Analytic dilation on complete manifolds with corners of codimension 2

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0.1 Introduction

In [30] a relation between the spectral analysis of many body Schrödinger operators and generalized Laplacians of complete manifolds with corner of codimension two is suggested. In this text we give a first step that make precise analogy between the spectral analysis of these two families of operators: we generalize the method of analytic dilation, coming from the analysis of many body Schrödinger operators, to the context of generalized Laplacians of complete manifolds with corner of codimension two. Using the method of analytic dilation we obtain the following results:

1) we find a meromorphic extension of the resolvent;

2) analytic dilation gives us a discrete set of possible accumulation points of the pure point spectrum;

3) we can prove the absence of singular spectrum for these Laplacians;

4) it provides us also with a theory of resonances.

All the above results have an equivalence in the context of Schrödinger operators. As for these operators, the method of analytic dilation describes the nature of the essential spectrum.

The method of analytic dilation was originally applied to N-particle Schrödinger operators and a classic reference in that setting is [15]. Also it has been applied to the black-box perturbations of the Euclidean Laplacian in the series of papers [38], [39], [40], [41]. In the paper [2] is used for studying Laplacians on hyperbolic manifolds. The analytic dilation has also been applied to the study of the spectral and scattering theory of quantum wave guides and Dirichlet boundary domains, some references in this setting are [10], [25]. It has also been applied to arbitrary symmetric spaces of noncompact types in the papers [26], [27], [28]. In each of these settings new ideas and new methods carry out. In this thesis we develop the analytic method for Laplacians on complete manifolds with corners of codimension 2.

Now we will explain the terminology and our main results more carefully. Let X_0 be a Riemannian manifold with boundary M. We assume that M is the union of two hypersurfaces, M_1 and M_2, intersected in a closed manifold Y, which is the corner in this case. Suppose that in small neighborhoods $M_1 \times [0, \epsilon)$ of M_1, $M_2 \times [0, \epsilon)$ of M_2, and $Y \times [0, \epsilon) \times [0, \epsilon)$ of Y, the Riemannian metric is the natural product type. We enlarge X_0 by gluing first

half-cylinders to the boundary M_i and then filling in $\mathbb{R}^2_+ \times Y$. In this way we construct a complete manifold, X, which is associated to X_0 canonically. Let $Z_i := M_i \cup_Y (\mathbb{R}_+ \times Y)$, $i = 1, 2$ be the manifold with cylindrical end obtained from M_i by attaching the half cylinder $\mathbb{R}_+ \times Y$ to its boundary. Observe that X is the union of $\mathbb{R}_+ \times Z_1$ and $\mathbb{R}_+ \times Z_2$. We call X a complete manifold with corner of codimension 2. In section 2.1 there are figures that represent a compact manifold with corner of codimension 2 and a complete manifold with corner of codimension 2.

Suppose that $\Delta : C^\infty(X, E) \to C^\infty(X, E)$ is a generalized Laplacian i.e. $\sigma_2(\Delta)(x, \xi) = |\xi|^2_{g_x} Id_{E_x}$. Δ is called compatible generalized Laplacian if it satisfies the following properties:

a) On $\mathbb{R}_+ \times Z_i$, Δ takes the form:

$$\Delta = -\frac{\partial^2}{\partial u_i^2} + A_i, \tag{1}$$

where A_i is a compatible generalized Laplacian on Z_i, i.e A_i is a generalized Laplacian and, it has the form:

$$A_i = -\frac{\partial^2}{\partial u_j^2} + \Delta_Y \tag{2}$$

on $\mathbb{R}_+ \times Y$, where Δ_Y is a generalized Laplacian on Y, $i, j \in \{1, 2\}$, and $i \neq j$.

b) Δ has the form:

$$\Delta = -\frac{\partial^2}{\partial u_2^2} - \frac{\partial^2}{\partial u_2^2} + \Delta_Y, \tag{3}$$

on $\mathbb{R}^2_+ \times Y$.

Examples of this kind of operators are the Laplacians associated to the Dirac operators analyzed in [30] and the metric Laplacian acting on functions.

Since X is a complete manifold $\Delta : C_c^\infty(X, E) \to L^2(X, E)$ is essentially self-adjoint. We denote H its self-adjoint extension. $A_i : C_c^\infty(Z_i, E_i) \to L^2(Z_i, E_i)$ is also essentially self-adjoint and we denote its self adjoint extension by $H^{(i)}$. Let b_i be the self-adjoint extension of $-\frac{\partial^2}{\partial u_i^2} : C_c^\infty(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ obtained with Von Neumann boundary conditions. We denote H_i the self-adjoint operator $1 \otimes b_i + 1 \otimes H^{(i)}$. Similarly, $H^{(3)}$ denotes

the self-adjoint operator associated to the essentially self-adjoint operator $\Delta_Y : C_c^\infty(Y,S) \to L^2(Y,S)$; and we denote by $H^{(3)}$, the self-adjoint operator $H_3 := 1 \otimes b_1 \otimes 1 + 1 \otimes 1 \otimes b_2 + H^{(3)} \otimes 1 \otimes 1$.

This notation is similar to the notation used in [19] and [20] for the spectral analysis of Schrödinger operators. There, one has a vector space W with an inner product, and a finite lattice of subspaces of W, $\mathscr{L}$. For the description we give here see [19], page 3454. The interacting Hamiltonian is given by $H = H_0 + \sum_{a \in \mathscr{L}} V_a$ where V_a is a function in $C_c^\infty(a)$ with a nice decaying at infinity (in a); and H_0 is the usual Laplacian on W. Given an element $a \in \mathscr{L}$ one define the operators H_a and H^a. The operator H^a acts on $a^\perp$, the orthogonal complement of a and is equal to $H_{0,a^\perp} + \sum V_{a^\perp}$, where $H_{0,a^\perp}$ is the free Hamiltonian on $a^\perp$, or in other words the usual Laplacian on $a^\perp$. Observe that $W = a \oplus a^\perp$ implies $L^2(W) = L^2(a) \otimes L^2(a^\perp)$. H_a acts on $L^2(W)$ as $H_a := H_{0,a} \otimes 1 + 1 \otimes H^a$, where $H_{0,a}$ denotes the free Hamiltonian acting on a.

Now we explain the method of analytic dilation applied to compatible generalized Laplacians on complete manifolds with corner of codimension 2. Using the dilation naturally defined in $\mathbb{R}_+^2 \times Y$ and $\mathbb{R}_+ \times Z_i$, we construct a family of unitary operators $\{U_\theta\}_{\theta \in \mathbb{R}_+}$ acting on $L^2(X,E)$, and a subset $\mathscr{V}$ of $L^2(X,E)$, satisfying the following basic properties:

i) $\mathscr{V}$ is a dense subset of $L^2(X,E)$.

ii) For all $\psi \in L^2(X,E)$, the function $\theta \mapsto U_\theta \psi$ has an analytic extension to the right half-plane.

iii) $U_\theta \mathscr{V}$ is dense in $L^2(X,E)$ for θ in the right-half plane.

iv) The family of operators $\{H_\theta := U_\theta H U_\theta^{-1}\}_{\theta \in \mathbb{R}_+}$ induces an holomorphic family of type A, H_θ, in a domain Γ (see 1.81). In other words, for $\theta \in \Gamma$, the operator $H_\theta : \mathscr{W}_2(X,E) \to L^2(X,E)$ is a closed operators with domain $\mathscr{W}_2(X,E)$ (the second Sobolev space, see (E.4)) and for all $\psi \in Dom(H)$ and $\phi \in L^2(X,E)$ the function $\theta \mapsto \langle H_\theta \phi, \psi \rangle_{L^2(X,E)}$ is holomorphic.

As in the analysis of Schrödinger operators, where the analytic dilation of the many-body Hamiltonian depend on the channel Hamiltonians, the analytic dilation of H depends on the analytic dilation of $H^{(i)}$ for $i = 1, 2$. For defining and studying the analytic dilation method on X, it turns out that one has to define and to study it over the manifolds with cylindrical end Z_1

and Z_2. In 1.1, we construct an *analytic dilation family* (see definition 3) for $H^{(i)}$, $U_{i,\theta}$, with their analytic vectors, $\mathscr{V}_i$, for $i = 1, 2$. We denote the operator $U_{i,\theta} H^{(i)} U_{i,\theta}^{-1}$ by $H^{(i),\theta}$. The method of analytic dilation for manifolds with cylindrical ends was recently developed by Kalvin in [23]; in fact in [23], it is developed not only for Riemannian metrics with cylindrical ends, but for *Riemannian metrics with axial analytic asymptotically cylindrical end.* The results of section 1.1 can be deduced from [23], however they were deduced independently by us, and were expected from the complex scaling in wave guides (see [10], [25]). In 2.1, we define U_θ and $\mathscr{V}$, an analytic dilation family for the operator H.

In [7] it is given a geometric characterization of the essential spectrum of certain well behaved closed operators (see theorem 36 and theorem 37); this characterization introduces a subset of the set of singular sequences associated to an operator, that we call *boundary Weyl sequences* (abbrev. b.W.s, see definition 11), that are more suitable to manipulate than the usual singular sequences. We adapt the characterization of [7] to operators in $L^2(X, E)$ in appendix D. We use it for proving in section 2.5 that the essential spectrum of H_θ is given by:

$$\begin{aligned} \sigma_{ess}(H_\theta) = & \bigcup_{i=1}^{2} \left(\cup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} (\lambda + \theta' \, \mathbb{R}_+) \right) \\ & \cup \left(\cup_{\mu \in \sigma(H^{(3)})} (\mu + \theta' \, \mathbb{R}_+) \right), \end{aligned} \tag{4}$$

where the parameter θ' is equal to $\theta' := \frac{1}{(1+\theta)^2}$. Apart of the geometric spectral techniques explained in appendix D, the Ichinose lemma is other important tool in the proof of (4) (see appendix C). Results similar to (4) are found in [27] for the Laplacian of $SL(3)/SO(3)$; for the Laplacian of hyperbolic manifolds in [2]; and for the Schrödinger operators in [19]. A version of (4) is proved in section 1.4 for generalized Laplacians on manifolds with cylindrical end; this version is also a consequence of results of [23].

$\sigma_{ess}(H_\theta)$ can be described in terms of the spectrum of $H^{(i)}$ for $i = 1, 2$. In fact, if $\mathscr{R}_\theta(H^{(i)})$ denote the set of resonances of $H^{(i)}$ inside the cone $\{z : 0 \leq z \leq arg(\theta')\}$, for $arg(\theta') > 0$ (or $\{z : 0 \geq z \geq arg(\theta')\}$, for $arg(\theta') < 0$), then:

$$\sigma_d(H^{(i),\theta}) = \sigma_{pp}(H^{(i)}) \dot{\cup} \mathscr{R}_\theta(H^{(i)}). \tag{5}$$

It is proved in [21], theorem 3.26, that $\mathscr{R}_\theta(H^{(i)}) \cap \mathbb{R} \subset \sigma(H^{(3)})$. Using equation (4) and the Aguilar-Balslev-Combes theory (see appendix B), we

obtain the following results (see theorem 11):

1) H has no singular spectrum.

2) We define the set of resonances of H:

$$\mathscr{R}_\theta(H) := \{\lambda \in \sigma_d(H_\theta) : \lambda \notin \sigma_{pp}(H)\}. \tag{6}$$

The set $\mathscr{R}_\theta(H)$ is in fact independent of θ in the sense that if $0 < arg(\theta_i') < \frac{\pi}{2}$, for $i = 1, 2$, and $arg(\theta_1') \geq arg(\theta_2')$ then:

$$\mathscr{R}_{\theta_1}(H) \subset \mathscr{R}_{\theta_2}(H). \tag{7}$$

3) The set of accumulation points of $\sigma_{pp}(H)$ is contained in:

$$\left(\bigcup_{i=1}^{2} \sigma_{pp}(H^{(i)})\right) \cup \sigma(H^{(3)}) \cup \{\infty\}. \tag{8}$$

In other words the pure point spectrum of H (if exists) could accumulate on points in the spectrum of $H^{(3)}$ or the pure point spectrum of $H^{(i)}$ for $i = 1, 2$.

It would be interesting also to study if it is possible to find examples of compatible generalized Laplacians with finite or infinite pure point spectrum. The conjecture is that, generically, a compatible generalized Laplacian has no pure point spectrum. We believe that it is also possible to prove that, if the pure point spectrum accumulates in one of the elements of (8), then it does it by below. We will study these problems in other texts.

Now we described some applications of the results of this thesis. They are part of a work in process and we hope to publish them soon. We study the time dependent scattering theory associated to the Hamiltonians H, H_k for $k = 1, 2, 3$. We point out that there is a natural generalization of Ruelle theorem (see [19], theorem 2.4) to this context. Let χ_R be a smooth extension of the characteristic function of the set X_R in the exhaustion of X defined in equation (2.6). We prove that, for $\varphi \in L^2_{ac}(X, E)$, $\varphi_t := e^{iHt}$ escapes of compact sets when $t \to \pm\infty$. This last claim in the ergodic sense, i.e.

$$\lim_{t\to\infty} t^{-1} \int_0^t vs||\chi_R e^{iHs}\varphi||^2 = 0. \tag{9}$$

This behavior of the absolutely continuous states contrasts with the behavior of the pure point spectrum, that we now explain. If $\varphi \in L^2(X, E)$ is an eigenvalue of H:

$$\lim_{R\to\infty} ||(1-\chi_R)e^{iHt}\varphi|| = 0 \text{ uniformly in } 0 \leq t < \infty. \tag{10}$$

Our version of Ruelle theorem claims that (9) and (10) characterize the absolutely continuous spectrum and the pure point spectrum respectively.

We introduce some notation. Let $L^2_{pp}(Z_k, E_k)$ be the space in $L^2(Z_k, E_k)$ generated by the L^2- eigenfunctions of $H^{(k)}$, for $k = 1, 2$. Recall that, for $k = 1, 2$, the self-adjoint operator $H^{(k)}$ splits $L^2(Z_k, E_k)$, as $L^2(Z_k, E_k) = L^2_{pp}(Z_k, E_k) \oplus L^2_{ac}(Z_k, E_k)$, in the discrete and absolutely continuous part of $H^{(k)}$. The discrete and the absolutely continuous spaces are $H^{(k)}$-invariant subspaces of $L^2(Z_k, E_k)$, hence $H^{(k)} = H^{(k),pp} \oplus H^{(k),ac}$. Associated to this splitting we have the operators

$$H_{k,pp} = b_k + H^{(k),pp} \text{ and } H_{k,ac} = b_k + H^{(k),ac} \tag{11}$$

acting on $L^2_{pp}(Z_k, E_k) \otimes L^2(\mathbb{R}_+)$ and $L^2_{ac}(Z_k, E_k) \otimes L^2(\mathbb{R}_+)$ respectively. In our work in progress, we show the existence of the following wave operators:

$$W_\pm(H, H_{1,pp}),\ W_\pm(H, H_{2,pp}),\ W_\pm(H, H_3),$$

and

$$W_\pm(H, H_{k,ac}))\left(W_+(H^{(k)}, b_j + H^{(3)}) \otimes (id)\right) \text{ for } k, j \in \{1, 2\} \text{ and } k \neq j.$$

We express $W_\pm(H, H_{k,pp})$ in terms of the generalized eigenfunctions associated to L^2-eigenfunctions of $H^{(k)}$ defined in section 3.1. Similarly, we express the operator

$$\Omega_\pm := \left(\sum_{k=1}^{2} W_\pm(H, H_{k,ac})\left(W_+(H^{(k)}, b_j + H^{(3)}) \otimes (id)\right)\right) - W_\pm(H, H_3), \tag{12}$$

in terms of the generalized eigenfunctions associated to eigenfunctions of $H^{(3)}$ of section 3.2. We prove that the images of $W_\pm(H, H_{1,d})$, $W_\pm(H, H_{2,d})$ and $\Omega_\pm$ are pairwise orthogonal. We have:

$$Im(W_\pm(H, H_{1,d})) \oplus Im(W_\pm(H, H_{2,d})) \oplus Im(\Omega_\pm) \subset L^2_{ac}(X, E). \tag{13}$$

By *asymptotic completeness* we mean that the following countenance holds:

$$Im(W_{\pm}(H, H_{1,d})) \oplus Im(W_{\pm}(H, H_{2,d})) \oplus Im(\Omega_{\pm}) = L^2_{ac}(X, E). \qquad (14)$$

We call *asymptotic clustering* the other inclusion, namely:

$$L^2_{ac}(X, E) \subset Im(W_{\pm}(H, H_{1,d})) \oplus Im(W_{\pm}(H, H_{2,d})) \oplus Im(\Omega_{\pm}). \qquad (15)$$

Our proof of asymptotic clustering is based in [19], [20]. The main ingredients of the proof are the Mourre's estimates, the Yafaev function and the knowledge of the spectral resolution of the operators $H^{(k)}$ for $k = 1, 2, 3$. We observe that the spectral resolution of $H^{(3)}$, a generalized Laplacian over the closed manifold Y, is well known. Similarly the spectral resolution of $H^{(k)}$, a Laplacian over a manifold with cylindrical end, is well known and described for example in [16] or [21]. We provide versions of Mourre estimates for compatible generalized Laplacians on complete manifolds with cylindrical end, and compatible generalized Laplacians on complete manifolds with corner of codimension 2. We also construct suitable Yafaev functions that together with our versions of Mourre estimate prove asymptotic clustering (in the sense of (15))

This thesis has 5 appendixes in which we described results that make the text more understandable. In appendix A we sketch a proof of the Weyl characterization of the essential spectrum (see theorem 22).

In appendix B, we apply the abstract Aguilar-Balslev-Combes theory to operators with a essential spectrum as $\sigma_{ess}(H_\theta)$ and $\sigma_{ess}(H_{k,\theta})$ for $k = 1, 2$. This appendix is based in [17]. In appendix C, we give a brief review of sectorial forms, being our goal to formulate the Ichisinose lemma (theorem 35) and introduce the basic results for applying it to the Laplacians that we are studying. Given A and B strictly m-sectorial operators, Ichinose lemma establishes that

$$\sigma(A \otimes 1 + 1 \otimes B) = \sigma(A) + \sigma(B). \qquad (16)$$

This is important for us because H looks like $b_k \otimes 1 + 1 \otimes H^{(k)}$ (in $L^2(Z_k) \otimes L^2(\mathbb{R}_+)$) for $k = 1, 2$, and $b_1 \otimes 1 \otimes 1 + 1 \otimes b_2 \otimes 1 + 1 \otimes 1 \otimes H^{(3)}$ (in $L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+) \otimes L^2(Y)$).

In appendix D we observe that the spectral geometric methods of [7] generalize to our context. These methods are used for proving theorem 10 and basically consist of making the Weyl characterization of the essential

spectrum (theorem 22) more geometric by the use of the Weyl boundary functions. Finally, in appendix E, we describe the basic results of the analysis of elliptic differential operators on manifolds with bounded geometry, that we use in this text.

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Chapter 1

Analytic dilation on complete manifolds with cylindrical end

In this section we generalize the method of analytic dilation to complete manifolds with cylindrical end. As we said in the introduction most of the results are consequence of [23] but they were obtained independently by us.

1.1 Manifolds with cylindrical end and their compatible Laplacians

Let Z_0 be a compact Riemannian manifold with boundary $Y := \partial Z_0$. Suppose that in a collar neighborhood, $Y \times (-\epsilon, 0]$, of the boundary Y, the Riemannian metric of Z_0 is a product metric i. e a metric of the form $g_Y + du \otimes du$ where g_Y is a Riemannian metric on Y and u is the variable on $(-\epsilon, 0]$. We make from Z_0 a complete manifold Z by attaching the infinite cylinder $Y \times \mathbb{R}_+$ to Z_0. We have then:

$$Z := Z_0 \cup_Y (Y \times \mathbb{R}_+), \tag{1.1}$$

where we are identifying the boundary of Z_0 with $Y \times \{0\}$. We extend the smooth structure and the Riemannian metric naturally.

Let E be a vector bundle over Z with an Hermitian metric. We assume that there exists E' an Hermitian vector bundle over Y such that $E|_{Y \times \mathbb{R}_+}$ is the pull back of E' by the projection $\pi : Y \times \mathbb{R}_+ \to Y$. We suppose that

the Hermitian metric of E is the pullback of the Hermitian metric of E'. Let Δ be a Laplacian on Z i.e. $\sigma_2(\Delta)(z,\xi) = |\xi|^2_{g_z}$. We assume furthermore that on $Y \times \mathbb{R}_+$:

$$\Delta = -\frac{\partial^2}{\partial u^2} + \Delta_Y, \tag{1.2}$$

where Δ_Y is a Laplacian acting on $C^\infty(Y, E')$. In fact, we will denote by Δ_Y the operator acting on distributions and the self-adjoint operator induced by $(\Delta_Y, C^\infty(Y, E'))$.

A Laplacian satisfying the previous assumptions is called **compatible Laplacian**. In this section we adapt the method of analytic dilation to compatible Laplacians.

1.2 The definition of U_θ

Let $0 < K < R$ and $\varphi \in C^\infty(\mathbb{R})$ with $\varphi' \geq 0$ such that:

$$\varphi(u) := \begin{cases} 0 & \text{for } 0 < u < K \\ 1 & \text{for } R < u < \infty. \end{cases} \tag{1.3}$$

Let $\theta \in \mathbb{R}_+$, define the function:

$$\psi_\theta(u) := (\varphi(u)\theta + 1)u = \varphi(u)u\theta + u, \tag{1.4}$$

for $u \in \mathbb{R}_+$. Observe that

$$\psi_\theta(u) = \begin{cases} u & u < K \\ (\theta+1)u & \text{for } u > R. \end{cases}$$

We calculate the first derivative of ψ_θ:

$$\psi'_\theta(u) := \frac{\partial}{\partial u}(\psi_\theta)(u) = \varphi'(u)u\theta + \varphi(u)\theta + 1. \tag{1.5}$$

The second derivative of ψ_θ is given by:

$$\psi''_\theta(u) := \frac{\partial^2}{\partial u^2}\psi_\theta(u) = \frac{\partial}{\partial u}(\varphi'(u)u\theta + \varphi(u)\theta + 1) = \varphi''(u)u\theta + 2\varphi'(u)\theta. \tag{1.6}$$

The third derivative of ψ_θ is equal to:

$$\psi'''_\theta(u) := \frac{\partial^3}{\partial u^3}\psi_\theta(u) = \frac{\partial}{\partial u}(\varphi''(u)u\theta + 2\varphi'(u)\theta) = \varphi'''(u)u\theta + 3\varphi''(u)\theta. \tag{1.7}$$

We define $U_\theta : L^2(Z,E) \to L^2(Z,E)$:

$$U_\theta f(z) = \begin{cases} f(z) & \text{for } z \in Z_0 \\ f(y, \psi_\theta(u))\psi'_\theta(u)^{\frac{1}{2}} & \text{for } z = (y,u) \in Y \times \mathbb{R}_+. \end{cases} \tag{1.8}$$

Observe that, for $\theta > 0$, the function ψ_θ is invertible (because $\psi'_\theta(u) \geq 1$ for $u \geq 0$). We will denote its inverse by α_θ.

For $\theta \in \mathbb{R}_+$ a natural inverse of U_θ is given by:

$$U_\theta^{-1} f(z_0) := f(z_0), \text{ for } z_0 \in Z_0. \tag{1.9}$$

$$U_\theta^{-1} f(y,u) := f(y, \alpha_\theta(u))\psi'_\theta(\alpha_\theta(u))^{-\frac{1}{2}}. \tag{1.10}$$

We check that in fact U_θ^{-1} is the inverse of U_θ:

$$\begin{aligned} U_\theta^{-1} U_\theta f(y,u) &= U_\theta^{-1}\left(f(y,\psi_\theta(u))\psi'_\theta(u)^{\frac{1}{2}}\right) \\ &= f(y,u)\psi'_\theta(\alpha_\theta(u))^{\frac{1}{2}}\psi'_\theta(\alpha_\theta(u))^{-\frac{1}{2}} = f(y,u). \end{aligned} \tag{1.11}$$

Notice that $U_0 = Id$. We observe that for $f \in C^\infty(Z,E)$, $U_\theta f$ belongs to $C^\infty(Z,E)$ and, if $f \in C_c^\infty(Z,E)$, then $U_\theta f \in C_c^\infty(Z,E)$.

Let $(\phi_i, \mu_i)_{i \in \mathbb{N}}$ be a spectral resolution of Δ_Y, that is $\phi_i \in Dom(\Delta_Y)$, $\Delta_Y \phi_i = \mu_i \phi_i$ and $(\phi_i)_{i \in \mathbb{N}}$ is an orthonormal basis for $L^2(Y, E')$.

Proposition 1 *For $\theta \in (0,\infty)$, U_θ induces a unitary operator acting on $L^2(Z,E)$.*

Proof:
We prove that for $f \in C_c^\infty(Z,E)$, $||U_\theta f|| = ||f||$. Observe that:

$$||U_\theta f||^2 = \int_{Z_0} |U_\theta f(z)|^2 dvol(z) + \int_{Y \times \mathbb{R}_+} |U_\theta f(z)|^2 dvol(z). \tag{1.12}$$

Since $\int_{Z_0} |U_\theta f(z)|^2 dvol(z) = \int_{Z_0} |f(z)|^2 dvol(z)$, we just have to show:

$$\int_{Y \times \mathbb{R}_+} |U_\theta f(z)|^2 dvol(z) = \int_{Y \times \mathbb{R}_+} |f(z)|^2 dvol(z).$$

Suppose that $f(y,u) = \sum_{i=0}^{\infty} f_i(u)\phi_i(y)$. Then we have:

$$\begin{aligned}
\int_{Y\times \mathbb{R}_+} |U_\theta f(z)|^2 dvol(z) &= \int_{Y\times \mathbb{R}_+} \sum_{i=1}^{\infty} f_i(\psi_\theta(u))\psi_\theta'^{\frac{1}{2}}(u)\phi_i(y)|^2 dydu \\
&= \int_{\mathbb{R}_+} \sum_{i=1}^{\infty} |f_i(\psi_\theta(u))\psi_\theta'^{\frac{1}{2}}(u)|^2 du \\
&= \int_{\mathbb{R}_+} \sum_{i=1}^{\infty} |f_i(v)|^2 dv \\
&= \sum_{i=1}^{\infty} \int_{Y\times \mathbb{R}_+} |f_i(u)\phi_i(y)|^2 dydu.
\end{aligned} \tag{1.13}$$

In the previous calculations, we did the change of coordinates $v := \psi_\theta(u)$.□

1.3 The family Δ_θ

For $\theta \in (0,\infty)$ we calculate $\Delta U_\theta^{-1} f$ restricted to $Y \times (0,\infty)$. For this, we observe:

$$\begin{aligned}
\frac{\partial}{\partial u} U_\theta^{-1} f(y,u) =& \frac{\partial}{\partial u}\left(f(y,\alpha_\theta(u))\psi_\theta'(\alpha_\theta(u))^{-\frac{1}{2}}\right) \\
&= (\frac{\partial}{\partial u} f)(y,\alpha_\theta(u))\alpha_\theta'(u)\psi_\theta'(\alpha_\theta(u))^{-\frac{1}{2}} \\
&- \frac{1}{2} f(y,\alpha_\theta(u))\psi_\theta'(\alpha_\theta(u))^{-\frac{3}{2}}\psi_\theta''(\alpha_\theta(u))\alpha_\theta'(u).
\end{aligned} \tag{1.14}$$

For the second derivative of $U_\theta^{-1} f(y,u)$ with respect to u, we calculate:

$$\begin{aligned}
\frac{\partial^2}{\partial u^2} U_\theta^{-1} f(y,u) =& \frac{\partial}{\partial u}\Bigg((\frac{\partial}{\partial u} f)(y,\alpha_\theta(u))\alpha_\theta'(u)\psi_\theta'(\alpha_\theta(u))^{-\frac{1}{2}} \\
&- \frac{1}{2} f(y,\alpha_\theta(u))\psi_\theta'(\alpha_\theta(u))^{-\frac{3}{2}}\psi_\theta''(\alpha_\theta(u))\alpha_\theta'(u)\Bigg).
\end{aligned}$$

Hence:

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2} U_\theta^{-1} f(y,u) = & (\frac{\partial^2}{\partial u^2} f))(y, \alpha_\theta(u)) \alpha'_\theta(u)^2 \psi'_\theta(\alpha_\theta(u))^{-\frac{1}{2}} \\
& + (\frac{\partial}{\partial u} f)(y, \alpha_\theta(u)) \alpha''_\theta(u) \psi'_\theta(\alpha_\theta(u))^{-\frac{1}{2}} \\
& - (\frac{\partial}{\partial u} f)(y, \alpha_\theta(u)) (\psi'_\theta(\alpha_\theta(u)))^{-\frac{3}{2}} \psi''_\theta(\alpha_\theta(u)) \alpha'_\theta(u)^2 \\
& + \frac{3}{4} f(y, \alpha_\theta(u)) \psi''_\theta(\alpha_\theta(u))^2 \psi'_\theta(\alpha_\theta(u))^{-\frac{5}{2}} \alpha'_\theta(u)^2 \\
& - \frac{1}{2} f(y, \alpha_\theta(u)) \psi'_\theta(\alpha_\theta(u))^{-\frac{3}{2}} \psi'''_\theta(\alpha_\theta(u)) \alpha'_\theta(u)^2 \\
& - \frac{1}{2} f(y, \alpha_\theta(u)) \psi'_\theta(\alpha_\theta(u))^{-\frac{3}{2}} \psi''_\theta(\alpha_\theta(u)) \alpha''_\theta(u).
\end{aligned}
\tag{1.15}
$$

Now we can calculate $\Delta_\theta f|_{Y \times \mathbb{R}_+} := U_\theta \Delta U_\theta^{-1} f_{Y \times \mathbb{R}_+}$:

$$
\begin{aligned}
\Delta_\theta f(y,u) = & \Delta_Y f(y,u) \\
& - \frac{\partial^2}{\partial u^2} f(y,u) (\alpha'_\theta(\psi_\theta(u)))^2 \\
& - \frac{\partial}{\partial u} f(y,u) \alpha''_\theta(\psi_\theta(u)) \\
& + \frac{\partial}{\partial u} f(y,u) (\psi'_\theta(u))^{-1} \psi''_\theta(u) (\alpha'_\theta(\psi_\theta(u)))^2 \\
& - \frac{3}{3} f(y,u) \psi'_\theta(u)^{-2} (\psi''_\theta(u))^2 (\alpha'_\theta(\psi_\theta(u)))^2 \\
& + \frac{1}{2} f(y,u) (\psi'_\theta(u))^{-1} \psi'''_\theta(u) (\alpha'(\psi_\theta(u)))^2 \\
& + \frac{1}{2} f(y,u) (\psi'_\theta(u))^{-1} \psi''_\theta(u) \alpha''_\theta(\psi_\theta(u)).
\end{aligned}
\tag{1.16}
$$

Observe that the coefficients of Id, $\frac{\partial}{\partial u}$, $\frac{\partial^2}{\partial u^2}$ are given in terms of $\alpha'_\theta(\psi_\theta(u))$ and $\alpha''_\theta(\psi_\theta(u))$. We describe these last terms more carefully. Since $\psi_\theta(\alpha_\theta(u)) = u$, we have the following formulas for the derivatives of α_θ with respect to

the variable u:

$$\alpha'_\theta(u) = \frac{1}{\psi'_\theta(\alpha_\theta(u))}; \tag{1.17}$$

$$\alpha''_\theta(u) = \frac{-\psi''_\theta(\alpha_\theta(u))\alpha'_\theta(u)}{(\psi'_\theta(\alpha_\theta(u)))^2}; \tag{1.18}$$

$$\begin{aligned}\alpha'''_\theta(u) = &-\frac{\psi'''_\theta(\alpha_\theta(u))\alpha'_\theta(u)}{(\psi'_\theta(\alpha_\theta(u)))^2} - \frac{\psi''_\theta(\alpha_\theta(u))\alpha''_\theta(u)}{(\psi'_\theta(\alpha_\theta(u)))^2} \\ &+ 2\frac{(\psi''_\theta(\alpha_\theta(u)))^2(\alpha'_\theta(u))^2}{(\psi'_\theta(\alpha_\theta(u)))^3}.\end{aligned} \tag{1.19}$$

Now we give $\alpha'_\theta(\psi_\theta(u))$, $\alpha''_\theta(\psi_\theta(u))$ and $\alpha''_\theta(\psi_\theta(u))$:

$$\alpha'_\theta(\psi_\theta(u)) = \frac{1}{\psi'_\theta(u)} \tag{1.20}$$

$$\alpha''_\theta(\psi_\theta(u)) = \frac{-\psi''_\theta(u)\alpha'_\theta(\psi_\theta(u))}{(\psi'_\theta(u))^2} = \frac{-\psi''_\theta(u)}{(\psi'_\theta(u))^3} \tag{1.21}$$

$$\begin{aligned}\alpha'''_\theta(\psi_\theta(u)) = &-\frac{\psi'''_\theta(u)\alpha'_\theta(\psi_\theta(u))}{(\psi'_\theta(u))^2} - \frac{\psi''_\theta(u)\alpha''_\theta(\psi_\theta(u))}{(\psi'_\theta(u))^2} \\ &+ 2\frac{(\psi''_\theta(u))^2(\alpha'_\theta(\psi_\theta(u)))^2}{(\psi'_\theta(u))^3}.\end{aligned} \tag{1.22}$$

As consequence of the above calculations we have:

Remark 1 *1) For fixed $u \in \mathbb{R}_+$, the following functions holomorphically extend from $\mathbb{R}_+$ to $\mathbb{C} - (-\infty, 0)$:*

i) $\theta \mapsto \alpha'_\theta(\psi_\theta(u))$.

ii) $\theta \mapsto \alpha''_\theta(\psi_\theta(u))$.

iii) $\theta \mapsto \alpha'''_\theta(\psi_\theta(u))$.

2) $\alpha'_\theta(\psi_\theta(u))$, $\alpha''_\theta(\psi_\theta(u))$ and $\alpha'''_\theta(\psi_\theta(u))$ are linear combinations of products of the functions $\frac{1}{\psi'_\theta(u)}$, $\psi''_\theta(u)$ and $\psi'''_\theta(u)$.

The previous remark allows us to define the differential operators Δ_θ for $Re(\theta) \geq 0$, because, for such a θ, $\psi'_\theta(u) = \varphi'(u)u\theta + \varphi(u)\theta + 1 \neq 0$, for all $u \in \mathbb{R}_+$, since $Re(\varphi'(u)u\theta + \varphi(u)\theta + 1) > 0$. One can see that, in fact, the operator Δ_θ is defined in a larger domain of θ, but we will restrict ourselves

to $Re(\theta) \geq 0$. Let $f \in C^\infty(Z, E)$ and $(y, u) \in Y \times \mathbb{R}_+$, we have:

$$\begin{aligned}\Delta_\theta f(y,u) :=& \Delta_Y f(y,u) - (\frac{\partial^2}{\partial u^2} f))(y,u)\alpha'_\theta(\psi_\theta(u))^2 \\ &- (\frac{\partial}{\partial u} f)(y,u)\alpha''_\theta(\psi_\theta(u)) \\ &+ (\frac{\partial}{\partial u} f)(y,u)\psi'_\theta(u)^{-1}\psi''_\theta(u)\alpha'_\theta(\psi_\theta(u))^2 \\ &- \frac{3}{3} f(y,u)\psi'_\theta(u)^{-3/2}\psi''_\theta(u)^2\alpha'_\theta(\psi_\theta(u))^2 \\ &+ \frac{1}{2} f(y,u)\psi'_\theta(u)^{-1}\psi'''_\theta(u)\alpha'_\theta(\psi_\theta(u))^2 \\ &+ \frac{1}{2} f(y,u)\psi'_\theta(u)^{-1}\psi''_\theta(u)\alpha''_\theta(\psi_\theta(u)).\end{aligned} \tag{1.23}$$

For $z \in Z_0$:

$$\Delta_\theta f(z) = \Delta f(z). \tag{1.24}$$

Observe that, on $Y \times \mathbb{R}_+$, we have:

$$\Delta_\theta = a_2(\theta, u)\frac{\partial^2}{\partial u^2} + a_1(\theta, u)\frac{\partial}{\partial u} + a_0(\theta, u) + \Delta_Y \tag{1.25}$$

where $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$ are given in equation (1.23) (see (1.31)). Observe that, by remark 1, $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$ are well defined and holomorphic for $Re(\theta) \geq 0$. We remark also that

$$a_k(\theta, .) \in C^\infty(\mathbb{R}_+)$$

for $k = 0, 1, 2$ and $Re(\theta) > 0$. We will continue denoting $a_2(\theta, u), a_1(\theta, u)$ and $a_0(\theta, u)$, the coefficients of $\frac{\partial^2}{\partial u^2}$, $\frac{\partial}{\partial u}$ and Id for the operator Δ_θ localized in $L^2(Y \times \mathbb{R}_+, E)$.

From now on, given $\theta \in \mathbb{C} - (-\infty, 0)$, we denote θ' to

$$\theta' := \frac{1}{(\theta + 1)^2}. \tag{1.26}$$

The parameter θ' naturally appears in the description of $\sigma_{ess}(\Delta_\theta)$ (see equation (1.44)) The next proposition follows easily from equation (1.23), remark 1 and the definition of ψ_θ in (1.4):

Proposition 2 *Let $f \in C^\infty(Z,E)$. For $Re(\theta) \geq 0$, the formula for Δ_θ reduces for $(y,u) \in Y \times (0,K)$ to:*

$$\Delta_\theta f(u,y) = -\frac{\partial^2}{\partial u^2} f(u,y) + \Delta_Y f(u,y); \tag{1.27}$$

and, for $(y,u) \in Y \times (R,\infty)$:

$$\Delta_\theta f(u,y) = -\theta' \frac{\partial^2}{\partial u^2} f(u,y) + \Delta_Y f(u,y). \tag{1.28}$$

The next proposition is a technical tool for proving that Δ_θ is a differential operator with bounded coefficients (see appendix E) and other important facts about the family Δ_θ for $\theta > 0$.

Proposition 3 *Let $a_0(\theta,u)$, $a_1(\theta,u)$ and $a_2(\theta,u)$ be the coefficients of f, $\frac{\partial}{\partial u}(f)$ and $\frac{\partial^2}{\partial u^2}(f)$ of $\Delta_\theta f$ in equation (1.23) (see (1.31) below). Let $N > 0$ be fixed. If $|\theta| < N$ and $Re(\theta) \geq 0$ then there exists a $C(N)$ independent of θ and $u \in \mathbb{R}_+$ such that for $i = 0,1,2$:*

$$|a_i(\theta,u)| \leq C(N), \tag{1.29}$$

and,

$$|\frac{\partial}{\partial \theta_i}(a_i)(\theta,u)| \leq C(N), \tag{1.30}$$

where $\theta := \theta_1 + i\theta_2$.

Proof:
We deduce from (1.23):

$$\begin{aligned}
a_2(\theta,u) &:= (\alpha'_\theta(\psi_\theta(u)))^2 = \frac{1}{(\psi'_\theta(u))^2}; \\
a_1(\theta,u) &:= \alpha''_\theta(\psi_\theta(u)) + (\psi'_\theta(u))^{-1}\psi''_\theta(u)\alpha'_\theta(\psi_\theta(u))^2; \\
a_0(\theta,u) &:= \frac{1}{2}\psi'_\theta(u)^{-1}\psi'''_\theta(u)\alpha'_\theta(\psi_\theta(u))^2 \\
&\quad + \frac{1}{2}\psi'_\theta(u)^{-1}\psi''_\theta(u)\alpha''_\theta(\psi_\theta(u)) \\
&\quad - \frac{3}{4}\psi'_\theta(u)^{-\frac{3}{2}}\psi''_\theta(u)^2\alpha'_\theta(\psi_\theta(u))^2.
\end{aligned} \tag{1.31}$$

By part 2 of remark 1 and (1.31), $a_0(\theta,u)$, $a_1(\theta,u)$ and $a_2(\theta,u)$ are given by a finite linear combination of products of $\frac{1}{(\psi'_\theta(u))^{\frac{1}{2}}}$, $\psi''_\theta(u)$ and $\psi'''_\theta(u)$. Using

equations (1.4),(1.5), (1.6) we observe that $\frac{1}{(\psi'_\theta(u))^{\frac{1}{2}}}, \psi''_\theta(u)$ and $\psi'''_\theta(u)$ are uniformly bounded in the set S, where $S := \{(\theta, u) \in \mathbb{C} \times \mathbb{R}_+ : |\theta| < N$, $Re(\theta) \geq 0$ and $u \in \mathbb{R}_+\}$. Hence, $a_0(\theta, u)$, $a_1(\theta, u)$ and $a_2(\theta, u)$ are uniformly bounded in S.

For showing that $\frac{\partial}{\partial\theta_i}(a_2)(u,\theta)$, $\frac{\partial}{\partial\theta_i}(a_1)(u,\theta)$, $\frac{\partial}{\partial\theta_i}(a_0)(u,\theta)$ are uniformly bounded in S, we prove that $\frac{\partial}{\partial\theta_i}$ of any finite linear combination of products of $\frac{1}{(\psi'_\theta(u))^{\frac{1}{2}}}, \psi''_\theta(u)$ and $\psi'''_\theta(u)$ is uniformly bounded in S. Let $\sum_{k,l,j} a_{k,l,j} \frac{1}{(\psi'_\theta(u))^{\frac{k}{2}}} \psi''_\theta(u)^j \psi'''_\theta(u)^l$ be a generic finite linear combination of products of $\frac{1}{(\psi'_\theta(u))^{\frac{1}{2}}}, \psi''_\theta(u)$ and $\psi'''_\theta(u)$. We have:

$$\begin{aligned}
&\frac{\partial}{\partial\theta_i}\left(\sum_{k,l,j} a_{k,l,j} \frac{1}{(\psi'_\theta(u))^{\frac{k}{2}}} \psi''_\theta(u)^j \psi'''_\theta(u)^l\right) = \\
&\sum_{k,l,j} a_{k,l,j} \left((-\frac{-k}{2}) \frac{1}{(\psi'_\theta(u))^{\frac{k}{2}-1}} \frac{\partial}{\partial\theta_i}(\psi'_\theta(u)) \psi''_\theta(u)^j \psi'''_\theta(u)^l \right. \\
&+ \frac{j}{(\psi'_\theta(u))^{\frac{k}{2}}} \psi''_\theta(u)^{j-1} \frac{\partial}{\partial\theta_i}(\psi''_\theta(u)) \psi'''_\theta(u)^l \\
&\left. + \frac{l}{(\psi'_\theta(u))^{\frac{k}{2}}} \psi''_\theta(u)^j \psi'''_\theta(u)^{l-1} \frac{\partial}{\partial\theta_i}(\psi'''_\theta(u)) \right).
\end{aligned} \tag{1.32}$$

Calculating $\frac{\partial}{\partial\theta_i}(\psi'_\theta(u))$, $\frac{\partial}{\partial\theta_i}(\psi''_\theta(u))$ and $\frac{\partial}{\partial\theta_i}(\psi'''_\theta(u))$ directly from (1.5), (1.6) and (1.7) and replacing that expressions on the right-side of (1.32), we can see that $\frac{\partial}{\partial\theta_i}\left(\sum_{k,l,j} a_{k,l,j} \frac{1}{(\psi'_\theta(u))^{\frac{k}{2}}} \psi''_\theta(u)^j \psi'''_\theta(u)^l\right)$ is uniformly bounded in the set S.□

We recall the definition of holomorphic families of type A.

Definition 1 *Let $U \subset \mathbb{C}$ be a domain. A family of operators $(A_\theta)_{\theta\in U}$ is* **an holomorphic family of type A** *iff*

i) The operators A_θ are all closed over the same domain $Dom(A_{\theta_0})$.

ii) For all $f \in Dom(A_{\theta_0})$ the map $\theta \mapsto A_\theta f$ is holomorphic.

Our next goal is to prove that Δ_θ is an holomorphic family of type A.

From theorem 40, $\Delta : C_c^\infty(Z,E) \to L^2(Z,E)$ is essentially self-adjoint. Abusing of the notation we denote by Δ the differential operator acting on distributions and the self-adjoint operator. We denote by Δ the distributional Laplacian and its self-adjoint extension. We have:

$$Dom(\Delta) = \{f \in L^2(Z,E) : \Delta f \in L^2(Z,E)\}, \tag{1.33}$$

where by $Dom(\Delta)$ we denote the domain where Δ is self-adjoint. Denote by $\mathscr{W}_2(Z,E)$ the closure of $C_c^\infty(Z,E)$ with respect to the norm $||f||_2 := ||f||+||\Delta f||$ for $f \in C_c^\infty(Z,E)$. Recall that we denote $\mathscr{W}_2(Z,E)$, the **second Sobolev space** (see (E.4)). The following proposition is a routine exercise:

Proposition 4 *The following equation holds:*

$$Dom(\Delta) := \{s \in L^2(Z,E) : \Delta_{dist}(s) \in L^2(Z,E)\} = \mathscr{W}_2(Z,E), \tag{1.34}$$

where Δ_{dist} denotes the Laplacian acting on distributions.

Proof:
We prove first that $\mathscr{W}_2(Z,E) \subset Dom(\Delta)$. Let $f \in \mathscr{W}_2(Z,E)$, then, by definition there exists $f_n \in C_c^\infty(Z,E)$, a Cauchy sequence in the norm $||.||_2$. $(C_c^\infty(Z,E), ||.||_2)$ is continuously included in $L^2(X,E)$. Since f_n converges in $||.||_2$, the sequence Δf_n converges in the norm $||.||_{L^2(Z,E)}$ to an element $g \in L^2(Z,E)$. Let $h \in C_c^\infty(Z,E)$, then:

$$\begin{aligned}\langle g,h\rangle_{L^2(Z,E)} &= \lim_{n\to\infty}\langle \Delta f_n, h\rangle_{L^2(Z,E)} = \lim_{n\to\infty}\langle f_n, \Delta h\rangle_{L^2(Z,E)} \\ &= \langle f, \Delta h\rangle_{L^2(Z,E)}.\end{aligned}$$

The previous calculations prove that $\Delta_{dist} f = g \in L^2(Z,E)$, hence $f \in Dom(\Delta)$.

For proving the other inclusion we observe that $C_c^\infty(Z,E) \subset Dom(\Delta)$ and that $Dom(\Delta)$ is closed under the norm $||.||_2$. $\square$

Remark 2 *As an application of theorem 39 in appendix E, if A is a bounded uniformly elliptic differential operator of order two, then, the norm $f \mapsto ||f||_{L^2(Z,E)} + ||Af||_{L^2(Z,E)}$ on $C_c^\infty(Z,E)$ is equivalent to the above defined $||.||_2$. Hence, $Dom(\Delta)$ can be defined as the closure of $C_c^\infty(Z,E)$ with the norm $f \mapsto ||f||_{L^2(Z,E)} + ||Af||_{L^2(Z,E)}$.*

Proposition 5 *For $Re(\theta) > 0$, the operator Δ_θ is an uniformly elliptic operator i.e.*

$$|\sigma_2(\Delta_\theta)^{-1}(z,\xi)|^{-1} \leq |\xi|^{-2}. \tag{1.35}$$

Proof:
The principal symbol of Δ_θ is given by:

$$\sigma_2(\Delta_\theta)((y,u),(\xi_u,\xi_Y)) = a_2(\theta,u)\xi_u^2 + |\xi_Y|^2_{g^{-1}}, \tag{1.36}$$

where $(\xi_u,\xi_Y) \in \mathbb{R}_+ \oplus T^*Y$. Observe that $a_2(\theta,u) = \frac{1}{h(u)\theta+1}$ for $h(u) := (\varphi(u) + \varphi'(u)u) \geq 0$. Since we are taking $Re(\theta) > 0$, $Re(a_2(\theta,u)) > 0$ and we can see that the operator is elliptic because $Re(a_2(\theta,u)\xi_u^2 + |\xi_Y|^2) > 0$ for $\xi \neq 0$. The following calculations show that in fact it is uniformly elliptic:

$$\begin{aligned} |(\sigma_2(\Delta_\theta)(y,\xi))^{-1}| =& \frac{1}{|a_2(\theta,u)\xi_u^2 + |\xi_Y|^2|} = \frac{1}{|\frac{1}{h(u)\theta+1}\xi_u^2 + |\xi_Y|^2|} \\ =& \frac{|h(u)\theta+1|}{|\xi_u^2 + (h(u)\theta+1)|\xi_Y|^2|} \leq \frac{1}{|\xi|^2}.\square \end{aligned} \tag{1.37}$$

We prove in the next proposition that $Dom(\Delta) = \mathscr{W}_2(Z,E)$ is a domain on which Δ_θ is closed:

Proposition 6 *For $Re(\theta) > 0$, the operator $\Delta_\theta : \mathscr{W}_2(Z,E) \to L^2(Z,E)$ is closed.*

Proof:
It is easy to see that Δ_θ is a C^∞-bounded operator (see definition 15 in appendix E). Using this fact and the coordinates and partition of unity as those given by theorem 38 in appendix E, one can prove, as for closed manifolds, that Δ_θ is a bounded operator from $\mathscr{W}_2(Z,E)$ to $L^2(Z,E)$. Hence, if $\varphi \in \mathscr{W}_2(Z,E)$, then $\Delta_\theta\varphi \in L^2(Z,E)$.

For proving that Δ_θ is closed in $\mathscr{W}_2(Z,E)$, consider $\varphi_n \in \mathscr{W}_2(Z,E)$ such that $(\varphi_n, \Delta_\theta\varphi_n)$ converges, in the norm of the Hilbert space $\mathscr{H} \times \mathscr{H}$, to $(\psi,\gamma) \in L^2(Z,E) \times L^2(Z,E)$. Let's denote by $||.||_P$ the norm of the product $\mathscr{H} \times \mathscr{H}$. Then $||\Delta_\theta\varphi_n - \gamma||_{L^2(Z,E)} \leq ||(\varphi_n, \Delta_\theta\varphi_n) - (\psi,\gamma)||_P$ and $||\varphi_n - \psi|| \leq ||(\varphi_n, \Delta_\theta\varphi_n) - (\psi,\gamma)||_P$. Hence φ_n and $\Delta_\theta\varphi_n$ converge in the L^2-norm to ψ and γ respectively. By proposition 5 and theorem 39, the norm $||.||_2$ is equivalent to the norm $\varphi \mapsto ||\varphi|| + ||\Delta_\theta\varphi||$; since φ_n is converging in this last norm, then φ_n converges also in $||.||_2$. Then $\psi \in \mathscr{W}_2(Z,E)$. $\Delta_\theta\psi = \gamma$ because $\Delta_\theta\psi = \Delta_\theta \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \Delta_\theta\varphi_n = \gamma$, where we use the continuity, mentioned above, of Δ_θ as an operator from $\mathscr{W}_2(Z,E)$ to $L^2(Z,E)$. $\square$

Remark 3 *The proof of the previous proposition can be generalized for proving that Δ is closed on $\mathscr{W}_2(Z,E)$.*

We prove next that Δ_θ is weakly holomorphic for $Re(\theta) > 0$. In order to do so we make use of the following lemma that can be found in [4].

Lemma 1 *([4], page 89) Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space. Let I be a non-degenerate (meaning containing more than one point) interval in $\mathbb{R}$, and $f : I \times \Omega \to \mathbb{R}$ be a function with the properties:*

a) $\omega \mapsto f(x,\omega)$ is μ-integrable for each $x \in I$.

b) $x \mapsto f(x,\omega)$ is differentiable on I for each $\omega \in \Omega$, the derivative at x being denoted by $f'(x,\omega)$;

c) There is a μ-integrable function $h \geq 0$ on Ω such that

$$|f'(x,\omega)| \leq h(\omega) \text{ for all } (x, \omega \in I \times \Omega). \tag{1.38}$$

Then the function defined by

$$\varphi(x) := \int f(x,\omega)\mu(d\omega) \tag{1.39}$$

is differentiable for each $x \in I$, the function $\omega \mapsto f'(x,\omega)$ is μ-integrable, and

$$\varphi'(x) = \int f'(x,\omega)\mu(d\omega), \text{ for every } x \in I. \tag{1.40}$$

The following proposition allows us to use the above lemma.

Proposition 7 *Let $N > 0$ be given. Suppose that $f \in Dom(\Delta)$ and $g \in L^2(Z,E)$. Then:*
1) There exists a h_1 in $L^1(Z)$ such that:

$$| < \Delta_\theta(\kappa f)(z), g(z) > | \leq h_1(z), \tag{1.41}$$

for almost all $z \in Z$ and $|\theta| \leq N$ with $Re(\theta) > 0$.
2) There exists a w_1, w_2 in $L^1(Z)$ such that:

$$|\frac{\partial}{\partial \theta_i}(< \Delta_\theta(\kappa f)(z), g(z) >)| \leq w_i(z), \tag{1.42}$$

where $i = 1, 2$ and the inequality holds for almost all $z \in Z$ and $|\theta| \leq N$ with $Re(\theta) > 0$.

Proof:
We begin proving 1). We have:

$$\begin{aligned}|\langle\Delta_\theta(\kappa f),g\rangle(z)| \leq & |\langle a_2(\theta,u)\frac{\partial^2}{\partial u^2}(\kappa f),g\rangle(z)| \\ & + |\langle a_1(\theta,u)\frac{\partial}{\partial u}(\kappa f),g\rangle(z)| \\ & + |\langle a_0(\theta,u)(\kappa f),g\rangle(z)| \\ & + |\langle\Delta_Y(\kappa f),g\rangle(z)|.\end{aligned}$$

Using proposition 3, we have the following inequality:

$$\begin{aligned}|\langle\Delta_\theta(\kappa f),g\rangle(z)| \leq & C\{|\langle\frac{\partial^2}{\partial u^2}(\kappa f),g\rangle(z)| \\ & + |\langle\frac{\partial}{\partial u}(\kappa f),g\rangle(z)| \\ & + |\langle\kappa f,g\rangle(z)|\} \\ & + |\langle\Delta_Y(\kappa f),g\rangle(z)|.\end{aligned}$$

The functions $\frac{\partial^2}{\partial u^2}(\kappa f)$, $\frac{\partial}{\partial u}(\kappa f)$ and $\Delta_Y(\kappa f)$ are in $L^2(Z,E)$. Hence, we can define:

$$\begin{aligned}h(z) := & C\{|\langle\frac{\partial^2}{\partial u^2}(\kappa f),g\rangle(z)| \\ & + |\langle\frac{\partial}{\partial u}(\kappa f),g\rangle(z)| \\ & + |\langle\kappa f,g\rangle(z)|\} \\ & + |\langle\Delta_Y(\kappa f),g\rangle(z)|,\end{aligned}$$

what finishes the proof of the proposition.

For proving 2), we estimate as for 1) and use proposition 3.□

Let $\eta \in C_c^\infty(Z)$ be such that $\eta(z) = 1$ for $z \in Z_{K-2}$ and $\eta(z) = 0$ for $z \in Y \times [K-1,\infty)$. Define $\kappa := 1-\eta$. The next theorem can be deduced from [23], page 21, theorem 5.4. In [2], [27] appear results similar to theorem 1 in other contexts.

Theorem 1 *The family $(\Delta_\theta)_{\theta\in\mathbb{R}_+}$ extends to an analytic family of type A for $Re(\theta) > 0$ i.e.*

i) Δ_θ are closed operators with $Dom(\Delta_\theta)$ independent of θ. More precisely, $Dom(\Delta_\theta) = Dom(\Delta)$.

ii) For every $f \in Dom(\Delta)$ the map $\theta \mapsto \Delta_\theta f$ is analytic for $Re(\theta) > 0$.

Proof: i) is proposition 6.

It is a known result that if a family $(f_\theta)_{\theta\in\Gamma\subset\mathbb{C}}$ in a Hilbert space $\mathscr{H}$ is weak analytic then it is strong analytic (see [22], page 365). We will show then that for $f \in Dom(\Delta_\theta)$ and $g \in L^2(Z,E)$, the function $\theta \mapsto \langle \Delta_\theta f, g\rangle_{L^2(Z,E)}$ is holomorphic. Proposition 7 shows that the conditions of lemma 1 hold. Then,

$$\frac{\partial}{\partial\theta_i}\left(\langle \Delta_\theta f, g\rangle_{L^2(Z,E)}\right) = \int \frac{\partial}{\partial\theta_i}\left(< \Delta_\theta f(z), g(z) >_z\right) dvol(z). \tag{1.43}$$

Since the function $\theta \mapsto < \Delta_\theta f(z), g(z) >_z$, for $z \in Z$ fixed, satisfies the Cauchy-Riemann equation, we finish the proof of the theorem. □

1.4 The essential spectrum of Δ_θ

Our next goal is to prove the equality:

$$\sigma_{ess}(\Delta_\theta) = \bigcup_{i=0}^{\infty}\left(\mu_i + \theta'\,\mathbb{R}_+\right), \tag{1.44}$$

where $\sigma_{ess}(\cdot)$ is defined in (A.2). Equation 1.44 is similar to part 1) of theorem 1.1 in [2], equation 3.2 in [27] and equation 5.19 in [19]. It could be deduced from part 3) of theorem 3.2 of [23]. The first step for proving (1.44) is to prove:

$$N_{ess}(\Delta_\theta) = \bigcup_{i=0}^{\infty}\left(\mu_i + \theta'\,\mathbb{R}_+\right), \tag{1.45}$$

where N_{ess} is defined in appendix D, definition 10. Theorem 36 and equation (1.45) imply

$$\sigma_{ess}(\Delta_\theta) = N_{ess}(\Delta_\theta), \tag{1.46}$$

and hence (1.44).

1.4.1 The perturbation of the operator $-\frac{d^2}{du^2}+\mu$ for $\mu \in \mathbb{R}_+$

For a careful treatment of the next results we refer to [32] (see page 144). Denote by $AC[0,\infty]^2$ the set of functions in $L^2(\mathbb{R}_+)$ whose first derivative is in $AC[0,\infty]$. Let $\mu > 0$. We denote $\Delta^0_{\theta,\mu}$ the operator $-\theta'\frac{d^2}{du^2}+\mu$ with domain $Dom(\Delta^0_{\theta,\mu}) := \{f \in L^2(\mathbb{R}_+) : f \in AC[0,\infty]^2, f(0) = 0\}$. Observe that if $\theta = 0$ then $\theta' = 1$. In particular, $\Delta^0_{0,\mu}$ is self-adjoint with this domain. We have that $\sigma(\Delta^0_{0,\mu}) = [\mu,\infty)$. The following corollary is a consequence of theorem 22.

Corollary 1 *Let $\mu > 0$. For $Re(\theta) > 0$ the following equation holds:*

$$\sigma_{ess}(\Delta^0_{\theta,\mu}) = \mu + \theta'\,\mathbb{R}_+. \tag{1.47}$$

Proof:
By theorem 22, $\lambda \in \sigma_{e2}(\Delta^0_{\theta,\mu})$ if and only if there exists a singular sequence $(f_n)_{n\in\mathbb{N}} \subset Dom(\Delta^0_{\theta,\mu})$ such that

$$\lim_{n\to\infty} ||(\Delta^0_{\theta,\mu} - \lambda)f_n|| = 0.$$

But,

$$0 = \lim_{n\to\infty} ||(\Delta^0_{\theta,\mu} - \lambda)f_n|| = \lim_{n\to\infty} ||(-\theta'\frac{d^2}{u^2} + \mu - \lambda)f_n|| =$$
$$\lim_{n\to\infty} ||(\theta'(-\frac{d^2}{u^2} + \theta'^{-1}\mu - \theta'^{-1}\lambda)f_n||.$$

Hence:

$$\begin{aligned}
&\lim_{n\to\infty} ||(-\frac{d^2}{u^2} + \theta'^{-1}\mu - \theta'^{-1}\lambda)f_n|| \\
&= \lim_{n\to\infty} ||(-\frac{d^2}{u^2} + \mu - \mu + \theta'^{-1}\mu - \theta'^{-1}\lambda)f_n|| \\
&= \lim_{n\to\infty} ||(\Delta^0_{0,\mu} - \mu + \theta'^{-1}\mu - \theta'^{-1}\lambda)f_n|| = 0.
\end{aligned}$$

Hence, using Weyl criterion, $\mu - \theta'^{-1}\mu + \theta'^{-1}\lambda = \mu + \theta'^{-1}(\lambda - \mu) \in \sigma_{e2}(\Delta^0_{0,\mu})$.

We have proved $\lambda \in \sigma_{e2}(\Delta^0_{\theta,\mu})$ if and only if $\mu - \theta'^{-1}\mu + \theta'^{-1}\lambda \in \sigma_{ess}(\Delta^0_{0,\mu})$. Since $\sigma_{ess}(\Delta^0_{0,\mu}) = \mu + \mathbb{R}_+$, we have that for all $s \in \mu + \mathbb{R}_+$ $\lambda = \mu + \theta'(s - \mu) \in \sigma_{ess}(\Delta^0_{\theta,\mu})$. Hence

$$\sigma_{e2}(\Delta^0_{\theta,\mu}) = \mu + \theta'\,\mathbb{R}_+. \tag{1.48}$$

Using the cosine transform we can prove:

$$\sigma(\Delta^0_{\theta,\mu}) = \mu + \theta' \, \mathbb{R}_+. \tag{1.49}$$

Corollary 11 implies $\sigma_{e2}(\Delta^0_{\theta,\mu}) \subset \sigma_{ess}(\Delta^0_{\theta,\mu})$. Hence, (1.48) and (1.49) prove the corollary. □

1.4.2 The inclusion $\bigcup_{i=0}^{\infty} (\mu_i + \theta' \, \mathbb{R}_+) \subset N_{ess}(\Delta_\theta)$

Denote $\Delta^0_{\theta,i} := \Delta^0_{\theta,\mu_i}$. In the following proofs we make use of the following consequence of Rellich theorem.

Lemma 2 *If $(f_n)_{n\in \mathbb{N}} \subset Dom(\Delta^0_{\theta,i}) := \overline{C_c^\infty((a,\infty))}$[1] is such that $\forall n \in \mathbb{N}$, support$f_n \subset K$ for K compact, and there exists a $C > 0$ such that $||f_n||_2 \leq C$ or $||f_n||_1 \leq C$, then f_n has a convergent subsequence in $L^2((a,\infty))$.*

Let $\kappa \in C^\infty(\mathbb{R})$ such that:

$$\kappa(s) = \begin{cases} 0 & \text{for } s \leq R \\ 1 & \text{for } s > R+1. \end{cases} \tag{1.50}$$

As always, κ induces a function on Z that abusing of the notation we will denote also κ.

Proposition 8 *Let $g \in \mathscr{W}_2(\mathbb{R}_+)$ and let $(\phi_i, \mu_i)_{i=0}^\infty$ be a spectral resolution for Δ_Y. Then $\kappa g \phi_i \in Dom(\Delta)$ for all $i \in \mathbb{N}$.*

Proof:
It is easy to see that $\kappa g \phi_i \in L^2(Z,E)$. We prove $\Delta(\kappa g \phi_i) \in L^2(Z,E)$. One has:

$$\Delta(\kappa g \phi_i) = \{-\frac{\partial^2}{\partial u^2}(\kappa) g - 2\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(g) - \kappa \frac{\partial^2}{\partial u^2} g\}\phi_i + \kappa g \Delta_Y(\phi_i). \tag{1.51}$$

It is clear that $-\frac{\partial^2}{\partial u^2}(\kappa) g \phi_i, \kappa \frac{\partial^2}{\partial u^2}(g)\phi_i$ and $\kappa g \Delta_Y(\phi_i)$ belong to $L^2(Z,E)$.

We have that $2\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(g)\phi_i \in L^2(Z,E)$, because $\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}$ is a bounded first order differential operator acting in $C_c^\infty(Z,E)$ and hence induces a continuous map from $\mathscr{W}_2(Z,E)$ to $\mathscr{W}_1(Z,E) \subset L^2(Z,E)$ (see appendix E). We have proved $\Delta(\kappa g \phi_i) \in L^2(Z,E)$. □

[1] We are taking the closure with respect to the second Sobolev norm $||.||_2$ in $\mathbb{R}$.

Let $\lambda \in \bigcup_i (\mu_i + \theta' \mathbb{R}_+)$, then $\lambda \in \sigma_{ess}(\Delta^0_{\theta,i})$ for some $i \in \mathbb{N}$. Hence there exists an orthonormal singular sequence (f_n) associated to λ and $\Delta^0_{\theta,i}$.

We denote $g_n := f_{2n+1} - f_{2n}$. Observe that g_n is also an orthogonal singular sequence associated to λ and $\Delta^0_{\theta,i}$. The reasons for considering g_n instead of f_n will be clarified in the proof of theorem 2.

Lemma 3 *There exists a subsequence s of $\mathbb{N}$ and a $c > 0$ such that*

$$||\kappa g_{s(n)}||_{L^2(\mathbb{R}_+)} > c.$$

Proof:
Suppose such c and s do not exist. Then we can find a subsequence s of $\mathbb{N}$ such that $||\kappa g_{s(n)}|| \to 0$. This implies that $\kappa g_{s(n)} \to 0$ in $L^2([a,\infty))$. From definition 2, one can conclude that $(1-\kappa)g_{s(n)}$ and $(\Delta-\lambda)(1-\kappa)g_{s(n)}$ have uniformly bounded L^2-norms. This fact and theorem 39 prove that $(1-\kappa)g_{s(n)}$ has uniformly bounded 2-Sobolev norm. Hence we can use lemma 2 for proving that $(1-\kappa)g_{s(n)}$ has a convergent subsequence in $L^2([a,\infty))$. Then we could construct a convergent subsequence of $g_n := f_{2n+1} - f_{2n}$, what is a contradiction, because g_n is an orthonormal sequence. □

Recall that ϕ_i is an eigenvector of Δ_Y with eigenvalue μ_i. Using the previous lemma we can suppose without loss of generality that there exists $c > 0$ such that $||\kappa g_n \phi_i||_{L^2(Z,E)} > c > 0$ for all n. Then we have:

Proposition 9 *The sequence $\kappa(u)g_n(u)\phi_i(y)$ has no convergent subsequence.*

Proof:
Suppose $\kappa(u)g_n(u)\phi_i$ has a convergent subsequence s. Then $\kappa(u)g_{s(n)}(u)$ is convergent. By lemma 2, $(1-\kappa(u))g_{s(n)}(u)$ has a convergent subsequence, so g_n has a convergent subsequence which is a contradiction. □

Taking g_n as above, we have:

Proposition 10 *For all $n \in \mathbb{N}$ and $Re(\theta) \geq 0$, $\kappa(u)g_n(u)\phi_i \in Dom(\Delta_\theta)$.*

Proof:
It is a consequence of proposition 8. □

Continuing with the notation of proposition 9, the following theorem is a synthesis of the previous lemmas and propositions.

Theorem 2 *Let $Re(\theta) \geq 0$ and let $(g_n)_{n\in \mathbb{N}}$ be an orthonormal singular sequence associated to λ and $-\theta'\frac{\partial^2}{\partial u^2} + \mu_i$. Then, there exists a subsequence of $h_n := (\kappa g_n \phi_i).||\kappa g_n \phi_i||^{-1}$ that is a singular sequence associated to the operator Δ_θ.*

Proof:
Since we have propositions 9 and 10, it only remains to check $lim_{n\to\infty}||(\Delta_\theta - \lambda)h_n|| = 0$. By lemma 3, $1 < ||\kappa(f_{2n+1} - f_{2n})\phi_i)||^{-1} < C$, then it is enough to prove $(\Delta_\theta - \lambda)\,(\kappa(f_{2n+1} - f_{2n})\phi_i) \to 0$. One has:

$$|| \left(-\theta'\frac{\partial^2}{\partial u^2} + \mu_i - \lambda\right)(\kappa(f_{2n+1} - f_{2n}))\,|| \leq A_n + B_n, \tag{1.52}$$

where A_n and B_n are defined below. A_n is defined and bounded as follows:

$$\begin{aligned} A_n &:= ||\kappa\left(-\theta'\frac{\partial^2}{\partial u^2} + \mu_i - \lambda\right)(f_{2n+1} - f_{2n})|| \\ &\leq C||\left(-\theta'\frac{\partial^2}{\partial u^2} + \mu_i - \lambda\right)(f_{2n+1} - f_{2n})||. \end{aligned} \tag{1.53}$$

For B_n, we have:

$$B_n := ||-\theta'\frac{\partial^2}{\partial u^2}(\kappa)(f_{2n+1} - f_{2n})|| + ||-2\theta'\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(f_{2n+1} - f_{2n})||. \tag{1.54}$$

One sees that A_n tends to 0 from the inequality (1.53) and the fact that f_n is Weyl sequence associated to the value λ and the operator $\frac{d^2}{du^2} + \mu_i$. Observe that the sequences, $\frac{\partial^2}{\partial u^2}(\kappa)f_n$ and $\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(f_n)$, are bounded in the second and first Sobolev norms, and the supports of their elements is contained in some fixed compact subset of $\mathbb{R}_+$. Hence, we can apply lemma 2 that guarantees that there exists a subsequence s of $\mathbb{N}$ such that $\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(f_{s(n)})$ and $\frac{\partial^2}{\partial u^2}(\kappa)f_{s(n)}$ are convergent. This implies that $\lim_{n\to\infty} B_{s(n)} = 0$ for some subsequence s, that is what we need. □

As a consequence of the previous theorem, we have:

$$\bigcup_{i=0}^{\infty}\left(\mu_i + \theta'\,\mathbb{R}_+\right) \subset N_{ess}(\Delta_\theta) \tag{1.55}$$

1.4.3 The inclusion $N_{ess}(\Delta_\theta) \subset \bigcup_{i=0}^{\infty}(\mu_i + \theta'\,\mathbb{R}_+)$

Let $\Delta_\theta^0 := -\theta'\frac{\partial^2}{\partial u^2} + \Delta_Y$ be the operator acting on $L^2(\mathbb{R}_+ \times Y, E)$ with domain given by functions $f \in L^2(Y \times \mathbb{R}_+, E)$ such that its Fourier expansion, $f(y,u) = \sum_{i=0}^{\infty} f_i(u)\phi_i(y)$, is such that $f_i \in Dom(\Delta^0_{\theta,\mu_i})$. We denote

$\Delta^0_{\theta,i} := \Delta^0_{\theta,\mu_i}$. Using the Fourier decomposition of $L^2(Y \times \mathbb{R}_+, E)$ associated to the operator Δ_Y, it is easy to see that

$$\Delta^0_\theta = \sum_{i=0}^{\infty} \Delta^0_{\theta,i}. \tag{1.56}$$

Let $\lambda \in N_{ess}(\Delta_\theta)$ and let $(g_n)_{n\in \mathbb{N}}$ be an orthonormal singular sequence associated to λ and the operator Δ_θ. Denote

$$h_n := g_{2n+1} - g_{2n}. \tag{1.57}$$

The reason for using h_n instead of g_n will be clear in the proof of theorem 3.

Proposition 11 *There exists a subsequence s of $\mathbb{N}$ and a $c > 0$ such that $||\kappa h_{s(n)}|| > c$ and $\kappa h_{s(n)}$ has no convergent subsequence.*

Proof:
Suppose κh_n has a convergent subsequence $\kappa h_{s(n)}$ (that is the case if $\lim_{n\to\infty} ||\kappa h_{s(n)}|| = 0$). Then, $(1-\kappa)h_{s(n)}$ has a convergent subsequence by Rellich theorem. Then we can construct a convergent subsequence of h_n, which is a contradiction. □

By the previous proposition we can suppose without loss of generality that $||\kappa h_n|| > c$ and κh_n has no convergent subsequence. From now on in this subsection we suppose these two things about κh_n.

Proposition 12 *Let $Re(\theta) \geq 0$. If $h \in Dom(\Delta)$ then $\kappa h \in Dom(\Delta^0_\theta)$.*

Proof:
Given $h \in Dom(\Delta)$, we have to prove that for all ϕ_i the function $f_i(u) := \langle(\kappa h)(u,.), \phi_i\rangle_{L^2(Y,E')}$ belongs to $Dom(\Delta^0_{\theta,i})$. The proof of the proposition follows by observing that $f_i \in L^2(\mathbb{R}_+)$, $\frac{d^2}{du^2}(f_i) \in L^2(\mathbb{R}_+)$ and $f_i(0) = 0$. □

The following theorem is a consequence of the previous propositions.

Theorem 3 *Let $Re(\theta) \geq 0$ and let g_n be an orthonormal singular sequence associated to the operator Δ_θ and the value λ. Then, there exists s subsequence of $\mathbb{N}$ such that $\kappa g_{s(n)}$ induces a singular sequence for the operator Δ^0_θ and the value λ.*

Proof
By propositions 11 and 12, the only thing that remains for proving that κg_n is a singular sequence for Δ_θ^0 and the value λ, is to prove $\lim_{n\to\infty} ||(\Delta_\theta^0 - \lambda)(\kappa h_n)|| = 0$. We have:

$$||(\theta' \frac{\partial^2}{\partial u^2} + \Delta_Y - \lambda)(\kappa h_n)|| \leq C(A_n + B_n + C_n), \tag{1.58}$$

where

$$A_n := ||\theta' \frac{\partial^2}{\partial u^2}(\kappa) h_n||, \tag{1.59}$$

$$B_n := ||\theta' \frac{\partial}{\partial u}(\kappa) \frac{\partial}{\partial u}(h_n)|| \tag{1.60}$$

and

$$C_n := ||\kappa(\theta' \frac{\partial^2}{\partial u^2} + \Delta_Y - \lambda)(h_n)||_{L^2(Z,E)} = ||\kappa(\Delta_\theta - \lambda)(h_n)||_{L^2(Z,E)}. \tag{1.61}$$

Since the sequences, $\frac{\partial^2}{\partial u^2}(\kappa)h_n$ and $\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(h_n)$, are bounded in the second and first Sobolev norms, and the supports of their elements are contained in the same compact subset of $Y \times \mathbb{R}_+$, we can apply the general Rellich theorem for Riemannian manifolds and show, using the definition of h_n in (1.57), that A_n and B_n tend to 0 when $n \to \infty$. It is also possible to deduce from lemma 2 that $\frac{\partial^2}{\partial u^2}(\kappa)h_n$ and $\frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(h_n)$ tend to 0, because lemma 2 implies that $\langle \frac{\partial^2}{\partial u^2}(\kappa)h_n, \phi_i\rangle_{L^2(Y,E')}$ and $\langle \frac{\partial}{\partial u}(\kappa)\frac{\partial}{\partial u}(h_n), \phi_i\rangle_{L^2(Y,E')}$ have convergent subsequences.

Since $0 \leq C_n \leq K||(\Delta_\theta - \lambda)g_n||$, we have proved $C_n \to 0$. $\square$

As a consequence of the previous theorem, we have:

Corollary 2 $N_{ess}(\Delta_\theta) \subset \bigcup_{i=0}^{\infty} (\mu_i + \theta' \mathbb{R}_+)$.

Proof:
Using the decomposition $\Delta_\theta^0 = \bigoplus_{\mu\in\sigma(\Delta_Y)}(-\frac{\partial^2}{\partial u_2} + \Delta_Y)$ and the cosine transform, one can prove $\sigma_{ess}(\Delta_\theta^0) = \sigma(\Delta_\theta^0) = \bigcup_{i=0}^{\infty} (\mu_i + \theta' \mathbb{R}_+)$.$\square$

The previous corollary together with corollary 2 prove

$$N_{ess}(\Delta_\theta) = \bigcup_{i=0}^{\infty} \left(\mu_i + \theta' \mathbb{R}_+\right). \tag{1.62}$$

Equation (1.62) together with theorem 36 imply

$$\sigma_{ess}(\Delta_\theta) = N_{ess}(\Delta_\theta), \tag{1.63}$$

as was noticed at the begining of this section.

1.5 The analytic vectors of U_θ

In this section we construct a subset, $\mathscr{V}$, of $L^2(Z, E)$ such that

i) $\mathscr{V}$ is a dense subset of $L^2(Z, E)$.

ii) For $f \in \mathscr{V}$ the function $\theta \mapsto U_\theta f \in L^2(Z, E)$ makes sense for $\theta \in \mathbb{C}$, $Re(\theta) > 0$.

iii) $U_\theta \mathscr{V}$ is dense in $L^2(Z, E)$ for $Re(\theta) > 0$.

The construction we give here is inspired in appendix 2 of [2]. Recall that we denote by $(\phi_i, \mu_i)_{i=1}^\infty$ a spectral resolution of the operator Δ_Y. Let $\kappa \in C^\infty(\mathbb{R}_+)$ be a function satisfying $0 \leq \kappa \leq 1$, $\kappa' \geq 0$ and

$$\kappa(u) = \begin{cases} 1 & K \leq u < \infty. \\ 0 & 1 < u \leq K - 1. \end{cases} \tag{1.64}$$

We extend κ to $Y \times \mathbb{R}_+$ defining $\kappa(y, u) := \kappa(u)$ for $(y, u) \in Y \times \mathbb{R}_+$. Making κ equal to 0 out of its support in $Y \times \mathbb{R}_+$, we extend κ to Z. Define the set $\mathscr{P}$ of elements $h \in L^2(Y \times \mathbb{R}_+, E)$ such that h has a Fourier expansion of the form $h(y, u) = \frac{1}{u^2} \sum_{i=0}^\infty p_i(\frac{1}{u})\phi_i(u)$ where $p_i(x) \in \mathbb{C}[x]$. Define:

$$\mathscr{V} := \{(1 - \kappa)g + \kappa h : g \in L^2(Z, E) \text{ and } h \in \mathscr{P}\}. \tag{1.65}$$

For proving i) and iii) we make use of the Stone-Weierstrass theorem that we enunciate now:

Theorem 4 *([13], page 139)(Complex Stone-Weierstrass theorem) Let X be a compact Hausdorff space. If $\mathscr{B}$ is a closed complex subalgebra of $C(X)$ that separates points and is closed under complex conjugation, then either $\mathscr{B} = C(X)$ or $\mathscr{B} = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$.*

We begin proving that $\mathscr{V}$ satisfies condition i).

Proposition 13 *$\mathscr{V}$ is dense in $L^2(Z, E)$.*

Proof:
Let $g \in L^2(Z,E)$, since $C_c^\infty(Z,E)$ is dense in $L^2(Z,E)$ we can assume without loss of generality that $g \in C_c^\infty(Z,E)$. Consider the following formal reasoning:

$$||g-(1-\kappa)g-\kappa\sum_{i=1}^{\infty}(\frac{1}{u^2}p_i(\frac{1}{u})\phi_i(y))||^2_{L^2(Z,E)} = ||\kappa\sum_{i=0}^{\infty}\left(g_i(u)-\frac{1}{u^2}p_i(\frac{1}{u})\right)\phi_i||^2_{L^2(Z,E)}$$

$$\leq C\sum_{i=1}^{\infty}||g_i(u)-\frac{1}{u^2}p_i(\frac{1}{u})||^2_{L^2([1,\infty))}.$$

Changing variables, we have that:

$$\begin{aligned} ||g_i(u)-\frac{1}{u^2}p_i(\frac{1}{u})||^2_{L^2([1,\infty))} &= \int_1^\infty |u^2 g_i(u)-p_i(u)|^2\frac{1}{u^4}du \\ &\leq \int_0^1 |\frac{1}{u^2}g_i(\frac{1}{u})-p_i(u)|^2u^2du. \end{aligned} \tag{1.66}$$

Since $g_i \in C_c([1,\infty))$ we observe $\frac{1}{u^2}g_i(\frac{1}{u})$ is continuous in $C([0,1])$ and, by Stone-Weierstrass theorem, theorem 4, we can choose $p_i(x) \in \mathbb{C}[x]$ in such a way that $||p_i(u)-\frac{1}{u^2}g_i(\frac{1}{u})||_\infty \leq \frac{\epsilon}{2^i}$. Hence, using (1.66), we have:

$$||g_i(u)-\frac{1}{u^2}p_i(\frac{1}{u})||^2_{L^2([1,\infty))} \leq \frac{\epsilon}{2^i}\int_0^1 u^2du \leq \frac{\epsilon}{2^i}. \tag{1.67}$$

We have proved that for all $\epsilon > 0$ we can find $p_i(x) \in \mathbb{C}[x]$ such that $||g-(1-\kappa)g+\kappa\sum_{i=1}^\infty \frac{1}{u^2}p_i(\frac{1}{u})\phi_i(y)||^2 < \frac{1}{(K-1)^3}\epsilon$. $\square$

The next proposition shows that $\mathscr{V}$ satisfies condition ii).

Proposition 14 *For all $f \in \mathscr{V}$, the map $\theta \mapsto U_\theta f$ has an analytic extension from $\mathbb{R}_+$ to $\theta \in \mathbb{C}$, $Re(\theta) > 0$, taking values in $L^2(Z,E)$.*

Proof:
The equation $U_\theta(\kappa g)(y,u) = \kappa(u)g(y,u)$ for $\theta \in \mathbb{R}_+$ motivates our definition of U_θ for $\theta \in \mathbb{C}-(-\infty,0)$ for $f \in \mathscr{V}$, $f := (1-\eta)g+\kappa\sum_{i=1}^\infty \frac{1}{u^2}p_i(\frac{1}{u})\phi_i(y)$:

$$\begin{aligned} &U_\theta\left((1-\kappa)g+\kappa\sum_{i=1}^{\infty}\frac{1}{u^2}p_i(\frac{1}{u})\phi_i(y))\right) := \\ &(1-\kappa)g+\kappa\sum_{i=1}^{\infty}\frac{1}{\psi_\theta^2(u)}p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}\phi_i(y). \end{aligned} \tag{1.68}$$

It is easy to see that for $f \in \mathscr{V}$, $U_\theta f \in L^2(Z,E)$ and $\theta \mapsto U_\theta f$ is an analytic function. $\square$

We denote by $C_0([1,\infty))$ the completion of the algebra of compactly supported function, $C_c([1,\infty))$, under the supremum norm, $||.||_\infty$. For $Re(\theta) \geq 0$, let $\mathscr{B}_\theta$ be the algebra obtained as the topological closure of the algebra $\{p(\frac{1}{\psi_\theta(u)}) : p(x) = xq(x) \text{ where } q(x) \in \mathbb{C}[x], q \neq 0\}$ in $C_0([1,\infty)), ||.||_\infty$. For proving that $U_\theta \mathscr{V}$ is dense in $L^2(Z,E)$ we will use the following lemma.

Lemma 4 *For $Re(\theta) \geq 0$, $\mathscr{B}_\theta$ is dense in $(C_0([1,\infty)), ||.||_\infty)$.*

Proof:
Let $\Theta : [1,\infty) \to \mathbb{C}$ be the function defined by $\Theta(u) := \frac{1}{\psi_\theta(u)}$ for $u \in [1,\infty)$. Observe that Θ is an homeomorphism between $[1,\infty)$ and $X := \Theta([1,\infty))$. We denote $[1,\infty] := [1,\infty) \cup \{\infty\}$ the one point compactification of $[1,\infty)$; and we denote by $\tilde{X}$ the one point compactification of X; $\tilde{X}$ is a curve in $\mathbb{C}$ homeomorphic to $[1,\infty]$ with initial point 1 and final point 0 (we are supposing $1 < K$).

Let $f \in C([1,\infty))$ and $p \in \mathbb{C}[x]$, $p(x) := q(x)x$ where $q \in \mathbb{C}[x]$, $q \neq 0$. We have:

$$\begin{aligned} ||f - p \circ \Theta||_\infty &= \sup_{v \in [1,\infty)} |f(v) - p \circ \Theta(v)| = \sup_{w \in X} |f(\Theta^{-1}(w)) - p(w)| \\ &= \sup_{w \in \tilde{X}} |f(\Theta^{-1}(w)) - wq(w)|. \end{aligned} \tag{1.69}$$

From the previous calculation, we deduce that, for proving the lemma, it is enough to prove that $x\mathbb{C}[x]$ is dense in $W := \{f \in C(\tilde{X}) : f(0) = 0\}$.

We have that $x\mathbb{C}[x] \subset W$; $x\mathbb{C}[x]$ separates points because, if $\lambda_1, \lambda_2 \in \tilde{X}$, $\lambda_1 \neq \lambda_2$, then $x \in xC[x]$ separates them. It is also obvious that $x\mathbb{C}[x]$ is closed under conjugation. Now we can apply theorem 4. Since the topological closure of $x\mathbb{C}[x]$, $\overline{x\mathbb{C}[x]}$, is not equal to $C(\tilde{X})$, we can conclude $\overline{x\mathbb{C}[x]} = W$. Observing that $f(\Theta^{-1}(w)) \in W$, calculation (1.69) finishes the proof of the lemma. $\square$

The following corollary proves condition iii).

Corollary 3 *For $Re(\theta) > 0$, $U_\theta \mathscr{V}$ is dense in $L^2(Z,E)$.*

Proof:
We have to prove that the set

$$\{(1-\kappa)g + U_\theta(\kappa h) : g \in L^2(Z,E) \text{ and } h \in \mathscr{P}\},$$

where h has the Fourier expansion $h(y,u) = \frac{1}{u^2}\sum_{i=0}^{\infty} p_i(\frac{1}{u})\phi_i(y)$, is dense in $L^2(Z,E)$. Observe that

$$U_\theta(\kappa h)(y,u) = \kappa \sum_{i=1}^{\infty} \frac{1}{\psi_\theta(u)^2} p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}\phi_i(y).$$

Let $g \in C_c^\infty(Z,E)$. We make the following formal reasoning:

$$||g - (1-\kappa)g + \kappa \sum_{i=1}^{\infty} \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta(u)})\psi_\theta'^{\frac{1}{2}}(u)\phi_i(y)||^2_{L^2(Z,E)} =$$

$$||\kappa \sum_{i=1}^{\infty} \left(g_i(u) - \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}\phi_i(y) \right) ||^2_{L^2(Z,E)} \leq$$

$$C \sum_{i=1}^{\infty} ||g_i(u) - \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}||^2_{L^2([1,\infty))}.$$

By the previous calculations, we prove the corollary if we show that we can choose $p_i \in \mathbb{C}[X]$ such that $||g_i(u) - \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}||_{L^2([1,\infty))}$ is as small as we want. We have that,

$$\begin{aligned} &||g_i(u) - \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta(u)})\psi'_\theta(u)^{\frac{1}{2}}||^2_{L^2([1,\infty))} = \\ &\int_1^\infty |\psi_\theta^{-2}(u)\psi'_\theta(u)|.|\psi_\theta(u)\psi'_\theta(u)^{-\frac{1}{2}}(u)g_i(u) - \frac{1}{\psi_\theta^2(u)} p_i(\frac{1}{\psi_\theta^2(u)})|^2 du. \end{aligned} \tag{1.70}$$

Since $\psi_\theta(u)\psi'_\theta(u)^{-\frac{1}{2}}(u)g_i(u) \in C_c([1,\infty))$, applying lemma 4, we can choose p_i such that:

$$\sup_{u\in[1,\infty)} |\psi_\theta(u)\psi'_\theta(u)^{-\frac{1}{2}}(u)g_i(u) - \frac{1}{\psi_\theta(u)} p_i(\frac{1}{\psi_\theta(u)})|^2 \leq \frac{\epsilon}{2^i}. \tag{1.71}$$

Furthermore $|\psi_\theta^{-2}(u)\psi'_\theta(u)| \in L^1([1,\infty))$, hence

$$\begin{aligned} &\int_1^\infty |\psi_\theta^{-2}(u)\psi'_\theta(u)| \cdot |\psi_\theta^{-2}(u)\psi'_\theta(u)^{-\frac{1}{2}}(u)g_i(u) - -\frac{1}{\psi_\theta(u)} p_i(\frac{1}{\psi_\theta(u)})|^2 du \\ &\leq \int_1^\infty |\psi_\theta^{-2}(u)\psi'_\theta(u)| du \cdot \left(\frac{\epsilon}{2^i}\right) \leq C\frac{\epsilon}{2^i}. \end{aligned} \tag{1.72}$$

We have proved the corollary. □

The following lemma shows that the L^2-eigenfunctions of Δ can be included in the set of analytic vectors.

Lemma 5 *Let φ be a L^2-eigenfunction of Δ with eigenvalue λ. The function $\theta \mapsto U_\theta \varphi$ extends meromorphically to $\theta \in \mathbb{C}$ such that $Re(\theta) > 0$.*

Proof:
For $(y,u) \in Y \times \mathbb{R}_+$, $\varphi(y,u) = \sum_{j=0}^{\infty} f_j(u)\phi_k(y)$, where $\{(\phi_k, \mu_k)\}_{k=0}^{\infty}$ is an spectral resolution of Δ_Y. Then, since $\Delta\varphi = \lambda\varphi$:

$$\varphi(y,u) = \sum_{\mu_k > \lambda}^{\infty} a_k e^{-\sqrt{\mu_k - \lambda}u}\phi_k(y).$$

For $\theta \in \Gamma$, we can define:

$$U_\theta \varphi(y,u) := \begin{cases} \varphi(z) & z \in Z_0. \\ \psi'_\theta(u)^{\frac{1}{2}} \sum_{k=0}^{\infty} a_k e^{-\sqrt{\mu_k - \lambda}\psi_\theta(u)}\phi_k(y) & z = (y,u) \in Y \times \mathbb{R}_+. \end{cases}$$

One can see that for $Re(\theta) > 0$, $U_\theta f \in C^\infty(Z,E) \cap L^2(Z,E)$. □

1.6 Consequences of Aguilar-Balslev-Combes theory

In section 1.4 we calculated the essential spectrum of Δ_θ. In this section we complete the description of $\sigma(\Delta_\theta)$ giving some information about $\sigma_d(\Delta_\theta)$ and $\sigma_{pp}(\Delta_\theta)$; most of the results that we compile here are consequences of the Aguilar-Balslev-Combes theory explained in appendix B. We have seen in the preceding sections that the dilation family $U_\theta, \mathscr{V}$ for the operator Δ satisfies assumptions 1), 2), 3) of appendix B; in this section we write the consequences of 1), 2), 3) of appendix B for the spectrum of Δ_θ, particularly for $\sigma_d(\Delta_\theta)$ and $\sigma_{pp}(\Delta_\theta)$, for $\theta \in \Gamma$, since $\sigma_{ess}(H_\theta)$ was already described in section 1.4 as was mentioned before.

The next theorem synthesizes the consequences of 1), 2) and 3) of appendix B for $\sigma(\Delta_\theta)$, $\theta \in \Gamma$. It should be compared theorem 1.1 in [2], theorem 16.4 in [17], with theorem 11 in this thesis, and the theorem in page 14 of [27].

Theorem 5 *We have:*

a) The set of non-threshold eigenvalues[2] *of* Δ *is equal to* $\sigma_d(\Delta_\theta) \cap \mathbb{R}$*, for all* $\theta \in \Gamma - \mathbb{R}_+$*. Moreover, given* λ_0 *non-threshold eigenvalue, the eigenspace* $E_{\lambda_0}(\Delta)$*, associated to* Δ *and* λ_0*, has finite dimension bounded by the degree of the pole* λ_0 *of the map* $\lambda \mapsto R(\lambda, \theta)$*. This algebraic multiplicity is independent of* $\theta \in \Gamma - \mathbb{R}_+$*.*

b) Fix $\theta \in \Gamma$*. For* $f, g \in \mathscr{V}$ *the function*

$$\lambda \mapsto \langle R(\lambda) f, g \rangle_{L^2(Z,E)}$$

has a meromorphic continuation from Λ *to* $\mathbb{C} - \big(\sigma_{ess}(\Delta_\theta) \cup \sigma_d(\Delta_\theta)\big)$*, where* $\sigma_{ess}(\Delta_\theta)$ *was calculated in 1.4.*

c) Δ *has no singular spectrum.*

d) Let $\theta_1, \theta_2 \in \Gamma$ *be such that* $arg(\theta_1') \geq arg(\theta_0')$ *for* $0 < arg(\theta_i') < \frac{\pi}{2}$*, we have:*

$$\sigma_d(\Delta_{\theta_0}) = \sigma_d(\Delta_{\theta_1}) \cap \sigma_d(\Delta_{\theta_0}). \tag{1.73}$$

e) Non-thresholds eigenvalues of Δ *are isolated (respect to the eigenvalues of* Δ*) and, in case they accumulate, they accumulate on the set of thresholds or on* ∞[3]

f) If the lowest eigenvalue, μ_0*, of* Δ_Y *is larger than* 0 *then* $\sigma_d(\Delta)$ *is a discrete subset of* $[0, \mu_0)$*. The unique possible accumulation point of* $\sigma_d(\Delta)$ *is* γ_0*. If* $\mu_0 = 0$*, then* $\sigma_d(\Delta) = \emptyset$*; in other words, all eigenvalues are embedded in the continuous spectrum.*

The next proposition pretends to give a little more information about $\sigma_{pp}(\Delta_\theta)$ for arbitrary $\theta \in \Gamma$. It is easy to prove from the definition of essential spectrum and from the form of $\sigma_{ess}(\Delta_\theta)$ (see section 1.4).

Proposition 15 *i) If* $\lambda \in \sigma_{pp}(\Delta_\theta)$ *and* $\lambda \notin \sigma_{ess}(\Delta_\theta)$*, then* λ *an isolated eigenvalue of finite multiplicity.*

ii) For $Re(\theta) > 0$*,* $\sigma_{pp}(\Delta_\theta)$ *accumulates in* $\sigma_{ess}(\Delta_\theta)$*. In particular, the real part of the pure point spectrum of* Δ_θ *accumulates only in* $\sigma(\Delta_Y)$*.*

Next we show that the unique possible accumulation point of $\sigma_{pp}(\Delta)$ is ∞. For that we use the following theorem of [9].

[2] The set of thresholds of Δ, $\tau(\Delta)$, is equal to $\sigma(\Delta_Y)$

[3] We prove in corollary 4 that ∞ is the unique possible accumulation point.

Theorem 6 *([9],pag. 352) If $N(\lambda)$ denotes the number of eigenvalues of Δ which are less than λ, then one has*

$$N(\lambda) \leq C\lambda^{m-\frac{1}{2}}. \tag{1.74}$$

In [9] the previous theorem is proved only for the Laplacian Δ acting on functions, but it generalizes easily to our context. As we have previously said theorem 6 implies the following corollary.

Corollary 4 *The unique possible accumulation point of $\sigma_{pp}(\Delta)$ is ∞.*

We recall some facts about the analytic extension of the resolvent, $R(\lambda)$, of Δ. First, we introduce some notation. Let Σ be the Riemann surface on which the functions $\sqrt{z-\mu_i}$ are defined. Observe that Σ is a ω-covering of $\mathbb{C}$, with ramification points $\{\mu_i : i \in \mathbb{N}\}$. Denote:

$$L^2{}_\delta(Z,E) := \\ \{\varphi : Z \to E : \text{ measurable section s.t. } \int_0^\infty \int_Y |\varphi|^2 e^{2\delta u} dvol(y) du < \infty\}. \tag{1.75}$$

In [21] the resolvent is extended as a function of $\lambda \in \Sigma$ taking values in the bounded operators from $L^2_{-\delta}(Z,E)$ to $L^2_\delta(Z,E)$.

We define resonances in the following way:

$$\mathscr{R}_\theta(\Delta) = \{\lambda \in \sigma_d(\Delta_\theta) : \lambda \notin \sigma_{pp}(\Delta)\}. \tag{1.76}$$

The parameter θ is simply uncovering new pure point spectrum in the sense of the following proposition.

Proposition 16 *Suppose $\theta_1, \theta_0 \in \Gamma$ and $0 < arg(\theta_0) < arg(\theta_1) < \frac{\pi}{2}$. Then*

$$\mathscr{R}_{\theta_0}(\Delta) \subset \mathscr{R}_{\theta_1}(\Delta). \tag{1.77}$$

Proof:
The proposition is a consequence of the fact that for $f, g \in \mathscr{V}$, for $Re(\lambda) < 0$,

$$\langle R(\lambda)f, g\rangle_{L^2(Z,E)} = \langle R(\lambda,\theta_0)U_{\theta_0}f, U_{\overline{\theta}_0}g\rangle_{L^2(Z,E)} = \langle R(\lambda,\theta_1)U_{\theta_1}f, U_{\overline{\theta}_1}g\rangle_{L^2(Z,E)}. \square$$

There is a natural version of the previous proposition for $\theta_1, \theta_0 \in \Gamma$ and $0 > arg(\theta_0) > arg(\theta_1) > -\frac{\pi}{2}$. The next theorem provides more information about the resonances of Δ.

Theorem 7 *Suppose that $\lambda \in \mathscr{R}_\theta(\Delta)$. Then:*

1) *Suppose $Im(\lambda) \neq 0$. Then, if $\lambda \notin \bigcup_{i\in \mathbb{N}} \mu_i + \theta' \mathbb{R}_+$ then $\lambda \in \sigma_d(\Delta_\theta)$. Under these conditions, if $0 < arg(\theta') < \frac{\pi}{2}$, then $Im(\lambda) > 0$; if $0 > arg(\theta') > -\frac{\pi}{2}$, then $Im(\lambda) < 0$.*

2) *There are not real resonances different than the set of thresholds $\tau(\Delta_\theta) = \sigma(\Delta_Y)$. If $\lambda = \mu_i$ for some $i \in \mathbb{N}$, then the resolvent, as a function from $L^2_\delta(Z,E)$ to $L^2_{-\delta}(Z,E)$, has a pole of at most second order. In fact, μ_i is a pole of second order always that it is a L^2-eigenvalue of Δ; in this case the leading part of the Laurent expansion is the orthogonal projection in the L^2-eigenspace space E_{μ_i}.*

Proof:
1) follows from the definition of σ_{ess} and (1.44). 2) is proved in theorem 3.26 of [21]. □

As a consequence of the previous theorem, we have the following corollary that completes the description of the spectrum of Δ of theorem 5.

Corollary 5 *The real resonances of Δ are contained in $\sigma(\Delta_Y)$.*

1.7 Δ_θ are m-sectorial

The theory of m-sectorial operators and forms that we use in this section is described in appendix C. Our goal in this section is to prove that the operators Δ_θ, for $\theta \in \Gamma$ (see 1.81), are m-sectorial (see definition 7). In the thesis, this result will be important when we calculate $\sigma_{ess}(H_\theta)$ for proving theorem 10.

Let $\eta_0 \in C^\infty(\mathbb{R}_+)$ be a positive real function such that $\eta_0(u) = 1$ for $u < 1$, $\eta_0(u) = 0$ for $u \in [K-1, \infty)$, and $\eta_0'(u) \leq 0$ for $u \in [1, K-1]$, where we are considering $K > 1$. Let $\eta_1 := 1 - \eta_0$. Both η_0 and η_1 induce functions on $Y \times \mathbb{R}_+$, defining $\eta_k(u,y) := \eta_k(u)$ for $k = 0, 1$; making η_k equal to 0 where it is not defined, we can extend it to all of Z. In this way we think η_0 and η_1 as functions in $C^\infty(Z)$. The proof of the next proposition is inspired in page 280, example 3.34 of [22].

Proposition 17 *For $Re(\theta) > 0$ there exist a $\gamma \geq 0$ such that*

$$\begin{aligned} Re\left(\langle a_0(\theta,u)f\ ,\eta_1 f\rangle_{L^2(Z,E)} + \gamma\langle f,f\rangle_{L^2(Z,E)}\right) \\ \geq |Im(\langle a_0(\theta,u)f, \eta_1 f\rangle_{L^2(Z,E)})| \end{aligned} \tag{1.78}$$

for all $f \in L^2(Z, E)$.

Proof:
Observe that, by definition, $\eta_0 + \eta_1 = 1$. Let $\gamma \geq 0$, then, for all $z \in Z$:

$$\begin{aligned} Re\,(\langle a_0(\theta,u)f, \eta_1 f\rangle(z) + \gamma\langle f,f\rangle(z)) - |Im(\langle a_0(\theta,u)f, \eta_1 f\rangle(z))| \\ \geq \eta_1(z)\langle f,f\rangle(z)\{Re(a_0(\theta,u)) + \gamma - |Im(a_0(\theta,u))|\} \\ + \gamma\eta_0(z)\langle f,f\rangle(z) \\ \geq \eta_1(z)\langle f,f\rangle(z)\{Re(a_0(\theta,u)) - |Im(a_0(\theta,u))| + \gamma\} \\ + \gamma\eta_0\langle f,f\rangle(z). \end{aligned} \tag{1.79}$$

Notice that in the previous calculations the inner product denote the Hermitian product in the fiber E_z. Since, for all $z \in Z$ $\eta_1(z)\langle f,f\rangle(z)$ and $\gamma\eta_0(z)\langle f,f\rangle(z)$ are both equal or larger than 0, then, it is enough to prove that there exist $\gamma > 0$ such that

$$(Re(a_0(\theta,u)) - |Im(a_0(\theta,u))| + \gamma \geq 0 \tag{1.80}$$

for all $u \in \mathbb{R}_+$. This is true because $\{a(\theta,u) : u \in \mathbb{R}_+\}$ is a compact subset of $\mathbb{R}^2$ (for θ fixed), any compact subset of $\mathbb{R}^2$ is inside a cube $[-n,n]^2$, and we can always find a N such that $[N-n, n+N]^2$ is inside a cone, with slope 1, included in a right-half-plane. □

Define the set

$$\Gamma := \{\theta := \theta_0 + i\theta_1 \in \mathbb{C} : \theta_0 > 0, \theta_0 \geq |\theta_1| \text{ and } \theta_1^2 < \frac{1}{2}\}. \tag{1.81}$$

The next is a sketch of Γ:

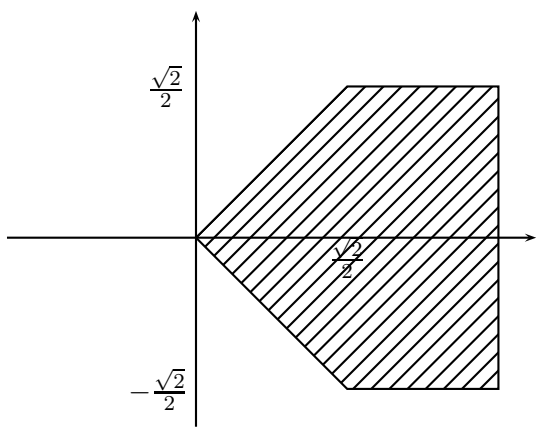

Figure 1. Sketch of Γ

The next theorem shows that the operators Δ_θ are m-sectorial for $\theta \in \Gamma$. We will use this fact when we calculate the essential spectrum of H_θ, a compatible Laplacian on a manifold with corner of codimension 2, in section 2.5.

Theorem 8 *For $\theta \in \Gamma$ there exists a $\gamma(\theta) \in \mathbb{R}_+$ such that the form with domain $\mathscr{W}_1(Z,E)$ defined by $f \mapsto \langle \Delta_\theta f, f\rangle_{L^2(Z,E)} + \gamma(\theta)\langle f, f\rangle_{L^2(Z,E)}$ is m-sectorial.*

Proof:
We prove that there exist $k > 0$ and $\gamma \in \mathbb{R}$ such that for all $f \in \mathscr{W}_2(Z,E)$:

$$Re(\langle \Delta_\theta f, f\rangle_{L^2(Z,E)} + \gamma\langle f, f\rangle_{L^2(Z,E)}) \geq k|Im\left(\langle \Delta_\theta f, f\rangle_{L^2(Z,E)}\right)|. \tag{1.82}$$

Observe that having the previous inequality, the theorems 31 and 32 imply that the form $f \mapsto \langle \Delta_\theta f, f\rangle_{L^2(Z,E)} + \gamma\langle f, f\rangle_{L^2(Z,E)}$ is strictly m-sectorial; and hence the form defined by Δ_θ is m-sectorial.

We prove the inequality 1.82. Since $f \in \mathscr{W}_1(Z,E)$ and by proposition 2, we have:

$$\begin{aligned}\langle \eta_0 \Delta_\theta(f), f\rangle_{L^2(Z,E)} &= \langle \nabla(f), f\frac{\partial}{\partial u}(\eta_0)du\rangle_{L^2(Z,E\otimes T^*Z)} \\ &\qquad + \langle \nabla(f), \eta_0\nabla(f)\rangle_{L^2(Z,E\otimes T^*Z)} \\ &= \langle \frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}(\eta_0)f\rangle_{L^2(Z,E)} + \langle \nabla(f), \eta_0\nabla(f)\rangle_{L^2(Z,E\otimes T^*Z)}.\end{aligned} \tag{1.83}$$

Let $a_2(\theta,u)\frac{\partial^2}{\partial u^2} + a_1(\theta,u)\frac{\partial}{\partial u} + a_0(\theta,u) + \Delta_Y$ be the local expression of the operator Δ_θ (see equation (1.23) for the precise expressions of a_i, $i = 0,1,2$). We have:

$$\begin{aligned}&\langle \eta_1 \Delta_\theta(f), f\rangle_{L^2(Z,E)} = \langle -\frac{\partial^2}{\partial u^2}(f), \overline{a}_2\eta_1 f\rangle_{L^2(Z,E)} \\ &+ \langle \frac{\partial}{\partial u}(f), \overline{a}_1\eta_1 f\rangle_{L^2(Z,E)} + \langle f, \overline{a}_0(u)\eta_1(f)\rangle_{L^2(Z,E)} + \langle \eta_1\Delta_Y(f), f\rangle_{L^2(Z,E)} \\ &= \langle \frac{\partial}{\partial u}(f), \frac{\partial}{\partial u}\left(\overline{a}_2\eta_1\right)f\rangle_{L^2(Z,E)} + \langle \frac{\partial}{\partial u}(f), \left(\overline{a}_2\eta_1\right)\frac{\partial}{\partial u}(f)\rangle_{L^2(Z,E)} \\ &+ \langle \frac{\partial}{\partial u}(f), \overline{a}_1\eta_1 f\rangle_{L^2(Z,E)} + \langle f, \overline{a}_0(u)\eta_1 f\rangle_{L^2(Z,E)} + \langle \eta_1\Delta_Y(f), f\rangle_{L^2(Z,E)}.\end{aligned} \tag{1.84}$$

Using calculations (1.83) and (1.84):

$$\begin{aligned}
&\langle \eta_0 \Delta_\theta(f), f\rangle_{L^2(Z,E)} + \langle \eta_1 \Delta_\theta(f), f\rangle_{L^2(Z,E)} \\
&= \langle \frac{\partial}{\partial u}(f), \left(\frac{\partial}{\partial u}(\eta_0) + \frac{\partial}{\partial u}\left(\overline{a}_2 \eta_1\right) + \overline{a}_1 \eta_1 \right) f\rangle_{L^2(Z,E)} \\
&\qquad + \langle \frac{\partial}{\partial u}(f), \left(\overline{a}_2 \eta_1\right) \frac{\partial}{\partial u}(f)\rangle_{L^2(Z,E)} \\
&\qquad + \langle \nabla(f), \eta_0 \nabla(f)\rangle_{L^2(Z,E)} \\
&\qquad + \langle f, \overline{a}_0 \eta_1 f > + \langle \eta_1 \Delta_Y(f), f\rangle_{L^2(Z,E)}.
\end{aligned} \tag{1.85}$$

Observe that

$$\begin{aligned}
\langle \nabla(f), \eta_0 \nabla(f)\rangle_{L^2(Z,E)} = &\int_{Z_0} \langle \nabla(f), \eta_0 \nabla(f)\rangle(z) dz \\
&+ \int_{Y\times \mathbb{R}_+} \langle \nabla_Y(f), \eta_0 \nabla_Y(f)\rangle(z) dz \\
&+ \int_{Y\times \mathbb{R}_+} \langle \frac{\partial}{\partial u}(f), \eta_0 \frac{\partial}{\partial u}(f)\rangle dz.
\end{aligned} \tag{1.86}$$

We have that the term

$$\begin{aligned}
s(f) := &\int_{Z_0} \langle \nabla(f), \eta_0 \nabla(f)\rangle(z) dz + \int_{Y\times \mathbb{R}_+} \langle \nabla_Y(f), \eta_0 \nabla_Y(f)\rangle(z) dz \\
&+ \langle \eta_1 \Delta_Y(f), (f)\rangle_{L^2(Z,E)}
\end{aligned} \tag{1.87}$$

is greater or equal to 0.

Define the two form $h(\theta)$, by:

$$\begin{aligned}
h(\theta)(f) :&= \langle \Delta_\theta f, f\rangle - s(f) - \langle f, a_0 \eta_1 f\rangle_{L^2(Z,E)} \\
&= \langle \frac{\partial}{\partial u}(f), \left(\frac{\partial}{\partial u}(\eta_0) + \frac{\partial}{\partial u}\left(\overline{a}_2(\theta) \eta_1\right) + \overline{a}_1(\theta) \eta_1 \right) f\rangle_{L^2(Z,E)} \\
&\qquad + \langle \frac{\partial}{\partial u}(f), \left(\overline{a}_2(\theta) \eta_1\right) \frac{\partial}{\partial u}(f)\rangle_{L^2(Z,E)} \\
&\qquad + \int_{Y\times \mathbb{R}_+} \langle \frac{\partial}{\partial u}(f), \eta_0 \frac{\partial}{\partial u}(f)\rangle(z) dz.
\end{aligned} \tag{1.88}$$

By theorem 33, proposition 17 and definition of $h(\theta)$, it only remains to prove that there exist $\gamma > 0$ and $k > 0$ that satisfy

$$Re(h(\theta)f) + \gamma \langle f, f\rangle_{L^2(Z,E)} \geq k|Im(h(\theta)f)|. \tag{1.89}$$

Observe that $\frac{\partial}{\partial u}(\eta_0)+\frac{\partial}{\partial u}(a_2(\theta)\eta_1)+a_1(\theta)\eta_1$ has support on $\mathbb{R}_+$ (as a function of u, θ fixed) and it is bounded there by a constant C (see proposition 3). Then:

$$\begin{aligned}
&Re(h(\theta)(f))-k|Im(h(\theta)(f))|\geq\\
&\qquad\int_{Y\times\mathbb{R}_+}\{(Re(a_2(\theta))-k|Im(a_2(\theta))|)\eta_1+\eta_0\}\langle\frac{\partial}{\partial u}f,\frac{\partial}{\partial u}f)\rangle(z)dz\\
&\qquad-C\int_{Y\times[0,\infty]}\langle|\frac{\partial}{\partial u}f|,|f|\rangle(z)dz.
\end{aligned}$$

Observe that $Re(a_2)\eta_1=\frac{1}{2}Re(a_2)\eta_1+\frac{1}{2}Re(a_2)-\frac{1}{2}Re(a_2)\eta_0$. Using the fact that for all ϵ,

$$\epsilon^2|u|^2+\frac{1}{4}\epsilon^2|v|^2\geq|u||v|,$$

we have for all $k\in\mathbb{R}_+$:

$$\begin{aligned}
&Re(h(\theta)(f))-k|Im(h(\theta)(f))|\geq\\
&\qquad\int_{Y\times\mathbb{R}_+}(\frac{1}{2}Re(a_2(\theta))-k|Im(a_2(\theta))|)\eta_1\langle\frac{\partial}{\partial u}(f),\frac{\partial}{\partial u}(f)\rangle(z)dz\\
&\qquad+\int_{Y\times[0,\infty]}\big(\frac{1}{2}Re(a_2(u,\theta))-C\epsilon^2\big)\langle\frac{\partial}{\partial u}(f),\frac{\partial}{\partial u}(f)\rangle(z)dz\\
&\qquad+\int_{Y\times[0,\infty]}\big(1-\frac{1}{2}Re(a_2(u,\theta))\big)\eta_0\langle\frac{\partial}{\partial u}(f),\frac{\partial}{\partial u}(f)\rangle(z)dz\\
&\qquad-\frac{C}{4\epsilon^2}\int_{Y\times[0,\infty]}\langle f,f\rangle(z)dz.
\end{aligned}\tag{1.90}$$

We recall that $a_2(\theta,u)=\frac{1}{\psi'_\theta(u)^2}$. Observe that ψ'_θ is bounded and $\psi'_\theta(u)=B(u)\theta+1$ where $B(u):=\varphi'(u)u+\varphi(u)\geq 0$ for all $u\in\mathbb{R}_+$. For $Re(\theta)>0$ and $Im(\theta)^2<\frac{1}{2}$ (as in the hypothesis), we have that

$$(B(u)Re(\theta)+1)^2-Im(\theta)^2\geq 1-Im(\theta)^2>\frac{1}{2}.\tag{1.91}$$

Then, there exists a constant $C_0>0$ such that

$$\begin{aligned}
&Re(a_2(u,\theta))=\frac{Re(\overline{\psi'_\theta}^2(u))}{|\psi'_\theta(u)|^4}\\
&=\frac{(B(u)Re(\theta)+1)^2-Im(\theta)^2}{|\psi'_\theta(u)|^4}\geq\frac{1}{2(max\{u\in\mathbb{R}_+:|\psi'_\theta(u)|\}^4}>C_0>0
\end{aligned}\tag{1.92}$$

for all $u \in \mathbb{R}_+$. Hence, we can find ϵ such that

$$(\frac{1}{2}Re(a_2(\theta,u)) - C\epsilon^2) > 0. \tag{1.93}$$

Now we show that for all $Re(\theta) \in \Gamma$ there exists k such that $Re(\psi'_\theta(u)^2) - k|Im(\psi'_\theta(u)^2| \geq 0$, for all $u \in \mathbb{R}_+$. Observe that $\psi'_\theta(u)^2 = B(u)^2\theta^2 + 2B(u)\theta + 1$. Suppose that $\theta := \theta_0 + i\theta_1$, for θ_0 and θ_1 real numbers. Denote by

$$M := \max\{B(u) : u \in \mathbb{R}_+\} = \max\{\varphi(u)u + \varphi'(u) : u \in \mathbb{R}_+\}. \tag{1.94}$$

Recall that for $\theta \in \Gamma$, $\theta_0^2 - |\theta_1|^2 \geq 0$, then:

$$\begin{aligned} &Re(\psi'_\theta(u)^2) - k|Im(\psi'_\theta(u)^2)| = \\ &\quad B(u)^2\{\theta_0^2 - |\theta_1|^2\} + 2B(u)\theta_0 + 1 - k|2B(u)^2\theta_1\theta_0 + 2B(u)\theta_1|\} \\ &\quad \geq 1 - k\{M^2|\theta_0\theta_1| + 2M|\theta_1|\}. \end{aligned} \tag{1.95}$$

The previous calculations show that, for fixed $\theta \in \Gamma$, we can always find a k such that

$$Re(\psi'_\theta(u)^2) - k|Im(\psi'_\theta(u)^2| \geq 0, \tag{1.96}$$

for all $u \in \mathbb{R}_+$. Finally, observe that:

$$1 - \frac{1}{2}Re(a_2(\theta,u)) = 1 - \frac{1}{2}\frac{Re(\overline{\psi'_\theta}^2(u))}{|\psi'_\theta(u)|^4} \geq 0, \tag{1.97}$$

because $2|\psi'_\theta(u)|^4 - Re(\overline{\psi'_\theta}^2(u)) \geq 0$. This last inequality is true because, for all $a \in \mathbb{C}$, $2|a|^2 \geq Re(\overline{a})$ (take $a := \psi'^2_\theta(u)$).

Observe that using (1.90), (1.93), (1.96) and (1.97), we have proved that there exists $K_0 \geq 0$:

$$Re(h(\theta)(f)) - k|Im(h(\theta)(f))| \geq K_0 - \frac{C}{4\epsilon^2}\int_{Y\times[0,\infty]} \langle f,f\rangle(z)dz. \tag{1.98}$$

Finally we can take γ large enough for having:

$$Re(h(\theta)(f)) - k|Im(h(\theta)(f))| + \gamma\langle f,f\rangle_{L^2(Z,E)} \geq 0, \tag{1.99}$$

what finishes the proof of (1.89), and with it the proof of the theorem.□

Chapter 2

Analytic dilation on complete manifolds with corners of codimension 2

In this section we generalize the method of analytic dilation to compatible Laplacians on manifolds with corners of codimension two.

2.1 Manifolds with corners of codimension 2

We describe the manifold with corner of codimension 2 in the same way that in [30], section 1. Let M be a closed oriented $n-1$-dimensional C^∞-manifold and let $Y \subset M$ be a closed oriented submanifold of codimension 1, which separates M in two submanifolds M_1 and M_2. Let X_0 be a compact oriented n-dimensional Riemannian manifold with boundary M. We assume that the Riemannian metric on X_0 has the following properties:

i) In a neighborhood $(-\epsilon, 0] \times M_i$ of the boundary component M_i, for $i = 1, 2$, X_0 is isometric to a product metric.

ii) In a neighborhood $(-\epsilon, 0]^2 \times Y$ of the corner Y, X_0 is isometric to to a product metric.

The next graphic provides an example of a compact manifold with corner of codimension 2:

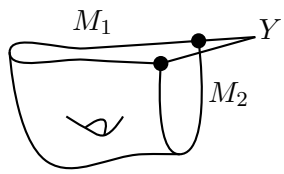

Figure 2. Compact mfld with corner of cod. 2

We canonically associate with X_0 a non-compact complete Riemannian manifold X as follows. Define the manifolds with cylindrical ends:

$$Z_i := M_i \cup_Y (\mathbb{R}_+ \times Y),\ i = 1, 2, \tag{2.1}$$

where the bottom $\{0\} \times Y$ of the half-cylinder is identified with $\partial M_i = Y$. Similarly, we define:

$$W_i := X_0 \cup_{M_i} (\mathbb{R}_+ \times M_i),\ i = 1, 2. \tag{2.2}$$

Observe that W_i is an n-dimensional manifold with boundary Z_i. Set:

$$X := W_1 \cup_{Z_1} (\mathbb{R}_+ \times Z_1) = W_2 \cup_{Z_2} (\mathbb{R}_+ \times Z_2), \tag{2.3}$$

where $\{0\} \times Z_i$ is identified with the boundary Z_i of W_i, $i = 1, 2$. We equip $\mathbb{R}_+ \times M_i$ and $\mathbb{R}_+ \times Z_i$ with the product metric and extend in this way the metric of X_0 to a geodesically complete C^∞ Riemannian metric on X. We call X a **a complete manifold with corner of codimension two at** Y. The next graphic gives an idea of how looks X:

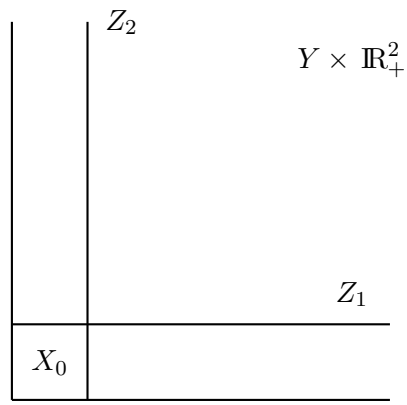

Figure 3. Complete mfld with corner of cod. 2

There exists a distinguished exhaustion of X by compact submanifolds X_T, $T \geq 0$ which we shall describe now. Let $T \geq 0$ be given and define:

$$Z_{i,T} := M_i \cup_Y ([0,T] \times Y), \tag{2.4}$$

here is understood that $\{0\} \times Y$ is identified with ∂M_i. $Z_{i,T}$ is a family of manifolds with boundary that exhaust Z_i. Set:

$$W_{i,T} := X_0 \cup_{M_i} ([0,T] \times M_i),\ T \geq 0 \tag{2.5}$$

where we are identifying $\{0\} \times M_i$ with M_i. Define:

$$X_T := W_{1,T} \cup_{Z_{1,T}} ([0,T] \times Z_1) = W_{2,T} \cup_{Z_{2,T}} ([0,T] \times Z_2),\ T \geq 0. \tag{2.6}$$

Observe that:

$$\partial X_T = Z_{1,T} \cup_Y -Z_{2,T}. \tag{2.7}$$

In the next section we describe the compatible Laplacians, the operators that we study in this thesis.

2.2 Compatible Laplacians

In this section we introduce a notation that makes more clear the analogy between compatible Laplacians and the many body Schrödinger operator.

Let X be a manifold with corner of codimension two as above. Let E be an Hermitian vector bundle over X. Let Δ be a a compatible Laplacian acting on $C^\infty(X,E)$. We suppose that Δ has the following properties:

i) There exists an Hermitian vector bundle E_i over Z_i such that $E|_{\mathbb{R}_+ \times Z_i}$ is the pull-back of E_i. We suppose also that the Hermitian metric of E is the pullback of the Hermitian metric of E_i. On $\mathbb{R}_+ \times Z_i$. we have:

$$\Delta = -\frac{\partial^2}{\partial u_k^2} + \Delta_{Z_i}, \tag{2.8}$$

where Δ_{Z_i} is a compatible Laplacian acting on $C^\infty(Z_i, E_i)$.

ii) There exists an Hermitian vector bundle S over Y such that $E|_{\mathbb{R}_+^2 \times Y}$ is the pull-back of S, and on $\mathbb{R}_+^2 \times Y$ we have,

$$\Delta = -\frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2} + \Delta_Y \tag{2.9}$$

where Δ_Y is a Laplacian acting on $C^\infty(Y,S)$.

We call a Laplacian satisfying i) and ii) a **compatible Laplacian** over X.

Since X is a complete manifold and the vector bundle E has bounded Hermitian metric, $\Delta : C_c^\infty(X,E) \to L^2(X,E)$ is essentially self-adjoint. We denote H its self-adjoint extension. For $i = 1,2$, $\Delta_{Z_i} : C_c^\infty(Z_i,E_i) \to L^2(Z_i,E_i)$ is also essentially self-adjoint and we denote its self adjoint extension by $H^{(i)}$. Let b_i be the self-adjoint extension of $-\partial_i^2 : C_c^\infty(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ obtained with Von Neumann boundary conditions. We denote H_i the self-adjoint operator $-b_i \otimes 1 + 1 \otimes H^{(i)}$ acting on $L^2(\mathbb{R}) \otimes L^2(Z_k,E_k)$. Similarly $H^{(3)}$ denotes the self-adjoint operator associated to the essentially self-adjoint operator $\Delta_Y : C_c^\infty(Y,S) \to L^2(Y,S)$; and we denote by H_3, the self-adjoint operator $H_3 := -b_1 \otimes 1 \otimes 1 - b_2 \otimes 1 \otimes 1 + H^{(3)}$ action $L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+) \otimes L^2(Y,S)$. The operators H_i are called **channel operators** for $i = 1,2,3$. H_1 and H_2 are channel operators with a free channel of dimension 1 (associated to b_1 and b_2 respectively), and H_3 is channel operator with a free channel of dimension 2 (associated to $b_1 + b_2$).

2.3 The definition of U_θ for $\theta \in \mathbb{R}_+$

For $i = 1,2$ and $\theta \in \mathbb{R}_+$, $U_{i,\theta} : L^2(Z_i,E_i) \to L^2(Z_i,E_i)$ denotes an analytic dilation operator associated to the compatible Laplacian $H^{(i)}$. $U_{i,\theta}$ was described in section 1.2. In this section we denote $H^{(i),\theta} := U_{i,\theta} H^{(i)} U_{i,\theta}^{-1}$.

For $\theta \in \mathbb{R}_+$, define:

$$U_\theta f(x) := \begin{cases} f(x_0) & \text{for } x = x_0 \in X_0, \\ f(m_i, \psi_\theta(u_i))\psi_\theta'^{1/2}(u_i) & \\ & \text{for } x = (m_i,u_i) \in M_i \times \mathbb{R}_+, i = 1,2, \\ f(y, \psi_\theta(u_1), \psi_\theta(u_2))\psi_\theta'^{1/2}(u_1)\psi_\theta'^{1/2}(u_2) & \\ & \text{for } x = (y,u_1,u_2) \in Y \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{2.10}$$

Proposition 18 *Let $\theta \in \mathbb{R}_+$ and $f \in C_c^\infty(X,E)$,*

i) $U_\theta f \in C_c^\infty(X,E)$.

ii) U_θ extends to a unitary operator in $L^2(X,E)$.

Proof:
For proving i), one checks differentiability easily in $Z_i \times \{0\}$, for $i = 1, 2$.

For $f \in C_c^\infty(X, E)$, we show $||U_\theta f||_{L^2(X,E)} = ||f||_{L^2(X,E)}$:

$$||U_\theta f||^2_{L^2(X,E)} = \int_X |U_\theta f|^2 dx = \sum_{i=1}^{2} \int_{M_i \times \mathbb{R}_+} |f(m_i, \psi_\theta(u_i))\psi_\theta'^{1/2}(u_i)|^2 dx +$$

$$\int_{Y \times \mathbb{R}_+^2} |f(y, \psi_\theta(u_1), \psi_\theta(u_2))\psi_\theta'^{1/2}(u_1)\psi_\theta'^{1/2}(u_2)|^2 dx.$$

Making the obvious change of variables, we observe :

$$\int_{M_i \times \mathbb{R}_+} |f(m_i, \psi_\theta(u_i))\psi_\theta'^{1/2}(u_i)|^2 dx = \int_{M_i \times \mathbb{R}_+} |f(m_i, v)|^2 dv$$

and similarly

$$\int_{Y \times \mathbb{R}_+^2} |f(y, \psi_\theta(u_1), \psi_\theta(u_2))\psi_\theta'^{1/2}(u_1)\psi_\theta'^{1/2}(u_2)|^2 dx$$
$$= \int_{Y \times \mathbb{R}_+^2} |f(y, v_1, v_2)|^2 dv. \square$$

U_θ^{-1} is given by:

$$U_\theta^{-1} f(x) := \begin{cases} f(x_0) & \text{for } x = x_0 \in X_0, \\ f(m_i, \alpha_\theta(u_i))\psi_\theta'^{-1/2}(\alpha_\theta(u_i)) & \\ & \text{for } x = (m_i, u_i) \in M_i \times \mathbb{R}_+, i = 1, 2 \\ f(y, \alpha_\theta(u_1), \alpha_\theta(u_2))\psi_\theta'^{-1/2}(\alpha_\theta(u_1))\psi_\theta'^{-1/2}(\alpha_\theta(u_2)) & \\ & \text{for } x = (y, u_1, u_2) \in Y \times \mathbb{R}_+ \times \mathbb{R}_+. \end{cases} \tag{2.11}$$

One can check the following proposition.

Proposition 19 *For $\theta \in \mathbb{R}_+$ and $i = 1, 2$, if $f \in C_c^\infty(X, E)$ and $(z_i, u_i) \in \mathbb{R}_+ \times Z_i$, then:*
i) $U_\theta f(u_i, z_i) = (U_{i,\theta} f)(\psi_\theta(u_i), z_i)\psi_\theta'^{1/2}(u_i)$.

ii) $U_\theta^{-1} f(u_i, z_i) = U_{i,\theta}^{-1} f(\alpha_\theta(u_i), z_i)\psi_\theta'^{-1/2}(\alpha_\theta(u_i))$.

Proof:
For $x = (m_i, u_i) \in \mathbb{R}_+ \times Z_i$,

$$(U_{i,\theta} f))(\psi_\theta(u_i), m_i)\psi_\theta'^{1/2}(u_i) = f(\psi_\theta(u_i), m_i)\psi_\theta'^{1/2}(u_i) = U_\theta f(u_i, m_i)$$

For $x = (y, u_i, u_j) \in (\mathbb{R}_+^2) \times Y$ with $j \in \{1, 2\}$, $i \neq j$,

$$\begin{aligned}(U_{i,\theta} f))(u_j, \psi_\theta(u_i), y)\psi_\theta'^{1/2}(u_i) &= f(y, \psi_\theta(u_j), \psi_\theta(u_i))\psi_\theta'^{1/2}(u_i)\psi_\theta'^{1/2}(u_j) \\ &= U_\theta f(u_i, u_j, y). \square\end{aligned}$$

2.4 The family H_θ for $\theta \in \mathbb{C} - (-\infty, 0)$

For $\theta \in \mathbb{R}_+$, define the operator $H_\theta := U_\theta H U_\theta^{-1}$.

Proposition 20 *For $\theta \in \mathbb{R}_+$ and $i = 1, 2$, if $f \in C_c^\infty(X, E)$, then for all $(z_i, u) \in Z_i \times \mathbb{R}_+$*

$$\begin{aligned}H_\theta f(u, z_i) =& H^{(i),\theta} f(u, z_i) - (\frac{\partial^2}{\partial u^2} f))(u, z_i)\alpha_\theta'(\psi_\theta(u))^2 \\ &- (\frac{\partial}{\partial u} f)(u, z_i)\alpha_\theta''(\psi_\theta(u)) \\ &+ (\frac{\partial}{\partial u} f)(u, z_i)\psi_\theta'(u)^{-1}\psi_\theta''(u)\alpha_\theta'(\psi_\theta(u))^2 \\ &- 3/4 f(u, z_i)\psi_\theta'(u)^{-3/2}\psi_\theta''(u)^2\alpha_\theta'(\psi_\theta(u))^2 \\ &+ 1/2 f(u, z_i)\psi_\theta'(u)^{-1}\psi_\theta'''(u)\alpha_\theta'(\psi_\theta(u))^2 \\ &+ 1/2 f(u, z_i)\psi_\theta'(\alpha_\theta(u))^{-1}\psi_\theta''(u)\alpha_\theta''(\psi_\theta(u)).\end{aligned} \tag{2.12}$$

In particular, if $u_i > R$, we have:

$$H_\theta f(u_i, z_i) = -\theta'(\frac{\partial^2}{\partial u_i^2} f)(u_i, z_i) + H^{(i),\theta} f(u_i, z_i). \tag{2.13}$$

If $u_i < K$,

$$H_\theta f(u_i, z_i) = -\frac{\partial^2}{\partial u_i^2} f(u_i, z_i) + H^{(i)} f(u_i, z_i). \tag{2.14}$$

Proof: By proposition 19, in $\mathbb{R}_+ \times Z_i$

$$U_\theta H U_\theta^{-1} f = U_\theta(-\frac{\partial^2}{\partial u_i^2} + H^{(i)}) \left(U_{i,\theta}^{-1}\big((u_i, z_i) \mapsto f(\alpha_\theta(u_i), z_i)\psi_\theta'^{-1/2}(\alpha_\theta(u_i))\big)\right).$$

We have:

$$\frac{\partial^2}{\partial u_i^2}(U_{i,\theta}^{-1} f(\alpha_\theta(u_i), z_i)\psi_\theta'^{-1/2}(\alpha_\theta(u_i)) = U_{i,\theta}^{-1}\frac{\partial^2}{\partial u_i^2}\left(f(\alpha_\theta(u_i), z_i)\psi_\theta'^{-1/2}(\alpha_\theta(u_i))\right).$$

Using equation (1.15),

$$\begin{aligned}
\frac{\partial^2}{\partial u_i^2}\left(f(\alpha_\theta(u_i), z_i)\psi_\theta'^{-1/2}(\alpha_\theta(u_i))\right) &= (\frac{\partial^2}{\partial u_i^2} f))(u_i, z_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&+ (\frac{\partial}{\partial u_i} f)(u_i, z_i)\alpha_\theta''(\psi_\theta(u_i)) \\
&- (\frac{\partial}{\partial u_i} f)(u_i, z_i)\psi_\theta'(u_i)^{-1}\psi_\theta''(u_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&+ 3/4 f(u_i, z_i)\psi_\theta'(u_i)^{-3/2}\psi_\theta''(u_i)^2\alpha_\theta'(\psi_\theta(u_i))^2 \\
&- 1/2 f(u_i, z_i)\psi_\theta'(u_i)^{-1}\psi_\theta'''(u_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&- 1/2 f(u_i, z_i)\psi_\theta'(\alpha_\theta(u_i))^{-1}\psi_\theta''(u_i)\alpha_\theta''(\psi_\theta(u_i)).
\end{aligned} \tag{2.15}$$

Using that $U_\theta f(u_i, z_i) = U_{i,\theta} f(\psi_\theta(u_i), z_i)\psi_\theta'(u_i)$,

$$\begin{aligned}
U_\theta U_{i,\theta}^{-1}\frac{\partial^2}{\partial u_i^2}\left(f(z_i, \alpha_\theta(u_i))\psi_\theta'^{-1/2}(\alpha_\theta(u_i))\right) &= (\frac{\partial^2}{\partial u_i^2} f)(z_i, u_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&+ (\frac{\partial}{\partial u_i} f)(z_i, u_i)\alpha_\theta''(\psi_\theta(u_i)) \\
&- (\frac{\partial}{\partial u_i} f)(z_i, u_i)\psi_\theta'(u_i)^{-1}\psi_\theta''(u_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&+ 3/4 f(z_i, u_i)\psi_\theta'(u_i)^{-3/2}\psi_\theta''(u_i)^2\alpha_\theta'(\psi_\theta(u_i))^2 \\
&- 1/2 f(z_i, u_i)\psi_\theta'(u_i)^{-1}\psi_\theta'''(u_i)\alpha_\theta'(\psi_\theta(u_i))^2 \\
&- 1/2 f(z_i, u_i)\psi_\theta'(\alpha_\theta(u_i))^{-1}\psi_\theta''(u_i)\alpha_\theta''(\psi_\theta(u_i)).
\end{aligned}$$

Finally, we observe that $(U_\theta H^{(i)} U_{i,\theta}^{-1})\left((z_i, u_i) \mapsto f(z_i, \alpha_\theta(u_i))\psi_\theta'^{-1/2}(\alpha_\theta(u_i))\right) = H^{(i),\theta} f$. $\square$

Proposition 21 *For $\theta \in \mathbb{R}_+$, for all $f \in C_c^\infty(X, E)$ and for all $(y, u_1, u_2) \in$*

$Y \times \mathbb{R}_+^2$

$$\begin{aligned} H_\theta f(y,u_1,u_2) =& H^{(3)} f(y,u_1,u_2)) - \sum_{i=1}^{2} \{ (\frac{\partial^2}{\partial u_i^2} f))(y,u_1,u_2) \alpha'_\theta(\psi_\theta(u_i))^2 \\ & - (\frac{\partial}{\partial u} f)(y,u_1,u_2) \alpha''_\theta(\psi_\theta(u_i)) \\ & + (\frac{\partial}{\partial u} f)(y,u_1,u_2) \psi'_\theta(u_i)^{-1} \psi''_\theta(u_i) \alpha'_\theta(\psi_\theta(u_i))^2 \\ & - 3/4 f(y,u_1,u_2) \psi'_\theta(u_i)^{-3/2} \psi''_\theta(u_i)^2 \alpha'_\theta(\psi_\theta(u_i))^2 \\ & + 1/2 f(y,u_1,u_2) \psi'_\theta(u_i)^{-1} \psi'''_\theta(u_i) \alpha'_\theta(\psi_\theta(u_i))^2 \\ & + 1/2 f(y,u_1,u_2) \psi'_\theta(\alpha_\theta(u_i))^{-1} \psi''_\theta(u_i) \alpha''_\theta(\psi_\theta(u_i)) \}. \end{aligned} \tag{2.16}$$

In particular, if $u_1, u_2 > R$, we have:

$$H_\theta f(y,u_1,u_2) = -\theta'(\frac{\partial^2}{\partial u_1^2} f)(y,u_1,u_2) - \theta'(\frac{\partial^2}{\partial u_2^2} f)(y,u_1,u_2) + H^{(3)} f(y,u_1,u_2). \tag{2.17}$$

an if $u_1, u_2 < K$ we have:

$$H_\theta f(y,u_1,u_2) = -(\frac{\partial^2}{\partial u_1^2} f)(y,u_1,u_2) - (\frac{\partial^2}{\partial u_2^2} f)(y,u_1,u_2) + H^{(3)} f(y,u_1,u_2). \tag{2.18}$$

Proof:
Using the fact that $U_\theta f(y,u_1,u_2) = U_{1,\theta} U_{2,\theta} f(y,u_1,u_2)$ and similar calculations to those that prove equation (2.12), we prove the proposition.□

Remark 4 *For all $f \in C_c^\infty(X,E)$ and $(y,u_1,u_2) \in Y \times \mathbb{R}_+^2$ we can write:*

$$\begin{aligned} H_\theta(f)(y,u_1,u_2) =& \sum_{i=1}^{2} \{ a_2(\theta,u_i) \frac{\partial^2}{\partial u_i^2} f(y,u_1,u_2) \\ & + a_1(\theta,u_i) \frac{\partial}{\partial u_i} f(y,u_1,u_2) + a_0(\theta,u_i) f(y,u_1,u_2) \} \end{aligned} \tag{2.19}$$

where the functions $(\theta,u) \mapsto a_i(\theta,u)$ are defined in (1.31) (see also (1.25)).

Our next goal is to prove that $(H_\theta)_{\theta \in \Gamma}$ is an holomorphic family of type A (see definition 1).

Proposition 22 *For all $\theta \in \Gamma$, the operator H_θ is an uniformly elliptic operator i.e.*

$$|\sigma_2(H_\theta)^{-1}(z,\xi)|^{-1} \leq |\xi|^{-2}. \tag{2.20}$$

Proof:
The principal symbol of H_θ in $\mathbb{R}_+ \times Z_i$ is given by:

$$\sigma_2(H_\theta)((z_i,u),(\xi_{z_i},\xi_u)) = a_2(\theta,u)\xi_u^2 + \sigma_2(H^{(i),\theta})(z_i,\xi_{z_i}). \tag{2.21}$$

where $a_2(\theta,u)$ is defined in (1.31). Recall that in proposition 5 was proved the following inequality:

$$|\sigma_2(H^{(i),\theta})(z_i,\xi_{z_i})| \geq |\xi_{z_i}|^2.$$

Using this, and the fact that $a_2(\theta,u) = \frac{1}{g(u)\theta+1}$ for $g(u) := (\varphi(u)+\varphi'(u)u) \geq 0$, we have:

$$\begin{aligned} |\sigma_2(H_\theta)^{-1}((z_i,u),(\xi_{z_i},\xi_u))| &= \frac{1}{|a_2(\theta,u)\xi_u^2 + \sigma_2(H^{(i),\theta})(z_i,\xi_{z_i})|} \\ &\leq \frac{|g(u)\theta+1|}{|\xi_u^2 + (g(u)\theta+1)\sigma_2(H^{(i),\theta})(z_i,\xi_{z_i})|} \leq |\xi|^{-2}. \end{aligned} \tag{2.22}$$

In the previous calculations we used that $Re(g(u)\theta) \geq 0$. $\square$

We can use proposition 22 and appendix E, in essentially the same way that we prove proposition 6, for proving the next corollary:

Corollary 6 *The operators H_θ are closed operators in the domain $\mathscr{W}_2(X,E)$.*

Given $f \in Dom(H)$ and $g \in L^2(X,E)$, the next step is to prove that the function $\theta \mapsto \langle H_\theta f, g\rangle_{L^2(X,E)}$ is holomorphic for $Re(\theta) > 0$. For doing that we proceed as in section 1.3, using lemma 1. Consider the partition of unity of X that we define now. Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\eta(u) = 1$ for $u \leq K-2$ and $\eta(u) = 0$ for $u \geq K-1$. Let $\kappa := 1-\eta$. We define the following functions with their natural extensions to the whole X. Let $(z_i,u_i) \in Z_i \times \mathbb{R}_+$

$$\kappa_i(z_i,u_i) := \kappa(u_i);\ \eta_i(z_i,u_i) := 1-\kappa_i.$$

Observe that $\eta_i + \kappa_i = 1$. In particular, we have that $(\eta_1+\kappa_1)(\eta_2+\kappa_2) = 1$. We study the functions $\theta \mapsto \kappa_1\kappa_2 H_\theta$ and $\theta \mapsto \eta_i\kappa_j H_\theta$.

Proposition 23 *Given $f \in Dom(H)$ and $g \in L^2(X,E)$, there exists $h \in L^1(X)$ such that:*

$$| < \kappa_1\kappa_2 H_\theta f, g > (x)| \leq h(x), \tag{2.23}$$

and

$$|\frac{\partial}{\partial\theta_i}(< \kappa_1\kappa_2 H_\theta f, g > (x))| \leq h(x), \tag{2.24}$$

where $\theta := \theta_0 + i\theta_1$.

Proof:
Using proposition 3 we have:

$$\begin{aligned}
| < \kappa_1\kappa_2 H_\theta f, g > | \leq & \sum_{i=1}^{2}\{|\kappa_1\kappa_2 < a_2(\theta,u_i)\frac{\partial^2}{\partial u_i^2}f, g > | \\
& + |\kappa_1\kappa_2 < a_1(\theta,u_i)\frac{\partial}{\partial u_i}f, g > | \\
& + |\kappa_1\kappa_2 < a_0(\theta,u_i)f, g > |\} + |\kappa_1\kappa_2 < H^{(3)}f, g > | \\
& \leq C\{\sum_{i=1}^{2}\{|\kappa_1\kappa_2\frac{\partial^2}{\partial u_i^2}f, g > | + |\kappa_1\kappa_2 < \frac{\partial}{\partial u_i}f, g > | \\
& + |\kappa_1\kappa_2 < f, g > |\} + |\kappa_1\kappa_2 < H^{(3)}Yf, g > |\} \in L^1(X).
\end{aligned} \tag{2.25}$$

The proof of equation (2.24) is similar. □

Proposition 24 *For $i,j \in \{1,2\}$, $i \neq j$, given $f \in Dom(H)$ and $g \in L^2(X,E)$, there exists $h \in L^1(X)$ such that:*

$$| < \kappa_i\eta_j H_\theta f, g > (x)| \leq h(x), \tag{2.26}$$

and

$$|\frac{\partial}{\partial\theta_i}(< \kappa_i\eta_j H_\theta f, g > (x))| \leq h(x). \tag{2.27}$$

Proof
For simplifying notation we make the proof for $i = 1$, $j = 2$. We use again

proposition 3 obtaining:

$$
\begin{aligned}
| < \kappa_1\eta_2 H_\theta f, g > | \leq \sum_{i=1}^{2}\{|\kappa_1\eta_2 < a_2(\theta,u_1)\frac{\partial^2}{\partial u_1^2}f, g > | \\
+ |\kappa_1\eta_2 < a_1(\theta,u_1)\frac{\partial}{\partial u_1}f, g > | \\
+ |\kappa_1\eta_2 < a_0(\theta,u_1)f, g > |\} + |\kappa_1\eta_2 < H^{(1)}f, g > | \\
\leq C\{\sum_{i=1}^{2}\{| < \kappa_1\eta_2\frac{\partial^2}{\partial u_1^2}f, g > | + |\kappa_1\eta_2 < \frac{\partial}{\partial u_1}f, g > | \\
+ |\kappa_1\eta_2 < f, g > |\} + |\kappa_1\eta_2 < H^{(1)}f, g > |\} \in L^1(X).
\end{aligned}
\tag{2.28}
$$

The proof for equation (2.27) is similar. □

Theorem 9 *Given $f \in Dom(H)$ and $g \in L^2(X,E)$, the function $\theta \mapsto \langle H_\theta f, g\rangle_{L^2(X,E)}$ is holomorphic.*

Proof:
Propositions 23 and 24 guarantee that we can use lemma 1 for introducing the partial derivatives $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ in the respective integrals. After this, the proof follows from Cauchy-Riemann equations. □

In the next section we calculate $\sigma_{ess}(H_\theta)$ for $\theta \in \Gamma$.

2.5 The essential spectrum of H_θ

The goal of this section is to prove the next theorem.

Theorem 10 *The essential spectrum of H_θ is given by:*

$$
\begin{aligned}
\sigma_{ess}(H_\theta) = \left(\bigcup_{i=1}^{2}\bigcup_{\lambda\in\sigma_{pp}(H^{(i),\theta})} \lambda + \theta'\,\mathbb{R}_+\right) \\
\cup \left(\bigcup_{\mu\in\sigma(H^{(3)})} \mu + \theta'\,\mathbb{R}_+\right).
\end{aligned}
\tag{2.29}
$$

Recall that the essential spectrum of a closed operator is defined in (A.2). (2.29), as equation (1.44) is similar to part 1) of theorem 1.1 in [2], equation

3.2 in [27] and equation 5.19 in [19]. It could be deduced from part 3) of theorem 3.2 of [23].

Define the set:

$$\begin{aligned}\mathscr{F}_\theta := &\left(\bigcup_{i=1}^{2}\bigcup_{\lambda\in\sigma_{pp}(H^{(i)})}\lambda+\theta'\,\mathbb{R}_+\right)\\ &\cup\left(\bigcup_{\mu\in\sigma(H^{(3)})}\mu+\theta'\,\mathbb{R}_+\right)\\ &\cup\left(\bigcup_{i=1}^{2}\bigcup_{\gamma\in\mathscr{R}(H^{(i),\theta})}\gamma+\theta'\,\mathbb{R}_+\right),\end{aligned}\tag{2.30}$$

where $\mathscr{R}(H^{(i),\theta})$ is defined by:

$$\mathscr{R}(H^{(i),\theta}) := \{\lambda\in\sigma_d(H^{(i),\theta}) : \lambda\notin\sigma_{pp}(H^{(i)})\}.\tag{2.31}$$

The elements of $\mathscr{R}(H^{(i),\theta})$ shall be called **resonances** (compare with (B.9) and (1.76)). Observe that the set $\mathscr{R}(H^{(i),\theta})$ is independent of θ in the sense that if $arg(\theta_1')\geq arg(\theta_2')$ then $\sigma_{pp}(H^{(i),\theta_2'})\subset\sigma_{pp}(H^{(i),\theta_1'})$ (see theorem 5, item d)). By part a) of theorem 5,

$$\sigma_{pp}(H^{(i)})\subset\Big(\sigma_d(H^{(i),\theta})\cap(\,\mathbb{R}-\tau(H^{(i)}))\Big)\cup\tau(H^{(i)})\subset\sigma_{pp}(H^{(i),\theta})\cup\sigma(H^{(3)}),\tag{2.32}$$

where $\tau(H^{(i)}) := \sigma(H^{(3)})$ is the set of thresholds of $H^{(i)}$ (see part a) of theorem 5).

Proposition 25 *For $\theta\in\Gamma$, the following equation holds:*

$$\begin{aligned}\mathscr{F}_\theta = &\left(\bigcup_{i=1}^{2}\bigcup_{\lambda\in\sigma_{pp}(H^{(i),\theta})}\lambda+\theta'\,\mathbb{R}_+\right)\\ &\cup\left(\bigcup_{\mu\in\sigma(H^{(3)})}\mu+\theta'\,\mathbb{R}_+\right).\end{aligned}\tag{2.33}$$

Proof:
Denote by $\mathscr{G}_\theta$ the right hand side of (2.33). Let $\lambda\in\sigma_{pp}(H^{(i),\theta})$. If $\lambda\in\sigma_{ess}(H^{(i),\theta})$, then, by equation (1.44), there exists a $s_0\in\mathbb{R}_+$ and a $\mu\in H^{(3)}$

such that $\lambda = \mu + \theta' s_0$. Then, for all $s \in [0, \infty)$, $\lambda + \theta' s = \mu + \theta'(s + s_0)$. For $\lambda \in \sigma_{pp}(H^{(i),\theta})$ and $\lambda \in \sigma_{ess}(H^{(i),\theta})$, we have proved:

$$\lambda + \theta' \, \mathbb{R}_+ \subset \mathscr{F}_\theta. \tag{2.34}$$

If $\lambda \in \sigma_{pp}(H^{(i),\theta})$ and $\lambda \notin \sigma_{ess}(H^{(i),\theta})$, then $\lambda \in \sigma_d(H^{(i),\theta})$. Hence, $\lambda \in \sigma_{pp}(H^{(i)})$ or $\lambda \in \mathscr{R}(H^{(i),\theta})$. This fact and (2.34), imply that for all $\lambda \in \sigma_{pp}(H^{(i),\theta})$:

$$\bigcup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} \left(\lambda + \theta' \, \mathbb{R}_+\right) \subset \bigcup_{\gamma \in \sigma_{pp}(H^{(i)}) \cup \mathscr{R}(H^{(i),\theta})} \left(\lambda + \theta' \, \mathbb{R}_+\right) \subset \mathscr{F}_\theta. \tag{2.35}$$

Since

$$\left(\bigcup_{\mu \in \sigma(H^{(3)})} \mu + \theta' \, \mathbb{R}_+ \right) \subset \mathscr{F}_\theta, \tag{2.36}$$

we have proved $\mathscr{G}_\theta \subset \mathscr{F}_\theta$.

Now we prove $\mathscr{F}_\theta \subset \mathscr{G}_\theta$. Let $\lambda \in \sigma_{pp}(H^{(i)})$, then, by (2.32), $\lambda \in \sigma_{pp}(H^{(i),\theta})$ or $\lambda \in \sigma(H^{(3)})$. Since $\mathscr{R}(H^{(i),\theta}) \subset \sigma_{pp}(H^{(i),\theta})$, we finish the proof of the proposition. $\square$

We will show $\sigma_{ess}(H_\theta) = \mathscr{F}_\theta$. We use the results of appendix D, about the sets $N_\infty(A)$ (see definition 11), $N_{ess}(A)$ (see definition 10) and $\sigma_{ess}(A)$, for proving $\sigma_{ess}(H_\theta) = \mathscr{F}_\theta$. The proof has two basic steps:

i) To prove that $N_\infty(H_\theta) = \mathscr{F}_\theta$.

ii) To prove that there exists $\eta_0^d \in C_c^\infty(X)$ such that $supp \eta_0^d \subset X_d$ and, for all $f \in C_c^\infty(X, E)$,

$$||[H_\theta, \eta_0^d] f||_{L^2(X,E)} \leq \epsilon(d)(||H_\theta f||_{L^2(X,E)} + ||f||_{L^2(X,E)})$$

with $\epsilon(d) \to 0$ as $d \to \infty$.

Using ii) we see that H_θ satisfies the conditions for applying theorem 37 and we get $N_{ess}(H_\theta) = N_\infty(H_\theta)$. Using i) and part three of theorem 36, we prove $\sigma_{ess}(H_\theta) = \mathscr{F}_\theta$.

In the next section we prove step i).

2.5.1 The equality $N_\infty(H_\theta) = \mathscr{F}_\theta$

Let $\mu \in \sigma(H^{(3)})$ and $\lambda \in \mu + \theta' \mathbb{R}_+$. The boundary Weyl sequence will play a very important role in the next proposition. They are defined in definition 11. Observe that we can apply theorems 36 and 37 to the operators $\theta' \frac{d^2}{du^2}$ and $\theta' \frac{d^2}{du^2} + \mu$. Then, there exist p_n a boundary Weyl sequence associated to 0 and the operator $\theta' \frac{\partial^2}{\partial u_1^2}$, and $q_n(u_2)$ a boundary Weyl sequence associated to λ and the operator $\theta' \frac{\partial^2}{\partial u_2^2} + \mu$. Let $\varphi \in C^\infty(Y, S)$ be an eigenfunction of $H^{(3)}$ associated to the eigenvalue μ.

Proposition 26 *Let $g_n \in C_c^\infty(Y \times \mathbb{R}_+^2, E)$ be defined by $g_n(u_1, u_2, y) = p_n(u_1) q_n(u_2) \varphi(y)$. Then (g_n) induces a boundary Weyl sequence for λ and the operator H_θ.*

Proof:
Since $p_n, q_n \in C_c^\infty((0, \infty))$ there exists a natural extension of $p_n q_n \varphi$ to $C_c^\infty(X, E)$. It is easy to check $g_n \in C_c^\infty(X, E)$, $||g_n|| = 1$ and that for all $K > 0$ there exists a N such that for all $n > N$ $supp(g_n) \cap X_K = \emptyset$. The next calculations prove $\lim_{n\to\infty} || (H_\theta - \lambda) g_n|| = 0$:

$$\begin{aligned} &|| \left(\theta' \frac{\partial^2}{\partial u_1^2} + \theta' \frac{\partial^2}{\partial u_2^2} + H^{(3)} - \lambda \right) g_n|| \leq \\ &C(||\theta' \frac{\partial^2}{\partial u_1^2} p_n|| + ||(\theta' \frac{\partial^2}{\partial u_2^2} + \mu - \lambda) q_n||). \end{aligned} \tag{2.37}$$

Since p_n is a boundary Weyl sequence of $\theta' \frac{\partial^2}{\partial u_1^2}$, and q_n is a boundary Weyl sequence of $\theta' \frac{\partial^2}{\partial u_2^2} + \mu$ and the value λ, the last terms of (2.37) tend to 0. □

Now let $\gamma \in \sigma_{pp}(H^{(i),\theta})$ and $\lambda \in \gamma + \theta' \mathbb{R}_+$. Let $\varphi \in C^\infty(Z_i, E_i)$ be an L^2-eigenfunction of $H^{(i),\theta}$ with eigenvalue γ. Let $\eta \in C^\infty(\mathbb{R})$ such that $\eta(u) = 1$, for $u \leq 1$; $\eta(u) = 0$, for $u > 2$, and $\eta'(u) \leq 0$. Define $\eta_n(u) := \eta(u/n)$. Let f_n be a boundary Weyl sequence associated to the operator $-\theta' \frac{d^2}{du^2}$ and 0.

Proposition 27 *We denote $i, j \in \{1, 2\}$ such that $i \neq j$*[1]*. Let $g_n \in C^\infty(Z_i \times \mathbb{R}_+, E)$ be defined by $g_n(u_1, u_2, z_i) := \frac{1}{||\eta_n(u_j) f_n(u_i) \varphi(z_1)||} \eta_n(u_j) f_n(u_i) \varphi(z_i)$. Then, (g_n) induces a boundary Weyl sequence. associated to H_θ and the value λ.*

[1] With this notation u_j is the real variable in the cylinder of Z_i.

Proof:
It is easy to check $g_n \in C_c^\infty(X,E)$, $||g_n|| = 1$, and that for all $K \in \mathbb{N}$ there exists an N such that for all $n \geq N$ $suppg_n \cap X_K = \emptyset$. Since $\lim_{n\to\infty} \frac{1}{||\eta_n(u_2)f_n(u_1)\varphi(z_1)||_{L^2(X,E)}} = 1$, it is enough to prove:

$$\lim_{n\to\infty} ||(\theta'\frac{\partial^2}{\partial u_i^2} + H^{(i),\theta})g_n|| = 0.$$

Observe that:

$$(H^{(i),\theta} - \gamma)(\eta_n(u_j)\varphi(z_i)) = -\theta'\frac{\partial^2}{\partial u_j^2}(\eta_n)\varphi - 2\theta'\frac{\partial}{\partial u_j}(\eta_n)\frac{\partial}{\partial u_j}(\varphi). \tag{2.38}$$

Since $\eta_n(u_j) = \eta(u_j/n)$, then $\frac{\partial}{\partial u_j}(\eta_n)(u_j) = \frac{1}{n}\frac{\partial}{\partial u_j}(\eta)(u_j/n)$ and $\frac{\partial^2}{\partial u_j^2}(\eta_n)(u_j) = \frac{1}{n^2}\frac{\partial^2}{\partial u_j^2}(\eta)(u_j/n)$. Hence, using equation (2.38):

$$||(\theta'\frac{\partial^2}{\partial u_i^2} + H^{(i),\theta} - \gamma)g_n|| \leq C(A_n + B_n + C_n),$$

where

$$\begin{aligned}
A_n &:= ||\eta_n(u_j)\varphi(z_i)(\theta'\frac{\partial^2}{\partial u_i^2})f_n(u_i)||;\\
B_n &:= ||\left(\theta'\frac{\partial^2}{\partial u_j^2}(\eta_n)(u_j)\right)\varphi(z_i)f_n(u_i)|| = ||\left(\theta'\frac{\partial^2}{\partial u_j^2}(\eta_n)(u_j)\right)\varphi(z_i)||;\\
C_n &:= ||\left(\theta'\frac{\partial}{\partial u_j}(\eta_n)(u_j)\right)\varphi(z_i)f_n(u_i)|| = ||\left(\theta'\frac{\partial}{\partial u_j}(\eta_n)(u_j)\right)\varphi(z_i)||.
\end{aligned} \tag{2.39}$$

For A_n, we have:

$$A_n \leq ||(\theta'\frac{\partial^2}{\partial u_i^2})f_n(u_i)|| \to 0. \tag{2.40}$$

For B_n, we estimate:

$$\begin{aligned}
B_n &\leq C(\theta)\left(\int_{Z_i} |\frac{\partial^2}{\partial u_j^2}(\eta_n)(u_j)\varphi(z_i)|^2 dz_i\right)^{1/2}\\
&\leq C(\theta)1/n^2\left(\int_{Z_i} |(\frac{\partial^2}{\partial u_j^2}\eta)(u_j/n)\varphi(z_i)|^2 dz_i\right)^{1/2}\\
&\leq 1/n^2||\varphi|| \to 0;
\end{aligned} \tag{2.41}$$

and, finally, for C_n:

$$\begin{aligned} C_n &\leq C(\theta)\left(\int_{Z_i} |\frac{\partial}{\partial u_j}(\eta_n)(u_j)\varphi(z_i)|^2 dz_i\right)^{1/2} \\ &\leq C(\theta)1/n\left(\int_{Z_i} |(\frac{\partial}{\partial u_j}\eta)(u_j/n)\varphi(z_i)|^2 dz_i\right)^{1/2} \\ &\leq 1/n||\varphi|| \to 0.\square \end{aligned} \tag{2.42}$$

In proposition 26 is proved that, for $\mu \in \sigma(H^{(3)})$, and for all $s \in \mathbb{R}_+$, $\mu + \theta' s \in N_\infty(H_\theta)$. In proposition 27 is proved that, for $\gamma \in \sigma_{pp}(H^{(i),\theta})$ and for all $s \in \mathbb{R}_+$, $\gamma + \theta' s \in N_\infty(H_\theta)$. These facts together with the equality (2.33) imply $\mathscr{F}_\theta \subset N_\infty(H_\theta)$.

Now we are going to prove the other inclusion; but before we establish some notation. Recall that we denote b_i the self-adjoint operator in $L^2(\mathbb{R}_+)$ obtained from $-\frac{\partial^2}{\partial u_i^2}$ with Neumann boundary conditions. We denote by $H_{i,\theta}$ the closed operator $-\theta' 1 \otimes b_i + H^{(i),\theta} \otimes 1$ acting on $L^2(Z_i \times \mathbb{R}_+, E) = L^2(Z_i, E) \otimes L^2(\mathbb{R}_+)$.

Proposition 28 *For $i = 1, 2$ and $\theta \in \Gamma$:*
i) $\sigma_{ess}(H_{i,\theta}) = N_\infty(H_{i,\theta})$.
ii) $N_\infty(H_{i,\theta})$ is given by:

$$N_\infty(H_{i,\theta}) = \left(\bigcup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} \{\lambda + \theta' \mathbb{R}_+\}\right) \cup \left(\bigcup_{\mu \in \sigma(H^{(3)})} \{\mu + \theta' \mathbb{R}_+\}\right). \tag{2.43}$$

Proof:
In the same way that we proved $\mathscr{F}_\theta \subset N_\infty(H_\theta)$, using propositions 26 and 27, we can prove

$$\left(\bigcup_{\lambda \in \sigma_{pp}(H^{(i),\theta})} \{\lambda + \theta' \mathbb{R}_+\}\right) \cup \left(\bigcup_{\mu \in \sigma(H^3)} \{\mu + \theta' \mathbb{R}_+\}\right) \subset N_\infty(H_{i,\theta}). \tag{2.44}$$

Denote by $\mathscr{F}_{i,\theta}$ the right-side of (2.43). Then, (2.44) implies $\mathscr{F}_{i,\theta} \subset N_\infty(H_{i,\theta})$, and by proposition 42, $\mathscr{F}_{i,\theta} \subset N_\infty(H_{i,\theta}) \subset \sigma_{ess}(H_{i,\theta})$. We know that the operator $H^{(i),\theta}$ is m-sectorial (theorem 8), then we can apply Ichinose lemma

(see theorem 35) in the next computations:

$$\begin{aligned}\sigma(H_{i,\theta}) &= \sigma(H^{(i),\theta}) + \sigma(-\theta'\frac{\partial^2}{\partial u_i^2}) \\ &= \Big(\theta'\,\mathbb{R}_+ + \sigma_{pp}(H^{(i),\theta})\Big) \cup \Big(\sigma(H^{(3)}) + \theta'\,\mathbb{R}_+\Big)\,.\end{aligned} \tag{2.45}$$

The above equation implies $N_\infty(H_{i,\theta}) \subset \mathscr{F}_{i,\theta}$ and $\sigma_{ess}(H_{i,\theta}) \subset \mathscr{F}_{i,\theta}$, that together with (2.44) implies $N_\infty(H_{i,\theta}) = \mathscr{F}_{i,\theta} = \sigma_{ess}(H_{i,\theta})$. We have proved the proposition. □

Next we prove $N_\infty(H_\theta) \subset \mathscr{F}_\theta$. Let $\lambda \in N_\infty(H_\theta)$ and let $f_n \in C_c^\infty(X,E)$ be a boundary Weyl sequence. associated to the operator H_θ and λ. The plan for proving $N_\infty(H_\theta) \subset \mathscr{F}_\theta$ is to construct, using f_n, a boundary Weyl sequence associated to λ and one of the operators $H_{i,\theta}$. Define $\kappa_n := 1 - \eta_n$.

Proposition 29 *There exists $c > 0$ and a subsequence s of $\mathbb{N}$ such that*

$$||\kappa_n(u_1) f_{s(n)}||_{L^2(X,E)} \geq c > 0 \text{ or } ||\kappa_n(u_2) f_{s(n)}||_{L^2(X,E)} \geq c > 0. \tag{2.46}$$

Proof:
We can choose s such that $\chi_n f_{s(n)} = f_{s(n)}$ where χ_n denotes the characteristic function of $X - X_{n^2+1}$. Observing that $\chi_n^2 \leq \kappa_n(u_1)^2 + \kappa_n(u_1)^2$,

$$1 = ||f_{s(n)}||^2_{L^2(X,E)} = ||\chi_n f_{s(n)}||^2_{L^2(X,E)} \leq ||\kappa_n(u_1) f_{s(n)}||^2_{L^2(X,E)} + ||\kappa_n(u_2) f_{s(n)}||^2_{L^2(X,E)},$$

which is a contradiction. □

The previous proposition allows us to suppose that $0 < c < ||\kappa_n(u_1) f_{s(n)}||_{L^2(X,E)}$.

Proposition 30 *Denote by g_n the function in $C^\infty(X,E)$, defined by $g_n := \frac{1}{||\kappa_n(u_1) f_{s(n)}||_{L^2(X,E)}} \kappa_n(u_1) f_{s(n)}$. g_n induces a boundary Weyl sequence associated to $H_{1,\theta}$ and λ.*

Proof:
It is easy to check $||g_n||_{L^2(X,E)} = 1$ and for all $T > 0$, there exists $N \in \mathbb{R}$ such that $\forall n \geq N$, $supp g_n \cap X_T = \emptyset$. Denoting $\kappa(u) := 1 - \eta(u)$, we have $\kappa_n(u_1) := \kappa(u_1/n)$, then $\frac{\partial}{\partial u_1}(\kappa_n)(u_1) = \frac{1}{n}\frac{\partial}{\partial u_1}(\kappa)(u_1/n)$ and $\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1) = \frac{1}{n^2}\frac{\partial^2}{\partial u_1^2}(\kappa)(u_1/n)$. Hence:

$$||\,(H_{1,\theta} - \lambda)\,(\kappa_n(u_1) f_{s(n)})||_{L^2(X,E)} \leq A_n + B_n + C_n \to 0, \tag{2.47}$$

where

$$
\begin{aligned}
A_n^2 &:= 4||\frac{\partial}{\partial u_1}(\kappa_n)(u_1)\frac{\partial}{\partial u_1}(f_{s(n)})||^2 \\
&\leq C\int_{Z_1\times \mathbb{R}_+} |\frac{\partial}{\partial u_1}(\kappa_n)(u_1))\frac{\partial}{\partial u_1}(f_{s(n)})|^2 dvol(x) \\
&\leq C/n^2\int_{Z_1\times \mathbb{R}_+} |\frac{\partial}{\partial u_1}(\kappa)(u_1/n)\frac{\partial}{\partial u_1}(f_{s(n)})|^2 dvol(x) \\
&\leq C/n^2||\frac{\partial}{\partial u_1}(f_{s(n)})||^2 \leq C||f_{s(n)}||_2^2\cdot 1/n^2 \leq C'1/n^2 \to 0.
\end{aligned}
$$

In the last inequalities we use that $||\frac{\partial}{\partial u_1}(f_{s(n)})|| \leq C||f_{s(n)}||_2$, that is true from the theory of bounded differential operators in manifolds with bounded geometry (see appendix E). We use also $||f_n||_2 \leq C$, that is true because $||f_n|| = 1$, $\lim_{n\to\infty}||(H_\theta - \lambda)f_n|| = 0$ and applying theorem 39. For B_n, we have:

$$
\begin{aligned}
B_n^2 := ||\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1)f_{s(n)}||^2 \leq & C\int_{Z_1\times \mathbb{R}_+} |\frac{\partial^2}{\partial u_1^2}(\kappa_n)(u_1)f_{s(n)}|^2 dvol(x) \\
&\leq C1/n^4\int_{Z_1\times \mathbb{R}_+} |\frac{\partial^2}{\partial u_1^2}(\kappa)(u_1/n)f_{s(n)}|^2 dvol(x) \\
&\leq C1/n^4||f_{s(n)}|| = C1/n^4 \to 0.
\end{aligned}
$$

Finally for C_n, we have

$$
C_n := ||\kappa_n(u_1)\left(H_{1,\theta} - \lambda\right)f_{s(n)}|| \leq C||\left(H_\theta - \lambda\right)f_{s(n)}|| \to 0,
$$

because $(f_{s(n)})$ is a b.W.s. for λ and H_θ. □

Next we prove step ii) of the proof of $\sigma_{ess}(H_\theta) = \mathscr{F}_\theta$. Recall that ii) is the claim that there exists $\eta_0^d \in C_c^\infty(X)$ such that $supp(\eta_0^d) \subset X_d$ and for all $f \in C_c^\infty(X,E)$, $||[H_\theta, \eta_0^d]f||_{L^2(X,E)} \leq \epsilon(d)(||H_\theta f||_{L^2(X,E)} + ||f||_{L^2(X,E)})$ with $\lim_{d\to\infty}\epsilon(d) = 0$. Let $\eta \in C^\infty(\mathbb{R})$ such that $\eta(u) = 1$ for $u \leq 1$, $\eta(u) = 0$ for $u > 2$ and $\eta'(u) \leq 0$. Denote $\eta_n(u) := \eta(u/n)$, and define

$$
\eta_0^d(u_1, u_2, y) := \eta_d(u_1)\eta_d(u_2). \tag{2.48}
$$

Proposition 31 *For all $f \in C_c^\infty(X,E)$,*

$$
||[H_\theta, \eta_0^d]f||_{L^2(X,E)} \leq \epsilon(d)(||H_\theta f||_{L^2(X,E)} + ||f||_{L^2(X,E)}), \tag{2.49}
$$

with $\lim_{d\to\infty}\epsilon(d) = 0$.

Proof:
Let $i, j \in \{1, 2\}$ and $i \neq j$. Observe that

$$\begin{aligned} H_\theta(\eta_0^d f) = \theta' \sum_{i=1}^{1} \{ & 2\eta_d(u_j)\frac{\partial}{\partial u_i}(\eta_d)(u_i)\frac{\partial}{\partial u_i}(f) \\ & + \eta_d(u_j)\frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i)f\} \\ & + \eta_d(u_j)\eta_d(u_i)H_\theta(f). \end{aligned} \tag{2.50}$$

Hence,

$$\begin{aligned} ||[H_\theta, \eta_0^d]f||_{L^2(X,E)} \leq & |\theta'| \cdot \sum_{i=1}^{1} \{ 2||\eta_d(u_j)\frac{\partial}{\partial u_i}(\eta_d)(u_i)\frac{\partial}{\partial u_i}(f)||_{L^2(X,E)} \\ & + ||\eta_d(u_j)\frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i)f||_{L^2(X,E)}\}. \end{aligned} \tag{2.51}$$

Since, by definition of η_d, $\eta_0^d(u_1, u_2, y) = \eta(u_1/d)\eta(u_2/d)$ then $\frac{\partial}{\partial u_1}(\eta_0^d)(u_1, u_2, y) = \frac{1}{d}\frac{\partial}{\partial u_1}(\eta)(u_1/d)\eta_d(u_2)$. Thus,

$$\begin{aligned} & ||\eta_d(u_j)\frac{\partial}{\partial u_i}(\eta_d(u_i))\frac{\partial}{\partial u_i}(f)||^2_{L^2(X,E)} \leq \int_X |\eta_d(u_j)\frac{\partial}{\partial u_i}(\eta_d)(u_i)\frac{\partial}{\partial u_i}(f)|^2 dx \\ & \leq 1/d^2 \int_{Y\times \mathbb{R}^2_+} |\frac{\partial}{\partial u_1}(\eta)(u_1/d)\eta_d(u_2)\frac{\partial}{\partial u_i}(f)|^2 dx \leq \frac{C}{d^2}(||f||_{L^2(X,E)} + ||H_\theta f||_{L^2(X,E)})^2 \to 0. \end{aligned} \tag{2.52}$$

In the last inequality we use that $\frac{\partial}{\partial u_1}(\eta)(u_1/d)\eta_d(u_2)\frac{\partial}{\partial u_i}(f)$ is a bounded differential operator of degree 1 (hence a continuous operator from $\mathscr{W}_2(X, E)$ to $L^2(X, E)$) and the fact that the norm $f \mapsto ||f||_{L^2(X,E)} + ||H_\theta f||_{L^2(X,E)}$ is equivalent to $||.||_2$ by theorem 39. We finish our proof of the proposition with the following calculation:

$$\begin{aligned} ||\eta_d(u_j)\frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i)f||^2_{L^2(X,E)} \leq & \int_X |\eta_d(u_j)\frac{\partial^2}{\partial u_i^2}(\eta_d)(u_i)f|^2 dx \\ & 1/d^4 \int_{Y\times \mathbb{R}^2_+} |f|^2 dx \to 0. \square \end{aligned} \tag{2.53}$$

2.6 Analytic vectors

Let $\mathscr{V}_i$ be the analytic vectors associated to $U_{i,\theta}$ (see equation (1.65)). Let $\eta \in C^\infty(\mathbb{R}_+)$ be such that $\eta' \geq 0$ and

$$\eta(u) = \begin{cases} 1 & K < u < \infty. \\ 0 & 0 < u \leq K-1. \end{cases}$$

For $i = 1, 2$, define $\eta_i(z_i, u_i) := \eta(u_i)$ and extend them to all X. Denote $\kappa := 1 - \eta_1 - \eta_2$. Define

$$\mathscr{V} := \{(\kappa g + \sum_{i=1}^{2} \eta_i \frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i) : g \in L^2(X, E), p_i(x) \in \mathbb{C}[x] \text{ and } f_i \in \mathscr{V}_i\}. \tag{2.54}$$

Proposition 32 *$\mathscr{V}$ is dense in $L^2(X, E)$.*

Proof:
Let $g \in L^2(X, E)$. We make the following formal calculation:

$$\begin{aligned} ||g - \kappa g + \sum_{i=1}^{2} \eta_i \frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i)||_{L^2(X,E)} &\leq \sum_{i=1}^{2} ||\eta_i \left(g - \frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i) \right) ||_{L^2(X,E)} \\ &\leq \sum_{i=1}^{2} ||g - \frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i)||_{L^2(Z_i, E_i \times \mathbb{R}_+)} \end{aligned} \tag{2.55}$$

In proposition 13 we proved that $\mathscr{V}_i$ is dense in $L^2(Z_i, E_i)$ and, implicitly, that $\{\frac{1}{u_i^2} p_i(\frac{1}{u_i}) : p_i(x) \in \mathbb{C}[x]\}$ is dense in $L^2(\mathbb{R}_+)$. Then, $\{\frac{1}{u_i^2} p_i(\frac{1}{u_i}) : p_i(x) \in \mathbb{C}[x]\} \otimes \mathscr{V}_i$ is dense in $L^2(\mathbb{R}_+) \otimes L^2(Z_i, E_i) = L^2(Z_i \times \mathbb{R}_+, E_i)$. Since $g|_{Z_i \times \mathbb{R}_+} \in L^2(Z_i \times \mathbb{R}_+, E_i)$, we can choose $\frac{1}{u_i^2} p_i(\frac{1}{u_i}) f_i(z_i)$ as near to $g|_{Z_i \times \mathbb{R}_+}$ as we want. $\square$

Proposition 33 *$\forall f \in \mathscr{V}$ the map $\theta \mapsto U_\theta f$ has an analytic extension from $\mathbb{R}_+$ to Γ with values in $L^2(X, E)$.*

Proof:
Observe that for $\theta \in \mathbb{R}_+$, using proposition 19, claim i), we have:

$$U_\theta \left(\eta_i(u_i)) 1/u_i^2 p(1/u_i) f_i(z_i) \right) = \psi'_\theta(u_i)^{1/2} \eta(\psi_\theta(u_i)) 1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i)) U_{i,\theta} f_i(z_i) =$$

$$\psi'_\theta(u_i)^{1/2}\eta(u_i)1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{i,\theta}f_i(z_i).$$

Using proposition 14 we see that the map $\theta \mapsto U_{i,\theta}f_i(z_i)$ extends to Γ, hence the previous term is also defined for $\theta \in \Gamma$.□

Proposition 34 *For $\theta \in \Gamma$, $U_\theta \mathscr{V}$ is dense in $L^2(X,E)$.*

Proof:
Observe that

$$\begin{aligned} U_\theta \mathscr{V} =& \{(\kappa g + \sum_{i=1}^{2} \psi'_\theta(u_i)^{1/2}\eta(u_i)1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{\theta,i}f_i(z_i) : g \in L^2(X,E), \\ & p_i(x) \in \mathbb{C}[x] \text{ and } f_i \in \mathscr{V}_i\}. \end{aligned} \tag{2.56}$$

Given $g \in L^2(X,E)$, we make the following formal calculation:

$$\begin{aligned} & ||g - \left(\kappa g + \sum_{i=1}^{2} \psi'_\theta(u_i)^{1/2}\eta(u_i)1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{i,\theta}f_i(z_i)\right)||_{L^2(X,E)} \\ & \leq \sum_{i=1}^{2} ||\eta_i(g - \psi'_\theta(u_i)^{1/2}1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{i,\theta}f_i(z_i))||_{L^2(X,E)} \\ & \leq \sum_{i=1}^{2} ||g - \psi'_\theta(u_i)^{1/2}1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{i,\theta}f_i(z_i)||_{L^2(Z_i,E_i\times \mathbb{R}_+)} \end{aligned} \tag{2.57}$$

Using the fact that $U_{i,\theta}\mathscr{V}_i$ is dense in $L^2(Z_i,E_i)$ (corollary 3) and lemma 4, we have that $(U_{i,\theta}(\mathscr{V}_i)) \otimes \mathscr{B}_\theta$ is dense in $L^2(\mathbb{R}_+) \otimes L^2(Z_i,E_i) = L^2(Z_i \times \mathbb{R}_+, E_i)$. Hence, we can take $\psi'_\theta(u_i)^{1/2}1/\psi_\theta(u_i)^2 p(1/\psi_\theta(u_i))U_{i,\theta}f_i(z_i)$ as near to $g|_{Z_i\times \mathbb{R}_+}$ as we want. □

2.7 Consequences of Aguilar-Balslev-Combes theory

In section 2.5 we calculated the essential spectrum of H_θ. In this section we complete the description of $\sigma(H_\theta)$ giving some information about $\sigma_d(H_\theta)$ and $\sigma_{pp}(H_\theta)$; most of the results that we compile here are consequences of the Aguilar-Balslev-Combes theory explained in appendix B. We have seen in the preceding sections that the dilation family $U_\theta, \mathscr{V}$ for the operator H

satisfies assumptions 1), 2), 3) of appendix B; in this section we write the consequences of 1), 2), 3) of appendix B for the spectrum of H_θ, particularly for $\sigma_d(H_\theta)$ and $\sigma_{pp}(H_\theta)$, for $\theta \in \Gamma$, since $\sigma_{ess}(H_\theta)$ was already described in section 2.5 as was mentioned before.

We define the set:

$$\tau(H) := \sigma_{pp}(H^{(1)}) \cup \sigma_{pp}(H^{(2)}) \cup \sigma_{pp}(H^{(3)}), \tag{2.58}$$

we call $\tau(H)$ the **set of thresholds**. The next theorem synthesizes the consequences of 1), 2) and 3) of appendix B for $\sigma(H_\theta)$, $\theta \in \Gamma$. As theorem 5 should be compared theorem 1.1 in [2], theorem 16.4 in [17], with theorem 11 in this thesis, and the theorem in page 14 of [27].

Theorem 11 *We have:*

a) The set of non-threshold eigenvalues of H (see equation (2.58)) is equal to $\sigma_d(H_\theta) \cap \mathbb{R}$, for all $\theta \in \Gamma - \mathbb{R}_+$. Moreover, given λ_0 non-threshold eigenvalue, the eigenspace $E_{\lambda_0}(H)$, associated to H and λ_0, has finite dimension bounded by the degree of the pole λ_0 of the map $\lambda \mapsto R(\lambda, \theta)$. This algebraic multiplicity is independent of $\theta \in \Gamma - \mathbb{R}_+$.

b) Fix $\theta \in \Gamma$. For $f, g \in \mathscr{V}$ the function

$$\lambda \mapsto \langle R(\lambda) f, g \rangle_{L^2(X,E)}$$

has a meromorphic continuation from Λ to $\mathbb{C} - (\sigma_{ess}(H_\theta) \cup \sigma_{pp}(H_\theta))$, where $\sigma_{ess}(H_\theta)$ was calculated in theorem 10.

c) H has no singular spectrum.

d) Let $\theta_1, \theta_2 \in \Gamma$ be such that $arg(\theta_1') \geq arg(\theta_0')$ for $0 < arg(\theta_i') < \pi/2$, we have:

$$\sigma_d(H_{\theta_0}) = \sigma_d(H_{\theta_1}) \cap \sigma_d(H_{\theta_0}). \tag{2.59}$$

e) Non-thresholds eigenvalues of H are isolated (respect to the eigenvalues of H) and, in case they accumulate, they accumulate on the set of thresholds or on ∞.

f) If the lowest threshold, γ_0, is larger than 0, then $\sigma_d(H)$ is a discrete subset of $[0, \gamma_0)$. In this case, the unique possible accumulation point of $\sigma_d(H)$ is γ_0. If $\gamma_0 = 0$, then $\sigma_d(H) = \emptyset$, in other words all eigenvalues are embedded in the continuous spectrum.

At the moment we do not know if there is a compatible Laplacian that has embedded eigenvalues. It seems that the natural conjecture is that, generically, there is not embedded eigenvalues. Similarly, we do not know if it is possible to find a compatible Laplacian that has embedded eigenvalues accumulating in one of the thresholds. We believe that it is possible to prove that $\sigma_{pp}(H_\theta)$ can only accumulate by below on $\tau(H)$, and we hope to show this in other paper. In particular, it would imply that 0 is not an accumulation point of $\sigma_{pp}(H_\theta)$.

The next proposition pretends to give a little more information about $\sigma_{pp}(H_\theta)$ for arbitrary $\theta \in \Gamma$. It is easy to prove from the definition of essential spectrum and from the form of $\sigma_{ess}(H_\theta)$ (see section 1.4).

Proposition 35 *i) If $\lambda \in \sigma_{pp}(H_\theta)$ and $\lambda \notin \sigma_{ess}(H_\theta)$, then λ an isolated eigenvalue of finite multiplicity.*

ii) For $Re(\theta) > 0$, the accumulation points of $\sigma_{pp}(H_\theta)$ are contained in $\sigma_{ess}(H_\theta)$. In particular, the real part of the pure point spectrum of H_θ can only accumulate in $\tau(H)$.

As an application of analytic dilation we define generalized eigenfunctions associated to L^2-eigenvectors of $H^{(k)}$, for $k = 1, 2, 3$. As we said in the introduction, we will prove in a forthcoming paper that these generalized eigenfunctions express the wave operators $W(H, H_{k,pp})$ (see (11)) for $k = 1, 2$ and the wave operator $\Omega_\pm$ (see (12)).

Chapter 3

Generalized eigenfunctions

In this section we define generalized eigenfunctions for H associated to L^2-eigenfunctions of $H^{(k)}$ for $k = 1, 2, 3$. Using the method of analytic dilation we meromorphically extend the domain where they are defined.

3.1 The generalized eigenfunctions associated to $H^{(k)}$ for $k = 1, 2$

In this subsection we define the generalized eigenfunctions associated to elements in the pure point part of $L^2(Z_k, E_k)$, that we denote by $L^2_{pp}(Z_k, E_k)$, for $k = 1, 2$. Along this subsection $k \in \{1, 2\}$. We begin by establishing some notation.

We define:

$$\sigma_0 := \min \left(\sigma(H^{(3)}) \cup \bigcup_{k=1}^{2} \sigma_{pp}(H^{(k)}) \right). \tag{3.1}$$

Denote by C_+ the cone $\{\lambda \in \mathbb{C} : 0 \leq \arg(\lambda - \sigma_0) \leq \pi/4\}$ and by C_- the cone $\{\lambda \in \mathbb{C} : -\pi/4 \leq \arg(\lambda - \sigma_0) \leq 0\}$. Define:

$$\mathscr{S}_{\pm\pi/4} := \left(C_\pm - \sigma(H_{\theta_\pm}) \right). \tag{3.2}$$

Observe that $\mathscr{S}_{\pm\pi/4}$ is a manifold with boundary $[\sigma_0, \infty)$. We think $\mathbb{C} - [\sigma_0, \infty)$ as an open subset of the 2-covering of $\mathbb{C}$ where $\sqrt{\cdot - \sigma_0}$ is defined. Observe that the boundary points of $\mathbb{C} - [\sigma_0, \infty)$ is naturally identified with the union of two copies of $[\sigma_0, \infty)$ identified only in the point σ_0. We denote $\mathscr{S}_0$ the union of $\mathbb{C} - [\sigma_0, \infty)$ and its boundary points. See the graphic below.

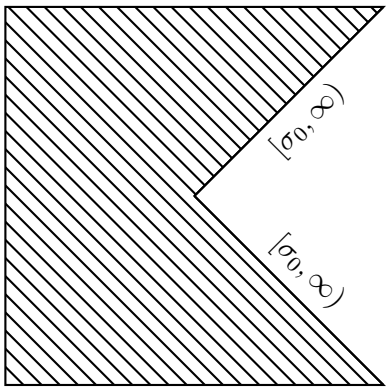

Figure 4. $\mathscr{S}_0$

Let $i : [\sigma_0, \infty) \to [\sigma_0, \infty)$ be the identity in $[\sigma_0, \infty)$. We define the surface $\mathscr{S}$ in two steps. First we joint to $\mathscr{S}_0$ the set $\mathscr{S}_{-\pi/4}$ identifying the copy $[\sigma_0, \infty)$ of $\mathscr{S}_0$ corresponding to the boundary points of the set $[\sigma_0, \infty) \times \mathbb{R}_+$ with $[\sigma_0, \infty) \subset \mathscr{S}_{-\pi/4}$. The next graphic describes this first step:

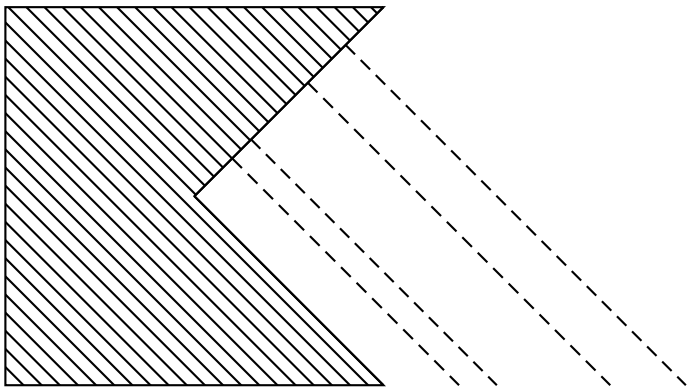

Figure 5. $\mathscr{S}_0 \cup \mathscr{S}_{-\pi/4}$

Observe that $\mathscr{S}_0 \cup_i \mathscr{S}_{-\pi/4}$ is a connected topological space that has a natural structure of complex manifold. In the second step we joint to the surface $\mathscr{S}_0 \cup_i \mathscr{S}_{-\pi/4}$ the set $\mathscr{S}_{\pi/4}$ identifying the copy $[\sigma_0, \infty)$ corresponding to the boundary points of the set $[\sigma_0, \infty) \times (-\infty, 0]$ of $\mathscr{S}_0$, with $[\sigma_0, \infty) \subset \mathscr{S}_{-\pi/4}$. We have defined:

$$\mathscr{S} := \mathscr{S}_0 \cup_i \mathscr{S}_{\pi/4} \cup_i \mathscr{S}_{-\pi/4}, \tag{3.3}$$

The next figure is a sketch of $\mathscr{S}$:

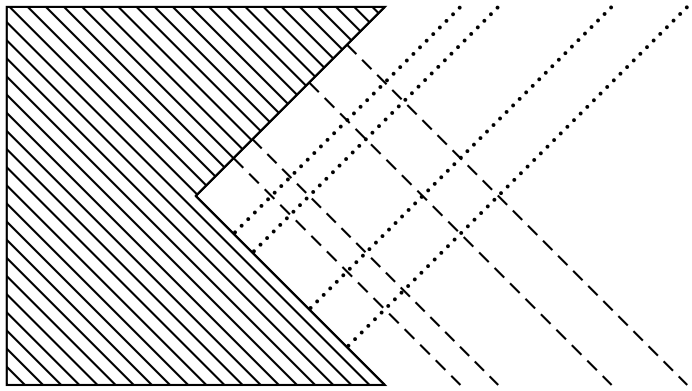

Figure 6. $\mathscr{S} := \mathscr{S}_0 \cup \mathscr{S}_{-\pi/4} \cup \mathscr{S}_{\pi/4}$

Observe that $\mathscr{S}$ has a natural complex structure. Recall that $\Lambda := \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$ and observe that it is included naturally in $\mathscr{S}$. Denote by $\pi_{\mathscr{S}} : \mathscr{S} \to \mathbb{C}$ the natural function induced by the inclusions of $\mathscr{S}_{\pm\pi/4}$ and $\mathscr{S}_0$ in $\mathbb{C}$.

The following equation shows that there are infinite $\theta_\pm$ in Γ (see (1.81)) that $arg(\frac{1}{(\theta_\pm+1)^2}) = \pm\pi/4$:

$$arg\left(\frac{1}{(\theta_\pm+1)^2}\right) = -2\arg(\theta_\pm+1) = \pm\frac{\pi}{4}. \tag{3.4}$$

As a consequence of this fact, we have the next theorem whose proof is similar to the proof of theorem 26:

Theorem 12 *Let $f, g \in \mathscr{V}$, the the function $\lambda \mapsto \langle R(\lambda)f, g\rangle_{L^2(X,E)}$ extends meromorphically from Λ to $\mathscr{S}$.*

Let $\varphi \in L^2(Z_k, E_k)$ be such that $H^{(k)}\varphi = \gamma\varphi$ and let $\lambda \in \mathbb{C}$. Our next goal is to associate with φ a generalized eigenfunction of the operator H. Define $h_k(\varphi, \lambda)$ by:

$$h_k(\varphi, \lambda, u_k, z_k) := e^{-\sqrt{\gamma-\lambda}u_k}\varphi(z_k), \tag{3.5}$$

where we are taking the branch of $\sqrt{\cdot}$ such that, if $s \in \mathbb{R}_+$, then $\sqrt{s} > 0$. This $\sqrt{\cdot}$ induces the function $\lambda \mapsto \sqrt{\gamma-\lambda}$ that has a natural extension to $\mathscr{S}$. Hence $h_k(\varphi, \lambda)$ is defined for $\lambda \in \mathscr{S}$. Note that $h_k(\varphi, \lambda)$ satisfies the equation:

$$\left(-\frac{\partial^2}{\partial u_k^2} + H^{(k)}\right)h_k(\varphi, \lambda) = \pi_{\mathscr{S}}(\lambda)h_k(\varphi, \lambda), \tag{3.6}$$

for all $\lambda \in \mathscr{S}$. Let κ be such that $\kappa(u) = 0$ for $u \leq 1$ and $\kappa(u) = 1$ for $u \geq 2$. Define $\kappa_k \in C^\infty(\mathbb{R}_+ \times Z_k)$ by $\kappa_k(z_k, u_k) := \kappa(u_k)$ (for $(z_k, u_k) \in Z_k \times \mathbb{R}_+$), and extend it naturally to all X. Recall that we denote by $\mathscr{S}(X, E)$ the intersection of all the Sobolev spaces (see (E.4)). We denote Δ the Laplacian acting on distributional sections of E. Observe that

$$L_k(\varphi, \lambda) := (\Delta - \lambda)(\kappa_k h_k(\varphi, \lambda)) = \{-\partial^2_{u_k}(\kappa_k) - 2 - \sqrt{\gamma - \lambda}\partial_{u_k}(\kappa_k)\}h_k(\varphi, \lambda) \tag{3.7}$$

belongs to $\mathscr{S}(X, E)$ (because $\varphi \in L^2(Z_k, E_k)$). Hence, for $\lambda \in \mathbb{C} - \mathbb{R}_+$, we can apply the resolvent $(H - \lambda)^{-1}$ to $L_k(\varphi, \lambda)$. For $\lambda \in \mathbb{C} - \mathbb{R}_+$ define:

$$F_k(\varphi, \lambda) := \kappa_k h_k(\varphi, \lambda) - (H - \lambda)^{-1}((\Delta - \lambda)(\kappa_k h_k(\varphi, \lambda))). \tag{3.8}$$

$F_k(\varphi, \lambda)$ satisfies:

$$\Delta(F_k(\varphi, \lambda)) = \lambda F_k(\varphi, \lambda), \tag{3.9}$$

for $\lambda \in \mathbb{C} - \mathbb{R}_+$.

The following lemma can be proved in the same way that proposition 5.

Lemma 6 *For $\theta \in \Gamma$, H_θ is uniformly elliptic.*

In order to show that $\lambda \mapsto F_k(\varphi, \lambda)$ extends to $\lambda \in \mathscr{S}$, we need the following lemma.

Lemma 7 *Let $\varphi \in L^2(Z_k, E_k)$ be an eigenfunction of $H^{(k)}$ with eigenvalue γ, $k = 1, 2$. Then:*

i) The function $\theta \mapsto U_{k,\theta}\varphi$ is defined for $\theta \in \mathbb{R}_+$ and extends meromorphically to all $\theta \in \Gamma$.

ii) $(H^{(k)}_\theta)^m U_{k,\theta}\varphi = \gamma^m U_{k,\theta}\varphi$ for all $\theta \in \Gamma$.

iii) $U_{k,\theta}\varphi \in \mathscr{S}(Z_k, E_k)$ (the intersection of all the Sobolev spaces, defined in (E.4)).

Proof:
Part i) is proved in lemma 5. For $(y, u) \in Y \times \mathbb{R}_+$, $\varphi(y, u) = \sum_{j=0}^\infty f_j(u)\phi_j(y)$, where $\{(\phi_j, \mu_j)\}_{j=0}^\infty$ is an spectral resolution of $H^{(3)}$. Then, since $H^{(k)}\varphi = \gamma\varphi$:

$$\varphi(y, u) = \sum_{\mu_j > \gamma}^\infty a_j e^{-\sqrt{\mu_j - \gamma}u}\phi_j(y).$$

For $\theta \in \Gamma$, we define:

$$U_{k,\theta}\varphi(y,u) := \begin{cases} \varphi(z) & z \in M_k. \\ \psi'_\theta(u)^{1/2} \sum_{\mu_j > \gamma}^{\infty} a_j e^{-\sqrt{\mu_j - \gamma}\psi_\theta(u)} \phi_j(y) & z = (y,u) \in Y \times \mathbb{R}_+. \end{cases} \tag{3.10}$$

For $\theta \in \mathbb{R}_+$ and for all $z \in Z$, we have: $H_\theta^{(k)} U_{k,\theta}\varphi(z) = \gamma U_{k,\theta}\varphi(z)$. Since the function $\theta \mapsto H_\theta^{(k)} U_{k,\theta}\varphi(z)$ is an holomorphic function in θ, then we have that $H_\theta^{(k)} U_{k,\theta}\varphi(z) = \gamma U_{k,\theta}\varphi(z)$ for all $\theta \in \Gamma$. Similarly one can prove

$$(H_\theta^{(k)})^m U_{k,\theta}\varphi = \gamma^m U_{k,\theta}\varphi \in L^2(Z_k, E_k) \text{for all } \theta \in \gamma. \tag{3.11}$$

Since $H_\theta^{(k)}$ is uniformly elliptic (proposition 5), then, by the theorem 39, we have the equivalence of the norms $f \mapsto ||H_\theta^{(k)} f||_{L^2(Z_k,E_k)} + ||f||_{L^2(Z_k,E_k)}$ and $f \mapsto ||H^{(k)} f||_{L^2(Z_k,E_k)} + ||f||_{L^2(Z_k,E_k)}$; this together with equation (3.11) imply that $U_{k,\theta}\varphi \in \mathscr{W}_{2m}(Z_k, E_k)$ for $\theta \in \Gamma$. Hence $U_{k,\theta}\varphi \in \mathscr{S}(Z_k, E_k)$.$\square$

Denote:

$$L_k(\varphi, \lambda) := (\Delta - \lambda)(\kappa_k h_k(\varphi, \lambda)). \tag{3.12}$$

Corollary 7 *Let $\varphi \in L^2(Z_k, E_k)$ be an eigenfunction of $H^{(k)}$ with eigenvalue γ. Let $\lambda \in S$ fixed. Then:*

i) The function $\theta \mapsto U_\theta L_k(\varphi, \lambda)$ is defined for $\theta \in \mathbb{R}_+$ and extends meromorphically to all $\theta \in \Gamma$.

ii) $U_\theta L_k(\varphi, \lambda) \in \mathscr{S}(X, E)$.

Proof:
Observe that:

$$\begin{aligned} L_k(\varphi, \lambda) &= (\Delta - \lambda)(\kappa_k h_k(\varphi, \lambda)) \\ &= (-\frac{\partial^2}{\partial u_k^2} + H^{(k)}(\kappa_k h_k(\varphi, \lambda)) \\ &= -\frac{\partial^2}{\partial u_k^2}(\kappa_k) h_k(\varphi, \lambda) - 2\frac{\partial}{\partial u_k}(\kappa_k)\frac{\partial}{\partial u_k} h_k(\varphi, \lambda). \end{aligned} \tag{3.13}$$

Then we can define:

$$\begin{aligned} &U_\theta\left((\Delta-\lambda)(\kappa_k h_k(\varphi,\lambda))\right)(z_k,u_k) := \\ &\quad \psi'_\theta(u_k)^{1/2}\big(\frac{\partial^2}{\partial u_k^2}(\kappa_k)(\psi_\theta(u_k))U_{k,\theta}h_k(\varphi,\lambda)(z_k,\psi_\theta(u_k)) \\ &\quad + 2\frac{\partial}{\partial u_k}(\kappa_k)(\psi_\theta(u_k))\frac{\partial}{\partial u_k}U_{k,\theta}h_k(\varphi,\lambda)(z_k,\psi_\theta(u_k))\big), \end{aligned} \tag{3.14}$$

for $\theta \in \mathbb{R}_+$. But,

$$\frac{\partial^2}{\partial u_k^2}(\kappa_k)(\psi_\theta(u_k)) = \frac{\partial^2}{\partial u_k^2}(\kappa_k)(u_k), \tag{3.15}$$

for $\theta \in \mathbb{R}_+$. Hence the equality (3.15) holds for all $\theta \in \Gamma$. Similarly $\frac{\partial}{\partial u_k}(\kappa_k)(\psi_\theta(u_k)) = \frac{\partial}{\partial u_k}(\kappa_k)(u_k)$. We have

$$U_\theta h_k(\varphi,\lambda)(z_k,\psi_\theta(u_k)) = e^{-\sqrt{\gamma-\lambda}\psi_\theta(u_k)}U_{k,\theta}\varphi(z_k). \tag{3.16}$$

We have seen in lemma 5 that $\theta \mapsto U_{k,\theta}\varphi$ has an analytic extension to $\theta \in \Gamma$. These remarks prove that we can use (3.14) for defining $U_\theta L_k(\varphi,\lambda)$.

That $U_\theta L_k(\varphi,\lambda) \in \mathscr{S}(X,E)$, for $\theta \in \Gamma$, follows from equation (3.13) and part iii) of lemma 7. $\square$

For $T > 1$ and $\theta \in \mathbb{R}_+$, we introduce the operator ${}^TU_\theta$ acting on $L^2(X,E)$ rescaling U_θ in such a way that ${}^TU_\theta$ is the identity in X_T. We define analytic vectors $\mathscr{V}_T$ in such a way that if $S \leq T$, then $\mathscr{V}_S \subset \mathscr{V}_T$. We suppose also that $\mathscr{V}_T$ contains the C^∞-sections of E whose support is inside X_T.

The following theorem generalizes theorem 6.5 in [30], because in our case $Ker(H^{(3)})$ can be different to 0.

Theorem 13 *Let φ be an L^2-eigenfunction of $H^{(k)}$ with eigenvalue γ. Then:*

1) For $\lambda \in \mathbb{C} - \mathbb{R}_+$ the function $\lambda \mapsto F_k(\varphi,\lambda)$, taking values on $C_c^\infty(X,E)'$, extends meromorphically to a function on $\mathscr{S}$ with values on $C_c^\infty(X,E)'$.

2) For $\lambda \in \mathscr{S}$, the distribution $F_k(\varphi,\lambda)$ is smooth in $x \in X$ and is a solution of the equation:

$$(\Delta - \pi_{\mathscr{S}}(\lambda))F_k(\varphi,\lambda) = 0. \tag{3.17}$$

Proof:
Let $g \in C_c^\infty(X,E)$. Then,

$$\langle F_k(\varphi,\lambda), g\rangle_{dist} = \langle \kappa_k h_k(\varphi,\lambda) - (H-\lambda)^{-1}((\Delta-\lambda)\kappa_k h_k(\varphi,\lambda)), g\rangle_{dist}. \quad (3.18)$$

Observe that the function $\lambda \mapsto \langle \kappa_k h_k(\varphi,\lambda), g\rangle_{dist}$ extends holomorphically to $\mathscr{S}$. Then we have to analyze the second term. Choose $T > 1$ in such a way that $supp(g) \subset X_T$. Using that U_θ^T is unitary for $\theta \in \mathbb{R}_+$, and that $g \in \mathscr{V}_T$ (for T large enough), we have for $\lambda \in \mathbb{C} - \mathbb{R}_+$, and $\theta \in \mathbb{R}_+$:

$$\begin{aligned} &\langle R(\lambda)(\Delta-\lambda)(\kappa_k h_k(\varphi,\lambda)), g\rangle_{dist} = \\ &\langle ({}^T U_\theta) R(\lambda) ({}^T U_\theta)^{-1} ({}^T U_\theta)((\Delta-\lambda)(\kappa_k h_k(\varphi,\lambda))), {}^T U_{\overline{\theta}} g\rangle_{L^2(X,E)} \\ &= \langle R(\lambda,\theta,T)({}^T U_\theta)((\Delta-\lambda)\kappa_k h_k(\varphi,\lambda)), ({}^T U_{\overline{\theta}}) g\rangle_{L^2(X,E)} \end{aligned} \quad (3.19)$$

Corollary 7 implies that we can take the analytic vectors $\mathscr{A}_T$ associated to U_θ^T in such a way that $(\Delta-\lambda)(\kappa_k h_k(\varphi,\lambda)) \in \mathscr{A}_T$. Recall also that, if $T_1 \leq T_2$ we suppose that $\mathscr{A}_{T_1} \subset \mathscr{A}_{T_2}$. Using the general Aguilar-Balslev-Combes theory (see theorem 30) for $\theta \in \Gamma$ fixed, this choice of $\mathscr{A}_T$ implies that the analytic extension of the function

$$\lambda \mapsto \langle R(\lambda,\theta) U_\theta^T((\Delta-\lambda)\kappa_k h_k(\varphi,\lambda)), U_\theta^T g\rangle_{L^2(X,E)}$$

is independent of T (for T large enough) and that the functional is continuous respect the Frechet structure of $C_0^\infty(X,E)$. This proves that $F_k(\varphi,\lambda)$ is a well defined distribution for $\lambda \in \mathscr{S}$.

For proving that $F_k(\varphi,\lambda) \in C^\infty(X,E)$, observe that it is a solution of equation (3.17) and H is a uniformly elliptic operator. □

3.2 The generalized eigenfunctions associated to $H^{(3)}$

Let $\phi \in L^2(Y,S)$ be an eigenfunction of $H^{(3)}$ with eigenvalue μ and such that $||\phi||_{L^2(Y,S)} = 1$. In this section, for $k = 1,2$, we consider the generalized eigenfunctions $E_k(\phi,\lambda)$ of $H^{(k)}$ parametrized in such a way that

$$H^{(k)} E_k(\phi,\lambda) = \lambda E_k(\phi,\lambda). \quad (3.20)$$

We observe that, from the results of [21], it follows that the functions $E_k(\phi,\lambda)$ are defined for $\lambda \in \mathscr{S}$ (see (3.3) for a definition of $\mathscr{S}$). We have the following theorem:

Theorem 14 *[21] Let b_i be the self-adjoint extension of $-\frac{\partial^2}{\partial u_i^2}$, acting on $L^2(\mathbb{R}_+)$ with Von Neumann boundary conditions, for $i = 1, 2$. Suppose that $j, k \in \{1, 2\}$ and $j \neq k$, then:*

i) The wave operators $W_{k,\pm}(H^{(k)}, b_j \otimes 1 + 1 \otimes H^{(3)})$ exists and are complete.

ii) $W_{k,\pm}(H^{(k)}, b_j \otimes 1 + 1 \otimes H^{(3)}) f(z_k) = \sum_{l=0}^{\infty} \int_{\mu_l}^{\infty} E_k(\phi_l, \lambda)(z_k) f(\lambda) \frac{d\lambda}{\sqrt{\lambda - \mu_l}}$, where $\{\phi_l, \mu_l\}_{l=0}^{\infty}$ is a spectral resolution of $H^{(3)}$.

In the next definitions and calculations we often use the index j, k in such a way that $j, k \in \{1, 2\}$ and $j \neq k$. We use this notation because on $Z_j \times \mathbb{R}_+$ the variable that belongs to $\mathbb{R}_+$ is denoted by u_j, and the real variable in the cylinder $Y \times \mathbb{R}_+$, by u_k. Observe that the term

$$\begin{aligned} P(\phi, \lambda_1, \lambda_2) :=& (\Delta - \pi_{\mathscr{S}}(\lambda_1) - \pi_{\mathscr{S}}(\lambda_2) - \mu)\{\sum_{k=1}^{2} \left(\kappa_k e^{-i\sqrt{\lambda_j} u_k} E_k(\phi, \lambda_k + \mu)\right) \\ & - \kappa_1 \kappa_2 e^{-i\sqrt{\lambda_1} u_2} e^{-i\sqrt{\lambda_2} u_1} \phi\}, \end{aligned}$$

makes sense for $\mathscr{S}$, since the functions $\lambda \mapsto \sqrt{\lambda}$ and $\lambda_k \mapsto E_k(\phi, \lambda_k + \mu)$ can be extended from $\mathbb{C} - \mathbb{R}_+$ to $\mathscr{S}$. We show that $P(\phi, \lambda_1, \lambda_2) \in L^2(X, E)$ for $Re(\lambda_k) < -\mu$. We have:

$$\begin{aligned} P(\phi, \lambda_1, \lambda_2) = & -\sum_{k=1}^{2} e^{-i\sqrt{\lambda_j} u_k} \{\partial_{kk}(\kappa_k) + 2\sqrt{\lambda_j} \partial_k(\kappa_k)\} \left(E_k(\phi, \lambda_k + \mu)\right. \\ & \left. - \kappa_j e^{-i\sqrt{\lambda_k} u_j} \phi\right). \end{aligned} \tag{3.21}$$

Observe that $E_k(\phi, \lambda_k + \mu) - \kappa_j e^{-i\sqrt{\lambda_k} u_j} \phi \in L^2(Z_k, E_k)$ for $Re(\lambda_k) < -\mu$, $k = 1, 2$. Hence

$$\begin{aligned} \partial_{kk}(\kappa_k) \left(E_k(\phi, \lambda_k + \mu) - \kappa_j e^{-i\sqrt{\lambda_k} u_j} \phi\right) \text{ and} \\ \partial_k(\kappa_k) \left(E_k(\phi, \lambda_k + \mu) - \kappa_j e^{-i\sqrt{\lambda_k} u_j} \phi\right) \text{ are in } L^2(X, E). \end{aligned}$$

What proves $P(\phi, \lambda_1, \lambda_2) \in L^2(X, E)$ for $Re(\lambda_k) < -\mu$, $k = 1, 2$.

For $Re(\lambda_k) < -\mu$, $k = 1, 2$, define:

$$\begin{aligned} F(\phi,\lambda_1,\lambda_2) := \sum_{k=1}^{2} \Big(& \kappa_k e^{-i\sqrt{\lambda_j}u_k} E_k(\phi, \lambda_k + \mu) \\ & -\kappa_k \kappa_j e^{-i\sqrt{\lambda_k+\mu}u_j} e^{-i\sqrt{\lambda_j}u_k} \phi \Big) \\ & - (H - \lambda_1 - \lambda_2 - \mu)^{-1} \left((\Delta - \lambda_1 - \lambda_2 - \mu) \right. \\ & \left. \{ \big(\sum_{k=1}^{2} \kappa_k e^{-i\sqrt{\lambda_j}u_k} E_k(\phi, \lambda_k + \mu) \big) - \kappa_1 \kappa_2 e^{-i\sqrt{\lambda_1+\mu}u_2} e^{-i\sqrt{\lambda_2}u_1} \phi \} \right) . \end{aligned} \tag{3.22}$$

Now we prove that $F(\phi, \lambda_1, \lambda_2)$ is well defined. For that, we observe that $\lambda_1 + \lambda_2 + \mu$ is in the resolvent set of H, because $Re(\lambda_1 + \lambda_2 + \mu) < 0$. Then $(H - \lambda_1 - \lambda_2 - \mu)^{-1}$ makes sense. Since:

$$\begin{aligned} F(\phi,\lambda_1, \lambda_2) = & \left(\sum_{k=1}^{2} \kappa_k e^{-i\sqrt{\lambda_j}u_k} E_k(\phi, \lambda_k + \mu) \right) \\ & - \kappa_1 \kappa_2 e^{-i\sqrt{\lambda_1}u_2} e^{-i\sqrt{\lambda_2}u_1} \phi \\ & - (H - \lambda_1 - \lambda_2 - \mu)^{-1} \left(P(\phi, \lambda_1, \lambda_2) \right), \end{aligned} \tag{3.23}$$

and above we proved that $P(\phi, \lambda_1, \lambda_2) \in L^2(X, E)$. Then, we can conclude that $F(\phi, \lambda_1, \lambda_2)$ is in fact well defined for $Re(\lambda_k) < -\mu$, $k = 1, 2$.

It is easy to see that $F(\phi, \lambda_1, \lambda_2)$ is a generalized eigenfunction of H, more explicitly:

$$\Delta(F(\phi, \lambda_1, \lambda_2)) = (\lambda_1 + \lambda_2 + \mu) F(\phi, \lambda_1, \lambda_2), \tag{3.24}$$

for $Re(\lambda_k) < -\mu$, $k = 1, 2$.

Denote by θ_- the complex number such that $\frac{1}{(\theta_- + 1)^2} = -\pi/8$. Define the set

$$\Lambda_- := +^{-1} \left(\mathbb{C} - \sigma(H_{\theta_-}) \right) \cap \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_k \notin \sigma(H_{\theta_-})\}. \tag{3.25}$$

where we are denoting by $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ the function $(\lambda_1, \lambda_2) \mapsto \lambda_1 + \lambda_2$.

We will extend $(\lambda_1, \lambda_2) \mapsto F(\phi, \lambda_1, \lambda_2)$ from the domain

$$\{(\lambda_1, \lambda_2) \in \mathbb{C} : \pi/4 < arg(\lambda_k) < \pi/2 \text{ for } k = 1, 2\}$$

to Λ_-. The following proposition shows that Λ_- is a domain.

Proposition 36 *Λ_- is an open path-connected set.*

Proof:
Λ_- is the intersection of the inverse, under the continuous function $+$, of the open set $\mathbb{C} - \sigma(H_{\theta_-})$, and the open set $\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_k \notin \sigma(H_{\theta_-})\}$; hence, it is open.

The proof that Λ_- is path connected is based in the three following observations. The first one is that the product of upper half-planes, $\mathbb{H} \times \mathbb{H}$, is contained in Λ_-. The second one is that $\mathbb{H} \times \mathbb{H}$ is path connected. The last observation is that all elements of Λ_- are path connected to an element of $\mathbb{H} \times \mathbb{H}$. Now we prove the third observation.

Take $(\lambda_1, \lambda_2) \in \Lambda_-$. Let $z_0 \in \mathbb{H}$ be such that $\lambda_1 + z_0 \notin \sigma(H_{\theta_-})$ and $\lambda_2 + z_0 \notin \sigma(H_{\theta_-})$. Let $f : [0,1] \to \mathbb{C}$ be a path joining λ_2 and z_0 in such a way that $z_0 + \lambda_1 + \lambda_2 - f(t) \notin \mathbb{C} - \sigma(H_{\theta_-})$ and $z_0 + \lambda_2 - f(t) \notin \sigma(H_{\theta_-})$. Such a path exists because we know $\sigma(H_{\theta_-})$, see equation (2.5). Define the path $\alpha_1(t) := (\lambda_1, z_0 + \lambda_2 - f(t))$ for $t \in [0,1]$; α_1 joins (λ_1, λ_2) and (λ_1, z_0). $\alpha_1(t) \in \Lambda_-$ by the way we chose $f(t)$.

Choose $g : [0,1] \to \mathbb{C}$, path joining z_0 and λ_1, in such a way that $z_0 + \lambda_1 + \lambda_2 - g(t) \notin \mathbb{C} - \sigma(H_{\theta_-})$ and $z_0 + \lambda_1 - g(t) \notin \sigma(H_{\theta_-})$. Define $\alpha_2(t) := (z_0 + \lambda_1 - g(t), z_0)$, α_2 joins (λ_1, z_0) and (z_0, z_0). Concatenating α_1 and α_2, we prove the third observation and with it the proposition. $\square$

We will apply the analytic dilation method for extending $F(\phi, \lambda_1, \lambda_2)$. The following proposition is necesary for taking that approach.

Proposition 37 *Suppose $(\lambda_1, \lambda_2) \in \Lambda_-$ and $k = 1, 2$. Then*

$$U_{k,\theta}\left(E_k(\phi, \lambda_k + \mu) - \kappa_j e^{i\sqrt{\lambda_k} u_j} \phi\right) \in L^2(Z_k, E_k) \tag{3.26}$$

for all θ such that $-\pi/8 < arg(\frac{1}{(\theta+1)^2}) < 3\pi/16$.

Proof:
Observe that for $\theta \in \mathbb{R}_+$ and $Re(\lambda_k) < -\mu$, we have:

$$\begin{aligned} &U_{k,\theta}\left(E_k(\phi, \lambda_k + \mu) - \kappa_j e^{i\sqrt{\lambda_k} u_j} \phi\right) \\ &= U_{k,\theta}\left((H^{(k)} - \lambda_k - \mu)^{-1}\{(\Delta_{Z_k} - \lambda_k - \mu)(\kappa_j e^{i\sqrt{\lambda_k} u_j} \phi)\}\right) \\ &= R_k(\lambda_k + \mu, \theta))^{-1}\{U_{k,\theta}(\Delta_{Z_k} - \lambda_k - \mu)(\kappa_j e^{i\sqrt{\lambda_k} u_j} \phi)\}, \end{aligned} \tag{3.27}$$

where $R_k(\lambda_k+\mu,\theta)$ denotes the operator $U_{k,\theta}((H^{(k)}-\lambda_k-\mu)^{-1}U_{k,\theta}^{-1}$, that is the resolvent of $H_\theta^{(k)}=U_{k,\theta}H^{(k)}U_{k,\theta}^{-1}$. Equations (3.27) hold for all $-\pi/8<arg(\frac{1}{(\theta+1)^2})<3\pi/16$ and $Re(\lambda_k)<-\mu$. In particular they hold for $Re(\lambda_k)<-\mu$ and θ_-, i.e.

$$\begin{aligned}&U_{k,\theta_-}\left(E_k(\phi,\lambda_k+\mu)-\kappa_j e^{i\sqrt{\lambda_k}u_j}\phi\right)\\&=R_k(\lambda_k+\mu,\theta_-))U_{k,\theta_-}\{(\Delta_{Z_k}-\lambda_k-\mu)(\kappa_j e^{i\sqrt{\lambda_k}u_j}\phi)\}.\end{aligned}\tag{3.28}$$

Observe that

$$U_{k,\theta}\{(\Delta_{Z_k}-\lambda_k-\mu)(\kappa_j e^{i\sqrt{\lambda_k}u_j}\phi)\}=(\Delta_{Z_k}-\lambda_k-\mu)(\kappa_j e^{i\sqrt{\lambda_k}u_j}\phi)\in L^2(Z_k,E_k),\tag{3.29}$$

taking $U_{k,\theta}=^T U_{k,\theta}$, for T large enough. Hence, the last term of (3.28) is well defined and, it is in $L^2(Z_k,E_k)$ for $(\lambda_1,\lambda_2)\in\Lambda_-$. □

From the equation (3.21) and the above proposition we deduce the following corollary.

Corollary 8 *Suppose $(\lambda_1,\lambda_2)\in\Lambda_-$. Then*

$$U_\theta P(\phi,\lambda_1,\lambda_2)\in L^2(X,E)\tag{3.30}$$

for all θ such that $-\pi/8<arg(\frac{1}{(\theta+1)^2})<3\pi/16$.

The next theorem provides the meromorphic extension of the function $(\lambda_1,\lambda_2)\mapsto F(\phi,\lambda_1,\lambda_2)$.

Theorem 15 *Let ϕ be an eigenfunction of $H^{(3)}$ with eigenvalue μ.*

1) The function $(\lambda_1,\lambda_2)\mapsto F(\phi,\lambda_1,\lambda_2)$, taking values on $C_c^\infty(X,E)'$, extends meromorphically from $Re(\lambda_i)<-\mu$ for $i=1,2$ to a function with domain Λ_- with values on $C_c^\infty(X,E)'$.

2) For $(\lambda_1,\lambda_2)\in\Lambda_-$, the distribution $F(\phi,\lambda_1,\lambda_2)$ is smooth in $x\in X$ and satisfies the equation:

$$(\Delta-\lambda_1-\lambda_2-\mu)F(\phi,\lambda_1,\lambda_2)=0.\tag{3.31}$$

Proof:
Let $g \in C_c^\infty(X,E)'$. Then,

$$\begin{aligned}\langle F(\phi,\lambda_1,\lambda_2),g\rangle_{dist} =& < \left(\sum_{k=1}^{2} \kappa_k e^{-i\sqrt{\lambda_j}u_k} E_k(\phi,\lambda_k+\mu)\right) \\ &- \kappa_1\kappa_2 e^{-i\sqrt{\lambda_1}u_2} e^{-i\sqrt{\lambda_2}u_1}\phi \\ &- (H-\lambda_1-\lambda_2-\mu)^{-1}\left(P(\phi,\lambda_1,\lambda_2)\right), g\rangle_{dist}.\end{aligned} \tag{3.32}$$

For $Re(\lambda_i) < -\mu$, $i = 1,2$, consider the function h:

$$\begin{aligned}h(\lambda_1,\lambda_2) :=& \langle \left(\sum_{k=1}^{2} \kappa_k e^{-i\sqrt{\lambda_j}u_k} E_k(\phi,\lambda_k+\mu)\right) \\ &- \kappa_1\kappa_2 e^{-i\sqrt{\lambda_1}u_2} e^{-i\sqrt{\lambda_2}u_1}\phi, g\rangle_{dist}\end{aligned} \tag{3.33}$$

In [21], $\lambda_k \mapsto E_k(\phi,\lambda_k)$ is extended meromorphically to a Riemann surface Σ. With this result we can prove that $(\lambda_1,\lambda_2) \mapsto E_k(\phi,\lambda_k)$ is meromorphic in Λ_-. Since the square roots are also defined on Λ_-, we have proved that h is meromorphic in Λ_-.

Our next goal is to prove that the function

$$(\lambda_1,\lambda_2) \mapsto \langle (H-\lambda_1-\lambda_2-\mu)^{-1}\left(P(\phi,\lambda_1,\lambda_2)\right), g\rangle_{L^2(X,E)} \tag{3.34}$$

extends meromorphically to Λ_-. For simplifying the notation we denote $({}^T U_\theta)$, considering T large enough, by U_θ. Observe that for $\theta \in \mathbb{R}_+$, $Re(\lambda_i) < -\mu$ for $i = 1,2$ and T large enough, the following calculation holds:

$$\begin{aligned}&\langle (H-\lambda_1-\lambda_2-\mu)^{-1}\left(P(\phi,\lambda_1,\lambda_2)\right), g\rangle_{L^2(X,E)} \\ &= \langle U_\theta (H-\lambda_1-\lambda_2-\mu)^{-1}\left(P(\phi,\lambda_1,\lambda_2)\right), U_{\overline{\theta}} g\rangle_{L^2(X,E)} \\ &= \langle U_\theta (H-\lambda_1-\lambda_2-\mu)^{-1} U_\theta^{-1} U_\theta \left(P(\phi,\lambda_1,\lambda_2)\right), U_{\overline{\theta}} g\rangle_{L^2(X,E)} \\ &= \langle R(\lambda_1+\lambda_2+\mu,\theta) U_\theta \left(P(\phi,\lambda_1,\lambda_2)\right), U_{\overline{\theta}} g\rangle_{L^2(X,E)}.\end{aligned} \tag{3.35}$$

For $\theta \in \mathbb{R}_+$ and $Re(\lambda_i) < -\mu$, for $i = 1,2$, corollary 8 proves that $U_\theta\left(P(\phi,\lambda_1,\lambda_2)\right) \in L^2(X,E)$. Hence the above calculation holds for $-3\pi/8 < arg(\frac{1}{(\theta+1)^2}) < 3\pi/8$ and $Re(\lambda_i) < -\mu$ for $i = 1,2$. In particular it holds for θ_-. The last term of (3.35) evaluated in $\theta = \theta_-$ is defined for $(\lambda_1,\lambda_2) \in \Lambda_-$. Hence, it provides the meromorphic extension of (3.34). The uniqueness

of the meromorphic extension implies that we can use (3.35) for defining $F(\varphi, \lambda_1, \lambda_2) \in C_c^\infty(X, E)'$ for $\lambda_1, \lambda_2 \in \Lambda_-$. We have proved part 1 of the theorem.

For proving that $F(\phi, \lambda_1, \lambda_2) \in C^\infty(X, E)$, we observe that $F(\phi, \lambda_1, \lambda_2)$ satisfies the equation

$$(\Delta - \lambda_1 - \lambda_2 - \mu)F(\phi, \lambda_1, \lambda_2) = 0$$

for $Re(\lambda_i) < -\mu$, and by analyticity, for $(\lambda_1, \lambda_2) \in \Lambda_-$. The ellipticity of Δ proves that $F(\phi, \lambda_1, \lambda_2)$ is in $C^\infty(X, E)$. $\square$

Let θ_+ be the complex number such that $\frac{1}{(\theta_+ + 1)^2} = -\pi/8$. Define:

$$\Lambda_+ := +^{-1}\left(\mathbb{C} - \sigma(H_{\theta_+})\right) \cap \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \lambda_k \notin \sigma(H_{\theta_+})\}. \tag{3.36}$$

Compare the above set with the set defined in 3.25. The following theorem is similar to theorem 15 but it rotates the essential spectrum in the opposite direction. It is proved in the same way.

Theorem 16 *Let ϕ be an eigenfunction of $H^{(3)}$ with eigenvalue μ.*

1) The function $(\lambda_1, \lambda_2) \mapsto F(\phi, \lambda_1, \lambda_2)$, taking values on $\Gamma_c(E)'$, extends meromorphically from $\lambda_i < -\mu$ for $i = 1, 2$ to a function with domain Λ_+ with values on $\Gamma_c(E)'$.

2) For $(\lambda_1, \lambda_2) \in \Lambda_+$, the distribution $F(\phi, \lambda_1, \lambda_2)$ is smooth in $x \in X$ and satisfies the equation:

$$(\Delta - \lambda_1 - \lambda_2 - \mu)F(\phi, \lambda_1, \lambda_2) = 0. \tag{3.37}$$

Appendix A

The essential spectrum of closed operators

In this appendix we collect some facts about the essential spectrum of a closed operators. Let $A : Dom(A) \subset \mathscr{H} \to \mathscr{H}$ be a closed operator in a Hilbert space $\mathscr{H}$. In the case that A is not self-adjoint, there are different versions of what should be considered the essential spectrum. According to [11] pag 415, there are 5 different types of essential spectrum. They are defined by $\sigma_{ek}(T) := \mathbb{C} - \Delta_k$ where:

- $\Delta_1(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is semifredholm$\}$.
- $\Delta_2(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is semifredholm and $dimKer(T - \lambda) < \infty\}$.
- $\Delta_3(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is semifredholm and its rank is finite codimensional $\}$.
- $\Delta_4(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is a Fredholm operator$\}$.
- $\Delta_5(T) :=$ union of all the components of $\Delta_1(T)$ which intersect the resolvent $\rho(T)$ of T.

Apart of this appendix, in this thesis we denote by $N_{ess}(T)$ the set $\sigma_{e2}(T)$. The sets $\sigma_{ek}(T)$ are essential in the sense that they remain unchanged under the perturbation of a compact operator, more precisely:

Theorem 17 *([11],page 418)Let T be a closed operator densely defined. Suppose that K is T-compact operator then:*

$$\sigma_{ek}(T + K) = \sigma_{ek}(T), \tag{A.1}$$

for $k = 1, 2, 3, 4, 5$.

Given a closed operator A, we recall the definitions of pure point, discrete, and essential spectrum:

$$\begin{aligned}
&\sigma_{pp}(A) := \{\lambda \in \mathbb{C} : \text{ is the set of eigenvalues of } A\}.\\
&\sigma_d(A) := \{\lambda \in \mathbb{C} : \ \lambda \text{ is an isolated eigenvalue of } A \text{ of finite algebraic multiplicity}\}.\\
&\sigma_{ess}(A) := \sigma(A) - \sigma_d(A).
\end{aligned} \tag{A.2}$$

Let $\lambda_0 \in \mathbb{C}$ be an isolated point of the spectrum $\sigma(A)$. Let Γ be a closed curve enclosing λ but no other point of $\sigma(A)$; we called such a contour **admissible**. Denote:

$$P_0 := \int_\Gamma R_A(z)dz. \tag{A.3}$$

In [17], the operators P_0 are called **Riez projections**. We collect, without a proof, some properties of Riez projection in the following theorem some:

Theorem 18 *([17], section 6.1) Let $\lambda_0 \in \sigma(A)$ be an isolated point of the spectrum $\sigma(A)$ of a closed operator A, and P_0 as in (A.3). Then:*

a) P_0 is independent of the admissible contour Γ i.e. Γ is a subset of $\rho(A)$, the resolvent of A, and Γ does not contain other point of $\sigma(A)$ but λ.

b) $P_0^2 = P_0$.

c) If E_0 denotes the space of eigenfunctions with eigenvalue λ_0, then

$$E_0 \text{ is a subspace of } P_0(\mathscr{H}). \tag{A.4}$$

d) If A is self-adjoint $E_0 = P_0(\mathscr{H})$.

We recall that the **algebraic multiplicity of** λ is the dimension of the space $X_0(\lambda) := P_\lambda(\mathscr{H})$. The following expression of the projection on the space of eigenvectors of a given eigenvalue is important in appendix B; it plays a role similar to the role that play the Riez projections in that appendix.

Theorem 19 *([17], page 67) Let A be a self-adjoint operator on the Hilbert space $\mathscr{H}$ with an embedded eigenvalue λ. The projection onto the eigenspace E_λ is given by:*

$$P_\lambda = s - \lim_{\epsilon \to 0^\pm} (-i\epsilon)(A - \lambda - i\epsilon)^{-1}. \tag{A.5}$$

We observe that P_λ in the above theorem does not depends on how we approach 0.

In sections 1.4 and 2.5 we calculate $\sigma_{e2}(\Delta_\theta)$ and $\sigma_{e2}(H_\theta)$. We find $\sigma_{ess}(\Delta_\theta)$ and $\sigma_{ess}(\Delta_\theta)$ using the methods described in D, and the calculations of $\sigma_{e2}(\Delta_\theta)$ and $\sigma_{e2}(H_\theta)$. In this appendix we described the relation between $\sigma_{e2}(A)$ and $\sigma_{ess}(A)$ for A a closed operator.

We begin by proving that $\sigma_{e2}(A)$ is the spectrum associated to singular sequences, that we define now:

Definition 2 *A sequence $(f_n)_{n\in\mathbb{N}} \subset Dom(A)$ is* **a singular sequence for A associated to the value $\lambda \in \mathbb{C}$** *if and only if*

i) $||f_n|| = 1$ *and* $(f_n)_{n\in\mathbb{N}}$ *has no convergent subsequence.*

ii) $\lim_{n\to\infty} ||(A-\lambda)f_n|| = 0$.

The singular sequences are also called **Weyl sequences**.

The following theorems and lemmas are classic; they contain information about closed operators that we will use in the characterization of the essential spectrum in terms of the singular sequences. The next lemma is implicitly proved in [22].

Lemma 8 *([22], page 231) Let $B : Dom(B) \subset \mathscr{H} \to \mathscr{H}$ be a closed operator, then*
i) $Ker(B)$ is closed.

ii)Let $\tilde{\mathscr{H}} := \mathscr{H}/ker(B)$. Define

$$Dom(\tilde{B}) := \{\tilde{u} \in \tilde{\mathscr{H}} : \text{ if } u \in \tilde{u} \text{ then } u \in Dom(B)\}.$$

Then: $\tilde{B} : Dom(\tilde{B}) \subset \tilde{\mathscr{H}} \to \mathscr{H}$ is a closed operator.

We have the following corollary.

Corollary 9 *Suppose that A is a closed operator with $Im(A)$ not closed. Then for all $n \in \mathbb{N}$ there exists φ such that $||A\varphi|| = 1$ and $||\varphi|| > n$.*

Proof:
Suppose that $Im(A)$ is not closed, then there exists a sequence $u_n \in Dom(A)$

such that Au_n is Cauchy but $\lim_{n\to\infty} Au_n \notin Im(A)$. Then, the $\tilde{A}^{-1} : Im(A) \to \tilde{\mathscr{H}}$ is not bounded. It is so because, if $\tilde{A}^{-1}$ were continuous, then:

$$|u_n - u_m| = |\tilde{A}^{-1}A(u_n - u_m)| = |A(u_n - u_m)|, \tag{A.6}$$

that proves that u_n is Cauchy, and hence $\lim_{n\to\infty} u_n = u$ exists. Since A is a closed operator, this would imply $Au = \lim_{n\to\infty} Au_n$, that is a contradiction.

Since $\tilde{A}^{-1}$ is not bounded, there exists $\psi \in \mathscr{H}$ such that $||\psi|| = 1$ and $\tilde{A}^{-1}\psi =: [\varphi]$ with $||[\varphi]|| > n$. Since

$$n < ||[\varphi]|| = \inf_{z\in KerA} ||\varphi - z|| \leq ||\varphi||,$$

we have proved the corollary. □

The restriction of a a closed operator to a closed subspace is closed, as is proved in the next proposition.

Proposition 38 *Let $B : \mathscr{H} \to \mathscr{H}$ be a closed operator. If $\Gamma \subset \mathscr{H}$ is a closed subspace of $\mathscr{H}$, then $B|_\Gamma$ is a closed operator from Γ to $\mathscr{H}$.*

Proof:
$Graph(B|_\Gamma) = Graph(B) \cap (\Gamma \times \mathscr{H})$. □

Let M, N be closed subspaces of $\mathscr{H}$. For $M \nsubseteq N$, we define (see [22] page 219):

$$\gamma(M, N) := \inf_{u\in M, u\notin N} \frac{dist(u, N)}{dist(u, M \cap N)}. \tag{A.7}$$

If $M \subset N$ we define $\gamma(M, N) := 1$. $\gamma(M, N)$ determines when the sum of two closed subspaces, $M + N$, is closed.

Theorem 20 *([22] page 219) $M + N$ is closed if and only if $\gamma(M, N) > 0$*

As a consequence of the above theorem we have:

Corollary 10 *Let Γ be a finite dimensional subspace of $\mathscr{H}$ and Λ be a closed subspaces of $\mathscr{H}$. Then $\Gamma + \Lambda$ is closed.*

Proof:
Let $\{\varphi_i\}_{i=0}^{\infty}$ be an orthonormal basis of $\mathscr{H}$ with $\{\varphi\}_{i=0}^{m}$ basis of $\Lambda \cap \Gamma$ and $\{\varphi_i\}_{i=0}^{N}$ basis of Γ. Let $u \in \Lambda, u \notin \Gamma$. Denote by P_Λ the orthogonal projection of $\mathscr{H}$ over Λ. Then, $P_\Lambda u = \sum_{i=0}^{N} < u, \varphi_i > \varphi_i$. Then:

$dist(u,\Gamma) = \sqrt{\sum_{i=N+1}^{\infty} | < u, \varphi_i > |^2}$. Similarly, we can prove $dist(u, \Gamma \cap \Lambda) = \sqrt{\sum_{i=m+1}^{\infty} | < u, \varphi_i > |^2}$. Hence:

$$\gamma(\Lambda,\Gamma) = \inf_{u\in\Lambda, u\notin\Gamma} \frac{dist(u,\Gamma)}{dist(u,\Gamma\cap\Lambda)} =$$

$$\inf_{u\in\Lambda, u\notin\Gamma} \sqrt{\frac{\sum_{i=m+1}^{N} | < u, \varphi_i > |^2 + \sum_{i=N+1}^{\infty} | < u, \varphi_i > |^2}{\sum_{i=N+1}^{\infty} | < u, \varphi_i >^2 |}} \geq 1,$$

and one can see $\gamma(\Lambda,\Gamma) = 1 > 0$. $\square$

Theorem 21 *Suppose $dimKer(A) < \infty$. Then there exists an orthonormal sequence φ_n such that $lim_{n\to\infty}||A\varphi_n|| = 0$ if and only if ImA is not closed.*

Proof:
Suppose that ImA is not closed. By corollary 9 there exists ψ_1 such that $||\psi_1|| > 1$ and $||A\psi_1|| = 1$. Suppose we have constructed $\psi_1, ..., \psi_N$ orthogonal with the properties $(\psi_i, \psi_j) = 0$ if $i \neq j$, $||\psi_i|| > i$ and $||T\psi|| = 1$. We construct ψ_{N+1} such that ψ_{N+1} is orthogonal to $\psi_1, ..., \psi_N$ and $||\psi_{N+1}|| > N+1$ and $||A\psi_{N+1}|| = 1$. Observe that $A|_{<\psi_1,...,\psi_N>^\perp}$ is a closed operator from $< \psi_1, ..., \psi_N >^\perp$ to $\mathscr{H}$ by proposition 38. Since $Im(A) = Im(A|_{<\psi_1,...,\psi_N>^\perp}) + Im(A|_{<\psi_1,...,\psi_N>})$, by corollary 10, $Im(A|_{<\psi_1,...,\psi_N>^\perp})$ is not closed. Hence we can find $\psi_{N+1} \in < \psi_1, ..., \psi_N >^\perp$ with the mentioned properties. Taking $\varphi_n := \frac{\psi_n}{||\psi_n||}$, we have $||\varphi_n|| = 1$ and $||A\varphi_n|| \leq \frac{1}{n} = 0$.

Suppose that there exists an orthonormal sequence such that $lim_{n\to\infty}||A\varphi_n|| = 0$. By the closed graph theorem is enough to prove $\tilde{A}^{-1}$ is not bounded. For n big enough, $\varphi_n \perp Ker(A)$

$$||\frac{\tilde{A}^{-1}(A\varphi_n)}{||A\varphi_n||}|| = ||[\frac{\varphi_n}{||A\varphi_n||}]|| = \inf_{z\in Ker(A)} ||\frac{\varphi_n}{||A\varphi_n||} + z|| \geq ||A\varphi_n||^{-1}.\square$$

We have the following criterion for $\sigma_{e2}(A)$. It is an easy corollary of the above theorem.

Theorem 22 *([11] page 415) (Weyl criterion) Let A be a closed operator. Then $\lambda \in \sigma_{e2}(A)$ if and only if there exists $(\psi_n)_{n\in\mathbb{N}} \subset Dom(A)$ singular sequence. Moreover, the singular sequence can be taken orthonormal.*

Proof:
Suppose $\lambda \in \sigma_{ess}(A)$ then $dim(Ker(A-\lambda))$ is infinite or $Im(A-\lambda)$ is not

closed. If $dim(Ker(A-\lambda))$ is infinite we can find an orthogonal basis φ_i of $Ker(A-\lambda)$, this sequence satisfies obviously $lim_{n\to\infty}||(A-\lambda)\varphi_n|| = 0$. If $dim(Ker(A-\lambda))$ is finite, then we can apply theorem 21 for finding a orthonormal singular sequence.

Suppose, there exists an orthonormal singular sequence associated to λ. If $dim(Ker(A-\lambda))$ is infinite we have that by definition $\lambda \in \sigma_{ess}(A)$. If $dim(Ker(A-\lambda))$ is finite we can apply again theorem 21. $\square$

Following [42], page 18, we define the **ascendant, descent, nullity** and **defect** of a closed operator $A : \mathscr{H} \to \mathscr{H}$. These numbers will be important for identifying the isolated elements of the spectrum that belong to the discrete spectrum (see theorems 24 and). In particular, for proving that $\sigma_{e2}(A)$ is a subset of $\sigma_{ess}(A)$. The **nullity of** A is the dimension of $Ker(A)$ and it shall be denote $n(A)$; the dimension $\mathscr{H}/Im(A)$ is the **defect** of A and it shall be denoted by $d(A)$. For defining the descent of of A consider the sequence $A, A^2, A^n, \cdots$. Given $n \in \mathbb{N}$ define $Dom(A^n) := \{x : Ax, \cdots, A^n x \in Dom(A)\}$. We can consider the image and kernel of these operator. According to [42], page 18, it is a known fact:

$$Ker(A^n) \subset Ker(A^{n+1}) \text{ and } Im(A^{n+1}) \subset Im(A^n), \tag{A.8}$$

for $n \geq 0$. The smallest non-negative integer m such that $Ker(A^m) = Ker(A^{m+1})$ is called the **ascent** of A and it shall be denoted $\alpha(A)$, if such smallest m does not exist we define $\alpha(A) = \infty$. Similarly the smallest non-negative integer such that $Im(A^{m+1}) = Im(A^m)$ is called **descent** and it shall be denoted $\delta(A)$. The following theorem characterizes poles of the resolvent of A in terms of the numbers ascendant, descent, nullity and defect.

Theorem 23 *([42], theorem 9.1, page 44) Let A be a closed operator. Suppose λ_0 is an isolated point of $\sigma(A)$ and let m be a positive integer. Then λ_0 is a pole of order m if and only if*

$$\alpha(A-\lambda_0) = \delta(A-\lambda_0) = m, \tag{A.9}$$

and $Im((A-\lambda_0)^m)$ is closed.

The next theorem characterizes poles of A in terms of the Riez projections and the ascendant, descent, nullity and defect.

Theorem 24 *([42], theorem 9.2, page 46) Suppose that λ_0 is an isolated point of $\sigma(A)$. Let P_0 be the projection associated to λ_0 as in (A.3). Suppose $dimP_0(\mathscr{H}) < \infty$. Then:*

$$\alpha(\lambda_0 - A) = \delta(\lambda_0 - A) < \infty \tag{A.10}$$

and

$$n(\lambda_0 - A) = d(\lambda_0 - A) < \infty. \tag{A.11}$$

Also $Im\left((\lambda_0 - A)^k\right)$ is closed for $k = 1, 2, 3....$ It then follows that λ_0 is a pole of $\lambda \mapsto (\lambda - A)^{-1}$.

As a corollary of the above theorem, we have:

Corollary 11 $\sigma_{e2}(A) \subset \sigma_{ess}(A)$.

Proof:
We prove $\sigma_{ess}(A)^c \subset \sigma_{e2}(A)^c$. Suppose $\lambda \in \sigma_{ess}(A)^c$, then $\lambda \in \sigma_d(A)$. Hence λ is an isolated point of $\sigma(A)$ and λ has finite algebraic multiplicity. Then, we can apply theorem 24 for proving that $A - \lambda$ is semi-Fredholm, and hence $\lambda \notin \sigma_{e2}(A)^c$.$\square$

In appendix D we give conditions for having $\sigma_{e2}(A) = \sigma_{ess}(A)$.

Appendix B

Aguilar-Balslev-Combes theory

In this appendix we adapt the method of analytic dilation to the setting on which it is applied in the thesis, i.e for describing the essential and pure point spectrum of compatible generalized Laplacians on manifolds with cylindrical ends and manifolds with corners of codimension two. Our main result is theorem 25.

In [2], the method of analytic dilation was adapted for describing the spectrum of the Laplacian on surfaces with cusps. In [27], the method was adapted for describing the spectrum of the Laplacian on $SL(3)/SO(3)$. Our results are analogous to those described in theorem 1.1 of [2] and to certain results of [3] and [27]. See also [17] for an explanation of the analytic dilation method similar to the given here.

Now we abstract the common properties of the analytic dilation of the operators H_θ, for H a generalized Laplacian on compatible structures on manifolds with cylindrical end, described in section 1, or on complete manifolds with corner of codimension 2, described in section 2. Before to give the main properties that the analytic dilations of these operators share, we establish some notation. Denote the set:

$$\Gamma := \{(x, y) \in \mathbb{C} : x > 0 \text{ and } |y| < x\}. \tag{B.1}$$

We recall that, for $\theta \in \mathbb{C} - (-\infty, 0]$, we define the parameter $\theta' := \frac{1}{(\theta+1)^2}$. We remark that taking the parameters θ in the set Γ guarantees that the essential spectrum of the operator Δ_θ (see section 1.4) and the essential spectrum

H_θ (see section 2.5) are included in the right half plane. In fact, the essential spectrum of these operators is given by a discrete union of positive translations of the half line $\theta' \mathbb{R}_+$, and $Re(\theta') = Re(\frac{1}{(\theta+1)^2}) = \frac{Re(\theta+1)^2 - Im(\theta)^2}{|(\theta+1)^2|} > 0$.

Let $A \geq 0$ be a self-adjoint operator with a dense domain $Dom(A) \subset \mathscr{H}$, $\mathscr{H}$ a Hilbert space. In the following hypothesis about the operator A, one should have in mind the properties of the analytic dilations of the operators Δ and H of the sections 1 and 2. Suppose A satisfies the following assumptions:

1) There exists a family of linear operators U_θ, $\theta \in \Gamma$ such that for $\theta \in \mathbb{R}_+$, U_θ is unitary and the family extends continuously to 0 with $U_0 = Id$. Furthermore there exists a dense set of vectors $\mathscr{V} \subset \mathscr{H}$ such that:

   1.1) For $\psi \in \mathscr{V}$ the map $\theta \mapsto U_\theta \psi$ is analytic on Γ with values in $\mathscr{H}$.

   1.2) For $\theta \in \Gamma$, $U_\theta(\mathscr{V})$ is dense in $\mathscr{H}$.

2) For $\theta \in \mathbb{R}_+$, the family of operators $A_\theta := U_\theta A U_\theta^{-1}$ extends analytically to a family of type A on Γ (see definition 1).

3) There exists an increasing sequence $(\gamma_i)_{i \in \mathbb{N}}$ in $\mathbb{R}_+$, $\lim_{i\to\infty} \gamma_i = \infty$, such that, for $\theta \in \Gamma$,

$$\sigma_{ess}(A_\theta) = \cup_{i=0}^{\infty} \left(\gamma_i + \theta' \mathbb{R}_+\right). \tag{B.2}$$

   where $\theta' := \frac{1}{(\theta+1)^2}$. We call the set $(\gamma_i)_{i \in \mathbb{N}}$, the **set of thresholds of** A**.** We denote it by $\tau(A)$.

Observe that, as a consequence of 3), $\sigma_{ess}(A_0) = [\gamma_0, \infty)$. All the results that we derive from 1), 2), and 3) hold if we allow complex numbers z, in the discrete sequence $(\gamma_i)_{i \in \mathbb{N}}$ of 3), with imaginary part positive or negative depending of $0 < arg(\theta') < \pi/2$ or $-\pi/2 < arg(\theta') < 0$.

Definition 3 *The family U_θ satisfying 1) and 2) above is called* **analytic dilation family for** A. *The dense set of vectors $\mathscr{V}$ is called* **analytic vectors for** U_θ.

In chapters 1 and 2 we construct analytic dilation families for the compatible generalized Laplacians of manifolds with cylindrical ends and complete manifolds with corner of codimension 2. For a compatible generalized Laplacian on a manifold with cylindrical end, 1) and 3) follow from (1.44). In

particular, in this case, $\gamma_0 > 0$ if and only if the Laplacian on the boundary, Δ_Y, is invertible, i.e. $Ker(\Delta_Y) = \{0\}$. Properties 2.1) and 2.2) are proved in section 1.5. 3) is proved in section 1.3.

Similarly, for a compatible generalized Laplacian on a complete manifold with corners of codimension 2, 1) and 4) follow from theorem 10. In particular, $\gamma_0 > 0$ if and only if the Laplacians on the manifolds with cylindrical ends, denoted by $H^{(1)}$ and $H^{(2)}$, and the Laplacian on the corner, denoted by $H^{(3)}$, satisfy $Ker(H^{(k)}) = \{0\}$ for $k = 1, 2, 3$. Properties 2.1) and 2.2) are proved in section 2.6. 3) is proved in section 2.2.

We observe that the parameter θ' appears naturally in the description of H_θ for the Laplacian H on both manifolds with cylindrical ends and manifolds with corners of codimension two, see propositions 2 and 20.

Define

$$\Lambda := \{(x, y) \in \mathbb{C} : x < 0\}. \tag{B.3}$$

Recall that $\Gamma := \{\theta := \theta_0 + i\theta_1 \in \mathbb{C} : \theta_0 > 0, \theta_0 \geq |\theta_1| \text{ and } \theta_1^2 < \frac{1}{2}\}$. The next theory can be deduced taking the parameter θ in different Γ's, however we use the set Γ defined before because it is the set used in section 1.7.

In this appendix we give the implications on the spectrum of A of properties 1), 2) and 3). The main result of this appendix is the next theorem.

Theorem 25 *Let U_θ and $\mathscr{V}$ be an analytic dilation family for A and its analytic vectors. Suppose $U_\theta, \mathscr{V}$ satisfies 1), 2), 3). Then*

a) The set of non-threshold eigenvalues, $\sigma_{pp}(A) \cap (\tau(A)^c)$, is equal to $\sigma_{pp}(A_\theta) \cap \mathbb{R}$, for all $\theta \in \Gamma$; moreover, if $Im(\theta') \neq 0$, then $\sigma_{pp}(A) \cap (\tau(A)^c) = \sigma_d(A_\theta) \cap \mathbb{R}$, and, given λ_0 non-threshold eigenvalue, the eigenspace $E_{\lambda_0}(A)$ associated to A and λ_0 has finite dimension bounded by the algebraic multiplicity of the pole λ_0 of $\lambda \mapsto R(\lambda, \theta)$. This algebraic multiplicity is independent of $\theta \in \Gamma - \mathbb{R}_+$.

b) Fix $\theta \in \Gamma$. For $f, g \in \mathscr{V}$ the function

$$\lambda \mapsto \langle R_A(\lambda) f, g \rangle$$

has a meromorphic continuation from Λ to $\mathbb{C} - \left(\bigcup(\gamma_i + \theta' \mathbb{R}_+) \cup \sigma_{pp}(A_\theta)\right)$.

c) A has no singular spectrum.

d) Let $\theta_1, \theta_2 \in \Gamma$ be such that $arg(\theta_1') \geq arg(\theta_0')$ for $0 < arg(\theta_i') < \pi/2$, we have:

$$\sigma_{pp}(A_{\theta_0}) = \sigma_{pp}(A_{\theta_1}) \cap \sigma_{pp}(A_{\theta_0}). \tag{B.4}$$

e) If $\lambda \in \sigma_{pp}(A)$ and $\lambda \notin \tau(A)$, i.e. if λ is an L^2-eigenvalue that is not threshold, then λ is isolated (respect to the eigenvalues of A). In case that non-threshold eigenvalues of A accumulate, they accumulate on the set of thresholds or on ∞.

f) If $\gamma_0 > 0$ then $\sigma_d(A)$ is a discrete subset of $[0, \gamma_0)$ and its unique possible accumulation point is γ_0. If $\gamma_0 = 0$, $\sigma_d(A) = \emptyset$.

We remark that the set $\sigma_{pp}(A)$ could be empty, finite or infinite.

The following propositions, lemmas and corollaries prove the above theorem, and give more information about the relations between different analytic dilation families associated to the same operator.

B.1 Meromorphic extension of the resolvent, resonances and absence of sing. spec.

Denote by $R(\lambda)$ the resolvent of A. We will denote $R(\lambda, \theta)$ the resolvent of A_θ.

Lemma 9 *Suppose that $f, g \in \mathscr{V}$ and denote $r(\lambda) := \langle R(\lambda)f, g\rangle$ and $s_\theta(\lambda) := \langle U_\theta R(\lambda, \theta) U_\theta f, g\rangle$. Then: $\forall(\theta, \lambda) \in \Gamma \times \Lambda$, $s_\theta(\lambda) = r(\lambda)$.*

Proof:
$\theta \in \Gamma$ implies that $Re(\theta') \geq 0$. Then, property 3) implies that all $\sigma(A_\theta)$, but the discrete subset $\sigma_d(A_\theta)$, is contained on $\mathbb{C} - \Lambda$. Then, $s_\theta(\lambda)$ is meromorphic on (θ, λ) in $\Gamma \times \Lambda$. This and the fact that $r(\lambda) = s_\theta(\lambda)$ on $\mathbb{R}_+ \times \Lambda$ finish the proof. □

The following theorem is a corollary of the above lemma and the uniqueness of the meromorphic extensions. We recall that $\sigma_{ess}(A_{\theta_0})$ is given in (B.2).

Theorem 26 *Suppose $f_1, f_2 \in \mathscr{V}$ and $\theta_0, \theta_1 \in \Gamma$. We have:*

i) If $\theta_0 \in \Gamma$, $0 < \theta_0' < \pi/2$, then the map $\lambda \mapsto \langle R(\lambda)f_1, g_1\rangle$ has a meromorphic extension from Λ to $\mathbb{C} - \sigma_{ess}(A_{\theta_0})$ whose poles are contained in

$\{\lambda \in \mathbb{C} - \sigma_{ess}(A_{\theta_0}) : Im(z) > 0\}$. *Similarly, if* $\theta_0 \in \Gamma$, $-\pi/2 < \theta_0' < 0$, *then* $\lambda \mapsto \langle R(\lambda)f_1, f_2\rangle$ *has a meromorphic extension to* $\mathbb{C} - \sigma_{ess}(A_{\theta_0})$ *whose poles are contained in* $\{\lambda \in \mathbb{C} - \sigma_{ess}(A_{\theta_0}) : Im(z) < 0\}$.

iii) Suppose $0 < \theta_0 \leq \theta_1 < \pi/2$. *Denote by* R_1 *the meromorphic extension of* $\lambda \mapsto \langle R(\lambda)f_1, f_2\rangle$ *to* $\mathbb{C} - \sigma_{ess}(A_{\theta_0})$; *denote by* R_2 *the meromorphic extension of* $\lambda \mapsto \langle R(\lambda)f_1, f_2\rangle$ *to* $\mathbb{C} - \sigma_{ess}(A_{\theta_1})$. *Then* $R_1(\lambda) = R_2(\lambda)$ *for* $\lambda \in \mathbb{C} - \sigma_{ess}(A_{\theta_0})$ *not a pole.*

In the next corollary we express i) of theorem 26 in terms of the Riemann surface $\mathscr{S}$ defined in (3.3).

Corollary 12 *If* $f, g \in \mathscr{V}$, *then map* $\lambda \mapsto \langle R(\lambda)f, g\rangle$ *has a meromorphic extension from* Λ *to* $\mathscr{S}$ *with poles contained in* $\mathscr{S}_{\pi/4} \cup \mathscr{S}_{-\pi/4}$, *where* $\mathscr{S}$, $\mathscr{S}_{\pi/4}$ *and* $\mathscr{S}_{-\pi/4}$ *were defined in (3.3).*

Proof:
The corollary follows from taking $\theta_0 = \pm\pi/4$ in i) of theorem 26.□

In order to show that A has no singular spectrum we make use of the following theorems:

Theorem 27 *([33],page 407) Let* H *be a self-adjoint operator with resolvent* $R(\lambda) := (H - \lambda)^{-1}$. *Let* (a, b) *be a bounded interval and* $\varphi \in \mathscr{H}$. *Suppose that there exists* $p > 1$ *for which:*

$$sup_{0<\epsilon<1} \int_a^b |Im(\varphi, R(x + i\epsilon)\varphi)|^p dx < \infty. \tag{B.5}$$

Then $E_{(a,b)}\varphi \in \mathscr{H}_{ac}$.

Theorem 28 *([33],page 408) Let* H *be a self-adjoint operator with resolvent* $R(\lambda) := (H - \lambda)^{-1}$. *Let* (a, b) *be a bounded interval. Suppose that there is a dense subset* D *in* $\mathscr{H}$ *so that for* $\varphi \in D$ *the inequality (B.5) holds for some* $p > 1$. *Then* H *has purely absolutely continuous spectrum on* (a, b).

We obtain:

Theorem 29 *A has no singular spectrum. We have* $\sigma(A) = \sigma_{ac}(A) \cup \sigma_{pp}(A)$ *where* $\sigma_{ac}(A) = \sigma_{ess}(A) = [\gamma_0, \infty)$.

Proof:
Let $\{\tilde{\gamma}_i\}_{i\in \mathbb{N}} := \{\gamma_i\} \cup \bigcap_{\theta\in U} \sigma_{pp}(A_\theta)$ and consider (a, b) a subinterval of

$(\tilde{\gamma}_i, \tilde{\gamma}_{i+1})$. Observe that theorem 26 guarantees that for all $f \in \mathscr{V}$ there exists a $p > 1$ for which:

$$sup_{0<\epsilon<1} \int_a^b |Im(f, R(x+i\epsilon)f)|^p dx < \infty.$$

Now we can use theorem 28 and conclude that A has no singular spectrum in (a, b). Hence, A has no singular spectrum. $\square$

From the assumption 3), we can deduce:

Corollary 13 *A_θ is Fredholm if and only if $0 \notin \{\gamma_i\}$.*

Corollary 14 *If $\lambda \in \sigma_{pp}(A)$, $\lambda > \gamma_0$ then $A - \lambda$ is not Fredholm.*

Proof:
Because $\sigma_{ess}(A) = [\gamma_0, \infty)$. $\square$

B.2 Eigenvalues and poles of A_θ for $\theta \in \Gamma$

The following proposition is a consequence of the definition of the essential spectrum in (A.2)

Proposition 39 *Suppose that $\theta \in \Gamma$. Then, the eigenvalues of A_θ can accumulate only at $\sigma_{ess}(A_\theta)$.*

The next proposition describes the real pure point spectrum of A_θ for $\theta \in \Gamma$.

Proposition 40 *Let $\theta \in \Gamma - \mathbb{R}_+$. Then,*

$$\{\textit{non-threshold eigenvalues of } A\} = \sigma(A_\theta) \cap (\mathbb{R}_+ - \tau(A)) = \sigma_{pp}(A_\theta) \cap \mathbb{R}. \tag{B.6}$$

Moreover, for λ_0 non-threshold eigenvalue of A, $\dim Ker(A_\theta - \lambda)$ is smaller than the degree of the pole λ_0 of the map $\lambda \mapsto \langle R(\lambda, \theta) U_\theta f, U_{\overline{\theta}} g\rangle$ for $f, g \in \mathscr{V}$ such that $\langle P_\lambda f, g\rangle \neq 0$ (P_λ the projection on the eigenspace associated to λ and H).

Proof:

The proof that we give here is adapted from [17], page 170. Suppose that $\lambda \in \sigma_{pp}(A)$ and $\lambda \notin \tau(A)$. Then, we can find a a contour Γ around λ that

does not intersect $\sigma(A_\theta)$ and such that the unique element of $\sigma(A_\theta)$ enclosed by Γ is λ.

According to theorem 19, the orthogonal projection P_λ over the eigenspace associated to λ satisfies:

$$P_\lambda = s - \lim_{\epsilon \to 0^\pm} (-i\epsilon)(A - \lambda - i\epsilon)^{-1}. \tag{B.7}$$

Let $f, g \in \mathscr{V}$, observe that

$$\langle P_\lambda f, g\rangle = \lim_{\epsilon \to 0^-} \langle (-i\epsilon) R(\lambda + i\epsilon) f, g\rangle = \lim_{\epsilon \to 0^-} \langle (-i\epsilon) R(\lambda + i\epsilon, \theta) U_\theta f, U_{\overline{\theta}} g\rangle, \tag{B.8}$$

for $\lambda \in \Lambda$ and $\theta \in \Gamma$. Then, the Riez projection $\int_\Gamma R(z, \theta) dz$ is different than 0, since we can find $f, g \in \mathscr{V}$ such that the left limit in (B.8) is different than 0 (because $\mathscr{V}$ and $U_\theta \mathscr{V}$ are dense). Hence, $\lambda \in \sigma_{pp}(A_\theta) \cap (\mathbb{R} - \tau(A))$. We have proved $\{$non-threshold eigenvalues of $A\} \subset \sigma_{pp}(A_\theta) \cap (\mathbb{R}_+ - \tau(A))$.

From (B.8) we can deduce also that, for λ_0 non-threshold eigenvalue of A, if θ is such that $(-\pi/2, \pi/2) \ni arg(\theta') \neq 0$, then $\dim Ker(A - \lambda_0)$ is smaller than the degree of the pole λ_0 of the map $\lambda \mapsto R(\lambda, \theta)$.

Now suppose that $\lambda_0 \in \sigma_d(A_\theta) \cap (\mathbb{R}_+ - \tau(A))$, and take $\theta \in \Gamma$ such that $(-\pi/2, \pi/2) \ni arg(\theta') \neq 0$. Then we can find $f, g \in \mathscr{V}$ such that the right side of (B.8) is not 0. Then the projector P_λ is different than 0, hence $\lambda \in \sigma_{pp}(A)$. $\square$

By the previous proposition, $\sigma_{pp}(A) \subset \tau(A) \cup \sigma_{pp}(A_\theta)$ for all $\theta \in \Gamma$. Then, we define **the set of resonances of A at an angle** $\theta_0 \in \Gamma$, $(-\pi/2, \pi/2) \ni arg(\theta_0') \neq 0$:

$$\mathscr{R}_{\theta_0}(A) := \{\lambda \in \sigma_d(A_{\theta_0}) : \lambda \notin \sigma_{pp}(A)\}. \tag{B.9}$$

By part iii) of theorem 26, the resonances of A are independent of θ_0.

We finish this section pointing out that theorem 19 plays a fundamental role in the proof that non-threshold eigenvalues of A have finite multiplicity. We do not find an argument for proving that eigenvalues embedded in the essential spectrum of A_θ, for $(-\pi/2, \pi/2) \ni arg(\theta) \neq 0$, have finite multiplicity.

B.3 Relations between different analytic dilation families

The following theorem gives information about the uniqueness of the meromorphic extension of the resolvent associated to two different analytic dilation families. Let $U_\theta, \mathscr{V}_1$ and $V_\theta, \mathscr{V}_2$ be two analytic dilation families for A with their respective sets of analytic vectors. Denote by $A_{1,\theta}$ the analytic extension of $U_\theta A U_\theta^{-1}$ and $A_{2,\theta}$ the analytic extension of $V_\theta A V_\theta^{-1}$. respectively.

Theorem 30 *Suppose that $\mathscr{V}_1 \cap \mathscr{V}_2$ is dense in $\mathscr{H}$. Let $f, g \in \mathscr{V}_1 \cap \mathscr{V}_2$. Then, we have:*

i) If $\tau_1(A)$ and $\tau_2(A)$ are the set of thresholds associated to the analytic dilation families U_θ and V_θ. Then:

$$\tau_1(A) = \tau_2(A). \tag{B.10}$$

ii) Let $f, g \in \mathscr{V}_1 \cap \mathscr{V}_2$ using i) of theorem 26, let $\lambda \mapsto \langle R_1(\lambda) f, g\rangle$ and $\lambda \mapsto \langle R_2(\lambda) f, g\rangle$ be the meromorphic extension of $\lambda \mapsto \langle R(\lambda) f, g\rangle$ obtained from $U_\theta, \mathscr{V}_1$ and $V_\theta, \mathscr{V}_2$. Then: $\langle R_1(\lambda) f, g\rangle = \langle R_2(\lambda) f, g\rangle$ on $\mathbb{C} - \sigma(A_\theta)$

Proof:
For $\lambda \in \Lambda$, we have:

$$\langle R_1(\lambda) f, g) = \langle R_2(\lambda) f, g\rangle = \langle R_1(\lambda, \theta) U_\theta f, U_{\overline{\theta}} g\rangle = \langle R_2(\lambda, \theta) f, g\rangle. \tag{B.11}$$

By uniqueness of the analytic extension and assumption 3), i) and ii) follow from (B.11).□

Appendix C

Ichinose lemma

In this appendix we recall some definitions and we formulate the Ichinose lemma (theorem 35). The following definitions follow [31] and [22], we refer there for a deeper study of the topic. Let q be a bilinear form on a Hilbert space $\mathscr{H}$ with domain $Q(q)$.

Definition 4 *q is* **closed** *if for sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $Q(q)$, if $\lim_{n\to\infty}\varphi_n = \varphi$ in norm, and*

$$lim_{n,m\to\infty} q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0$$

imply $\varphi \in Q(q)$ and $q(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$.

Definition 5 *A quadratic form q is* **sectorial** *if there exists a θ, $0 < \theta < \pi/2$ with $|arg(q(\varphi,\varphi))| \leq \theta$ for all $\varphi \in Q(q)$.*

Definition 6 *A quadratic form q is called* **strictly m-accretive** *if it is closed and sectorial.*

Definition 7 *A form q is called* **strictly m-sectorial** *if there are complex numbers z and $e^{i\alpha}$, with α real, so that $e^{i\alpha}q + z$ is strictly m-accretive. The operator T associated to q is also called* **strictly m-sectorial**.

Observe that in order to prove that q is strictly m-sectorial it is enough to show that there exists $\gamma \in \mathbb{R}$ and $k \in \mathbb{R}_+$ such that for all $f \in Q(q)$

$$kRe(q(f)) - |Im(q(f))| \geq \gamma(f,f). \tag{C.1}$$

Every closed operator T defines a dense form $q(T)$ by

$$q(t)(\varphi,\psi) := (\varphi, T\psi), \tag{C.2}$$

for $\varphi, \psi \in D(T)$.

Definition 8 *An operator T is sectorial if there is a θ, $0 < \theta < \pi/2$ such that its numerical range, $\Theta(T)$, is a subset of a sector $\{z \in \mathbb{C} : |arg(z)| \leq \theta\}$.*

The following theorems are important in section 1.7.

Theorem 31 *([22], page 318) A sectorial operator T is form closable, that is, the form $q(T)$ defined by (C.2) has an extension that is closed in the sense of definition 4.*

Theorem 32 *([22],page 316) Let $\tilde{q}$ be the closure of a densely defined form q. The numerical range $\Theta(q)$ of q is a dense subset of the numerical range $\Theta(\tilde{q})$ of $\tilde{q}$.*

The next theorem is also used in section 1.7.

Theorem 33 *([22], page 319) Let $q_1, ...q_s$ be sectorial forms in $\mathscr{H}$ and let $q := q_1 + ... + q_s$ [with $D(q) := D(q_1) \cap ... \cap D(q_s)$]. Then q is sectorial. If all q_j are closed, so is q. If all the q_j are closable so is q and*

$$\tilde{q} \subset \tilde{q_1} + ... + \tilde{q_s}.$$

The following theorem naturally associates to strictly m-accretive quadratic forms a unique operator T.

Theorem 34 *([31], page 281) Let q be a strictly m-accretive quadratic form with domain $Q(q)$. Then there is a unique operator T on $\mathscr{H}$ such that:*

a) T is closed.

b) $D(T) \subset Q(q)$ and if $\varphi, \psi \in D(T)$, then $q(\varphi, \psi) = (\varphi, T\psi)$. Further, $D(T)$ is a form core for q.

c) $D(T^*) \subset Q(q)$ and if $\varphi, \psi \in D(T)$, then $q(\varphi, \psi) = (T^*\varphi, \psi)$.Further, $D(T^*)$ is a form core for q.

From this theorem we can define.

Definition 9 *A closed operator T is called* **strictly m-sectorial operator** *if there exists q strictly m-sectorial such that q and T satisfy properties a),b) c) of the above theorem.*

Now we can formulate the Ichinose lemma.

Theorem 35 *([32],page 183) (Ichinose's lemma) Let $\overline{S}_{\omega,\varphi,\theta}$ denote the sector $\{z|\varphi - \theta \leq arg(z - \omega) \leq \varphi + \theta; \theta > \pi/2\}$. Let A and B be strictly m-sectorial operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with sectors $\overline{S}_{\omega_1,\varphi,\theta_1}$ and $\overline{S}_{\omega_2,\varphi,\theta_2}$ (same φ!). Let C denote the closure of $A \otimes I + I \otimes B$ on $D(A) \otimes D(B)$. Then C is a strictly m-sectorial operator with sector $\overline{S}_{\omega_1+\omega_2,\varphi,min\{\theta_1,\theta_2\}}$ and $\sigma(C) = \sigma(A) + \sigma(B)$.*

Appendix D

Geometric spectral analysis of σ_{ess}

In this appendix we generalize the definitions and results of section 3 of the paper [7] in order to apply them in our context. Most of the proofs and theorems in this appendix are essentially the same that those of [7]. However, we transcript them for making this text more understandable, and also for remarking that the generalization that we need holds.

D.1 Geometric spectral methods

In this section we distinguish between different types of singular sequences of a geometric operator and we describe some relations between them. They define different subsets of the essential spectrum defined in definition **??**.

Let $A : Dom(A) \subset L^2(X) \to L^2(X)$ be a closed operator.

Definition 10 *Define the set $N_{ess}(A)$ of $\lambda \in \mathbb{C}$ such that there exists a sequence $(u_n)_{n\in \mathbb{N}} \subset Dom(A)$ such that $||u_n|| = 1$, $u_n \to 0$ (weakly) and $||(\lambda - A)u_n|| \to 0$.*

Observe that if $\lambda \in N_{ess}(A)$, then the sequence $(u_n)_{n\in \mathbb{N}}$ associated to λ is a singular sequence in the sense of definition 2. Then, corollary 11 implies $N_{ess}(A) \subset \sigma_{e2}(A) \subset \sigma_{ess}(A)$, where σ_{e2} is defined in appendix A. In fact, we have:

Proposition 41 *[7] Let A be a closed operator. Then, $N_{ess}(A) = \sigma_{e2}(A)$.*

Proof:
We proved above $N_{ess}(A) \subset \sigma_{e2}(A)$. Suppose that $\lambda \in \sigma_{e2}(A)$, then, by theorem 22, there exists an orthonormal singular sequence, φ_n in $L^2(X)$, associated to the value λ and the operator A. For all $\gamma \in L^2(X)$, we have:

$$\lim_{n\to\infty} \langle \gamma, \varphi_n \rangle_{L^2(X)} = 0, \text{ and } ||\varphi_n|| = 1. \tag{D.1}$$

The above equation proves that $\lambda \in N_{ess}(A)$. $\square$

We remark that the above proposition holds for A a closed operator in an arbitrary Hilbert space $\mathscr{H}$, not necessarily $L^2(X)$. Now we define other important class of singular sequences $N_\infty(A)$.

Definition 11 *([7], page 10) Let $N_\infty(A)$ be the set of $\lambda \in \mathbb{C}$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(X)$*

- $||u_n|| = 1$.
- $||(A-\lambda)u_n|| \to 0$
- *For every K compact subset of M there exists a N such that for $n > N$, $supp u_n \cap K = \emptyset$.*

We will call the sequence u_n a **boundary Weyl sequence** *(abbr. bWs).*

Observe that if $\lambda \in N_\infty(A)$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence as in definition 11, then $(u_n)_{n\in\mathbb{N}}$ is a singular sequence associated to A and the value λ. From corollary 11, we deduce $N_\infty(A) \subset \sigma_{e2}(A) \subset \sigma_{ess}(A)$. We have the next proposition:

Proposition 42 *([7], page 9) i) $N_{ess}(A) \subset \sigma_{ess}(A)$.*

ii) $N_{ess}(A)$ is closed.

The following theorem gives conditions for the equality of $\sigma_{ess}(A)$ and $N_{ess}(A)$.

Theorem 36 *([7], theorem 3.1)(Weyl's criterion for $\sigma_{ess}(A)$) Let A be a closed operator on a Hilbert space $\mathscr{H}$ with non-empty resolvent set. Then:*

i) $N_{ess}(A) \subset \sigma_{ess}(A)$.

ii) The boundary of $\sigma_{ess}(A)$ is contained in $N_{ess}(A)$.

iii) $N_{ess}(A) = \sigma_{ess}(A)$ *if and only if each connected component of the complement of* $N_{ess}(A)$ *contains a point of* $\rho(A)$.

The next theorem gives conditions for the equality of $N_\infty(A)$ and $N_{ess}(A)$. We will use the notation X_0 and X_d for the manifolds defined in (2.6) for $T = 0$ and $T = d$ respectively.

Theorem 37 *([7], theorem 3.2) Let* A *be a closed operator on* $L^2(X)$ *with non-empty resolvent set, having* $C_c^\infty(X)$ *as a core. Let* $\eta_0 \in C_c^\infty(X)$ *such that* $\eta_0(x) = 1$ *for* $x \in X_0$. *Let* $\eta_0^d \in C_c^\infty(X)$ *such that* $\eta_0^d(x) = 1$ *for* $x \in X_d$, *and* $0 \leq \eta_0^d(x) \leq 1$ *for* $x \in X$. *Suppose* $\forall d$ $\eta_0^d(z-A)^{-1}$ *is compact for some* $z \in \rho(A)$ *and that for all* $u \in C_c^\infty(X)$,

$$||[A, \eta_0^d]u|| \leq \epsilon(d)(||Au|| + ||u||), \tag{D.2}$$

with $\epsilon(d) \to 0$ *as* $d \to \infty$. *Then* $N_\infty(A) = N_{ess}(A)$.

Proof:
Let $\lambda \in N_{ess}(A)$, then, by definition, there exists $u_n \in Dom(A)$, $||u_n|| = 1$, $u_n \to 0$ (weakly) $||(\lambda - A)u_n|| \to 0$. Since $C_c^\infty(X)$ is a core of A we can even choose $u_n \in C_c^\infty(X)$. We divide the proof in three steps.

Step 1: For $z \in \rho(A)$, $\lim_{n\to\infty} ||\eta_0^d u_n|| = \lim_{n\to\infty} ||\eta_0^d(z-A)^{-1}(z-A)u_n|| = 0$.

Observe that for all $v \in L^2(X)$,

$$|\langle (z-A)u_n, v\rangle| \leq |z-\lambda|.|\langle u_n, v\rangle| + |\langle (\lambda - A)u_n, v\rangle|. \tag{D.3}$$

This implies that $(z-A)u_n \to 0$ (weakly). Since weakly convergent sequences are bounded and $\eta_0^d(z-A)^{-1}$ is compact, then there exists a L^2-norm convergent subsequence of $v_n := \eta_0^d(z-A)^{-1}(z-A)u_n$. Without loss of generality, we assume that v_n is L^2-norm convergent. Since $v_n \to 0$ (weakly) and converges in norm then $v_n \to 0$ in norm. We have proved step 1.

Step 2: $||(\lambda - A)(1-\eta_0^d)u_n|| \leq C||(\lambda - A)u_n|| + \epsilon(d)(||Au_n|| + 1)$.

Observe:

$$||(\lambda - A)(1-\eta_0^d)u_n|| \leq ||(\lambda - A)u_n|| + ||(\lambda\eta_0^d - A\eta_0^d)u_n||,$$

and, by (D.2),

$$\begin{aligned} ||(\lambda\eta_0^d - A\eta_0^d)u_n|| &= ||(\lambda\eta_0^d - A\eta_0^d + \eta_0^d A - \eta_0^d A)u_n|| = ||\eta_0^d(\lambda - A)u_n + [\eta_0^d, A]u_n|| \\ &\leq C||(\lambda - A)u_n|| + \epsilon(d)(||Au_n|| + ||u_n||), \end{aligned}$$

that proves step 2.

Step 3: There exists a subsequence $(n(d))_{d\in \mathbb{N}}$ of $\mathbb{N}$ such that there exists $k > 0$ such that $||(1-\eta_0^d)u_{n(d)}|| \geq k > 0$ for all $d \in \mathbb{N}$. Observe that:

$$||(1-\eta_0^d)u_{n(d)}||_{L^2(X)} \geq 1 - ||\eta_0^d u_{n(d)}||_{L^2(X)}. \tag{D.4}$$

Step 1 and (D.4) prove step 3.

Steps 1,2 and 3 imply that there exists a subsequence $(n(d))_{d\in \mathbb{N}}$ of $\mathbb{N}$ such that $||(1-\eta_0^d)u_{n(d)}||^{-1}(1-\eta_0^d)u_{n(d)}$ is a b.W.s associated to the value λ and the operator A.$\square$

Appendix E

Elliptic differential operators on manifolds with bounded geometry

In this appendix we describe results of the global analysis on manifolds of bounded geometry that we use in the thesis. We are based mainly in [37] and [12] and we refer to them for details.

Definition 12 *Let (M, g) be a Riemannian manifold. M, g is a manifold with C^k-bounded geometry if and only if:*

a) The injectivity radius, r_{inj}, is strictly positive.

b) If R denotes the curvature associated to the Levi-Civita connection, $|\nabla^i R|_g \leq C_i$ for $i \leq k$.

We fix the following set of charts in M. Let $0 < r < r_{inj}$. Then there exists an open set $U_{x,r} \subset M$ such that $exp : B_r(0) \subset T_xM \to U_{x,r}$ is a diffeomorphism. b) in the above definition can be reformulated in the following way:

b') *If $U_{x,r} \cap U_{x',r} \neq \emptyset$ and y denotes the coordinates associated to $U_{x,r}$ and y' denotes the coordinates associated to $U_{x',r}$ then:*

$$\frac{\partial^\alpha}{\partial y^\alpha}(y'^{-1} \circ y) \leq C \text{ for all } |\alpha| \leq k \text{ and all } U_{x,r},\ U_{x',r} \text{ satisfiyng } U_{x,r} \cap U_{x',r} \neq \emptyset.$$

The following definitions are straightforward:

Definition 13 *([37], appendix 1) $f : M \to \mathbb{C}$ is C^k-bounded if and only if*

$$|\frac{\partial^\alpha}{\partial y^\alpha} f(y)| \leq C_\alpha, \text{ for } |\alpha| \leq k, \tag{E.1}$$

in any of the charts $exp : B_r(0) \to U_{x,r}$ where r is shorter than the injectivity radius of M.

The above local definition is equivalent to the following global version:

$$|\nabla^i f(x)|_{g_x} \leq C \text{ for } i \leq k \text{ and all } x \in M. \tag{E.2}$$

If M, g is a manifold with C^k-bounded geometry for all $k \in \mathbb{N}$, then we will call it simply manifold with bounded geometry. Observe that the complete manifolds with cylindrical end and the complete manifolds with corner of codimension 2, described in sections 1.1 and 2.1, are manifolds with bounded geometry. The following theorem provides nice charts and partition of unity for manifolds with bounded geometry.

Theorem 38 *([37], appendix 1, lemmas 1.2 and 1.3) Let M, g be a manifold with bounded geometry.*

1) *There exists ϵ_0 such that if $\epsilon \in (0, \epsilon_0)$ then there exists a countable covering of M by balls of radius ϵ: $M = B_\epsilon(x_i)$ such that the covering of M by the balls $B_{2\epsilon}(x_i)$ has finite multiplicity (the maximal number of balls with non-empty intersection is finite).*

2) *For every $\epsilon \in (0, \epsilon_0)$ there exists a partition of unity $\{\varphi_i\}_{i=1}^N$ on M, $1 = \sum_{i=1}^N \varphi_i$ such that:*

   - *$supp\varphi_i \subset B_{2\epsilon}(x_i)$.*
   - *$|\frac{\partial^\alpha}{\partial y^\alpha}\varphi_i(y)| \leq C_\alpha$ in any chart $exp : B_r(x_i) \subset T_{x_i}M \to B_{2\epsilon}(x_i)$, where r is shorter than the injectivity radius of M.*

Natural vector bundles over M are vector bundles of bounded geometry that we define next.

Definition 14 *([37], appendix 1) Let E be a vector bundle on M. We shall say that E is a* **vector bundle of bounded geometry** *if and only if on any canonical coordinate neighborhoods $B_r(x_i)$ and $B_r(x_i')$ the transitions functions $g|_{B_r(x_i)\cap B_r(x_i')}$ are C^∞-bounded i.e. all their derivatives $\partial_y^\alpha g|_{B_r(x_i)}\cap B_r(x_i')$ are bounded with bound C_α independent of the pair $B_r(x_i)$, $B_r(x_i')$.*

Consider a partition of unity, $\{\varphi_i\}_{i=1}^N$, as 2) of theorem 38 and an associated trivialization of E i. e. $\psi_i : E|_{\exp(B_r(x_i))} \to \mathbb{C}^d$ ($d := \dim E_{x_i}$). Define the following Sobolev norm for $u \in C_c^\infty(M, E)$:

$$||u||_s^2 := \sum_{i=1}^N ||\psi_i \circ (\varphi_i u) \circ \exp||_{s,x_i}^2, \tag{E.3}$$

where $||\cdot||_{s,x_i,r}^2$ denotes the s-Sobolev norm on $(T_{x_i}M, g_{x_i})$ (observe that $\psi_i \circ ((\varphi_i u) \circ \exp)$ is a function from $B_r(x_i) \subset T_{x_i}M$ to $\mathbb{C}^d$). We define the s-Sobolev space by:

$$\mathscr{W}_s(M, E) := \text{ completion of } C_c^\infty(M, E) \text{ with respect to the norm } ||\cdot||_s. \tag{E.4}$$

Definition 15 *Let $A : C^\infty(M, E) \to C^\infty(M, E)$ be a differential operator or order m with C^∞-coefficients:*

*1) A is C^∞-**bounded** if and only if in any of the charts $exp : B_r(0) \to U_{x,r}$:*

$$A = \sum_{|\alpha| \leq m} a_\alpha(y) \frac{\partial^\alpha}{\partial y^\alpha} \text{ and } |\frac{\partial^\beta}{\partial y^\beta}(a_\alpha)(y)| \leq C_\beta. \tag{E.5}$$

*2) A is called **uniformly elliptic** if it is elliptic and there exists $C > 0$ such that:*

$$|(a_m^{-1}(y, \xi)| \leq C|\xi|^{-m}, \ (y, \xi) \in T^*M, \xi \neq 0, \tag{E.6}$$

where $a_m(y, \xi)$ is the principal symbol of A.

We have the following lemma:

Lemma 10 *([37], lemma 1.4, appendix 1) Let A be a C^∞-bounded uniformly elliptic differential operator of order m. Then for any $s, t \in \mathbb{R}$ there exists $C > 0$ such that:*

$$||u||_s \leq C(||Au||_{s-m} + ||u||_t) \text{ for } u \in C_c^\infty(M, E). \tag{E.7}$$

We denote

$$||f||^2 := \int_M |f|^2 dvol(x). \tag{E.8}$$

The above lemma implies the following theorem

Theorem 39 *The following norms on $C_c^\infty(M, E)$ are equivalent:*

- $||.||_m$ *as defined above.*
- $f \mapsto ||f||+||Af||$ *where* A *is* C^∞*-bounded uniformly elliptic differential operator of order* m*.*
- $f \mapsto \sum_{i=0}^{m} ||\nabla^i f||$.

The next theorem gives sufficient conditions for essentially self-adjointness of C^∞-differential operators.

Theorem 40 *([37],corollary 4.2) Let* M *be a manifold of bounded geometry,* E *a vector bundle of bounded geometry. Let* $A : C^\infty(M,E) \to C^\infty(M,E)$ *be a* C^∞*-bounded uniformly elliptic differential operator and suppose that* A *is formally self-adjoint. Then, the operator* $A : C_c^\infty(M,E) \to L^2(M,E)$ *is essentially self-adjoint.*

Bibliography

[1] M. Atiyah, K. Patodi, I. Singer, *Spectral asymmetry and Riemannian geometry. I.*. Math. Proc. Camb. Phil. Soc. **77**, pp. 43-69, 1975.

[2] E. Balslev, *Spectral deformations of Laplacians on hyperbolic manifolds.* Comm. Analysis and Geometry **5**, pp. 213-247, 1997.

[3] E. Balslev, J. Combes, *Spectral properties of many-body Schrödinger operators with dilation-analytic interactions.* Commun.math. Phys. **22**, pp. 280-294, 1997.

[4] H. Bauer *Measure and integration theory*, Walter de Gruyter, Berlin-New York, 2001.

[5] F.E. Browder, *On the spectral theory of elliptic differential operators. I.*Math. Annalen **142**, pp.22-130, 1961.

[6] T. Christiansen, *Some upper bounds on the number of resonances for manifolds with infinite cylindrical ends* Ann. Henri Poincar?e 3 (2002) 895-920.

[7] P. Deift, W. Hunziker, B. Simon, E. Vock, *Point-wise bounds on eigenfunctions and wave packets in N-body quantum systems IV* Commun. math. Phys. **64**, pp. 1-34, 1978.

[8] H. Donelly *Eigenvalues estimates for certain non compact manifolds* Michigan Math J., Vol. 31, 1984, 349-357.

[9] H. Donelly and P. Li, *Pure point spectrum and negative curvature for non compact manifolds* Duke Mathematical Journal, Vol.46, No 3 1979, 497-503.

[10] P. Duclos, P. Exner, B. Meller, *Exponential bounds on curvature-induced resonances in a two-dimensional Dirichlet tube* Helv. Phys. Acta 71 (1998), 133-162.

[11] D. E. Edmunds, W.D. Evans, *Spectral theory and differential operators* Clarendon Press, Oxford, 1987.

[12] J. Eichhorn, *Global analysis on open manifolds* Nova Science Publisher, 2007.

[13] G. Folland, *Real analysis* A Wiley-Inter-science Publication, 2nd edition, 1999.

[14] R. Froese, I. Herbst, *Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators* Commun. Math. Phys.87, 429-337, 1982.

[15] C. Gérard, *Distortion analyticity for N-particles Hamiltonians* Helv. Phys. Acta 66, 216-225 (1993)

[16] L. Guillopé, *Théorie spectrale de quelques variérés à bouts* Annales scientifiques de l' É.N.S. 4e, tome 22,n^0 1 (1989),p. 137-160.

[17] P.D. Hislop, I.M. Sigal, *Introduction to spectral theory* Springer-Verlag, New York, 1996.

[18] P.D. Hislop *Spectral analysis of non-compact manifolds using commutator methods* Séminaire Équations aux dérivées partielles (Polythenique) (1987-1988), exp. $n^0$8, p. 1-11.

[19] W. Hunziker, I.M. Sigal *The quantum N-body problem.* Journal of Mathematical Physics **41**, pp. 3348-3510, 2000.

[20] W. Hunziker, I.M. Sigal *Time-dependent scattering theory of N-body quantum systems.* Reviews in Mathematical Physics **12** No 8, pp. 1033-1084, 2000.

[21] R. Husseini, *Zur Spektraltheorie verallgemeinter Laplace-Operatoren auf Mannigfaltigkeiten mit zilindrischen Enden*, Diplome Thesis, Rheinischen Friedrich Whilhelms-Universitaet Bonn.

[22] T. Kato, *Perturbation theory for linear operators* Springer-Verlag, Berlin Heidelberg, 1976.

[23] V. Kalvin, *The Aguilar-Baslev-Combes theorem for the Laplacian on a manifold with an axial analytic asymptotically cylindrical end* Arxiv 10032538v2 [math-ph] 18 Mar 2010.

[24] S. Kobayashi, K. Nomizu, *Foundations of differential geometry Vol. 1*Inter-science Publishers,1963.

[25] H. Kovarik, A. Sacheti, *Resonances in twisted quantum wave guides* J. Phys. A: Math. Theor. 40 (2007) 8371-8384

[26] R. Mazzeo, A. Vasy, *Resolvents and Martin boundaries of product spaces.* Geometric and Functional Analysis **12**, pp. 1018-1079, 2002.

[27] R. Mazzeo, A. Vasy, *Analytic continuation of the resolvent of the Laplacian on $Sl(3)/SO(3)$.* American Journal of Mathematics **126**, pp. 821-844, 2004.

[28] R. Mazzeo, A. Vasy, *Scattering theory on $SL(3)/SO(3)$: connections with quantum 3-Body scattering* Proceeding of the London Mathematical Society. American Journal of Mathematics **126**, pp. 821-844, 2004.

[29] R. Melrose, *The Atiyah Patodi Singer index theorem* webpage, www-math.mit.edu/ rbm/book.html

[30] Mueller, W. *On the L^2-index of Dirac operators on manifolds with corners of codimension two. I.* J. Differential Geometry **44**, pp. 97-177, 1996.

[31] Reed, M. Barry, S. *Functional analysis.* Academic press, 1980.

[32] M. Reed, S. Barry, *Fourier analysis, self-adjointness.* Academic press,1980.

[33] M. Reed, S. Barry, *Scattering theory.* Academic press,1980.

[34] M. Reed, S. Barry, *Analysis of operators.* Academic press,1980.

[35] G. Salomonsen, *Equivalence of Sobolev norms* Results Math. 39 (2001) 115-130.

[36] M. Schechter, *On the essential spectrum of an arbitrary operator I* J. Math. Anal. Appl. 13 1966, 205-215.

[37] M.A. Shubin, Spectral theory of elliptic operators on non-compact manifolds. Paper on lectures Summer School on Semi-classical Methods, Nantes, 1991.

[38] J. Sjöstrand, M. Zworski *Lower bounds on the number of scattering poles*, Comm.PDE 18 (1993), 847-858.

[39] J. Sjöstrand, M. Zworski *Estimates on the number of scattering poles near the real axis for strictly convex obstacles*, Ann. Inst. Fourier 43(3)(1993), 769-790.

[40] J. Sjöstrand, M. Zworski *Lower bounds on the number of scattering poles II.*, J. Func. Anal. 123(2)(1994), 336-367.

[41] J. Sjöstrand, M. Zworski *The complex scaling method for scattering by strictly convex obstacles.*, Ark. för Math. 33(1)(1995), 135-172.

[42] A. Taylor, *Theorems on ascent, descent, nullity and defect of linear operators.* Math. Annalen **163** , pp. 18-49, 1966.

[43] D.Yafaev, *Radiation conditions and scattering theory for N-particle Hamiltonians.* Commun.Math.Phys. **154**, pp. 523-554, 1993.

[44] D. R. Yafaev, *Mathematical scattering theory.* Providence, RI, American Mathematical Society, 1992.

Zusammenfassung
Analytic dilation on complete manifolds with corners of codimension 2

Leonardo A. Cano García

The method of analytic dilation was originally applied to N-particle Schrödinger operators and a classic reference in that setting is [4]. Also it has been applied to the black-box perturbations of the Euclidean Laplacian in the series of papers [12], [13], [14], [15]. In the paper [1] is used for studying Laplacians on hyperbolic manifolds. The analytic dilation has also been applied to the study of the spectral and scattering theory of quantum wave guides and Dirichlet boundary domains, some references in this setting are [3], [8]. It has also been applied to arbitrary symmetric spaces of noncompact types in the papers [9], [10], [11]. In each of these settings new ideas and new methods carry out. In this thesis we develop the analytic method for Laplacians on complete manifolds with corners of codimension 2.

Let X_0 be a compact manifold with boundary M. X_0 has a corner of codimension 2 if:

i) There exists a hypersurface Y of M that divides M in two manifolds with boundary M_1 and M_2. More explicitly, $M = M_1 \cup M_2$ and $Y = M_1 \cap M_2$.

ii) X_0 is endowed with a Riemannian metric g that is a product metric on small neighborhoods of the M_i's and the corner Y.

The following is a figure of X_0:

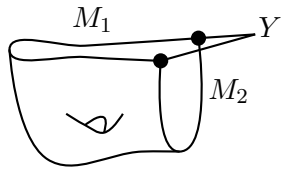

Figure 1. compact mfld with corner of cod. 2

We construct from X_0 a complete manifold X by attaching $(\mathbb{R}_+ \times M_1)$, $(\mathbb{R}_+ \times M_2)$ and filling the rest with $(\mathbb{R}_+ \times \mathbb{R}_+ \times Y)$. As a set,

$$X := X_0 \cup (\mathbb{R}_+ \times M_1) \cup (\mathbb{R}_+ \times M_2) \cup (\mathbb{R}_+ \times \mathbb{R}_+ \times Y), \tag{1}$$

and it has the natural differential structure and Riemannian metric that are compatible with the product structures at the boundary of X_0 . For each $T \in \mathbb{R}_+$, X has two submanifolds with cylindrical ends namely $M_i \times \{T\} \cup (Y \times \{T\}) \times [0, \infty)$, $i = 1, 2$. We denote these manifolds by Z_i. The next figure is a sketch of a complete manifold with one corner of codimension two, Y, and its submanifolds Z_i:

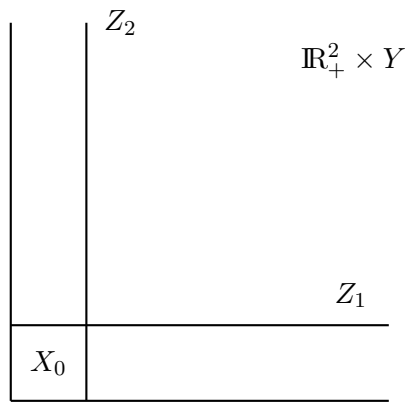

Figure 2. Complete mfld with corner of cod. 2

We consider the operator $\Delta := d^*d : C_c^\infty(X) \to L^2(X)$. It is well known that Δ is essentially self-adjoint and we denote by H its self-adjoint extension.

For $i = 1, 2$, since Z_i is a complete manifold, the Laplacian $\Delta : C_c^\infty(Z_i) \to L^2(Z_i)$ is essentially self-adjoint; we denote by $H^{(i)}$ its self-adjoint extension. Similarly $\Delta : C^\infty(Y) \to L^2(Y)$ is also essentially self-adjoint, and we denote its self-adjoint extension by $H^{(3)}$. The analytic dilation of a many-body Schrödinger operator depends on the analytic dilation of its clusters operators. In a similar way the analytic dilation of H is described in terms of the spectral theory of the operators $H^{(1)}$, $H^{(2)}$ and $H^{(3)}$.

For $\theta > 0$, $U_{i,\theta} : L^2(Z_i) \to L^2(Z_i)$ essentially is the dilation operator by

$\theta+1$ up to a compact set. More precisely:

$$U_{i,\theta}f(x)=\begin{cases} f(x) & \text{for } x\in M_i. \\ (\theta+1)^{1/2}f((\theta+1)u,y) & \text{for } x=(u,y)\in \mathbb{R}_+\times Y \\ & \text{for } u \text{ big enough,} \end{cases} \tag{2}$$

and $U_{i,\theta}f$ is extended to the whole Z_i in such a way that it sends $C_c^\infty(Z_i)$ into $C_c^\infty(Z_i)$, and it becomes a unitary operator on $L^2(Z_i)$. Similarly, the operators $U_\theta : L^2(X)\to L^2(X)$ are defined by

$$U_\theta f(x)=\begin{cases} f(x) & \text{for } x\in X_0. \\ (\theta+1)^{1/2}U_{i,\theta}f((\theta+1)u_i,z_i) & \text{for } x=(u_i,z_i)\in \mathbb{R}_+\times Z_i \\ & \text{for } u_i \text{ big enough.} \end{cases} \tag{3}$$

$U_\theta f$ is extended to the whole X in such a way that, for $f\in C_c^\infty(X)$, $U_\theta f\in C_c^\infty(X)$, and that U_θ becomes a unitary operator in $L^2(X)$.

For $\theta\in\mathbb{R}_+$, define $H_\theta := U_\theta H U_\theta^{-1}$ an unbounded operator with domain

$$\mathscr{W}_2(X):=\{f\in L^2(X):\Delta_{dist}f\in L^2(X)\}. \tag{4}$$

the second Sobolev space associated to (X,g). This definition of Sobolev space is a good definition because (X,g) is a manifold with bounded geometry and its Laplacian is a uniformly elliptic operator (see [16]). Our first result is:

Theorem 1 *(cs. [1], [10],[7]) The family $(H_\theta)_{\theta\in\mathbb{R}_+}$ extends to an holomorphic family for $\theta\in\mathbb{C}$, $Re(\theta)>0$, which satisfies:*

1) H_θ is a closed operator in $\mathscr{W}_2(X)$ for all $Re(\theta)>0$.

2) For $\varphi\in\mathscr{W}_2(X)$ the map $\theta\mapsto H_\theta\varphi$ is holomorphic in $Re(\theta)>0$.

An holomorphic family of operators satisfying (1) and (2) will be called a holomorphic family of type A. This theorem is proved using the corresponding version for $H^{(i),\theta}$, the operator $U_{i,\theta}\Delta_{Z_i}U_{i,\theta}^{-1}$ with domain

$$\mathscr{W}_2(Z_i):=\{f\in L^2(Z_i):\Delta_{dist}(f)\in L^2(Z_i)\}, \tag{5}$$

the second Sobolev space associated to (Z_i,g_i).

We define

$$\theta' := \frac{1}{(\theta+1)^2}. \tag{6}$$

The parameter θ' is very important in the description of the essential spectrum of H_θ as we can see in the next theorem that is our second result:

Theorem 2 *(cs. [1], [10],[7]) For $Re(\theta) \geq 0$,*

$$\begin{aligned}\sigma_{ess}(H_\theta) = & \bigcup_{\mu \in \sigma(H^{(3)})} (\mu + \theta'[0,\infty)) \\ & \cup \bigcup_{\lambda_1 \in \sigma_{pp}(H^{(1),\theta})} \left(\lambda_1 + \theta'[0,\infty)\right) \\ & \cup \bigcup_{\lambda_2 \in \sigma_{pp}(H^{(2,\theta)})} \left(\lambda_2 + \theta'[0,\infty)\right).\end{aligned} \tag{7}$$

We associate to $(U_\theta)_{\theta \in \mathbb{R}_+}$ a set $\mathscr{V} \subset \mathscr{W}_2(X)$ that satisfies:

i) $\mathscr{V}$ is dense in $L^2(X)$;

ii) for $\varphi \in \mathscr{V}$, $U_\theta \varphi$ is defined for all $Re(\theta) > 0$ and iii) $U_\theta \mathscr{V}$ is dense in $L^2(X)$.

The elements of a subset of $\mathscr{W}_2(X)$ that satisfies i) and ii) will be called **analytic vectors**. We define

$$\Lambda := \{(x,y) \in \mathbb{C} : x < 0\}. \tag{8}$$

Using the analytic vectors, the next theorem describes more carefully the spectrum of H; it is a consequence of theorem 2 and the general analytic dilation theory of Aguilar-Balslev-Combes (see [1]).

Theorem 3 *(cs. [1], [10],[7]) 1) For $f, g \in \mathscr{A}$ the function $\lambda \mapsto \langle R(\lambda) f, g\rangle_{L^2(X)}$ extends for $\mathbb{C}$ to the surface $\mathbb{C} - \sigma(H_\theta)$.*

2) For all θ with $Re(\theta) \geq 0$, H_θ has not singular spectrum.

3) The accumulation points of $\sigma_{pp}(H)$ are contained in $\{\infty\} \cup \sigma(H^{(3)}) \cup \cup_{i=1}^2 \sigma_{pp}(H^{(i)})$.

The meromorphic extension of the resolvent entries is used by us for extending generalized eigenfunctions associated to L^2-eigenfunctions of $H^{(i)}$ for $i = 1, 2, 3$. It is part of a work in progress to prove that these generalized eigenfunctions describe natural wave operators whose image is the complete set of absolute continuous states associated to H.

In order to prove theorem 2, we based on [2]. In [2] the notion of singular spectrum is refined geometrically introducing the notion of what we called boundary Weyl sequences. In our work we observe that the methods of [2] apply for compatible Laplacians on manifolds with corners of codimension 2.

Finally we include the bibliography used in this summary. The bibliography on which we based our research is larger and is included in the thesis.

References

[1] E. Balslev, *Spectral deformations of Laplacians on hyperbolic manifolds.* Comm. Analysis and Geometry **5**, pp. 213-247, 1997.

[2] P. Deift, W. Hunziker, B. Simon, E. Vock, *Point-wise bounds on eigenfunctions and wave packets in N-body quantum systems IV* Commun. math. Phys. **64**, pp. 1-34, 1978.

[3] P. Duclos, P. Exner, B. Meller, *Exponential bounds on curvature-induced resonances in a two-dimensional Dirichlet tube* Helv. Phys. Acta 71 (1998), 133-162.

[4] C. Gérard, *Distortion analyticity for N-particles Hamiltonians* Helv. Phys. Acta 66, 216-225 (1993)

[5] P.D. Hislop, I.M. Sigal, *Introduction to spectral theory* Springer-Verlag, New York, 1996.

[6] W. Hunziker, I.M. Sigal, *The quantum N-body problem.* Journal of Mathematical Physics **41**, pp. 3348-3510, 2000.

[7] V. Kalvin *The Aguilar-Baslev-Combes theorem for the Laplacian on a manifold with an axial analytic asymptotically cylindrical end* Arxiv 10032538v2 [math-ph] 18 Mar 2010.

[8] H. Kovarik, A. Sacheti, *Resonances in twisted quantum wave guides* J. Phys. A: Math. Theor. 40 (2007) 8371-8384

[9] R. Mazzeo, A. Vasy, *Resolvents and Martin boundaries of product spaces.* Geometric and Functional Analysis **12**, pp. 1018-1079, 2002.

[10] R. Mazzeo, A. Vasy, *Analytic continuation of the resolvent of the Laplacian on $Sl(3)/SO(3)$*. American Journal of Mathematics **126**, pp. 821-844, 2004.

[11] R. Mazzeo, A. Vasy, *Scattering theory on $SL(3)/SO(3)$: connections with quantum 3-Body scattering* Proceeding of the London Mathematical Society. American Journal of Mathematics **126**, pp. 821-844, 2004.

[12] J. Sjöstrand, M. Zworski *Lower bounds on the number of scattering poles*, Comm.PDE 18 (1993), 847-858.

[13] J. Sjöstrand, M. Zworski *Estimates on the number of scattering poles near the real axis for strictly convex obstacles*, Ann. Inst. Fourier 43(3)(1993), 769-790.

[14] J. Sjöstrand, M. Zworski *Lower bounds on the number of scattering poles II.*, J. Func. Anal. 123(2)(1994), 336-367.

[15] J. Sjöstrand, M. Zworski *The complex scaling method for scattering by strictly convex obstacles.*, Ark. för Math. 33(1)(1995), 135-172.

[16] M.A. Shubin, *Spectral theory of elliptic operators on non-compact manifolds*. Paper on lectures Summer School on Semiclassical Methods, Nantes, 1991.