Stratifolds And Equivariant Cohomology Theories

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Haggai Tene

aus

Jerusalem, Israel

Bonn, August 2010

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

1. Referent: Prof. Dr. Matthias Kreck (Bonn)
2. Referent: Prof. Dr. Carl-Friedrich Bödigheimer (Bonn)
Tag der Promotion: 11.10.2010
Erscheinungsjahr: 2011

Contents

CHAPTER 1

Introduction

Equivariant (co)homology groups are an important tool for studying G-spaces. These (co)homology groups are defined via the Borel construction. For discrete group they can also be defined using projective resolutions. One can also define Tate (co)homology groups for G-spaces in a similar way. There is a map from equivariant cohomology to Tate cohomology. In such a situation one naturally asks two questions:

- Can one say something about the kernel and cokernel of this map?

- Can one define Tate cohomology groups for spaces when G is not a finite group but a compact Lie group?

To both questions we give an answer in this thesis. The answer to the first question is given by defining a third (co)homology theory called backwards (co)homology and an exact sequence relating all three (co)homology theories. This new theory is a straightforward generalization of the construction of equivariant (co)homology and Tate (co)homology in terms of resolutions. Of course, this only works for finite groups.

Kreck has given a geometric bordism description of singular homology groups and - for smooth manifolds - of singular cohomology groups using stratifolds. This can be used to give a bordism description of equivariant homology groups defined via the Borel construction. This works for compact Lie groups and is the starting point for the answer to the second question.

Before we come to this we address another question. For ordinary (co)homology one of the most important results and tools is Poincaré duality. This does not hold for equivariant (co)homology as one can see for the simplest G-manifold: the point. The equivariant homology and cohomology groups of a point are trivial in negative dimensions but in general non-trivial in positive degree giving no room for Poincaré duality. Thus the following question is very natural:

- Can one define new (co)homology groups which are Poincaré dual to the groups given via the Borel construction?

In the case of finite groups the new (co)homology groups give an answer to this question, the new cohomology theory is Poincaré dual to the homology of the Borel construction whereas the new homology theory is Poincaré dual to the cohomology of the Borel construction. This is a very natural since the Tate cohomology groups are self dual (with the expected dimension shift). We also give a bordism interpretation of the new groups which extends to actions of compact Lie groups. This gives the answer to the second question.

The question of Poincaré duality has been dealt already before by Greenlees and May, who construct equivariant spectra allowing duality by general principles. But this answer is rather abstract and - although this should be the case - it is not obvious whether their groups agree with ours. We did not manage to decide this question.

Besides allowing a generalization to actions of compact Lie groups one of the main motivation for defining equivariant (co)homology groups by bordism groups is that this might be helpful for computations. In particular this might be interesting for the computation of equivariant Tate cohomology groups which is very hard. The exact sequence mentioned above relating classical equivariant (co)homology, the new equivariant (co)homology groups and the equivariant Tate (co)homology groups shows that the equivariant Tate (co)homology groups measure the failure of Poincaré duality between equivariant homology and cohomology groups of the Borel construction. The Tate groups vanish if and only if this duality holds. This is one reason why the computation of the equivariant Tate (co)homology groups of a G-manifold is interesting (and difficult). We use our geometric definitions to compute such groups for certain actions on the 3-sphere, just to indicate how such computations can be done. There are other good reasons to compute Tate groups as we will indicate later in this introduction.

The Tate cohomology groups (of a group G, not a G-space) have a ring structure given by the cup product. As in ordinary cohomology the computation of cup products can be very difficult. In the case of a smooth manifold the cup product often is computed geometrically using representatives given by manifolds or stratifolds. Kreck has constructed a geometric product on negative Tate cohomology groups and asked for the relation to the cup product. We show that these products agree, if G is a finite group.

These are the main themes and indications of the answers of this thesis. We now summarize the results in more detail.

Let G be a discrete group and X a $G{-}CW$ complex. One defines the equivariant (co)homology of X as the (co)homology of the Borel construction $EG \times_G X$, and denotes it by $H_*^G(X)$ and $H_G^*(X)$ resp. If G is finite one can also define the Tate (co)homology of X, denoted by $\hat{H}_*^G(X)$ and $\hat{H}_G^*(X)$ resp. An important property of $\hat{H}_G^*$ is that if X is a finite dimensional $G-CW$ complex and Σ_X is the subcomplex consisting of all points with non trivial stabilizer then the inclusion induces an isomorphism $\hat{H}_G^*(X) \to \hat{H}_G^*(\Sigma_X)$ and similarly in homology. In particular, $\hat{H}_G^*(X)$ vanishes if G acts freely on X.

There is a natural transformation $H_G^*(X) \to \hat{H}_G^*(X)$. One can wonder whether there is a third cohomology theory and natural transformations to $H_G^*(X)$ and from $\hat{H}_G^{*-1}(X)$ to this new theory such that the corresponding sequence is exact. We construct such an equivariant cohomology theory, which we denote by $DH_G^*(X)$ and call the **backwards cohomology**. We have the following:

THEOREM. *(5.22) For every finite group G we construct an equivariant cohomology theory DH_G^* on the category of finite dimensional $G-CW$ complexes and equivariant cellular maps and a natural exact sequence:*

$$(1) \qquad ... \to DH_G^k(X) \to H_G^k(X) \to \hat{H}_G^k(X) \to DH_G^{k+1}(X) \to ...$$

A similar construction also exists in homology where we denote the groups by $DH_k^G(X)$ and the sequence looks like:

$$... \to DH_{k+1}^G(X) \to \hat{H}_k^G(X) \to H_k^G(X) \to DH_k^G(X) \to ...$$

This construction is both simple and natural but we could not find it in the literature. It answers a natural problem. Obviously equivariant (co)homology does not fulfill Poincaré duality as one can see in the case X a point. Thus one can wonder about Poincaré dual theories. The groups $DH_G^*(X)$ and $DH_*^G(X)$ fill this gap. We have chosen the notation DH for that reason, "D" stands for Poincaré duality. In the following we assume that all actions on oriented smooth manifolds are smooth and orientation preserving.

THEOREM. *(5.40) (Poincaré duality) Let M be a closed oriented smooth manifold of dimension m with an action of a finite group G. We have the following isomorphisms:*

$$H_G^k(M) \to DH_{m-k}^G(M), \qquad DH_G^k(M) \to H_{m-k}^G(M), \qquad \hat{H}_G^k(M) \to \hat{H}_{m-k-1}^G(M).$$

For such M the map $DH_G^k(M) \to H_G^k(M)$ together with Poincaré duality gives a map $H_{m-k}^G(M) \to H_G^k(M)$. This map is not an isomorphism in general, $\hat{H}_G^*(M)$ is an obstruction for that. Poincaré duality between $H_*^G(M)$ and $H_G^*(M)$ holds if and only if $\hat{H}_G^*(M)$ is zero. Note that $\hat{H}_G^*(M)$ vanishes if and only if the action is free, in which case Poincaré duality is ordinary Poincaré duality of the quotient.

If we will be able to compute the kernel and the cokernel of the map $DH_G^k(X) \to H_G^k(X)$ we will be able to compute $\hat{H}_G^*(X)$ up to extension. If M is a closed oriented G manifold then for $k > m$ the map $H_G^k(M) \to \hat{H}_G^k(M)$ is an isomorphism and for $k < -1$ the map $\hat{H}_G^k(M) \to H_{m-k-1}^G(M)$ is an isomorphism. The group $\hat{H}_G^{-1}(M)$ is mapped isomorphically to the torsion part of $H_m^G(M)$. When G has periodic cohomology (for example if there is an orientation preserving free G action on a sphere) then computing $\hat{H}_G^*(M)$ is easier then computing $H_G^*(M)$ and $H_*^G(M)$ and this might help in computing the map $H_{m-k}^G(M) \to H_G^k(M)$.

The Borel construction can be applied also to compact Lie groups and so one has equivariant (co)homology theories generalizing the case of finite groups. The construction of the backwards theories for finite groups is based on homological algebra. This does not generalize immediately to compact Lie groups. In this situation we look at a new construction of theories isomorphic to the (co)homology of the Borel construction for arbitrary compact Lie groups for which we can also define the backwards theory. This is done by a geometric construction of equivariant (co)homology theories as certain bordism groups. The theories corresponding to the (co)homology of the Borel construction are denoted by $SH_*^G(X)$ and $SH_G^*(X)$ resp. where in the case of cohomology we have to assume that X is a smooth (in general non compact) oriented manifold with a smooth and orientation preserving action. In this geometric context we define backwards theories $DSH_*^G(M)$ and $DSH_G^*(M)$ and Tate groups $\widehat{SH}_*^G(X)$ and $\widehat{SH}_G^*(X)$ where again X is a smooth manifold when we consider cohomology. The exact sequences above generalize to compact Lie groups:

THEOREM. *(6.51) For every compact Lie group G we construct equivariant cohomology theories on the category of smooth oriented G-manifolds and equivariant smooth maps, denoted by $DSH_G^*(M)$, $SH_G^*(M)$ and $\widehat{SH}_G^*(M)$ and a natural exact sequence:*

$$(2) \qquad ... \to DSH_G^k(M) \to SH_G^k(M) \to \widehat{SH}_G^k(M) \to DSH_G^{k+1}(M) \to ...$$

There are corresponding equivariant homology theories on the category of finite dimensional $G-CW$ complexes and an exact sequence:

$$... \to SH_k^G(X) \to DSH_k^G(X) \to \widehat{SH}_k^G(X) \to SH_{k-1}^G(X) \to ...$$

For G finite one would expect that the theories occurring in the first theorem and in the last theorem are naturally isomorphic. For G the trivial group this was proved by Kreck applying the characterization of ordinary (co)homology theories by the Eilenberg Steenrod axioms. To generalize this to the equivariant case we give explicit isomorphisms which requires a lot of effort. Unfortunately we can only prove this for the theories SH_G^*, SH_*^G, DSH_G^*, DSH_*^G and $\widehat{SH}_*^G(X)$. Nevertheless, for Tate cohomology we construct an isomorphism, only naturality is a problem. We summarize this as:

THEOREM. *(6.63) There are natural isomorphisms $DSH_G^k(M) \to DH_G^k(M)$, $SH_G^k(M) \to H_G^k(M)$ and an isomorphism $\widehat{SH}_G^k(M) \to \hat{H}_G^k(M)$ such that the following diagram commutes:*

$$\begin{array}{ccccccc} DSH_G^k(M) & \to & SH_G^k(M) & \to & \widehat{SH}_G^k(M) & \to & DSH_G^{k+1}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ DH_G^k(M) & \to & H_G^k(M) & \to & \hat{H}_G^k(M) & \to & DH_G^{k+1}(M) \end{array}$$

We also have in homology natural isomorphisms $SH_k^G(X) \to H_k^G(X)$, $DSH_k^G(X) \to DH_k^G(X)$, $\widehat{SH}_k^G(X) \to \hat{H}_{k-1}^G(X)$ such that the following diagram commutes (6.61):

$$\begin{array}{ccccccc} SH_k^G(X) & \to & DSH_k^G(X) & \to & \widehat{SH}_k^G(X) & \to & SH_{k-1}^G(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_k^G(X) & \to & DH_k^G(X) & \to & \hat{H}_{k-1}^G(X) & \to & H_{k-1}^G(X) \end{array}$$

The new cohomology theories and their geometric generalizations to compact Lie groups might be useful in computing the "classical groups" for example the Tate groups. For this, one would exploit the exact sequence (2) by computing $DSH_G^*(M)$ and $SH_G^*(M)$ and the induced maps. It seems that even in very simple cases the computation of the Tate cohomology of a space is very difficult. This view is supported by the following: If Γ is a group of finite virtual cohomological dimension then its Farrell integral cohomology is isomorphic to $\hat{H}_G^*(X)$ for a finite quotient of Γ denoted by G and some $G-CW$ complex X (see [**1**]). Often one can take X to be a closed oriented manifold. Although the following example is of different nature it illustrates how such a computation of the Tate cohomology groups can be done:

THEOREM. *Let $G = \mathbb{Z}/n$ act on $S^3 \subseteq \mathbb{C}^2$ by $(x,y) \mapsto (\theta^k x, \theta^l y)$ where θ is the generator of G considered as a subgroup of S^1, then $\hat{H}_G^r(S^3) \cong \mathbb{Z}/gcd(n, k \cdot l)$ for all $r \in \mathbb{Z}$.*

We believe that it is possible to make other computations with similar methods.

In the last part we give a simple geometric interpretation of the cup product in negative Tate cohomology of a finite group using the join of cycles, which generalizes to compact Lie groups. For a compact Lie group G, the elements in $DSH_G^*(pt)$ are bordism classes of compact oriented regular p-stratifolds with a free and orientation preserving G action. The cup product is given, up to sign, by the Cartesian product with the diagonal action - $[S,\rho] \otimes [S',\rho'] \to [S \times S', \Delta]$ (ρ, ρ', Δ denote the G action). When $dim(S), dim(S') > 0$ this product vanishes since it is the boundary of $[CS \times S', \tilde{\rho}]$ where $\tilde{\rho}$ is the obvious extension of the action Δ, but it is also the boundary of $[S \times CS', \hat{\rho}]$ (up to sign) where $\hat{\rho}$ is the obvious extension of the action Δ. The Kreck product, denoted by $*$, is a secondary product defined by gluing both bordisms along the boundary $[S,\rho] * [S',\rho'] = [S * S', \rho * \rho']$ (note that after the gluing what we get is the join of the two p-stratifolds). This product $DSH^n_G(pt) \otimes DSH^m_G(pt) \to DSH^{n+m-1}_G(pt)$ does not vanish in general, for example when G finite cyclic or more generally for every group acting freely and orientation preserving on some sphere like $G = S^1$ and $G = S^3$. For these groups the product has a very simple geometric interpretation.

By Poincaré duality and the isomorphism $SH^G_n(X) \to H_{n-dim(G)}(EG \times_G X)$ this gives a product $H_n(BG) \otimes H_m(BG) \to H_{n+m+1+dim(G)}(BG)$, again denoted by $*$. We prove the following:

THEOREM. *(7.13) Let G be a finite group, then there is a natural isomorphism $\varphi : SH^G_*(pt) \to \hat{H}^{-*-1}(G,\mathbb{Z})$ for $* > 0$ and $\varphi(\alpha * \beta) = \varphi(\alpha) \cup \varphi(\beta)$ for all $\alpha \in SH^G_n(pt)$ and $\beta \in SH^G_m(pt)$ where $n, m > 0$.*

There is another approach for defining dual equivariant theories by Greenlees and May which appears in [**15**]. They do it in stable homotopy theory using equivariant spectra and so it applies to more general (co)homology theories. It would be interesting to study the relations between their theories and ours.

In this thesis we consider only compact Lie groups. An attempt to generalize $SH^G_*(M)$ to non compact Lie groups is strait forward but for $DSH^*_G(M)$ there are fundamental problems (we can define induced maps only for proper maps and it is no longer a multiplicative theory). The fact that we can only define it for proper maps makes it impossible to define the natural transformation $DSH^*_G(M) \to SH^*_G(M)$ and thus we cannot define a generalization of Tate cohomology for non compact Lie groups so we decide not to deal with this case.

Organization of the paper.

Chapter 2 is a short exposition about stratifolds. We discuss some properties of stratifolds and give some examples.

Chapter 3 deals with (non equivariant) homology theories defined using bordism maps from stratifolds. We present the following homology theories:

- SH_* - Stratifold homology
- SH^{lf}_* - Locally finite stratifold homology
- SH^{∞}_* - Stratifold end homology

where SH_* was defined by Kreck and the other two are new. These theories are related by a long exact sequence:

$$... \to SH_k(X) \to SH^{lf}_k(X) \to SH^{\infty}_k(X) \to SH_{k-1}(X) \to ...$$

We construct natural isomorphisms between these theories and their singular equivalents: $SH_* \to H_*$, $SH^{lf}_* \to H^{lf}_*$, $SH^{\infty}_* \to H^{\infty}_*$.

Chapter 4 deals with (non equivariant) cohomology theories defined using bordism maps from stratifolds. We present the following cohomology theories defined on the category of smooth oriented manifolds and smooth maps (proper smooth maps in the latter two cases):

- SH^* - Stratifold cohomology
- SH_c^* - Stratifold cohomology with compact support
- SH_∞^* - Stratifold end cohomology

where SH^* was defined by Kreck and the other two are new. These theories are related by a long exact sequence:

$$... \to SH_c^k \to SH^k \to SH_\infty^k \to SH_c^{k+1} \to ...$$

We construct natural isomorphisms $SH^* \to H^*$, $SH_c^* \to H_c^*$, $SH_\infty^* \to H_\infty^*$.

Chapter 5 is a survey about homological algebra. For a finite group G we define $DH_G^*(X)$, called the backwards cohomology, discussed before.

Chapter 6 deals with equivariant homology and cohomology theories defined using stratifolds where the groups are compact Lie groups. We present equivariant stratifold homology SH_*^G and construct a natural isomorphism $SH_*^G \to H_{*-dim(G)}^G$ where $dim(G)$ is the dimension of G and H_*^G is the homology of the Borel construction.

stratifold backwards cohomology $DSH_G^*(M)$ is defined for smooth oriented manifolds with a smooth and orientation preserving action of G. It has the property that for a compact oriented smooth manifold of dimension m with a smooth and orientation preserving action of G there is a Poincaré duality isomorphism $DSH_G^k(M) \to SH_{m-k}^G(M)$. For a finite group G we construct a natural isomorphism $DSH_G^*(X) \to DH_G^*(X)$. Using this we define a geometric version of Tate cohomology for compact Lie groups, denoted by $\widehat{SH}_G^k$.

Chapter 7 deals with the cup product in the negative part of $\hat{H}^*(G,\mathbb{Z})$. By duality, this product gives a product structure on the integral homology of BG with a dimension shift: $H_k(BG,\mathbb{Z}) \otimes H_l(BG,\mathbb{Z}) \to H_{k+l+1}(BG,\mathbb{Z})$. We give a geometric construction of a product with the same grading, introduced by Kreck, and prove that those products coincide.

Acknowledgement. First of all I would like to thank my doktorvater Prof. Matthias Kreck. He inspired me both on a professional level and on a personal level. His endless patience and personal approach made me feel at home. Without his support and encouragement I would not have been able to write this thesis.

During the time of my PhD studies I was in the Hausdorff Research Institute for Mathematics (HIM). This is a wonderful place for conducting research in mathematics. I would like to thank all the administrative staff for making me feel at home and took care of everything my family and I needed.

I was a member of the Bonn International Graduate School in Mathematics (BIGS) and the Graduiertenkolleg 1150 Homotopy and Cohomology, I would like to thank them for their financial support.

I would like to thank my Professors from Israel who taught me and supported me while I was there, in particular Prof. Emmanuel Farjoun who taught me a big part of the topology I know and helped me for a long time while I was a student in the Hebrew University in Jerusalem, Prof. Patrick Iglesias-Zemmour, which helped me in a very critical point in my mathematical way, Prof. David Blanc who is always happy to assist, Prof. Yakov Varshavsky who read some version of chapter 7 and gave me some comments.

I would like to thank my friends Dr. Johannes Ebert and Irakli Patchkoria for reading the algebraic parts of this thesis and giving me some comments.

I would like to thank all the people who listened to me and gave me comments, Prof. Wolfgang Lück, Prof. Peter Teichner, Prof. Andrew Ranicki, Prof. John Klein, Dr. Diarmuid Crowley and Prof. Jim Davis who was always very patient. I thank especially Prof. Alejandro Adem, who directed me to some interesting questions. I also thank Prof. Carl-Friedrich Bödigheimer for accepting my request to be a second referee of this thesis.

I would like to thank my friends from the university of Bonn, especially to Anna Abczynski, who helped me during my stay in the wonderful city of Bonn.

I would like to dedicate this thesis to my wife Orly and my daughter Rotem whom I love so much.

CHAPTER 2

Stratifolds and Parametrized Stratifolds

Abstract. In this chapter we collect some fundamental properties of stratifolds, used later on.

Stratifolds were introduced by Kreck in [**23**], as a generalization of manifolds. Briefly, a stratifold is a pair, consisting of a topological space S together with a subsheaf of the sheaf of continuous real functions on S. S is assumed to be locally compact, Hausdorff and second countable and thus paracompact. The sheaf structure is assumed to present S as a union of strata which are smooth manifolds. For a stratifold S, we will denote by S^k its k^{th} stratum and by S_k its k^{th} skeleton. The sheaf is supposed to fulfill certain axioms, which we won't present here but appear in [**23**].

A stratifold is said to be **oriented** if its top stratum is oriented and the stratum of codimension one is empty.

A stratifold S is said to be **regular** if for each $x \in S^k$ there is an open neighborhood U of x in S, a stratifold F with F^0 a single point pt, an open subset V of S^k, and an isomorphism $\phi : V \times F \to U$, whose restriction to $V \times pt$ is the projection.

There is also a notion of a stratifold with boundary, which is called a c-stratifold since a part of its structure is a collar. The main relations between the two is that for two c-stratifolds (T, S) and (T', S') and an isomorphism $f : S \to S'$ there is a well defined stratifold structure on the space $T \cup_f T'$ which is called the gluing, and for a smooth map $g : T \to \mathbb{R}$ such that there is a neighborhood of 0 which consists only of regular values the preimages $g^{-1}((-\infty, 0]) = T'$ and $g^{-1}([0, \infty)) = T''$ are c-stratifolds and T is equal to the gluing $T' \cup_{Id} T''$.

Among the examples of stratifolds are manifolds, real and complex algebraic varieties [**16**], and the one point compactification of a smooth manifold. The cone over a stratifold and the product of two stratifolds are again stratifolds.

If S is a stratifold and $g : S \to \mathbb{R}$ is a smooth map such that there is a neighborhood of 0 which consists only of regular values then $S' = g^{-1}(0)$ has a natural structure of a stratifold. If S is oriented then we orient S' in the following way: Look at the top stratum S^k, there is an embedding $i : (-1, 1) \times S'^k \hookrightarrow S^k$ with the property that $g \circ i = \pi_1$ where $\pi_1 : (-1, 1) \times S'^k \to (-1, 1)$ is the projection on the first factor. We orient S' such that i will be orientation reversing (outward normal first, this is the same convention as in [**29**]). We call this the induced orientation on S'.

A parametrized stratifold, or a p-stratifold, is a kind of a stratifold constructed inductively by gluing manifolds with boundary and a collar in a process similar to the construction of a CW complex, but the attaching maps are supposed to be proper and smooth. The sheaf of functions consists of all functions which are smooth when restricted to all manifolds and commute with a germ of the collars. A p-stratifold of dimension n is oriented if and only if in the $n-1$ step we don't attach any smooth manifold and in the n^{th} step we attach an oriented smooth manifold along its boundary.

There is also a parametrized version of c-stratifolds, which we refer to as p-stratifolds with boundary. It is also constructed inductively. We will talk about it later.

We will use three properties of p-stratifolds:

(1) The cone over a p-stratifold has a p-stratifold structure so each p-stratifold is the boundary a p-stratifold with boundary.
(2) If S is a p-stratifold and $f : S \to \mathbb{R}$ is a smooth map then the preimage of a regular value is naturally a p-stratifold.
(3) P-stratifolds have the homotopy type of a CW-complex:

PROPOSITION 2.1. *Let (T, S) be p-stratifolds with boundary then it has the (proper) homotopy type of a CW pair (X,A) with $dim(X) \leq dim(T)$.*

PROOF. This can be proved by induction, where the inductive step uses the fact that (T, S) is constructed by gluing manifolds along their boundary, as will be explained later, which are known to be CW pairs. □

Not every stratifold is isomorphic to a p-stratifold, for example the one point compactification of the surface obtained by an infinite connected sum of tori [**23**]. This stratifold does not have the homotopy type of a CW-complex, thus it doesn't have a p-stratifold structure.

In this paper we will only use p-stratifolds.

CHAPTER 3

Stratifold Homology Theories

ABSTRACT. In this chapter we summarize definitions and properties of various homology theories and introduce new homology theories: SH_*^{lf} and SH_*^{∞} and identify them with the corresponding homology theories.

3.1. Report about stratifold homology

Stratifold homology was defined by Kreck in [**23**]. We will describe here a variant of this theory called parametrized stratifold homology, which is naturally isomorphic to it for CW complexes. In this thesis we will refer to parametrized stratifold homology just as stratifold homology and use the same notation for it.

(parametrized) Stratifold homology is a homology theory, denoted by SH_*. We will construct a natural isomorphism $\Phi : SH_* \to H_*$. It gives a new geometric point of view on integral homology, and has some advantages, some of which we will view later.

DEFINITION 3.1. Let X be a topological space and $k \geq 0$, define $SH_k(X)$ to be $\{g : S \to X\} / \sim$ i.e., bordism classes of maps $g : S \to X$ where S is a compact oriented regular p-stratifold of dimension k and g is a continuous map. We often denote the class $[g : S \to X]$ by $[S, g]$ or by $[S \to X]$. $SH_k(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \to Y$ is a continuous map than we can define an induced map by composition $f_* : SH_k(X) \to SH_k(Y)$.

A triple (U, V, X) consists of X which is a topological space and $U, V \subseteq X$ which are two closed subspaces such that their interiors cover X. For each triple there is a natural boundary operator $\partial : SH_k(X) \to SH_{k-1}(U \cap V)$. We define it for $X = S$, a compact oriented regular p-stratifold of dimension k, and the element $[S, Id]$ and extend it to all other triples by naturality. Choose a smooth map $f : S \to \mathbb{R}$ such that $f|_{S \setminus U} = -1$ and $f|_{S \setminus V} = 1$ and a regular value $-1 < x < 1$. Denote by $S' = f^{-1}(x)$, then S' is a compact regular p-stratifold of dimension $k-1$ and we give it the induced orientation discussed before. Define $\partial([S, Id]) = [S', i]$ where i is the inclusion $S' \xrightarrow{i} U \cap V$. The fact that it is well defined and the following appears in [**23**]:

THEOREM 3.2. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH_k(U \cap V) \to SH_k(U) \oplus SH_k(V) \to SH_k(X) \xrightarrow{\partial} SH_{k-1}(U \cap V) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

REMARK 3.3. In [**23**], Mayer-Vietoris Theorem is stated for open subsets U, V, but the same proof holds in our case.

We define the cross product in SH_* - $\times : SH_k(X) \otimes SH_l(Y) \to SH_{k+l}(X \times Y)$ by $[g_1 : S \to X] \times [g_2 : T \to Y] = [g_1 \times g_2 : S \times T \to X \times Y]$. This product is bilinear and natural.
SH_* with the boundary operator and the cross product is a multiplicative homology theory.

A natural isomorphism between SH_* and H_*.
We are going to construct a natural isomorphism $\Phi : SH_* \to H_*$, where H_* is integral homology. In order to do so we want to associate to each compact oriented regular p-stratifold S of dimension k a fundamental class which we denote by $[S] \in H_k(S)$.

LEMMA 3.4. *Let S be a p-stratifold of dimension k then $H_l(S)$ vanishes for $l > k$.*

PROOF. This can be proved by induction. The inductive step uses the Mayer-Vietoris long exact sequence and the fact that for M^k, a compact k dimensional smooth manifold (with boundary), $H_l(M^k)$ vanishes for $l > k$. □

Let S be a compact oriented regular p-stratifold of dimension k and denote by $(M^k, \partial M^k)$ the smooth manifold we attach as a top stratum.

The map $H_k(M^k, \partial M^k) \xrightarrow{\cong} H_k(S, S_{k-2})$ is an isomorphism by excision. The map $H_k(S) \xrightarrow{\cong} H_k(S, S_{k-2})$ is an isomorphism by the long exact sequence for the pair (S, S_{k-2}) and the fact that $H_l(S_{k-2})$ vanish for $l = k-1, k$ by the previous lemma.

DEFINITION 3.5. Define $[S] \in H_k(S)$ to be the image of $[M^k, \partial M^k]$ (the fundamental class of $(M^k, \partial M^k)$) under the composition $H_k(M^k, \partial M^k) \xrightarrow{\cong} H_k(S, S_{k-2}) \xrightarrow{\cong} H_k(S)$. We call $[S]$ the fundamental class of S. Note that $[S \amalg S'] = [S] + [S']$ and $[-S] = -[S]$.

Recall the notion of a p-stratifold with boundary:

DEFINITION 3.6. A k dimensional p-stratifold with boundary $(T, \partial T)$ is a pair of topological spaces where $\overset{\circ}{T} = T \setminus \partial T$ is a k dimensional p-stratifold and ∂T is a $k-1$ dimensional p-stratifold, which is a closed subspace of T together with a germ of collar $[c]$. We call ∂T the boundary of T. A smooth map from T to $\mathbb{R}$ is a continuous function f whose restrictions to $\overset{\circ}{T}$ and to ∂T are smooth and commutes with an appropriate representative of the germ of collars, i.e., there is a $\delta > 0$ such that $fc(x,t) = f(x)$ for all $x \in \partial T$ and $t < \delta$.

Let (T, S) be a $k+1$ dimensional p-stratifold with boundary. We have the following:

LEMMA 3.7. *As a topological space, (T, S) is constructed inductively, where in the k^{th} stage we have a p-stratifold with boundary (T_k, S_{k-1}). (T_{k+1}, S_k) is obtained from (T_k, S_{k-1}) together with a smooth manifold with boundary and collar $(M^{k+1}, \partial M^{k+1})$ such that $\partial M^{k+1} = \partial_+ \cup \partial_-$ and both ∂_+ and ∂_- are k dimensional manifolds with boundary, $\partial_+ \cup \partial_-$ is obtained by gluing them along their boundary, and a continuous map $f_{k+1} : \partial_+ \to T_k$ sending $\partial_+ \cap \partial_-$ to S_{k-1}. That is $(T_{k+1}, S_k) = (T_k, S_{k-1}) \cup_{\partial_+} (M^{k+1}, \partial)$.*

PROOF. We do this by induction on the dimension of (T,S). S is a p-stratifold thus $S = S_{k-1} \cup_{\partial N} N$ for some k dimensional smooth manifold N with boundary ∂N. $\overset{\circ}{T}$ is a p-stratifold thus $\overset{\circ}{T} = \overset{\circ}{T}_k \cup_{\partial P} P$ for some $k+1$ dimensional smooth manifold P with boundary ∂P. The collar c gives us an embedding $N \times (0,1) \to P$. Denote by M the space $N \times [0,1) \cup_c P$. M is a $k+1$ dimensional topological manifold with boundary $\partial M = \partial P \cup_{\partial N} N$. We will get the same notations as above if we set $N = \partial_-$ and $\partial P \cup_c \partial N = \partial_+$. □

Let (T,S) be a compact oriented regular p-stratifold of dimension $k+1$ with boundary. The map $H_{k+1}(M^{k+1}, \partial M^{k+1}) \xrightarrow{\cong} H_{k+1}(T, T_{k-1} \cup S)$ is an isomorphism by excision. The map $H_{k+1}(T,S) \xrightarrow{\cong} H_{k+1}(T, T_{k-1} \cup S)$ is an isomorphism by the long exact sequence for the triple $(T, T_{k-1} \cup S, S)$ and the fact that $H_l(T_{k-1} \cup S, S) \cong H_l(T_{k-1}, S_{k-2})$ by excision which vanishes for $l = k, k+1$.

DEFINITION 3.8. Define $[T,S] \in H_{k+1}(T,S)$ to be the image of $[M^{k+1}, \partial M^{k+1}]$ under the composition $H_{k+1}(M^{k+1}, \partial M^{k+1}) \xrightarrow{\cong} H_{k+1}(T, T_{k-1} \cup S) \xrightarrow{\cong} H_{k+1}(T,S)$.

LEMMA 3.9. *Let (T,S) be a compact oriented regular p-stratifold of dimension $k+1$ with boundary, then $\partial[T,S] = [S]$.*

PROOF. For compact oriented manifolds with boundary $(M, \partial M)$ it is proved in appendix 1 that $\partial[M, \partial M] = [\partial M]$ (this is a subtle question of orienting the boundary in a way that this equation will hold).

The following diagram is commutative:

$$\begin{array}{ccccccc}
H_{k+1}(M^{k+1}, \partial M^{k+1}) & \xrightarrow{\partial} & H_k(\partial M^{k+1}) & \to & H_k(\partial M^{k+1}, \partial_+) & \xrightarrow{\cong} & H_k(\partial_-, \partial_+ \cap \partial_-) \\
\downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
H_{k+1}(T, T_{k-1} \cup S) & \xrightarrow{\partial} & H_k(T_{k-1} \cup S) & \to & H_k(T_{k-1} \cup S, T_{k-1}) & \xrightarrow{\cong} & H_k(S, S_{k-2}) \\
\uparrow\cong & & \uparrow & & \uparrow\cong & & \uparrow\cong \\
H_{k+1}(T,S) & \xrightarrow{\partial} & H_k(S) & \xrightarrow{Id} & H_k(S) & \xrightarrow{Id} & H_k(S)
\end{array}$$

We follow the image of $[M^{k+1}, \partial M^{k+1}]$. Its image in $H_k(\partial_-, \partial_+ \cap \partial_-)$ is the generator that by the definition is mapped to $[S]$. On the other hand, as defined before, $[M^{k+1}, \partial M^{k+1}]$ is mapped to $[T,S] \in H_{k+1}(T,S)$, so by the commutativity of the diagram we conclude that $\partial[T,S] = [S]$. □

COROLLARY 3.10. *Let (T,S) be a compact oriented regular p-stratifold of dimension $k+1$ with boundary. Denote the inclusion of S in T by i then $i_*([S]) = 0$.*

PROOF. This follows from the previous lemma and the exactness of the sequence for the pair - $H_{k+1}(T,S) \xrightarrow{\partial} H_k(S) \xrightarrow{i_*} H_k(T)$. □

Define a natural transformation $\Phi : SH_k(X) \to H_k(X)$ by $\Phi([S,g]) = g_*([S])$.
Φ is well defined: If (S,g) and (S',g') are bordant, then there is a $k+1$ dimensional p-stratifold with boundary $(T, S \amalg -S')$ and a map $\widetilde{g} : T \to X$ such that $\widetilde{g}|_S = g$ and $\widetilde{g}|_{S'} = g'$. Denote the inclusion of $S \amalg -S'$ in T by i, then by the lemma above we have $i_*([S \amalg -S']) = 0$. We deduce that:
$0 = \widetilde{g}_*(i_*([S \amalg -S'])) = \widetilde{g}_*(i_*([S]) - i_*([S'])) = \widetilde{g}_*(i_*([S])) - \widetilde{g}_*(i_*([S'])) = g_*([S]) - g'_*([S'])$, therefore $\Phi(S,g) = \Phi(S',g')$.
Φ is a group homomorphism: This follows from the fact that $[S \amalg S'] = [S] + [S']$ and $[-S] = -[S]$
Φ is natural: It follows from the functoriality of singular homology.

Φ commutes with boundary: We have to show that for a triple (U, V, X) the following diagram commutes:

$$\begin{array}{ccc} SH_k(X) & \xrightarrow{\partial} & SH_{k-1}(U \cap V) \\ \downarrow \Phi & & \downarrow \Phi \\ H_k(X) & \xrightarrow{\partial} & H_{k-1}(U \cap V) \end{array}$$

By functoriality, it is enough to prove this for S, a compact oriented regular p-stratifold of dimension k and the element $[S, Id] \in SH_k(S)$. Assume we have such a p-stratifold S and $U, V \subseteq S$ two closed subspaces such that their interiors cover S. Let $f : S \to \mathbb{R}$ be a smooth map such that $f|_{S \setminus U} = -1, f|_{S \setminus V} = 1$ and suppose 0 is a regular value and its preimage $S' = f^{-1}(0)$ has a (closed) bicollar $S' \times [-\varepsilon, +\varepsilon]$ $(0 < \varepsilon < 1)$ such that $f(s, t) = t$ (this can be done since by compactness regular values are open). Denote by $U' = f^{-1}[-\varepsilon, \infty) \subseteq U$ $V' = f^{-1}(-\infty, \varepsilon] \subseteq V$, then there is a map of triples $(U', V', S) \to (U, V, S)$. By functoriality, again, it is enough to prove the claim for the triple (U', V', S) where $U' \cap V' = S' \times [-\varepsilon, +\varepsilon]$. The boundary map $SH_k(S) \to SH_k(S' \times [-\varepsilon, +\varepsilon])$ maps $[S, Id]$ to the inclusion of S' in $S' \times [-\varepsilon, +\varepsilon]$ as the zero section. Thus we have to show that the same is true for H_k.

The following diagram commutes:

$$\begin{array}{ccccccc} H_k(S) & \to & H_k(S, S \setminus \overset{\circ}{U'}) & \to & H_k(U', S' \times \{\varepsilon\}) & \xrightarrow{\partial} & H_{k-1}(S' \times \{\varepsilon\}) \\ \downarrow Id & & \downarrow & & \downarrow\cong & & \downarrow \\ H_k(S) & \to & H_k(S, V') & \to & H_k(U', U' \cap V') & \xrightarrow{\partial} & H_{k-1}(U' \cap V') \end{array}$$

The boundary in singular homology is the composition $H_k(S) \to H_k(S, V') \to H_k(U', U' \cap V') \to H_{k-1}(U' \cap V')$ ([**10**] III,8.11). We want to show that $\partial[S] = [S']$. To do so we have to follow the image of $[S] \in H_k(S)$ in the lower row. Since the diagram is commutative, we can follow its image in the upper row. $[S] \in H_k(S)$ is mapped to $[U', S' \times \{\varepsilon\}] \in H_k(U', S' \times \{\varepsilon\})$ and as we saw before $\partial[U', S' \times \{\varepsilon\}] = [S' \times \{\varepsilon\}]$ so we deduce that $\partial[S] = i_*[S' \times \{\varepsilon\}] = i_*[S'] \in H_{k-1}(U' \cap V')$.

Φ commutes with the cross product: We have to show that $\Phi(\alpha \times \beta) = \Phi(\alpha) \times \Phi(\beta)$. By the naturality of the cross product in H_* and in SH_* it is enough to show that for any two compact oriented regular p-stratifolds S, T of dimension k and l we have $[S \times T] = [S] \times [T]$. If k or l are equal to 0 then it is clear, so we can assume that $k, l > 0$. In each component of the top strata we choose a single point - $\{s_1 ... s_p\}$ and $\{t_1 ... t_q\}$. By the naturality of the cross product we have (we use the notation $H_*(X|x)$ instead of $H_*(X, X \setminus \{x\})$ for brevity):

$$\begin{array}{ccccc} H_k(S) \otimes H_l(T) & \xrightarrow{\cong} & H_k(S, \{s_1 ... s_p\}) \otimes H_l(T, \{t_1 ... t_q\}) & \xleftarrow{\cong} & \oplus H_k(\mathbb{R}^k|0) \otimes H_l(\mathbb{R}^l|0) \\ \downarrow & & \downarrow & & \downarrow \\ H_{k+l}(S \times T) & \xrightarrow{\cong} & H_{k+l}(S \times T, \{s_i \times t_j\}) & \xleftarrow{\cong} & \oplus H_{k+l}(\mathbb{R}^{k+l}|0) \end{array}$$

Which reduces this to the fact which is proved in appendix 1 that the cross product of the generators in $H_k(\mathbb{R}^k|0)$ and $H_l(\mathbb{R}^l|0)$ is the generator of $H_{k+l}(\mathbb{R}^{k+l}|0)$ with the standard orientations.

Φ is a natural isomorphism: For a one point space it is easy to show that $\Phi : SH_0(pt) \to H_0(pt)$ is an isomorphism. For $k > 0$ the map $\Phi : SH_k(pt) \to H_k(pt)$ is also an isomorphism since both groups vanish (every compact oriented regular p-stratifold of positive dimension is the boundary of its cone which is also compact and orientable. For a zero dimensional p-stratifold the cone is non orientable since its codimension 1 stratum is non empty).

We have the $Mayer-Vietoris$ long exact sequence for both SH_* and H_*, Φ commutes with the boundary therefore by the five lemma we have the following:

LEMMA 3.11. *Φ is an isomorphism for finite dimensional CW complexes.*

The following is more general

LEMMA 3.12. *Φ is an isomorphism for all CW complexes.*

This follows from the fact that for every CW complex X we have $H_k(X) = \underrightarrow{lim}(H_k(X_\alpha))$ and $SH_k(X) = \underrightarrow{lim}(SH_k(X_\alpha))$ where both limits are taken over all finite subcomplexes of X (the fact that $SH_k(X) = \underrightarrow{lim}(SH_k(X_\alpha))$ follows from the fact that we use compact p-stratifolds).

THEOREM 3.13. *Φ is an isomorphism for all spaces.*

PROOF. Let X be a topological space and $f : X_{CW} \to X$ be its cellular approximation. f is a weak equivalence thus if we show that $f_* : SH_k(X_{CW}) \to SH_k(X)$ is an isomorphism we will conclude that $\Phi : SH_k(X) \to H_k(X)$ is an isomorphism. The last statement follows from the fact that p-stratifolds have the homotopy type of a CW complex and therefore all maps from a p-stratifold to X factor, up to homotopy, through X_{CW}. $\square$

Here are two corollaries of this theorem:

COROLLARY 3.14. *Let X be a topological space. Every $\alpha \in H_2(X, \mathbb{Z})$ can be represented by a map from a closed oriented two dimensional manifold, that is there exists a closed oriented surface M^2 with a fundamental class $[M]$ and a map $f : M \to X$ such that $f_*([M]) = \alpha$.*

PROOF. This follows from the classification of compact oriented p-stratifolds of dimension two. Let S be a compact oriented p-stratifold of dimension two. By definition $S = M^2 \cup_{\partial M^2} P$ where P is a finite discrete set of points and M^2 is a compact oriented surface with boundary. Take the manifold M to be $M^2 \cup_{\partial M^2} \amalg D^2$ where to each boundary component of M^2, which is a circle, we glue a disc along the boundary. Thus M is a compact oriented surface and the quotient map $q : M \to S$ maps the fundamental class of M to the fundamental class of S. $\square$

REMARK. This fact is well known by other methods.

COROLLARY 3.15. *Let X be a CW complex. Every $\alpha \in H_k(X, \mathbb{Z})$ can be represented by a map from a compact oriented smooth manifold with boundary, in the sense that there exists a compact oriented smooth manifold M^k of dimension k with boundary and a map $g : (M^k, \partial M^k) \to (X, X_{k-2})$ such that $g_*([M^k, \partial M^k]) = \tilde{\alpha}$ where $\tilde{\alpha}$ is the image of α under the isomorphism $H_k(X) \to H_k(X, X_{k-2})$ we get from the long exact sequence for a pair.*

PROOF. We use the isomorphism $\Phi : SH_k(X) \to H_k(X)$ to represent α by a pair (S, g), that is $g_*([S]) = \alpha$. We can choose g such that $g(S_{k-2}) \subseteq X_{k-2}$ by cellular approximation. By definition $[S]$ is the image of $[M^k, \partial M^k]$ under the composition, mentioned before, $H_k(M^k, \partial M^k) \xrightarrow{i_*} H_k(S, S_{k-2}) \xrightarrow{\cong} H_k(S)$. Look at the following commutative diagram:

$$\begin{array}{ccccc} & & H_k(S) & \xrightarrow{g_*} & H_k(X) \\ & & \downarrow p_1 & & \downarrow p_2 \\ H_k(M^k, \partial M^k) & \xrightarrow{i_*} & H_k(S, S_{k-2}) & \xrightarrow{g_*} & H_k(X, X_{k-2}) \end{array}$$

By the long exact sequence for pairs the vertical maps are isomorphisms. We have $i_*([M^k, \partial M^k]) = p_1([S])$ thus $g_* \circ i_*([M^k, \partial M^k]) = g_* \circ p_1([S]) = p_2 \circ g_*([S]) = p_2(\alpha) = \tilde{\alpha}$ as stated in the corollary. □

3.2. Report about locally finite homology

REMARK. This section is based mainly on chapter 3 in [**20**].

Given a topological space X, its singular homology is the homology of the chain complex $S_*(X)$ where $S_k(X)$ is the free Abelian group generated by singular k simplices. It is sometimes useful to look at chains which are formal sums of infinitely many singular simplices. An example of this is the generalization of Poincaré duality to non compact manifolds, which we will talk about later. If we wish to look at chains of arbitrary formal sums $\Sigma_{\sigma \in I} n_\sigma \sigma$ we will have a problem to define the boundary map since a singular simplex may be the boundary of infinitely many singular simplices of higher dimension. In order to avoid this problem we have the following definition of locally finite homology ([**20**] 3.1):

DEFINITION 3.16. Let X be a space, define the locally finite chain complex $S_k^{lf}(X)$ to be the set of all formal sums of singular k simplices $\Sigma_{\sigma \in I} n_\sigma \sigma$ such that for every $x \in X$ there is an open neighborhood U such that $\{\sigma \in I | n_\sigma \neq 0 \, and \, |\sigma| \cap U \neq \emptyset\}$ is finite where $|\sigma|$ is the image of σ. $S_*^{lf}(X)$ is a chain complex. Its homology is called the locally finite homology of X and it is denoted by $H_*^{lf}(X)$.

REMARK. The condition that for every $x \in X$ there is an open neighborhood U such that $\{\sigma \in I | n_\sigma \neq 0 \, and \, |\sigma| \cap U \neq \emptyset\}$ is finite is equivalent to the condition that for every compact subset $K \subseteq X$ there is an open neighborhood U such that $\{\sigma \in I | n_\sigma \neq 0 \, and \, |\sigma| \cap U \neq \emptyset\}$ is finite. For locally compact spaces it is equivalent to the condition that every compact subset meets only finitely many simplices.

If $f : X \to Y$ is a continuous map than the image of a locally finite chain is not necessarily locally finite as one can see in the example where X is an infinite discrete set and Y is a point.

DEFINITION 3.17. A continuous map $f : X \to Y$ between two topological spaces is called proper if for every compact subset $K \subseteq Y$ its preimage $f^{-1}(K)$ is compact.

LEMMA 3.18. *Let $f : X \to Y$ be a proper map where Y is a locally compact Hausdorff space, then f is closed.*

PROOF. It is enough to show that $f(X)$ is closed. Let $y \in Y$ be a point which is not in the image, and let $K \subseteq Y$ be a compact neighbourhood of y. Since f is proper $f^{-1}(K)$ is compact. $A = f(f^{-1}(K))$ is compact and Y is Hausdorff so it is closed. $\overset{\circ}{K} \setminus f(X) = \overset{\circ}{K} \setminus A$ is open and contains y thus $f(X)$ is closed. □

Assume f is proper and Y is locally compact. Take $\Sigma_{\sigma \in I} n_\sigma \sigma$ a locally finite chain, $y \in Y$ and let K be a compact neighborhood of y. Since f is proper $f^{-1}(K)$ is compact in X, thus it has a neighborhood U such that $\{\sigma \in I | n_\sigma \neq 0 \, and \, |\sigma| \cap U \neq \emptyset\}$

is finite, hence $\{\sigma \in I | n_\sigma \neq 0 \, and \, |f(\sigma)| \cap K \neq \emptyset\}$ is finite and $\Sigma_{\sigma \in I} n_\sigma f(\sigma)$ is a locally finite chain. Thus there is a map $f_* : S_k^{lf}(X) \to S_k^{lf}(Y)$ compatible with the differential which induces a map in locally finite homology $f_* : H_k^{lf}(X) \to H_k^{lf}(Y)$. If f is properly homotopic to g then $f_* = g_*$. Therefore, locally finite homology is a proper homotopy invariant.

For a closed subset $A \subseteq X$ we have a short exact sequence:

$$0 \to S_*^{lf}(A) \to S_*^{lf}(X) \to S_*^{lf}(X)/S_*^{lf}(A) \to 0$$

that gives a long exact sequence in homology, just like in singular homology. The homology of the third term is called the relative locally finite homology and is denoted by $H_*^{lf}(X, A)$.

Locally finite homology fulfills the axioms of a homology theory on the category of locally compact Hausdorff spaces and proper maps. All the proofs are standard, except maybe for excision where we refer to **[31]** 7.1. The proof is the same as for singular homology. Once one has excision one can define the Mayer Vietoris sequence.

We note the following simple observations: For a disjoint union $\amalg X_\alpha$ we have isomorphisms $S_*^{lf}(\amalg X_\alpha) \to \Pi_\alpha S_*^{lf}(X_\alpha)$ and $H_*^{lf}(\amalg X_\alpha) \to \Pi_\alpha H_*^{lf}(X_\alpha)$. For every space X there is a chain map $S_*(X) \to S_*^{lf}(X)$ which induces a map in homology $H_*(X) \to H_*^{lf}(X)$. For a compact space X the maps $S_*(X) \to S_*^{lf}(X)$ and $H_*(X) \to H_*^{lf}(X)$ are isomorphisms (even identities).

The following proposition relates the locally finite homology and the singular homology of a space:

PROPOSITION 3.19. *(**[20]** 3.16) 1) Let X be a topological space then $S_*^{lf}(X) = \underleftarrow{lim} S_*(X, X \setminus K)$ where the inverse limit is taken over all compact subsets $K \subseteq X$. 2) Let X be a σ compact space and let $X_k \subseteq X$ be compact subsets such that $X_k \subseteq \overset{\circ}{X}_{k+1}$ and $X = \cup_k \overset{\circ}{X}_k$ then the following is an exact sequence for every k:*

$$0 \to \underleftarrow{lim}^1 H_{k+1}(X, X \setminus X_i) \to H_k^{lf}(X) \to \underleftarrow{lim} H_k(X, X \setminus X_i) \to 0$$

In particular every manifold can be presented this way since we always assume that the manifolds are second countable.

PROOF. 1) For each such $K \subseteq X$ there is a natural map $S_*^{lf}(X) \to S_*(X, X \setminus K)$ which induces a map to the inverse limit $\varphi_X : S_*^{lf}(X) \to \underleftarrow{lim} S_*(X, X \setminus K)$.

φ_X is injective - if $\varphi_X(\Sigma_{\sigma \in I} n_\sigma \sigma) = 0$ then the image of this element vanishes when restricted to $S_*(X, X \setminus K)$ for every $K \subseteq X$. Taking $K = |\sigma|$ we get that $n_\sigma = 0$, doing this for all σ we get $\Sigma_{\sigma \in I} n_\sigma \sigma = 0$.

φ_X is surjective - an element in $\underleftarrow{lim} S_*(X, X \setminus K)$ is of the form $(a_K) \in \Pi_K S_*(X, X \setminus K)$ such that for every $K \subseteq K'$ we have $a_{K'} \mapsto a_K$. We can choose representatives for all a_K of the sort $\Sigma_{\sigma \in I_K} n_\sigma \sigma$ where I_K is finite and all $\sigma \in I_K$ meet K. In this description $I_k \subseteq I_{k'}$ and the map $a_{K'} \mapsto a_K$ will be of the form $\Sigma_{\sigma \in I_{K'}} n_\sigma \sigma \mapsto \Sigma_{\sigma \in I_K} n_\sigma \sigma$. Define $\Sigma_{\sigma \in I} n_\sigma \sigma$ where $I = \cup I_k$. $\varphi_X(\Sigma_{\sigma \in I} n_\sigma \sigma)$ has a finite support when restricted to each compact subset and hence $\Sigma_{\sigma \in I} n_\sigma \sigma$ is a locally finite chain. 2) In this case the inverse limit $\underleftarrow{lim} S_*(X, X \setminus K)$ taken over all compact subsets $K \subseteq X$, is equal to $\underleftarrow{lim} S_*(X, X \setminus X_k)$ since each compact subset is included in some X_k. To prove the exactness of the above sequence one needs the following proposition and the fact that the tower $\ldots \to S_*(X, X \setminus X_2) \to S_*(X, X \setminus X_1)$ satisfies the Mittag Leffler condition since all the maps are surjective. $\square$

PROPOSITION 3.20. *([**33**] 3.5.8) Let ($\ldots \to C_2 \to C_1 \to C_0$) be a tower of chain complexes of Abelian groups satisfying the Mittag Leffler condition. Set C to be the inverse limit of this tower, then the following is an exact sequence for every k:*

$$0 \to \varprojlim{}^1 H_{k+1}(C_i) \to H_k(C) \to \varprojlim H_k(C_i) \to 0$$

COROLLARY 3.21. *Let M be a connected oriented manifold of dimension m then $H_m^{lf}(M)$ is infinite cyclic with generator which we denote by $[M]^{lf}$ such that for every point $x \in X$ we have $[M]^{lf} \mapsto [M, M/\{x\}]$.*

PROOF. Present $M = \cup_k \overset{\circ}{M}_k$ where M_k are connected, compact and $M_k \subseteq \overset{\circ}{M}_{k+1}$. In this case $H_{m+1}(M, M \setminus K)$ vanish for all K ([**10**] VIII,4.1) so the map $H_m^{lf}(M) \to \varprojlim H_m(M, M \setminus M_i)$ is an isomorphism. The statement holds since $\varprojlim H_m(M, M \setminus M_i) = \mathbb{Z}$. $\square$

LEMMA 3.22. *Let M be a connected oriented manifold of dimension m, X a topological space and $f : X \to M$ a proper map. If f is not surjective then $f_* : H_m^{lf}(X) \to H_m^{lf}(M)$ is the zero map.*

PROOF. M is locally compact and Hausdorff and f is proper so by lemma 3.18 $Im(f)$ is closed. Choose a small disc $D \subseteq M \setminus Im(f)$ and look at the following commutative diagram:

$$\begin{array}{ccc} H_m^{lf}(X) & \to & H_m^{lf}(M) \\ \downarrow 0 & & \downarrow\cong \\ H_m^{lf}(X, X) & \to & H_m^{lf}(M, M \setminus D) \end{array}$$

$\square$

COROLLARY 3.23. *Let $(M, \partial M)$ be a connected oriented manifold of dimension m with non empty boundary then $H_m^{lf}(M)$ vanishes.*

PROOF. Denote by DM the double of M, that is $M \cup_{\partial M} M$. The inclusion $i : M \to DM$ is a proper retract thus the map $i_* : H_m^{lf}(M) \to H_m^{lf}(DM)$ is injective. By the lemma, i_* is the zero map so $H_m^{lf}(M)$ must vanish. $\square$

COROLLARY 3.24. *Let $(M, \partial M)$ be a connected oriented manifold of dimension m with non empty boundary then $H_m^{lf}(M, \partial M)$ is infinite cyclic with generator which we denote by $[M, \partial M]^{lf}$ and $\partial[M, \partial M]^{lf} = [\partial M]^{lf}$.*

PROOF. Denote by DM the double of M and by M^- the other copy of M. We have the following commutative diagram:

$$\begin{array}{ccccccccc} H_m^{lf}(\partial M) & \to & H_m^{lf}(M) & \to & H_m^{lf}(M, \partial M) & \to & H_{m-1}^{lf}(\partial M) & \to & H_{m-1}^{lf}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_m^{lf}(M^-) & \to & H_m^{lf}(DM) & \to & H_m^{lf}(DM, M^-) & \to & H_{m-1}^{lf}(M^-) & \to & H_{m-1}^{lf}(DM) \end{array}$$

$H_m^{lf}(M^-)$ and $H_m^{lf}(M)$ vanish since M and M^- are both connected oriented manifolds of dimension m with non empty boundary. There is homeomorphism $M \to M^-$ which preserve ∂M thus an element in $H_{m-1}^{lf}(\partial M)$ is mapped to zero in $H_{m-1}^{lf}(M)$ if and only if it is mapped to zero in $H_{m-1}^{lf}(M^-)$. By exactness the map the map $H_m^{lf}(M,\partial M) \to H_{m-1}^{lf}(M)$ is the zero map so also the map $H_m^{lf}(M,\partial M) \to H_{m-1}^{lf}(M^-)$ is the zero map. By excision $H_m^{lf}(M,\partial M) \to H_m^{lf}(DM,M^-)$ is an isomorphism, so the map $H_m^{lf}(DM,M^-) \to H_{m-1}^{lf}(M^-)$ is the zero map. We conclude that the map $H_m^{lf}(DM) \to H_m^{lf}(DM,M^-)$ is an isomorphism. Since $H_m^{lf}(DM)$ is infinite cyclic we know that the same is true for $H_m^{lf}(DM,M^-)$ and for $H_m^{lf}(M,\partial M)$. We Denote by $[M,\partial M]^{lf}$ the element corresponds to $[DM]^{lf}$.

The fact that $\partial[M,\partial M]^{lf} = [\partial M]^{lf}$ is a local question since by definition these classes are detected locally. Therefore this follows from the compact case which is proved in appendix 1. $\square$

For a discussion about locally finite homology the reader is referred to [**20**], [**25**] and [**31**].

Locally finite homology for CW complexes.

For CW complexes the following is the analog of local compactness:

Definition 3.25. A CW complex X is called locally finite if the set of closed cells is locally finite, that is every point has a neighborhood which meets only finitely many closed cells.

Proposition 3.26. *A CW complex is locally finite if and only if it is locally compact.*

Proof. If X is a locally finite CW complex then every $x \in X$ has a neighborhood which meets only finitely many closed cells, and thus contained in a compact subset which implies that X is locally compact. If X is locally compact then every point has a compact neighborhood which meets only finitely many cells and thus X is locally finite. $\square$

There is also a cellular version of locally finite homology. Let X be a CW complex we define $C_k^{lf}(X) = H_k^{lf}(X_k, X_{k-1})$ with the differential coming from the long exact sequence for the triple (X_k, X_{k-1}, X_{k-2}). For locally finite CW complexes we have by the properties above:

$H_k^{lf}(X_k, X_{k-1}) \cong H_k^{lf}(\amalg_I D^k, \amalg_I S^{k-1}) \cong \Pi_I H_k^{lf}(D^k, S^{k-1}) \cong \Pi_I \mathbb{Z}$

where I is the set of k cells of X.

In general the locally finite cellular homology and the locally finite singular homology are different. For a certain class of CW complexes they agree.

Definition 3.27. A CW complex X is called strongly locally finite if it is the union of finite subcomplexes such that every point in X has a neighborhood which meets only finitely many of them.

Clearly, a strongly locally finite CW complex is locally finite but a locally finite CW complex needs not be strongly locally finite. An example for this is the space $X = e^0 \cup e^1 \cup e^2 ...$ where we attach each k-cell e^k to $e^0 \cup e^1 \cup e^2 ... \cup e^{k-1}$ by collapsing its boundary to a point in the interior of e^{k-1}. X is not strongly locally finite since e^0 is contained in any subcomplex (see [**11**] 1.8).

We have the following propositions regarding strongly locally finite CW complexes:

Proposition 3.28. *([**11**] 1.4) Every locally finite, finite dimensional CW complex is strongly locally finite.*

And the following:

PROPOSITION 3.29. *([**20**] 4.7) For a (countable) strongly locally finite CW complex X the singular chain $S_*^{lf}(X)$ and the cellular chain $C_*^{lf}(X)$ are homology equivalent so $H_*^{lf}(X) = H_*(S_*^{lf}(X)) = H_*(C_*^{lf}(X))$.*

REMARK 3.30. The fact that X is countable plays no role since every component of a locally finite CW complex is countable ([**13**] 11.4.3) and it is enough to prove it for connected spaces.

In [**13**] there is a full treatment of the cellular version.

Poincaré duality for non compact manifolds.
Poincaré duality is a deep theorem about manifolds. In its most common version it states that a closed oriented manifold of dimension m has a fundamental class $[M] \in H_m(M)$ and there is an isomorphism $PD_M : H^k(M) \to H_{m-k}(M)$ given by $\varphi \mapsto \varphi \cap [M]$. There is a similar result for smooth manifolds in locally finite homology ([**25**] 3.1):

THEOREM 3.31. *Let M be a smooth oriented manifold of dimension m, not necessarily compact, then M has a fundamental class $[M]^{lf} \in H_m^{lf}(M)$ and there is an isomorphism $PD_M : H^k(M) \to H_{m-k}^{lf}(M)$ given by $\varphi \mapsto \varphi \cap [M]^{lf}$.*

3.3. Locally finite stratifold homology

DEFINITION 3.32. Let X be a topological space and $k \geq 0$, define $SH_k^{lf}(X)$ to be $\{g : S \to X\} / \sim$ i.e., bordism classes of maps $g : S \to X$ where S is an oriented regular p-stratifold of dimension k and g is a continuous proper map. $SH_k^{lf}(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \to Y$ is a proper map than we can define an induced map by composition $f_* : SH_k^{lf}(X) \to SH_k^{lf}(Y)$.

For each triple there is a boundary operator $\partial : SH_k^{lf}(X) \to SH_{k-1}^{lf}(U \cap V)$. We define it for $X = S$, an oriented regular p-stratifold of dimension k, and the element $[S, Id]$ and extend it to all other triples by naturality. Choose a smooth map $f : S \to \mathbb{R}$ such that $f|_{S \setminus U} = -1$ and $f|_{S \setminus V} = 1$ and a regular value $x \in \mathbb{R}$. Denote by $S' = f^{-1}(x)$ with the induced orientation discussed before. Define $\partial([S, Id]) = [S', i]$ where i is the inclusion $S' \xrightarrow{i} U \cap V$. The fact that it is well defined and the following is similar to what we had before (see App. B in [**23**] for a more subtle discussion):

THEOREM 3.33. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH_k^{lf}(U \cap V) \to SH_k^{lf}(U) \oplus SH_k^{lf}(V) \to SH_k^{lf}(X) \xrightarrow{\partial} SH_{k-1}^{lf}(U \cap V) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

We define the cross product in SH_*^{lf} - $\times : SH_k^{lf}(X) \otimes SH_l^{lf}(Y) \to SH_{k+l}^{lf}(X \times Y)$ by $[g_1 : S \to X] \times [g_2 : T \to Y] = [g_1 \times g_2 : S \times T \to X \times Y]$. This product is bilinear and natural.

SH_*^{lf} with this boundary operator and the cross product is a multiplicative homology theory on the category of topological spaces and proper maps. We call it locally finite (parametrized) stratifold homology.

REMARK 3.34. Recall that SH_k was defined as bordism classes of maps $g : S \to X$ where S is a compact oriented regular p-stratifold of dimension k and g is a continuous map. That is, in the definition of SH_k^{lf} we use proper maps rather than compact p-stratifolds. Since a continuous map from a compact space to a Hausdorff space is proper there is a natural transformation $SH_k(X) \to SH_k^{lf}(X)$.

A natural isomorphism between SH_*^{lf} and H_*^{lf}.

We are going to construct a natural isomorphism $\Phi^{lf} : SH_k^{lf} \to H_k^{lf}$. The construction of Φ^{lf} and the proof that it is an isomorphism is similar to what we did for singular homology. We will give only the outline and stress the differences.

In order to do so we want to associate to each oriented regular p-stratifold S of dimension k a fundamental class which we denote by $[S]^{lf} \in H_k^{lf}(S)$.

LEMMA 3.35. *Let S be an oriented regular p-stratifold of dimension k then $H_l^{lf}(S)$ vanish for $l > k$.*

PROOF. This can be proved by induction. The inductive step uses the Mayer-Vietoris long exact sequence and the fact that for M^k, a k dimensional smooth manifold (with boundary), $H_l^{lf}(M^k)$ vanish for $l > k$ (when we use MV we use the fact that all p-stratifolds are locally compact). □

Let S be an oriented regular p-stratifold of dimension k. The map $H_k^{lf}(M^k, \partial M^k) \xrightarrow{\cong} H_k^{lf}(S, S_{k-2})$ is an isomorphism by excision. The map $H_k^{lf}(S) \xrightarrow{\cong} H_k^{lf}(S, S_{k-2})$ is an isomorphism by the long exact sequence for the pair (S, S_{k-2}) and the fact that $H_l^{lf}(S_{k-2})$ vanish for $l = k-1, k$ by the previous lemma.

DEFINITION 3.36. Define $[S]^{lf} \in H_k(S)$ to be the image of $[M^k, \partial M^k]^{lf}$ under the composition $H_k^{lf}(M^k, \partial M^k) \xrightarrow{\cong} H_k^{lf}(S, S_{k-2}) \xrightarrow{\cong} H_k^{lf}(S)$. We call $[S]^{lf}$ the fundamental class of S. Note that $[S \amalg S']^{lf} = [S]^{lf} + [S']^{lf}$ and $[-S]^{lf} = -[S]^{lf}$.

Let (T, S) be an oriented regular p-stratifold of dimension $k+1$ with boundary. The map $H_{k+1}^{lf}(M^{k+1}, \partial M^{k+1}) \xrightarrow{\cong} H_{k+1}^{lf}(T, T_{k-1} \cup S)$ is an isomorphism by excision. The map $H_{k+1}^{lf}(T, S) \xrightarrow{\cong} H_{k+1}^{lf}(T, T_{k-1} \cup S)$ is an isomorphism by the long exact sequence for the triple $(T, T_{k-1} \cup S, S)$ and the fact that $H_l^{lf}(T_{k-1} \cup S, S) \cong H_l^{lf}(T_{k-1}, S_{k-2})$ by excision which vanish for $l = k, k+1$.

DEFINITION 3.37. Define $[T, S]^{lf} \in H_{k+1}^{lf}(T, S)$ to be the image of $[M^{k+1}, \partial M^{k+1}]^{lf}$ under the composition $H_{k+1}^{lf}(M^{k+1}, \partial M^{k+1}) \xrightarrow{\cong} H_{k+1}^{lf}(T, T_{k-1} \cup S) \xrightarrow{\cong} H_{k+1}^{lf}(T, S)$.

LEMMA 3.38. *Let (T, S) be an oriented regular p-stratifold of dimension $k+1$ with boundary, then $\partial [T, S]^{lf} = [S]^{lf}$.*

PROOF. The same proof as for SH_*. □

COROLLARY 3.39. *Let (T, S) be an oriented regular p-stratifold of dimension $k+1$ with boundary. Denote the inclusion of S in T by i then $i_*([S]^{lf}) = 0$.*

PROOF. This follows from the previous lemma and the exactness of the sequence for the pair - $H_{k+1}^{lf}(T, S) \xrightarrow{\partial} H_k^{lf}(S) \xrightarrow{i_*} H_k^{lf}(T)$ □

Define a natural transformation $\Phi^{lf} : SH_k^{lf}(X) \to H_k^{lf}(X)$ by $\Phi^{lf}([S, g]) = g_*([S]^{lf})$.

Φ^{lf} **is well defined:** As before, it follows from the fact that $\partial[T,S]^{lf}=[S]^{lf}$.
Φ^{lf} **is a group homomorphism:** This follows from the fact that $[S \amalg S']^{lf} = [S]^{lf}+[S']^{lf}$ and $[-S]^{lf}=-[S]^{lf}$.
Φ^{lf} **is natural:** It follows from the functoriality of H_*^{lf}.
Φ^{lf} **commutes with boundary:** The boundary is defined in a similar way to that in SH_*. Again the same proof holds, just note the differences. For the proof that we can find a bicollar for S we used the fact that T was compact. Instead we use the fact that we can always choose a map with a regular value such that its preimage will have a bicollar (**[23]** appendix B).
Φ^{lf} **commutes with the cross product:** We have to show that $\Phi^{lf}(\alpha\times\beta)=\Phi^{lf}(\alpha)\times\Phi^{lf}(\beta)$. By the naturality of the cross product, it is enough to show that for any two oriented regular p-stratifolds S,T of dimension k and l we have $[S\times T]^{lf}=[S]^{lf}\times[T]^{lf}$. If k or l are equal to 0 then it is clear, so we can assume that $k,l>0$. In each component of the top strata we choose a small closed disc, D_α in S and D_β in T. By the naturality of the cross product we have (we use the notation $H_k(X|A)$ instead of $H_k(X,X\setminus A)$ for brevity):

$$\begin{array}{ccccc}
H_k^{lf}(S)\otimes H_l^{lf}(T) & \xrightarrow{\cong} & H_k^{lf}(S|\amalg \overset{\circ}{D}_\alpha)\otimes H_l^{lf}(T|\amalg \overset{\circ}{D}_\beta) & \xleftarrow{\cong} & \oplus H_k^{lf}(D^k,S^{k-1})\otimes H_l^{lf}(D^l,S^{l-1}) \\
\downarrow & & \downarrow & & \downarrow \\
H_{k+l}^{lf}(S\times T) & \xrightarrow{\cong} & H_{k+l}^{lf}(S\times T|\amalg \overset{\circ}{D}_\alpha\times \overset{\circ}{D}_\beta) & \xleftarrow{\cong} & \oplus H_{k+l}^{lf}(D^{k+l},S^{k+l-1})
\end{array}$$

Since the spaces on the right side are compact the locally finite homology is equal to the singular homology. In this case we have

$$\begin{array}{ccc}
\oplus H_k(D^k,S^{k-1})\otimes H_l(D^l,S^{l-1}) & \xrightarrow{\cong} & \oplus H_k(\mathbb{R}^k|0)\otimes H_l(\mathbb{R}^l|0) \\
\downarrow & & \downarrow \\
\oplus H_{k+l}(D^{k+l},S^{k+l-1}) & \xrightarrow{\cong} & \oplus H_{k+l}(\mathbb{R}^{k+l}|0)
\end{array}$$

Which reduces this to the fact which is proved in appendix 1 that the cross product of the generators in $H_k(\mathbb{R}^k|0)$ and $H_l(\mathbb{R}^l|0)$ is the generator of $H_{k+l}(\mathbb{R}^{k+l}|0)$ with the standard orientations.
Φ^{lf} **is a natural isomorphism:** We have the $Mayer\,Vietoris$ long exact sequence for both SH_*^{lf} and H_*^{lf}, Φ^{lf} commutes with boundary and it is an isomorphism on a one point space. This implies the following:

LEMMA 3.40. *Φ^{lf} is an isomorphism for locally finite, finite dimensional CW complexes.*

PROOF. We do it by induction on the dimension of X. We know it is true for 0 dimensional CW complexes. Assume it is true for all locally finite CW complexes of dimension $<n$ and let X be an n dimensional locally finite CW complex. From $Mayer\,Vietoris$ sequence we have:

$$\begin{array}{ccccccc}
SH_k^{lf}(\amalg_\alpha S^{n-1}) \to & SH_k^{lf}(\amalg_\alpha e^n)\oplus SH_k^{lf}(X_{n-1}) \to & SH_k^{lf}(X) & \xrightarrow{\partial} & \dots \\
\downarrow\cong & \downarrow\cong & \downarrow & & \\
H_k^{lf}(\amalg_\alpha S^{n-1}) \to & H_k^{lf}(\amalg_\alpha e^n)\oplus H_k^{lf}(X_{n-1}) \to & H_k^{lf}(X) & \xrightarrow{\partial} & \dots
\end{array}$$

$$\begin{array}{cccc}
\dots & \xrightarrow{\partial} & SH_{k-1}^{lf}(\amalg_\alpha S^{n-1}) \to & SH_{k-1}^{lf}(\amalg_\alpha e^n)\oplus SH_{k-1}^{lf}(X_{n-1}) \\
 & & \downarrow\cong & \downarrow\cong \\
\dots & \xrightarrow{\partial} & H_{k-1}^{lf}(\amalg_\alpha S^{n-1}) \to & H_{k-1}^{lf}(\amalg_\alpha e^n)\oplus H_{k-1}^{lf}(X_{n-1})
\end{array}$$

This diagram is commutative from the naturality of Φ^{lf} and the fact it commutes with ∂. All the vertical arrows except the middle one are isomorphisms by induction

(e^n are properly contractible) so by the five lemma it is also true for the middle one (when we write X_{n-1} we actually mean $X_{n-1} \cup_{f_\alpha} I \times S_\alpha^{n-1}$ - the $n-1$ skeleton with a collar of the attaching maps of the n cells thus it is properly homotopy equivalent to X_{n-1} since X is locally compact). □

To deal with the infinite dimensional case we have to use the following:

PROPOSITION 3.41. *([**11**] 1.7)(Cellular approximation theorem) Let K and M be strongly locally finite CW complexes, L a subcomplex of K and $f : K \to M$ a proper map with $f|_L$ is cellular; then f is properly homotopic to a cellular map through a homotopy fixed on L.*

COROLLARY 3.42. *Φ^{lf} is an isomorphism for all strongly locally finite CW complexes.*

PROOF. p-stratifolds with boundary have the proper homotopy type of a CW pair which is finite dimensional and locally finite (since p-stratifolds are locally compact), hence strongly locally finite by proposition 3.28. This implies that for a map from a p-stratifold with boundary to a strongly locally finite CW complex X we can use cellular approximation and therefore we have an isomorphism $SH_k^{lf}(X) = \varinjlim(SH_k^{lf}(X_n))$. We also have $H_k^{lf}(X) = \varinjlim(H_k^{lf}(X_n))$, since we can compute both sides by the cellular chain (both limits are taken over the skeleta X_n). Thus the statement follows from the finite dimensional case. □

3.4. Stratifold end homology

Let X be a locally compact topological space. The short exact sequence $0 \to S_*(X) \to S_*^{lf}(X) \to S_*^{lf}(X)/S_*(X) \to 0$ gives a long exact sequence in homology. The homology of the third term is called the end homology and is denoted by $H_*^\infty(X)$. The long exact sequence has the form:

$$... \to H_k(X) \to H_k^{lf}(X) \to H_k^\infty(X) \to H_{k-1}(X) \to ...$$

We note few properties of H_k^∞:

- Like H_*^{lf} the end homology are functors from the category of locally compact topological spaces and proper maps to Abelian groups.
- Since both singular homology and locally finite homology are invariant of the proper homotopy type so is end homology, by the five lemma.
- For a compact space the map $S_k(X) \to S_k^{lf}(X)$ is an isomorphism and so is $H_k(X) \to H_k^{lf}(X)$ thus $H_*^\infty(X)$ vanish, this is not the case in general.

We define stratifold end homology denoted by SH_*^∞ such that the following is a long exact sequence:

$$... \to SH_k(X) \to SH_k^{lf}(X) \to SH_k^\infty(X) \to SH_{k-1}(X) \to ...$$

and we construct a natural isomorphism $\Phi^\infty : SH_*^\infty \to H_*^\infty$ such that the map between the two long exact sequences will be a chain map.

DEFINITION 3.43. Let X be a topological space and $k \geq 0$, define $SH_k^\infty(X)$ to be $\{g : (T,S) \to X\} / \sim$ i.e., bordism classes of maps $g : (T,S) \to X$ where (T,S) is a p-stratifold with boundary which is oriented regular of dimension k, S is compact and g is a continuous proper map. The bordism relation is defined the following way: $g : (T,S) \to X$ is bordant to $g' : (T',S') \to X$ if and only if $g|_S : S \to X$ is bordant to $g'|_{S'} : S' \to X$ via a compact bordism T'' and $g'' : T \cup T'' \cup T' \to X$ is null bordant (note that T'' might be non empty even if S and S' are empty).

LEMMA 3.44. *This is an equivalence relation.*

PROOF. The relation is reflexive: Given a map $((T, S), g)$ one can give the structure of a p-stratifold with boundary to $T \times I$ such that its boundary will be equal to $S' = T \times \{0\} \cup S \times I \cup T \times \{1\}$ by a similar procedure to the one appears in **[23]** in appendix A. This implies that $((T, S), g)$ is equivalent to itself.
The relation is symmetric: This is clear.
The relation is transitive: In order to prove this we have to know how to glue two p-stratifolds along a part of their boundary. This is proved in **[23]** in appendix A. $\square$

$SH^{\infty}_k(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. There is a natural transformation $SH^{lf}_k(X) \to SH^{\infty}_k(X)$ given by $[T, g] \mapsto [(T, \emptyset), g]$ and a boundary operator $SH^{\infty}_k(X) \to SH_{k-1}(X)$ given by $[(T, S), g] \mapsto [S, g|_S]$.

PROPOSITION 3.45. The following is a long exact sequence

$$... \to SH_k(X) \to SH^{lf}_k(X) \to SH^{\infty}_k(X) \to SH_{k-1}(X) \to ...$$

PROOF. Clearly, the composition of every two maps is the zero map.

Exactness in $SH_k(X)$ - If $[S, g] \in SH_k(X)$ is mapped to zero in $SH^{lf}_k(X)$ than S is the boundary of some T and g can be extended to a proper map $\tilde{g} : T \to X$, which means that $[S, g] = \partial[(T, S), \tilde{g}]$.

Exactness in $SH^{lf}_k(X)$ is by the definition of the bordism relation in $SH^{\infty}_k(X)$.

Exactness in $SH^{\infty}_k(X)$ - Assume $[(T, S), g] \in SH^{\infty}_k(X)$ and $[S, g|_S] = 0 \in SH_{k-1}(X)$. Take a (compact) bordism of it and glue it to $(T, S) \to X$ and you will get a map from a boundaryless p-stratifold to X. It only left to see that gluing a compact element doesn't change the bordism class which is clear by the definition. $\square$

A natural isomorphism between SH^{∞}_* and H^{∞}_*.

Let (T, S) be a p-stratifold with boundary which is oriented regular of dimension k, and S is compact. There is a long exact sequence for the pair (T, S) - $H^{\infty}_k(S) \to H^{\infty}_k(T) \to H^{\infty}_k(T, S) \to H^{\infty}_{k-1}(S)$. Since S is compact its end homology vanishes, hence the map $H^{\infty}_k(T) \to H^{\infty}_k(T, S)$ is an isomorphism. Take $[T, S]^{lf}$ and push it forward to $H^{\infty}_k(T, S)$ using the map $H^{lf}_k(T, S) \to H^{\infty}_k(T, S)$, and use the isomorphism $H^{\infty}_k(T) \to H^{\infty}_k(T, S)$ to define the fundamental class $[T, S]^{\infty} \in H^{\infty}_k(T)$. Note that if we don't require anything about T this element might be the zero element. If we assume that $H_k(T, S)$ is trivial (or that the fundamental class in $H^{lf}_k(T, S)$ is not in the image of the map $H_k(T, S) \to H^{lf}_k(T, S)$) then it implies that $[T, S]^{\infty}$ is non zero. This suggests that we can look only on maps from pairs (T, S) with this property, for example when T is non compact, or better, has no compact components. As before we have $[T \amalg T', S \amalg S']^{\infty} = [T, S]^{\infty} + [T', S']^{\infty}$ and $[-T, -S]^{\infty} = -[T, S]^{\infty}$.

LEMMA 3.46. *In the previous notation, for the map $\partial : H^{\infty}_k(T) \to H_{k-1}(T)$, we have $\partial[T, S]^{\infty} = i_*([S])$ where i denotes the inclusion of S in T.*

PROOF. By the definition of $[T,S]^\infty$, there is a representative which is an infinite cycle modulo S, that is its boundary is finite and is contained in S. $\partial[T,S]^\infty$ is defined as the class of this boundary, considered as a class in $H_{k-1}(T)$. If we look at the same representative as an element of $H_k^{lf}(T,S)$, its boundary in $H_k^{lf}(S)=H_k(S)$ is exactly the same boundary (even though it lies in a different group). As we saw before, this element is $[S]$, and therefore $\partial[T,S]^\infty = i_*([S])$. $\square$

LEMMA 3.47. *Let (T,S) be a p-stratifold with boundary which is oriented regular of dimension k, and S is compact. Let $(T',-S)$ be a p-stratifold with boundary which is oriented regular of dimension k, and T' is compact. Denote by T'' the gluing of both p-stratifolds along their boundary and the inclusion of T in T'' by i then $i_*([T]^\infty)=[T'']^\infty$.*

PROOF. We have the following commutative diagrams and a natural transformation between them:

$$\begin{array}{ccc} H_k^{lf}(T) & \to & H_k^{lf}(T,S) \\ \downarrow & & \downarrow \\ H_k^{lf}(T'') & \to & H_k^{lf}(T'',T') \end{array} \qquad \begin{array}{ccc} H_k^{\infty}(T) & \xrightarrow{\cong} & H_k^{\infty}(T,S) \\ i_*\downarrow & & \downarrow \\ H_k^{\infty}(T'') & \xrightarrow[j_*]{\cong} & H_k^{\infty}(T'',T') \end{array}$$

T' is compact and therefore the map $j_*: H_k^\infty(T'')\to H_k^\infty(T'',T')$ is an isomorphism so it is enough to prove that $j_*\circ i_*([T]^\infty)=j_*([T'']^\infty)$. We start with $j_*\circ i_*([T]^\infty)$. By the commutativity of the right diagram this equals to the image of $[T,S]^{lf}$ under the composition $H_k^{lf}(T,S)\to H_k^\infty(T,S)\to H_k^\infty(T'',T')$ or by commutativity to its image under the composition $H_k^{lf}(T,S)\to H_k^{lf}(T'',T')\to H_k^\infty(T'',T')$. The first map is an isomorphism by excision and $[T,S]^{lf}$ is mapped to the image of $[T'']^{lf}$ under the map $H_k^{lf}(T'')\to H_k^{lf}(T'',T')$. Therefore the result follows from the commutativity of the following diagram:

$$\begin{array}{ccc} H_k^{lf}(T'') & \to & H_k^{lf}(T'',T') \\ \downarrow & & \downarrow \\ H_k^{\infty}(T'') & \to & H_k^{\infty}(T'',T') \end{array}$$

$\square$

Define a natural transformation $\Phi^\infty : SH_*^\infty \to H_*^\infty$ by $\Phi^\infty([(T,S),g]) = g_*([T,S]^\infty)$.

PROPOSITION 3.48. *Φ^∞ is a well defined natural transformation.*

PROOF. Φ^∞ **is well defined:** It is enough to prove that $g_*([T,S]^\infty)=0$ for the inclusion $g:(T,S)\to L$ where L is a null bordism, that is L is an oriented regular p-stratifold with boundary equal to $T''=T\cup_S T'$ where T' is compact. By the lemma above $i_*([T]^\infty)=[T'']^\infty$. $[T'']^\infty$ is by definition the image of $[T'']^{lf}$. The result follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} & & H_k^{lf}(T'') & \to & H_k^{lf}(L) \\ & & \downarrow & & \downarrow \\ H_k^{\infty}(T) & \to & H_k^{\infty}(T'') & \to & H_k^{\infty}(L) \end{array}$$

Φ^∞ **is a group homomorphism:** This follows from the fact that $[T\amalg T', S\amalg S']^\infty=[T,S]^\infty+[T',S']^\infty$ and $[-T,-S]^\infty=-[T,S]^\infty$
Φ^∞ **is natural:** It follows from the functoriality of H_*^∞. $\square$

PROPOSITION 3.49. *The following diagram commutes:*

$$\begin{array}{ccccccc} SH_k(X) & \to & SH_k^{lf}(X) & \to & SH_k^{\infty}(X) & \to & SH_{k-1}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_k(X) & \to & H_k^{lf}(X) & \to & H_k^{\infty}(X) & \to & H_{k-1}(X) \end{array}$$

PROOF. The left square commutes: It is enough to prove it for $X = S$ a compact oriented regular p-stratifold of dimension k and the element $[S, Id]$. In this case it is trivial since both horizontal maps are actually the identity maps.

The middle square commutes: It is enough to prove it for $X = S$ an oriented regular p-stratifold of dimension k and the element $[S, Id]$ and this is clear.

The right square commutes: It is enough to prove that in the following case: $X = (T, S)$ a regular oriented p-stratifold of dimension k with boundary where S is compact. Here we have to show that the fundamental class $[T, S]^{\infty} \in H_k^{\infty}(T)$ is mapped to the class $i_*([S]) \in H_{k-1}(T)$ which is the case by lemma 3.46. $\square$

COROLLARY 3.50. *The natural transformation $\Phi^{\infty} : SH_*^{\infty} \to H_*^{\infty}$ is an isomorphism for all strongly locally finite CW complexes.*

PROOF. This follows from the fact that the natural transformations $\Phi : SH_* \to H_*$ and $\Phi^{lf} : SH_*^{lf} \to H_*^{lf}$ are isomorphisms and the five lemma. $\square$

CHAPTER 4

Stratifold Cohomology Theories

ABSTRACT. In this chapter we summarize definitions and properties of various cohomology theories and introduce new cohomology theories: SH_c^* and SH_∞^* and identify them with the corresponding ordinary cohomology theories. We also construct an explicit natural isomorphism between stratifold cohomology and singular cohomology. The existence of such an isomorphism was known but the construction is new.

4.1. Report about stratifold cohomology

Stratifold cohomology was defined by Kreck in [**23**]. We will describe here a variant of this theory called parametrized stratifold cohomology, which is naturally isomorphic to it. In this paper we will refer to parametrized stratifold cohomology just as stratifold cohomology.

(parametrized) Stratifold cohomology, denoted by SH^*, is an ordinary cohomology theory defined on the category of smooth oriented manifolds and smooth maps. We will construct a natural isomorphism $\Theta : SH^* \to H^*$. It gives a new geometric point of view on integral cohomology, and has some advantages, some of which we will view later. Poincaré duality for a closed oriented smooth manifold M of dimension m is given by $SH^k(M) \xrightarrow{\cong} SH_{m-k}(M)$ which is trivial.

DEFINITION 4.1. Let M be a smooth oriented manifold of dimension m (not necessarily compact) and $k \geq 0$, define $SH^k(M)$ to be $\{g : S \to M\} / \sim$ i.e., bordism classes of maps $g : S \to M$ where S is an oriented regular p-stratifold of dimension $m-k$ and g is a smooth proper map. $SH^k(M)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : N \to M$ is a smooth map than we can define an induced map $f^* : SH^k(M) \to SH^k(N)$ by pullback (after making f transversal to g). See [**23**] for details on how do we orient this pullback. It can be shown that for the projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ we have $\pi_M^*([S,g]) = [S \times N, g \times Id]$ and $\pi_N^*([T,g']) = (-1)^{ml}[M \times T, Id \times g']$ ($[T,g'] \in SH^l(N)$) where we orient the products by the product orientation.

A triple (U, V, M) consists of M which is a smooth oriented manifold and $U, V \subseteq M$ which are two open subspaces cover M, with the orientation induced by M. For each triple there is a natural coboundary operator $\delta : SH^k(U \cap V) \to SH^{k+1}(M)$ which is defined in [**23**] in a similar way to ∂ in SH_k (but in the opposite direction). We will define it later, note that we add a sign so it will be consistent with the definition in [**10**]. The following is proved in [**23**]:

THEOREM 4.2. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH^k(M) \to SH^k(U) \oplus SH^k(V) \to SH^k(U \cap V) \xrightarrow{\delta} SH^{k+1}(M) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

SH^* is a multiplicative theory. The cross product $\times : SH^k(M) \otimes SH^l(N) \to SH^{k+l}(M \times N)$ is given by $[g_1 : S \to M] \times [g_2 : T \to N] = (-1)^{ml}[g_1 \times g_2 : S \times T \to M \times N]$ (again, the sign differs from the one in [**23**]). This product is bilinear and natural. The cup product is given by $\alpha \cup \beta = \Delta^*(\alpha \times \beta)$ where $\Delta : M \to M \times M$ is the diagonal map. It can be shown that the cup product is also given by transversal intersection. We denote by 1_M or just 1 the element $[M, Id]$, we will see that it is the unit element.

Here are several properties of the cross product which are easily verified (M and N are two smooth oriented manifolds of dimension m and n respectively):

(1) The cross product is associative (this is a simple sign check).
(2) Let $\tau : M \times N \to N \times M$ be the flip map defined by $\tau(x, y) = (y, x)$ then $\tau^*(\alpha \times \beta) = (-1)^{kl} \beta \times \alpha$ for every $\alpha \in SH^l(N)$ and $\beta \in SH^k(M)$ (similar to [**23**]).
(3) $\pi_M^*(\alpha) = \alpha \times 1_N$ and $\pi_N^*(\beta) = 1_M \times \beta$ for every $\alpha \in SH^k(M)$ and $\beta \in SH^l(N)$ where the maps are the projections (this follows from the computation of $\pi_M^*(\alpha)$ above).
(4) $1 \cup \alpha = \alpha \cup 1 = \alpha$ for every $\alpha \in SH^k(M)$ ($1 \cup \alpha = \Delta^*(1 \times \alpha) = \Delta^*(\pi_M^*(\alpha)) = \alpha$ since $\pi_M \circ \Delta = Id$).
(5) $\alpha \times \beta = \pi_M^*(\alpha) \cup \pi_N^*(\beta)$ for every $\alpha \in SH^k(M)$ and $\beta \in SH^l(N)$.

PROOF. This follows from properties 1-4 using the relation $\Delta_{M\times N} = T \circ (\Delta_M \times \Delta_N)$ where:
Δ_M, Δ_N and $\Delta_{M\times N}$ are the diagonal maps of M, N and $M \times N$.
τ the flip map defined above.
$T : M \times M \times N \times N \to M \times N \times M \times N$ defined by $Id_M \times \tau \times Id_N$. □

SH^* with the coboundary operator and the cross product is a multiplicative cohomology theory. We call it (parametrized) stratifold cohomology.

Poincaré duality.

There are two forms of duality called Poincaré duality:

THEOREM 4.3. *Let M be a closed oriented smooth manifold of dimension m then there is an isomorphism $PD_M : SH^k(M) \to SH_{m-k}(M)$.*

THEOREM 4.4. *Let M be a smooth oriented manifold of dimension m then there is an isomorphism $PD_M : SH^k(M) \to SH^{lf}_{m-k}(M)$.*

Both proofs use the following approximation proposition ([**23**] 12.4):

PROPOSITION 4.5. *Let $g : T \to M$ be a continuous map from a smooth c-stratifold T to a smooth manifold M, whose restriction to ∂T is a smooth map. Then g is homotopic rel. boundary to a smooth map.*

The first one also uses the fact that a continuous map to a compact space is proper if and only if the domain is compact.

A natural isomorphism between SH^* and H^*.

Let M be an oriented manifold of dimension m. We have the following isomorphisms:

(1) Poincaré duality $PD_M : SH^k(M) \to SH^{lf}_{m-k}(M)$
(2) $\Phi^{lf} : SH^{lf}_{m-k}(M) \to H^{lf}_{m-k}(M)$
(3) Poincaré duality $PD_M^{-1} : H^{lf}_{m-k}(M) \to H^k(M)$ which is well defined since M is oriented.

The composition $SH^k(M) \to SH^{lf}_{m-k}(M) \to H^{lf}_{m-k}(M) \to H^k(M)$ is an isomorphism of groups for all oriented manifolds, denote it by Θ. We would like to show that Θ is a natural isomorphism.

Before we do that, here are some properties of the cap product we will use later. The signs in the formulas are a subject of convention, and change from one book to the other, depending on the way we define the various products. We follow the one used in [**10**] (see also appendix 1).

We have the following:

- Naturality - Let $f : X \to Y$ be a continuous map then $f_*(f^*(\varphi) \cap \alpha) = \varphi \cap f_*(\alpha)$ for all $\varphi \in H^k(Y)$ and $\alpha \in H_m(X)$ ([**10**] VII,12.6).
- Associativity - $\varphi \cap (\psi \cap \alpha) = (\varphi \cup \psi) \cap \alpha$ for all $\varphi \in H^k(X)$, $\psi \in H^l(X)$ and $\alpha \in H_m(X)$ ([**10**] VII,12.7).
- Unit - Denote by $1_X \in H^0(X)$ the identity element in cohomology then $1_X \cap \alpha = \alpha$ for all $\alpha \in H_m(X)$ ([**10**] VII,12.9).
- Relation with cross product - $(\varphi \times \psi) \cap (\alpha \times \beta) = (-1)^{ml}(\varphi \cap \alpha) \times (\psi \cap \beta)$ for all $\varphi \in H^k(X)$, $\psi \in H^l(Y)$, $\alpha \in H_m(X)$ and $\beta \in H_n(Y)$([**10**] VII,12.17).

The first three formulas hold on chain level so they remain true if we switch H_* with H^{lf}_* or H^* with H^*_c for the last one we will need the following:

For a CW complex X, one can define the cap product in cellular homology. First choose a cellular approximation for the diagonal map $\Delta : X \to X \times X$ denoted by Δ'. This induces a map $\Delta'_* : C_*(X) \to C_*(X \times X) \xrightarrow{\cong} C_*(X) \otimes C_*(X)$. Let $\alpha \in C_m(X)$ be a cellular chain and $\varphi \in C^k(X)$ a cellular cochain. Denote $\Delta'_*(\alpha) = \sum \alpha^1_i \otimes \alpha^2_{m-i}$ and define $\varphi \cap \alpha = (-1)^{k\cdot(m-k)}\varphi(\alpha^2_k) \cdot \alpha^1_{m-k}$. This definition does not depend on the choice of Δ' after passing to homology. More about it can be found in [**27**]. The same thing can be done for locally finite homology for strongly locally finite CW complexes.

LEMMA 4.6. *Let X and Y be strongly locally finite CW complexes. The relation above: $(\varphi \times \psi) \cap (\alpha \times \beta) = (-1)^{ml}(\varphi \cap \alpha) \times (\psi \cap \beta)$ also holds for locally finite homology, that is for , $\varphi \in H^k(X)$, $\psi \in H^l(Y)$, $\alpha \in H^{lf}_m(X)$ and $\beta \in H^{lf}_n(Y)$.*

PROOF. We prove it using cellular homology. We choose cellular approximations to the diagonal maps which we denote by $\Delta'_X : X \to X \times X$ and $\Delta'_Y : Y \to Y \times Y$. The map $\Delta'_X \times \Delta'_Y$ is also cellular. Let $\tau : X \times Y \to Y \times X$ be the flip map then $\Delta'_{X\times Y} = (Id_X \times \tau \times Id_Y) \circ (\Delta'_X \times \Delta'_Y)$ is a cellular approximation for the diagonal map $\Delta_{X\times Y} : X \times Y \to X \times Y \times X \times Y$.

Let $\alpha \in C_m(X)$ and $\beta \in C_n(Y)$ be cellular chains and $\varphi \in C^k(X)$ and $\psi \in C^l(Y)$ be cellular cochains. Denote $\Delta'_{X*}(\alpha) = \sum \alpha^1_i \otimes \alpha^2_{m-i}$ and $\Delta'_{Y*}(\beta) = \sum \beta^1_j \otimes \beta^2_{n-j}$ then:

$\Delta'_{X\times Y*}(\alpha \times \beta) = (Id_X \times \tau \times Id_Y)_* \circ (\Delta_X \times \Delta_Y)_* (\alpha \times \beta)$
$= (Id_X \times \tau \times Id_Y)_* \left(\sum \alpha^1_i \otimes \alpha^2_{m-i} \otimes \beta^1_j \otimes \beta^2_{n-j}\right)$

$= \sum(-1)^{(m-i)j}\left(\alpha_i^1 \otimes \beta_j^1\right) \otimes \left(\alpha_{m-i}^2 \otimes \beta_{n-j}^2\right)$
We conclude:
$(\varphi \times \psi) \cap (\alpha \times \beta) = (-1)^{k(n-l)+(k+l)(m+n-k-l)} \varphi \times \psi(\alpha_k^2 \otimes \beta_l^2) \cdot \alpha_{m-k}^1 \otimes \beta_{n-l}^1$
$= (-1)^{k(n-l)+(k+l)(m+n-k-l)+kl} \varphi(\alpha_k^2)\psi(\beta_l^2) \cdot \alpha_{m-k}^1 \otimes \beta_{n-l}^1$
$= (-1)^{km-kk+lm+ln-ll} \varphi(\alpha_k^2)\psi(\beta_l^2) \cdot \alpha_{m-k}^1 \otimes \beta_{n-l}^1$
$= (-1)^{ml}(-1)^{k(m-k)} \varphi(\alpha_k^2) \cdot \alpha_{m-k}^1 \otimes (-1)^{l(n-l)} \psi(\beta_l^2)\beta_{n-l}^1$
$= (-1)^{ml} \varphi \cap \alpha \times \psi \cap \beta$
The same relation holds after passing to homology. □

LEMMA 4.7. $\Theta(1_M) = 1_M$.

PROOF. This follows from the fact that $1_M \cap [M]^{lf} = [M]^{lf}$ (Unit). □

PROPOSITION 4.8. *Θ is natural, that is for every smooth map $f : N \to M$ between two smooth oriented manifolds of dimension n and m resp. the following diagram commutes:*

$$\begin{array}{ccc} SH^k(M) & \xrightarrow{\Theta} & H^k(M) \\ f^* \downarrow & & \downarrow f^* \\ SH^k(N) & \xrightarrow{\Theta} & H^k(N) \end{array}$$

PROOF. **First case** - $f : N \hookrightarrow M$ is an embedding of N as a closed submanifold of M:

Take an element $\alpha = [S, g] \in SH^k(M)$. We can assume that g is transversal to f, thus we can find a (closed) tubular neighborhood U of N with boundary ∂U (also transversal to g) and a projection map $\pi_N : U \to N$ with the property that the pullback of U will be a tubular neighbourhood $\pi_S : S \pitchfork U \to S \pitchfork N$ (see appendix 1). This can be done when S is a smooth manifold and for a p-stratifold this can be done inductively. We claim that the following diagram commutes:

$$\begin{array}{ccccc} H_{m-k}^{lf}(S) & \xrightarrow{g_*} & H_{m-k}^{lf}(M) & \xrightarrow[\cong]{PD_M^{-1}} & H^k(M) \\ \downarrow & (1) & \downarrow & & \\ H_{m-k}^{lf}(S, S \setminus g^{-1}(\mathring{U})) & \xrightarrow{g_*} & H_{m-k}^{lf}(M, M \setminus \mathring{U}) & & \\ \cong\uparrow Excision & (2) & \cong\uparrow Excision & & \downarrow f^* \\ H_{m-k}^{lf}(S \pitchfork U, S \pitchfork \partial U) & \xrightarrow{g_*} & H_{m-k}^{lf}(U, \partial U) & & \\ \cong\downarrow \varepsilon \cdot \tau_{\pi_S} \cap - & (3) & \cong\downarrow \varepsilon \cdot \tau_{\pi_N} \cap - & & \\ H_{n-k}^{lf}(S \pitchfork N) & \xrightarrow{g_*} & H_{n-k}^{lf}(N) & \xrightarrow[\cong]{PD_N^{-1}} & H^k(N) \end{array}$$

Where $\varepsilon = (-1)^{(n-k)(m-n)}$. The fact that the right side commutes is proved in appendix 1. Squares (1) and (2) commute by the functoriality of H_*^{lf}. Square (3) commutes by the naturality of the Thom class and the fact that the bundle $\pi_S : S \pitchfork U \to S \pitchfork N$ is the pullback of the bundle $\pi_N : U \to N$:
$g_* \circ \pi_{S*}(\varepsilon \cdot \tau_{\pi_S} \cap \alpha) = \varepsilon \cdot \pi_{N*} \circ g_*(\tau_{\pi_S} \cap \alpha) = \varepsilon \cdot \pi_{N*} \circ g_*(g^*(\tau_{\pi_N}) \cap \alpha) = \pi_{N*}(\varepsilon \cdot \tau_{\pi_N} \cap g_*(\alpha))$
We follow both images of $[S]^{lf} \in H_{m-k}^{lf}(S)$. By definition, the image of $[S]^{lf}$ in the top row is $\Theta(\alpha)$, which is mapped in the right column to $f^*(\Theta(\alpha))$. The composition of the maps on the left is denoted in appendix 1 by $(-1)^{(n-k)(m-n)} \cdot \phi$ and it is proved there that $(-1)^{(n-k)(m-n)} \cdot \phi([S]^{lf}) = [S \pitchfork N]^{lf}$. By definition its image in the bottom row is equal to $\Theta(f^*(\alpha))$ using the fact that $[S \pitchfork N, g] = f^*(\alpha)$. Since the diagram commutes we conclude that $f^*(\Theta(\alpha)) = \Theta(f^*(\alpha))$.

The general case - $f : N \to M$ is an arbitrary smooth map:

We embed $i : N \hookrightarrow \mathbb{R}^p$ as a closed submanifold for some p. f is equal to the composition $N \xrightarrow{f\times i} M \times \mathbb{R}^p \xrightarrow{\pi_M} M$. $f \times i$ is an embedding of N as a closed submanifold. π_M has an inverse up to homotopy which is an embedding - $M \xrightarrow{Id\times 0} M \times \mathbb{R}^p$ hence this follows from the previous case. □

PROPOSITION 4.9. *Θ commutes with the coboundary operator in the Mayer-Vietoris sequence.*

PROOF. Let (U, V, M) be a triple where M is a smooth oriented manifold of dimension m and U, V are two open sets in M. For $k \geq 0$ the coboundary operator $\delta : SH^k(U\cap V) \to SH^{k+1}(M)$ is given in the following way: For $[S, g] \in SH^k(U\cap V)$ we choose a smooth map $f : M \to \mathbb{R}$ such that $f|_{M\setminus U} = -1$ and $f|_{M\setminus V} = 1$ and a regular value $-1 < x < 1$ of the composition $f \circ g$. Denote by $S' = (f \circ g)^{-1}(x)$, then S' is a regular p-stratifold of dimension $m - k - 1$ and we give it the induced orientation. The map $g' : S' \to M$ is proper, we define $\delta([S, g]) = (-1)^{k+1}[S', g']$.

We can choose x to be a regular value both of $f\circ g$ and f. Denote $M_\partial = f^{-1}(x)$, then M_∂ is a closed submanifold of M of dimension $m - 1$ which is included in $U \cap V$, we give it the induced orientation. Denote the inclusions $i : M_\partial \to U \cap V$ and $j : M_\partial \to M$ Then $(-1)^{k+1}\delta$ equals to the composition:

$$SH^k(U \cap V) \xrightarrow{i^*} SH^k(M_\partial) \xrightarrow{PD_{M_\partial}} SH^{lf}_{m-k-1}(M_\partial) \xrightarrow{j_*} SH^{lf}_{m-k-1}(M) \xrightarrow{PD_M^{-1}} SH^{k+1}(M)$$

By what we showed so far the following diagram commutes:

$$\begin{array}{ccccccccc}
SH^k(U\cap V) & \xrightarrow{i^*} & SH^k(M_\partial) & \xrightarrow{PD_{M_\partial}} & SH^{lf}_{m-k-1}(M_\partial) & \xrightarrow{j_*} & SH^{lf}_{m-k-1}(M) & \xrightarrow{PD_M^{-1}} & SH^{k+1}(M) \\
\Theta\downarrow & & \Theta\downarrow & & \Phi^{lf}\downarrow & & \Phi^{lf}\downarrow & & \Theta\downarrow \\
H^k(U\cap V) & \xrightarrow{i^*} & H^k(M_\partial) & \xrightarrow{PD_{M_\partial}} & H^{lf}_{m-k-1}(M_\partial) & \xrightarrow{j_*} & H^{lf}_{m-k-1}(M) & \xrightarrow{PD_M^{-1}} & H^{k+1}(M)
\end{array}$$

Denote by $\rho : H^k(U \cap V) \to H^{k+1}(M)$ the composition of the maps in the bottom row. By commutativity of the diagram $\Theta(\delta\alpha) = (-1)^{k+1}\rho\Theta(\alpha)$. If we show that $(-1)^{k+1}\rho$ equals to δ we will deduce that $\Theta(\delta\alpha) = \delta\Theta(\alpha)$.

Denote by $\tilde{U} = f^{-1}([x, \infty))$ and $\tilde{V} = f^{-1}((\infty, x])$ then $\tilde{U} \cap \tilde{V} = M_\partial$. There is a map of triples $s : (\tilde{U}, \tilde{V}, M) \to (U, V, M)$. We claim that the following diagram commutes up to sign $(-1)^{k+1}$ (here δ is the connecting homomorphism in the sequence of a pair):

$$\begin{array}{ccccc}
H^k(U\cap V) & \xrightarrow{i^*} & H^k(M_\partial) & \xrightarrow{PD_{M_\partial}} & H^{lf}_{m-k-1}(M_\partial) \\
\delta\downarrow & (1) & \delta\downarrow & (4) & i_*\downarrow \\
H^{k+1}(U, U\cap V) & \xrightarrow{i^*} & H^{k+1}(\tilde{U}, M_\partial) & \xrightarrow{PD_{\tilde{U}}} & H^{lf}_{m-k-1}(\tilde{U}) \\
excision\downarrow & (2) & excision\downarrow & (5) & s_*\downarrow \\
H^{k+1}(M, V) & \xrightarrow{i*} & H^{k+1}(M, \tilde{V}) & \xrightarrow{-\cap[M,\tilde{V}]^{lf}} & H^{lf}_{m-k-1}(M) \\
\downarrow & (3) & \downarrow & (6) & \downarrow \\
H^{k+1}(M) & \xrightarrow{Id} & H^{k+1}(M) & \xrightarrow{PD_M} & H^{lf}_{m-k-1}(M)
\end{array}$$

(1) Commutes by the naturality of the connecting homomorphism for pairs.
(2) Commutes by the functoriality H^*.
(3) Commutes by the functoriality H^*.
(4) Commutes up to sign $(-1)^{k+1}$ by ([**10**] VIII, 9.1).
(5) Commutes by the naturality of the cap product:
Take $\varphi \in H^{k+1}(M, \tilde{V})$ then the composition:

$H^{k+1}(M,\tilde{V}) \xrightarrow{s^*} H^{k+1}(\tilde{U},M_\partial) \xrightarrow{PD_{\tilde{U}}} H^{lf}_{m-k-1}(\tilde{U}) \xrightarrow{s_*} H^{lf}_{m-k-1}(M)$ equals to:
$\varphi \mapsto s^*(\varphi) \mapsto s^*(\varphi) \cap [\tilde{U},M_\partial]^{lf} \mapsto s_*(s^*(\varphi) \cap [\tilde{U},M_\partial]^{lf})$
$= \varphi \cap s_*([\tilde{U},M_\partial]^{lf}) = \varphi \cap [M,\tilde{V}]^{lf}$
which equals to the image of φ under the map:
$H^{k+1}(M,\tilde{V}) \xrightarrow{-\cap[M,\tilde{V}]^{lf}} H^{lf}_{m-k-1}(M)$.

(6) Commutes by the naturality of the cap product.

We conclude that $(-1)^{k+1}\rho$ equals to the composition:
$H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(U,U\cap V) \xrightarrow{excision} H^{k+1}(M,V) \xrightarrow{i*} H^{k+1}(M)$
We would like to show that this composition is equal to the coboundary map in the Mayer Vietoris sequence. To see that we just note that both maps take an element $[\varphi] \in H^k(U\cap V)$ and map it in the following steps:

(1) Extend φ to U denoted by $\bar{\varphi}$ by defining $\bar{\varphi}(\sigma) = \varphi(\sigma)$ if σ is a simplex in $U\cap V$ and 0 else.
(2) Take its differential $\delta(\bar{\varphi}) = (-1)^{k+1}\bar{\varphi}\circ\partial$ and notice that it lies in $C^{k+1}(U,U\cap V)$.
(3) Extend this cochain by zero to an element in $C^{k+1}(M,V)$ which is also an element in $C^{k+1}(M)$.

□

Lemma 4.10. *Θ commutes with the cross product, that is for every two smooth oriented manifolds M and N of dimension m and n respectively the following diagram commutes:*

$$\begin{array}{ccc} SH^k(M)\otimes SH^l(N) & \xrightarrow{\Theta\otimes\Theta} & H^k(M)\otimes H^l(N) \\ \times\downarrow & & \downarrow\times \\ SH^{k+l}(M\times N) & \xrightarrow{\Theta} & H^{k+l}(M\times N) \end{array}$$

Proof. This diagram is equal to the composition of the following three diagrams:

The first diagram commutes up to sign $(-1)^{ml}$ since the horizontal maps are identities and the vertical maps are equal up to that exact sign:

$$\begin{array}{ccc} SH^k(M)\otimes SH^l(N) & \xrightarrow{PD_M\otimes PD_N} & SH^{lf}_{m-k}(M)\otimes SH^{lf}_{n-l}(N) \\ \times\downarrow & & \downarrow\times \\ SH^{k+l}(M\times N) & \xrightarrow{PD_{M\times N}} & SH^{lf}_{m+n-k-l}(M\times N) \end{array}$$

The second diagram commutes:

$$\begin{array}{ccc} SH^{lf}_{m-k}(M)\otimes SH^{lf}_{n-l}(N) & \xrightarrow{\Phi^{lf}\otimes\Phi^{lf}} & H^{lf}_{m-k}(M)\otimes H^{lf}_{n-l}(N) \\ \times\downarrow & & \downarrow\times \\ SH^{lf}_{m+n-k-l}(M\times N) & \xrightarrow{\Phi^{lf}} & H^{lf}_{m+n-k-l}(M\times N) \end{array}$$

since Φ^{lf} commutes with cross product (as was shown before).
And the third diagram commutes up to sign $(-1)^{ml}$:

$$\begin{array}{ccc} H^{lf}_{m-k}(M)\otimes H^{lf}_{n-l}(N) & \xrightarrow{PD_M^{-1}\otimes PD_N^{-1}} & H^k(M)\otimes H^l(N) \\ \times\downarrow & & \downarrow\times \\ H^{lf}_{m+n-k-l}(M\times N) & \xrightarrow{PD^{-1}_{M\times N}} & H^{k+l}(M\times N) \end{array}$$

This can be seen by the formula $\varphi\times\psi\cap[M]^{lf}\times[N]^{lf} = (-1)^{ml}\varphi\cap[M]^{lf}\times\psi\cap[N]^{lf}$ and the fact that $[M]^{lf}\times[N]^{lf} = [M\times N]^{lf}$.
Composing the three diagrams we get the commutativity of the original diagram. □

We proved that $\Theta : SH^* \to H^*$ is a natural isomorphism of graded groups, it commutes with the coboundary operator in the $Mayer-Vietoris$ sequence and with the cross product, thus we proved the following:

THEOREM 4.11. *Θ is a natural isomorphism of multiplicative cohomology theories.*

4.2. Stratifold cohomology with compact support

Stratifold cohomology with compact support, denoted by SH^*_c, is a multiplicative theory defined on the category of smooth oriented manifolds and smooth proper maps between them. It is given by bordism classes of smooth maps from compact oriented regular p-stratifolds. The definitions are similar to those of stratifold cohomology so we will not repeat them.

Let (U,V,M) be a triple, that is U and V are open subspaces in M. Since the inclusion of an open subspace is not proper we don't have induced maps $SH^*_c(M)\to SH^*_c(U)$ and $SH^*_c(M)\to SH^*_c(V)$ so we don't have a $Mayer-Vietoris$ sequence like we had for SH^*. For an open subspace of a manifold we can define an induced map in the other direction by composition so we will get maps $SH^*_c(U)\to SH^*_c(M)$ and $SH^*_c(V)\to SH^*_c(M)$. We can define $\partial : SH^k_c(M)\to SH^{k+1}_c(U\cap V)$ like the one we had for SH_* and we will get the following $Mayer-Vietoris$ sequence in SH^*_c:
$... \to SH^k_c(U\cap V)\to SH^k_c(U)\oplus SH^k_c(V)\to SH^k_c(M)\xrightarrow{\partial} SH^{k+1}_c(U\cap V)\to ...$
We have the following duality which is also called Poincaré duality:

THEOREM 4.12. *Let M be a smooth oriented manifold of dimension m then there is an isomorphism $PD_M : SH^k_c(M)\to SH_{m-k}(M)$.*

PROOF. This follows from the approximation proposition we stated before, that every map from a stratifold to a smooth manifold is homotopic to a smooth map relative its boundary. □

A natural isomorphism between SH^*_c and H^*_c.
Like we did before, for a smooth oriented manifold M of dimension m we have group isomorphisms $SH^k_c(M)\to SH_{m-k}(M)\to H_{m-k}(M)\to H^k_c(M)$ where the last isomorphism is given by Poincaré duality. We denote the composition by Θ_c.

LEMMA 4.13. *Let M and N be two smooth oriented manifolds of dimension m and n respectively, then for every $\varphi\in H^k_c(M)$ and $\psi\in H^l_c(N)$ we have:*
$\varphi\times\psi\cap[M\times N]^{lf} = (-1)^{ml}\varphi\cap[M]^{lf}\times\psi\cap[N]^{lf}$

PROOF. $H^k_c(M) = \varinjlim H^k(M,M\setminus K)$ and $H^k_c(N) = \varinjlim H^k(N,N\setminus L)$ where the limits are taken over all compact subsets ([**17**] p. 244). Let $\varphi_0\in H^k(M,M\setminus K_0)$ and $\psi_0\in H^k(N,N\setminus L_0)$ be two classes that are mapped to φ and ψ. Note that also $\varphi_0\times\psi_0$ is mapped to $\varphi\times\psi$. This means that:

(1) $\varphi_0\cap[M]_{K_0} = \varphi\cap[M]^{lf}$
(2) $\psi_0\cap[N]_{L_0} = \psi\cap[N]^{lf}$

(3) $\varphi_0 \times \psi_0 \cap [M \times N]_{K_0 \times L_0} = \varphi \times \psi \cap [M \times N]^{lf}$

Combining this with the fact that $[M \times N]_{K_0 \times L_0} = [M]_{K_0} \times [N]_{L_0}$ we get:
$\varphi \times \psi \cap [M \times N]^{lf} = \varphi_0 \times \psi_0 \cap [M \times N]_{K_0 \times L_0} = \varphi_0 \times \psi_0 \cap [M]_{K_0} \times [N]_{L_0} =$
$(-1)^{ml}\varphi_0 \cap [M]_{K_0} \times \psi_0 \cap [N]_{L_0} = (-1)^{ml}\varphi \cap [M]^{lf} \times \psi \cap [N]^{lf}$ $\square$

Proposition 4.14. *Θ_c commutes with the cross product.*

Proof. It is proved in a similar way to what we had for Θ, using the lemma above. $\square$

Proposition 4.15. *Θ_c is natural.*

Proof. **First case** - $f : N \hookrightarrow M$ is an embedding of N as a closed submanifold of M:
This is proved just like what we had for Θ.
Second case - $f = \pi_M : M \times N \to M$ is a projection (for example on the first factor) with N compact:

$$\begin{array}{ccc} SH_c^k(M) & \xrightarrow{\Theta_c} & H_c^k(M) \\ \pi_M^* \downarrow & & \downarrow \pi_M^* \\ SH_c^k(M \times N) & \xrightarrow{\Theta_c} & H_c^k(M \times N) \end{array}$$

Let $\alpha \in SH_c^k(M)$ then $\Theta_c \circ \pi_M^*(\alpha) = \Theta_c(\alpha \times 1_N) = \Theta_c(\alpha) \times \Theta_c(1_N) = \Theta_c(\alpha) \times 1_N = \pi_M^*(\Theta_c(\alpha))$.
We used here the following facts proved before:

(1) $\pi_M^*(\alpha) = \alpha \times 1_N$ which is true both for SH_c^* and for H_c^*.
(2) $\Theta_c(1_N) = 1_N$
(3) Θ_c commutes with the cross product.

The general case - $f : N \to M$ is an arbitrary smooth map:
We cannot use the factorization $N \xrightarrow{f \times i} M \times \mathbb{R}^p \xrightarrow{\pi_M} M$ since π_M is not proper. Instead we embed $N \hookrightarrow \mathbb{R}^p$ and compose it with the map in $\mathbb{R}^p \to S^p$. We get an injective map $g : N \to S^p$ which is not proper but the map $f \times g : N \to M \times S^p$ is proper (it is enough that f is proper) and the map $\pi_M : M \times S^p \to M$ is proper. Hence the general case follows from the previous cases. $\square$

Proposition 4.16. *Θ_c commutes with the induced maps for inclusions of open subspaces and with ∂.*

Proof. Unlike the situation in ordinary cohomology, in cohomology with compact support all the functors $SH_c^k(M), SH_{m-k}(M), H_{m-k}(M), H_c^k(M)$ are covariant with respect to inclusions of open subspaces and ∂ maps in the same direction. To show that Θ_c commutes with the induced maps for inclusions of open subspaces and with ∂ we will show that each of the natural transformations above does.
$PD_M : SH_c^k(M) \to SH_{m-k}(M)$ commutes with induced maps of inclusions of open subspaces and with ∂ by definition.
Φ commutes with induced maps of inclusions of open subspaces and with ∂ as was shown before.
$PD_M^{-1} : H_{m-k}(M) \to H_c^k(M)$ commutes with induced maps of inclusions of open subspaces and with ∂ (in the proof of [**29**] A.9). $\square$

4.3. Stratifold end cohomology

Stratifold end cohomology, denoted by SH^*_∞, is defined in a similar way to stratifold end homology and the induced maps are defined like those in stratifold cohomology. It is defined on the category of smooth oriented manifolds and proper smooth maps between them. It is given by bordism classes of proper maps from oriented regular p-stratifolds with boundary which is compact. The definitions are similar to those of stratifold end homology so we will not repeat them. Note that it is not multiplicative. Again we have a long exact sequence:

$$... \to SH^k_c \to SH^k \to SH^k_\infty \to SH^{k+1}_c \to ...$$

We can construct $\Theta_\infty : SH^*_\infty \to H^*_\infty$ as the composition $SH^*_\infty(M) \to SH^\infty_{m-*}(M) \to H^\infty_{m-*}(M) \to H^*_\infty(M)$. We will have the following:

Proposition 4.17. *Let M be a smooth oriented manifold of dimension m then the following diagram commutes:*

$$\begin{array}{ccccccc} SH^k_c(M) & \to & SH^k(M) & \to & SH^k_\infty(M) & \to & SH^{k+1}_c(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^k_c(M) & \to & H^k(M) & \to & H^k_\infty(M) & \to & H^{k+1}_c(M) \end{array}$$

Proof. The following diagram clearly commutes:

$$\begin{array}{ccccccc} SH^k_c(M) & \to & SH^k(M) & \to & SH^k_\infty(M) & \to & SH^{k+1}_c(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ SH_{m-k}(M) & \to & SH^{lf}_{m-k}(M) & \to & SH^\infty_{m-k}(M) & \to & SH_{m-k-1}(M) \end{array}$$

We proved before that the following diagram commutes:

$$\begin{array}{ccccccc} SH_{m-k}(M) & \to & SH^{lf}_{m-k}(M) & \to & SH^\infty_{m-k}(M) & \to & SH_{m-k-1}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{m-k}(M) & \to & H^{lf}_{m-k}(M) & \to & H^\infty_{m-k}(M) & \to & H_{m-k-1}(M) \end{array}$$

And the following diagram commutes by ([**25**] 3.1):

$$\begin{array}{ccccccc} H_{m-k}(M) & \to & H^{lf}_{m-k}(M) & \to & H^\infty_{m-k}(M) & \to & H_{m-k-1}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^k_c(M) & \to & H^k(M) & \to & H^k_\infty(M) & \to & H^{k+1}_c(M) \end{array}$$

The composition of those diagrams gives us the diagram above by the construction of Θ, Θ_c and Θ_∞. □

CHAPTER 5

Backwards (Co)Homology and Equivariant Poincaré Duality

ABSTRACT. Let G be a finite group. In this chapter we introduce a new cohomology theory for $G-CW$ complexes called backwards (co)homology, relate it to ordinary equivariant (co)homology and Tate (co)homology and prove equivariant Poincaré duality for closed oriented manifolds with a smooth and orientation preserving G action.

5.1. Group (co)homology with coefficients in a chain complex

We follow the presentation of (co)homology in [**7**] and [**3**]. To do this we need some basic constructions. In this chapter the group G is assumed to be finite unless stated otherwise and all modules are assumed to be left modules unless stated otherwise. Moreover, since each left $\mathbb{Z}[G]$ module has a natural structure of a right $\mathbb{Z}[G]$ module and vice versa we will not pay attention to the difference between them.

We fix a group G. We start by introducing group homology and cohomology which are functors from the category of $\mathbb{Z}[G]$ modules to the category of grades Abelian groups. Note that both group homology and group cohomology are covariant considered this way. Before we do that, here are some preliminaries:

Let M be a $\mathbb{Z}[G]$ module. A projective resolution of M is a sequence of projective $\mathbb{Z}[G]$ modules $... \to Q_1 \to Q_0$ with a map $Q_0 \to M$ such that $Q_* \to M \to 0$ is exact. There is a functorial way to construct projective resolutions.

The following implies the uniqueness of projective resolutions:

PROPOSITION 5.1. *([**7**] I,7.5) Let Q_* and Q'_* be two projective resolutions of a $\mathbb{Z}[G]$ module M then there exists an augmentation preserving chain map $f : Q_* \to Q'_*$ which is unique up to homotopy and f is a homotopy equivalence.*

We will later need the following (here R is a ring):

THEOREM 5.2. *(Duality [**7**] I,8.3) Let P be a finitely generated projective (left) R module and denote $P^* = Hom_R(P, R)$ then:*
1) P^* is a finitely generated projective (right) R module.
2) For every (left) R module M there is an isomorphism $P^* \otimes_R M \to Hom_R(P, M)$ of Abelian groups.
3) For every (right) R module M there is an isomorphism $M \otimes_R P \to Hom_R(P^*, M)$ of Abelian groups.
4) There is an isomorphism $P \to P^{**}$ of (left) R modules.

PROPOSITION 5.3. *([7] VI,3.4) For any finite group G and any module M there is a natural isomorphism of $\mathbb{Z}[G]$ modules $\psi : Hom_{\mathbb{Z}}(M,\mathbb{Z}) \to M^* = Hom_{\mathbb{Z}[G]}(M,\mathbb{Z}[G])$ given by $\psi(u)(m) = \Sigma_{g\in G} u(g^{-1}m)\cdot g$*

Group homology and cohomology.

Let P_* be a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$ module. The homology of the group G with coefficients in the $\mathbb{Z}[G]$ module M, denoted by $H_*(G,M)$, is the homology of the chain complex $P_* \otimes_{\mathbb{Z}[G]} M$. The cohomology of G with coefficients in M, denoted by $H^*(G,M)$, is the homology of the cochain complex $Hom_{\mathbb{Z}[G]}(P_*,M)$ (we use the sign convention as in [7] $(\delta u)(x) = (-1)^{n+1}u(\partial x)$ for all $u \in Hom_{\mathbb{Z}[G]}(P_n,M)$ and $x \in P_{n+1}$). A consequence of proposition 5.1 is that the definition is independent of the choice of the projective resolution.

REMARK 5.4. From now on we will assume that G is finite(!)

A backwards projective resolution of $\mathbb{Z}$ is a sequence of projective $\mathbb{Z}[G]$ modules $P_0 \to P_{-1} \to ...$ with a map $\mathbb{Z} \to P_0$ such that $0 \to \mathbb{Z} \to P_*$ is exact. The following appears in ([7] VI,3.5) and it implies the existence and uniqueness of backwards resolutions:

PROPOSITION 5.5. *Let G be a finite group and $P_* \to \mathbb{Z}$ a finite type (all P_k are finitely generated) projective resolution then the dual cochain complex $\mathbb{Z} \to Hom_{\mathbb{Z}[G]}(P_0,\mathbb{Z}[G]) \to Hom_{\mathbb{Z}[G]}(P_1,\mathbb{Z}[G]) \to ...$ is a backwards projective resolution. Every finite type backwards projective resolution is obtained this way up to isomorphism.*

REMARK 5.6. 1) The condition that P_* is of finite type can always be obtained, even in a functorial way.
2) In the dual chain, we use the sign convention for Hom complexes.

Let P_*^+ be a projective resolution and P_*^- a backwards projective resolution. By splicing together P_*^+ and $P_*^-[-1]$ (P_*^- with a dimension shift) we get what is called a complete (projective) resolution $..P_1 \to P_0 \to P_{-1} \to ...$ (for $k < 0$ we define $P_k = P_{k+1}^-$). We denote the whole sequence by P_*. The map $P_0 \to P_{-1}$ is given by the composition $P_0 \to \mathbb{Z} \to P_0^-$ which is a part of the data of a complete resolution. From now on, we use the notation P_*^+, P_*^-, P_* for a projective resolution, a backwards projective resolution and a complete resolution respectively.

REMARK 5.7. We use the convention that the boundary operator in $P_*^-[-1]$ is minus the boundary operator in P_*^-.

Tate homology and cohomology.

Tate homology and cohomology are defined for finite groups. The Tate homology (cohomology) of the group G with coefficients in the $\mathbb{Z}[G]$ module M, denoted by $\hat{H}_*(G,M)$ ($\hat{H}^*(G,M)$), is the homology of the chain complex $P_* \otimes_{\mathbb{Z}[G]} M$ ($Hom_{\mathbb{Z}[G]}(P_*,M)$) where P_* is a complete resolution. One might show that the definition is independent of the choice of the complete resolution ([7] VI,3.3).

Backwards homology and cohomology.

We define the backwards homology and cohomology for finite groups. The backwards homology (cohomology) of the group G with coefficients in the $\mathbb{Z}[G]$ module M, denoted by $DH_*(G,M)$ ($DH^*(G,M)$), is the homology of the chain complex $P_*^- \otimes_{\mathbb{Z}[G]} M$ ($Hom_{\mathbb{Z}[G]}(P_*^-, M)$) where P_*^- is a backwards projective resolution. One might show that the definition is independent of the choice of the backwards projective resolution (similar to **[7]** VI,3.3).

At first look this doesn't look interesting due to the following:

PROPOSITION 5.8. *There are natural isomorphisms:*
$DH_k(G,M) \to H^{-k}(G,M)$ *and* $DH^k(G,M) \to H_{-k}(G,M)$.

PROOF. $DH_k(G,M)$ is the homology of the chain complex $P_*^- \otimes_{\mathbb{Z}[G]} M$. We may assume that all the modules in the complex P_*^- are finitely generated and are equal to $Hom_{\mathbb{Z}[G]}(P_{-k}, \mathbb{Z}[G])$. By the duality theorem we have a natural isomorphism $Hom_{\mathbb{Z}[G]}(P_{-k}, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} M \to Hom_{\mathbb{Z}[G]}(P_{-k}, M)$ since P_{-k} is finitely generated. Thus we get that $DH_k(G,M)$ is naturally isomorphic to the homology of the complex $Hom_{\mathbb{Z}[G]}(P_{-k}, M)$ which is equal to $H^{-k}(G,M)$. The other statement is proved in a similar way. $\square$

REMARK 5.9. This isomorphism is natural in M but not in G.

PROPOSITION 5.10. *(**[7]** III,6.1) Let* $0 \to M \to M' \to M'' \to 0$ *be an exact sequence of* $\mathbb{Z}[G]$ *modules then the following is exact:*
$... \to H_k(G,M) \to H_k(G,M') \to H_k(G,M'') \to H_{k-1}(G,M) \to ...$
We also have similar results for $\hat{H}_*$ *and* DH_* *and also in cohomology.*

There is a short exact sequence of complexes $0 \to P_*^-[-1] \to P_* \to P_*^+ \to 0$ since for $k < 0$ we have $0 \to P_k \xrightarrow{Id} P_k \to 0 \to 0$ and for $0 \le k$ we have $0 \to 0 \to P_k \xrightarrow{Id} P_k \to 0$. This implies that for every $\mathbb{Z}[G]$ module M the following are short exact sequences:
$0 \to P_*^-[-1] \otimes_{\mathbb{Z}[G]} M \to P_* \otimes_{\mathbb{Z}[G]} M \to P_*^+ \otimes_{\mathbb{Z}[G]} M \to 0$
$0 \to Hom_{\mathbb{Z}[G]}(P_*^+, M) \to Hom_{\mathbb{Z}[G]}(P_*, M) \to Hom_{\mathbb{Z}[G]}(P_*^-[-1], M) \to 0$
Thus we have the following:

PROPOSITION 5.11. *The following are exact:*
$... \to DH_{k+1}(G,M) \to \hat{H}_k(G,M) \to H_k(G,M) \to DH_k(G,M) \to ...$
$... \to \hat{H}^{k-1}(G,M) \to DH^k(G,M) \to H^k(G,M) \to \hat{H}^k(G,M) \to ...$
for every finite group G *and* $\mathbb{Z}[G]$ *module* M.

REMARK 5.12. The boundary maps $H_k(G,M) \to DH_k(G,M)$ and $DH^k(G,M) \to H^k(G,M)$ are induced by the chain map $P_*^+ \to P_*^-$.

The only interesting case is when $k = 0$ since in all other cases the maps are isomorphisms or the zero map. This gives us the following:

COROLLARY 5.13. *The following is exact:*
$0 \to \hat{H}^{-1}(G,M) \to H_0(G,M) \to H^0(G,M) \to \hat{H}^0(G,M) \to 0$.

PROOF. This follows from the fact that $DH^k(G,M)$ vanishes for $k = 1$, $H^k(G,M)$ vanishes for $k = -1$ and the isomorphism $DH^0(G,M) \to H_0(G,M)$. $\square$

(Co)homology with coefficients in a chain complex.

Before we start talking about homology and cohomology with coefficients in a chain complex we recall the basic operations on chain complexes, that is the tensor product and the Hom complex. In this section all chain complexes, tensor products and Hom complexes will be over R, for some ring R, but we will omit it from the notation.

Let C_* and D_* be two chain complexes of R modules. We define their tensor product to be $(C_* \otimes D_*)_n = \oplus_{k+l=n} C_k \otimes D_l$ with the following differential: $\partial : \oplus_{k+l=n} C_k \otimes D_l \to \oplus_{i+j=n-1} C_i \otimes D_j$ given by $\partial(c \otimes d) = \partial c \otimes d + (-1)^k (c \otimes \partial d)$. It is easily verified that $C_* \otimes D_*$ is a chain complex.

The tensor product of chain complexes is functorial in the sense that if we have chain maps $f : C_* \to C'_*$ and $g : D_* \to D'_*$ then there is an induced map $f \otimes g : (C_* \otimes D_*) \to (C'_* \otimes D'_*)$.

Let C_* be a chain complex and D^* a cochain complex of R modules. We define the Hom complex of C_* and D^* to be $Hom(C_*, D^*)_n = \oplus_{k+l=n} Hom(C_k, D^l)$ where the differential $\oplus_{k+l=n} Hom(C_k, D^l) \to \oplus_{i+j=n+1} Hom(C_i, D^j)$ is given by $\delta\varphi = \delta_D \circ \varphi + (-1)^{n+1} \varphi \circ \partial_C$. It is easily verified that $Hom(C_*, D^*)$ is a cochain complex.

The Hom complex of a chain complex and a cochain complex is functorial in the sense that if we have chain maps $f : C'_* \to C_*$ and $g : D^* \to D'^*$ then there is an induced map $Hom(f, g) : Hom(C_*, D^*) \to Hom(C'_*, D'^*)$, that is contravariant in the first factor and covariant in the second.

REMARK 5.14. We can associate to a chain complex A_* a cochain complex A^{-*} and vise versa. In this way we can define for example the tensor product of cochain complexes or the Hom complex of maps between chain complexes, after making the right adjustments to the indexing of the sums.

We have the following:

PROPOSITION 5.15. *([**7**] I,0 ex. 6) Let C_1 and C_2 be complexes of $\mathbb{Z}[G]$ modules, and C_3 a complex of $\mathbb{Z}$ modules. There is a natural isomorphism:*
$Hom_{\mathbb{Z}}(C_1 \otimes_{\mathbb{Z}[G]} C_2, C_3) \to Hom_{\mathbb{Z}[G]}(C_1, Hom_{\mathbb{Z}}(C_2, C_3))$ which is a chain map.

We call this map the adjunction map.

REMARK 5.16. Here $Hom_{\mathbb{Z}}(C_*, D^*) = \Pi_{k+l=n} Hom(C_k, D^l)$ and not as mentioned before. We will only use the adjunction map when there is no difference between the direct sum and the direct product.

It is a simple check of signs that the following isomorphism is a chain map:

PROPOSITION 5.17. *Let R be a ring and C_1 and C_2 be complexes of R modules. If C_1 consists of finitely generated projective modules then the duality map introduced before induces a natural isomorphism:*
$\varphi : Hom_R(C_1, R) \otimes C_2 \to Hom_R(C_1, C_2)$ given by $\varphi(f \otimes c_2)(c_1) = (-1)^{|f||c_2|} f(c_1) \cdot c_2$ which is a chain map.

We call this map the duality map.

Now, it is easy to define the homology (cohomology) of a group G with coefficients in a chain (cochain) complex M_* (M^*) to be the homology of the chain (cochain) complex $P^+_* \otimes_{\mathbb{Z}[G]} M_*$ ($Hom_{\mathbb{Z}[G]}(P^+_*, M^*)$) where P^+_* is a projective resolution of $\mathbb{Z}$ as we had before. We denote it by $H_*(G, M_*)$ ($H^*(G, M^*)$). In a similar way we define the Tate homology and cohomology with coefficients in a chain/cochain complex and the backwards homology and cohomology with coefficients in a chain/cochain complex for a finite group.

PROPOSITION 5.18. *Let $0 \to M_* \to M'_* \to M''_* \to 0$ be an exact sequence of $\mathbb{Z}[G]$ chain complexes then the following is exact:*

$... \to H_k(G, M_*) \to H_k(G, M'_*) \to H_k(G, M''_*) \to H_{k-1}(G, M_*) \to ...$
We also have similar results for $\hat{H}_*$ *and* DH_* *and also in cohomology.*

PROOF. The same proof as before. □

PROPOSITION 5.19. *The following are exact:*
$... \to DH_{k+1}(G, M_*) \to \hat{H}_k(G, M_*) \to H_k(G, M_*) \to DH_k(G, M_*) \to ...$
$... \to \hat{H}^{k-1}(G, M^*) \to DH^k(G, M^*) \to H^k(G, M^*) \to \hat{H}^k(G, M^*) \to ...$
for every finite group G *and every chain complex* M_* *and cochain complex* M^**.*

PROOF. The same proof as before. □

REMARK 5.20. As before, the boundary maps in the long exact sequences $H_k(G, M_*) \to DH_k(G, M_*)$ and $DH^k(G, M^*) \to H^k(G, M^*)$ are induced by the chain map $P_*^+ \to P_*^-$. Since $P_*^+ \to P_*^-$ factors through $\mathbb{Z}$ we can study this map as the composition of two simpler maps.

Product structure.

Let M be a $\mathbb{Z}[G]$ module and N a $\mathbb{Z}[G']$ module then $M \otimes_{\mathbb{Z}} N$ is a $\mathbb{Z}[G \times G']$ module. In case $G = G'$ restriction along the diagonal map $\Delta : G \to G \times G$ makes it into a $\mathbb{Z}[G]$ module. The action of G will be the diagonal action $g(x \otimes y) = gx \otimes gy$. This also holds for chain complexes.

If P_*^+ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ and $\bar{P}_*^+$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G']$ then $P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times G']$ ([**7**] V,1.1). If $G = G'$ then $P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$ ([**7**] V,1.2). Therefore, there are augmentation preserving homotopy equivalences $P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+ \to P_*^+$ and $P_*^+ \to P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+$ which are unique up to homotopy.

Homology cross product - The map:
$\left(P_*^+ \otimes_{\mathbb{Z}[G]} M_*\right) \otimes_{\mathbb{Z}} \left(\bar{P}_*^+ \otimes_{\mathbb{Z}[G']} N_*\right) \to \left(P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+\right) \otimes_{\mathbb{Z}[G \times G']} (M_* \otimes_{\mathbb{Z}} N_*)$
given by $(x \otimes m) \otimes (\bar{x} \otimes n) \mapsto (-1)^{|\bar{x}| \cdot |m|} (x \otimes \bar{x}) \otimes (m \otimes n)$
induces a map:
$\times : H_k(G, M_*) \otimes_{\mathbb{Z}} H_l(G', N_*) \xrightarrow{\times} H_{k+l}(G \times G', M_* \otimes_{\mathbb{Z}} N_*)$
If $G = G'$ we can compose it with the transfer map $H_{k+l}(G \times G, M_* \otimes_{\mathbb{Z}} N_*) \to H_{k+l}(G, M_* \otimes_{\mathbb{Z}} N_*)$ and get the cross product:
$\times : H_k(G, M_*) \otimes_{\mathbb{Z}} H_l(G, N_*) \to H_{k+l}(G, M_* \otimes_{\mathbb{Z}} N_*)$
Cohomology cross product - The map:
$Hom_{\mathbb{Z}[G]}(P_*^+, M^*) \otimes_{\mathbb{Z}} Hom_{\mathbb{Z}[G']}(\bar{P}_*^+, N^*) \xrightarrow{\times} Hom_{\mathbb{Z}[G \times G']}(P_*^+ \otimes_{\mathbb{Z}} \bar{P}_*^+, M^* \otimes_{\mathbb{Z}} N^*)$
given by $\langle u \times \bar{u}, x \otimes \bar{x} \rangle = (-1)^{|x||\bar{u}|} \langle u, x \rangle \otimes \langle \bar{u}, \bar{x} \rangle$
induces a map:
$\times : H^k(G, M^*) \otimes_{\mathbb{Z}} H^l(G', N^*) \to H^{k+l}(G \times G', M^* \otimes_{\mathbb{Z}} N^*)$
If $G = G'$ we can compose it with the restriction map $H^{k+l}(G \times G, M_* \otimes_{\mathbb{Z}} N_*) \to H^{k+l}(G, M_* \otimes_{\mathbb{Z}} N_*)$ and get the cross product:
$\times : H^k(G, M^*) \otimes_{\mathbb{Z}} H^l(G, N^*) \to H^{k+l}(G, M^* \otimes_{\mathbb{Z}} N^*)$
Cup product - If M^* has a product, that is a map $M^* \otimes_{\mathbb{Z}} M^* \to M^*$ then we have a cup product:
$\cup : H^k(G, M^*) \otimes_{\mathbb{Z}} H^l(G, M^*) \to H^{k+l}(G, M^*)$

Similar results are obtained for backwards resolutions so we have similar products in $DH_*(G, -)$ and $DH^*(G, -)$. For Tate cohomology this require a bit more work, and we refer to ([**7**] VI,5).

5.2. Backwards (co)homology

Let G be a discrete group and X a $G-CW$ complex. Denote by $C_*(X)$ the cellular chain complex of X. This is a chain complex of $\mathbb{Z}[G]$ modules. We denote $H_*(G, C_*(X))$ by $H_*^G(X)$ and call it the equivariant homology of X. Denote by $C^*(X)$ the cellular cochain of X (that is $Hom_{\mathbb{Z}}(C_*(X), \mathbb{Z})$) then we denote $H^*(G, C^*(X))$ by $H_G^*(X)$ and call it the equivariant cohomology of X.

Let EG be a contractible $G-CW$ complex with a free G action. The fact that the action is free implies that $C_*(EG)$ consists of free (and hence projective) $\mathbb{Z}[G]$ modules. The fact that EG is contractible implies that $C_*(EG) \to \mathbb{Z}$ is acyclic (where the map to $\mathbb{Z}$ is the augmentation map). We conclude that $C_*(EG)$ is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$ module, denote it by P_*^+. Recall that the Borel construction on X is the quotient space $EG \times_G X$. We have the following:

PROPOSITION 5.21. *There are natural isomorphisms $H_*(EG \times_G X) \cong H_*^G(X)$ and $H^*(EG \times_G X) \cong H_G^*(X)$ which commute with the cross product.*

PROOF. 1) $H_*(EG \times_G X) = H_*(C_*(EG \times_G X)) \cong H_*(C_*(EG) \otimes_{\mathbb{Z}[G]} C_*(X)) =$
$= H_*(P_*^+ \otimes_{\mathbb{Z}[G]} C_*(X)) = H_*^G(X)$
Using the natural isomorphism $C_*(EG \times_G X) \cong C_*(EG) \otimes_{\mathbb{Z}[G]} C_*(X)$.
2) $H^*(EG \times_G X) \cong H_*(Hom_{\mathbb{Z}}(C_*(EG) \otimes_{\mathbb{Z}[G]} C_*(X), \mathbb{Z}) \cong Hom_{\mathbb{Z}[G]}(C_*(EG), Hom_{\mathbb{Z}}(C_*(X), \mathbb{Z}))$
$= H_*(Hom_{\mathbb{Z}[G]}(P_*^+, C^*(X)) = H_G^*(X)$ where the second isomorphism is the adjunction. The fact that the natural isomorphisms commute with the cross product follows from the fact that $EG \times_G X \times EG' \times_{G'} Y \cong E(G \times G') \times_{G \times G'} X \times Y$. $\square$

In a similar way, for a finite group G we define Tate homology and cohomology of a $G-CW$ complex which we denote by $\hat{H}_*^G(X)$ and $\hat{H}_G^*(X)$ and the backwards homology and cohomology of a $G-CW$ complex which we denote by $DH_*^G(X)$ and $DH_G^*(X)$. Note that the equivariant homology theories are covariant in X and the equivariant cohomology theories are contravariant in X (although they are all covariant in the chain complex of coefficients). As before we have:

THEOREM 5.22. *The following are exact:*
$... \to DH_{k+1}^G(X) \to \hat{H}_k^G(X) \to H_k^G(X) \to DH_k^G(X) \to ...$
$... \to \hat{H}_G^{k-1}(X) \to DH_G^k(X) \to H_G^k(X) \to \hat{H}_G^k(X) \to ...$
for every finite group G and every $G-CW$ complex.

REMARK 5.23. We could use the locally finite cellular chain complex of a locally finite $G-CW$ complex to define analogs of these theories in the locally finite setting which we will denote by $H_*^{lf,G}(X)$ for example. The same can be done for cohomology with compact support, end homology and end cohomology. We will get long exact sequences similar to those we had before. We can actually associate to each $G-CW$ complex a lattice of groups where each row and each column is exact, both for homology and cohomology.

The spectral sequences.

REMARK 5.24. In this section we use the language of [**8**] for spectral sequences.

The homology and cohomology theories described before are the homology and cohomology of the total complex of a double complex. The total complex of a double complex $A^n = \oplus A^{k,n-k}$ has two natural filtrations: by columns $F_I^p A^n = \oplus_{k \geq p} A^{k,n-k}$ and by rows $F_{II}^q A = \oplus_{k \geq p} A^{n-k,k}$. Each filtration gives rise to a spectral sequence. In several cases these spectral sequences strongly converge to the homology of the total complex.

Equivariant homology and cohomology - The two double complexes are first quadrant and therefore the spectral sequences are bounded. Therefore we have the following:

THEOREM 5.25. *The following spectral sequences strongly converge: ([**7**] VII,5.3,5.6):*
$E^2_{pq} = H_p(G, H_q(X)) \Longrightarrow H^G_{p+q}(X)$ *and* $E_2^{pq} = H^p(G, H^q(X)) \Longrightarrow H_G^{p+q}(X)$
$E^1_{pq} = H_q(G, C_p(X)) \Longrightarrow H^G_{p+q}(X)$ *and* $E_1^{pq} = H^q(G, C^p(X)) \Longrightarrow H_G^{p+q}(X)$

Tate cohomology and backwards cohomology - In both cases the first filtration is regular and hence strongly convergent ([**8**] XV,4.1):

THEOREM 5.26. *The following spectral sequences strongly converge:*
$E_2^{pq} = \hat{H}^p(G, H^q(X)) \Longrightarrow \hat{H}_G^{p+q}(X)$ *and* $E_2^{pq} = DH^p(G, H^q(X)) \Longrightarrow DH_G^{p+q}(X)$

In case X is finite dimensional the spectral sequences associated to both filtrations are bounded and therefore strongly converge:

THEOREM 5.27. *Let X be a finite dimensional $G-CW$ complex. The following spectral sequences strongly converge:*
$E^2_{pq} = \hat{H}_p(G, H_q(X)) \Longrightarrow \hat{H}^G_{p+q}(X)$ *and* $E^2_{pq} = DH_p(G, H_q(X)) \Longrightarrow DH^G_{p+q}(X)$
$E^1_{pq} = \hat{H}_q(G, C_p(X)) \Longrightarrow \hat{H}^G_{p+q}(X)$ *and* $E_1^{pq} = \hat{H}^q(G, C^p(X)) \Longrightarrow \hat{H}_G^{p+q}(X)$
$E^1_{pq} = DH_q(G, C_p(X)) \Longrightarrow DH^G_{p+q}(X)$ *and* $E_1^{pq} = DH^q(G, C^p(X)) \Longrightarrow DH_G^{p+q}(X)$

COROLLARY 5.28. *([**7**] VII,7.3) Let $f : X \to Y$ be an equivariant cellular map between two $G-CW$ complexes. If $f_* : H_n(X) \to H_n(Y)$ ($f_* : H^n(Y) \to H^n(X)$) is an isomorphism then the maps $H^G_n(X) \to H^G_n(Y)$ ($H^n_G(Y) \to H^n_G(X)$) are isomorphisms. Similar results can be obtained for Tate and backwards homology and cohomology (in the case of homology we have to assume that X and Y are finite dimensional $G-CW$ complexes).*

We use the following ([**10**] V,1):

PROPOSITION 5.29. *Let X be a CW complex. Denote by $S_*(X)$ the singular chain of X and by $C_*(X)$ the cellular chain of X. There is a chain complex of $\mathbb{Z}$ modules $D_*(X)$ which is natural in X and weak equivalences (maps which induce isomorphism on homology) $C_*(X) \leftarrow D_*(X) \to S_*(X)$.*

From the construction of $D_*(X)$ it follows that when X is a $G-CW$ complex then $D_*(X)$ is a chain complex of $\mathbb{Z}[G]$ modules and all the maps are $\mathbb{Z}[G]$ chain maps.
The following appears partially in ([**3**] 4.6.12) :

COROLLARY 5.30. *There is an isomorphism $H_*(G, S_*(X)) \cong H_*(G, C_*(X))$ which is natural in X. A similar result holds in cohomology and in Tate and backwards cohomology.*

PROOF. Look at the spectral sequences associated to the first filtration. We have an isomorphism on E^2_{pq} term. Since those sequences are strongly convergent this implies the isomorphism $H_*(G, C_*(X)) \xleftarrow{\cong} H_*(G, D_*(X)) \xrightarrow{\cong} H_*(G, S_*(X))$ ([**8**] XV,3.2). For cohomology use the argument in ([**3**] 4.6.12) to show that there is a weak equivalence $C^*(X) \to S^*(X)$ which is natural in X up to homotopy and then use the spectral sequence. $\square$

COROLLARY 5.31. *1)* $H_*^G(X)$ *and* $H_G^*(X)$ *are independent of the* $G-CW$ *structure.*
2) $\hat{H}_G^*(X)$ *and* $DH_G^*(X)$ *are independent of the* $G-CW$ *structure.*

REMARK 5.32. It is actually shown in the proof that the isomorphism is natural in X. This proof can be used also to the case of locally finite homology for strongly locally finite $G-CW$ complexes by the methods of [**20**] in appendix A.

Properties of the (co)homology theories.

Equivariant homology and cohomology are well known, as well are Tate homology and cohomology. We mention few facts about them, and deduce some facts about the backwards homology. In the following propositions X is a $G-CW$ complex. We state the results in cohomology but they stay the same for homology:

PROPOSITION 5.33. *For* $k<0$ *the group* $H_G^k(X)$ *vanishes. If* X *is finite dimensional then for* $k>dim(X)$ *the group* $DH_G^k(X)$ *vanishes.*

COROLLARY 5.34. *For* $k<0$ *we have* $\hat{H}_G^{k-1}(X)\cong DH_G^k(X)$*. If* X *is finite dimensional then for* $k>dim(X)$ *we have* $H_G^k(X)\cong\hat{H}_G^k(X)$*.*

PROPOSITION 5.35. *(*[**7**] *VII,7.3) Let* X *be finite dimensional and* Y *its singular part, that is all points with non trivial stabilizer, then* $\hat{H}_G^k(X)\to\hat{H}_G^k(Y)$ *is an isomorphism. Therefore, if* X *is free,* $\hat{H}_G^k(X)$ *vanishes.*

COROLLARY 5.36. *If* Y *is the singular part of* X *(a finite dimensional* $G-CW$ *complex), then for* $k<0$ *the map* $DH_G^k(X)\to DH_G^k(Y)$ *is an isomorphism.*

PROPOSITION 5.37. *If the action of* G *is free then* $H_G^k(X)\cong H^k(X/G)$*.*

PROOF. This follows from the fact that $H_G^k(X)=H^k(EG\times_G X)$ and when X is free the map $EG\times_G X\to X/G$ is a homotopy equivalence. □

COROLLARY 5.38. *If* X *is finite dimensional and the action of* G *is free then* $DH_G^k(X)\cong H^k(X/G)$*.*

PROOF. Since $\hat{H}_G^k(X)$ vanishes, by the long exact sequence in cohomology we have $DH_G^k(X)\cong H_G^k(X)\cong H^k(X/G)$. □

5.3. Equivariant Poincaré duality

Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving action of a finite group G. There is an equivariant triangulation of M ([**22**]) which gives M a structure of a $G-CW$ complex. Let $\sigma_M\in C_m^{lf}(M)$ be a representative of $[M]^{lf}$. For any $g\in G$ we know that $g\cdot\sigma_M$ is also a representative of $[M]^{lf}$ since the action is orientation preserving . Since $C_{m+1}^{lf}(M)=0$ we deduce that $g\cdot\sigma_M=\sigma_M$.

PROPOSITION 5.39. *The map* $T:C^k(M)\to C_{m-k}^{lf}(M)$ *defined by* $T(\varphi)=\varphi\cap\sigma_M$ *is a* $\mathbb{Z}[G]$ *chain map.* T *induces an isomorphism* $DH_G^k(M)\to H_{m-k}^{lf,G}(M)$*.*

PROOF. Defining the cap product on cellular chains requires a choice of a proper cellular approximation of the diagonal map $\Delta:M\to M\times M$. We can choose it to be equivariant. Since σ_M is invariant T is actually a $\mathbb{Z}[G]$ chain map:
$g\cdot T(\varphi)=g_*(\varphi\cap\sigma_M)=g_*((g^*\circ(g^{-1})^*\varphi)\cap\sigma_M)=(g^{-1})^*\varphi\cap g_*\sigma_M=(g\cdot\varphi)\cap\sigma_M=T(g\cdot\varphi)$
T commutes with the boundary due to the following formula which is proved by a

sign check:
$\partial(\varphi \cap \sigma) = \delta\varphi \cap \sigma + (-1)^{|\varphi|}\varphi \cap \partial\sigma$
which implies that:
$\partial(T(\varphi)) = \partial(\varphi \cap \sigma_M) = \delta\varphi \cap \sigma_M + (-1)^{|\varphi|}\varphi \cap \partial\sigma_M = \delta\varphi \cap \sigma_M = T(\delta\varphi)$
By Poincaré duality T is a weak equivalence.
Choose a projective resolution P_*^+ of finite type and the dual backwards resolution P_*^- defined by $P_i^- = Hom_{\mathbb{Z}[G]}(P_{-i}^+, \mathbb{Z}[G])$.
We have the duality isomorphism:
$Hom_{\mathbb{Z}[G]}(P_*^-, C^*(M)) \to Hom_{\mathbb{Z}[G]}(P_*^-, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M) = P_{-*}^+ \otimes_{\mathbb{Z}[G]} C^*(M)$.
We also have the map induced by T:
$Id \otimes T : P_{-*}^+ \otimes_{\mathbb{Z}[G]} C^*(M) \to P_{-*}^+ \otimes_{\mathbb{Z}[G]} C_{m-*}^{lf}(M)$.
$Id \otimes T$ induces an isomorphism on the E_2^{pq} term of the spectral sequences associated with the first filtration. Since these spectral sequences strongly converge this map is also a weak equivalence. The composition of those two weak equivalences induces an isomorphism in homology $DH_G^k(M) \to H_{m-k}^{lf,G}(M)$. Note that $DH_G^k(M)$ is independent of the $G-CW$ structure as we noted so the same is true for $H_{m-k}^{lf,G}(M)$.
The following theorem is proved similarly: □

THEOREM 5.40. *Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving action of a finite group G. We have the following isomorphisms:*
$PD_M : H_G^k(M) \to DH_{m-k}^{lf,G}(M)$ *and if* M *is compact* $PD_M : H_G^k(M) \to DH_{m-k}^G(M)$.
$PD_M : DH_G^k(M) \to H_{m-k}^{lf,G}(M)$ *and if* M *is compact* $PD_M : DH_G^k(M) \to H_{m-k}^G(M)$.
$PD_M : \hat{H}_G^k(M) \to \hat{H}_{m-k-1}^{lf,G}(M)$ *and if* M *is compact* $PD_M : \hat{H}_G^k(M) \to \hat{H}_{m-k-1}^G(M)$.
where the lf *stands for using the locally finite cellular chain complex.*
Moreover, the following diagram commutes:

$$\begin{array}{ccccccccc}
\dots \to & \hat{H}_G^{k-1}(M) & \to & DH_G^k(M) & \to & H_G^k(M) & \to & \hat{H}_G^k(M) & \to \dots \\
& PD_M \downarrow & & PD_M \downarrow & (1) & PD_M \downarrow & & PD_M \downarrow & \\
\dots \to & \hat{H}_{m-k}^{lf,G}(M) & \to & H_{m-k}^{lf,G}(M) & \to & DH_{m-k}^{lf,G}(M) & \to & \hat{H}_{m-k-1}^{lf,G}(M) & \to \dots
\end{array}$$

Let M be as before and, for simplicity, let us assume that it is compact. We look at (1), composing the isomorphism $H_{m-k}^G(M) \xrightarrow{PD_M^{-1}} DH_G^k(M)$ with the map $DH_G^k(M) \to H_G^k(M)$ gives us a map $H_{m-k}^G(M) \to H_G^k(M)$. This map is an isomorphism if and only if the map $DH_G^k(M) \to H_G^k(M)$ is an isomorphism. By exactness we deduce that the map $H_{m-k}^G(M) \to H_G^k(M)$ is an isomorphism for all k if and only if $\hat{H}_G^k(M)$ vanish for every k, for example when G acts freely on M.

If we were able compute this map we would have been able to compute $\hat{H}_G^*(M)$ up to extension. We give some information about this map, a more concrete way of computing is given in 6.65.

Let $P_*^+ \to \mathbb{Z}$ and $\mathbb{Z} \to P_*^-$ be projective and backwards resolutions respectively. There are also natural maps $\mathbb{Z}[G] \to P_*^+$ and $P_*^- \to \mathbb{Z}[G]$ given by the map $G \to EG$. We can view $\mathbb{Z}$ and $\mathbb{Z}[G]$ as chain complexes concentrated in dimension zero then the maps $\mathbb{Z}[G] \to P_*^+ \to \mathbb{Z} \to P_*^- \to \mathbb{Z}[G]$ can be considered as chain maps. The composition $\mathbb{Z}[G] \to \mathbb{Z}[G]$ is given by $1 \mapsto N$ where $N = \Sigma_{g \in G} g$ is the norm element. This induces chain maps:
$Hom(\mathbb{Z}[G], C^*(M)) \to Hom(P_*^-, C^*(M)) \to Hom(\mathbb{Z}, C^*(M)) \to Hom(P_*^+, C^*(M)) \to Hom(\mathbb{Z}[G], C^*(M))$
where all Hom complexes are over $\mathbb{Z}[G]$. And similarly in homology:

$\mathbb{Z}[G] \otimes C_*(M) \to P_*^+ \otimes C_*(M) \to \mathbb{Z} \otimes C_*(M) \to P_*^- \otimes C_*(M) \to \mathbb{Z}[G] \otimes C_*(M)$
where all tensor products are over $\mathbb{Z}[G]$.
We get the following commutative diagram:

$$\begin{array}{ccccccccc}
H^k(M) & \to & DH_G^k(M) & \to & H^k(M/G) & \to & H_G^k(M) & \to & H^k(M) \\
PD_M \downarrow\cong & & PD_M \downarrow\cong & & & & PD_M \downarrow\cong & & PD_M \downarrow\cong \\
H_{m-k}(M) & \to & H_{m-k}^G(M) & \to & H_{m-k}(M/G) & \to & DH_{m-k}^G(M) & \to & H_{m-k}(M)
\end{array}$$

The reason we don't have a map $H^k(M/G) \to H_{m-k}(M/G)$ is that the map $Hom_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M) \to Hom_{\mathbb{Z}[G]}(\mathbb{Z}, C^*(M))$ is not an isomorphism in general since $\mathbb{Z}$ is not a projective $\mathbb{Z}[G]$ module. If G acts freely on M then this map is an isomorphism and we do get a map $H^k(M/G) \to H_{m-k}(M/G)$ commuting with the rest of the diagram, which is also an isomorphism. The composition $H_{m-k}(M) \to H_{m-k}^G(M) \to H_G^k(M) \to H^k(M)$ is equal to $N \cdot PD_M^{-1}$ where N is the map induced by the multiplication with N.

CHAPTER 6

Equivariant Stratifold Homology and Cohomology Theories for Compact Lie Groups

ABSTRACT. This chapter is the center of this thesis. We define equivariant stratifold (co)homology theories of various types, relate them to each other and prove some fundamental properties: we relate those theories by a long exact sequence, identify them with the theories introduced in chapter 5 in the case the group is finite and prove Poincaré duality. Furthermore, we demonstrate their potential for computation by looking at a concrete example.

6.1. Equivariant stratifold homology

Group actions and equivariant maps.

Let G be a compact Lie group of dimension $dim(G)$ and X a topological space. A continuous map $\rho : G \times X \to X$ which induces a homomorphism $\tilde{\rho} : G \to Homeo(X)$ is called an action of G on X. We denote $\rho(g,x)$ by $g \cdot x$ and $\tilde{\rho}(g)$ by ρ_g. A space with a G action is called a G space. If X is a smooth manifold, or more generally a stratifold, we say that the action is smooth if the map $\rho : G \times X \to X$ is smooth. In this case all maps $\rho_g : X \to X$ are diffeomorphisms. If X is a stratifold this will imply that G acts on each strata separately. For p-stratifolds we would like to require a bit more, we ask that the action will be by isomorphisms of p-stratifolds, that is if $S = M_0 \cup_{\partial M_1} M_1 ... \cup_{\partial M_n} M_n$ then the action can be extended to every manifold with boundary $(M_k, \partial M_k)$ separately and the gluing maps will be equivariant.

A map $f : X \to Y$ between two G spaces is called equivariant if it commutes with the action of G, that is for each $g \in G$ and $x \in X$ we have $f(g \cdot x) = g \cdot f(x)$.

Here is a theorem which we will often use:

THEOREM 6.1. *(**[6]** 5.8) Suppose X is a completely regular G space, G compact Lie, and that all the orbits have type G/H. Then the orbit map $X \to X/G$ is the projection in a fiber bundle with fiber G/H and a structure group $N(H)/H$ (acting by right translations on G/H). Conversely, every such bundle comes from such an action.*

In particular, in the case of $H = \{1\}$, the theorem states that if the action is free then the map $X \to X/G$ is a principal G bundle and every principal G bundle comes from a free action on the total space X.

Let G be a group, a contractible CW complex with a free cellular G action is denoted by EG. EG has the following universal property: for every paracompact space X with a free G action there is a continuous equivariant map $f : X \to EG$ and f is unique up to G-homotopy. This implies the uniqueness of EG up to G-homotopy equivalence. The quotient space EG/G is called the classifying space of principal G bundles and is denoted by BG.

There are several ways to construct EG, we just note that it is possible to construct EG as a union of closed oriented manifolds $EG_n \subseteq EG_{n+1} \subseteq ...$ (inclusion as submanifolds) with a free and orientation preserving G action and with $\pi_k(EG_n) = \{0\}$ for $k \leq n$. Denote $BG_n = EG_n/G$, oriented as we will explain later, then the quotient map $EG_n \to BG_n$ is a principal G bundle. This bundle has the universal property that for every principal bundle $p : E \to B$ where B is (homotopy equivalent to) a CW complex of dimension less then or equal to n there is a bundle map

$$\begin{array}{ccc} E & \to & EG_n \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & EG_n/G \end{array}$$

and f is unique up to homotopy ([**21**] 13.1 chapter 4). We will call the map $E \to EG_n$ the classifying map. We set one copy of EG for each G and work with it all along. More about that appears in [**9**] and [**21**].

REMARK 6.2. From now on we will assume that all spaces are paracompact. This is not very restrictive for us since it includes all manifolds, or more generally, p-stratifolds, and all CW complexes.

Smooth actions on p-stratifolds.

Let G be a compact Lie group. An action of G on a p-stratifold S is called regular if for each $x \in S^i$ there is an open invariant neighborhood U of x in S, a p-stratifold F with F^0 a single point pt, an open invariant subspace V of S^i, and an equivariant isomorphism $\phi : V \times F \to U$, whose restriction to $V \times pt$ is the projection (where the action on $V \times F$ is given by $g \cdot (v, f) = (g \cdot v, f)$). Clearly, if the action of G on S is regular then S is regular.

Here are some properties of p-stratifolds with a free action and their quotients:

LEMMA 6.3. *Let S be a p-stratifold of dimension k (with boundary) with a smooth G action.*

(1) *If the action is free then S/G has a unique structure of a p-stratifold of dimension $k - dim(G)$ (with boundary), such that the map $\pi : S \to S/G$ is smooth.*
(2) *If the action is regular then S/G is regular.*
(3) *If S is compact then so is S/G.*
(4) *If S is oriented and the action is orientation preserving then there is a natural orientation for S/G.*
(5) *If the action is free then $\pi : S \to S/G$ is a principal G bundle.*

PROOF. Assume S is as above then:

(1) We say that a map $f : S/G \to \mathbb{R}$ is smooth if and only if the composition $S \to S/G \to \mathbb{R}$ is smooth. If S is a smooth manifold with boundary this gives a structure of a smooth manifold with boundary to S/G ([**9**] I,5.2) and for a p-stratifold this can be proved by induction. (The same proof holds for G non compact if we assume that the action is also proper, a condition which is always fulfilled when G is a compact Lie group. We will discuss that later).
(2) This is clear.
(3) This is clear.

(4) We start by fixing an orientation on G such that the action of G on itself by multiplication on the left is orientation preserving (if G is discrete then we give it the positive orientation). For a point $x \in S/G$ we choose some $y \in S$ such that $\pi(y) = x$ and we orient S/G at a point x as the quotient $T_yEG_n/T_eG \xrightarrow{\cong} T_xBG_n$. This is independent of the choice of y since the action is orientation preserving.
(5) This follows from the fact that p-stratifolds are completely regular (since they are paracompact).

□

LEMMA 6.4. *Let S be a p-stratifold of dimension k and $\widetilde{S} \to S$ a principal G-bundle.*

(1) *There is a unique p-stratifold structure on $\widetilde{S}$ of dimension $k + dim(G)$ such that the quotient is smooth.*
(2) *If S and G are compact then so is $\widetilde{S}$.*
(3) *If S is oriented there is a way to give $\widetilde{S}$ an orientation such that the action will be orientation preserving and the map $\widetilde{S}/G \to S$ will be orientation preserving using the convention above.*
(4) *If S is regular then the action of G is regular.*

PROOF. Assume S is as above then:

(1) Look at the classifying map $S \to BG$. BG is the union of $BG_n = EG_n/G$ which are closed oriented manifolds, since S has the homotopy type of a finite dimensional CW complex it is homotopic to a map which factors smoothly through some BG_n. The following is a pullback square:
$$\begin{array}{ccc} \widetilde{S} & \to & EG_n \\ \downarrow & & \downarrow \\ S & \to & BG_n \end{array}$$
Since the map on the right is a smooth submersion of manifolds the maps are transversal thus $\widetilde{S}$ has a p-stratifold structure. The structure of $\widetilde{S}$ does not depend on the choice of EG_n by the uniqueness of the classifying maps up to homotopy and the functoriality of the pullback.
(2) This is proved like the fact that a product of compact spaces is compact since the bundle is locally trivial.
(3) As above, just in the opposite way.
(4) This is clear.

□

LEMMA 6.5. *Let X be a completely regular space with a free G action where G is a compact Lie group. Let $S \xrightarrow{g} X/G$ be a map where S is an oriented regular p-stratifold of dimension k then in the pullback diagram*
$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{g}} & X \\ \pi' \downarrow & & \downarrow \pi \\ S & \xrightarrow{g} & X/G \end{array}$$

(1) *$\widetilde{S}$ has a natural structure of an oriented p-stratifold of dimension $k + dim(G)$ with a free orientation preserving regular G action.*
(2) *$\tilde{S}$ is compact if and only if S is compact.*

(3) *The map $g : S \to X/G$ is proper if and only if the map $\widetilde{g} : \widetilde{S} \to X$ is proper.*

PROOF. The first two assertions follows from the previous lemma and the fact that the pullback of a principal bundle is a principal bundle.

For the third, assume $g : S \to X/G$ is proper. Take a compact subset $K \subseteq X$ then $\pi(K) \subseteq X/G$ is also compact and therefore $g^{-1}(\pi(K))$ is compact (since g is proper). K is closed in X (since X is Hausdorff) thus $\tilde{g}^{-1}(K)$ is closed in $g^{-1}(\pi(K)) \times K$ hence compact.

Assume $\tilde{g} : \tilde{S} \to X$ is proper. Take a compact subset $K \subseteq X/G$ then $\pi^{-1}(K) \subseteq X$ is also compact ([**6**] I,3.1). Since $\tilde{g} : \tilde{S} \to X$ is proper $\tilde{g}^{-1}(\pi^{-1}(K))$ is compact and so is its image under the map π', which is equal by the commutativity of the diagram and the fact that π' is surjective, to $g^{-1}(K)$. $\square$

Equivariant stratifold homology.

Equivariant stratifold homology was defined in [**24**] and is denoted by SH^G_*. This is an equivariant homology theory defined on the category of G spaces and equivariant maps, where G is a compact Lie group of dimension $dim(G)$. This equivariant homology theory is naturally isomorphic to the homology of the Borel construction after a dimension shift - $SH^G_*(X) \cong H_{*-dim(G)}(EG \times_G X)$ (when X is completely regular). If G is finite this implies by what we showed before that $SH^G_*(X) \cong H^G_*(X)$.

DEFINITION 6.6. Let G be a compact Lie group, X a G space and $k \geq 0$, define $SH^G_k(X) = \{g : S \to X\}_G / \sim$ i.e., bordism classes of equivariant maps $g : S \to X$ where:

- S is a compact oriented p-stratifold of dimension k with a G action.
- The action of G on S is free, smooth, orientation preserving and regular.
- g is a continuous equivariant map.
- The bordism relation has to fulfill the same properties as S does. In particular the action on the cobordism should be free and extend the action on the boundary.

$SH^G_k(X)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \to Y$ is a continuous equivariant map than we can define an induced map by composition $f_* : SH^G_k(X) \to SH^G_k(Y)$.

A triple (U, V, X) consists of X which is a G space and $U, V \subseteq X$ which are two equivariant closed subspaces such that their interiors cover X. The boundary map is defined in a similar way to the boundary in SH_* we just have to choose an equivariant map $f : S \to \mathbb{R}$ so the preimage of every point will be invariant. This way we will get a well defined G action on the boundary. This can be done by pulling back any smooth map from S/G. We then have:

THEOREM 6.7. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH^G_k(U \cap V) \to SH^G_k(U) \oplus SH^G_k(V) \to SH^G_k(X) \xrightarrow{\partial} SH^G_{k-1}(U \cap V) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

If H is a closed subgroup of G then every G space has a natural structure of an H space. We define the transfer map $tr_H^G : SH_k^G(X) \to SH_k^H(X)$ by $[S \to X] \mapsto [S \to X]$ where on the right side the spaces are considered as spaces with an action of H.

The cross product $\times : SH_k^G(X) \otimes SH_l^{G'}(Y) \to SH_{k+l}^{G\times G'}(X \times Y)$ is defined by $[g_1 : S \to X] \times [g_2 : T \to Y] = (-1)^{dim(G)(l-dim(G'))}[g_1 \times g_2 : S \times T \to X \times Y]$. This product is bilinear and natural. If $G = G'$ we can use the diagonal $\Delta : G \to G\times G$ and compose this product with the transfer map - $tr_{\Delta(G)}^{G\times G} : SH_{k+l}^{G\times G}(X \times Y) \to SH_{k+l}^G(X \times Y)$. This gives us a cross product $\times : SH_k^G(X) \otimes SH_l^G(Y) \to SH_{k+l}^G(X \times Y)$.
SH_*^G with the boundary operator and the cross product is a multiplicative equivariant homology theory. We call it equivariant (parametrized) stratifold homology.

A natural isomorphism between SH_*^G and $H_{*-dim(G)}(EG \times_G -)$.

LEMMA 6.8. *Let X be a G space, the projection $\pi_X : EG \times X \to X$ induces an isomorphism $\pi_{X*} : SH_*^G(EG \times X) \to SH_*^G(X)$.*

PROOF. The inverse of π_{X*} is given by $[g : S \to X] \mapsto [f \times g : S \to EG \times X]$ where $f : S \to EG$ is the classifying map defined by the universal property of EG and the fact that the action on S is free. f is unique up to homotopy thus the map is well defined. □

LEMMA 6.9. *Let X be a completely regular topological space with a free G action, then there is a natural isomorphism $SH_*^G(X) \to SH_{*-dim(G)}(X/G)$.*

PROOF. We define this map by $[S \to X] \mapsto [S/G \to X/G]$. This is well defined by lemmas 6.3 and it has an inverse $[S \to X/G] \mapsto [\widetilde{S} \to X]$ which is also well defined by lemmas 6.5. □

REMARK 6.10. For a fiber bundle $\pi : E \to B$ with fiber F which is a compact oriented manifold one can define a dimension shifting transfer $SH_*(B) \xrightarrow{tr} SH_{*+dim(F)}(E)$ by pullback. Using the natural isomorphism $SH_* \to H_*$ we can define it in singular homology. When F is discrete this agrees with the ordinary notion of transfer.

THEOREM 6.11. *Let X be a completely regular G space, there is a natural isomorphism $\Phi_0^G : SH_*^G(X) \to H_{*-dim(G)}(EG \times_G X)$. Φ_0^G commutes with the boundary, with the transfer and with the cross product.*

PROOF. We define Φ_0^G to be the composition:
$SH_*^G(X) \to SH_*^G(EG \times X) \to SH_{*-dim(G)}(EG \times_G X) \to H_{*-dim(G)}(EG \times_G X)$
where the second isomorphism is well defined since the action on $EG \times X$ is free. Clearly, it is natural.

Φ_0^G commutes with the boundary:
The first two maps clearly commute with the boundary and we also proved that the third one does. Therefore, Φ_0^G commutes with the boundary.

Φ_0^G commutes with the transfer:
Let G be a Lie group and H a closed subgroup. We would like to show that the following diagram commutes:

$$\begin{array}{ccc} SH_*^G(X) & \xrightarrow{tr_H^G} & SH_*^H(X) \\ \Phi_0^G \downarrow & & \downarrow \Phi_0^G \\ H_{*-dim(G)}(EG \times_G X) & \xrightarrow{tr_H^G} & H_{*-dim(H)}(EG \times_H X) \end{array}$$

Given $\alpha = [g : S \to X] \in SH_*^G(X)$ then $\Phi_0^G(\alpha) = (f \times_G g)_*([S/G]) \in H_{*-dim(G)}(EG \times_G X)$ $tr_H^G([S/G]) = [S/H]$ so by functoriality of the transfer we get $tr_H^G \circ \Phi_0^G(\alpha) = \Phi_0^G \circ tr_H^G(\alpha)$.

Φ_0^G commutes with the cross product:
We first show that the following diagram commutes:

$$\begin{array}{ccc} SH_k^G(X) \otimes SH_l^{G'}(Y) & \to & SH_{k-dim(G)}(EG \times_G X) \otimes SH_{l-dim(G')}(EG' \times_{G'} Y) \\ & & \times \downarrow \\ \times \downarrow & & SH_{k+l-dim(G)-dim(G')}(EG \times_G X \times EG' \times_{G'} Y) \\ & & \downarrow \\ SH_{k+l}^{G \times G'}(X \times Y) & \to & SH_{k+l-dim(G)-dim(G')}(E(G \times G') \times_{G \times G'} X \times Y) \end{array}$$

Given $\alpha = [S \to X] \in SH_k^G(X)$ and $\beta = [T \to Y] \in SH_l^{G'}(Y)$, we follow the image of $\alpha \otimes \beta$. If we first go down and then right we get:
$(-1)^{dim(G)(l-dim(G'))}[S \times T/G \times G' \to E(G \times G') \times_{G \times G'} X \times Y]$
If we first go right and then down we get:
$[S/G \times T/G' \to E(G \times G') \times_{G \times G'} X \times Y]$
$S \times T/G \times G'$ and $S/G \times T/G'$ are isomorphic as p-stratifolds. The orientations on these p-stratifolds differs by the sign $(-1)^{dim(G)(l-dim(G'))}$. Therefore the diagram commutes.
The following diagram also commutes since Φ is natural and multiplicative:

$$\begin{array}{ccc} SH_{k-dim(G)}(EG \times_G X) \otimes SH_{l-dim(G')}(EG' \times_{G'} Y) & \to & H_{k-dim(G)}(EG \times_G X) \otimes H_{l-dim(G')}(EG' \times_{G'} Y) \\ \times \downarrow & & \times \downarrow \\ SH_{k+l-dim(G)-dim(G')}(EG \times_G X \times EG' \times_{G'} Y) & \to & H_{k+l-dim(G)-dim(G')}(EG \times_G X \times EG' \times_{G'} Y) \\ \downarrow & & \downarrow \\ SH_{k+l-dim(G)-dim(G')}(E(G \times G') \times_{G \times G'} X \times Y) & \to & H_{k+l-dim(G)-dim(G')}(E(G \times G') \times_{G \times G'} X \times Y) \end{array}$$

We conclude that the composition commutes:

$$\begin{array}{ccc} SH_k^G(X) \otimes SH_l^{G'}(Y) & \to & H_{k-dim(G)}(EG \times_G X) \otimes H_{l-dim(G')}(EG' \times_{G'} Y) \\ & & \times \downarrow \\ \times \downarrow & & H_{k+l-dim(G)-dim(G')}(EG \times_G X \times EG' \times_{G'} Y) \\ & & \downarrow \\ SH_{k+l}^{G \times G'}(X \times Y) & \to & H_{k+l-dim(G)-dim(G')}(E(G \times G') \times_{G \times G'} X \times Y) \end{array}$$

Since Φ_0^G commutes with the transfer we deduce that it also commutes with the cross product $\times : SH_k^G(X) \otimes SH_l^G(Y) \to SH_{k+l}^G(X \times Y)$. $\square$

If G is finite and X is a $G - CW$ complex we can compose Φ_0^G with the isomorphism $H_*(EG \times_G X) \to H_*^G(X)$. Denote the composition $SH_*^G(X) \to H_*^G(X)$ by Φ^G, then we have the following:

THEOREM 6.12. *Let X be a $G - CW$ complex, the map $\Phi^G : SH_*^G(X) \to H_*^G(X)$ is a natural isomorphism. Φ^G commutes with the boundary, with the transfer and with the cross product .*

PROOF. The map $H_*(EG \times_G X) \to H_*^G(X)$ is a natural isomorphism and commutes with the boundary (see also **[3]** 1.2.8). It also commutes with tr_H^G and the cross product since the following diagram commutes:

$$\begin{array}{ccc} H_k(EG \times_G X) \otimes H_l(EG' \times_{G'} Y) & \to & H_k^G(X) \otimes H_l^G(Y) \\ \times \downarrow & & \\ H_{k+l}(EG \times_G X \times EG' \times_{G'} Y) & & \times \downarrow \\ \downarrow & & \\ H_{k+l}(E(G \times G') \times_{G \times G'} X \times Y) & \to & H_{k+l}^{G \times G'}(X \times Y) \end{array}$$

So we get the commutativity of the following diagram:

$$\begin{array}{ccc} SH_k^G(X) \otimes SH_l^{G'}(Y) & \to & H_k^G(X) \otimes H_l^G(Y) \\ \times \downarrow & & \times \downarrow \\ SH_{k+l}^{G\times G'}(X \times Y) & \to & H_{k+l}^{G\times G'}(X \times Y) \end{array}$$

Since Φ^G commutes with the transfer we deduce that it also commutes with the cross product $\times : SH_k^G(X) \otimes SH_l^G(Y) \to SH_{k+l}^G(X \times Y)$. □

Locally finite equivariant stratifold homology.

Locally finite equivariant stratifold homology, denoted by $SH_*^{lf,G}$, is defined in a similar way to SH_*^G, but instead of compact p-stratifolds we use proper maps from arbitrary p-stratifolds. We will not get into all the details, since they are similar to what we had before, we just stress the differences.

PROPOSITION 6.13. *Let X be a G space and EG_n an n connected closed oriented manifold with an orientation preserving free G action. The projection $\pi_X : EG_n \times X \to X$ is proper and induces an isomorphism $\pi_{X*} : SH_k^{lf,G}(EG_n \times X) \to SH_k^{lf,G}(X)$ for $n > k+1$.*

PROOF. We cannot follow the same proof we used before since EG is not compact thus the projection $EG \times X \to X$ is not proper. Therefore we approximate EG by EG_n for n big enough. The inverse of π_{X*} is given by $[g : S \to X] \mapsto [f \times g : S \to EG_n \times X]$ where $f : S \to EG_n$ is the classifying map defined by the universal property of EG_n and the fact that the action on S is free and S has the homotopy type of a CW complex of dimension $\leq k$. f is unique up to homotopy thus the map is well defined. Note that since g is proper so is $f \times g$. □

PROPOSITION 6.14. *Let X be a completely regular topological space with a free G action, then there is a natural isomorphism $SH_k^{lf,G}(X) \to SH_{k-dim(G)}^{lf}(X/G)$.*

PROOF. We define this map by $[g : S \to X] \mapsto [g/G : S/G \to X/G]$. This is well defined by lemma 6.3 and the fact that if g is proper and G is compact then so is g/G (proved in lemma 6.5). It has an inverse $[g : S \to X/G] \mapsto [\widetilde{g} : \widetilde{S} \to X]$ which is also well defined by lemma 6.5. Again, if g is proper then so is $\tilde{g}$. □

COROLLARY 6.15. *Let X be a strongly locally finite $G-CW$ complex, there is a natural isomorphism $\Phi_n^{lf,G} : SH_k^{lf,G}(X) \to H_{k-dim(G)}^{lf}(EG_n \times_G X)$ where $n > k+1$. $\Phi_n^{lf,G}$ commutes with the boundary, the transfer and with the cross product.*

PROOF. We define this isomorphism to be the composition:
$SH_k^{lf,G}(X) \to SH_k^{lf,G}(EG_n \times X) \to SH_{k-dim(G)}^{lf}(EG_n \times_G X) \to H_{k-dim(G)}^{lf}(EG_n \times_G X)$
where the second isomorphism is well defined since the action on $EG_n \times X$ is free. $\Phi_n^{lf,G}$ commutes with the boundary, the transfer and the cross product for the same reason Φ^G does. □

COROLLARY 6.16. *Let G be a finite group and X a strongly locally finite $G-CW$ complex. There is a natural isomorphism $\Phi^{lf,G} : SH_k^{lf,G}(X) \to H_k^{lf,G}(X)$, which commutes with the boundary, the transfer and the cross product.*

PROOF. EG_n are compact so we can identify $C_*^{lf}(EG_n)$ with $C_*(EG_n)$. Denote $P_*^+ = \varinjlim(C_*(EG_n))$ then $P_*^+ = C_*(\varinjlim(EG_n)) = C_*(EG)$ so it is a projective resolution. Since colimits commute with homology and with tensor products we have:

$\underrightarrow{lim}(H_k^{lf}(EG_n \times_G X)) = H_k(\underrightarrow{lim}(C_k^{lf}(EG_n \times_G X))) \cong H_k(P_*^+ \otimes C_*^{lf}(X)) = H_k^{lf,G}(X)$
We saw before that $\Phi_n^{lf,G}$ is an isomorphism for n large enough therefore $\Phi^{lf,G}$ is also an isomorphism. $\Phi_n^{lf,G}$ commutes with boundary and with the cross product for all n and also the maps $H_k^{lf}(EG_n \times_G X) \to H_k^{lf}(EG_{n+1} \times_G X)$ which imply the same thing for the colimit (both the boundary and the cross product commute with the colimit since in order to compute them we only need the group $H_k^{lf}(EG_n \times_G X)$ for $k < n$ and those stabilize. $\square$

6.2. Stratifold backwards cohomology

Stratifold backwards cohomology was defined in [**24**] and is denoted by DSH_G^* (in [**24**] it is denoted SH_G^* and called equivariant stratifold cohomology). This is an equivariant cohomology theory defined on the category of smooth oriented manifolds with a smooth, orientation preserving G action and equivariant maps between them, where G is a compact Lie group. Poincaré duality for a closed oriented smooth manifold M of dimension m with an orientation preserving G action is given by $PD_M : DSH_G^k(M) \xrightarrow{\cong} SH_{m-k}^G(M)$ which is trivial.

Definition 6.17. Let G be a compact Lie group and M a smooth oriented manifold of dimension m with an orientation preserving smooth G action. For $k \leq m$, define $DSH_G^k(M) = \{g : S \to M\}_G / \sim$ i.e., bordism classes of equivariant maps $g : S \to M$ where:

- S is an oriented p-stratifold of dimension $m-k$ with a G action.
- The action of G on S is orientation preserving, regular, smooth and free.
- g is a smooth equivariant proper map.
- The bordism relation has to fulfill the same properties as the p-stratifolds and the action does. In particular the action on the cobordism should be free and extend the action on the boundary.

$DSH_G^k(M)$ has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : M \to N$ is a continuous equivariant map than we can define an induced map by pullback $f^* : DSH_G^k(N) \to DSH_G^k(M)$ (for equivariant transversality see [**24**] lemma 5).

A triple (U, V, M) consists of M which is a smooth oriented manifold with a smooth G action and $U, V \subseteq M$ which are two equivariant open subspaces which cover M. The coboundary map is defined in a similar way to the coboundary in SH^*. For an element $[S \to M] \in DSH_G^*(M)$ choose an equivariant map $f : S \to \mathbb{R}$ so that we will get a well defined G action on the preimage of every point. This can be done by taking any smooth map $S/G \to \mathbb{R}$ and pulling it back to S. We then have:

Theorem 6.18. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to DSH_G^k(M) \to DSH_G^k(U) \oplus DSH_G^k(V) \to DSH_G^k(U \cap V) \xrightarrow{\delta} DSH_G^{k+1}(M) \to ...$$

where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

Let G be a compact Lie group and H a closed subgroup. Every G manifold has a natural structure of an H manifold. Define the restriction map:

$res^G_H : DSH^k_G(M) \to DSH^k_H(M)$ by $[S \to X] \mapsto [S \to X]$ where on the right side the spaces are considered as spaces with an action of H.

There is a cross product $DSH^k_G(M) \otimes DSH^l_{G'}(N) \to DSH^{k+l}_{G\times G'}(M \times N)$ given by: $[g_1 : S \to M] \times [g_2 : T \to N] = (-1)^{(m+dim(G))l}[g_1 \times g_2 : S \times T \to M \times N]$. This product is bilinear and natural.

If $G = G'$ we can use the diagonal $\Delta : G \to G \times G$ and compose this product with the restriction map $res^{G\times G}_{\Delta(G)} : DSH^{k+l}_{G\times G}(M \times N) \to DSH^{k+l}_G(M \times N)$, and to get a cross product $\times : DSH^k_G(M) \otimes DSH^l_G(N) \to DSH^{k+l}_G(M \times N)$. The cup product is given by $\alpha \cup \beta = \Delta^*(\alpha \times \beta)$ where $\Delta : M \to M \times M$ is the diagonal map.

DSH^*_G with the coboundary operator and the cross product is a multiplicative equivariant cohomology theory. We call it (parametrized) stratifold backwards cohomology.

Here are some properties of stratifold backwards cohomology:
The main property is Poincaré duality. As before, there are obvious forms of equivariant duality:

Theorem 6.19. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism $PD_M : DSH^k_G(M) \to SH^G_{m-k}(M)$.*

Theorem 6.20. *Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism:*
$PD_M : DSH^k_G(M) \to SH^{lf,G}_{m-k}(M)$.

In both proofs we have to use an equivariant analog to the approximation proposition we used before. This can be deduced from ([**6**] VI,4.2)

Theorem 6.21. *For a smooth oriented manifold M with a free orientation preserving G action we have a natural isomorphism $\Upsilon_G : DSH^*_G(M) \to SH^*(M/G)$ (notice that in cohomology there is no dimension shift).*

Proof. We define Υ_G by $[g : S \to M] \mapsto [g/G : S/G \to M/G]$. The proof that it is a well defined isomorphism is similar to what we had before.
Υ_G commutes with the cross product:
Let $\alpha = [S \to M] \in DSH^k_G(M)$ and $\beta = [T \to N] \in DSH^l_{G'}(N)$ then:
$\Upsilon_{G\times G'}(\alpha \times \beta) = \Upsilon_{G\times G'}((-1)^{(m+dim(G))l}[S \times T \to M \times N]) =$
$= (-1)^{(m+dim(G))l}[S \times T/G \times G' \to M \times N/G \times G']$
The diffeomorphism $S/G \times T/G' \cong S \times T/G \times G'$ switches the orientation by $(-1)^{dim(G)\cdot(dim(T)-dim(G'))}$, and the diffeomorphism $M/G \times N/G' \cong M \times N/G \times G'$ switches the orientation by $(-1)^{dim(G)\cdot(n-dim(G'))}$,thus the above is equal to:
$= (-1)^{ml}[S/G \times T/G' \to M/G \times N/G'] = [S/G \to M/G] \times [T/G' \to N/G'] = \Upsilon_G(\alpha) \times \Upsilon_{G'}(\beta)$
□

The coefficients $DSH^*_G(G/H)$ are determined completely by Poincaré duality (where H is a closed subgroup of dimension $dim(H)$). All the maps are isomorphisms:

$DSH^k_G(G/H) \xrightarrow{PD} SH^G_{dim(G)-dim(H)-k}(G/H) \xrightarrow{\Phi^G_0} H_{-dim(H)-k}(EG\times_G G/H) \to H_{-dim(H)-k}(BH)$
$(= H_{-k}(H,\mathbb{Z})$ if H is discrete). This is isomorphic to $SH^H_{-k}(pt) \to DSH^k_H(pt)$

A natural isomorphism between DSH_G^* and DH_G^*.

Let G be a fixed finite group and M a smooth oriented manifold of dimension m with a smooth and orientation preserving G action. The composition $DSH_G^k(M) \to SH_{m-k}^{lf,G}(M) \to H_{m-k}^{lf,G}(M) \to DH_G^k(M)$ is an isomorphism of groups for all oriented manifolds, denote it by Θ_G^D. We would like to show that Θ_G^D is a natural isomorphism of multiplicative equivariant cohomology theories.

We choose $n > m-k+1$ and let EG_n be an n connected closed oriented smooth manifold of dimension d_n with a free and orientation preserving G action.

Remark 6.22. From now on we will assume, without loss of generality, that all d_n are even. This way we avoid some sign problems that would have occurred otherwise.

Lemma 6.23. *There is a natural isomorphism:*

$$C^*(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) \xrightarrow{\times_G} C^*(EG_n \times_G M)$$

Proof. Since G is finite we have an isomorphism:
$C^*(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) \xrightarrow{\cong} Hom_{\mathbb{Z}[G]}(C_*(EG_n), \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M)$
$C_*(EG_n)$ are finitely generated since EG_n is compact and projective since G acts freely on EG_n. So, by the duality theorem, we have:
$\xrightarrow{\cong} Hom_{\mathbb{Z}[G]}(C_*(EG_n), C^*(M))$
By adjunction we have:
$\xrightarrow{\cong} Hom_{\mathbb{Z}}(C_*(EG_n) \otimes_{\mathbb{Z}[G]} C_*(M), \mathbb{Z}) \xrightarrow{\cong} Hom_{\mathbb{Z}}(C_*(EG_n \times_G M), \mathbb{Z}) = C^*(EG_n \times_G M)$.
One might check that the composition is given by:
$\varphi \times_G \psi(e \otimes m) = (-1)^{|\varphi|\cdot|\psi|}\Sigma_{g\in G}\varphi(ge)\cdot\psi(gm)$ □

Lemma 6.24. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} C^{d_n+*}(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) & \underset{\cong}{\xrightarrow{\times_G}} & C^{d_n+*}(EG_n \times_G M) \\ PD_{EG_n} \otimes PD_M \downarrow & & \downarrow PD_{EG_n \times_G M} \\ C_{-*}(EG_n) \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) & \underset{\cong}{\xrightarrow{\times_G}} & C^{lf}_{m-*}(EG_n \times_G M) \end{array}$$

Proof. Let $\sigma_{EG_n} = \sum e_k \in C_{d_n}(EG_n)$ and $\sigma_M = \sum m_l \in C^{lf}_m(M)$ be the representatives of the fundamental classes of EG_n and M respectively. Let $\sigma_{EG_n,G} = \sum e_{k'} \in C_{d_n}(EG_n)$ be a chain with the property that $\sum_{g\in G} g \cdot \sigma_{EG_n,G} = \sigma_{EG_n}$ then $\sigma_{EG_n,G} \otimes \sigma_M$ is the representative of the fundamental class of $EG_n \times_G M$ in $C^{lf}_{d_n+m}(EG_n \times_G M) \cong C_{d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C^{lf}_m(M)$. Choose equivariant cellular approximations to the diagonal maps: $\Delta_1 : EG_n \to EG_n^2$, $\Delta_2 : M \to M^2$ and using these maps choose $\Delta : EG_n \times_G M \to (EG_n \times_G M)^2$. Denote $\Delta_{1*}(\sigma_{EG_n,G}) = \sum_i e^1_i \otimes e^2_{d_n-i}$ and $\Delta_{2*}(\sigma_M) = \sum_j m^1_j \otimes m^2_{m-j}$ then $\Delta_*(\sigma_{EG_n,G} \otimes \sigma_M) = \sum_{i,j}(-1)^{(d_n-i)j} e^1_i \otimes_G m^1_j \otimes e^2_{d_n-i} \otimes_G m^2_{m-j}$.

We follow the image of an element $\varphi \otimes \psi \in C^{d_n+*}(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M)$. If we first go down and then right we get:
$(\varphi \cap \sigma_{EG_n}) \otimes (\psi \cap \sigma_M) = \varphi \cap (\Sigma_g g \cdot \sigma_{EG_n,G}) \otimes_G \psi \cap \sigma_M$
$= \Sigma_g (\varphi \cap (g \cdot \sigma_{EG_n,G}) \otimes_G \psi \cap (g \cdot \sigma_M))$ (σ_M is invariant - $g \cdot \sigma_M = \sigma_M$)
$= \Sigma_g \left((-1)^{|\varphi|(d_n-|\varphi|)}\varphi(ge^2_{|\varphi|}) \cdot ge^1_{d_n-|\varphi|} \otimes_G (-1)^{|\psi|(m-|\psi|)}\psi(gm^2_{|\psi|}) \cdot gm^1_{m-|\psi|}\right)$
$= (-1)^{|\varphi|(d_n-|\varphi|)+|\psi|(m-|\psi|)} \left(\Sigma_g \varphi(ge^2_{|\varphi|}) \cdot \psi(gm^2_{|\psi|})\right) \cdot e^1_{d_n-|\varphi|} \otimes_G m^1_{m-|\psi|}$
$= (-1)^{|\varphi|(d_n-|\varphi|)+|\psi|(m-|\psi|)+|\varphi||\psi|}\varphi \times_G \psi(e^2_{|\varphi|} \otimes_G m^2_{|\psi|}) \cdot e^1_{d_n-|\varphi|} \otimes_G m^1_{m-|\psi|}$

$\varphi \times_G \psi \cap \sigma_{EG_n,G} \otimes \sigma_M$
which is equal to the image if we first go right and then down. □

Define a map $H^{k+d_n}(EG_n \times_G M) \to DH^k_G(M)$ on chain level as the composition:
$C^{*+d_n}(EG_n \times_G M) \to C^{*+d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) \to P^+_{-*} \otimes_{\mathbb{Z}[G]} C^*(M) \to$
$\to Hom_{\mathbb{Z}[G]}(P^-_*, \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} C^*(M) \to Hom_{\mathbb{Z}[G]}(P^-_*, C^*(M)) = DH^k_G(M)$
Where $C^{*+d_n}(EG_n)$ has an augmentation $C^{d_n}(EG_n) \to \mathbb{Z}[G]$ given by evaluation on the top class and the map $C^{*+d_n}(EG_n) \to P^+_{-*}$ is the unique (up to homotopy) augmentation preserving map given by the universal property of P^+_* (**[7]** I,7.4).

Lemma 6.25. *The following diagram commutes:*

$$\begin{array}{ccc} H^{k+d_n}(EG_n \times_G M) & \to & DH^k_G(M) \\ \downarrow PD_{EG\times_G M} & & \downarrow PD_M \\ H^{lf}_{m-k}(EG_n \times_G M) & \to & H^{lf,G}_{m-k}(M) \end{array}$$

Moreover, this diagram can be defined on chain level and there it commutes up to homotopy.

Proof. We prove that the following diagram commutes up to homotopy:

$$\begin{array}{ccccc} C^{d_n+*}(EG_n \times_G M) & \to & C^{d_n+*}(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) & \to & P^+_{-*} \otimes_{\mathbb{Z}[G]} C^*(M) \\ \downarrow PD_{EG_n \times_G M} & (1) & PD_{EG_n} \otimes PD_M \downarrow & (2) & PD_M \downarrow \\ C^{lf}_{m-*}(EG_n \times_G M) & \to & C_{-*}(EG_n) \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) & \to & P^+_{-*} \otimes_{\mathbb{Z}[G]} C^{lf}_{m-*}(M) \end{array}$$

(1) This is lemma 6.24.
(2) We have to show that $C^{d_n+*}(EG_n) \xrightarrow{PD_{EG_n}} C_{-*}(EG_n)$ is augmentation preserving. Take a cochain $\varphi_j \in C^{d_n}(EG_n)$ with the property that $\varphi_j(e_j) = 1$ for one n-cell e_j and zero else then $\varphi_j \cap \sigma_{EG_n} = \varphi_j \cap e_j = e_0$ which is a single one cell so the image of e_0 under the map $C_0(EG_n) \to \mathbb{Z}$ is 1.

This proves the lemma since the right vertical map induces the map $DH^k_G(M) \xrightarrow{PD_M} H^{lf,G}_{m-k}(M)$. □

Proposition 6.26. *Θ^D_G is natural, that is for every smooth equivariant map $f : N \to M$ between two smooth oriented manifolds of dimension n and m resp. with a smooth and orientation preserving G action the following diagram commutes:*

$$\begin{array}{ccc} DSH^k_G(M) & \xrightarrow{\Theta^D_G} & DH^k_G(M) \\ f^* \downarrow & & \downarrow f^* \\ DSH^k_G(N) & \xrightarrow{\Theta^D_G} & DH^k_G(N) \end{array}$$

Proof. **First case** - $f : N \hookrightarrow M$ is an embedding of N as a closed invariant submanifold of M:

Take an element $\alpha = [S, g] \in DSH^k_G(M)$. We can assume that g is transversal to f, thus we can find a (closed) invariant tubular neighborhood U of N with boundary ∂U (**[6]** VI,2.2) (also transversal to g) and a projection map $\pi_N : U \to N$ with the property that the pullback of U will be a tubular neighbourhood $\pi_S : S \pitchfork U \to S \pitchfork N$ (similar to what we did in chapter 3).

Let EG_n be as above, we claim that the following diagram commutes:

$$\begin{array}{ccccccc}
H^{lf}_{m-k}(S/G) & \xrightarrow{(h\times_G g)_*} & H^{lf}_{m-k}(EG_n \times_G M) & \rightarrow & H^{lf,G}_{m-k}(M) & \xrightarrow{PD_M^{-1}} & DH^k_G(M) \\
\varepsilon\cdot\phi\downarrow & (1) & \downarrow \varepsilon\cdot\phi & & (2) & & \downarrow f^* \\
H^{lf}_{n-k}(S \pitchfork N/G) & \xrightarrow{(h\times_G g)_*} & H^{lf}_{n-k}(EG_n \times_G N) & \rightarrow & H^{lf,G}_{n-k}(N) & \xrightarrow{PD_N^{-1}} & DH^k_G(N)
\end{array}$$

i) $h : S \rightarrow EG_n$ is the classifying map defined by the universal property of EG_n and the fact that S has the homotopy type of a CW complex of dimension $\leq m-k < n$ and has a free G action.
ii) $H^{lf}_{m-k}(EG_n \times_G M) \xrightarrow{\phi} H^{lf}_{n-k}(EG_n \times_G N)$ and $H^{lf}_{m-k}(S/G) \xrightarrow{\phi} H^{lf}_{n-k}(S \pitchfork N/G)$ are defined in appendix 1 and $\varepsilon = (-1)^{(n-k)(m-n)}$. Both maps are defined using the Thom isomorphism.
(1) commutes by the naturality of the Thom class. This was also proved in chapter 3. To prove that (2) commutes it is enough to show that the following diagram commutes (using lemma 6.25):

$$\begin{array}{ccccc}
H^{lf}_{m-k}(EG_n \times_G M) & \rightarrow & H^{k+d_n}(EG_n \times_G M) & \rightarrow & DH^k_G(M) \\
\downarrow \varepsilon\cdot\phi & & \downarrow f^* & & \downarrow f^* \\
H^{lf}_{n-k}(EG_n \times_G N) & \rightarrow & H^{k+d_n}(EG_n \times_G N) & \rightarrow & DH^k_G(N)
\end{array}$$

The left side commutes by proposition 7.21, the right side clearly commutes.

Remark. It is actually true that all the maps in both diagrams can be defined on chain level and the it will commute up to homotopy. We will use that later.

We follow both images of $[S/G]^{lf} \in H^{lf}_{m-k}(S/G)$ in the original diagram. By definition, the image of $[S/G]^{lf}$ in the top row is $\Theta^D_G(\alpha)$, which is mapped in the right column to $f^*(\Theta^D_G(\alpha))$. As seen before $\varepsilon \cdot \phi([S/G]^{lf}) = [S \pitchfork N/G]^{lf}$. By definition its image in the bottom row is equal to $\Theta^D_G(f^*(\alpha))$ using the fact that $[S \pitchfork N/G, g]^{lf} = f^*(\alpha)$. Since the diagram commutes we conclude that $f^*(\Theta^D_G(\alpha)) = \Theta^D_G(f^*(\alpha))$.

The general case - $f : N \rightarrow M$ is an arbitrary smooth map:

There exists a finite dimensional representation V^G and a smooth equivariant embedding $i : N \hookrightarrow V^G$ as a closed submanifold [**30**]. f is equal to the composition $N \xrightarrow{f\times i} M \times V^G \xrightarrow{\pi_M} M$. $f \times i$ is an embedding of N as a closed invariant submanifold. π_M has an inverse up to G homotopy which is an embedding - $M \xrightarrow{Id\times 0} M \times V^G$ hence this follows from the previous case.

□

Lemma 6.27. *Θ^D_G commutes with the cross product, that is for every two smooth oriented manifolds M and N of dimension m and n respectively with a smooth and orientation preserving G action the following diagram commutes:*

$$\begin{array}{ccc}
DSH^k_G(M) \otimes DSH^l_{G'}(N) & \xrightarrow{\Theta^D_G \otimes \Theta^D_{G'}} & DH^k_G(M) \otimes DH^l_{G'}(N) \\
\times\downarrow & & \times\downarrow \\
DSH^{k+l}_{G\times G'}(M \times N) & \xrightarrow{\Theta^D_{G\times G'}} & DH^{k+l}_{G\times G'}(M \times N)
\end{array}$$

PROOF. The following diagram commutes up to sign $(-1)^{ml}$ since the horizontal maps are identities and the vertical maps are equal up to that exact sign:

$$\begin{array}{ccc} DSH^k_G(M)\otimes DSH^l_{G'}(N) & \xrightarrow{PD_M\otimes PD_N} & SH^{lf,G}_{m-k}(M)\otimes SH^{lf,G'}_{n-l}(N) \\ \times\downarrow & & \downarrow\times \\ DSH^{k+l}_{G\times G'}(M\times N) & \xrightarrow{PD_{M\times N}} & SH^{lf,G\times G'}_{m+n-k-l}(M\times N) \end{array}$$

The following diagram commutes:

$$\begin{array}{ccc} SH^{lf,G}_{m-k}(M)\otimes SH^{lf,G'}_{n-l}(N) & \xrightarrow{\Phi^{lf}_G\otimes\Phi^{lf}_{G}} & H^{lf,G}_{m-k}(M)\otimes H^{lf,G'}_{n-l}(N) \\ \downarrow\times & & \downarrow\times \\ SH^{lf,G\times G'}_{m+n-k-l}(M\times N) & \xrightarrow{\Phi^{lf}_{G\times G'}} & H^{lf,G\times G'}_{m+n-k-l}(M\times N) \end{array}$$

since Φ^{lf}_G commutes with the cross product (as was shown before).
And last, the following diagram commutes up to sign $(-1)^{ml}$:

$$\begin{array}{ccc} H^{lf,G}_{m-k}(M)\otimes H^{lf,G'}_{n-l}(N) & \xrightarrow{PD^{-1}_M\otimes PD^{-1}_N} & DH^k_G(M)\otimes DH^l_{G'}(N) \\ \downarrow\times & & \times\downarrow \\ H^{lf,G\times G'}_{m+n-k-l}(M\times N) & \xrightarrow{PD^{-1}_{M\times N}} & DH^{k+l}_{G\times G'}(M\times N) \end{array}$$

To see that take an element $p\otimes\varphi\otimes q\otimes\psi\in P^+_G\otimes C^*(M)\otimes P^+_{G'}\otimes C^*(N)$. If we first go left and then down we get:
$p\otimes\varphi\otimes q\otimes\psi\mapsto p\otimes(\varphi\cap\sigma_M)\otimes q\otimes(\psi\cap\sigma_N)\mapsto(-1)^{|q|\cdot|\varphi\cap\sigma_M|}p\otimes q\otimes(\varphi\cap\sigma_M)\otimes(\psi\cap\sigma_N)$
where σ_M and σ_N are the representatives of the fundamental classes of M and N.
If we first go down and then left we get:
$p\otimes\varphi\otimes q\otimes\psi\mapsto(-1)^{|q|\cdot|\varphi|}p\otimes q\otimes\varphi\otimes\psi\mapsto(-1)^{|q|\cdot|\varphi|}p\otimes q\otimes(\varphi\otimes\psi)\cap(\sigma_M\otimes\sigma_N)$
($\sigma_M\otimes\sigma_N$ is the representative of the fundamental class of $M\times N$).
$=(-1)^{|q|\cdot|\varphi|+m\cdot|\psi|}p\otimes q\otimes(\varphi\cap\sigma_M)\otimes(\psi\cap\sigma_M)$
Comparing the signs we get that the diagram commutes up to the sign:
$(-1)^{|q|\cdot|\varphi|+m\cdot|\psi|}\cdot(-1)^{|q|\cdot|\varphi\cap\sigma_M|}$ which is equal to $(-1)^{m\cdot(|\psi|+|q|)}=(-1)^{m\cdot l}$
Combining the three diagrams we get the commutativity of the original diagram. □

LEMMA 6.28. *Θ^D_G commutes with the coboundary operator.*

PROOF. This is similar to what we did for Θ. □

LEMMA 6.29. *Θ^D_G commutes with the restriction map.*

PROOF. This follows from the fact that the maps $DSH^k_G(M)\to SH^{lf,G}_{m-k}(M)\to H^{lf,G}_{m-k}(M)\to DH^k_G(M)$ commute with the restriction in cohomology and transfer in homology. □

We proved that $\Theta^D_G : DSH^*_G(M)\to DH^*_G(M)$ is a natural isomorphism of graded groups, it commutes with the coboundary operator in the $Mayer-Vietoris$ sequence, with the cross product and the restriction map, thus we proved the following:

THEOREM 6.30. *$\Theta^D_G : DSH^*_G(M)\to DH^*_G(M)$ is a natural isomorphism of multiplicative equivariant cohomology theories.*

6.3. Stratifold Borel cohomology

Let G be a compact Lie group and M a smooth manifold of dimension m with an orientation preserving smooth G action. Define the stratifold Borel cohomology $SH^k_G(M)$ to be $\varprojlim DSH^k_G(EG_n \times M)$ with respect to the maps induced by the inclusions $EG_n \times M \hookrightarrow EG_{n+1} \times M$.

REMARK 6.31. This inverse limit is different from $\varprojlim DSH^*_G(EG_n \times M)$ in the category of rings!

This inverse limit stabilizes due to the following:

LEMMA 6.32. *For n large enough the maps $DSH^k_G(EG_{n+1} \times M) \to DSH^k_G(EG_n \times M)$ are isomorphisms.*

PROOF. The following diagram commutes and all the maps are isomorphisms:

$$\begin{array}{ccccc} DSH^k_G(EG_{n+1} \times M) & \xrightarrow{\cong} & SH^k(EG_{n+1} \times_G M) & \xrightarrow{\cong} & H^k(EG_{n+1} \times_G M) \\ \downarrow & & \downarrow & & \downarrow\cong \\ DSH^k_G(EG_n \times M) & \xrightarrow{\cong} & SH^k(EG_n \times_G M) & \xrightarrow{\cong} & H^k(EG_n \times_G M) \end{array}$$

□

$SH^*_G(M)$ has a natural structure of an Abelian group. As we saw before, one can define triples and a boundary map for a triple and we will have the following:

THEOREM 6.33. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to SH^k_G(M) \to SH^k_G(U) \oplus SH^k_G(V) \to SH^k_G(U \cap V) \xrightarrow{\delta} SH^{k+1}_G(M) \to ...$$

PROOF. This follows from the fact that if (U, V, M) is a triple then also $(EG_n \times U, EG_n \times V, EG_n \times M)$ are triples so we can use the Mayer-Vietoris for DSH^*_G. □

We can also define the cross product. SH^*_G with the boundary operator and the product is a multiplicative equivariant cohomology theory.

A natural isomorphism between SH^*_G and H^*_G.
A natural isomorphism $\Theta_G : SH^k_G \to H^k_G$ given by the composition:
$SH^k_G = \varprojlim DSH^k_G(EG_n \times -) \to \varprojlim SH^k(EG_n \times_G -) \to \varprojlim H^k(EG_n \times_G -) \to H^k_G$
Θ_G is a natural isomorphism of graded groups, it commutes with the coboundary operator in the $Mayer-Vietoris$ sequence and with the cross product since it is a composition of such natural isomorphisms. This proves the following:

THEOREM 6.34. *$\Theta_G : SH^*_G \to H^*_G$ is a natural isomorphism of multiplicative equivariant cohomology theories.*

In order to see the relation with Θ^D_G we present the dual homology theory:

6.4. Stratifold backwards homology

Let G be a fixed compact Lie group, we introduce the homology theory dual to stratifold Borel cohomology, which we call stratifold backwards homology (since it is naturally isomorphic to the backwards homology when G is finite) and denote it by DSH^G_*. It is defined on the category of finite dimensional $G-CW$ complexes, and equivariant cellular maps between them.

Let X be a finite dimensional $G-CW$ complex. For EG_n and EG_{n+1} as before we define a map:

$i^! : SH^G_{k+d_{n+1}}(EG_{n+1} \times X) \to SH^G_{k+d_n}(EG_n \times X)$ the following way: For an element $\alpha = [S, g] \in SH^G_{k+d_{n+1}}(EG_{n+1} \times X)$ we can change g up to homotopy such that the composition $\pi_{EG_{n+1}} \circ g : S \to EG_{n+1}$ will be transversal to the inclusion $EG_n \hookrightarrow EG_{n+1}$. We define $i^!(\alpha)$ to be the pullback of $g : S \to EG_{n+1} \times X$ (after changing g) along the inclusion $i : EG_n \times X \hookrightarrow EG_{n+1} \times X$. The fact that it is well defined is similar to the corresponding result in DSH^*_G keeping in mind that the following diagram is a pullback diagram:

$$\begin{array}{ccccc} S' & \to & EG_n \times X & \xrightarrow{\pi_{EG_n}} & EG_n \\ \downarrow & & \downarrow & & \downarrow \\ S & \to & EG_{n+1} \times X & \xrightarrow{\pi_{EG_{n+1}}} & EG_{n+1} \end{array}$$

REMARK 6.35. In case X is a smooth oriented manifold with a smooth and orientation preserving G action $i^!$ is the umkehr map.

LEMMA 6.36. *$i^!$ commutes with induced maps, that is for $f : X \to Y$, an equivariant continuous map between two finite dimensional $G-CW$ complexes, the following diagram commutes:*

$$\begin{array}{ccc} SH^G_{k+d_{n+1}}(EG_{n+1} \times X) & \xrightarrow{i^!} & SH^G_{k+d_n}(EG_n \times X) \\ \downarrow f_* & & \downarrow f_* \\ SH^G_{k+d_{n+1}}(EG_{n+1} \times Y) & \xrightarrow{i^!} & SH^G_{k+d_n}(EG_n \times Y) \end{array}$$

PROOF. Let $[g : S \to EG_{n+1} \times X]$ be an element in $SH^G_{k+d_{n+1}}(EG_{n+1} \times X)$. The commutativity of the diagram follows from the fact that the following diagram is a composition of pullback squares:

$$\begin{array}{ccccc} S' & \to & EG_n \times X & \to & EG_n \times Y \\ \downarrow & & \downarrow & & \downarrow \\ S & \to & EG_{n+1} \times X & \to & EG_{n+1} \times Y \end{array}$$

□

Define the stratifold backwards homology $DSH^G_k(X)$ to be $\varprojlim SH^G_{k+d_n}(EG_n \times X)$ with respect to the maps $i^!$. The induced maps are defined to be the inverse limit of the induced maps in SH^G_*. This is well defined by the lemma above.

REMARK 6.37. This idea is quit general, we can define homology theories using a filtration of a space by a sequence of manifolds $M_n \hookrightarrow M_{n+1}$ (with or without a group action).

DSH^G_* has a natural structure of an Abelian group. Like before, one can define triples and a boundary map for a triple and we will have the Mayer-Vietoris long exact sequence.

REMARK 6.38. Similarly we can define a locally finite version of this theory $DSH^{lf,G}_k(X)$.

As before, there are obvious forms of equivariant duality we call Poincaré duality:

THEOREM 6.39. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism $PD_M : SH^k_G(M) \to DSH^G_{m-k}(M)$.*

THEOREM 6.40. *Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism:* $PD_M : SH^k_G(M) \to DSH^{lf,G}_{m-k}(M)$.

A natural isomorphism between $DSH^{lf,G}_*$ and $DH^{lf,G}_*$.

We now restrict to the case of G finite. Let X be a locally finite, finite dimensional $G-CW$ complex. We would like to construct a natural isomorphism $\Phi^{lf,G}_D : DSH^{lf,G}_*(X) \to DH^{lf,G}_*(X)$.

LEMMA 6.41. *The following diagram commutes:*

$$\begin{array}{ccc} SH^{lf,G}_{k+d_{n+1}}(EG_{n+1}\times X) & \to & H^{lf,G}_{k+d_{n+1}}(EG_{n+1}\times_G X) \\ i^! \downarrow & & \downarrow \phi \\ SH^{lf,G}_{k+d_n}(EG_n \times X) & \to & H^{lf,G}_{k+d_n}(EG_n \times_G X) \end{array}$$

Where ϕ is defined as in appendix 1.

PROOF. Similar to the proof of 6.26, since we did not use the fact that X was a manifold (we use here the fact that $d_{n+1}-d_n$ is even). $\square$

LEMMA 6.42. *The map* $C_{*+d_{n+1}}(EG_{n+1}) \otimes_{\mathbb{Z}[G]} C^{lf}_*(X) \xrightarrow{\phi\otimes Id} C_{*+d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C^{lf}_*(X)$ induces in homology the map $H_{k+d_{n+1}}(EG_{n+1}\times_G X) \xrightarrow{\phi} H_{k+d_n}(EG_n \times_G X)$.

PROOF. Consider EG_n as an invariant submanifold of EG_{n+1} and find an invariant tubular neighborhood U with boundary ∂U. Let τ be a representative of the Thom class of $\pi : U \to EG_n$. It can be chosen to be invariant by pulling back a representative of the Thom class of $\pi : U/G \to EG_n/G$ using the fact that G acts freely on EG_n). The pullback of the Thom class of the bundle $U\times_G X \to EG_n \times_G X$ along the quotient map to the bundle $U \times X \to EG_n \times X$ is equal to $\tau \times 1_X$. We can define $\phi : C^{lf}_{*+d_{n+1}}(EG_{n+1}\times_G X) \to C^{lf}_{*+d_n}(EG_n \times_G X)$ on chain level by $e\otimes x \mapsto (\pi\times_G Id)_*(\tau_{EG_n\times_G X} \cap \bar{e}\otimes x)$ where $\bar{e} = \begin{cases} 0 & if\ e \notin \mathring{U} \\ e & else \end{cases}$

The fact that the pullback of $\tau_{EG_n\times_G X}$ to $EG_n \times X$ is equal to $\tau\times 1_X$ implies that that $e\otimes x \mapsto \phi(e)\otimes x$. Note that if u denotes the representative of $[U,\partial U]^{lf}$ and σ_{EG_k} denotes the representative of the fundamental class of EG_k then $\tau\cap u = i_*(\sigma_{EG_n})$ therefore $\phi(\sigma_{EG_{n+1}}) = \pi_*(\tau\cap u) = \sigma_{EG_n}$. $\square$

We construct a natural isomorphism $\Phi^{lf,G}_D : DSH^{lf,G}_k(X) \to DH^{lf,G}_k(X)$ as the composition of two natural isomorphisms:

1) The limit map: $\varprojlim SH^{lf,G}_{k+d_n}(EG_n\times X) \to \varprojlim H_k(C_{*+d_n}(EG_n)\otimes_{\mathbb{Z}[G]} C^{lf}_*(X))$

2) The map $\varprojlim H_k(C_{*+d_n}(EG_n)\otimes_{\mathbb{Z}[G]} C^{lf}_*(X)) \to DH^{lf,G}_k(X)$ which is the inverse of the following map: Let P^-_* be a backwards projective resolution. By ([**7**] VI,2.4) there are chain maps $\iota_n : P^-_* \to C_{*+d_n}(EG_n)$ which commute with the coaugmentation maps $\mathbb{Z} \to P^-_0$ and $\mathbb{Z} \to C_{d_n}(EG_n)$ where the latter map is given by $k \mapsto k\cdot\sigma_{EG_n}$, ($\iota_n$ are unique up to homotopy). ϕ is co-augmentation preserving thus $\phi\circ\iota_{n+1} \sim \iota_n$ by uniqueness up to homotopy of co-augmentation preserving maps. If we truncate both chain complexes under $k=-n$ the map will be a homotopy equivalence (truncate, but leave the images of the last map). The ι_n's induce maps $\iota^*_n : DH^{lf,G}_k(X) \to H_k(C_{*+d_n}(EG_n)\otimes_{\mathbb{Z}[G]} C^{lf}_*(X))$ which are compatible with the

maps ϕ, so they induce a map $DH_k^{lf,G}(X) \to \varprojlim H_k(C_{*+d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C_*^{lf}(X))$. ι_n^* are isomorphisms for n large enough (here we use the fact that X is finite dimensional) so the limit map is also an isomorphism.

REMARK 6.43. Given an element $\alpha = [S, g] \in SH_*^{lf,G}(EG_n \times X)$ one can look at the composition: $C_*^{lf}(S/G) \xrightarrow{g_*} C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) \xrightarrow{\iota_n' \otimes Id} P_*^- \otimes C_*^{lf}(X)$ then $\Phi_D^{lf,G}(\alpha)$ will be image of $[S/G]^{lf}$ (ι_n' is a map $C_{*+d_n}(EG_n) \xrightarrow{\iota_n'} P_*^-$ which is defined only for $* > -n$ but this is not a problem since we are only interested in n large, here we use the fact that X is of finite dimension).

PROPOSITION 6.44. *The natural isomorphism* $\Theta_G : SH_G^k(M) \to H_G^k(M)$ *is equal to the composition* $SH_G^k(M) \xrightarrow{PD} DSH_{m-k}^{lf,G}(M) \xrightarrow{\Phi_D^{lf,G}} DH_{m-k}^{lf,G}(M) \xrightarrow{PD} H_G^k(M)$.

PROOF. Look at the following diagram:

$$\begin{array}{ccccc} \varprojlim DSH_G^k(EG_n \times M) & \to & \varprojlim H^k(EG_n \times_G M) & \to & H_G^k(M) \\ \downarrow PD_{EG_n \times M} & (1) & \downarrow PD_{EG_n \times M} & (2) & \downarrow PD_M \\ \varprojlim SH_{m-k+d_n}^{lf,G}(EG_n \times M) & \to & \varprojlim H_{m-k+d_n}^{lf}(EG_n \times_G M) & \to & DH_{m-k}^{lf,G}(M) \end{array}$$

(1) This square commutes for all n by definition thus it also commutes in the limit.
(2) Look at the following diagram:

$$\begin{array}{ccc} C^*(EG_n) \otimes_{\mathbb{Z}[G]} C^*(M) & \leftarrow & P_{-*}^- \otimes_{\mathbb{Z}[G]} C^*(M) \\ \downarrow PD_{EG_n} \otimes PD_M & & \downarrow PD_M \\ C_{d_n-*}(EG_n) \otimes_{\mathbb{Z}[G]} C_{m-*}^{lf}(M) & \leftarrow & P_*^- \otimes_{\mathbb{Z}[G]} C_{m-*}^{lf}(M) \end{array}$$

This diagram commutes after passing to homology since $1_{EG_n} \cap \sigma_{EG_n} = \sigma_{EG_n}$ (so the map $C^*(EG_n) \to C_{d_n-*}(EG_n)$ is co-augmentation preserving). This implies the commutativity of the diagram (2) by lemma 6.24. $\square$

PROPOSITION 6.45. *The following diagram commutes:*

$$\begin{array}{ccc} SH_k^{lf,G}(X) & \xrightarrow{\Phi^{lf,G}} & H_k^{lf,G}(X) \\ \downarrow & & \downarrow \\ DSH_k^{lf,G}(X) & \xrightarrow{\Phi_D^{lf,G}} & DH_k^{lf,G}(X) \end{array}$$

PROOF. Let EG_n be as before, denote by $\sigma_{EG_n} \in C_{d_n}(EG_n)$ the representatives of the fundamental class of EG_n. It is unique and thus invariant since we are using cellular chains. Choose augmentation preserving chain maps $l^+ : C_*(EG_n) \to P_*^+$ and $l^- : C_{*+d_n}(EG_n) \to P_*^-$ (the second map is defined only for $* > -n$ but this is not a problem since we are only interested in n large, here we use the fact that X is of finite dimension).

Define a degree d_n chain map: $\rho : C_*(EG_n) \to C_{*+d_n}(EG_n)$ the following way: $\rho : C_0(EG_n) \to C_{d_n}(EG_n)$ is defined on generators by $\rho(e_0) = \sigma_{EG_n}$ (this is a map of $\mathbb{Z}[G]$-modules since σ_{EG_n} is invariant). For $k \neq 0$ let $\rho : C_k(EG_n) \to C_{k+d_n}(EG_n)$ be zero map. We have the following commutative diagram:

$$\begin{array}{ccccc} C_*(EG_n) & \to & \mathbb{Z} & \to & C_{*+d_n}(EG_n) \\ \downarrow l^+ & & \downarrow Id & & \downarrow l^- \\ P_*^+ & \to & \mathbb{Z} & \to & P_*^- \end{array}$$

Where the composition of the top row is ρ and the composition of the bottom row is the map $P_*^+ \to P_*^-$ we had before (l^- is the map ι_n' we had before).

Let $[S,g]$ be an element in $SH_k^{lf,G}(X)$. By approximation we can assume, without loss of generality, that S is a CW complex. Denote by $\sigma_S \in C_k^{lf}(S)$ the representative of the fundamental classes of S. Again, it is invariant. Choose an equivariant map $S \to EG_n$, it is unique up to G-homotopy. It gives us a map to the product $EG_n \times S$ which we approximate by a cellular map $h : S \to EG_n \times S$. The map $f : S \to X$ induces a map $S/G \xrightarrow{h/G} EG_n \times_G S \xrightarrow{Id \times_G f} EG_n \times_G X$. We have the following (strictly) commutative diagram:

$$\begin{array}{ccccccc} C_*^{lf}(S/G) & \xrightarrow{h_*} & C_*(EG_n) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes f_*} & C_*(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^+ \otimes Id} & P_*^+ \otimes C_*^{lf}(X) \\ & & \downarrow \rho \otimes Id & & \downarrow \rho \otimes Id & & \downarrow \\ & & C_{*+d_n}(EG_n) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes f_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^- \otimes Id} & P_*^- \otimes C_*^{lf}(X) \end{array}$$

where all tensor products are over $\mathbb{Z}[G]$.

The composition $S \xrightarrow{h} EG_n \times S \xrightarrow{\pi_S} S$ is homotopic to the identity, thus the same holds for the composition $C_*^{lf}(S/G) \xrightarrow{h_*} C_*(EG_n) \otimes_{\mathbb{Z}[G]} C_*^{lf}(S) \xrightarrow{\pi_{S*}} C_*^{lf}(S/G)$. Therefore $[S/G]^{lf}$ is mapped to itself and by uniqueness the same is true for its representative $\sigma_{S/G}$. This implies that $h_*(\sigma_{S/G})$ is of the form:
$e_0 \otimes \sigma_{S,G} + \Sigma_{0<i} e_i \otimes s_{k-i}$ where the image of e_0 under the map $C_0(EG_n) \to \mathbb{Z}$ is 1 and $\sigma_{S,G}$ is a lift of $\sigma_{S/G}$ (any element with the property $\Sigma_{g \in G} g \cdot \sigma_{S,G} = \sigma_S$). Thus $(\rho \otimes Id) \circ h_*(\sigma_{S/G}) = \sigma_{EG_n} \otimes \sigma_{S,G}$. The proposition follows from the fact that the image of $[S,g]$ under the map $SH_k^{lf,G}(X) \to DSH_k^{lf,G}(X)$ is $[Id \times g : EG_n \times S \to EG_n \times X]$ and $\sigma_{EG_n} \otimes \sigma_{S,G}$ is the representative of the fundamental class of $EG_n \times_G S$. □

Corollary 6.46. *The following diagram commutes:*

$$\begin{array}{ccc} DSH_G^k(M) & \xrightarrow{\Theta_G^D} & DH_G^k(M) \\ \downarrow & & \downarrow \\ SH_G^k(M) & \xrightarrow{\Theta_G} & H_G^k(M) \end{array}$$

Proof. This follows from the proposition above together with the commutativity of the following two diagrams:

$$\begin{array}{ccc} DSH_G^k(M) & \xrightarrow{PD_M} & SH_{m-k}^{lf,G}(M) \\ \downarrow & & \downarrow \\ SH_G^k(M) & \xrightarrow{PD_M} & DSH_{m-k}^{lf,G}(M) \end{array} \qquad \begin{array}{ccc} H_{m-k}^{lf,G}(M) & \xrightarrow{PD_M} & DH_G^k(M) \\ \downarrow & & \downarrow \\ DH_{m-k}^{lf,G}(M) & \xrightarrow{PD_M} & H_G^k(M) \end{array}$$

□

6.5. Stratifold Tate homology and cohomology

Stratifold Tate cohomology.

Let G be a compact Lie group and M a smooth manifold of dimension m with an orientation preserving smooth G action. For every n we define $\widehat{SH}_G^k(M)_n$ to be $\{(T, S, g, i_\partial)\} / \sim$ where:

- T is an oriented p-stratifold of dimension $m + d_n - k$ (where d_n is the dimension of EG_n) with boundary together with a G action which is orientation preserving, regular, smooth and free.
- S is an oriented p-stratifold of dimension $m - k - 1$ together with a G action which is orientation preserving, regular, smooth and free.
- $i_\partial : EG_n \times S \to \partial T$ is an orientation preserving, equivariant isomorphism.
- $g : T \to EG_n \times M$ is a proper equivariant smooth map and the composition $g \circ i_\partial : EG_n \times S \to EG_n \times M$ is of the form $Id \times \tilde{g}$ for some equivariant smooth map $\tilde{g} : S \to M$.

The bordism relation is defined the following way: We say that (T, S, g, i_∂) is bordant to $(T', S', g', i'_\partial)$ if the following conditions hold:

- There exists a bordism $(B, S \amalg -S', h)$ between $\tilde{g} : S \to M$ and $\tilde{g}' : S' \to M$ (i.e., $[S, \tilde{g}] = [S', \tilde{g}']$ as elements in $DSH_G^{k+1}(M)$).
  Thus $(EG_n \times B, EG_n \times (S \amalg S'), Id \times h)$ is a bordism between $Id \times \tilde{g} : EG_n \times S \to EG_n \times M$ and $Id \times \tilde{g}' : EG_n \times S' \to EG_n \times M$. (Note that B might be non empty even if $S = S' = \emptyset$).
- By gluing $(EG_n \times B, EG_n \times S \amalg EG_n \times S')$ to $(T, \partial T) + (T', \partial T')$ along $i_\partial + i'_\partial$ we obtain an oriented p-stratifold of dimension $m + d_n - k$ together with a G action which is orientation preserving, regular, smooth and free, which we denote by $\tilde{B}$. We require that the element $[g + g' + Id \times h : \tilde{B} \to EG_n \times M] \in DSH_G^k(EG_n \times M)$ will be the zero element.

Proposition 6.47. *The bordism relation is an equivalence relation.*

Proof. **The relation is reflexive:** Given an element (T, S, g, i_∂) one can give $T \times I$ the structure of a p-stratifold with boundary such that its boundary will be equal to $T \times \{0\} \cup \partial T \times I \cup T \times \{1\}$ by a similar procedure to the one appears in [**23**] in appendix A. This implies that (T, S, g, i_∂) is equivalent to itself.
The relation is symmetric: This is clear.
The relation is transitive: In order to prove this we have to know how to glue two p-stratifolds along a part of their boundary. This is proved in [**23**] in appendix A. $\square$

The maps $i : EG_n \to EG_{n+1}$ induce maps $i^* : \widehat{SH}_G^k(M)_{n+1} \to \widehat{SH}_G^k(M)_n$ by transversal pullback. Notice that the pullback of $Id \times \widetilde{g} : EG_{n+1} \times S \to EG_{n+1} \times M$ along the map $EG_n \times M \to EG_{n+1} \times M$ is equal to $Id \times \widetilde{g} : EG_n \times S \to EG_n \times M$, hence the map is well defined. Define the stratifold Tate cohomology to be $\varprojlim \widehat{SH}_G^k(M)_n$. This limit stabilizes due to the following:

Lemma 6.48. *The maps $i^* : \widehat{SH}_G^k(M)_{n+1} \to \widehat{SH}_G^k(M)_n$ are isomorphisms for n large enough.*

Proof. This follows from the following diagram and the five lemma :

$$\begin{array}{ccccccccc}
DSH^k_G(M) & \to & DSH^k_G(EG_{n+1}\times M) & \to & \widehat{SH}^k_G(M)_{n+1} & \to & DSH^{k+1}_G(M) & \to & DSH^{k+1}_G(EG_{n+1}\times M) \\
\downarrow Id & & \downarrow i^* & & \downarrow i^* & & \downarrow Id & & \downarrow i^* \\
DSH^k_G(M) & \to & DSH^k_G(EG_n\times M) & \to & \widehat{SH}^k_G(M)_n & \to & DSH^{k+1}_G(M) & \to & DSH^{k+1}_G(EG_n\times M)
\end{array}$$

□

REMARK 6.49. The exactness of the rows and the definitions of the natural transformations are explained below.

$\widehat{SH}^*_G(M)$ has a natural structure of an Abelian group. As we saw before, one can define triples and a boundary map for a triple and we will have the following:

THEOREM 6.50. *(Mayer-Vietoris) The following sequence is exact:*

$$... \to \widehat{SH}^k_G(M) \to \widehat{SH}^k_G(U) \oplus \widehat{SH}^k_G(U) \to \widehat{SH}^k_G(U \cap V) \xrightarrow{\delta} \widehat{SH}^{k+1}_G(M) \to ...$$

Define the cross product of $[(T,S,g,i_\partial)] \in \widehat{SH}^k_G(M)$ and $[(T',S',g',i'_\partial)] \in \widehat{SH}^l_G(N)$ by:
$[(-1)^{(m+dim(G))l}(T \times T', T \times \partial T' \cup \partial T \times T', g \times g', Id \times \tilde{g}' \cup \tilde{g} \times Id)]$
after smoothing the corners in $T \times T'$ and then pulling it back along the map $EG_n \times M \times N \to EG_n \times M \times EG_n \times N$. $\widehat{SH}^*_G$ with the boundary operator and the cup product is a multiplicative equivariant cohomology theory.

There are natural transformations:

- $DSH^*_G \to SH^*_G$ given by $[S \to M] \mapsto [EG_n \times S \to EG_n \times M]$ which is induced by π^* where $\pi : EG_n \times M \to M$ is the projection.
- $SH^*_G \to \widehat{SH}^*_G$ given by $[(T,g)] \mapsto [(T,\emptyset,g,\emptyset)]$.
- $\widehat{SH}^*_G \to DSH^{*+1}_G$ given by $[(T,S,g,i_\partial)] \mapsto [(S,\tilde{g})]$.

Note that the first two natural transformations are multiplicative.

We have the following:

THEOREM 6.51. *The following is a long exact sequence:*
$... \to DSH^*_G \to SH^*_G \to \widehat{SH}^*_G \to DSH^{*+1}_G \to ...$

PROOF. This follows easily from the definition of the bordism relation in $\widehat{SH}^*_G$. To prove exactness in $\widehat{SH}^*_G$ one might use the following: □

LEMMA 6.52. *Let (T,S,g,i_∂) be as before and assume that the map $\tilde{g} : S \to M$ bounds the map $\bar{g} : (S',S) \to M$. Denote by $(T',\emptyset,\hat{g},\emptyset) = (T,\hat{g})$ the element we get by gluing (T,S,g,i_∂) and $(EG_n \times S', S, \hat{g}, i)$ where i is the inclusion of $EG_n \times S$ in $EG_n \times S'$. Then (T,S,g,i_∂) is bordant to $(T',\emptyset,\hat{g},\emptyset) = (T,\hat{g})$.*

PROOF. This is trivial from the definition of the bordism relation. □

COROLLARY 6.53. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action. $DSH^k_G(M)$ vanishes for $k > m - dim(G)$ and $SH^k_G(M)$ vanishes for $k < 0$. Therefore the map $H^k(EG \times_G M) \to \widehat{SH}^k_G(M)$ is an isomorphism for $k > m - dim(G)$ and the map $\widehat{SH}^k_G(M) \to H_{m-k-1-dim(G)}(EG \times_G M)$ is an isomorphism for $k < -1$. If $dim(G) > m$ we deduce that $\widehat{SH}^k_G(M)$ is isomorphic to $H^k(EG \times_G M)$ when $k \geq 0$ and to $H_{m-k-1-dim(G)}(EG \times_G M)$ when $k < 0$.*

EXAMPLE 6.54. Suppose $M = pt$ (consists of one point) then:
$DSH^k_G(pt) = SH^G_{-k}(pt) = H_{-k-dim(G)}(BG)$ is concentrated in the non positive part.
$SH^k_G(pt) = H^k(BG)$ is concentrated in the non negative part.
The long exact sequence consists only of isomorphisms and zero maps aside of maybe in $k = 0$, what makes the computation of $\widehat{SH}^*_G(pt)$ easy. We would like to see that geometrically.

If $k \geq 0$ then $\widehat{SH}^k_G(pt)$ consists of elements of the form $[(T, \emptyset, g, \emptyset)]$ that is without boundary since otherwise the dimension of S would have been negative. Therefore the map $SH^k_G(pt) \to \widehat{SH}^k_G(pt)$ is surjective and if $k > 0$ it is actually an isomorphism.

For $k < 0$ take any element $[S \to pt] \in DSH^{k+1}_G(pt)$. S is the boundary of the cone CS. CS has a natural G action which is not free. Nevertheless, $EG_n \times CS$ has a free G action and its boundary is $EG_n \times S$ and the projection $\pi : EG_n \times CS \to EG_n$ give an element $[(EG_n \times CS, S, \pi, i)] \in \widehat{SH}^k_G(pt)$ so the map $\widehat{SH}^k_G(pt) \to DSH^{k+1}_G(pt)$ is surjective. It is actually an isomorphism since $DSH^k_G(EG_n \times pt)$ vanishes.

The map $\widehat{SH}^*_G(M) \to DSH^{*+1}_G(M)$ is not multiplicative since it shifts dimension. In the case of $M = pt$ the isomorphism $\widehat{SH}^{k-1}_G(pt) \to DSH^k_G(pt)$ for $k < 0$ gives an interesting product $DSH^{-k}_G(pt) \otimes DSH^{-l}_G(pt) \to DSH^{-k-l-1}_G(pt)$ $(k, l > 0)$. This induces by Poincaré duality a product $SH^G_k(pt) \otimes SH^G_l(pt) \to SH^G_{k+l+1}(pt)$ or in singular homology $H_k(BG) \otimes H_l(BG) \to H_{k+l+1+d_n}(BG)$. More about that appears in chapter 7.

Stratifold Tate homology. Let G be a fixed compact Lie group, we introduce stratifold Tate homology and denote it by $\widehat{SH}^G_*$. It is defined on the category of finite dimensional $G-CW$ complexes and equivariant cellular maps between them. The definition is like for stratifold Tate cohomology so we will not repeat it. Induced maps are defined by composition.

$\widehat{SH}^G_*(X)$ has a natural structure of an Abelian group. Like before, one can define triples and a boundary map for a triple and we will have the Mayer-Vietoris long exact sequence.

As before, there are obvious forms of equivariant duality we call Poincaré duality:

THEOREM 6.55. *Let M be a closed oriented smooth manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism* $PD_M : \widehat{SH}^k_G(M) \to \widehat{SH}^G_{m-k}(M)$.

THEOREM 6.56. *Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving G action then there is an isomorphism:* $PD_M : \widehat{SH}^k_G(M) \to \widehat{SH}^{lf,G}_{m-k}(M)$.

There are natural transformations:

- $SH^G_* \to DSH^G_*$ given by $[S \to X] \mapsto [EG_n \times S \to EG_n \times X]$.
- $DSH^G_* \to \widehat{SH}^G_*$ given by $[T, g] \mapsto [T, \emptyset, g, \emptyset]$.
- $\widehat{SH}^G_* \to SH^G_{k-1}$ given by $[T, S, g, i_\partial] \mapsto [S, \tilde{g}]$.

We have the following (the proof is the same as for cohomology):

PROPOSITION 6.57. *The following is a long exact sequence:*

$$... \to SH_*^G \to DSH_*^G \to \widehat{SH}_*^G \to SH_{*-1}^G \to ...$$

A natural isomorphism between $\widehat{SH}_*^{lf,G}$ and $\hat{H}_{*-1}^{lf,G}$.

The algebraic mapping cone: Let A and B be two chain complexes and $f : A \to B$ be a chain map. The algebraic mapping cone (or just mapping cone) of f, denoted by C_f, is the chain complex $C_{f,k} = B_{k+1} \oplus A_k$ with the differential given by $\partial(b, a) = (f(a) - \partial b, \partial a)$. It is easy to check that C_f is a chain complex.

Here are some properties of the mapping cone. We omit the proofs, which are left as an easy exercise.
1) The maps $B[-1] \to C_f$, given by $b \mapsto (b, 0)$, and $C_f \to A$, given by $(b, a) \mapsto a$, are chain maps and the following is an exact sequence:

$$0 \to B[-1] \to C_f \to A \to 0$$

2) This sequence induces a long exact sequence in homology:

$$... \to H_k(A) \xrightarrow{f_*} H_k(B) \to H_{k-1}(C_f) \to H_{k-1}(A) \xrightarrow{f_*} H_{k-1}(B) \to ...$$

and the connecting homomorphism ∂ is equal to f_*.
3) The mapping cone is functorial in the category of chain complexes and chain maps. That means that given a square of chain complexes which strictly commutes:

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ h_A \downarrow & & \downarrow h_B \\ A' & \xrightarrow{f'} & B' \end{array}$$

then the map $h_C : C_f \to C_{f'}$ given by $h_C(b, a) = (h_B(b), h_A(a))$ is a chain map and the following diagram commutes:

$$\begin{array}{ccccc} B[-1] & \to & C_f & \to & A \\ \downarrow h_B & & \downarrow h_C & & h_A \downarrow \\ B'[-1] & \to & C_{f'} & \to & A' \end{array}$$

inducing a map between the long exact sequences mentioned in 2).
4) Given a square of chain complexes which commutes up to homotopy s:

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ h_A \downarrow & & \downarrow h_B \\ A' & \xrightarrow{f'} & B' \end{array}$$

then the map $h_C : C_f \to C_{f'}$ given by $h_C(b, a) = (h_B(b) + s(a), h_A(a))$ is a chain map and the following diagram commutes:

$$\begin{array}{ccccc} B[-1] & \to & C_f & \to & A \\ \downarrow h_B & & \downarrow h_C & & h_A \downarrow \\ B'[-1] & \to & C_{f'} & \to & A' \end{array}$$

inducing a map between the long exact sequences mentioned in 2).

REMARK 6.58. h_C depends on the choice of s, even after passing to homology. Replacing any of the maps by another chain map which is chain homotopic to it might change the map induced in homology by h_C. These facts make the mapping cone non functorial in the homotopy category of chain complexes, which leads to a great deal of trouble. We will have to overcome this trouble again and again, since most of our maps are defined only up to chain homotopy, depending on choices that we made.

A fundamental class for a pair $(T, EG_n \times S)$.

Let G be a finite group, S a regular oriented p-stratifold of dimension $k-1$ together with a free and orientation preserving G action and $(T, EG_n \times S)$ a regular oriented p-stratifold of dimension $k + d_n$ with boundary together with a free and orientation preserving G action and $i_\partial : EG_n \times S \to T$ the inclusion which is assumed to be equivariant. Look at the map $\bar{\rho} : C_*^{lf}(S/G) \to C_{*+d_n}^{lf}(T/G)$ defined as the composition:

$$C_*^{lf}(S/G) \to C_*(EG_n) \otimes C_*^{lf}(S) \xrightarrow{\rho \otimes Id} C_{*+d_n}(EG_n) \otimes C_*^{lf}(S) \xrightarrow{i_{\partial *}} C_{*+d_n}^{lf}(T/G)$$

as defined in proposition 6.45. Denote the mapping cone of this map by $C_*^{lf}(T, S)$.

For a regular oriented p-stratifold P (with boundary) of dimension l denote by $\sigma_P \in C_l^{lf}(P)$ the representatives of its fundamental class. For example we have $\sigma_S \in C_{k-1}^{lf}(S)$, $\sigma_{EG_n} \in C_{d_n}(EG_n)$ and $\sigma_T \in C_{k+d_n}^{lf}(T)$. Define the fundamental class $\widehat{[T, S]}^{lf}$ to be the class of $(\sigma_{T/G}, \sigma_{S/G}) \in C_{k-1}^{lf}(T, S)$. This is a cycle since:
$\partial(\sigma_{T/G}, \sigma_{S/G}) = (i_{\partial *}(\sigma_{EG_n \times_G S}) - \partial\sigma_{T/G}, \partial\sigma_{S/G}) = (0, 0)$.

Let $(T, EG_n \times S)$ be as above and let (T'', T') be a null bordism, that is a regular oriented p-stratifold of dimension $k + d_n + 1$ with boundary, together with a free and orientation preserving G action, such that its boundary is obtained by gluing $(T, EG_n \times S)$ with an element of the form $(S' \times EG_n, S \times EG_n)$ along the boundary.

LEMMA 6.59. *The inclusion induces a commutative square:*

$$\begin{array}{ccc} C_*(S/G) & \to & C_*(S'/G) \\ \downarrow & & \downarrow \\ C_{*+d_n}(T/G) & \to & C_{*+d_n}(T''/G) \end{array}$$

which induces a map between the mapping cones, mapping $\widehat{[T, S]}^{lf}$ *to zero.*

PROOF. This follows from the fact that:
$\partial(-\sigma_{T''/G}, -\sigma_{S'/G}) = (\partial\sigma_{T''/G} - \sigma_{S' \times_G EG_n}, -\partial\sigma_{S'/G}) = (\sigma_{T/G}, \sigma_{S/G})$. □

Construction of the natural isomorphism.

Let X be a locally finite, finite dimensional $G-CW$ complex where G is a finite group. We would like to construct a natural isomorphism $\widehat{\Phi}^{lf,G} : \widehat{SH}_*^{lf,G}(X) \to \hat{H}_{*-1}^{lf,G}(X)$. Choose $n > dim(X) - k + 1$. Let $[T, S, g, i_\partial]$ be an element in $\widehat{SH}_k^{lf,G}(X)_n$ and choose a representative (T, S, g, i_∂). Recall the diagram we had before:

$$\begin{array}{ccccccc} C_*^{lf}(S/G) & \xrightarrow{h_*} & C_*(EG_n) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes g_*} & C_*(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^+ \otimes Id} & P_*^+ \otimes C_*^{lf}(X) \\ & & \downarrow \rho \otimes Id & & \downarrow \rho \otimes Id & & \downarrow \\ & & C_{*+d_n}(EG_n) \otimes C_*^{lf}(S) & \xrightarrow{Id \otimes g_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^- \otimes Id} & P_*^- \otimes C_*^{lf}(X) \end{array}$$

where all tensor products are over $\mathbb{Z}[G]$. Note that the map l^- is defined only for $j > -n$ so the map $l^- \otimes Id$ is defined only for $j > dim(X) - n$, This is the reason we have to choose $n > dim(X) - k + 1$. The map $EG_n \times_G S \to EG_n \times_G X$ factors through T/G thus the map $C_{*+d_n}(EG_n) \otimes C_*^{lf}(S) \xrightarrow{Id \otimes g_*} C_{*+d_n}(EG_n) \otimes C_*^{lf}(X)$ factors through $C_{*+d_n}^{lf}(T/G)$. We get the following commutative diagram:

$$\begin{array}{ccccc} C_*^{lf}(S/G) & \xrightarrow{g_*} & C_*(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^+ \otimes Id} & P_*^+ \otimes C_*^{lf}(X) \\ \bar{\rho} \downarrow & & \downarrow \rho \otimes Id & & \downarrow \\ C_{*+d_n}(T/G) & \xrightarrow{g_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l^- \otimes Id} & P_*^- \otimes C_*^{lf}(X) \end{array}$$

This induces a map between the mapping cones. The mapping cone of $\bar{\rho}$ was denoted by $C_*^{lf}(T,S)$, and the mapping cone of the map $P_*^+ \otimes C_*^{lf}(X) \to P_*^- \otimes C_*^{lf}(X)$ is naturally isomorphic to $P_* \otimes C_*^{lf}(X)$. We get a map $g_* : C_*^{lf}(T,S) \to P_* \otimes C_*^{lf}(X)$. Define $\widehat{\Phi}^{lf,G}([T,S,g,i_\partial]) = g_*(\widehat{[T,S]}^{lf})$.

There were some choices involved in the definitions of the maps in the above diagram. Since maps between mapping cones depend on the actual maps and not only on the homotopy classes of the maps we will be careful and show that $g_*(\widehat{[T,S]}^{lf})$ does not depend on those choices:

1) The choice of the representative (T,S,g,i_∂) and the choices of the cellular approximations $C_*^{lf}(S/G) \xrightarrow{g_*} C_*(EG_n) \otimes C_*^{lf}(X)$ and $C_*^{lf}(T/G) \xrightarrow{g_*} C_*(EG_n) \otimes C_*^{lf}(X)$:

This follows from lemma 6.59.

2) The choices of the maps $C_*(EG_n) \xrightarrow{l^+} P_*^+$ and $C_{*+d_n}(EG_n) \xrightarrow{l^-} P_*^-$:

We choose those maps once and for all.

3) The choice of the map $h_* : C_*^{lf}(S/G) \to C_*(EG_n) \otimes C_*^{lf}(S)$:

Let $h_*^1, h_*^2 : C_*^{lf}(S/G) \to C_*(EG_n) \otimes C_*^{lf}(S)$ be two cellular approximations, then there is a chain homotopy between them, denote it by s. Since s rises the dimension $\rho \otimes Id\left(s(\sigma_{S/G})\right) = 0$. We get two diagrams for $i = 1, 2$:

$$\begin{array}{ccc} C_*^{lf}(S/G) & \xrightarrow{Id \otimes g_* \circ h_*^i} & C_*(EG_n) \otimes C_*^{lf}(X) \\ \bar{\rho}^i \downarrow & & \downarrow \rho \otimes Id \\ C_*^{lf}(T/G) & \xrightarrow{g_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) \end{array}$$

We follow both images of $\widehat{[T,S]}^{lf}$. They are equal to $(g_*(\sigma_{T/G}), Id \otimes g_* \circ h_*^i(\sigma_{S/G}))$. We show that their difference is a boundary:

$(0, Id \otimes g_* \circ (h_*^2 - h_*^1)(\sigma_{S/G})) = \left(0, Id \otimes g_* \circ \left(s\partial(\sigma_{S/G}) + \partial s(\sigma_{S/G})\right)\right) = \left(0, Id \otimes g_* \left(\partial s(\sigma_{S/G})\right)\right)$
$= \left(0, \partial\left(Id \otimes g_* \left(s(\sigma_{S/G})\right)\right)\right) = \partial\left(0, Id \otimes g_* \left(s(\sigma_{S/G})\right)\right) - \left(\rho \otimes Id \circ Id \otimes g_* \left(s(\sigma_{S/G})\right), 0\right)$
$= \partial\left(0, Id \otimes g_* \left(s(\sigma_{S/G})\right)\right) - \left(Id \otimes g_* \circ \rho \otimes Id \left(s(\sigma_{S/G})\right), 0\right) = \partial\left(0, Id \otimes g_* \left(s(\sigma_{S/G})\right)\right)$

Which is a boundary.

4) The choice of n:

Let $[T', S', g', i'_\partial]$ be an element in $\widehat{SH}_k^{lf,G}(X)_{n+1}$ and $[T,S,g,i_\partial]$ an element in $\widehat{SH}_k^{lf,G}(X)_n$ such that $[T,S,g,i_\partial] = i^*([T', S', g', i'_\partial])$. We would like to show that $\widehat{\Phi}^{lf,G}([T,S,g,i_\partial]) = \widehat{\Phi}^{lf,G}([T', S', g', i'_\partial])$. We may assume that (T,S,g,i_∂) is given as the transversal intersection of $(T', S', g', i'_\partial)$ with $EG_n \times X$ (in the sense mentioned before) so $S' = S$. $\widehat{\Phi}^{lf,G}([T', S, g', i'_\partial])$ is the class of the image of

$(\sigma_{T'/G}, \sigma_{S/G})$, and $\widehat{\Phi}^{lf,G}([T,S,g,i_\partial])$ is the class of the image of $(\sigma_{T/G}, \sigma_{S/G})$, we would like to show that their images differ by a boundary.

The first thing we notice is that by choosing the maps $l^+ : C_*(EG_n) \to P_*^+$ and $l^+ : C_*(EG_{n+1}) \to P_*^+$ compatible with the map $C_*(EG_n) \to C_*(EG_{n+1})$ the images of $\sigma_{S/G}$ in $P_*^+ \otimes C_*^{lf}(X)$ will be equal. We can do it by defining P_*^+ to be the colimit of $C_*(EG_i)$ and the maps l^+ to be the limit maps. Therefore, it is enough to show the following:

LEMMA 6.60. *Denote by γ and γ' the images of $\sigma_{T/G}$ and $\sigma_{T'/G}$ resp. in $P_*^- \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)$ then $\gamma - \gamma'$ is a boundary.*

PROOF. In lemma 6.42 we constructed a map:
$\phi : C_{*+d_{n+1}}(EG_{n+1}) \otimes_{\mathbb{Z}[G]} C_*^{lf}(X) \to C_{*+d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)$
using a representative τ of the Thom class. This map depends on the choice of a diagonal approximation. By making the right choice this map will be of the form:
$\phi \otimes Id : C_{*+d_{n+1}}(EG_{n+1}) \otimes_{\mathbb{Z}[G]} C_*^{lf}(X) \to C_{*+d_n}(EG_n) \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)$
By pulling back τ and the tubular neighbourhood we can define in a similar way a map $\phi : C^{lf}_{*+d_{n+1}}(T'/G) \to C^{lf}_{*+d_n}(T/G)$. Again it depends on a diagonal approximation. It is shown in appendix 1 that $\phi([T'/G, \partial T'/G]) = [T/G, \partial T/G]$ so the same is true for the representatives $\phi(\sigma_{T'/G}) = \sigma_{T/G}$. This is true for every choice of a diagonal approximation. Look at the following diagram:

$$\begin{array}{ccccc}
C^{lf}_{*+d_{n+1}}(T'/G) & \xrightarrow{g_*} & C_{*+d_{n+1}}(EG_{n+1}) \otimes C_*^{lf}(X) & \xrightarrow{l' \otimes Id} & P_*^- \otimes_{\mathbb{Z}[G]} C_*^{lf}(X) \\
\phi \downarrow & & \downarrow \phi \otimes Id & & \downarrow Id \\
C^{lf}_{*+d_n}(T/G) & \xrightarrow{g_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) & \xrightarrow{l' \otimes Id} & P_*^- \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)
\end{array}$$

Denote by γ'' the image of $\sigma_{T'/G}$ under the composition:
$C^{lf}_{*+d_{n+1}}(T'/G) \to C_{*+d_{n+1}}(EG_{n+1}) \otimes C_*^{lf}(X) \to C_{*+d_n}(EG_n) \otimes C_*^{lf}(X) \to P_*^- \otimes_{\mathbb{Z}[G]} C_*^{lf}(X)$
It is enough to show that $\gamma'' - \gamma'$ and $\gamma - \gamma''$ are boundaries:
1) $\gamma' - \gamma''$ is a boundary - The right square commutes up to a homotopy of the form $H = h \otimes Id$, so $\gamma' - \gamma'' = H\partial g_*(\sigma_{T'/G}) + \partial H g_*(\sigma_{T'/G})$. It is enough to show that $H\partial g_*(\sigma_{T'/G}) = 0$. But $H\partial g_*(\sigma_{T'/G}) = H(\sigma_{EG_{n+1}} \otimes g_*(\sigma_{S,G})) = h(\sigma_{EG_{n+1}}) \otimes g_*(\sigma_{S,G}) = 0$ since P_1^- vanishes.
2) $\gamma - \gamma''$ is a boundary - It is enough to show that the difference between the two images of $\sigma_{T'/G}$ in $C_{*+d_n}(EG_n) \otimes C_*^{lf}(X)$ is a boundary. We prove that the left square commutes up to homotopy, denoted by h, and by choosing the diagonal approximations in a certain way we will get $h\partial(\sigma_{T'/G}) = 0$ thus:
$g_*(\sigma_{T'/G}) - g_*(\sigma_{T/G}) = h\partial(\sigma_{T'/G}) + \partial h(\sigma_{T'/G}) = \partial h(\sigma_{T'/G})$.

Denote by CT' the mapping cylinder of the map $T'/G \xrightarrow{g} EG_{n+1} \times_G X$. Since both T'/G and $EG_{n+1} \times_G X$ are subcomplexes of CT' we can use the homotopy extension lemma to choose a diagonal approximation for CT' which is compatible with those of T'/G and $EG_{n+1} \times_G X$. In a similar way to what we had before we can construct the following diagram:

$$\begin{array}{ccccc}
C^{lf}_{*+d_{n+1}}(T'/G) & \xrightarrow{i_*} & C_{*+d_{n+1}}(CT') & \xleftarrow{i_*} & C_{*+d_{n+1}}(EG_{n+1}) \otimes C_*^{lf}(X) \\
\phi \downarrow & & \downarrow \phi & & \downarrow \phi \otimes Id \\
C^{lf}_{*+d_n}(T/G) & \xrightarrow{i_*} & C_{*+d_n}(CT) & \xleftarrow{i_*} & C_{*+d_n}(EG_n) \otimes C_*^{lf}(X)
\end{array}$$

Where CT is the mapping cylinder of the map $T/G \xrightarrow{g} EG_n \times_G X$ and i denote the various inclusions into the mapping cylinders. Since the diagonal approximation are compatible the diagram strictly commutes. Look at the following diagram where π denotes the projection from CT' to $EG_{n+1} \times_G X$ and form CT to $EG_n \times_G X$:

$$\begin{array}{ccccc} C^{lf}_{*+d_{n+1}}(T'/G) & \xrightarrow{i_*} & C_{*+d_{n+1}}(CT') & \xrightarrow{\pi_*} & C_{*+d_{n+1}}(EG_{n+1}) \otimes C^{lf}_*(X) \\ \phi \downarrow & & \downarrow \phi & & \downarrow \phi \otimes Id \\ C^{lf}_{*+d_n}(T/G) & \xrightarrow{i_*} & C_{*+d_n}(CT) & \xrightarrow{\pi_*} & C_{*+d_n}(EG_n) \otimes C^{lf}_*(X) \end{array}$$

The composition of the horizontal maps is equal to g_*. The left side commutes as before. Denote by h the homotopy between $i_* \circ \pi_*$ and the identity in CT', then for every $a \in C_{*+d_{n+1}}(CT')$ we will have:
$\phi \otimes Id \circ \pi_*(a) = \pi_* \circ i_* \circ \phi \otimes Id \circ \pi_*(a) = \pi_* \circ \phi \circ i_* \circ \pi_*(a) = \pi_* \circ \phi(a) + \pi_* \circ \phi\left(h\partial a + \partial h a\right)$
By taking $a = i_*(\sigma_{T'/G})$ we get that:
$\phi \otimes Id(g_*(\sigma_{T'/G})) - g_*(\sigma_{T/G}) = \pi_* \circ \phi\left(h\partial i_*(\sigma_{T'/G})\right) + \partial\left(\pi_* \circ \phi\left(h i_*(\sigma_{T'/G})\right)\right)$
The fact that $\pi_* \circ \phi\left(h\partial i_*(\sigma_{T'/G})\right) = 0$ follows from the fact that it factors through $C^{lf}_{d_n+k}(EG_n \times_G K_{k-1}) = \{0\}$ where X_{k-1} be the $k-1$ skeleton of X. The reason for this is that the computation can be reduced to the mapping cylinder of the map $EG_{n+1} \times_G S \xrightarrow{g} EG_{n+1} \times_G X_k$ since the map is cellular.

We conclude that $\widehat{\Phi}^{lf,G}([T, S, g, i_\partial]) = \widehat{\Phi}^{lf,G}([T', S, g', i'_\partial])$. □

THEOREM 6.61. *The following diagram commutes:*

$$\begin{array}{ccccccc} SH^{lf,G}_* & \to & DSH^{lf,G}_* & \to & \widehat{SH}^{lf,G}_* & \to & SH^{lf,G}_{*-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{lf,G}_* & \to & DH^{lf,G}_* & \to & \hat{H}^{lf,G}_{*-1} & \to & H^{lf,G}_{*-1} \end{array}$$

PROOF. This is clear from the construction of the natural transformation. □

COROLLARY 6.62. $\widehat{\Phi}^{lf,G} : \widehat{SH}^{lf,G}_k \to \hat{H}^{lf,G}_{k-1}$ *is a natural isomorphism.*

PROOF. Naturality is clear. The fact that it is an isomorphism follows from the fact that we know that $SH^{lf,G}_* \to H^{lf,G}_*$ and $DSH^{lf,G}_* \to DH^{lf,G}_*$ are natural isomorphisms and by the 5 lemma. □

An isomorphism between $\widehat{SH}^*_G$ and $\hat{H}^*_G$.

Let G be a fixed finite group and M a smooth oriented manifold of dimension m with a smooth and orientation preserving G action. The composition $\widehat{SH}^k_G(M) \to \widehat{SH}^{lf,G}_{m-k}(M) \to \hat{H}^{lf,G}_{m-k-1}(M) \to \hat{H}^k_G(M)$ is an isomorphism of groups for all oriented manifolds, denote it by $\widehat{\Theta}_G$. We do not prove that it is natural. Trying to prove it in a way similar to what we had before runs into some technical problems, nevertheless it seems like it is possible.

COROLLARY 6.63. *The following diagram commutes:*

$$\begin{array}{ccccccc} DSH^*_G & \to & SH^*_G & \to & \widehat{SH}^*_G & \to & DSH^{*+1}_G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ DH^*_G & \to & H^*_G & \to & \hat{H}^*_G & \to & DH^{*+1}_G \end{array}$$

PROOF. This follows from the analog diagram for homology. □

Lemma 6.64. *Let M be a smooth oriented manifold of dimension m with a smooth and orientation preserving action of a finite group G. Choose $n > k$ and $n' > m - k + d_n$, then the following diagram commutes:*

$$\begin{array}{ccc} DSH^k_G(M) & \xrightarrow{\pi^*} & DSH^k_G(EG_n \times M) \\ \cong\downarrow & (1) & \downarrow\cong \\ DSH^{k+d_{n'}}_G(EG_{n'} \times M) & \xrightarrow{\pi^*} & DSH^{k+d_{n'}}_G(EG_{n'} \times EG_n \times M) \\ \cong\downarrow & (2) & \downarrow\cong \\ H^{k+d_{n'}}(EG_{n'} \times_G M) & \xrightarrow{\pi^*} & H^{k+d_{n'}}(EG_{n'} \times EG_n \times M/G) \end{array}$$

where the horizontal maps are induced by the projection.

Proof. (1) The left vertical map is defined by $[S \to M] \mapsto [S \to EG_{n'} \times M]$ where the map $S \to EG_{n'}$ is the classifying map given by the fact that G acts freely on S which is of dimension $\leq m-k$ and $n' > m-k$. This is an isomorphism since it has an inverse given by $[S \to EG_{n'} \times M] \mapsto [S \to M]$ (note that the map $S \to EG_{n'} \times M$ is proper if and only if the map $S \to M$ is proper since $EG_{n'}$ is compact). The right vertical map is given in a similar way, and it is also an isomorphism.
Take $[S \to M] \in DSH^k_G(M)$, its image in $DSH^k_G(EG_n \times M)$ is equal to $[EG_n \times S \to EG_n \times M]$, which is mapped to $[EG_n \times S \to EG_{n'} \times EG_n \times M]$. On the other hand, the image of $[S \to M]$ in $DSH^{k+d_{n'}}_G(EG_{n'} \times M)$ is equal to $[S \to EG_{n'} \times M]$ which is mapped to $[EG_n \times S \to EG_{n'} \times EG_n \times M]$.
(2) commutes since G acts freely on both spaces. $\square$

Corollary 6.65. *Let M, EG_n and $EG_{n'}$ be as in lemma 6.64. There are maps φ, ψ such that the following sequence is exact:*

$$H^{k+d_{n'}}(M'/G) \xrightarrow{\pi^*} H^{k+d_{n'}}(EG_n \times_G M') \xrightarrow{\varphi} \hat{H}^k_G(M) \xrightarrow{\psi} H^{k+d_{n'}+1}(M'/G) \xrightarrow{\pi^*} H^{k+d_{n'}+1}(EG_n \times_G M')$$

where π^* is induced by the projection map and $M' = EG_{n'} \times M$.

6.6. Some computations

We give a simple computational example which is rather geometric. It is in the spirit of [**1**]. Let Σ^m be an odd dimensional homology sphere, that is a closed oriented m dimensional smooth manifold having the same homology as S^m, with an orientation preserving action of $G = \mathbb{Z}/n$.

Lemma 6.66. *There exists b which divides n such that $\hat{H}^r_G(\Sigma^m) \cong \mathbb{Z}/b$ for all $r \in \mathbb{Z}$. $b = 1$ if and only if the action is free.*

Proof. G is cyclic so it has a two periodic complete resolution. This implies that $\hat{H}^*_G(\Sigma^m)$ is also two periodic. We have a spectral sequence $E_2^{pq} = \hat{H}^p(G, H^q(\Sigma^m)) \Rightarrow \hat{H}^{p+q}_G(\Sigma^m)$, the only non vanishing rows are $q = 0, m$ where we have $\hat{H}^*(G, \mathbb{Z})$ which is equal to G if $*$ is even and 0 else. By looking at the differential $d : E_2^{0,m} \to E_2^{m+1,0}$ we get an exact sequence: $0 \to \hat{H}^m_G(\Sigma^m) \to G \xrightarrow{d} G \to \hat{H}^{m+1}_G(\Sigma^m) \to 0$
Regardless to what the differential d is, the order of $\hat{H}^m_G(\Sigma^m)$ and $\hat{H}^{m+1}_G(\Sigma^m)$ is equal and since they are both finite cyclic groups this means that they are isomorphic. $\square$

LEMMA 6.67. *Let G be a finite group and M a closed oriented smooth manifold of dimension m with an orientation preserving smooth G action. Assume that G acts transitively on $\pi_0(M)$ then $H^0_G(M)$ and the G-invariant part of $H_m(M)$ are infinite cyclic and the degrees of the maps $DH^0_G(M) \to H^0_G(M)$ and $H^G_m(M) \to H_m(M)$ are equal (the map $H^G_m(M) \to H_m(M)$ is equivariant so the image lies in the G-invariant part).*

PROOF. Let $[S \to M]$ be an element in $DH^0_G(M) \cong H^G_m(M)$, its image in $H^0_G(M)$ is given by $[EG_n \times S \to EG_n \times M]$. The degree of this element in $H^0_G(M)$ is the degree when we forget the G action which is equal to the degree of $[S \to M]$ as an element in $H_m(M)$ when we forget the action. $\square$

LEMMA 6.68. *$\hat{H}^{-1}_G(\Sigma^m) \cong Tor(H^G_m(\Sigma^m))$ and $\hat{H}^0_G(\Sigma^m) \cong \mathbb{Z}/deg$ where deg is equal to the degree of the map $H^G_m(M) \to H_m(M)$.*

PROOF. We have the long exact sequence:
$0 \to \hat{H}^{-1}_G(\Sigma^m) \to DH^0_G(\Sigma^m) \to H^0_G(\Sigma^m) \to \hat{H}^0_G(\Sigma^m) \to DH^1_G(\Sigma^m)$
$\hat{H}^*_G(\Sigma^m)$ are torsion groups (by a transfer map argument) and $H^0_G(\Sigma^m) \cong \mathbb{Z}$ is torsion free so $\hat{H}^{-1}_G(\Sigma^m) \cong Tor(DH^0_G(\Sigma^m)) \cong Tor(H^G_m(\Sigma^m))$ using Poincaré duality. Since the $\hat{H}^0_G(\Sigma^m)$ is also torsion we can deduce that the free part in $H^G_m(\Sigma^m)$ is $\mathbb{Z}$. The second part follows from the fact that $DH^1_G(\Sigma^m) \cong H^G_{m-1}(\Sigma^m) = 0$ which we deduce from the spectral sequence. $\square$

We use this to compute two specific examples:

THEOREM 6.69. *Let $G = \mathbb{Z}/n$ act on $S^1 \subseteq \mathbb{C}$ by $x \mapsto \theta^k x$ where θ is the generator of G considered as a subgroup of S^1 and k is some integer which divides n, then $\hat{H}^r_G(S^1) \cong \mathbb{Z}/k$ for all $r \in \mathbb{Z}$.*

PROOF. We compute $Tor(H^G_1(S^1))$ which is isomorphic to $\hat{H}^r_G(S^1)$ for all $r \in \mathbb{Z}$. The subgroup $H = \mathbb{Z}/k$ acts trivially on S^1 so $H^G_1(S^1) \cong H_1(EG \times_G S^1) \cong H_1(BH \times_{G/H} S^1)$
Denote by $X = BH \times_{G/H} S^1$ then we have a fibration $BH \to X \to S^1$. We have the Wang sequence:
$H_1(BH) \xrightarrow{Id-m_*} H_1(BH) \to H_1(X) \to H_0(BH) \xrightarrow{Id-m_*} H_0(BH)$
Where m_* is the monodromy map. In this case the monodromy is homotopic to the identity since the action on BH extends to an action of S^1 so $m_* = Id$ and we get an exact sequence $0 \to H_1(BH) \to H_1(X) \to H_0(BH) \to 0$ which splits since $H_0(BH) \cong \mathbb{Z}$. Therefore the map $H_1(BH) \to Tor(H_1(X))$ is an isomorphism so $Tor(H^G_1(S^1)) \cong Tor(H_1(X)) \cong \mathbb{Z}/k$ and $\hat{H}^r_G(S^1) \cong \mathbb{Z}/k$ for all $r \in \mathbb{Z}$. $\square$

THEOREM 6.70. *Let $G = \mathbb{Z}/n$ act on $S^3 \subseteq \mathbb{C}^2$ by $(x,y) \mapsto (\theta^k x, \theta^l y)$ where θ is the generator of G considered as a subgroup of S^1, then $\hat{H}^r_G(S^3) \cong \mathbb{Z}/gcd(|G|, k \cdot l)$ for all $r \in \mathbb{Z}$.*

PROOF. By corollary 6.68 we have to compute the image of the map $H^G_m(S^3) \to H_m(S^3)$. We take another copy of S^3 with the free action given by $(x,y) \mapsto (\theta x, \theta y)$. To avoid confusion we denote this copy by S^3_f. $S^3_f \times_G S^3$ contains the 3 skeleton of $S^\infty \times_G S^3$ so the inclusion induces a surjection $H_3(S^3_f \times_G S^3) \twoheadrightarrow H^G_3(S^3)$ so it is enough to compute the image of the composition $H_3(S^3_f \times_G S^3) \twoheadrightarrow H^G_3(S^3) \to$

$H_3(S^3)$. We look at the fibration $S^3 \to S^3_f \times_G S^3 \to L^3$, by the Serre spectral sequence we get that $H_3(S^3_f \times_G S^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ where one generator is given by the fundamental class of the fiber. Another generator is given by a section - $(x,y) \to (x,y,x^k,y^l)$ ((x^k,y^l) is not of norm one but we can normalize it). We use the following commutative diagram in order to compute the images of the generators.

$$\begin{array}{ccccc}
S^3_f & \to & S^3_f \times S^3 & \to & S^\infty \times S^3 \simeq S^3 \\
\downarrow & & \downarrow & & \downarrow \\
L^3 & \to & S^3_f \times_G S^3 & \to & S^\infty \times_G S^3
\end{array}$$

The map $H^G_3(S^3) \to H_3(S^3)$ is the transfer map $H_3(S^\infty \times_G S^3) \to H_3(S^\infty \times S^3)$. The first generator is the image of the fundamental class of S^3 which is mapped by the transfer to $|G| \cdot [S^3]$. To compute the image of the second generator note that the composition $H_3(L^3) \to H^G_3(S^3) \to H_3(S^3)$ is equal to the composition $H_3(L^3) \to H_3(S^3_f) \to H_3(S^3)$. The first map is an isomorphism, the second map $S^3_f \to S^3$ is given by $(x,y) \to (x^k,y^l)$ which is of degree $k \cdot l$. We conclude that the image is generated by $|G|$ and $k \cdot l$, so $deg = gcd(|G|, k \cdot l)$ and $\hat{H}^r_G(S^3) \cong \mathbb{Z}/gcd(|G|, k \cdot l)$ for all $r \in \mathbb{Z}$. If for example k and l are coprime then it is equal to $k \cdot l$. In this case G acts freely outside of $S^1 \times \{0\}$ and $\{0\} \times S^1$ so the inclusion of the two circles induces an isomorphism $\mathbb{Z}/k \cdot l \cong \hat{H}^r_G(S^3) \to \hat{H}^r_G(S^1 \times \{0\}) \oplus \hat{H}^r_G(\{0\} \times S^1) \cong \mathbb{Z}/k \oplus \mathbb{Z}/l$ which fits with the previous example. $\square$

REMARK 6.71. From the proof one sees that in the general case, in order to compute the Tate groups $\hat{H}^*_G(\Sigma^m)$ it is enough to construct a section $L^m \to S^m \times_G \Sigma^m$ and to follow the image of the fundamental class of L^m.

CHAPTER 7

On the Product in Negative Tate Cohomology for Finite Groups

ABSTRACT. Our aim in this chapter is to give a geometric interpretation of the cup product in Tate cohomology in negative degrees. By duality it corresponds to a product in ordinary homology of BG - $H_n(BG,\mathbb{Z})\otimes H_m(BG,\mathbb{Z}) \to H_{n+m+1}(BG,\mathbb{Z})$ for $n,m>0$. We first interpret this product as join of cycles, which explains the shift in dimensions. Our motivation came from the product defined by Kreck using stratifold backwards cohomology for compact Lie groups. We then prove that for finite groups the cup product in negative Tate cohomology and the Kreck product coincide.

7.1. Another description of the cup product in Tate cohomology

In chapter 5 we gave a definition of Tate (co)homology using complete resolution. In this chapter we give another definition of Tate cohomology and the cup product which appears in [**5**]. To do so we introduce the language taken from the stable module category. We will not go into details, for a formal treatment the reader is referred to appendix 2.

Let M,N be two R-modules, we denote by $\underline{Hom}_R(M,N)$ the quotient of $Hom_R(M,N)$ by the maps that factor through some projective module.

DEFINITION 7.1. Given an R-module M, denote by $\Omega^k M$ the following module: take any partial projective resolution of M, $P_{k-1} \xrightarrow{d_{k-1}} P_{k-2}...P_0 \to M$ then $\Omega^k M = ker(P_{k-1} \xrightarrow{d_{k-1}} P_{k-2})$. This module clearly depends on the choice of the resolution. Nevertheless, as proved in appendix 2, the modules $\underline{Hom}_R(\Omega^k M, \Omega^l N)$ do not depend on the choice of resolutions i.e., they are well defined up to canonical isomorphisms. If we would like to stress the dependency in P we would use the notation $\Omega^k_P M$.

Note that there is a natural map $\Psi : \underline{Hom}_R(M,N) \to \underline{Hom}_R(\Omega M, \Omega N)$ $(\Omega M = \Omega^1 M)$.

DEFINITION 7.2. We define the Tate cohomology of G with coefficients in a $\mathbb{Z}[G]$-module M to be $\hat{H}^n(G,M) = \widehat{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},M) = \varinjlim_m \underline{Hom}_{\mathbb{Z}[G]}(\Omega^{n+m}\mathbb{Z}, \Omega^m M)$ (if $n<0$ we start this sequence from $m=-n$).

In our case where G is finite we have the following proposition which is proved in appendix 2:

PROPOSITION 7.3. *If G is a finite group and M is a $\mathbb{Z}[G]$-module which is projective as a $\mathbb{Z}$-module then the homomorphism $\Psi : \underline{Hom}_{\mathbb{Z}[G]}(M,N) \to \underline{Hom}_{\mathbb{Z}[G]}(\Omega M, \Omega N)$ is an isomorphism.*

Therefore, since $\mathbb{Z}$ and $\Omega^k\mathbb{Z}$ are projective as $\mathbb{Z}$-modules this limit equals to $\hat{H}^n(G,M) = \underline{Hom}_{\mathbb{Z}[G]}(\Omega^n\mathbb{Z}, M)$ if $n \geq 0$ or $\hat{H}^n(G,M) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{-n} M)$ if $n<0$. Our main interest will be the second case, especially when $M=\mathbb{Z}$.

EXAMPLE 7.4. For $n = -1$ we have $\hat{H}^{-1}(G,\mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},\Omega^1\mathbb{Z})$. Take the following exact sequence $0 \to I \to \mathbb{Z}[G] \xrightarrow{f} \mathbb{Z} \to 0$ where the map f is the augmentation map and I is the augmentation ideal. $I = \Omega^1\mathbb{Z}$ thus $(\Omega^1\mathbb{Z})^G = \{0\}$ and therefore $\hat{H}^{-1}(G,\mathbb{Z}) = \{0\}$.

Let G be a finite group. We construct a natural isomorphism $\hat{H}^{-n-1}(G,\mathbb{Z}) \to H_n(G,\mathbb{Z})$ for $n \geq 1$. Before that we prove a small lemma.

LEMMA 7.5. *Let G be a finite group and P a projective $\mathbb{Z}[G]$-module, then for every element $x \in P$ we have:*
1) $x \in P^G \Leftrightarrow \exists y \in P, x = Ny$
2) $y \otimes 1 = y' \otimes 1 \in P \otimes_{\mathbb{Z}[G]} \mathbb{Z} \Leftrightarrow Ny = Ny'$
Where P^G are the invariants of P under the action of G, N is the norm homomorphism defined by multiplication by the element $N = \sum\limits_{g\in G} g \in \mathbb{Z}[G]$.

PROOF. For every $\mathbb{Z}[G]$-module M we showed before the following exact sequence $0 \to \hat{H}^{-1}(G,M) \to H_0(G,M) \to H^0(G,M) \to \hat{H}^0(G,M) \to 0$, where the map $H_0(G,M) \to H^0(G,M)$ is given by $N : M \otimes \mathbb{Z} \to M^G$ $(N(x \otimes k) = kNx)$ ([**7**] VI,4). If M is projective then $\hat{H}^m(G,M) = 0$ for all $m \in \mathbb{Z}$, hence N is an isomorphism. We conclude:

1) For a projective module P the map $N : P \otimes \mathbb{Z} \to P^G$ is surjective and thus $x \in P^G \Leftrightarrow \exists y \in P, x = Ny$.

2) For a projective module P the map $N : P \otimes \mathbb{Z} \to P^G$ is injective and thus for every $y \in P$ we have $y \otimes 1 = y' \otimes 1 \Leftrightarrow Ny = Ny'$.

□

PROPOSITION 7.6. *Let G be a finite group then there is a natural isomorphism between $\hat{H}^{-n-1}(G,\mathbb{Z})$ and $H_n(G,\mathbb{Z})$ for $n \geq 1$.*

PROOF. Let G be a finite group. We define a map $\Phi : \hat{H}^{-n-1}(G,\mathbb{Z}) \to H_n(G,\mathbb{Z})$ the following way. We take a projective resolution of $\mathbb{Z}$ $\cdots \to P_n \xrightarrow{d_n} P_{n-1} ... \to P_0 \to \mathbb{Z}$, taking the tensor of it with $\mathbb{Z}$ gives us the chain complex for the homology of G which we denote by $C_*(G)$. We define a map from $Hom_{\mathbb{Z}[G]}(\mathbb{Z},\Omega^{n+1}\mathbb{Z})$ to $C_n(G)$ the following way: Given a homomorphism $f : \mathbb{Z} \to \Omega^{n+1}\mathbb{Z}$, $f(1) = x$ is an invariant element in P_n. By the lemma, since P_n is projective and x is invariant there is some $y \in P_n$ such that $x = Ny$. We define $\Phi(f) = y \otimes 1$. This doesn't depend on the choice of y since $Ny = Ny' \Leftrightarrow y \otimes 1 = y' \otimes 1$ by the lemma. We know that $Nd_n(y) = d_n(Ny) = d_n(x) = 0$ and by the lemma this implies that $d_n(y) \otimes 1 = 0$ (P_{n-1} is projective and here we use the fact that $n \geq 1$). We deduce that $y \otimes 1 \in Z_n(G)$. The map described now $Hom_{\mathbb{Z}[G]}(\mathbb{Z},\Omega^{n+1}\mathbb{Z}) \to Z_n(G)$ is surjective since given an element $y \otimes 1 \in C_n(G)$ such that $d_n(y) \otimes 1 = 0$ so as before this implies that $Nd_n(y) = 0$, so we define $f(k) = kNy$, this is well defined since Ny is invariant and in the kernel of d_n.

We now have a surjective homomorphism $\Phi : Hom_{\mathbb{Z}[G]}(\mathbb{Z},\Omega^{n+1}\mathbb{Z}) \to H_n(G,\mathbb{Z})$. If $f \in ker(\Phi)$ then there exist an element $s \in P_{n+1}$ such that $\Phi(f) = y \otimes 1 = d_{n+1}(s) \otimes 1$ then the map $f : \mathbb{Z} \to \Omega^{n+1}\mathbb{Z}$ factors through P_{n+1}, which is projective, by $1 \mapsto Ns$. On the other hand if f factors through a projective module, w.l.o.g. P_{n+1}, then $Ny = f(1) = d_{n+1}(Ns)$ (every invariant element in P_{n+1} is of the form Ns by the lemma). This implies that $Nd_{n+1}(s) = Ny \Leftrightarrow d_{n+1}(s \otimes 1) = d_{n+1}(s) \otimes 1 = y \otimes 1$.

We conclude that the induced map $\Phi : \hat{H}^{-n-1}(G, \mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}\mathbb{Z}) \to H_n(G, \mathbb{Z})$ is an isomorphism for all $n \geq 1$. $\square$

The product structure.

The cup product in Tate cohomology $\hat{H}^{-n}(G,\mathbb{Z}) \otimes \hat{H}^{-m}(G,\mathbb{Z}) \to \hat{H}^{-n-m}(G,\mathbb{Z})$ is given by composition (this is also called the Yoneda composition product): Given $[f] \in \hat{H}^{-n}(G,\mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n\mathbb{Z})$, $[g] \in \hat{H}^{-m}(G,\mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m\mathbb{Z}) \cong \underline{Hom}_{\mathbb{Z}[G]}(\Omega^n\mathbb{Z}, \Omega^{n+m}\mathbb{Z})$ we compose them to get a map $[g \circ f] \in \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+m}\mathbb{Z})$. Since for $n, m \geq 2$ we have $\hat{H}^{-n}(G,\mathbb{Z}) \cong H_{n-1}(G,\mathbb{Z})$, $\hat{H}^{-m}(G,\mathbb{Z}) \cong H_{m-1}(G,\mathbb{Z})$ we have a product $H_{n-1}(G,\mathbb{Z}) \otimes H_{m-1}(G,\mathbb{Z}) \to H_{n+m-1}(G,\mathbb{Z})$. Our main interest will be to show that this product is the same product as the one defined by Kreck. What we have to understand is the isomorphism $\underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m\mathbb{Z}) \cong \underline{Hom}_{\mathbb{Z}[G]}(\Omega^n\mathbb{Z}, \Omega^{n+m}\mathbb{Z})$. In order to understand it we will use the following construction:

The join of augmented chain complexes.

Let G be a finite group and let P and Q be the following augmented chain complexes over $\mathbb{Z}[G]$ - $\ldots \to P_2 \to P_1 \to P_0 \to \mathbb{Z}$ and $\ldots \to Q_2 \to Q_1 \to Q_0 \to \mathbb{Z}$. We define the join of those two chain complexes to be $P * Q = (\Sigma P) \otimes_{\mathbb{Z}} Q$ that is the suspension of the tensor product over $\mathbb{Z}$ (with a diagonal G action). To be more specific $(P * Q)_n = \bigoplus_{0 \leq k \leq n+1} P_{k-1} \otimes_{\mathbb{Z}} Q_{n-k}$. $P * Q$ is an augmented $\mathbb{Z}[G]$ chain complex in a natural way.

PROPOSITION 7.7. *If both P and Q are projective and acyclic augmented $\mathbb{Z}[G]$ chain complexes then $P * Q$ is projective and acyclic augmented $\mathbb{Z}[G]$ chain complex.*

PROOF. $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module thus every projective $\mathbb{Z}[G]$-module is also a projective $\mathbb{Z}$-module, so both P and Q are projective acyclic chain complexes over $\mathbb{Z}$ so the same is true for their tensor product, by the *Künneth* formula (here we use the fact that the modules are projective over $\mathbb{Z}$ and that $\mathbb{Z}$ is a PID). $(P * Q)_n$ is projective over $\mathbb{Z}[G]$ for $n \geq 0$ since each of the modules $P_{k-1} \otimes_{\mathbb{Z}} Q_{n-k}$ is projective. $\square$

LEMMA 7.8. *Let P and Q be* two resolutions of $\mathbb{Z}$ *over $\mathbb{Z}[G]$, and let $s \in Q_{n-1}$ be an element, $n > 1$. Define a map $s_* : P_{k-1} \to (P * Q)_{k+n-1}$ by $s_*(x) = x \otimes s$ called the join with s. Then we have:*
1) s_* is a group homomorphism.
2) If s is G-invariant then the map s_* is a homomorphism over $\mathbb{Z}[G]$.
3) If $s \in ker(Q_{n-1} \to Q_{n-2})$ then s_* commutes with the boundary so it will be a chain map of degree n.

PROOF. 1) Follows from the properties of the tensor product.
2) If s is G-invariant then for every $g \in G$ we have:
$g(s_*(x)) = g(x \otimes s) = g(x) \otimes g(s) = g(x) \otimes s = s_*(g(x))$
3) If $s \in ker(Q_{n-1} \to Q_{n-2})$ then:
$\partial(s_*(x)) = \partial(x \otimes s) = \partial(x) \otimes s + (-1)^{|x|+1} x \otimes \partial s = \partial(x) \otimes s = s_*(\partial(x))$ $\square$

This implies the following theorem:

THEOREM 7.9. *Let $n, m > 0$, the product $\hat{H}^{-n}(G,\mathbb{Z}) \otimes \hat{H}^{-m}(G,\mathbb{Z}) \to \hat{H}^{-n-m}(G,\mathbb{Z})$ is given by $[f] \cdot [g] = [f * g]$ where $(f * g)(k) = k \cdot f(1) \otimes g(1) \in \Omega^{m+n}_{P*P}\mathbb{Z}$ and $k \in \mathbb{Z}$.*

PROOF. Take a projective resolution P for $\mathbb{Z}$ over $\mathbb{Z}[G]$. Let $[f] \in \hat{H}^{-n}(G,\mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n\mathbb{Z})$, $[g] \in \hat{H}^{-m}(G,\mathbb{Z}) = \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m\mathbb{Z}) \cong \underline{Hom}_{\mathbb{Z}[G]}(\Omega^n\mathbb{Z}, \Omega^{n+m}\mathbb{Z})$. Choose representatives f, g and define a degree m map $P \to P * P$ by $x \mapsto x \otimes g(1)$. Since $g(1)$ is invariant and in the kernel this map is a chain map of $\mathbb{Z}[G]$ chain complexes of degree m. This gives us a concrete construction of the isomorphism $\underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m\mathbb{Z}) \cong \underline{Hom}_{\mathbb{Z}[G]}(\Omega^n\mathbb{Z}, \Omega^{n+m}\mathbb{Z})$. The composition is therefore $g \circ f(1) = f(1) * g(1)$. $\square$

7.2. An interpretation of the product by joins of cycles

We now consider resolutions which come from singular chains of spaces. Let G be a finite group, recall that a contractible $G-CW$ complex with a free G action is denoted by EG and the quotient space EG/G is called the classifying space of G principal bundles and is denoted by BG.

We consider now the augmented singular chain of EG denoted by $C_*(EG)$. We noted before that $C_*(EG)$ is projective ($n \geq 0$) and acyclic.

As we saw before every element of $H_n(G,\mathbb{Z})$ can be considered as an invariant cycle in $C_n(EG)$ (modulo invariant boundary), we will show that the product can be considered as the join of the two such cycles, which is naturally an invariant cycle in $C_*(EG * EG)$ where $EG * EG$ is the join of two copies of EG.

PROPOSITION 7.10. *The space $EG * EG$ is contractible, has a natural free G action so its augmented singular chain complex is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.*

PROOF. $EG * EG$ is contractible since the join of contractible spaces is a contractible space. The action of G on $EG * EG$ is defined by $g(x,y,t) = (gx, gy, t)$. This action is free since it is free on both copies of EG. $\square$

We now associate the join of chain complexes to the join of spaces.

PROPOSITION 7.11. *Let A and B be two spaces and let $C_*(A)$ and $C_*(B)$ be their augmented (!) singular chain complexes, then there is a natural chain map $h : C_*(A) * C_*(B) \to C_*(A * B)$. If G acts on A and B then it also acts on $A * B$ and the chain complexes are complexes over $\mathbb{Z}[G]$ and h is a map of $\mathbb{Z}[G]$ chain complexes*

PROOF. We first note that for $n, m \geq 0$, for every two singular simplices $\sigma \in C_n(A)$ and $\tau \in C_m(B)$ there is a canonical singular chain $\sigma * \tau \in C_{n+m+1}(A * B)$ and this definition can be extended in a bilinear way to chains. Define h the following way:
Given an element $s \otimes t \in C_n(A) \otimes C_m(B)$, if $n, m \geq 0$ then $h(s \otimes t) = s * t$, else $n = -1$ (or $m = -1$) then s is an integer, denote it by k then $h(s \otimes t) = h(k \otimes t) = k \cdot t$ where t is the chain induced by the inclusion of B in $A * B$ (and similarly for $m = -1$).

We have to show that h is a chain map. For two simplices of positive (!) dimension we have the formula $\partial(\sigma * \tau) = \partial(\sigma) * \tau + (-1)^{dim(\sigma)+1} \sigma * \partial(\tau)$. The formula extends to chains, so we have:
$\partial h(s \otimes t) = \partial(s * t) = \partial(s) * t + (-1)^{|s|+1} s * \partial(t) = h(\partial(s) \otimes t + (-1)^{|s|+1} s \otimes \partial(t)) = h(\partial(s \otimes t))$.
For σ, a simplex of dimension 0 (a point), $\sigma * \tau$ is the cone over τ and its boundary is given by $\partial(\sigma * \tau) = \tau + (-1)^{dim(\sigma)+1} \sigma * \partial(\tau)$. Since the boundary map $C_0(A) \to \mathbb{Z}$ is the augmentation map we see indeed that also in this case h commutes with the boundary (with respect to the way we have defined $h(k \otimes t)$).

The boundary formula is not (!) true when one of the simplices is zero dimensional due to the non symmetric way we define the faces of a zero simplex (the n simplex has $n+1$ faces while the zero simplex has no faces). If we wanted to be consistent with the boundaries of the higher simplices we should have used only augmented chain complexes. For a detailed discussion in this direction see [**12**].

When there is a G action on both spaces then clearly all the complexes are complexes over $\mathbb{Z}[G]$. h is a $\mathbb{Z}[G]$ chain map since for every $g \in G$ we have $h(g(s \otimes t)) = h(gs \otimes gt)) = gs * gt = g(s * t) = g(h(s \otimes t))$. □

THEOREM 7.12. *The cup product in negative Tate cohomology gives a product $H_n(G, \mathbb{Z}) \otimes H_m(G, \mathbb{Z}) \to H_{n+m+1}(G, \mathbb{Z})$ $(n, m > 0)$. Each homology class in $H_n(G, \mathbb{Z})$ is represented by an invariant cycle in EG. The product of two classes is given by the join of those cycles, which is an invariant cycle in $EG * EG$.*

PROOF. We already saw that the product can be interpreted by the join of resolutions. By the proposition above there is a degree zero chain map $C_*(EG) * C_*(EG) \to C_*(EG * EG)$. The image of $f(1) \otimes g(1)$ under this map is the join of $f(1)$ with $g(1)$. This gives a more concrete model where the cycles are actual invariant singular cycles of the space $EG * EG$. □

7.3. Comparing Kreck's product and the cup product

Let G be a compact Lie group. The cup product in $DSH^*_G(pt)$ is given, up to sign, by the Cartesian product with the diagonal action - $[S, \rho] \otimes [S', \rho'] \to [S \times S', \Delta]$ (here, instead of writing the map to pt we denote the G action). When $n, m < 0$ this product vanishes since it is the boundary of $[CS \times S', \tilde{\rho}]$ where $\tilde{\rho}$ is the obvious extension of the action Δ, but it is also the boundary of $[S \times CS', \hat{\rho}]$ (up to sign) where $\hat{\rho}$ is the obvious extension of the action Δ.

The Kreck product, denoted by $*$, is a secondary product defined by gluing both along the boundary $[S, \rho] \otimes [S', \rho'] \to [S * S', \rho * \rho']$ (note that after the gluing what we get is the join of the two p-stratifolds). This product $DSH^n_G(pt) \otimes DSH^m_G(pt) \to DSH^{n+m-1}_G(pt)$ does not vanish in general, for example when G cyclic or more generally for every group acting freely and orientation preserving on some sphere like S^1 and S^3. When $G = \mathbb{Z}/2$ then $DSH^*_G(pt)$ is zero in positive dimensions and in even dimensions, infinite cyclic when $* = 0$ and G for negative odds. The generators in negative odd dimensions can be taken to be odd dimensional spheres with the antipodal action. In this case the product of generators is again a generator. A similar construction will hold for S^1 and S^3.

By Poincaré duality this gives a product $SH^G_n(pt) \otimes SH^G_m(pt) \to SH^G_{n+m+1}(pt)$. For a finite group G we constructed an isomorphism $\Psi' : SH^G_n(pt) \to H_n(G, \mathbb{Z})$, we conclude that there is an isomorphism $\Psi : SH^G_n(pt) \to \hat{H}^{-n-1}(G, \mathbb{Z})$ (for $n > 0$).

It it easy to show that this isomorphism is given the following way: Take some model for EG. Its singular chain complex $C_*(EG)$ is a projective resolution for $\mathbb{Z}$ over $\mathbb{Z}[G]$. Let $[(S, \rho)]$ be an element in $SH^G_n(pt)$, then there is an equivariant map $f : S \to EG$ called the classifying map, which is unique up to G homotopy. f induces a $\mathbb{Z}[G]$ chain map between the singular chain complexes - $C_*(S) \xrightarrow{f_*} C_*(EG)$. S has a fundamental class, we take some representative of it which is G invariant (we can do that by lifting a fundamental class of S/G) and denote it by s. We get an element $f_*(s) \in C_n(EG)$ which is both invariant and a cycle thus we get an element in $Hom_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n\mathbb{Z})$. As before different choices of S, s and f will give elements that differ by a map which factors through a projective (the fundamental class of the bordism is mapped into $C_{n+1}(EG)$ which is projective), hence gives a homomorphism $SH_n^G(pt) \to \underline{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n\mathbb{Z}) = \hat{H}^{-n-1}(G, \mathbb{Z})$ which is exactly the isomorphism explained above.

Given two such p-stratifolds S and S', the join of their fundamental classes is a fundamental class of $S * S'$. We give $S * S'$ this orientation thus the map $SH_*^G(pt) \to \hat{H}^{-*-1}(G, \mathbb{Z})$ commutes with the product. We have thus proved the following:

THEOREM 7.13. *Let G be a finite group, there is a natural isomorphism $\varphi : SH_*^G(pt) \to \hat{H}^{-*-1}(G, \mathbb{Z})$ for $* > 0$ and $\varphi(\alpha * \beta) = \varphi(\alpha) \cup \varphi(\beta)$ for all $\alpha \in SH_n^G(pt)$ and $\beta \in SH_m^G(pt)$ where $k, l > 0$.*

In other words, the product in group homology defined by Kreck using stratifold homology and the join is the same product as the cup product in negative Tate cohomology.

Appendix 1 - Homology, orientation and sign conventions

A great deal of this paper is dedicated to the construction of natural transformations. In order to make those natural transformations as simple as possible and to avoid getting an extra sign we have to choose the right set of definitions and this is our goal in this appendix. Text books in algebraic topology like [**31, 10, 17, 29**] use different sign conventions. For example for the cup and cap products and for the boundary operator for cochains. Therefore, the signs in the formulas change from one book to the other, depending on the way the various products are defined.

We follow the definitions of [**10**] and [**29**] since they are well suited for working on chain level (those definitions also agree with the ones appear in [**7**]). The definitions regarding orientation of vector spaces and vector bundles are the ones used in [**29**]. Note that the definitions for the cap product agree with the one as in [**17**] after passing to homology, but the cup product differs by a sign.

We also give proofs to certain propositions which are related to those definitions, mostly the ones which use the Thom isomorphism.

Homology cross product.

Let Δ^n denote the standard n simplex we define two maps $\lambda_p : \Delta^p \to \Delta^n$ and $\rho_q : \Delta^q \to \Delta^n$ by $\lambda_p(t_0, ...t_p) = (t_0, ...t_p, 0...0)$ and $\rho_q(t_0, ...t_q) = (0, ...0, t_0, ...t_p)$. Let X and Y be two topological spaces, the Alexander Whitney map $A : S_*(X \times Y) \to S_*(X) \otimes S_*(Y)$ is given by mapping the generators $(\sigma_X, \sigma_Y) \in S_n(X \times Y)$ to $A(\sigma_X, \sigma_Y) = \Sigma \sigma_X \circ \lambda_k \otimes \sigma_Y \circ \rho_{n-k}$. This map is a chain equivalence with an inverse called the Eilenberg-Zilber map $EZ : S_*(X) \otimes S_*(Y) \to S_*(X \times Y)$. We define the cross product in homology to be the composition $H_*(X) \otimes H_*(Y) \to H_*(S_*(X) \otimes S_*(Y)) \xrightarrow{EZ} H_*(X \times Y)$.

Orientation of vector spaces and manifolds.

Let V be an n dimensional real vector space. An orientation of V is a choice of a generator $\tau_V \in H_n(V, V_0)$ where $V_0 = V \setminus \{0\}$. Equivalently, we can choose a generator $\tau^V \in H^n(V, V_0)$ and we can switch from one to the other by requiring that $\left\langle \tau^V, \tau_V \right\rangle = 1$. Denote by $o_1 \in H_1(\mathbb{R}, \mathbb{R}_0)$ the class of $\sigma : \Delta^1 \to \mathbb{R}$ given by $\sigma(t_0, t_1) = t_1 - t_0$, then $\partial o_1(x) = \{1\} - \{-1\} \in H_0(\mathbb{R}_0)$. Denote $o_n = o_1 \times o_1 \times ..o_1$ which is a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$. We call it the standard orientation of $\mathbb{R}^n$. By associativity $o_m \times o_n = o_{m+n}$. We denote by $o^n \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ the unique element with the property $\langle o^n, o_n \rangle = 1$.

Let V be an n dimensional real vector space with an ordered base $(v_1, ...v_n)$. Denote by $f : \mathbb{R}^n \to V$ the map $f(x_1, ...x_n) = \Sigma x_i \cdot v_i$ then we give V the orientation $f_*(o_n)$. If V has 2 bases that differ by a linear map with a positive determinant then both maps will be homotopic as maps of pairs since $GL_n(\mathbb{R})_+$ is path connected so this procedure will give V the same orientation.

Let $\xi : E \to B$ be an n-dimensional vector bundle (whenever we say a bundle we mean a locally trivial bundle). We denote by E_0 the set of non zero elements in E, and for each $b \in B$ we denote $V^b = \xi^{-1}(b)$ so $V_0^b = V^b \cap E_0$. An orientation of ξ is a choice of orientations τ_b to each V^b (or τ^b) such that each $b \in B$ has a neighbourhood $U \subseteq B$ such that $\xi|_U$ is a trivial bundle and there is an orientation preserving bundle map (that is it preserves orientation in every fiber):

$$\begin{array}{ccc} \xi^{-1}(U) & \to & U \times \mathbb{R}^n \\ \downarrow & & \downarrow \\ U & \xrightarrow{Id} & U \end{array}$$

where we take the standard orientation of $\mathbb{R}^n$.

Let M be a smooth manifold of dimension m. A local orientation of M at $x \in M$ is a choice of a generator $\tau_x \in H_m(M, M \setminus \{x\})$ and an orientation of M is local orientation of M at each point such that every $x \in M$ has a neighbourhood U and an element $\tau_U \in H_m(M, M \setminus U)$ such that for every $y \in U$, τ_U is mapped to τ_y. We have the following:

THEOREM 7.14. *([**29**] A.8) Let M be a smooth oriented manifold of dimension m and $K \subseteq M$ a compact subset then there is a unique class $\tau_K \in H_m(M, M \setminus K)$ that for every $x \in K$ is mapped to τ_x. If M is compact we can take $K = M$ then $\tau \in H_m(M)$ is called the fundamental class of M.*

For a smooth manifold M there is a way of identifying the tangent space over a point $x \in M$ with a neighbourhood of x. This gives a way to associate to an orientation of the tangent bundle an orientation of M and vice versa.

PROPOSITION 7.15. *Let M and N be two closed smooth manifolds with an orientation of their tangent bundle and denote the corresponding fundamental classes by $[M]$ and $[N]$. We give the tangent bundle of $M \times N$ the product orientation and denote the corresponding fundamental class by $[M \times N]$, then $[M] \times [N] = [M \times N]$. The same holds if one of the manifolds has a boundary and we take the fundamental class relative its boundary.*

PROOF. Given a point $(x, y) \in M \times N$, we look at the composition $H_{m+n}(M \times N) \to H_{m+n}(M \times N, M \times N \setminus \{(x, y)\}) \to H_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{(0, 0)\})$ Where the latter map is induced by excision. The image of $[M \times N]$ is o_{m+n} by the way we oriented $M \times N$. By the naturality of the cross product the image of $[M] \times [N]$ is $o_m \times o_n$ so the proposition follows from the fact that $o_m \times o_n = o_{m+n}$. The same proof holds in case one of the manifolds has a boundary. $\square$

Let M be a closed oriented smooth manifold with boundary ∂M then we orient ∂M in a way that the inclusion of the collar $(0, 1) \times \partial M$ is orientation reversing (outward normal first, this is the same convention as in [**29**]). Then we have:

PROPOSITION 7.16. *1) $\partial(M \times N) = (\partial M) \times N$ is an orientation preserving diffeomorphism.*
2) Orient $I = [0, 1]$ by the induced orientation from $\mathbb{R}$ then $\partial[I, \partial I] = \{1\} - \{0\}$.
3) $\partial[M, \partial M] = [\partial M]$

PROOF. 1) Clear by the definition of the orientation.
2) Follows from the definition of o_1.
3) We use the following notation - $C(\partial M)$ is the cone over ∂M, that is $I\times\partial M/\{0\}\times\partial M$. The subspace $\{1\}\times\partial M$ is denoted by ∂C and the cone point is denoted by $*$. The cylinder $I\times\partial M$ is denoted by $\Sigma(\partial M)$ and its boundary is denoted by $\partial\Sigma$. The collar $c:[0,1]\times\partial M\to M$ is orientation reversing so we define the map $c':[0,1]\times\partial M\to M$ by $c'(t,x)=c(1-t,x)$ then c' is orientation preserving. It induces a map $(M,\partial M)\to(C(\partial M),\partial C\cup *)$ which is orientation preserving and maps ∂M to ∂C. c' gives us the following maps of pairs:
$(M,\partial M)\to(C(\partial M),\partial C\cup *)\leftarrow(\Sigma(\partial M),\partial\Sigma)$
We get the following commutative diagram in homology:

$$\begin{array}{ccccc} H_m(M,\partial M) & \to & H_m(C(\partial M),\partial C\cup *) & \overset{\cong}{\leftarrow} & H_m(\Sigma(\partial M),\partial\Sigma) \\ \downarrow & & \downarrow & & \downarrow \\ H_{m-1}(\partial M) & \to & H_{m-1}(\partial C\cup *) & \leftarrow & H_{m-1}(\partial\Sigma) \end{array}$$

In the upper row $[M,\partial M]$ is mapped $[\Sigma(\partial M),\partial\Sigma]$, since c' is orientation preserving. In the lower row $[\partial M]$ is mapped to $[\{1\}\times\partial M]$ in $H_{m-1}(\partial C\cup *)$ and in $H_{m-1}(\partial\Sigma)$. Thus the result follows from the fact that:
$\partial[\Sigma\partial M,\partial\Sigma]=\partial[I\times\partial M,\partial I\times\partial M]=\partial[I,\partial I]\times[\partial M]=[\{1\}\times\partial M]-[\{0\}\times[\partial M]$.
We conclude that $\partial[M,\partial M]=[\partial M]$. $\square$

Thom isomorphism.

For an oriented vector bundle we have the following ([**29**] 9.1):

THEOREM 7.17. *(Thom isomorphism) Let $\xi:E\to B$ be an n dimensional oriented vector bundle, then there exists a unique class $\tau\in H^n(E,E_0)$ such that $\tau|(V^b,V_0^b)=\tau^b$ for every $b\in B$. τ is called the Thom class of ξ. The map $H^k(E)\to H^{n+k}(E,E_0)$ defined by $\varphi\mapsto\varphi\cup\tau$ is an isomorphism for all k. Precomposing it with the isomorphism $\xi^*:H^k(B)\to H^k(E)$ gives us an isomorphism $\tau^\xi:H^k(B)\to H^{n+k}(E,E_0)$. τ^ξ is called the Thom isomorphism.*

There is also a homological version to this theorem ([**29**] 10.7):

THEOREM 7.18. *Let $\xi:E\to B$ be an n dimensional oriented vector bundle. The map $H_{n+k}(E,E_0)\to H_k(E)$ defined by $\alpha\mapsto\tau\cap\alpha$ is an isomorphism for all k. Composing it with the isomorphism $\xi_*:H_k(E)\to H_k(B)$ gives us an isomorphism $\tau_\xi:H_{n+k}(E,E_0)\to H_k(B)$. τ_ξ is called the Thom isomorphism.*

Closed submanifolds.

Let M be a smooth oriented manifold of dimension m and N a smooth oriented submanifold of dimension n which is closed. N has a tubular neighbourhood $U\subseteq M$ with projection $p:U\to N$ ([**29**] 11.1). We orient the normal bundle of N in a way that the map from $T_N\oplus\nu_N$ to $T_M|_N$ is orientation preserving where T denotes the tangent bundle and ν the normal bundle. Denote by ϕ the composition:
$H_{m+k}(M)\to H_{m+k}(M,M\setminus N)\xrightarrow{excision}H_{m+k}(U,U\setminus N)\xrightarrow{\tau_p}H_{n+k}(N)$

PROPOSITION 7.19. *Let M be a closed oriented smooth manifold of dimension m and N a smooth oriented submanifold of dimension n which is closed (and hence compact) then $\phi([M])=(-1)^{n(m-n)}[N]$.*

Proof. Choose an orientation preserving local coordinates $f : \mathbb{R}^n \to N$ and pullback the bundle $p : U \to N$:

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{f'} & U \\ \pi \downarrow & & \downarrow p \\ \mathbb{R}^n & \xrightarrow{f} & N \end{array}$$

Where π is the projection on the first n coordinates. We orient this bundle by identifying the fibers with $\mathbb{R}^{m-n}$, we can choose f' to be orientation preserving (otherwise we take $-f'$). This implies that $\tau' = f'^*(\tau)$ is the Thom class of $\pi : \mathbb{R}^m \to \mathbb{R}^n$ where τ is the Thom class of $p : U \to N$. For each $x \in N$ we have the following diagram:

$$\begin{array}{ccccc} H_m(U, U \setminus N) & \to & H_m(U, U \setminus \{x\}) & \xleftarrow{\cong} & H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \\ \tau\cap \downarrow & & \tau\cap \downarrow & & \tau'\cap \downarrow \\ H_n(U) & \to & H_n(U, U \setminus V^x) & \xleftarrow{\cong} & H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\} \times \mathbb{R}^{m-n}) \end{array}$$

The vertical maps are defined using the cap product with the Thom class:

- $H_m(U, U \setminus N) \otimes H^{m-n}(U, U \setminus N) \to H_n(U)$
- $H_m(U, U \setminus \{x\}) \otimes H^{m-n}(U, U \setminus N) \to H_n(U \setminus V^x)$
- $H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \otimes H^{m-n}(\mathbb{R}^m, \mathbb{R}^m \setminus \mathbb{R}^n \times \{0\}) \to H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\} \times \mathbb{R}^{m-n})$

The diagram commutes by the naturality of the cap product and the fact that $\tau' = f'^*(\tau)$. Composition with the maps induced by $p : U \to N$ and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ gives the following commutative diagram:

$$\begin{array}{ccccc} H_m(U, U \setminus N) & \to & H_m(U, U \setminus \{x\}) & \xleftarrow{\cong} & H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \\ \tau_p \downarrow & & \tau_p \downarrow & & \tau_\pi \downarrow \\ H_n(N) & \to & H_n(N, N \setminus \{x\}) & \xleftarrow{\cong} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \end{array}$$

The image of $[N]$ is o_n since f is orientation preserving.
f' is orientation preserving as a bundle map so by our orientation convention for normal bundle we deduce that f' is also orientation preserving as a map between manifolds. Thus the image $[U, U \setminus N]$ is o_m.
By commutativity of the diagram it is enough to show that $\tau_\pi(o_m) = (-1)^{n(m-n)} o_n$.

Denote by $\pi' : \mathbb{R}^m \to \mathbb{R}^{m-n}$ the projection on the last $m - n$ coordinates, then $\pi'^*(o^{m-n}) = 1^n \times o^{m-n}$ is the Thom class of the bundle $\pi : \mathbb{R}^m \to \mathbb{R}^n$ since clearly its restriction to each fiber is the standard generator. Thus we get:
$\tau_\pi(o_m) = \pi_*(\tau' \cap o_m) = \pi_*(\pi'^*(o^{m-n}) \cap o_m) = \pi_*(1^n \times o^{m-n} \cap o_n \times o_{m-n})) =$
$= (-1)^{n(m-n)} \pi_* \left((1^n \cap o_n) \times (o^{m-n} \cap o_{m-n})\right) = (-1)^{n(m-n)} o_n$ $\square$

A similar proof can be applied in locally finite homology. We just have to take care that all pairs are of a space and a closed subspace. We can do this, for example, by using $(M, M \setminus U)$ instead of using $(M, M \setminus N)$:

Proposition 7.20. *Let M be a smooth oriented manifold of dimension m and N a smooth oriented submanifold of dimension n which is closed then $\phi([M]^{lf}) = (-1)^{n(m-n)}[N]^{lf}$.*

We conclude the following:

PROPOSITION 7.21. *Let M be a smooth oriented manifold of dimension m and N a smooth oriented submanifold of dimension n which is closed and denote $f : N \to M$ the embedding. Let $f_! : H^{lf}_{m-k}(M) \to H^{lf}_{n-k}(N)$ be the umkehr map $PD_N \circ f^* \circ PD_M^{-1}$ and $\phi : H^{lf}_{m-k}(M) \to H^{lf}_{n-k}(N)$ the map defined above then $f_! = (-1)^{(n+k)(m-n)}\phi$.*

PROOF. We have to show that the following diagram commutes:

$$\begin{array}{ccc} H^k(M) & \xrightarrow{f^*} & H^k(N) \\ PD_M \downarrow & & PD_N \downarrow \\ H^{lf}_{m-k}(M) & \xrightarrow{(-1)^{(n+k)(m-n)}\phi} & H^{lf}_{n-k}(N) \end{array}$$

Given $\varphi \in H^k(M)$ we follow its image. If we first go down and then right we get:
$(-1)^{(n+k)(m-n)}\phi(\varphi \cap [M]^{lf}) = (-1)^{(n+k)(m-n)}\pi_* \left(\tau \cap \left(\varphi \cap [U, \partial U]^{lf}\right)\right)$
$= (-1)^{n(m-n)}\pi_* \left(\varphi \cap \left(\tau \cap [U, \partial U]^{lf}\right)\right) = \pi_* \left(\varphi \cap \left(f_*[N]^{lf}\right)\right)$
$= \pi_* \circ f_* \left(f^*(\varphi) \cap [N]^{lf}\right) = f^*(\varphi) \cap [N]^{lf}$. □

Consider the following situation: Let M be an oriented manifold of dimension M and N a closed submanifold of dimension n with a tubular neighbourhood U. Let $g : S \to M$ be a proper map where S is a p-stratifold and assume that g and f are transversal, where $f : N \hookrightarrow M$ is the inclusion. Consider the pullback diagram:

$$\begin{array}{ccccc} S' & \hookrightarrow & U' & \hookrightarrow & S \\ \downarrow & & \downarrow & & \downarrow g \\ N & \hookrightarrow & U & \hookrightarrow & M \end{array}$$

Since f and g are transversal S' is a p-stratifold. The pullback of the tubular neighbourhood U of N which we denote by U' is a tubular neighbourhood of S' in S in the sense that there is a vector bundle $V' \to S'$ and an isomorphism $V' \to U'$ that maps the zero section isomorphically onto S'.

In order to construct the map ϕ we only used the fact that N is a closed subset of M and it has a tubular neighbourhood in this sense. Therefore, we can construct a map $\phi : H_{m+k}(S) \to H_{n+k}(S')$ in a similar way to what we did for M and N. A similar proof will show the following:

PROPOSITION 7.22. *Let S be a compact oriented regular p-stratifold of dimension l and S' a compact regular oriented p-stratifold of dimension k with an inclusion $S' \hookrightarrow S$ and a tubular neighbourhood as explained above then $\phi([S]) = (-1)^{k(l-k)}[S']$.*

And a version in locally finite homology:

PROPOSITION 7.23. *Let S be a regular oriented p-stratifold of dimension l and S' a regular oriented p-stratifold of dimension k with an inclusion $S' \hookrightarrow S$ and a tubular neighbourhood as explained above then $\phi([S]^{lf}) = (-1)^{k(l-k)}[S']^{lf}$.*

Similarly to what we had before, assume that S is a regular oriented p-stratifold of dimension l with boundary mapped to M as before and N is a closed submanifold which is transversal both to S and to ∂S. Denote by $(S', \partial S')$ the intersection of $(S, \partial S)$ and N. By the previous propositions we can define a map $\phi : H^{lf}_l(S, \partial S) \xrightarrow{\phi} H^{lf}_{l+n-m}(S', \partial S')$ then we will have:

PROPOSITION 7.24. $\phi([S, \partial S]^{lf}) = (-1)^{(l+n-m)(m-n)}[S', \partial S']^{lf}$

PROOF. Denote by DS the double of S, that is $S \cup_\partial -S$. The map $S \to M$ factors through DS. We know that the assertion is true for DS an therefore is is true for the relative case so we prove it by excision:

$$\begin{array}{ccc} H_l^{lf}(S, \partial S) & \xrightarrow{\cong} & H_l^{lf}(DS, S^-) \\ \phi \downarrow & & \phi \downarrow \\ H_{l+n-m}^{lf}(S', \partial S') & \xrightarrow{\cong} & H_{l+n-m}^{lf}(DS', S'^-) \end{array}$$

□

Appendix 2 - The stable module category

In this appendix we give the background needed for the construction we used for Tate cohomology in chapter 7. Again R is ring with a unit and all modules are left R-modules.

The stable category $St-mod(R)$.

Let M and N be two R-modules, denote by $PHom_R(M,N)$ the set of R-homomorphisms $M \xrightarrow{f} N$ that factor through a projective R-module, i.e., there exists a projective R-module P and two maps $M \xrightarrow{f_1} P \xrightarrow{f_2} N$ such that $f = f_2 \circ f_1$. The following proposition is left as an easy exercise:

PROPOSITION 7.25. *$PHom_R(M,N)$ is a submodule of $Hom_R(M,N)$ and the composition of two homomorphisms such that one of them factors through a projective module also factors through a projective module.*

By the proposition above we can define $\underline{Hom}_R(M,N) = Hom_R(M,N)/PHom_R(M,N)$ which is an R-module, and a composition $\underline{Hom}_R(N,K)\times\underline{Hom}_R(M,N) \to \underline{Hom}_R(M,K)$ which is R-bilinear.

DEFINITION 7.26. Let R be a ring, denote by $St-mod(R)$ the category whose objects are all R-modules and the morphisms between each M and N are $\underline{Hom}_R(M,N)$. This category is called the stable module category.

The functor Ω.

For every R-module M choose (once and for all) a projective cover, that is a surjective map $\pi_M : P_M \to M$ where P_M is a projective R-module (for example the canonical free cover).

Define a functor $\Omega : St-mod(R) \to St-mod(R)$ the following way: For an object M define $\Omega(M) = Ker(\pi_M)$. For a morphism $[f] \in \underline{Hom}_R(M,N)$ choose some representative $f : M \to N$, use the fact that P_M is projective and π_N is surjective to define a map $\widetilde{f} : P_M \to P_N$ such that the following diagram become commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow \widetilde{f}|_{\Omega(M)} & & \downarrow \widetilde{f} & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Now take $\Omega(f)$ to be the class of the induced map $\widetilde{f}|_{\Omega(f)} : \Omega(M) \to \Omega(N)$. This is well defined by the following lemma:

LEMMA 7.27. *1) In the previous notations, if $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of $f \circ \pi_M$ then $\tilde{f}_1|_{\Omega(M)}$ and $\tilde{f}_2|_{\Omega(M)}$ represent the same element in $\underline{Hom}_R(\Omega M, \Omega N)$.*
2) The map $Hom_R(M,N) \to \underline{Hom}_R(\Omega M, \Omega N)$ is a homomorphism.

3) If f factors through a projective then also $\widetilde{f}|_{\Omega(f)}$ does, thus we get a homomorphism $\underline{Hom}_R(M,N) \to \underline{Hom}_R(\Omega M, \Omega N)$.

Proof. 1) Assume we have two such lifts $\tilde{f}_1$ and $\tilde{f}_2$ then the following diagram is commutative (where $h = \tilde{f}_1|_{\Omega(M)} - \tilde{f}_2|_{\Omega(M)}$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow h & & \downarrow \tilde{f}_1 - \tilde{f}_2 & & \downarrow 0 & & \downarrow \\ 0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

It will be enough to show that h factors through P_M which is projective. This follows from the fact that the image of the map $\tilde{f}_1 - \tilde{f}_2$ is contained in $\Omega(N)$ by the commutativity of the diagram.
2) Choose the lifting of $a \cdot f + b \cdot g$ to be $a \cdot \widetilde{f} + b \cdot \widetilde{g}$.
3) Assume f factors through a projective module P. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow s & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The map $s : P \to P_N$ can be defined using the fact that P is projective and the map $P_N \to N$ is surjective. We get that the induced map $\Omega(M) \to \Omega(N)$ is the zero map. $\square$

The following is important for the definition of Tate cohomology:

Proposition 7.28. *Let G be a finite group and $R = \mathbb{Z}[G]$. If M is a $\mathbb{Z}[G]$-module which is projective as an Abelian group then the map $\underline{Hom}_R(M,N) \to \underline{Hom}_R(\Omega M, \Omega N)$ is an isomorphism.*

Proof. Before we start recall ([**7**] VI,2) that a $\mathbb{Z}[G]$-module Q is called relatively injective if for every injection $A \hookrightarrow B$ of $\mathbb{Z}[G]$-modules which splits as an injection of Abelian groups and every $\mathbb{Z}[G]$ homomorphism $A \to Q$ there exists an extension to a $\mathbb{Z}[G]$ homomorphism $B \to Q$, and that if G is a finite group every projective module is relatively injective ([**7**] VI,2.3).

We construct an inverse to this map. Given a map $f : \Omega M \to \Omega N$. We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & & & & & \\ 0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Since M is projective as an Abelian group the upper row splits as Abelian groups. This means that $\Omega(M) \longrightarrow P_M$ is a split injection as Abelian groups. P_N is projective and hence relatively injective therefore we can extend the homomorphism $\Omega(M) \longrightarrow P_N$ to a homomorphism $\tilde{f} : P_M \to P_N$ such that the diagram will commute. This induces a homomorphism $\overline{f} : M \to N$. Of course $\overline{f}$ depends on the choice of $\tilde{f}$. Suppose that $\tilde{f}_1, \tilde{f}_2$ are two extensions then $\tilde{f}_1 - \tilde{f}_2$ vanishes on $\Omega(M)$ hence the map $\overline{f}_1 - \overline{f}_2 : M \to N$ factors through P_N which is projective. This gives a well defined homomorphism $Hom_R(\Omega M, \Omega N) \to \underline{Hom}_R(M,N)$. Assume

$f : \Omega M \to \Omega N$ factors through a projective P then we can choose $\tilde{f}$ to factor through P again since it is relatively injective and get that $\overline{f}$ is the zero map:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega(M) & \longrightarrow & P_M & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & P & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow s & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Omega(N) & \longrightarrow & P_N & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Hence we get a homomorphism $\underline{Hom}_R(\Omega M, \Omega N) \to \underline{Hom}_R(M, N)$ which is easily seen to be the inverse of the homomorphism $\underline{Hom}_R(M, N) \to \underline{Hom}_R(\Omega M, \Omega N)$. $\square$

We have defined the endofunctor Ω. We define Ω^n by induction: $\Omega^0 = Id$ and $\Omega^n = \Omega \circ \Omega^{n-1}$.

PROPOSITION 7.29. *Let M be an R-module and let $... \to Q_{n-1} \to ... \to Q_0 \to M$ be any projective resolution of M, then $\Omega^n(M)$ can be identified with $Ker(Q_{n-1} \to Q_{n-2})$, that is there is a canonical map $Ker(Q_{n-1} \to Q_{n-2}) \to \Omega^n(M)$ which is an isomorphism in the category $St - mod(R)$.*

PROOF. Given an R-module M we construct a canonical projective resolution of it using the projective covers we have chosen before. We do it by induction where P_n is defined to be the projective cover of $Ker(P_{n-1} \to P_{n-2})$ with the induced map $P_n \to P_{n-1}$, which clearly make this into a projective resolution. Notice that by the definition of Ω we have $\Omega^n(M) = Ker(P_{n-1} \to P_{n-2})$, and for a map $f : M \to N$ the map $\Omega^n(f)$ can be be constructed by extending the map f to a chain map between the two resolutions. In order to prove the proposition it will suffice to show that given two projective resolutions of M $... \to Q_{n-1} \to ... \to Q_0 \to M$ and $... \to P_{n-1} \to ... \to P_0 \to M$ there is a canonical isomorphism $Ker(Q_{n-1} \to Q_{n-2}) \to Ker(P_{n-1} \to P_{n-2})$. This follows directly by induction from what we have already showed in the case of a the projective cover of M. $\square$

REMARK 7.30. By similar reasons we can compute the induced maps $\Omega^n(f)$ for any map $f : M \to N$ by taking any two resolutions for M and for N and extending f into a chain map between the two resolutions.

Bibliography

[1] A, Adem, On the Exponent of Cohomology of Discrete Groups. Bull. London Math. Soc. 21, no. 6, 1989

[2] A. Adem, R. J. Milgram, Cohomology of Finite Groups, Springer, 1994

[3] C. Allday, V. Puppe, Cohomological Methods in Transformation Groups, Cambridge University Press, 1993

[4] D.J.Benson, Representations and Cohomology, Cambridge Studies in Advanced Mathematics, 1991

[5] D.J.Benson and J.F.Carlson, Products in Negative Cohomology. J. Pure & Applied Algebra 82, 1992

[6] G. E. Bredon, Introduction to Compact Transformation Groups, Academic Press, 1972

[7] K. Brown, Cohomology of Groups, Springer-Verlag, 1982

[8] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, 1956

[9] T. Dieck, Transformation Groups, De Gruyter Studies in Mathematics, Vol. 8, 1987

[10] A. Dold, Lectures on Algebraic Topology, Springer, 1972

[11] F. T. Farrell, L.Taylor and J.Wagoner, The Whitehead Theorem in the Proper Category, Compositio Math. 27, 1973

[12] G. Fors, On the Foundation of Algebraic Topology, Arxiv preprint math/0412552, 2004

[13] R. Geoghegan, Topological Methods in Group Theory, Springer, Graduate Texts in Mathematics , Vol. 243, 2008

[14] M. J. Greenberg, Lectures on Algebraic Topology, Benjamin inc. 1967

[15] J. P. C. Greenlees, J. P. May, Generalized Tate Cohomology, Memoir American Math. Soc. No. 543, 1995

[16] A. Grinberg, Resolution of Stratifolds and Connection to Mather's Abstract Pre-stratified Spaces. Dissertation Ruprecht-Karls-Universität Heidelberg, www.ub.uni-heidelberg.de/archiv/3127, 2003

[17] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002

[18] P. J. Hilton, Homotopy Theory and Duality, Gordon and Breach 1965

[19] P. J. Hilton and U. Stammbach, A Course in Homological Algebra, Springer 1996

[20] B. Hughes, A. Ranicki, Ends of Complexes, Cambridge Tracts in Mathematics, 1996

[21] D. Husemöller, Fibre bundles, Springer, 1994

[22] S, Illman, Smooth Equivariant Triangulations of G-Manifolds for G a Finite Group, Math. Ann. 233 no. 3, 1978

[23] M. Kreck, Differential Algebraic Topology: From Stratifolds to Exotic Spheres, Graduate Studies in Mathematics, Vol. 110, 2010

[24] M. Kreck, Equivariant Stratifold Cohomology, Equivariant Poincaré Duality and Equivariant Characteristic Classes, Preprint

[25] E. Laitinen, End Homology and Duality, Forum Mathematicum Vol.8 NO.1, 1996

[26] J. M. Lee, Introduction to smooth manifolds, Springer, 2002

[27] J. P. May, A Concise Course in Algebraic Topology, Chicago Lectures in Mathematics, 1999

[28] J. W. Milnor, Construction of Universal Bundles, II, Ann. of Math., Vol 63 No. 3, 1956

[29] J. W. Milnor, J. D. Stasheff, Characteristic classes, Princeton University Press, (AM-76), Vol. 76, 1974

[30] G. D. Mostow, Equivariant Embeddings in Euclidean Space, The Annals of Mathematics, Vol. 65, No. 3, 1957

[31] E. Spanier, Singular Homology and Cohomology with Local Coefficients and Duality for Manifolds, Pacific Journal of Mathematics Vol. 160, No. 1, 1993

[32] H. Tene, On the Product in Negative Tate Cohomology for Finite Groups, Arxiv preprint arXiv:0911.3014, 2009

[33] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press, 2002