Quantum cluster algebras and the dual canonical basis

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Zusammenfassung

Sei Q ein Dynkinköcher vom Typ A mit alternierender Orientierung oder der Kroneckerköcher. Wir betrachten die direkte Summe M der unzerlegbaren, injektiven Moduln über der Wegealgebra des Köchers und ihrer Auslander-Reiten-Verschiebungen. Sei $\mathfrak{g}$ die zu Q assoziierte komplexe Liealgebra. Sie besitzt eine Dreieckszerlegung $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Hierbei ist $\mathfrak{n}$ eine maximal nilpotente Unterliealgebra von $\mathfrak{g}$. Zu M gehört in natürlicher Weise ein Element w in der Weylgruppe des gleichen Typs. Die Dimensionsvektoren der unzerlegbaren, injektiven Moduln und ihrer Verschiebungen entsprechen nach dem Satz von Gabriel bzw. nach dem Satz von Kac positiven Wurzeln im Wurzelsystem der Liealgebra $\mathfrak{g}$. Ihre Anordnung erbringt einen reduzierten Ausdruck für das Weylgruppenelement w.

Die vorliegende Arbeit erbringt den Nachweis, dass geschickt gewählte Erzeuger der zu w gehörigen Unteralgebra $U_q^+(w)$ der quantisierten universellen einhüllenden Algebra $U_q(\mathfrak{n})$ in sich überlappende Mengen, sogenannte Cluster, gruppiert werden können, so dass $U_q^+(w)$ die Struktur einer quantisierten Clusteralgebra erhält.

Aus der Konstruktion der quantisierten Clusteralgebrenstruktur auf $U_q^+(w)$ und der Wahl der Erzeuger ergeben sich folgende Eigenschaften:

Erstens: Die quantisierten Clustervariablen stimmen jeweils bis auf eine Potenz des Deformationsparameters q mit einem Element im Dualen der von Lusztig definierten kanonischen Basis unter Kashiwaras Bilinearform überein.

Zweitens: Geiß-Leclerc-Schröer haben für ein derartiges w eine azyklische Clusteralgebra konstruiert. Sie wird realisiert als Unteralgbera der graduiert dualen Hopfalgebra der universellen einhüllenden Algebra $U(\mathfrak{n})$. Unsere quantisierte Clusteralgebra degeneriert zu Geiß-Leclerc-Schröers Clusteralgebra im klassischen Limes $q = 1$. Sowohl die quantisierte als auch die gewöhnliche Clusteralgebra hat eingefrorene sowie mutierbare Clustervariablen.

Drittens: Bestimmte Elemente in der dualen kanonischen Basis erfüllen nennerfreie Rekursionsgleichungen. Die Rekursionsgleichungen implizieren die Austauschrelationen für quantisierte Clusteralgebren.

Die Arbeit ist in zwei Teile gegliedert. Der erste Teil behandelt den Fall des Dynkinköchers vom Typ A mit alternierender Orientierung, der zweite Teil behandelt den Kroneckerköcher. Nach einem Satz von Lusztig besitzt die Algebra $U_q^+(w)$ in beiden Fällen eine Poincaré-Birkhoff-Witt-Basis, die sich aus dem reduzierten Ausdruck für w ergibt. Die duale kanonische Basis kann mit Hilfe der Poincaré-Birkhoff-Witt-Basis über eine Invarianz- und eine Gittereigenschaft charakterisiert werden.

Wir beschreiben zunächst die Begradigungsrelationen der Erzeuger von $U_q^+(w)$ aus der Poincaré-Birkhoff-Witt-Basis. Sodann leiten wir nennerfreie Rekursionsgleichungen für bestimmte Elemente in der dualen kanonischen Basis her. Die Nennerfreiheit erlaubt es, die Invarianz- und die Gittereigenschaft nachzuweisen. Es folgen die Austauschrelationen der quantisierten Clusteralgebra.

Contents

1 Introduction

Cluster algebras are commutative algebras created in 2000 by Fomin-Zelevinsky [14] in the hope to obtain a combinatorial description of the dual of Lusztig's *canonical basis* of a quantum group.

A cluster algebra of rank n (for some natural number n) is a subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of rational functions in n variables. Its generators are called *cluster variables*. Each cluster variable belongs to several overlapping *clusters*. Every cluster, and hence every cluster variable, is obtained from an initial cluster by a sequence of *mutations*. Every mutation replaces an element in a cluster by an explicitly defined rational function in the variables of that cluster. A cluster together with the exchange matrix that describes the mutation rule is called a *seed*. We refer to Fomin-Zelevinsky [14] for definitions and to Fomin-Zelevinsky [16] for a good survey about cluster algebras.

A cluster algebra is said to be of finite type if it exhibits only finitely many seeds. Fomin-Zelevinsky [15] classified cluster algebras of finite type. They are parametrized by the same Cartan-Killing types as semisimple Lie algebras. The classification of cluster algebras of finite type indicates a strong connection to Lie theory, and in fact it quickly turned out that Fomin-Zelevinsky's theory of cluster algebras has many interesting applications and coheres with various mathematical objects. Let us mention the representation theory of quivers and finite-dimensional algebras, the representation theory of preprojective algebras, root systems of Kac-Moody algebras, Calabi-Yau categories, quantum groups, and Lusztig's canonical basis of universal enveloping algebras.

A momentous step in the development was the *categorification* of acyclic cluster algebras by *cluster categories*. Cluster categories were defined by Buan-Marsh-Reineke-Reiten-Todorov [4]. The cluster category $\mathcal{C}_Q$ associated with a quiver Q is an orbit category of the bounded derived category of the category of representations of a quiver. Keller [29] proved that cluster categories are triangulated categories. Key ingredients for the verification of the categorification of acyclic cluster algebras by cluster categories are due to Geiß-Leclerc-Schröer [23, 24] and Caldero-Keller [8, 9]. The process of mutation in the cluster algebra resembles tilting in the cluster category. Hence, we obtain a link between quiver representations and triangulated categories on one side and a large class of cluster algebras on the other side.

Furthermore, Geiß-Leclerc-Schröer [21] provided a categorification of cluster algebras by *Kac-Moody groups* und *unipotent cells*. In this construction the categorified cluster algebras are not necessarily acyclic. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be its triangular decomposition. Geiß-Leclerc-Schröer's construction is related to preprojective algebras Λ associated with quivers Q. Buan-Iyama-Reiten-Scott [3] attached to every element w in the Weyl group of corresponding type a subcategory $\mathcal{C}_w \subset \mathrm{mod}(\lambda)$. The category $\mathcal{C}_w$ is a Frobenius category, so we can construct its stable category $\underline{\mathcal{C}}_w$. It is a Calabi-Yau category of dimension two. Geiß-Leclerc-Schröer [21] prove that the categories $\mathcal{C}_w$ categorify cluster algebras. They endow the coordinate ring $\mathbb{C}[N(w)]$ of the unipotent group $N(w)$ with the structure of a cluster algebra. Here, N denotes the pro-unipotent pro-group associated with the completion $\widehat{\mathfrak{n}}$ and $N(w) = N \cap (w^{-1} N_- w)$. The coordinate ring $\mathbb{C}[N(w)]$ is naturally isomorphic to a subalgebra of the graded dual $U(\mathfrak{n})^*_{gr}$ of the universal enveloping algebra of $\mathfrak{n}$. All cluster monomials lie in the dual semicanonical basis.

In this thesis we transfer to the quantized setup and investigate quantum cluster algebra structures on subalgebras of the *quantized* universal enveloping algebra $U_q(\mathfrak{n})$

of $\mathfrak{n}$ attached to Weyl group elements w.

Let us mention that cluster algebras also gained popularity in other branches of mathematics, for example Poisson geometry, see Gekhtman-Shapiro-Vainshtein, Teichmüller theory, see Fock-Goncharov [13], combinatorics, see Musiker-Propp [45], integrable systems, see Fomin-Zelevinsky [18], etc.

Now we give a more detailed description of the structure and the results of the thesis. Let $Q = (Q_0, Q_1)$ be an *acyclic quiver*, i.e., a directed graph without oriented cycles whose set of vertices is denoted by Q_0 and whose set of arrows is denoted by Q_1. The thesis focuses on two examples, namely *alternating quivers of type A* and the *Kronecker quiver*.

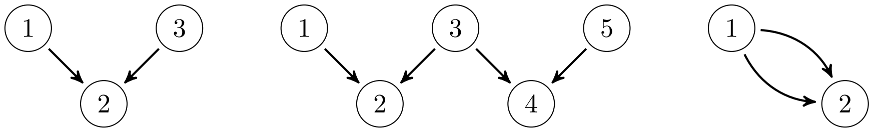


Figure 1: Alternating quivers of type A_3 and A_5 and the Kronecker quiver

Let k be a field. A *representation* M of Q is defined to be a collection $M = ((V_i)_{i\in Q_0}, (\phi_a)_{a\in Q_1})$ of k-vector spaces V_i associated with every vertex i and k-linear maps $\phi_a\colon V_i \to V_j$ associated with every arrow $a\colon i \to j$ in Q_1. A *morphism* $F\colon M \to N$ from M to another representation $N = ((W_i)_{i\in Q_0}, (\psi_a)_{a\in Q_1})$ is a collection of k-linear maps $F_i\colon V_i \to W_i$ such that the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{\phi_a} & V_j \\ {\scriptstyle F_i}\downarrow & & \downarrow{\scriptstyle F_j} \\ W_i & \xrightarrow[\psi_a]{} & W_j \end{array}$$

commutes for every $a\colon i \to j$ in Q_1. The finite-dimensional representations of Q together with their morphisms form an abelian category $\mathrm{rep}_k(Q)$. In particular, there is a direct sum $M\oplus N$ for every two representations M and N. The category $\mathrm{rep}_k(Q)$ is equivalent to the category $\mathrm{mod}(kQ)$, where kQ is the *path algebra* of the quiver, given as follows: As a vector space kQ is generated by all paths in the quiver (including a path of length zero for every $i \in Q_0$). The product of two paths is the concatenation of paths if possible and zero otherwise. The vector $\underline{\dim}(M) = (\dim V_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ is called the *dimension vector* of M.

The quiver Q is called *representation-finite* if Q admits only finitely many (isomorphism classes of) indecomposable representations. Gabriel's theorem [19] asserts that Q is representation-finite if and only if Q is an orientation of a *Dynkin diagram* of type A, D, or E. For example, the alternating quiver of type A_3 form above is representation-finite. It admits six (isomorphism classes of) indecomposable representations. The *Auslander-Reiten quiver* in Figure 2 epitomizes the category $\mathrm{rep}_k(Q)$. Representations are displayed by their dimension vectors.

On the other hand the Kronecker quiver is representation-infinite. But it is a *tame quiver*. Albeit there are infinitely many (isomorphism classes of) indecomposable kQ-modules, the indecomposable kQ-modules can be classified. The three kinds of indecomposables are called *preprojective*, *preinjective*, and *regular*. A part of the prein-

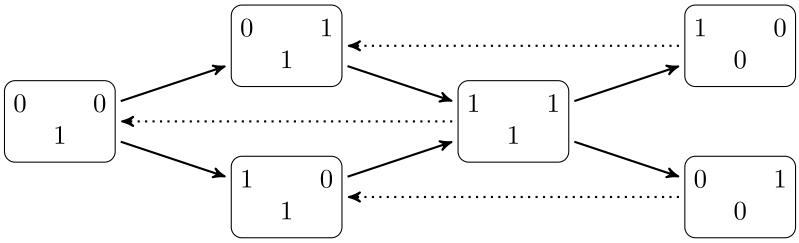


Figure 2: The Auslander-Reiten quiver of type A_3

jective component of the Auslander-Reiten quiver of the Kronecker quiver is shown in Figure 3.

We are interested in the injective modules I_i (for $i \in Q_0$) and their *Auslander-Reiten translates* $\tau(I_i)$ (for $i \in Q_0$). Put $Q_0 = \{1, 2, \ldots, n\}$. Thus, we consider $2n$ modules. In the examples of Figures 2 and 3 these are all indecomposable modules in the Auslander-Reiten quiver of type A_3 and the four modules at rightmost position in the preinjective component of the Auslander-Reiten quiver of the Kronecker quiver. The direct sum M of the injective modules and their Auslander-Reiten translates is a *terminal* kQ-module in the sense of Geiß-Leclerc-Schröer [21]. With the $2n$ modules we associate an element w in the *Weyl group* of the *Kac-Moody Lie algebra* of corresponding type together with a reduced expression for it.

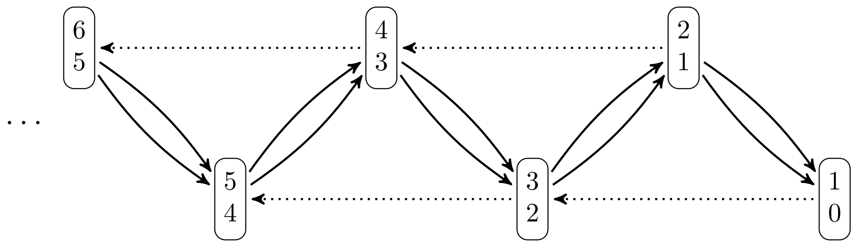
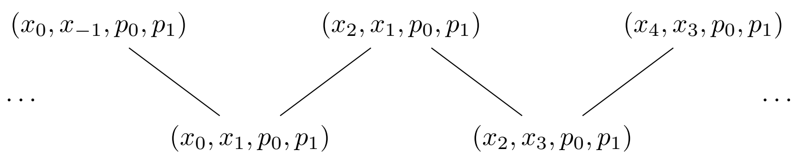


Figure 3: A part of the preinjective component of the AR quiver of the Kronecker quiver

Geiß-Leclerc-Schröer [21] attached to w a *cluster algebra* $\mathcal{A}(w)$ of rank n. For the terminal module M from above the initial seed of Geiß-Leclerc-Schröer's cluster algebra $\mathcal{A}(w)$ contains n *mutatable* and n *frozen* cluster variables. In the case of A_n there are only finitely many cluster variables. In the particular example of A_3 from above the exchange graph is a *Stasheff polyhedron*. The cluster algebra attached to M in the Kronecker case is generated by two frozen cluster variables p_0, p_1 and a sequence $(x_n)_{n \in \mathbb{Z}}$ of mutatable cluster variables. Starting with an initial cluster (x_0, x_1, p_0, p_1) we get all further cluster variables by a sequence of mutations.



If we put $p_0 = p_1 = 1$, then the exchange relation which allows to switch between adjacent clusters becomes $x_{k+1}x_{k-1} = x_k^2 + 1$ for $k \in \mathbb{Z}$. If we furthermore specialize $x_0 = x_1 = 1$, then we get $x_2 = 2$, $x_3 = 5$, $x_4 = 13$, $x_5 = 34$, $x_6 = 89$, etc. Every term in the sequence is a natural number. (In fact, the sequence is every other Fibonacci number.) The integrality is an instance of the Fomin-Zelevinsky's *Laurent phenomenon* [14]: Every cluster variable is a Laurent polynomial in x_0 and x_1. Caldero-Zelevinsky [10] gave an explicit formula for the cluster variables in terms of binomial coefficients by interpreting coefficients as Euler characteristics of quiver Grassmannians arising in the Caldero-Chapoton map [7].

A monomial in the cluster variable of a single cluster is called *cluster monomial*.

The representation theory of the path algebra kQ is closely related to the representation theory of the corresponding *preprojective algebra* Λ. Ringel [48] proved that the category $\mathrm{mod}(\Lambda)$ is isomorphic to a category called $C(1,\tau)$. The objects in the category $C(1,\tau)$ are pairs (X,f) consisting of a kQ-module X and a kQ-module homomorphism $f\colon X \to \tau(X)$ from X to its translate $\tau(X)$; morphisms in $C(1,\tau)$ from a pair (X,f) to a pair (Y,g) are given by a kQ-module homomorphism $h\colon X \to Y$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \tau(X) & \xrightarrow{\;\tau(h)\;} & \tau(Y) \end{array}$$

commutes. The algebra Λ is finite-dimensional if and only if Q is an orientation of a Dynkin diagram, see Reiten [47, Theorem 2.2a].

To construct the cluster algebra $\mathcal{A}(w)$, Geiß-Leclerc-Schröer [21] attached to every terminal $\mathbb{C}Q$-module M a natural subcategory $\mathcal{C}_M \subseteq \mathrm{nil}(\Lambda)$ of nilpotent Λ-modules. The category $\mathcal{C}_M$ is a Frobenius category. The stable category $\underline{\mathcal{C}}_M$ is triangulated by a theorem of Happel [26, Section 2.6]. Geiß-Leclerc-Schröer [21, Theorem 11.1] showed that if $M = I \oplus \tau(I)$ where I is the direct sum of all indecomposable injective representations, then there is an equivalence of triangulated categories $\underline{\mathcal{C}}_M \simeq \mathcal{C}_Q$ between $\underline{\mathcal{C}}_M$ and the cluster category $\mathcal{C}_Q$ as defined by Buan-Marsh-Reineke-Reiten-Todorov [4] to be the orbit category $\mathcal{D}^b\left(\mathrm{mod}(kQ)\right)/\tau_{\mathcal{D}}^{-1} \circ [1]$.

Geiß-Leclerc-Schröer [21, Section 4] implemented the cluster algebra $\mathcal{A}(w)$ as a subalgebra of the graded dual of the universal enveloping algebra $U(\mathfrak{n})$ of the maximal nilpotent subalgebra $\mathfrak{n}$ of the symmetric Kac-Moody Lie algebra $\mathfrak{g}$ attached to the quiver Q, i.e., $\mathcal{A}(\mathcal{C}_M) \subseteq U(\mathfrak{n})^*_{gr}$. Moreover, Geiß-Leclerc-Schröer [21] also proved that all cluster monomials are in the dual of Lusztig's semicanonical basis. There is an isomorphism between $U(\mathfrak{n})$ and an algebra $\mathcal{M}$ of $\mathbb{C}$-valued functions on Λ. We refer to [21] for a precise definition of $\mathcal{M}$. It is generated by functions $d_{\mathbf{i}}$ that map a Λ-module X to the Euler characteristic of the flag variety of X of type $\mathbf{i}$. Prominent elements in $\mathcal{A}(\mathcal{C}_M)$ are (under the described isomorphism) the δ-functions of certain rigid Λ-modules. Additionally, there is an isomorphism $\mathcal{A}(\mathcal{C}_M) \simeq \mathbb{C}[N(w)]$ where $\mathbb{C}[N(w)]$ is the coordinate ring of the unipotent subgroup $N(w)$ attached to the adaptable Weyl group element w of M. Therefore, we may call the $\mathcal{C}_M$ a *categorification* of the cluster algebra $\mathcal{A}(\mathcal{C}_M)$.

We transfer to the quantized setup. The *quantized universal enveloping algebra* $U_q(\mathfrak{n})$ is a self-dual *Hopf algebra*. Following Lusztig [42] we attach to w a subalgebra $U_q^+(w)$ of $U_q(\mathfrak{n})$. The subalgebra is generated by $2n$ elements that satisfy straightening relations; it degenerates to a commutative algebra in the classical limit $q = 1$.

The generators are constructed via Lusztig's T-automorphisms. The algebra $U_q^+(w)$ possesses four distinguished bases, a *Poincaré-Birkhoff-Witt basis*, a *canonical basis*, and their duals. The thesis concerns the dual of Lusztig's canonical basis of the subalgebra $U_q^+(w)$ under Kashiwara's bilinear form [28]. The dual canonical basis elements can be described as linear combinations of dual Poincaré-Birkhoff-Witt basis elements satisfying a lattice property and an invariance property.

It is conjectured (see for example [33]) that the quantized coordinate ring $\mathbb{C}_q[N(w)]$ is *quantum cluster algebra* $\mathcal{A}_q(w)$ in the sense of Berenstein-Zelevinsky [6] and that the set $\mathcal{M}_q$ of all quantum cluster monomials, taken up to powers of q, is a subset of the dual canonical basis $\mathcal{B}^*$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_q(w) & \longrightarrow & \mathbb{C}_q[N(w)] \subset U_q(\mathfrak{n})^*_{gr} \\ \uparrow & & \uparrow \\ \mathcal{M}_q & \hookrightarrow & \mathcal{B}^* \end{array}$$

The thesis is divided into two parts. The first part concerns alternating quivers of type A_n, the second part concerns the Kronecker quiver. In both cases, we prove recursions for dual canonical basis elements in $U_q^+(w)$. The recursions imply quantum exchange relations so that the integral form of $U_q^+(w)$ becomes (after extending coefficients) a quantum cluster algebra $\mathcal{A}_q(w)$. It follows that the quantum cluster variables are, up to a power of q, elements in the dual of Lusztig's canonical basis under Kashiwara's bilinear form.

The proof relies on the exact form of the straightening relations. In the case A_n, the description of the straightening relations features (besides Lusztig's T-automorphisms) Leclerc's embedding [36] of $U_q(\mathfrak{n})$ in the *quantum shuffle algebra*. The straightening relations for the Kronecker case are due to Leclerc [37]. The exact form of the straightening relations enables us to verify that recursively defined variables satisfy the lattice property and the invariance property of the dual canonical basis.

In the case of the Kronecker quiver, we give explicit formulae for the quantum cluster variables that quantize Caldero-Zelevinsky's formulae [10] for the ordinary cluster variables. In this case we also provide formulae for expansions of products of dual canonical basis elements.

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2 Quantum cluster algebras of type A

2.1 Introduction

Cluster algebras are commutative rings defined by Fomin-Zelevinsky [14] to investigate *total positivity* and *canonical bases*. The study of cluster algebras promptly extended over various branches of mathematics. One of the two original motivations, namely the connection between cluster algebras and canonical bases, has only been observed in a few cases. The passage from cluster algebras to canonical bases features Berenstein-Zelevinsky's *quantum cluster algebras* [6].

To give a more detailed description of this connection we introduce the following notations from Lie theory: Let $\mathfrak{g}$ be a *complex Kac-Moody Lie algebra* with *Cartan matrix* C. It admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. There exist quantizations of the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{n}$, called $U_v(\mathfrak{g})$ and $U_v(\mathfrak{n})$, respectively. With every Weyl group element $w \in W$ Lusztig [42] associates a subalgebra $U_v^+(w) \subset U_v(\mathfrak{n})$. Lusztig's construction [42] of $U_v^+(w)$ involves the evaluation of T-automorphisms at initial subsequences of a reduced expression $\underline{i} = (i_r, \ldots, i_1)$ for w. According to Lusztig [42] $U_v^+(w)$ possesses several bases: For every reduced expression $\underline{i}$ of w there is a *Poincaré-Birkhoff-Witt basis*. Furthermore, there is the *canonical basis*. It is conjectured that the integral form of the subalgebra $U_v^+(w) \subset U_v(\mathfrak{n})$ is (after extending coefficients) a quantum cluster algebra.

The conjecture has only been verified in very few cases, see Berenstein-Zelevinsky [5] for type A_2 and A_3, and the author [34] for an example of Kronecker type. Therefore, particular instances are worthwile. In this note we focus on type A_n (for a natural number n) and a particular Weyl group element w of length $2n$. We are going to prove that $U_v^+(w)$ carries indeed a quantum cluster algebra structure. In this case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ is the *complex semi-simple Lie algebra* of traceless $(n+1) \times (n+1)$ matrices and $\mathfrak{n}$ consists of all strictly upper triangular matrices.

The topic is truely linked with the *representation theory of quivers*. The case $\mathfrak{g} = \mathfrak{sl}_{n+1}$ is related to *Dynkin quivers* of type A_n. We choose a particular orientation: Let $Q = (Q_0, Q_1)$ be a Dynkin quiver of type A_n with an alternating orientation beginning with a source. We denote the set of vertices by $Q_0 = \{1, 2, \ldots, n\}$. Figure 4 illustrates the example $n = 13$. The choice of the orientation matches the choice of the Weyl group element w. The reduced expression of w (that is used to construct $U_v^+(w)$) and its initial subsequences (that are used to construct the generators $U_v^+(w)$) are related to the indecomposable injective modules over the *path algebra* of Q and their *Auslander-Reiten translates*, respectively.

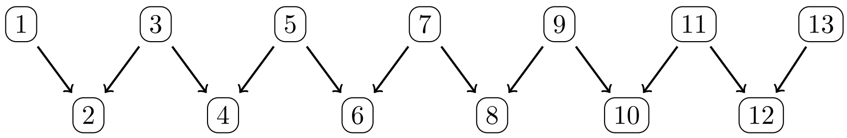


Figure 4: The quiver Q of type A_{13}

We denote the resulting quantum cluster algebra by $\mathcal{A}_v(w)$. It is a deformation of the cluster algebra $\mathcal{A}(w)$ Geiß-Leclerc-Schröer [21] attached to w. In Geiß-Leclerc-Schröer's setting, the cluster variables are δ-functions of rigid modules over the *preprojective algebra* of Q. The cluster algebra $\mathcal{A}(w)$, just as the quantum cluster algebra $\mathcal{A}_v(w)$, is of type A_n. Every cluster contains n frozen and n mutatable cluster variables. Altogether there are $n + \frac{n(n+1)}{2}$ mutatable and n frozen cluster variables. Most of the cluster variables can be realized as minors of certain matrices, see Section 2.6. The structure of these minors implies that there is (besides the usual cluster exchange relation) a recursive way to compute these cluster variables avoiding denominators. Theorem 2.70, the main theorem, asserts that the recursion can be quantized to a recursion for the corresponding quantum cluster variables. The quantized recursions imply quantum exchange relations so that the integral form $U_v^+(w)_{\mathbb{Z}}$ of $U_v^+(w)$ becomes (after extending coefficients) a quantum cluster algebra.

Furthermore, it follows from our construction that the quantum cluster variables are (up to a power of v) elements in the dual of Lusztig's canonical basis under Kashiwara's bilinear form [28].

2.2 Representation theory of the quiver of type A and cluster algebras

Let k be a field. In what follows we study the category $\mathrm{rep}_k(Q)$ of finite-dimensional k-representations of Q over the field k. (For more detailed information on representations of quivers see for example Crawley-Boevey [11].) The category $\mathrm{rep}_k(Q)$ is equivalent to the category $\mathrm{mod}(kQ)$ of finite-dimensional modules over the *path algebra* kQ. Gabriel's theorem [19] asserts that the quiver Q admits (up to isomorphism) only finitely many indecomposable representations. In fact there are $\frac{(n+1)n}{2}$ indecomposable representations (up to isomorphism) and they are in bijection with the set of intervals $[i,j] = \{i, i+1, i+2, \ldots j\} \subset \mathbb{Z}$ with $1 \leq i \leq j \leq n$. The indecomposable representation associated with the interval $[i,j]$ is $V_{[i,j]} = ((V_s)_{s\in Q_0}, (V_a)_{a\in Q_1})$ defined by k-vector spaces

$$V_s = \begin{cases} k, & \text{if } i \leq s \leq j; \\ 0, & \text{otherwise}; \end{cases}$$

associated with vertices s, and k-linear maps

$$V_a = \begin{cases} 1, & \text{if } V_s = V_t = k; \\ 0, & \text{otherwise}; \end{cases}$$

associated with arrows $a\colon s \to t$.

All further considerations will basically depend on the parity of n. For a compact and effective handling of all cases we make the assumption that n is odd. Denote by $Q' = (Q'_0, Q'_1)$ to be the quiver obtained from Q by removing the vertex n. The quiver Q' is of type A_{n-1}, and the examination of both Q' and Q covers all cases. Note that every representation of Q' can be viewed as a representation of Q supported on the first $n-1$ vertices. An example of the quiver Q' is shown in Figure 5.

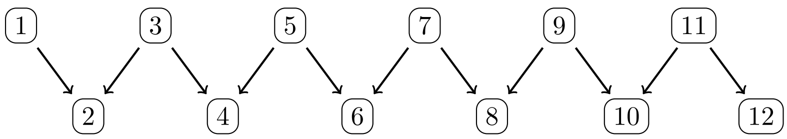


Figure 5: The quiver Q' of type A_{12}

If $n = 1$, i.e., the quiver has one vertex and no arrows, then the category $\mathrm{rep}_k(Q)$ can easily be described. In this case modules over the path algebra kQ are k-vector

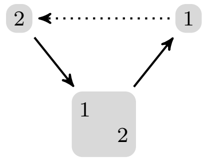


Figure 6: The Auslander-Reiten quiver for $n = 2$

spaces, and the k-vector space k of dimension 1 is the only irreducible module. In the other cases, the most suggestive way to illustrate the category $\mathrm{rep}_k(Q)$ is given by its *Auslander-Reiten quiver*. For an introduction to Auslander-Reiten theory we refer to Assem-Simson-Skowronski [1, Chapter IV].

The simplest non-trivial example is the Auslander-Reiten quiver of type A_2 which can be seen in Figure 6. In this case there are (up to isomorphism) three indecomposable representations, two of which are injective. The representations are displayed by their graded dimension vectors. The solid arrows represent *irreducible maps*; the dashed arrow represents the *Auslander-Reiten translation*. Note that the Auslander-Reiten translate of the injective representation associated with vertex 2 is the zero representation.

In what follows we are interested in the indecomposable injective kQ-modules I_i associated with vertices $i \in Q_0$ and their Auslander-Reiten translates $\tau_{kQ}(I_i)$. Similarly, we are interested in the indecomposable injective kQ'-modules I'_i associated with vertices $i \in Q'_0$ and their Auslander-Reiten translates $\tau_{kQ'}(I'_i)$. (For simplicity, we drop from now on the index attached to τ whenever it is clear which algebra we are referring to.) The choice of the alternating orientions of the quivers Q and Q' ensure that from type A_3 onwards we have $\tau(I) \neq 0$ for every indecomposable injective kQ-module. (This would not be true for the linear orientation of the Dynkin diagram A_n. The Auslander-Reiten translate of the indecomposable injective representation corresponding to the sink would be zero in this case.) The direct sum $M = \bigoplus_{i=1}^{n} I_i \oplus \tau(I_i)$ is a *terminal* kQ-module in the sense of Geiß-Leclerc-Schröer [21, Section 2.2], and so is the kQ' module $M' = \bigoplus_{i=1}^{n-1} I'_i \oplus \tau(I'_i)$.

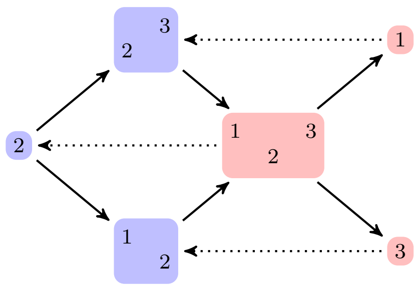


Figure 7: The Auslander-Reiten quiver for $n = 3$

The small cases A_3 and A_4 will have to be treated separately. Figure 7 and Figure 8 display the indecomposable injective modules (red), their Auslander-Reiten translates (blue), and irreducible maps between them for the case $n = 3$ and $n = 4$, respectively. We visualize the modules by their graded dimension vectors.

If $n = 3$, then M is the direct sum of all indecomposable kQ-modules, i.e.,

$\operatorname{mod}(kQ) = \operatorname{add}(M)$.

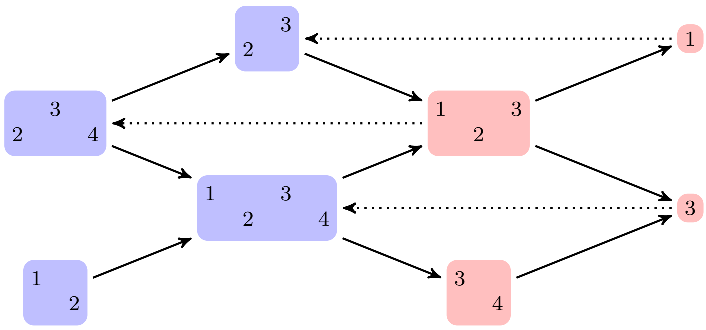


Figure 8: A part of the Auslander-Reiten quiver for $n = 4$

From A_5 onwards a uniform description is possible. For type A_n (remember that n is assumed to be odd) the indecomposable components of M can be written down explicitly:

$$\begin{aligned}
I_i &= V_{[i,i]}, && \text{if i is odd and } 1 \le i \le n, \\
I_i &= V_{[i-1,i+1]}, && \text{if i is even and } 2 \le i \le n, \\
\tau(I_1) &= V_{[2,3]}, && \\
\tau(I_i) &= V_{[i-2,i+2]}, && \text{if i is odd and } 3 \le i \le n-2, \\
\tau(I_n) &= V_{[n-2,n-1]}, && \\
\tau(I_2) &= V_{[2,5]}, && \\
\tau(I_i) &= V_{[i-3,i+3]}, && \text{if i is even and } 4 \le i \le n-3, \\
\tau(I_{n-1}) &= V_{[n-4,n-1]}. &&
\end{aligned}$$

We display the relevant part of the Auslander-Reiten quiver of A_n in Figure 9 for the case $n = 13$. As above, the indecomposable injective modules are colored red, their Auslander-Reiten translates blue.

There are only a few changes if we restrict Q to Q'. Observe that $I'_i = I_i$ for $i \in \{1, 2, \dots, n-3\}$, and that $\tau(I'_i) = \tau(I_i)$ for $i \in \{1, 2, \dots, n-3\}$. Note that the latter modules are kQ-modules supported on the first $n-1$ vertices and may therefore be viewed as kQ'-modules. Furthermore, we have

$$\begin{aligned}
I'_{n-1} &= V_{[n-2,n-1]}, \\
\tau(I'_{n-3}) &= V_{[n-6,n-1]}, \\
\tau(I'_{n-2}) &= V_{[n-4,n-1]}, \\
\tau(I'_{n-1}) &= V_{[n-4,n-3]}.
\end{aligned}$$

An example of type A_{12} is illustrated in Figure 10.

2.3 The preprojective algebra and rigid modules

The representation theory of the path algebra kQ is closely related to the representation theory of the corresponding *preprojective algebra* Λ defined as follows. For every arrow $a\colon s \to t$ in Q_1 introduce an additional arrow $a^*\colon t \to s$ in reverse direction and

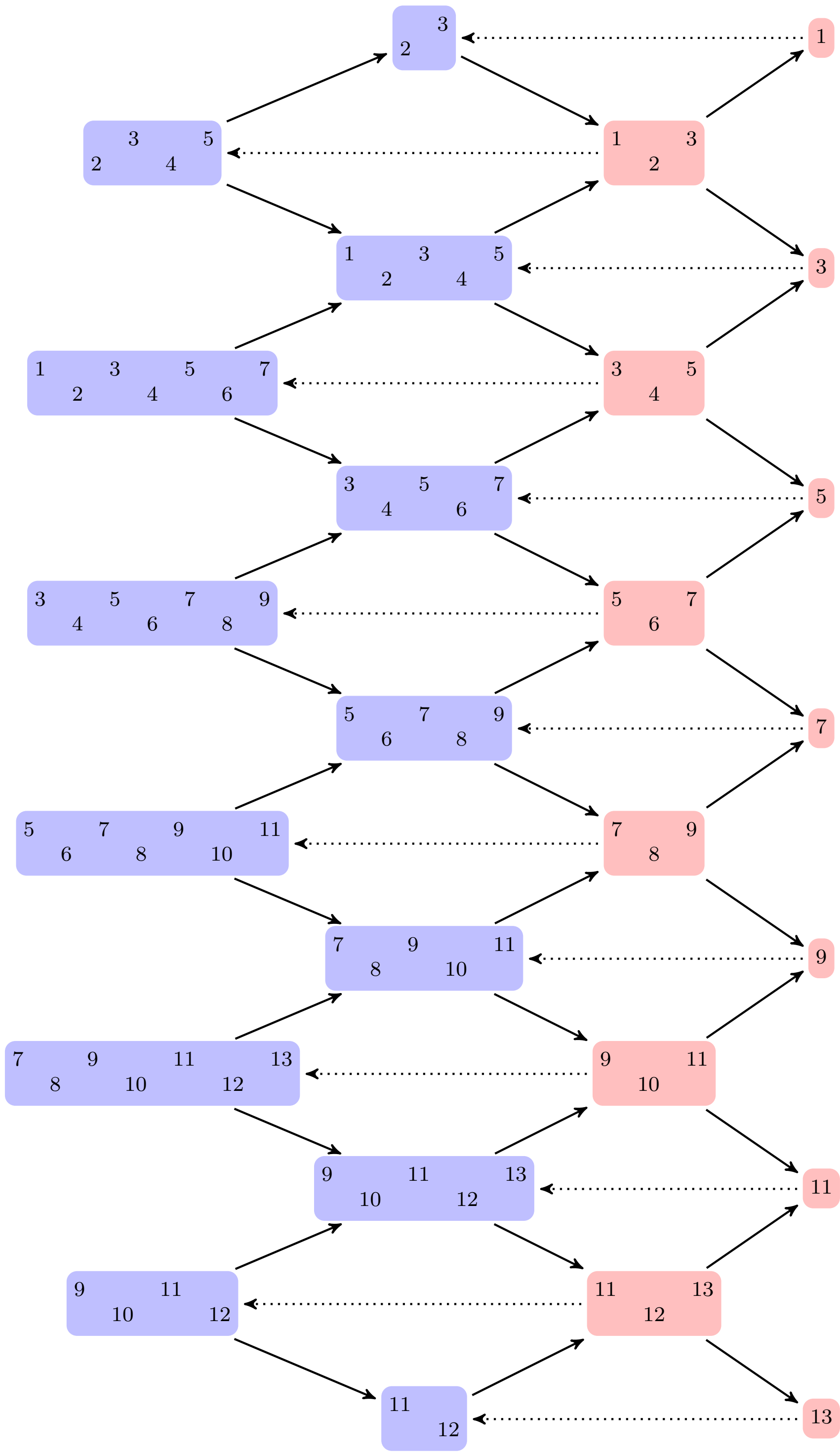


Figure 9: A part of the Auslander-Reiten quiver of $\mathrm{mod}(kQ)$

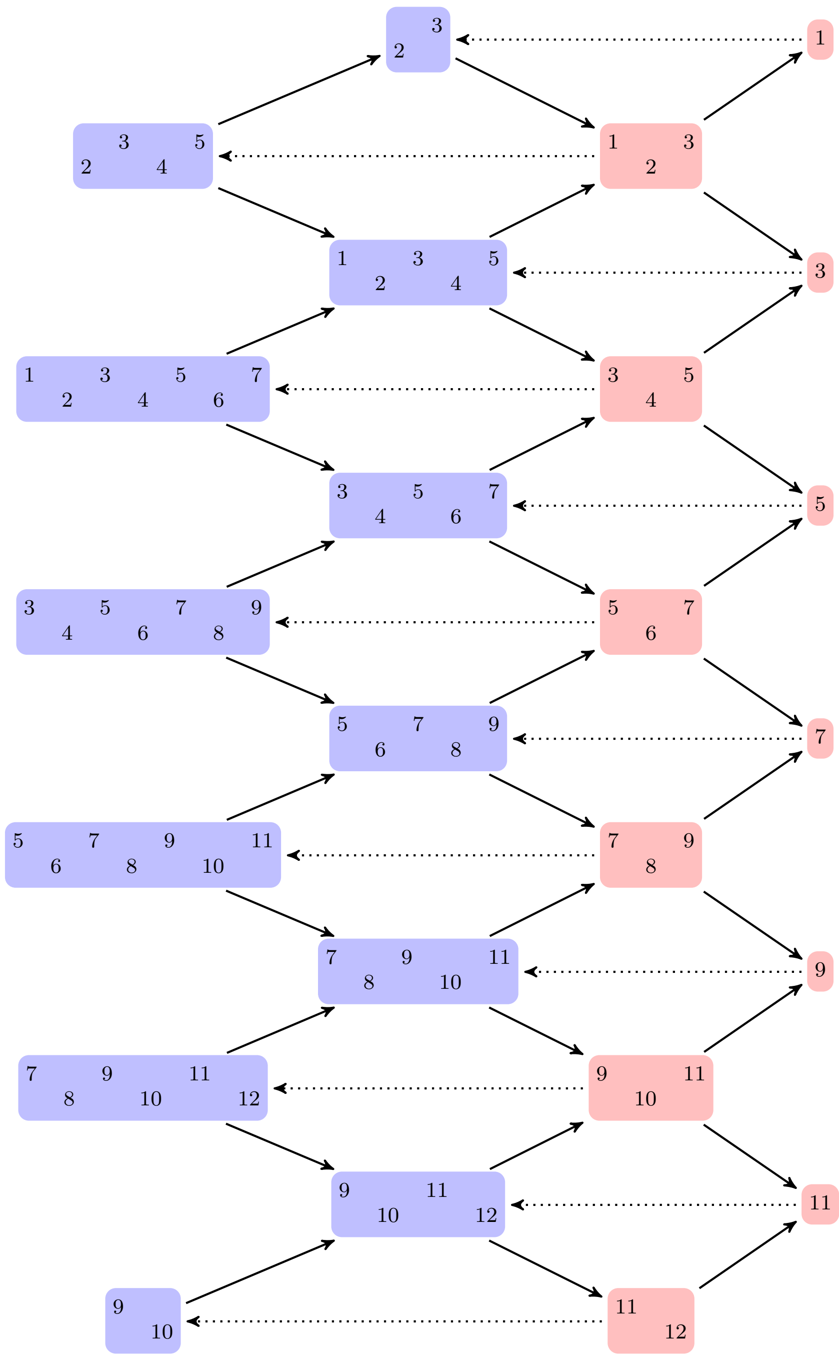


Figure 10: A part of the Auslander-Reiten quiver of $\mathrm{mod}(kQ')$

denote by $Q_1^* = \{a^* \colon a \in Q_1\}$ the set of all reversed arrows. The *double quiver* of Q is by defined to be the quiver $\overline{Q} = (\overline{Q}_0, \overline{Q}_1)$ given by a vertex set $\overline{Q}_0 = Q_0$ and an arrow set $\overline{Q}_1 = Q_1 \cup Q_1^*$. The preprojective algebra is defined to be

$$\Lambda = k\overline{Q}/(c)$$

where the ideal (c) is the two-sided ideal generated by the element

$$c = \sum_{a \in Q_1} (a^*a - aa^*) \in k\overline{Q}.$$

The algebra Λ is finite- dimensional, since Q is an orientation of a Dynkin diagram, see Reiten [47, Theorem 2.2a]. The category $\mathrm{mod}(\Lambda)$ of finite-dimensional Λ-modules is equivalent to the category $\mathrm{rep}_k(\overline{Q}, (c))$ of finite-dimensional representations $M = ((M_s)_{s \in Q_0}, (M_a)_{a \in \overline{Q}_1})$ of $\overline{Q}$ such that for any two vertices $s, t \in Q_0$ and any linear combination $\sum_{i=1}^m \lambda_i p_i \in (c)$ of paths $p_i \colon s \to t$ with scalars $\lambda_i \in k$ the associated linear map $\sum_{i=1}^m \lambda_i M_{p_i}$ is zero.

There is a *restriction functor* $\pi_Q \colon \mathrm{mod}(\Lambda) \to \mathrm{mod}(kQ)$ given by forgetting the linear maps associated with a^* for all $a \in Q_1$ in the corresponding representation of the quiver $\overline{Q}$. Ringel [48, Theorem B] proved that the category $\mathrm{mod}(\Lambda)$ is isomorphic to a category called $C(1, \tau)$. The objects in the category $C(1, \tau)$ are pairs (X, f) consisting of a kQ-module X and a kQ-module homomorphism $f \colon X \to \tau(X)$ from X to its translate $\tau(X)$; morphisms in $C(1, \tau)$ from a pair (X, f) to a pair (Y, g) are given by a kQ-module homomorphism $h \colon X \to Y$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \tau(X) & \xrightarrow{\ \tau(h)\ } & \tau(Y) \end{array}$$

commutes.

Using the correspondence from above Geiß-Leclerc-Schröer [21, Section 7.1] constructed for every $i \in Q_0$ and any natural numbers a, b satisfying $0 \le a \le b \le 1$ a Λ-module $T_{i,[a,b]} = \left(I_{i,[a,b]}, e_{i,[a,b]}\right)$ where $I_{i,[a,b]} = \bigoplus_{j=a}^{b} \tau^j(I_i)$ and the map

$$e_{i,[a,b]} \colon I_{i,[a,b]} \to \tau\left(I_{i,[a,b]}\right) = \bigoplus_{j=a+1}^{b+1} \tau^j(I_i)$$

is identity on every $\tau^j(I_i)$ for $a + 1 \le j \le b$ and zero otherwise. We study Λ-modules $T_{i,[a,b]}$ for $i \in Q_0$ and $0 \le a, b \le 1$. We display the modules by their graded dimension vectors in Figures 11, 12, 13.

The modules $T_{i,[a,b]}$ for $i \in Q_0$ and $0 \le a, b \le 1$ are *rigid* and *nilpotent*. Recall that a Λ-module T is said to be rigid if $\mathrm{Ext}^1_\Lambda(T, T) = 0$ and it is said to be nilpotent if there exists an integer $N > 0$ such that for each path $a_1 a_2 \cdots a_N$ of length N in $\overline{Q}$ the associated linear map $T_{a_1} T_{a_2} \cdots T_{a_N}$ is zero. Rigidity follows from Geiß-Leclerc-Schröer [21, Lemma 7.1]; nilpotency follows from Lusztig [44, Proposition 14.2].

Similarly, the representation theory of the path algebra kQ' is closely related to the representation theory of the corresponding preprojective algebra Λ'.

$$
T_{i,[0,0]} = \begin{cases}
\boxed{i} & \text{if } i \text{ is odd and } 1 \leq i \leq n \\[2ex]
\boxed{\begin{array}{ccc} i-1 & & i+1 \\ & i & \end{array}} & \text{if } i \text{ is even and } 2 \leq i \leq n-1
\end{cases}
$$

Figure 11: The modules $T_{i,[0,0]}$

$$
T_{i,[0,1]} = \begin{cases}
\boxed{\begin{array}{ccc} & & 3 \\ & 2 & \\ 1 & & \end{array}} & \text{if } i = 1 \\[2ex]
\boxed{\begin{array}{ccccc} & & 3 & & 5 \\ & 2 & & 4 & \\ 1 & & 3 & & \\ & 2 & & & \end{array}} & \text{if } i = 2 \\[2ex]
\boxed{\begin{array}{ccccc} i-2 & & i & & i+2 \\ & i-1 & & i+1 & \\ & & i & & \end{array}} & \begin{array}{c}\text{if } i \text{ is odd and} \\ 3 \leq i \leq n-2\end{array} \\[2ex]
\boxed{\begin{array}{ccccccc} i-3 & & i-1 & & i+1 & & i+3 \\ & i-2 & & i & & i+2 & \\ & & i-1 & & i+1 & & \\ & & & i & & & \end{array}} & \begin{array}{c}\text{if } i \text{ is even and} \\ 4 \leq i \leq n-3\end{array} \\[2ex]
\boxed{\begin{array}{ccccc} n-4 & & n-2 & & \\ & n-3 & & n-1 & \\ & & n-2 & & n \\ & & & n-1 & \end{array}} & \text{if } i = n-1 \\[2ex]
\boxed{\begin{array}{ccc} n-2 & & \\ & n-1 & \\ & & n \end{array}} & \text{if } i = 2
\end{cases}
$$

Figure 12: The modules $T_{i,[0,1]}$

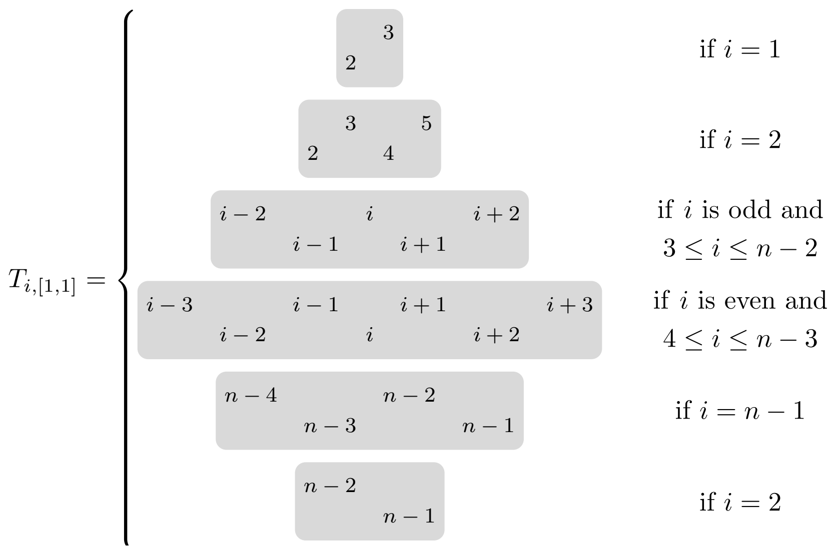


Figure 13: The modules $T_{i,[1,1]}$

2.4 Notations from Lie theory

The representation theory of the quiver Q is related with Lie theory. Let $k = \mathbb{C}$. The Lie algebra associated with the Dynkin diagram A_n is $\mathfrak{g} = \mathfrak{sl}_{n+1}$, i.e., the Lie algebra of $(n+1) \times (n+1)$ matrices with complex entries and vanishing trace. It admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Here, $\mathfrak{n}$ and $\mathfrak{n}_-$ denote the Lie algebras of strictly upper and strictly lower triangular $(n+1) \times (n+1)$ matrices, respectively, and $\mathfrak{h}$ denotes the Lie algebra of $(n+1) \times (n+1)$ diagonal matrices. The Lie algebra $\mathfrak{n}$ is called the *positive part* of $\mathfrak{g}$.

Let $C = (a_{ij})_{1 \leq i,j \leq n}$ be the *Cartan matrix* associated with the quiver Q; its entries are:

$$a_{ij} = \begin{cases} 2, & \text{if } i = j; \\ -1, & \text{if} |i-j| = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The Lie algebra $\mathfrak{g}$ is studied by its *roots*. The *root lattice* Q is defined to be the free abelian group generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. The elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called *simple roots*. (By an abuse of notation the variable Q is double assigned, but it should be clear from the context whether Q denotes the quiver or the root lattice.) By $Q^+ \subset Q$ we denote the set of all linear combinations $\sum_{i=1}^n c_i \alpha_i$ with $c_i \in \mathbb{N}^+$. There is a symmetric bilinear form $(\cdot,\cdot)\colon Q \times Q \to \mathbb{R}$ which is on generators given by $(\alpha_i, \alpha_j) = a_{ij}$ for $1 \leq i,j \leq n$. By $\Delta^+ \subseteq Q$ we denote the set of *positive roots* of the corresponding root system. Then $\Delta^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \colon 1 \leq i \leq j \leq n\}$. Under the bijection of Gabriel's theorem, a positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ with $1 \leq i \leq j \leq n$ is mapped to the indecomposable representation $V_{[i,j]}$ from Section 2.2.

The simple reflections $s_1, s_2, \dots, s_n \colon \mathfrak{h}^* \to \mathfrak{h}^*$ act on the simple roots by

$$s_i(\alpha_j) = \begin{cases} -\alpha_i, & \text{if } i = j; \\ \alpha_i + \alpha_j, & \text{if } |i-j| = 1; \\ \alpha_j, & \text{otherwise.} \end{cases}$$

The group W generated by the simple reflections is called the *Weyl group* of type $\mathfrak{g}$. The simple reflection satisfy the following relations

$$s_i s_j = s_j s_i, \qquad \text{if } |i-j| \geq 2; \tag{1}$$

$$s_i s_j s_i = s_j s_i s_j, \qquad \text{if } |i-j| = 1; \tag{2}$$

$$s_i^2 = 1, \tag{3}$$

for all $1 \leq i, j \leq n$. Therefore, the Weyl group W is isomorphic to the symmetric group S_n.

To every terminal kQ-module Geiß-Leclerc-Schröer [21, Section 3.7] attach a Q^{op}-adapted Weyl group element. The Q^{op}-adapted Weyl group element associated with the terminal module M from Section 2.2 is

$$w = s_1 s_3 s_5 \cdots s_n s_2 s_4 s_6 \cdots s_{n-1} s_1 s_3 s_5 \cdots s_n s_2 s_4 s_6 \cdots s_{n-1}. \tag{4}$$

The given expression for w is reduced. Let $j_1, j_2, \dots, j_{2n} \in [1, n]$ such that for the reduced expression for w from above we have $w = s_{j_1} s_{j_2} \cdots s_{j_{2n}}$. We abbreviate $\beta_k = s_{j_1} s_{j_2} \cdots s_{j_{k-1}}(\alpha_{j_k})$ for $1 \leq k \leq 2n$. Denote by $\Delta_w^+ = \{\beta_1, \beta_2, \dots, \beta_{2n}\} \subseteq \Delta^+$ the set of all β_k with $1 \leq k \leq 2n$. Note that the notation is well-defined. If we choose another reduced expression $w = s_{j'_1} s_{j'_2} \cdots s_{j'_{2n}}$ for w, then

$$\left\{ s_{j'_1} s_{j'_2} \cdots s_{j'_{k-1}}(\alpha_{j'_k}) \colon 1 \leq k \leq 2n \right\} = \{\beta_1, \beta_2, \dots, \beta_{2n}\}.$$

Furthermore, notice that under the bijection of Gabriel's theorem, the $2n$ positive roots β_k with $1 \leq k \leq 2n$, correspond to the dimension vectors of the indecomposable direct summands of M (compare Figure 4). More precisely, for $n \geq 5$,

$$\begin{aligned} \Delta_w^+ = {} & \{\alpha_i \colon \text{ is odd and } 1 \leq i \leq n\} \\ & \cup \{\alpha_{i-1} + \alpha_i + \alpha_{i+1} \colon \text{ is even and } 2 \leq i \leq n-1\} \\ & \cup \{\alpha_2 + \alpha_3\} \cup \{\alpha_{n-2} + \alpha_{n-1}\} \\ & \cup \{\alpha_{i-2} + \cdots + \alpha_{i+2} \colon \text{ is odd and } 3 \leq i \leq n-3\} \\ & \cup \{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\} \cup \{\alpha_{n-4} + \alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}\} \\ & \cup \{\alpha_{i-3} + \cdots + \alpha_{i+3} \colon \text{ is even and } 4 \leq i \leq n-4\}. \end{aligned}$$

The universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ is the associative $\mathbb{C}$-algebra generated by E_i $(1 \leq i \leq n)$ subject to the relations

$$E_i E_j = E_j E_i, \qquad \text{for } |i-j| \geq 2, \tag{5}$$

$$E_i^2 E_j - 2 E_i E_j E_i + E_j E_i^2 = 0, \qquad \text{for } |i-j| = 1. \tag{6}$$

The last relation is called *Serre relation*.

Similarly, the representation theory of the quiver Q' of type A_{n-1} is linked with the Lie algebra $\mathfrak{g}' = \mathfrak{sl}_n$ with Weyl group W'. The Lie algebra $\mathfrak{g}' = \mathfrak{sl}_n$ similarly admits a triangular decomposition $\mathfrak{g}' = \mathfrak{n}'_- \oplus \mathfrak{h}' \oplus \mathfrak{n}'$. The Weyl group element associated with M' is $w' = s_1 s_3 s_5 \cdots s_{n-2} s_2 s_4 s_6 \cdots s_{n-1} s_1 s_3 s_5 \cdots s_{n-2} s_2 s_4 s_6 \cdots s_{n-1} \in W'$. The universal enveloping algebra $U(\mathfrak{n}')$ may be viewed as the subalgebra of $U(\mathfrak{n})$ generated by E_i $(1 \leq i \leq n-1)$.

2.5 The cluster algebra attached to the terminal module

To the terminal $\mathbb{C}Q$-module M from Section 2.2 Geiß-Leclerc-Schröer ([21, Section 4]) attached a category $\mathcal{C}_M \subseteq \mathrm{nil}(\Lambda)$ of nilpotent Λ-modules. The projective and injective objects in $\mathcal{C}_M$ coincide, so $\mathcal{C}_M$ is a *Frobenius category* and there is a stable category $\underline{\mathcal{C}}_M$. By a theorem of Happel [26, Section 2.6] the stable category $\underline{\mathcal{C}}_M$ is a *triangulated* category. Furthermore, Geiß-Leclerc-Schröer [21, Theorem 11.1] showed that there is an equivalence of triangulated categories $\underline{\mathcal{C}}_M \simeq \mathcal{C}_Q$ between $\underline{\mathcal{C}}_M$ and the *cluster category* $\mathcal{C}_Q$ as defined by Buan-Marsh-Reineke-Reiten-Todorov [4] to be the orbit category $\mathcal{D}^b\left(\mathrm{mod}(kQ)\right)/\tau_{\mathcal{D}}^{-1}\circ[1]$. The category $\mathcal{C}_Q$ is indeed triangulated by a result of Keller [29].

With every $\mathcal{C}_M$ Geiß-Leclerc-Schröer [21, Section 4] associated a cluster algebra $\mathcal{A}(\mathcal{C}_M)$; it is constructed as a subalgebra of the graded dual of the universal enveloping algebra of the positive part of the corresponding Lie algebra, i.e., $\mathcal{A}(\mathcal{C}_M) \subseteq U(\mathfrak{n})^*_{gr}$. For a definition of and a general introduction to cluster algebras see Fomin-Zelevinsky [17]. The cluster algebra $\mathcal{A}(\mathcal{C}_M)$ is also called $\mathcal{A}(w)$.

There is an isomorphism between $U(\mathfrak{n})$ and an algebra $\mathcal{M}$ of $\mathbb{C}$-valued functions on Λ. We refer to Geiß-Leclerc-Schröer [21] for a precise definition of $\mathcal{M}$. It is generated by functions $d_{\mathbf{i}}$ that map a Λ-module X to the Euler characteristic of the flag variety of X of type $\mathbf{i}$. Prominent elements in $\mathcal{A}(\mathcal{C}_M)$ are (under the described isomorphism) the δ-functions of the rigid Λ-modules $T_{i,[a,b]}$ with $i \in Q_0$ and $0 \le a \le b \le 1$. For $1 \le i \le n$ put

$$P_i = \delta_{T_{i,[0,1]}};$$
$$Y_i = \begin{cases} \delta_{T_{i,[0,0]}}, & \text{if } i \text{ is odd;} \\ \delta_{T_{i,[1,1]}}, & \text{if } i \text{ is even;} \end{cases}$$
$$Z_i = \begin{cases} \delta_{T_{i,[0,0]}}, & \text{if } i \text{ is even;} \\ \delta_{T_{i,[1,1]}}, & \text{if } i \text{ is odd.} \end{cases}$$

The initial seed of the cluster algebra $\mathcal{A}(\mathcal{C}_M)$ for the case $n = 9$ is shown in Figure 14. The vertices represent the cluster variables in the initial cluster, the arrows describe the initial exchange matrix. Just as in Keller's mutation applet [30], the blue vertices are frozen, the red vertices are mutatable. The frozen variables $P_1, P_2, \ldots, P_n$ may be viewed as coefficients in the sense of Fomin-Zelevinsky [16]. The cluster algebra $\mathcal{A}(\mathcal{C}_M)$ is of type A_n, and therefore of finite type. Besides the n frozen variables there are $n + \frac{n(n+1)}{2}$ mutatable cluster variables grouped into $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$ clusters, where C_{n+1} denotes the $(n+1)^{\text{th}}$ Catalan number (see Fomin-Zelevinsky [15, Section 12]). The Catalan number C_{n+1} is the number of triangulations of a convex polygon with $n+3$ sides using only diagonals.

The δ-functions of P_i, Y_i, and Z_i for $i \in Q_0$ are not algebraically independent. For example, the equation

$$P_i = Y_i Z_i - Z_{i-1} Z_{i+1} \tag{7}$$

holds for every $i \in Q_0$. The equations are due to Geiß-Leclerc-Schröer [21, Theorem 18.1] and called *determinantal identities*. Here and in what follows we use the convention $Z_0 = Z_{n+1} = 1$.

Similarly, we can construct a cluster algebra $\mathcal{A}(\mathcal{C}_{M'})$ assocítated with the terminal $\mathbb{C}Q'$-module M' from Section 2.2. The initial seed of $\mathcal{A}(\mathcal{C}_{M'})$ is obtained from the

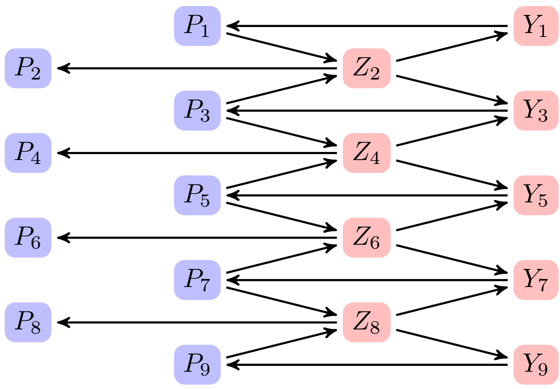


Figure 14: The initial seed for the case $n = 9$

initial seed of $\mathcal{A}(\mathcal{C}_M)$ by ignoring the vertices Y_n and P_n and all incident arrows. We denote the corresponding cluster variables of $\mathcal{A}(\mathcal{C}_{M'})$ by P_i', Y_i', and Z_i' (for $1 \leq i \leq n-1$).

2.6 The description of cluster variables

In this subsection we describe the cluster variables explicitly. Note that our desription of cluster variables differs from the explicit description of Geiß-Leclerc-Schröer [21, Section 18.2] due to a different choice of orientation of the quiver. Put $c_i = \frac{Z_{i-1}Z_{i+1}}{Z_i}$ for $1 \leq i \leq n$.

Definition 2.1. For two natural numbers i, j with $1 \leq i \leq j \leq n$ put $\Delta_{i,j} = c_i c_{i+1} \cdots c_j \det(M_{ij})$ where $M_{ij} = ((M_{ij})_{rs})_{i \leq r,s \leq j}$ is the $(j-i+1) \times (j-i+1)$ matrix defined by

$$(M_{ij})_{rs} = \begin{cases} \frac{y_r}{c_r}, & \text{if } r = s; \\ 1, & \text{if } s > r \text{ or } r = s+1; \\ 0, & \text{otherwise;} \end{cases}$$

i.e., $\Delta_{i,j}$ is given by the following determinant

$$\Delta_{i,j} = c_i c_{i+1} \cdots c_j \begin{vmatrix} \frac{Y_i}{c_i} & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \frac{Y_{i+1}}{c_{i+1}} & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & \frac{Y_{i+2}}{c_{i+2}} & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{Y_{i+3}}{c_{i+3}} & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{Y_{j-2}}{c_{j-2}} & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{Y_{j-1}}{c_{j-1}} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \frac{Y_j}{c_j} \end{vmatrix}.$$

Remark 2.2. Note that

$$\Delta_{i,j} = \begin{vmatrix} Y_i & c_i & c_i & c_i & \cdots & c_i & c_i & c_i \\ c_{i+1} & Y_{i+1} & c_{i+1} & c_{i+1} & \cdots & c_{i+1} & c_{i+1} & c_{i+1} \\ 0 & c_{i+2} & Y_{i+2} & c_{i+2} & \cdots & c_{i+2} & c_{i+2} & c_{i+2} \\ 0 & 0 & c_{i+3} & Y_{i+3} & \cdots & c_{i+3} & c_{i+3} & c_{i+3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Y_{j-2} & c_{j-2} & c_{j-2} \\ 0 & 0 & 0 & 0 & \cdots & c_{j-1} & Y_{j-1} & c_{j-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_j & Y_j \end{vmatrix}$$

for $1 \le i \le j \le n$. It follows that each $\Delta_{i,j}$ $(1 \le i, j \le n)$ is actually a polynomial in Y_i $(1 \le i \le n)$ and Z_i $(1 \le i \le n)$, i.e., $\Delta_{i,j} \in \mathbb{Z}[Y_k, Z_k \colon 1 \le k \le n]$ for all $i \le j$. Polynomiality follows from Geiß-Leclerc-Schröer [21, Theorem 3.4], but is also follows directly from the formula above once we notice that $c_i c_{i+1} \cdots c_j \in \mathbb{Z}[Z_k \colon 1 \le k \le n]$ for all i, j with $1 \le i < j \le n$.

Proposition 2.3. For all i, j with $1 \le i \le j \le n$ and $j - i \ge 3$ the equation $\Delta_{i,j} = Y_j \Delta_{i,j-1} - Z_{j+1} P_{j-2} \Delta_{i,j-3}$ holds.

Proof. Perform a Laplace expansion of the determinant on the last row. The last row has only two non-zero entries and it is easy to see that the two occuring summands in the Laplace expansion are the two summands in the recursion formula. □

For $1 \le i \le n$ let $\Delta_{i,i-1}$, $\Delta_{i,i-2}$, and $\Delta_{i,i-3}$ be the unique elements from $\mathbb{Q}(Y_k, Z_k \colon 1 \le k \le n)$ such that the recursion formula from Proposition 2.3 also holds for $j = i+2$, $j = i+1$, and $j = i$. Explicitly, we put $\Delta_{i,i-1} = 1$, $\Delta_{i,i-2} = \frac{1}{y_{i-1} - c_{i-1}}$, and $\Delta_{i,i-3} = 0$. The next lemma follows easily from Proposition 2.3.

Lemma 2.4. For all i, j with $1 \le i \le j \le n$ the equation $\Delta_{i,j} Z_j = P_j \Delta_{i,j-1} + Z_{j+1} P_{j-1} \Delta_{i,j-2}$ holds.

Proof. Fix i. We prove prove Lemma 2.4 by induction on j. The statement is true for $j = i$ since $Y_i Z_i = P_i + Z_{i+1} Z_{i-1}$. If the statement is true for $j - 1$, then by Proposition 2.3

$$\begin{aligned} \Delta_{i,j} Z_j &= Y_j Z_j \Delta_{i,j-1} - Z_{j+1} Z_j P_{j-2} \Delta_{i,j-3} \\ &= P_j \Delta_{i,j-1} + Z_{j+1} Z_{j-1} \Delta_{i,j-1} - Z_{j+1} Z_j P_{j-2} \Delta_{i,j-3} \\ &= P_j \Delta_{i,j-1} + Z_{j+1} P_{j-1} \Delta_{i,j-2}, \end{aligned}$$

and the statement is true for j. □

Lemma 2.5. The mutatable cluster variables are $Z_1, Z_2, Z_3, \ldots, Z_n$ and $\Delta_{i,j}$ for $1 \le i \le j \le n$.

Proof. Starting with the initial seed (which is shown in Figure 14 for the case $n = 9$) perform mutations at the odd vertices $1, 3, 5, \ldots, n$, consecutively. In each step, because of the equation $Y_i Z_i = P_i + Z_{i-1} Z_{i+1}$, the cluster variable Y_i for odd i with $1 \le i \le n$ is replaced by the cluster variable Z_i. Therefore, the mutations generate a seed whose mutatable cluster variables are $Z_1, Z_2, Z_3, \ldots, Z_n$. We refer to that seed as the *base seed*. The exchange matrix of the base seed is described by the associated quiver. By the rules of quiver mutation the mutatable vertices of the base seed form an

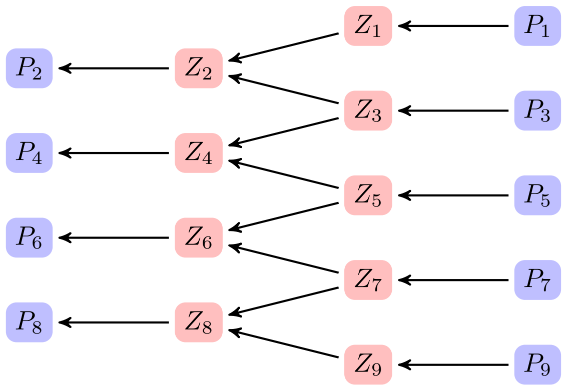


Figure 15: The base seed for the case $n = 9$

alternating quiver of type A_n isomorphic to Q. The only other arrows are the following. For every i with $1 \leq i \leq n$ there is an arrow between Z_i and P_i starting in P_i if i is odd and starting in Z_i if i is even. The quiver of the base seed for the example $n = 9$ is shown in Figure 15.

We now claim that starting from the base seed the cluster variable obtained by consecutive mutation at $i, i+1, i+2, \ldots, j$ is $\Delta_{i,j}$ for all $1 \leq i \leq j \leq n$. The equation $\Delta_{i,j} Z_j = P_j \Delta_{i,j-1} + Z_{j+1} P_{j-1} \Delta_{i,j-2}$ from Lemma 2.4 is the exchange relation. For a proof consider the mutation of the quiver of the base seed. Fix i. Wlog assume that i is odd. (If i is even reverse all arrows in the following argumentation.) We prove the statement by induction on j. The statement is true for $i = j$ since mutation at i yields $(P_i + Z_{i-1}Z_{i+1})/Z_i = Y_i = \Delta_{i,i}$. It is also true for $j = i+1$ because $\Delta_{i,i+1} Z_{i+1} = P_{i+1}\Delta_{i,i} + Z_{i+2}P_i\Delta_{i,i-1} = P_{i+1}\Delta_{i,i} + Z_{i+2}P_i$.

Now assume that $j \geq i+2$ and that mutation at $i, i+1, i+2, \ldots, j-1$ obtains cluster variables $\Delta_{i,i}, \Delta_{i,i+1}, \ldots, \Delta_{i,j-1}$. Let us describe the quiver after these mutations; let us first concentrate on the subquiver given by all mutatable vertices. It is easy to see that the subquiver supported on vertices $(Z_1, Z_2, \ldots, Z_{i-1})$ is the same as in the base quiver; similarly, the subquiver supported on vertices $(Z_j, Z_{j+1}, \ldots, Z_n)$ is unchanged. The description of the other remaining part depends on the parity of j. If j is even, then it contains of the two sequences $Z_{i-1} \to \Delta_{i,i} \to \Delta_{i,i+2} \to \Delta_{i,i+4} \to \cdots \to \Delta_{i,j-1}$ and $\Delta_{i,j-2} \to \Delta_{i,j-4} \to \cdots \to \Delta_{i,i+3} \to \Delta_{i,i+1}$ and a triangle $Z_j \to \Delta_{i,j-1} \to \Delta_{i,j-2} \to Z_j$. If j is odd, then it contains of the two sequences $Z_{i-1} \to \Delta_{i,i} \to \Delta_{i,i+2} \to \Delta_{i,i+4} \to \cdots \to \Delta_{i,j-2}$ and $\Delta_{i,j-1} \to \Delta_{i,j-3} \to \cdots \to \Delta_{i,i+3} \to \Delta_{i,i+1}$ and a triangle $Z_j \to \Delta_{i,j-2} \to \Delta_{i,j-1} \to Z_j$.

Now let us consider frozen vertices. Consider a natural number k with $i \leq k \leq j$. We are interested in the vertices Z_l resp. $\Delta_{i,l}$ with $k \leq l \leq j-1$ to which P_k is connected. In the base seed the vertex P_k is only connected with Z_k. Wlog let us assume that k is even. (If k is odd reverse all arrows in the following argumentation.) We have an arrow $Z_k \to P_k$ in the base seed. The adjacency relations for P_k remain unaffected by mutations at $i, i+1 \ldots, k-1$. After mutation at k the arrows reverses (and Z_k is replaced by $\Delta_{i,k}$) and we get an additional arrow $Z_{k+1} \to P_k$. Mutation at $k+1$ cancels the arrow $P_k \to \Delta_{i,k}$ whereas the arrow $Z_{k+1} \to P_k$ is replaced by an arrow $P_k \to \Delta_{i,k+1}$. Afterwards all adjacency relations for P_k with vertices Z_l for $k \leq l$ remain unaffected.

The adjacency relations for the vertices together with the induction hypothesis and

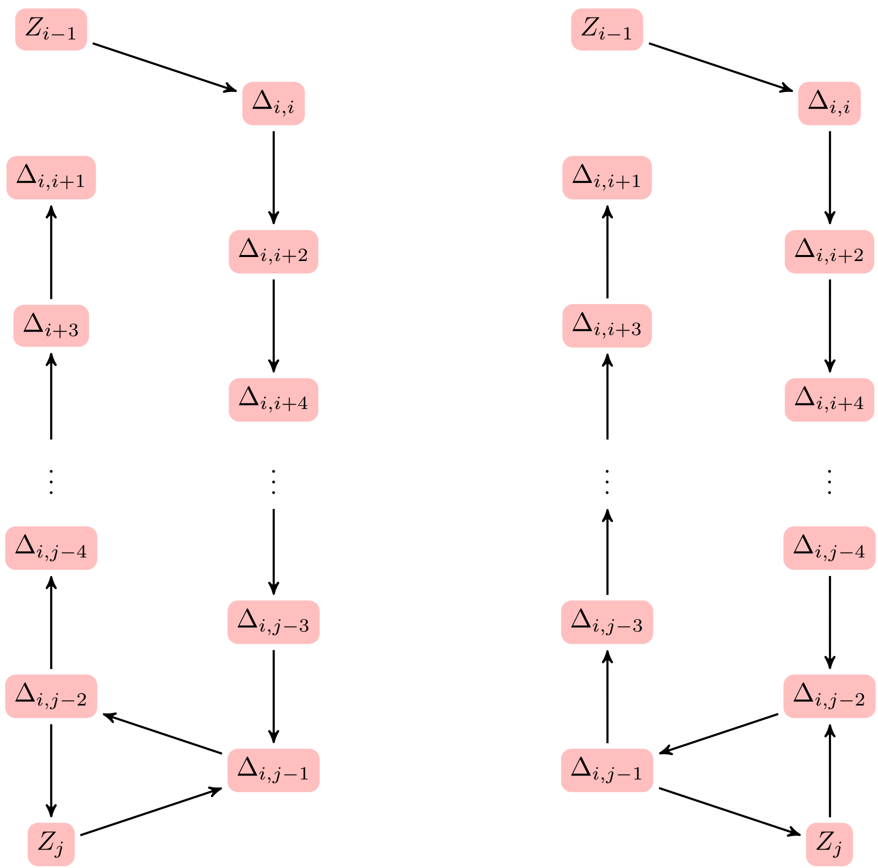


Figure 16: The mutated seed for even j (left) and odd j (right)

the mutation rule for cluster variables imply that $(P_j\Delta_{i,j-1}+Z_{j+1}P_{j-1}\Delta_{i,j-2})/Z_j$ is the cluster variable obtained from consecutive mutation at $i, i+1, \dots, j$. By Lemma 2.4 it is equal to $\Delta_{i,j}$.

The number of mutatable cluster variables of a cluster algebra of finite type is the sum of the rank of the cluster algebra and the number of positive roots of the associated root system. Since the $n+\frac{n(n+1)}{2}$ cluster variables $Z_1, Z_2, Z_3, \dots, Z_n$ and $\Delta_{i,j}$ for $1 \le i \le j \le n$ are all distinct these must be all mutatable cluster variables. □

By Lemma 2.5 the recursion provided by Proposition 2.3 allows to compute iteratively every cluster variable in terms of the Y_i and Z_i $(1 \le i \le n)$.

Example 2.6. Let us look at an example. We put $n = 3$. The initial cluster contains three mutatable and three frozen variables. It is $(P_1, P_2, P_3, Y_1, Z_2, Y_3)$. One can check, by hand or by using Keller's mutation applet [30], that the following figure describes the exchange graph of the cluster algebra $\mathcal{A}(\mathcal{C}_M)$ in the case $n = 3$. This particular exchange graph is known as *associahedron* or *Stasheff polytope* K_5. The mutatable cluster variables are colored red, the frozen cluster variables blue. Beside the $3+3$ initial cluster variables there are 6 further cluster variables, namely Z_1, Y_2, Z_3,

$$\Delta_{1,2} = \begin{vmatrix} Y_1 & c_1 \\ c_2 & Y_2 \end{vmatrix} = Y_1Y_2 - Z_3, \quad \Delta_{2,3} = \begin{vmatrix} Y_2 & c_2 \\ c_3 & Y_3 \end{vmatrix} = Y_2Y_3 - Z_1,$$

$$\Delta_{1,3} = \begin{vmatrix} Y_1 & c_1 & c_1 \\ c_2 & Y_2 & c_2 \\ 0 & c_3 & Y_3 \end{vmatrix} = Y_1Y_2Y_3 - Y_1Z_1 - Y_3Z_3 + Z_2.$$

The cluster variables are grouped into $C_4 = 14$ clusters.

Remark 2.7. Formulae for cluster variables in $\mathcal{A}(\mathcal{C}_{M'})$ can be obtained from these formulae by setting $Y_n = Z_n = P_n = 1$.

Remark 2.8. The cluster variables correspond to δ-functions of indecomposable rigid objects. The indecomposable rigid objects in $\mathcal{C}_M$ for this case have classified by Rohleder [49, Theorem 7.3]. Besides the $3n$ objects of the form $T_{i,[a,b]}$ for $1 \le i \le n$ and $0 \le a \le b \le 1$, these are (when viewed as elements in $C(1,\tau)$) the objects $M_{i,j}$

$$\bigoplus_{\substack{i<r<j \\ r \text{ odd}}} I_i \oplus \bigoplus_{\substack{i<r<j \\ r \text{ even}}} \tau(I_i) \xrightarrow{f} \bigoplus_{\substack{i<r<j \\ r \text{ odd}}} \tau(I_i) \oplus \bigoplus_{\substack{i<r<j \\ r \text{ even}}} \tau^2(I_i)$$

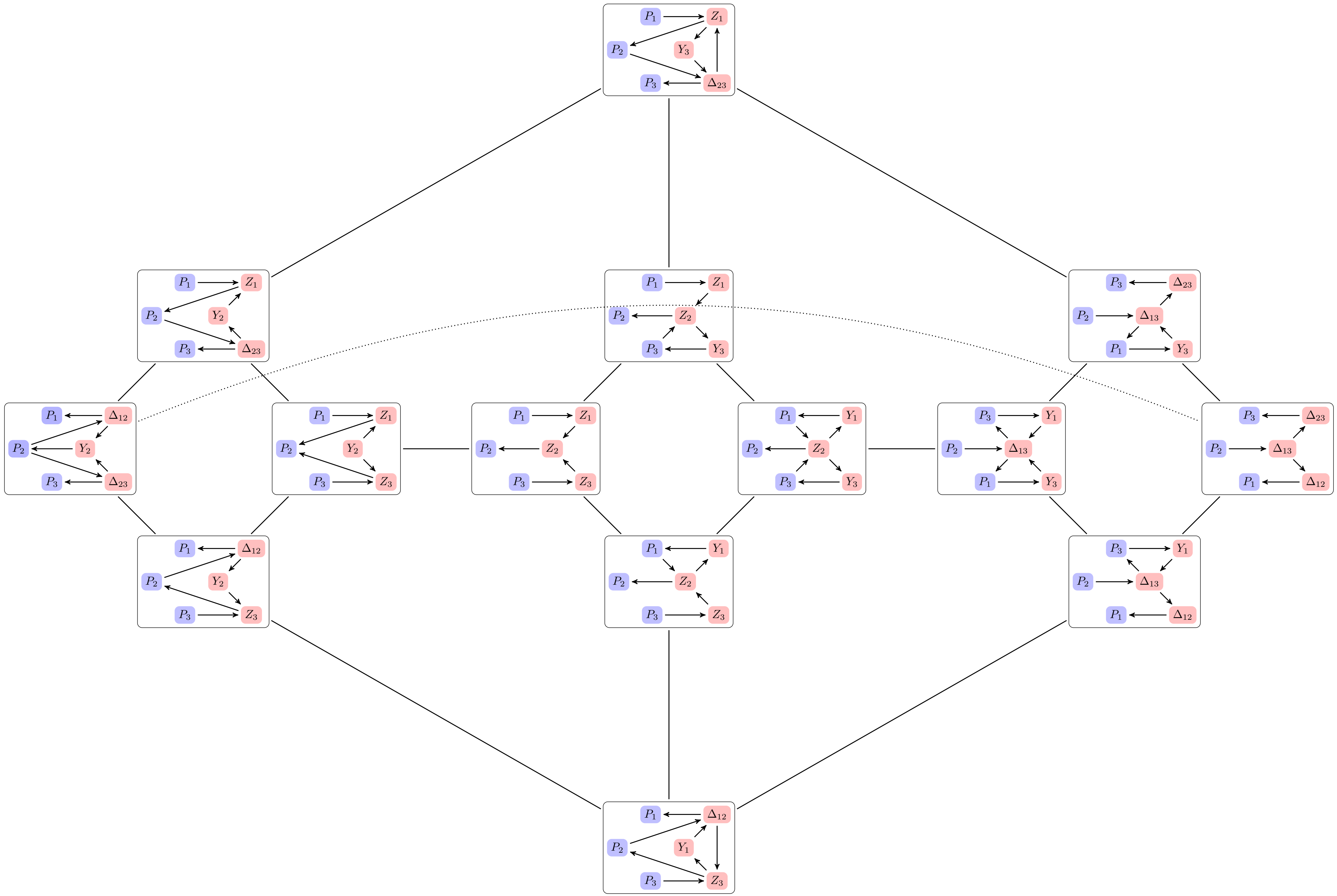
for $1 < j \le n$ where $f|_{\tau(I_i)}^{\tau(I_{i\pm1})} = 1$ for all even i and $f|_x^y = 0$ for all other direct summands X, Y.

2.7 Definition of the quantized enveloping algebra

The quantized universal enveloping algebra $U_v(\mathfrak{g})$ is a deformation of the ordinary universal enveloping algebra $U(\mathfrak{g})$. To describe this construction we introduce quantized integers and quantized binomial coefficients.

Definition 2.9. For a natural number k, denote by

$$[k] = \frac{v^k - v^{-k}}{v - v^{-1}} \in \mathbb{Q}(v)$$


P_1 Z_1 P_2 Y_3 P_3 Δ_{23}
P_1 Z_1 P_2 Y_2 P_3 Δ_{23}
P_1 Z_1 P_2 Z_2 P_3 Y_3
P_3 Δ_{23} P_2 Δ_{13} P_1 Y_3
P_1 Δ_{12} P_2 Y_2 P_3 Δ_{23}
P_1 Z_1 P_2 Y_2 P_3 Z_3
P_1 Z_1 P_2 Z_2 P_3 Z_3
P_1 Y_1 P_2 Z_2 P_3 Y_3
P_3 Y_1 P_2 Δ_{13} P_1 Y_3
P_3 Δ_{23} P_2 Δ_{13} P_1 Δ_{12}
P_1 Δ_{12} P_2 Y_2 P_3 Z_3
P_1 Y_1 P_2 Z_2 P_3 Z_3
P_3 Y_1 P_2 Δ_{13} P_1 Δ_{12}
P_1 Δ_{12} P_2 Y_1 P_3 Z_3

the quantum integer and by $[k]! = [k][k-1]\cdots[1]$ the quantized factorial. For two natural numbers k and l, define the quantum binomial coefficient by

$$\begin{bmatrix} k \\ l \end{bmatrix} = \frac{[k]!}{[l]!\,[k-l]!} \in \mathbb{Q}(v)\,.$$

Remark 2.10. Both $[k]$ and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ are actually Laurent polynomials in v. If we specialize $v = 1$, then $[k] = k$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \binom{n}{k}$, and $[k]! = k!$. Some authors such as Kac-Cheung [27] use a different convention for quantum integers.

Definition 2.11. The *quantized enveloping algebra* $U_v(\mathfrak{g})$ is the $\mathbb{Q}(v)$-algebra generated by E_i, F_i, K_i, K_i^{-1} for $i = 1, 2, \ldots, n$, subject to the following relations

$$K_iK_j = K_jK_i, \qquad (i \neq j) \tag{8}$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad (i = 1, 2, \ldots, n) \tag{9}$$
$$K_iE_jK_i^{-1} = v^{a_{ij}}E_j, \qquad (1 \leq i, j \leq n) \tag{10}$$
$$K_iF_jK_i^{-1} = v^{-a_{ij}}F_j, \qquad (1 \leq i, j \leq n) \tag{11}$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v - v^{-1}}, \qquad (1 \leq i, j \leq n) \tag{12}$$
$$E_i^2E_j - [2]E_iE_jE_i + E_jE_i^2 = 0, \qquad |i-j| = 1, \tag{13}$$
$$F_i^2F_j - [2]F_iF_jF_i + F_jF_i^2 = 0, \qquad |i-j| = 1, \tag{14}$$
$$E_iE_j = E_jE_i, \qquad |i-j| \geq 2, \tag{15}$$
$$F_iF_j = F_jF_i, \qquad |i-j| \geq 2, \tag{16}$$

where δ_{ij} is the Kronecker delta function. Note that $[2] = v + v^{-1}$, so we may write equation (13) as $E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 = 0$.

Definition 2.12. The subalgebra generated by E_i for $i = 1, 2, \ldots, n$ is called the *quantized enveloping algebra* $U_v(\mathfrak{n})$.

The only relations in $U_v(\mathfrak{n})$ remain (13) and (15). These are called *quantized Serre relations*. The algebra $U_v(\mathfrak{n})$ specializes to $U(\mathfrak{n})$ in the limit $v = 1$.

Remark 2.13. The algebra $U_v(\mathfrak{g})$ is a *graded algebra*. It is graded by the root lattice Q if we set $\deg(E_i) = \alpha_i$, $\deg(F_i) = -\alpha_i$, and $\deg(K_i) = 0$ for all $1 \leq i \leq n$. Note that $\deg(A) \in Q^+$ for all $A \in U_v(\mathfrak{n})$. We also use the abbreviation $\deg(A) = |A|$ for $A \in U_v(\mathfrak{n})$.

Remark 2.14. Put $\sigma(v) = v^{-1}$ and $\sigma(E_i) = E_i$ for all i with $1 \leq i \leq n$. By the symmetry of the relations (13) and (15) in $U_v(\mathfrak{n})$ the map σ extends to an algebra antihomomorphism $\sigma\colon U_v(\mathfrak{n}) \to U_v(\mathfrak{n})$, i.e., a $\mathbb{Q}$-linear map $\sigma\colon U_v(\mathfrak{n}) \to U_v(\mathfrak{n})$ such that $\sigma(AB) = \sigma(B)\sigma(A)$ for all $A, B \in U_v(\mathfrak{n})$. By construction σ is an antiautomorphism and an involution, i.e., $\sigma^2(A) = A$ for all $A \in U_v(\mathfrak{n})$. Compare the construction of the antiautomorphism σ with Lusztig's bar involution [42, Section 1.2.10].

Remark 2.15. In literature the deformation parameter v is sometimes called q. There are also different sign conventions for the exponent of the deformation parameter. We adopt Lusztig's convention [42]. It matches Leclerc's usage [36] if we set $q = v^{-1}$.

Remark 2.16. The quantized enveloping algebra $U_v(\mathfrak{g}') = U_v(\mathfrak{sl}_n)$ associated with Q' is defined similarly and may be regarded as the subalgebra of $U_v(\mathfrak{g}) = U_v(\mathfrak{sl}_{n+1})$ generated by the elements E_i, F_i, K_i, K_i^{-1} for $1 \leq i \leq n-1$.

2.8 The subalgebra $U_v^+(w)$ and the Poincaré-Birkhoff-Witt basis

We introduce Lusztig's T-automorphisms. For $1 \leq i \leq j$ put

$$T_i(E_j) = \begin{cases} -K_i^{-1}F_i, & \text{if } i = j; \\ E_jE_i - v^{-1}E_iE_j, & \text{if } |i-j| = 1; \\ E_j, & \text{if } |i-j| \geq 2; \end{cases}$$

$$T_i(F_j) = \begin{cases} -E_iK_i, & \text{if } i = j; \\ F_iF_j - vF_jF_i, & \text{if } |i-j| = 1; \\ E_j, & \text{if } |i-j| \geq 2; \end{cases}$$

$$T_i(K_j) = K_jK_i^{-a_{ij}}.$$

Lusztig [42, Chapter 37] shows that every T_i can be extended to an $\mathbb{Q}(v)$-algebra homomorphism $T_i\colon U_v(\mathfrak{g}) \to U_v(\mathfrak{g})$. (In Lusztig's book [42] it is called $T'_{i,-1}$.) In fact, every T_i is an $\mathbb{Q}(v)$-algebra automorphism. The images of the generators of $U_v(\mathfrak{g})$ under the inverse T_i^{-1} are given by

$$T_i^{-1}(E_j) = \begin{cases} -F_iK_i, & \text{if } i = j; \\ E_iE_j - v^{-1}E_jE_i, & \text{if } |i-j| = 1; \\ E_j, & \text{if } |i-j| \geq 2; \end{cases}$$

$$T_i^{-1}(F_j) = \begin{cases} -K_i^{-1}E_i, & \text{if } i = j; \\ F_jF_i - vF_iF_j, & \text{if } |i-j| = 1; \\ E_j, & \text{if } |i-j| \geq 2; \end{cases}$$

$$T_i^{-1}(K_j) = K_jK_i^{-a_{ij}}.$$

Remark 2.17. If $g \in U_v(\mathfrak{g})$ is homogeneous of degree β, then $T_i(g)$ is homogeneous of degree $s_i(\beta)$.

Remark 2.18. Furthermore, the T_i satisfy *braid relations*. For brevity we write T_iT_j for $T_i \circ T_j$ for all $i, j \in Q_0$. The braid relations are

$$\begin{aligned} T_iT_j &= T_jT_i, && \text{if } |i-j| \geq 2; \\ T_iT_jT_i &= T_jT_iT_j, && \text{if } |i-j| = 1. \end{aligned}$$

Definition 2.19. To $w = s_1s_3s_5 \cdots s_ns_2s_4s_6 \cdots s_{n-1}s_1s_3s_5 \cdots s_ns_2s_4s_6 \cdots s_{n-1}$ we attach elements in $U_v(\mathfrak{g})$. If $j_1, j_2, \ldots, j_{2n} \in [1, n]$ are indices such that for the reduced expression from above we have $w = s_{j_1}s_{j_2} \cdots s_{j_{2n}}$, then we consider the elements $T_{j_1}T_{j_2} \cdots T_{j_{k-1}}(E_{j_k})$ for $1 \leq k \leq 2n$. Since

$$\deg(T_{j_1}T_{j_2} \cdots T_{j_{k-1}}(E_{j_k})) = s_{j_1}s_{j_2} \cdots s_{j_{k-1}}(\alpha_{j_k}) = \beta_k$$

for all k, we introduce the shorthand notation $E(\beta_k) = T_{j_1}T_{j_2} \cdots T_{j_{k-1}}(E_{j_k})$ for all $1 \leq k \leq 2n$.

Definition 2.20. For $i \in Q_0$ and $a \in \mathbb{N}$ put $E_i^{(a)} = \frac{1}{[a]!}E_i^a \in U_v(\mathfrak{n})$. For a natural number k with $1 \leq k \leq 2n$ and $a \in \mathbb{N}$ put $E^{(a)}(\beta_k) = T_{j_1}T_{j_2} \cdots T_{j_{k-1}}(E_{j_k}^{(a)}) = \frac{1}{[a]!}E(\beta)^a$.

The following theorem is due to Lusztig [42, Theorem 40.2.1]. Theorem 2.21 also contains the definition of the subalgebra $U_v^+(w)$ which is crucial for our further studies; moreover, it enables us to define the Poincaré-Birkhoff-Witt basis of $U_v^+(w)$. For an idea of a proof different from Lusztig's [42] see Bergman's diamond lemma [2].

Theorem 2.21. The set

$$\mathcal{P} = \left\{ E^{(a_1)}(\beta_1)E^{(a_2)}(\beta_2)\cdots E^{(a_{2n})}(\beta_{2n})\colon (a_1,a_2,\ldots,a_{2n}) \in \mathbb{N}^{2n} \right\}$$

is linearly independent over $\mathbb{Q}(v)$. It forms a basis of a $\mathbb{Q}(v)$-subalgebra $U_v^+(w) \subset U_v(\mathfrak{n})$. Moreover, $U_v^+(w)$ is well-defined in the sense that it is independent of the choice of the reduced expression for w. If we choose another reduced expression $w = s_{j_1'}s_{j_2'}\cdots s_{j_{2n}'}$ for w, then the set of all

$$E_{j_1'}^{(a_1)} \cdot T_{j_1'}(E_{j_2'}^{(a_2)}) \cdots T_{j_1'}T_{j_2'}\cdots T_{j_{2n-1}'}(E_{j_{2n}'}^{(a_{2n})})$$

for all sequences $(a_1,a_2,\ldots,a_{2n}) \in \mathbb{N}^{2n}$ is also a basis of the same subalgebra $U_v^+(w) \subset U_v(\mathfrak{n})$.

Remark 2.22. The basis $\mathcal{P}$ is called the *Poincaré-Birkhoff-Witt basis* of $U_v^+(w)$ associated with the reduced expression (4). Unlike the canonical basis which we will define later the Poincaré-Birkhoff-Witt $\mathcal{P}$ basis depends on the choice of the reduced expression for w. Every choice of a reduced expression for w induces a bijection between $\mathbb{N}^{2n}$ and a basis for $U_v^+(w)$. In this sense $\mathcal{P}$ is not canonical. The various bijections are called *Lusztig parametrizations*.

Remark 2.23. Theorem 2.21 particularly implies that $E(\beta_k) \in U_v(\mathfrak{n})$ for every $1 \leq k \leq 2n$ which is not abvious from the definition of the T-automorphisms.

For any $\mathbf{a} = (a_1,a_2,\ldots,a_{2n}) \in \mathbb{N}^{2n}$ we introduce the shorthand notation $E[\mathbf{a}]$ for $E^{(a_1)}(\beta_1)E^{(a_2)}(\beta_2)\cdots E^{(a_{2n})}(\beta_{2n})$. We also use a different notation for $E(\beta_k)$ with $1 \leq k \leq 2n$; namely we put

$$\begin{aligned}
u_i &= T_1T_3T_5\cdots T_{i-1}(E_i), && \text{for odd } i \text{ with } 1 \leq i \leq n,\\
v_i &= T_1T_3T_5\cdots T_nT_2T_4\cdots T_{i-2}(E_i), && \text{for even } i \text{ with } 2 \leq i \leq n-1,\\
w_i &= TT_1T_3T_5\cdots T_{i-1}(E_i), && \text{for odd } i \text{ with } 1 \leq i \leq n,\\
x_i &= TT_1T_3T_5\cdots T_nT_2T_4\cdots T_{i-2}(E_i), && \text{for even } i \text{ with } 2 \leq i \leq n-1,
\end{aligned}$$

where $T = T_1T_3T_5\cdots T_nT_2T_4T_6\cdots T_{n-1}$. In what follows we use the convention $T_i = \mathrm{id}_{U_v(\mathfrak{g})}$ for $i \notin Q_0$. Because of the braid relation $T_iT_j = T_jT_i$ for $|i-j| \geq 2$ and $T_i(E_j) = E_j$, $T_i(F_j) = F_j$, and $T_i(K_j) = K_j$ for $|i-j| \geq 2$ the formulae simplify to

$$\begin{aligned}
u_i &= E_i, && \text{for odd } i \text{ s.t. } 1 \leq i \leq n,\\
v_i &= T_{i-1}T_{i+1}(E_i), && \text{for even } i \text{ s.t. } 2 \leq i \leq n-1,\\
w_i &= T_{i-2}T_iT_{i+2}T_{i-1}T_{i+1}(E_i), && \text{for odd } i \text{ s.t. } 1 \leq i \leq n,\\
x_i &= T_{i-3}T_{i-1}T_{i+1}T_{i+3}T_{i-2}T_iT_{i+2}T_{i-1}T_{i+1}(E_i), && \text{for even } i \text{ s.t. } 2 \leq i \leq n-1.
\end{aligned}$$

Note that $w_i = Tu_i$ for all odd i with $1 \leq i \leq n$ and $x_i = Tv_i$ for all even i with $2 \leq i \leq n-1$.

Remark 2.24. The degrees of these variables are the dimension vectors of the indecomposable direct summands of the terminal module T, compare Figure 9.

Remark 2.25. Similarly, we can associate elements $E(\beta_k) \in U_v(\mathfrak{n}') \subset U_v(\mathfrak{n})$ for $1 \leq k \leq 2(n-1)$ to the reduced expression $s_1 s_3 s_5 \cdots s_{n-2} s_2 s_4 s_6 \cdots s_{n-1}\, s_1 s_3 s_5 \cdots s_{n-2} \cdot s_2 s_4 s_6 \cdots s_{n-1}$ of the Weyl group element $w' \in W'$. The elements generate an algebra $U_v^+(w') \subset U_v(\mathfrak{n}')$, and the set of all ordered products of the $E(\beta_k)$ is a Poincaré-Birkhoff-Witt basis of $U_v^+(w')$ just as above. Elements u_i', w_i' (for odd i with $1 \leq i \leq n-2$) and v_i', x_i' (for even i with $2 \leq i \leq n-1$) in $U_v(\mathfrak{n}')$ are defined analogously. Under the inclusion $U_v(\mathfrak{n}') \subset U_v(\mathfrak{n})$ they are literally the same as the corresponding elements except for

$$\begin{aligned}
v_{n-1}' &= T_{n-2}(E_{n-1}),\\
w_{n-2}' &= T_{n-4}T_{n-2}T_{n-3}T_{n-1}(E_{n-2}),\\
x_{n-3}' &= T_{n-6}T_{n-4}T_{n-2}T_{n-5}T_{n-3}T_{n-1}T_{n-4}T_{n-2}(E_{n-3}),\\
x_{n-1}' &= T_{n-4}T_{n-2}T_{n-3}T_{n-1}T_{n-2}(E_{n-1}).
\end{aligned}$$

2.9 The quantum shuffle algebra and Euler numbers

In this section we study the *quantum shuffle algebra* $(\mathcal{F}, *)$. The quantum shuffle algebra is defined in combinatorial terms. Leclerc [36, Section 2.5, 2.6] shows there is an embedding $U_v(\mathfrak{n}) \hookrightarrow \mathcal{F}$. For some calculations it will be useful to view $U_v(\mathfrak{n})$ as a subalgebra of $\mathcal{F}$.

Definition 2.26. Let r, s be natural numbers. A permutation $\pi \in S_{r+s}$ is called a *shuffle* of type (r, s) if $\pi(1) < \pi(2) < \cdots < \pi(r)$ and $\pi(r+1) < \pi(r+2) < \cdots < \pi(r+s)$.

The following definition is due to Leclerc [36, Section 2.5].

Definition 2.27. For every sequence $(i_1, i_2, \ldots, i_r) \in Q_0^r$ of elements in Q_0 of length $r \geq 0$ define a symbol $w[i_1, i_2, \ldots, i_r]$. (Especially, we have a symbol $w[\,]$ for the empty sequence.) Let $\mathcal{F}$ be the $\mathbb{Q}(v)$-vector space generated by all $w[i_1, i_2, \ldots, i_r]$ for all $r \geq 0$. Define the *quantum shuffle product* on two basis elements by

$$w[i_1, i_2, \ldots, i_r] * w[i_{r+1}, i_{r+2}, \ldots, i_s] = \sum_{\substack{\pi \text{ shuffle} \\ \text{of type } (r,s)}} v^{e(\pi)} w[i_{\pi(1)}, i_{\pi(2)}, \ldots, i_{\pi(r+s)}],$$

where the function $e\colon S_{r+s} \to \mathbb{Z}$ is defined as

$$e(\pi) = \sum_{k \leq r < l, \pi(k) < \pi(l)} (\alpha_{i_{\pi k}}, \alpha_{i_{\pi(l)}}).$$

It is easy to see that the product is associative. We extend the product bilinearly to map $*\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$. The algebra $(\mathcal{F}, *)$ is called the *quantum shuffle algebra*.

Remark 2.28. The shuffle product $*$ on the quantum shuffle algebra $\mathcal{F}$ is associative, but it is not commutative. Hence, $\mathcal{F}$ is an associative $\mathbb{Q}(v)$-algebra. The quantum shuffle algebra $\mathcal{F}$ degenerates to the classical shuffle algebra when we specialize $v = 1$. The quantum shuffle algebra $\mathcal{F}$ is graded by the root lattice if we set $\deg(w[i]) = \alpha_i$ for all $i \in \{1, 2, \ldots, n\}$.

Lemma 2.29. The map $E_i \to w[i]$ extends to an embedding of graded algebras $U_v(\mathfrak{n}) \hookrightarrow \mathcal{F}$. In other words, $U_v(\mathfrak{n})$ is isomorphic to the subalgebra of $\mathcal{F}$ generated by all $w[i]$ for $i \in Q_0$.

Proof. See Leclerc [36, Theorem 4]. □

From now on we view $U_v(\mathfrak{n})$ as a subalgebra of $\mathcal{F}$. In the rest of the section we expand the generators $u_i, v_i, w_i, x_i \in U_v^+(w)$ in the shuffle basis. The following elements will be important for the description.

Definition 2.30. For integers i, j such that $1 \le i \le j \le n$ put

$$X_{i,j} = T_i^{(-1)^{i-1}} T_{i+1}^{(-1)^i} \cdots T_{j-1}^{(-1)^{j-2}}(E_j).$$

By definition, $X_{i,j} \in U_v(\mathfrak{n}) \subset \mathcal{F}$.

Example 2.31. Let us give some examples of $X_{i,j}$ expanded in the shuffle basis: First of all, we have $X_{1,1} = E_1 = w_1$. Moreover,

$$\begin{aligned} X_{1,2} = T_1(E_2) &= E_2E_1 - v^{-1}E_1E_2 \\ &= w[2] * w[1] - v^{-1}w[1] * w[2] \\ &= w[1,2] + v^{-1}w[2,1] - v^{-1}\Big(w[2,1] + v^{-1}w[1,2]\Big) \\ &= (1 - v^{-2})w[1,2] \end{aligned}$$

is a second example.

The next lemma shows that we can compute the expansion of $X_{i,j}$ for all pairs (i, j) in the shuffle basis explicitly.

Lemma 2.32. Let i, j be integers such that $1 \le i \le j \le n$. Then

$$X_{i,j} = (1 - v^{-2})^{j-i} \sum_{\pi} w[\pi(i), \pi(i+1), \ldots, \pi(j)]$$

where the sum runs over all permutations π of $\{i, i+1, \ldots, j\}$ such that for every even number k with $i \le k \le j-1$ we have $\pi^{-1}(k) > \pi^{-1}(k+1)$ and for every even number k with $i+1 \le k \le j$ we have $\pi^{-1}(k) > \pi^{-1}(k-1)$.

Proof. By backwards induction on i we see that the $X_{i,j}$ (for $1 \le i < j \le n$) satisfy the following recursion:

$$X_{i,j} = \begin{cases} E_jX_{i,j-1} - v^{-1}X_{i,j-1}E_j, & \text{if } j \text{ is even;} \\ X_{i,j-1}E_j - v^{-1}E_jX_{i,j-1}, & \text{if } j \text{ is odd.} \end{cases}$$

Now fix i. We proceed by induction on $j - i$. The statement is trivial for $j = i$. Suppose that $j > i$ and that

$$X_{i,j-1} = (1 - v^{-2})^{j-1-i} \sum_{\pi} w[\pi(i), \pi(i+1), \cdots, \pi(j-1)],$$

where sum is taken over all permutations of $\{i, i+1, \ldots, j-1\}$ such that for every even number k with $i \le k \le j-2$ we have $\pi^{-1}(k) > \pi^{-1}(k+1)$ and for every even number k with $i+1 \le k \le j-1$ we have $\pi^{-1}(k) > \pi^{-1}(k-1)$.

We consider two cases. First of all, assume that j is even. Let π be a permutation of $\{i, i+1, \ldots, j-1\}$ as above. When shuffling the sequence (j) of length 1 with the sequence $(\pi(i), \pi(i+1), \ldots, \pi(j-1))$ of length $j - i$, we get $j - i + 1$

permutations of $\{i, i+1, \ldots, j\}$. Among these we distinguish two kinds of permutations. The permutations π_1 where j comes after $j-1$ satisfy $\pi_1^{-1}(k) > \pi_1^{-1}(k+1)$ and $\pi_1^{-1}(k) > \pi_1^{-1}(k-1)$ for all even numbers k such that $i \leq k \leq j-1$ or $i+1 \leq k \leq j$, respectively. Conversely, every permutation π_1 of $\{i, i+1, \ldots, j\}$ satisfying these conditions is uniquely obtained from shuffling (j) with a such a sequence $(\pi(i), \pi(i+1), \ldots, \pi(j-1))$ such that j comes after $j-1$.

We also get permuations π_2 where j occurs before $j-1$. Now we see that

$$w[j] * w[\pi(i), \pi(i+1), \ldots, \pi(j-1)] = \sum_{\pi_1} w[\pi_1] + v^{-1} \sum_{\pi_2} w[\pi_1],$$
$$w[\pi(i), \pi(i+1), \ldots, \pi(j-1)] * w[j] = v^{-1} \sum_{\pi_1} w[\pi_1] + \sum_{\pi_2} w[\pi_2].$$

It follows by induction hypothesis that $X_{i,j} = w[j] * X_{i,j-1} - v^{-1} X_{i,j-1} * w[j] = (1-v^{-2})^{j-i} \sum_{\pi_1} w[\pi_1]$.

The other case where j is odd is proved similarly. □

Remark 2.33. The number $a(i,j)$ of permuations of $\{i, i+1, \ldots, j-1\}$ such that for every even number k with $i \leq k \leq j-2$ we have $\pi^{-1}(k) > \pi^{-1}(k+1)$ and for every even number k with $i+1 \leq k \leq j-1$ we have $\pi^{-1}(k) > \pi^{-1}(k-1)$ only depends on $j-i$. The table displays some values of $a(i,j)$.

j-i	0	1	2	3	4	5	6
$a(i,j)$	1	1	2	5	16	61	272

The sequence is known as *Euler numbers*. It is listed as A000111 in Sloane's Encyclopedia of Integer Sequences [53]. Its exponential generating function is $\sec(x)+\tan(x)$.

Lemma 2.34. The following formulae for the generators of $U_v^+(w)$ are valid:

$$\begin{aligned}
u_i &= E_i, && \text{for odd } i \text{ with } 1 \leq i \leq n;\\
v_i &= T_{i-1}T_i^{-1}(E_{i+1}), && \text{for even } i \text{ with } 2 \leq i \leq n-2;\\
w_1 &= T_2^{-1}(E_3);\\
w_i &= T_{i-2}T_{i-1}^{-1}T_iT_{i+1}^{-1}(E_{i+2}), && \text{for odd } i \text{ with } 3 \leq i \leq n-3;\\
w_n &= T_{n-2}(E_{n-1});\\
x_2 &= T_2^{-1}T_3T_4^{-1}(E_5);\\
x_i &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_i^{-1}T_{i+1}T_{i+2}^{-1}(E_{i+3}), && \text{for even } i \text{ with } 4 \leq i \leq n-4;\\
x_{n-1} &= T_{n-4}T_{n-3}^{-1}T_{n-2}(E_{n-1});
\end{aligned}$$

Proof. The equation $u_i = E_i$ for odd i follows from definition. Note that by definition of Lusztig's T-automorphisms we have $T_{i+1}(E_i) = T_i^{-1}(E_{i+1})$ for $1 \leq i \leq n-1$ and that $T_{i-1}(E_i) = T_i^{-1}(E_{i-1})$ for $2 \leq i \leq n$. The first equation is equivalent to $T_iT_{i+1}(E_i) = E_{i+1}$, the second one is equivalent to $T_iT_{i-1}(E_i) = E_{i-1}$.

Therefore, for even i we have $v_i = T_{i-1}T_{i+1}(E_i) = T_{i-1}T_i^{-1}(E_{i+1})$.

For all further calculations the formula

$$T_iT_{i-1}T_{i+1}(E_i) = T_{i-1}^{-1}T_i(E_{i+1}) = T_{i+1}^{-1}T_i(E_{i-1}) \tag{17}$$

which holds for $2 \leq i \leq n-1$ will be crucial. To verify formula (17) note that by the braid relation we have $T_{i-1}T_iT_{i-1}T_{i+1}(E_i) = T_iT_{i-1}T_iT_{i+1}(E_i) = T_iT_{i-1}(E_{i+1}) =$

$T_i(E_{i+1})$. Application of T_{i-1}^{-1} gives the first part of equation (17), the second part is proved analogously.

Now we compute $w_1 = T_1T_3T_2(E_1) = T_3T_1T_2(E_1) = T_3(E_2) = T_2^{-1}(E_3)$. Similarly, $w_n = T_{n-2}T_nT_{n-1}(E_n) = T_{n-2}(E_{n-1})$. Furthermore, for odd i with $3 \leq i \leq n-2$ we compute

$$\begin{aligned} w_i &= T_{i-2}T_iT_{i+2}T_{i-1}T_{i+1}(E_i) = T_{i-2}T_{i+2}T_iT_{i-1}T_{i+1}(E_i) \\ &= T_{i-2}T_{i+2}T_{i-1}^{-1}T_i(E_{i+1}) = T_{i-2}T_{i-1}^{-1}T_iT_{i+2}(E_{i+1}) \\ &= T_{i-2}T_{i-1}^{-1}T_iT_{i+1}^{-1}(E_{i+2}). \end{aligned}$$

Moreover, we have $x_2 = T_1T_3T_5T_2T_4T_1T_3(E_2) = T_1T_3T_5T_4T_1^{-1}T_2(E_3) = T_3T_5T_4T_2(E_3) = T_5T_2^{-1}T_3(E_4) = T_2^{-1}T_3T_5(E_4) = T_2^{-1}T_3T_4^{-1}(E_5)$, and

$$\begin{aligned} x_{n-1} &= T_{n-4}T_{n-2}T_nT_{n-3}T_{n-1}T_{n-2}T_n(E_{n-1}) \\ &= T_{n-4}T_{n-2}T_nT_{n-3}T_n^{-1}T_{n-1}(E_{n-2}) \\ &= T_{n-4}T_{n-2}T_{n-3}T_{n-1}(E_{n-2}) \\ &= T_{n-4}T_{n-3}^{-1}T_{n-2}(E_{n-1}). \end{aligned}$$

Finally, for even i with $4 \leq i \leq n-3$, the equation

$$\begin{aligned} x_i &= T_{i-3}T_{i-1}T_{i+1}T_{i+3}T_{i-2}T_iT_{i+2}T_{i-1}T_{i+1}(E_i) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-2}T_{i-1}T_{i-2}T_{i+1}T_{i+3}T_{i+2}T_iT_{i-1}T_{i+1}(E_i) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_{i-2}T_{i-1}T_{i+1}T_{i+3}T_{i+2}T_{i-1}^{-1}T_i(E_{i+1}) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_{i-2}T_{i+1}T_{i+3}T_{i+2}T_i(E_{i+1}) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_{i-2}T_{i+3}T_i^{-1}T_{i+1}(E_{i+2}) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_i^{-1}T_{i-2}T_{i+3}T_{i+1}(E_{i+2}) \\ &= T_{i-3}T_{i-2}^{-1}T_{i-1}T_i^{-1}T_{i+1}T_{i+2}^{-1}(E_{i+3}). \end{aligned}$$

holds which is the last equation to be checked. □

Remark 2.35. Lemma 2.34 shows all generators u_i, v_i, w_i, x_i (for appropriate i) of $U_v^+(w)$ are of the form $X_{i',j'}$ (for appropriate i', j'). Hence, the formula of Lemma 2.32 applies. In each case, $V_{[i',j']}$ is the associated kQ-module from Figure 4. In other words, in each case $\deg(X_{i',j'}) = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$.

Remark 2.36. With the same argument we can conclude that $v'_{n-1} = T_{n-2}(E_{n-1})$, $w'_{n-2} = T_{n-4}T_{n-3}^{-1}T_{n-2}(E_{n-1})$, $x'_{n-1} = T_{n-4}(E_{n-3})$, and

$$x'_{n-3} = T_{n-6}T_{n-5}^{-1}T_{n-4}T_{n-3}^{-1}T_{n-2}(E_{n-1}).$$

Hence, Lemma 2.34 and Remark 2.35 also true for the generators u'_i, v'_i, w'_i, x'_i (for appropriate i) of $U_v^+(w')$.

2.10 The straightening relations for the generators of $U_v^+(w)$

The following lemma expands every $E(\beta_j)E(\beta_i)$ with $1 \leq i < j \leq 2n$ in the Poincaré-Birkhoff-Witt basis $\mathcal{P}$. The relations of Lemma 2.37 are called *straightening relations*. Iterative use of the straightening relations from Lemma 2.37 allows us to write an arbitrarily ordered monomial in the generators $E(\beta_k)$ with $1 \leq k \leq 2n$ (and hence every element in $U_v^+(w)$) as a linear combination of Poincaré-Birkhoff-Witt basis elements $E[\mathbf{a}]$ with $\mathbf{a} \in \mathbb{N}^{2n}$.

Lemma 2.37. The generators of $U_v^+(w)$ satisfy the following relations

$$
\begin{aligned}
&v_{i+1}u_i = vu_iv_{i+1}, && \text{for } i = 1, 3, \cdots, n-2,\\
&v_{i-1}u_i = vu_iv_{i-1}, && \text{for } i = 3, 5, \ldots, n,\\
\\
&w_{i+2}u_i = vu_iw_{i+2}, && \text{for } i = 1, 3, \ldots, n-2,\\
&w_{i-2}u_i = vu_iw_{i-2}, && \text{for } i = 3, 5, \ldots, n,\\
&w_1u_1 = v^{-1}u_1w_1 + v_2, &&\\
&w_iu_i = u_iw_i + (v - v^{-1})v_{i-1}v_{i+1}, && \text{for } i = 3, 5, \cdots, n-2,\\
&w_nu_n = v^{-1}u_nw_n + v_{n-1}, &&\\
\\
&x_{i+3}u_i = vu_ix_{i+3}, && \text{for } i = 1, 3, \ldots, n-4,\\
&x_{i-3}u_i = vu_ix_{i-3}, && \text{for } i = 5, 7, \ldots, n,\\
&x_{i-1}u_i = u_ix_{i-1} + (v - v^{-1})v_{i+1}w_{i-2}, && \text{for } i = 3, 5, \ldots, n-2,\\
&x_{n-1}u_n = v^{-1}u_nx_{n-1} + w_{n-2}, &&\\
&x_2u_1 = v^{-1}u_1x_2 + w_3, &&\\
&x_{i+1}u_i = u_ix_{i+1} + (v - v^{-1})v_{i-1}w_{i+2}, && \text{for } i = 3, 5, \ldots, n-2,\\
\\
&w_{i+1}v_i = vv_iw_{i+1}, && \text{for } i = 2, 4, \ldots, n-1,\\
&w_{i-1}v_i = vv_iw_{i-1}, && \text{for } i = 2, 4, \ldots, n-1,\\
\\
&x_{i+2}v_i = vv_ix_{i+2}, && \text{for } i = 2, 4, \ldots, n-3,\\
&x_{i-2}v_i = vv_ix_{i-2}, && \text{for } i = 4, 6, \ldots, n-1,\\
&x_iv_i = v_ix_i + (v - v^{-1})w_{i-1}w_{i+1}, && \text{for } i = 2, 4, \ldots, n-1,\\
\\
&x_{i+1}w_i = vw_ix_{i+1}, && \text{for } i = 1, 3, \ldots, n-2,\\
&x_{i-1}w_i = vw_ix_{i-1}, && \text{for } i = 3, 5, \ldots, n.
\end{aligned}
$$

For every i, j with $1 \leq i < j \leq 2n$ such that $E(\beta_j)E(\beta_i)$ is not listed on the left-hand side above the commutativity relation $E(\beta_j)E(\beta_i) = E(\beta_i)E(\beta_j)$ holds.

Proof. Let i be an integer with $1 \leq i \leq n-1$. We have

$$
\begin{aligned}
T_iT_{i+2}(E_{i+1}) &= T_i(E_{i+1}E_{i+2} - v^{-1}E_{i+2}E_{i+1})\\
&= (E_{i+1}E_i - v^{-1}E_iE_{i+1})E_{i+2} - v^{-1}E_{i+2}(E_{i+1}E_i - v^{-1}E_iE_{i+1})\\
&= E_{i+1}E_iE_{i+2} - v^{-1}E_iE_{i+1}E_{i+2}\\
&\quad - v^{-1}E_{i+2}E_{i+1}E_i + v^{-2}E_iE_{i+2}E_{i+1}. \qquad (18)
\end{aligned}
$$

Now let i be an odd integer with $1 \leq i \leq n-2$. Then $u_i = E_i$ and $v_{i+1} =$

$T_iT_{i+2}(E_{i+1})$. By equation (18) the following relations hold:

$$v_{i+1}u_i = \left(E_{i+1}E_i^2 - v^{-1}E_iE_{i+1}E_i\right)E_{i+2} + E_{i+2}\left(v^{-2}E_iE_{i+1}E_i - v^{-1}E_{i+1}E_i^2\right),$$
$$u_iv_{i+1} = \left(E_iE_{i+1}E_i - v^{-1}E_i^2E_{i+1}\right)E_{i+2} + E_{i+2}\left(v^{-2}E_i^2E_{i+1} - v^{-1}E_iE_{i+1}E_i\right).$$

By equation (36) we get $v_{i+1}u_i - vu_iv_{i+1} = 0$. The equation $v_{i-1}u_i = vu_iv_{i-1}$, for odd i with $3 \leq i \leq n$, is proved analogously. Application of T to the last two equations yields $x_{i+1}w_i = vw_ix_{i+1}$, for odd integers i with $1 \leq i \leq n-2$, and $x_{i-1}w_i = vw_ix_{i-1}$, for odd integers i with $3 \leq i \leq n$.

Now let i be an even integer with $2 \leq i \leq n-1$. Then $v_i = T_{i-1}T_{i+1}(E_i)$ and $w_{i+1} = T_{i-3}T_{i-1}T_{i+1}T_{i+3}T_iT_{i+2}(E_{i+1})$. With he same argument as above we have $T_iT_{i+2}(E_{i+1})E_i = vE_iT_iT_{i+2}(E_{i+1})$. Let us apply the the automorphism $T_{i-3}T_{i-1}T_{i+1}T_{i+3}$ to the last equation. Using the equation $T_{i-3}T_{i-1}T_{i+1}T_{i+3}(E_i) = T_{i-1}T_{i+1}(E_i) = v_i$ we get $w_{i+1}v_i = vv_iw_{i+1}$. Similarly, the equation $w_{i-1}v_i = vv_iw_{i-1}$ holds.

Let i be an integer with $1 \leq i \leq n-1$. We have

$$\begin{aligned}-T_{i+1}(E_{i+2})F_iK_i &= -v^{-1}(E_{i+2}E_{i+1} - v^{-1}E_{i+1}E_{i+2})F_iK_i\\ &= -F_iK_i(E_{i+2}E_{i+1} - v^{-1}E_{i+1}E_{i+2}) = -F_iK_iT_{i+1}(E_{i+2}).\end{aligned}$$

Application of the composition of automorphisms $T_iT_{i+2}T_{i+4}T_{i+3}$ yields

$$T_iT_{i+2}T_{i+4}T_{i+1}T_{i+3}(E_{i+2})E_i = vE_iT_iT_{i+2}T_{i+4}T_{i+1}T_{i+3}(E_{i+2}). \tag{19}$$

Now let i be more specifically an odd integer with $1 \leq i \leq n-2$. The previous equation (19) asserts that $w_{i+2}u_i = vu_iw_{i+2}$. The equation $w_{i-2}u_i = vu_iw_{i-2}$, for odd i with $3 \leq i \leq n$, is proved analogously. Now let i be an even integer with $2 \leq i \leq n-3$. From (19) we see after application of $T_{i-1}T_{i+1}T_{i+3}T_{i+5}$ that $x_{i+2}v_i = vv_ix_{i+2}$. The equation $x_{i-2}v_i = vv_ix_{i-2}$, for even i with $4 \leq i \leq n-1$, is proved analogously.

Furthermore, there holds:

$$\begin{aligned}&-F_1K_1(E_1E_2 - v^{-1}E_2E_1)\\ &= -vF_1E_1E_2K_1 + F_1E_2E_1K_1\\ &= v\left(\frac{K_1 - K_1^{-1}}{v - v^{-1}} - E_1F_1\right)E_2K_1 + E_2\left(E_1F_1 - \frac{K_1 - K_1^{-1}}{v - v^{-1}}\right)K_1\\ &= -v(E_1E_2 - v^{-1}E_2E_1)F_1K_1 + \frac{1}{v - v^{-1}}(E_2K_1^2 - v^2E_2 - E_2K_1^2 + E_2)\\ &= -v(E_1E_2 - v^{-1}E_2E_1)F_1K_1 - vE_2.\end{aligned}$$

Application of the map T_1T_3 yields to $u_1w_1 = vw_1u_1 - vv_2$ which is equivalent to $w_1u_1 = v^{-1}u_1w_1 + v_2$. The next equation $w_nu_n = v^{-1}u_nw_n + v_{n-1}$ is proved analogously.

Let i be an integer with $2 \leq i \leq n-1$. Put $S = T_{i-1}T_{i+1}(E_i) = E_iE_{i-1}E_{i+1} - v^{-1}E_{i-1}E_iE_{i+1} - v^{-1}E_{i+1}E_iE_{i-1} + v^{-2}E_{i-1}E_{i+1}E_i$. We have $K_iS = SK_i$, so:

$$\begin{aligned}-F_iK_iS &= (-F_iE_iE_{i-1}E_{i+1} + v^{-1}E_{i-1}F_iE_iE_{i+1}\\ &\quad + v^{-1}E_{i+1}F_iE_iE_{i-1} - v^{-2}E_{i-1}E_{i+1}F_iE_i)K_i\\ &= -SF_iK_i + \frac{1}{v - v^{-1}}\left[(K_i - K_i^{-1})E_{i-1}E_{i+1}\right.\\ &\quad -v^{-1}E_{i-1}(K_i - K_i^{-1})E_{i+1} - v^{-1}E_{i+1}(K_i - K_i^{-1})E_{i-1}\\ &\quad \left.+v^{-2}E_{i-1}E_{i+1}(K_i - K_i^{-1})\right]\end{aligned}$$

$$
\begin{aligned}
&= -SF_iK_i + \frac{1}{v-v^{-1}}\left[(v^{-2}-v^{-2}-v^{-2}+v^{-2})E_{i-1}E_{i+1}K_i^2\right.\\
&\quad \left.+(-v^2+2-v^{-2})E_{i-1}E_{i+1}\right]\\
&= -SF_iK_i + (v^{-1}-v)E_{i+1}E_{i-1}. \qquad (20)
\end{aligned}
$$

Now let i be more specifically an odd integer with $3 \le i \le n-2$. After application of $T_{i-2}T_iT_{i+2}$ the previous equation (20) asserts that $u_iw_i = w_iu_i + (v^{-1} - v)v_{i+1}v_{i-1}$. If i is an even integer with $2 \le i \le n-1$, then application of the composition $T_{i-3}T_{i-1}T_{i+1}T_{i+3}T_{i-2}T_iT_{i+2}$ to equation (20) yields $v_ix_i = x_iv_i + (v^{-1} - v)w_{i+1}w_{i-1}$.

Let i be an odd integer with $1 \le i \le n-4$. Since $T_{i+1}T_{i+2}T_{i+4}(E_{i+3})$ is a linear combination of monomials in E_{i+1}, E_{i+2}, E_{i+3}, and E_{i+4} with each factor appearing once, we see that

$$
-T_{i+1}T_{i+2}T_{i+4}(E_{i+3})F_iK_i = -vF_iK_iT_{i+1}T_{i+2}T_{i+4}(E_{i+3}). \qquad (21)
$$

Applying $T_iT_{i+2}T_{i+4}T_{i+6}T_{i+3}T_{i+5}$ to (21) yields $x_{i+3}u_i = vu_ix_{i+3}$. The equation $x_{i-3}u_i = vu_ix_{i-3}$, for odd i with $5 \le i \le n$, is proved analogously.

Consider the three elements $T_2^{-1}(-F_1K_1)$, $T_1T_3(E_2)$, and E_3. We abbreviate $X = T_2^{-1}(-F_1K_1) = (vF_2F_1 - F_1F_2)K_1K_2$. We have

$$
\begin{aligned}
&(vF_2F_1 - F_1F_2)E_1\\
&= vF_2\left(E_1F_1 - \frac{K_1-K_1^{-1}}{v-v^{-1}}\right) - \left(E_1F_1 - \frac{K_1-K_1^{-1}}{v-v^{-1}}\right)F_2\\
&= E_1(vF_2F_1 - F_1F_2) + \frac{1}{v-v^{-1}}\left[-vF_2(K_1-K_1^{-1}) + (K_1-K_1^{-1})F_2\right]\\
&= E_1(vF_2F_1 - F_1F_2) + F_2K_1^{-1}.
\end{aligned}
$$

Therefore, $XE_1 = vE_1X + vF_2K_2$. Similarly, $XE_2 = vE_2X + vF_1K_1K_2^2$. Furthermore, $XE_3 = v^{-1}E_3X$. Hence

$$
\begin{aligned}
&X(E_2E_1 - v^{-1}E_1E_2)\\
&= (vE_2X + vF_1K_1K_2^2)E_1 - (E_1X + F_2K_2)E_2\\
&= v(vE_1X + vF_2K_2) + v^2F_1K_1E_1 - E_1(vE_2X + vF_1K_1K_2^2) - F_2K_2E_2\\
&= v^2(E_2E_1 - v^{-1}E_1E_2)X + v^2(E_2F_2 - F_2E_2)K_2 + v(F_1E_1 - E_1F_1)K_1K_2^2.
\end{aligned}
$$

Abbreviate $Y = E_2E_1 - v^{-1}E_1E_2$ and

$$
\begin{aligned}
R &= v^2(E_2F_2 - F_2E_2)K_2 + v(F_1E_1 - E_1F_1)K_1K_2^2\\
&= \frac{1}{v-v^{-1}}\left[v^2K_2^2 - v^2 - vK_1^2K_2^2 + vK_2^2\right].
\end{aligned}
$$

Then $RE_3 - v^{-2}E_3R = -vE_3$. Note that $T_1T_3(E_2) = YE_3 - v^{-1}E_3Y$ is equal to a v-commutator. Hence

$$
\begin{aligned}
XT_1T_3(E_2) &= X(YE_3 - v^{-1}E_3Y)\\
&= (v^2YX + R)E_3 - v^{-2}E_3(v^2Yx + R)\\
&= v(YE_3 - v^{-1}E_3Y) - RE_3 - v^{-2}E_3R\\
&= vT_1T_3(E_2) - vE_3. \qquad (22)
\end{aligned}
$$

Application of $T_1T_3T_5T_2T_4$ to equation (22) yields to $u_1x_2 = vx_2u_1 - vw_3$ which is equivalent to $x_2u_1 = v^{-1}u_1x_2 + w_3$. The equation $x_{n-1}u_n = v^{-1}u_nx_{n-1} + w_{n-2}$ is proved analogously.

Now let i be an odd integer with $3 \le i \le n-2$. Let us consider the four elements $T_{i+1}^{-1}(-F_iK_i)$, $T_{i-1}T_iT_{i+2}(E_{i+1})$, E_{i+2}, and E_{i-1}. Now denote by X the element $T_{i+1}^{-1}(-F_iK_i)$. Similarly as above, we have $XE_i = vE_iX + vF_{i+1}K_{i+1}$, $XE_{i+1} = vE_{i+1}X + vF_iK_iK_{i+1}^2$, $XE_{i+2} = v^{-1}E_{i+2}X$. Furthermore, we have $XE_{i-1} = (vF_{i+1}F_i - F_iF_{i+1})K_iK_{i+1}E_{i-1} = v^{-1}E_{i-1}X$. Note that

$$\begin{aligned}&T_{i-1}T_iT_{i+2}(E_{i+1})\\&= T_{i-1}(E_{i+1}E_iE_{i+2} - v^{-1}E_iE_{i+1}E_{i+2} - v^{-1}E_{i+2}E_{i+1}E_i + v^{-2}E_iE_{i+2}E_{i+1})\\&= T_iT_{i+2}(E_{i+1})E_{i-1} - v^{-1}E_{i-1}T_iT_{i+2}(E_{i+1}).\end{aligned}$$

With the same argument as above one can prove that

$$XT_iT_{i+2}(E_{i+1}) = vT_iT_{i+2}(E_{i+1})X - vE_{i+2}.$$

From this equation it follows that

$$\begin{aligned}&XT_{i-1}T_iT_{i+2}(E_{i+1})\\&= (vT_iT_{i+2}(E_{i+1})X - vE_{i+2})E_{i-1} - v^{-2}E_{i-1}(vT_iT_{i+2}(E_{i+1})X - vE_{i+2})\\&= (T_iT_{i+2}(E_{i+1})E_{i-1} - v^{-1}E_{i-1}T_iT_{i+2}(E_{i+1}))X + (v^{-1} - v)E_{i+2}E_{i-1}. \quad (23)\end{aligned}$$

Application of the automorphism $T_{i-2}T_iT_{i+2}T_{i+4}T_{i+1}T_{i+3}$ to equation (23) gives $u_ix_{i+1} = x_{i+1}u_i + (v^{-1} - v)w_{i+2}v_{i-1}$. The equation $u_ix_{i-1} = x_{i-1}u_i + (v^{-1} - v)w_{i-2}v_{i+1}$ is proved analogously.

After multiplying with appropriate T-automorphisms, all others pairs $E(\beta_i)$ and $E(\beta_j)$ of generators become $\mathbb{Q}(v)$-linear combinations of momomials $E_{i_1}E_{i_2}\cdots E_{i_k}$ and $E_{i'_1}E_{i'_2}\cdots E_{i'_{k'}}$, respectively, where the two occuring sequences $(i_1, i_2, \ldots, i_k)$ and $(i'_1, i'_2, \ldots, i'_{k'})$ of indices come from two intervals of distance at least two. Hence they commute. □

Remark 2.38. In the straightening relations of Lemma 2.37, for all i, j with $1 \le i < j \le 2n$, the coefficient in front of $E(\beta_i)E(\beta_j)$ in the expansion of $E(\beta_j)E(\beta_i)$ in the Poincaré-Birkhoff-Witt basis is $v^{(\beta_i,\beta_j)}$.

Remark 2.39. Similarly, there are straightening relations for the generators u'_i, v'_i, w'_i, x'_i (for appropriate i) of $U_v^+(w')$ that enable us to expand every element of $U_v^+(w')$ in the Poincaré-Birkhoff-Witt basis. Let us describe them. First of all, note that $v'_{n-1} = T_n^{-1}(v_{n-1})$, $w'_{n-1} = T_n^{-1}w_{n-1}$, $x'_{n-3} = T_n^{-1}x_{n-3}$, and that T_n^{-1} leaves all generators invariant except for v'_{n-1}, w'_{n-1}, x'_{n-3}, and x'_{n-1}. Therefore, the straightening relations of $U_v^+(w')$ are the same as the ones for $U_v^+(w)$ except for the straightening relations involving x'_{n-1}.

Calculations similar to those in Lemma 2.37 show that the straightening relations involving x'_{n-1} are commutativity relations except for:

$$\begin{aligned}x'_{n-1}w'_{n-2} &= vw'_{n-2}x'_{n-1},\\x'_{n-1}v'_{n-3} &= vv'_{n-3}x'_{n-1},\\x'_{n-1}v'_{n-1} &= v^{-1}v'_{n-1}x'_{n-1} + w'_{n-2},\\x'_{n-1}u'_{n-2} &= v^{-1}u'_{n-2}x'_{n-1} + v'_{n-3},\\x'_{n-1}u'_{n-4} &= vu'_{n-4}x'_{n-1}.\end{aligned}$$

Note that Remark 2.38 is also true in this case.

Remark 2.40. The commutation exponent of Remark 2.38 and a weaker (non-explicit) form the straightening relations of Lemma 2.37 is given by the Lemma of Levendorkiĭ-Soibelman [41, Proposition 5.5.2]. See also Kimura [33, Theorem 4.24].

2.11 The dual Poincaré-Birkhoff-Witt basis

Kashiwara [28] introduced operators $E_i' \in \mathrm{End}(U_v(\mathfrak{n}))$ for $1 \leq i \leq n$ such that the following two properties hold: First of all, $E_i'(E_j) = \delta_{i,j}$ for all i, j. Secondly, the Leibniz rule $E_i'(xy) = E_i'(x)y + v^{(\alpha_i,|x|)}xE_i'(y)$ holds for all i and all homogeneous elements $x, y \in U_v(\mathfrak{n})$. Furthermore, Kashiwara [28] introduced a non-degenerate symmetric bilinear form

$$(\cdot,\cdot) : U_v(\mathfrak{n}) \times U_v(\mathfrak{n}) \to \mathbb{Q}(v).$$

It is characterized by the assumption that the endomorphism E_i' of $U_v(\mathfrak{n})$ is adjoint to the left multiplication with E_i, i.e., $(E_i'(x), v) = (x, E_i y)$ for all $x, y \in U_v(\mathfrak{n})$ and $i \in \{1, 2, \ldots, n\}$.

The algebra $U_v(\mathfrak{n})$ is a *Hopf algebra*. The E_i' may be viewed as elements in the graded dual Hopf algbera $U_v(\mathfrak{n})^*_{gr}$. We refer to Berenstein-Zelevinsky [5, Appendix] for details.

Lusztig [42, Section 1.2] defined a different non-degenerate symmetric bilinear form $(\cdot,\cdot)_L : U_v(\mathfrak{n}) \times U_v(\mathfrak{n}) \to \mathbb{Q}(v)$. Both bilinear resemble each other. In this paper we use Kashiwara's form. Both forms can be compared, see Leclerc [36, Section 2.2].

According to Lusztig [42, Proposition 38.2.3] the Poincaré-Birkhoff-Witt basis is orthogonal with respect to Lusztig's bilinear form. The comparison between both forms shows that it is also orthogonal with respect to Kashiwara's bilinear form. More precisely, for all $\beta, \gamma \in \Delta^+$, we have (see Leclerc [36, Equation 21] where the author uses the variable q^{-1} intead of v)

$$\begin{aligned} &(E(\beta), E(\gamma)) = 0, && \text{if } \beta \neq \gamma; \\ &(E(\beta), E(\beta)) = \frac{\prod_{i=1}^n (1 - v^{-(\alpha_i,\alpha_i)})^{c_i}}{1 - v^{-(\beta,\beta)}}, && \text{if } \beta = \sum_{i=1}^n c_i\alpha_i \text{ with } c_i \in \mathbb{N}. \end{aligned}$$

Compare also with Kimura [33, Proposition 4.18].

Remark 2.41. Every $\beta \in \Delta^+$ fulfills $(\beta, \beta) = 2$; every α_i with $i \in Q_0$ fulfills $(\alpha_i, \alpha_i) = 2$. Hence, if $\beta = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \in \Delta^+$, then $(E(\beta), E(\beta)) = (1 - v^{-2})^{j-i}$.

Definition 2.42. The *dual Poincaré-Birkhoff-Witt basis* $\mathcal{P}^*$ of $U_v^+(w)$ is defined to be the basis adjoint to the Poincaré-Birkhoff-Witt basis with respect to Kashiwara's form. For every natural number k with $1 \leq k \leq 2n$ we denote by $E^*(\beta_k) \in \mathcal{P}^*$ the dual of $E(\beta_k) \in \mathcal{P}$, i.e., the unique scalar multiple of $E(\beta_k)$ such that $(E(\beta_k), E^*(\beta_k)) = 1$.

Remark 2.43. Assume that k is an integer with $1 \leq k \leq 2n$. Let i, j be the integers with $1 \leq i \leq j \leq n$ such that we can write $\beta_k \in \Delta^+$ as $\beta_k = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$. By Lemma 2.32 we have

$$E^*(\beta_k) = (1 - v^{-2})^{i-j} X_{i,j} = \sum_\pi w[\pi(i), \pi(i+1), \ldots, \pi(j)]$$

where the sum runs over all permutations π of $\{i, i+1, \ldots, j\}$ such that for every even number k with $i \le k \le j-1$ we have $\pi^{-1}(k) > \pi^{-1}(k+1)$ and for every even number k with $i+1 \le k \le j$ we have $\pi^{-1}(k) > \pi^{-1}(k-1)$.

Definition 2.44. We also introduce a shorthand notation for $E^*(\beta_k)$ with $1 \le k \le 2n$. To reflect similarities with the cluster algebra from Section 2.5 we put

$$\begin{aligned}
y_i &= u_i^*, && \text{for odd } i \text{ with } 1 \le i \le n,\\
z_i &= v_i^*, && \text{for even } i \text{ with } 2 \le i \le n-1,\\
z_i &= w_i^*, && \text{for odd } i \text{ with } 1 \le i \le n,\\
y_i &= x_i^*, && \text{for even } i \text{ with } 2 \le i \le n-1.
\end{aligned}$$

Remark 2.45. The straightening relations of Lemma 2.37 now become

$$\begin{aligned}
&z_{i+1}y_i = v y_i z_{i+1}, && \text{for } i \text{ odd with } 1 \le i \le n-2,\\
&z_{i-1}y_i = v y_i z_{i-1}, && \text{for } i \text{ odd with } 3 \le i \le n,\\
\\
&z_{i+2}y_i = v y_i z_{i+2}, && \text{for } i \text{ odd with } 1 \le i \le n-2,\\
&z_{i-2}y_i = v y_i z_{i-2}, && \text{for } i \text{ odd with } 3 \le i \le n,\\
&z_1 y_1 = v^{-1} y_1 z_1 + (1-v^{-2}) z_2, &&\\
&z_i y_i = y_i z_i + (v - v^{-1}) z_{i-1} z_{i+1}, && \text{for } i \text{ odd with } 3 \le i \le n-2,\\
&z_n y_n = v^{-1} y_n z_n + (1-v^{-2}) z_{n-1}, &&\\
\\
&y_{i+3}y_i = v y_i y_{i+3}, && \text{for } i \text{ odd with } 1 \le i \le n-4,\\
&y_{i-3}y_i = v y_i y_{i-3}, && \text{for } i \text{ odd with } 5 \le i \le n,\\
&y_{i-1}y_i = y_i y_{i-1} + (v - v^{-1}) z_{i+1} z_{i-2}, && \text{for } i \text{ odd with } 3 \le i \le n-2,\\
&y_{n-1}y_n = v^{-1} y_n y_{n-1} + (1-v^{-2}) z_{n-2}, &&\\
&y_2 y_1 = v^{-1} y_1 y_2 + (1-v^{-2}) z_3, &&\\
&y_{i+1}y_i = y_i y_{i+1} + (v - v^{-1}) z_{i-1} z_{i+2}, && \text{for } i \text{ odd with } 3 \le i \le n-2,\\
\\
&z_{i+1}z_i = v z_i z_{i+1}, && \text{for } i \text{ even with } 2 \le i \le n-1,\\
&z_{i-1}z_i = v z_i z_{i-1}, && \text{for } i \text{ even with } 2 \le i \le n-1,\\
\\
&y_{i+2}z_i = v z_i y_{i+2}, && \text{for } i \text{ even with } 2 \le i \le n-3,\\
&y_{i-2}z_i = v z_i y_{i-2}, && \text{for } i \text{ even with } 4 \le i \le n-1,\\
&y_i z_i = z_i y_i + (v - v^{-1}) z_{i-1} z_{i+1}, && \text{for } i \text{ even with } 2 \le i \le n-1,\\
\\
&y_{i+1}z_i = v z_i y_{i+1}, && \text{for } i \text{ odd with } 1 \le i \le n-2,\\
&y_{i-1}z_i = v z_i y_{i-1}, && \text{for } i \text{ odd with } 3 \le i \le n.
\end{aligned}$$

Definition 2.46. Consider $U_v^+(w)_{\mathbb{Z}} = \bigoplus_{\mathbf{a}\in\mathbb{N}^{2n}} \mathbb{Z}[v, v^{-1}]E[\mathbf{a}]^*$, the integral form of $U_v^+(w)$. Furthermore, put $\mathcal{A}(w)_1 = \mathbb{Q} \otimes_{\mathbb{Z}[v,v^{-1}]} U_v^+(w)_{\mathbb{Z}}$; we call the algebra $\mathcal{A}(w)_1$ the classical limit of $U_v^+(w)$ or the specialization of $U_v^+(w)$ at $v = 1$. Furthermore, the $\mathbb{Z}[v^{\pm\frac{1}{2}}]$-algebra $\mathcal{A}_v(w) = \bigoplus_{\mathbf{a}\in\mathbb{N}^{2n}} \mathbb{Z}[v^{\pm\frac{1}{2}}]E[\mathbf{a}]^*$ is the integral form of the algebra $\mathbb{Q}[v^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[v,v^{-1}]} \mathcal{A}(w)$ and will be useful in further considerations.

Remark 2.47. Note that, by the form of the straightening relations for the dual variables from above, $\mathcal{A}(w)_1$ is a commutative algebra.

Definition 2.48. Define a function $b\colon \mathbb{N}^{2n} \to \mathbb{Z}$ by $b(\mathbf{a}) = \sum_{k=1}^{2n} \binom{a_k}{2}$ for a sequence $\mathbf{a} = (a_1, a_2, \ldots, a_{2n}) \in \mathbb{N}^{2n}$.

Proposition 2.49. For every sequence $\mathbf{a} = (a_1, a_2, \ldots, a_{2n}) \in \mathbb{N}^{2n}$ we have $E[\mathbf{a}]^* = v^{-b[\mathbf{a}]}E^*(\beta_1)^{a_1}E^*(\beta_2)^{a_2}\cdots E^*(\beta_{2n})^{a_{2n}}$.

Proof. Follows from Lusztig's evaluation [42, Proposition 38.2.3] for of the bilinear form at Poincaré-Birkhoff-Witt basis elements together with Leclerc's conversion formula [36, Section 2.2]. (Compare with the argument from Leclerc [36, Section 5.5.3].) □

2.12 The dual canonical basis

In this section we present the dual canonical basis of $U_v^+(w)$. It is the dual of Lusztig's canonical basis from Lusztig [42, Theorem 14.2.3]. We need some auxiliary notations. Compare the following definitions with Leclerc [36, Section 2.7].

Definition 2.50. For $\mathbf{a} = (a_1, a_2, \ldots, a_{2n}) \in \mathbb{N}^{2n}$ write $\deg(E[\mathbf{a}]^*) = \sum_{k=1}^{2n} a_k\beta_k \in Q^+$ as a $\mathbb{N}$-linear combination in the simple roots, i.e., $\deg(E[\mathbf{a}]^*) = \sum_{k=1}^{2n} a_k\beta_k = \sum_{i=1}^{n} c_i\alpha_i$ with $c_i \in \mathbb{Z}$. Put

$$N(\mathbf{a}) = \frac{1}{2}\Big(\deg(E[\mathbf{a}]^*), \deg(E[\mathbf{a}]^*)\Big) - \sum_{i=1}^{n} c_i.$$

We call N the *norm* of the sequence $\mathbf{a} \in \mathbb{N}^{2n}$. We also use the convention

$$N(\sum_{i=1}^{n} c_i\alpha_i) = \frac{1}{2}\Big(\sum_{i=1}^{n} c_i\alpha_i, \sum_{i=1}^{n} c_i\alpha_i\Big) - \sum_{i=1}^{n} c_i$$

for elements in the root lattice.

Proposition 2.51. For every natural number k with $1 \leq k \leq 2n$ the following equation holds

$$\sigma(E^*(\beta_k)) = v^{N(\beta_k)}E^*(\beta_k).$$

Proof. Leclerc [36, Lemma 7] proves that a homogeneous element $f \in \mathcal{F}$ satisfies $\sigma(f) = v^{N(|f|)}f$ if and only if all coefficients in the expansion of f in the basis of shuffles are invariant under σ. By Remark 2.43 all coefficients are 0 or 1 in the case of $E^*(\beta_k)$. □

Definition 2.52. We define a partial order $\lhd$ on the parametrizing set $\mathbb{N}^{2n}$ of dual Poincaré-Birkhoff-Witt basis elements. For every k with $1 \leq k \leq 2n$ let $\mathbf{e}_k \in \mathbb{N}^{2n}$ be the vector satisfying $(\mathbf{e}_k)_l = \delta_{k,l}$ for all l. For every integer i with $1 \leq i \leq n$ let $k_i, l_i, m_i, n_i \in \{1, 2, \ldots, 2n\}$ be the indices for which $|y_i| = \beta_{k_i}$, $|z_{i-1}| = \beta_{l_i}$, $|z_{i+1}| = \beta_{m_i}$, $|z_i| = \beta_{n_i}$. (Note that k_1 and n_n are not defined. We put $\mathbf{e}_{k_1} = \mathbf{e}_{n_n} = 0$.) Put $\mathbf{v}_i = \mathbf{e}_{k_i} - \mathbf{e}_{l_i} - \mathbf{e}_{m_i} + \mathbf{e}_{n_i}$. Now we say that $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{2n}$ satisfy $\mathbf{a} \lhd \mathbf{b}$ if and only if $\mathbf{b} - \mathbf{a} = \bigoplus_{i=1}^{n} \mathbb{N}\mathbf{v}_i$.

Remark 2.53. The straightening relations of Lemma 2.37 imply the following fact: If $\mathbf{a} \in \mathbb{N}^{2n}$ and we expand $E^*(\beta_{2n})^{a_{2n}} E^*(\beta_{2n-1})^{a_{2n-1}} \cdots E^*(\beta_1)^{a_1}$ in the dual Poincaré-Birkhoff-Witt basis, then we get a $\mathbb{Q}(q)$-linear combination of $E[\mathbf{b}]^*$ with $\mathbf{b} \in S(\mathbf{a}) \cup \{\mathbf{a}\}$.

Definition 2.54. For $\mathbf{a} \in \mathbb{N}^{2n}$ put $S(\mathbf{a}) = \{\mathbf{b} \in \mathbb{N}^{2n} \colon \mathbf{a} \lhd \mathbf{b}, \mathbf{a} \neq \mathbf{b}\}$.

Theorem 2.55. There exist elements $B[\mathbf{a}]^* \in U_v^+(w)$ parametrized by sequences $\mathbf{a} \in \mathbb{N}^{2n}$ such that the set $\mathcal{B}^* = \{B[\mathbf{a}]^* \colon \mathbf{a} \in \mathbb{N}^{2n}\}$ is a basis of $U_v^+(w)$ and the following two properties hold.

(1) For every $\mathbf{a} \in \mathbb{N}^{2n}$ we have $B[\mathbf{a}]^* - E[\mathbf{a}]^* \in \bigoplus_{\mathbf{b} \in S(\mathbf{a})} v^{-1}\mathbb{Z}[v^{-1}]E[\mathbf{b}]^*$.

(2) For every $\mathbf{a} \in \mathbb{N}^{2n}$ we have $\sigma(B[\mathbf{a}]^*) = v^{N(\mathbf{a})} B[\mathbf{a}]^*$.

The elements $B[\mathbf{a}]^* \in U_v^+(w)$ are uniquely determined by these two properties.

Proof. Note that if $\mathbf{a} \lhd \mathbf{b}$, then $|E[\mathbf{a}]^*| = \sum_{k=1}^{2n} a_k\beta_k = \sum_{k=1}^{2n} b_k\beta_k = |E[\mathbf{b}]^*|$ since the straightening relations are relations in a Q-graded algebra. For $\gamma \in Q$ in the root lattice, consider the (finite) set $S_\gamma \subset \mathbb{N}^{2n}$ of all $\mathbf{a} = (a_i)_{1 \leq i \leq 2n}$ with $\sum_{k=1}^{2n} a_k\beta_k = \gamma$. We extend the partial order $\lhd$ on S_γ to a total order $<$. Let $\mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_m$ be the elements of S_γ written in increasing order. Now we prove by backward induction that for every $k = m, m-1, \ldots, 2, 1$ there exist linearly indpendent $B[\mathbf{a}_k]$, $B[\mathbf{a}_{k+1}], \ldots, B[\mathbf{a}_m]$ satisfying (1) and (2).

Put $B[\mathbf{a}_m] = E[\mathbf{a}_m]$. It clearly satisfies property (1). Let $\mathbf{a}_m = (a_1, a_2, \ldots, a_{2n})$. Since there are no $\mathbf{b} \in S_\gamma$ such that $\mathbf{a}_m < \mathbf{b}$, the dual Poincaré-Birkhoff-Witt element $E[\mathbf{a}_m]^*$ cannot be straightened, i.e., all $E^*(\beta_k)$ for with $a_k \neq 0$ are v-commutative. Therefore, by Remark 2.38 we have

$$\begin{aligned}
\sigma(E[\mathbf{a}_m]^*) &= \sigma(v^{-b[\mathbf{a}]} E^*(\beta_1)^{a_1} E^*(\beta_2)^{a_2} \cdots E^*(\beta_{2n})^{a_{2n}}) \\
&= v^{b[\mathbf{a}]} v^{\sum_{k=1}^{2n} a_k N(\beta_k)} E^*(\beta_{2n})^{a_{2n}} \cdots E^*(\beta_2)^{a_2} E^*(\beta_1)^{a_1} \\
&= v^{b[\mathbf{a}]} v^{\sum_{k=1}^{2n} \frac{1}{2} a_k(\beta_k,\beta_k) - \sum_{k=1}^{2n} a_k \|\beta_k\|} \\
&\quad \cdot v^{\sum_{k<l}(\beta_k,\beta_l)} E^*(\beta_1)^{a_1} E^*(\beta_2)^{a_2} \cdots E^*(\beta_{2n})^{a_{2n}} \\
&= v^{\frac{1}{2}(\sum_{k=1}^{2n} \beta_k, \sum_{k=1}^{2n} \beta_k) - \sum_{k=1}^{2n} a_k \|\beta_k\|} E[\mathbf{a}_m]^* \\
&= v^{N(\mathbf{a}_m)} E[\mathbf{a}_m]^*.
\end{aligned}$$

Here $\|\beta_k\|$ denotes the sum of the coefficients of β_k when expanded as a $\mathbb{Z}$-linear combination of simple roots as in Definition 2.50. Hence, the variable $E[\mathbf{a}_m]$ also satisfies property (2).

Now let $1 \leq k < m$ and assume that properties (1) and (2) hold for $\mathbf{a}_{k+1}$, $\mathbf{a}_{k+2}, \ldots, \mathbf{a}_m$. We expand $\sigma(E[\mathbf{a}_k]^*)$ in the dual Poincaré-Birkhoff-Witt basis. Note that by the same argument as above (and ignoring terms of lower order) we see that the coefficient of the leading term $E[\mathbf{a}_k]^*$ is $v^{N(\mathbf{a}_k)}$. Thus, we have

$$\sigma(E[\mathbf{a}_k]^*) = v^{N(\mathbf{a}_k)} E[\mathbf{a}_k]^* + \sum_{k<l\leq m} f_l E[\mathbf{a}_l]^*$$

for some $f_l \in \mathbb{Z}[v, v^{-1}]$. By induction hypothesis every $B[\mathbf{a}_l]^*$ with $l > k$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of $E[\mathbf{a}_l']^*$ with $l' > l$. By solving an upper triangular linear system of equations we see that every $E[\mathbf{a}_l]^*$ with $l > k$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of $B[\mathbf{a}'_l]^*$ with $l' > l$. Hence, we may write

$$\sigma(E[\mathbf{a}_k]^*) = v^{N(\mathbf{a}_k)} E[\mathbf{a}_k]^* + \sum_{k<l\leq m} g_l B[\mathbf{a}_l]^*$$

for some $g_l \in \mathbb{Z}[v, v^{-1}]$. We apply the antiinvolution σ to the last equation:

$$E[\mathbf{a}_k]^* = v^{-N(\mathbf{a}_k)} \sigma(E[\mathbf{a}_k]^*) + \sum_{k<l\leq m} v^{N(\mathbf{a}_l)} \sigma(g_l) B[\mathbf{a}_l]^*.$$

Note that $N(\mathbf{a}_k) = N(\mathbf{a}_l)$ for all l. Comparing coefficients yields $v^{2N(\mathbf{a}_l)}\sigma(g_l) = -g_l$. It follows that $\sigma(v^{-N(\mathbf{a}_l)} g_l) = -v^{-N(\mathbf{a}_l)} g_l$. Thus, we my write $v^{-N(\mathbf{a}_l)} g_l = h_l - \sigma(h_l)$ for some $h_l \in v^{-1}\mathbb{Z}[v^{-1}]$. Now put

$$B[\mathbf{a}_k]^* = E[\mathbf{a}_k]^* + \sum_{k<l\leq m} h_l B[\mathbf{a}_l]^*.$$

It is easy to see that properties (1) and (2) are true for $B[\mathbf{a}_k]^*$ and that $B[\mathbf{a}_k]^*, B[\mathbf{a}_{k+1}]^*, \ldots, B[\mathbf{a}_m]^*$ are linearly independent.

For the uniqueness, suppose that k is some index such that there are variables $B[\mathbf{a}_k]^*_1$ and $B[\mathbf{a}_k]^*_2$ fulfilling the two properties of the theorem. Then their difference $B[\mathbf{a}_k]^*_1 - B[\mathbf{a}_k]^*_2 \in \bigoplus_{l>k} v^{-1}\mathbb{Z}[v^{-1}]B[\mathbf{a}_l]^*$. Application of σ and multiplication with $v^{-N(\mathbf{a}_k)}$ afterwards yields $B[\mathbf{a}_k]^*_1 - B[\mathbf{a}_k]^*_2 \in \bigoplus_{l>k} v\mathbb{Z}[v]B[\mathbf{a}_l]^*$, so $B[\mathbf{a}_k]^*_1 = B[\mathbf{a}_k]^*_2$. □

Remark 2.56. It is known that the dual of Lusztig's canonical basis under Kashiwara's bilinear form obeys the two properties of Theorem 2.55, compare Leclerc [36, Proposition 39]. By uniqueness, the set $\mathcal{B}^* = \{B[\mathbf{a}]^* \colon \mathbf{a} \in \mathbb{N}^{2n}\}$ is the dual of Lusztig's canonical basis, or the *dual canonical basis* for short.

We call $E[\mathbf{a}]^*$ (for $\mathbf{a} \in \mathbb{N}^{2n}$) the *leading term* in the expansion of $B[\mathbf{a}]^*$ in the Poincaré-Birkhoff-Witt basis. In what follows we use the convention $z_0 = z_{n+1} = 1$. Prominent elements in $\mathcal{B}^*$ are

$$\begin{aligned} p_i &= y_i z_i - v^{-1} z_{i-1} z_{i+1}, && \text{for } i \text{ odd with } 1 \leq i \leq n, \\ p_i &= z_i y_i - v^{-1} z_{i-1} z_{i+1}, && \text{for } i \text{ even with } 2 \leq i \leq n-1. \end{aligned}$$

The first property of Theorem 2.55 is obvious and the second follows easily from a calculation using the straightening relations. The variables are v-deformations of the δ-functions of the $\mathcal{C}_M$-projective rigid Λ-modules from Section 2.3. The non-deformed δ-function associated with these modules are frozen cluster variables in Geiß-Leclerc-Schröer's cluster algebra $\mathcal{A}(\mathcal{C}_M)$, compare Section 2.5.

Lemma 2.57. For every pair (i, k) of natural numbers such that $1 \leq i \leq n$ and $1 \leq k \leq 2n$ the elements $p_i, E^*(\beta_k) \in U_v^+(w)$ are v-commutative, i.e., there is an integer a such that $p_i E^*(\beta_k) = v^a E^*(\beta_k) p_i$.

Proof. Let i be an odd integer such that $1 \leq i \leq n$. Note that

$$p_i = y_i z_i - v^{-1} z_{i-1} z_{i+1} = v^{-(|z_i|,|y_i|)} z_i y_i - v z_{i-1} z_{i+1}$$

by property (2) of Theorem 2.55. From the straightening relations of Lemma 2.37 it is clear that p_i commutes with every y_j with $|j-i| \geq 4$ and with every z_j with $|j-i| \geq 3$. If $i \geq 5$, then $y_{i-3}p_i = y_{i-3}y_iz_i - v^{-1}y_{i-3}z_{i-1}z_{i+1} = vy_iz_iy_{i-3} - z_{i-1}z_{i+1}y_{i-3} = vp_iy_{i-3}$. Now assume that $i \geq 3$. We have $y_{i-2}p_i = y_{i-2}y_iz_i - v^{-1}y_{i-2}z_{i-1}z_{i+1} = v^{-1}y_iz_iy_{i-2} - v^{-2}z_{i-1}z_{i+1}y_{i-2} = v^{-1}p_iy_{i-2}$. Furthermore, the equation $z_{i-2}p_i = z_{i-2}y_iz_i - v^{-1}z_{i-2}z_{i-1}z_{i+1} = vy_iz_iz_{i-2} - z_{i-1}z_{i+1}z_{i-2} = vp_iz_{i-2}$ holds. The calculation

$$\begin{aligned} y_{i-1}p_i &= y_{i-1}y_iz_i - v^{-1}y_{i-1}z_{i-1}z_{i+1} \\ &= v^{(|y_{i-1}|,|y_i|)}\Big(y_iy_{i-1} + (v-v^{-1})z_{i+1}z_{i-2}\Big)z_i \\ &\quad - v^{-1}\Big(z_{i-1}y_{i-1} + (v-v^{-1})z_iz_{i-2}\Big)z_{i+1} \\ &= v^{(|y_{i-1}|,|y_i|)}(vy_iz_iy_{i-1} - z_{i-1}z_{i+1}y_{i-1}) = v^{1+(|y_{i-1}|,|y_i|)}p_iy_{i-1} \end{aligned}$$

shows that p_i also v-commutes with y_{i-1}. It also v-commutes with z_{i-1} as the calculation shows: $z_{i-1}p_i = z_{i-1}y_iz_i - v^{-1}z_{i-1}^2z_{i+1} = y_iz_iz_{i-1} - v^{-1}z_{i-1}z_{i+1}z_{i-1} = p_iz_{i-1}$. Now assume that $i \geq 1$.The following equation is true:

$$\begin{aligned} y_ip_i &= y_i^2z_i - v^{-1}y_iz_{i-1}z_{i+1} \\ &= y_i\Big(v^{-(|y_i|,|z_i|)}z_iy_i + (v^{-1}-v)z_{i-1}z_{i+1}\Big) - v^{-1}y_iz_{i-1}z_{i+1} \\ &= v^{-(|y_i|,|z_i|)}y_iz_iz_i - vy_iz_{i-1}z_{i+1} \\ &= v^{-(|y_i|,|z_i|)}(y_iz_i - v^{-1}z_{i-1}z_{i+1})y_i = v^{-(|y_i|,|z_i|)}p_iy_i. \end{aligned}$$

Finally we see that $z_ip_i = z_iy_iz_i - v^{-1}z_iz_{i-1}z_{i+1} = v^{(|y_i|,|z_i|)}(v^{-(|y_i|,|z_i|)}z_iy_i - vz_{i-1}z_{i+1})z_i = v^{(|y_i|,|z_i|)}p_iz_i$. The v-commutativity relations of p_i is elements with index $j > i$ are proved in the same way.

The case of even i can be handled with similar arguments. □

Remark 2.58.

1. The v-commutativity relations of Lemma 2.57 will crucial for the construction of the initial quantum seed of $\mathcal{A}(w)_v$. Another verification of v-commutativity relations for the initial seed is due to Kimura [33, Section 6].
2. Multiplicative properties of (dual) canonical basis elements have also been studied by Reineke [46].
3. As observed by Leclerc, the v-deformations of the δ-functions of $\mathcal{C}_M$-projective rigid Λ-modules also v-commutate with the generators of $U_v^+(w)$ in the Kronecker cases for w of length 4, see the author [34, Section 4.1].
4. A consideration of the leading terms of the two occuring variables in Lemma 2.57 is sufficient to determine the integer a.

Remark 2.59. The techniques in this section work with the same proofs for the case $U_v^+(w')$ as well. We use a similar notation $y_i', z_i' \in U_v^+(w')$ for the elements dual to $u_i', v_i', w_i', x_i' \in U_v^+(w)$ (for appropriate indices i). The straightening relations for the dual variables can be computed using the same methods.

2.13 The quantum cluster algebra structure induced by the dual canonical basis

In this section we are going to prove that the integral form $\bigoplus_{\mathbf{a}\in\mathbb{N}^{2n}} \mathbb{Z}[v^{\pm\frac{1}{2}}]E[\mathbf{a}]^*$ is a quantum cluster algebra in the sense of Bereinstein-Zelevinsky [6]. The corresponding non-quantized cluster algebra is Geiß-Leclerc-Schröer's [21] cluster algebra $\mathcal{A}(w)$.

Natural Quantum cluster algebra structures have only been observed in very few cases, see for example Grabowski-Launois [20], Rupel [50], and the author [34]. For a study of bases of quantum cluster algebras of type $\tilde{A}_1^{(1)}$ see Ding-Xu [12].

Definition 2.60. For $1 \leq i \leq j \leq n$ define $\Delta^v_{i,j} \in \mathcal{B}^*$ to be the dual canonical basis element with leading term $\prod_{i\leq r\leq j, r \text{ odd}} y_r \prod_{i\leq r\leq j,\ r\text{even}} y_r$.

Remark 2.61. We provide some examples of elements in the dual canonical basis $\mathcal{B}^*$ of the form $\Delta^v_{i,j}$ with $1 \leq i \leq j \leq n$. We focus on examples where the interval $[i,j]$ is small, i.e., $j \leq i+2$. First of all, we clearly have $\Delta^v_{i,i} = y_i$ for all i. Furthermore, an elementary calculation using the straightening relations shows that: $\Delta^v_{1,2} = y_1y_2 - v^{-1}z_3 = vy_2y_1 - vz_3$, $\Delta^v_{n-1,n} = y_ny_{n-1} - v^{-1}z_{n-2} = vy_{n-1}y_n - vz_{n-2}$, and that for $2 \leq i \leq n-2$

$$\Delta^v_{i,i+1} = \begin{cases} y_iy_{i+1} - v^{-1}z_{i-1}z_{i+2} = y_{i+1}y_i - vz_{i+2}z_{i-1}, & \text{if } i \text{ is odd;} \\ y_{i+1}y_i - v^{-1}z_{i+2}z_{i-1} = y_iy_{i+1} - vz_{i-1}z_{i+2}, & \text{if } i \text{ is even.} \end{cases}$$

Recall the convention $z_0 = z_{n+1} = 1$. The formulae simplify to $\Delta^v_{i,i+1} = y_iy_{i+1} - v^{-1}z_{i-1}z_{i+2}$ for odd i and $\Delta^v_{i,i+1} = y_{i+1}y_i - v^{-1}z_{i+2}z_{i-1}$ for even i. With the same convention we can compute:

$$\begin{aligned}\Delta_{1,3} &= y_1y_3y_2 - v^{-1}y_1z_4z_1 - v^{-1}y_3z_3 + v^{-2}z_2z_4 \\ &= vy_2y_1y_3 - vz_3y_3 - v^2z_1z_4y_1 + v^2z_2z_4,\end{aligned}$$

$$\begin{aligned}\Delta_{n-2,n} &= y_ny_{n-2}y_{n-1} - v^{-1}y_nz_{n-3}z_n - v^{-1}y_{n-2}z_{n-2} + v^{-2}z_{n-1}z_{n-3} \\ &= vy_{n-1}y_ny_{n-2} - vz_{n-2}y_{n-2} - v^2z_nz_{n-3}y_n + v^2z_{n-1}z_{n-3}.\end{aligned}$$

For odd i such that $3 \leq i \leq n-4$ we have:

$$\begin{aligned}\Delta_{i,i+2} &= y_iy_{i+2}y_{i+1} - v^{-1}y_iz_{i+3}z_i - v^{-1}y_{i+2}z_{i-1}z_{i+2} + v^{-2}z_{i-1}z_{i+1}z_{i+3} \\ &= y_{i+1}y_{i+2}y_i - vz_iz_{i+3}y_i - vz_{i+2}z_{i-1}y_{i+2} + v^2z_{i-1}z_{i+1}z_{i+3}.\end{aligned}$$

For odd i such that $3 \leq i \leq n-4$ we have:

$$\begin{aligned}\Delta_{i,i+2} &= y_{i+1}y_iy_{i+2} - v^{-1}z_iz_{i+3}y_i - v^{-1}z_{i+2}z_{i-1}y_{i+2} + v^2z_{i-1}z_{i+1}z_{i+3} \\ &= y_iy_{i+2}y_{i+1} - vy_iz_{i+3}z_i - vy_{i+2}z_{i+2}z_{i-1} + v^2z_{i-1}z_{i+1}z_{i+3}.\end{aligned}$$

Remark 2.62. In what follows we prove recursions for the $\Delta^v_{i,j}$ with $1 \leq i \leq j \leq n$. The formulae turn out to be quantized versions of the formulae for the $\Delta_{i,j}$ from Section 2.6. The quantized recursions will be crucial for the verification of the quantum cluster algebra struture on $U_v^+(w)$ in the next section.

To formulate the recursion effectively the following definitions are helpful. First of all we introduce the convention $\Delta^v_{i,i-1} = 1$ for all i. Furthermore, we give the following two definitions.

Definition 2.63. For $1 \leq i, j \leq n$ put

$$s_{i,j} = \sum_{i \leq s \leq j} |y_s|; \quad o_{i,j} = \sum_{\substack{i \leq s \leq j \\ s \text{ odd}}} |y_s|; \quad e_{i,j} = \sum_{\substack{i \leq s \leq j \\ s \text{ even}}} |y_s|.$$

Definition 2.64. For all i, j with $1 \leq i, j \leq n$ and $j - i \geq 2$ define

$$A_{i,j} = \begin{cases} -1, & \text{if } j - i \leq 3; \\ -2, & \text{if } j - i \geq 4. \end{cases}$$

Proposition 2.65. For all pairs (i, j) of integers such that $1 \leq i \leq j \leq n$ the following equation is true:

$$N(s_{i,j}) - N(s_{i,j-1}) - N(|y_j|) = \begin{cases} (|y_j|, o_{i,j-1}), & \text{if } j \text{ is even;} \\ (|y_j|, e_{i,j-1}), & \text{if } j \text{ is odd.} \end{cases}$$

Proof. The proposition follows easily from the observation $\frac{1}{2}(s_{i,j}, s_{i,j}) = \frac{1}{2}(e_{i,j} + o_{i,j}, e_{i,j} + o_{i,j}) = (j - i + 1) + (e_{i,j}, o_{i,j})$. □

Proposition 2.66. Let i, j be integers such that $1 \leq i \leq j \leq n$ and $j \geq i + 2$. The following equation holds:

$$N(s_{i,j}) - N(s_{i,j-3}) - N(|p_{j-2}|) - N(|z_{j+1}|) = \begin{cases} (|y_j| + |y_{j-2}|, o_{i,j-3}) + (|y_{j-1}|, e_{i,j-4}), & \text{if } j \text{ is even;} \\ (|y_j| + |y_{j-2}|, e_{i,j-3}) + (|y_{j-1}|, o_{i,j-4}), & \text{if } j \text{ is odd.} \end{cases}$$

Proof. The elements $y_j, y_{j-1}, z_{j+1}, z_{j-2} \in Q^+$ satisfy the equation $|y_j| + |y_{j-1}| = |z_{j+1}| + |z_{j-2}|$, compare Figure 18. With this fact the proposition follows just as above from the observation $\frac{1}{2}(s_{i,j}, s_{i,j}) = \frac{1}{2}(e_{i,j} + o_{i,j}, e_{i,j} + o_{i,j}) = (j - i + 1) + (e_{i,j}, o_{i,j})$. □

Remark 2.67. By definition, the leading term of $\Delta^v_{i,j}$ (where $1 \leq i \leq j \leq n$) is $\prod_{i \leq r \leq j, r \text{ odd}} y_r \prod_{i \leq r \leq j,\ r \text{even}} y_r$. Therefore, $\Delta^v_{i,j}$ is a $\mathbb{Z}[v, v^{-1}]$-linear combination of terms of the form

$$\prod_{\substack{i \leq r \leq j \\ r \text{ odd}}} y_r^{1-a_r} \prod_{\substack{i-1 \leq r \leq j+1 \\ r \text{ even}}} z_r^{a_{r+1}+a_{r-1}-a_r} \prod_{\substack{i-1 \leq r \leq j+1 \\ r \text{ odd}}} z_r^{a_{r+1}+a_{r-1}-a_r} \prod_{\substack{i \leq r \leq j \\ r \text{ even}}} y_r^{1-a_r}$$

for some $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$ such that $a_1 = a_2 = \ldots = a_{i-1} = 0$, $a_{j+1} = a_{j+1} = \ldots = a_n = 0$, and $a_{r+1} + a_{r-1} - a_r \geq 0$ for all r. We denote this term by $\Delta^v_{i,j}[\mathbf{a}]$.

We know that p_{j+1} commutes, up to a power of v, with every $\Delta^v_{i,j}[\mathbf{a}]$. The following proposition shows that much more is true: The commutation exponent only depends on the pair (i, j), but not on $\mathbf{a} \in \{0, 1\}^n$.

Proposition 2.68. Let i, j be integers such that $1 \leq i \leq j \leq n - 1$. The variables $\Delta^v_{i,j}[\mathbf{a}]$ and p_{j+1} are v-commutative. More precisely: If j is even, then $\Delta^v_{i,j}[\mathbf{a}]p_{j+1} = v^{(|y_{j+1}|, e_{i,j}) + (|z_{j+1}|, e_{i,j}) - (|z_{j+1}|, o_{i,j-1})} p_{j+1} \Delta^v_{i,j}[\mathbf{a}]$, and if j is odd, then $\Delta^v_{i,j}[\mathbf{a}]p_{j+1} = v^{-(|y_{j+1}|, o_{i,j}) + (|z_{j+1}|, e_{i,j}) - (|z_{j+1}|, o_{i,j-1})} p_{j+1} \Delta^v_{i,j}[\mathbf{a}]$.

Proof. Assume that j be even. By Lemma 2.57 we know that p_{j+1} commutes, up to a power of v, with every $\Delta^v_{i,j}[\mathbf{a}]$. To determine the appropriate power of v, we compare the leading terms. The leading term of p_{j+1} is $y_{j+1}z_{j+1}$. By Remark 2.38 we see that

$$\begin{aligned}\Delta^v_{i,j}[\mathbf{a}]p_{j+1} =& v^{a_j(|y_{j+1}|,|z_{j+1}|+|z_{j-1}|-|y_j|-|-z_j|)}\\ &\cdot v^{a_{j-1}(|y_{j+1}|,|z_j|+|z_{j-2}|-|z_{j-1}|)}\\ &\cdot v^{a_{j-2}(|y_{j+1}|,|z_{j-1}|-|z_{j-2}|)}\\ &\cdot v^{a_j(|z_{j+1}|,|y_j|-|z_j|)}\\ &\cdot v^{a_{j-1}(|z_{j+1}|,|y_{j-1}|-|z_j|+|z_{j-2}|)}\\ &\cdot v^{(|y_{j+1}|,e_{i,j})+(|z_{j+1}|,e_{i,j})-(|z_{j+1}|,o_{i,j-1})}p_{j+1}\Delta^v_{i,j}[\mathbf{a}].\end{aligned}$$

Note that $(|y_k|,|y_{k-2}|) = (|y_k|,|y_{k-4}|) = 0$ for all k, $(|y_k|,|y_l|) = 0$ for $|k-l| \geq 4$, $(|y_k|,|z_l|) = 0$ for $|k-l| \geq 3$, and that $|y_j| + |z_j| = |z_{j+1}| + |z_{j-1}|$ for all k. Furthermore, notice that $(|z_{k+1}|,|y_k|) = (|z_{k+1},|z_k|)$ for all k. Hence, we have $\Delta^v_{i,j}[\mathbf{a}]p_{j+1} = v^{(|y_{j+1}|,e_{i,j})+(|z_{j+1}|,e_{i,j})-(|z_{j+1}|,o_{i,j-1})}p_{j+1}\Delta^v_{i,j}[\mathbf{a}]$.

By a similar argument we can show that for odd j the equation $\Delta^v_{i,j}[\mathbf{a}]p_{j+1} = v^{-(|y_{j+1}|,o_{i,j})+(|z_{j+1}|,e_{i,j})-(|z_{j+1}|,o_{i,j-1})}p_{j+1}\Delta^v_{i,j}[\mathbf{a}]$ holds. □

The independence of the commutation exponent of $\mathbf{a} \in \{0,1\}^n$ implies the corollary.

Corollary 2.69. Let i,j be integers as above. Then the variables $\Delta^v_{i,j}$ and p_{j+1} are v-commutative. More precisely: If j is even, then

$$\Delta^v_{i,j}p_{j+1} = v^{(|y_{j+1}|,e_{i,j})+(|z_{j+1}|,e_{i,j})-(|z_{j+1}|,o_{i,j-1})}p_{j+1}\Delta^v_{i,j},$$

and if j is odd, then

$$\Delta^v_{i,j}p_{j+1} = v^{-(|y_{j+1}|,o_{i,j})+(|z_{j+1}|,e_{i,j})-(|z_{j+1}|,o_{i,j-1})}p_{j+1}\Delta^v_{i,j}.$$

Having established these conventions, definitions and proposistions, we are now able to formulate a theorem that provides a recursion for the $\Delta^v_{i,j}$ and a quantized version of the exchange relation. These are the parts (a) and (b) of Theorem 2.70. For a proof of the theorem, we proceed by induction. For a functioning induction step we include also the v-commutator relations in part (c) and (d).

Theorem 2.70. Let i,j be integers such that $1 \leq i,j \leq n$ and $j-i \geq 2$.

(a) The dual canonical basis element $\Delta^v_{i,j}$ can be computed recursively from elements $\Delta^v_{i,j'}$ with $j' < j$. More precisely, if j is even, then we have:

$$\begin{aligned}\Delta^v_{i,j} &= \Delta^v_{i,j-1}y_j - v^{A_{i,j}}\Delta^v_{i,j-3}p_{j-2}z_{j+1}\\ &= v^{-(|y_j|,o_{i,j-1})}y_j\Delta^v_{i,j-1}\\ &\quad - v^{-A_{i,j}-(|y_j|+|y_{j-2}|,o_{i,j-3})-(|y_{j-1}|,e_{i,j-4})}z_{j+1}p_{j-2}\Delta^v_{i,j-3}.\end{aligned}$$

If j is odd, the we have:

$$\begin{aligned}\Delta^v_{i,j} &= y_j\Delta^v_{i,j-1} - v^{A_{i,j}}z_{j+1}p_{j-2}\Delta^v_{i,j-3}\\ &= v^{-(|y_j|,e_{i,j-1})}\Delta^v_{i,j-1}y_j\\ &\quad - v^{-A_{i,j}-(|y_j|+|y_{j-2}|,e_{i,j-3})-(|y_{j-1}|,o_{i,j-4})}\Delta^v_{i,j-3}p_{j-2}z_{j+1}.\end{aligned}$$

(b) Furthermore, the following quantum cluster exchange relation holds:

$$\Delta^v_{i,j} z_j = \begin{cases} \Delta^v_{i,j-1} p_j + v^{1-(|y_{j-1}|, e_{i,j-2})} \Delta^v_{i,j-2} p_{j-1} z_{j+1}, & \text{if } j \text{ is even;} \\ v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} p_j + v^{-1-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-2} p_{j-1} z_{j+1}, & \text{if } j \text{ is odd.} \end{cases}$$

(c) If $j+1 \leq n$, then the following v-commutator relation holds. If j is even, then

$$y_{j+1} \Delta^v_{i,j} = v^{-(|y_{j+1}|, e_{i,j})} \Delta_{i,j} y_{j+1} + v^{-1-(|y_{j-1}|, e_{i,j-2})} (v^{-1} - v) \Delta^v_{i,j-2} p_{j-1} z_{j+2}.$$

If j is odd, then

$$y_{j+1} \Delta^v_{i,j} = v \Delta^v_{i,j} y_{j+1} + v^{1-(|y_j|, e_{i,j-3})} (v - v^{-1}) \Delta^v_{i,j-2} p_{j-1} z_{j+2}.$$

(d) If $j+2 \leq n$, then $\Delta^v_{i,j}$ and y_{j+2} are v-commutative. More precisely, if j is even, then $\Delta^v_{i,j} y_{j+2} = v^{-1} y_{j+2} \Delta^v_{i,j}$, and if j is odd, then $\Delta^v_{i,j} y_{j+2} = v y_{j+2} \Delta^v_{i,j}$.

Proof. We proceed by induction on $j - i$. Using the explicit formulae provided by Remark 2.61 it is easy to see that Theorem 2.70 is true for $j - i = 2$. Now let $j - i \geq 3$ and assume that Theorem 2.70 is true for all smaller values of $j - i$. We distinguish two cases.

Assume that j is even. It follows that $4 \leq j \leq n - 1$. Put

$$A = \Delta^v_{i,j-1} y_j - v^{A_{i,j}} \Delta^v_{i,j-3} p_{j-2} z_{j+1}.$$

We have to prove that $A = \Delta^v_{i,j}$, i.e., we have to show that A satisfies properties (1) and (2) of Theorem 2.55. First of all, we verify property (1). We expand the dual canonical basis elements $\Delta^v_{i,j-1}, \Delta^v_{i,j-3}$ according to Remark 2.67. We see that

$$\Delta^v_{i,j-1} = \sum_{\mathbf{a}} f_{\mathbf{a}} v^{\sum -\binom{a_{r+1}+a_{r-1}-a_r}{2}} \prod_{\substack{i \leq r \leq j-3 \\ r \text{ odd}}} y_r^{1-a_r} \prod_{\substack{i-1 \leq r \leq j \\ r \text{ even}}} z_r^{a_{r+1}+a_{r-1}-a_r} \cdot \prod_{\substack{i-1 \leq r \leq j \\ r \text{ odd}}} z_r^{a_{r+1}+a_{r-1}-a_r} \prod_{\substack{i \leq r \leq j-1 \\ r \text{ even}}} y_r^{1-a_r}$$

where the sum is taken over all the admissible sequences $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ as in Remark 2.67 and $f_{\mathbf{a}} \in v^{-1}\mathbb{Z}[v^{-1}]$ except for $f_0 = 1$. Here, we have used that $-\binom{1-a_r}{2} = 0$ for all terms a_r in such a sequence. It is clear that $\Delta^v_{i,j-1} y_j - \Delta^v_{i,j-1}[0] y_j \in \bigoplus_{\mathbf{b} \in S(\mathbf{a})} v^{-1}\mathbb{Z}[v^{-1}] E[\mathbf{b}]^*$ and that $\Delta^v_{i,j-1}[0] y_j$ is the dual Poincaré-Birkhoff-Witt basis element from Definition 2.60 that serves as leading term.

Furthermore, we have

$$\Delta^v_{i,j-3} = \sum_{\mathbf{a}} g_{\mathbf{a}} v^{\sum -\binom{a_{r+1}+a_{r-1}-a_r}{2}} \prod_{\substack{i \leq r \leq j-3 \\ r \text{ odd}}} y_r^{1-a_r} \prod_{\substack{i-1 \leq r \leq j-2 \\ r \text{ even}}} z_r^{a_{r+1}+a_{r-1}-a_r} \cdot \prod_{\substack{i-1 \leq r \leq j-3 \\ r \text{ odd}}} z_r^{a_{r+1}+a_{r-1}-a_r} \prod_{\substack{i \leq r \leq j-4 \\ r \text{ even}}} y_r^{1-a_r}$$

where the sum is taken over all the admissible sequences $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ as in Remark 2.67 and $g_{\mathbf{a}} \in v^{-1}\mathbb{Z}[v^{-1}]$ except for $g_0 = 1$. Now we consider

$v^{A_{i,j}}\Delta^v_{i,j-3}p_{j-2}z_{j+1}$. Note that the generator z_{j+1} commutes with all occuring terms in this expansion. For p_{j-2}, by Lemma 2.57 the v-commutativity relation

$$\left(\prod_{\substack{i\le r\le j-4\\ r \text{ even}}} y_r^{1-a_r}\right)p_{j-2} = v^{(1-a_{j-4})(|z_{j-2}|,|y_{j-4}|)}p_{j-2}\left(\prod_{\substack{i\le r\le j-4\\ r \text{ even}}} y_r^{1-a_r}\right)$$

holds. Hence, in each summand we can transfer p_{j-2} to the left of the product. We get a factor $v^{(1-a_{j-4})}$ since $(|z_{j-2}|,|y_{j-4}|)=1$ for all j. We concentrate on a single summand. Write $p_{j-2}=z_{j-2}y_{j-2}-v^{-1}z_{j-3}z_{j-1}$. This decomposition splits the sum into two parts. Consider the summand coming from $z_{j-2}y_{j-2}$. To write this term in the dual PBW basis we have to tranfer z_{j-2} to the left of the product of the odd z_r. We get a factor $v^{a_{j-4}-a_{j-3}}$. Now all the monomials are in the right order. The generators z_{j+1} and y_{j-2} do not occur in the expansion of $\Delta^v_{i,j-3}$ in the dual PBW basis, but z_{j-2} may. In the summands where z_{j-2} occurs, we have increased the exponent from 1 to 2. So all coefficients in the dual PBW expansion of these summands have the form

$$g_{\mathbf{a}}v^{A_{i,j}}v^{1-a_{j-4}}v^{a_{j-4}-a_{j-3}}v^{\binom{1+a_{j-3}}{2}}.$$

Note that $A_{i,j}\le -1$ and that $\binom{1+a_{j-3}}{2}-a_{j-3}=0$ for $a_{j-3}\in\{0,1\}$. Hence, all coefficients are in $v^{-1}\mathbb{Z}[v^{-1}]$. Now consider the summand coming from $z_{j-3}z_{j-1}$. The monomials are already in the right order. We may have increased the exponent of z_{j-3} from 1 to 2. It is easy to see that all coefficients are in $v^{-1}\mathbb{Z}[v^{-1}]$. This shows that A satisfies property (1).

To conclude to $A=\Delta^v_{i,j}$, we have to verify property (2) of Theorem 2.70. We use parts (c) and (d) of the induction hypothesis for the for the pair $(i,j-1)$ to obtain the following equation. Note that if $j-i\le 3$, then $(|y_{j-1}|,|e_{i,j-4}|)=0$, and if $j-i\ge 4$, then $(|y_{j-1}|,|e_{i,j-4}|)=1$, therefore

$$\begin{aligned}A &= \Delta^v_{i,j-1}y_j - v^{A_{i,j}}\Delta^v_{i,j-3}p_{j-2}z_{j+1}\\ &= v^{-1}y_j\Delta^v_{i,j-1}+v^{-(|y_{j-1}|,|e_{i,j-4}|)}(v^{-1}-v)\Delta^v_{i,j-3}p_{j-2}z_{j+1}\\ &\quad - v^{A_{i,j}}\Delta^v_{i,j-3}p_{j-2}z_{j+1}\\ &= v^{-1}y_j\Delta^v_{i,j-1}+v^{1-(|y_{j-1}|,|e_{i,j-4}|)}\Delta^v_{i,j-3}p_{j-2}z_{j+1}.\end{aligned}$$

It is easy to see that z_{j+1} commutes with p_{j-2}. Furthermore, it commutes with $\Delta^v_{i,j-3}$ since it commutes with every $E^*(\beta_k)$ in every summand in the dual PBW expansion of $\Delta^v_{i,j-3}$ according to Remark 2.67. Lemma 2.57 implies

$$\Delta^v_{i,j-3}p_{j-2}=v^{-(|y_{j-2}|,o_{i,j-3})-(|z_{j-2}|,o_{i,j-3})+(|z_{j-2}|,e_{i,j-4})}p_{j-2}\Delta^v_{i,j-3}.$$

The assumptions $j-i\ge 3$ and $j<n$ imply $(|y_j|,o_{i,j-1})=(|y_j|,o_{i,j-3})=1$ and $(|z_{j-2}|,o_{i,j-3})=1$. The observation $(|z_{j-2}|,e_{i,j-4})=0$ for $j-i\le 3$, and $(|z_{j-2}|,e_{i,j-4})=1$ for $j-i\ge 4$ yields $(|z_{j-2}|,e_{i,j-4})=-A_{i,j}-1$. It follows that

$$\begin{aligned}A &= v^{-(|y_j|,o_{i,j-1})}y_j\Delta^v_{i,j-1}\\ &\quad - v^{-A_{i,j}-(|y_j|+|y_{j-2}|,o_{i,j-3})-(|y_{j-1}|,e_{i,j-4})}z_{j+1}p_{j-2}\Delta^v_{i,j-3}.\end{aligned}$$

By Propositions 2.65 and 2.66 the last equation is equivalent to property (2). Thus $A=\Delta^v_{i,j}$. Incidentally, we have verified part (a) for the pair (i,j).

Now we prove that the new defined $\Delta^v_{i,j}$ satisfies the quantized cluster recursion (b) of Theorem 2.70 using the induction hypothesis for part (b):

$$\begin{aligned}\Delta^v_{i,j} z_j &= \Delta^v_{i,j-1}(p_j + v z_{j-1} z_{j+1}) - v^{A_{i,j}} \Delta^v_{i,j-3} p_{j-2} z_{j+1} z_j \\ &= \Delta^v_{i,j-1} p_j + \Big(v\Delta^v_{i,j-1} z_{j-1} - v^{1+A_{i,j}} \Delta^v_{i,j-3} p_{j-2} z_j \Big) z_{j+1} \\ &= \Delta^v_{i,j-1} p_j + \Big(v\Delta^v_{i,j-1} z_{j-1} - v^{-(|y_{j-1}|, e_{i,j-2})} \Delta^v_{i,j-3} p_{j-2} z_j \Big) z_{j+1} \\ &= \Delta^v_{i,j-1} p_j + v^{1-(|y_{j-1}|, e_{i,j-2})} \Delta^v_{i,j-2} p_{j-1} z_{j+1}.\end{aligned}$$

Now we prove that the new defined $\Delta^v_{i,j}$ satisfies property (c) of Theorem 2.70. By induction hypothesis we know that part (c) is true for the pair $(i, j-1)$. Note that $(|y_{j+1}|, |z_{j+1}|) = (|y_j|, |y_{j+1}|) = -\delta_{j,n-1}$. The following calculation verifies part (c) of the theorem:

$$\begin{aligned}y_{j+1}\Delta^v_{i,j} &= y_{j+1}\Delta^v_{i,j-1} y_j - v^{A_{i,j}} y_{j+1} \Delta^v_{i,j-3} p_{j-2} z_{j+1} \\ &= v^{-1}\Delta^v_{i,j-1}\Big(v^{-(|y_j|,|y_{j+1}|)} y_j y_{j+1} + (v^{-1} - v) z_{j+2} z_{j-1} \Big) \\ &\quad - v^{A_{i,j}-1} \Delta^v_{i,j-3} p_{j-2} \Big(v^{-(|y_{j+1}|,|z_{j+1}|)} z_{j+1} y_{j+1} + (v^{-1} - v) z_j z_{j+2} \Big) \\ &= v^{-(|y_{j+1}|, e_{i,j})} \Delta_{i,j} y_{j+1} \\ &\quad + v^{-1}(v^{-1} - v) \Big(\Delta^v_{i,j-1} z_{j-1} - v^{A_{i,j}} \Delta^v_{i,j-3} p_{j-2} z_j \Big) z_{j+2} \\ &= v^{-(|y_{j+1}|, e_{i,j})} \Delta_{i,j} y_{j+1} + (v^{-1} - v) v^{-1-(|y_{j-1}|, e_{i,j-2})} \Delta^v_{i,j-2} p_{j-1} z_{j+1}.\end{aligned}$$

It remains to verify part (d). Note that in each monomial of the dual PBW expansion of $\Delta^v_{i,j-1}$ either y_{j-1} or z_j occurs (depending on whether a_{j-1} in Remark 2.67 is zero or one). The variable y_{j+2} commutes with all other terms. Thus, we have $\Delta^v_{i,j-1} y_{j+2} = v^{-1} y_{j+2} \Delta^v_{i,j-1}$ It follows that $\Delta_{i,j} y_{j+2} = y_j \Delta^v_{i,j-1} y_{j+2} - v^{A_{i,j}} z_{j+1} p_{j-2} \Delta^v_{i,j-3} y_{j+2} = v^{-1} y_j y_{j+2\Delta^v_{i,j-1}} - v^{-1+A_{i,j}} y_{j+2} z_{j+1} p_{j-2} \Delta^v_{i,j-3} = v^{-1} y_{j+2} \Delta_{i,j}$.

Now assume that j is odd. Put $A = y_j \Delta^v_{i,j-1} - v^{A_{i,j}} z_{j+1} p_{j-2} \Delta^v_{i,j-3}$. We prove that $A = \Delta^v_{i,j}$. By the same arguments as above A fulfills property (1) of the theorem. To conclude to $A = \Delta^v_{i,j}$, we have to verify property (2) of Theorem 2.70. We use parts (c) and (d) of the induction hypothesis for the pair $(i, j-1)$ to obtain the following equation:

$$\begin{aligned}A &= y_j \Delta^v_{i,j-1} - v^{A_{i,j}} z_{j+1} p_{j-2} \Delta^v_{i,j-3} \\ &= v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} y_j + v^{-1-(|y_{j-2}|, e_{i,j-3})} (v^{-1} - v) \Delta^v_{i,j-3} p_{j-2} z_{j+1} \\ &\quad - v^{A_{i,j}} v^{-(|y_{j-2}|, e_{i,j-3}) - (|z_{j-2}|, e_{i,j-3}) + (|z_{j-2}|, o_{i,j-4})} \Delta^v_{i,j-3} p_{j-2} z_{j+1}.\end{aligned}$$

Note that $(|z_{j-2}|, e_{i,j-3}) = 1$, and that $(|z_{j-2}|, o_{i,j-4}) = 0$ for $j - i \le 3$ and $(|z_{j-2}|, o_{i,j-4}) = 1$ for $j - i \ge 4$. Hence, $A_{i,j} - (|z_{j-2}|, o_{i,j-4})(|z_{j-2}|, o_{i,j-4}) = -2$ yielding

$$A = v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} y_j - v^{-(|y_{j-2}|, e_{i,j-3})} \Delta^v_{i,j-3} p_{j-2} z_{j+1}.$$

The equation $-A_{i,j} - (|y_j|, e_{i,j-3}) - (|y_{j-1}|, o_{i,j-4})$ finishes the proof the second equation of part (a). By Propositions 2.65 and 2.66 the last equation is equivalent to property (2). Hence, $A = \Delta^v_{i,j}$.

Now we prove that the new defined $\Delta^v_{i,j}$ satisfies property (b) of Theorem 2.70. First of all assume that $j < n$. We obtain:

$$\begin{aligned}\Delta^v_{i,j} z_j &= v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1}(p_j + v^{-1} z_{j-1} z_{j+1}) \\ &\quad - v^{-(|y_{j-2}|, e_{i,j-3})} \Delta^v_{i,j-3} p_{j-2} z_{j+1} z_{j-1}. \\ &= v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} p_j \\ &\quad + \Big(v^{-1-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} z_{j-1} - v^{-1-(|y_{j-2}|, e_{i,j-3})} \Delta^v_{i,j-3} p_{j-2} z_{j-1} \Big) z_{j+1} \\ &= v^{-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-1} p_j + v^{-1-(|y_j|, e_{i,j-1})} \Delta^v_{i,j-2} p_{j-1} z_{j+1}.\end{aligned}$$

Here we used he fact that $(|y_j|, e_{i,j-1}) = 1$ to adopt the induction hypothesis. For $j = n$ we have $(|y_j|, e_{i,j-1}) = 0$, but this defect is compensated by the relation $z_{j-1} z_{j+1} = z_{j+1} z_{j-1}$ (due to $z_{j+1} = 1$) instead of $z_{j-1} z_{j+1} = v^{-1} z_{j+1} z_{j-1}$.

Now we prove that the new defined $\Delta^v_{i,j}$ satisfies property (c) of Theorem 2.70. By induction hypothesis we know that part (b) is true for the pair $(i, j-1)$. Note that $(|y_j|, e_{i,j-3}) = (|y_j|, e_{i,j-1})$ and that $1 = -A_{i,j} - (|y_{j-1}|, o_{i,j-4})$. We also use the fact that z_{j+1} commutes with $\Delta^v_{i,j-3}$ since it commutes with every factor of every monomial in the PBW expansion of $\Delta^v_{i,j-3}$. The following equation is true:

$$\begin{aligned} y_{j+1} \Delta^v_{i,j} &= v^{-(|y_j|, e_{i,j-1})} y_{j+1} \Delta^v_{i,j-1} y_j \\ &\quad - v^{-A_{i,j} - (|y_j| + |y_{j-2}|, e_{i,j-3}) - (|y_{j-1}|, o_{i,j-4})} y_{j+1} \Delta^v_{i,j-3} p_{j-2} z_{j+1} \\ &= v^{-(|y_j|, e_{i,j-3}) + 1} \Big[\Delta^v_{i,j-1} \Big(y_j y_{j+1} + (v - v^{-1}) z_{j-1} z_{j+2} \Big) \\ &\quad - v^{1-(|y_{j-2}|, e_{i,j-3})} \Delta^v_{i,j-3} p_{j-2} \Big(z_{j+1} y_{j+1} + (v - v^{-1}) z_j z_{j+2} \Big) \Big] \\ &= v \Delta^v_{i,j} y_{j+1} + v^{-(|y_j|, e_{i,j-3}) + 1} (v - v^{-1}) \Big[\Delta^v_{i,j-1} z_{j-1} \\ &\qquad - v^{1-(|y_{j-2}|, e_{i,j-3})} \Delta^v_{i,j-3} p_{j-2} z_j \Big] z_{j+2} \\ &= v \Delta^v_{i,j} y_{j+1} + v^{-(|y_j|, e_{i,j-3}) + 1} (v - v^{-1}) \Delta^v_{i,j-2} p_{j-1} z_{j+2}. \end{aligned}$$

Part (d) is proved similarly as in the case where j is even. □

Remark 2.71. By symmetry there is also a recursion for every $\Delta^v_{i,j}$ (with $j - i \geq 2$) in terms of various $\Delta^v_{i',j}$ with $i' > i$.

Now we conclude to the quantum cluster algebra structure on the $\mathbb{Z}[v^{\pm\frac{1}{2}}]$-algebra $\mathcal{A}_v(w) = \bigoplus_{\mathbf{a} \in \mathbb{N}^{2n}} \mathbb{Z}[v^{\pm\frac{1}{2}}] E[\mathbf{a}]^*$

Theorem 2.72. The $\mathbb{Z}[v^{\pm\frac{1}{2}}]$-algebra $\mathcal{A}_v(w) = \bigoplus_{\mathbf{a} \in \mathbb{N}^{2n}} \mathbb{Z}[v^{\pm\frac{1}{2}}] E[\mathbf{a}]^*$ is a quantum cluster algebra in the sense of Berenstein-Zelevinsky [6]. The mutatable quantum cluster variables are $v^{\frac{1}{2}} z_i$ for $1 \leq i \leq n$, and $v^{\frac{1}{4}(s_{i,j}, s_{i,j})} \Delta^v_{i,j}$ for $1 \leq i \leq j \leq n$. The frozen quantum cluster variables are $v^{\frac{1}{4}(|y_i| + |z_i|, |y_i| + |z_i|)} p_i$ for $1 \leq i \leq n$.

Note that $\frac{1}{2}(s_{i,j}, s_{i,j}) = j - i + 1 + (e_{i,j-1}, o_{i,j-1}) \in \mathbb{Z}$.

Proof. We quantize the proof of Lemma 2.5. We construct a seed of a quantum cluster algebra similar to the base seed in Figure (15). The cluster variables Z_i, P_i (for $1 \leq i \leq n$) are replaced by the quantum cluster variables $v^{\frac{1}{2}} z_i, v^{\frac{1}{4}(|y_i| + |z_i|, |y_i| + |z_i|)} p_i$ (for $1 \leq i \leq n$). Notably, every pair of quantum cluster variables in the base seed forms a quantum torus, i.e., it satisfies a v-commutativity relation. (The v-commutativity relations among the z_i follow from the straigthening relations; the v-commutativity relations between the P_i and the z_i follow from Lemma 2.57; the v-commutativity relations among the P_i can be checked using the straightening relations.) These relations are strict commutativity relations except for the following v-commutativity relations:

$$\begin{aligned}
z_i z_j &= v^{-(|z_i|,|z_j|)} z_j z_i, && i \text{ even}, j \text{ odd},\\
z_i p_j &= v^{(|z_i|,|y_j|)} p_j z_i, && i, j \text{ even},\\
z_i p_j &= v^{-(|z_i|,|y_j|)} p_j z_i, && i, j \text{ odd},\\
p_i p_j &= v^{-(|z_i|,|z_j|)+(|z_i|,|y_j|)+(|y_i|,|z_j|)+(|y_i|,|y_j|)} p_j z_i, && i \text{ even}, j \text{ odd}.\\
p_i p_j &= v^{-(|z_i|,|y_j|)+(|y_i|,|z_j|)} p_j z_i, && i, j \text{ even},\\
p_i p_j &= v^{(|z_i|,|y_j|)-(|y_i|,|z_j|)} p_j z_i, && i, j \text{ odd}.
\end{aligned}$$

Now it is easy to see that the B-matrix induced from the quiver in Figure 15 and the Λ-matrix induced from the v-commutativity relations form a *compatible pair* in the sense of Berenstein-Zelevinsky [6, Definition 3.1]. Hence, the base seed is a valid initial quantum cluster.

Now fix an integer i such that $1 \leq i \leq n$. Beginning with the base we perform mutations at vertices $i, i+1, \ldots, j$, consecutively, as in the proof of Lemma 2.5. We prove by induction on j that the new quantum cluster variable X that occurs after the sequence of mutations from above is equal to $\Delta^v_{i,j}$. The case $i = j$ is trivial. Note that part (c) of Theorem 2.70 makes also sense for $j = i + 1$. We distinguish two cases. First of all, assume that j is even. By Berenstein-Zelevinsky [6, Proposition 4.9] we have

$$X = M' + M''$$

where definition of M' and M'' involves v-commutativity relations among the quantum cluster variables in the previous seed. To describe M' we compute

$$\Delta^v_{i,j-1} p_j z_j^{-1} = v^{-(|y_j|,o_{i,j-1})} z_j^{-1} p_j \Delta^v_{i,j-1}.$$

Therefore, the summand M' is given by the following equation:

$$\begin{aligned}
M' &= v^{\frac{1}{2}(|y_j|,o_{i,j-1})} \cdot v^{\frac{1}{2}(j-i)+\frac{1}{2}(e_{i,j-2},o_{i,j-1})} \Delta^v_{i,j-1} \cdot v p_j \cdot v^{-\frac{1}{2}} z_j^{-1}\\
&= v^{\frac{1}{2}(j-i+1)+\frac{1}{2}(e_{i,j},o_{i,j-1})} \Delta^v_{i,j-1} p_j z_j^{-1}.
\end{aligned}$$

Furthermore, to describe M'' we obtain:

$$\begin{aligned}
\Delta^v_{i,j-2} p_{j-1} z_{j+1} z_j^{-1} &= v^{-(|z_j|,|z_{j+1}|)-(|z_j|,|z_{j-1}|)+(|z_j|,|y_{j-1}|)-(|z_{j-1}|,|y_{j-1}|)}\\
&\quad \cdot v^{(e_{i,j-2},|y_{j-1}|+|z_{j-1}|-|z_j|)}\\
&\quad \cdot v^{(o_{i,j-3},-|y_{j-1}|-|z_{j-1}|+|z_j|)} z_j^{-1} z_{j+1} p_{j-1} \Delta^v_{i,j-2}\\
&= v^{-2+(e_{i,j-2},|y_{j-1}|)-(o_{i,j-3},|y_j|)} z_j^{-1} z_{j+1} p_{j-1} \Delta^v_{i,j-2}
\end{aligned}$$

Therefore, the summand M' is given by the following equation:

$$\begin{aligned}
M'' &= v^{1-\frac{1}{2}(e_{i,j-2},|y_{j-1}|)+\frac{1}{2}(o_{i,j-3},|y_j|)} \cdot v^{\frac{1}{2}(j-i-1)+\frac{1}{2}(e_{i,j-2},o_{i,j-3})} \Delta^v_{i,j-2}\\
&\quad \cdot v^{1-\frac{1}{2}(|y_j|,|y_{j-1}|)} \cdot v^{\frac{1}{2}} z_{j+1} \cdot v^{-\frac{1}{2}} z_j^{-1}\\
&= v^{1-(|y_{j-1}|,e_{i,j-2})} v^{\frac{1}{2}(j-i+1)+\frac{1}{2}(e_{i,j},o_{i,j-1})} \Delta^v_{i,j-2} p_{j-1} z_{j+1} z_j^{-1}.
\end{aligned}$$

A comparison with the formula in part (c) of Theorem 2.70 shows that $X = \Delta^v_{i,j}$ which completes the induction step.

The case with odd j is treated similarly. □

Remark 2.73. By adjusting the forms we similarly equip $U_v^+(w)$ with a quantum cluster algebra structure.

Remark 2.74. Note that $\deg(\Delta_{i,j}) = \underline{\dim}(M_{i,j}) = s_{i,j}$ for all $1 < i < j < n$. Thus, the object $M_{i,j}$ from Remark 2.8 is the indecomposable rigid object in $\mathcal{A}(\mathcal{C}_M)$ corresponding to the cluster variable $\Delta_{i,j}$. The corresponding quantum cluster variable is the dual canonical basis $\Delta^v_{i,j}$ scaled by a factor $v^{\frac{1}{4}(s_{i,j},s_{i,j})}$. By [22, Lemma 3.12] the exponent can be interpreted as $\frac{1}{4}(s_{i,j}, s_{i,j}) = \frac{1}{2}\dim(\mathrm{End}(M_{i,j}))$. The same relation is true for the other (mutatable and frozen) quantum cluster variables.

3 A quantum cluster algebra of Kronecker type

3.1 Representation theory of the Kronecker quiver

The quiver $Q = (Q_0, Q_1)$ with vertex set $Q_0 = \{0, 1\}$ and arrow set $Q_1 = \{a_1, a_2\}$ with $a_1, a_2 \colon 0 \to 1$ is called the *Kronecker quiver* (see Figure 17).

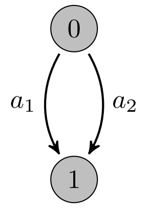
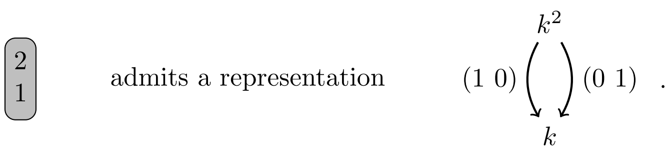


Figure 17: The Kronecker quiver

Let k be a field. The category $\mathrm{rep}_k(Q)$ of finite-dimensional representations of Q can be identified with the category $\mathrm{mod}(kQ)$ of finite-dimensional modules over the *path algebra* kQ. (For more information on representations of quivers, see, for example Crawley-Boevey [11].)

The Kronecker quiver is a *tame* quiver. There are infinitely many indecomposable kQ-modules which are classified as *preprojective*, *preinjective* or *regular*. A part of the preinjective component of the *Auslander-Reiten quiver* of $\mathrm{mod}(kQ)$ is shown in Figure 18. The modules are represented by their dimension vectors. For example, the dimension vector



The maps are given by 1×2 matrices, if we choose a basis of the vector spaces. The solid arrows display the space of *irreducible maps*; the dotted arrows display the *Auslander-Reiten translation* τ.

We consider the direct sum $M = I_0 \oplus I_1 \oplus \tau(I_0) \oplus \tau(I_1)$ of the four gray modules. These four modules are the two indecomposable injective modules I_0 and I_1 asssociated with the vertices 0 and 1 and their Auslander-Reiten translates, $\tau(I_0)$ and

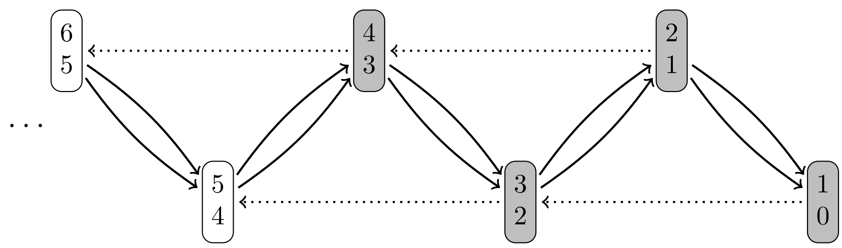


Figure 18: A part of the preinjective component of $\mathrm{mod}(kQ)$

$\tau(I_1)$. The module M is a *terminal kQ-module* in the sense of Geiß-Leclerc-Schröer [21, Section 2.2]. According to Geiß-Leclerc-Schröer [21, Theorem 3.3] the terminal kQ-module M gives rise to a cluster algebra structure. To explain this theorem, let us introduce some notation.

3.2 The preprojective algebra

Let Λ be the *preprojective algebra*; it is defined as $\Lambda = k\overline{Q}/(c)$. Here, $\overline{Q}$ denotes *double quiver* of Q, which is by definition given by a vertex set $\overline{Q}_0 = Q_0$ and an arrow set $\overline{Q}_1 = Q_1 \cup \{a_1^*, a_2^*\}$ with two additional arrows $a_1^*, a_2^* \colon 1 \to 0$. The ideal (c) is the two-sided ideal generated by the element

$$c = \sum_{a \in Q_1} (a^* a - a a^*) \in k\overline{Q}.$$

The algebra Λ is infinite-dimensional, because Q is not an orientation of a Dynkin diagram. There is a *restriction functor* $\pi_Q \colon \mathrm{mod}(\Lambda) \to \mathrm{mod}(kQ)$ given by forgetting the linear maps associated with a_1^* and a_2^* in a Λ-module, i.e., a representation of $\overline{Q}$ such that the linear maps satisfy relations corresponding to the ideal (c). Ringel [48, Theorem B] proved that the category $\mathrm{mod}(\Lambda)$ is isomorphic to a category called $C(1,\tau)$. The objects in the category $C(1,\tau)$ are pairs (X, f) consisting of a kQ-module X and a kQ-module homomorphism $f \colon X \to \tau(X)$ from X to its translate $\tau(X)$; morphisms in $C(1,\tau)$ from a pair (X, f) to a pair (Y, g) are given by a kQ-module homomorphism $h \colon X \to Y$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \tau(X) & \xrightarrow{\ \tau(h)\ } & \tau(Y) \end{array}$$

commutes. Using this correspondence, Geiß-Leclerc-Schröer [21, Section 7.1] constructed, for every $i = 0, 1$, and any natural numbers $a \leq b$ a Λ-module $T_{i,[a,b]} = \big(I_{i,[a,b]}, e_{i,[a,b]}\big)$, where $I_{i,[a,b]} = \bigoplus_{j=a}^{b} \tau^j(I_i)$, and the map

$$e_{i,[a,b]} \colon I_{i,[a,b]} \to \tau\left(I_{i,[a,b]}\right) = \bigoplus_{j=a+1}^{b+1} \tau^j(I_i)$$

is identity on every $\tau^j(I_i)$ for $a + 1 \leq j \leq b$ and zero otherwise. We are interested in the six Λ-modules $T_{i,[a,b]}$ for $i = 0, 1$ and $0 \leq a, b \leq 1$. We display the modules by their graded dimension vectors.

All six modules are *rigid* and *nilpotent*.

3.3 The δ-functions and the cluster algebra structure

In this section, let $k = \mathbb{C}$. The δ-functions of the modules $T_{i,[a,b]}$ satisfy *generalized determinantal identities*, see Geiß-Leclerc-Schröer [21, Theorem 18.1]. Let U_0, U_1, U_2, and U_3 be the δ-functions of $T_{0,[0,0]}, T_{1,[0,0]}, T_{0,[1,1]}$ and $T_{1,[1,1]}$ respectively; let P_0 and P_1 be the δ-functions of $T_{0,[0,1]}$ and $T_{1,[0,1]}$, respectively. The determinantal identities

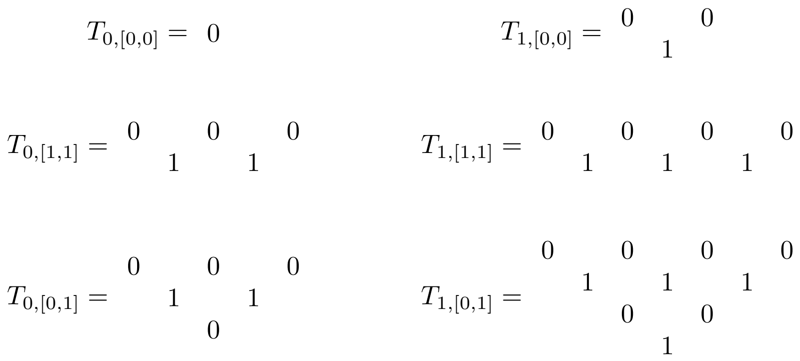


Figure 19: The modules $T_{i,[a,b]}$

in this case read as follows:

$$P_0 = U_2U_0 - U_1^2, \tag{24}$$

$$P_1 = U_3U_1 - U_2^2. \tag{25}$$

These relations may be regarded as first exchange relations of a cluster algebra, called $\mathcal{A}(\mathcal{C}_M)$ in Geiß-Leclerc-Schröer [21], with initial cluster (U_0, U_1, P_0, P_1), initial exchange matrix visualized by the the quiver in Figure 20 and the frozen variables P_0 and P_1. The frozen variables P_0 and P_1 may be regarded as coefficients of the cluster algebra, see Fomin-Zelevinsky [16].

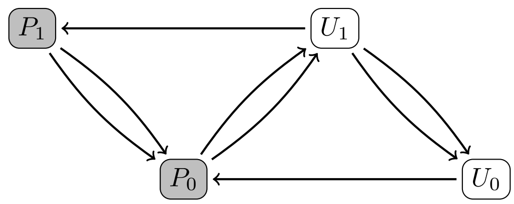


Figure 20: Initial cluster

Consecutive mutations at U_0 and U_1 yield to the new cluster variables U_2 and U_3, namely $U_2 = \frac{U_1^2+P_0}{U_0}$ and $U_3 = \frac{U_2^2+P_1}{U_1}$. The associated quiver encodes the exchange relation; it also gets mutated. The mutated quivers are shown in Figure 21.

If we specialize the coefficients $P_0 = P_1 = 1$, then we obtain a coefficient-free cluster algebra of rank 2 with initial exchange matrix $B = \left(\begin{smallmatrix} 0 & -2 \\ 2 & 0 \end{smallmatrix}\right)$; the specialized cluster variables $U_n (n \in \mathbb{Z})$ satisfy the recursion $U_{n+1}U_{n-1} = U_n^2 + 1$ for every $n \in \mathbb{Z}$. Caldero-Zelevinsky [10, Theorem 4.1] proved that

$$U_{n+2} = \frac{1}{U_1^n U_0^{n+1}} \sum_{k,l} \binom{n-k}{l} \binom{n+1-l}{k} U_1^{2k} U_0^{2l} \tag{26}$$

where the sum is taken over all $k, l \in \mathbb{N}$ such that either $k+l \leq n$ or $(k,l) = (n+1, 0)$. Musiker and Propp [45] gave nice combinatorial descriptions of the coeffcients. The

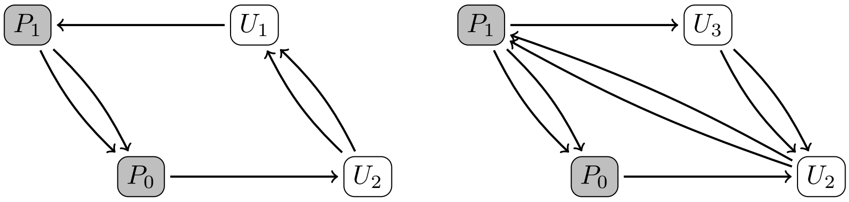


Figure 21: The cluster after mutation at U_0 and U_1, consecutively

author [35] gives a different formula for the coefficients. Szántó [54] shows that a quantized version of the formula is related to the number of points in a Grassmannian over a finite field $\mathbb{F}_q$ in the context of *Hall algebras*.

Equation (26) illustrates the Fomin-Zelevinsky's *Laurent phenomenon* [14]: every cluster variable $U_n (n \in \mathbb{Z})$ is a Laurent polynomial in U_1 and U_0.

Caldero-Zelevinsky [10] derive formula (26) using the *Caldero-Chapoton map* [7] and computing Euler characteristics of Grassmannians of quiver representations. Later Zelevinsky [55] gave a simpler proof for formula (26). He observed that the expression

$$T = \frac{1 + U_n^2 + U_{n+1}^2}{U_n U_{n+1}}$$

is invariant of n. Thus, the non-linear exchange relation $U_{n+1}U_{n-1} = U_n^2 + 1$ may be replaced by a linear three-term recursion $U_{n+1} = TU_n - U_{n-1}$, $(n \in \mathbb{Z})$, when we define

$$T = \frac{1 + U_1^2 + U_2^2}{U_1 U_2}.$$

Note that $T = U_3 U_0 - U_2 U_1$.

Recently, Keller-Scherotzke [32] observed that linear exchange relations exist for cluster variables of affine quivers in general.

In analogy to these formulae, the cluster variables U_n of the cluster algebra with initial exchange matrix

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ 0 & -1 \\ 1 & 0 \end{pmatrix},$$

initial seed (U_1, U_2, P_0, P_1), and frozen variables P_0 and P_1 satisfy the exchange relation $U_{n+1}U_{n-1} = U_n^2 + P_1^{n-1}P_0^{n-4}$ for $n \geq 4$. The cluster variables are explicitly given by

$$U_{n+3} = \frac{1}{U_1^{n+1}U_2^n} \sum_{k,l} \binom{n-k}{l} \binom{n+1-l}{k} P_1^{n+1-k} U_2^{2k} U_1^{2l} P_0^{n-l}, \qquad (27)$$

where the sum is taken over the same index set as above. For reasons that will become clear later on we have switched our initial seed from (U_0, U_1) to (U_1, U_2).

Geiß-Leclerc-Schröer realized the cluster algebra $\mathcal{A}(\mathcal{C}_M)$ as a subalgebra of the graded dual $U(\mathfrak{n})_{gr}^*$ of the positive part $\mathfrak{n}$ of the universal enveloping algebra of the Kac-Moody Lie algebra of type $A_1^{(1)}$.

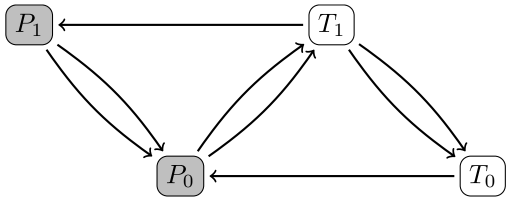


Figure 22: The quiver of T

A striking feature is *polynomiality*: every U_n is actually a polynomial in U_3, U_2, U_1, U_0. For example, one may check that $U_4 = U_3^2 U_0 - 2U_3 U_2 U_1 + U_2^3$. A priori the cluster variables U_n are only rational functions in U_3, U_2, U_1, U_0. If we plug $P_0 = U_2 U_0 - U_1^2$ and $P_1 = U_3 U_1 - U_2^2$ in equation (27) and use the binomial theorem we see that

$$U_{n+3} = \sum_{a,b} c_{n,a,b} U_3^a U_2^{n+2-2a+b} U_1^{n-1-2b+a} U_0^b \tag{28}$$

with coefficients

$$c_{n,a,b} = \sum_{k,l} (-1)^{k+l+a+b+1} \binom{n-k}{l} \binom{n+1-l}{k} \binom{n+1-k}{a} \binom{n-l}{b}. \tag{29}$$

Polynomiality implies the combinatorially non-trivial binomial identity $c_{n,a,b} = 0$ if either $n + 2 - 2a + b < 0$ or $n - 1 - 2b + a < 0$.

3.4 Mutations of rigid modules

In the following subsection we describe the mutation of rigid Λ-modules. We use the abbreviations $T_0 = T_{0,[0,0]}$, $T_1 = T_{1,[0,0]}$, $P_0 = T_{0,[0,1]}$, and $P_1 = T_{1,[0,1]}$; recall that the Λ-modules on the right hand sides are displayed in Figure 19.

The Λ-module $T = T_0 \oplus T_1 \oplus P_0 \oplus P_1$ is rigid. Moreover, T is *maximal rigid*, i.e., every indecomposable Λ-module $\tilde{T}$ for which $T \oplus \tilde{T}$ is rigid is isomorphic to a direct summand of T. The quiver of T is given in Figure 22. Note the similarity with Figure 20.

The dimension of the 16 homomorphism spaces between the direct summands of T are put in a matrix C_T called *Cartan matrix*. In our example we have

$$C_T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 4 \end{pmatrix},$$

where rows and columns are ordered in accordance with the order T_0, T_1, P_0, P_1. For example $\dim(\mathrm{Hom}(T_0, T_0)) = 1$, $\dim(\mathrm{Hom}(T_1, T_0)) = 2$, $\dim(\mathrm{Hom}(P_0, T_0)) = 3$, etc.

There is a mutation process for maximal rigid Λ-modules analogous to the mutation process for cluster algebras. We refer to Geiß-Leclerc-Schrer [22] for a detailed exposition. (Geiß-Leclerc-Schrer [22] work with Dynkin quivers Q, but same procedures apply to the general setup as well.) The modules P_0 and P_1 are projective-injective and cannot be mutated; they correspond to frozen variables in the cluster algebra. Both T_0 and T_1 can be mutated. Let us describe the mutation for T_0. There is a (unique up to isomorphism) Λ-module T_2 such that $T_2 \not\cong T_0$ and the module $T_2 \oplus T/T_0 \cong T_2 \oplus T_1 \oplus P_0 \oplus P_1$ is again maximal rigid; two short exact sequences

$$\begin{array}{c} 0 \longrightarrow T_0 \longrightarrow P_0 \longrightarrow T_2 \longrightarrow 0 \\ 0 \longrightarrow T_2 \to T_1 \oplus T_1 \to T_0 \longrightarrow 0 \end{array}$$

characterize T_2. The appearence of P_0 and $T_1 \oplus T_1$ as middle terms is an incarnation of the fact that there is one arrow form T_0 to P_0 in the quiver of T and two arrows form T_1 to T_0. We see that $T_2 = T_{0,[1,1]}$.

Denote the mutated module by $T' = \mu_{T_0}(T) = T_2 \oplus T_1 \oplus P_0 \oplus P_1$. The Cartan matrix and the quiver of T' are shown in Figure 23. The quiver of T' is obtained from the previous quiver by quiver mutation. A combinatorial recursion for the Cartan matrices is given in [22, Proposition 7.5].

$$C_{T'} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 4 & 6 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

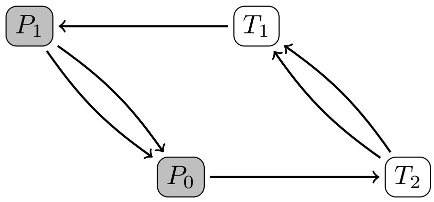


Figure 23: The Cartan matrix and the quiver of T'

Mutation of rigid modules is involutive just as mutation of cluster algebras, i.e., $\mu_{T_2}(T') = T$. Since P_0 and P_1 are not mutatable, the only non-trivial further mutation is $T'' = \mu_{T_1}(T')$. Using short exact sequences we see that $T'' = T_2 \oplus T_3 \oplus P_0 \oplus P_1$ with $T_3 = T_{1,[1,1]}$. Iteration of the mutation process gives rise to a sequence of Λ-modules $(T_n)_{n\in\mathbb{N}}$. Similarly to cluster variables, one obtains a sequence $(T_n)_{n\in\mathbb{N}^-}$ of Λ-modules by starting with mutation of T at T_1.

$$C_{T'} = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 3 & 4 & 4 & 6 \\ 2 & 3 & 2 & 4 \end{pmatrix}$$

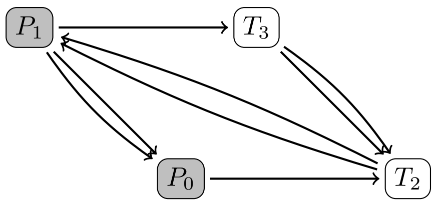


Figure 24: The Cartan matrix and the quiver of T''

Using [22, Proposition 7.5] and one proves by mathematical induction that for $n \geq 3$ the Cartan matrix of $T_n \oplus T_{n+1} \oplus P_0 \oplus P_1$ is

$$\begin{pmatrix} (2n-5)^2 & (2n-4)^2 - 2 & 3n-9 & 5n-14 \\ (2n-4)^2 & (2n-3)^3 & 3n-6 & 5n-9 \\ 5n-11 & 5n-6 & 4 & 6 \\ 3n-6 & 3n-3 & 2 & 4 \end{pmatrix}.$$

Especially, we have $\dim(\mathrm{End}(T_n)) = (2n-5)^2$.

3.5 Bases of the cluster algebra $\mathcal{A}(\mathcal{C}_M)$

Several authors studied various bases of the cluster algebra $\mathcal{A}(\mathcal{C}_M)$. In this subsection we describe the *semicanonical basis* of Caldero-Zelevinsky [10], the *canonical basis* of Sherman-Zelevinsky [51], and the *dual semicanonical basis* of Geiß-Leclerc-Schrer [21]. To define these bases, we introduce the *normalized Chebyshev polynomials of the first and second kinds*.

Definition 3.1. Define a sequence $(T_k)_{k=0}^{\infty}$ of polynomials $T_k \in \mathbb{Z}[X]$ recursively by $T_0 = 2$, $T_1 = X$, and $T_{k+1} = XT_k - T_{k-1}$ for $k \geq 1$; another sequence $(S_k)_{k=0}^{\infty}$ of polynomials $S_k \in \mathbb{Z}[X]$ is defined recursively by $S_0 = 1$, $S_1 = X$, and $S_{k+1} = XS_k - S_{k-1}$ for $k \geq 1$.

k	0	1	2	3	4
T_k	2	x	x^2-2	x^3-3x	x^4-4x^2+2
S_k	1	x	x^2-1	x^3-2x	x^4-3x^2+1

Figure 25: Normalized Chebyshev polynomials of the first and second kind

The polynomial T_k is called the k^{th} *normalized Chebyshev polynomial of the first kind*; the polynomial S_k is called the k^{th} *normalized Chebyshev polynomial of the second kind*. Figure 25 displays the Chebyshev polynomials with lowest indices.

A monomial $U_{n+1}^{a_1}U_n^{a_2}P_1^{a_3}P_0^{a_4}$ (with $a_1, a_2, a_3, a_4 \in \mathbb{N}$) in the cluster variables of a *single* cluster is called a *cluster monomial*. Let $\underline{Mono}$ be the set of all cluster monomials. Put $z = U_3U_0 - U_2U_1 = \frac{P_1P_0+P_1U_1^2+P_0U_2^2}{U_1U_2}$ and

- $\underline{\mathcal{B}} = \underline{Mono} \cup \left\{ (P_1P_0)^{\frac{k}{2}} T_k \left(z(P_1P_0)^{-\frac{1}{2}} \right) | k \geq 1 \right\}$,
- $\underline{\mathcal{S}} = \underline{Mono} \cup \left\{ (P_1P_0)^{\frac{k}{2}} S_k \left(z(P_1P_0)^{-\frac{1}{2}} \right) | k \geq 1 \right\}$,
- $\underline{\Sigma} = \underline{Mono} \cup \left\{ z^k | k \geq 1 \right\}$.

The elements $s_k = (P_1P_0)^{\frac{k}{2}} S_k(z(P_1P_0)^{-\frac{1}{2}}) \in \underline{\mathcal{S}}$ obey the relation $s_{k+1} = zs_k - P_1P_0s_{k-1}$ for $k \geq 2$ which resembles the relation $U_{k+1} = zU_k - P_1P_0U_{k-1}$ for $k \geq 4$ for cluster variables.

Theorem 3.2 ([10, 51, 21])**.** Each of $\underline{\mathcal{B}}$, $\underline{\mathcal{S}}$, and $\underline{\Sigma}$ is a $\mathbb{Q}$-basis of $\mathcal{A}(\mathcal{C}_M)$.

The basis $\underline{\mathcal{B}}$ is known as the *canonical basis* of $\mathcal{A}(\mathcal{C}_M)$, $\underline{\mathcal{S}}$ is the *semicanonical basis* of $\mathcal{A}(\mathcal{C}_M)$, and $\underline{\Sigma}$ is the *dual semicanonical basis* of $\mathcal{A}(\mathcal{C}_M)$.

3.6 The quantized universal enveloping algebra $U_q(\mathfrak{g})$ of type $A_1^{(1)}$

Let $C = (a_{ij})_{1 \leq i,j \leq 2}$ be the *Cartan matrix* associated with the Kronecker quiver Q, i.e.,

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Furthermore, let $\mathfrak{g}$ be the *Kac-Moody Lie algebra* of type C; it admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. The Lie algebra $\mathfrak{n}$ is called the *positive part* of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is studied by its *root lattice*. There are two *simple roots*, α_1 and α_2. By Δ^+ we denote the set of *positive roots*. There are two kinds of positive roots called *real* and *imaginary* roots, i.e., $\Delta^+ = \Delta^+_{re} \cup \Delta^+_{im}$ with real roots $\Delta^+_{re} = \{(n+1)\alpha_1 + n\alpha_2 \colon n \in \mathbb{N}\} \cup \{n\alpha_1 + (n+1)n\alpha_2 \colon n \in \mathbb{N}\}$ and imaginary roots $\Delta^+_{im} = \{n\alpha_1 + n\alpha_2 \colon n \in \mathbb{N}^+\}$. Note that the real positive roots correspond to dimension vectors which admit a unique irreducible kQ-module. Examples are displayed in Figure 18. They are also the *g-vectors* of the cluster algebra of type $A_1^{(1)}$ introduced above, see [16].

Let W be the *Weyl group* of type $\mathfrak{g}$; it is generated by two simple reflections $s_1, s_2 \in W$ which act on the simple roots by $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 2\alpha_1$, $s_2(\alpha_1) = \alpha_1 + 2\alpha_2$, and $s_2(\alpha_2) = -\alpha_2$.

In Section 3 we mentioned the *universal enveloping algebra* $U(\mathfrak{n})$ of $\mathfrak{n}$. It is the $\mathbb{C}$-algebra generated by E_1 and E_2 subject to the *Serre relations*

$$E_1^3E_2 - 3E_1^2E_2E_1 + 3E_1E_2E_1^2 - E_2E_1^3 = 0,$$
$$E_2^3E_1 - 3E_2^2E_1E_2 + 3E_2E_1E_2^2 - E_1E_2^3 = 0.$$

It is known that $U(\mathfrak{n})$ can be endowed with a comultiplication $\Delta \colon U(\mathfrak{n}) \to U(\mathfrak{n}) \otimes U(\mathfrak{n})$ defined by $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{n}$ (using the canonical embedding $\iota \colon \mathfrak{n} \to U(\mathfrak{n})$) and an antipode so that $U(\mathfrak{n})$ becomes a cocommutative Hopf algebra. It is graded by the root lattice. The graded dual of $U(\mathfrak{n})$, the Hopf algebra $U(\mathfrak{n})^*_{gr}$, is a commutative $\mathbb{C}$-algebra.

By introducing a deformation parameter q one can construct a series of Hopf algebras $U_q(\mathfrak{n})$ that are not cocommutative but specialize to $U(\mathfrak{n})$ if we set $q = 1$. To describe this construction we introduce quantized integers and quantized binomial coefficients.

Remarkably, $U_q(\mathfrak{n}) \cong U_q(\mathfrak{n})^*_{gr}$ is a self-dual Hopf algebra whereas $U(\mathfrak{n}) \ncong U(\mathfrak{n})^*$.

Definition 3.3. For two integers n, k, let

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} \in \mathbb{Q}(q), \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]\,[n-1]\cdots[n-k+1]}{[k]\,[k-1]\cdots[1]} \in \mathbb{Q}(q) \tag{30}$$

denote the quantum integer and the quantum binomial coefficient. Furthermore, for a natural number k, let $[k]! = [k][k-1]\cdots[1]$ denote the quantized factorial.

Both $[k]$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ are actually Laurent polynomials in q. Note that $[k] = k$, $\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$, and $[k]! = k!$ if we specialize $q = 1$. Examples of quantum integers include $[0] = 0$, $[1] = 1$, $[2] = q + q^{-1}$, and $[3] = q^2 + 1 + q^{-2}$.

Note that some authors, for example [27], use a different notation for quantum integers. Note also that quantum binomial coefficients, just as ordinary binomial coefficients, are defined for negative integers n, k as well. For example, $\begin{bmatrix} -2 \\ 1 \end{bmatrix} = -q - q^{-1}$, but $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $k < 0$.

Quantized integers are related with the normalized Chebyshev polynomials S_k, for $k \geq 0$, of the second kind from Subsection 3.5. More precisely, there holds $[k] = S_{k-1}([2]) = S_{k-1}(q + q^{-1})$ for $k \geq 1$.

Definition 3.4. The quantized enveloping algebra $U_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-algebra generated by E_i, $(i = 1, 2)$, F_i, $(i = 1, 2)$, and K_i, K_i^{-1}, $(i = 1, 2)$ subject to the following relations

$$K_iK_j = K_jK_i, \qquad (i \neq j) \tag{31}$$
$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \qquad (i = 1, 2) \tag{32}$$
$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \qquad (1 \leq i, j \leq 2) \tag{33}$$
$$K_iF_jK_i^{-1} = q^{-a_{ij}}F_j, \qquad (1 \leq i, j \leq 2) \tag{34}$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \qquad (1 \leq i, j \leq 2) \tag{35}$$
$$E_1^3E_2 - [3]E_1^2E_2E_1 + [3]E_1E_2E_1^2 - E_2E_1^3 = 0, \tag{36}$$
$$E_2^3E_1 - [3]E_2^2E_1E_2 + [3]E_2E_1E_2^2 - E_1E_2^3 = 0, \tag{37}$$
$$F_1^3F_2 - [3]F_1^2F_2F_1 + [3]F_1F_2F_1^2 - F_2F_1^3 = 0, \tag{38}$$
$$F_2^3F_1 - [3]F_2^2F_1F_2 + [3]F_2F_1F_2^2 - F_1F_2^3 = 0, \tag{39}$$

where δ_{ij} is the Kronecker delta function.

The subalgebra generated by E_1 and E_2 is called the *quantized enveloping algebra* $U_q(\mathfrak{n})$. The only relations in $U_q(\mathfrak{n})$ remain (36) and (37). These are called *quantized Serre relations*. The algebra $U_q(\mathfrak{n})$ specializes to $U(\mathfrak{n})$ in the limit $q = 1$.

3.7 The Poincaré-Birkhoff-Witt basis

To construct a basis of $U_q(\mathfrak{n})$ Lusztig [42, Chapter 37] defines *T-automorphisms*. We will use the notation $E_i^{(k)} = E_i^k/[k]!$ for $i = 1, 2$ and the similar notation for F_i. For every $i = 1, 2$ define

- $T_i(E_i) = -K_i^{-1}F_i$,
- $T_i(F_i) = -E_iK_i$,
- $T_i(E_j) = \sum_{r+s=2}(-1)^r q^{-r}E_i^{(r)}E_jE_i^{(s)}$ for $j \neq i$,
- $T_i(F_j) = \sum_{r+s=2}(-1)^r q^{r}F_i^{(s)}F_jF_i^{(r)}$ for $j \neq i$,
- $T_i(K_j) = K_jK_i^{-a_{ij}}$ for $i = 1, 2$.

Lusztig shows that T_i can be extended to an algebra homomorphism $T_i\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. (It is denoted $T'_{i,-1}$ in Lusztig's book [42, Chapter 37] where the variable q is called v instead.) Furthermore, T_i is an algebra automorphism. The images of the generators under the inverse T_i^{-1} are given by (see Lusztig [42, Chapter 37])

- $T_i^{-1}(E_i) = -F_iK_i$,
- $T_i^{-1}(F_i) = -K_i^{-1}E_i$,
- $T_i^{-1}(E_j) = \sum_{r+s=2}(-1)^r q^{-r}E_i^{(s)}E_jE_i^{(r)}$ for $j \neq i$,
- $T_i^{-1}(F_j) = \sum_{r+s=2}(-1)^r q^{r}F_i^{(r)}F_jF_i^{(s)}$ for $j \neq i$,

- $T_i^{-1}(K_j) = K_j K_i^{-a_{ij}}$ for $i = 1, 2$.

The automorphisms T_i are sometimes called *braid operators*. This terminology comes from the fact that the operators T_i can be defined for abitrary quivers Q and that they satisfy braid group relations. In this particular case it means that $T_1 T_2$ has infinite order because $s_1 s_2 \in W$ has infinite order.

For every reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ of a Weyl group element $w \in W$ Lusztig [42, Proposition 40.2.1] constructs a *Poincaré-Birkhoff-Witt basis*.

Theorem 3.5 (Lusztig)**.** Let $w \in W$ and let $s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression for w. Then all elements

$$p_i(\mathbf{c}) := (T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_{k-1}})(E_{i_k}^{(c_k)}) \cdots (T_{i_1} \circ T_{i_2})(E_{i_3}^{(c_3)}) T_{i_1}(E_{i_2}^{(c_2)}) E_{i_1}^{(c_1)}$$

parametrized by sequences $\mathbf{c} = (c_1, \cdots, c_k) \in \mathbb{N}^k$, form a $\mathbb{Q}(q)$-basis of a subalgebra called $U_q^+(w)$ of $U_q(\mathfrak{n})$ which does only depend on w but not on the choice of the reduced expression for w.

Let us make some remarks.

1. The basis $\{p_i(\mathbf{c}) \colon \mathbf{c} \in \mathbb{N}^k\}$ is called a *PBW-type basis* of $U_q^+(w)$.

2. The basis is not canonical in the sense that it depends on the choice of the reduced expression for w. Every choice of a reduced expression gives a bijection between $\mathbb{N}^k$ and a basis of $U_q^+(w)$. The bijections are known as *Lusztig parametrizations*.

3. The same theorem holds for other quivers. For a Dynkin quiver Q there is a unique longest element $w_0 \in W$. In this case $U_q^+(w_0) = U_q(\mathfrak{n})$. Thus, in the Dynkin case we get a PBW basis of the whole algebra $U_q(\mathfrak{n})$ whereas in the present case we have to restrict ourselves to an appropriate subalgebra.

4. It is not obvious that $(T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_{l-1}})(E_{i_l}^{(c_l)}) \in U_q(\mathfrak{n})$ for all $1 \leq l \leq k$ since the T-automorphisms are maps $T_i \colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$.

5. The algebra $U_q(\mathfrak{n})$ is graded by the root lattice R if we set $\deg(E_i) = \alpha_i$ for $i = 1, 2$. Then $\deg((T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_{l-1}})(E_{i_l}^{(c_l)})) = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l})$ for all $1 \leq l \leq k$.

Let us consider the reduced expression $s_1 s_2 s_1 s_2$ associated with the terminal kQ-module M defined in Section 3. It is an Q^{op}-adapted reduced expression for the given orientation. Note that under the bijection between positive roots and dimension vectors of indecomposable modules given by *Kac's theorem*, the positive roots α_1, $s_1(\alpha_2) = 2\alpha_1 + \alpha_2$, $s_1 s_2(\alpha_1) = 3\alpha_1 + 2\alpha_2$, and $s_1 s_2 s_1(\alpha_2) = 4\alpha_1 + 3\alpha_2$ correspond to the dimension vectors of the four preinjective modules I_0, I_1, $\tau(I_0)$, $\tau(I_1)$ that are the direct summands of the terminal kQ-module M from Section 3. Therefore we introduce the notation

$$v_0 = E_1, \quad v_1 = T_1(E_2), \quad v_2 = (T_1 \circ T_2)(E_1), \quad v_3 = (T_1 \circ T_2 \circ T_1)(E_2).$$

Monomials of the form $v[\mathbf{a}] = v_3^{(a_3)} v_2^{(a_2)} v_1^{(a_1)} v_0^{(a_0)}$ with $\mathbf{a} = (a_3, a_2, a_1, a_0) \in \mathbb{N}^4$ form a basis of the subalgebra $U_q^+(s_1 s_2 s_1 s_2)$. Call the basis $\mathcal{P} = \{v[\mathbf{a}] \colon \mathbf{a} \in \mathbb{N}^4\}$. We call the basis $\mathcal{P}$ the *PBW basis* of $U_q^+(s_1 s_2 s_1 s_2)$.

3.8 The derivation of the straightening relations

The aim of this subsection is to write arbitrary monomials in the elements v_3, v_2, v_1, v_0, for example $v_0^7 v_2^3$, as a $\mathbb{Q}(q)$-linear combination of basis elements $v[\mathbf{a}]$. Clearly, $v_0^7 v_2^3 \in U_q^+(s_1 s_2 s_1 s_2)$ but $v_0^7 v_2^3$ is not in $\mathcal{P}$ since v_0 and v_2 are multiplied in a different order.

First of all let us compute v_1. We have

$$\begin{aligned} v_1 = T_1(E_2) &= E_2 E_1^{(2)} - q^{-1} E_1 E_2 E_1 + q^{-2} E_1^{(2)} E_2 \\ &= \frac{1}{[2]} \bigg(E_2 E_1^2 - (q^{-2}+1) E_1 E_2 E_1 + q^{-2} E_1^2 E_2 \bigg). \end{aligned}$$

It follows that

$$\begin{aligned} v_1 v_0 - q^2 v_0 v_1 = \frac{1}{[2]} \bigg(& E_2 E_1^3 - (q^{-2}+1) E_1 E_2 E_1^2 + q^{-2} E_1^2 E_2 E_1 \\ & - q^2 E_1 E_2 E_1^2 + (1+q^2) E_1^2 E_2 E_1 - E_1^3 E_2 \bigg) = 0 \end{aligned}$$

by the quantum Serre relation. Thus we know that $v_0 v_1 = q^{-2} v_1 v_0$. Applying the composition $T_1 \circ T_2$ to this equation we get $v_2 v_3 = q^{-2} v_3 v_2$. By an analogous argument with interchanged role of E_1 and E_2 we get $v_1 v_2 = q^{-2} v_2 v_1$.

Note that we my write v_1 as a commutator $v_1 = \frac{1}{[2]}(E_2 E_1 - q^{-2} E_1 E_2) E_1 - E_1 \frac{1}{[2]}(E_2 E_1 - q^{-2} E_1 E_2)$ of E_1 and an element which we will abbreviate as $A = \frac{1}{[2]}(E_2 E_1 - q^{-2} E_1 E_2)$. For symmetry let us introduce an element $B = \frac{1}{[2]}(E_1 E_2 - q^{-2} E_2 E_1)$. The next lemma is useful for further computations.

Lemma 3.6. The equation $T_1(A) = B$ holds. Similarly, $T_2(B) = A$.

Proof. We only prove the first statement. The second is similar. Note that

$$\begin{aligned} T_1(E_1 E_2 - q^{-2} E_2 E_1) &= -K_1^{-1} F_1 \Big(E_2 E_1^2 - (q^{-2}+1) E_1 E_2 E_1 + q^{-2} E_1^2 E_2 \Big) \\ &\quad + q^{-2} \Big(E_2 E_1^2 - (q^{-2}+1) E_1 E_2 E_1 + q^{-2} E_1^2 E_2 \Big) K_1^{-1} F_1 \\ &= -K_1^{-1} F_1 \Big(E_2 E_1^2 - (q^{-2}+1) E_1 E_2 E_1 + q^{-2} E_1^2 E_2 \Big) \\ &\quad + K_1^{-1} \Big(E_2 E_1^2 - (q^{-2}+1) E_1 E_2 E_1 + q^{-2} E_1^2 E_2 \Big) F_1. \end{aligned} \tag{40}$$

because $E_2 K_1^{-1} = q^{-2} K_1^{-1} E_2$ and $E_1 K_1^{-1} = q^2 K_1^{-1} E_1$ by relation (33). Now we use (35) to deduce that

$$E_2 E_1^2 F_1 = F_1 E_2 E_1^2 + E_2 \frac{K_1 - K_1^{-1}}{q - q^{-1}} E_1 + E_2 E_1 \frac{K_1 - K_1^{-1}}{q - q^{-1}}, \tag{41}$$

$$E_1 E_2 E_1 F_1 = F_1 E_1 E_2 E_1 + \frac{K_1 - K_1^{-1}}{q - q^{-1}} E_2 E_1 + E_1 E_2 \frac{K_1 - K_1^{-1}}{q - q^{-1}}, \tag{42}$$

$$E_1^2 E_2 F_1 = F_1 E_1^2 E_2 + \frac{K_1 - K_1^{-1}}{q - q^{-1}} E_1 E_2 + E_1 \frac{K_1 - K_1^{-1}}{q - q^{-1}} E_2, \tag{43}$$

The first terms on the RHS of equations (41),(42), and (43) cancel out with corresponding terms if we substitute in equation (40). We sum up the appropriate linear combinations of the remaining terms on the RHS of equations (41),(42) and (43). Using again relation (33) we get

$$\begin{aligned}
&T_1(E_1E_2 - q^{-2}E_2E_1)\\
&= \frac{K_1^{-1}K_1}{q-q^{-1}}\Bigg(q^2E_2E_1 + E_2E_1 - (q^{-2}+1)E_2E_1\\
&\qquad\qquad - (q^{-2}+1)E_1E_2 + q^{-2}E_1E_2 + q^{-4}E_1E_2\Bigg)\\
&\quad - \frac{K_1^{-1}K_1^{-1}}{q-q^{-1}}\Bigg(q^{-2}E_2E_1 + E_2E_1 - (q^{-2}+1)E_2E_1\\
&\qquad\qquad - (q^{-2}+1)E_1E_2 + q^{-2}E_1E_2 + E_1E_2\Bigg)\\
&= E_2E_1 - q^{-2}E_1E_2.
\end{aligned}$$

Multiplying with $\frac{1}{[2]}$ gives $T_1(A) = B$. □

We know that $v_1 = Av_0 - v_0A$. Similarly, $T_2(E_1) = BE_2 - E_2B$. Applying T_1 yields to $v_2 = Av_1 - v_1A$. Applying $T_1 \circ T_2$ to the first equation yields to $v_3 = Av_2 - v_2A$.

Thus, every v_i, for $1 \le i \le 3$, satifies the commutator relation $v_i = Av_{i-1} - v_{i-1}A$.

Lemma 3.7. The equation $v_iv_{i+1} = q^{-2}v_{i+1}v_i$ holds for $0 \le i \le 2$, the equation $v_iv_{i+2} = q^{-2}v_{i+2}v_i + (q^{-2}-1)v_{i+1}^2$ holds for $0 \le i \le 1$, and the equation $v_iv_{i+3} = q^{-2}v_{i+3}v_i + (q^{-4}-1)v_{i+2}v_{i+1}$ holds for $i = 0$.

Proof. The equations in the first line have already been checked. Now

$$\begin{aligned}
v_0v_2 &= v_0Av_1 - v_0v_1A = (Av_0 - v_1)v_1 - q^{-2}v_1(Av_0 - v_1)\\
&= q^{-2}Av_1v_0 - v_1^2 - q^{-2}v_1Av_0 + q^{-2}v_1^2 = q^{-2}v_2v_0 + (q^{-2}-1)v_1^2.
\end{aligned}$$

By interchanging the role of E_1 and E_2 and applying T_1 we also get $v_1v_3 = q^{-2}v_3v_1 + (q^{-2}-1)v_2^2$. These are the equations in the second line of the Lemma. Furthermore

$$\begin{aligned}
v_0v_3 &= v_0Av_2 - v_0v_2A = (Av_0 - v_1)v_2 - (q^{-2}v_2v_0 + (q^{-2}-1)v_1^2)A\\
&= Av_0v_2 - v_1v_2 - q^{-2}v_2v_0A + (q^{-2}-1)v_1^2A\\
&= A(q^{-2}v_2v_0 + (q^{-2}-1)v_1^2) - q^{-2}v_2v_1 - q^{-2}v_2v_0A - (q^{-2}-1)v_1^2A\\
&= q^{-2}(Av_2 - v_2A)v_0 + (q^{-2}-1)Av_1^2 - (q^{-2}-1)v_1^2A.
\end{aligned}$$

Now the rest of the lemma follows from $Av_1^2 - v_1^2A = (v_1A + v_2)v_1 - v_1(Av_1 - v_2) = v_2v_1 + v_1v_2 = (q^{-2}+1)v_2v_1$. □

These relations are called *straightening relations*; they enable us to write every element in $U_q^+(s_1s_2s_1s_2)$ as a $\mathbb{Q}(q)$-linear combination of basis elements in $\mathcal{P}$, i.e. elements of the form $v_3^{(a_3)}v_2^{(a_2)}v_1^{(a_1)}v_0^{(a_0)}$ with $a_3, a_2, a_1, a_0 \in \mathbb{N}$ and coefficients in $\mathbb{Q}(q)$.

The straightening relations tell us that $U_q^+(s_1s_2s_1s_2)$ becomes a *commutative* subalgebra of $U(\mathfrak{n})$ if we specialize $q = 1$. This is remarkable because $U(\mathfrak{n})$ is a noncommutative algebra. (For instance, $E_1E_2 \neq E_2E_1$ in $U(\mathfrak{n})$.) The specialization $q = 1$ is sometimes called the *classical limit*.

3.9 The dual canonical basis

The definitions, results, and proofs from Subsection 3.9 and Lemma 3.9 from Subsection 3.10 are due to Leclerc ([37]). Since [37] is not published we give a brief sketch.

To study the algebra $U_q^+(s_1s_2s_1s_2)$ Lusztig [42] and Kashiwara [28] introduced (slightly different) non-degenerate bilinear forms $(.,.) : U_q(\mathfrak{n}) \times U_q(\mathfrak{n}) \to \mathbb{Q}(q)$. We work with Kashiwara's form. As described in [36] the *dual PBW basis* is defined to be the basis adjoint to $\mathcal{P}$ with respect to the bilinear form. The generators v_i, for $0 \leq i \leq 3$, satisfy (compare [36, Section 4.7] and note the difference in sign conventions)

$$(v_i, v_i) = \big(E_{(i+1)\alpha_1+i\alpha_2}, E_{(i+1)\alpha_1+i\alpha_2}\big) = \frac{(1-q^{-2})^{2i+1}}{1-q^{-2}} = (1-q^{-2})^{2i}.$$

Therefore, the duals are given by $u_i = \frac{1}{(1-q^{-2})^{2i}} v_i$. We see that the u_i, $0 \leq i \leq 3$, satisfy the *same* straightening relations,

$$\begin{aligned}
u_iu_{i+1} &= q^{-2}u_{i+1}u_i, && (0 \leq i \leq 2),\\
u_iu_{i+2} &= q^{-2}u_{i+2}u_i + (q^{-2}-1)u_{i+1}^2, && (0 \leq i \leq 1),\\
u_iu_{i+3} &= q^{-2}u_{i+3}u_i + (q^{-4}-1)u_{i+2}u_{i+1}, && (i = 0).
\end{aligned}$$

It is also possible to derive the straightening relations using Leclerc's algorithm from [36] which features quantum shuffles.

To study the algebra $U_q^+(s_1s_2s_1s_2)$ Leclerc [37] introduced the following structures:

- a ring anti-automorphism $\sigma\colon U_q^+(s_1s_2s_1s_2) \to U_q^+(s_1s_2s_1s_2)$ by $\sigma(q) = q^{-1}$ and $\sigma(u_i) = q^{2i}u_i$ for $i \in \{0,1,2,3\}$,
- a norm $N\colon \mathbb{N}^4 \to \mathbb{Z}$ by $N(a_3,a_2,a_1,a_0) = (a_3+a_2+a_1+a_0)^2 - 7a_3 - 5a_2 - 3a_1 - a_0$,
- a partial order $\lhd$ on $\mathbb{N}^4$ by saying that $\mathbf{a},\mathbf{b} \in \mathbb{N}^4$ satisfy $\mathbf{a} \lhd \mathbf{b}$ if and only if $\mathbf{b} - \mathbf{a} \in \mathbb{N}\,(-1,2,-1,0) \bigoplus \mathbb{N}\,(0,-1,2,-1)$,
- a set $S(\mathbf{a}) = \big\{\mathbf{b} \in \mathbb{N}^4 \colon \mathbf{a} \lhd \mathbf{b} \text{ and } \mathbf{a} \neq \mathbf{b}\big\}$,
- a function $b\colon \mathbb{N}^4 \to \mathbb{Z}$ by $b(a_3,a_2,a_1,a_0) = \binom{a_3}{2} + \binom{a_2}{2} + \binom{a_1}{2} + \binom{a_0}{2}$.

Using these definitions one can describe the dual PBW basis and construct another basis of $U_q^+(w)$, the *dual canonical basis*. Both bases are parametrized by $\mathbb{N}^4$. For every $\mathbf{a} = (a_3,a_2,a_1,a_0) \in \mathbb{N}^4$ the dual PBW basis element corresponding to $\mathbf{a}$, $E[\mathbf{a}]$, is given by $E[\mathbf{a}] = q^{b(\mathbf{a})}u_3^{a_3}u_2^{a_2}u_1^{a_1}u_0^{a_0}$, see [36, Section 5.5]. It is a rescaling of the PBW basis and for every $\mathbf{a} \in \mathbb{N}^4$ we have $(E[\mathbf{a}], v[\mathbf{a}]) = 1$. In what follows we often use the fact that $N(\mathbf{a}) = N(\mathbf{b})$ if $\mathbf{b} \lhd \mathbf{a}$.

Theorem 3.8 (Leclerc, [37])**.** There is a basis $\big\{B[\mathbf{a}] \colon \mathbf{a} \in \mathbb{N}^4\big\}$ of $U_q^+(w)$ such that for every $\mathbf{a} \in \mathbb{N}^4$ the following two conditions hold

- $B[\mathbf{a}] - E[\mathbf{a}] \in \bigoplus_{\mathbf{b}\in S(\mathbf{a})} q\mathbb{Z}[q]E[\mathbf{b}]$,
- $\sigma(B[\mathbf{a}]) = q^{-N(\mathbf{a})}B[\mathbf{a}]$.

Proof. For every $k \in \mathbb{N}$ consider the set $W_k = \{\mathbf{a} \in \mathbb{N}^4 : a_3 + a_2 + a_1 + a_0 = k\}$. Extend the partial order $\lhd$ on W_k to a total order $<$ so that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_l$ are the elements of W_k written in increasing order.

We induct backwards. We can start with $B[\mathbf{a}_l] = E[\mathbf{a}_l]$. For the induction step, suppose that $\mathbf{a}_{m+1}, \mathbf{a}_{m+2}, \dots, \mathbf{a}_l$ satisfy the two conditions of Theorem 3.8. Expand $\sigma(E[\mathbf{a}_m])$ in the dual PBW basis using the straightening relations. We get a $\mathbb{Z}[q, q^{-1}]$-linear combination of basis elements $E[\mathbf{b}]$ with $b \in \{\mathbf{a}_m, \mathbf{a}_{m+1}, \dots, \mathbf{a}_l\}$, so that

$$\sigma(E[\mathbf{a}_m]) = \sum_{i=m}^{l} c_i E[\mathbf{a}_i]$$

with $c_i \in \mathbb{Z}[q, q^{-1}]$. A short calculation shows that $c_m = q^{-N(\mathbf{a})}$; to get $E[\mathbf{a}_m]$ you always have to choose the first summand when straightening a monomial.

By induction hypothesis we know that each $B[\mathbf{a}_i]$, for $m + 1 \le i \le l$, is a $\mathbb{Z}[q, q^{-1}]$-linear combination in the elements $E[\mathbf{a}_i], E[\mathbf{a}_{i+1}], \dots, E[\mathbf{a}_l]$. The vector $(B[\mathbf{a}_{m+1}], B[\mathbf{a}_{m+2}], \dots, B[\mathbf{a}_l])$ is obtained from $(E[\mathbf{a}_{m+1}], \dots, E[\mathbf{a}_l])$ by multiplication with an upper triangular matrix with diagonal entries 1. By inverting we may write each $B[\mathbf{a}_i]$, for $m + 1 \le i \le l$, as a $\mathbb{Z}[q, q^{-1}]$-linear combination of $B[\mathbf{a}_i]$, $B[\mathbf{a}_{i+1}], \dots, B[\mathbf{a}_l]$. Thus,

$$\sigma(E[\mathbf{a}_m]) = q^{-N(\mathbf{a}_m)}E[\mathbf{a}_m] + \sum_{i=m+1}^{l} d_i B[\mathbf{a}_i]$$

for some $d_i \in \mathbb{Z}[q, q^{-1}]$. Apply σ, an involution, to get

$$E[\mathbf{a}_m] = q^{N(\mathbf{a}_m)}\sigma(E[\mathbf{a}_m]) + \sum_{i=m+1}^{l} \sigma(d_i) q^{-N(\mathbf{a}_i)} B[\mathbf{a}_i].$$

The $B[\mathbf{a}_i]$, for $m + 1 \le i \le l$, are linearly independent. Multiply the first equation with $q^{N(\mathbf{a}_i)}$ (and remember that $N(\mathbf{a}_i) = N(\mathbf{a}_m)$) to get $q^{N(\mathbf{a}_i)}d_i = -q^{-N(\mathbf{a}_i)}\sigma(d_i) = -\sigma(q^{N(\mathbf{a}_i)}d_i)$. Therefore, there are polynomials $\phi_i \in q\mathbb{Z}[q]$ such that $q^{N(\mathbf{a}_i)}d_i = \phi_i(q) - \phi_i(q^{-1})$. Now $B[\mathbf{a}_m] = E[\mathbf{a}_m] + \sum_{i=m+1}^{l} \phi_i B[\mathbf{a}_i]$ satisfies the two conditions of Theorem 3.8. □

The two conditions of Theorem 3.8 imply that the basis $\{B[\mathbf{a}] : \mathbf{a} \in \mathbb{N}^4\}$ is adjoint to a basis $\{b[\mathbf{a}] : \mathbf{a} \in \mathbb{N}^4\}$ with respect to the bilinear form from above that satisfies the following two properties. On one hand we have $(b[\mathbf{a}], b[\mathbf{a}]) \in 1 + q\mathbb{Z}[[q]]$ for every $\mathbf{a} \in \mathbb{N}^4$. On the other hand we have $\overline{b[\mathbf{a}]} = b[\mathbf{a}]$ for every $\mathbf{a} \in \mathbb{N}^4$. Here, the symbol $\bar{\ }$ denotes the bar involution from [36, Proposition 6]. It follows from [42, Theorem 14.2.3] that $\{b[\mathbf{a}] : \mathbf{a} \in \mathbb{N}^4\}$ is Lusztig's canonical basis. Therefore, $\{B[\mathbf{a}] : \mathbf{a} \in \mathbb{N}^4\}$ is the dual of the canonical basis, or the *dual canonical basis* to put it shortly.

The two conditions of Theorem 3.8 uniquely determine the dual canonical basis.

The simplest elements in the dual canonical basis are given by $B[1, 0, 0, 0] = u_3$, $B[0, 1, 0, 0] = u_2$, $B[0, 0, 1, 0] = u_1$, and $B[0, 0, 0, 1] = u_0$. Further examples include

- $B[1, 0, 1, 0] = u_3u_1 - q^2u_2^2$,

- $B[0,1,0,1] = u_2u_0 - q^2u_1^2$,
- $B[1,0,0,1] = u_3u_0 - q^2u_2u_1$,
- $B[2,0,0,1] = qu_3^2u_0 - (q+q^3)u_3u_2u_1 + q^5u_2^3$,
- $B[1,0,0,2] = qu_3u_0^2 - (q+q^3)u_2u_1u_0 + q^5u_1^3$,
- $B[2,0,0,2] = q^2u_3^2u_0^2 - (2q^3+q^4)u_3u_2u_1u_0 - q^6u_2^3u_0 - q^6u_3u_1^3 + q^8u_2^2u_1^2$.

Note that $B[1,0,1,0]$ and $B[0,1,0,1]$ are q-deformations of the elements P_1 and P_0. We introduce the abbreviations $p_1 = u_3u_1 - q^2u_2^2$ and $p_0 = u_2u_0 - q^2u_1^2$. As observed by Leclerc, the elements p_0 and p_1 have the remarkable property that they q-commute with each other and with each of the generators u_3, u_2, u_1 and u_0. More precisely, there holds $p_0p_1 = q^{-4}p_1p_0$ and $p_0u_0 = q^2u_0p_0$, $p_0u_1 = u_1p_0$, $p_0u_2 = q^{-2}u_2p_0$, $p_0u_3 = q^{-4}u_3p_0$, $p_1u_0 = q^4u_0p_1$, $p_1u_1 = q^2u_1p_1$, $p_1u_2 = u_2p_1$, $p_1u_3 = q^{-2}u_3p_1$. These relations can be checked by elementary calculations using the straightening relations.

3.10 A recursion for dual canonical basis elements

Let us introduce the convention that $B[\mathbf{a}] = 0$ for some $\mathbf{a} = (a_3,a_2,a_1,a_0) \in \mathbb{Z}^4$ if there is an $i \in \{0,1,2,3\}$ such that $a_i < 0$. Note that $B[0,0,0,0] = 1$.

Lemma 3.9 (Leclerc, [37])**.** For every $\mathbf{a} = (a_3,a_2,a_1,a_0) \in \mathbb{N}^4$ the equations

$$B[a_3, a_2+1, a_1, a_0+1] = q^{a_2+2a_1+3a_0}B[\mathbf{a}]p_0 = q^{4a_3+3a_2+2a_1+a_0}p_0B[\mathbf{a}],$$
$$B[a_3+1, a_2, a_1+1, a_0] = q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}] = q^{a_3+2a_2+3a_1+4a_0}B[\mathbf{a}]p_1$$

hold.

Proof. We only prove the equation $B[a_3+1,a_2,a_1+1,a_0] = q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}]$. The other equations are similar. We prove that $q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}]$ satisfies the two conditions of Theorem 3.8. Let us expand $B[\mathbf{a}]$ in the dual PBW basis, i.e. $B[\mathbf{a}] = \sum c_{b_3,b_2,b_1,b_0} q^{b(b_3,b_2,b_1,b_0)} u_3^{b_3}u_2^{b_2}u_1^{b_1}u_0^{b_0}$ where the sum is taken over all vectors $\mathbf{b} = (b_3,b_2,b_1,b_0) \in \mathbb{N}^4$ such that $\mathbf{b} \lhd \mathbf{a}$ and $c_{\mathbf{b}} \in \mathbb{Z}[q]$. If $\mathbf{b} \neq \mathbf{a}$, then $c_{\mathbf{b}} \in q\mathbb{Z}[q]$.

Next, $u_3^{b_3}u_2^{b_2}u_1^{b_1}u_0^{b_0}p_1 = q^{2b_3-2b_1-4b_0}p_1u_3^{b_3}u_2^{b_2}u_1^{b_1}u_0^{b_0}$. If $\mathbf{b} \lhd \mathbf{a}$, then $2b_3 - 2b_1 - 4b_0 = 2a_3 - 2a_1 - 4a_0$. Therefore, we can conclude that $p_1B[\mathbf{a}]$ is invariant under σ up to a power of q. More precisely, we have

$$\begin{aligned}\sigma\left(q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}]\right) &= q^{-3a_3-2a_2-a_1}q^{-N(1,0,1,0)}q^{-N(\mathbf{a})}B[\mathbf{a}]p_1\\ &= q^{-3a_3-2a_2-a_1}q^{-N(1,0,1,0)}q^{-N(\mathbf{a})}q^{2a_3-2a_1-4a_0}p_1B[\mathbf{a}]\\ &= q^{-N(a_3+1,a_2,a_1+1,a_0)}q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}].\end{aligned}$$

Recall that $p_1 = u_3u_1 - q^2u_2^2$. Thus, $q^{3a_3+2a_2+a_1}p_1B[\mathbf{a}]$ is equal to

$$\sum c_{\mathbf{b}}q^{b(\mathbf{b})}q^{3a_3+2a_2+a_1}\left(u_3u_1u_3^{b_3}u_2^{b_2}u_1^{b_1}u_0^{b_0} - q^2u_2^2u_3^{b_3}u_2^{b_2}u_1^{b_1}u_0^{b_0}\right).$$

One can check by induction that for every positive integer l there holds $u_1u_3^l = q^{-2l}u_3^lu_1 + \left(q^{-4l+2} - q^{-2l+2}\right)u_3^{l-1}u_2^2$. The sum above simplifies to

$$\begin{aligned}&\sum c_{\mathbf{b}}q^{b(\mathbf{b})}q^{3a_3+2a_2+a_1}\left(q^{-2b_3}u_3^{b_3+1}u_1u_2^{b_2}u_1^{b_1}u_0^{b_0} - q^{-2b_3+2}u_3^{b_3}u_2^{b_2+2}u_1^{b_1}u_0^{b_0}\right)\\ &\quad= \sum c_{\mathbf{b}}q^{b(b_3+1,b_2,b_1+1,b_0)}u_3^{b_3+1}u_2^{b_2}u_1^{b_1+1}u_0^{b_0}\\ &\qquad- \sum c_{\mathbf{b}}q^{b_3+b_1+1}q^{b(b_3,b_2+2,b_1,b_0)}u_3^{b_3}u_2^{b_2+2}u_1^{b_1}u_0^{b_0}.\end{aligned}$$

The coefficients $c_{\mathbf{b}}$ are in $q\mathbb{Z}[q]$ except for $c_{\mathbf{a}} = 1$. □

The preceding lemma enables us two write every dual canonical basis element $B[a_3, a_2, a_1, a_0]$ as a product of powers of dual canonical basis elements p_1, p_0, a power of the parameter q and an element of the form $B[a_3, a_2, 0, 0]$, $B[0, a_2, a_1, 0]$, $B[0, 0, a_1, a_0]$ or $B[a_3, 0, 0, a_0]$.

We have $B[a_3, a_2, 0, 0] = E[a_3, a_2, 0, 0]$, $B[0, a_2, a_1, 0] = E[0, a_2, a_1, 0]$ and $B[0, 0, a_1, a_0] = E[0, 0, a_1, a_0]$, because these sequences are maximal elements with respect to the partial order $\lhd$. Therefore, dual canonical basis elements of the form $B[a_3, 0, 0, a_0]$ are particularly interesting. The elements $B[a_3, 0, 0, a_0]$ with $|a_3 - a_0| \leq 1$ can be computed recursively.

Theorem 3.10. For every $n \geq 1$ the following recursions

$$\begin{aligned} B[n,0,0,n-1] &= q^{n-1}u_3 B[n-1,0,0,n-1] - q^{2n-1}u_2 B[n-1,0,1,n-2] \\ &= q^{3n-3} B[n-1,0,0,n-1]\, u_3 - q^{2n-3} B[n-1,0,1,n-2]\, u_2, \end{aligned}$$

$$\begin{aligned} B[n-1,0,0,n] &= q^{n-1} B[n-1,0,0,n-1]\, u_0 - q^{2n-1} B[n-2,1,0,n-1]\, u_1 \\ &= q^{3n-3} u_0 B[n-1,0,0,n-1] - q^{2n-3} u_1 B[n-2,1,0,n-1], \end{aligned}$$

$$\begin{aligned} B[n,0,0,n] &= q^{n-1} B[n,0,0,n-1]\, u_0 - q^{2n} B[n-1,1,0,n-1]\, u_1 \\ &= q^{3n-1} u_0 B[n,0,0,n-1] - q^{2n-2} u_1 B[n-1,1,0,n-1] \\ &= q^{n-1} u_3 B[n-1,0,0,n] - q^{2n} u_2 B[n-1,0,1,n-1] \\ &= q^{3n-1} B[n-1,0,0,n]\, u_3 - q^{2n-2} B[n-1,0,1,n]\, u_2, \end{aligned}$$

for the dual canonical basis elements parametrized by $(n, 0, 0, n-1)$, $(n-1, 0, 0, n)$, and $(n, 0, 0, n)$ hold.

The recursions allow us to compute the dual canonical basis elements $B[n, 0, 0, n-1]$, $B[n-1, 0, 0, n]$, and $B[n, 0, 0, n]$ from the likewise elements with lower indices.

Proof. We prove the three statements simultaneously by mathematical induction on n. For $n = 1$ the equations become $B[1, 0, 0, 0] = u_3 = u_3$, $B[0, 0, 0, 1] = u_0 = u_0$, and $B[1, 0, 0, 1] = u_3 u_0 - q^2 u_2 u_1 = q^2 u_0 u_3 - u_1 u_2$. Using the explicit formulae for $B[2, 0, 0, 1], B[1, 0, 0, 2]$ and $B[2, 0, 0, 2]$ from above and the straightening relations one can check that the equations are true for $n = 2$. Suppose that $n \geq 3$ and that the three statements are true for all smaller n. Define $f = q^{n-1} u_3 B[n-1, 0, 0, n-1] - q^{2n-1} u_2 B[n-1, 0, 1, n-2]$. We claim that

(i) $f - E[n, 0, 0, n-1] \in \bigoplus_{\mathbf{b} \in S((n,0,0,n-1))} q\mathbb{Z}E[\mathbf{b}]$,

(ii) $\sigma(f) = q^{-N(n,0,0,n-1)} f$.

The two claims imply that $f = B[n, 0, 0, n-1]$.

Let us expand $B[n-1, 0, 0, n-1]$ in the dual PBW basis, i.e. $B[n-1, 0, 0, n-1] = \sum c_{a_3,a_2,a_1,a_0} q^{b(a_3,a_2,a_1,a_0)} u_3^{a_3} u_2^{a_2} u_1^{a_1} u_0^{a_0}$ where the sum is taken over all $\mathbf{a} = (a_3, a_2, a_1, a_0) \in \mathbb{N}^4$ such that $(n-1, 0, 0, n-1) \lhd \mathbf{a}$. This implies that $a_3 \leq n-1$. By definition of the dual canonical basis all coefficients obey $c_{a_3,a_2,a_1,a_0} \in q\mathbb{Z}[q]$ except for $c_{n-1,0,0,n-1} = 1$. Then

$$\begin{aligned} &q^{n-1} u_3 B[n-1, 0, 0, n-1] \\ &= \sum c_{\mathbf{a}} q^{-\binom{a_3+1}{2} + \binom{a_3}{2} + n - 1} q^{b(a_3+1,a_2,a_1,a_0)} u_3^{a_3+1} u_2^{a_2} u_1^{a_1} u_0^{a_0}. \end{aligned}$$

But $q^{-\binom{a_3+1}{2}+\binom{a_3}{2}+n-1} = q^{n-1-a_3} \in \mathbb{Z}[q]$, so $c_{\mathbf{a}}q^{n-1-a_3} \in q\mathbb{Z}[q]$ except for the coefficient $q^{n-1-(n-1)}c_{n-1,0,0,n-1} = 1$.

Now let us expand $B[n-1,0,1,n-2]$ in the dual PBW basis, i.e. $B[n-1,0,1,n-2] = \sum d_{a_3,a_2,a_1,a_0} q^{b(a_3,a_2,a_1,a_0)} u_3^{a_3}u_2^{a_2}u_1^{a_1}u_0^{a_0}$ where the sum is taken over all $\mathbf{a} = (a_3,a_2,a_1,a_0) \in \mathbb{N}^4$ such that $(n-1,0,1,n-2) \lhd \mathbf{a}$. This implies that $a_3 \leq n-1$. By definition of the dual canonical basis all coefficients obey $d_{a_3,a_2,a_1,a_0} \in \mathbb{Z}[q]$. Then

$$\begin{aligned}&q^{2n-1}u_2 B[n-1,0,1,n-2]\\&= \sum d_{\mathbf{a}} q^{-a_2+2n-1-2a_3} q^{b(a_3,a_2+1,a_1,a_0)} u_3^{a_3}u_2^{a_2+1}u_1^{a_1}u_0^{a_0}.\end{aligned}$$

Since $(n-1,0,1,n-2) \lhd \mathbf{a}$, there exists non-negative integers r,s such that

$$(a_3,a_2,a_1,a_0) = (n-1,0,1,n-2) + s(-1,2,-1,0) + r(0,-1,2,-1).$$

Thus, $a_2 = 2s-r$ and $a_3 = n-1-s$ so that $-a_2+2n-1-2a_3 = r+1$. So $d_{\mathbf{a}}q^{-a_2+2n-1-2a_3} \in q\mathbb{Z}[q]$. So the first claim is satisfied.

Note that $N(n,0,0,n-1) = 4n^2-12n+2$. Put $N = 4n^2-12n+2$. We apply the anti-automorphism σ to f and use the fact that $B[n-1,0,0,n-1]$ and $B[n-1,0,1,n-2]$ are, up to a power of q, invariant under σ to get

$$\begin{aligned}&q^N\sigma(f) =\\&= q^{3n-3}B[n-1,0,0,n-1]u_3 - q^{2n-3}B[n-1,0,1,n-1]u_2\\&= q^{3n-3}B[n-1,0,0,n-1]u_3 - q^{5n-9}p_1B[n-2,0,0,n-2]u_2\\&= q^{3n-3}\left(q^{n-2}u_3B[n-2,0,0,n-1] - q^{2n-2}u_2B[n-2,0,1,n-2]\right)u_3\\&\quad - q^{5n-9}p_1\left(q^{n-3}u_3B[n-3,0,0,n-2] - q^{2n-4}u_2B[n-3,0,1,n-3]\right)u_2\\&= q^{4n-5}u_3B[n-2,0,0,n-1]u_3 - q^{5n-5}u_2B[n-2,0,1,n-2]u_3\\&\quad - q^{6n-12}p_1u_3B[n-3,0,0,n-2] - q^{7n-13}p_1u_2B[n-3,0,1,n-3]u_2.\end{aligned}$$

The first and the third summand in the last expression add up to

$$\begin{aligned}&q^{n-1}u_3\left(q^{3n-4}B[n-2,0,0,n-1]u_3 - q^{5n-13}p_1B[n-3,0,0,n-2]u_2\right)\\&= q^{n-1}u_3\left(q^{3n-4}B[n-2,0,0,n-1]u_3 - q^{2n-4}p_1B[n-2,0,1,n-2]u_2\right)\\&= q^{n-1}u_3B[n-1,0,0,n-1];\end{aligned}$$

whereas the second and the fourth summand add up to

$$\begin{aligned}&q^{5n-5}u_2B[n-2,0,1,n-2]u_3 - q^{7n-13}p_1u_2B[n-3,0,1,n-3]u_2\\&= q^{8n-14}u_2p_1B[n-3,0,0,n-2]u_3 - q^{7n-13}p_1u_2B[n-3,0,1,n-3]u_2\\&= q^{5n-7}u_2p_1\left(q^{3n-7}B[n-3,0,0,n-1]u_3 - q^{2n-6}B[n-3,0,1,n-3]u_2\right)\\&= q^{5n-7}u_2p_1B[n-2,0,0,n-2] = q^{2n-1}u_2B[n-1,0,1,n-2].\end{aligned}$$

Altogether we get $q^N\sigma(f) = q^{n-1}u_3B[n-1,0,0,n-1] - q^{2n-1}u_2B[n-1,0,1,n-2] = f$. We see that $f = B[n,0,0,n-1]$ and that the equations of Theorem 3.10 hold.

By the same argument one can show that

$$\begin{aligned} B[n-1,0,0,n] &= q^{n-1}B\left[n-1,0,0,n-1\right]u_0 - q^{2n-1}B\left[n-2,1,0,n-1\right]u_1 \\ &= q^{3n-3}u_0B\left[n-1,0,0,n-1\right] - q^{2n-3}u_1B\left[n-2,1,0,n-1\right]. \end{aligned}$$

By a very similar argument with the established recursion for $B[n-1,0,0,n]$ and by the inductively known recursion for $B[n-2,0,0,n-1]$ one can show just as above that an element $g = q^{n-1}u_3B\left[n-1,0,0,n\right] - q^{2n}u_2B\left[n-1,0,1,n-1\right]$ is indeed the dual canonical basis element $B[n,0,0,n]$ and derive the equations

$$\begin{aligned} B[n,0,0,n] &= q^{n-1}u_3B\left[n-1,0,0,n\right] - q^{2n}u_2B\left[n-1,0,1,n-1\right] \\ &= q^{3n-1}B\left[n-1,0,0,n\right]u_3 - q^{2n-2}B\left[n-1,0,1,n\right]u_2; \end{aligned}$$

with the same technique one can derive the other equations

$$\begin{aligned} B[n,0,0,n] &= q^{n-1}u_3B\left[n-1,0,0,n\right] - q^{2n}u_2B\left[n-1,0,1,n-1\right] \\ &= q^{3n-1}B\left[n-1,0,0,n\right]u_3 - q^{2n-2}B\left[n-1,0,1,n\right]u_2. \end{aligned}$$

□

From the previous lemma we get a corollary.

Corollary 3.11. The following equations

$$\begin{aligned} B[n,0,0,n-1]B[1,0,0,1] &= q^{3-4n}B[n+1,0,0,n] + q^{4-4n}B[n,1,1,n-1], \\ B[1,0,0,1]B[n,0,0,n-1] &= q^{1-4n}B[n+1,0,0,n] + q^{-4n}B[n,1,1,n-1], \\ B[n,0,0,n]B[1,0,0,1] &= q^{-4n}B[n+1,0,0,n+1] + q^{-4n}B[n,1,1,n], \\ B[1,0,0,1]B[n,0,0,n] &= q^{-4n}B[n+1,0,0,n+1] + q^{-4n}B[n,1,1,n]. \end{aligned}$$

hold for every integer $n \geq 1$.

The last two equations were conjectured by Leclerc.

Proof. We prove the statement by mathematical induction. The case $n = 1$ can be checked in a short calculation using the straightening relations. Note also that the last two equations in Corollary 3.11 make sense and are true for $n = 0$. Suppose that the equations hold for n and all smaller numbers. Let us write down the first equation of Theorem (3.10) for three consecutive integers $n+1$, n, and $n-1$,

$$B[n+1,0,0,n] = q^nu_3B[n,0,0,n] - q^{2n+1}u_2B[n,0,1,n-1], \tag{44}$$

$$B[n,0,0,n-1] = q^{n-1}u_3B[n-1,0,0,n-1] - q^{2n-1}u_2B[n-1,0,1,n-2], \tag{45}$$

$$B[n-1,0,0,n-2] = q^{n-2}u_3B[n-2,0,0,n-2] - q^{2n-3}u_2B[n-2,0,1,n-3]. \tag{46}$$

Multiply (46) from the right by $q^{4n-5}p_0p_1$to get

$$q^{4-4n}B[n,1,1,n-1] = q^{-3n+3}u_3B[n-1,1,1,n-1] - q^{n+1}u_2p_1B[n-2,1,1,n-2],$$

multiply (45) from the right by $B[1,0,0,1]$ to get

$$\begin{aligned} & B[n,0,0,n-1]B[1,0,0,1] \\ & = q^{n-1}u_3 B[n-1,0,0,n-1]B[1,0,0,1] \quad (47) \\ & - q^{5n-3}u_2 p_1 B[n-2,0,0,n-2]B[1,0,0,1], \quad (48) \end{aligned}$$

multiply (44) by q^{3-4n} to get

$$q^{3-4n}B[n+1,0,0,n] = q^{3-3n}u_3 B[n,0,0,n] - q^{n+1}u_2 p_1 B[n-1,0,0,n-1].$$

Using the induction hypothesis for $n-1$ and n we see that

$$B[n,0,0,n-1]B[1,0,0,1] = q^{3-4n}B[n+1,0,0,n] + q^{4-4n}B[n,1,1,n-1].$$

The other equations in Corollary 3.11 can be proved in a similar way. □

The last two equations in Corollary 3.11 imply that the dual canonical basis element $B[1,0,0,1]$ commutes with every $B[n,0,0,n]$ $(n \in \mathbb{Z})$. A conjecture by Berenstein and Zelevinsky (see [5]) says that the product of two dual canonical basis elements b_1 and b_2 is, up to a power of q, again a dual canonical basis element if and only if b_1 and b_2 q-commute, that is $b_1 b_2 = q^s b_2 b_1$ for some $s \in \mathbb{Z}$. The conjecture turns out to be wrong. Using his quantum shuffle algorithm from [36], in [38] Leclerc constructs five counterexamples. The last two equations of Corollary 3.11 give infinitely many counterexamples to Berenstein and Zelevinsky's conjecture. The commutativity of $B[1,0,0,1]$ and $B[n,0,0,n]$ implies q-commutativity, but the product $B[n,0,0,n]B[1,0,0,1]$ is a linear combination of *two* dual canonical basis elements.

Several authors, e.g. Reineke in [46], emphasized the importance of multiplicative properties of dual canoncial basis elements. Corollary 3.11 gives four series of expansions of products of dual canonical basis elements in the dual canonical basis.

3.11 The quantized version of the explicit formula for cluster variables

We may view Corollary 3.11 as a quantization of linear exchange relation for cluster variables from Section 3.5. Define the integral form $\mathcal{A}_{\mathbb{Z}}$ of $U_q^+(w)$ as $\mathcal{A}_{\mathbb{Z}} = \bigoplus_{\mathbf{a}\in\mathbb{N}^4} \mathbb{Z}[q,q^{-1}]u[\mathbf{a}]$; define an algebra $\mathcal{A}_1$ to be $\mathcal{A}_1 = \mathbb{Q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{A}_{\mathbb{Z}}$. Then $\mathcal{A}_1 = \mathbb{Q}[U_0,U_1,U_2,U_3]$ with $U_i = 1 \otimes u_i$ for $i = 0,1,2,3$. We see that $\mathcal{A}_1 = \mathcal{A}(\mathcal{C}_M)$.

Note that $B[1,0,0,1] = u_3 u_0 - q^2 u_2 u_1 \in \mathcal{A}_{\mathbb{Z}}$ becomes $1 \otimes B[1,0,0,1] = z \in \mathcal{A}_1$ in the specialization $q = 1$. The elements $p_1 = u_3 u_1 - q^2 u_2^2$ and $p_0 = u_2 u_0 - q^2 u_1^2$ specialize to $P_1 = U_3 U_1 - U_2^2$ and $P_0 = U_2 U_0 - U_1^2$. Corollary 3.11 means that the elements $B[n,0,0,n-1]$ are quantized cluster variables, because $B[n,1,1,n-1]$ is equal to $B[n-1,0,0,n-2]p_1 p_0$ up to a power of q. Similarly, the specialization of $B[n,0,0,n]$ at $q=1$ is an element in Caldero-Zelevinsky's semicanonical basis of $\mathcal{A}(\mathcal{C}_M)$, because Corollary 3.11 provides a quantized version of the Chebyshev recursion $s_{k+1} = z s_k - P_1 P_0 s_{k-1}$ for $k \geq 2$.

In the rest of this section we want to study the quantized cluster algebra structure of $U_q^+(w)$. We will work with quantum binomial coefficients instead of ordinary binomial coefficients as in Section 3.

Proposition 3.12. The following quantized version of the addition rule in Pascal's triangle

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{k-n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = q^{-k} \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

for quantum binomial coefficients holds.

Proof. See [27], p. 17-18. □

Definition 3.13. Define two functions $f, g\colon \mathbb{Z}^3 \to \mathbb{Z}$ by $f(n,k,l) = n(n-2) + k(n+2) + l(n+1) - 2kl$, and $g(n,k,l) = n(n-3) + k(n+1) + l(n+1) - 2kl$.

Theorem 3.14. For every natural number $n \geq 0$ we have

$$u_2^n B\left[n+1,0,0,n\right] u_1^{n+1} = \sum_{k,l} q^{f(n,k,l)} \begin{bmatrix} n-k \\ l \end{bmatrix} \begin{bmatrix} n+1-l \\ k \end{bmatrix} p_1^{n+1-k} u_2^{2k} u_1^{2l} p_0^{n-l}, \tag{49}$$

$$u_2^n B\left[n,0,0,n\right] u_1^n = \sum_{k,l} q^{g(n,k,l)} \begin{bmatrix} n-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-l}. \tag{50}$$

The summation in the first case runs over all pairs $(k,l) \in \mathbb{N}^2$ such that $k + l \leq n$ or $(k,l) = (n+1,0)$; the summation in the second case runs over all pairs $(k,l) \in \mathbb{N}^2$ such that $k + l \leq n$.

Proof. We prove the theorem by mathematical induction. One can check that both equations hold for $n = 0$ and $n = 1$. Let $n \geq 2$ and suppose that the equations hold for all smaller values of n. By Theorem 3.10 we have

$$\begin{aligned} B[n,0,0,n] &= q^{n-1} B[n,0,0,n-1] u_0 - q^{2n} B[n-1,1,0,n-1] u_1 \\ &= q^{n-1} B[n,0,0,n-1] u_0 - q^{5n-6} B[n-1,0,0,n-2] p_0 u_1. \end{aligned}$$

In the following calculations we intensively use the fact that the four variables p_1, p_0, u_2, u_1 q-commute with each other, see Subsection 3.9. By induction hypothesis we can assume that

$$\begin{aligned} & u_2^n q^{n-1} B[n,0,0,n-1] u_0 u_1^n \\ &= q^{-n-1} u_2^n B[n,0,0,n-1] u_1^n u_0 \\ &= \sum_{k,l} q^{f(n-1,k,l)-n-1} \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} u_2 p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-1-l} u_0 \\ &= \sum_{k,l} q^{f(n-1,k,l)+n-3+2l} \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-1-l} u_2 u_0. \end{aligned} \tag{51}$$

Now we use the identity $u_2 u_0 = p_0 + q^2 u_1^2$. The sum (51) splits into two summmands, namely

$$\sum_{k,l} q^{f(n-1,k,l)+n-3+2l} \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-l}, \tag{52}$$

and

$$\sum_{k,l} q^{f(n-1,k,l)+n-1+2l} \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l+2} p_0^{n-1-l}$$
$$= \sum_{k,l} q^{f(n-1,k,l)+n-3+2l} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n+1-l \\ k \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-l}. \qquad (53)$$

In the last step we shifted the index from l to $l-1$. Again by induction hypothesis we have

$$u_2^n q^{5n-6} B[n-1,0,0,n-1] p_0 u_1^n$$
$$= \sum_{k,l} q^{f(n-2,k,l)+5n-6} \begin{bmatrix} n-2-k \\ l \end{bmatrix} \begin{bmatrix} n-1-l \\ k \end{bmatrix} p_1^{n-1-k} u_2^{2k+2} u_1^{2l+2} p_0^{n-1-l}$$
$$= \sum_{k,l} q^{f(n-2,k-1,l-1)+5n-6} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n-l \\ k-1 \end{bmatrix} p_1^{n-k} u_2^{2k} u_1^{2l} p_0^{n-l}. \qquad (54)$$

A calculation shows that $f(n-1,k,l) - g(n,k,l) = -n+3-l$, $f(n-1,k,l-1) - g(n,k,l) = -2n+3-l$ and $f(n-2,k-1,l-1) - g(n,k,l) = -5n+7+k$. Thus, by comparing coefficients in (52), (53) and (54) it is enough to show that

$$\begin{bmatrix} n-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} = q^l \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} + q^{-n+l} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n+1-l \\ k-1 \end{bmatrix}$$
$$- q^{k+1} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n-l \\ k-1 \end{bmatrix}.$$

But, by Proposition 3.12,

$$\begin{bmatrix} n-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} - q^l \begin{bmatrix} n-1-k \\ l \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix} = \begin{bmatrix} n-l \\ k \end{bmatrix} \left(\begin{bmatrix} n-k \\ l \end{bmatrix} - q^l \begin{bmatrix} n-k \\ l \end{bmatrix} \right)$$
$$= q^{l+k-n} \begin{bmatrix} n-l \\ k \end{bmatrix} \begin{bmatrix} n-k-1 \\ l-1 \end{bmatrix},$$

and

$$q^{-n+l} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n+1-l \\ k \end{bmatrix} - q^{k+1} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n-l \\ k-1 \end{bmatrix}$$
$$= q^{n-l} \begin{bmatrix} n-k-1 \\ l-1 \end{bmatrix} \left(\begin{bmatrix} n+1-l \\ k \end{bmatrix} - q^{l+k-n+1} \begin{bmatrix} n-l \\ k-1 \end{bmatrix} \right)$$
$$= q^{k+l-n} \begin{bmatrix} n-1-k \\ l-1 \end{bmatrix} \begin{bmatrix} n-l \\ k \end{bmatrix}.$$

This proves (49). Once we have established equation (49), equation (50) is proved similarly using a recursion for $B[n+1,0,0,n]$ from Theorem 3.10 involving $B[n,0,0,n]$ and $B[n,0,1,n-1]$. □

Theorem 3.14 is a quantized version of the formula (27) for the cluster variables in Section 3. There is an analogous formula for $B[n,0,0,n+1]$.

It is possible to give an explicit expansion of dual canonical elements of the form $B[n,0,0,n-1]$ in the (dual) PBW basis. Therefore we have to write powers of p_1 and

p_0 in the (dual) PBW basis. The following relations can be proved by induction. For every natural number k the relations

$$p_1^k = \sum_{i=0}^{k} (-1)^i q^{2i^2-ik-k^2+i+k} \begin{bmatrix} k \\ i \end{bmatrix} u_3^{k-i} u_2^{2i} u_1^{k-i}, \tag{55}$$

$$p_0^k = \sum_{i=0}^{k} (-1)^i q^{2i^2-ik-k^2+i+k} \begin{bmatrix} k \\ i \end{bmatrix} u_2^{k-i} u_1^{2i} u_0^{k-i} \tag{56}$$

hold. Substituting (55) and (56) in (49) yields to

$$\begin{aligned} B[n+1,0,0,n] = \sum_{k,l,r,s} (-1)^{k+l+s+r+1} \begin{bmatrix} n-k \\ l \end{bmatrix} \begin{bmatrix} n+1-l \\ k \end{bmatrix} \begin{bmatrix} n+1-k \\ s \end{bmatrix} \begin{bmatrix} n-l \\ r \end{bmatrix} \\ \cdot q^{-l-2kl+2n+kn+ln-3r-lr-r^2+s-ks+2rs-s^2} \\ \cdot E[s, n+2-2s+r, n-1-2r+s, r], \end{aligned}$$

here the sum is taken over all $k, l, r, s \in \mathbb{N}$ such that $0 \le s \le n+1-k$, $0 \le r \le n-l$ and either $k+l \le n$ or $(k,l) = (n+1, 0)$. The formula is an q-analogue of (29).

3.12 The quasi-commutativity of adjacent quantized cluster variables and the quantum exchange relation

We prove that adjacent quantized cluster quasi-commute, i.e., they are commutative up to a power of q.

Lemma 3.15. For every $n \ge 1$ the elements $B[n+1,0,0,n]$ and $B[n,0,0,n-1]$ are q-commutative. More precisely, $B[n,0,0,n-1]B[n+1,0,0,n] = q^2 B[n+1,0,0,n]B[n,0,0,n-1]$.

Proof. We prove the theorem by mathematical induction. We can verify the statement for $n = 1$ in a short calculation using the straightening relations. Suppose that the statement holds for $n-1$. Combine Lemma 3.9 with Corollary 3.11 to get

$$\begin{aligned} B[n+1,0,0,n] &= q^{4n-1} B[1,0,0,1] B[n,0,0,n-1] \\ &\quad - q^{8n-14} B[n-1,0,0,n-2] p_1 p_0 \\ &= q^{4n-3} B[n,0,0,n-1] B[1,0,0,1] \\ &\quad - q^{8n-12} B[n-1,0,0,n-2] p_1 p_0. \end{aligned}$$

Multiply the first expression for $B[n+1,0,0,n]$ from the left and the second from the right with $B[n,0,0,n-1]$. It remains to show that $B[n,0,0,n-1]B[n-1,0,0,n-2]p_1p_0 = q^4 B[n-1,0,0,n-2]p_1p_0B[n,0,0,n-1]$, which follows from the induction hypothesis and the relation $B[n,0,0,n-1]p_1p_0 = q^6 p_1p_0B[n,0,0,n-1]$, which follows from Lemma 3.9. ☐

Lemma 3.15 says that two adjacent quantized cluster variables $B[n+1,0,0,n]$ and $B[n,0,0,n-1]$ form a *quantum torus*. If we specialize $q = 1$, the elements $B[n+1,0,0,n]$ and $B[n,0,0,n-1]$ become cluster variables in the same cluster of $\mathcal{A}(\mathcal{C}_M)$.

Lemma 3.16. For $n \geq 2$ we have

$$B[n+1,0,0,n]B[n-1,0,0,n-2] = q^2 B[n,0,0,n-1]^2 + q^{2n^2-6n+8} p_1^{n+1} p_0^{n-2}.$$

Proof. The statement is true in the case $n = 2$. We use mathematical induction. Consider the product $p = B[n,0,0,n-1]B[1,0,0,1]B[n-1,0,0,n-2]$. We evaluate p according to Corollary 3.11 in two different ways. On one hand we get

$$\begin{aligned} p &= B[n,0,0,n-1]\left(q^{5-4n}B[n,0,0,n-1] + q^{4-4n}B[n-1,0,0,n-2]\right) \\ &= q^{5-4n}B[n,0,0,n-1]^2 + q^{4n-17}B[n,0,0,n-1]B[n-2,0,0,n-3]p_1p_0 \\ &= q^{5-4n}B[n,0,0,n-1]^2 \\ &\quad + q^{4n-17}\left(q^2 B[n-1,0,0,n-2]^2 + q^{2n^2-10n+16}p_1^n p_0^{n-3}\right) p_1 p_0. \end{aligned}$$

In the above equations we have used Lemma 3.9 and the induction hypothesis. On the other hand

$$\begin{aligned} p &= \left(q^{3-4n}B[n+1,0,0,n] + q^{4-4n}B[n,1,1,n-1]\right) B[n-1,0,0,n-2] \\ &= q^{3-4n}B[n+1,0,0,n]B[n-1,0,0,n-2] + q^{4n-15}B[n-1,0,0,n-2]^2 p_1 p_0. \end{aligned}$$

Comparing both expressions for p gives $B[n+1,0,0,n]B[n-1,0,0,n-2] = q^2 B[n,0,0,n-1]^2 + q^{2n^2-6n+8} p_1^{n+1} p_0^{n-2}$. ☐

Lemma 3.16 is a quantized version of the exchange relation for the cluster algebra.

3.13 Conclusion

In the last subsection we draw the conclusion that $U_q^+(s_1 s_2 s_1 s_2)$ carries a quantum cluster algebra structure as defined by Berenstein-Zelevinsky in [6].

To be in accord with [6] we rescale our quantized cluster variables. Recall that the generators u_0, u_1, u_2, and u_3 of $U_q^+(s_1 s_2 s_1 s_2)$ correspond to the Λ-modules T_0, T_1, T_2, and T_3 of Subsection 3.4, the dual canonical basis elements $B[n-2,0,0,n-3]$ corresponds to the Λ-module T_n, and p_0 and p_1 correspond to the Λ-modules P_1 and P_0. We rescale each of the above elements by a power of q; the exponent is $-\frac{1}{2}$ times the dimension of the endomorphism algebra of the associated Λ-module. More precisely, introduce elements

- $X_0 = q^{-\frac{1}{2}} u_0$, $X_1 = q^{-\frac{1}{2}} u_1$, $X_2 = q^{-\frac{1}{2}} u_2$, $X_3 = q^{-\frac{1}{2}} u_3$,
- $Y_0 = q^{-2} p_0$, $Y_1 = q^{-2} p_1$,
- $X_n = q^{-\frac{1}{2}(2n-5)^2} B[n-2,0,0,n-3]$ for $n \geq 3$

in the the algebra $\mathbb{Q}[q^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[q,q^{-1}]} U_q^+(w)$ and analogous elements X_n for $n < 0$. Here we have enlarged the field of coefficients $\mathbb{Q}(q)$ of $U_q^+(w)$ to contain a square root of q.

Note that in the above examples, the dimension of the endomorphism algebra of the Λ-module corresponding to $B[a_3, a_2, a_1, a_0]$ is equal to $(a_3 + a_2 + a_1 + a_0)^2$. The exact form of the rescaling exponent was suggested by Leclerc.

For $n \geq 3$ consider four variables (X_n, X_{n+1}, Y_0, Y_1) which we group into a cluster. By Lemma 3.9 and Lemma 3.15 the variables q-commute; more precisely, we have $X_n X_{n+1} = q^2 X_{n+1} X_n$, $X_n Y_0 = q^{2n-2} Y_0 X_n$, $X_n Y_1 = q^{-2n+8} Y_1 X_n$,

$X_{n+1}Y_0 = q^{2n}Y_0X_{n+1}$, $X_{n+1}Y_1 = q^{-2n+6}Y_1X_{n+1}$, and $Y_0Y_1 = q^{-4}Y_1Y_0$. The matrix

$$L = \begin{pmatrix} 0 & 2 & 2n-2 & -2n+8 \\ -2 & 0 & 2n & -2n+6 \\ -2n+2 & -2n & 0 & -4 \\ 2n-8 & 2n-6 & 4 & 0 \end{pmatrix}$$

describes the exponents that occur in the commutation relations. The matrix L, the exchange matrix

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \\ n-3 & -n+4 \\ n & -n+1 \end{pmatrix},$$

(which is the same as the exchange matrix for the ordinary cluster algebra $\mathcal{A}(\mathcal{C}_M)$) and the cluster (X_n, X_{n+1}, Y_0, Y_1) form a *quantum seed* (compare [6, Definition 4.5]). With every $\mathbf{a} = (a_4, a_3, a_2, a_1) \in \mathbb{Z}^4$ Berenstein-Zelevinsky (see [6, Equation 4.19]) associate an expression

$$\begin{aligned} M(a_1, a_2, a_3, a_4) &= q^{\frac{1}{2}\sum_{i>j} a_i a_j L_{ij}} X_n^{a_1} X_{n+1}^{a_2} Y_0^{a_3} Y_1^{a_4} \\ &= q^{-\frac{1}{2}\sum_{i>j} a_i a_j L_{ij}} Y_1^{a_4} Y_0^{a_3} X_{n+1}^{a_2} X_n^{a_1}. \end{aligned}$$

We have $\frac{1}{2}\sum_{i>j} a_i a_j L_{ij} = -a_1a_2 - (n-1)a_1a_3 + (n-4)a_1a_4 - na_2a_3 + (n-3)a_2a_4 + 2a_3a_4$. Lemma 3.16, written in terms of X_n, X_{n+1}, Y_0, and Y_1, says that

$$X_{n+2}X_n = q^{-2}X_{n+1}^2 + q^{-2n^2+6n-3}Y_1^nY_0^{n-3}$$

Thus, we get an equation for the quantized cluster variable X_{n+2}. There holds

$$\begin{aligned} X_{n+2} &= q^{-2}X_{n+1}^2X_n^{-1} + q^{-2n^2+6n-3}Y_1^nY_0^{n-3}X_n^{-1} \\ &= M(-1, 2, 0, 0) + M(-1, 0, n-3, n). \end{aligned} \tag{57}$$

Equation (57) is equal to the exchange relation [6, Equation 4.23] of Berenstein and Zelevinsky.

The direct sum $\bigoplus_{\mathbf{a}\in\mathbb{N}^4} \mathbb{Z}[q^{\pm\frac{1}{2}}]u[\mathbf{a}]$ is a $\mathbb{Z}[q, q^{-1}]$-algebra, since the straightening relations involve only polynomials in q. It is an integral form of the algebra $\mathbb{Q}[q^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[q,q^{-1}]} U_q^+(w)$ defined above. It is generated by X_0, X_1, X_2, and X_3 and furthermore contains each X_n for $n \in \mathbb{Z}$. Therefore, it coincides with the $\mathbb{Z}[q, q^{-1}]$-algebra generated by all X_n with $n \in \mathbb{Z}$ which is by definition equal to the quantum cluster algebra as defined in [6].

We conclude with the theorem.

Theorem 3.17. The $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^4} \mathbb{Z}[q^{\pm\frac{1}{2}}]u[\mathbf{a}] \subseteq \mathbb{Q}[q^{\pm\frac{1}{2}}] \otimes_{\mathbb{Z}[q,q^{-1}]} U_q^+(w)$$

is a quantum cluster algebra.

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