Equilibrium dynamics of continuous unbounded spin systems

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Contents

Summary

In the introduction of the dissertation we define Glauber and Kawasaki dynamics of a lattice system of continuous unbounded spins. Both dynamics can be understood as stochastically perturbed gradient flows w.r.t. the energy landscape given by the Hamiltonian H of the system. The main difference between them is that Kawasaki dynamics conserve the mean spin m of the system in contrast to Glauber dynamics. We identify natural candidates for the equilibrium state of the dynamics, which are closely connected to the Hamiltonian H. For Glauber dynamics this state is the grand canonical ensemble μ. For Kawasaki dynamics this state is the canonical ensemble $\mu_{N,m}$. Additionally, we motivate the use of functional inequalities – namely the spectral gap (SG), the logarithmic Sobolev inequality (LSI), and the transport-information inequality (WI) – for the analysis of the relaxation to equilibrium of the dynamics. Roughly speaking, the SG, LSI, and WI constants characterize the exponential rate of convergence to equilibrium. The main focus of Chapter 1 is laid on Glauber dynamics, whereas the main focus of Chapter 2 and Chapter 3 is laid on Kawasaki dynamics.

In Section 1.1 we introduce some standard criteria for the SG, the LSI, and the WI. In Section 1.2 we derive a new covariance estimate that can be naturally applied to our spin system with weak interaction. Here, the Hamiltonian H of the system of N spins is given by

$$H(x) = \sum_{i=1}^{N} \psi(x_i) + \sum_{1 \leq i < j \leq N} m_{ij} x_i x_j$$

for a single-site potential ψ and small real-valued numbers m_{ij} determining the interaction. The algebraic structure of this estimate is close to the Brascamp-Lieb inequality [7], but the assumption of the convexity of the Hamiltonian is relaxed. The estimate also yields a weighted covariance estimate due to Helffer [30], which was applied to derive decay of correlations. However, our result applies to general weak (not just nearest neighbor) interaction and is optimal for quadratic Hamiltonians with attractive interaction. The proof is based on a new directional SG. In Section 1.3 we derive this directional inequality on the level of the WI. The latter yields a non-linear version of the covariance estimate and a criterion for the WI similar to the Otto & Reznikoff criterion for LSI [46]. The proof of the directional SG is based on ideas of Helffer [28] and Ledoux [40], whereas the proof of the directional WI follows the proof of the Otto & Reznikoff criterion.

In Chapter 2 we consider the LSI for the canonical ensemble $\mu_{N,m}$ in the case of a non-interacting Hamiltonian H given by a sum of single-site potentials ψ i.e.

$$H(x) = \sum_{i=1}^{N} \psi(x_i).$$

Even if there is no interaction term in the Hamiltonian H, there is long-range interaction in the system due to the conservation of the mean spin m. We show that the LSI holds uniformly in the system size N and the mean spin m, if the single-site potential ψ is a bounded perturbation of a strictly convex function; more precisely, if there is a splitting $\psi = \psi_c + \delta\psi$ such that

$$\psi_c'' \gtrsim 1 \qquad \text{and} \qquad |\delta\psi| + |\delta\psi'| \lesssim 1.$$

This verifies a conjecture of Landim, Panizo, and Yau [38] and simultaneously answers a question Varadhan [53] posed in 1993. The argument is independent of the geometric structure and adapts the two-scale approach of Grunewald, Otto, Westdickenberg, and Villani [22] from the quadratic to the super-quadratic case. Compared to the proof of [22] there are three major changes:

- Instead of coarse-graining of big blocks, we consider iterated coarse-graining of pairs.
- The latter allows to apply a new asymmetric Brascamp-Lieb type inequality for covariances, because the situation is reduced to one dimension. The asymmetric Brascamp-Lieb inequality can be applied to perturbed strictly convex single-site potentials ψ in contrast to the classical covariance estimate that was used in [22].
- This procedure reduces the task of deriving a uniform LSI for $\mu_{N,m}$ to the convexification of the coarse-grained Hamiltonian, which follows from a new local Cramér theorem for perturbed strictly convex single-site potentials ψ.

In Chapter 3 we consider the LSI for the canonical ensemble $\mu_{N,m}$ in the case of weak interaction. Here, the Hamiltonian H is given by

$$H(x) = \sum_{i=1}^{N} (\psi(x_i) + s_i x_i) \; + \sum_{1 \leq i < j \leq N} m_{ij} x_i x_j.$$

The linear term – given by the vector s – models the interaction of the spins with the boundary data. Due to technical reasons, we assume that ψ has the same structure as in [22]; namely ψ is a bounded perturbation of a quadratic potential

$$\psi(x_i) = \frac{1}{2} x_i^2 + \delta\psi(x_i) \qquad \text{and} \qquad |\delta\psi| + |\delta\psi'| + |\delta\psi''| \lesssim 1.$$

Provided the interaction is small in a certain sense, we derive the LSI for the canonical ensemble $\mu_{N,m}$ uniformly in the system size N, the mean spin m, and the boundary data s. The argument is independent of the geometric structure of the system. In contrast to Chapter 2, the proof consists of an application of the original two-scale approach [22]. Several ideas are needed to solve new technical difficulties due to the interaction:

- The interaction between blocks is controlled by an application of the covariance estimate of Section 1.2.

- The convexification of the coarse-grained Hamiltonian is deduced using a conditioning technique and a perturbation argument.
- The interaction with the boundary data s induces a natural dependence of the single-site potentials $(\psi(x_i) + s_i x_i)$ on the site i. Therefore, we have to generalize the local Cramér theorem of [22] to the case of inhomogeneous single-site potentials.

It remains to note that the contents of Section 1.2 and Chapter 2 emerged from joint projects of Prof. Felix Otto and the author. The content of Chapter 3 is contained in the preprint [44] of the author, which has been recommended for publication in the journal *Communications in Mathematical Physics*.

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Introduction

In the dissertation we study classical lattice systems of continuous unbounded spins. These systems appear in the literature in several situations:

- as a generalization of discrete spin systems like the Ising or Potts model [41];
- as a modeling and computational tool in physics, as for example in the description of magnetic materials [34, 37, 45] and phase separation [19, 18, 14];
- in statistical mechanics and in Euclidean quantum field theory [39, 23].

Let us introduce the basic concepts of the spin system considered in the dissertation. The set Λ consists of finitely many *sites*. For example, Λ can be a finite part of a lattice or a finite graph. We index the elements of Λ and identify Λ with the set $\{1, \ldots, N\}$. A real-valued *spin* $x_i \in \mathbb{R}$ is associated to each site $i \in \{1, \ldots, N\}$. Compared to the Ising model, where the spin values are bounded and discrete (i.e. $x_i \in \{-1, 1\}$), considering real-valued spins leads to a technical advantage on the one side and to a technical challenge on the other side:

- The advantage is that because the spin value x_i is continuous one can use analytic tools as for example differentiation and gradients.
- The challenge is that because the spin value x_i is unbounded a lot of arguments known for the bounded case cannot be used.

A *state* of the spin system is given by a vector $x \in \mathbb{R}^N$. The *Hamiltonian* H assigns to each state $x \in \mathbb{R}^N$ a certain amount of energy $H(x) \in \mathbb{R}$. We assume that the Hamiltonian H is smooth. The *Gibbs measure* μ is a probability measure on the state space $\mathbb{R}^N$ given by the density

$$\mu(dx) = \frac{1}{Z}\, \exp(-H(x))\, dx. \tag{1}$$

Here and later on, Z denotes a generic normalization constant. The definition of μ shows that the occurrence of states with high energies is penalized in an exponentially strong way. Sometimes, we call μ the *grand canonical ensemble*.

Even if the study of phase transitions in spin systems has attracted a lot of interest [3, 41, 48], we will concentrate on aspects of equilibrium dynamics in the one phase region. We consider a stochastic process $\xi = \xi(t) \in \mathbb{R}^N$ satisfying the stochastic differential equation

$$d\xi = -A\nabla H(\xi)\, dt + \sqrt{2A}\, dB_t. \tag{2}$$

Here, ∇ denotes the gradient determined by the standard Euclidean structure on $\mathbb{R}^N$ and the noise $B_t \in \mathbb{R}^N$ consists of N independent standard Brownian motions. The $N \times N$ matrix A is chosen in two different ways:

- In the case of *Glauber* dynamics, the matrix A is given by the identity matrix. This choice corresponds to spin-flip dynamics in the Ising model (cf. [43, 59]).

- In the case of *Kawasaki* dynamics, the matrix A is given by the discrete second-order difference operator. This choice corresponds to spin-exchange dynamics in the Ising model (cf. [9, 38]). Note that the matrix A depends on the geometric structure of the sites Λ i.e. on the notion of nearest neighbor. For simplicity, we assume that Λ is a periodic one-dimensional lattice of size N. Then the elements A_{ij} of the $N \times N$ matrix A are given by

$$\frac{1}{N^2} A_{ij} = \begin{cases} 2, & \text{if } \ i = j, \\ -1, & \text{if } \ |i-j| \in \{1, N-1\}, \\ 0, & \text{else.} \end{cases} \tag{3}$$

  Even if we only consider the periodic one-dimensional lattice explicitly, adapted statements of our results for Kawasaki dynamics also hold for general lattices and graphs (cf. [22, Remark 15]).

The main difference between Glauber and Kawasaki dynamics is that Glauber dynamics are non-conservative and Kawasaki dynamics are conservative. The latter means that the initial *mean spin* m of the system is conserved over time by the process i.e. for all times $t \geq 0$ we have

$$m := \frac{1}{N} \sum_{i=1}^{N} \xi_i(t=0) = \frac{1}{N} \sum_{i=1}^{N} \xi_i(t).$$

The last identity follows from the fact $\sum_{i=1}^{N} d\xi_i = 0$, which is verified by a straight forward calculation using the stochastic differential equation (2) and the definition (3) of A. Hence, for Kawasaki dynamics the state space $\mathbb{R}^N$ can be restricted to the $(N-1)$ dimensional hypersurface

$$X_{N,m} := \left\{ x \in \mathbb{R}^N, \ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\}. \tag{4}$$

The restriction of the Gibbs measure μ to the new state space $X_{N,m}$ is called the *canonical ensemble* $\mu_{N,m}$. More precisely, $\mu_{N,m}$ is given by the density

$$\mu_{N,m}(dx) := \frac{1}{Z} \ \exp\left(-H(x)\right) \ \mathcal{H}^{N-1}_{\lfloor X_{N,m}}(dx), \tag{5}$$

where $\mathcal{H}^{N-1}_{\lfloor X_{N,m}}$ denotes the $(N-1)$ dimensional Hausdorff measure restricted to $X_{N,m}$.

We assume that the initial distribution of the stochastic process ξ is given by a smooth positive density. Then standard probability theory yields that the process ξ is distributed at time t according to the density given by the time-evolution

$$\frac{d}{dt}(f_t \mu) = \nabla \cdot (\mu \ A \nabla f_t) \tag{6}$$

in the case of Glauber dynamics and

$$\frac{d}{dt}(f_t \mu_{N,m}) = \nabla \cdot (\mu_{N,m}\, A \nabla f_t) \tag{7}$$

in the case of Kawasaki dynamics. Both equations have to be understood in the weak sense. For example, equation (6) means that for any smooth test function ζ

$$\frac{d}{dt}\int \zeta(x) f_t(x)\mu(dx) = -\int \nabla\zeta(x)\cdot A\nabla f_t(x)\mu(dx).$$

We pose the following questions on the dynamics:

- Is there an equilibrium state?
- If yes, do the dynamics converge to equilibrium, in which sense, and how fast?

We can immediately give an answer to the first question: By using the time-evolution (6) and (7) one sees that

$$\frac{d}{dt}\mu = 0 \qquad \text{and} \qquad \frac{d}{dt}\mu_{N,m} = 0.$$

It follows that:

- For Glauber dynamics the Gibbs measure μ is a stationary distribution and therefore a natural candidate for an equilibrium state.
- For Kawasaki dynamics the canonical ensemble $\mu_{N,m}$ is a stationary distribution and therefore a natural candidate for an equilibrium state.

Let us turn to the second question, which we approach with the help of functional inequalities. We introduce the spectral gap (SG), which is also called Poincaré inequality in the literature, and the logarithmic Sobolev inequality (LSI):

Definition 0.1 (SG). *Let X be a Euclidean space. A Borel probability measure μ on X satisfies the SG(ϱ) with constant $\varrho > 0$, if for all functions f*

$$\operatorname{var}_\mu(f) := \int \left(f^2 - \int f d\mu\right)^2 d\mu \le \frac{1}{\varrho}\int |\nabla f|^2 d\mu.$$

Here, ∇ denotes the gradient determined by the Euclidean structure of X.

Definition 0.2 (LSI). *Let X be a Euclidean space. A Borel probability measure μ on X satisfies the LSI(ϱ) with constant $\varrho > 0$, if for all functions $f > 0$*

$$\operatorname{Ent}(f\mu,\mu) := \int f\log f\, d\mu - \int f\, d\mu \log\int f\, d\mu \le \frac{1}{2\varrho}\int \frac{|\nabla f|^2}{f} d\mu. \tag{8}$$

Here, ∇ denotes the gradient determined by the Euclidean structure of X. If $\int f d\mu = 1$, the relative entropy of the probability measure $f\mu$ w.r.t. μ is given by $\operatorname{Ent}(f\mu,\mu)$.

Remark 0.3 (Gradient on $X_{N,m}$). *Because $X_{N,m}$ inherits the standard Euclidean structure of $\mathbb{R}^N$, one can calculate $|\nabla f|^2$ in the following way: Extend $f : X_{N,m} \to \mathbb{R}$ to be constant on the direction normal to $X_{N,m}$, then*

$$|\nabla f|^2 = \sum_{i=1}^{N} \left| \frac{d}{dx_i} f \right|^2 .$$

In our framework the functional inequalities SG and LSI are useful, because they yield exponential convergence of ξ to the equilibrium state of the dynamics (cf. [50, 51, 52, 59, 62]):

Lemma 0.4. *Let μ denote the grand canonical ensemble given by* (1) *and let $f_t\mu$ denote the distribution of the Glauber dynamics given by* (6). *It holds:*

1. *If μ satisfies SG(ϱ), then* $\quad \operatorname{var}_\mu(f_t) \leq \exp(-2\varrho t)\ \operatorname{var}_\mu(f_0)$.
2. *If μ satisfies LSI(ϱ), then* $\quad \operatorname{Ent}(f_t\mu, \mu) \leq \exp(-2\varrho t)\ \operatorname{Ent}(f_0\mu, \mu)$.

Let $\mu_{N,m}$ denote the canonical ensemble given by (5) *and let $f_t\mu_{N,m}$ denote the distribution of the Kawasaki dynamics given by* (7). *Then there is a constant $C > 0$ such that:*

1. *If $\mu_{N,m}$ satisfies SG(ϱ), then* $\quad \operatorname{var}_{\mu_{N,m}}(f_t) \leq \exp(-2C^{-1}N^{-2}\varrho t)\ \operatorname{var}_{\mu_{N,m}}(f_0)$.
2. *If $\mu_{N,m}$ satisfies LSI(ϱ), then*

$$\operatorname{Ent}(f_t\mu_{N,m}, \mu_{N,m}) \leq \exp(-2C^{-1}N^{-2}\varrho t) \operatorname{Ent}(f_0\mu_{N,m}, \mu_{N,m}).$$

Proof of Lemma 0.4. We start with considering Glauber dynamics. It follows from (6) that

$$\frac{d}{dt} \int f d\mu = 0.$$

A direct calculation using the last identity and (6) reveals

$$\frac{d}{dt} \operatorname{var}_\mu(f_t) = -2 \int |\nabla f_t|^2 d\mu \qquad \text{and} \qquad \frac{d}{dt} \operatorname{Ent}_\mu(f_t\mu, \mu) = - \int \frac{|\nabla f_t|^2}{f_t} d\mu.$$

An application of the SG(ϱ) and the LSI(ϱ) yields

$$\frac{d}{dt} \operatorname{var}_\mu(f_t) \leq -2\varrho\ \operatorname{var}_\mu(f_t) \qquad \text{and} \qquad \frac{d}{dt} \operatorname{Ent}_\mu(f_t\mu, \mu) \leq -2\varrho \operatorname{Ent}_\mu(f_t\mu, \mu).$$

Hence, the desired statement follows from an application of the differential inequality.
The argument for Kawasaki dynamics is almost the same. Using the time-evolution (7) one sees that for Kawasaki dynamics

$$\frac{d}{dt} \operatorname{var}_{\mu_{N,m}}(f_t) = -2 \int |\sqrt{A}\nabla f_t|^2 d\mu_{N,m} \qquad \text{and}$$
$$\frac{d}{dt} \operatorname{Ent}_{\mu_{N,m}}(f_t\mu, \mu) = - \int \frac{|\sqrt{A}\nabla f_t|^2}{f_t} d\mu_{N,m}.$$

On the right hand side of the last equation one applies the discrete Poincaré inequality (cf. [12, 22]), which states that for some constant C

$$|\nabla f|^2 \leq CN^2 |\sqrt{A}\nabla f|^2.$$

One concludes the proof by applying the SG(ϱ), the LSI(ϱ), and the differential inequality in the same way as for the Glauber dynamics. □

The last lemma also characterizes the rate of convergence in terms of the SG and LSI constant ϱ. The rate for Kawasaki dynamics depends diffusively on the system size N, which is the optimal scaling behavior (cf. [57]). This dependence on the system size N is natural: By the definition of the matrix A, only nearest neighbors are allowed to interchange their spin values in order to equilibrate.

The SG yields convergence to equilibrium in the sense of variances, whereas the LSI yields convergence in the sense of relative entropies. As the next remark shows, we prefer the convergence in the sense of relative entropies, because it is better adapted to the hydrodynamic limit i.e. sending the system size N to infinity.

Remark 0.5. *Let us consider the scaling behavior of* $\operatorname{var}_\mu(f)$ *and* $\operatorname{Ent}_\mu(f\mu,\mu)$ *in the system size* N *for a simple example: Let* ν *be a probability measure on* $\mathbb{R}$ *with* $\int z\ \nu(dz) = 1$. *If the grand canonical ensemble* μ *is the product measure* $\mu(dx) = \otimes_{i=1}^N \nu(dx_i)$ *on* $\mathbb{R}^N$, *then a direct calculation yields for* $f(x) = \Pi_{i=1}^N x_i$

$$\operatorname{var}_\mu(f) = (\operatorname{var}_\nu(\mathrm{id}_\mathbb{R}) + 1)^N - 1 \qquad \textit{and} \qquad \operatorname{Ent}_\mu(f\mu,\mu) = N \operatorname{Ent}_\nu(\mathrm{id}_\mathbb{R}\,\nu,\nu).$$

Hence, the term $\operatorname{var}_\mu(f)$ *diverges exponentially fast for* $N \to \infty$. *The term* $\operatorname{Ent}_\mu(f\mu,\mu)$ *only increases linearly. The latter shows that it makes more sense to consider the relative entropy per site than to consider the variance per site.*

The SG constant ϱ also determines the rate of convergence of the empirical time-average of a bounded random variable u to its ensemble average.

Lemma 0.6. *Let* μ *denote the Gibbs measure given by* (1) *and let* ξ *denote the Glauber dynamics given by* (2). *If* μ *satisfies SG(ϱ), then any* $\varepsilon > 0$ *and* $t > 0$ *it holds for any bounded function* u

$$\mathbb{P}_{f_0}\left(\frac{1}{t}\int_0^t u(\xi(s))ds - \int u\, d\mu \geq \varepsilon\right) \leq \|f_0\|_{L^2(\mu)}\ \exp\left(-\frac{t\varepsilon^2\varrho}{(\operatorname{osc} u)^2}\right).$$

Here, $\mathbb{P}_{f_0}$ *denotes the probability of Glauber dynamics with initial distribution* $f_0\mu$ *and* $\operatorname{osc} u := \sup_x u(x) - \inf_x u(x)$ *is the oscillation of* u.

For the proof of the last statement we refer the reader to [24][Theorem 3.1]. Note that Lemma 0.6 only holds for bounded random variables u. For this reason we introduce two more functional inequalities. Using these inequalities one is able to consider Lipschitz continuous random variables u (cf. Lemma 0.9 below).

Definition 0.7 (Wasserstein distance)**.** *Let ν and μ be probability measures on a Euclidean space X. For $p \in \{1,2\}$ the L^p-Wasserstein distance $W_p(\nu,\mu)$ is given by*

$$W_p(\nu,\mu) = \left(\int |x-y|^p \; \pi(dx,dy) \right)^{\frac{1}{p}},$$

where π is the optimal transference plan of ν and μ. More precisely, π minimizes the expression

$$\int |x-y|^p \; \tilde{\pi}(dx,dy)$$

over all joint probability measures $\tilde{\pi}$ with marginals ν and μ, which means

$$\int \xi(x)\tilde{\pi}(dx,dy) = \int \xi(x)\nu(dx) \qquad \text{and} \qquad \int \xi(y)\tilde{\pi}(dx,dy) = \int \xi(y)\mu(dy)$$

for all functions ξ. In the rest of the dissertation, all transference plans correspond to the choice $p=2$.

For an introduction to the Wasserstein distance and optimal transport in general we refer the reader to Villani's books [54] and [55].

Definition 0.8 (WI)**.** *Let X be a Euclidean space and $p \in \{1,2\}$. A Borel probability measure μ on X satisfies the $W_pI(\varrho)$ with constant $\varrho > 0$, if for all functions $f > 0$ with $\int f d\mu = 1$*

$$W_p^2(f\mu,\mu) \leq \frac{1}{\varrho^2} \int \frac{|\nabla f|^2}{f} d\mu$$

For convenience, we write WI(ϱ) for $W_2I(\varrho)$. In the abbreviation WI, "W" stands for Wasserstein distance and "I" stands for Fisher information, which is the name of the term on the r.h.s. of the last inequality.

In the literature, this type of functional inequality is called transportation-information inequality. In our framework, the W_1I is interesting because of the following equivalent characterization (cf. [24, Corollary 2.5]):

Lemma 0.9. *Let μ denote the Gibbs measure given by* (1) *and let ξ denote the Glauber dynamics given by* (2)*. Then μ satisfies $W_1I(\varrho)$ if and only if for any initial distribution $f_0\mu$, $\varepsilon > 0$, $t > 0$, and Lipschitz function u*

$$\mathbb{P}_{f_0}\left(\frac{1}{t}\int_0^t u(\xi(s))ds - \int u \, d\mu \geq \varepsilon \right) \leq \; \|f_0\|_{L^2(\mu)} \; \exp\left(-\frac{t\varepsilon^2\varrho^2}{\|u\|^2_{\mathtt{Lip}}} \right).$$

Here, $\mathbb{P}_{f_0}$ denotes the probability of Glauber dynamics with initial distribution $f_0\mu$.

Remark 0.10. *Similar results of Lemma 0.6 and Lemma 0.9 also hold for Kawasaki dynamics ξ and the canonical ensemble $\mu_{N,m}$. One only has to exchange the constant ϱ with the constant $C^{-1}N^{-2}\varrho$ (cf. proof of Lemma 0.4).*

In the rest of the dissertation we will only consider the WI(ϱ), which implies the W_1I(ϱ) by Hoelder's inequality. The purpose of the introduction was to motivate the use of the functional inequalities SG, LSI, and WI for the analysis of equilibrium dynamics. In the main part of the dissertation, we will consider the question if the functional inequalities SG, LSI, and WI hold for the grand canonical ensemble μ and the canonical ensemble $\mu_{N,m}$.

1 Functional inequalities for Glauber dynamics

1.1 Standard criteria for the LSI, the WI, and the SG

In this section we recall some standard criteria for the LSI, the WI, and the SG. For a general introduction to the SG and LSI we refer to [40, 49, 25]. For more background information about the WI we refer the reader to [47] and [24]. We start with the interplay between the functional inequalities LSI, the WI, and the SG, which was first observed by Otto & Villani in [47].

Lemma 1.1.1. *Let μ be a probability measure on a Euclidean space X. Then:*

$$\mu \text{ satisfies LSI}(\varrho) \quad \Rightarrow \quad \mu \text{ satisfies WI}(\varrho) \quad \Rightarrow \quad \mu \text{ satisfies SG}(\varrho).$$

Remark 1.1.2. *Note that the implications of the last lemma are strict. This was shown in [11] for the first implication. For the second implication we consider the probability measure $d\mu = Z^{-1}\exp(-|x|)dx$ on the real line: On the one hand [21, Theorem 6] yields that μ does not satisfy the WI, on the other hand the measure μ satisfies the SG because it is log-concave by a result of Bobkov [4].*

The first criterion shows that the functional inequalities LSI, WI, and SG are compatible with products (cf. for example [25, Theorem 4.4]).

Theorem 1.1.3 (Tensorization principle)**.** *Let μ_1 and μ_2 be probability measures on Euclidean spaces X_1 and X_2 respectively. If μ_1 and μ_2 satisfy LSI(ϱ_1) and LSI(ϱ_2) respectively, then the product measure $\mu_1 \otimes \mu_2$ satisfies LSI($\min\{\varrho_1, \varrho_2\}$).*

Note that the last statement also holds for the WI (cf. [24, Theorem 2.7]) and the SG (cf. [25, Theorem 2.5.]). The next criterion shows, how the LSI constant behaves under perturbations (cf. [33, p. 1184]).

Theorem 1.1.4 (Criterion of Holley & Stroock)**.** *Let μ be a probability measure on a Euclidean space X and let $\delta\psi : X \to \mathbb{R}$ be a bounded function. Let the probability measure $\tilde{\mu}$ be defined as*

$$\tilde{\mu}(dx) = \frac{1}{Z}\exp\left(-\delta\psi(x)\right)\ \mu(dx).$$

If μ satisfies LSI(ϱ), then $\tilde{\mu}$ satisfies LSI($\tilde{\varrho}$) with constant $\tilde{\varrho} = \varrho\ \exp\left(-\operatorname{osc}\delta\psi\right)$.

The last statement also holds in the case of the SG. Because of its perturbative nature, the criterion of Holley & Stroock is not well adapted for high dimensions. For the proof we refer the reader to [40, Lemma 1.2]. Now, we state the criterion of Bakry & Émery, which connects the convexity of the Hamiltonian to the LSI constant (cf. [1, Proposition 3 and Corollary 2] or [40, Corollary 1.6]).

Theorem 1.1.5 (Criterion of Bakry & Émery)**.** *Let* $d\mu := Z^{-1} \exp(-H(x))\, dx$ *be a probability measure on a Euclidean spaces* X*. If there is a constant* $\varrho > 0$ *such that in the sense of quadratic forms*

$$\operatorname{Hess} H(x) \geq \varrho$$

uniformly in $x \in X$*, then* μ *satisfies LSI(*ϱ*).*

A proof using semigroup methods can be found in [40, Corollary 1.6]. There is also a nice heuristic interpretation of the criterion of Bakry & Émery on a formal Riemannian structure on the space of probability measures (cf. [47, Section 3]).

We illustrate the criteria from above with some examples. Let μ denote the Gibbs measure associated to the Hamiltonian H i.e.

$$\mu(dx) = \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

Using the criterion of Bakry & Émery one directly sees that for $H(x) = \frac{1}{2}x^2$, $x \in \mathbb{R}$, the associated Gibbs measure μ satisfies LSI(1). Let us consider the Ginzburg-Landau single-site potential $H(x) = \frac{1}{4}(x^2-1)^2$, $x \in \mathbb{R}$, which is very important in the study of continuous phase-transitions (cf. [27, Chapter 13]). One can split $H(x) = \frac{1}{4}(x^2-1)^2$ into

$$H(x) = \psi_c(x) + \delta\psi(x) \qquad \text{such that} \qquad \psi_c''(x) \gtrsim 1 \quad \text{and} \quad |\delta\psi| \lesssim 1.$$

The relations $\sim$ and $\lesssim$ are defined in the Chapter Conventions at the end of the dissertation. A combination of the criterion of Bakry & Émery and the criterion of Holley & Stroock yields that the associated Gibbs measure μ satisfies the LSI(ϱ) for some constant $\varrho > 0$. Together with the tensorization principle from above this implies that the Gibbs measure μ on $\mathbb{R}^N$ associated to the Hamiltonian $H(x) = \sum_{i=1}^{N} \frac{1}{4}(x_i^2-1)^2$, $x \in \mathbb{R}^N$, satisfies LSI(ϱ) with the same constant $\varrho > 0$ uniformly in the system size N.

The situation becomes more complex if one adds an interaction term to the Hamiltonian. Let us consider for example the Hamiltonian

$$H(x) = \sum_{i=1}^{N} \frac{1}{4}(x_i^2-1)^2 + J \sum_{|i-j|=1}^{N} x_i x_j, \qquad \text{for} \quad x \in \mathbb{R}^N \quad \text{and} \quad |J| \ll 1.$$

For this type of Hamiltonian, deriving the LSI(ϱ) with constant $\varrho > 0$ uniformly in the system size N is a well-studied problem in the literature (cf. [5, 49, 58, 40]). More recently, Otto & Reznikoff [46] deduced a criterion for LSI that covers this situation without any further analysis. Before we formulate the criterion of Otto & Reznikoff, let us recall the disintegration of probability measures into conditional measures and the marginal:

Definition 1.1.6. *Let $\mathcal{P}(X)$ denote the space of probability measures on a Euclidean space X. We consider an arbitrary probability measure $\mu(dx_1, dx_2) \in \mathcal{P}(X_1 \times X_2)$. Then the marginal $\bar{\mu}(dx_1) \in \mathcal{P}(X_1)$ and the family of conditional measures*

$$\{\mu(dx_2|x_1) \in \mathcal{P}(X_2)\}_{x_1 \in X_1}$$

are defined via

$$\forall\, \zeta(x_1, x_2) \qquad \int \zeta(x_1, x_2)\, \mu(dx_1, dx_2) = \int \int \zeta(x_1, x_2)\, \mu(dx_2|x_1)\, \bar{\mu}(dx_1).$$

For convenience, we will use the notation $\bar{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ that erases the i-th coordinate of the vector $x = (x_1, \ldots, x_N)$.

Theorem 1.1.7 (Criterion of Otto & Reznikoff)**.** *Let $d\mu := Z^{-1} \exp(-H(x))\, dx$ be a probability measure on a direct product of Euclidean spaces $X = X_1 \times \cdots \times X_N$. We assume that*

- *the conditional measures $\mu(dx_i|\bar{x}_i)$, $1 \leq i \leq N$, satisfy a uniform LSI(ϱ_i).*
- *the numbers κ_{ij}, $1 \leq i \neq j \leq N$, satisfy*

$$|\nabla_i \nabla_j H(x)| \leq \kappa_{ij} < \infty$$

  uniformly in $x \in X$. Here, $|\cdot|$ denotes the operator norm of a bilinear form.
- *the symmetric matrix $A = (A_{ij})_{N \times N}$ defined by*

$$A_{ij} = \begin{cases} \varrho_i, & \textit{if } i = j, \\ -\kappa_{ij}, & \textit{if } i < j, \end{cases}$$

  satisfies in the sense of quadratic forms

$$A \geq \varrho\, \mathrm{Id} \qquad \textit{for a constant } \varrho > 0. \tag{1.1}$$

Then μ satisfies LSI(ϱ).

By [46, Remark 5], the last statement is optimal for ferromagnetic Gaussian Hamiltonians given by

$$H(x) = \frac{1}{2} \sum_{1 \leq i,j \leq N} x_i A_{ij} x_j + \sum_{1 \leq i \leq N} b_i x_i, \qquad A_{ij}, b_j \in \mathbb{R}, \tag{1.2}$$

where ferromagnetic means that the coupling is attractive i.e.

$$A_{ij} = A_{ji} \leq 0 \quad \text{for } i < j \in \{1, \ldots, N\}.$$

In Section 1.3 we derive an analog version of the criterion of Otto & Reznikoff on the level of the WI (see Theorem 1.3.3). On the level of the SG there is not only an analog version but also a relaxed one (cf. [40, Proposition 3.1], [46, Remark 4], and Remark 1.2.9):

Theorem 1.1.8. *Let* $d\mu := Z^{-1} \exp(-H(x))\, dx$ *be a probability measure on* $\mathbb{R}^N$. *Assume that*

- *the conditional measures* $\mu(dx_i|\bar{x}_i)$, $1 \leq i \leq N$, *satisfy a uniform LSI(*ϱ_i*).*
- *the matrix* $\mathcal{A}(x) = (\mathcal{A}_{ij}(x))_{N\times N}$ *given by*

$$\mathcal{A}_{ij}(x) = \begin{cases} \varrho_i, & \text{if } i = j, \\ \nabla_i \nabla_j H(x), & \text{else}, \end{cases}$$

satisfies in the sense of quadratic forms and uniformly in x

$$\mathcal{A}(x) \geq \varrho \,\mathrm{Id} \qquad \text{for a constant } \varrho > 0. \tag{1.3}$$

Then μ *satisfies SG(*ϱ*).*

It is an open question if the assumption (1.1) of Theorem 1.1.7 can also be relaxed similar to the assumption (1.3) of Theorem 1.1.8.

As we have illustrated with examples, the standard criteria are very useful for deriving the LSI, the WI, and the SG for the grand canonical ensemble μ. As we will explain in Chapter 2 below, one cannot directly apply the standard criteria to the canonical ensemble $\mu_{N,m}$ for a non-convex Hamiltonian H. In the remaining part of Chapter 1, we continue to consider functional inequalities for the grand canonical ensemble μ. In Chapter 2 and Chapter 3, we will have a closer look at the question of deriving the LSI for the canonical ensemble $\mu_{N,m}$.

1.2 A Brascamp-Lieb type covariance estimate

In this section we derive a new covariance estimate for a certain class of Gibbs measures

$$\mu(dx) = \frac{1}{Z} \exp\left(-H(x)\right) dx,$$

on a finite-dimensional Euclidean space X (see Theorem 1.2.4). The covariance estimate can be seen as an analogon of the Brascamp-Lieb inequality (BLI), which estimates variances. The BLI was originally introduced by Brascamp & Lieb in [7]:

Theorem 1.2.1 (Brascamp & Lieb)**.** *Let* H *be strictly convex. Then for all functions* f

$$\mathrm{var}_\mu(f) := \int \left(f - \int f \, d\mu \right)^2 d\mu \leq \int \left\langle \nabla f, (\mathrm{Hess}\, H)^{-1} \nabla f \right\rangle d\mu. \tag{1.4}$$

The main difference between our estimate and the BLI is that

- our estimate applies to covariances,
- it also handles non-convex Hamiltonians,
- in the convex case the bound is slightly weaker than in the BLI.

The estimate also implies a well-known weighted covariance estimate due to Helffer (see Theorem 1.2.8, [30, Section 4] or [40, Proposition 2.1 or 3.1]), which yields exponential decay of correlations for unbounded spin systems with a non-convex single-site potential and a weak finite-range interaction (see [30, Theorem 2.1], [5, Theorem 1.1], [6, Theorem 3.1] or [40, Proposition 6.2]). On the other hand our estimate already yields the decay as a simple consequence (see Corollary 1.2.10 and Proposition 1.2.11). Decay of correlations is often used to derive the LSI or the SG (see for example [61, 62, 30, 5, 58, 60] or [5] for an overview). Hence, it is not surprising that our covariance estimate is one of the key ingredients to derive the LSI for the canonical ensemble $\mu_{N,m}$ in the case of a weak two-body interaction (cf. Chapter 3). We deduce the covariance estimate from a new inequality called *directional* SG (see Theorem 1.2.12). The proof the directional SG is based on ideas, which were outlined by Ledoux for the proof of the weighted covariance estimate (cf. [40] and Theorem 1.2.8).

We consider a finite dimensional Euclidean space X. Norms $|\cdot|$ and gradients ∇ are derived from the Euclidean structure. If a probability measure μ on X satisfies the SG, we directly obtain the following standard covariance estimate:

Lemma 1.2.2. *Assume μ satisfies SG(ϱ). Then for any function f and g we have*

$$\operatorname{cov}_{\mu}(f,g) \leq \frac{1}{\varrho} \left(\int |\nabla f|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int |\nabla g|^2 \, d\mu \right)^{\frac{1}{2}}. \tag{1.5}$$

Even if the estimate (1.5) is optimal (cf. [46, Remark 4]), it does not yield information about the dependence of the covariance on the specific coordinates. Hence, the estimate (1.5) is useless for deducing decay of covariances. For example, let us consider a Gaussian Gibbs measure

$$\mu(dx) = \frac{1}{Z} \exp\left(-x \cdot Ax\right) \, dx$$

on $\mathbb{R}^N$ with a symmetric and positive definite $N \times N$- Matrix A. Then it is known that

$$\operatorname{cov}_{\mu}(x_n, x_k) = \left(A^{-1}\right)_{nk} \leq \frac{1}{\varrho}. \tag{1.6}$$

Therefore, we can hope for a finer estimate than (1.5) that is also sensitive to the dependence of the functions f and g on the specific coordinates x_i. Our covariance estimate shows this feature:

Assumption 1.2.3. *We assume that the Hamiltonian H of the Gibbs measure μ is convex at infinity i.e. H is a bounded perturbation of a convex function. It follows from the observation by Bobkov [4] – all log-concave measures satisfy SG – and the perturbation lemma of Holley & Stroock [33] (cf. Lemma 1.1.4), that μ satisfies SG with an unspecified constant $\tilde{\varrho} > 0$.*

Theorem 1.2.4 (Covariance estimate)**.** *Let $d\mu := Z^{-1} \exp(-H(x)) \, dx$ be a probability measure on a direct product of Euclidean spaces $X = X_1 \times \cdots \times X_N$. We assume that*

- *the conditional measures $\mu(dx_i|\bar{x}_i)$, $1 \leq i \leq N$, satisfy a uniform SG(ϱ_i).*

- *the numbers* κ_{ij}, $1 \leq i \neq j \leq N$, *satisfy*

$$|\nabla_i \nabla_j H(x)| \leq \kappa_{ij} < \infty$$

  uniformly in $x \in X$. *Here,* $|\cdot|$ *denotes the operator norm of a bilinear form.*

- *the symmetric matrix* $A = (A_{ij})_{N\times N}$ *defined by*

$$A_{ij} = \begin{cases} \varrho_i, & \text{if } i = j, \\ -\kappa_{ij}, & \text{if } i < j, \end{cases} \tag{1.7}$$

  is positive definite.

Then for all functions f *and* g

$$\operatorname{cov}_\mu(f,g) \leq \sum_{i,j=1}^{N} \left(A^{-1}\right)_{ij} \left(\int |\nabla_i f|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2 \, d\mu\right)^{\frac{1}{2}}. \tag{1.8}$$

The structure of the estimate in Theorem 1.2.4 is related to the BLI in the sense that variance is replaced by covariance and that $\operatorname{Hess} H$ is replaced by A.

Remark 1.2.5 (Connection to BLI)**.** *We assume* $X_i = \mathbb{R}$ *for* $i \in \{1, \ldots, N\}$ *and let* A *be a symmetric positive definite* $N \times N$*- matrix. We consider a ferromagnetic Gaussian Hamiltonian given by* (1.2)*. Then the covariance estimate* (1.8) *coincides with the BLI given by* (1.4) *provided the function* $f = g$ *is an affine function.*

The next remark considers the optimality of Theorem 1.2.4.

Remark 1.2.6 (Optimality)**.** *Provided the Hamiltonian H is ferromagnetic Gaussian, the estimate of Theorem 1.2.4 is optimal. This remark is verified by setting* $f(x_n) = x_n$ *and* $g(x_k) = x_k$ *and using* (1.6)*.*

Remark 1.2.7 (Criterion for SG)**.** *Theorem 1.2.4 contains a well-known criterion for SG i.e.*

$$A \geq \varrho \operatorname{Id}, \quad \varrho > 0 \qquad \Rightarrow \qquad \mu \text{ satisfies SG}(\varrho).$$

As we have seen in the last section, this criterion also holds in a more relaxed version (cf. Theorem 1.1.8 and Remark 1.2.9).

The assumption under which Theorem 1.2.4 holds has the same algebraic structure as the assumption in the Otto & Reznikoff criterion for LSI (cf. Theorem 1.1.7). The only difference is that the uniform LSI constant for the single-site conditional measures is replaced by the uniform SG constant. Starting point of the proof of Theorem 1.2.4 is a representation of the covariance, which was used by Helffer [28] to give another proof of the BLI. More precisely, one can express the covariance of the measure μ as

$$\operatorname{cov}_\mu(f,g) = \int \nabla\varphi \cdot \nabla g \, d\mu, \tag{1.9}$$

where the potential φ is defined as the solution of the elliptic equation

$$-\nabla \cdot (\mu \nabla \varphi) = \left(f - \int f \, d\mu \right) \mu. \tag{1.10}$$

Here we used the convention, that μ also denotes the Lebesgue density of the probability measure μ. As a solution of (1.10) we understand any $\varphi \in H^1(\mu)$ such that for all $\zeta \in H^1(\mu)$

$$\int \nabla \zeta \cdot \nabla \varphi \, d\mu = \int \zeta \left(f - \int f \, d\mu \right) \, d\mu. \tag{1.11}$$

The existence of such solutions follows directly from the Riez representation theorem applied to

$$\mathcal{H} = H^1(\mu) \cap \left\{ \varphi, \int \varphi d\mu = 0 \right\}$$

equipped with the inner product

$$\int \nabla \zeta \cdot \nabla \varphi \, d\mu. \tag{1.12}$$

The completeness of $\mathcal{H}$ w.r.t. the chosen inner product follows from the fact that μ satisfies some SG, which is guaranteed by our Assumption 1.2.3.

Let us return to the sketch of the proof of Theorem 1.2.4. After applying the Cauchy-Schwarz inequality to (1.9), the main step of the argument (see Theorem 1.2.12) is an estimation of

$$\left(\int |\nabla_i \varphi|^2 \, d\mu \right)^{\frac{1}{2}} \tag{1.13}$$

for $i \in \{1, \ldots, N\}$, where the upper bound on (1.13) is given in terms of weighted components of

$$\left(\int |\nabla_j f|^2 \, d\mu \right)^{\frac{1}{2}}, \qquad j \in \{1, \ldots, N\}.$$

The full argument of the proof is outlined in Section 1.2.2.

1.2.1 Decay of correlations

In this section we compare the covariance estimate of Theorem 1.2.4 with a well known weighted covariance estimate due to Helffer [30], which is often applied to derive exponential decay of correlations of certain spin systems (cf. [5] and [6]). For this purpose we follow the presentation of Ledoux [40, Proposition 3.1], but rephrase the estimate in our framework.

Theorem 1.2.8 (Helffer, Ledoux)**.** *We assume that the conditions of Theorem 1.2.4 are satisfied. Additionally, we consider positive weights* $d_i > 0$, $i \in \{1, \ldots N\}$. *Let the diagonal* $N \times N$*- matrix* D *be defined as*

$$D := \operatorname{diag}(d_1 \ldots, d_N).$$

We assume that there exists $\varrho > 0$ *such that in the sense of quadratic forms*

$$DAD^{-1} \geq \varrho \, \mathrm{Id} \, . \tag{1.14}$$

Then the matrix A *is positive definite and for all functions* f *and* g*,*

$$\mathrm{cov}_\mu(f,g) \leq \frac{1}{\varrho} \left(\int |D \nabla f|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int |D^{-1} \nabla g|^2 \, d\mu \right)^{\frac{1}{2}} . \tag{1.15}$$

In fact, we will show that this estimate is a direct consequence of our covariance estimate of Theorem 1.2.4. Hence, our covariance estimate is consistent with the existing literature.

Remark 1.2.9. *For the sake of completeness we will give another proof of Theorem 1.2.8 in Section 1.2.2, which just relies on the ideas of Helffer [29, 30] and Ledoux [40]. This argument shows that condition* (1.14) *can be relaxed by a weaker condition, which was already observed in [13, Proposition 3.2]. More precisely, let the symmetric* $N \times N$*-matrix* $\mathcal{A}(x) = (\mathcal{A}_{ij}(x))$ *be defined by*

$$\mathcal{A}_{ij}(x) = \begin{cases} \varrho_i, & \text{if } i = j, \\ \nabla_i \nabla_j H(x), & \text{if } i < j. \end{cases} \tag{1.16}$$

Assume that there is $\varrho > 0$ *such that for all* $x \in X$

$$D\mathcal{A}(x)D^{-1} \geq \varrho \, \mathrm{Id} \, . \tag{1.17}$$

Note that the last condition applied to $D = \mathrm{Id}$ *yields the criterion for SG of Theorem 1.1.8.*

Let us recapitulate the method of Helffer to deduce exponential decay of correlations. One considers a metric $\delta(\cdot, \cdot)$ on the set of sites $\{1, \ldots, N\}$ of the spin system. For an arbitrary but fixed site $l \in \{1, \ldots, N\}$ one chooses

$$d_i := \exp\left(-\delta(i,l)\right)$$

as weights in Theorem 1.2.8. Because the triangle inequality implies

$$\frac{d_i}{d_j} = \exp\left(\delta(j,l) - \delta(i,l)\right) \leq \exp\left(\delta(j,i)\right),$$

a direct application of Theorem 1.2.8 yields the following criterion for exponential decay of correlations.

Corollary 1.2.10 (Helffer & Ledoux)**.** *Assume that the conditions of Theorem 1.2.4 are satisfied. Additionally, we consider a metric* $\delta(\cdot, \cdot)$ *on the set* $\{1, \ldots, N\}$ *and the symmetric* $N \times N$*- matrix* $\tilde{A} = (\tilde{A}_{ij})$ *defined by*

$$\tilde{A}_{ij} = \begin{cases} \varrho_i, & \text{if } i = j, \\ -\exp\left(\delta(i,j)\right) \kappa_{ij}, & \text{if } i < j. \end{cases} \tag{1.18}$$

We assume that there exists $\tilde{\varrho} > 0$ *such that in the sense of quadratic forms*

$$\tilde{A} \geq \tilde{\varrho}\, \mathrm{Id}\,. \tag{1.19}$$

Then for all functions $f = f(x_i)$ *and* $g = g(x_j)$, $i, j \in \{1, \dots, N\}$,

$$\mathrm{cov}_\mu(f, g) \leq \frac{1}{\tilde{\varrho}}\, \exp\left(-\delta(i,j)\right) \left(\int |\nabla_i f|^2\, d\mu\right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2\, d\mu\right)^{\frac{1}{2}}. \tag{1.20}$$

This criterion may also be stated more generally for functions with arbitrary disjoint supports. It is implicitly contained in the prelude of [40, Proposition 6.2]. In Section 1.2.2 we will give another proof of Corollary 1.2.10, which is just based on our covariance estimate of Theorem 1.2.4.

Now, let us give an example how Corollary 1.2.10 can be applied. For that purpose we consider a two-dimensional lattice system with non-convex single-site potential and weak nearest-neighbor interaction. The same type of argument would also work for any dimension and finite-range interaction. Let X denote a two-dimensional periodic lattice of N-sites and let $\delta(\cdot,\cdot)$ denote the graph distance on it. We assume that $\mu \in \mathcal{P}(X)$ has the Hamiltonian

$$H(x) = \sum_i \psi(x_i) - \varepsilon \sum_{\delta(i,j)=1} x_i x_j, \tag{1.21}$$

where the smooth potential ψ is a bounded perturbation of a Gaussian in the sense that

$$\psi(x) = \frac{1}{2}x^2 + \delta\psi(x) \qquad \text{and} \qquad \sup_{\mathbb{R}} |\delta\psi(x)| < \infty.$$

By the criterion of Holley & Stroock (cf. Theorem 1.1.4) all conditional measures $\mu(dx_i|\bar{x}_i)$ satisfy a uniform LSI with constant $\Delta := \exp\left(-\operatorname{osc}\delta\psi\right)$. From (1.21) we see that

$$\kappa_{ij} = \sup_x |\nabla_i \nabla_j H(x)| = \varepsilon.$$

Hence, we know that if the interaction is sufficiently weak in the sense of $\varepsilon < \frac{\Delta}{4}$, the matrix A of Theorem 1.2.4 satisfies

$$A \geq (\Delta - 4\varepsilon)\, \mathrm{Id}\,.$$

Analogously one obtains that if $\varepsilon < \frac{\Delta}{4}e^{-1}$, the matrix $\tilde{A}$ of Corollary 1.2.10 satisfies

$$\tilde{A} \geq (\Delta - 4\varepsilon e)\, \mathrm{Id}\,.$$

Therefore, an application of Corollary 1.2.10 yields exponential decay of correlations:

Proposition 1.2.11. *Assume that* $\varepsilon < \frac{\Delta}{4}e^{-1}$. *Then for any functions* $f = f(x_i)$ *and* $g = g(x_j)$, $i, j \in \{1, \dots, N\}$,

$$\mathrm{cov}_\mu(f, g) \leq \frac{1}{\Delta - 4\varepsilon e}\, \exp\left(-\delta(i,j)\right) \left(\int |\nabla_i f|^2\, d\mu\right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2\, d\mu\right)^{\frac{1}{2}}.$$

This statement reproduces the correlation bounds established by Helffer [30] and reproved by Ledoux in [40, Proposition 6.2].

1.2.2 Proof of the Brascamp-Lieb type covariance estimate

Behind our covariance estimate of Theorem 1.2.4 stands a stronger inequality. In fact we deduce the following theorem, from which the main result follows as a simple consequence.

Theorem 1.2.12 (Directional SG)**.** *Assume that the conditions of Theorem 1.2.4 are satisfied. For any function f let the potential φ be a solution of* (1.10)*. Then for all $i \in \{1, \ldots, N\}$*

$$\left(\int |\nabla_i \varphi|^2 d\mu \right)^{\frac{1}{2}} \leq \sum_{j=1}^{N} \left(A^{-1}\right)_{ij} \left(\int |\nabla_j f|^2 d\mu \right)^{\frac{1}{2}}. \tag{1.22}$$

In order to understand inequality (1.22) better, we recall the dual formulation of the SG (cf. for example [47]).

Lemma 1.2.13 (Dual formulation of the SG)**.** *A probability measure μ satisfies SG(ϱ) if and only if for any function f and the solution φ of* (1.10)

$$\left(\int |\nabla \varphi|^2 \, d\mu \right)^{\frac{1}{2}} \leq \frac{1}{\varrho} \left(\int |\nabla f|^2 d\mu \right)^{\frac{1}{2}}. \tag{1.23}$$

Because the directional SG given by (1.22) estimates each coordinate of the gradient separately, it is a refinement of the dual formulation of the SG given by (1.23). As in [47] we can interpret the function φ as the infinitesimal optimal displacement transporting μ into $(1+\varepsilon f)\mu$. Therefore, the left hand side of (1.22) measures the average flux of mass into the direction of the i-th coordinate against a weighted gradient of f. For this reason we call (1.22) directional spectral gap. One can also interpret the estimate (1.22) in terms of the Witten complex (for a nice overview see [31]). At least formally one can introduce the Witten-Laplacian A_1^{-1} as

$$A_1^{-1} \, \nabla f := \nabla \varphi,$$

which maps the gradient of some function f onto the gradient of the solution φ of the equation (1.10). Let Π_i denote the projection onto the space X_i, $i \in \{1, \ldots, N\}$. Then the estimate (1.22) becomes a weighted estimate of the L^2-operator norm of $\Pi_i A_1^{-1}$. The proof of Theorem 1.2.12 is very basic. It combines the core inequality of Ledoux's argument for [40, Proposition 3.1] with linear algebra that was used in the argument of [46, Theorem 1].

Proof of Theorem 1.2.12. To make the main ideas of the argument more visible, we assume that the Euclidean spaces X_i, $i \in \{1, \ldots, N\}$, are one dimensional i.e. $X_i = \mathbb{R}$. The argument for general Euclidean spaces X_i is almost the same. Then the product space $X = X_1 \times \cdots \times X_N$ becomes $\mathbb{R}^N$. The gradient ∇_i on X_i is just the partial derivative ∂_i w.r.t. the i-th coordinate. The first ingredient of the proof is the basic estimate for $j \in \{1, \ldots, N\}$

$$\int \left(|\partial_j \partial_j \varphi|^2 + \partial_j \varphi \, \partial_j \partial_j H \, \partial_j \varphi \right) \mu(dx_j|\bar{x}_j) \geq \varrho_j \int |\partial_j \varphi|^2 \mu(dx_j|\bar{x}_j), \tag{1.24}$$

which is just an equivalent formulation of the SG(ϱ_i) for the single-site measure $\mu(dx_j|\bar{x}_j)$ (cf. [40, Proposition 1.3, (1.8)] or [32, 29]). The second ingredient of the proof is the identity

$$\int \partial_j \varphi \, \partial_j f d\mu = \int \sum_{k=1}^{N} \left(|\partial_j \partial_k \varphi|^2 + \partial_j \varphi \, \partial_j \partial_k H \, \partial_k \varphi \right) d\mu. \tag{1.25}$$

Indeed, by partial integration one sees that

$$\int \partial_j\varphi \, \partial_j f d\mu = -\int \partial_j\partial_j\varphi \, \left(f - \int f d\mu\right) d\mu + \int \partial_j\varphi \, \partial_j H \left(f - \int f d\mu\right) d\mu.$$

Applying now (1.11) on the terms of the r.h.s. yields the identitiy

$$\int \partial_j\varphi \, \partial_j f \, d\mu = -\int \sum_{k=1}^N \partial_k\partial_j\partial_j\varphi \, \partial_k\varphi \, d\mu + \int \sum_{k=1}^N \partial_k\partial_j\varphi \, \partial_j H \, \partial_k\varphi \, d\mu + \int \sum_{k=1}^N \partial_j\varphi \, \partial_k\partial_j H \, \partial_k\varphi \, d\mu.$$

Let us have a closer look at the second term on the r.h.s of the last identity. It follows from the definition of μ that

$$\int \sum_{k=1}^N \partial_k\partial_j\varphi \, \partial_j H \, \partial_k\varphi \, d\mu = -\frac{1}{Z}\int \sum_{k=1}^N \partial_k\partial_j\varphi(x) \;\; \partial_k\varphi(x) \, \partial_j \exp\left(-H(x)\right) \, dx$$
$$= \int \sum_{k=1}^N \partial_j\partial_k\partial_j\varphi \;\; \partial_k\varphi \, d\mu + \int \sum_{k=1}^N \partial_k\partial_j\varphi \, \partial_j\partial_k\varphi \, d\mu$$

A combination of the last two formulas yields the desired identity (1.25).

Now, we turn to the proof of (1.22). A combination of (1.24) and (1.25) yields the estimate

$$\int \partial_j\varphi \, \partial_j f \, d\mu \geq \varrho_j \int |\partial_j\varphi|^2 d\mu + \int \sum_{k=1,\ k\neq j}^N \partial_j\varphi \, \partial_j\partial_k H \, \partial_k\varphi \, d\mu$$
$$\geq \varrho_j \int |\partial_j\varphi|^2 d\mu - \sum_{k=1,\ k\neq j}^N \kappa_{jk} \int \partial_j\varphi \, \partial_k\varphi \, d\mu.$$

Applying Cauchy-Schwarz on the last estimate yields for all $j \in \{1, \ldots, N\}$

$$\left(\int |\partial_j f|^2 d\mu\right)^{\frac{1}{2}} \geq \varrho_j \left(\int |\partial_j\varphi|^2 d\mu\right)^{\frac{1}{2}} - \sum_{k=1,\ k\neq j}^N \kappa_{jk} \left(\int |\partial_k\varphi|^2 d\mu\right)^{\frac{1}{2}}$$
$$= \sum_{k=1}^N A_{jk} \left(\int |\partial_k\varphi|^2 d\mu\right)^{\frac{1}{2}}. \tag{1.26}$$

A simple linear algebra argument outlined in [46, Lemma 9] shows that the elements of the inverse of A are non negative i.e. $\left(A^{-1}\right)_{ij} \geq 0$ for all $i, j \in \{1, \ldots, N\}$. Hence, (1.26) yields

$$\sum_{j=1}^N \left(A^{-1}\right)_{ij} \left(\int |\partial_j f|^2 d\mu\right)^{\frac{1}{2}} \geq \sum_{j=1}^N \left(A^{-1}\right)_{ij} \sum_{k=1}^N A_{jk} \left(\int |\partial_k\varphi|^2 d\mu\right)^{\frac{1}{2}}$$
$$= \delta_{ik} \left(\int |\partial_k\varphi|^2 d\mu\right)^{\frac{1}{2}} = \left(\int |\partial_i\varphi|^2 d\mu\right)^{\frac{1}{2}}.$$

□

The proof of Theorem 1.2.4 is just a direct application of Theorem 1.2.12.

Proof of Theorem 1.2.4. Using the definition of φ, cf. (1.10), we obtain the following estimate of the covariance

$$\begin{aligned}\operatorname{cov}_\mu(f,g) &= \int f\left(g - \int g\ \mu\right) d\mu\\ &= \int \nabla\varphi\cdot\nabla g\ d\mu\\ &\leq \sum_{j=1}^N \left(\int |\nabla_j\varphi|^2 d\mu\right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2 d\mu\right)^{\frac{1}{2}}\end{aligned}$$

Now, the statement follows directly from Theorem 1.2.12. □

Proof of Theorem 1.2.8 using Theorem 1.2.4. We start with deducing that A is positive definite. Because A is a symmetric Matrix, it suffices to show that every eigenvalue of A is positive. Let $\lambda\in\mathbb{R}$ be an eigenvalue of A with eigenvector x i.e.

$$Ax = \lambda x.$$

An application of (1.14) to the vector Dx yields

$$\lambda|Dx|^2 = Dx\cdot DAx = Dx\cdot DAD^{-1}Dx \geq \varrho|Dx^2| > 0,$$

which implies $\lambda > 0$.
Now, we will deduce (1.15). Because A is symmetric, the inverse A^{-1} also is symmetric. Therefore, an application of Theorem 1.2.4 yields the estimate

$$\begin{aligned}\operatorname{cov}_\mu(f,g) &\leq \sum_{i,j=1}^N \left(A^{-1}\right)_{ij} \left(\int |\nabla_i f|^2\ d\mu\right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2\ d\mu\right)^{\frac{1}{2}}\\ &= \sum_{i,j=1}^N d_j \left(A^{-1}\right)_{ji} d_i^{-1} \left(\int |d_i\nabla_i f|^2\ d\mu\right)^{\frac{1}{2}} \left(\int |d_j^{-1}\nabla_j g|^2\ d\mu\right)^{\frac{1}{2}}\\ &= DA^{-1}D^{-1}z\cdot\tilde{z}\\ &\leq |DA^{-1}D^{-1}z|\ |\tilde{z}|,\end{aligned}$$

where the vectors $z,\tilde{z}\in\mathbb{R}^N$ are defined for $i,j\in\{1,\ldots,N\}$ by

$$z_i := \left(\int |d_i\nabla_i f|^2\ d\mu\right)^{\frac{1}{2}} \qquad \text{and} \qquad \tilde{z}_j := \left(\int |d_j^{-1}\nabla_j g|^2\ d\mu\right)^{\frac{1}{2}}.$$

Therefore, (1.15) is verified provided

$$|DA^{-1}D^{-1}z| \leq \frac{1}{\varrho}\ |z| \tag{1.27}$$

holds for any $z \in \mathbb{R}^N$. From the hypothesis (1.14) it follows that

$$\begin{aligned} \varrho\, z \cdot z &\leq DAD^{-1}z \cdot z \\ &\leq |DAD^{-1}z|\,|z|. \end{aligned}$$

Hence, we have

$$|z| \leq \frac{1}{\varrho}\,|DAD^{-1}z|,$$

which immediately yields (1.27). □

Now, we give a direct argument for Theorem 1.2.8. The proof is based on the estimate (1.24) and the identity (1.25), which were the core elements of the proof of Theorem 1.2.12 and Theorem 1.2.4.

Proof of Theorem 1.2.8. As in the proof of Theorem 1.2.4 we estimate the covariance with the help of the potential φ defined by (1.10) as

$$\begin{aligned} \operatorname{cov}_\mu(f,g) &= \int f\left(g - \int g\,\mu\right) d\mu \\ &= \int \nabla\varphi \cdot \nabla g\, d\mu \\ &= \int D\nabla\varphi \cdot D^{-1}\nabla g\, d\mu \\ &\leq \int |D\nabla\varphi|\,|D^{-1}\nabla g|\, d\mu \\ &\leq \left(\int |D\nabla\varphi|^2 d\mu\right)^{\frac{1}{2}} \left(\int |D^{-1}\nabla g|^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

The proof is finished if we show

$$\left(\int |D\nabla\varphi|^2 d\mu\right)^{\frac{1}{2}} \leq \frac{1}{\varrho}\left(\int |D\nabla f|^2 d\mu\right)^{\frac{1}{2}}. \tag{1.28}$$

To verify (1.28) we need two observations. The first one is that (1.14) is equivalent to

$$D^2 A \geq \varrho D^2 \tag{1.29}$$

in the sense of quadratic forms. The second observation is that for $\mathcal{A}$ given by (1.16)

$$D^2\mathcal{A} \geq D^2 A \tag{1.30}$$

in the sense of quadratic forms. Because $d_i \geq 0$, the estimates (1.24), (1.25), (1.29), and (1.30) yield

$$\begin{aligned} \int D\nabla\varphi D\nabla f d\mu &\geq \int \nabla\varphi D^2\mathcal{A}\nabla\varphi d\mu \\ &\geq \varrho \int |D\nabla\varphi|^2 d\mu. \end{aligned}$$

Applying now Cauchy-Schwarz yields the estimate (1.28). □

Now, we will deduce Corollary 1.2.10 from Theorem 1.2.4.

Proof of Corollary 1.2.10. Let us fix two indices $i, j \in \{1, \dots, N\}$. Let f and g be arbitrary functions just depending on x_i and x_j respectively. We apply Theorem 1.2.4 and get

$$\operatorname{cov}_\mu(f,g) \leq \left(A^{-1}\right)_{ij} \left(\int |\nabla_i f|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int |\nabla_j g|^2 \, d\mu \right)^{\frac{1}{2}}, \tag{1.31}$$

where A is defined as in (1.7). Therefore, it remains to estimate the element $\left(A^{-1}\right)_{ij}$. By Neumann series (also called the random walk expansion of A^{-1} (cf. [8]) we have

$$\begin{aligned}\left(A^{-1}\right)_{ij} &= \delta_{ij}\frac{1}{\varrho_i} + \frac{\kappa_{ij}}{\varrho_i\varrho_j} + \sum_{s=1}^{N} \frac{\kappa_{is}\kappa_{sj}}{\varrho_i\varrho_s\varrho_j} + \sum_{s,l=1}^{N} \frac{\kappa_{is}\kappa_{sl}\kappa_{lj}}{\varrho_i\varrho_s\varrho_l\varrho_j} + \cdots\cdots \\ &= \delta_{ij}\frac{1}{\varrho_i} + \frac{e^{-\delta(i,j)}}{e^{-\delta(i,j)}}\frac{\kappa_{ij}}{\varrho_i\varrho_j} + \sum_{s=1}^{N} \frac{e^{-\delta(i,s)}e^{-\delta(s,j)}}{e^{-\delta(i,s)}e^{-\delta(s,j)}}\frac{\kappa_{is}\kappa_{sj}}{\varrho_i\varrho_s\varrho_j} \\ &\quad + \sum_{s,l=1}^{N} \frac{e^{-\delta(i,s)}e^{-\delta(s,l)}e^{-\delta(l,j)}}{e^{-\delta(i,s)}e^{-\delta(s,l)}e^{-\delta(l,j)}}\frac{\kappa_{is}\kappa_{sl}\kappa_{lj}}{\varrho_i\varrho_s\varrho_l\varrho_j} + \cdots\cdots .\end{aligned} \tag{1.32}$$

By the triangle inequality we get

$$e^{-\delta(i,s)}e^{-\delta(s,j)} \leq e^{-\delta(i,j)}$$

for all $i, s, j \in \{1, \dots, N\}$. Hence, we can continue the estimation of (1.32) as

$$\left(A^{-1}\right)_{ij} \leq e^{-\delta(i,j)} \left(\tilde{A}^{-1}\right)_{ij}, \tag{1.33}$$

where $\tilde{A}$ is defined as in (1.18). By (1.19) we have the bound

$$\left(\tilde{A}^{-1}\right)_{ij} \leq \frac{1}{\tilde{\varrho}},$$

which together with (1.31) and (1.33) finishes the proof. □

1.3 The directional WI and two applications

In this section we derive a similar statement of the directional SG (see Theorem 1.2.12) on the level of the WI (see Theorem 1.3.1 below). A first application yields a criterion for the WI, which is an analog version of the Otto & Reznikoff criterion for the LSI (see Theorem 1.1.7). A second application yields a non-linear version of the covariance estimate of Theorem 1.2.4. Both applications are again optimal for ferromagnetic Gaussian Hamiltonians given by (1.2). It remains to mention that this part was originally motivated by a preprint of Gao & Wu [20], who among other things generalized the to the WI to some extent (cf. Remark 1.3.5 below). The main result of this section is:

Theorem 1.3.1 (Directional WI). *Let $d\mu := Z^{-1}\exp(-H(x))\,dx$ be a probability measure on a direct product of Euclidean spaces $X = X_1 \times \cdots \times X_N$. We assume that*

- *the conditional measures $\mu(dx_i|\bar{x}_i)$, $1 \le i \le N$, satisfy a uniform WI(ϱ_i).*
- *the numbers κ_{ij}, $1 \le i \ne j \le N$, satisfy*

$$|\nabla_i \nabla_j H(x)| \le \kappa_{ij} < \infty$$

  uniformly in $x \in X$. Here, $|\cdot|$ denotes the operator norm of a bilinear form.

- *the symmetric matrix $A = (A_{ij})_{N\times N}$ defined by*

$$A_{ij} = \begin{cases} \varrho_i, & \text{if } i = j, \\ -\kappa_{ij}, & \text{if } i < j, \end{cases} \tag{1.34}$$

  is positive definite.

Then for all $i \in \{1, \ldots, N\}$ and all functions $f > 0$ satisfying $\int f\, d\mu = 1$ holds

$$\left(\int |x_i - y_i|^2\, \pi(dx, dy)\right)^{\frac{1}{2}} \le \sum_{j=1}^{N} \left(A^{-1}\right)_{ij} \left(\int \frac{|\nabla_j f|^2}{f} d\mu\right)^{\frac{1}{2}}, \tag{1.35}$$

where π denotes the optimal transference plan of $f\mu$ and μ (cf. Definition 0.7).

Remark 1.3.2. *In* (1.35) *the Wasserstein transportation cost in one direction is estimated by a weighted Fisher information. Therefore, we call the inequality* (1.35) *directional WI. It is the non-linear analogon of* (1.22)*.*

The assumption under which Theorem 1.2.4 holds has the same algebraic structure as the assumption in the Otto & Reznikoff criterion for LSI (cf. Theorem 1.1.7). The only difference is that the uniform LSI constant for the single-site conditional measures is replaced by the uniform WI constant. The structure of the proof of Theorem 1.3.1 is similar to the structure of the proof of the Otto & Reznikoff criterion for LSI. In particular, we use a similar induction in the dimension. For the proof of Theorem 1.3.1, which is outlined in Section 1.3.4, we need some auxiliary results. They are stated in Section 1.3.2 and verified in Section 1.3.3.

Application 1: A new criterion for the transportation-information inequality

In the first application of Theorem 1.3.1 we deduce a criterion for the WI inequality.

Theorem 1.3.3 (Criterion for WI). *We assume that the conditions of Theorem 1.3.1 are satisfied. Additionally, we assume that there is $\varrho > 0$ such that in the sense of quadratic forms*

$$A \ge \varrho\, \mathrm{Id}\,. \tag{1.36}$$

Then the Gibbs measure μ satisfies the WI(ϱ).

Note that Theorem 1.3.3 is formulated in the same way as the Otto & Reznikoff criterion for LSI (cf. Theorem 1.1.7).

Remark 1.3.4. *Theorem 1.3.3 is optimal for ferromagnetic Gaussian Hamiltonians in the sense of* (1.2)*: Recall that Lemma 1.1.1 states*

$$\mu \text{ satisfies LSI}(\varrho) \quad \Rightarrow \quad \mu \text{ satisfies WI}(\varrho) \quad \Rightarrow \quad \mu \text{ satisfies SG}(\varrho).$$

Hence, the argument for optimality is the same as for the Otto & Reznikoff criterion for LSI formulated in Theorem 1.1.7 (cf. [46, Remark 4]).

Remark 1.3.5. *As already mentioned before, Gao & Wu derived a similar criterion for the WI with a different approach (cf. [20][Theorem 5.3]). If one translates their statement into our setting and applies some simplification, it becomes exactly the same statement as Theorem 1.3.3. There is only one difference: Instead of considering the symmetric matrix A given by* (1.34)*, Gao & Wu consider the symmetric matrix $\tilde{A} = (\tilde{A}_{ij})_{N \times N}$ given by*

$$\tilde{A}_{ij} = \begin{cases} \min_{1 \le k \le N} \varrho_k, & \text{if } i = j, \\ -\kappa_{ij}, & \text{if } i < j. \end{cases}$$

Note that A and $\tilde{A}$ coincide except of the terms on the main diagonal and $A \ge \tilde{A}$ in the sense of quadratic forms.

Application 2: A new non-linear covariance estimate

The second application is a non-linear version of the covariance estimate of Theorem 1.2.4.

Theorem 1.3.6 (Non-linear covariance estimate)**.** *Assume that the conditions of Theorem 1.3.1 are satisfied. Then for all functions $\tilde{f} > 0$, f, and g holds*

$$a) \qquad \operatorname{cov}_\mu(f,g) \le \sum_{i,j=1}^{N} \left(A^{-1}\right)_{ij} \|\nabla_i f\|_{L^2(\mu)} \|\nabla_j g\|_{L^2(\mu)},$$

$$b) \qquad \operatorname{cov}_\mu(\tilde{f},g) \le \sum_{i,j=1}^{N} \left(A^{-1}\right)_{ij} \left(\int \tilde{f}\, d\mu\right)^{\frac{1}{2}} \left(\int \frac{|\nabla_i \tilde{f}|^2}{\tilde{f}}\, d\mu\right)^{\frac{1}{2}} \|\nabla_j g\|_{L^\infty(\mu)}.$$

Note that part $a)$ of Theorem 1.3.6 trivially follows from a combination of Theorem 1.2.4 and the fact that WI(ϱ) implies SG(ϱ). In order to show self-consistency, we will give a direct proof of part $a)$ that is only based on the directional WI. Obviously, Theorem 1.3.6 is optimal for ferromagnetic Gaussian systems (cf. Remark 1.2.6).

1.3.1 Proof of the applications

Proof of Theorem 1.3.3. From the hypothesis (1.36) one directly gets

$$\left\langle x, A^{-1}A^{-1}x \right\rangle \le \frac{1}{\varrho^2} \left\langle x, x \right\rangle. \tag{1.37}$$

By using Theorem 1.3.1 we can estimate

$$
\begin{aligned}
W_2^2\left(f\mu,\mu\right) &= \sum_{i=1}^{N}\int |x_i-y_i|^2\ \pi(dx,dy) \\
&\leq \sum_{i=1}^{N}\left[\sum_{j=1}^{N}\left(A^{-1}\right)_{ij}\left(\int \frac{|\nabla_j f|^2}{f}d\mu\right)^{\frac{1}{2}}\right]^2 \\
&= \sum_{i=1}^{N}\sum_{k,j=1}^{N}\left(A^{-1}\right)_{ik}\left(A^{-1}\right)_{ij}\left(\int \frac{|\nabla_k f|^2}{f}d\mu\right)^{\frac{1}{2}}\left(\int \frac{|\nabla_j f|^2}{f}d\mu\right)^{\frac{1}{2}} \\
&= \sum_{k,j=1}^{N}\sum_{i=1}^{N}\left(A^{-1}\right)_{ki}\left(A^{-1}\right)_{ij}\left(\int \frac{|\nabla_k f|^2}{f}d\mu\right)^{\frac{1}{2}}\left(\int \frac{|\nabla_j f|^2}{f}d\mu\right)^{\frac{1}{2}}.
\end{aligned}
$$

Applying now (1.37) directly yields

$$
W_2^2\left(f\mu,\mu\right) \leq \frac{1}{\varrho^2}\sum_{i=1}^{N}\int \frac{|\nabla_i f|^2}{f}d\mu = \frac{1}{\varrho^2}\int \frac{|\nabla f|^2}{f}d\mu.
$$

□

Proof of Theorem 1.3.6. Argument for $a)$: We assume that the functions f and g are smooth and have compact support. Without restriction $\int f d\ \mu = 0$, else consider the function $\tilde{f} := f - \int f\ d\mu$. For an arbitrary $\varepsilon > 0$ we consider the measure $\mu_\varepsilon := (1+\varepsilon f)\mu$. Then

$$
\begin{aligned}
\operatorname{cov}_\mu(f,g) &= \int f\left(g - \int g\ d\mu\right)\ d\mu = \int \left(g - \int g\ d\mu\right)\ d\ \frac{\mu_\varepsilon - \mu}{\varepsilon} \\
&= \frac{1}{\varepsilon}\int g(x) - g(y)\ \pi_\varepsilon(dx,dy).
\end{aligned}
$$

Here $\pi_\varepsilon(dx,dy)$ denotes the optimal transference plan between $\mu_\varepsilon(dx)$ and $\mu(dy)$. We know by Taylor formula that

$$
g(x) - g(y) \leq \sum_{j=1}^{N}|\nabla_j g(y)||x_j - y_j| + C|x-y|^2.
$$

Therefore, we can estimate

$$
\begin{aligned}
\operatorname{cov}_\mu(f,g) &\leq \frac{1}{\varepsilon}\int\sum_{j=1}^{N}|\nabla_j g(y)||x_j-y_j|\ \pi_\varepsilon(dx,dy) + \frac{C}{\varepsilon}\int |x-y|^2\ d\pi_\varepsilon(dx,dy) \\
&\leq \sum_{j=1}^{N}\|\nabla_j g\|_{L^2(\mu)}\ \frac{1}{\varepsilon}\left(\int |x_j-y_j|^2\ \pi_\varepsilon(dx,dy)\right)^{\frac{1}{2}} + \frac{C}{\varepsilon}\int |x-y|^2\ \pi_\varepsilon(dx,dy).
\end{aligned}
$$

On the first term of the r.h.s. we apply Theorem 1.3.1 and on the second term we apply Theorem 1.3.3 i.e.

$$\begin{aligned}\operatorname{cov}_\mu(f,g) \leq \sum_{i,j=1}^N (A^{-1})_{ij}\, \|\nabla_j g\|_{L^2(\mu)} \left(\int \frac{|\nabla_i(1+\varepsilon f)|^2}{\varepsilon^2(1+\varepsilon f)}\, d\mu\right)^{\frac{1}{2}} \\ + \frac{C}{\varepsilon\varrho^2}\int \frac{|\nabla(1+\varepsilon f)|^2}{1+\varepsilon f}\, d\mu.\end{aligned}$$

For $\varepsilon \to 0$ the first term on the r.h.s. converges to

$$\left(\int \frac{|\nabla_i \varepsilon f|^2}{\varepsilon^2(1+\varepsilon f)}\, d\mu\right)^{\frac{1}{2}} \longrightarrow \|\nabla_i f\|_{L^2(\mu)}$$

and for the second term converges to

$$\frac{C}{\varepsilon\varrho^2}\int \frac{|\nabla \varepsilon f|^2}{1+\varepsilon f}\, d\mu \longrightarrow 0.$$

Using now a standard approximation argument one can get rid of the assumptions of smoothness and compact support on f and g.

Argument for $b)$: We assume w.l.o.g. $\int \tilde{f}\, d\mu = 1$. A direct calculation yields

$$\begin{aligned}\operatorname{cov}_\mu(\tilde{f},g) &= \int \tilde{f} g\, d\mu - \int \tilde{f}\, d\mu \int g\, d\mu \\ &= \int \tilde{f} g\, d\mu - \int g\, d\mu \\ &= \int g(x) - g(y)\, \pi(dx,dy)\ ,\end{aligned}$$

where $\pi(dx,dy)$ denotes the optimal transference plan of the measures $\tilde{f}\mu(dx)$ and $\mu(dy)$. Because

$$\begin{aligned}g(x)-g(y) &= \int_0^1 \nabla g\,(tx+(1-t)y)\cdot(x-y)\, dt \\ &= \int_0^1 \sum_{j=1}^N \nabla_j g\,(tx_j+(1-t)y_j)\,(x_j-y_j)\, dt\end{aligned}$$

we get the estimate

$$\begin{aligned}\operatorname{cov}_\mu(\tilde{f},g) \ &\leq\ \sum_{j=1}^N \|\nabla_j g\|_\infty \int |x_j-y_j|\, \pi(dx,dy) \\ &\leq\ \sum_{j=1}^N \|\nabla_j g\|_\infty \left(\int |x_j-y_j|^2\, \pi(dx,dy)\right)^{\frac{1}{2}}.\end{aligned}$$

Now, an application of Proposition 1.3.1 yields the desired statement. □

1.3.2 Auxiliary results

For the proof of Theorem 1.3.1 we need some auxiliary results. We start with recalling a basic fact for the optimal transport, which was observed for example by Gao & Wu in their proof of [20, Theorem 3.1]:

Lemma 1.3.7. *For an arbitrary function $f > 0$ with $\int f d\mu = 1$, let $\pi(dx, dy)$ denote the optimal transference plan between the measures $f\mu(dx)$ and $\mu(dy)$. Then for every $i \in \{1, \ldots, N\}$ and every vector $\bar{x}_i$ and $\bar{y}_i$ (cf. the Chapter Conventions), the conditional transference plan $\pi(dx_i, dy_i|\bar{x}_i, \bar{y}_i)$ is the optimal transference plan of the conditional measures $\frac{f\mu(dx_i|\bar{x}_i)}{\bar{f}(\bar{x}_i)}$ and $\mu(dy_i|\bar{y}_i)$ i.e.*

$$W_2\left(\frac{f\mu(\cdot|\bar{x}_i)}{\bar{f}(\bar{x}_i)}, \mu(\cdot|\bar{y}_i)\right) = \left(\int |x_i - y_i|^2 \pi(dx_i, dy_i|\bar{x}_i, \bar{y}_i)\right)^{\frac{1}{2}}.$$

Here, we used the notation

$$\bar{f}(\bar{x}_i) := \int f(x)\mu(dx_i|\bar{x}_i).$$

The last statement is used to deduce the following estimate for the optimal transport:

Lemma 1.3.8. *For an arbitrary function $f > 0$ with $\int f d\mu = 1$, let $\pi(dx, dy)$ denote the optimal transference plan between $f\mu(dx)$ and $\mu(dy)$. For $i \in \{1, \ldots, N\}$ let $\bar{\mu}_i(d\bar{x}_i)$ denote the marginal measure of μ w.r.t. the conditional measures $\mu(dx_i|\bar{x}_i)$. Additionally, let $\tilde{\pi}(d\bar{x}_i, d\bar{y}_i)$ denote the optimal transference plan between the marginals $\bar{f}\bar{\mu}_i(d\bar{x}_i)$ and $\bar{\mu}_i(d\bar{y}_i)$. Then*

$$\int |x_i - y_i|^2 \, \pi(dx, dy) \leq \int W_2^2 \left(\frac{f\mu(\cdot|\bar{x}_i)}{\bar{f}(\bar{x}_i)}, \mu(\cdot|\bar{y}_i)\right) \, \tilde{\pi}(d\bar{x}_i, d\bar{y}_i).$$

Because we follow the approach of Otto & Reznikoff [46], the remaining auxiliary results are almost the same as in [46]. There is only one difference: In our case the statements are formulated on the level of the WI and not on the level of the LSI.

Lemma 1.3.9. *[Analogue of Lemma 5 in [46]] Let $\mu(dx)$ be a probability measure on a Euclidean space X. We assume that there exists $\varrho > 0$ such that*

$$\mu \text{ satisfies WI}(\varrho).$$

Then we have for arbitrary $f > 0$ and g:

$$|\mathrm{cov}_\mu(g, f)| \leq \frac{1}{\varrho} \, \sup_x |\nabla g| \, \left(\int f \, d\mu \int \frac{1}{f}|\nabla f|^2 \, d\mu\right)^{\frac{1}{2}}.$$

We also need a linearized version of Lemma 1.3.9.

Corollary 1.3.10 (Analogue of Corollary 1 in [46]). *Let $\mu(dx)$ be a probability measure on a Euclidean space X. We assume that there exists $\varrho > 0$ such that*

$$\mu \text{ satisfies WI}(\varrho).$$

Then we have for arbitrary f and g:

$$\begin{aligned} |\mathrm{cov}_\mu(g,f)| &\le \frac{1}{\varrho}\ \sup_x |\nabla g| \ \left(\int |\nabla f|^2 \, d\mu\right)^{\frac{1}{2}} \\ &\le \frac{1}{\varrho}\ \sup_x |\nabla g|\ \sup_x |\nabla f|. \end{aligned}$$

Lemma 1.3.9 is used to establish the following result.

Lemma 1.3.11 (Analogue of Lemma 6 in [46]). *Let X_1, X_2 be two Euclidean spaces and $\mu(dx_1, dx_2)$ a probability measure on the product space $X_1 \times X_2$ with a smooth positive Lebesgue density $\frac{d\mu}{d\mathcal{L}}$.*
We assume that there exists $\kappa_{12} < \infty$ such that the Hamiltonian $H(x_1,x_2) = -\log \frac{d\mu}{d\mathcal{L}}$ satisfies

$$\forall \ (x_1,x_2) \quad |\nabla_1 \nabla_1 H(x_1,x_2)| \le \kappa_{12}.$$

We assume that there exists $\varrho_2 > 0$ such that we have for the conditional measure

$$\forall \ x_1 \quad \mu(dx_2|x_1) \quad \text{satisfies WI}(\varrho_2).$$

For arbitrary $f(x_1,x_2) \ge 0$ consider

$$\bar{f}(x_1) = \int f(x_1,x_2)\mu(dx_2|x_1).$$

Then we obtain for the marginal $\bar{\mu}(dx_1)$

$$\left(\int \frac{1}{\bar{f}}\ |\nabla_1 \bar{f}|^2 \ \bar{\mu}(dx_1)\right)^{\frac{1}{2}} \le \left(\int \frac{1}{f}\ |\nabla_1 f|^2 \ d\mu\right)^{\frac{1}{2}} + \frac{\kappa_{12}}{\varrho_2} \left(\int \frac{1}{f}\ |\nabla_2 f|^2 \ d\mu\right)^{\frac{1}{2}}.$$

Lemma 1.3.12 (Analogue of Lemma 7 in [46]). *Let X_1, X_2 be two Euclidean spaces and $\mu(dx_1,dx_2)$ a probability measure on the product space $X_1 \times X_2$ with smooth positive Lebesgue density $\frac{d\mu}{d\mathcal{L}}$.*
We assume that there exists $\varrho_2, \bar{\varrho}_1 > 0$ such that we have for the conditional measure and marginal

$$\begin{aligned} \forall \ x_1 \qquad & \mu(dx_2|x_1) \text{ satisfies WI}(\varrho_2), \\ & \bar{\mu}(dx_1) \text{ satisfies WI}(\bar{\varrho}_1). \end{aligned}$$

Then we obtain for the marginal $\bar{\mu}(dx_2)$

$$\bar{\mu}(dx_2) \text{ satisfies WI}(\bar{\varrho}_2)$$

with

$$\frac{1}{\bar{\varrho}_2} \le \frac{1}{\varrho_2} + \frac{1}{\bar{\varrho}_1}\frac{\kappa_{12}^2}{\varrho_2^2}.$$

Corollary 1.3.13 (Analogue of Corollary 2 in [46])**.** *Let X_1, X_2 be two Euclidean spaces and $\mu(dx_1, dx_2)$ a probability measure on the product space $X_1 \times X_2$ with smooth positive Lebesgue density $\frac{d\mu}{d\mathcal{L}}$.*
We assume that there exists $\varrho_1, \varrho_2 > 0$ such that we have for the conditional measures

$$\begin{aligned} \forall\, x_2 \qquad & \mu(dx_1|x_2) \text{ satisfies WI}(\varrho_1), \\ \forall\, x_1 \qquad & \mu(dx_2|x_1) \text{ satisfies WI}(\varrho_2). \end{aligned}$$

We assume that

$$\varrho_1\varrho_2 - \kappa_{12}^2 > 0.$$

Then we obtain for the marginal $\bar{\mu}(dx_1)$

$$\bar{\mu}(dx_1) \text{ satisfies WI}(\bar{\varrho}_1)$$

with

$$\bar{\varrho}_1 \geq \varrho_1 - \frac{\kappa_{12}^2}{\varrho_2}.$$

Lemma 1.3.14 (Analogue of Lemma 8 in [46])**.** *Let X_1, X_2, X_3 be Euclidean spaces and $\mu(dx_1, dx_2, dx_3)$ a probability measure on the product space $X_1 \times X_2 \times X_3$ with a smooth positive Lebesgue density $\frac{d\mu}{d\mathcal{L}}$.*
We assume that for $i < j \in \{1, 2, 3\}$ there exists $\kappa_{ij} < \infty$ such that the Hamiltonian $H(x_1, x_2, x_3) = -\log \frac{d\mu}{d\mathcal{L}}$ satisfies

$$\forall\, (x_1, x_2, x_3) \quad |\nabla_i \nabla_j H(x_1, x_2, x_3)| \leq \kappa_{ij}.$$

We assume that there exists $\varrho_3 > 0$ such that we have for the conditional measures

$$\forall\, (x_1, x_2) \quad \mu(dx_3|x_1, x_2) \text{ satisfies WI}(\varrho_3).$$

Consider the Hamiltonian $\bar{H}(x_1, x_2)$ belonging to the marginal $\bar{\mu}(dx_1, dx_2)$, i.e.

$$\bar{H}(x_1, x_2) = -\log \int \exp(-H(x_1, x_2, x_3))\, dx_3.$$

It satisfies

$$\forall\, (x_1, x_2) \quad |\nabla_1 \nabla_2 \bar{H}(x_1, x_2)| \leq \bar{\kappa}_{12}$$

with

$$\bar{\kappa}_{12} \leq \kappa_{12} + \frac{\kappa_{13}\kappa_{23}}{\varrho_3}.$$

1.3.3 Proof of the auxiliary results

In this section we will proof the auxiliary results of Section 1.3.2.

Proof of Lemma 1.3.8. A direct calculation yields that $\pi(dx_i, dy_i|\bar{x}_i, \bar{y}_i)\tilde{\pi}(d\bar{x}_i, d\bar{y}_i)$ is a transference plan of $f\mu(dx)$ and $\mu(dy)$. Therefore, we can estimate by using the optimality of π that

$$\begin{aligned}\int \sum_{j=1}^{N} |x_j - y_j|^2 \, \pi(dx, dy) &\leq \int \int \sum_{j=1}^{N} |x_j - y_j|^2 \, \pi(dx_i, dy_i|\bar{x}_i, \bar{y}_i) \, \tilde{\pi}(d\bar{x}_i, d\bar{y}_i) \\ &= \int \sum_{j=1,\ j\neq i}^{N} |x_j - y_j|^2 \, \tilde{\pi}(d\bar{x}_i, d\bar{y}_i) \\ &\quad + \int \int |x_i - y_i|^2 \, \pi(dx_i, dy_i|\bar{x}_i, \bar{y}_i) \, \tilde{\pi}(d\bar{x}_i, d\bar{y}_i). \quad (1.38)\end{aligned}$$

Let $\bar{\pi}(d\bar{x}_i, d\bar{y}_i)$ be the marginal of $\pi(dx, dy)$ w.r.t. $(\bar{x}_i, \bar{y}_i)$. Another direct calculation yields that $\bar{\pi}(d\bar{x}_i, d\bar{y}_i)$ is a transference plan of $\bar{f}\bar{\mu}_i(d\bar{x}_i)$ and $\bar{\mu}_i(d\bar{y}_i)$. Hence, by optimality of $\tilde{\pi}$ we can estimate

$$\begin{aligned}&\int \sum_{j=1,\ j\neq i}^{N} |x_j - y_j|^2 \, \tilde{\pi}(d\bar{x}_i, d\bar{y}_i) + \int |x_i - y_i|^2 \, \pi(dx, dy) \\ &\qquad \leq \int \sum_{j=1,\ j\neq i}^{N} |x_j - y_j|^2 \, \bar{\pi}(d\bar{x}_i, d\bar{y}_i) + \int |x_i - y_i|^2 \, \pi(dx, dy) \\ &\qquad = \int \sum_{j=1}^{N} |x_j - y_j|^2 \, \pi(dx, dy). \qquad (1.39)\end{aligned}$$

A combination of (1.38), (1.39), and Lemma 1.3.7 yields the desired statement. □

Proof of Lemma 1.3.9. Let us assume w.l.o.g. $\int f d\mu = 1$. Recall from the proof of Theorem 1.3.6 that

$$\operatorname{cov}_\mu(f, g) = \int g(x) - g(y) \, \pi(dx, dy).$$

Here, π is the optimal transference plan of the measures $f\mu$ and μ. Because

$$g(x) - g(y) = \int_0^1 \nabla g \left(tx + (1-t)y\right) \cdot (x - y) \, dt$$

we can estimate

$$\operatorname{cov}_\mu(f, g) \ \leq \ \sup |\nabla g| \int |x - y| \, \pi(dx, dy) \ \leq \ \sup |\nabla g| \left(\int |x - y|^2 \, \pi(dx, dy) \right)^{\frac{1}{2}},$$

which yields the desired statement by applying WI(ϱ). □

Proof of Corollary 1.3.10. The statement follows from Lemma 1.3.9 by linearization (see also the proof of Theorem 1.3.6). □

Proof of Lemma 1.3.11. The statement is an analogue formulation of [46][Lemma 6]. Therefore, one can directly copy the proof, because the argument just relies on Lemma 1.3.9. □

Proof of Lemma 1.3.12. For convenience we will use the notation

$$\bar{f}(x_1) := \int f(x_2)\, \mu(dx_2|x_1).$$

Let $\tilde{\pi}(dx_1, dy_1)$ denote the optimal transference plan between $\bar{f}(x_1)\bar{\mu}(dx_1)$ and $\bar{\mu}(dy_1)$. Let ξ be a test function on the Euclidean space X_2, then

$$\begin{aligned}
&\int_{X_1\times Y_1}\int_{X_2} \xi(x_2)\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1)\ \tilde{\pi}(dx_1, dy_1) \\
&\qquad = \int_{X_1}\int_{X_2} \xi(x_2)\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1)\ \bar{f}(x_1)\bar{\mu}(dx_1) \\
&\qquad = \int_{X_2} \xi(x_2) f(x_2)\mu(dx_2) \\
&\qquad = \int_{X_2} \xi(x_2) f(x_2)\ \bar{\mu}(dx_2).
\end{aligned}$$

Also let ζ be a test function on the Euclidean space Y_2 then

$$\begin{aligned}
&\int_{X_1\times Y_1}\int_{Y_2} \zeta(y_2)\ \mu(dy_2|y_1)\ \tilde{\pi}(dx_1, dy_1) \\
&\qquad = \int_{Y_1}\int_{Y_2} \zeta(y_2)\ \mu(dy_2|y_1)\ \bar{\mu}(dy_1) \\
&\qquad = \int_{Y_2} \zeta(y_2)\ \bar{\mu}(dy_2).
\end{aligned}$$

Therefore, $(f(x_2)\bar{\mu}(dx_2)\ ,\ \bar{\mu}(dx_2))$ is a convex combination of

$$\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1)\ ,\ \mu(dy_2|y_1)\right)$$

with respect to $\tilde{\pi}(dx_1, dy_1)$. Hence, we get by the convexity of the Wasserstein distance that

$$W_2^2\left(f(x_2)\bar{\mu}(dx_2), \bar{\mu}(dy_2)\right) \leq \int W_2^2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1), \mu(dy_2|y_1)\right)\ \tilde{\pi}(dx_1, dy_1). \tag{1.40}$$

By using the triangle inequality we get

$$\begin{aligned}
&W_2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1), \mu(dy_2|y_1)\right) \\
&\quad \leq W_2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1), \mu(dy_2|x_1)\right) + W_2\left(\mu(dy_2|x_1), \mu(dy_2|y_1)\right).
\end{aligned} \tag{1.41}$$

The first term on the r.h.s. is estimated by applying the WI(ϱ_2) for $\mu(dx_2|x_1)$ as

$$W_2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1),\mu(dy_2|x_1)\right)\leq\frac{1}{\varrho_2}\left(\frac{1}{\bar{f}(x_1)}\int\frac{1}{f(x_2)}\left|\nabla_{x_2}f(x_2)\right|^2\mu(dx_2|x_1)\right)^{\frac{1}{2}}. \tag{1.42}$$

We put now our attention on the second term on the r.h.s. of (1.41). Let

$$Z_1:=\int\exp\left(-H(x_1,y_2)\right)\,dy_2\quad\text{and}\quad Z_2:=\int\exp\left(-H(y_1,y_2)\right)\,dy_2.$$

Notice that

$$\begin{aligned}\mu(dy_2|x_1)&=Z_1^{-1}\exp\left(-H(x_1,y_2)\right)\,dy_2\\&=\underbrace{Z_1^{-1}Z_2\exp\left(-H(x_1,y_2)+H(y_1,y_2)\right)}_{=:g(y_2)}\,\mu(dy_2|y_1).\end{aligned}$$

Therefore, we can estimate by applying the WI(ϱ_2) to $\mu(dy_2|y_1)$ that

$$\begin{aligned}W_2\left(\mu(dy_2|x_1),\mu(dy_2|y_1)\right)&=W_2\left(g(y_2)\mu(dy_2|y_1),\mu(dy_2|y_1)\right)\\&\leq\frac{1}{\varrho_2}\left(\int\frac{1}{g(y_2)}\left|\nabla_{y_2}g(y_2)\right|^2\mu(dy_2|y_1)\right)^{\frac{1}{2}}\\&=\frac{1}{\varrho_2}\left(\int\left|\nabla_{y_2}\ln g(y_2)\right|^2g(y_2)\,\mu(dy_2|y_1)\right)^{\frac{1}{2}}.\end{aligned}\tag{1.43}$$

Notice that

$$\begin{aligned}\left|\nabla_{y_2}\ln g(y_2)\right|&=\left|\nabla_{y_2}H(x_1,y_2)-\nabla_{y_2}H(y_1,y_2)\right|\\&\leq\sup\,\left|\nabla_1\nabla_2H\right|\left|x_1-y_1\right|=\kappa_{12}\left|x_1-y_1\right|\end{aligned}$$

and

$$g(y_2)\,\mu(dy_2|y_1)=\mu(dy_2|x_1).$$

Therefore, we get from (1.43) that

$$W_2\left(\mu(dy_1|x_1),\mu(dy_2|y_1)\right)\leq\frac{\kappa_{12}}{\varrho_2}\left|x_1-y_1\right|\underbrace{\int\mu(dy_2|x_1)}_{=1}. \tag{1.44}$$

By applying now the L^2-triangle inequality we get from (1.41) that

$$\begin{aligned}&\int W_2^2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1),\mu(dy_2|y_1)\right)\,\tilde{\pi}(dx_1,dx_2)\\&\quad\leq\left[\left(\int W_2^2\left(\frac{f(x_2)}{\bar{f}(x_1)}\mu(dx_2|x_1),\mu(dy_2|x_1)\right)\,\tilde{\pi}(dx_1,dy_1)\right)^{\frac{1}{2}}\right.\\&\qquad\left.+\left(\int W_2^2\left(\mu(dy_2|x_1),\mu(dy_2|y_1)\right)\,\tilde{\pi}(dx_1,dy_1)\right)^{\frac{1}{2}}\right]^2.\end{aligned}\tag{1.45}$$

We estimate now the first term on the r.h.s. of (1.45) by using (1.42) as

$$\begin{aligned}
&\left(\int W_2^2\left(\frac{f(x_2)}{\bar f(x_1)}\mu(dx_2|x_1),\mu(dy_2|x_1)\right)\ \tilde\pi(dx_1,dy_1)\right)^{\frac12}\\
&\quad\le \frac{1}{\varrho_2}\left(\int \frac{1}{\bar f(x_1)}\int \frac{1}{f(x_2)}\,|\nabla_2 f(x_2)|^2\ \mu(dx_2|x_1)\ \tilde\pi(dx_1,dy_1)\right)^{\frac12}\\
&\quad= \frac{1}{\varrho_2}\left(\int \frac{1}{\bar f(x_1)}\int \frac{1}{f(x_2)}\,|\nabla_2 f(x_2)|^2\ \mu(dx_2|x_1)\ \bar f(x_1)\ \bar\mu(dx_1)\right)^{\frac12}\\
&\quad= \frac{1}{\varrho_2}\left(\int \frac{1}{f(x_2)}\,|\nabla_2 f(x_2)|^2\ \bar\mu(dx_2)\right)^{\frac12}.
\end{aligned}$$

By using (1.44) we can estimate the second term on the r.h.s. of (1.45) as

$$\begin{aligned}
\left(\int W_2^2\left(\mu(dy_2|x_1),\mu(dy_2|y_1)\right)\ \tilde\pi(dx_1,dy_1)\right)^{\frac12} &\le \frac{\kappa_{12}}{\varrho_2}\ \left(\int |x_1-y_1|^2\ \tilde\pi(dx_1,dy_1)\right)^{\frac12}\\
&= \frac{\kappa_{12}}{\varrho_2}\ W_2\left(\bar f(x_1)\bar\mu(dx_1),\bar\mu(dy_1)\right).
\end{aligned}$$

We use now that $\bar\mu(dy_1)$ satisfies WI($\bar\varrho_1$) and apply Lemma 1.3.11 in the to get

$$\begin{aligned}
W_2\left(\bar f(x_1)\bar\mu(dx_1),\bar\mu(dy_1)\right) &\le \frac{1}{\bar\varrho_1}\ \left(\int \frac{|\nabla_{x_1}\bar f|^2}{\bar f}\ \bar\mu(dx_1)\right)^{\frac12}\\
&\le \frac{1}{\bar\varrho_1}\frac{\kappa_{12}}{\varrho_2}\left(\int \frac{1}{f}\ |\nabla_{x_2} f|^2\ \bar\mu(dx_2)\right)^{\frac12}.
\end{aligned}$$

Therefore, we overall get by a combination of (1.40) and the last four estimates that

$$\begin{aligned}
W_2^2\left(f(x_2)\bar\mu(dx_2),\bar\mu(dy_2)\right) \le \Bigg[&\frac{1}{\varrho_2}\left(\int \frac{1}{f(x_2)}\ |\nabla_2 f(x_2)|^2\ \bar\mu(dx_2)\right)^{\frac12}\\
&+\frac{1}{\bar\varrho_1}\frac{\kappa_{12}^2}{\varrho_2^2}\left(\int \frac{1}{f(x_2)}\ |\nabla_2 f(x_2)|^2\ \bar\mu(dx_2)\right)^{\frac12}\Bigg]^2, \qquad (1.46)
\end{aligned}$$

which yields the desired statement. □

Proof of Corollary 1.3.13. Note that one can take over the proof of [46][Corollary 2] using Lemma 1.3.12 as the main ingredient. □

Proof of Lemma 1.3.14. Note that one can take over the proof of [46][Lemma 8] using Corollary 1.3.10 as the main ingredient. □

1.3.4 Proof of the directional WI inequality

For the proof of Proposition 1.3.1 we adapt the argument of the proof of the Otto & Reznikoff criterion for LSI (cf. [46, Theorem 2]). Therefore, we show (1.35) by induction. For $N = 1$ the statement (1.35) is a trivial consequence of our assumptions. Now, let us assume that (1.35) holds for a system with $(N-1)$ components. We will show that it also holds for system with N components. Let $\kappa_N := (\kappa_{1N}, \ldots, \kappa_{NN})^t$. As in [46] we introduce the block decomposition of A as

$$A = \begin{pmatrix} A' & -\kappa_N \\ -\kappa_N^t & \varrho_N \end{pmatrix}.$$

Let $\bar{A}$ denote the $(N-1) \times (N-1)$ matrix defined by

$$\bar{A} = A' - \frac{1}{\varrho_N} \kappa_N \otimes \kappa_N.$$

Note that $\bar{A}$ inherits our assumptions on A: It is symmetric and positive definite.

Now, we consider the system $\bar{\mu}(dx_1, \ldots, x_{N-1})$ i.e. the marginal of $\mu(dx_1, \ldots, dx_N)$ on $X_1 \times \cdots \times X_{N-1}$. Its Hamiltonian is given by

$$\bar{H}(x_1, \ldots, x_{N-1}) = -\log \int \exp(-H(x_1, \ldots, x_{N-1}, x_N)))\, dx_N.$$

Analog to [46] we apply Lemma 1.3.14 to $\mu(dx_i, dx_j, dx_N | \cdots)$ and get for $i \neq j$

$$-\bar{\kappa}_{ij} \geq -\kappa_{ij} - \frac{\kappa_{iN}\kappa_{jN}}{\varrho_N} = \bar{A}_{ij}.$$

As in [46] we apply Corollary 1.3.13 to $\mu(dx_i, dx_N | \cdots)$ and get for any $i \in \{1, \ldots, N-1\}$ and $\bar{x}_{iN} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-1})$ that

$$\bar{\mu}(dx_i | \bar{x}_{iN}) \text{ satisfies WI}(\bar{\varrho}_i)$$

with

$$\bar{\varrho}_i \geq \varrho_i - \frac{\kappa_{iN}^2}{\varrho_N} = \bar{A}_{ii}.$$

Recall the convention $\bar{x}_N := (x_1, \ldots, x_{N-1})$. Hence, we may apply the induction hypothesis to $\bar{\mu}(d\bar{x}_N)$ and $\bar{A}$ and get for any $j \in \{1, \ldots, N-1\}$ and $\bar{f}(\bar{x}_N) > 0$ satisfying $\int \bar{f} d\bar{\mu} = 1$ that

$$\left(\int |x_j - y_j|^2 \, \tilde{\pi}(d\bar{x}_N, d\bar{y}_N) \right)^{\frac{1}{2}} \leq \sum_{k=1}^{N-1} \left(\bar{A}^{-1}\right)_{jk} \left(\int \frac{|\nabla_k \bar{f}|^2}{\bar{f}} \, d\bar{\mu} \right)^{\frac{1}{2}}. \tag{1.47}$$

Here, $\tilde{\pi}(d\bar{x}_N, d\bar{y}_N)$ denotes the optimal transference plan between $\bar{f}\bar{\mu}(d\bar{x}_N)$ and $\bar{\mu}(d\bar{y}_N)$.

Now, we state the induction step for (1.35) in the case $i = N$. In the case $i \in \{1, \dots, N-1\}$, one could re-numerate the basis such that $i \mapsto N$ and carry out the same argument. From [46] we know that the inverse of A can be written as

$$A^{-1} = \begin{pmatrix} \bar{A}^{-1} & \frac{\bar{A}^{-1}\kappa_N}{\varrho_N} \\ \left(\frac{\bar{A}^{-1}\kappa_N}{\varrho_N}\right)^t & \frac{1}{\varrho_N} + \frac{\kappa_N \cdot \bar{A}^{-1}\kappa_N}{\varrho_N^2} \end{pmatrix}. \tag{1.48}$$

For convenience, we introduce the probability measures ν_j, $0 \le j \le N$, on the Euclidean space X_N according to

$$\nu_j(dx_N) = \begin{cases} \mu(dx_N|\bar{x}_N), & \text{if } j = 0, \\ \mu(dx_N|y_1, \dots y_j, x_{j+1}, \dots x_N), & \text{if } 1 \le j \le N-1, \\ \mu(dx_N|\bar{y}_N), & \text{if } j = N. \end{cases}$$

Recalling the definition $\bar{f}(\bar{x}_N) := \int f(x)\mu(dx_N|\bar{x}_N)$ we get by applying the triangle inequality for the Wasserstein distance twice that

$$\begin{aligned} W_2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \mu(\cdot|\bar{y}_N)\right) &= W_2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \nu_N\right) \\ &\le W_2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \nu_0\right) + W_2(\nu_0, \nu_N) \\ &\le W_2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \mu(\cdot|\bar{x}_N)\right) + \sum_{j=1}^N W_2(\nu_{j-1}, \nu_j). \end{aligned}$$

By Lemma 1.3.8 we have the estimate

$$\left(\int |x_N - y_N|^2 \, \pi(dx, dy)\right)^{\frac{1}{2}} \le \left(\int W_2^2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \mu(\cdot|\bar{y}_N)\right) \tilde{\pi}(d\bar{x}_N, d\bar{y}_N)\right)^{\frac{1}{2}},$$

where $\tilde{\pi}(d\bar{x}_N, d\bar{y}_N)$ is the optimal transference plan of $\bar{f}(\bar{x}_N)\bar{\mu}(d\bar{x}_N)$ and $\bar{\mu}(d\bar{y}_N)$. Applying now the triangle inequality for the L^2-norm yields

$$\begin{aligned} \left(\int |x_N - y_N|^2 \, \pi(dx, dy)\right)^{\frac{1}{2}} &\le \left(\int W_2^2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \mu(\cdot|\bar{x}_N)\right) \tilde{\pi}(d\bar{x}_N, d\bar{y}_N)\right)^{\frac{1}{2}} \\ &\quad + \sum_{j=1}^N \left(\int W_2^2(\nu_{j-1}, \nu_j) \; \tilde{\pi}(d\bar{x}_N, d\bar{y}_N)\right)^{\frac{1}{2}}. \end{aligned} \tag{1.49}$$

The first term on the r.h.s. of (1.49) is estimated by applying the WI(ϱ_N) for $\mu(dx_N|\bar{x}_N)$ as

$$\begin{aligned} &\left(\int W_2^2\left(\frac{f\mu(\cdot|\bar{x}_N)}{\bar{f}(\bar{x}_N)}, \mu(\cdot|\bar{x}_N)\right) \tilde{\pi}(d\bar{x}_N, d\bar{y}_N)\right)^{\frac{1}{2}} \\ &\le \left(\int \frac{1}{\varrho_N^2} \int \frac{|\nabla_N f(x)|^2}{f(x)} \, \mu(dx_N|\bar{x}_N) \, \frac{1}{\bar{f}(\bar{x}_N)} \, \tilde{\pi}(d\bar{x}_N, d\bar{y}_N)\right)^{\frac{1}{2}} \\ &= \frac{1}{\varrho_N}\left(\int \frac{|\nabla_N f|^2}{f} \, d\mu\right)^{\frac{1}{2}}. \end{aligned} \tag{1.50}$$

Let us turn to the remaining terms of (1.49). Note that for $1 \leq j \leq N$ the vectors

$$(y_1, \ldots, y_{j-1}, x_j, \ldots, x_N) \qquad \text{and} \qquad (y_1, \ldots, y_j, x_{j+1}, \ldots, x_N)$$

only differ in the j-th entry. Hence, the same argument as in the proof of Lemma 1.3.12 applied to the measures ν_{j-1} and ν_j yields (cf. equation (1.44))

$$\left(\int W_2^2 \left(\nu_{j-1}, \nu_j \right) \ \tilde{\pi}(d\bar{x}_N, d\bar{y}_N) \right)^{\frac{1}{2}} \leq \frac{\kappa_{jN}}{\varrho_N} \left(\int |x_j - y_j|^2 \ \tilde{\pi}(d\bar{x}_N, d\bar{y}_N) \right)^{\frac{1}{2}} .$$

Now, we apply the induction hypothesis (1.47):

$$\left(\int W_2^2 \left(\nu_{j-1}, \nu_j \right) \ \tilde{\pi}(d\bar{x}_N, d\bar{y}_N) \right)^{\frac{1}{2}} \leq \frac{\kappa_{jN}}{\varrho_N} \sum_{k=1}^{N-1} \left(\bar{A}^{-1} \right)_{jk} \left(\int \frac{|\nabla_k \bar{f}|^2}{\bar{f}} \ d\bar{\mu} \right)^{\frac{1}{2}} .$$

On the integral on the r.h.s. we apply Lemma 1.3.11 and get

$$\begin{aligned} &\left(\int W_2^2 \left(\nu_{j-1}, \nu_j \right) \ \tilde{\pi}(d\bar{x}_N, d\bar{y}_N) \right)^{\frac{1}{2}} \\ &\quad \leq \frac{\kappa_{jN}}{\varrho_N} \sum_{k=1}^{N-1} \left(\bar{A}^{-1} \right)_{jk} \left[\left(\int \frac{|\nabla_k f|^2}{f} \ d\mu \right)^{\frac{1}{2}} + \frac{\kappa_{kN}}{\varrho_N} \left(\int \frac{|\nabla_N f|^2}{f} \ d\mu \right)^{\frac{1}{2}} \right] . \end{aligned} \tag{1.51}$$

Now, we can perform the final step: Inserting (1.51) and (1.50) into (1.49) yields

$$\begin{aligned} \left(\int |x_N - y_N|^2 \ d\pi \right)^{\frac{1}{2}} &\leq \sum_{k=1}^{N-1} \sum_{j=1}^{N-1} \left(\bar{A}^{-1} \right)_{kj} \frac{\kappa_{jN}}{\varrho_N} \left(\int \frac{|\nabla_k f|^2}{f} \ d\mu \right)^{\frac{1}{2}} \\ &\quad + \left[\frac{1}{\varrho_N} + \frac{\kappa_N \cdot \bar{A}^{-1} \kappa_N}{\varrho_N^2} \right] \left(\int \frac{|\nabla_N f|^2}{f} \ d\mu \right)^{\frac{1}{2}} . \end{aligned}$$

It follows from (1.48) that the r.h.s. of the last inequality can be written as

$$\left(\int |x_N - y_N|^2 \ d\pi \right)^{\frac{1}{2}} \leq \sum_{k=1}^{N} \left(A^{-1} \right)_{Nk} \left(\int \frac{|\nabla_k f|^2}{f} \ d\mu \right)^{\frac{1}{2}} ,$$

which verifies (1.35) in the case $i = N$. Therefore, the proof of Proposition 1.3.1 is complete.

2 Uniform LSI for Kawasaki dynamics: the non-interacting case

We start with recalling the definition of the grand canonical ensemble μ and the canonical ensemble $\mu_{N,m}$ (cf. the Chapter Introduction). The grand canonical ensemble μ is a probability measure on $\mathbb{R}^N$ given by

$$\mu(dx) := \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

In the non-interacting case, the Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$ is given by a sum of single-site potentials $\psi : \mathbb{R} \to \mathbb{R}$ that are specified later i.e.

$$H\left(x\right) := \sum_{i=1}^{N} \psi(x_i). \tag{2.1}$$

For a real number m we consider the $(N-1)$ dimensional hyper-plane $X_{N,m}$ given by

$$X_{N,m} := \left\{ x \in \mathbb{R}^N,\ \frac{1}{N} \sum_{i=1}^{N} x_i = m \right\}.$$

We equip $X_{N,m}$ with the standard scalar product induced by $\mathbb{R}^N$, namely

$$\langle x, \tilde{x} \rangle := \sum_{i=1}^{N} x_i \tilde{x}_i.$$

The restriction of μ to $X_{N,m}$ is called canonical ensemble $\mu_{N,m}$. It is given by the density

$$\mu_{N,m}(dx) := \frac{1}{Z}\ \exp\left(-H(x)\right)\ \mathcal{H}^{N-1}_{\lfloor X_{N,m}}(dx). \tag{2.2}$$

Here, $\mathcal{H}^{N-1}_{\lfloor X_{N,m}}$ denotes the $(N-1)$ dimensional Hausdorff measure restricted to the hyper-plane $X_{N,m}$. Recall the notation

$a \lesssim b$	$\Leftrightarrow$	there is a uniform constant $C > 0$ such that $a \leq Cb$,
$a \sim b$	$\Leftrightarrow$	it holds that $a \lesssim b$ and $b \lesssim a$.

In 1993, Varadhan [53] posed the question for which kind of single-site potential ψ the canonical ensemble $\mu_{N,m}$ satisfies the SG(ϱ) with constant $\varrho > 0$ uniformly in the system size N and the mean spin m. A partial answer was given by Caputo [10]:

Theorem 2.0.15 (Caputo)**.** *Assume that for the single-site potential ψ exists a splitting $\psi = \psi_o + \delta\psi$ and constants β_-, $\beta_+ \in [0,\infty)$ such that for all $x \in [0,\infty)$*

$$\psi_0''(x) \sim |x|^{\beta_+} + 1, \quad \psi_0''(-x) \sim |x|^{\beta_-} + 1, \quad \textit{and} \quad |\delta\psi| + \left|\delta\psi'\right| + \left|\delta\psi''\right| \lesssim 1. \tag{2.3}$$

Then the canonical ensemble $\mu_{N,m}$ satisfies the SG(ϱ) with constant $\varrho > 0$ uniformly in the system size N and the mean spin m.

In this chapter, we give a full answer to the question by Varadhan [53] and also consider the question if the statement of the last theorem can be strengthened to the LSI. We consider three cases of single-site potentials: sub-quadratic, quadratic, and super-quadratic potentials. In the case of sub-quadratic single-site potentials, Barthe and Wolff [2] gave a counterexample where the scaling in the system size of the SG and the LSI constant of the canonical ensemble differs in the system size. More precisely, they showed:

Theorem 2.0.16 (Barthe & Wolff)**.** *Assume that the single-site potential ψ is given by*

$$\psi(x) = \begin{cases} x, & \textit{for } x > 0, \\ \infty, & \textit{else.} \end{cases}$$

Then the SG constant ϱ_1 and the LSI constant ϱ_2 of the canonical ensemble $\mu_{N,m}$ satisfy

$$\varrho_1 \sim \frac{1}{m^2} \qquad \textit{and} \qquad \varrho_2 \sim \frac{1}{Nm^2}.$$

In the case of perturbed quadratic single-site potentials it is known that Theorem 2.0.15 can be improved to the LSI. More precisely, several authors (cf. [42, 38, 12, 22]) deduced the following statement by different methods:

Theorem 2.0.17 (Landim, Panizo, and Yau)**.** *Assume that the single-site potential ψ is perturbed quadratic in the following sense: There exists a splitting $\psi = \psi_o + \delta\psi$ such that*

$$\psi_0'' = 1 \quad \textit{and} \quad |\delta\psi| + \left|\delta\psi'\right| + \left|\delta\psi''\right| \lesssim 1. \tag{2.4}$$

Then the canonical ensemble $\mu_{N,m}$ satisfies the LSI(ϱ) with constant $\varrho > 0$ uniformly in the system size N and the mean spin m.

There is only left to consider the super-quadratic case. It is conjectured that the optimal scaling LSI also holds, if the single-site potential ψ is a bounded perturbation of a strictly convex function (cf. [38, p. 741], [12, Theorem 0.3 f.], and [10, p. 226]). Heuristically, this conjecture seems reasonable: Because the LSI is closely linked to convexity (consider for example the criterion of Bakry & Émery formulated in Theorem 1.1.5), a perturbed strictly convex potential should behave no worse than a perturbed quadratic one. However technically, the methods for the quadratic case are not able to handle the perturbed strictly convex case, because they require an upper bound on the second derivative of the Hamiltonian. In the main result of the article we show that the conjecture from above is true:

Theorem 2.0.18. *Assume that the single-site potential ψ is perturbed strictly convex in the sense that there is a splitting $\psi = \psi_c + \delta\psi$ such that*

$$\psi_c'' \gtrsim 1 \qquad \text{and} \qquad |\delta\psi| + |\delta\psi'| \lesssim 1. \tag{2.5}$$

Then the canonical ensemble $\mu_{N,m}$ satisfies the LSI(ϱ) with constant $\varrho > 0$ uniformly in the system size N and the mean spin m.

Note that the standard criteria for the SG and LSI (cf. Section 1.1) fail for the canonical ensemble $\mu_{N,m}$:

- The **tensorization principle** (cf. Theorem 1.1.3) for SG and LSI does not apply because of the restriction to the hyper-plane $X_{N,m}$.
- The criterion of **Bakry & Émery** (cf. Theorem 1.1.5) does not apply because the Hamiltonian H is not strictly convex.
- The criterion of **Holley & Stroock** (cf. Theorem 1.1.4) does not help because the LSI constant ϱ has to be independent of the system size N.

Therefore, a more elaborated machinery was needed for the proof of Theorem 2.0.15 and Theorem 2.0.17. The approach of Caputo to Theorem 2.0.15 seems to be restricted to the SG, because it relies on the spectral nature of the SG. The most common approach for the proof of Theorem 2.0.17 is the Lu-Yau martingale method (see [42, 38, 12]). Recently, Grunewald, Otto, Villani, and Westdickenberg [22] provided a new technique for deducing Theorem 2.0.17 called the two-scale approach. We follow this approach in the proof of Theorem 2.0.18.

The limiting factor for extending Theorem 2.0.17 to more general single-site potentials is almost the same for the Lu-Yau martingale method and for the two-scale approach: It is the estimation of a covariance term w.r.t. the measure $\mu_{N,m}$ conditioned on a special event (cf. [38, (4.6)] and [22, (42)]). In the two-scale approach one has to estimate for some large but fixed $K \gg 1$ and any non-negative function f the covariance

$$\left| \operatorname{cov}_{\mu_{K,m}} \left(f, \frac{1}{K} \sum_{i=1}^{K} \psi'(x_i) \right) \right| .$$

In [22] this term term was estimated by using a standard estimate, which only can be applied to perturbed quadratic single-site potentials ψ (cf. Lemma 1.2.2, Lemma 2.1.9, and [22] [Lemma 22]). We get around this difficulty by making the following adaptations: Instead of one-time coarse-graining of big blocks we consider iterative coarse-graining of pairs. As a consequence we only have to estimate the covariance term from above in the case $K = 2$. Because $\mu_{2,m}$ is a one-dimensional measure, we are able to apply the more robust asymmetric Brascamp-Lieb inequality (cf. Lemma 2.1.10), which can also be applied for perturbed strictly convex single-site potentials ψ.

As we will see in Chapter 3, the optimal scaling LSI also holds in the case of a weakly-interacting Hamiltonian H given by

$$H(x) = \sum_{i=1}^{N} \psi(x_i) + \varepsilon \sum_{1\leq i<j\leq N} b_{ij} x_i x_j,$$

provided the single-site potential ψ is perturbed quadratic in the sense of (2.4). Because the original two-scale approach is used, it is an interesting question if one could extend this result to perturbed strictly convex single-site potentials. A direct transfer of the argument for perturbed strictly convex single-site potentials ψ fails, because of the iterative structure of the proof of Theorem 2.0.18.

The remaining part of this chapter is organized as follows. In Section 2.1.1 we prove of the main result. The auxiliary results of Section 2.1.1 are proved in Section 2.1.2. There is one exception: The convexification of the single-site potential by iterated renormalization (see Theorem 2.1.6) is proved in Section 2.2.

2.1 The adapted two-scale approach

2.1.1 Proof of the main result of Chapter 2

In this section we state the proof of Theorem 2.0.18, which is based based on an adaptation of the two-scale approach of [22]. We start with introducing the concept of coarse-graining of pairs. We recommend to read Chapter 2.1 of [22] as a guideline. We assume that the number N of sites is given by $N = 2^K$ for some large number $K \in \mathbb{N}$. The step to arbitrary N is not difficult (cf. Remark 2.1.7 below).

We decompose the spin system into blocks each containing two spins. The coarse-graining operator $P : X_{N,m} \to X_{\frac{N}{2},m}$ assigns to each block the mean spin of the block. More precisely, P is given by

$$P(x) := \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_3 + x_4), \ldots, \frac{1}{2}(x_{N-1} + x_N)\right). \tag{2.6}$$

Due to the coarse-graining operator P we can decompose the canonical ensemble $\mu_{N,m}$ into

$$\mu_{N,m}(dx) = \mu(dx|y)\bar{\mu}(dy), \tag{2.7}$$

where $\bar{\mu} := P_{\#}\mu_{N,m}$ denotes the push forward of the Gibbs measure μ under P and $\mu(dx|y)$ is the conditional measure of x given $Px = y$. The last equation has to be understood in a weak sense i.e. for any test function ξ

$$\int \xi(x)\, \mu_{N,m}(dx) = \int_Y \left(\int_{\{Px=y\}} \xi(x)\, \mu(dx|y) \right) \bar{\mu}(dy).$$

Now, we are able to state the first ingredient of the proof of Theorem 2.0.18.

Proposition 2.1.1 (Hierarchic criterion for LSI)**.** *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. If the marginal $\bar{\mu}$ satisfies the LSI(ϱ_1) with constant $\varrho_1 > 0$ uniformly in the system size N and the mean spin m, then the canonical ensemble $\mu_{N,m}$ also satisfies the LSI(ϱ_2) with constant $\varrho_2 > 0$ uniformly in the system size N and the mean spin m.*

The proof of this statement is given in Section 2.1.2. Due to the last proposition it suffices to deduce the LSI for the marginal $\bar{\mu}$. Hence, let us have a closer look at the structure of $\bar{\mu}$. We will characterize the Hamiltonian of the marginal $\bar{\mu}$ with the help the of the renormalization operator $\mathcal{R}$, which is introduced as follows.

Definition 2.1.2. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a single-site potential. Then the renormalized single-site potential $\mathcal{R}\psi$ is defined as*

$$\mathcal{R}\psi(y) := -\log \int \exp\left(-\psi(x+y) - \psi(-x+y)\right)\, dx \qquad \text{for } y \in \mathbb{R}. \tag{2.8}$$

Remark 2.1.3. *The renormalized single-site potential $\mathcal{R}\psi$ can be interpreted in the following way: A change of variables (cf. [16, Section 3.3.3]) and the invariance of the Hausdorff measure under translation yield the identity*

$$\begin{aligned}\exp\left(-\mathcal{R}\psi(y)\right) &= \int \exp\left(-\psi(x+m) - \psi(-x+m)\right) dx \\ &= \frac{1}{\sqrt{2}} \int \exp\left(-\psi(x_1) - \psi(x_2)\right) \mathcal{H}^1_{\lfloor\{x_1+x_2=2y\}}(dx).\end{aligned}$$

Therefore, the renormalized single-site potential $\mathcal{R}\psi$ describes the free energy of two independent spins X_1 and X_2 (identically distributed according to $Z^{-1}\exp(-\psi)$) conditioned on a fixed mean value $\frac{1}{2}(X_1 + X_2) = y$.

Lemma 2.1.4 (Invariance under renormalization)**.** *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. Then the renormalized Hamiltonian $\mathcal{R}\psi$ is also perturbed strictly convex in the sense of* (2.5)*.*

A direct calculation using the coarea formula (cf. [16, Section 3.4.2]) reveals the following structure of the marginal $\bar{\mu}$.

Lemma 2.1.5. *The marginal $\bar{\mu}$ is given by*

$$\bar{\mu}(dy) := \frac{1}{Z}\, \exp\left(-\sum_{i=1}^{\frac{N}{2}} \mathcal{R}\psi(y_i)\right) \mathcal{H}^{\frac{N}{2}-1}_{\lfloor X_{\frac{N}{2},m}}(dy).$$

It follows from the last two lemmas that the marginal $\bar{\mu}$ has the same structure as the canonical ensemble $\mu_{N,m}$. The single-site potential of $\bar{\mu}$ is given by the renormalized single-site potential $\mathcal{R}\psi$. Hence, one can iterate the coarse-graining of pairs. The next statement shows that after finitely many iterations the renormalized single-site potential $\mathcal{R}^M\psi$ becomes uniformly strictly convex. Therefore, the criterion of Bakry & Émery (cf. Theorem 1.1.5) yields that the corresponding marginal satisfies the LSI with constant $\tilde{\varrho} > 0$, uniformly in the system size N and the mean spin m. Then an iterated application of the hierarchic criterion of LSI (cf. Proposition 2.1.1) yields Theorem 2.0.18 in the case $N = 2^K$.

Theorem 2.1.6 (Convexification by renormalization)**.** *Let ψ be a perturbed strictly convex single-site potential in the sense of* (2.5)*. Then there is an integer M_0 such that for all $M \geq M_0$ the $M-$times renormalized single-site potential $\mathcal{R}^M \psi$ is uniformly strictly convex independently of the system size N and the mean spin m.*

We conclude this section with some remarks and pointing out the central tools needed for the proof of the auxiliary results. The next remark shows how Theorem 2.0.18 is proved in the case of an arbitrary number N of sites.

Remark 2.1.7. *Note that an arbitrary number of sites N can be written as*

$$N = \tilde{K} 2^K + R$$

for some number $\tilde{K}$, a large but fixed number K, and a bounded number $R < 2^K$. Hence, one can decompose the spin system into $\tilde{K}$ blocks of 2^K spins and one block of R spins. The big blocks of 2^K spins are coarse-grained by pairs, whereas the small block of R spins is not coarse-grained at all. After iterating this procedure sufficiently often, the renormalized single-site potentials of the big blocks are uniformly strictly convex. On the remaining R spins, the corresponding single-site potentials are unchanged. Because ψ is a bounded perturbation of a strictly convex function, it follows from a combination of the criterion of Bakry & Émery (cf. Theorem 1.1.5) and the criterion of Holley & Stroock (cf. Theorem 1.1.4) that the marginal of the whole system satisfies the LSI(ϱ) with constant

$$\varrho \gtrsim \exp(-R \operatorname{osc} \delta\psi),$$

which is independent on N and m. Therefore, an iterated application of the hierarchic criterion of LSI (cf. Proposition 2.1.1) yields Theorem 2.0.18 for an arbitrary number of sites N.

The proof of Proposition 2.1.1 and Lemma 2.1.4 is given in Section 2.1.2, whereas the proof of Theorem 2.1.6 is stated in Section 2.2.

Starting point for the proof of Theorem 2.1.6 is the observation that the M-times renormalized single-site potential $\mathcal{R}^M \psi$ corresponds to the coarse-grained Hamiltonian related to coarse-graining with block size 2^M (cf. [22]).

Lemma 2.1.8. *For $K \in \mathbb{N}$ let the coarse-grained Hamiltonian $\bar{H}_K$ be defined by*

$$\bar{H}_K(m) = -\frac{1}{K} \log \int \exp(-H(x))\, \mathcal{H}^{K-1}_{\lfloor X_{K,m}}(dx). \tag{2.9}$$

Let $M \in \mathbb{N}$. Then there is a constant $0 < C(2^M) < \infty$ depending only on 2^M such that

$$\mathcal{R}^M \psi = 2^M \bar{H}_{2^M} + C(2^M).$$

Because the last statement is verified by a straight forward application of the area and coarea formula, we omit the proof. In Lemma 2.1.8 one could easily determine the exact value of the constant $C(2^M)$. Because we are only interested in the convexity of $\mathcal{R}^M \psi$, this is not important. In [22] the convexification of $\bar{H}_K$ was deduced from a local Cramér theorem (cf. [22][Proposition 31]). For the proof of Theorem 2.1.6 we follow the same strategy generalizing the argument to perturbed strictly convex single-site potentials ψ.

Now, we make some comments on the proof of Proposition 2.1.1 and Lemma 2.1.4. One of the limiting factors in the proof of Theorem 2.0.17 is the application of a classical covariance estimate (cf. [22][Lemma 22]). In our framework this estimate can be formulated as:

Lemma 2.1.9. *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. Let ν be a probability measure on $\mathbb{R}$ given by*

$$\nu(dx) = \frac{1}{Z} \exp\left(-\psi(x)\right) dx.$$

Then for any function $f \geq 0$ and g

$$|\operatorname{cov}_\nu(f,g)| \lesssim \sup_x \left|g'(x)\right| \left(\int f d\nu\right)^{\frac{1}{2}} \left(\int \frac{|f'|^2}{f} d\nu\right)^{\frac{1}{2}}.$$

In [22], the last estimate was applied to the function $g = \psi'$. Note that $|g'(x)| = |\psi''(x)|$ is only bounded in the case of a perturbed quadratic single-site potential ψ. The main new ingredient for the proof of the hierarchic criterion for LSI (cf. Proposition 2.1.1) and the invariance principle (cf. Lemma 2.1.4) is an asymmetric Brascamp-Lieb inequality, which does not exhibit this restriction.

Lemma 2.1.10. *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. Let ν be a probability measure on $\mathbb{R}$ given by*

$$\nu(dx) = \frac{1}{Z} \exp\left(-\psi(x)\right) dx.$$

Then for any function f and g

$$|\operatorname{cov}_\nu(f,g)| \leq \exp\left(-3 \operatorname{osc} \delta\psi\right) \sup_x \left|\frac{g'(x)}{\psi_c''(x)}\right| \int |f'| d\nu,$$

where $\operatorname{osc} \delta\psi := \sup_x \delta\psi(x) - \inf_x \delta\psi(x)$.

We call the last inequality asymmetric, because compared to the original Brascamp-Lieb inequality [7] $L^2 \times L^2$ is replaced by $L^1 \times L^\infty$ and the factor $\frac{1}{\sqrt{\psi_c''}}$ is not evenly distributed. It is an interesting question if an analog statement also holds for higher dimensions. The proof of Lemma 2.1.10 is based on a kernel representation of the covariance. All steps are elementary.

Proof of Lemma 2.1.10. Let μ be a Gibbs measure on $\mathbb{R}$ associated to a Hamiltonian $H : \mathbb{R} \to \mathbb{R}$. More precisely, μ is given by

$$\mu(dx) := \frac{1}{Z} \exp\left(-H(x)\right) dx.$$

We start by deriving the following integral representation of the covariance of μ:

$$\mathrm{cov}_\mu(f,g) \;=\; \int\int f'(x)K_\mu(x,y)g'(y)\,dx\,dy, \tag{2.10}$$

where the non-negative kernel $K_\mu(x,y)$ is given by

$$K_\mu(x,y) \;:=\; \left\{ \begin{array}{lll} M_\mu(x)(1-M_\mu)(y) & \text{for} & y\geq x \\ (1-M_\mu)(x)M_\mu(y) & \text{for} & y\leq x \end{array} \right\},$$

and $M_\mu(x) \;:=\; \mu((-\infty,x))$ so that $(1-M_\mu)(x)=\mu((x,\infty))$. Indeed, we start by noting that

$$\mathrm{cov}_\mu(\mathrm{f},\mathrm{g}) \;=\; \int\int(f(z)-f(x))\mu(x)\,dx\int(g(z)-g(y))\mu(y)\,dy\,\mu(z)\,dz, \tag{2.11}$$

where we don't distinguish between the measure $\mu(dx)$ and its Lebesgue density $\mu(x)$ in our notation. Using $M_\mu'(x) = \mu(x)$, we can use integration by parts to rewrite each factor in terms of the derivative:

$$\begin{aligned}
&\int(f(z)-f(x))\mu(x)\,dx \\
&= \int_{-\infty}^{z}(f(z)-f(x))M_\mu'(x)\,dx - \int_{z}^{\infty}(f(z)-f(x))(1-M_\mu)'(x)\,dx \\
&= \int_{-\infty}^{z} f'(x)M_\mu(x)\,dx - \int_{z}^{\infty} f'(x)(1-M_\mu)(x)\,dx \\
&= \int f'(x)\big(I(x<z)M_\mu(x) - I(x>z)(1-M_\mu)(x)\big)\,dx,
\end{aligned}$$

where $I(x<z)$ assumes the value 1 if $x<z$ and zero otherwise. Inserting this, and the corresponding identity for $g(y)$, into (2.11), we obtain

$$\begin{aligned}
&\mathrm{cov}_\mu(f,g) \\
&\quad= \int\int f'(x)\big(I(x<z)M_\mu(x) - I(x>z)(1-M_\mu)(x)\big)\,dx \\
&\qquad\times\int g'(y)\big(I(y<z)M_\mu(y) - I(y>z)(1-M_\mu)(y)\big)\,dy\mu(z)dz \\
&\quad= \int\int f'(x)K_\mu(x,y)g'(y)\,dx\,dy
\end{aligned} \tag{2.12}$$

with kernel $K_\mu(x,y)$ as desired given by

$$\begin{aligned}
&K_\mu(x,y)\\
&= M_\mu(x)M_\mu(y)\textstyle\int I(x<z)I(y<z)\mu(z)\,dz\\
&- M_\mu(x)(1-M_\mu)(y)\textstyle\int I(x<z)I(y>z)\mu(z)\,dz\\
&- (1-M_\mu)(x)M_\mu(y)\textstyle\int I(x>z)I(y<z)\mu(z)\,dz\\
&+ (1-M_\mu)(x)(1-M_\mu)(y)\textstyle\int I(x>z)I(y>z)\mu(z)\,dz\\
&= M_\mu(x)M_\mu(y)(1-M_\mu)(\max\{x,y\})\\
&- M_\mu(x)(1-M_\mu)(y)I(y>x)(M_\mu(y)-M_\mu(x))\\
&- (1-M_\mu)(x)M_\mu(y)I(y<x)(M_\mu(x)-M_\mu(y))\\
&+ (1-M_\mu)(x)(1-M_\mu)(y)M_\mu(\min\{x,y\})\\
&= I(y>x)\big(M_\mu(x)M_\mu(y)(1-M_\mu)(y)-M_\mu(x)(1-M_\mu)(y)(M_\mu(y)-M_\mu(x))\\
&\quad +(1-M_\mu)(x)(1-M_\mu)(y)M_\mu(x)\big)\\
&+ I(y\le x)\big(M_\mu(x)M_\mu(y)(1-M_\mu)(x)-(1-M_\mu)(x)M_\mu(y)(M_\mu(x)-M_\mu(y))\\
&\quad +(1-M_\mu)(x)(1-M_\mu)(y)M_\mu(y)\big)\\
&= I(y>x)M_\mu(x)(1-M_\mu)(y)+I(y\le x)(1-M_\mu)(x)M_\mu(y).
\end{aligned}$$

We now establish the following identity for the above kernel:

$$\int K_\mu(x,y)H''(y)dy \;=\; \mu(x). \tag{2.13}$$

Indeed, we have by integrations by part

$$\begin{aligned}
&\int K_\mu(x,y)H''(y)\,dy\\
&= (1-M_\mu)(x)\int_{-\infty}^{x} M_\mu(y)H''(y)\,dy + M_\mu(x)\int_x^\infty (1-M_\mu)(y)H''(y)\,dy\\
&= (1-M_\mu)(x)\left(M_\mu(x)H'(x)-\int_{-\infty}^{x} M'_\mu(y)H'(y)\,dy\right)\\
&+ M_\mu(x)\left(-(1-M_\mu)(x)H'(x)+\int_x^\infty M'_\mu(y)H'(y)\,dy\right)\\
&= -(1-M_\mu)(x)\int_{-\infty}^{x}\exp(-H(y))H'(y)\,dy\\
&+ M_\mu(x)\int_x^\infty \exp(-H(y))H'(y)\,dy\\
&= (1-M_\mu)(x)\mu(x)+M_\mu(x)\mu(x) \;=\; \mu(x).
\end{aligned}$$

Let us now consider the Gibbs measures $\nu(dx)$ and $\nu_c(dx)$ given by

$$\nu(dx)=\frac{1}{Z}\exp\left(-\psi_c(x)-\delta\psi(x)\right)dx \qquad\text{and}\qquad \nu_c(dx)=\frac{1}{Z}\exp\left(-\psi_c(x)\right)dx.$$

By the integral representation (2.10) of the covariance we have the estimate

$$|\mathrm{cov}_\nu(f,g)| \ \le\ \int\int \left|f'(x)\right| K_\nu(x,y) \left|g'(y)\right| \, dx\, dy.$$

By a straight forward calculation we can estimate

$$\begin{aligned} M_\nu(x) &= \frac{\int_{-\infty}^x \exp(-\psi_c(x) - \delta\psi(x))dx}{\int \exp(-\psi_c(x) - \delta\psi(x))dx} \\ &\le \exp(-\operatorname{osc}\delta\psi)\ \frac{\int_{-\infty}^x \exp(-\psi_c(x))dx}{\int \exp(-\psi_c(x))dx} \\ &= \exp(-\operatorname{osc}\delta\psi)\ M_{\nu_c}(x). \end{aligned}$$

Together with a similar estimate for $(1 - M_\nu(y))$, this yields the kernel estimate

$$K_\nu(x,y) \le \exp(-2\operatorname{osc}\delta\psi)\ K_{\nu_c}(x,y).$$

Applying this to the covariance estimate from above yields

$$|\mathrm{cov}_\nu(f,g)| \ \le\ \exp(-2\operatorname{osc}\delta\psi)\ \int\int \left|f'(x)\right| K_{\nu_c}(x,y) \left|g'(y)\right| \, dx\, dy.$$

Using the identity (2.13) for $\mu = \nu_c$ we may easily conclude:

$$\begin{aligned} |\mathrm{cov}_\nu(f,g)| \ &\le\ \exp(-2\operatorname{osc}\delta\psi)\ \sup_y \frac{|g'(y)|}{\psi_c''(y)}\ \int \left|f'(x)\right| \int K_{\nu_c}(x,y)\psi_c''(y)\, dy\, dx \\ &=\ \exp(-2\operatorname{osc}\delta\psi)\ \sup_y \frac{|g'(y)|}{\psi_c''(y)}\ \int \left|f'(x)\right| \nu_c(dx) \\ &\le\ \exp(-3\operatorname{osc}\delta\psi)\ \sup_y \frac{|g'(y)|}{\psi_c''(y)}\ \int \left|f'(x)\right| \nu(dx). \end{aligned}$$

□

For the entertainment of the reader, let us now argue how the identity (2.13) also yields the traditional Brascamp-Lieb inequality in the case $H''(y) > 0$. Indeed, by the symmetry of the kernel $K_\mu(x,y)$ the identity (2.13) yields for all x and y

$$\int K_\mu(x,y)H''(y)\, dy \ =\ \mu(x) \quad \text{and} \quad \int K_\mu(x,y)H''(x)\, dx \ =\ \mu(y). \tag{2.14}$$

The integral representation of the covariance (2.10) yields

$$\begin{aligned} \mathrm{var}_\mu(f) &= \int\int f'(x) K_\mu(x,y) f'(y)\, dx\, dy \\ &= \int\int f'(x) \left(\frac{K_\mu(x,y)\ H''(y)}{H''(x)}\right)^{\frac12}\ f'(y) \left(\frac{K_\mu(x,y)\ H''(x)}{H''(y)}\right)^{\frac12}\ dx\, dy. \end{aligned}$$

Then Hoelder's inequality and the identity (2.14) for the kernel $K_\mu(x,y)$ yield the Brascamp-Lieb inequality:

$$\begin{aligned}&\operatorname{var}_\mu(f)\\ &\leq \left(\int\int \frac{|f'(x)|^2}{H''(x)}\, K_\mu(x,y)H''(y)dydx\right)^{\frac{1}{2}} \left(\int\int \frac{|f'(y)|^2}{H''(y)}\, K_\mu(x,y)H''(x)dxdy\right)^{\frac{1}{2}}\\ &= \left(\int \frac{|f'(x)|^2}{H''(x)}\mu(x)dx\right)^{\frac{1}{2}} \left(\int \frac{|f'(y)|^2}{H''(y)}\mu(y)dy\right)^{\frac{1}{2}}\\ &= \int \frac{|f'(x)|^2}{H''(x)}\mu(x)dx. \end{aligned} \tag{2.15}$$

2.1.2 Proof of the auxiliary results

In this section we outline the proof of Proposition 2.1.1 and Lemma 2.1.4. We start with Proposition 2.1.1, which is the hierarchic criterion for LSI. Unfortunately, we cannot directly apply the two-scale criterion of [22][Theorem 3]. The reason is that the number

$$\kappa := \left\{\langle \operatorname{Hess} H(x)u, v\rangle\,,\ u\in \operatorname{im}(2P^tP),\ v\in \operatorname{im}(\operatorname{id}_X - 2P^tP);\ |u|=|v|=1\right\}, \tag{2.16}$$

which measures the interaction between the microscopic and macroscopic scales, can be infinite for a perturbed strictly convex single-site potential ψ. However, we follow the proof of [22][Theorem 3] with only one major difference: Instead of applying the classical covariance estimate (cf. Lemma 2.1.9) we apply the asymmetric Brascamp-Lieb inequality (cf. Lemma 2.1.10). Let us assume for the rest of this section that the single-site potential ψ is perturbed strictly convex in the sense of (2.5).

For convenience we set $X := X_{N,m}$ and $Y := X_{\frac{N}{2},m}$. We choose on X and Y the standard Euclidean structure given by

$$\langle x, y\rangle = \sum_{i=1}^N x_i y_i.$$

The coarse-graining operator $P : X \to Y$ given by (2.6) satisfies the identity

$$2PP^t = id_Y,$$

where $P^t : Y \to X$ is the adjoint operator of P. Note that our P^t differs from the P^t of [22], because the Euclidean structure on Y differs from the Euclidean structure used in [22]. The last identity yields that $2P^tP$ is the orthogonal projection of X to $\operatorname{im} P^t$. Hence, one can decompose X into the orthogonal sum of *microscopic fluctuations* and *macroscopic variables* according to

$$\begin{aligned} X &= \ker P \oplus \operatorname{im} P^t \quad \text{and}\\ x &= \left(\operatorname{id}_X - 2P^tP\right)x + 2P^tPx. \end{aligned}$$

We apply this decomposition to the gradient ∇f of a smooth function f on X. The gradient ∇f is decomposed into a macroscopic gradient and a fluctuation gradient satisfying

$$\nabla f(x) = \left(\mathrm{id}_X - 2P^t P\right) \nabla f(x) + 2P^t P \nabla f(x) \qquad \text{and}$$
$$|\nabla f(x)|^2 = \left|\left(\mathrm{id}_X - 2P^t P\right) \nabla f(x)\right|^2 + \left|2P^t P \nabla f(x)\right|^2. \tag{2.17}$$

Note that $\ker P$ is the tangent space of the fiber $\{Px = y\}$. Hence, the gradient of f on $\{Px = y\}$ is given by $\left(\mathrm{id}_X - 2P^t P\right) \nabla f(x)$. The first main ingredient of the proof of Proposition 2.1.1 is the following statement.

Lemma 2.1.11. *The conditional measure $\mu(dx|y)$ given by* (2.7) *satisfies the LSI(ϱ) with constant $\varrho > 0$ uniformly in the system size N, the macroscopic profile y, and the mean spin m. More precisely, for any non-negative function f*

$$\int f \log f \mu(dx|y) - \int f \mu(dx|y) \log\left(\int f \mu(dx|y)\right) \leq \frac{1}{2\varrho} \int \frac{|\left(\mathrm{id}_X - 2P^t P\right) \nabla f|^2}{f} \mu(dx|y).$$

Proof of Lemma 2.1.11. Observe that the conditional measures $\mu(dx|y)$ have a product structure: We decompose $\{Px = y\}$ into a product of Euclidean spaces. Namely for

$$X_{2,y_i} := \left\{(x_{2i-1}, x_{2i}) \in \mathbb{R}^2,\ x_{2i-1} + x_{2i} = 2y_i\right\}, \qquad i \in \left\{1, \ldots, \frac{N}{2}\right\}$$

we have

$$\{Px = y\} = X_{2,y_1} \times \cdots \times X_{2,y_{\frac{N}{2}}}.$$

It follows from the coarea formula (cf. [16, Section 3.4.2]) that

$$\int_{\{Px=y\}} f(x) \mu(dx|y)$$
$$= \int f(x) \bigotimes_{i=1}^{\frac{N}{2}} \frac{1}{Z} \exp\left(-\psi(x_{2i-1}) - \psi(x_{2i})\right)\ \mathcal{H}^1_{\lfloor X_{2,y_i}}(dx_{2i-1}, dx_{2i}).$$

Hence, $\mu(dx|y)$ is the product measure

$$\mu(dx|y) = \bigotimes_{i=1}^{\frac{N}{2}} \mu_{2,y_i}(dx_{2i-1}, dx_{2i}), \tag{2.18}$$

where we make use of the notation introduced in (2.2). Because the single-site potential ψ is perturbed strictly convex in the sense of (2.5), a combination of the criterion of Bakry & Émery (cf. Theorem 1.1.5) and the criterion of Holley & Stroock (cf. Theorem 1.1.4) yield that the measure $\mu_{2,m}(dx_1, dx_2)$ satisfies the LSI(ϱ) with constant $\varrho > 0$ uniformly in m. Then the tensorization principle (cf. Theorem 1.1.3) implies the desired statement. $\square$

For convenience, let us introduce the following notation: Let f be an arbitrary function. Then its conditional expectation $\bar{f}$ is defined by

$$\bar{f}(y) := \int f(x)\mu(dx|y).$$

The second main ingredient of the proof of Proposition 2.1.1 is the following proposition, which is the analogue statement of [22, Proposition 20].

Proposition 2.1.12. *Assume that the marginal $\bar{\mu}(dy)$ given by* (2.7) *satisfies the LSI(λ) with constant $\lambda > 0$ uniformly in the system size N and the mean spin m. Then for any non-negative function f*

$$\frac{|\nabla \bar{f}(y)|^2}{\bar{f}(y)} \lesssim \int \frac{|\nabla f(x)|^2}{f(x)} \, \mu(dx|y),$$

uniformly in the macroscopic profile y and the system size N.

Before we will verify Proposition 2.1.12, let us show how it can be used in the proof of Proposition 2.1.1.

Proof of Proposition 2.1.1. Under the assumption that Lemma 2.1.11 and Proposition 2.1.12 hold, the argument is exactly the same as in the proof of [22, Theorem 3]: Let ϕ denote the function

$$\phi(x) := x \log x.$$

First, the additive property of the entropy implies

$$\begin{aligned}\int \phi(f) d\mu_{N,m} - \phi\left(\int f d\mu_{N,m}\right) = \int \left[\int \phi\left(f(x)\right)\mu(dx|y) - \phi\left(\bar{f}(y)\right)\right]\bar{\mu}(dy) \\ + \left[\int \phi\left(\bar{f}(y)\right)\bar{\mu}(dy) - \phi\left(\int \bar{f}(y)\bar{\mu}(dy)\right)\right].\end{aligned}$$

An application of Lemma 2.1.11 yields the estimate

$$\begin{aligned}&\int \left[\int \phi\left(f(x)\right)\mu(dx|y) - \phi\left(\bar{f}(y)\right)\right]\bar{\mu}(dy) \\ &\qquad \leq \frac{1}{2\varrho}\int\int \frac{|\left(\mathrm{id}_X - 2P^tP\right)\nabla f(x)|^2}{f(x)}\mu(dx|y)\bar{\mu}(dy).\end{aligned}$$

By assumption the marginal $\bar{\mu}$ satisfies LSI(λ) with constant $\lambda > 0$. Together with Proposition 2.1.12 this yields the estimate

$$\begin{aligned}\int \phi\left(\bar{f}(y)\right)\bar{\mu}(dy) - \phi\left(\int \bar{f}(y)\bar{\mu}(dy)\right) &\leq \frac{1}{2\lambda}\int \frac{|\nabla \bar{f}(y)|^2}{\bar{f}(y)}\,\bar{\mu}(dy) \\ &\lesssim \int\int \frac{|\nabla f(x)|^2}{f(x)}\,\mu(dx|y)\bar{\mu}(dy).\end{aligned}$$

A combination of the last three formulas and the observations (2.7) and (2.17) yield

$$\begin{aligned}
&\int \phi(f) d\mu_{N,m} - \phi\left(\int f d\mu_{N,m}\right) \\
&\quad \lesssim \int \frac{|\left(\mathrm{id}_X - 2P^t P\right) \nabla f(x)|^2}{f(x)} \mu_{N,m}(dx) + \int \frac{|\nabla f(x)|^2}{f(x)} \mu_{N,m}(dx) \\
&\quad \lesssim \int \frac{|\nabla f(x)|^2}{f(x)} \mu_{N,m}(dx),
\end{aligned}$$

uniformly in the system size N and the mean spin m. □

Because the hierarchic criterion for LSI is an important ingredient in the proof of the main result, we outline the proof of Proposition 2.1.12 in full detail. We follow the proof of [22][Proposition 20], which is based on two lemmas. We directly take over the first lemma (cf. [22, Lemma 21]), which in our notation becomes:

Lemma 2.1.13. *For any function f on X and any $y \in Y$ it holds*

$$\int P \nabla f(x) \mu(dx|y) = \frac{1}{2} \nabla \bar{f}(y) + P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H).$$

Remark 2.1.14. *The notational difference compared to [22, Lemma 21] is based on our choice of the Euclidean structure on $Y = X_{\frac{N}{2},m}$. Compared to the notation in Lemma 21 of [22] we have*

$$\nabla_Y \bar{f}(y) = \frac{N}{2} \nabla \bar{f}(y).$$

Hence, we omit the proof, which is a straight forward calculation.

The more interesting ingredient of the proof of [22, Proposition 20] is the estimate (see [22, (42),(43)])

$$|2P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H)|^2 \leq \frac{\sqrt{2}\kappa^2}{\varrho^2} \, \bar{f}(y) \, \int \frac{|(\mathrm{id}_X - 2P^t P) \nabla f(x)|^2}{f(x)} \mu(dx|y). \qquad (2.19)$$

The estimate (2.19) follows in [22] by direct calculation from the standard covariance estimate given by Lemma 2.1.9. In contrast to [22], we cannot use the estimate (2.19) because the constant κ given by (2.16) maybe infinite for perturbed strictly convex single-site potentials ψ. We avoid this problem by applying the more robust asymmetric Brascamp-Lieb inequality given by Lemma 2.1.10. Our substitute for (2.19) is:

Lemma 2.1.15. *For any non-negative function f*

$$|2P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H)|^2 \lesssim \bar{f}(y) \, \int \frac{|\nabla f(x)|^2}{f(x)} \mu(dx|y),$$

uniformly in the system size N, the macroscopic profile y, and the mean spin m.

We postpone the proof of Lemma 2.1.15 and show how it is used in the proof of Proposition 2.1.12 (cf. proof of [22][Proposition 20]).

Proof of Proposition 2.1.12. Note that because for any $a, b \in \mathbb{R}$

$$\frac{1}{2}(a+b)^2 \leq a^2 + b^2,$$

it follows form the definition (2.6) of P that

$$|Px|^2 \leq |x^2|. \tag{2.20}$$

By successively using Lemma 2.1.13 and Jensen's inequality (with the convex function $(a,b) \mapsto |b|^2/a$), we have

$$\begin{aligned}
\frac{|\nabla \bar{f}(y)|^2}{\bar{f}(y)} &= \frac{4}{\bar{f}(y)} \left| P \int \nabla f(x) \mu(dx|y) - P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H) \right|^2 \\
&\lesssim \frac{1}{\bar{f}(y)} \left| \int P \nabla f(x) \mu(dx|y) \right|^2 + \frac{1}{\bar{f}(y)} \left| P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H) \right|^2 \\
&\lesssim \int \frac{|P \nabla f(x)|^2}{f(x)} \mu(dx|y) + \frac{1}{\bar{f}(y)} \left| 2P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H) \right|^2 .
\end{aligned}$$

On the first term on the r.h.s. we apply the estimate (2.20). On the second term we apply Lemma 2.1.15, which yields the desired estimate. □

Now, we state the proof of Lemma 2.1.15, which also represents one of the main differences compared to the two-scale approach of [22]. The main ingredients are the product structure (2.18) of $\mu(dx|y)$ and the asymmetric Brascamp-Lieb inequality (cf. Lemma 2.1.10).

Proof of Lemma 2.1.15. We have to estimate the covariance

$$|2P \operatorname{cov}_{\mu(dx|y)}(f, \nabla H)|^2 = \sum_{j=1}^{\frac{N}{2}} |\operatorname{cov}_{\mu(dx|y)}\left(f, (2P\nabla H)_j\right)|^2. \tag{2.21}$$

Therefore, let us consider for $j \in \left\{1, \dots \frac{N}{2}\right\}$ the term $\operatorname{cov}_{\mu(dx|y)}\left(f, (2P\nabla H)_j\right)$. Note that the function

$$(2P\nabla H(x))_j = \psi'(x_{2j-1}) + \psi'(x_{2_j})$$

only depends of the variables x_{2j-1} and x_{2j}. Hence, the product structure (2.18) of $\mu(dx|y)$ yields the identity

$$\begin{aligned}
&\operatorname{cov}_{\mu(dx|y)}(f, 2\,(P\nabla H)_j) \\
&\quad = \int \operatorname{cov}_{\mu_{2,y_j}(dx_{2j-1}, dx_{2_j})}(f, (2P\nabla H)_j) \bigotimes_{i=1, i\neq j}^{\frac{N}{2}} \mu_{2,y_i}(dx_{2i-1}, dx_{2_i}).
\end{aligned}$$

As we will show below, we obtain by using the asymmetric Brascamp-Lieb inequality of Lemma 2.1.10 and the Csiszár-Kullback-Pinsker inequality the estimate

$$\left|\operatorname{cov}_{\mu_{2,y_j}(dx_{2j-1},dx_{2_j})}(f,(2P\nabla H)_j)\right| \lesssim \left(\int f(x)\mu_{2,y_j}(dx_{2j-1},dx_{2_j})\right)^{\frac{1}{2}} \times \left(\int \frac{|\frac{d}{dx_{2j-1}}f(x)|^2+|\frac{d}{dx_{2j}}f(x)|^2}{f(x)}\mu_{2,y_j}(dx_{2j-1},dx_{2_j})\right)^{\frac{1}{2}} \tag{2.22}$$

uniformly in j and y_j. Therefore, a combination of the identity from above, the last estimate, and Hölder's inequality yield

$$|\operatorname{cov}_{\mu(dx|y)}(f,(2P\nabla H)_j)|^2 \lesssim \int f(x)\mu(dx|y)\int \frac{|\frac{d}{dx_{2j-1}}f(x)|^2+|\frac{d}{dx_{2j}}f(x)|^2}{f(x)}\mu(dx|y),$$

which implies the desired estimate by the identity (2.21). It is only left to deduce the estimate (2.22). We assume w.l.o.g. $j=1$. Recall the splitting $\psi=\psi_c+\delta\psi$ given by (2.5). We use the bound on $|\delta\psi'|$ to estimate

$$\left|\operatorname{cov}_{\mu_{2,y_1}(dx_1,dx_2)}(f,(2P\nabla H)_1)\right| \lesssim \left|\operatorname{cov}_{\mu_{2.y_1}(dx_1,dx_2)}\left(f,\psi_c'(x_1)+\psi_c'(x_2)\right)\right| + \int\left|f-\int f\mu_{2,y_1}(dx_1,dx_2)\right|\mu_{2,y_1}(dx_1,dx_2). \tag{2.23}$$

Now, we consider the first term on the r.h.s. of the last estimate. For $y_1\in\mathbb{R}$ let the one-dimensional probability measure $\nu(dz|y_1)$ be defined by the density

$$\nu(dz|y_1):=\frac{1}{Z}\,\exp\left(-\left(\psi(z+y_1)+\psi(-z+y_1)\right)\right)\,dz.$$

A reparametrization of the one-dimensional Hausdorff measure implies

$$\int\xi(x_1,x_2)\mu_{2,y_1}(dx_1,dx_2)=\int\xi(-z+y_1,z+y_1)\nu(dz|y_1) \tag{2.24}$$

for any measurable function ξ. We may assume w.l.o.g. that the function $f(x)=f(x_1,x_2)$ just depends on the variables x_1 and x_2. Hence, for

$$\tilde f(z,y_1):=f(-z+y_1,z+y_1) \qquad \text{and} \qquad \tilde g(z,y_1):=\psi_c'(-z+y_1)+\psi_c'(z+y_1)$$

the last identity yields

$$\operatorname{cov}_{\mu_{2,y_1}(dx_1,dx_2)}\left(f,\psi_c'(x_1)+\psi_c'(x_2)\right)=\operatorname{cov}_{\nu(dz|y_1)}(\tilde f,\tilde g).$$

Because

$$\left|\frac{\frac{d}{dz}\tilde g(z,y_1)}{\psi_c''(-z+y_1)+\psi_c''(z+y_1)}\right| = \left|\frac{-\psi_c''(-z+y_1)+\psi_c''(z+y_1)}{\psi_c''(-z+y_1)+\psi_c''(z+y_1)}\right|\leq 2,$$

an application of the asymmetric Brascamp-Lieb inequality (cf. Lemma 2.1.10) yields

$$\left|\operatorname{cov}_{\nu(dz|y_1)}(\tilde{f},\tilde{g})\right| \lesssim \int |\frac{d}{dz}\tilde{f}|\nu(dz|y_1) \lesssim \left(\int \tilde{f}\ \nu(dz|y_1)\right)^{\frac{1}{2}} \left(\int \frac{|\frac{d}{dz}\tilde{f}|^2}{\tilde{f}}\ \nu(dz|y_1)\right)^{\frac{1}{2}}.$$

From the last inequality and (2.24) follows the estimate

$$\begin{aligned}&\left|\operatorname{cov}_{\mu_{2,y_1}(dx_1,dx_2)}\left(f, \left(\psi_c'(x_1)+\psi_c'(x_2)\right)\right)\right| \\ &\lesssim \left(\int f\ \mu_{2,y_1}(dx_1,dx_2)\right)^{\frac{1}{2}} \left(\int \frac{|\frac{d}{dx_1}f|^2+|\frac{d}{dx_2}f|^2}{f}\ \mu_{2,y_1}(dx_1,dx_2)\right)^{\frac{1}{2}}. \qquad (2.25)\end{aligned}$$

We turn to the second term on the r.h.s. of (2.23). For convenience we write

$$\tilde{f}(y_1) := \int f \mu_{2,y_1}(dx_1,dx_2).$$

An application of the (well-known) Csiszár-Kullback-Pinsker inequality (cf. [15, 36]) yields

$$\begin{aligned}\int \left|f-\tilde{f}(y_1)\right| \mu_{2,y_1}(dx_1,dx_2) &= \tilde{f}(y_1)\ \int \left|\frac{f}{\tilde{f}(y_1)}-1\right| \mu_{2,y_1}(dx_1,dx_2) \\ &\lesssim \tilde{f}(y_1)\ \left(\int \frac{f}{\tilde{f}(y_1)} \log \frac{f}{\tilde{f}(y_1)}\ \mu_{2,y_1}(dx_1,dx_2)\right)^{\frac{1}{2}}.\end{aligned}$$

An application of the LSI for the measure $\mu_{2,y_1}(dx_1,dx_2)$ implies

$$\begin{aligned}&\int \left|f - \int f\mu_{2,y_1}(dx_1,dx_2)\right|\ \mu_{2,y_1}(dx_1,dx_2) \\ &\lesssim \left(\int f\mu_{2,y_1}(dx_1,dx_2)\right)^{\frac{1}{2}} \left(\int \frac{|\frac{d}{dx_1}f|^2+|\frac{d}{dx_2}f|^2}{f}\mu_{2,y_1}(dx_1,dx_2)\right)^{\frac{1}{2}}.\end{aligned}$$

A combination of (2.23), (2.25), and the last inequality yield the desired estimate (2.22). □

We turn to the proof of Lemma 2.1.4. Again, the main ingredient of the proof is the asymmetric Brascamp-Lieb inequality.

Proof of Lemma 2.1.4. We define

$$\overline{\psi}_c(m) := -\frac{1}{2}\log\int \exp\left(-\psi_c(-x+m)-\psi_c\left(x+m\right)\right)dx$$

and

$$\begin{aligned}\overline{\delta\psi}(m) :&= -\frac{1}{2}\log\int \exp\left(-\psi(-x+m)-\psi\left(x+m\right)\right)dx \\ &\quad + \frac{1}{2}\log\int \exp\left(-\psi_c(-x+m)-\psi_c\left(x+m\right)\right)dx.\end{aligned}$$

We show that the splitting $\mathcal{R}\psi = \overline{\psi}_c + \overline{\delta\psi}$ satisfies the conditions given by (2.5). Using the strict convexity of ψ_c it follows by a standard argument based on the Brascamp-Lieb inequality (cf. [7] and (2.15)) that the first condition is preserved i.e.

$$\overline{\psi}_c'' \gtrsim 1.$$

We turn to the perturbation $\overline{\delta\psi}$. For convenience, we introduce the measures

$$\nu(dx) := \frac{1}{Z} \exp\left(-\psi(-x+m) - \psi\left(x+m\right)\right) dx$$

and

$$\nu_c(dx) := \frac{1}{Z} \exp\left(-\psi_c(-x+m) - \psi_c\left(x+m\right)\right) dx$$

so that

$$\overline{\delta\psi}(m) = -\frac{1}{2} \log \int \exp\left(-\delta\psi(-x+m) - \delta\psi\left(x+m\right)\right) \nu_c(dx).$$

A direct calculation using the bound $|\delta\psi| \lesssim 1$ yields

$$|\overline{\delta\psi}(m)| \lesssim 1.$$

We turn to the first derivative of $\overline{\delta\psi}$. A direct calculation based on the definition of $\overline{\delta\psi}$ yields

$$\begin{aligned} 2\overline{\delta\psi}'(m) = &\int \big(\psi'(-x+m) + \psi'\left(x+m\right)\big)\, \nu(dx) \\ &- \int \big(\psi_c'(-x+m) + \psi_c'\left(x+m\right)\big)\, \nu_c(dx). \end{aligned}$$

For $s \in [0,1]$ we define the measure

$$\nu^s(dx) := \frac{1}{Z} \exp\left(-\psi_c(-x+m) - \psi_c\left(x+m\right) - s\delta\psi(-x+m) - s\delta\psi\left(x+m\right)\right) dx$$

that interpolates between $\nu^0 = \nu_c$ and $\nu^1 = \nu$. By the mean-value theorem there is $s \in [0,1]$ such that

$$\begin{aligned} 2\overline{\delta\psi}'(m) &= \frac{d}{ds} \int \big(\psi_c'(-x+m) + \psi_c'\left(x+m\right) + s\delta\psi'(-x+m) + s\delta\psi'\left(x+m\right)\big)\, \nu^s(dx) \\ &= \int \big(\delta\psi'(-x+m) + \delta\psi'\left(x+m\right)\big)\, \nu^s(dx) \\ &\quad + \operatorname{cov}_{\nu^s}\Big(\psi_c'(-x+m) + \psi_c'\left(x+m\right),\ \delta\psi(-x+m) + \delta\psi\left(x+m\right)\Big) \\ &\quad + \operatorname{cov}_{\nu^s}\Big(s\delta\psi'(-x+m) + s\delta\psi'\left(x+m\right),\ \delta\psi(-x+m) + \delta\psi\left(x+m\right)\Big). \end{aligned}$$

The first term on the r.h.s. is controlled by the assumption $|\delta\psi'| \lesssim 1$. We turn to the estimation of the first covariance term. An application of the asymmetric Brascamp-Lieb inequality of Lemma 2.1.10 and $|\delta\psi| + |\delta\psi'| \lesssim 1$ yield the estimate

$$\begin{aligned}
&\left|\operatorname{cov}_{\nu^s}\left(\psi_c'(-x+m)+\psi_c'(x+m)\,,\;\delta\psi(-x+m)+\delta\psi(x+m)\right)\right| \\
&\quad\lesssim \sup_x \left|\frac{\psi_c''(-x+m)-\psi_c''(x+m)}{\psi_c''(-x+m)+\psi_c''(x+m)}\right| \int \left|-\delta\psi'(-x+m)+\delta\psi'(x+m)\right| \nu^s(dx) \\
&\quad\lesssim 1.
\end{aligned}$$

The second covariance term can be estimated using the assumption $|\delta\psi| + |\delta\psi'| \lesssim 1$. Summing up, we have deduced the desired estimate $|\overline{\delta\psi}'| \lesssim 1$. □

2.2 Convexification by iterated renormalization

In this section we will prove Theorem 2.1.6 that states the convexification of a perturbed strictly convex single-site potential ψ by iterated renormalization. The proof relies on a local Cramér theorem and some auxiliary results. The proof of Theorem 2.1.6 is given in the Subsection 2.2.1. The proofs of the auxiliary results are given in the Subsection 2.2.2.

2.2.1 Proof of Theorem 2.1.6

In view of Lemma 2.1.8 it suffices to show the strict convexity of the coarse-grained Hamiltonian $\bar{H}_K$ defined by (2.9) for large $K \gg 1$. The strategy is the same as in [22, Proposition 31]. Let φ denote the Cramér transform of ψ, namely

$$\varphi(m) := \sup_{\sigma\in\mathbb{R}} \left(\sigma m - \log \int \exp(\sigma x - \psi(x))dx\right).$$

Because φ is the Legendre transform of the strictly convex function

$$\varphi^*(\sigma) = \log \int \exp(\sigma x - \psi(x))dx, \tag{2.26}$$

there exists for any $m \in \mathbb{R}$ a unique $\sigma = \sigma(m)$ such that

$$\varphi(m) = \sigma m - \varphi^*(\sigma). \tag{2.27}$$

From basic properties of the Legendre transform it follows that the σ is determined by the equation

$$\frac{d}{d\sigma}\varphi^*(\sigma) = \frac{\int x \exp(\sigma x - \psi(x))dx}{\int \exp(\sigma x - \psi(x))dx} = m. \tag{2.28}$$

The starting point of the proof of the convexification of the coarse-grained Hamiltonian $\bar{H}_K(m)$ is the explicit representation

$$\tilde{g}_{K,m}(0) = \exp\left(K\varphi(m) - K\,\bar{H}_K(m)\right). \tag{2.29}$$

Here, $\tilde{g}_{K,m}$ denotes the Lebesgue density of the distribution of the random variable

$$\frac{1}{\sqrt{K}} \sum_{i=1}^{K} (X_i - m)\,,$$

where X_i are K real-valued independent random variables identically distributed as

$$\mu^\sigma(dx) := \exp\left(-\varphi^*(\sigma) + \sigma x - \psi(x)\right) dx. \tag{2.30}$$

We note that in view of (2.28) the mean of X_i is m. As in [22, (125)] the Cramér representation (2.29) follows from direct substitution and the coarea formula. As we will see in the proof of Lemma 2.2.3, the Cramér transform φ is strictly convex. The main idea of the proof is to transfer the convexity from φ to $\bar{H}_K$ using the representation (2.29) and a local central limit type theorem for the density $\tilde{g}_{K,m}$, which is formulated in the next statement.

Proposition 2.2.1. *Let $\psi(x)$ be a smooth function that is increasing sufficiently fast as $|x| \uparrow \infty$ for all subsequent integrals to exist. Note that the probability measure μ^σ defined by* (2.30) *depends on the field strength σ. We introduce its mean m and variance s^2*

$$m \;:=\; \int x \mu^\sigma(dx) \quad \textit{and} \quad s^2 \;:=\; \int (x-m)^2 \mu^\sigma(dx). \tag{2.31}$$

We assume that uniformly in the field strength σ, the probability measure μ^σ has its standard deviation s as unique length scale in the sense that

$$\int |x-m|^k \mu^\sigma(dx) \;\lesssim\; s^k \quad \textit{for } k = 1, \cdots, 5, \tag{2.32}$$

$$\left| \int \exp(ix\xi) \mu^\sigma(dx) \right| \;\lesssim\; |s\xi|^{-1} \quad \textit{for all } \xi \in \mathbb{R}. \tag{2.33}$$

Consider K independent random variables $X_1, \cdots, X_K$ identically distributed according to μ^σ. Let $g_{K,\sigma}$ denote the Lebesgue density of the distribution of the normalized sum $\frac{1}{\sqrt{K}} \sum_{i=1}^{K} \frac{X_i - m}{s}$.

Then $g_{K,\sigma}(0)$ converges for $K \uparrow \infty$ to the corresponding value for the normalized Gaussian. This convergence is uniform in m, of order $\frac{1}{\sqrt{K}}$, and C^2 in σ:

$$|g_{K,\sigma}(0) - \frac{1}{\sqrt{2\pi}}| \;\lesssim\; \frac{1}{\sqrt{K}}, \tag{2.34}$$

$$|\frac{1}{s}\frac{d}{d\sigma} g_{K,\sigma}(0)| \;\lesssim\; \frac{1}{\sqrt{K}}, \tag{2.35}$$

$$|(\frac{1}{s}\frac{d}{d\sigma})^2 g_{K,\sigma}(0)| \;\lesssim\; \frac{1}{\sqrt{K}}. \tag{2.36}$$

Let us comment a bit on this result: Quantitative versions of the central limit theorem like (2.34) are abundant in the literature, see for instance [17][Chapter XVI], [35][Appendix 2],

[26][Section 3], and [38][p. 752 an Section 5]. In his work on the spectral gap, Caputo appeals even to a finer estimate that makes the first terms in an error expansion in $\frac{1}{\sqrt{K}}$ explicit [10, Theorem 2.1]. The coefficients of the higher order terms are expressed in terms of moments of μ^σ. However, following [22, Proposition 31], for our two-scale argument we need *pointwise* control of the Lebesgue density $g_{K,\sigma}$ (in form of $g_{K,\sigma}(0)$) and, in addition, control of derivatives of $g_{K,\sigma}$ w.r.t. the field parameter σ, cf. (2.35), (2.36). Note that the derivative $\frac{d}{d\sigma}$ has units of length (because σ, which multiplies x in the Hamiltonian, cf. (2.30), has units of inverse length) so that $\frac{1}{s}\frac{d}{d\sigma}$ is the properly non-dimensionalized derivative. Pointwise control means that control of the moments, cf. (2.32), is not sufficient. One also needs to know that μ^σ has no fine structure on scales much smaller than s. This property is ensured the upper bound (2.33).

As opposed to [22, Proposition 31], the Hamiltonian ψ we want Proposition 2.2.1 apply to is not a perturbation of the quadratic $\frac{1}{2}x^2$ but of a general strictly convex potential ψ. As a consequence, the variance s^2 can be a strongly varying function of the field strength σ. Nevertheless, Lemma 2.2.2 from below shows that every element μ^σ in the family of measures is characterized by the single length scale s, uniformly in σ in the sense of (2.32) and (2.33). For the verification of (2.32) and (2.33) in Lemma 2.2.2 we provide a self-contained argument just using basic calculus of one variable. The merit of Proposition 2.2.1 consists in providing a version of the central limit theorem that is C^2 in the field strength σ even if the variance s^2 varies strongly with σ.

Lemma 2.2.2. *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. Then $s \lesssim 1$ uniformly in m, and the conditions* (2.32) *and* (2.33) *of Proposition 2.2.1 are satisfied.*

Using Proposition 2.2.1, Lemma 2.2.2, and the Cramér representation (2.29) we could easily deduce a local Cramér theorem (cf. [22, Proposition 31]) for general perturbed strictly convex potentials ψ. However, because we are just interested in the convexification of $\bar{H}_K$ we just consider the convergence of the second derivatives of φ and $\bar{H}_K$.

Lemma 2.2.3. *Assume that the single-site potential ψ is perturbed strictly convex in the sense of* (2.5)*. Then for all $m \in \mathbb{R}$ it holds*

$$\left| \frac{d^2}{dm^2}\,\varphi(m) - \frac{d^2}{dm^2}\,\bar{H}_K(m) \right| \lesssim \frac{1}{Ks^2},$$

where s^2 is defined as in Proposition 2.2.1.

Proof of Theorem 2.1.6. Because of Lemma 2.1.8 it suffices to show that there exists $\delta > 0$ and $K_0 \in \mathbb{N}$ such that for all $K \geq K_0$ and $m \in \mathbb{R}$

$$\frac{d^2}{dm^2}\bar{H}_K(m) \geq \delta.$$

We start with some formulas on the derivatives of φ. Differentiation of the identity (2.27) yields

$$\begin{aligned}\frac{d}{dm}\varphi &= \frac{d}{dm}\sigma\, m + \sigma - \frac{d}{d\sigma}\varphi^*\, \frac{d}{dm}\sigma \\ &\overset{(2.28)}{=} \frac{d}{dm}\sigma\, m + \sigma - m\, \frac{d}{dm}\sigma \\ &= \sigma.\end{aligned}$$

A direct calculation reveals that (see (2.60) below)

$$\frac{d}{d\sigma}m = s^2,$$

where s^2 is defined as in Proposition 2.2.1. Hence, a second differentiation of φ yields the identity

$$\frac{d^2}{dm^2}\varphi = \frac{d}{dm}\sigma = \left(\frac{d}{d\sigma}m\right)^{-1} = \frac{1}{s^2}. \tag{2.37}$$

By Lemma 2.2.3 we thus have

$$\begin{aligned}\frac{d^2}{dm^2}\bar{H}_K &= \frac{d^2}{dm^2}\varphi + \frac{d^2}{dm^2}\left(\bar{H}_K - \varphi\right) \\ &\geq \frac{1}{s^2} - \frac{C}{K}\frac{1}{s^2} \\ &\geq \frac{1}{2}\,\frac{1}{s^2},\end{aligned}$$

if $K \geq K_0$ for some large K_0. The statement follows from the uniform bound $s \lesssim 1$ provided by Lemma 2.2.2. □

2.2.2 Proof of Theorem 2.2.1 and of the auxiliary results

In this section we prove the auxiliary statements of the last subsection. Before turning to the proof of Proposition 2.2.1 we sketch the strategy. For convenience we introduce the notation

$$\langle f \rangle \ := \int f(x)\mu^\sigma(dx) = \int f(x) \exp(-\varphi^*(\sigma) + \sigma x - \psi(x))\, dx. \tag{2.38}$$

The definition of $g_{K,\sigma}$ (cf. Proposition 2.2.1) suggests to introduce the shifted and rescaled variable

$$\hat{x} \ := \ \frac{x-m}{s}. \tag{2.39}$$

We note that by (2.31) the first and second moment in $\hat{x}$ are normalized

$$\langle \hat{x} \rangle \ = \ 0, \quad \langle \hat{x}^2 \rangle \ = \ 1 \tag{2.40}$$

and that (2.32) turns into

$$\sum_{k=1}^{5} \langle |\hat{x}|^k \rangle \ \lesssim \ 1. \tag{2.41}$$

Proposition 2.2.1 is a version of the central limit theorem, that, like most others, is best proved with help of the Fourier transform. Indeed, since the random variables $\hat{X}_1 := \frac{X_1-m}{s}, \cdots, \hat{X}_K := \frac{X_K-m}{s}$ in the statement of Proposition 2.2.1 are independent and identically distributed, the distribution of their sum is the K-fold convolution of the distribution of $\hat{X}_1$. Therefore, the Fourier transform of the distribution of the $\sum_{n=1}^{K} \hat{X}_n$ is the K-th power of the Fourier transform of the distribution of $\hat{X}$. The latter is given by

$$\langle \exp(i\hat{x}\hat{\xi})\rangle,$$

where $\hat{\xi}$ denotes the variable dual to $\hat{x}$. Hence, the Fourier transform of the distribution of the normalized sum $\frac{1}{\sqrt{K}} \sum_{n=1}^{K} \hat{X}_K$ is given by $\langle \exp(i\hat{x}\frac{1}{\sqrt{K}}\hat{\xi})\rangle^K$. Applying the inverse Fourier transform, we obtain the representation

$$2\pi\, g_{K,\sigma}(0) \;=\; \int \langle \exp(i\hat{x}\frac{1}{\sqrt{K}}\hat{\xi})\rangle^K d\hat{\xi}. \tag{2.42}$$

In order to make use of formula (2.42), we need estimates on $\langle \exp(i\hat{x}\hat{\xi})\rangle$. Because of

$$\frac{d^k}{d\hat{\xi}^k}\langle \exp(i\hat{x}\hat{\xi})\rangle \;=\; i^k \langle \hat{x}^k \exp(i\hat{x}\hat{\xi})\rangle, \tag{2.43}$$

the moment bounds (2.41) translate into control of $\langle \exp(i\hat{x}\hat{\xi})\rangle$ for $|\hat{\xi}| \ll 1$. Together with the normalization (2.40), we obtain in particular by Taylor

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle - (1-\frac{1}{2}\hat{\xi}^2)| \;\lesssim\; |\hat{\xi}|^3. \tag{2.44}$$

We will use the latter in the following form: There exists a complex-valued function $h(\hat{\xi})$ such that for $|\hat{\xi}| \ll 1$:

$$\langle \exp(i\hat{x}\hat{\xi})\rangle \;=\; \exp(-h(\hat{\xi})) \quad \text{with} \quad |h(\hat{\xi}) - \frac{1}{2}\hat{\xi}^2| \;\lesssim\; |\hat{\xi}|^3. \tag{2.45}$$

This estimate, showing that the Fourier transform of the normalized probability $\langle\cdot\rangle$ is close for $|\hat{\xi}| \ll 1$ to the Fourier transform of the normalized Gaussian, is at the core of most proofs of the central limit theorem.

Estimate (2.45) provides good control over $\langle \exp(i\hat{x}\hat{\xi})\rangle$ for $|\hat{\xi}| \ll 1$. Another key ingredient is uniform decay for $|\hat{\xi}| \gg 1$. In our new variables, (2.33) takes on the form

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\lesssim\; |\hat{\xi}|^{-1}. \tag{2.46}$$

As usual in central limit theorems, we also need control of the characteristic function for intermediate values of $|\hat{\xi}|$. This can be inferred from (2.41) and (2.46) by a soft argument (in particular, it does not require the more intricate argument for [10, (2.10)] from [10, Lemma 2.5]):

Lemma 2.2.4. *Under the assumptions of Proposition 2.2.1 and for any $\delta > 0$ there exists $\lambda < 1$ such that for all σ*

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\leq\; \lambda \quad \textit{for all } |\hat{\xi}| \geq \delta.$$

So far, the strategy is standard; now comes the new ingredient: In view of formula (2.42), in order to control σ-derivatives of $g_{K,\sigma}(0)$, we need to control $\frac{1}{s}\frac{1}{d\sigma}\langle \exp(i\hat{x}\hat{\xi})\rangle$. Relying on the identities

$$\frac{1}{s}\frac{1}{d\sigma}\langle f(x)\rangle \;=\; \langle \hat{x} f(x)\rangle, \tag{2.47}$$

$$\frac{1}{s}\frac{1}{d\sigma}\hat{x} \;=\; -1 - \frac{1}{2}\langle \hat{x}^3\rangle \hat{x}, \tag{2.48}$$

that will be established in the proof of Lemma 2.2.5 below, we see that the estimate again follow from the moment control (2.41). Lemma 2.2.5 is the only new element of our analysis.

Lemma 2.2.5. *Under the assumptions of Proposition 2.2.1 we have*

$$|\frac{1}{s}\frac{1}{d\sigma}\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\lesssim\; (1+|\hat{\xi}|)|\hat{\xi}|^3, \tag{2.49}$$

$$|(\frac{1}{s}\frac{1}{d\sigma})^2\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\lesssim\; (1+\hat{\xi}^2)|\hat{\xi}|^3. \tag{2.50}$$

Before turning to the proof of Proposition 2.2.1, we prove Lemma 2.2.4 and Lemma 2.2.5.

Proof of Lemma 2.2.4. In view of (2.41) and (2.46), it suffices to show: For any $C < \infty$ and $\delta > 0$ there exists $\lambda < 1$ with the following property: Suppose $\langle\cdot\rangle$ is a probability measure (in $\hat{x}$) such that

$$\langle |\hat{x}|\rangle \;\leq\; C, \tag{2.51}$$

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\leq\; \frac{C}{|\hat{\xi}|} \quad \text{for all } \hat{\xi}. \tag{2.52}$$

Then

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\leq\; \lambda \quad \text{for all } |\hat{\xi}| \geq \delta.$$

In view of (2.52), it is enough to show

$$|\langle \exp(i\hat{x}\hat{\xi})\rangle| \;\leq\; \lambda \quad \text{for all } \delta \leq |\hat{\xi}| \leq \frac{1}{\delta}.$$

We will give an indirect argument for this statement and thus assume that there is a sequence $\{\langle\cdot\rangle_\nu\}$ of probability measures satisfying (2.51) & (2.52) and a sequence $\{\hat{\xi}_\nu\}$ of numbers in $[\delta, \frac{1}{\delta}]$ such that

$$\liminf_{\nu\uparrow\infty} |\langle \exp(i\hat{x}\hat{\xi}_\nu)\rangle_\nu| \;\geq\; 1. \tag{2.53}$$

In view of (2.51), after passage to a subsequence, we may assume that there exists a probability measure $\langle\cdot\rangle_\infty$ and a number $\hat\xi_\infty > 0$ such that

$$\lim_{\nu\uparrow\infty}\langle f\rangle_\nu \;=\; \langle f\rangle_\infty \quad \text{for all bounded and continuous } f(\hat x), \tag{2.54}$$

$$\lim_{\nu\uparrow\infty}\hat\xi_\nu \;=\; \hat\xi_\infty. \tag{2.55}$$

Since $|\exp(i\hat x\hat\xi_\nu)-\exp(i\hat x\hat\xi_\infty)|\le|\hat x||\hat\xi_\nu-\hat\xi_\infty|$, we obtain from (2.51), (2.54) & (2.55):

$$\lim_{\nu\uparrow\infty}\langle\exp(i\hat x\hat\xi_\nu)\rangle_\nu \;=\; \langle\exp(i\hat x\hat\xi_\infty)\rangle_\infty,$$

so that (2.53) saturates to

$$|\langle\exp(i\hat x\hat\xi_\infty)\rangle_\infty| \;\ge\; 1. \tag{2.56}$$

On the other hand, (2.52) is preserved under (2.54) so that we have in particular

$$\lim_{|\hat\xi|\uparrow\infty}|\langle\exp(i\hat x\hat\xi)\rangle_\infty| \;=\; 0. \tag{2.57}$$

We claim that (2.56) and (2.57) contradict each other. Indeed, since $\hat x\mapsto\exp(i\hat x\hat\xi_\infty)$ is S^1-valued, it follows from (2.56) that there is a fixed $\zeta\in S^1$ such that

$$\exp(i\hat x\hat\xi_\infty) \;=\; \zeta \quad \text{for } \langle\cdot\rangle_\infty - a.\,e.\ \hat x.$$

This implies for every $n\in\mathbb{N}$

$$\exp(i\hat x(n\hat\xi_\infty)) \;=\; \zeta^n \quad \text{for } \langle\cdot\rangle_\infty - a.\,e.\ \hat x$$

and thus

$$|\langle\exp(i\hat x(n\hat\xi_\infty))\rangle_\infty| \;=\; |\zeta^n| \;=\; 1, \tag{2.58}$$

which in view of $\hat\xi_\infty\neq 0$ and thus $|n\hat\xi_\infty|\uparrow\infty$ as $n\uparrow\infty$ contradicts (2.57). □

Proof of Lemma 2.2.5. We restrict our attention to estimate (2.50); estimate (2.49) is easier and can be derived by the same arguments. We start with the identities (2.47) and (2.48). Deriving (2.38) w.r.t. σ yields

$$\frac{d}{d\sigma}\langle f(x)\rangle \;=\; \langle(x-\frac{d\varphi^*}{d\sigma})f(x)\rangle \;\overset{(2.28)}{=}\; \langle(x-m)f(x)\rangle. \tag{2.59}$$

In view of definition (2.39), the latter turns into (2.47).

We now turn to identity (2.48) and note that in view of definitions (2.31) and (2.39), (2.59) yields in particular

$$\frac{d}{d\sigma}m \;\overset{(2.31),(2.59)}{=}\; \langle(x-m)x\rangle \;\overset{(2.31)}{=}\; \langle(x-m)^2\rangle \;\overset{(2.31)}{=}\; s^2, \tag{2.60}$$

$$\frac{d}{d\sigma}s^2 \;\overset{(2.31),(2.59)}{=}\; \langle(x-m)(x-m)^2\rangle \;\overset{(2.39)}{=}\; s^3\langle\hat x^3\rangle, \tag{2.61}$$

which we rewrite as

$$\begin{aligned} \frac{1}{s}\frac{d}{d\sigma}m &= s, \\ \frac{1}{s}\frac{d}{d\sigma}s &= \frac{1}{2}s\langle\hat{x}^3\rangle. \end{aligned} \tag{2.62}$$

These formulas imply as desired

$$\frac{1}{s}\frac{d}{d\sigma}\hat{x} \overset{(2.39)}{=} \frac{1}{s}\frac{d}{d\sigma}\frac{x-m}{s} = -1-\frac{1}{2}\langle\hat{x}^3\rangle\hat{x}.$$

We now combine formulas (2.47) and (2.48) to express derivatives of $\langle f(\hat{x})\rangle$. We start with the first derivative:

$$\begin{aligned} \frac{1}{s}\frac{d}{d\sigma}\langle f(\hat{x})\rangle &\overset{(2.47)}{=} \langle\frac{df}{d\hat{x}}(\hat{x})\frac{1}{s}\frac{d}{d\sigma}\hat{x}+f(\hat{x})\hat{x}\rangle \\ &\overset{(2.48)}{=} -\langle\frac{df}{d\hat{x}}(\hat{x})\rangle-\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}\frac{df}{d\hat{x}}(\hat{x})\rangle+\langle\hat{x}f(\hat{x})\rangle. \end{aligned} \tag{2.63}$$

(As a consistency check we note that $\frac{1}{s}\frac{d}{d\sigma}\langle f(\hat{x})\rangle \overset{(2.63)}{=} -\langle(\frac{d}{d\hat{x}}-\hat{x})f\rangle-\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}\frac{df}{d\hat{x}}\rangle$ vanishes if ψ is quadratic since then the distribution of $\hat{x}$ under $\langle\cdot\rangle$ is the normalized Gaussian so that both $\langle(\frac{d}{d\hat{x}}-\hat{x})f\rangle=0$ and $\langle\hat{x}^3\rangle=0$.)
Iterating this formula, we obtain for the second derivative

$$\begin{aligned} (\frac{1}{s}\frac{d}{d\sigma})^2\langle f(\hat{x})\rangle &\overset{(2.63)}{=} -\frac{1}{s}\frac{d}{d\sigma}\langle\frac{df}{d\hat{x}}(\hat{x})\rangle-\frac{1}{2}\left(\frac{1}{s}\frac{d}{d\sigma}\langle\hat{x}^3\rangle\right)\langle\hat{x}\frac{df}{d\hat{x}}(\hat{x})\rangle \\ &\quad -\frac{1}{2}\langle\hat{x}^3\rangle\left(\frac{1}{s}\frac{d}{d\sigma}\langle\hat{x}\frac{df}{d\hat{x}}(\hat{x})\rangle\right)+\frac{1}{s}\frac{d}{d\sigma}\langle\hat{x}f(\hat{x})\rangle \\ &\overset{(2.63)}{=} \langle\frac{d^2f}{d\hat{x}^2}\rangle+\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}\frac{d^2f}{d\hat{x}^2}\rangle-\langle\hat{x}\frac{df}{d\hat{x}}\rangle \\ &\quad +\frac{1}{2}\left(3\langle\hat{x}^2\rangle+\frac{3}{2}\langle\hat{x}^3\rangle^2-\langle\hat{x}^4\rangle\right)\langle\hat{x}\frac{df}{d\hat{x}}\rangle \\ &\quad +\frac{1}{2}\langle\hat{x}^3\rangle\left(\langle\frac{df}{d\hat{x}}+\hat{x}\frac{d^2f}{d\hat{x}^2}\rangle+\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}\frac{df}{d\hat{x}}+\hat{x}^2\frac{d^2f}{d\hat{x}^2}\rangle-\langle\hat{x}^2\frac{df}{d\hat{x}}\rangle\right) \\ &\quad -\langle f+\hat{x}\frac{df}{d\hat{x}}\rangle-\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}f+\hat{x}^2\frac{df}{d\hat{x}}\rangle+\langle\hat{x}^2f\rangle \\ &= \langle\frac{d^2f}{d\hat{x}^2}\rangle+\langle\hat{x}^3\rangle\langle\hat{x}\frac{d^2f}{d\hat{x}^2}\rangle+\frac{1}{4}\langle\hat{x}^3\rangle^2\langle\hat{x}^2\frac{d^2f}{d\hat{x}^2}\rangle \\ &\quad +\frac{1}{2}\langle\hat{x}^3\rangle\langle\frac{df}{d\hat{x}}\rangle-\frac{1}{2}(1-2\langle\hat{x}^3\rangle^2+\langle\hat{x}^4\rangle)\langle\hat{x}\frac{df}{d\hat{x}}\rangle-\langle\hat{x}^3\rangle\langle\hat{x}^2\frac{df}{d\hat{x}}\rangle \\ &\quad -\langle f\rangle-\frac{1}{2}\langle\hat{x}^3\rangle\langle\hat{x}f\rangle+\langle\hat{x}^2f\rangle. \end{aligned}$$

Because of (2.43) we have for any $k\in\mathbb{N}$

$$\frac{d^k}{d\hat{\xi}^k}(\frac{1}{s}\frac{d}{d\sigma})^2\langle\exp(i\hat{\xi}\hat{x})\rangle = (\frac{1}{s}\frac{d}{d\sigma})^2\frac{d^k}{d\hat{\xi}^k}\langle\exp(i\hat{\xi}\hat{x})\rangle = i^k(\frac{1}{s}\frac{d}{d\sigma})^2\langle\hat{x}^k\exp(i\hat{\xi}\hat{x})\rangle. \tag{2.64}$$

This formula and the normalization (2.40) yield that $(\frac{1}{s}\frac{d}{d\sigma})^2\langle\exp(i\hat\xi\hat x)\rangle$ vanishes to second order in $\hat\xi$. More precisely, for $k\in\{0,1,2\}$

$$\frac{d^k}{d\hat\xi^k}\Big|_{\hat\xi=0}(\frac{1}{s}\frac{d}{d\sigma})^2\langle\exp(i\hat\xi\hat x)\rangle \;=\; i^k(\frac{1}{s}\frac{d}{d\sigma})^2\langle\hat x^k\rangle=0. \tag{2.65}$$

Therefore, we consider the third derivative w.r.t. $\hat\xi$ given by (2.64). For this purpose we apply the formula for $(\frac{1}{s}\frac{d}{d\sigma})^2\langle f(\hat x)\rangle$ from above to the function $f=\hat x^3\exp(i\hat\xi\hat x)$. Using the abbreviation $e:=\exp(i\hat\xi\hat x)$ we obtain

$$\begin{aligned}\frac{d^3}{d\hat\xi^3}(\frac{1}{s}\frac{d}{d\sigma})^2\langle e\rangle &= i^3(\frac{1}{s}\frac{d}{d\sigma})^2\langle\hat x^3 e\rangle\\ &= i^3\Bigg(6\left\langle\hat x e\right\rangle+i6\hat\xi\left\langle\hat x^2e\right\rangle-\hat\xi^2\left\langle\hat x^3e\right\rangle\\ &\qquad+\left\langle\hat x^3\right\rangle\Big(6\left\langle\hat x^2e\right\rangle+i6\hat\xi\left\langle x^3e\right\rangle-\xi^2\left\langle\hat x^4e\right\rangle\Big)\\ &\qquad+\frac14\left\langle x^3\right\rangle^2\Big(6\left\langle\hat x^3e\right\rangle+i6\hat\xi\left\langle\hat x^4e\right\rangle-\hat\xi^2\left\langle\hat x^5e\right\rangle\Big)\\ &\qquad+\frac12\left\langle\hat x^3\right\rangle\Big(3\left\langle\hat x^2e\right\rangle+i\hat\xi\left\langle\hat x^3e\right\rangle\Big)\\ &\qquad-\frac12\Big(1-2\left\langle\hat x^3\right\rangle^2+\left\langle\hat x^4\right\rangle\Big)\Big(3\left\langle\hat x^3e\right\rangle+i\hat\xi\left\langle\hat x^4e\right\rangle\Big)\\ &\qquad-\left\langle\hat x^3\right\rangle\Big(3\left\langle\hat x^4e\right\rangle+i\hat\xi\left\langle\hat x^5e\right\rangle\Big)\\ &\qquad-\left\langle\hat x^3e\right\rangle-\frac12\left\langle\hat x^3\right\rangle\left\langle\hat x^4e\right\rangle+\left\langle\hat x^5e\right\rangle\Bigg).\end{aligned}$$

From this formula and the moment estimates (2.41) we obtain the estimate

$$|\frac{d^3}{d\hat\xi^3}(\frac{1}{s}\frac{d}{d\sigma})^2\langle e\rangle|\;\lesssim\;1+\hat\xi^2.$$

In combination with (2.65), this estimate yields (2.50). □

Proof of Proposition 2.2.1. We focus on (2.34) and (2.36). The intermediate (2.35) can be established as (2.36).

We start with (2.34). Fix a $\delta>0$ so small such that the expansion (2.45) of $\langle\exp(i\hat x\hat\xi)\rangle$ holds for $|\hat\xi|\le\delta$. We split the integral representation (2.42) accordingly:

$$\begin{aligned}2\pi g_{K,\sigma}(0) &= \int_{\{|\frac{1}{\sqrt K}\hat\xi|\le\delta\}}\langle\exp(i\hat x\frac{1}{\sqrt K}\hat\xi)\rangle^K d\hat\xi\\ &\quad+\int_{\{|\frac{1}{\sqrt K}\hat\xi|>\delta\}}\langle\exp(i\hat x\frac{1}{\sqrt K}\hat\xi)\rangle^K d\hat\xi.\end{aligned} \tag{2.66}$$

We consider the first term I on the r.h.s. of (2.66), which will turn out to be of leading order. Since δ is so small that (2.45) holds, we may rewrite it as

$$I \ := \ \int_{\{|\frac{1}{\sqrt{K}}\hat\xi|\le\delta\}} \langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle^K d\hat\xi \ = \ \int_{\{|\frac{1}{\sqrt{K}}\hat\xi|\le\delta\}} \exp(-Kh(\frac{1}{\sqrt{K}}\hat\xi))d\hat\xi. \qquad (2.67)$$

We note that for $|\frac{1}{\sqrt{K}}\hat\xi|\le\delta$ we have by (2.45),

$$|Kh(\frac{1}{\sqrt{K}}\hat\xi)-\frac{1}{2}\hat\xi^2| \ \lesssim \ \frac{1}{\sqrt{K}}|\hat\xi|^3, \qquad (2.68)$$

in particular for δ small enough

$$\mathrm{Re}\left(Kh(\frac{1}{\sqrt{K}}\hat\xi)\right) \ \ge \ \frac{1}{4}\hat\xi^2, \qquad (2.69)$$

so that (2.68) implies by the Lipschitz continuity of $\mathbb{C}\ni y\mapsto\exp(y)\in\mathbb{C}$ on $\mathrm{Re}\, y\le-\frac14\hat\xi^2$ with constant $\exp(-\frac14\hat\xi^2)$:

$$|\exp(-Kh(\frac{1}{\sqrt{K}}\hat\xi))-\exp(-\frac12\hat\xi^2)| \ \lesssim \ \frac{1}{\sqrt{K}}|\hat\xi|^3\exp(-\frac14\hat\xi^2).$$

Inserting this estimate into (2.67) we obtain

$$\begin{aligned} |I-\int_{\{|\frac{1}{\sqrt{K}}\hat\xi|\le\delta\}}\exp(-\frac12\hat\xi^2)d\hat\xi| \ &\lesssim \ \frac{1}{\sqrt{K}}\int_{\{|\frac{1}{\sqrt{K}}\hat\xi|\le\delta\}}|\hat\xi|^3\exp(-\frac14\hat\xi^2)d\hat\xi \\ &\lesssim \ \frac{1}{\sqrt{K}}\int|\hat\xi|^3\exp(-\frac14\hat\xi^2)d\hat\xi \\ &\lesssim \ \frac{1}{\sqrt{K}}. \end{aligned}$$

The latter turns as desired into

$$\begin{aligned} |I-\sqrt{2\pi}| \ &= \ |I-\int\exp(-\frac12\hat\xi^2)d\hat\xi| \\ &\lesssim \ \frac{1}{\sqrt{K}}+\int_{\{|\frac{1}{\sqrt{K}}\hat\xi|>\delta\}}\exp(-\frac12\hat\xi^2)d\hat\xi \\ &\lesssim \ \frac{1}{\sqrt{K}}, \end{aligned}$$

since $\int_{\{|\frac{1}{\sqrt{K}}\hat\xi|>\delta\}}\exp(-\frac12\hat\xi^2)d\hat\xi$ is exponentially small in K.

We now address the second term II on the r.h.s. of (2.66). On the integrand we apply Lemma 2.2.4 (on $K-2$ of the K factors) and (2.46) (on the remaining 2 factors):

$$\begin{aligned} |\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^K \ &\lesssim \ \lambda^{K-2}\left(\frac{1}{1+\frac{1}{\sqrt{K}}|\hat\xi|}\right)^2 \\ &\lesssim \ K\,\lambda^{K-2}\,\frac{1}{K+\hat\xi^2} \ \lesssim \ K\,\lambda^{K-2}\,\frac{1}{1+\hat\xi^2}. \end{aligned}$$

It follows that the second term II on the r.h.s. of (2.66) is exponentially small and thus higher order:

$$\begin{aligned} \left|\int_{\{|\frac{1}{\sqrt{K}}\hat\xi|>\delta\}} \langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle^K d\hat\xi\right| &\lesssim K\,\lambda^{K-2}\int\frac{1}{1+\hat\xi^2}d\hat\xi \\ &\lesssim K\,\lambda^{K-2} \overset{\lambda<1}{\ll} \frac{1}{\sqrt{K}}. \end{aligned}$$

We now turn to (2.36). We take the second σ-derivative of the integral representation (2.42):

$$\begin{aligned} &2\pi(\frac{1}{s}\frac{d}{d\sigma})^2 g_{K,\sigma}(0) \\ &= \int\Big(K(K-1)\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle^{K-2}(\frac{1}{s}\frac{d}{d\sigma}\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle)^2 \\ &\quad +K\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle^{K-1}(\frac{1}{s}\frac{d}{d\sigma})^2\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle\Big)d\hat\xi \end{aligned} \tag{2.70}$$

and use Lemma 2.2.5:

$$\begin{aligned} &\left|(\frac{1}{s}\frac{d}{d\sigma})^2 g_{K,\sigma}(0)\right| \\ &\lesssim \int\Big(K^2|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(1+|\frac{1}{\sqrt{K}}\hat\xi|^2)|\frac{1}{\sqrt{K}}\hat\xi|^6 \\ &\quad +K|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-1}(1+|\frac{1}{\sqrt{K}}\hat\xi|^2)|\frac{1}{\sqrt{K}}\hat\xi|^3\Big)d\hat\xi \\ &\lesssim \frac{1}{\sqrt{K}}\int|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(1+|\frac{1}{\sqrt{K}}\hat\xi|^2)(|\hat\xi|^6+1)d\hat\xi. \end{aligned} \tag{2.71}$$

As for (2.34), we split the integral representation (2.71) according to δ:

$$\begin{aligned} &\left|(\frac{1}{s}\frac{d}{d\sigma})^2 g_{K,\sigma}(0)\right| \\ &\lesssim \frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat\xi|\le\delta\}}|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(1+|\frac{1}{\sqrt{K}}\hat\xi|^2)(\hat\xi^6+1)d\hat\xi \\ &\quad +\frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat\xi|>\delta\}}|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(1+|\frac{1}{\sqrt{K}}\hat\xi|^2)(\hat\xi^6+1)d\hat\xi \\ &\lesssim \frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat\xi|\le\delta\}}|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(\hat\xi^6+1)d\hat\xi \\ &\quad +\frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat\xi|>\delta\}}|\langle\exp(i\hat x\frac{1}{\sqrt{K}}\hat\xi)\rangle|^{K-2}(\hat\xi^8+1)d\hat\xi. \end{aligned} \tag{2.72}$$

On the first r.h.s. term we use (2.69):

$$\begin{aligned}
&\frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat{\xi}|\leq\delta\}}|\langle\exp(i\hat{x}\frac{1}{\sqrt{K}}\hat{\xi})\rangle|^{K-2}(\hat{\xi}^6+1)d\hat{\xi}\\
&\lesssim \quad \frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat{\xi}|\leq\delta\}}\exp(-(K-2)\frac{1}{4}(\frac{1}{\sqrt{K}}\hat{\xi})^2)(\hat{\xi}^6+1)d\hat{\xi}\\
&\overset{K\gg 1}{\lesssim} \quad \frac{1}{\sqrt{K}}\int\exp(-\frac{1}{8}\hat{\xi}^2)(\hat{\xi}^6+1)d\hat{\xi}\\
&\lesssim \quad \frac{1}{\sqrt{K}}.
\end{aligned} \tag{2.73}$$

On the integrand of the second r.h.s. term in (2.72) we use Lemma 2.2.4 (on $K-12$ of the $K-2$ factors) and (2.46) (on the remaining 10 factors):

$$\begin{aligned}
|\langle\exp(i\hat{x}\frac{1}{\sqrt{K}}\hat{\xi})\rangle|^{K-2}(\hat{\xi}^8+1) \quad &\lesssim \quad \lambda^{K-12}\left(\frac{1}{1+\frac{1}{\sqrt{K}}|\xi|}\right)^{10}(\hat{\xi}^8+1)\\
&\lesssim \quad K^5\lambda^{K-12}\,\frac{1}{K^5+\hat{\xi}^{10}}\,(\hat{\xi}^8+1)\\
&\lesssim \quad K^5\lambda^{K-12}\frac{1}{1+\hat{\xi}^2}.
\end{aligned}$$

Hence, we see that this second term in (2.72) is exponentially small and thus higher order:

$$\begin{aligned}
&\frac{1}{\sqrt{K}}\int_{\{\frac{1}{\sqrt{K}}|\hat{\xi}|>\delta\}}|\langle\exp(i\hat{x}\frac{1}{\sqrt{K}}\hat{\xi})\rangle|^{K-2}(|\hat{\xi}|^8+1)d\hat{\xi}\\
&\lesssim \quad K^{9/2}\lambda^{K-12}\int\frac{1}{1+\hat{\xi}^2}d\hat{\xi}\\
&\lesssim \quad K^{9/2}\lambda^{K-12}\overset{\lambda<1}{\ll}\frac{1}{\sqrt{K}}.
\end{aligned}$$

□

For the proof of Lemma 2.2.2 we need the following auxiliary statement, based on elementary calculus.

Lemma 2.2.6. *Assume that the single-site potential $\psi:\mathbb{R}\to\mathbb{R}$ is convex. We consider the corresponding Gibbs measure*

$$\nu(dx)=\frac{1}{Z}\,\exp(-\psi(x))dx.$$

Let M denote the maximum of the density of ν i.e.

$$M:=\max_x\frac{1}{Z}\exp(-\psi(x)).$$

Then we have for all $k \in \mathbb{N}$

$$\int |x|^k \, \nu(dx) \lesssim \frac{1}{M^k}$$

for some constant only depending on k.

Proof of Lemma 2.2.6. We may assume w.l.o.g. that

$$Z = \int \exp(-\psi(x))dx = 1 \tag{2.74}$$

and $M := \sup_x \exp(-\psi(x))$ is attained at $x = 0$, which means

$$M = \exp(-\psi(0)). \tag{2.75}$$

It follows from convexity of ψ that

$$\psi'(x) \leq 0 \text{ for } x \leq 0 \qquad \text{and} \qquad \psi'(x) \geq 0 \text{ for } x \geq 0. \tag{2.76}$$

We start with an analysis of the convex single-site potential ψ. We first argue that

$$\psi\left(\pm\frac{e}{M}\right) \geq -\log M + \log e. \tag{2.77}$$

Indeed in view of the monotonicity (2.76) we have

$$1 \overset{(2.74)}{\geq} \int_0^{\frac{e}{M}} \exp(-\psi(y))dy \overset{(2.76)}{\geq} \frac{e}{M} \exp\left(-\psi\left(\frac{e}{M}\right)\right)$$

and

$$1 \overset{(2.74)}{\geq} \int_{-\frac{e}{M}}^{0} \exp(-\psi(y))dy \overset{(2.76)}{\geq} \frac{e}{M} \exp\left(-\psi\left(-\frac{e}{M}\right)\right).$$

We now argue that for $|x| \geq \frac{e}{M}$

$$\psi(x) \geq \frac{M}{e}\left(|x| - \frac{e}{M}\right) - \log M. \tag{2.78}$$

W.l.o.g. we may restrict ourselves to $x \geq \frac{e}{M}$. By the mean-value theorem there is $0 \leq \xi \leq \frac{e}{M}$ such that

$$\psi'(\xi) = \frac{\psi\left(\frac{e}{M}\right) - \psi(0)}{\frac{e}{M}}.$$

Using once again the monotonicity of ψ', (2.75), and (2.77) yields the estimate

$$\psi'\left(\frac{e}{M}\right) \geq \psi'(\xi) \overset{(2.75)}{=} \frac{\psi\left(\frac{e}{M}\right) + \log M}{\frac{e}{M}} \overset{(2.77)}{\geq} \frac{M}{e}.$$

The convexity of ψ, the last estimate, and (2.77) yield for $x \geq \frac{e}{M}$ as desired

$$\begin{aligned}\psi(x) &\geq \psi'\left(\frac{e}{M}\right)\left(x-\frac{e}{M}\right)+\psi\left(\frac{e}{M}\right)\\ &\geq \frac{M}{e}\left(x-\frac{e}{M}\right)-\log M.\end{aligned}$$

We finished the analysis on ψ and turn to the verification of the first estimate of Lemma 2.2.6. We split the integral according to

$$\int |x|^k \exp(-\psi(x))dx = \int_{-\infty}^{0} |x|^k \exp(-\psi(x))dx + \int_{0}^{\infty} |x|^k \exp(-\psi(x))dx.$$

We will now deduce the estimate

$$\int_0^\infty |x|^k \exp(-\psi(x))dx \lesssim \frac{1}{M^k}.$$

A similar estimate for the integral $\int_{-\infty}^0 |x|^k \exp(-\psi(x))dx$ follows from the same argument by symmetry. We split the integral:

$$\int_0^\infty |x|^k \exp(-\psi(x))dx = \int_0^{\frac{e}{M}} |x|^k \exp(-\psi(x))dx + \int_{\frac{e}{M}}^\infty |x|^k \exp(-\psi(x))dx.$$

The first integral on the r.h.s. can be estimated as

$$\int_0^{\frac{e}{M}} |x|^k \exp(-\psi(x))dx \leq \frac{e^k}{M^k}\int \exp(-\psi(x))dx \overset{(2.74)}{=} \frac{e^k}{M^k}.$$

For the estimation of the second integral we apply (2.78), which yields by the change of variables $\frac{M}{e}\left(x-\frac{e}{M}\right)=\hat{x}$

$$\begin{aligned}\int_{\frac{e}{M}}^\infty |x|^k \exp(-\psi(x))dx &\leq \int_{\frac{e}{M}}^\infty |x|^k \exp\left(-\frac{M}{e}\left(x-\frac{e}{M}\right)+\log M\right)dx\\ &= M\ \frac{e}{M}\int_0^\infty \left|\frac{e}{M}\hat{x}+\frac{e}{M}\right|^k \exp\left(-\hat{x}\right)d\hat{x}\\ &= e\ \left(\frac{e}{M}\right)^k \int_0^\infty |\hat{x}+1|^k \exp\left(-\hat{x}\right)d\hat{x}\\ &\lesssim \frac{1}{M^k}.\end{aligned}$$

□

Equipped with Lemma 2.2.6 we are able to give an elementary proof of Lemma 2.2.2:

Proof of Lemma 2.2.2. We argue that $s \lesssim 1$. Because ψ is a bounded perturbation of a uniformly strictly convex function, the measure μ^σ given by (2.30) has a spectral gap with constant independently of σ. This implies in particular

$$s^2 = \mathrm{var}_{\mu^\sigma}(x) \lesssim \int \left(\frac{d}{dx}\, x\right)^2 d\mu^\sigma \lesssim 1 \tag{2.79}$$

uniformly in σ and thus in m.
Now, we verify (2.32). Using $|\delta\psi| \lesssim 1$ to pass from ψ to ψ_c, we may assume that ψ is strictly convex. In fact, we can give up *strict* convexity of ψ and may only assume that ψ is convex. By the change of variables $\hat{x} = \frac{x-m}{s}$ we have for any $k \in \mathbb{N}$

$$\frac{\int |x-m|^k d\mu}{s^k} = \int |\hat{x}|^k \exp(-\hat{\psi}(\hat{x}))d\hat{x}$$

for some convex function $\hat{\psi}$, which is normalized in the sense that

$$\int \exp(-\hat{\psi}(\hat{x}))d\hat{x} = 1 \qquad \text{and} \qquad \int \hat{x}^2 \exp(-\hat{\psi}(\hat{x}))d\hat{x} = 1. \tag{2.80}$$

An application of Lemma 2.2.6 yields the estimate

$$\frac{\int |x-m|^k d\mu}{s^k} \leq \int |\hat{x}|^k \exp(-\hat{\psi}(\hat{x}))d\hat{x} \lesssim \frac{1}{M^k},$$

where M is given by $M := \max_{\hat{x}} \exp(-\hat{\psi}(\hat{x}))$. Now, we argue that due to the normalization of $\hat{\psi}$ we have

$$M \geq C$$

for some universal constant $C > 0$, which verifies the desired estimate (2.32). Indeed the normalization (2.80) implies

$$\begin{aligned}
\int_{(-2,2)} \exp(-\psi(x))dx \quad &\overset{(2.80)}{=} \quad 1 - \int_{\mathbb{R}-(-2,2)} \exp(-\psi(x))dx \\
&\geq \quad 1 - \frac{1}{4}\int x^2 \exp(-\psi(x))dx \overset{(2.80)}{\geq} \frac{3}{4}.
\end{aligned}$$

Hence, there exists an $x_0 \in (-2,2)$ such that $\exp(-\psi(x_0)) \geq \frac{3}{8}$, which yields

$$M = \max_{\hat{x}} \exp(-\hat{\psi}(\hat{x})) \geq \exp(-\psi(x_0)) \geq \frac{3}{8}.$$

Let us turn to the statement (2.33) of Proposition 2.2.1. Writing

$$\exp\left(ix\xi\right) = \frac{d}{dx}\left(-i\,\frac{1}{\xi}\,\exp(ix\xi)\right)$$

we obtain by integration by parts that

$$\begin{aligned}\langle\exp(ix\xi)\rangle &= i\,\frac{1}{\xi}\int \exp(ix\xi)\;\frac{d}{dx}\left(\exp\left(-\varphi^*(\sigma)+\sigma x-\psi(x)\right)\right)dx\\ &= i\,\frac{1}{\xi}\int \exp(ix\xi)\;\big(\sigma-\psi'(x)\big)\exp\left(-\varphi^*(\sigma)+\sigma x-\psi(x)\right)dx.\end{aligned}$$

The splitting $\psi=\psi_c+\delta\psi$ with $|\delta\psi|$, $|\delta\psi'|\lesssim 1$ and definition (2.26) of φ^* yield the estimate

$$|\langle\exp(ix\xi)\rangle|\lesssim\;\frac{1}{s|\xi|}\;\frac{s\int|\sigma-\psi_c'(x)|\exp\left(\sigma x-\psi_c(x)\right)dx}{\int\exp\left(\sigma x-\psi_c(x)\right)dx}+\frac{1}{s|\xi|}\;s,$$

where s is defined as in Proposition 2.2.1. Because $s\lesssim 1$ by (2.79), we only have to consider the first term of the r.h.s. of the last inequality. We argue that for

$$M:=\max_x\frac{\exp\left(\sigma x-\psi_c(x)\right)}{\int\exp\left(\sigma x-\psi_c(x)\right)dx}$$

it holds

$$2M=\frac{\int|\sigma-\psi_c'(x)|\exp\left(\sigma x-\psi_c(x)\right)dx}{\int\exp\left(\sigma x-\psi_c(x)\right)dx}. \tag{2.81}$$

For the proof of the last statement, we only need the fact that the function $H(x)=-\sigma x+\psi_c(x)$ is convex. W.l.o.g. we may assume that $\int\exp(-H(x))dx=1$ and that M is attained at $x=0$, which means

$$M=\exp(-H(0)).$$

It follows from convexity of H that

$$H'(x)\leq 0\;\text{ for }\;x\leq 0\qquad\text{and}\qquad H'(x)\geq 0\;\text{ for }\;x\geq 0.$$

Therefore, we get

$$\begin{aligned}\int|H'(x)|\exp(-H(x))dx &= -\int_{-\infty}^{0}H'(x)\exp(-H(x))dx+\int_0^\infty H'(x)\exp(H(x))dx\\ &= \int_{-\infty}^{0}\exp(-H(x))'dx-\int_0^\infty\exp(-H(x))'dx\\ &= 2\exp(-H(0))=2M.\end{aligned}$$

Because the mean of a measure μ is optimal in the sense that for all $c\in\mathbb{R}$

$$\begin{aligned}\int(x-c)^2\,\mu(dx) &= \int x^2\mu(dx)-2c\int x\mu(dx)+c^2\\ &\geq \int x^2\mu(dx)-\left(\int x\mu(dx)\right)^2\\ &= \int\left(x-\int y\mu(dy)\right)^2\mu(dx), \end{aligned} \tag{2.82}$$

we can estimate

$$s^2 \quad \leq \quad \frac{\int x^2 \exp(\sigma x - \psi(x))\, dx}{\int \exp(\sigma x - \psi(x))\, dx} \overset{|\delta\psi|\lesssim 1}{\lesssim} \frac{\int x^2 \exp(\sigma x - \psi_c(x))\, dx}{\int \exp(\sigma x - \psi_c(x))\, dx}. \tag{2.83}$$

Therefore, Lemma 2.2.6 applied to $k=2$ and ψ replaced by $-\sigma x + \psi_c$ yields

$$\frac{s \int |\sigma - \psi_c'(x)| \exp(\sigma x - \psi_c(x))\, dx}{\int \exp(\sigma x - \psi_c(x))\, dx} \overset{(2.81),(2.83)}{\lesssim} \left(\frac{\int x^2 \exp(\sigma x - \psi_c(x))\, dx}{\int \exp(\sigma x - \psi_c(x))\, dx} \right)^{\frac{1}{2}} M \lesssim 1,$$

which verifies (2.33) of Proposition 2.2.1. □

Before we turn to the proof of Lemma 2.2.3 we will deduce the following auxiliary result.

Lemma 2.2.7. *Assume that* (2.32) *of Proposition 2.2.1 is satisfied. Then, using the notation of Proposition 2.2.1, it holds:*

$$(i) \quad \left| \frac{d}{dm} s \right| \lesssim 1 \qquad \textit{and} \qquad (ii) \quad \left| \frac{d^2}{dm^2} s \right| \lesssim \frac{1}{s}.$$

Proof of Lemma 2.2.7. We start with restating some basic identities (cf. (2.60) and (2.61)): It holds that

$$\frac{d}{d\sigma} m = s^2, \tag{2.84}$$

$$\frac{d^2}{d\sigma^2} m = \frac{d}{d\sigma} s^2 = \int (x-m)^3 \, \mu^\sigma(dx), \tag{2.85}$$

$$\frac{d^3}{d\sigma^3} m = \int (x-m)^4 \, \mu^\sigma(dx). \tag{2.86}$$

Let us consider (i): It follows from (2.84) and (2.85) that

$$\begin{aligned} \frac{d}{dm} s^2 &= \frac{d}{d\sigma} s^2 \, \frac{d}{dm} \sigma \\ &= \int (x-m)^3 \, \mu^\sigma(dx) \left(\frac{d}{d\sigma} m \right)^{-1} \\ &= \frac{\int (x-m)^3 \, \mu^\sigma(dx)}{s^3} \, s, \end{aligned}$$

which yields by assumption (2.32) of Proposition 2.2.1 the estimate

$$\left| \frac{d}{dm} s^2 \right| \lesssim s.$$

The statement of (i) is a direct consequence of the last estimate and the identity

$$\frac{d}{dm} s = \frac{1}{2s} \frac{d}{dm} s^2.$$

We turn to the statement (ii): Differentiating the last identity yields

$$\frac{d^2}{dm^2}s = -\frac{1}{2}\frac{1}{s^2}\frac{d}{dm}s\frac{d}{dm}s^2 + \frac{1}{2s}\frac{d^2}{dm^2}s^2.$$

The estimation of the first term on the r.h.s. follows from the estimates

$$\left|\frac{d}{dm}s^2\right| \lesssim s \qquad \text{and} \qquad \left|\frac{d}{dm}s\right| \lesssim 1,$$

which we have deduced in the first step of the proof. We turn to the estimation of the second term. A direct calculation using (2.84) yields the identity

$$\frac{d^2}{dm^2}s^2 = \frac{d^2}{dm^2}\frac{d}{d\sigma}m = \frac{d}{dm}\left(\frac{d^2}{d\sigma^2}m\frac{d}{dm}\sigma\right) = \frac{d^3}{d\sigma^3}m\left(\frac{d}{dm}\sigma\right)^2 + \frac{d^2}{d\sigma^2}m\frac{d^2}{dm^2}\sigma. \quad (2.87)$$

Considering the first term on the r.h.s. we get from the identities (2.84) and (2.86), and the assumption (2.32) of Proposition 2.2.1 that

$$\left|\frac{d^3}{d\sigma^3}m\left(\frac{d}{dm}\sigma\right)^2\right| = \frac{\int (x-m)^4\,\mu^\sigma(dx)}{s^4} \lesssim 1.$$

Before we consider the second term of the r.h.s. of (2.87) we establish the following estimate:

$$\left|\frac{d^2}{dm^2}\sigma\right| \lesssim \frac{1}{s^3}. \quad (2.88)$$

Indeed, direct calculation using (2.84) and (2.85) yields

$$\begin{aligned}\frac{d^2}{dm^2}\sigma &= \left(\frac{d}{d\sigma}\frac{d}{dm}\sigma\right)\frac{d}{dm}\sigma \\ &= \left(\frac{d}{d\sigma}\left(\frac{d}{d\sigma}m\right)^{-1}\right)\left(\frac{d}{d\sigma}m\right)^{-1} \\ &= -\left(\frac{d}{d\sigma}m\right)^{-3}\frac{d^2}{d\sigma^2}m \\ &= -\frac{1}{s^3}\frac{\int (x-m)^3\,\mu^\sigma(dx)}{s^3}.\end{aligned}$$

The last identity yields (2.88) using the assumption (2.32) of Proposition 2.2.1. Using (2.88) and (2.85) we can estimate the second term of the r.h.s. of (2.87) as

$$\left|\frac{d^2}{d\sigma^2}m\,\frac{d^2}{dm^2}\sigma\right| \lesssim \frac{1}{s^3}\,\left|\int (x-m)^3\,\mu^\sigma(dx)\right|.$$

By applying the assumption (2.32) of Proposition 2.2.1 this yields

$$\left|\frac{d^2}{d\sigma^2}m\,\frac{d^2}{dm^2}\sigma\right| \lesssim 1,$$

which concludes the argument for (ii). $\square$

Proof of Lemma 2.2.3. Recall the representation (2.29) i.e.

$$\tilde{g}_{K,m}(0) = \exp\left(K\varphi(m) - K\bar{H}_K(m)\right).$$

Here $\tilde{g}_{K,m}(\xi)$ denotes the Lebesgue density of the random variable $\frac{1}{\sqrt{K}}\sum_{i=1}^{K}(X_i - m)$, where X_i are real-valued independent random variables identically distributed according to μ^σ (cf. (2.30)). Let $g_{K,\sigma}$ denote the density of the normalized random variable $\frac{X}{s}$, where s is given by (2.31). Then the densities are related by

$$\frac{1}{s} g_{K,\sigma}\left(\frac{x}{s}\right) = \tilde{g}_{K,m}(x).$$

It follows from (2.29) that

$$K\varphi(m) - K\bar{H}_K(m) = \log g_{K,\sigma}(0) - \log s.$$

In order to deduce the desired estimate it thus suffices to show

$$\left|\frac{d^2}{dm^2}\log s\right| \lesssim \frac{1}{s^2} \tag{2.89}$$

and

$$\left|\frac{d^2}{dm^2}\log g_{K,\sigma}(0)\right| \lesssim \frac{1}{s^2}. \tag{2.90}$$

The first estimate follows directly from the identity

$$\frac{d^2}{dm^2}\log s = \frac{d}{dm}\left(\frac{1}{s}\frac{d}{dm}s\right) = -\frac{1}{s^2}\left(\frac{d}{dm}s\right)^2 + \frac{1}{s}\frac{d^2}{dm^2}s$$

and the estimates provided by Lemma 2.2.7. We turn to the second estimate. The identity

$$\frac{d^2}{dm^2}\log g_{K,\sigma} = -\frac{1}{g_{K,\sigma}^2}\left(\frac{d}{dm}g_{K,\sigma}\right)^2 + \frac{1}{g_{K,\sigma}}\frac{d^2}{dm^2}g_{K,\sigma}$$

and (2.34) yield for large K the estimate

$$\left|\frac{d^2}{dm^2}\log g_{K,\sigma}(0)\right| \lesssim \left(\frac{d}{dm}g_{K,\sigma}(0)\right)^2 + \left|\frac{d^2}{dm^2}g_{K,\sigma}(0)\right|.$$

The estimation of the first term on the r.h.s. follows from the estimate (2.35) of Proposition 2.2.1 and the identity

$$\frac{1}{s}\frac{d}{d\sigma} = s\frac{d}{dm}, \tag{2.91}$$

which is a direct consequence of (2.60). Let us consider the second term. The identity

$$\left(\frac{1}{s}\frac{d}{d\sigma}\right)^2 \overset{(2.91)}{=} \left(s\frac{d}{dm}\right)\left(s\frac{d}{dm}\right) = s^2\frac{d^2}{dm^2} + s\left(\frac{d}{dm}s\right)\frac{d}{dm},$$

which we rewrite as

$$s^2 \frac{d^2}{dm^2} = \left(\frac{1}{s}\frac{d}{d\sigma}\right)^2 - \left(\frac{d}{dm}s\right)\,\frac{1}{s}\frac{d}{d\sigma}$$

yields

$$\frac{d^2}{dm^2} g_{K,\sigma}(0) = \frac{1}{s^2}\left(\left(\frac{1}{s}\frac{d}{d\sigma}\right)^2 g_{K,\sigma}(0) - \left(\frac{d}{dm}s\right)\frac{1}{s}\frac{d}{d\sigma} g_{K,\sigma}(0)\right).$$

Now, the estimates (2.35) and (2.36) of Proposition 2.2.1 and Lemma 2.2.7 yield the desired estimate (2.90). □

3 Uniform LSI for Kawasaki dynamics: the weakly-interacting case

Once again, we recall the definition (5) of the canonical ensemble

$$\mu_{N,m}(dx) := \frac{1}{Z}\, \exp\left(-H(x)\right)\, \mathcal{H}^{N-1}_{\lfloor\{\frac{1}{N}\sum_{i=1}^{N} x_i = m\}}(dx).$$

In Chapter 2, we showed that the canonical ensemble $\mu_{N,m}$ satisfies an optimal scaling LSI provided the Hamiltonian

$$H(x) = \sum_{i=1}^{N} \psi(x_i)$$

is non-interacting and the single site potential ψ is a bounded perturbation of a strictly convex potential (cf. Theorem 2.0.18). In this chapter, we consider the question if the optimal scaling LSI still holds when adding a small interaction term to the Hamiltonian. In the case of discrete spins, this question was already positively answered assuming finite-range interaction and a mixing condition (cf. [57] and [9]). We show that the LSI also holds in the case of unbounded continuous spins and a weak two-body interaction provided the single-site potential ψ is perturbed quadratic in the sense of (3.1) below. The interaction is not restricted to finite range. Any two spins of the system are allowed to interact. The LSI constant is uniform in the boundary data and scales optimally in the system size. Compared to the discrete case we have to deal with new technical difficulties due to the fact that the spin values and the range of interaction are unbounded. Because we apply the original two-scale approach of [22], it is also possible to derive the hydrodynamic limit with the same method as outlined in [22]. However, the hydrodynamic limit is not considered in the dissertation. Note that for existing results on the hydrodynamic limit (cf. [26, 56]) there are restrictions to lattices of certain dimensions or nearest neighbor interaction, whereas our approach is independent of the geometrical structure of the system.

Let us take a closer look at the Hamiltonian H considered in this chapter. There are three contributions to the Hamiltonian H:

- for each site $i \in \{1, \ldots, N\}$, a Ginzburg-Landau type single-site potential $\psi_i : \mathbb{R} \to \mathbb{R}$ satisfying uniformly in i

$$\psi_i(x) = \frac{1}{2}\, x^2 + \delta\psi_i(x) \quad \text{and} \quad \|\delta\psi_i\|_{C^2} \leq c_1 < \infty. \tag{3.1}$$

- a two-body interaction given by a real-valued symmetric matrix $M = (m_{ij})_{N\times N}$ with zero diagonal $m_{ii} = 0$;

◦ a linear term given by a vector $s \in \mathbb{R}^N$. This term models the interaction of the sites with the boundary data of the spin system.

Explicitly, the Hamiltonian of the system is given by

$$H(x) := \sum_{i=1}^{N} \psi_i(x_i) + \frac{1}{2} \sum_{i,j=1}^{N} m_{ij} x_i x_j + \sum_{i=1}^{N} s_i x_i. \tag{3.2}$$

Note that in contrast to [22] and Chapter 2 we do not consider homogeneous single-site potentials $\psi_i = \psi$, $i \in \{1, \ldots, N\}$. The reason is that the linear term in the definition of H naturally induces a dependence of the single-site potentials on the site i. The value of $|m_{ij}|$ determines the strength of the interaction between the spin x_i and x_j. The sign of m_{ij} determines if the interaction is repulsive or attractive. To avoid phase transition, it is natural to assume that the interaction is small in a certain sense. Our substitute for the mixing condition in the discrete case is:

Definition 3.0.8 (Condition of smallness)**.** *The interaction matrix M satisfies the smallness condition CS(ε) with $\varepsilon > 0$, if for all $x \in \mathbb{R}^N$*

$$\sum_{i,j=1}^{N} x_i \, |m_{ij}| \, x_j \leq \varepsilon \sum_{i=1}^{N} x_i^2. \tag{CS(ε)}$$

Later, we will use the condition CS(ε) to apply the covariance estimate of Theorem 1.2.4. This proceeding is similar to the discrete case, where the mixing condition was used to deduce a decay of correlations. Note that the condition CS(ε) does not impose finite-range interaction as for example the condition used by Yoshida [60] (cf. Remark 3.0.10). The main result of this chapter is:

Theorem 3.0.9. *Assume that the Hamiltonian H is given by* (3.2) *and that the single-site potentials ψ_i satisfy* (3.1) *with a constant $c_1 < \infty$ independent of the site i, the system size $N \in \mathbb{N}$, the mean spin $m \in \mathbb{R}$, and the boundary data $s \in \mathbb{R}^N$.*
Then there exist $\varepsilon > 0$ and $\varrho > 0$ depending only on c_1 such that: If the interaction matrix M satisfies CS(ε), then the canonical ensemble $\mu_{N,m}$ satisfies LSI(ϱ) independent of N, m, and s.

For the proof of Theorem 3.0.9 we apply the original two-scale approach of Grunewald, Otto, Westdickenberg and Villani [22]. Hence, we consider coarse-graining of big blocks and not iterated coarse-graining of pairs as in Chapter 2. Additionally, we apply the original two-scale criterion for LSI (cf. [22, Theorem 3]) that only holds for perturbed quadratic single-site potentials ψ_i in the sense of (3.1) (cf. Remark 3.1.3). Therefore compared to Chapter 2, we are not able to consider the whole class of perturbed strictly convex single-site potentials ψ_i in the sense of (2.5) but only the relatively small subclass of perturbed quadratic single-site potentials ψ_i. Because we allow for interaction $M \neq 0$, we have to deal with new technical difficulties compared to [22] and Chapter 2:

- The interaction between blocks is controlled by the covariance estimate of Theorem 1.2.4.

- The convexification of the coarse-grained Hamiltonian with interaction is attained by a conditioning technique (that artificially reduces the system size) and a non standard perturbation argument.

- The local Cramér theorem (cf. [22, Proposition 31]) is generalized to inhomogeneous single-site potentials ψ_i.

The unboundedness of the spins and of the range of interaction also leads to new difficulties compared to the discrete and bounded case (cf. [57, 9]):

- In the case of finite-range interaction one could use the covariance estimate due to Helffer (cf. Theorem 1.2.8, Corollary 1.2.10, and [30, 40]) to deduce exponential decay of covariances (see also [5, 6] and Section 1.2.1). The application of the covariance estimate of Theorem 1.2.4 makes it possible to consider infinite-range interaction (cf. proof of Lemma 3.1.9).

- The perturbation argument used in the proof of Lemma 3.1.13 is a lot easier in the case of bounded spins and finite-range interaction. The proof becomes a lot more delicate in the case of unbounded spins and infinite-range interaction (cf. comments after (3.31)). Additionally, we require for the argument that the single-site potentials ψ_i are perturbed quadratic in the sense of (3.1).

The rest of Chapter 3 is organized in the following way. Section 3.1 is devoted to the two-scale approach. In Section 3.1.2, we state the proof of Theorem 3.0.9 directly after the formulation of the two-scale criterion for LSI (see Theorem 3.1.2), which is the main tool of the argument. In the remaining part of Section 3.1, the ingredients of the two-scale criterion are verified: The microscopic LSI is deduced in Section 3.1.2 and the macroscopic LSI is deduced in Section 3.1.3. For the proof of the macroscopic LSI we need a generalized version of the local Cramér theorem, which we state and prove in Section 3.2. We conclude this section with a remark on the condition CS(ε).

Remark 3.0.10 (Alternative condition of smallness)**.** *Note that the condition CS(ε) is weaker than the condition Yoshida used in [60], namely*

$$\max_{j=1\ldots N} \sum_{i=1}^{N} |m_{ij}| \leq \varepsilon \qquad \text{and} \qquad m_{ij} = 0, \quad \text{if} \quad |i-j| \geq R,$$

for some fixed $R \in \mathbb{N}$. There is an obvious difference between both conditions: the CS(ε) allows infinite-range interaction and Yoshida's condition not. Even if infinite-range interaction is allowed in Yoshida's condition, we give an example to distinguish both conditions: Let us consider the interaction matrix $M = (m_{ij})_{N\times N}$ given by

$$m_{ij} = \begin{cases} \frac{\varepsilon}{2\sqrt{N}}, & \text{if } i = 1 \text{ and } j \neq 1, \\ \frac{\varepsilon}{2\sqrt{N}}, & \text{if } j = 1 \text{ and } i \neq 1, \\ 0, & \text{else}. \end{cases}$$

By Cauchy-Schwarz we have

$$\sum_{j=1}^{N} |x_j| \leq \left(\sum_{j=1}^{N} 1\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N} |x_j|^2\right)^{\frac{1}{2}} = \sqrt{N} \left(\sum_{j=1}^{N} |x_j|^2\right)^{\frac{1}{2}} .$$

Then a direct calculation reveals that

$$\sum_{i,j=1}^{N} x_i \, |m_{ij}| \, x_j = \frac{\varepsilon}{\sqrt{N}} \, |x_1| \sum_{j=1}^{N} |x_j|$$
$$\leq \varepsilon \, |x_1| \left(\sum_{j=1}^{N} |x_j|^2\right)^{\frac{1}{2}} \leq \varepsilon \sum_{j=1}^{N} |x_j|^2,$$

which yields that the matrix M satisfies CS(ε).
Considering Yoshida's condition one directly sees that

$$\max_{j=1 \dots N} \sum_{i=1}^{N} |m_{ij}| = \sum_{i=1}^{N} |m_{i1}| = \frac{\varepsilon}{2} \left(\sqrt{N} - \frac{1}{\sqrt{N}}\right).$$

This bound is not uniform in the system size N.

3.1 The original two-scale approach

We make the following assumption and convention for the Section 3.1 .

Assumption 3.1.1. *We assume that the Hamiltonian H is given by* (3.2) *and the single-site potentials ψ_i satisfy* (3.1) *with a constant $c_1 < \infty$ independent of the site i, the system size N, the mean spin m, and the boundary data s.*

Convention. *For convenience, we write on μ for the canonical ensemble $\mu_{N,m}$.*

3.1.1 Proof of the main result of Chapter 3

In this section we state the proof of Theorem 3.0.9. For that reason we explain the two-scale approach, point out the new difficulties arising from the interaction, and explain how they are solved. We use the same notation as in [22, Subsection 2.1 and 5.1]. We decompose the spin system of N sites into L blocks each containing K sites (note that $N = KL$). The index set of the l-th block, $l \in \{1, \dots, L\}$, is given by (cf. Figure 3.1)

$$B(l) := \{(l-1)K + 1, \dots, l\}.$$

The spin values inside the block $B(l)$ are denoted by $x^l := (x_i)_{i \in B(l)}$. Hence, a configuration $x \in X_{N,m}$ of the spin system can be written as

$$x = (x^1, \dots, x^L). \tag{3.3}$$

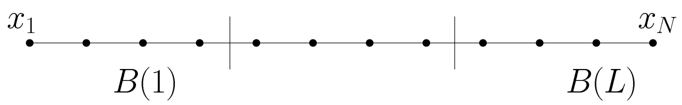

Figure 3.1: Block decomposition of the spin system

Note that the block decomposition is arbitrary and has no geometric significance. The coarse-graining operator $P : X_{N,m} \to X_{L,m} =: Y$ assigns to each block its mean spin i.e.

$$P(x) := \left(\frac{1}{K} \sum_{i \in B(1)} x_i \,, \ldots, \, \frac{1}{K} \sum_{i \in B(L)} x_i \right) . \tag{3.4}$$

In contrast to Section 2.1.2, we endow Y with the same scalar product as in [22] i.e.

$$\langle y, z \rangle_Y := \frac{1}{L} \sum_{i=1}^{L} y_i z_i, \qquad \text{for } y, z \in Y. \tag{3.5}$$

Let $P^* : Y \to X_{N,m}$ denote the adjoint operator of P. More precisely, P^* is given by

$$P^*(y_1, \ldots y_L) = \frac{1}{N} (\underbrace{y_1, \ldots, y_1}_{K \text{ times}}, \ldots\ldots\ldots\ldots, \underbrace{y_L, \ldots, y_L}_{K \text{ times}}).$$

The orthogonal projection of $X_{N,m}$ on $\ker P$ is given by $\mathrm{Id} - NP^*P$, which can be seen using the identity

$$PNP^* = \mathrm{Id}_Y \,.$$

Hence, we can decompose $x \in X_{N,m}$ into a macroscopic profile and a microscopic fluctuation according to

$$x = \underbrace{(NP^*P)\, x}_{\in (\ker P)^\perp} + \underbrace{(\mathrm{Id} - NP^*P) x}_{\in \, \ker P} . \tag{3.6}$$

The coarse-graining also induces a natural decomposition of measures. Recall that μ denotes the canonical ensemble given by (5) associated to the Hamiltonian H and the mean spin m. Let $\bar{\mu} := P_{\#}\mu$ be the push forward of μ under P and let $\mu(dx|y)$ denote the conditional measure of μ given $Px = y$. Then by disintegration

$$\mu(dx) = \mu(dx|y)\bar{\mu}(dy). \tag{3.7}$$

This equation has to be understood in a weak sense i.e. for any test function ξ

$$\int \xi \, d\mu = \int_Y \left(\int_{\{Px=y\}} \xi \; \mu(dx|y) \right) \bar{\mu}(dy).$$

By the coarea formula one can determine the density of $\bar{\mu}(dy)$ as

$$\bar{\mu}(dy) = \exp(-N\bar{H}(y)) \, dy,$$

where the coarse-grained Hamiltonian $\bar{H}$ is given by

$$\bar{H}(y) := -\frac{1}{N} \log \int \exp(-H(x)) \, \mathcal{H}^{N-L}_{\lfloor \{Px=y\}}(dx). \tag{3.8}$$

Note that this definition of the coarse-grained Hamiltonian $\bar{H}$ differs slightly from the definition (2.9) in Chapter 2. The coarse-grained Hamiltonian $\bar{H}(y)$ represents the energy of a macroscopic profile y. Overall, we observe the system at two different scales:

- the microscopic scale $\mu(dx|y)$ considers all fluctuations of the system around a macroscopic profile $y \in Y$, and
- the macroscopic scale $\bar{\mu}(dy)$ considers the macroscopic profiles and neglects all fluctuations.

We will apply the two-scale criterion for LSI (see [22, Theorem 3]) to derive the LSI for the canonical ensemble μ. In our setting the two-scale criterion becomes

Theorem 3.1.2 (Two-scale criterion)**.** *Assume that the canonical ensemble μ given by* (5) *is decomposed by* (3.7)*. Additionally, assume that:*

(i) There is $\varrho > 0$ such that for all N, m, s, and $y \in Y$ the conditional measures $\mu(dx|y)$ satisfy LSI(ϱ).

(ii) There is $\lambda > 0$ such that for all N, m, and s the marginal $\bar{\mu}$ satisfies LSI(λN).

Then μ satisfies LSI($\hat{\varrho}$) with $\hat{\varrho}$ independent of N, m, and s.

Remark 3.1.3. *The two-scale criterion in [22] also contains an explicit representation of the LSI constant $\hat{\varrho}$ in terms of ϱ, λ, and a constant κ given by* (2.16)*, which represents the strength of the coupling between the microscopic and macroscopic scale. However, for our purpose it is just important that $\hat{\varrho}$ is independent of the system size N, the mean spin m, and the boundary data s. Additionally, note that the constant κ can be infinite for a perturbed strictly convex single-site potential ψ in the sense of* (2.5)*.*

Proof of Theorem 3.0.9. We carry out the coarse-graining procedure with a large but fixed block size $K \geq K_0$, where K_0 is determined by Proposition 3.1.5 below. Note that K_0 is independent of the system size N, the mean spin m, and the boundary data s. The ingredients of the two-scale criterion of Theorem 3.1.2, namely the microscopic LSI and the macroscopic LSI, are verified by Proposition 3.1.4 and Corollary 3.1.6 respectively. Then Theorem 3.0.9 follows directly from an application of Theorem 3.1.2. □

Now, we discuss how the ingredients of Theorem 3.1.2 are verified. The microscopic LSI (cf. Proposition 3.1.4) follows directly from the Otto & Reznikoff criterion for LSI (cf. Theorem 1.1.7) using the condition CS(ε). Difficulties arise deducing the macroscopic LSI (cf. Proposition 3.1.5 and Corollary 3.1.6). We follow the strategy of [22] and want to show that $\bar{H}$ is uniformly strictly convex provided the block size K is large enough and the interaction ε is small enough. The uniform strict convexity of $\bar{H}$ would yield the macroscopic

LSI by the criterion of Bakry & Émery (see Theorem 1.1.5). Due to the interaction between blocks we lose the product structure of $\bar{\mu}$ (cf. [22, (63)]), that was crucial for the argument of [22]. As a consequence, the off-diagonal entries of the Hessian of $\bar{H}$ become non trivial (see (3.17)) i.e. for $l \neq n$

$$h_{ln} := \left(\mathrm{Hess}_Y \, \bar{H}(y)\right)_{ln} \neq 0.$$

However, applying the covariance estimate of Theorem 1.2.4 yields sufficient control of h_{ln}, $l \neq n$, in terms of ε (see Subsection 3.1.3).

The main difficulty of the proof is encountered checking the positivity of the diagonal elements h_{ll} of the Hessian of $\bar{H}$. It is not possible to transfer the positivity of h_{ll} from the case of $\varepsilon = 0$ to the case of small ε by a simple perturbation argument. The reason is that due to the loss of the product structure h_{ll} depends on all spins of system. In the case $\varepsilon = 0$ the diagonal elements h_{ll} depend only on the spins of the l-th block, which has size K. Hence, one could not choose ε independent from the system size N and the LSI constant would depend on N. We avoid this problem by conditioning on all spins except of a single block (see Subsection 3.1.3). This procedure artificially reduces the system size to the number K and introduces new boundary data, which is expressed by an additional linear term in the Hamiltonian (cf. proof of Proposition 3.1.4). Independently, we observe in Proposition 3.2.1 that for $\varepsilon = 0$ the positivity of h_{ll} for large K is untouched by adding a linear term to the Hamiltonian. Therefore, we are able to apply a perturbation argument to transfer the positivity of h_{ll} to small ε depending only on K and not on the total system size N (see Lemma 3.1.12 and Lemma 3.1.13).

3.1.2 The microscopic LSI

In this subsection we will prove the following statement.

Proposition 3.1.4 (Microscopic LSI). *There is $0 < \varepsilon$ independent of N, m, s, and $y \in Y$ (depending only on the block size K and c_1) such that:*
If M satisfies CS(ε), then the conditional measures $\mu(dx|y)$ given by (3.7) *satisfy LSI(ϱ) with $\varrho > 0$ independent of N, m, s, and y (depending only on $K$and c_1).*

Proof of Proposition 3.1.4. The statement follows from an application of the Otto & Reznikoff criterion for LSI (see Theorem 1.1.7). Let us consider an arbitrary but fixed macroscopic profile $y = (y_1, \ldots, y_L) \in Y$. We start with decomposing the Euclidean space $\{Px = y\}$ into a finite product of Euclidean spaces. It follows from the definition (3.4) of the coarse-graining operator P that

$$\{x \in \mathbb{R}^N, \; Px = y\} = X_{K,y_1} \times \ldots \times X_{K,y_L},$$

where the hyperplane X_{K,y_l}, $1 \leq l \leq L$, given by (4) is identified with

$$X_{K,y_l} = \left\{ x^l \in \mathbb{R}^{B(l)}, \; \frac{1}{K} \sum_{i \in B(l)} x_i = y_l \right\}.$$

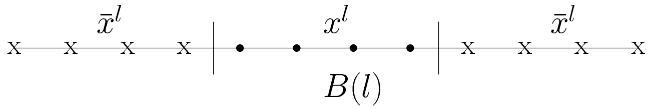

Figure 3.2: Conditioning on spins outside of the block $B(l)$

Hence, we can decompose a configuration $x \in \{Px = y\}$ into

$$x = (x^1, \dots, x^L) \quad \text{with} \quad x^l = (x_i)_{i \in B(l)} \in X_{K,y_l}.$$

The spin values outside the block $B(l)$ (or rather X_{K,y_l}) are denoted by $\bar{x}^l := (x_i)_{i \notin B(l)}$ for convenience. Disintegration of the microscopic measure $\mu(dx|y)$ with respect to x^l for a fixed $1 \leq l \leq L$ yields

$$\mu(dx|y) = \mu(dx^l|\bar{x}^l, y)\; \bar{\mu}(d\bar{x}^l|y),$$

where $\mu(dx^l|\bar{x}^l, y)$ and $\bar{\mu}(d\bar{x}^l|y)$ denotes the conditional measure and the corresponding marginal respectively (cf. Figure 3.2). More precisely, we have for all test functions $\xi : \{Px = y\} \to \mathbb{R}$

$$\int \xi(x)\mu(dx|y) = \int \int \xi(x^l, \bar{x}^l)\mu(dx^l|\bar{x}^l, y)\bar{\mu}(d\bar{x}^l|y). \tag{3.9}$$

For the first requirement of Theorem 1.1.7 we have to show that on X_{K,y_l}, $1 \leq l \leq L$, the conditional measures $\mu(dx^l|\bar{x}^l, y)$ satisfy the LSI($\tilde{\varrho}$) with constant $\tilde{\varrho} > 0$ independent of N, m, s, y, l, and $\bar{x}^l$. For this purpose let us have a closer look at the Hamiltonian of the conditional measure $\mu(dx^l|\bar{x}^l, y)$:

For an arbitrary vector $s^* \in \mathbb{R}^{B(l)}$ we define the Hamiltonian $H(x^l|M, s^*)$ by

$$H(x^l|M, s^*) \;=\; \sum_{i \in B(l)} \psi_i(x_i) + \frac{1}{2} \sum_{i,j \in B(l)} m_{ij} x_i x_j + \sum_{i \in B(l)} s_i^* x_i.$$

The definition (3.2) of the Hamiltonian H yields

$$\begin{aligned}
H(x) &= \sum_{i=1}^{N} \psi_i(x_i) + \frac{1}{2} \sum_{i,j=1}^{N} m_{ij} x_i x_j + \sum_{i=1}^{N} s_i x_i \\
&= \sum_{i \in B(l)} \psi_i(x_i) + \frac{1}{2} \sum_{i,j \in B(l)} m_{ij} x_i x_j + \sum_{i \in B(l)} \left(s_i + \sum_{j \notin B(l)} m_{ij} x_j \right) x_i \\
&\quad + \sum_{i \notin B(l)} \psi_i(x_i) + \frac{1}{2} \sum_{i,j \notin B(l)} m_{ij} x_i x_j + \sum_{i \notin B(l)} s_i x_i \\
&= H\left(x^l|M, s_c\right) + \sum_{i \notin B(l)} \psi_i(x_i) + \frac{1}{2} \sum_{i,j \notin B(l)} m_{ij} x_i x_j + \sum_{i \notin B(l)} s_i x_i,
\end{aligned}$$

where the vector $s_c = s_c(s, M, \bar{x}^l) \in \mathbb{R}^{B(l)}$ is defined for $i \in B(l)$ by the elements

$$s_{c,i} := s_i + \sum_{j \notin B(l)} m_{ij} x_j.$$

Because one can cancel all terms that are independent of $x^l = (x_i)_{i \in B(l)}$ with terms of the normalization constant Z, the effective Hamiltonian of the conditional measure $\mu(dx^l|\bar{x}^l, y)$ is given by $H(x^l|M, s_c)$. More precisely,

$$\mu(dx^l|\bar{x}^l, y) = \frac{1}{Z} \exp\left(-H(x^l|M, s_c)\right) \mathcal{H}^{K-1}_{\lfloor X_{K,y_l}}(dx).$$

Using the assumption (3.1) on the single-site potentials ψ_i we can write $H(x^l|M, s_c)$ as the sum of

$$H(x^l|M, s_c) = H_1(x^l|M, s_c) + H_2(x^l|M, s_c),$$

where $H_1(x^l|M, s_c)$ and $H_2(x^l|M, s_c)$ are given by

$$\begin{aligned} H_1(x^l|M, s_c) &= \sum_{i \in B(l)} \left[\frac{x_i^2}{2} + \left(s_i + \sum_{j \notin B(l)} m_{ij} x_j \right) x_i \right] + \frac{1}{2} \sum_{i,j \in B(l)} m_{ij} x_i x_j, \\ H_2(x^l|M, s_c) &= \sum_{i \in B(l)} \delta\psi_i(x_i). \end{aligned}$$

Using CS(ε) it follows that

$$\sum_{i,j \in B(l)} m_{ij} x_i x_j \leq \varepsilon |x^l|^2.$$

Hence, if ε is small enough, then $H_1(x^l|M, s_c)$ is a uniformly strictly convex function with constant $\lambda \geq \frac{1}{4}$. By the assumption (3.1) on the functions $\delta\psi_i$ it follows that $H_2(x^l|M, s_c)$ is a bounded function satisfying

$$\left| \sup_{x^l \in X_{K,y_l}} H_2(x^l|M, s_c) - \inf_{x^l \in X_{K,y_l}} H_2(x^l|M, s_c) \right| \leq 2Kc_1.$$

Therefore, a combination of the criterion of Bakry & Émery (see Theorem 1.1.5) and of the criterion of Holley & Stroock (see Theorem 1.1.4) yields that the conditional measures $\mu(dx^l|\bar{x}^l, y)$ satisfy a uniform LSI with constant

$$\tilde{\varrho} = \exp(-2Kc_1) \frac{1}{4}. \tag{3.10}$$

Note that $\tilde{\varrho}$ is independent of N, m, s, y, l, and $\bar{x}^l$ (depending only on the block size K and the constant c_1 given by (3.1)).

Now, we verify the remaining ingredients of the criterion of Otto & Reznikoff. For $n, m \in \{1, \ldots, L\}$ let M_{nm} denote the $K \times K$ matrix given by

$$M_{nm} = (m_{ij})_{i \in B(n),\, j \in B(m)}. \tag{3.11}$$

Let $\|M_{nm}\|$ be defined as the operator norm of M_{nm} as a bilinear form i.e.

$$\|M_{nm}\| = \max \left\{ \sum_{i \in B(n),\, j \in B(m)} \frac{x_i m_{ij} y_j}{|x|\,|y|}, \;\; x \in \mathbb{R}^{B(n)}, y \in \mathbb{R}^{B(m)} \right\}. \tag{3.12}$$

Let the matrix $A = (a_{nm})_{K \times K}$ be defined by the elements

$$a_{nm} = \begin{cases} \tilde{\varrho}, & \text{if} \quad n = m, \\ -\|M_{nm}\|, & \text{if} \quad n \neq m, \end{cases} \qquad n, m \in \{1, \ldots, K\}. \tag{3.13}$$

We will show that A satisfies in the sense of quadratic forms

$$A \geq \varrho \, \mathrm{Id}$$

for some $\varrho > 0$ independently of N, m,s, y, l, and $\bar{x}^l$. For the rest of the proof let $C < \infty$ denote a generic constant that only depends on K. Firstly, we will show that

$$(\|M_{nm}\|)_{L \times L} \leq C\varepsilon \;\, \mathrm{Id}. \tag{3.14}$$

in the sense of quadratic forms. Because of the equivalence of norms in finite dimensional vector spaces we have for $n, m \in \{1, \ldots, L\}$

$$\|M_{nm}\| \leq C \sum_{i \in B(n), j \in B(m)} |m_{ij}|.$$

For any vector $x \in \mathbb{R}^L$ we have

$$\begin{aligned} \sum_{n,m=1}^{L} x_n \, \|M_{nm}\| \, x_m \quad &\leq \quad C \sum_{n,m=1}^{L} \sum_{i \in B(n), j \in B(m)} |x_n| \, |m_{ij}| \, |x_m| \\ &\overset{\mathrm{CS}(\varepsilon)}{\leq} \quad C\varepsilon \sum_{n=1}^{L} x_n^2. \end{aligned}$$

This inequality already yields (3.14). Because $\tilde{\varrho}$ only depends on the block size K and c_1, we can choose $\varepsilon \leq \frac{\tilde{\varrho}}{2C}$ independently of N, m, s, and y such that

$$\begin{aligned} A &= \tilde{\varrho} \, \mathrm{Id} - (\|M_{nm}\|)_{L \times L} + \mathrm{diag}\,(\|M_{11}\|, \ldots, \|M_{LL}\|) \\ &\geq \tilde{\varrho} \, \mathrm{Id} - (\|M_{nm}\|)_{L \times L} \\ &\geq (\tilde{\varrho} - C\varepsilon) \, \mathrm{Id} \\ &\geq \frac{\tilde{\varrho}}{2} \, \mathrm{Id}. \end{aligned} \tag{3.15}$$

Hence, we can apply the criterion of Otto & Reznikoff and the proof is finished. □

3.1.3 The macroscopic LSI

In this section we will derive the macroscopic LSI. More precisely, we will prove that $\bar{H}$ becomes uniformly convex for large K and small ε.

Proposition 3.1.5. *Let $\bar{H}$ denote the coarse-grained Hamiltonian defined by* (3.8). *Let* $\operatorname{Hess}_Y \bar{H}$ *denote the Hessian of $\bar{H}$ w.r.t. the Euclidean structure $\langle\cdot,\cdot\rangle_Y$ on Y given by* (3.5). *Then there exists $K_0 \in \mathbb{N}$ depending only on c_1 such that:*
If the block size $K \geq K_0$ and the interaction matrix M satisfies $\mathrm{CS}(\varepsilon)$*, then there are constants $\lambda > 0$ and $C < \infty$ independent of N, m, and s (depending only on K and c_1) such that for all $y \in Y$*

$$\operatorname{Hess}_Y \bar{H}(y) \geq (\lambda - C\varepsilon)\,\mathrm{Id}$$

in the sense of quadratic forms.

By the definition (3.8) of $\bar{H}$ we have

$$\bar{\mu}(dy) = \exp(-N\bar{H}(y))\mathcal{H}^{L-1}_{\lfloor Y}(dy).$$

Hence, the macroscopic LSI is a direct consequence of Proposition 3.1.5 and the criterion of Bakry & Émery (see Theorem 1.1.5), if we choose ε small enough. More precisely, we have

Corollary 3.1.6 (Macroscopic LSI)**.** *Choose a fixed block size $K \geq K_0$, where K_0 is given by Proposition 3.1.5. Consider the marginal $\bar{\mu}$ defined by* (3.7)*. Then there exist $\varepsilon > 0$ and $\lambda > 0$ independent of N, m, and s (depending only on K and c_1) such that:*
If the interaction matrix M satisfies $\mathrm{CS}(\varepsilon)$*, then $\bar{\mu}$ satisfies LSI(λN).*

The proof of Proposition 3.1.5 consists of three steps. In the next subsection we will deduce a formula for the elements of $\operatorname{Hess}_Y \bar{H}$. In Subsection 3.1.3 we will show that the off-diagonal elements of $\operatorname{Hess}_Y \bar{H}$ are small in a certain sense (cf. Lemma 3.1.9). In Subsection 3.1.3 we will show that the diagonal elements of $\operatorname{Hess}_Y \bar{H}$ are uniformly positive for large K and small ε (cf. Lemma 3.1.11).

Proof of Proposition 3.1.5. We decompose the $\operatorname{Hess}_Y \bar{H}(y)$ into its diagonal matrix and its remainder i.e.

$$\begin{aligned}\operatorname{Hess}_Y \bar{H}(y) = \operatorname{diag}\left(\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{11}, \ldots, \left(\operatorname{Hess}_Y \bar{H}(y)\right)_{LL}\right)\\ + \left[\operatorname{Hess}_Y \bar{H}(y) - \operatorname{diag}\left(\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{11}, \ldots, \left(\operatorname{Hess}_Y \bar{H}(y)\right)_{LL}\right)\right]\end{aligned}$$

A combination of Lemma 3.1.9 and Lemma 3.1.11 from below yields the statement. □

Formula for the elements of the Hessian of $\bar{H}$. Before we derive the formula for the elements of the Hessian of $\bar{H}$, we state an alternative representation of the coarse-grained Hamiltonian $\bar{H}$.

Lemma 3.1.7. *Assume that the Hamiltonian H and the coarse-grained Hamiltonian $\bar{H}$ are given by* (3.2) *and* (3.8) *respectively. For $x \in \{Px = 0\}$ and $y \in Y$ let $H_M(x,y)$ be defined by*

$$H_M(x,y) := \frac{1}{2}\langle x, (\mathrm{Id} + M)x\rangle + \langle x, MNP^*y\rangle + \langle s, x\rangle + \sum_{i=1}^{N} \delta\psi_i(x_i + (NP^*y)_i).$$

Then

$$\bar{H}(y) = \frac{1}{2}\langle y, (\mathrm{Id} + PMNP^*)y\rangle_Y + \langle Ps, y\rangle_Y - \frac{1}{N}\log\int \exp\left(-H_M(x,y)\right)\mathcal{H}^{N-L}_{\lfloor\{Px=0\}}(dx), \tag{3.16}$$

where the scalar product $\langle\cdot,\cdot\rangle_Y$ is given by (3.5).

The last lemma is verified by a straight forward calculation: One applies the linear transformation $x \mapsto x - NP^*y$ to the integral in the definition (3.8) of $\bar{H}(y)$. Additionally, one has to use the fact that by orthogonality $\langle x, NP^*y\rangle = 0$ for any $x \in \ker P$ and $NP^*y \in (\ker P)^\perp$ (cf. (3.6)).

The last statement is used to deduce the following representation of the Hessian of $\bar{H}$, which is the base of our argument for the convexity of the coarse-grained Hamiltonian $\bar{H}$.

Lemma 3.1.8. *Assume that the Hamiltonian H and the coarse-grained Hamiltonian $\bar{H}$ are given by* (3.2) *and* (3.8) *respectively. Recall that the conditional measures $\mu(dx|y)$ are defined by* (3.7). *For $1 \leq l, n \leq L$ we have*

$$\left(\mathrm{Hess}_Y\,\bar{H}(y)\right)_{ln} = \delta_{ln} + \delta_{ln}\frac{1}{K}\int \sum_{i\in B(l)} \delta\psi_i''(x_i)\,\mu(dx|y) + \frac{1}{K}\sum_{i\in B(l),\ j\in B(n)} m_{ij} - \frac{1}{K}\mathrm{cov}_{\mu(dx|y)}\left(\sum_{j\in B(l)}\left(\sum_{i=1}^{N} m_{ij}x_i\right) + \delta\psi_j'(x_j),\ \sum_{j\in B(n)}\left(\sum_{i=1}^{N} m_{ij}x_i\right) + \delta\psi_j'(x_j)\right). \tag{3.17}$$

The last lemma is easily deduced by differentiating (3.16). Additionally, one has to apply the inverse translation $x + NP^*y$ to the occurring integrals, consider the orthogonality of $NP^*y \in (\ker P)^\perp$, and apply the fact that covariances are invariant under adding constant functions. Because every step of the proof is very basic, we will omit the details.

Estimation of the off-diagonal elements of the Hessian of $\bar{H}$. In this section we will show, that the off-diagonal elements of the Hessian of $\bar{H}$ are controlled by ε. Explicitly, we will prove the following statement.

Lemma 3.1.9. *If the interaction matrix M satisfies CS(ε), then there is a constant $0 \leq C < \infty$ independent of N, m, and s (depending only on the block size K and c_1) such that*

$$\mathrm{Hess}_Y\,\bar{H}(y) - \mathrm{diag}\left(\left(\mathrm{Hess}_Y\,\bar{H}(y)\right)_{11}, \ldots, \left(\mathrm{Hess}_Y\,\bar{H}(y)\right)_{LL}\right) \geq -C\varepsilon\,\mathrm{Id}$$

in the sense of quadratic forms.

This lemma is not obvious. Considering (3.17) one has to estimate for example the covariance

$$\operatorname{cov}_{\mu(dx|y)}\left(\sum_{j\in B(l)} \delta\psi_j'(x_j), \sum_{j\in B(n)} \delta\psi_j'(x_j)\right)$$

for $1 \leq l \neq n \leq L$. It is not clear how to exploit the control CS(ε) on the last expression. The key observation is that the first function only depends on spins of the block $B(l)$, whereas the second function only depends on spins of block $B(n)$. One hopes that the covariance is decaying in the distance of the blocks, if ε is small enough. It turns out, that the covariance estimate of Theorem 1.2.4 is optimally adapted for this purpose. We will use Theorem 1.2.4 to deduce the following auxiliary lemma, which is the main ingredient in the proof of Lemma 3.1.9.

Lemma 3.1.10. *The following statements hold:*

(i) The conditional measures $\mu(dx|y)$ given by (3.7) *satisfy the covariance estimate* (1.8) *with the matrix A given by* (3.13)*.*

(ii) Assume that $\tilde{\varrho}$ is given by (3.10) *and that the elements $\|M_{s_1 s_2}\|$ of the $L \times L$- Matrix $(\|M_{s_1 s_2}\|)_{L\times L}$ are given by* (3.12)*. Then in the sense of quadratic forms:*

$$A^{-1} - \operatorname{diag}\left(\left(A^{-1}\right)_{11}, \ldots, \left(A^{-1}\right)_{LL}\right) \leq \frac{1}{\tilde{\varrho}} \frac{\varepsilon}{\tilde{\varrho} - \varepsilon} \operatorname{Id}, \tag{3.18}$$

$$(\|M_{s_1 s_2}\|)_{L\times L} A^{-1} (\|M_{s_1 s_2}\|)_{L\times L} \leq \frac{1}{\tilde{\varrho}} \frac{\varepsilon^2}{\tilde{\varrho} - \varepsilon} \operatorname{Id}. \tag{3.19}$$

Proof of Lemma 3.1.10. Argument for (i): The LSI(ϱ) implies the SG(ϱ) by Lemma 1.1.1. Hence, the hypotheses of Theorem 1.2.4 are weaker than the hypotheses of the criterion of Otto & Reznikoff (cf. Theorem 1.1.7), which were already verified for the conditional measures $\mu(dx|y)$ in the proof of Proposition 3.1.4. Thus the statement follows from a direct application of Theorem 1.2.4.

Argument for (ii): Using the Neumann representation of A^{-1} one sees that

$$\operatorname{diag}\left(\left(A^{-1}\right)_{11}, \ldots, \left(A^{-1}\right)_{LL}\right) \geq \frac{1}{\tilde{\varrho}} \operatorname{Id}, \tag{3.20}$$

in the sense of quadratic forms. Because for sufficiently small ε (cf. (3.15))

$$A \geq \tilde{\varrho} \operatorname{Id} - (\|M_{s_1 s_2}\|)_{L\times L} > 0,$$

it follows that

$$A^{-1} \leq \left(\tilde{\varrho} \operatorname{Id} - (\|M_{s_1 s_2}\|)_{L\times L}\right)^{-1} = \frac{1}{\tilde{\varrho}} \sum_{k=0}^{\infty} \left(\frac{(\|M_{s_1 s_2}\|)_{L\times L}}{\tilde{\varrho}}\right)^k. \tag{3.21}$$

A combination of (3.20) and (3.21) yields

$$A^{-1} - \operatorname{diag}\left(\left(A^{-1}\right)_{11}, \ldots, \left(A^{-1}\right)_{LL}\right) \leq \frac{1}{\tilde{\varrho}} \sum_{k=1}^{\infty} \left(\frac{(\|M_{s_1 s_2}\|)_{L\times L}}{\tilde{\varrho}} \right)^k ,$$

which implies the desired estimate (3.18) by using (3.14). By (3.21) we have

$$(\|M_{s_1 s_2}\|)_{L\times L}\, A^{-1}\, (\|M_{s_1 s_2}\|)_{L\times L} \leq \frac{1}{\tilde{\varrho}} \sum_{k=2}^{\infty} \left(\frac{(\|M_{s_1 s_2}\|)_{L\times L}}{\tilde{\varrho}} \right)^k ,$$

which implies the desired estimate (3.19) by using (3.14). □

Proof of Lemma 3.1.9. Because of (3.17) we can write

$$\operatorname{Hess}_Y \bar{H}(y) - \operatorname{diag}\left(\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{11}, \ldots, \left(\operatorname{Hess}_Y \bar{H}(y)\right)_{LL}\right) = W_1 + W_2,$$

where the matrix W_1 is given by

$$(W_1)_{ln} = \begin{cases} \frac{1}{K} \sum_{i\in B(l), j \in B(n)} m_{ij}, & \text{if } 1 \leq n \neq l \leq L, \\ 0, & \text{if } l = n, \end{cases}$$

and the elements of the matrix W_2 are defined for $1 \leq n \neq l \leq L$ by

$$(W_2)_{ln} = \\ -\frac{1}{K} \operatorname{cov}_{\mu(dx|y)} \left(\sum_{j\in B(l)} \left(\sum_{i=1}^{N} m_{ij} x_i \right) + \delta \psi_j'(x_j) , \sum_{j\in B(n)} \left(\sum_{i=1}^{N} m_{ij} x_i \right) + \delta \psi_j'(x_j) \right)$$

and for $l = n$ by $(W_2)_{ll} = 0$. By using CS(ε) we can estimate

$$W_1 \geq -\varepsilon \ \mathrm{Id}$$

in the sense of quadratic forms. The estimation of W_2 is a little bit more subtle. By bilinearity of the covariance the matrix W_2 can be rewritten as

$$W_2 = W_3 + W_4 + W_5 + W_6,$$

where the elements of the matrices $W_1, \ldots, W_6$ are defined for $1 \leq l \neq n \leq L$ by

$$(W_3)_{ln} = -\frac{1}{K} \operatorname{cov}_{\mu(dx|y)} \left(\sum_{j\in B(l)} \left(\sum_{i=1}^{N} m_{ij} x_i \right), \sum_{j\in B(n)} \left(\sum_{i=1}^{N} m_{ij} x_i \right) \right),$$

$$(W_4)_{ln} = -\frac{1}{K} \operatorname{cov}_{\mu(dx|y)} \left(\sum_{j\in B(l)} \delta \psi_j'(x_j) , \sum_{j\in B(n)} \delta \psi_j'(x_j) \right),$$

$$(W_5)_{ln} = -\frac{1}{K} \operatorname{cov}_{\mu(dx|y)} \left(\sum_{j\in B(l)} \left(\sum_{i=1}^{N} m_{ij} x_i \right), \sum_{j\in B(n)} \delta \psi_j'(x_j) \right),$$

$$(W_6)_{ln} = -\frac{1}{K} \operatorname{cov}_{\mu(dx|y)} \left(\sum_{j\in B(l)} \delta \psi_j'(x_j) , \sum_{j\in B(n)} \left(\sum_{i=1}^{N} m_{ij} x_i \right) \right),$$

and for $l = n$ by

$$(W_3)_{ll} = 0, \qquad (W_4)_{ll} = 0, \qquad (W_5)_{ll} = 0, \qquad (W_6)_{ll} = 0.$$

We estimate each matrix separately and start with W_3. A simple linear algebra argument outlined in [46, Lemma 9] shows that the elements of the inverse of A are non negative i.e. $(A^{-1})_{s_1 s_2} \geq 0$ for all $s_1, s_2 \in \{1, \ldots, L\}$. Hence, Lemma 3.1.10 (i) and the equivalence of norms in finite dimensional vector spaces yield for $1 \leq l \neq n \leq L$ the estimate

$$\begin{aligned} -(W_3)_{ln} &\leq \sum_{s_1,s_2=1}^{L} \left(A^{-1}\right)_{s_1 s_2} \left(\sum_{i\in B(l), j\in B(s_1)} m_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{i\in B(n), j\in B(s_2)} m_{ij}^2 \right)^{\frac{1}{2}} \\ &\leq C \sum_{s_1,s_2=1}^{L} \|M_{ls_1}\| \left(A^{-1}\right)_{s_1 s_2} \|M_{s_2 n}\|, \end{aligned}$$

where the matrix A is defined by (3.13) and $\|M_{ls_1}\|$ is defined by (3.12). Here and later on in this proof, $0 < C < \infty$ denotes a generic constant depending only on K and c_1. It follows from the last estimate and (3.19) that

$$-W_3 \leq \left(\|M_{s_1 s_2}\|\right)_{L\times L} A^{-1} \left(\|M_{s_1 s_2}\|\right)_{L\times L} \leq C\varepsilon$$

in the sense of quadratic forms.
Let us turn to the estimation of W_4. An application of Lemma 3.1.10 (i) implies the estimate

$$-(W_4)_{ln} \leq \left(A^{-1}\right)_{ln} \max_{i\in\{1,\ldots,N\}} \max_{x\in\mathbb{R}} |\delta\psi_i''(x)|^2$$

for $1 \leq l \neq n \leq L$. Hence, (3.18) yields in the sense of quadratic forms

$$-W_4 \leq A^{-1} - \operatorname{diag}\left(\left(A^{-1}\right)_{11}, \ldots, \left(A^{-1}\right)_{LL}\right) \leq C\varepsilon.$$

With an similar argument one can estimate the matrices W_5 and W_6 as

$$-W_5 - W_6 \leq C\varepsilon$$

in the sense of quadratic forms, which together with the estimates of W_3 and W_4 yields

$$-W_2 \leq C\varepsilon$$

in the sense of quadratic forms. □

Estimation of the diagonal elements of the Hessian of $\bar{H}$. In this section we will deduce the strict positivity of the diagonal elements of the Hessian of $\bar{H}$ for sufficiently large block sizes K and sufficiently small interaction ε. More precisely, we will show the following statement.

Lemma 3.1.11. *There exist $K_0 \in \mathbb{N}$ depending only on c_1 such that:
If the block size $K \geq K_0$ and the interaction matrix M satisfies CS(ε), then there are constants $\lambda > 0$ and $C < \infty$ independent of N, m, and s (depending only on K and c_1) such that for all $1 \leq l \leq L$ and $y \in Y$*

$$\left(\mathrm{Hess}_Y\, \bar{H}(y)\right)_{ll} \geq \lambda - C\varepsilon.$$

Therefore,

$$\mathrm{diag}\left(\left(\mathrm{Hess}_Y\, \bar{H}(y)\right)_{11}, \ldots, \left(\mathrm{Hess}_Y\, \bar{H}(y)\right)_{LL}\right) \geq (\lambda - C\varepsilon)\,\mathrm{Id}$$

in the sense of quadratic forms.

For the proof of Lemma 3.1.11 we use a conditioning technique, which allows us to apply a perturbation argument for small ε independently of N, m, and s. Let us consider an arbitrary but fixed block $B(l)$, $1 \leq l \leq L$. Recall that the spin values inside the block $B(l)$ are denoted by $x^l := (x_i)_{i \in B(l)}$ and the spin values outside the block $B(l)$ are denoted by $\bar{x}^l := (x_i)_{i \notin B(l)}$. As in the proof of Proposition 3.1.4, disintegration of the measure $\mu(dx|y)$ with respect to x^l yields (cf. Figure 3.2)

$$\mu(dx|y) = \mu(dx^l|\bar{x}^l, y)\; \bar{\mu}(d\bar{x}^l|y),$$

where $\mu(dx^l|\bar{x}^l, y)$ and $\bar{\mu}(d\bar{x}^l|y)$ denote the conditional measure and the corresponding marginal respectively (cf. (3.9)). Recall the definition of $H(x^l|M, s^*)$ for an arbitrary vector $s^* \in \mathbb{R}^{B(l)}$ i.e.

$$H(x^l|M, s^*) := \sum_{i \in B(l)} \psi_i(x_i) + \frac{1}{2} \sum_{i,j \in B(l)} m_{ij} x_i x_j + \sum_{i \in B(l)} s_i^* x_i. \tag{3.22}$$

In the proof of Proposition 3.1.4 we have shown that the conditional measures $\mu(dx^l|\bar{x}^l, y)$ are given by

$$\mu(dx^l|\bar{x}^l, y) = \frac{1}{Z}\, \exp\left(-H(x^l|M, s_c)\right) \mathcal{H}^{K-1}_{\lfloor X_{K,y_l}}(dx), \tag{3.23}$$

where the vector $s_c = s_c(M, s) \in \mathbb{R}^{B(l)}$ defined by

$$s_{c,i} := s_i + \sum_{j \notin B(l)} m_{ij} x_j \qquad \text{for } i \in B(l) \tag{3.24}$$

and the integration space X_{K,y_l} is identified with

$$X_{K,y_l} = \left\{ x^l \in \mathbb{R}^{B(l)} \mid \frac{1}{K} \sum_{i \in B(l)} x_i = y_l \right\}. \tag{3.25}$$

We introduce the coarse-grained Hamiltonian of $H(x^l|M, s^*)$ as usual i.e. for $y_l \in \mathbb{R}$

$$\bar{H}(y_l|M, s^*) := -\frac{1}{K} \log \int \exp\left(-H(x^l|M, s^*)\right) \mathcal{H}^{K-1}_{\lfloor X_{K,y_l}}(dx^l). \tag{3.26}$$

The next lemma shows that uniform positivity of

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|M,s^*)$$

yields uniform positivity of $(\mathrm{Hess}_Y\ \bar{H}(y))_{ll}$ for small ε . This observation is one of the main insights in order to apply a perturbation argument for small ε independently of the system size N. The advantage of $\bar{H}(y_l|M,s^*)$ over $\bar{H}(y)$ is that in (3.26) one integrates only over sites of the block $B(l)$, whereas in the definition (3.8) of the coarse-grained Hamiltonian $\bar{H}(y)$ one integrates over all sites of the spin system.

Lemma 3.1.12. *Assume that the vector s_c and the Hamiltonian $H(x^l|M,s_c)$ are given by* (3.24) *and* (3.22) *respectively. Then:*
If the interaction matrix M satisfies CS(ε), then for all $1 \le l \le L$ and $y \in Y$

$$(\mathrm{Hess}_Y\ \bar{H}(y))_{ll} \ge \int \frac{d^2}{dy_l^2}\bar{H}(y_l|M,s_c)\bar{\mu}(d\bar{x}^l|y) - C\varepsilon,$$

where the constant $C < \infty$ is independent of N, m, and s (depending only on the block size K and c_1).

The proof of Lemma 3.1.12 consists of two steps. In the first step we show that the disintegration (3.9) yields the identity

$$\begin{aligned}\big(\mathrm{Hess}_Y\ \bar{H}(y)\big)_{ll} &= \int \frac{d^2}{dy_l^2}\bar{H}(y_l|M,s_c)\ \bar{\mu}(d\bar{x}^l|y)\\ &- \frac{1}{K}\,\mathrm{var}_{\bar{\mu}(d\bar{x}^l|y)}\left(\int \sum_{j\in B(l)}\left(\sum_{i=1}^{N} m_{ij}x_i\right) + \delta\psi_j'(x_j)\ \mu(dx^l|\bar{x}^l,y)\right). \end{aligned}\tag{3.27}$$

In the second step we show that the variance term on the right hand side can be estimated by using the covariance estimate of Theorem 1.2.4 as

$$\frac{1}{K}\,\mathrm{var}_{\bar{\mu}(d\bar{x}^l|y)}\left(\int \sum_{j\in B(l)}\left(\sum_{i=1}^{N} m_{ij}x_i\right) + \delta\psi_j'(x_j)\ \mu(dx^l|\bar{x}^l,y)\right) \le C\varepsilon. \tag{3.28}$$

We will state the full proof of Lemma 3.1.12 below. The next lemma provides the last remaining ingredient of the proof of Lemma 3.1.11, which is the uniform positivity of $\frac{d^2}{dy_l^2}\bar{H}(y_l|M,s^*)$.

Lemma 3.1.13. *There is $K_0 \in \mathbb{N}$ such that:*
If the block size $K \ge K_0$ and the interaction matrix M satisfies CS(ε), then there are constants $\lambda > 0$ and $C < \infty$ independent of N, m, and s (depending only on K and c_1) such that for all $1 \le l \le L$, $y_l \in \mathbb{R}$, and $s^* \in \mathbb{R}^{B(l)}$

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|M,s^*) \ge \lambda - C\varepsilon. \tag{3.29}$$

For the proof of Lemma 3.1.13 we apply the following strategy. If the block size K is large enough, the generalized local Cramér theorem (cf. Proposition 3.2.1 and Theorem 3.2.2) yields

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|0,\tilde{s}) \geq \lambda > 0 \tag{3.30}$$

for all $y_l \in \mathbb{R}$ and $\tilde{s} \in \mathbb{R}^{B(l)}$. We want to derive (3.29) from (3.30) by a perturbation argument. More precisely, we will show that for a specific choice of $\tilde{s} = \tilde{s}(s^*) \in \mathbb{R}^{B(l)}$ given by (3.43)

$$\left| \frac{d^2}{dy_l^2}\bar{H}(y_l|M,s^*) - \frac{d^2}{dy_l^2}\bar{H}(y_l|0,\tilde{s}) \right| \leq C\varepsilon. \tag{3.31}$$

The constant $C < \infty$ just depends on K and c_1. For the proof of Lemma 3.1.11 it is crucial that the last inequality holds uniformly in $s^* \in \mathbb{R}^{B(l)}$ and y_l. Because we consider unbounded spins with quadratic interaction, this is difficult and leads to the specific choice of $\tilde{s} = \tilde{s}(s^*)$. It would be a lot easier to derive (3.31) for bounded spin-values with finite-range interaction. In this case one could also deduce the estimate (3.31) choosing $\tilde{s} = 0$. Then, the standard version of the local Cramér theorem [22, Proposition 31] would be sufficient for the perturbation argument at least for homogeneous single-site potentials $\psi_i = \psi$. The reason is that [22, Proposition 31] yields in this case

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|0,0) \geq \lambda > 0.$$

We will state the full proof of Lemma 3.1.13 below.

Proof of Lemma 3.1.11. The desired statement follows from a combination of Lemma 3.1.12 and Lemma 3.1.13. □

Proof of Lemma 3.1.12. Let us deduce the identity (3.27). Recall that by Lemma 3.1.8 we have

$$\begin{aligned}\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{ll} = 1 + \frac{1}{K}\sum_{i,j\in B(l)} m_{ij} + \frac{1}{K}\int \sum_{j\in B(l)} \delta\psi_j''(x_j)\mu(dx|y) \\ - \frac{1}{K}\operatorname{var}_{\mu(dx|y)}\left(\sum_{j\in B(l)}\left(\sum_{i=1}^{N} m_{ij}x_i\right) + \delta\psi_j'(x_j)\right).\end{aligned}$$

The disintegration rule (3.9) and the additive property of variances yield the identity

$$
\begin{aligned}
&\left(\operatorname{Hess}_Y \bar{H}(y)\right)_{ll} = \\
&\int \Bigg[\int \Bigg(1 + \frac{1}{K} \sum_{i,j \in B(l)} m_{ij} + \frac{1}{K} \int \sum_{j \in B(l)} \delta\psi_j''(x_j)\Bigg) \mu(dx^l|\bar{x}^l, y) \\
&\qquad - \frac{1}{K} \operatorname{var}_{\mu(dx^l|\bar{x}^l,y)} \Bigg(\sum_{j \in B(l)} \Bigg(\sum_{i=1}^{N} m_{ij} x_i \Bigg) + \delta\psi_j'(x_j) \Bigg) \Bigg] \bar{\mu}(d\bar{x}^l|y) \\
&\qquad - \frac{1}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \Bigg(\int \Bigg[\sum_{j \in B(l)} \Bigg(\sum_{i=1}^{N} m_{ij} x_i \Bigg) + \delta\psi_j'(x_j) \Bigg] \mu(dx^l|\bar{x}^l, y) \Bigg).
\end{aligned}
$$

Note that the Hamiltonian $H(x^l|M, s^*)$ defined by (3.22) has the same structure as the Hamiltonian $H(x)$ given by (3.2). Therefore, an application of Lemma 3.1.8 yields that

$$
\begin{aligned}
\frac{d^2}{dy_l^2} \bar{H}(y_l|M, s_c) &= 1 + \frac{1}{K} \sum_{i,j \in B(l)} m_{ij} + \frac{1}{K} \int \sum_{j \in B(l)} \delta\psi_j''(x_j) \mu(dx^l|\bar{x}^l, y) \\
&\qquad - \frac{1}{K} \operatorname{var}_{\mu(dx^l|\bar{x}^l,y)} \Bigg(\sum_{j \in B(l)} \Bigg(\sum_{i \in B(l)} m_{ij} x_i \Bigg) + \delta\psi_j'(x_j) \Bigg). \qquad (3.32)
\end{aligned}
$$

The desired identity (3.27) follows from the last two equations and the fact that adding constant functions does not change variances.

It remains to derive the estimate (3.28) of the variance term of the right hand side of (3.27). By Young's inequality

$$
\begin{aligned}
&\frac{1}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \Bigg(\int \Bigg[\sum_{j \in B(l)} \Bigg(\sum_{i=1}^{N} m_{ij} x_i \Bigg) + \delta\psi_j'(x_j) \Bigg] \mu(dx^l|\bar{x}^l, y) \Bigg) \\
&\quad \leq \frac{2}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \Bigg(\int \sum_{j \in B(l)} \sum_{i=1}^{N} m_{ij} x_i \, \mu(dx^l|\bar{x}^l, y) \Bigg) \\
&\qquad + \frac{2}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \Bigg(\int \sum_{j \in B(l)} \delta\psi_j'(x_j) \, \mu(dx^l|\bar{x}^l, y) \Bigg). \qquad (3.33)
\end{aligned}
$$

Let us consider the first term of the right hand side of (3.33). By the disintegration rule (3.9) we have for any function $\xi(\bar{x}^l)$

$$
\int \xi(\bar{x}^l) \bar{\mu}(d\bar{x}^l|y) = \int \xi(\bar{x}^l) \underbrace{\int 1 \, \mu(dx^l|\bar{x}^l, y)}_{=1} \, \bar{\mu}(d\bar{x}^l|y) = \int \xi(\bar{x}^l) \mu(dx|y).
$$

It follows that

$$
\frac{2}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \left(\xi(\bar{x}^l) \right) = \frac{2}{K} \operatorname{var}_{\mu(dx|y)} \left(\xi(\bar{x}^l) \right).
$$

Therefore, an application of Theorem 1.2.4 to the measure $\mu(dx|y)$ yields

$$\begin{aligned}
&\frac{2}{K}\operatorname{var}_{\bar\mu(d\bar x^l|y)}\left(\int \sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i \,\mu(dx^l|\bar x^l,y)\right)\\
&\quad\leq \frac{2}{\varrho K}\sum_{s_1,s_2=1}^{L}\left(A^{-1}\right)_{s_1s_2}\\
&\quad\times\left(\int\sum_{k\in B(s_1)}\left|\frac{d}{dx_k}\int\sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i\,\mu(dx^l|\bar x^l,y)\right|^2\mu(dx|y)\right)^{\frac12}\\
&\quad\times\left(\int\sum_{k\in B(s_2)}\left|\frac{d}{dx_k}\int\sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i\,\mu(dx^l|\bar x^l,y)\right|^2\mu(dx|y)\right)^{\frac12}. \qquad (3.34)
\end{aligned}$$

It follows from the definition $x^l=(x_k)_{k\in B(l)}$ that for $k\in B(l)$

$$\frac{d}{dx_k}\left(\int\sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i\,\mu(dx^l|\bar x^l,y)\right)=0. \qquad (3.35)$$

Using the definition (3.22) of $H(x^l|M,s_c)$ direct calculation shows that

$$\begin{aligned}
&\frac{d}{dx_k}\int\sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i\,\mu(dx^l|\bar x^l,y)\\
&\quad=\sum_{j\in B(l)} m_{kj}-\operatorname{cov}_{\mu(dx^l|\bar x^l,y)}\left(\sum_{j\in B(l)}\sum_{i=1}^{N} m_{ij}x_i\ ,\ \frac{d}{dx_k}H(x^l|M,s_c)\right)
\end{aligned}$$

for $k\notin B(l)$. From now on, let $C<\infty$ denote a generic constant depending only on K and c_1. Because $\mu(dx^l|\bar x^l,y)$ satisfies LSI($\tilde\varrho$) with $\tilde\varrho>0$ depending only on K and c_1 (cf. proof of Proposition 3.1.4), the measure $\mu(dx^l|\bar x^l,y)$ also satisfies the SG($\tilde\varrho$) by Lemma 1.1.1. Hence, an application of the standard covariance estimate of Lemma 1.2.2 and the equivalence of norms in finite-dimensional vector spaces yield

$$\begin{aligned}
&\left| \frac{d}{dx_k} \int \sum_{j\in B(l)} \sum_{i=1}^{N} m_{ij} x_i \, \mu(dx^l|\bar{x}^l, y) \right| \\
&\quad \leq \; C \left(\sum_{j\in B(l)} m_{kj}^2 \right)^{\frac{1}{2}} + \frac{1}{\tilde{\varrho}} \underbrace{\left(\sum_{i,j\in B(l)} m_{ij}^2 \right)^{\frac{1}{2}}}_{\leq C\|M_{ll}\|} \left(\sum_{j\in B(l)} m_{kj}^2 \right)^{\frac{1}{2}} \\
&\quad \overset{(3.14)}{\leq} \left(C + \frac{C}{\tilde{\varrho}}\, \varepsilon \right) \left(\sum_{j\in B(l)} m_{kj}^2 \right)^{\frac{1}{2}} . \qquad (3.36)
\end{aligned}$$

A combination of the estimates (3.34), (3.35) and (3.36) yields the estimate of the first term on the right hand side of (3.33). More precisely,

$$\begin{aligned}
&\frac{2}{K} \operatorname{var}_{\bar{\mu}(d\bar{x}^l|y)} \left(\int \sum_{j\in B(l)} \sum_{i=1}^{N} m_{ij} x_i \, \mu(dx^l|\bar{x}^l, y) \right) \\
&\quad \leq C \sum_{s_1,s_2=1}^{L} \left(A^{-1}\right)_{s_1 s_2} \left(\sum_{i\in B(s_1),\, j\in B(l)} m_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{i\in B(s_2),\, j\in B(l)} m_{ij}^2 \right)^{\frac{1}{2}} \\
&\quad \leq C \sum_{s_1,s_2=1}^{L} \left(A^{-1}\right)_{s_1 s_2} \|M_{ls_1}\| \, \|M_{s_2 l}\| \overset{(3.19)}{\leq} C\varepsilon.
\end{aligned}$$

The second term on the right hand side of (3.33) can be estimated with the same argument as we used for the first term. The only different ingredient is the estimation of

$$\begin{aligned}
&\left| \frac{d}{dx_k} \int \sum_{j\in B(l)} \delta\psi_j'(x_j) \, \mu(dx^l|\bar{x}^l, y) \right| \\
&\quad = \left| \operatorname{cov}_{\mu(dx^l|\bar{x}^l,y)} \left(\sum_{j\in B(l)} \delta\psi_j'(x_j) \, , \sum_{s\in B(l)} m_{ks} x_s \right) \right| \leq \frac{C}{\tilde{\varrho}} \left(\sum_{j\in B(l)} m_{kj}^2 \right)^{\frac{1}{2}} ,
\end{aligned}$$

where we applied Lemma 1.2.2 and the uniform bound (3.1) of the functions $\delta\psi_i$. □

Proof of Lemma 3.1.13. Note that the estimate (3.30) follows directly from the generalized local Cramér theorem (cf. Proposition 3.2.1 and Theorem 3.2.2). Hence, it is only left to deduce (3.31). Let $\nu(dx^l|M, s^*)$ denote the Gibbs measure on X_{K,y_l} (see (3.25)) associated to the Hamiltonian $H(x^l|M, s^*)$ i.e.

$$\nu(dx^l|M, s^*) = \frac{1}{Z} \, \exp(-H(x^l|M, s^*)) \mathcal{H}^{K-1}_{\lfloor X_{K,y_l}}(dx^l).$$

The same reason as for (3.32) yields that

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|M,s^*) = 1 + \frac{1}{K}\sum_{i\in B(l),\, j\in B(l)} m_{ij} + \int \frac{1}{K}\sum_{j\in B(l)} \delta\psi_j''(x_j)\, \nu(dx^l|M,s^*) - \frac{1}{K}\operatorname{var}_{\nu(dx^l|B,s^*)}\left(\sum_{j\in B(l)}\left(\sum_{i\in B(l)} m_{ij}x_i\right) + \delta\psi_j'(x_j)\right).$$

An application of this formula to $\bar{H}(y_l|0,\tilde{s})$ with arbitrary $\tilde{s}\in\mathbb{R}^{B(l)}$ yields

$$\frac{d^2}{dy_l^2}\bar{H}(y_l|0,\tilde{s}) = 1 + \int \frac{1}{K}\sum_{j\in B(l)} \delta\psi_j''(x_j)\, \nu(dx^l|0,\tilde{s}) - \frac{1}{K}\operatorname{var}_{\nu(dx^l|0,\tilde{s})}\left(\sum_{j\in B(l)}\delta\psi_j'(x_j)\right).$$

It follows from the last two equations and the bilinearity of the covariance that

$$\left|\frac{d^2}{dy_l^2}\bar{H}(y_l|M,s) - \frac{d^2}{dy_l^2}\bar{H}(y_l|0,\tilde{s})\right| \le T_1+T_2+T_3+T_4+T_5 \tag{3.37}$$

where the terms T_1, T_2, and T_4 are given by

$$T_1 := \frac{1}{K}\left|\sum_{i,j\in B(l)} m_{ij}\right|, \qquad T_2 := \frac{1}{K}\left|\operatorname{var}_{\nu(dx^l|M,s^*)}\left(\sum_{i,j\in B(l)} m_{ij}x_i\right)\right|,$$

$$T_3 := \frac{2}{K}\left|\operatorname{cov}_{\nu(dx^l|M,s^*)}\left(\sum_{i,j\in B(l)} m_{ij}x_i\,,\ \delta\psi_j'(x_j)\right)\right|,$$

and the terms T_4 and T_5 are given by

$$T_4 := \frac{1}{K}\left|\int\sum_{j\in B(l)}\delta\psi_j''(x_j)\,\nu(dx^l|M,s^*) - \int\sum_{j\in B(l)}\delta\psi_j''(x_j)\,\nu(dx^l|0,\tilde{s})\right|,$$

$$T_5 := \frac{1}{K}\left|\operatorname{var}_{\nu(dx^l|M,s^*)}\left(\sum_{j\in B(l)}\delta\psi_j'(x_j)\right) - \operatorname{var}_{\nu(dx^l|0,\tilde{s})}\left(\sum_{j\in B(l)}\delta\psi_j'(x_j)\right)\right|.$$

Note that the measure $\nu(dx^l|M,s^*)$ has the same structure as the measure $\mu(dx^l|\bar{x}^l,y)$. Therefore, it follows by the same argument as in the proof of Proposition 3.1.4 that the measure $\nu(dx^l|M,s^*)$ satisfies LSI($\tilde{\varrho}$) with $\tilde{\varrho}>0$ depending only on K and c_1. Hence, the measure $\nu(dx^l|M,s^*)$ also satisfies the SG($\tilde{\varrho}$) by Lemma 1.1.1. It is easy to deduce by using CS(ε) and the basic covariance estimate of Lemma 1.2.2 that

$$T_1+T_2+T_3 \le C\varepsilon$$

for a constant $C < \infty$ depending only on K and c_1.

The interesting part is the estimation of T_4 and T_5. The right choice of $\tilde{s} = \tilde{s}(s^*) \in \mathbb{R}^{B(l)}$ plays an important role. Therefore, let us motivate how to choose $\tilde{s} = \tilde{s}(s^*)$ for a given vector $s^* \in \mathbb{R}^{B(l)}$. The structure of T_4 and T_5 is given by

$$\left| \int \xi(x^l)\, \nu(dx^l|M, s^*) - \int \xi(x^l)\, \nu(dx^l|0, \tilde{s}) \right|$$

for a bounded function $\xi : \mathbb{R}^{B(l)} \to \mathbb{R}$. We want to estimate the last expression uniformly in the unbounded parameters $y_l \in \mathbb{R}$ and $s^* \in \mathbb{R}^{B(l)}$. Therefore, let us take a closer look at the dependence of

$$\int \xi(x^l)\, \nu(dx^l|M, s^*) = \frac{1}{Z} \int \xi(x^l) \exp\Big(-H(x^l|M, s^*)\Big)\, \mathcal{H}^{K-1}_{\lfloor X_{K,y_l}}(dx^l) \tag{3.38}$$

on the parameters y_l and s^*. On the block $B(l)$ the coarse-graining operator $P_l : \mathbb{R}^{B(l)} \to \mathbb{R}$ is defined by $P_l x^l = \frac{1}{K} \sum_{i \in B(l)} x_i$. Let P_l^* denote the adjoint operator of P i.e. for $y_l \in \mathbb{R}$

$$P_l^*(y_l) := \frac{1}{K}(y_l, \ldots, y_l) \in \mathbb{R}^{B(l)}.$$

By using the identity $P_l K P_l^* = \mathrm{Id}_{\mathbb{R}}$ one sees that the orthogonal projection Π of $\mathbb{R}^{B(l)}$ on $\ker P_l = X_{K,0}$ is given by

$$\Pi = \mathrm{Id} - K P_l^* P_l. \tag{3.39}$$

Consider the right hand side of (3.38). The dependence of the integration space X_{K,y_l} on y_l is abolished by the translation $x^l \mapsto \tilde{z} = \Pi x^l$, which maps X_{K,y_l} onto $X_{K,0}$ and yields the identity

$$\begin{aligned} \int \xi(x^l)\nu(dx^l|M, s^*) &= \frac{1}{Z} \int \xi(\tilde{z} + K P_l^* y_l) \times \qquad (3.40)\\ \exp\left(-\frac{1}{2} \langle \tilde{z}, (\mathrm{Id} + M_{ll})\tilde{z} \rangle - \langle s^* + M_{ll} K P_l^* y_l, \tilde{z} \rangle - \sum_{i \in B(l)} \delta\psi_i(z_i + y_l) \right) &\mathcal{H}^{K-1}_{\lfloor X_{K,0}}(d\tilde{z}), \end{aligned}$$

where the matrix M_{ll} is given by (3.11). Deriving the last identity consists of a straight forward calculation, where one has to consider the definition (3.22) of $H(x^l|M, s^*)$, cancel all terms that are independent of $\tilde{z}$ with terms of the normalization constant Z, and apply the fact that $\langle K P_l^* y_l, \tilde{z} \rangle = 0$ for $\tilde{z} \in X_{K,0}$. Note that in (3.40) only the linear term $\langle s^* + M_{ll} K P_l^* y_l, \tilde{z} \rangle$ depends on the parameters y_l and s^*. The idea is to get rid of this term by a second translation $\tilde{z} \mapsto \tilde{z} + v$, which leaves the integration space $X_{K,0}$ invariant. Because $\tilde{z} \in X_{K,0} = \ker P_l$, we can rewrite the Gaussian part of the Hamiltonian in (3.40) as

$$\begin{aligned} &\frac{1}{2} \langle \tilde{z}, (\mathrm{Id} + M_{ll})\tilde{z} \rangle + \langle s^* + M_{ll} K P_l^* y_l, \tilde{z} \rangle \\ &\qquad = \frac{1}{2} \langle \tilde{z}, (\mathrm{Id} + \Pi M_{ll})\tilde{z} \rangle + \langle \Pi s^* + \Pi M_{ll} K P_l^* y_l), \tilde{z} \rangle . \end{aligned}$$

Because M satisfies CS(ε) with $\varepsilon < 1$, the map $(\mathrm{Id} + \Pi M_{ll}) : X_{K,0} \to X_{K,0}$ is invertible. We define v by

$$v = (\mathrm{Id} - \Pi M_{ll})^{-1} (\Pi s^* + \Pi M_{ll} K P_l^* y_l). \tag{3.41}$$

A direct calculation using the definition of v yields

$$\begin{aligned} &\frac{1}{2} \langle \tilde{z}, (\mathrm{Id} + \Pi M_{ll}) \tilde{z} \rangle + \langle \Pi s^* + \Pi M_{ll} K P_l^* y_l, \tilde{z} \rangle \\ &= \frac{1}{2} \langle z, (\mathrm{Id} + \Pi M_{ll}) z \rangle - \langle \Pi s^* + \Pi M_{ll} K P_l^* y_l, v \rangle + \frac{1}{2} \langle v, (\mathrm{Id} + \Pi M_{ll}) v \rangle . \end{aligned}$$

Because $v \in X_{K,0}$, the transformation $\tilde{z} \mapsto z = \tilde{z} + v$ leaves the integration space $X_{K,0}$ on the right hand side of (3.40) invariant and yields by using the last identity that

$$\begin{aligned} \int \xi(x^l) \, \nu(dx^l | M, s^*) &= \frac{1}{Z} \int \xi(z + N P^* y_l - v) \\ &\times \exp\left(-\frac{1}{2} \langle z, (\mathrm{Id} + M_{ll}) z \rangle - \sum_{i \in B(l)} \delta\psi_i (z_i + y_l - v_i) \right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dz), \end{aligned} \tag{3.42}$$

where we have canceled the terms that are independent of z with terms of the normalization constant Z. Note that we have gained compactness by this representation: The unbounded parameters y_l and s^* only enter (3.42) as an argument of the bounded functions ξ and $\delta\psi_i$. This observation is crucial for the estimation of T_4 and T_5. The derivation of (3.42) reveals that it is natural to choose

$$\tilde{s}(s^*) = \Pi s^* + \Pi M_{ll} K P_l^* y_l = (\mathrm{Id} - K P_l^* P_l) \left(s^* + M_{ll} K P_l^* y_l \right), \tag{3.43}$$

where the matrix M_{ll} is given by (3.11). The reason is that carrying out the two translations from above yields

$$\begin{aligned} \int \xi(x^l) \, \nu(dx^l | 0, \tilde{s}) &= \frac{1}{Z} \int \xi(z + K P_l^* y_l - v) \\ &\times \exp\left(-\frac{1}{2} \langle z, z \rangle - \sum_{i \in B(l)} \delta\psi_i (z_i + y_l - v_i) \right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dz). \end{aligned} \tag{3.44}$$

The right hand side of (3.42) and (3.44) coincide except of the interaction term $\left\langle x^l, M_{ll} x^l \right\rangle$. The latter is very helpful to apply a perturbation argument for the uniform estimation of T_4 and T_5.

Now, we will estimate T_4 and T_5. Let us choose $\tilde{s} = \tilde{s}(s^*)$ as in (3.43). For $0 \leq \lambda \leq 1$ we define the probability measure ν_λ on $X_{K,0}$ (see (3.25)) by

$$\nu_\lambda(dz) := \frac{1}{Z} \; \exp\left(-\frac{1}{2} \langle z, (\mathrm{Id} + \lambda M_{ll}) z \rangle - \sum_{j \in B(l)} \delta\psi_j \left(z_j + y_l - v_j \right) \right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dz),$$

where the vector v is defined by (3.41). Applying the translation $x^l \mapsto z = \Pi x^l + v$ on the integrals of T_4 yields (cf. (3.42), and (3.44))

$$\begin{aligned} T_4 &= \frac{1}{K}\left| \int \sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\nu_1(dz) - \int \sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\nu_0(dz) \right| \\ &\le \frac{1}{K} \sup_{0\le\lambda\le 1} \left| \frac{d}{d\lambda} \int \sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\, \nu_\lambda(dz) \right|. \end{aligned} \tag{3.45}$$

Because M satisfies CS(ε), we may assume w.l.o.g. that

$$-\frac{1}{2}\ \mathrm{Id} \le M_{ll} \le \frac{1}{2}\ \mathrm{Id}\,. \tag{3.46}$$

By direct calculation we get that for any $0 \le \lambda \le 1$

$$\begin{aligned} &\frac{d}{d\lambda} \int \sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\, \nu_\lambda(dz) \\ &= \frac{1}{2} \operatorname{cov}_{\nu_\lambda(dz)} \left(\sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\ ,\ \langle z, M_{ll} z\rangle \right) \\ &= \frac{1}{2} \int \left(\sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j) - \int \delta\psi_j''(z_j + y_l - v_j)\nu_\lambda(dz) \right) \langle z, M_{ll} z\rangle \nu_\lambda(dz). \end{aligned}$$

Let $C < \infty$ denote a generic constant depending only on K and c_1. From the last identity we can deduce the estimate

$$\begin{aligned} &\left| \frac{d}{d\lambda} \int \sum_{j\in B(l)} \delta\psi_j''(z_j + y_l - v_j)\, \nu_\lambda(dz) \right| \\ &\le\ K \max_{j\in B(l)} \sup_{x\in\mathbb{R}} \left|\delta\psi_j''(x)\right| \int |\langle z, M_{ll} z\rangle|\, \nu_\lambda(dz) \\ &\overset{(3.14)}{\le}\ C\varepsilon\, \frac{\int |z|^2 \exp\left(-\frac{1}{2}\langle z, (\mathrm{Id} + \lambda M_{ll})\, z\rangle - \sum_{j\in B(l)} \delta\psi_j\,(z_j + y_l - v_j)\right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dx)}{\int \exp\left(-\frac{1}{2}\langle z, (\mathrm{Id} + \lambda M_{ll})\, z\rangle - \sum_{j\in B(l)} \delta\psi_j\,(z_j + y_l - v_j)\right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dx)} \\ &\overset{(3.46)}{\le}\ C\varepsilon\, \exp\left(2K \max_{j\in B(l)} \sup_x |\delta\psi_j(x)|\right) \frac{\int |z|^2 \exp\left(-\frac{1}{2}\langle z, z\rangle\right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dx)}{\int \exp\left(-\frac{3}{2}\langle z, z\rangle\right) \mathcal{H}^{K-1}_{\lfloor X_{K,0}}(dx)} \\ &\le\ C\varepsilon. \end{aligned} \tag{3.47}$$

A combination of (3.45) and (3.47) yields the estimate

$$T_4 \le C\varepsilon.$$

The same argument also yields

$$T_5 \le C\varepsilon.$$

Compared to the estimation of T_4 one has to take a closer look at the term

$$\frac{d}{d\lambda}\operatorname{var}_{\nu_\lambda(dz)}\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)\right)$$
$$=\frac{d}{d\lambda}\int\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)-\int\delta\psi_j'(z_j+y_l-v_j)\,\nu_\lambda(dz)\right)^2\nu_\lambda(dz).$$

Because

$$\int\left[\frac{d}{d\lambda}\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)-\int\delta\psi_j'(z_j+y_l-v_j)\,\nu_\lambda(dz)\right)^2\right]\nu_\lambda(dz)$$
$$=-2\int\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)-\int\delta\psi_j'(z_j+y_l-v_j)\,\nu_\lambda(dz)\right)\nu_\lambda(dz)$$
$$\times\ \frac{d}{d\lambda}\int\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)\,\nu_\lambda(dz)$$
$$=0,$$

it follows by direct calculation that

$$\frac{d}{d\lambda}\operatorname{var}_{\nu_\lambda(dz)}\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)\right)$$
$$=\int\left(\sum_{j\in B(l)}\delta\psi_j'(z_j+y_l-v_j)-\int\delta\psi_j'(z_j+y_l-v_j)\nu_\lambda(dz)\right)^2\left(\frac{d}{d\lambda}\nu_\lambda(dz)\right)$$
$$=\frac{1}{2}\operatorname{cov}_{\nu_\lambda(dz)}\left(\left(\sum_{j\in B(l)}\delta\psi_j'(\cdots)-\int\delta\psi_j'(\cdots)\nu_\lambda(dz)\right)^2,\langle z,M_{ll}z\rangle\right).$$

However, the covariance term on the right hand side can be estimated in the same way as in (3.47). Therefore, we have deduced (3.31) uniformly in $y_l\in\mathbb{R}$ and $s^*\in\mathbb{R}^{B(l)}$, which completes the proof of Lemma 3.1.13. □

3.2 The local Cramér theorem for inhomogeneous single-site potentials

The main goal of this section is to deduce a convexification result that is one of the central ingredients for the macroscopic LSI (cf. Proposition 3.1.5 and Lemma 3.1.13):

Proposition 3.2.1. *Assume that the Hamiltonian* $H : \mathbb{R}^K \to \mathbb{R}$ *is given by*

$$H(x) := \sum_{j=1}^{K} \frac{1}{2} x_j^2 + s_j x_j + \delta\psi_j(x_j) \tag{3.48}$$

for some arbitrary vector $s \in \mathbb{R}^K$ *and some functions* $\delta\psi_j : \mathbb{R} \to \mathbb{R}$ *satisfying the uniform bound* (3.1) *i.e. for all* $j \in \{1, \ldots, K\}$.

$$\|\delta\psi_j\|_{C^2} \le c_1 < \infty.$$

Let $\bar{H}_K$ *denote the coarse-grained Hamiltonian of* H *associated to coarse-graining the whole system. More precisely, for* $m \in \mathbb{R}$

$$\bar{H}_K(m) := -\frac{1}{K} \log \int_{\left\{\frac{1}{K}\sum_{j=1}^{K} x_j = m\right\}} \exp\left(-H\left(x\right)\right) \, \mathcal{H}\left(dx\right). \tag{3.49}$$

Then there is K_0 *and* $\lambda > 0$ *such that for all* $K \ge K_0$, s, *and* m

$$\frac{d^2}{dm^2} \bar{H}_K(m) \ge \lambda.$$

Like the convexification result of Theorem 2.1.6 in Chapter 2 and [22][Lemma 29], the statement of Proposition 3.2.1 is a direct consequence of the a local Cramér theorem, namely:

Theorem 3.2.2 (Local Cramér theorem)**.** *Assume that the Hamiltonian* H *is given by* (3.48). *Let* $\varphi_K(m)$ *be defined as the Cramér transform of* H, *namely*

$$\varphi_K(m) := \sup_{\sigma \in \mathbb{R}} \left(\sigma m - \frac{1}{K} \log \int_{\mathbb{R}^K} \exp\left(-H(x) + \sum_{j=1}^{K} \sigma x_j \right) \, dx \right). \tag{3.50}$$

Then φ_K *is strictly convex independently of* s, m, *and* K. *Additionally, it holds*

$$\|\bar{H}_K(m) - \varphi_K(m)\|_{C^2} \to 0 \qquad \text{as } K \to \infty,$$

The convergence only depends on the constant c_1 *given by* (3.1).

In Section 2.2 we have implicitly generalized the local Cramér theorem to Hamiltonians given by (cf. comment after Lemma 2.2.2)

$$H(x) := \sum_{j=1}^{K} \psi(x_j)$$

for an arbitrary perturbed strictly convex single-site potential ψ in the sense of (2.5). Now, we have to generalize it to Hamiltonians of the form

$$H(x) := \sum_{j=1}^{K} \psi_j(x_j).$$

The difference to [22] and Section 2.2 is that the single-site potentials ψ_j are allowed to depend on the site $j \in \{1, \ldots, K\}$. Because we want to apply the local Cramér theorem to single-site potentials given by

$$\psi_j(x_j) = \frac{1}{2}x_j^2 + s_j x_j + \delta\psi_j(x_j),$$

we only consider this nice class of potentials making the proof of the local Cramér theorem less complex than in Section 2.2.

As usual, the proof of the local Cramér theorem is based on two ingredients. The first one is Cramér's representation of the difference $(\bar{H}_K(m) - \varphi_K(m))$ (cf. [22, (125)]):

Lemma 3.2.3. *For $j \in \{1, \ldots, K\}$ we consider the one-dimensional probability measure μ_j^σ given by*

$$\mu_j^\sigma(dx_j) := \exp\left(-\varphi_{K,j}^*(\sigma) + \sigma x_j - \frac{1}{2}x_j^2 - s_j x_j - \delta\psi_j(x_j)\right)\, dx_j,$$

where

$$\varphi_{K,j}^*(\sigma) := \log \int \exp\left(\sigma x_j - \frac{1}{2}x_j^2 - s_j x_j - \delta\psi_j(x_j)\right)\, dx_j.$$

We introduce the mean m_j and variance ς_j^2 of the measure μ_j^σ

$$m_j := \int x_j \mu_j^\sigma(dx_j) \qquad \textit{and} \qquad \varsigma_j^2 := \int (x_j - m_j)^2 \mu_j^\sigma(dx_j).$$

Assume that X_j, $j \in \{1, \ldots, K\}$, are independent random variables distributed according to μ_j^σ. Let $g_{K,m}(\xi)$ denote the Lebesgue density of the distribution of the random variable

$$\frac{1}{\sqrt{K}} \sum_{j=1}^{K} X_j - m_j.$$

Then

$$g_{K,m}(0) = \exp(K\varphi_K(m) - K\bar{H}_K(m)). \tag{3.51}$$

The second ingredient is a local central limit type theorem for the density $g_{K,m}$. The generalization of the local Cramér theorem by Theorem 3.2.2 is not surprising: For the classical central limit theorem it is not important that the random variables X_j are identically distributed. It suffices that the standard deviation ς_j of X_j is uniformly bounded. The latter is guaranteed by the uniform control $\|\delta\psi_j\| \leq c_1$ (cf. Lemma 3.2.4 below). As a consequence we can proceed with the same strategy as for the classical local Cramér theorem (cf. [22, Proposition 31]). We just have to pay attention that every step does not rely on the specific form of ψ_j but on the uniform bound of ς_j. Because the complete proof of Theorem 3.2.2 is elementary but a bit lengthy, we will state the details in the next section.

Lemma 3.2.4. *Assume that* $\|\delta\psi_j\|_{C^2} \leq c_1 < \infty$ *uniformly in* $j \in \{1, \ldots, K\}$. *Then there is a constant* $0 < c < \infty$ *such that for any* σ *and* j

$$\frac{1}{c} \leq \varsigma_j \leq c, \tag{3.52}$$

where ς_j *is defined as in Lemma 3.2.3.*

We conclude this chapter with the proof of Lemma 3.2.3 and Lemma 3.2.4.

Lemma 3.2.3. Because φ_K is the Legendre transform of the strictly convex function

$$\varphi_K^*(\sigma) := \frac{1}{K} \log \int_{\mathbb{R}^K} \exp\left(-H(x) + \sum_{j=1}^{K} \sigma x_j\right) dx,$$

there exits for every $m \in \mathbb{R}$ a unique $\sigma = \sigma(m) \in \mathbb{R}$ such that

$$\varphi_K(m) = \sigma m - \varphi_K^*(\sigma). \tag{3.53}$$

It is well-known that σ is determined by the equation

$$m = \frac{d}{d\sigma}\varphi_K^*(\sigma). \tag{3.54}$$

Now, we will show that φ_K^* and m can be decomposed according to

$$\varphi_K^*(\sigma) = \frac{1}{K}\sum_{j=1}^{K} \varphi_{K,j}^*(\sigma) \qquad \text{and} \qquad m = \frac{1}{K}\sum_{j=1}^{K} m_j. \tag{3.55}$$

Indeed, the decomposition of φ_K^* directly follows from definitions. Observe that

$$m_j = \int x_j \mu_j^\sigma(dx_j) = \frac{d}{d\sigma}\, \varphi_{K,j}^*(\sigma).$$

Then, the decomposition of m follows from (3.54) and the decomposition of φ_K^*. More precisely,

$$m = \frac{d}{d\sigma}\varphi_K^*(\sigma) = \frac{1}{K}\sum_{j=1}^{K} \frac{d}{d\sigma}\varphi_{K,j}^*(\sigma) = \frac{1}{K}\sum_{j=1}^{K} m_j.$$

Now, we will deduce Cramér's representation (3.51). The density $g_{K,m}(\xi)$ at $\xi = 0$ can be written as

$$g_{K,m}(0) = \int_{\left\{K^{-\frac{1}{2}} \sum_{j=1}^{K} x_j - m_j = 0\right\}} \exp\left(\sum_{j=1}^{K} -\varphi_{K,j}^*(\sigma) + \sigma x_j - \psi_j(x_j)\right) \mathcal{H}(dx).$$

By (3.55) we get

$$g_{K,m}(0) = \int_{X_{K,m}} \exp\left(-K\varphi_K^*(\sigma) + K\sigma m - \sum_{j=1}^{K} \psi_j(x_j) \right) \mathcal{H}(dx).$$

Using (3.53) the right hand side becomes

$$g_{K,m}(0) = \exp\left(K\varphi_K(m)\right) \int_{X_{K,m}} \exp\left(-\sum_{j=1}^{K} \psi_j(x_j) \right) \mathcal{H}(dx).$$

Applying the definition (3.49) of $\bar{H}_K(m)$ yields the desired formula. □

Lemma 3.2.4. Observe that the variance of a one-dimensional Gaussian measure is invariant under adding a linear term to the Hamiltonian i.e. for any $\tilde{\sigma} \in \mathbb{R}$

$$\begin{aligned} \varsigma^2 := &\int \left(x - \frac{\int x \exp(-\frac{x^2}{2}) dx}{\int \exp(-\frac{x^2}{2}) dx} \right)^2 \frac{\exp(-\frac{x^2}{2})}{\int \exp(-\frac{x^2}{2}) dx} \, dx \\ = &\int \left(x - \frac{\int x \exp(\tilde{\sigma}x - \frac{x^2}{2}) dx}{\int \exp(\tilde{\sigma}x - \frac{x^2}{2}) dx} \right)^2 \frac{\exp(\tilde{\sigma}x - \frac{x^2}{2})}{\int \exp(\tilde{\sigma}x - \frac{x^2}{2}) dx} \, dx. \end{aligned}$$

Let us consider the upper bound of (3.52). Because the mean of a probability measure ν is optimal in the sense that for all $c \in \mathbb{R}$

$$\begin{aligned} \int (x-c)^2 \, \nu(dx) &= \int x^2 \nu(dx) - 2c \int x\nu(dx) + c^2 \\ &\geq \int x^2 \nu(dx) - \left(\int x\nu(dx) \right)^2 \\ &= \int \left(x - \int x\nu(dx) \right)^2 \nu(dx), \end{aligned}$$

we have by using the uniform bound $\|\delta\psi_j\|_{C^2} \leq c_1 < \infty$ and $\tilde{\sigma} = \sigma - s_j$

$$\begin{aligned} \varsigma_j^2 &= \int (x_j - m_j)^2 \, \frac{\exp(\tilde{\sigma}x_j - \frac{x_j^2}{2} - \delta\psi_j(x_j))}{\int \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2} - \delta\psi_j(x_j)) dx_j} \, dx_j \\ &\leq \int \left(x_j - \frac{\int x_j \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2}) dx_j}{\int \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2}) dx_j} \right)^2 \frac{\exp(\tilde{\sigma}x_j - \frac{x_j^2}{2} - \delta\psi_j(x_j))}{\int \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2} - \delta\psi_j(x_j)) dx_j} \, dx_j \\ &\leq \exp(2c_1) \int \left(x_j - \frac{\int x_j \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2}) dx_j}{\int \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2}) dx_j} \right)^2 \frac{\exp(\tilde{\sigma}x_j - \frac{x_j^2}{2})}{\int \exp(\tilde{\sigma}x_j - \frac{x_j^2}{2}) dx_j} \, dx_j \\ &= \exp(2c_1) \, \varsigma^2. \end{aligned}$$

The lower bound of (3.52) is deduced by the same type of argument, namely

$$\begin{aligned}
\varsigma_j^2 &\geq \exp(-2c_1) \int (x_j - m_j)^2 \, \frac{\exp(\tilde{\sigma} x_j - \frac{x_j^2}{2})}{\int \exp(\tilde{\sigma} x_j - \frac{x_j^2}{2}) dx_j} \, dx_j \\
&\geq \exp(-2c_1) \int \left(x_j - \frac{\int x_j \exp(\tilde{\sigma} x_j - \frac{x_j^2}{2}) dx_j}{\int \exp(\tilde{\sigma} x_j - \frac{x_j^2}{2}) dx_j} \right)^2 \frac{\exp(\tilde{\sigma} x_j - \frac{x_j^2}{2})}{\int \exp(\tilde{\sigma} x_j - \frac{x_j^2}{2}) dx_j} \, dx_j \\
&= \exp(-2c_1) \, \varsigma^2.
\end{aligned}$$

□

3.2.1 Proof of the local Cramér theorem

As in Section 2.2, the main tool for the proof of Theorem 3.2.2 is a local central limit type result for the density $\tilde{g}_{K,m}$. Even if we use some auxiliary results of Section 2.2, we cannot apply the local central limit result of Theorem 2.2.1 because it is only formulated for the case of homogeneous single-site potentials $\psi_j = \psi$, $j \in \{1, \ldots, K\}$. In another aspect, the setting of this section is not as complex as the setting of Section 2.2, because we have the uniform control (3.52) on the standard deviation ς_j. Therefore, we can apply a simpler argument than the one of Theorem 2.2.1. Because the proceeding is more or less standard, some elements of the proof may also be found in [17, Chapter XVI], [35, Appendix 2], [26, Section 3], [38, p. 752 and Section 5] and [22, Appendix: Local Cramér theorem].

Convention. *For the rest of Section 3.2.1, we assume that the index j is given by some number $j \in \{1, \ldots, K\}$. Additionally, we introduce the notation*

$$\langle f \rangle_j := \int f(x_j) \mu_j^{\sigma}(dx_j).$$

The definition of $\tilde{g}_{K,m}$ suggests to introduce for the shifted variables

$$\tilde{x}_j := x_j - m_j,$$

which yields that the mean of $\tilde{x}_j$ is normalized i.e. $\langle \tilde{x}_j \rangle_j = 0$. The following auxiliary lemma provides tools needed for the proof of Theorem 3.2.2.

Lemma 3.2.5. *There is a constant $0 < C < \infty$ such that the following statements are true:*

(i) For any $k \in \{1, \ldots, 5\}$ and j it holds: $\left\langle |\tilde{x}_j|^k \right\rangle_j \leq C.$

(ii) For any $\xi \in \mathbb{R}$ and j it holds: $|\langle \exp(i\tilde{x}_j \xi) \rangle_j| \leq C|\xi|^{-1}.$

(iii) For any $\delta > 0$ there is $\lambda < 1$ such that for all σ, $|\xi| \geq \delta$, and j it holds:

$$\left| \langle \exp(i\tilde{x}_j \xi) \rangle_j \right| \leq \lambda.$$

(iv) For any $\delta > 0$ there is $0 < C_\delta < \infty$ such that for all σ, $|\xi| \geq \delta$, and j it holds:

$$\left| \langle \exp\left(i\tilde{x}_j \xi\right)\rangle_j \right| \leq C_\delta \, \frac{1}{1+|\xi|}.$$

(v) For any j it holds:

$$\left| \frac{d}{dm} \, \langle \exp(i\tilde{x}_j\xi)\rangle_j \right| \leq C \, (1+|\xi|) \, |\xi|^3,$$
$$\left| \frac{d^2}{dm^2} \, \langle \exp(i\tilde{x}_j\xi)\rangle_j \right| \leq C \, \left(1+|\xi|^2\right) \, |\xi|^3.$$

(vi) There exists a complex-valued function $h_j(\xi)$ such that for $|\xi| \ll 1$:

$$\langle \exp(i\tilde{x}_j\xi)\rangle_j = \exp(-h_j(\xi)) \qquad \textit{with} \qquad \left| h_j(\xi) - \frac{1}{2}\, \varsigma_j^2 \xi^2 \right| \lesssim |\xi|^3.$$

The proof of the last lemma is straight forward using the auxiliary results of Section 2.2 and the uniform bound (3.52).

Proof of Lemma 3.2.5. The statements (i) and (ii) follow from a combination of the uniform bound (3.52) and Lemma 2.2.2.

The statement (iii) follows from an application of Lemma 2.2.4 and the observation that the constant λ only depends on the upper bound of the statements (i) and (ii), which is uniform in j.

The statement (iv) follows directly from a combination of (ii) and (iii).

Now, let us deduce the statement (v). We need the fact that by (3.55) we have

$$\frac{d}{d\sigma}\, m = \frac{1}{K}\sum_{j=1}^{K} \frac{d}{d\sigma}\, m_j \overset{(2.84)}{=} \frac{1}{K}\sum_{j=1}^{K} \varsigma_j^2 \overset{(3.52)}{\leq} c \tag{3.56}$$

and

$$\frac{d^2}{d\sigma^2}\, m = \frac{1}{K}\sum_{j=1}^{K} \frac{d}{d\sigma}\, \varsigma_j^2 \overset{(2.85)}{\leq} \frac{1}{K}\sum_{j=1}^{K} \langle |\tilde{x}_j|^3 \rangle_j \overset{(i)}{\leq} C. \tag{3.57}$$

We fix the index j. Then an application of Lemma 2.2.5 yields (observing $\hat{x} = \frac{x-m_j}{\varsigma_j} = \frac{\tilde{x}}{\varsigma_j}$ and $\hat{\xi} = \varsigma_j \xi$)

$$\begin{aligned}
\left| \frac{d}{dm} \, \langle \exp(i\tilde{x}_j\xi)\rangle_j \right| &= \left| \frac{d}{d\sigma} \, \langle \exp(i\tilde{x}_j\xi)\rangle_j \right| \left| \frac{d}{d\sigma}\, m \right| \\
&\overset{(3.56)}{\leq} \varsigma_j \, c \, (1+|\varsigma_j\xi|) \, (|\varsigma_j\xi|)^2 \\
&\overset{(3.52)}{\leq} c^4 \max(1,c) \, (1+|\xi|) \, (|\xi|)^2.
\end{aligned}$$

We turn to the second statement of (v). A direct calculation reveals

$$\begin{aligned}
\frac{d^2}{dm^2}\ \langle\exp(i\tilde{x}_j\xi)\rangle_j &= \frac{d}{dm}\left(\frac{d}{d\sigma}\ \langle\exp(i\tilde{x}_j\xi)\rangle_j\ \ \frac{d}{d\sigma}\ m\right)\\
&= \frac{d}{d\sigma}\left(\frac{1}{\varsigma_j}\frac{d}{d\sigma}\ \langle\exp(i\tilde{x}_j\xi)\rangle_j\ \ \varsigma_j\ \frac{d}{d\sigma}\ m\right)\ \frac{d}{d\sigma}\ m\\
&= \varsigma_j^2\left[\left(\frac{1}{\varsigma_j}\ \frac{d}{d\sigma}\right)\ \langle\exp(i\tilde{x}_j\xi)\rangle_j\right]\ \left(\frac{d}{d\sigma}\ m\right)^2\\
&\quad+\left(\frac{1}{\varsigma_j}\frac{d}{d\sigma}\ \langle\exp(i\tilde{x}_j\xi)\rangle_j\right)\ \frac{d}{d\sigma}\varsigma_j\ \left(\frac{d}{d\sigma}\ m\right)^2\\
&\quad+\left(\frac{1}{\varsigma_j}\frac{d}{d\sigma}\ \langle\exp(i\tilde{x}_j\xi)\rangle_j\right)\ \varsigma_j\ \frac{d^2}{d\sigma^2}\ m\ \ \frac{d}{d\sigma}\ m.
\end{aligned}$$

Now, the desired estimate can be achieved by an application of Lemma 2.2.5, (3.56), (3.57), and some basic estimates.

Finally, let us deduce the statement (vi). We fix the index j. Recalling that $\hat{x} = \frac{x-m_j}{\varsigma_j} = \frac{\tilde{x}}{\varsigma_j}$ and $\hat{\xi} = \varsigma_j\xi$, the statement follows from the uniform bound (3.52) and the observation (2.44). □

Proof of Theorem 3.2.2. We start with deducing the strict convexity of φ_K for any K. With the same argument as for (2.37) we get

$$\frac{d^2}{dm^2}\varphi_K(m) = \left(\frac{d}{d\sigma}\ m\right)^{-1},$$

which yields the desired statement by using the estimate (3.56).

Now, let us consider the convergence of $\|\varphi_K(m) - \psi_K(m)\|_{C^2}$. Because the random variables $\tilde{X}_j := X_j - m_j$ of Lemma 3.2.3 are independent, it follows by the same argument as for (2.42) that

$$2\pi\,\tilde{g}_{K,m}(0) \ = \ \int\prod_{j=1}^{K}\langle\exp(i\tilde{x}_j\frac{1}{\sqrt{K}}\xi)\rangle_j d\xi, \tag{3.58}$$

where $\tilde{g}_{K,m}(\xi)$ denotes the Lebesgue density of the distribution of the sum $\frac{1}{\sqrt{K}}\sum_{j=1}^{K}\tilde{X}_j$. Assume that the following estimates hold uniformly in K and m:

$$\left|\int\prod_{j=1}^{K}\langle\exp(i\tilde{x}_j\frac{1}{\sqrt{K}}\xi)\rangle_j d\xi\right| \sim 1, \tag{3.59}$$

$$\left|\frac{d}{dm}\int\prod_{j=1}^{K}\langle\exp(i\tilde{x}_j\frac{1}{\sqrt{K}}\xi)\rangle_j d\xi\right| \lesssim 1, \tag{3.60}$$

$$\left|\frac{d^2}{dm^2}\int\prod_{j=1}^{K}\langle\exp(i\tilde{x}_j\frac{1}{\sqrt{K}}\xi)\rangle_j d\xi\right| \lesssim 1. \tag{3.61}$$

Then a combination of the formula (3.58) and Cramér's representation (3.51) yields the desired result

$$\|\psi_K(m) - \varphi_K(m)\|_{C^2} \to 0 \qquad \text{as } K \to \infty.$$

It remains to establish the estimates from above. Note that the intermediate estimate (3.60) follows from the estimates (3.59) and (3.61) by interpolation. Using the tools of Lemma 3.2.5, we can deduce (3.59) and (3.61) with the same strategy as in the proof of Theorem 2.2.1.

Argument for (3.59): We start with deducing the upper bound

$$\left| \int \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi) \rangle_j d\xi \right| \lesssim 1. \tag{3.62}$$

For some fixed $0 < \delta \ll 1$ we split the integral according to

$$\begin{aligned} \int \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi) \rangle_j d\xi = & \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta\right\}} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi) \rangle_j d\xi \\ & + \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \geq \delta\right\}} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi) \rangle_j d\xi. \end{aligned}$$

Let us consider the inner integral. We can choose δ is so small that the statement (vi) of Lemma 3.2.5 applies. Hence, we may rewrite the inner integral as

$$I := \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta\right\}} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi) \rangle_j d\xi = \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta\right\}} \exp\left(-\sum_{j=1}^{K} h_j \left(\frac{1}{\sqrt{K}}\xi \right) \right) d\xi.$$

Note that for $\left|\frac{1}{\sqrt{K}}\xi\right| \leq \delta$ the statement (vi) of Lemma 3.2.5 yields

$$\left| \sum_{j=1}^{K} h_j \left(\frac{1}{\sqrt{K}}\ \xi \right) - \sum_{j=1}^{K} \frac{\varsigma_j^2}{2K}\xi^2 \right| \lesssim \frac{1}{\sqrt{K}}\ |\xi|^3. \tag{3.63}$$

In particular for δ small enough this implies by using the assumption (3.52)

$$\operatorname{Re}\left(\sum_{j=1}^{K} h_j \left(\frac{1}{\sqrt{K}}\ \xi \right) \right) \geq \frac{1}{4}\ \sum_{j=1}^{K} \frac{\varsigma_j^2}{K}\xi^2 \geq \frac{1}{4c^2}\xi^2, \tag{3.64}$$

where the constant $0 \leq c < \infty$ is given by (3.52). The last statement yields the estimate

$$\begin{aligned} |I| &\leq \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta\right\}} \left| \exp\left(-\sum_{j=1}^{K} h_j \left(\frac{1}{\sqrt{K}}\xi \right) \right) \right| d\xi \\ &\leq \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta\right\}} \exp\left(-\frac{1}{4c^2}\xi^2 \right) d\xi \lesssim 1. \end{aligned}$$

Now, let us consider the outer integral

$$II := \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\geq\delta\right\}} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi.$$

On the integrand we apply the statement (iii) of Lemma 3.2.5 (on $K-2$ of the K factors) and the statement (iv) of Lemma 3.2.5 (on the remaining 2 factors):

$$\begin{aligned}\left|\prod_{j=1}^{K} \langle \exp\left(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi\right)\rangle_j\right| &\lesssim \lambda^{K-2}\left(\frac{1}{1+\frac{1}{\sqrt{K}}|\xi|}\right)^2 \\ &\lesssim K\ \lambda^{K-2}\ \frac{1}{K+\xi^2} \lesssim K\ \lambda^{K-2}\ \frac{1}{1+\xi^2}.\end{aligned}$$

It follows that the second term II is exponentially small:

$$\begin{aligned}|II| = \left|\int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\geq\delta\right\}} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi\right| &\lesssim K\ \lambda^{K-2} \int \frac{1}{1+\xi^2} d\xi \\ &\lesssim K\ \lambda^{K-2} \to 0 \qquad \text{as } K\to\infty.\end{aligned}$$

Together with the estimate of $|I|$ from above, this yields the desired upper bound (3.62).

We turn to the lower bound

$$\left|\int \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi\right| = |I+II| \gtrsim 1.$$

Applying the triangle inequality yields

$$|I+II| \gtrsim |I| - |II|.$$

Because $|II|\to 0$ as $K\to\infty$ it suffices to show

$$|I| \gtrsim 1.$$

Recall that for $\left|\frac{1}{\sqrt{K}}\ \xi\right| \leq \delta$ we have (cf. (3.64))

$$\operatorname{Re}\left(\sum_{j=1}^{K} h_j\left(\frac{1}{\sqrt{K}}\ \xi\right)\right) \geq \frac{1}{4c^2}\xi^2.$$

Note that the function $\mathbb{C} \ni y \mapsto \exp(y) \in \mathbb{C}$ is Lipschitz continuous on $\operatorname{Re} y \leq -\frac{1}{4c^2}\xi^2$ with constant $\exp(-\frac{1}{4c^2}\xi^2)$. Therefore (3.63) yields the estimate

$$\left|\exp\left(-\sum_{j=1}^{K} h_j\left(\frac{1}{\sqrt{K}}\ \xi\right)\right) - \exp\left(-\sum_{j=1}^{K}\frac{\varsigma_j^2}{2K}\xi^2\right)\right| \lesssim \frac{1}{\sqrt{K}}\ |\xi|^3\ \exp\left(-\frac{1}{4c^2}\xi^2\right).$$

The last estimate implies

$$\left| I - \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\leq\delta\right\}} \exp\left(-\sum_{j=1}^{K} \frac{\varsigma_j^2}{2K}\xi^2\right) d\xi \right| \lesssim \frac{1}{\sqrt{K}} \int |\xi|^3 \exp\left(-\frac{1}{4c^2}\xi^2\right) d\xi \to 0$$

as $K \to \infty$. Additionally, we observe that by the assumption (3.52)

$$III := \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\leq\delta\right\}} \exp\left(-\sum_{j=1}^{K} \frac{\varsigma_j^2}{2K}\xi^2\right) d\xi \gtrsim \int_{\{|\xi|\leq\delta\}} \exp\left(-\frac{c}{2}\xi^2\right) d\xi \gtrsim 1.$$

Hence, we may conclude that

$$|I| = |I - III + III| \geq |III| - |I - III| \gtrsim 1$$

for $K \gg 1$ large enough.

Argument for (3.61): We split the integral according to

$$\begin{aligned} \frac{d^2}{dm^2} \int \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi &= \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\leq\delta\right\}} \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi \\ &\quad + \int_{\left\{\left|\frac{1}{\sqrt{K}}\ \xi\right|\geq\delta\right\}} \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}}\xi)\rangle_j d\xi \\ &=: IV + V. \end{aligned}$$

Let us consider the inner integral IV. An application of the chain rule for differentiation yields

$$\frac{d}{dm} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j\xi)\rangle_j = \sum_{j=1}^{K} \frac{d}{dm} \langle \exp(i\tilde{x}_j\xi)\rangle_j \prod_{\substack{k\in\{1,\ldots,K\},\\ k\neq j}} \langle \exp(i\tilde{x}_k\xi)\rangle_k.$$

A second differentiation yields

$$\begin{aligned} \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j\xi)\rangle_j = \sum_{j=1}^{K} \Bigg[&\frac{d^2}{dm^2} \langle \exp(i\tilde{x}_j\xi)\rangle_j \prod_{\substack{k\in\{1,\ldots,K\},\\ k\neq j}} \langle \exp(i\tilde{x}_k\xi)\rangle_k \\ + \frac{d}{dm} \langle \exp(i\tilde{x}_j\xi)\rangle_j &\sum_{\substack{n\in\{1,\ldots K\},\\ n\neq j}}^{K} \frac{d}{dm} \langle \exp(i\tilde{x}_n\xi)\rangle_n \prod_{\substack{l\in\{1,\ldots,K\},\\ l\neq j,\ l\neq n}} \langle \exp(i\tilde{x}_l\xi)\rangle_l \Bigg]. \end{aligned} \tag{3.65}$$

The same argument as for (3.64) yields that for $\left|\frac{1}{\sqrt{K}}\xi\right| \leq \delta$ with δ small enough

$$\begin{aligned} \left| \prod_{\substack{l\in\{1,\ldots,K\},\\ l\neq j,\ l\neq n}} \langle \exp(i\tilde{x}_l \frac{1}{\sqrt{K}}\xi)\rangle_l \right| &= \left| \exp\left(\sum_{\substack{l\in\{1,\ldots,K\},\\ l\neq j,\ l\neq n}} h_j\left(\frac{1}{\sqrt{K}}\xi\right)\right)\right| \\ &\leq \exp\left(-\frac{1}{4c^2}\xi^2\right). \end{aligned} \tag{3.66}$$

Hence, a combination of the identity (3.65), the estimate (3.66), and the estimates of Lemma 3.2.5 (v) yields

$$\left| \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \frac{1}{\sqrt{K}} \xi) \rangle_j \right|$$
$$\leq \left[\frac{1}{\sqrt{K}} \left(1 + \frac{|\xi|^2}{K} \right) |\xi|^3 + \frac{1}{K} \left(1 + \frac{|\xi|^2}{K} \right) |\xi|^6 \right] \exp\left(-\frac{1}{4c^2} \xi^2 \right).$$

The desired estimate directly follows from the last estimate, i.e.

$$|IV| \lesssim \int_{\left\{ \left| \frac{1}{\sqrt{K}} \xi \right| \leq \delta \right\}} \left(1 + |\xi|^2 \right) \left(|\xi|^3 + |\xi|^6 \right) \exp\left(-\frac{1}{4c} \xi^2 \right) d\xi \lesssim 1.$$

Now, we turn to the outer integral V. By substitution we have

$$V = \sqrt{K} \int_{\{|\xi| \geq \delta\}} \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \xi) \rangle_j d\xi.$$

On the identity (3.65), we apply the estimates of Lemma 3.2.5 (v) in a first step and $|\langle \exp(i\tilde{x}_j \xi) \rangle_j| \leq 1$ in a second step:

$$\left| \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \xi) \rangle_j \right|$$
$$\lesssim \sum_{j=1}^{K} \left[(1 + |\xi|^2)\, |\xi|^3 \prod_{\substack{k \in \{1,\ldots,K\}, \\ k \neq j}} |\langle \exp(i\tilde{x}_k \xi) \rangle_k| \right.$$
$$\left. + (1 + |\xi|^2)\, |\xi|^6 \sum_{\substack{n \in \{1,\ldots,K\}, \\ n \neq j}} \prod_{\substack{l \in \{1,\ldots,K\}, \\ l \neq j,\, l \neq n}} |\langle \exp(i\tilde{x}_l \xi) \rangle_l| \right]$$
$$\lesssim (1 + |\xi|^8) \sum_{j \in \{1,\ldots,K\}} \sum_{\substack{n \in \{1,\ldots,K\}, \\ n \neq j}} \prod_{\substack{l \in \{1,\ldots,K\}, \\ l \neq j,\, l \neq n}} |\langle \exp(i\tilde{x}_l \xi) \rangle_l|.$$

We use Lemma 3.2.5 (iii) (on $K - 12$ of the $K - 2$ factors $|\langle \exp(i\tilde{x}_l \xi) \rangle_l|$) and Lemma 3.2.5 (iv) (on the remaining 10 factors $|\langle \exp(i\tilde{x}_l \xi) \rangle_l|$):

$$\left| \frac{d^2}{dm^2} \prod_{j=1}^{K} \langle \exp(i\tilde{x}_j \xi) \rangle_j \right| \lesssim K^2\, (1 + |\xi|^8)\, \lambda^{K-12} \left(\frac{1}{1 + |\xi|} \right)^{10}$$
$$\lesssim K^2\, \lambda^{K-12} \frac{1}{1 + |\xi|^2}.$$

Hence, we see that the term $|V|$ is exponentially small i.e.

$$|V| \lesssim \sqrt{K}\, K^2\, \lambda^{K-12} \int \frac{1}{1+|\xi|^2}\, d\xi \to 0 \qquad \text{as } K \to \infty.$$

Together with the estimate for $|IV|$ from above, the latter yields (3.61). □

Conventions

In addition to standard notation we use the following conventions:

- $a \lesssim b$ means that there is a uniform constant $C > 0$ such that $a \leq Cb$,
  $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$.
- $\bar{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$ erases the i-th entry of $x = (x_1, \ldots, x_N)$.
- $\operatorname{osc} f = \sup_x f(x) - \inf_x f(x)$ is the oscillation of f.
- $\frac{d}{dx_i} f$ stands for the partial derivative of f w.r.t. the variable x_i.
- $\langle \cdot, \cdot \rangle$ denotes the scalar product, $| \cdot |$ denotes the norm, ∇ denotes the gradient, and Hess denotes the Hessian of a Euclidean space X. If nothing else is written, the standard Euclidean structure is considered on $\mathbb{R}^N$ i.e. $x \cdot y = \langle x, y \rangle = \sum_{i=1}^N x_i y_i$.
- $\int f(x)dx$ denotes the integration of f w.r.t. the Lebesgue measure in the according dimension.
- $\mathcal{H}^K$ denotes the K-dimensional Hausdorff measure,
  $\mathcal{H}^K_{\lfloor A}(dx)$ denotes the K-dimensional Hausdorff measure restricted to the set A.
- $\mathcal{P}(X)$ denotes the space of probability measures on a Euclidean space X.
- Z denotes a generic normalization constant of a probability measure. Its value may change from line to line or even within a line. For example, if $\mu(dx) = \frac{1}{Z} \exp(-H(x))\, dx$, then $Z = \int \exp(-H(x))dx$.
- We do not distinguish between the measure $\mu(dx)$ and its Lebesgue density $\mu(x)$.
- $f\mu$ denotes the measure given by the density $f(x)\mu(dx)$.
- $\operatorname{cov}_\mu(f, g) = \int \left(f - \int f d\mu\right) \left(g - \int g d\mu\right) d\mu$ denotes the covariance of f and g,
  $\operatorname{var}_\mu(f) = \operatorname{cov}_\mu(f, f)$ denotes the variance of f w.r.t. the probability measure μ.
- $\operatorname{Ent}(f\mu, \mu) = \int f \log f d\mu - \int f d\mu \ \log \int f d\mu$ coincides with the relative entropy of $f\mu$ w.r.t. μ provided $\int f d\mu = 1$.

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