kummer surface $Km(E \times E')$ of the product of non-isogenous elliptic curves E and E'. By Proposition 3.5.4, there exists an elliptic fibration on X whose Jacobian surface is not a Kummer surface and $\mathrm{disc}(\mathrm{Pic}(J(X))) = 4$ (this is also obtained in the proof of Prop. 2.5., [26]). Thus, there exists a element $\alpha \in \mathrm{Br}(J(X))$ and an equivalence $\Phi : \mathrm{D}^b(J(X), \alpha) \cong \mathrm{D}^b(X)$. Since $16 = \mathrm{disc}(\mathrm{Pic}\, X) = \mathrm{disc}\, T(X)$, one has $\mathrm{ord}(\alpha) = 2$, because $16 = \mathrm{disc}\, T(X) = \mathrm{disc}\, T(J(X), \alpha) = \mathrm{ord}(\alpha)^2 \,\mathrm{disc}\, T(J(X))$. □

Proposition 3.5.7. *Let X be an algebraic Kummer surface, Y an algebraic surface and $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ an equivalence. Suppose*

(1) $\rho(Y) = 20$ and $\mathrm{disc}\,\mathrm{Pic}(Y) < 48$, or

(2) $\rho(Y) = 19$ and $\mathrm{disc}\,\mathrm{Pic}(Y) < 64$,

Then $\alpha = 1$ in $\mathrm{Br}(X)$. Moreover $X \cong Y$.

Proof. Suppose α nontrivial. From the equivalence Φ, we get an isometry $T(X, \alpha) \cong T(Y)$. Thus,

$$\mathrm{ord}(\alpha)^2 \,\mathrm{disc}(\mathrm{Pic}(X)) = \mathrm{ord}(\alpha)^2 \,\mathrm{disc}(T(X)) = \mathrm{disc}(T(X, \alpha)) = \mathrm{disc}(\mathrm{Pic}(Y))$$

and so $\mathrm{disc}(\mathrm{Pic}(X)) < 12$ or $\mathrm{disc}(\mathrm{Pic}(X)) < 16$ in the first ($\rho(Y) = 20$) and second case ($\rho(Y) = 19$), respectively. Hence X is not a Kummer surface by Corollary 3.5.2, a contradiction. □

Example 3.5.8. Let X be a K3 an surface of type Barth-Peters (This was introduced in [36]). Thus, $NS(X) = U(2) \oplus E_8^{\oplus 2}$ and the number of Enriques quotients is at most 1. (This was claimed by Ohashi in [36], page 200). Suppose $\Phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y, \alpha)$ is an equivalence with Y an algebraic K3 surface. Since $\operatorname{disc} NS(X) = 4$, then $\operatorname{ord}(\alpha) \leq 2$. If $\operatorname{ord}(\alpha) = 2$, $\operatorname{disc}(T(Y)) = \pm 1$, i.e. $T(Y)$ is unimodular. Thus $\mathrm{NS}(Y) = U \oplus E_8 \oplus E_8$ (Lemma 4.1, [27]).

Example 3.5.9. Let X be the Kummer surface $Km(E_{\tau_3} \times E_{\tau_3})$ with period $\tau_3 =$ 3-th rooth of unity, so $\operatorname{disc}(\operatorname{Pic}(X)) = 12$. Suppose $\Phi : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y, \alpha)$ is an equivalence where Y is an elliptic K3 surface. From the equivalence we obtain that $\operatorname{ord}(\alpha) \leq 2$. If $\operatorname{ord}(\alpha) = 2$, then $\operatorname{disc}(T(Y)) = 3$ and then Y is a Jacobian fibration (it has a section). On the other hand $X \cong Y$, and so Y is also a Jacobian fibration.

$\square$

Bibliography

[1] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces*, 2nd Edition, Modern Surveys in Mathematics, vol. 4, Springer Verlag, Berlin, 2004.

[2] A. Beauville, *On the Brauer group of Enriques surfaces*, Math. Research Letters **16** (2009), 927–934.

[3] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Comp. Math. **125** (2001), 327–344.

[4] T. Bridgeland and A. Maciocia, *Fourier–Mukai transforms for quotient varieties*, arXiv:9811101v1.

[5] ______, *Complex surfaces with equivalent derived categories*, Math. Z. **236** (2001), no. 4, 677–697.

[6] A. Canonaco and P. Stellari, *Twisted Fourier–Mukai functors*, Adv. Math. **212** (2007), no. 2, 484–503.

[7] J. L. Colliot-Thélène, *Algèbres simples centrales sur les corps de fonctions de deux variables*, Séminaire Bourbaki **949** (2005).

[8] A. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University, 2000.

[9] F. DeMeyer and T. Ford, *On the Brauer group of toric varieties*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 559–577.

[10] I. Dolgachev and M. Gross, *Elliptic threefolds I. Ogg-Shafarevich theory*, J. Alg. Geom. **3** (1994), no. 1, 39–80.

[11] R. Donagi and T. Pantev, *Torus fibrations, gerbes and duality*, Mem. Amer. Math. Soc. **193** (2008), no. 901.

[12] R. Friedman and J. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse Math. Grengeb., vol. 27, Springer-Verlag, Berlin, 1994.

[13] O. Gabber, *Some theorems on Azumaya algebras, The Brauer group*, vol. 844, Springer, Berlin-New York, 1981.

[14] A. Grothendieck, *Eléments de géométrie algébrique*, Publ. Math. I.H.E.S.

[15] ______, *Le groupe de Brauer I. Algèbres d'Azumaya et interprétations diverses*, Dix esposés sur la cohomologie des schémas, North–Holland, Amsterdam (1968), 46–66.

[16] D. Harari and A. Skorobogatov, *Non-abelian descent and the arithmetic of Enriques surfaces*, Int. Math. Res. Not. (2005), no. 52, 3203–3228.

[17] R. Hartshorne, *Residues and Duality*, LNM, no. 20, Springer-Verlag, Berlin-New York, 1966.

[18] ———, *Algebraic geometry*, Graduate Text in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg, 1977.

[19] R.T. Hoobler, *Brauer groups of abelian schemes*, Ann. Sci. l'École Nor. Sup. **4** (1972), 45–70.

[20] S. Hosono, B. Lian, K. Oguiso, and S.T. Yau, *Fourier–Mukai number of a K3 surface*, CRM Proc. Lect. Notes, Amer. Math. Soc., Providence **38** (2004), 177–192.

[21] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2006.

[22] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[23] D. Huybrechts and P. Stellari, *Equivalences of twisted K3 surfaces*, Math. Ann. **332** (2005), no. 4, 901–936.

[24] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York, 1990.

[25] Y. Kawamata, *D-equivalence and K-equivalence*, J. Diff. Geom. **61** (2002), no. 1, 147–171.

[26] J. Keum, *A note on elliptic K3 surfaces*, Trans. Amer. Math. Soc. **352** (2000), no. 5, 2077–2086.

[27] S. Kondo, *Automorphisms of algebraic K3 surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), no. 1, 75–98.

[28] J. Milne, *Étale cohomology*, Princeton Mathematical series, vol. 33, Princeton University Press, Princeton, 1980.

[29] J.S. Milne, *Lectures on Étale Cohomology. http://www.jmilne.org/math*.

[30] D. Morrison, *On K3 surfaces with large Picard number*, Invent. Math. **75** (1984), no. 1, 105–121.

[31] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its applications to Picard sheaves*, Nagoya Math. J (1981), no. 81, 153–175.

[32] ———, *On the moduli space of bundles on K3 surfaces, I*, Vector bundles on algebraic varieties (Bombay) (1984), 341–413.

[33] V. V. Nikulin, *Integral Symmetric Bilinear Forms*, Math USSR Izvestija **14** (1980), no. 1, 103–167.

[34] V.V. Nikulin, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections*, (English Translation) J. Soviet Math. **22** (1983), 1401–1476.

[35] H. Ohashi, *Counting Enriques quotients of a K3 surface*, http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1609.pdf.

[36] ———, *On the number of Enriques quotients of a K3 surface*, Publ. Res. Inst. Math. Sci. **43** (2007), no. 1, 181–200.

[37] D. Orlov, *Derived categories of coherent sheaves and equivalences between them*, Russian Math. Surveys **58** (2003), no. 3, 511–591.

[38] A. Polishschuk, *Abelian varieties, theta functions and the Fourier transform*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2003.

[39] S. Schröer, *There are enough Azumaya algebras on surfaces*, Math. Ann. **321** (2001), no. 2, 439–454.

[40] R.L.E. Schwarzenberger, *Vector bundles on algebraic surfaces*, Proc. London Math. Soc. **3** (1961), no. 11, 601–622.

[41] A. S. Sertöz, *Which singular K3 surfaces cover an Enriques surface*, Proc. Amer. Math. Soc. **133** (2005), 43–50.

[42] P. Stellari, *A Twisted Derived Torelli Theorem for K3 surfaces*, Ph.D. thesis, Universita degli Studi di Milano, 2006.

[43] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge University Press, Cambridge, 1994.

Abstract

In this thesis we study Fourier–Mukai transforms between derived categories of twisted sheaves. We show that some well known results about the classification of surfaces under derived categories extend to the derived category of twisted sheaves. In particular, we study the relationship between the derived category of twisted sheaves $\mathrm{D}^b(Y, \alpha)$ for an Enriques surface Y and the derived category of twisted sheaves $\mathrm{D}^b(X, \pi^*\alpha)$ where $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$ is the induced homomorphism obtained from the K3 cover of Y: $\pi : X \to Y$. We also study the injectivity of the morphism $\pi^* : \mathrm{Br}'(Y) \to \mathrm{Br}'(X)$.