

Holomorphic Couplings In Non-Perturbative String Compactifications

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Denis Marco Klevers
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Referent: Prof. Dr. Albrecht Klemm
Korreferent: Prof. Dr. Hans-Peter Nilles
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Abstract

In this thesis we present an analysis of several aspects of four-dimensional, non-perturbative $\mathcal{N} = 1$ compactifications of string theory. Our focus is on the study of brane dynamics and their effective physics as encoded in the holomorphic couplings of the low-energy $\mathcal{N} = 1$ effective action, most prominently the superpotential W . The thesis is divided into three parts. In part one we derive the effective action of a spacetime-filling D5-brane in generic Type IIB Calabi-Yau orientifold compactifications. In the second part we invoke tools from string dualities, namely from F-theory, heterotic/F-theory duality and mirror symmetry, for a more elaborate study of the dynamics of (p, q) 7-branes and heterotic five-branes. In this context we demonstrate exact computations of the complete perturbative effective superpotential, both due to branes and background fluxes. Finally, in the third part we present a novel geometric description of five-branes in Type IIB and heterotic M-theory Calabi-Yau compactifications via a non-Calabi-Yau threefold \hat{Z}_3 , that is canonically constructed from the original five-brane and Calabi-Yau threefold Z_3 via a blow-up. We exploit the use of the blow-up threefold \hat{Z}_3 as a tool to derive open-closed Picard-Fuchs differential equations, that govern the complete effective brane and flux superpotential. In addition, we present first evidence to interpret \hat{Z}_3 as a flux compactification dual to the original five-brane by defining an $SU(3)$ -structure on \hat{Z}_3 , that is generated dynamically by the five-brane backreaction.

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“Worte können all dem nicht gleichkommen, was ich euch verdanke!”

To my mother Monika

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Chapter 1

Introduction

It is “The Unreasonable Effectiveness of Mathematics in the Natural Sciences”, to say it in the words of Wigner’s famous article [1], that explains the tremendous success of the current mathematical description of the laws of nature in today’s theories. It expresses the astonishing and countless verified observation that these laws can be consistently mapped to the clear and elegant structures of mathematics. This in particular applies to theoretical physics, founded by the works of Galileo and Newton. Contemporary theoretical physics nowadays provides an extremely accurate description of almost all processes in nature and has led to countless predictions that already have been verified or will hopefully be verified in the (near) future. This huge success of the synthesis of physics and mathematics has led to the rise of a new branch in science, denoted mathematical physics. It appreciates the strengths of both fields and exploits the synergies of their combined study. One especially interesting topic in mathematical physics is String Theory, which is at the final frontier of our understanding of fundamental physics, aiming at the unification of all matter and forces in a single theory - a theory of everything. Some aspects of this enormously rich theory are the content of this work, which addresses naturally both physical but mathematical questions.

1.1 String Theory and its Implications

The foundation of contemporary physics consists of two fundamental theories, Quantum Theory, governing the physics on subatomic scales, and General Relativity, that determines the large scale structure of our universe.

Quantum Theory is applied and verified in almost all areas of physics and is widely accepted without any doubt. Its most advanced formulation is relativistic quantum field theory being a combination of special relativity, classical field and gauge theory as well as quantum mechanics. It provides the appropriate framework to analyze interactions at high energies and has led to the formulation of a theory of what is currently known as elementary particles and their interactions, the Standard Model (SM) of elementary particle physics. The SM describes all dynamics of the fundamental particles, the six leptons and six quarks, that are governed by strong, weak and electro-magnetic interactions, mathematically encoded by the SM gauge group $SU(3) \times SU(2) \times U_Y(1)$. The origin of particle masses is explained by the

Higgs mechanism exploiting vacuum degeneracy and broken gauge symmetry which implies the existence of the Higgs particle. Except for the Higgs particle the SM has been tested and verified to high precision in modern experiments, with the Large Hadron Collider in Geneva currently at the frontier, which is also designed to discover the Higgs within the next years. However, the SM suffers from several experimental as well as conceptual drawbacks that to our current knowledge seem to rule it out as a fundamental theory of nature. First of all it is an experimental and conceptual drawback that the SM does not explain the existence of dark matter, which is required to explain the rotational curves of galaxies. Since dark matter has not been directly observed it has to consist of weakly interacting particles, that might not be covered by the SM particle content. In addition it is conceptually unsatisfying that the SM contains a number of 19 free parameters that are not predicted by the theory itself and have to be fixed by experiment. Furthermore the SM suffers from a hierarchy problem addressing the question of its naturalness, namely why the electroweak breaking scale $M_{\text{weak}} = 246\text{GeV}$ is so much smaller than the cutoff-scale Λ of the quantum field theory. This is a particularly crucial problem if one assumes that no new physics happens up to the Planck scale M_{Pl} where gravitational interactions, which are not covered in the SM, become relevant and invalidate the SM description and prediction. Then $\Lambda = M_{\text{Pl}}$ and since for example the corrections to the Higgs mass at one loop are proportional to Λ^2 , an enormous fine-tuning would be necessary in order to reproduce the theoretically favored Higgs mass around about 120 GeV.

A compelling solution to some of these problems is supersymmetry at the TeV scale. By the introduction of superpartners to every SM-particle, dark matter candidates can be provided and the quadratic divergence in the Higgs mass is traded by a logarithmic one. In addition the cut-off Λ is then at the TeV-scale making a fine-tuning obsolete. For more conceptual reasons a supersymmetric extension of the SM, the simplest one being the Minimal Supersymmetric Standard Model (MSSM), is furthermore favored since it supports the idea of a Grand Unified Theory (GUT) that unifies all SM-interactions into a bigger, more fundamental gauge theory¹. However, even the MSSM suffers from theoretical problems like a small hierarchy or μ -problem of the Higgs mass parameter μ and an even bigger number of free parameters. More conceptually, also the MSSM is a quantum field theory which does not include gravity and thus can not be, even if one accepts the concept of renormalization in a microscopic theory, a fundamental and complete theory.

In this light it is an even more unfortunate fact, that canonically quantized Einstein gravity is non-renormalizable² and thus fails to be a predictive quantum field theory by the requirement of introducing infinitely many free parameters to absorb all divergences. One way out is to view the MSSM as well as gravity as an effective theory that is valid up to a scale Λ . This cuts off all momenta above Λ and thus trivially regularizes all quantum loops

¹In order to avoid the singularity of α due to the Landau-pole of QED the idea to replace the SM gauge theory by an asymptotically free GUT gauge theory at a high scale $\sim 10^{15}\text{GeV}$ is also favorable.

²We note the asymptotic safety program which pursues the formulation of Einstein gravity as a valid quantum theory with a UV fixed point.

restoring the predictive power of the quantum field theory. This, however, is equivalent to the statement that the MSSM and Einstein gravity are not valid above the cut-off scale and that new physics applies that is not yet discovered. Another argument for this conclusion can be provided within classical General Relativity (GR). On the one hand also classical GR is at least as predictive as the SM. It has strong experimental verifications mainly in an astrophysical and cosmological context like its early checks by light deflection, time dilatation, gravitational lensing, gravitational red shifting, its more sophisticated implication of Hubble's expanding universe and the Big Bang Theory. On the other hand, however, also GR seems to be incomplete as a fundamental theory in the sense that it predicts its own breakdown due to spacetime singularities. Starting with a massive object of radius smaller than its Schwarzschild radius, a gravitational collapse occurs yielding a black hole as the final state of the evolution. This is a formal spacetime singularity that exceeds the description in terms of classical differential geometry. Also physically this poses a severe problem that is known as the information loss paradox. Since the interior of the black hole is screened from an exterior observer by a horizon, all information about its initial state and the matter falling into the black hole is lost. Furthermore the black hole is known, following the semi-classical description of Bekenstein and Hawking, to be a thermodynamical object emitting thermal radiation, denoted Hawking radiation. Consequently the formation of a black hole violates unitarity as it describes the evolution of a pure state into a mixed state which yields a further tension between GR and quantum mechanics. This implies the necessity to obtain a more fundamental and microscopic understanding of black holes. It requires, even from the point of view of pure gravity, the introduction of quantum gravity in which the evolution of an initial state, for example a star, into a black hole state is described in a unitary theory which allows to determine the statistical origin of black hole thermodynamics. Together with the drawbacks of our description of particle physics this inevitably requires the invention of novel physical and mathematical ideas and concepts.

In summary we have collected known arguments for the observation that the SM as a quantum field theory and GR both as a classical geometrical theory and as a quantum theory can not be fundamental. They have to be replaced at a fundamental high scale Λ , which is at most the Planck scale M_{Pl} , by a microscopic, fundamental theory of nature that UV-completes their incomplete physical description. The basic requirement of this theory is of course to contain, in a well-defined limit, both theories at low-energies/large lengths compared to Λ .

It is the ultimate goal of String Theory to overcome all these obstacles of nowadays theories in one unique mathematical formulation of nature that is valid on the most fundamental scales [2–5]. The basic idea of String Theory to simultaneously avoid the inconsistencies both due to infinities in quantum loops and due to a lack of a quantum gravity description is comparatively simple. String Theory just introduces a new, fundamental length scale $\sqrt{\alpha'}$ in a way that the concepts of Quantum Theory and general covariance are maintained. This is achieved by replacing the common point-particle description of elementary particles by a description via

one-dimensional strings of characteristic size $\sqrt{\alpha'}$. This in a certain way minimal length scale smooths the interaction vertices in quantum field theory as well as the black hole and big bang singularities in spacetime in a general covariant way. Thus both infinities in quantum loops and in spacetime itself are naturally regularized. Stated differently String Theory introduces a new paradigm: Not the spacetime physics is at the core of any physical question but the physics on the two-dimensional world-sheet that the string sweeps out in spacetime. Indeed, as long as this microscopic description of String Theory makes sense, all emerging spacetime physics is consistent as well, even if the spacetime contains singularities. More conceptually String Theory provides a novel way to probe geometries, that fundamentally differs from the way point-particles probe geometries. This in particular alters the concepts of classical differential and algebraic geometry to what is usually referred to as quantum or stringy geometry. Consistency of microscopic quantum strings and their interactions then naturally leads to a unified description of gauge fields *and* gravity. This happens in the sense, that whenever a massless gauge field is present, which is an element in the Fock space of quantized open strings or dual closed heterotic strings, also a massless spin two particle or graviton has to be present as well, which is always a closed string state³. Moreover, consistency of the quantum string, which means vanishing Weyl anomaly of the underlying two-dimensional conformal field theory, implies the spacetime physics of gauge theory and general relativity. However, the minimal Maxwell and Einstein equations are reproduced only in the point-particle limit $\alpha' \rightarrow 0$ of the full string equations of motion. Higher α' -corrections indicate hints how String Theory changes both conventional gauge theory and gravity that allows for a reconciliation of them at quantum level. Furthermore, consistency at one-loop, in more detail modular invariance, or equivalently the absence of spacetime tachyons requires spacetime supersymmetry. A milestone of the quantum gravity description via String Theory for example is the explanation of the Bekenstein-Hawking entropy of specific supersymmetric black holes [6]. This all is a first demonstration of the appealing feature of String Theory, namely its strong predictive power, which has almost all of the mentioned concepts of contemporary physics as a consequence of consistency, not an ambiguous choice.

However, it happens that String Theory is not quite as predictive as it seems at first glance due to the choice of a compactification geometry. The string equations of motion require a ten-dimensional spacetime, at least in the formulation of what is called critical string theory. In order to make contact with the real, four-dimensional world requires to “hide” the additional six dimensions in a certain way. One very prominent and old idea first proposed by Kaluza and Klein is the concept of compactification⁴. Here the unwanted extra dimensions are hidden in a tiny, compact six-dimensional “internal” manifold of size scale ℓ_c , which has to be chosen sufficiently small in order to be consistent with today’s experiments.

³Here we omit more sophisticated versions of quantum gravity via the description in terms of quantum gauge theories in the context of AdS/CFT duality.

⁴An alternative to compactification is given by realizing our four-dimensional spacetime as a slice in a higher dimensional space [7], so called brane-world scenarios. This idea can be further used to solve the hierarchy problem [8] which is naturally realized in string theory on the deformed conifold [9].

Indeed such a spacetime configuration can arise as a vacuum state of String Theory and the allowed compactification geometries maintaining four-dimensional supersymmetry, to address problems like the hierarchy problem, have been determined [10] to be Calabi-Yau manifolds [2, 11]. In this course the heavy String and Kaluza-Klein modes of masses of order $1/\ell_s$, $1/\ell_c$ can be consistently integrated out yielding a four-dimensional Wilsonian effective action which is then viewed as the Lagrangian from that one builds up a four-dimensional effective quantum field theory. Furthermore, in this dimensional reduction process from ten to four dimensions the four-dimensional physics is encoded by key geometrical quantities of the internal Calabi-Yau manifold, which is denoted as geometrization of physics. The laws of classical geometry that mathematically encode these quantities have however in general to be replaced by their stringy geometry counterparts, like in the context of mirror symmetry discussed below, and are often constraint from strong string theoretic consistency conditions like for example tadpoles.

However, this complete geometrization of four-dimensional physics has also one immediate and very serious drawback. In general a topologically fixed internal compactification geometry can undergo shape and size deformations without a cost of energy. This reflects in the effective four-dimensional theory by the existence of massless scalar fields, so called moduli [2, 12, 13], whose VEV determines the shape and size of the compactification geometry. Since no massless scalar field has yet been observed in nature and since the same fields also determine the four-dimensional couplings like the fine-structure constant, that would then be allowed to slowly vary - another contradiction with experiment, in particular cosmology - the moduli have to be fixed by a potential, generated for example by fluxes as noted first in [14] and thoroughly reviewed in [15–17]. It is thus one crucial task of every attempt for predictive string model building to give a mechanism that dynamically generates such a potential. Once this conceptual problem is solved, one can try to search a four-dimensional string vacuum reproducing all aspects of the MSSM, for example. However, although remarkably good models are known a single model meeting all criteria to be accepted as a valid theory of particle physics has not been found yet. Luckily there is, at least to our current understanding of string dynamics, a vast landscape [18, 19] of string vacua⁵. It is the goal of String Phenomenology to single out the most promising string vacua to reproduce the observable particles and interaction. Prominent corners in the landscape for performing explicit searches are the heterotic string, both on smooth Calabi-Yau manifolds as pioneered in [10, 22], see e.g. for more elaborate SM models [23], and on orbifolds starting with [24, 25], cf. [26] for a review, Type II setups with D-branes [16, 17, 27, 28] and most recently also F-theory compactifications [29–31]. It is furthermore expected that a more elaborate understanding of string dynamics, in particular in a non-perturbative formulation of string theory by M-theory or using string dualities, sheds further light on the phenomenologically appealing corners in the string landscape or even dynamically selects MSSM string models, probably when non-perturbative effects are taken into

⁵A first counting of flux vacua led to the prominent finite result of the order of 10^{500} string vacua [20]. The search of a realistic model out of these seems to be theoretical impossible without a better understanding of the dynamics or without an intelligent searching strategy [21].

account. Indeed there are prominent examples how non-perturbative string and field theory physics naturally reproduces aspects of four-dimensional physics like de Sitter vacua [32] and confinement [33]. In particular the inclusion of non-perturbative physics drastically affects even qualitative features of physics⁶.

Conceptually, all these approaches of string phenomenology have in common that they first appreciate the reduction of the complexity of the full string dynamics to a comparably trivial subsector of light modes and then exploit the translation of the four-dimensional physics to the structure of the internal geometry. This reduction of the complicated string dynamics giving us back a four-dimensional effective theory is one principle we follow. Concretely there are two central aims of this work. One aim is the detailed study of a wide class of compactification geometries and the second aim is to analyze the dynamics of additional fundamental objects in string theory, the dynamics of branes. Branes often have to be included in a string compactification for consistency and crucially influence string dynamics and phenomenology. The results of this analysis provide on the one hand a better conceptual understanding of the background geometries in string theory and on the other hand, however up to now only in toy models, exact potentials to make quantitative statements about moduli stabilization.

Finally we note that issues addressed in the context of studying a particular physical problem in string theory are often related to deep mathematical questions and structures. Both fields, mathematics and string theory, influence each other severely, and underlying mathematical structures often even make it possible to perform specific physical calculations, as demonstrated on the basis of selected examples in this thesis. Conversely, physical motivation can be very successful in making mathematical predictions as in enumerative geometry where specific symplectic invariants, the Gromov-Witten invariants that are related to the counting of holomorphic curves, are calculated using mirror symmetry [35]. A further example for the interplay of mathematics and physics is the question of finding supersymmetric vacua in string theory. This translates, for example, to the mathematical problem solved by Yau of finding a Ricci-flat metric, which is a solution to the closed string equations of motion, on a given manifold with vanishing first Chern class and fixed Kähler class. For supporting a supersymmetric open string sector this question corresponds to the task of finding calibrated cycles of minimal area in their homology class [36]. In this context the key physical coupling functions of effective field theories are geometrized by key geometrical quantities. The $\mathcal{N} = 1$ superpotential, for example, is related to period integrals in the closed string case or to chain integrals, more generally to the holomorphic Chern-Simons functional, in the open string case. Finally string theory sometimes even guides interesting ways to extend the notion of classical geometry for example. Directions include generalized geometry, special structure manifolds, like $SU(3)$ -structure manifolds, and mirror symmetry, that relates classical complex geometry with a quantum deformation of symplectic geometry in the closed string case or classical deformation theory and quantum deformation theory in the open string case.

⁶See also the reasoning of Dine, Seiberg that the string vacuum of our universe is likely strongly coupled [34].

1.2 D-Branes

The relevance of D-branes for a fundamental formulation of String Theory has been discovered during the second superstring revolution in 1995 starting with the seminal work of [37], see e.g. [38,39] for a review. They were noted to be dynamical objects of String Theory carrying R–R-charge that are as fundamental as the string itself. However D-branes are intrinsically non-perturbative and thus not visible as a perturbative string state, which explains their relatively late discovery. This can be directly seen by their tension $T \sim \frac{1}{g_s} = \frac{1}{g_o^2}$ identifying them as open string solitons⁷. BPS D-branes provide precisely those BPS-states that are necessary for a complete understanding of string dualities [41]. D-branes yield new particle states [41,42] in an effective theory and contribute as non-perturbative effects, so-called D-instantons, to correlation functions, see e.g. [43] for a review. Prominent examples in string dualities are the rise of non-perturbatively enhanced gauge symmetry in the Type IIA string due to massless D-branes wrapping vanishing cycles in the geometry that matches the enhanced gauge symmetry of a dual heterotic string [41] or Type I/heterotic duality [41], where the fundamental heterotic string at small coupling has to be viewed as the Type I D1-brane at strong coupling. The study of D-branes has further led to the famous AdS/CFT-correspondence [44–46] that is the first explicit realization of a holographic description of gravity in terms of a supersymmetric gauge theory. From a very conceptual point of view the discovery of D-branes even gives rise to the question, what the fundamental formulation of string theory really is - a theory of fundamental strings and branes, of branes only, like in M-theory [41,47], or a more abstract theory e.g. of matrices [48,49].

In general D-branes have often to be included in a given string compactification for consistency. Either they are required to cancel tadpoles due to orientifold planes and fluxes or they are produced dynamically via a conifold transition turning flux into D-branes and vice versa [50,51]. Most important for direct model building applications however is the case of spacetime-filling D-branes in a background yielding an $\mathcal{N} = 1$ supersymmetric effective theory in four dimensions. Then an effective action description of the D-brane dynamics can be invoked to analyze their light modes. These are described by a supersymmetric gauge theory that is localized to the D-brane world-volume [52] and further reduced to four dimensions. The most obvious case to consider are D-branes in flat space, so called brane world scenarios, where the real four-dimensional world is located on intersections of branes with the standard model gauge symmetry and particle content, see [17,27,28,53,54] for a review. Generally these setups are defined by a Calabi-Yau background geometry Z_3 with orientifold planes in order to cancel unphysical tadpoles, so called Calabi-Yau orientifolds, in which the D-branes wrap specific calibrated submanifolds to preserve supersymmetry [36]. In this context the realization of the standard model gauge group or a sensible GUT-sector with the correct spectrum of charged matter is of most immediate relevance. Recently substantial progress

⁷There are also closed string solitons with tension $T \sim \frac{1}{g_s^2}$, the so-called NS5-brane, cf. the discussion in [40].

has been made by the formulation of realistic GUT models with enhanced gauge symmetry to exceptional gauge groups⁸ in the non-perturbative formulation of Type IIB via F-theory, see [30, 31] for reviews. These setups exploit the localization of the brane gauge dynamics to cycles inside the Calabi-Yau, which may allow to consistently decouple the complex physics of gravity in a compact geometry⁹ and to work entirely in an (ultra)local model, to address more model independent and detailed phenomenological questions [55–57]. In addition model building with branes in Type IIB has on the one hand side led to a realization of hierarchies in warped compactifications with deep throats [9] which provides an embedding of RS-scenarios [8] into String Theory. On the other hand scenarios for controlled supersymmetry breaking via anti-D3-branes to a metastable de Sitter vacuum with fixed moduli [32] as well as string embeddings of cosmological models including inflation have been realized, see e.g. [58] for a review.

For all these phenomenological applications an adequate description of D-brane dynamics is essential. However, in general the complete dynamics of quantum D-brane can only be indirectly described via string dualities or in specific parameter regions where for example a perturbative string theory description in terms of Dirichlet-branes or a supergravity description in terms of p -branes is valid. The thorough analysis of brane dynamics is the central aim of this work, where we focus mostly on the dynamics of spacetime-filling D5-branes, NS5-branes and seven-brane. For this analysis methods from basically three different fields are applied, that are methods from the study of Effective Actions, from String Dualities like F-theory, heterotic/F-theory and mirror symmetry and finally from purely geometrical consideration, namely Blow-Up Geometries and $SU(3)$ -structure Manifolds.

1.3 Effective Actions

A very direct way to analyze and to make quantitative statements about the dynamics of a D-brane in a given string background is the determination of the general D-brane effective action. Generically a consistent and controllable string compactification has to be supersymmetric. The setups we initially study are defined by a Calabi-Yau threefold Z_3 with supersymmetric D-branes and orientifold planes or heterotic bundles. Since these backgrounds respect a minimal account of supersymmetry we are dealing with four-dimensional $\mathcal{N} = 1$ string vacua.

In general every effective four-dimensional $\mathcal{N} = 1$ supergravity theory contains one supergravity multiplet, a number of vector and of chiral multiplets [59]. The complete action is determined by three coupling functions¹⁰ of the chiral superfields Φ , the Kähler potential

⁸Before exceptional gauge groups have only been accessible in the heterotic string or in M-theory with ADE-singularities.

⁹Although appealing due to the obvious simplifications of decoupled gravity, global questions like the embedding of the local setup into a global string vacuum or the moduli problem can not be addressed.

¹⁰We neglect Fayet-Iliopoulos terms for our discussion.

$K(\Phi, \bar{\Phi})$, the gauge kinetic function $f(\Phi)$ and the superpotential $W(\Phi)$ [59]. In this thesis the derivation of the full effective action, i.e. of the three coupling functions K , f , and W , is performed for the case of a spacetime-filling D5-brane [60]. This completes the list of $\mathcal{N} = 1$ effective actions for compactifications with D-branes, see [61] for the D3-brane, [62, 63] for the D7-brane and finally [64] for the D6-brane action.

Besides the immediate application of the effective action for a detailed analysis of a specific D5-brane model of particle physics, one can address more model-independent questions. One natural such direction is to focus on an universal sector of the brane theory, that is given by a number of chiral multiplets corresponding to the position modes of the D5-brane. The scalars from this geometrical sector of the D5-brane effective action can remain as massless fields in the four-dimensional effective action and yield a serious moduli problem, that potentially spoils every D-brane model right at the beginning. Thus, it is of essential importance to ask the question whether these fields remain massless at any order in the fields and in α' . Out of the three coupling functions of the $\mathcal{N} = 1$ effective theory, the perturbative brane superpotential depending on these moduli is of central importance for this question [65–67]. Thus, together with the flux superpotential a complete understanding of the Type IIB superpotential allows to systematically study moduli stabilization in the Type IIB string with D5-branes, that are complex structure moduli and the dilaton in the closed string and the brane moduli in the open string sector¹¹. We refer to [15–17, 29] for a review of moduli stabilization in flux compactifications.

Furthermore, the superpotential is in general, due to its holomorphy, a controllable coupling since it is preserved by a non-renormalization theorem of $\mathcal{N} = 1$ supersymmetry from perturbative and non-perturbative corrections, see e.g. [68]. Thus we can trust our expressions for it obtained for example by the dimensional reduction of the tree-level D5-brane effective action. Having realized this conceptual importance of the superpotential, we will in the remainder of the thesis calculate explicitly the flux and brane superpotential using techniques from string dualities. This way, we obtain *exact* string theoretic results for a coupling function in the four-dimensional effective theory. In the context of string dualities we will also view the superpotential as a test field for the more elaborate formulations of non-perturbative string and brane dynamics like F-theory, heterotic/F-theory duality and the blow-up geometry.

1.4 String Dualities

In order to understand string theory more thoroughly beyond perturbation theory including non-perturbative effects, string dualities can be applied, see [69–71] for a selection of review articles. Dualities in general rely on the observation that identical physics can come in two or more seemingly different disguises like for example in the mentioned AdS/CFT-

¹¹The Kähler moduli are fixed by non-perturbative effects, cf. the pioneering paper [32].

correspondence relating gauge with a gravity theory. An important class of dualities are S-dualities that relate strong coupling physics in one theory to weak coupling physics in another or even the same theory. Examples are S-duality between the Type I and the heterotic $SO(32)$ string or of the Type IIB string with itself, which is exploited for the formulation of F-theory [72].

Besides S-duality F-theory also incorporates a second concept, namely geometrization. This term is in general used to express that a given theory is mapped to a complete geometric description, so that the underlying geometrical structures are exploited to study the dynamical processes in the theory. In F-theory both the description of a varying coupling constant and the physics of non-perturbative seven-branes in the Type IIB theory are geometrized by an elliptic curve, a two-torus, fibered over the spacetime manifold [72]. This provides on the one hand one of the few known duality invariant description of physics, in this case of S-duality in Type IIB, and on the other hand allows to study non-perturbative seven-brane dynamics. Furthermore an F-theory compactifications on elliptic Calabi-Yau manifolds [72–74] correspond to fully consistent supersymmetric Type IIB compactifications with automatically canceled tadpoles due to a consistent inclusion of orientifold planes [75, 76] and even allows for background R–R, NS–NS and brane fluxes¹². Thus F-theory provides the ideal setup to study many aspects of Type IIB compactifications and to answer questions about non-perturbative brane dynamics and about moduli stabilization, cf. [29]. For the latter issue one in particular exploits the calculability of the F-theory superpotential in compactifications to four dimensions which unifies the Type IIB flux and brane superpotential of the geometrized seven-branes¹³. This is a major strategy we follow in this thesis and in [79].

A second example of geometrization is heterotic/M-theory duality [41], which implies, using further string dualities, heterotic/Type II duality [41, 80] and its close cousin, heterotic/F-theory duality [72–74]. Focusing on the latter we observe a geometrization of the data of a heterotic compactification in terms of the purely geometrical data defining an F-theory compactification. In particular heterotic gauge dynamics as well as the dynamics of distinguished heterotic five-branes, so called horizontal five-branes, are completely geometrized in F-theory. This provides yet another route to analyze five-brane dynamics and to calculate brane superpotentials via string dualities that is of essential use for this thesis [79, 81].

One of the oldest string dualities is mirror symmetry, see e.g. [82] for a review. Conceptually it can be understood as a consequence of probing the spacetime geometry with one-dimensional strings instead of point-particles [83], which changes the notions of classical geometry to string geometry, a special form of quantum geometry. A toy version of mirror symmetry is T-duality¹⁴ that is well-known for the closed bosonic string where it relates the

¹²The only additional consistency condition is the M-theory tadpole [77]. Furthermore fluxes induce a warping, however, the internal metric is still conformally Calabi-Yau [78].

¹³We note that the seven-brane superpotential is formally a five-brane superpotential.

¹⁴See [84, 85] for a detailed review of T-duality.

compactification on a circle of radius R to a different compactification on a circle of radius $1/R$ by exchanging windings and momenta [2–5]. Mirror symmetry for Calabi-Yau threefold compactifications of Type II string theory was noted first from a CFT point of view [86, 87] and is indeed conjectured to be a generalization of T-duality [88]. It is most thoroughly formulated for the topological versions of Type II string theory, the A- and B-model, on Calabi-Yau manifolds [89], where it states the equivalence of the A-model on a given Calabi-Yau threefold \tilde{Z}_3 with the B-model on its mirror Calabi-Yau threefold Z_3 . Geometrically it identifies the classical complex geometry of Z_3 , i.e. the complex structure moduli space, with the quantum corrected symplectic geometry¹⁵ of \tilde{Z}_3 , i.e. the Kähler structure on \tilde{Z}_3 . The equivalence of the full Type IIA theory on \tilde{Z}_3 with the Type IIB theory on Z_3 is only conjectured, first by Kontsevich and then by [36, 88, 92], since it requires the equivalence of all perturbative but most notably non-perturbative effects like BPS-particles and D-instantons. Physically mirror symmetry allows the calculation of the world-sheet instanton corrected¹⁶ holomorphic $\mathcal{N} = 2$ prepotential in the Type IIA vector multiplet sector of the effective four-dimensional theory by matching with the classical Type IIB prepotential [35].

Mirror symmetry can be extended in two directions. One direction is the extension to higher-dimensional Calabi-Yau manifolds like Calabi-Yau fourfolds [93–95] and the other direction is to open string mirror symmetry defined by the inclusion of D-branes and a corresponding open string sector. The latter was formulated first by Kontsevich as the homological mirror symmetry conjecture and was interpreted in more physical terms in [96]. In physical setups both extensions yield effective theories with four supercharges. Closed mirror symmetry for fourfolds naturally occurs in two-dimensional Type II, three-dimensional M-theory and four-dimensional F-theory compactifications with an $\mathcal{N} = 1$ effective theory, which is the case of primary interest in this thesis [79]. Open mirror symmetry as used in Type II theories breaks $\mathcal{N} = 2$ supersymmetry of closed string mirror symmetry to $\mathcal{N} = 1$ by the inclusion of D-branes. It extends the purely geometrical closed mirror symmetry to D-branes identifying BPS D6-branes wrapping special Lagrangian three-manifolds in \tilde{Z}_3 in Type IIA with BPS D3-, D5-, D7- and D9-branes in Type IIB¹⁷. Both generalizations of mirror symmetry allow to calculate exact holomorphic couplings in the $\mathcal{N} = 1$ effective theory. For mirror symmetry on fourfolds this is the effective flux superpotential, e.g. of the F-theory compactification, which is classical in one theory and receives instanton corrections in the dual theory [97]. In the open string case mirror symmetry also relates the corresponding brane superpotentials on both sides, which allows as in the closed string case to compute the disk instanton corrected superpotential on the IIA side by the classical superpotential in Type IIB [98]. Furthermore it can be physically motivated in F-theory, that under certain circumstances, i.e. for specific flux choices, the fourfold flux superpotential agrees with the brane superpotential in a dual Type

¹⁵This is defined by a certain deformation quantization of the classical intersection theory on \tilde{Z}_3 due to world-sheet instantons [90, 91], where \hbar is identified with the string scale α' .

¹⁶There are no perturbative nor non-perturbative g_s -corrections, since g_s resides in a hypermultiplet [42].

¹⁷See [64] for a detailed discussion of open mirror symmetry at the classical level.

II or heterotic compactification [79, 81, 99]. Detailed discussions and explicit calculations in favor of this observation are the core of the technical part of this thesis following [79, 81, 100].

1.5 Blow-Up Geometries and $SU(3)$ -Structure Manifolds

As we have just discussed, string dualities often unexpectedly relate two different physical descriptions. It is the essential goal of the last part of this work to collect hints for a novel duality between a compactification with a five-brane supported on a curve Σ and a dual blow-up threefold geometry carrying an $SU(3)$ -structure, on which the five-brane has dissolved into flux and a canonical non-Kähler form [100]. This duality should apply equally to Type IIB Calabi-Yau orientifold compactifications with D5-branes [60, 100] as well as to heterotic or M-theory Calabi-Yau threefold compactifications with M5-branes [79, 81].

This dual description was first noted in [60] in the context of the study of the D5-brane superpotential as a tool to analyze its in general complicated dependence on both complex structure moduli of Z_3 and on the deformations of Σ . It is argued that the natural setup for this study is provided by a different threefold \hat{Z}_3 , that is obtained by blowing up along Σ in Z_3 , since the deformations of the curve Σ in Z_3 are unified with the complex structure moduli of Z_3 as pure complex structure deformations of \hat{Z}_3 ¹⁸. It is crucial for this unification that there are no branes present on \hat{Z}_3 hinting to a complete geometrization of the D5-brane. Indeed it could be argued [60] that the geometry of \hat{Z}_3 can at least be used to calculate the D5-brane and flux superpotential of the original Type IIB compactification on Z_3 .

It was later realized in [81] that this blow-up procedure is also applicable to five-branes on curves Σ in the heterotic string, for which the brane superpotential is formally identical to that of a D5-brane and should also be calculable from \hat{Z}_3 . In the context of heterotic/F-theory duality the description of the five-brane dynamics via \hat{Z}_3 is further confirmed by the agreement with the usual geometrization of a horizontal five-brane as a blow-up in the dual F-theory fourfold X_4 [73, 101–105]. This is shown and exploited for concrete calculations in [81] which are cross-checked by the calculations of F-theory flux superpotentials in [79] in combination with mirror symmetry for Calabi-Yau fourfolds and branes.

Finally the blow-up \hat{Z}_3 is used in [100] for direct calculations of both the five-brane and flux superpotential, that are again cross-checked via open mirror symmetry and shown to be in agreement with the available results in the literature [99, 106–127] that they partially extend. Furthermore, first arguments are provided for the blow-up threefold \hat{Z}_3 to define a consistent Type IIB or heterotic compactification by defining an $SU(3)$ -structure on \hat{Z}_3 . Most remarkably, a canonical non-Kähler form is defined that seems to naturally encode the five-brane in the original setup. This together with the performed calculational checks should

¹⁸We thank D. Huybrechts for pointing us in that direction and for patient explanation of mathematical details of this mapping.

be viewed as a first step to consistently treat the backreaction of five-branes and to establish a physical duality between the pure closed string compactification on the blow-up threefold \hat{Z}_3 and the original string compactification with five-branes. Although complete evidence is still lacking it is tempting and will hopefully be justified further by some future publication to view the blow-up geometry \hat{Z}_3 as a full dual description, that can probably be interpreted as a compact version of the geometric transition of [50, 51]. In this regard the aim of this thesis is to provide a self-contained presentation of the idea, calculational use and the current physical interpretation of the blow-up threefold \hat{Z}_3 .

1.6 Outline

The thesis is split into three logically different and self-contained parts, part I on Effective Actions, part II on String Dualities and part III on Blow-Up Geometries and $SU(3)$ -Structure Manifolds. The three parts are mainly independent, however, the explicit calculations of part III partially rely on mirror symmetry and enumerative geometry as discussed in part II.

We start in this work with the discussion of effective actions in part I. In order to prepare for our presentation we begin with a review of orientifold compactifications of Type II string theory in chapter 2. We start with a brief discussion of the classification of possible Type II orientifolds in section 2.1. Then we proceed with a review of the $\mathcal{N} = 1$ effective action of generic Type IIB Calabi-Yau orientifold compactifications Z_3/\mathcal{O} in section 2.2, that are $O3/O7$ - and $O5/O9$ -orientifold compactifications, where the latter is of particular relevance for this work. The chapter is concluded by determining the $\mathcal{N} = 1$ characteristic data.

In chapter 3 we begin the analysis of brane dynamics by the computation of the effective action of a spacetime-filling D5-brane in generic $O5/O9$ -Calabi-Yau orientifolds following [60]. In section 3.1 we present a general review of supersymmetric D-branes in Calabi-Yau manifolds focusing on their low-energy effective dynamics and on the geometric calibration conditions on BPS D-branes. The calculation of the D5-brane effective action is performed in section 3.2 by a purely bosonic reduction. We put special emphasis on the universal sector of the D5-brane action that is given by the open-closed geometric moduli as well as on the determination of the scalar potential. The calculation of the scalar potential reveals the ad-hoc surprising relevance of four-dimensional non-dynamical three-forms in order to recover the complete F-term potential in a purely bosonic reduction. In section 3.3 we work out the influence of the presence of the D5-brane on the $\mathcal{N} = 1$ characteristic data, in particular in comparison with the pure $O5/O9$ -orientifold compactification of chapter 2. The D5-brane induces a brane superpotential, that is given by a chain integral, as well as in an additional gauging of a chiral field of the bulk by the $U(1)$ -gauge boson of the D5-brane and the corresponding D-term. The discussion of the effective action is concluded in section 3.4 where the scalar potential is determined on the space of all geometric deformations of the D5-brane, including also massive

deformations. This is, as we will further discuss in chapter 7, of conceptual importance.

Part II is dedicated to a more sophisticated treatment of brane dynamics via string dualities. We start our discussion in chapter 4 by a detailed presentation of heterotic/F-theory duality and its use to analyze non-perturbative seven-branes in F-theory respectively five-branes in the heterotic string. Firstly, in section 4.1 we review heterotic string compactification in their most complete formulation via heterotic M-theory. We begin with a general presentation of heterotic M-theory and its use to calculate corrections to heterotic string compactifications. Special emphasis is put on the effect of five-branes influencing e.g. the heterotic B-field and the tadpole condition. In particular we note, following [81], the interpretation of heterotic five-branes as a singular bundle effect, a so-called small instanton, and its use to formally obtain a chain-integral superpotential for the five-brane when evaluating the holomorphic Chern-Simons functional on the small instanton. A brief review of the essentialities of the spectral cover construction of heterotic vector bundles concludes the heterotic string section. Then in section 4.2 we turn to F-theory. We review Vafa's basic idea of F-theory, readily turn to the construction of F-theory compactifications via elliptic Calabi-Yau manifolds and finally comment on the F-theory flux superpotential in Calabi-Yau fourfold compactifications. There we emphasize that this flux superpotential contains both the Type IIB flux and geometrized seven-brane superpotential and can consequently be used, as we explicitly demonstrate in this thesis, to calculate open-closed superpotentials in a unified framework. Finally we present a discussion of heterotic/F-theory duality, first of the fundamental eight-dimensional duality and then in lower dimensions. Of most conceptual importance for our study of brane dynamics is the duality map of heterotic five-branes, some of which are geometrized in F-theory by blow-ups in the F-theory fourfold, which is of particular use for concrete calculations.

The actual tools for many calculations performed in this thesis are provided in chapter 5. Here the basic geometric techniques of mirror symmetry are introduced. We begin by reviewing toric geometry in section 5.1 which is the main tool to generate concrete examples of Calabi-Yau manifolds, their mirrors and also branes. Then in section 5.2 we present a basic account on Calabi-Yau threefold mirror symmetry in order to prepare for its higher-dimensional analogue for Calabi-Yau fourfolds in section 5.3. For both sections the study of the complex structure moduli space via Picard-Fuchs differential equations is essential. In particular the existence of these differential equations allows the study of global properties of the complex structure moduli space as performed, following [79], for Calabi-Yau fourfolds yielding a novel behavior of the fourfold periods at the universal conifold. The chapter is concluded in section 5.4 by reviewing the enumerative meaning of mirror symmetry for the A-model. In particular the interpretation of the flux and brane superpotentials as generating functions for world-sheet- and disk-instantons is emphasized, since these, but also the classical terms, are used as cross-checks for our results obtained in the B-model calculations.

Finally in chapter 6 the methods from heterotic/F-theory duality, F-theory and mirror symmetry are applied for calculations of effective superpotentials. Firstly in section 6.1 the

F-theory flux superpotential is calculated for selected Calabi-Yau fourfold geometries that possess a clear Type IIB interpretation and a canonical heterotic dual, cf. [79]. Both the Type IIB flux and seven-brane superpotentials are determined and a first interpretation in terms of the heterotic dual is given. A more systematic study of the application of mirror symmetry for heterotic/F-theory duality with five-branes is presented in section 6.2. The central strategy there is to calculate the heterotic flux and five-brane superpotential from the F-theory dual geometry. Two examples emphasizing different aspects of heterotic/F-theory duality with five-branes are discussed.

In part III of this thesis, the novel prescription of the dynamics of five-branes on curves Σ via blow-up threefolds \hat{Z}_3 is discussed and applied for calculations [60]. Chapter 7 begins with a conceptual discussion of five-brane dynamics in section 7.1. There the formal treatment of the backreaction of five-branes naturally leads to the consideration of the open manifold $Z_3 - \Sigma$, which is replaced by the blow-up $\hat{Z}_3 - E$ in section 7.2, where E denotes the exceptional divisor of the blow-up of Σ . The unification of closed and open deformations of (Z_3, Σ) is discussed in detail and probed via the pullback $\hat{\Omega}$ of the Calabi-Yau three-form Ω from Z_3 to \hat{Z}_3 in section 7.3. Finally, the flux and five-brane superpotentials are lifted to \hat{Z}_3 in section 7.4 and shown to obey Picard-Fuchs equations for the complex structure moduli of \hat{Z}_3 .

Then in chapter 8 the blow-up \hat{Z}_3 is exploited to perform calculations of the effective superpotentials. These calculations are performed for two examples, branes in the quintic in section 8.1 and branes in an elliptic fibration over \mathbb{P}^2 in section 8.3. In both cases the existence of a toric GKZ-system, in general constructed in section 8.2, is used to derive open-closed Picard-Fuchs equations, that both contain the flux and the five-brane superpotential as solutions. Finally in section 8.4 the geometrization of five-branes via the blow-up \hat{Z}_3 is compared to the geometrization of five-branes in heterotic/F-theory duality. This comparison is applied for an explicit matching of the geometric structures and the five-brane superpotentials.

In the concluding chapter 9 first steps to view the threefold \hat{Z}_3 as the background of a string compactification are provided. For this purpose an $SU(3)$ -structure, as reviewed in section 9.1, is constructed on \hat{Z}_3 , that can be interpreted to be generated by the five-brane backreaction. Firstly the classical Kähler geometry on \hat{Z}_3 is discussed in section 9.2. Then, a minimal non-Kähler deformation of the Kähler structure on \hat{Z}_3 is presented in section 9.3, which can be physically viewed as a dissolution of the original five-brane on Σ into conventional three-flux localized around the divisor E and into additional geometrical flux.

Finally in part IV, we present in chapter 10 our conclusions and an outlook on future work. In addition this thesis has three appendices. Appendix A contains further background material on the calculation of the effective action, appendix B summarizes a basic account of geometrical properties of elliptic Calabi-Yau manifolds, ruled threefolds and the blow-up threefold \hat{Z}_3 and finally appendix C provides further details of the calculations performed in section 6.1 and in chapter 8, as well as two further fourfold examples.

1.7 List of Publications

Parts of this thesis have been published in scientific journals and on the preprint server arxiv.org:

- T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, “The D5-brane effective action and superpotential in N=1 compactifications,” Nucl. Phys. B **816** (2009) 139 [arXiv:0811.2996 [hep-th]].
- T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, “Computing Brane and Flux Superpotentials in F-theory Compactifications,” JHEP **1004** (2010) 015 [arXiv:0909.2025 [hep-th]].
- T. W. Grimm, T. W. Ha, A. Klemm and D. Klevers, “Five-Brane Superpotentials and Heterotic/F-theory Duality,” Nucl. Phys. B **838** (2010) 458 [arXiv:0912.3250 [hep-th]].
- T. W. Grimm, A. Klemm and D. Klevers, “Five-Brane Superpotentials, Blow-Up Geometries and SU(3) Structure Manifolds,” arXiv:1011.6375 [hep-th].

Part I

Effective Actions

Chapter 2

Orientifold Compactifications

In this chapter we review compactifications of Type II string theory on Calabi-Yau threefold orientifolds Z_3/\mathcal{O} . Our main focus is on Type IIB string theory and the corresponding four-dimensional effective action, where we closely follow the detailed discussion in the original works [128–130]. This provides the necessary background for our computation of the D5-brane effective action in chapter 3.

We start our presentation in section 2.1 with a brief review of the classification of possible Type II orientifolds, first from the perspective of the two-dimensional string sigma-model, then from the spacetime point of view of Type II string theory, where we readily focus on the geometry of orientifold planes. Then we proceed with a review of the $\mathcal{N} = 1$ effective action of generic Type IIB Calabi-Yau orientifold compactifications Z_3/\mathcal{O} in section 2.2.2, that are $O3/O7$ - and the $O5/O9$ -orientifold compactifications where the latter are of particular relevance for this work. We first discuss the expected massless spectrum in four dimensions, then present the actual Kaluza-Klein reduction to obtain the effective action and conclude with the organization of the fields and couplings into the $\mathcal{N} = 1$ characteristic data.

2.1 Classification of Orientifolds

In order to emphasize the concept of orientifolds we start in section 2.1.1 with a brief summary of the construction of an orientifold theory from the point of view of an $\mathcal{N} = (2, 2)$ supersymmetric field theory in two-dimensions. Then in a second step we specialize to sigma-models with a Calabi-Yau target space X and understand the orientifolded SCFT as part of a Type II string theory compactified on X in section 2.1.2. We conclude in section 2.1.3 with an analysis of the geometry of orientifold planes as calibrated submanifolds in a Calabi-Yau threefold Z_3 .

2.1.1 Orientifolds in $\mathcal{N} = (2, 2)$ Two-dimensional Field Theories

In general an orientifold theory of a given two-dimensional $\mathcal{N} = (2, 2)$ field theory is defined by dividing out a symmetry group of the theory, the orientifold action. This is an involutive parity symmetry P of the $\mathcal{N} = (2, 2)$ theory where P is a combination $P = \mathcal{T} \circ \Omega$ of the parity

action Ω of the $\mathcal{N} = (2, 2)$ superspace with a canonically induced action on the superfields and an internal action \mathcal{T} on the superfields so that the $\mathcal{N} = (2, 2)$ Lagrangian is invariant. There are two different parity operations on $\mathcal{N} = (2, 2)$ superspace, denoted A- and B-parity Ω_A, Ω_B [131], that both contain world-sheet parity Ω_p exchanging left- and right-movers but act differently on the $\mathcal{N} = (2, 2)$ -Grassmann variables. Both parities break half but different supersymmetries and are interchanged by mirror symmetry¹. In the case of a single chiral superfield $\Phi(x, \theta)$ with canonical kinetic term $\bar{\Phi}\Phi$ the internal action on the superfields is trivial, i.e. Φ is transformed as a function on superspace

$$P_A : \quad \Phi(x, \theta) \mapsto \overline{\Phi(\Omega_A(x, \theta))}, \quad P_B : \quad \Phi(x, \theta) \mapsto \Phi(\Omega_B(x, \theta)). \quad (2.1)$$

For non-linear sigma-models on a Kähler target manifold X , in particular for Calabi-Yau backgrounds, the two possible parities have to be combined with a non-trivial internal transformation on the chiral superfields [131]. From invariance of the Kähler potential K on X , which determines the kinetic term of the Φ as $K(\Phi, \bar{\Phi})$, it follows that the internal symmetry acts as an (anti-)holomorphic and isometric diffeomorphism² f on the coordinates z^i of X , which are the chiral superfields in the language of the sigma-model, for (A-) B-parities,

$$P_A : \quad \Phi^i(x, \theta) \mapsto \overline{h^i(\Phi(\Omega_A(x, \theta)))}, \quad P_B : \quad \Phi^i(x, \theta) \mapsto f^i(\Phi(\Omega_B(x, \theta))). \quad (2.2)$$

Here, $f : z^i \mapsto f^i(z)$ and $h : z^i \mapsto h^i(z)$ are holomorphic functions with $K(f(z), \bar{f}(z)) = K(z, \bar{z})$ and $K(h(z), h(z)) = K(z, \bar{z})$ up to Kähler transformations. In the case of an orientifold the map f is involutive $f \equiv \sigma$ so that P takes the form $P = \sigma \circ \Omega$ with $\mathcal{P}^2 = \mathbf{1}$. The geometrical fix-point locus of σ is the orientifold plane. Exploiting further the consistency of the Sigma-model at quantum level, in particular ensuring anomaly freedom of the parity symmetry P , one concludes in general that [131]

$$f^*[B_2] = -[B_2] \quad (2.3)$$

for the cohomology class $[B_2]$ of the B-field.

2.1.2 Orientifolds of Type II Superstring Theory

When we consider this orientifolded $\mathcal{N} = (2, 2)$ as part of full string theory we are more interested in spacetime physics. In the following we consider compactifications of Type II string theory on $(\mathbb{R}^{1,3} \times Z_3)/\sigma$ with σ acting trivially on Minkowski space $\mathbb{R}^{1,3}$ and as (2.2) on the Calabi-Yau threefold Z_3 . Thus, the orientifold planes are space-time filling. In this context the quotient by the orientifold, denoted by \mathcal{O} in the physical string, can be understood as a consequence of the inclusion of unoriented strings. Moreover, consistency of the full string

¹In the two-dimensional field theory mirror symmetry is just the exchange of supercharges Q_- and \bar{Q}_- .

²For Kähler manifolds with Kähler form J an (anti-)holomorphic and isometric diffeomorphism is characterized by $f^*J = J$ ($f^*J = -J$).

theory, most importantly cancellation of tadpoles, and spacetime supersymmetry are more restrictive than preserving $\mathcal{N} = 2$ world-sheet supersymmetry.

Firstly, the requirement of $\mathcal{N} = 1$ supersymmetry in four dimensions puts no additional restriction on a possible B-parity orientifolds. However A-parity preserves spacetime supersymmetry only if σ acts on the holomorphic three-form Ω on Z_3 by complex conjugation modulo a phase. This is a consequence of the condition $J^3 \cong \Omega \wedge \bar{\Omega}$. Thus, we arrive at the following transformation properties of the key geometrical objects on Z_3 ,

$$P_A : \quad \sigma^* \Omega = e^{2i\theta} \bar{\Omega}, \quad \sigma^* J = -J, \quad P_B : \quad \sigma^* \Omega = \pm \Omega, \quad \sigma^* J = J, \quad (2.4)$$

where we followed the convention of [128–130]. The sign in the transformation P_B will correspond to the two possible orientifolds with a plus sign for $\mathcal{O}5/\mathcal{O}9$ -planes and a minus sign for $\mathcal{O}3/\mathcal{O}7$ -planes, respectively.

Secondly, in Type II string theory orientifold planes are charged with respect to the R–R-fields with possible negative charge. The cancellation of tadpoles requires net charge zero which can be achieved without breaking supersymmetry by the inclusion of calibrated D-branes. The maintenance of supersymmetry can already be seen in the $\mathcal{N} = (2, 2)$ theory since the linear combinations of the $\mathcal{N} = (2, 2)$ world-sheet supercharges preserved by A- and B-parity are also preserved by A- and B-branes, respectively. Thus, for a stable compactification with spacetime supersymmetry of Type IIA respectively Type IIB string theory only A-parity respectively B-parity can be used. However, we ignore the effect of D-branes, which will be discussed in chapter 3, for the moment and focus here exclusively on the closed strings. For the two parities in (2.4) the orientifold action \mathcal{O} , lifted to the Type II theory, reads

$$IIA : \quad \mathcal{O} = \sigma \circ (-1)^{F_L} \Omega_p, \quad IIB : \quad (\mathcal{O}3/\mathcal{O}7) \mathcal{O} = \sigma \circ (-1)^{F_L} \Omega_p, \quad (\mathcal{O}5/\mathcal{O}9) \mathcal{O} = \sigma \circ \Omega_p \quad (2.5)$$

which makes it more convenient to infer the action on all spacetime fields in the theory. Here F_L is the spacetime fermion number of the left moving sector, see e.g. [132] for a review.

The action of the orientifold (2.5) on the massless bosonic fields in the Type II theory is determined by first noting that [4, 132, 133]

$$\Omega_p : \quad +1 : \phi, g, C_1, C_2, \quad -1 : C_0, B_2, C_3, C_4, \quad (2.6)$$

$$(-1)^{F_L} : \quad +1 : \phi, g, B_2, \quad -1 : C_0, C_1, C_2, C_3, C_4, \quad (2.7)$$

where we denoted the eigenvalue ± 1 under corresponding transformation. Then, from this together with (2.5) we obtain the results of table 2.1, where one can further use (2.3) and that σ is isometric. Dividing out the orientifold symmetry projects on the \mathcal{O} -invariant states of the theory, $\mathcal{O}(\Phi) = \pm \sigma^* \Phi \stackrel{!}{=} \Phi$, where the sign is fixed in table 2.1. Thus, table 2.1 determines also the parity with respect to σ^* of the remaining massless fields by the requirement of invariance, as summarized below in table 2.2. On the level of the four-dimensional effective action this is reflected in an $\mathcal{N} = 1$ theory truncated from the $\mathcal{N} = 2$ theory obtained from Type II string theory compactified on $\mathbb{R}^{1,3} \times Z_3$, as demonstrated below.

	\mathcal{O} -image
O6	$\sigma^*\phi, -\sigma^*B_2, \sigma^*g, -\sigma^*C_1, \sigma^*C_3$
O3/O7	$\sigma^*\phi, -\sigma^*B_2, \sigma^*g, \sigma^*C_0, -\sigma^*C_2, \sigma^*C_4$
O5/O9	$\sigma^*\phi, -\sigma^*B_2, \sigma^*g, -\sigma^*C_0, \sigma^*C_2, -\sigma^*C_4$

Table 2.1: Transformation of massless bulk fields.

2.1.3 The Geometry of Orientifold Planes

As mentioned above the fixed-point set of the involution σ determines the orientifold planes. In both Type II theories the geometry of orientifold planes is specified by the calibration of their volume and the vanishing of characteristic differential forms of $\mathbb{R}^{1,3} \times Z_3$. In type IIA the fixed point set of σ is a real three-cycle L in Z_3 times $\mathbb{R}^{1,3}$ and the corresponding orientifold planes are $O6$ -planes. This follows in local coordinates z^i from (2.4) since $\Omega = g(z)dz^1 \wedge dz^2 \wedge dz^3$ which identifies the action of σ as the real involution $\sigma(z^i) = \bar{z}^i$. In Type IIB the involution σ respects the complex structure on Z_3 and the fixed point set are holomorphic cycles C_{Op} in Z_3 times $\mathbb{R}^{1,3}$. Again we check in local coordinates that the two possible phases for the transformation of Ω in (2.4) allow either (minus sign) for fixed points C_{O3} and fixed divisors C_{O7} in Z_3 , which are $O3/O7$ -planes, or (plus-sign) for fixed curves C_{O5} and the entire Calabi-Yau fixed, $C_{O9} = \mathbb{R}^{1,3} \times Z_3$, which are $O5/O9$ -planes. Equivalently, the orientifold planes can be specified by the geometric conditions

$$\begin{aligned}
O6 : \quad & (J + iB_2)|_L = 0, \quad \text{Im}(e^{-i\theta}\Omega)|_L = 0, \quad \text{Re}(e^{-i\theta}\Omega)|_L = \frac{\sqrt{\det(g + B_2)}}{\sqrt{\det(g)}} \text{vol}_L \\
O_p : \quad & \Omega|_L = 0, \quad e^{J+iB_2}|_{C_{Op}} = \frac{\sqrt{\det(g + B_2)}}{\sqrt{\det(g)}} \text{vol}_{C_{Op}}, \quad p = 3, 5, 7, 9, \quad (2.8)
\end{aligned}$$

where we included a background B-field [134] and denoted the pullback to the fixed point loci L , respectively C_{Op} , by $|_L$, respectively $|_{C_{Op}}$. The volume forms on the corresponding cycles are denoted vol_L , $\text{vol}_{C_{Op}}$. In general, cycles obeying (2.8) are of minimal volume in their homology class. This directly identifies L as a special Lagrangian submanifold and the cycles C_{Op} as holomorphic submanifolds of Z_3 . They are calibrated with respect to the calibration forms $\text{Re}(e^{-i\theta}\Omega)$, $(J + iB_2)^{\frac{p-3}{2}}$, respectively. This is very similar to the BPS-bound obeyed by massive particles in a theory with enhanced supersymmetry with a saturation of the BPS-bound for BPS-particles. Indeed, the calibration conditions (2.8) are indeed derived as the condition for BPS stable D-branes in background B-fields with trivial worldvolume gauge flux [134].

We conclude with a qualitative argument to infer the existence of the various D-branes and O-planes in Type II theory. For the case of $O9$ -planes the geometric involution σ in the orientifold in (2.5) acts trivially on ten-dimensional spacetime and only world-sheet parity

is divided out. Thus, orientifolds with $O9$ -planes just add unoriented strings to Type IIB theory in ten dimensions. However, it is well-known that this theory is inconsistent due to tadpoles that have to be canceled by the inclusion of 32 D9-branes. This introduces an open string sector and the theory is identified with Type I string theory [39, 132]. By compactifying this theory on tori T^n and by application of T-duality consistent toroidal string theory compactifications are obtained. Moreover, T-duality interchanges Dirichlet and Neumann boundary conditions and furthermore world-sheet and spacetime parity [38, 39, 132]. This way D-branes of all possible dimension and O-planes sitting at the edge of a hypercube $T^n/(\mathbb{Z}_2)^n$ are obtained.

2.2 Effective Action of Type IIB Calabi-Yau Orientifolds

In this section we review the $\mathcal{N} = 1$ effective action of Type IIB orientifold compactifications, that are $O3/O7$ - and $O5/O9$ -orientifolds, where we closely follow [128, 130]. We demonstrate a purely bosonic reduction and infer the fermionic action and fields by spacetime supersymmetry. In section 2.2.1 we summarize the massless four-dimensional spectrum as obtained from a Kaluza-Klein ansatz of the ten-dimensional Type IIB supergravity fields. Then in section 2.2.2 we present the bosonic Lagrangian of the full effective action, that we recast into the standard $\mathcal{N} = 1$ form in section 2.2.3. There we extract the $\mathcal{N} = 1$ chiral coordinates and the three coupling functions, the Kähler potential, the gauge kinetic function and the superpotential, which is trivial due to the absence of fluxes.

Since the main focus of this thesis is on Type IIB string compactifications we omit a complete discussion of the mirror dual Type IIA orientifolds and their effective actions and refer the interested reader to the literature [129, 130].

2.2.1 The Spectrum in Four Dimensions

We determine the massless fluctuations of Type IIB theory around the $\mathcal{N} = 1$ supersymmetric background $(\mathbb{R}^{1,3} \times Z_3)/\mathcal{O}$. Since a ten-dimensional massive field remains massive in four-dimensions we only consider the massless fields in ten dimensions, which agrees with the field content of ten-dimensional Type IIB supergravity truncated by the orientifold \mathcal{O} . Dimensional reduction yields the field content of the four-dimensional effective theory that will be assembled to $\mathcal{N} = 1$ superfields. These $\mathcal{N} = 1$ fields are obtained by demanding invariance under the orientifold action yielding a consistent truncation from the $\mathcal{N} = 2$ fields of Type IIB on $\mathbb{R}^{1,3} \times Z_3$. The requirement of invariance immediately fixes the parity under σ^* of the ten-dimensional massless fields. From table 2.1 and (2.3) we infer the action as given in table 2.2. The fields C_p denote the form-fields comprising the R-R-sector. In the democratic formulation of Type IIB supergravity these are C_0, C_2, C_4, C_6 and C_8 with field

	R-R	NS-NS
O3/O7	$\sigma^*(C_p) = (-1)^{p/2}C_p$	$\sigma^*\phi = \phi, \sigma^*g = g,$
O5/O9	$\sigma^*(C_p) = (-1)^{(p+2)/2}C_p$	$\sigma^*B_2 = -B_2$

Table 2.2: The required σ^* -parity of the \mathcal{O} -invariant states.

strengths [135]

$$G^{(p)} = \begin{cases} dC_0 & p = 1, \\ dC_{p-1} - dB_2 \wedge C_{p-3} & \text{else .} \end{cases} \quad (2.9)$$

In addition the duality constraints $G^{(n)} = (-1)^{\lfloor \frac{n}{2} \rfloor} *_{10} G^{(10-n)}$ are imposed, which read [135]

$$G^{(1)} = *_{10}G^{(9)}, \quad G^{(2)} = -*_{10}G^{(8)}, \quad G^{(3)} = -*_{10}G^{(7)}, \quad G^{(4)} = *_{10}G^{(6)}, \quad G^{(5)} = *_{10}G^{(5)}, \quad (2.10)$$

where $*_{10}$ denotes the ten-dimensional Hodge star and $\lfloor \frac{n}{2} \rfloor$ the integer part of $\frac{n}{2}$. These conditions are necessary in order to drop the redundant degrees of freedom in the democratic formulation of Type IIB supergravity [135].

Having identified the invariant fields in ten-dimensions we perform the Kaluza-Klein reduction. As a first step in the dimensional reduction process we specify the background values of the fields in the theory. Since the standard Kaluza-Klein ansatz for the metric field of a direct product form is not a solution to the supergravity equations of motion in the presence of source terms induced by branes or fluxes [9, 78, 136, 137] a warped product with warp factor $A(z)$ has to be considered,

$$ds_{10}^2 = e^{2A(z)} \eta_{\mu\nu}^{\text{SF}} dx^\mu dx^\nu + 2e^{-2A(z)} g_{i\bar{j}}(z) dz^i d\bar{z}^{\bar{j}}. \quad (2.11)$$

Here we choose the string frame (SF) and the Calabi-Yau metric $g_{i\bar{j}}$ on Z_3 . However, we neglect the effect of the warp factor in this work which might be argued for in the large radius limit of Z_3 where A approaches unity [9, 138, 139]. In addition we exclude any background fluxes for the moment and refer to section 3.2.4 and to [128, 130] for a complete discussion.

Next we make a reduction ansatz for the fluctuations of the ten-dimensional fields of Type IIB theory around their background vacuum expectation values (VEV). For this purpose we expand by harmonic forms $\mathcal{H}^p(Z_3, g)$ on Z_3 that are in the kernel of the Laplacian on Z_3 and thus correspond to massless modes in four-dimensions. Fortunately the number of harmonic forms is topological, i.e. does not depend on the metric g , and is counted by the cohomology groups³ $H^p(Z_3) \cong \mathcal{H}^p(Z_3, g)$. According to Hodge theory on compact Kähler manifolds these groups admit a decomposition into Dolbeault cohomology groups, $H^p(Z_3, \mathbb{C}) = \sum_{i+j=p} H_{\bar{\partial}}^{i,j}(Z_3)$ for $d = \bar{\partial} + \partial$, which is independent of the Kähler structure.

³More precisely, this isomorphism can be characterized by saying that for every cohomology class $[\alpha]$ there exists a unique representative α_h with $\Delta\alpha_h = 0$ or equivalently with $\|\alpha_h\|^2 = \int \alpha_h \wedge \star\alpha_h$ minimal in its class.

In order to accommodate for the transformation properties of table 2.2 we have to introduce the ± 1 -eigenspaces $H_{\pm}^{(i,j)}$ of σ^* in $H^{(i,j)}$,

$$H^{(i,j)}(Z_3) = H_+^{(i,j)}(Z_3) \oplus H_-^{(i,j)}(Z_3). \quad (2.12)$$

This is possible since the action of σ is closed on the Dolbeault cohomology as a holomorphic map and since $(\sigma^*)^2 = \mathbf{1}$. For a Calabi-Yau threefold the only non-vanishing Hodge numbers are $h^{(0,0)} = h^{(3,3)} = h^{(3,0)} = h^{(0,3)} = 1$, $h^{(1,1)} = h^{(2,2)}$ and $h^{(2,1)} = h^{(1,2)}$. In terms of these numbers the Euler characteristic reads $\chi(Z_3) = 2(h^{(1,1)} - h^{(2,1)})$. We introduce a basis $(\omega_\alpha, \omega_a)$ of $H_+^{(1,1)}$, $H_-^{(1,1)}$ with dual basis $(\tilde{\omega}^\alpha, \tilde{\omega}^a)$ of $H_+^{(2,2)}$, $H_-^{(2,2)}$ such that

$$\int_{Z_3} \omega_\alpha \wedge \tilde{\omega}^\beta = \delta_\alpha^\beta, \quad \int_{Z_3} \omega_a \wedge \tilde{\omega}^b = \delta_a^b, \quad (2.13)$$

where $\alpha, \beta = 1, \dots, h_+^{(1,1)}$ and $a, b = 1, \dots, h_-^{(1,1)}$. These vector spaces can still be dual since σ commutes with the Hodge star $*$, which implies $h_{\pm}^{(1,1)} = h_{\pm}^{(2,2)}$. Moreover, we denote by (α_K, β^K) , $(\alpha_{\tilde{K}}, \beta^{\tilde{K}})$ a real symplectic basis of H_+^3 and H_-^3 , respectively. This basis is chosen such that the intersection pairings take the form

$$\int_{Z_3} \alpha_K \wedge \beta^L = \delta_K^L, \quad \int_{Z_3} \alpha_{\tilde{K}} \wedge \beta^{\tilde{L}} = \delta_{\tilde{K}}^{\tilde{L}}, \quad (2.14)$$

and vanish otherwise. Note that the holomorphic three-form Ω is contained in $H_+^3(Z_3)$, hence, $K = 0, \dots, h_+^{(2,1)}$ but $\tilde{K} = 1, \dots, h_-^{(2,1)}$ for $O5/O9$ orientifolds. Conversely for $O3/O7$ -orientifolds Ω is an element of $H^-(Z_3)$, thus $\tilde{K} = 0, \dots, h_-^{(2,1)}$, but $K = 1, \dots, h_+^{(2,1)}$. Thus, to indicate the difference of the dimensionality we introduce different labels of the cohomology basis of H_{\pm}^3 in the $O5/O9$ and $O3/O7$ case. Furthermore, we introduced a basis χ_κ , χ_k of $H_{\pm}^{(2,1)}$, respectively. We summarize our conventions in table 2.3. We note that b_{\pm}^3 is always even dimensional as in the Calabi-Yau case.

Type	(± 1)-Eigenspace		Basis		Dimension	
$O3/O7$	$H_+^3(Z_3)$	$H_-^3(Z_3)$	$\alpha_\kappa, \beta^\kappa$	$\alpha_{\tilde{K}}, \beta^{\tilde{K}}$	$2h_+^{(2,1)}$	$2h_-^{(2,1)} + 2$
$O5/O9$	$H_+^3(Z_3)$	$H_-^3(Z_3)$	α_K, β^K	α_k, β^k	$2h_+^{(2,1)} + 2$	$2h_-^{(2,1)}$
Both	$H_+^{(1,1)}(Z_3)$	$H_-^{(1,1)}(Z_3)$	ω_α	ω_a	$h_+^{(1,1)}$	$h_-^{(1,1)}$
	$H_+^{(2,1)}(Z_3)$	$H_-^{(2,1)}(Z_3)$	χ_κ	χ_k	$h_+^{(2,1)}$	$h_-^{(2,1)}$
	$H_+^{(2,2)}(Z_3)$	$H_-^{(2,2)}(Z_3)$	$\tilde{\omega}^\alpha$	$\tilde{\omega}^a$	$h_+^{(2,2)}$	$h_-^{(2,2)}$

Table 2.3: Basis of the cohomology groups.

To determine the four-dimensional bulk spectrum we use this cohomology basis and expand the NS-NS as well as the R-R fields. We start with the NS-NS sector which is universal for

both the $O3/O7$ - and $O5/O9$ -orientifold. According to table 2.1 the ten-dimensional dilaton ϕ is expanded by harmonic, σ -invariant functions on Z_3 . For a compact manifold Z_3 only the constant function is harmonic and we obtain one massless mode, the four-dimensional dilaton $\phi(x)$. The ten-dimensional B-field is odd under σ for both orientifolds. Consequently its zero-modes $b_a(\underline{x})$ enter the expansion by forms $H_-^{(1,1)}(Z_3)$ as

$$B_2 = b^a(\underline{x})\omega_a . \quad (2.15)$$

The reduction of the metric tensor g is most efficiently organized by the number of spacetime indices μ, ν . For two spacetime indices μ, ν we obtain the four-dimensional metric tensor, which is proportional to the flat Minkowski metric $\eta_{\mu\nu}$. Since a Calabi-Yau threefold does not admit continuous isometries, i.e. there are no Killing vector fields, no four-dimensional massless vector fields are obtained from $g_{i\mu}$. Finally, variations of the internal metric yield four-dimensional scalars. These scalars take values in the Calabi-Yau orientifold moduli space⁴. This moduli space splits into two parts related to deformations of the complex structure and of the Kähler structure, that are encoded by variations of the holomorphic three-form Ω and the Kähler class $[J]$, respectively, as we discuss next.

The three-form Ω is expanded in accord with (2.4) as

$$O3/O7 : \Omega = X^{\tilde{K}}(\underline{z})\alpha_{\tilde{K}} - \mathcal{F}_{\tilde{K}}(\underline{z})\beta^{\tilde{K}}, \quad O5/O9 : \Omega = X^K(z)\alpha_K - \mathcal{F}_K(z)\beta^K, \quad (2.16)$$

where the \underline{z} denote the complex structure moduli of Z_3/σ . In the effective four-dimensional theory the $\underline{z}(x)$ are complex scalar fields taking values in the complex structure moduli space \mathcal{M}^{cs} . We note that due to (2.4) there are only $h_{\pm}^{(2,1)}$ allowed complex structure deformations for $O5/O9$ - respectively $O3/O7$ -orientifolds. The coefficient functions $(\underline{X}, \underline{\mathcal{F}})$ are the periods of Ω . They can be expressed as period integrals over the symplectic homology basis (A_K, B^K) dual to (α_K, β^K) as $\int_{A_L} \alpha_K = \delta_K^L = -\int_{B^K} \beta^L$,

$$X^K = \int_{A^K} \Omega, \quad \mathcal{F}_K = \int_{B^K} \Omega \quad (2.17)$$

for $O5/O9$ - and analogously for $O3/O7$ -orientifolds. Then the first order variation of Ω under a change of complex structure reads

$$\partial_{z^\kappa} \Omega = \chi_\kappa - K_\kappa \Omega, \quad (2.18)$$

for $O5/O9$ -orientifolds and with $\kappa \rightarrow k$ also for $O3/O7$ -orientifolds where K_κ is the first derivative of the Kähler potential on \mathcal{M}^{cs} . The periods may still be used to define special coordinates as $t^k = X^k/X^0$ for $O3/O7$ -orientifolds respectively $t^\kappa = X^\kappa/X^0$ for $O5/O9$ -orientifolds. Similarly we expand the Kähler form as

$$J = v^\alpha(\underline{x})\omega_\alpha \quad (2.19)$$

⁴It was argued in [131] that the Calabi-Yau orientifold moduli space forms a smooth submanifold of the Calabi-Yau moduli space.

where the $v^\alpha(x)$ are real fields taking values in the Kähler moduli space \mathcal{M}^K of Z_3 . At classical level \mathcal{M}^K is a cone specified by positivity conditions on the \underline{v} . We emphasize that the Kähler form is not complexified by the B-field on Z_3/σ due to the different parity under σ^* .

The infinitesimal variations of the internal Calabi-Yau metric of mixed respectively of purely holomorphic type similarly describe variations of the Kähler and complex structure. They are related to the variations of the forms J , Ω as

$$\begin{aligned} \delta g_{i\bar{j}} &= -i(\omega_\alpha)_{i\bar{j}} \delta v^\alpha, & \alpha &= 1, \dots, h_+^{(1,1)}, & (2.20) \\ O3/O7 &: \delta g_{ij} = \frac{i\mathcal{V}}{\int \Omega \wedge \bar{\Omega}} \Omega_j^{\bar{i}\bar{j}} (\bar{\chi}_{\bar{k}})_{\bar{i}\bar{j}} \delta \bar{z}^{\bar{k}}, & k &= 1, \dots, h_-^{(2,1)}, \\ O5/O9 &: \delta g_{ij} = \frac{i\mathcal{V}}{\int \Omega \wedge \bar{\Omega}} \Omega_j^{\bar{i}\bar{j}} (\bar{\chi}_{\bar{k}})_{\bar{i}\bar{j}} \delta \bar{z}^{\bar{k}}, & \kappa &= 1, \dots, h_+^{(2,1)}, \end{aligned}$$

where we introduced the string-frame volume $\mathcal{V} = \int_{Z_3} d^6y \sqrt{g}$ of Z_3 .

Let us now turn to the R–R sector. The various R–R-fields are expanded according to table 2.2. For $O5/O9$ -orientifolds the expansion into invariant fields reads

$$\begin{aligned} C_6 &= A_{(3)}^K \wedge \alpha_K + \tilde{A}_K^{(3)} \wedge \beta^K + \tilde{c}_\alpha^{(2)} \wedge \tilde{\omega}^\alpha + h m_6, & (2.21) \\ C_4 &= V^k \wedge \alpha_k + U_k \wedge \beta^k + \tilde{\rho}_{(2)}^a \wedge \omega_a + \rho_a \tilde{\omega}^a, \\ C_2 &= \mathcal{C}_{(2)} + c^\alpha \omega_\alpha, \end{aligned}$$

where $m_6 = \Omega \wedge \bar{\Omega} / \int_{Z_3} \Omega \wedge \bar{\Omega}$ is a top form on Z_3 normalized such that $\int_{Z_3} m_6 = 1$. Similarly for $O3/O7$ -orientifolds we obtain

$$\begin{aligned} C_8 &= \mathcal{L}_{(2)} \wedge m_6, & (2.22) \\ C_6 &= A_{(3)}^{\tilde{K}} \wedge \alpha_{\tilde{K}} + \tilde{A}_{\tilde{K}}^{(3)} \wedge \beta^{\tilde{K}} + \tilde{c}_a^{(2)} \wedge \tilde{\omega}^a, \\ C_4 &= V^\kappa \wedge \alpha_\kappa + U_\kappa \wedge \beta^\kappa + \tilde{\rho}_{(2)}^\alpha \wedge \omega_\alpha + \rho_\alpha \tilde{\omega}^\alpha, \\ C_2 &= c^a \omega_a, \quad C_0 = l. \end{aligned}$$

Here the fields $(\underline{A}_{(3)}, \tilde{\underline{A}}^{(3)})$ are three-forms, $(\tilde{\underline{c}}^{(2)}, \tilde{\underline{\rho}}_{(2)}, \mathcal{C}_{(2)}, \mathcal{L}_{(2)})$ are two-forms, $(\underline{V}, \underline{U})$ are vectors and $(h, \underline{\rho}, \underline{c}, l)$ are scalars in spacetime $\mathbb{R}^{1,3}$. We note that h and \mathcal{C}_2 are projected out in $O3/O7$ -orientifolds since C_2 , C_6 are odd under σ , whereas the scalar l and its dual $\mathcal{L}_{(2)}$ are projected out in $O5/O9$ -orientifolds as C_0 , C_8 are odd. The scalar l can be arranged as the four-dimensional axio-dilaton

$$\tau = l + i e^{-\phi} \quad (2.23)$$

which is of convenience in the study of the non-perturbative $SL(2, \mathbb{Z})$ -symmetry in setups with $O3/O7$ -planes and F-theory. This non-perturbative symmetry is inherited from the ten-dimensional non-perturbative $SL(2, \mathbb{Z})$ -invariance of Type IIB string theory. It is of particular importance for the understanding of F-theory discussed in chapter 4 since it requires

new objects, (p, q) -strings and (p, q) 7-branes. For the effective action derived for $O3/O7$ -orientifolds we expect invariance of the $\mathcal{N} = 1$ data under this duality group⁵ which is explicitly checked in [128].

Let us comment on the expansions (2.21), (2.22). Note that due to the duality constraints (2.10) not all degrees of freedom of the R–R-forms are physical. On the level of the four-dimensional effective action one can eliminate half of the degrees of freedom in (2.21), (2.22) by introducing Lagrange multiplier terms. For example the duality of the gauge potentials \underline{V} , \underline{U} amounts to standard $\mathcal{N} = 1$ electric-magnetic duality in four dimensions. However, in order to couple the bulk fields to the D-brane sector as demonstrated in chapter 3, it turns out to be useful to work with the democratic formulation. Only at the end of our analysis we choose a set of physical degrees of freedom and eliminate the remaining fields using (2.10).

In addition, we note that the expansions (2.21), (2.22) also contain massless three-form fields $(\underline{A}_{(3)}, \tilde{\underline{A}}^{(3)})$ that are clearly non-dynamical in four space-time dimensions. However, we show in section 3.2.4 that the inclusion of these three-forms is crucial to determine the scalar potential of compactifications with background fluxes and D-branes in a purely bosonic reduction. In case these are omitted a fermionic reduction has to be applied to derive the induced brane and flux superpotential as done, for example, for D5- and D7-branes in [63, 140].

2.2.2 The Orientifold Effective Action

In this section we determine the effective action for Type IIB $O3/O7$ - and $O5/O9$ -orientifold compactifications. It is obtained from the Type IIB supergravity theory in a bosonic Kaluza-Klein reduction on Z_3/σ . Here we mainly follow the steps and conventions used in [128, 130].

The string-frame Type IIB action is used in its democratic form [135]

$$S_{\text{IIB}}^{\text{SF}} = \int \frac{1}{2} e^{-2\phi} R *_{10} 1 - \frac{1}{4} e^{-2\phi} (8d\phi \wedge *_{10} d\phi - H \wedge *_{10} H) + \frac{1}{8} \sum_{p \text{ odd}} G^{(p)} \cdot G^{(p)}, \quad (2.24)$$

where $H = dB_2$ denotes the NS–NS field strength, $G^{(p)} \cdot G^{(p)} = \pm G^{(p)} \wedge *_{10} G^{(p)}$ ⁶ and the R–R field strengths have been introduced in (2.9). The corresponding action in ten-dimensional Einstein-frame with canonically normalized Einstein-Hilbert term is obtained by the Weyl rescaling $g_{10}^{\text{SF}} = e^{\frac{\phi}{2}} g_{10}^{\text{EF}}$ [4]. We can use the duality constraints (2.10) imposed on the level of the equations of motion and the conventions from [135] like $(*_{10})^2 = (-1)^{k+1}$ on k -forms to obtain the common Type IIB action from (2.24).

Clearly the effective action is expected to differ in the R–R-sector for the $O3/O7$ - and the $O5/O9$ -setup. The NS–NS-sector will be qualitatively identical for both orientifolds. In

⁵Since the effective action will be derived from classical supergravity we obtain an even bigger duality group $SL(2, \mathbb{R})$. This is broken to the modular group by D(–1)-instantons.

⁶According to the conventions in [135] a sign has to be included for a non-vanishing kinetic term of the R–R form fields upon application the duality constraint (2.10). It is +1, for $p < 5$, and –1 for $p > 4$.

particular, the dilaton is invariant in both cases as well as the expansion of the B-field (2.3) and the Kähler form J (2.19) are identical whereas the complex structure variations (2.20) of the metric are mapped onto each other by the exchange $H_+^{(2,1)} \leftrightarrow H_-^{(2,1)}$. We first present the complete actions in both cases, then comment on their derivation and the differences.

We begin with the $O5/O9$ -setup. We readily insert the expansions of the fields in the NS–NS and in the R–R sector in (2.21) into the bulk action (2.24) to obtain the action in four-dimensional Einstein-frame as

$$\begin{aligned}
S_{O5/O9}^{(4)} = & \int -\frac{1}{2}R * \mathbf{1} - G_{\kappa\bar{\lambda}} dz^\kappa \wedge *d\bar{z}^\lambda + \frac{1}{4}\text{Im } \mathcal{M}_{kl} F^k \wedge *F^l + \frac{1}{4}\text{Re } \mathcal{M}_{kl} F^k \wedge F^l \quad (2.25) \\
& - G_{\alpha\beta} dv^\alpha \wedge *dv^\beta - G_{ab} db^a \wedge *db^b - dD \wedge *dD \\
& - \frac{1}{4\mathcal{V}}e^{2D}(dh + \frac{1}{2}(d\rho_a b^a - \rho_a db^a)) \wedge *(dh + \frac{1}{2}(d\rho_a b^a - \rho_a db^a)) - e^{2D}\mathcal{V}G_{\alpha\beta} dc^\alpha \wedge *dc^\beta \\
& - \frac{1}{16\mathcal{V}}e^{2D}G^{ab}(d\rho_a - \mathcal{K}_{ac\alpha}c^\alpha db^c) \wedge *(d\rho_b - \mathcal{K}_{bd\beta}c^\beta db^d) .
\end{aligned}$$

Here we have introduced the four-dimensional dilaton

$$e^D = e^\phi \mathcal{V}^{-1/2} . \quad (2.26)$$

Similarly for $O3/O7$ -orientifolds we use the same expansions of the fields in the NS–NS sector and the expansion (2.22) of the R–R-fields to obtain

$$\begin{aligned}
S_{O3/O7}^{(4)} = & \int -\frac{1}{2}R * \mathbf{1} - G_{k\bar{l}} dz^k \wedge *d\bar{z}^l + \frac{1}{4}\text{Im } \mathcal{M}_{\kappa\lambda} F^\kappa \wedge *F^\lambda + \frac{1}{4}\text{Re } \mathcal{M}_{\kappa\lambda} F^\kappa \wedge F^\lambda \\
& - G_{\alpha\beta} dv^\alpha \wedge *dv^\beta - G_{ab} db^a \wedge *db^b - dD \wedge *dD - \frac{1}{4}e^{2D}\mathcal{V} dl \wedge *dl \\
& - e^{2D}\mathcal{V}G_{ab}(dc^a - l db^a) \wedge *(dc^b - l db^b) \\
& - \frac{1}{16\mathcal{V}}e^{2D}G^{\alpha\beta}(d\rho_\alpha - \mathcal{K}_{\alpha ab}c^a db^b) \wedge *(d\rho_\beta - \mathcal{K}_{\beta cd}c^c db^d) . \quad (2.27)
\end{aligned}$$

There are several remarks in order.

In both cases the four-dimensional two-forms in (2.21) respectively (2.22) are traded for their dual scalars to express the action only in terms of chiral multiplets instead of linear multiplets. In technical terms this is achieved by adding Lagrange multipliers [141]

$$S_{O5/O9}^{\text{Lag}} = \frac{1}{4} \left(dV^k \wedge dU_k + d\mathcal{C}_{(2)} \wedge dh + d\tilde{\rho}_{(2)}^a \wedge d\rho_a + d\tilde{c}_\alpha^{(2)} \wedge dc^\alpha \right) , \quad (2.28)$$

$$S_{O3/O7}^{\text{Lag}} = \frac{1}{4} \left(dV^\kappa \wedge dU_\kappa + d\mathcal{L}_{(2)} \wedge dl + d\tilde{\rho}_{(2)}^\alpha \wedge d\rho_\alpha + d\tilde{c}_a^{(2)} \wedge dc^a \right) , \quad (2.29)$$

which add an irrelevant total derivative to the action. Then the equations of motion of the four-dimensional tensor fields consistently reproduce the duality constraints (2.10) after dimensional reduction. We refer to appendix A.3 for a more detailed presentation of the dualization between the associated linear and chiral multiplets in a different context.

To further discuss the actions (2.25), (2.27) it is convenient to organize the various terms according to their origin in the reduction from the four-dimensional $\mathcal{N} = 2$ theory obtained by the reduction of Type IIB on $\mathbb{R}^{1,3} \times Z_3$ to the $\mathcal{N} = 1$ orientifold theory.

The first term is the standard Einstein-Hilbert action in four-dimensions that is obtained by a Weyl rescaling $\mathcal{V}^{-1}\eta^{\text{EF}}e^{-2\phi} = \eta^{\text{SF}}$ between the four-dimensional String-frame (SF) and Einstein-frame (EF) metric. Note that C_4 is even (odd) for $O3/O7$ ($O5/O9$) setups but $h_+^{(3,0)} = 0$ ($h_-^{(3,0)} = 0$). Thus, the four-dimensional graviphoton V^0 which is obtained from the reduction of $C_4 \supset V^0 \wedge \alpha_0$ in the $\mathcal{N} = 2$ theory is projected out in both orientifolds. The second to fourth term are the reduction of kinetic and instanton action of the $\mathcal{N} = 2$ vector multiplets to $\mathcal{N} = 1$ chiral and vector multiplets. However, the $\mathcal{N} = 2$ vector multiplets do not simply split into an equal number of $\mathcal{N} = 1$ chiral and vector multiplets. Of each $\mathcal{N} = 2$ vector multiplet either the chiral multiplet or the vector multiplet are projected out. Since the vector fields in the $\mathcal{N} = 2$ vector multiplet are obtained from the reduction of C_4 whereas the complex scalars are the zero-modes in the complex structure variation of the internal metric captured by Ω , the parity of C_4 and Ω with respect to the orientifold \mathcal{O} will determine the remaining fields in the $\mathcal{N} = 1$ orientifold theory. For $O5/O9$ -orientifolds $h_+^{(2,1)}$ chiral multiplets with complex scalars z^κ remain from the $h^{(2,1)} = h_+^{(2,1)} + h_-^{(2,1)}$ vector multiplets of the $\mathcal{N} = 2$ theory while the corresponding $\mathcal{N} = 1$ vectors are projected out. Conversely $h_-^{(2,1)}$ vector multiplets remain while the corresponding $\mathcal{N} = 1$ chiral multiplets are projected out. For $O3/O7$ -orientifolds the opposite $\mathcal{N} = 1$ multiplets survive. The scalar metric G and the gauge kinetic and instanton couplings, encoded in the complex matrix \mathcal{M} [142, 143], are intimately related to the geometry of the complex structure moduli space \mathcal{M}^{cs} of the orientifold Z_3/σ . The scalar metric G is a restriction of the $\mathcal{N} = 2$ metric on the complex structure moduli space of Z_3 . It is obtained from the Kähler potential K_{cs} as

$$G_{\kappa\bar{\lambda}} = \frac{\partial}{\partial z^\kappa} \frac{\partial}{\partial \bar{z}^\lambda} K_{\text{cs}} , \quad K_{\text{cs}} = -\ln \left[-i \int_{Z_3} \Omega \wedge \bar{\Omega} \right] = -\ln i \left[Z^K \bar{\mathcal{F}}_K - \bar{Z}^K \mathcal{F}_K \right] , \quad (2.30)$$

for $O5/O9$ -orientifolds and with $(\kappa, \lambda) \rightarrow (k, l)$ and $K \rightarrow \tilde{K}$ for $O3/O7$ -orientifolds where we used the expansions (2.16). We note that the periods (X^K, \mathcal{F}_K) of Z_3 have to be evaluated on the subvariety $z^k = 0$ ($z^\kappa = 0$) for $O5/O9$ ($O3/O7$) setups. The definition of the complex matrix \mathcal{M} and its relation with the periods is summarized in appendix A.1.

Similarly the remaining terms are related to reduction of the $\mathcal{N} = 2$ double-tensor [144–146] or its dual chiral multiplet containing the dilaton ϕ and of the quaternionic Kähler manifold of the hypermultiplets. The double-tensor multiplet is composed of the four-dimensional two-forms and scalars $(B_2, \mathcal{C}_2, \phi, C_0)$. For $O5/O9$ -orientifolds B_2 and C_0 are projected out whereas B_2 and \mathcal{C}_2 do not survive in $O3/O7$ -orientifolds. In the quaternionic sector the parity of the corresponding R–R-forms determines which one of the two $\mathcal{N} = 1$ chiral multiplets contained in the $\mathcal{N} = 2$ hypermultiplet remains. This and the complete orientifold spectrum are summarized in table 2.4. The coupling functions for the remaining terms in the actions (2.25), (2.27) are encoded by the Kähler sector of Z_3/σ , which are the classical intersection numbers \mathcal{K}_{ijk} of three divisors in Z_3 or their duals in $H^{(1,1)}(Z_3)$ and the metric on the Kähler moduli space \mathcal{M}^{K} . The restriction imposed by the orientifold are universal for both setups since $\sigma^*J = J$ and $\sigma^*B_2 = -B_2$ do hold for all B-parities, c.f. (2.3) and (2.4). The decomposition $H^{(1,1)} = H_+^{(1,1)} \oplus H_-^{(1,1)}$ results in a decomposition of the

$\mathcal{N} = 2$ multiplet	Type IIB		$\mathcal{N} = 1$ multiplet	$O3/O7$		$O5/O9$	
gravity	1	$(g_{\mu\nu}, V^0)$	gravity	1	$g_{\mu\nu}$	1	$g_{\mu\nu}$
vector	$h^{(2,1)}$	(V^K, z^K)	vector	$h_+^{(2,1)}$	V^λ	$h_-^{(2,1)}$	V^k
			chiral	$h_-^{(2,1)}$	z^k	$h_+^{(2,1)}$	z^λ
hyper	$h^{(1,1)}$	(v^i, b^i, c^i, ρ^i)	chiral/linear	$h_+^{(1,1)}$	$t^\alpha = (v^\alpha, \rho_\alpha)$	$h_-^{(1,1)}$	$P^a = (b^a, \rho_a)$
			chiral	$h_-^{(1,1)}$	$G^a = (b^a, c^a)$	$h_+^{(1,1)}$	$t^\alpha = (v^\alpha, c^\alpha)$
double- tensor	1	(B_2, C_2, ϕ, l)	chiral/linear	1	$\tau = (\phi, l)$		
						1	$S = (\phi, C_2)$

Table 2.4: The $\mathcal{N} = 1$ spectrum of the Type IIB orientifold compactifications. The split of the $\mathcal{N} = 2$ multiplet to $\mathcal{N} = 1$ multiplets is indicated. The labels of the $\mathcal{N} = 2$ hypermultiplets correspond to a basis ω^i of $H^{(1,1)}(Z_3)$.

intersection numbers \mathcal{K}_{ijk} . In the basis of table 2.3 only $\mathcal{K}_{\alpha\beta\gamma}$ and \mathcal{K}_{abc} can be non-zero while

$$\mathcal{K}_{\alpha\beta c} = \mathcal{K}_{abc} = \mathcal{K}_{\alpha b} = \mathcal{K}_a = 0, \quad (2.31)$$

is imposed by the orientifold projection. Here we used the definition $\mathcal{K}_{ab} = v^\beta \mathcal{K}_{\alpha\beta b}$, $\mathcal{K}_a = v^\alpha v^\beta \mathcal{K}_{\alpha\beta a}$. In general the Kähler metric on Z_3 is in general given by [12, 147]

$$G_{ij} = \frac{1}{4\mathcal{V}} \int_{Z_3} \omega_i \wedge * \omega_j = -\frac{\mathcal{K}_{ij}}{4\mathcal{V}} + \frac{\mathcal{K}_i \mathcal{K}_j}{16\mathcal{V}^2}, \quad (2.32)$$

where $*$ denotes the Hodge-star on Z_3 and ω_i a basis of $H^{(1,1)}(Z_3)$. It can be derived from the Kähler potential K^K given by

$$K^K = -\ln(8\mathcal{V}) = -\ln\left(\frac{i}{6} \mathcal{K}_{ijk} (t + \bar{t})^i (t + \bar{t})^j (t + \bar{t})^k\right), \quad (2.33)$$

where the $t^i = v^i + i b^i$ denote the complexified coordinates on the Kähler moduli space of Z_3 . The metric on the Kähler moduli space \mathcal{M}^K of the orientifold is then obtained as

$$G_{\alpha\beta} = -\frac{\mathcal{K}_{\alpha\beta}}{4\mathcal{V}} + \frac{\mathcal{K}_\alpha \mathcal{K}_\beta}{16\mathcal{V}^2}, \quad G_{ab} = -\frac{\mathcal{K}_{ab}}{4\mathcal{V}}, \quad G_{\alpha b} = G_{a\beta} = 0, \quad (2.34)$$

where we introduced the intersections

$$\mathcal{K}_{\alpha\beta} = \mathcal{K}_{\alpha\beta\gamma} v^\gamma, \quad \mathcal{K}_{ab} = \mathcal{K}_{ab\gamma} v^\gamma, \quad \mathcal{K}_\alpha = \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma, \quad \mathcal{K} = \mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma. \quad (2.35)$$

Its inverse is given by

$$G^{\alpha\beta} = -\frac{2}{3}\mathcal{K}\mathcal{K}^{\alpha\beta} + 2v^\alpha v^\beta, \quad G^{ab} = -\frac{2}{3}\mathcal{K}\mathcal{K}^{ab}, \quad (2.36)$$

where $\mathcal{K}^{\alpha\beta}$ and \mathcal{K}^{ab} are the formal inverses of $\mathcal{K}_{\alpha\beta}$ and \mathcal{K}_{ab} , respectively.

Having defined the couplings of the reduced quaternionic sector we note that the corresponding orientifold actions (2.25), (2.27) for the $O5/O9$ - and the $O3/O7$ -setup differ since the massless spectrum from the R-R-sector differs, which in particular yields different couplings of the B-field zero modes b^a .

2.2.3 The $\mathcal{N} = 1$ Couplings and Coordinates

In this section we rewrite the four-dimensional effective action of the orientifold theory into the standard $\mathcal{N} = 1$ supergravity form. We first determine the complex coordinates M^I forming the bosonic part of the $\mathcal{N} = 1$ chiral multiplets. Their kinetic terms are expressed by a Kähler potential $K(M, \bar{M})$, while their F-term scalar potential is encoded by a holomorphic superpotential $W(M)$. The couplings of the $\mathcal{N} = 1$ vector multiplets are encoded by holomorphic gauge-kinetic coupling functions $f(M)$. Possible gaugings of the scalars M^I will induce a D-term potential. The general form of the bosonic $\mathcal{N} = 1$ action is given by [59, 148]

$$S^{(4)} = - \int \frac{1}{2}R * 1 + K_{I\bar{J}} \mathcal{D}M^I \wedge * \mathcal{D}\bar{M}^{\bar{J}} + \frac{1}{2} \text{Re} f_{\kappa\lambda} F^\kappa \wedge * F^\lambda + \frac{1}{2} \text{Im} f_{\kappa\lambda} F^\kappa \wedge F^\lambda + V * 1, \quad (2.37)$$

where the scalar potential reads

$$V = e^K (K^{I\bar{J}} D_I W D_{\bar{J}} \bar{W} - 3|W|^2) + \frac{1}{2} (\text{Re } f)^{-1 \kappa\lambda} D_\kappa D_\lambda. \quad (2.38)$$

Here we set the four-dimensional gravitational coupling $\kappa_4 = 1$ for simplicity. We denote by $K_{I\bar{J}}$ and $K^{I\bar{J}}$ the Kähler metric and its inverse, where the former is locally given as $K_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} K(M, \bar{M})$. The scalar potential is expressed in terms of the Kähler-covariant derivative $D_I W = \partial_I W + (\partial_I K)W$ which promotes $\partial_I K$ to a connection one-form on the scalar manifold.

Before we begin our discussion we note that in case that the action (2.37) is deduced from a higher-dimensional theory the formal expression for the Kähler potential can in general already be inferred from Weyl rescaling arguments, e.g. from the factor e^K in front of the $\mathcal{N} = 1$ F-term potential in (2.38). However, the explicit form as a function of the $\mathcal{N} = 1$ chiral superfields depends on the setup under consideration and is in general not analytically computable. We also note that we will not obtain, for the case of orientifold compactifications of Type IIB, the most general $\mathcal{N} = 1$ effective action since we did not include fluxes and thus do not obtain an F-term scalar potential, i.e. we find $W = 0$. We will include fluxes and a non-trivial superpotential in section 3.2.

The Kähler potential and $\mathcal{N} = 1$ coordinates

As a first step to identify the $\mathcal{N} = 1$ characteristic data we have to find the appropriate complex coordinates for which the scalar metrics are manifestly Kähler. First, we note that the complex structure moduli z^k , respectively $z^{\bar{k}}$, already define Kähler coordinates for both orientifolds since the scalar metric in the actions (2.25), (2.27) is already Kähler with Kähler potential (2.30). The other Kähler coordinates are different for the two setups and are discussed next.

For $O5/O9$ -orientifolds the additional complex fields in the chiral multiplets read [128, 130]

$$t^\alpha = e^{-\phi} v^\alpha - i c^\alpha, \quad P_a = \Theta_{ab} b^b + i \rho_a, \quad S = e^{-\phi} \mathcal{V} + i \tilde{h} - \frac{1}{4} (\text{Re} \Theta)^{ab} P_a (P + \bar{P})_b, \quad (2.39)$$

where $v^\alpha, b^a, c^\alpha, \rho_a$ and $\tilde{h} = h - \frac{1}{2} \rho_a b^a$ are given in (2.15), (2.19) and (2.21). The complex symmetric tensor Θ appearing in (2.39) is given by $\Theta_{ab} = \mathcal{K}_{ab\alpha} t^\alpha$ and $(\text{Re} \Theta)^{ab}$ denotes the inverse of $\text{Re} \Theta_{ab}$. More conceptually, the coordinates (2.39) can be obtained by probing the cycles in Z_3/σ by D-branes and are then formally extracted from [128, 130, 149–151]

$$\text{Im}(\varphi^{ev}) - i\mathcal{A} = t^\alpha \omega_\alpha - P_b \tilde{\omega}^b - S m_6. \quad (2.40)$$

Here we introduced the polyforms of even forms $\varphi^{ev}, \mathcal{A}$ in the NS–NS- respectively R–R-sector,

$$\varphi^{ev} = e^{-\phi} e^{-B_2 + iJ}, \quad \mathcal{A} = e^{-B_2} \wedge \sum_{q=0,2,4,6,8} C_q, \quad (2.41)$$

which have to be interpreted, as (2.40), as formal polynomials of forms in the even cohomology $H^{ev}(Z_3)$ of different degrees. Then the reduction ansätze (2.15), (2.19) and (2.21) have to be inserted to obtain the coordinates (2.39) as the coefficients of a basis expansion of (2.40). Roughly, the forms φ^{ev} and \mathcal{A} are the Dirac-Born-Infeld and Chern-Simons actions of Dp -instantons, $p = 1, 3, 5$, wrapping the even-dimensional cycles and the coordinates (2.39) are the complexified volumes of the corresponding cycles in the Type IIB $O5/O9$ -background.

For $O3/O7$ -orientifolds besides the complex structure moduli z^k the additional Kähler coordinates read [128, 130],

$$\tau = l + i e^{-\phi}, \quad G^a = c^a - \tau b^a, \quad T_\alpha = i(\rho_\alpha - \frac{1}{2} \mathcal{K}_{\alpha ab} c^a b^b) + \frac{1}{2} e^{-\phi} \mathcal{K}_\alpha + \frac{i}{2(\tau - \bar{\tau})} \mathcal{K}_{abc} G^b (G - \bar{G})^c, \quad (2.42)$$

where \mathcal{K}_α is introduced in (2.35). This data is again summarized as

$$\text{Re}(\varphi^{ev}) - i\mathcal{A} = -i\tau - iG^a \omega_a - T_\alpha \tilde{\omega}^\alpha, \quad (2.43)$$

which is interpreted as the complexified volume of even dimensional cycles wrapped by Dp -instantons, $p = -1, 1, 3$, in the Type IIB $O3/O7$ -background.

The definition of the appropriate $\mathcal{N} = 1$ coordinates prepares us for the determination of the $\mathcal{N} = 1$ coupling functions. As mentioned before, we do not obtain superpotential a W

due to the absence of fluxes and D-branes, $W = 0$. We first determine the Kähler potential K . In general, the full $\mathcal{N} = 1$ Kähler potential is determined by integrating the kinetic terms of the complex scalars M^I .

For $O5/O9$ -orientifolds it takes the form [128, 130]

$$K = K_{\text{cs}}(z, \bar{z}) + K^{\text{q}}(S, t, P), \quad K_{\text{cs}} = -\ln \left[-i \int \Omega \wedge \bar{\Omega} \right] \quad (2.44)$$

where we introduce the Kähler potential

$$\begin{aligned} K^{\text{q}} &= -\ln \left[\frac{1}{48} \mathcal{K}_{\alpha\beta\gamma} (t + \bar{t})^\alpha (t + \bar{t})^\beta (t + \bar{t})^\gamma \right] - \ln \left[S + \bar{S} + \frac{1}{4} (P + \bar{P})_a (\text{Re} \Theta^{-1})^{ab} (P + \bar{P})_b \right] \\ &= -2 \ln \left[\sqrt{2} e^{-2\phi} \mathcal{V} \right]. \end{aligned} \quad (2.45)$$

We note that the scalar metric corresponding to K is block-diagonal in the fields \underline{z} and $(S, \underline{P}, \underline{t})$. Thus, we infer that the scalar manifold of the $O5/O9$ -orientifold effective theory is locally a direct product $\mathcal{M}^{\text{cs}} \times \mathcal{M}^{\text{q}}$. The first factor is the special Kähler manifold with local coordinates z^k . It is realized as a submanifold of the complex structure moduli space of Z_3 , which itself is special Kähler. The second factor \mathcal{M}^{q} is a Kähler submanifold of dimension $h^{(1,1)} + 1$ in the quaternionic manifold of the $\mathcal{N} = 2$ hypermultiplets. Locally it is a fibration of the scalar manifold of \underline{P}, S over the Kähler moduli space of the orientifold Z_3/σ .

For $O3/O7$ -orientifolds the Kähler potential reads [128, 130]

$$K = K_{\text{cs}}(z, \bar{z}) + K^{\text{q}}(\tau, T, G), \quad K_{\text{cs}} = -\ln \left[-i \int \Omega \wedge \bar{\Omega} \right] \quad (2.46)$$

where we introduce the Kähler potential

$$K^{\text{q}} = -\ln \left[-i(\tau - \bar{\tau}) \right] - 2 \ln \left[\frac{1}{6} e^{-\frac{3}{2}\phi} \mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma \right] = -2 \ln \left[\sqrt{2} e^{-2\phi} \mathcal{V} \right]. \quad (2.47)$$

We note that the scalar manifold of the $O3/O7$ -orientifold effective theory is again locally a direct product $\mathcal{M}^{\text{cs}} \times \mathcal{M}^{\text{q}}$. However, the first factor is a different special Kähler submanifold with local coordinates z^k of the complex structure moduli space of Z_3 than in the $O5/O9$ case. The second factor \mathcal{M}^{q} is a Kähler submanifold of the quaternionic manifold of the $\mathcal{N} = 2$ hypermultiplets. Locally it is a fibration of the scalar manifold of $\underline{T}, \underline{G}$ over the dilaton moduli space. We note that the full Kähler potential K is invariant, up to a Kähler transformation, under the action of $SL(2, \mathbb{R})$ on τ as required by $SL(2, \mathbb{R})$ -duality of the ten-dimensional Type IIB theory.

In general a first check of the consistency of the Kähler coordinates (2.39), (2.39) with the corresponding Kähler potentials (2.44), (2.46) is provided by a comparison with the expected result from Weyl rescaling,

$$K = -2 \ln \left[\sqrt{2} e^{-2\phi} \mathcal{V} \int \Omega \wedge \bar{\Omega} \right]. \quad (2.48)$$

This is in perfect agreement with the above results. However, we emphasize that the real difficulty in the determination of the $\mathcal{N} = 1$ characteristic data is to express the general expression (2.48) as a function of appropriate coordinates so that all scalar kinetic terms, as well as some mixing terms, in the action are reproduced.

We conclude by rephrasing the $\mathcal{N} = 1$ Kähler potential in a more formal and compact form for both orientifolds. We note that both Kähler potentials (2.45), (2.47) can be written in a unified way as [128, 130]

$$K^q = -2 \ln \Phi_B, \quad \Phi_B := i \langle \varphi^{\text{ev}}, \bar{\varphi}^{\text{ev}} \rangle \quad (2.49)$$

up to a Kähler transformation where we introduced the pairing [152]

$$\langle \varphi, \psi \rangle = \int_{Z_3} \sum_m (-1)^m \varphi_{2m} \wedge \psi_{6-2m} \quad (2.50)$$

on the space of polyforms. The only difference between the two orientifold setups then reduces to the choice of variables on which (2.49) depends. For the case of $O5/O9$ -orientifolds K^q is a function of $\text{Im}(\varphi^{\text{ev}})$ only, in particular the real part has to be understood as a function of $\text{Im}(\varphi^{\text{ev}})$. For $O3/O7$ -orientifolds, however, (2.49) is a function of $\text{Re}(\varphi^{\text{ev}})$ and the imaginary parts is determined by $\text{Re}(\varphi^{\text{ev}})$.

The Gauge Kinetic Function

We conclude with the determination of the gauge kinetic coupling $f_{\kappa\lambda}$ which is a holomorphic function of the chiral superfields. By comparison of the actions (2.25), (2.27) with the general $\mathcal{N} = 1$ action (2.37) we obtain [128, 130]

$$O5/O9 : f_{kl} = -\frac{i}{2} \bar{\mathcal{M}}_{kl} \Big|_{z^k=0=\bar{z}^l}, \quad O3/O7 : f_{\kappa\lambda} = -\frac{i}{2} \bar{\mathcal{M}}_{\kappa\lambda} \Big|_{z^\kappa=0=\bar{z}^\lambda} \quad (2.51)$$

where \mathcal{M} is introduced in appendix A.1 and a submatrix of the $\mathcal{N} = 2$ gauge kinetic coupling matrix. It has to be evaluated on the submanifold $z^k = \bar{z}^l = 0$ or $z^\kappa = \bar{z}^\lambda = 0$, respectively, in the complex structure moduli space of Z_3 . Its holomorphicity in the complex structure deformations z^κ, z^k follows since f can be rewritten for both orientifolds in terms of the holomorphic $\mathcal{N} = 2$ prepotential $\mathcal{F}(z)$ evaluated on the submanifold \mathcal{M}^{cs} [128, 130],

$$O5/O9 : f_{kl}(z^\kappa) = -\frac{i}{2} \mathcal{F}_{kl} \Big|_{z^k=0=\bar{z}^l}, \quad O3/O7 : f_{\kappa\lambda}(z^k) = -\frac{i}{2} \mathcal{F}_{\kappa\lambda} \Big|_{z^\kappa=0=\bar{z}^\lambda}. \quad (2.52)$$

Chapter 3

The D5-Brane Effective Action

In this chapter we begin with the analysis of brane dynamics for the example of D5-branes in generic $O5/O9$ -Calabi-Yau orientifolds. We present a detailed computation of the $\mathcal{N} = 1$ effective action of a spacetime-filling D5-brane wrapping a curve Σ , mainly following the original work [60]. First in section 3.1 we start with a general review of supersymmetric D-branes in Calabi-Yau manifolds, where we focus on their low-energy effective dynamics as encoded by the Dirac-Born-Infeld and Chern-Simons action and on the geometric calibration conditions on the internal cycles supporting a BPS D-brane. Then the actual calculation of the D5-brane effective action is performed in section 3.2 by a purely bosonic reduction. This is divided into four parts, in which we first count the bosonic four-dimensional massless spectrum both of the closed and open string sector, then present some special identities on the space of open-closed geometric moduli before we actually derive the D5-brane effective action, where we put special emphasis on the determination of the scalar potential. The calculation of the scalar potential reveals the ad-hoc surprising relevance of keeping non-dynamical three-forms in the four-dimensional action in order to recover the complete F-term potential in a purely bosonic reduction. Next in section 3.3 we work out the $\mathcal{N} = 1$ characteristic data of the D5-brane action, in particular in comparison with $O5/O9$ -orientifold result of chapter 2. We determine new $\mathcal{N} = 1$ chiral coordinates and the Kähler potential that is derived explicitly at large volume of Z_3 . Then the effective superpotential is read off, which is a projection of the familiar Type IIB flux superpotential and the D5-brane superpotential, that is given as a chain integral. The $\mathcal{N} = 1$ data is completed by the determination of the D5-brane gauge kinetic function, the gaugings of chiral fields and the corresponding D-terms. Finally in section 3.4 we extend, following [100], the reduction of [60] to the full geometric deformation space of the D5-brane which is infinite dimensional and does include massive fields. This is proposed to be the natural domain of the D5-brane superpotential which encodes the general deformation theory of the curve Σ in Calabi-Yau threefold Z_3 . We concluded by a calculation of the corresponding F-term scalar potential as a functional of the infinite dimensional space of massive modes. Further details on some calculations performed in the context of the derivation of the D5-brane action can be found in appendix A.

3.1 BPS D-Branes in Calabi-Yau Manifolds

In general the existence of D-branes as dynamical objects in string theory can be inferred from first principles, namely the spectrum of BPS-particles and string dualities. It is a basic fact found in textbooks, see e.g. [5], that the fundamental string state with two oscillator modes, winding number m and no momentum along a compact spacetime direction, here an S^1 , has a mass proportional to m . The winding number m is the quantized charge with respect to the NS–NS B-field and indeed this state is identified as a BPS-state in the theory [132]. In combination with the non-perturbative $SL(2, \mathbb{Z})$ -symmetry of Type IIB theory, that acts on the B-field B_2 and the R–R-form C_2 as a doublet, this predicts a whole family of BPS-states with charge (p, q) under (B_2, C_2) for any pair of relative primes p, q [52]. In particular one expects fundamental states with charge $(0, 1)$ that are charged under C_2 . Since C_2 couples electrically to string-like objects the fundamental object corresponding to these states is called a D-string. Analogously, the strings associated to BPS-states with charge (p, q) are denoted (p, q) -strings. Then by T-duality one expects BPS-states charged under all R–R-forms C_p in the theory which correspond to quantized $p + 1$ -dimensional membranes. However, these states do not have a perturbative description in terms of quantized fundamental string excitations since all perturbative states do not couple to the R–R-forms but their field strengths. Consequently, these states are invisible in the spectrum of the perturbative string. However, they have convenient descriptions in various limits of string theory.

On the one hand, at low-energies membranes with non-zero electric R–R-charge are found in the effective supergravity theory as solitons, i.e. stable finite energy solutions of the classical equations of motions with a conserved (topological) charge called p -branes, see e.g. [5] for a summary. Also the fundamental string occurs on an equal footing as a solitonic solution with non-zero NS–NS-charge. It can further be shown that these solutions indeed obey a BPS-bound. On the other hand, membranes are visible at small coupling in the perturbative string as $9 - p$ Dirichlet boundary conditions on the string endpoints, denoted by Dp -branes.

Both descriptions of membranes have different regimes of validity, however. The requirement of the supergravity description is low curvature, which in more technical terms means, that the Schwarzschild radius r_S of the p -brane solution is big compared to the string scale $\sqrt{\alpha'}$. This requirement yields [40]

$$n_p g_S \gg 1 \tag{3.1}$$

where n_p is the integral number of fundamental p -brane charge of the solution. The requirement for the validity of the perturbative string is of course small string coupling g_S so that we obtain

$$n_p g_S \ll 1, \tag{3.2}$$

which is the opposite to (3.1). Despite their different validity, however, it is believed that both the p -brane and the Dp -brane describe one and the same fundamental object simply denoted

as a Dp -brane. This conjecture is well established by various checks invoking techniques from different corners of string theory starting with the CFT analysis in the groundbreaking work [37], see [16, 17, 28, 29] for reviews.

For our purposes we treat spacetime-filling D-branes and their dynamics as a BPS-background state in Type IIB string theory with small fluctuations around it. Here we will mainly be restricted to low energies which means that we are working with Type IIB supergravity and to scales larger than the thickness of the D-brane where the description in terms of the D-brane effective action is valid. Then the BPS condition on the D-brane is translated into BPS calibration conditions [36] on the embedding of the D-brane worldvolume \mathcal{W}_{p+1} and the D-brane worldvolume fields into the Type IIB background under consideration. By additional consistency conditions of the setup imposed on the one hand by $\mathcal{N} = 1$ supersymmetry in extended four-dimensional Minkowski space and on the other hand by cancellations of tadpoles we will consider Calabi-Yau threefolds Z_3 with $O5/O9$ -orientifolds.

Following this program we start by introducing the effective action of a single D-brane. The spectrum of light fields on the D-brane can in general be thought of as a reduction of the $\mathcal{N} = 1$ Super-Yang-Mills theory in ten dimensions to the worldvolume \mathcal{W}_{p+1} [52]. It consists of an $\mathcal{N} = 1$ supersymmetric $U(1)$ -gauge theory on the submanifold \mathcal{W}_{p+1} . In addition there are adjoint-valued scalars corresponding to fluctuations $\delta\iota$ of the embedding $\iota : \mathcal{W}_{p+1} \rightarrow M_{10}$ into ten-dimensional spacetime. The effective action of these fields consists of two parts, one part being the Dirac-Born-Infeld action [4, 38, 39]

$$S_{\text{DBI}}^{\text{SF}} = -T_p \int_{\mathcal{W}_{p+1}} e^{-\phi} \sqrt{\det \iota^*(g + B_2) - \ell F} \quad (3.3)$$

in string-frame units, where T_p denotes the Dp -brane tension and $\ell = 2\pi\alpha'$. The second part of the effective action is the Chern-Simons action encoding the coupling of the D-brane to the R-R-sector,

$$S_{\text{CS}} = -\mu_p \int_{\mathcal{W}_{p+1}} \sum_q C_q \wedge e^{\ell F - B_2}, \quad (3.4)$$

where μ_p denotes the D-brane charge. The formal sum over all R-R-potentials indicates a coupling of the D-brane to forms of degree lower than C_{p+1} in a background of worldvolume flux/instantons [153]. We note that the presence of the B-field B_2 in the combination $\mathcal{B} := B_2 - \ell F$ can be understood from gauge invariance in the bulk. The string sigma-model action for world-sheets \mathcal{W}_2 with boundary contains the terms

$$S_\sigma \supset \frac{1}{\ell} \int_{\mathcal{W}_2} B_2 + \int_{\partial\mathcal{W}_2} A, \quad (3.5)$$

where A denotes the gauge field on the D-brane. Since this is not invariant under the gauge transformation $\delta B_2 = d\Lambda$ alone, one has to demand $\delta A = \frac{1}{\ell}\Lambda$. Thus, \mathcal{B} is the only gauge invariant combination [52].

Next we formulate the BPS conditions a D-brane in a Calabi-Yau manifold has to meet to define a supersymmetric background [36, 134]. The most immediate condition is of course

$\mu_p = T_p$ which is enforced by the SUSY algebra for every BPS state. For vanishing background fields the BPS-conditions further imply for a spacetime-filling D6-brane in Type IIA to wrap a special Lagrangian submanifold L , which can roughly be thought of as a real locus of real dimensions d in a complex manifold of complex dimension d . In terms of the complex structure I on Z_3 this is expressed by $I(TL) \subset NL$. A trivial example is a special Lagrangian L in \mathbb{C} which is just e.g. the real axis, where I is the usual multiplication by i . Equivalently L is defined as the volume-minimizing representative within its homology class such that the volume form vol_L on L is calibrated as [134]

$$\iota^*(J) = 0, \quad e^{-i\theta_{D_6}} \iota^*(\Omega) = \frac{\sqrt{\det(g + \mathcal{B})}}{\sqrt{\det(g)}} \text{vol}_L. \quad (3.6)$$

The constant angle θ_{D_6} is a priori free and determines the covariantly constant spinor corresponding to the supercharge conserved by the BPS D-brane. More precisely θ_{D_6} determines the linear combination of the two covariantly constant spinors of the Type II compactification on Z_3 [36] that is unbroken by the D-brane background and thus yields a supercharge in four dimensions generating $\mathcal{N} = 1$ spacetime supersymmetry. All additional D-branes in the setup have to be calibrated with respect to the same calibration form $e^{-i\theta_{D_p}} \iota^*(\Omega)$ in order to preserve the same covariantly constant spinor and thus preserve $\mathcal{N} = 1$ spacetime supersymmetry. The identical argumentation applies to orientifold planes, which are $O6$ -planes in the Type IIA case, that have to be included for tadpole cancellation discussed at the end of this section. Then, the free angle θ_{D_6} has to equal the angle θ in the orientifold calibration (2.8).

In Type IIB a spacetime-filling BPS D-brane is supported along holomorphic submanifolds in Z_3 [36]. This in particular yields a natural choice of complex structure on Σ by aligning it with the ambient complex structure using the embedding ι which then defines a holomorphic map obeying $\partial\bar{z}^i(u^a)/\partial u = \partial z^i(u^a)/\partial \bar{u} = 0$, where we introduced complex coordinates \underline{u} on the worldvolume \mathcal{W}_{p+1} . In other words, D-branes in Type IIB test the even dimensional homology of Z_3 , that are D3-branes on points, D5-branes on holomorphic curves Σ , D7-branes on holomorphic divisors D or D9-branes on the entire Calabi-Yau threefold Z_3 . A holomorphic submanifold is volume minimizing since its volume form is just proportional to (powers of) the pullback of J , a well-known mathematical fact for complex submanifolds of Kähler manifolds [154]. In the presence of background fields this is replaced by the more general calibration conditions [134]

$$\iota^*(\Omega) = 0, \quad e^{-i\theta_{D_p}} \iota^* e^{J+i\mathcal{B}}|_{D_p} = \frac{\sqrt{\det(g + \mathcal{B})}}{\sqrt{\det(g)}} \text{vol}_{D_p}, \quad p = 3, 5, 7, 9. \quad (3.7)$$

where vol_{D_p} denotes the volume form on the analytic cycle wrapped by the Dp -brane. Again we note that in the presence of O_p -orientifold planes, a necessary ingredient to cancel tadpoles, the angle θ_{D_p} has to equal the angle in the orientifold calibration (2.8), which is $\theta_{D_p} = 0$. Then the calibrations (3.7) are even more restrictive. Explicitly, we evaluate the real and

imaginary part of the calibration conditions to obtain

$$\begin{aligned}
D5 & : \quad du^2 \sqrt{-\det(\iota^*(g_{10} + \mathcal{B}))} = \iota^* J, & \iota^* \mathcal{B} &= 0, \\
D7 & : \quad du^4 \sqrt{-\det(\iota^*(g_{10} + \mathcal{B}))} = \frac{1}{2} \iota^*(J^2 - \mathcal{B}^2), & \iota^*(\mathcal{B} \wedge J) &= 0, \\
D9 & : \quad du^6 \sqrt{-\det(\iota^*(g_{10} + \mathcal{B}))} = \frac{1}{6} \iota^*(J^3 - J \wedge \mathcal{B}^2), & \iota^*(\mathcal{B} \wedge J^2 - \mathcal{B}^3) &= 0.
\end{aligned} \tag{3.8}$$

Once again, these formulas are given in the string frame and can be translated to the Einstein frame by multiplying the first column of equation in (3.8) by e^ϕ .

Clearly, the calibration conditions (3.6), (3.7) only hold in the supersymmetric background configuration or vacuum. Consequently, in order to understand the *dynamics* of D-branes we have to allow for small variations of the fields about this vacuum that explicitly violate the conditions (3.6), (3.7). This, however, is part of the derivation of a low-energy effective action as performed in section 3.2 for D5-branes.

We conclude this section by a brief discussion of tadpoles. Dp -branes couple electrically via the Chern-Simons-action (3.4) to C_{p+1} . This yields localized source terms in the equation of motion of C_{p+1} of the form

$$dF_{8-p} = \sum_i \mu_p \delta_i^{(9-p)} + \text{induced charges} \tag{3.9}$$

for each Dp -brane on the support of $\delta_i^{(9-p)}$, that is a $(9-p)$ -form¹ on the normal space to the cycle wrapped by the D-brane. Here we also included possible charges induced from higher-dimensional Dq -branes, $q > p$, due to worldvolume fluxes [153]. Since the left-hand-side of (3.9) is exact, the sum on the right has to vanish in cohomology, which goes by the name of tadpole cancellation. This implies that we have to include negatively charged orientifolds planes in the same homology class as the D-branes to meet the requirement of tadpole cancellation in a supersymmetric fashion.

3.2 Dynamics of D5-Branes in Calabi-Yau Orientifolds

In this section we derive the effective action of a spacetime-filling D5-brane wrapped on a curve Σ . We turn on bulk and brane fluctuations around the vacuum defined in section 3.1 yielding fields in the four-dimensional action. We discuss this spectrum of fields in detail in section 3.2.1, where we put special emphasis on the universal sector of D5-brane fluctuations associated to geometric deformations of the curve Σ . As a first step to understand the structure of the combined geometrical open-closed deformation modes we derive some special identities between light open-closed fields in section 3.2.2. Then, in section 3.2.3, we perform the actual reduction of the D5-brane action, where we restrict to bosonic fields only and infer the fermionic action from supersymmetry. We conclude the dimensional reduction in 3.2.4

¹The analogue of delta-function as a k -form is properly called a delta-current, as introduced in chapter 7.

by a detailed derivation of the scalar potential, for which the couplings of non-dynamical four-dimensional three-forms are essential.

3.2.1 The Four-dimensional Spectrum

Here we discuss the four-dimensional spectrum emerging from compactification of Type IIB theory with D5-branes. We start our discussion by fixing the background geometry of our setup. As before in section 2.2.2 we consider the direct product of a compact Calabi-Yau orientifold Z_3/σ and flat Minkowski space $\mathbb{R}^{1,3}$ for which the metric in the string frame reads

$$ds_{10}^2 = \eta_{\mu\nu}^{\text{SF}} dx^\mu dx^\nu + 2g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} . \quad (3.10)$$

Since we are interested in compactifications which allow the inclusion of space-time filling D5-branes we have to add $O5$ -planes to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions. This fixes the orientifold projection to the $O5/O9$ case in (2.5) with the central properties

$$\mathcal{O} = \Omega_p \sigma^* , \quad \sigma^* J = J , \quad \sigma^* \Omega = \Omega , \quad (3.11)$$

for an isometric, holomorphic involution σ . The spectrum consists of two classes of fields. Firstly, there are zero modes arising from the expansion of the ten-dimensional closed string fields into harmonics on Z_3 . Secondly, one finds zero modes arising from the D5-brane effective actions (3.3) and (3.4). In the following we discuss both sets of fields in turn, where we only briefly summarize the closed string content and refer to section 2.2.1 for a more detailed discussion.

Closed String Spectrum

We start by recalling the following transformation behavior of the bulk fields under the orientifold from table 2.2 of section 2.2.1

$$\sigma^* g_{10} = g_{10} , \quad \sigma^* B_2 = -B_2 , \quad \sigma^* \phi = \phi , \quad \sigma^* C_p = (-1)^{(p+2)/2} C_p . \quad (3.12)$$

This yields the following reduction ansatz for the massless four-dimensional fields in terms of the cohomology groups of table 2.3,

$$\begin{aligned} \delta g_{i\bar{j}} &= -i(\omega_\alpha)_{i\bar{j}} \delta v^\alpha \quad (\alpha = 1, \dots, h_+^{(1,1)}) , & \delta g_{ij} &= \frac{i\nu}{\int \Omega \wedge \bar{\Omega}} \Omega_j^{\bar{i}\bar{j}} (\bar{\chi}_{\bar{k}})_{\bar{i}\bar{j}} \delta \bar{z}^{\bar{k}} \quad (\kappa = 1, \dots, h_+^{(2,1)}) , \\ J &= v^\alpha(\underline{x}) \omega_\alpha , & B_2 &= b^a(\underline{x}) \omega_a , & \phi &= \phi(\underline{x}) , \end{aligned} \quad (3.13)$$

in the NS–NS-sector and accordingly in the R–R-sector

$$\begin{aligned} C_6 &= A_{(3)}^K \wedge \alpha_K + \tilde{A}_K^{(3)} \wedge \beta^K + \tilde{c}_\alpha^{(2)} \wedge \tilde{\omega}^\alpha + h m_6 , & (3.14) \\ C_4 &= V^k \wedge \alpha_k + U_k \wedge \beta^k + \tilde{\rho}_{(2)}^a \wedge \omega_a + \rho_a \tilde{\omega}^a , \\ C_2 &= \mathcal{C}_{(2)} + c^\alpha \omega_\alpha . \end{aligned}$$

Here the fields $(\underline{A}_{(3)}, \tilde{\underline{A}}^{(3)})$ are three-forms, $(\tilde{\underline{c}}^{(2)}, \tilde{\underline{\rho}}_{(2)}, \underline{\mathcal{C}}_{(2)})$ are two-forms, $(\underline{V}, \underline{U})$ are vectors and $(h, \underline{\rho}, \underline{c}, \underline{v}, \underline{z}, \underline{b}, \phi)$ are scalars in four-dimensional Minkowski space $\mathbb{R}^{1,3}$.

Open String Spectrum

Let us now include space-time filling D5-branes into our setup. In general, they can be arranged in a complicated way as long as the consistency constraints for the compactification are met. We consider a stack of N D5-branes on a curve Σ in Z_3 . If Σ is in the fix-point set of the involution σ , the D5-branes lie on top of an orientifold five-plane and Σ is its own image with respect to σ . More generally Σ can be mapped to a curve $\Sigma' = \sigma(\Sigma)$ which is not pointwise identical to Σ .

In this work we will mostly focus on the simplest situation, for which $N = 1$, $\Sigma \cap \Sigma' = \emptyset$ and Σ, Σ' are in different homology classes. Hence, we consider one D5-brane on Σ and its image brane on Σ' . For this situation the pair of the D5-brane and its image D5-brane is merely an auxiliary description of a single smooth D5-brane wrapping a cycle in the orientifold Z_3/\mathcal{O} . On Z_3 it is natural to define the curves

$$\Sigma_+ = \Sigma + \Sigma' , \quad \Sigma_- = \Sigma - \Sigma' , \quad (3.15)$$

where Σ_+ is the union of Σ and Σ' while Σ_- contains the orientation reversed cycle Σ' . Clearly, one finds that $\sigma(\Sigma_{\pm}) = \pm\Sigma_{\pm}$.

We first discuss the degrees of freedom from the gauge theory on the D5-brane due to $U(1)$ Wilson lines $a_I(x)$. They arise from non-trivial one-cycles on the six-dimensional D5-brane world-volume. These enter the expansion of the $U(1)$ gauge boson $A(\xi)$ on the D5-brane as

$$A(\underline{x}, \underline{u}) = A_{\mu}(\underline{x})dx^{\mu}P_{-}(\underline{u}) + a_I(\underline{x})A^I(\underline{u}) + \bar{a}_{\bar{I}}(\underline{x})\bar{A}^{\bar{I}}(\underline{u}) , \quad (3.16)$$

where $A_{\mu}(\underline{x})$ denotes the four-dimensional gauge-field. Here we introduce real coordinates ξ on the world-volume of the D5-brane where we distinguish coordinates $\xi = (\underline{x}, \underline{u})$ for the Minkowski space and the two-cycle Σ_+ , respectively. We denote complex coordinates for Σ_+ by u, \bar{u} in the complex structure induced by the ambient space Z_3 . The one-forms $A^I = A_u^I d\bar{u}$, $\bar{A}^{\bar{I}} = \bar{A}_{\bar{u}}^{\bar{I}} du$ denote a basis of the Dolbeault cohomology $H_{-}^{0,1}(\Sigma_+)$ and $H_{-}^{1,0}(\Sigma_+)$, respectively, and P_{-} is the step function equaling 1 on Σ and -1 on Σ' . Note that generally the $U(1)$ field strength $F = dA$ can admit a background flux $\langle F \rangle = f$. Since F is negative under σ , it enjoys the expansion

$$f = f^a \iota^* \omega_a = f^a (\iota^* \omega_a)_{u\bar{u}} du \wedge d\bar{u} , \quad (3.17)$$

where $\iota^* \omega_a$ denotes the pullback of the basis elements ω_a of $H_{-}^{(1,1)}(Z_3)$ introduced in Table 2.3. As we already recalled in section 3.1, F naturally combines with the NS-NS B-field into the combination $\mathcal{B} = \iota^*(B_2) - \ell F$ with $\ell = 2\pi\alpha'$. Thus it is convenient to consider variations of \mathcal{B} around the background value (3.8) in the calibrated vacuum. We denote them by

$$\mathcal{B}^a(\underline{x}) = b^a(\underline{x}) - \ell f^a , \quad \mathcal{B}^{\Sigma} = \mathcal{B}^a \int_{\Sigma_-} \iota^* \omega_a , \quad (3.18)$$

multiplet	closed		open	
chiral	$h_+^{(1,1)}$	$t^\alpha = (v^\alpha, c^\alpha)$	$h_+^0(\Sigma_+, N\Sigma_+)$	ζ^A
	$h_-^{(1,1)}$	$P_a = (b^a, \rho_a)$		
	1	$S = (\phi, h)$	$h^{(1,0)}(\Sigma_+)$	a_I
	$h_+^{(2,1)}$	z^κ		
vector	$h_-^{(2,1)}$	$V\tilde{K}$	1	A

Table 3.1: The $\mathcal{N} = 1$ spectrum of the D5-brane effective theory.

where we singled out the combination \mathcal{B}^Σ of that four-dimensional field which corresponds to the class of the cycle Σ_- .

There is a universal sector in the worldvolume theory governing the dynamics of the D5-brane. It contains the geometric fluctuations of the embedding map $\iota : \Sigma_+ \hookrightarrow Z_3$. These fluctuations are described by sections $\hat{\zeta}$ of the normal bundle to Σ_+ . To first order massless fields $\zeta(\underline{x})$ correspond to holomorphic sections of the holomorphic normal bundle $H^0(\Sigma_+, N_{Z_3}\Sigma_+)$ of Σ_+ and their conjugates $\bar{\zeta}$. This gives rise to the expansion

$$\hat{\zeta} = \zeta + \bar{\zeta} = \zeta^A(\underline{x}) s_A + \bar{\zeta}^{\bar{A}}(\underline{x}) \bar{s}_{\bar{A}} = \hat{\zeta}^A \hat{s}_A, \quad (3.19)$$

where we introduced a basis \hat{s}_A of real sections $\hat{\zeta}$ on $H^0(\Sigma_+, N_{Z_3}\Sigma_+) \oplus \overline{H^0(\Sigma_+, N_{Z_3}\Sigma_+)}$ for convenience. We note that the group of holomorphic sections $H^0(\Sigma_+, N_{Z_3}\Sigma_+)$ is finite due to the strong requirement of holomorphicity. This is in contrast to the fact that a real vector bundle E has in general infinitely many real sections in $\mathcal{C}^\infty(E)$. However, a real section of $N_{\mathbb{R}}\Sigma_+$ in general corresponds to a massive mode and is therefore neglected in our description of the effective action. However, as we will see in chapter 7, the question for the appropriate truncation of $\mathcal{C}^\infty(N_{\mathbb{R}}\Sigma_+)$ is mathematically sophisticated. On the one hand side there is in general a brane superpotential for the modes ζ that may obstruct them at a higher order. On the other hand, when varying the complex structure on Z_3 a non-holomorphic section in $\mathcal{C}^\infty(N_{\mathbb{R}}\Sigma_+)$ may become massless at some point in the complex structure moduli space and contribute to the effective action. Due to this complications we restrict our considerations in this chapter, with the exception of section 3.4, to the finite group $H^0(\Sigma_+, N_{Z_3}\Sigma_+)$ to count first order massless fields and postpone the more difficult question of a global open-closed moduli space. The discussion of the complete open-closed moduli space and the brane superpotential will be of central importance in this work though, in particular in chapter 7.

We conclude by summarizing the $\mathcal{N} = 1$ field content in four dimensions emerging from the bulk and the brane sector in Table 3.1. The precise organization of these fields into $\mathcal{N} = 1$ complex coordinates is performed in section 3.3.

3.2.2 Special Relations on the $\mathcal{N} = 1$ Moduli Space

In this section we discuss a subtlety in the decomposition (3.19). The notion of ζ^A being a complex scalar field depends on the background complex structure chosen on the ambient Calabi-Yau Z_3 , i.e. on the split (3.19), $N^{\mathbb{R}}_{Z_3}\Sigma_+ \otimes \mathbb{C} = N_{Z_3}\Sigma_+ \oplus \overline{N_{Z_3}\Sigma_+}$, into holomorphic and anti-holomorphic parts. To explore this dependence further it is natural to consider the contractions of the s_A with the holomorphic (3,0)-form Ω , the (2,1)-forms χ_κ introduced in (3.13) and their complex conjugates. In the background complex structure defined at z_0 we find, in the cohomology of Z_3 as well as in the cohomology of Σ , that

$$s_A \lrcorner \Omega(z_0) = 0, \quad s_A \lrcorner \bar{\chi}_\kappa(z_0) = 0, \quad s_A \lrcorner \bar{\Omega}(z_0) = 0. \quad (3.20)$$

These contractions vanish on Z_3 since there are no non-trivial (2,0)-forms in $H^2(Z_3)$. Moreover, they also vanish on Σ for a supersymmetrically embedded D5-brane. In addition, since every two-form pulled back to Σ has to be proportional to the (1,1)-Kähler form J , only $s_A \lrcorner \chi_\kappa$ can be a non-trivial (1,1)-form on Σ . Note, however, that also $s_A \lrcorner \chi_\kappa$ is trivial in the cohomology of Z_3 due to the primitivity of $H^{(2,1)}(Z_3)$.

However, in the four-dimensional effective theory we also have to allow for possible fluctuations around the supersymmetric background configuration, including those corresponding to complex structure deformations of Z_3 . The holomorphic three-form Ω as well as the complex scalars ζ are then functions of the complex structure parameters z^κ . In a different complex structure on Z_3 , the notion of holomorphic and anti-holomorphic coordinates on Z_3 , encoded by the type of $\Omega(z)$, has not to be aligned with the splitting into complex scalars (3.19) in general. To measure this discrepancy, we consider the pullback $\iota^*(s_A \lrcorner \Omega(z))$ on Σ . For $z = z_0 + \delta z$ near a background complex structure z_0 we expand $\Omega(z)$ to linear order in δz as

$$\iota^*(s_A \lrcorner \Omega(z)) = (1 - K_\kappa \delta z^\kappa) \iota^*(s_A \lrcorner \Omega(z_0)) + \iota^*(s_A \lrcorner \chi_\kappa(z_0)) \delta z^\kappa = \iota^*(s_A \lrcorner \chi_\kappa(z_0)) \delta z^\kappa, \quad (3.21)$$

where we used (2.18) and (3.20). In other words, the form $s_A \lrcorner \Omega$ is a (2,0)-form on Σ in the complex structure z but a (1,1)-form on Σ in the complex structure z_0 to linear order in the complex structure variation δz . However, a similar argument shows that

$$(s_A \lrcorner \bar{\Omega})(z) = (s_A \lrcorner \bar{\chi})(z^\kappa) = 0, \quad (3.22)$$

even to linear order in δz^κ . These forms only appear at higher order in the complex structure variations.

The above considerations allow us to describe the metric deformations of the induced metric ι^*g on the two-cycle Σ_+ . In general, both the complex structure deformations of Z_3 and the fluctuations of the embedding map ι contribute. Here, we discuss only those variations $\delta(\iota^*g)$ originating from complex structure deformations and postpone the analysis of all possible metric variations to section 3.2.3. Analogously to (3.13) the complex structure

deformations on Σ_+ are encoded in the purely holomorphic metric variation

$$\iota^*(\delta g)_{uu} = \frac{2iv^\Sigma}{\int \Omega \wedge \bar{\Omega}} \iota^*(s_{A \lrcorner} \Omega)_{uu} (\iota^* g)^{u\bar{u}} \iota^*(\bar{s}_{\bar{B} \lrcorner} \bar{\chi}_{\bar{k}})_{\bar{u}u} \mathcal{G}^{A\bar{B}} \delta z^{\bar{k}}. \quad (3.23)$$

Here we have introduced the volume of the holomorphic two-cycle Σ_+ as

$$v^\Sigma = \int_{\Sigma_+} d^2u \sqrt{g} = \int_{\Sigma_+} \iota^* J \quad (3.24)$$

and a natural hermitian metric $\mathcal{G}_{A\bar{B}}$ given by

$$\mathcal{G}_{A\bar{B}} = -\frac{i}{\mathcal{V}} \int_{\Sigma_+} s_{A \lrcorner} \bar{s}_{\bar{B} \lrcorner} (J) \iota^* J. \quad (3.25)$$

We will show later on that $\mathcal{G}_{A\bar{B}}$ can be obtained by dimensional reduction, cf. section 3.2.3. It is identified with the metric for the moduli $\underline{\zeta}$ on the open string moduli space and is independent of the coordinates u, \bar{u} on Σ .

The metric variation (3.23) can be explained by application of some useful formulas for the open string moduli space. First, we use the fact that $H^{(1,1)}(\Sigma_+)$ is spanned by the pullback $\iota^* J$. This is exploited to rewrite the pullback of any closed $(1,1)$ -form ω to Σ_+ in cohomology, cf. (3.45). In particular we obtain

$$\iota^*(s_{A \lrcorner} \chi_\kappa) = \frac{\iota^* J}{v^\Sigma} \int_{\Sigma_+} \iota^*(s_{A \lrcorner} \chi_\kappa), \quad (3.26)$$

which can be written after multiplication with $\mathcal{V}^{-1} \mathcal{G}^{A\bar{B}} g(s_C, \bar{s}_{\bar{B}})$ and by using (3.25) as

$$\int_{\Sigma_+} \iota^*(s_{A \lrcorner} \chi_\kappa) = -\frac{v^\Sigma}{\mathcal{V}} \int_{\Sigma_+} g(s_A, \bar{s}_{\bar{B}}) \mathcal{G}^{\bar{B}C} \iota^*(s_{C \lrcorner} \chi_\kappa). \quad (3.27)$$

We evaluate this for every choice of s_A and compare the coefficients on both sides to relate the metric on the normal bundle $N_{Z_3} \Sigma$ and the metric $\mathcal{G}^{A\bar{B}}$.

Thus, the identity (3.27) allows us to infer the metric variations (3.23) from the complex structure deformations on Z_3 . First, we consider the pullback to Σ_+ of the metric variations δg_{ij} in (3.13) of the ambient Calabi-Yau Z_3

$$\iota^*(\delta g)_{uu} = \frac{i\mathcal{V}}{\int \Omega \wedge \bar{\Omega}} \Omega_u^{\bar{i}\bar{j}} (\bar{\chi}_{\bar{k}})_{\bar{i}\bar{j}u} \delta z^{\bar{k}}. \quad (3.28)$$

Then we replace, motivated by (3.27), the inverse metric $g^{i\bar{j}}$ occurring in the contraction of $\bar{\chi}_{\bar{k}}$ and Ω by $s_A^i s_{\bar{B}}^{\bar{j}} \mathcal{G}^{A\bar{B}}$ to obtain our ansatz for the induced metric deformation on Σ_+ given in (3.23).

However, there are some remarks in order. Since there are no $(2,0)$ -forms on Σ_+ in the background complex structure z_0 , the form $\iota^*(s_{A \lrcorner} \Omega)$ should vanish identically. Thus, in order to make sense of the metric variation (3.23) we have to evaluate it, following the logic of (3.21),

in the complex structure $z = z_0 + \delta z$. Applying this to (3.23) we expand $\delta(\iota^*g)$ to linear order in δz , i.e. $\iota^*(\delta g)_{uu}(z) = \iota^*(\delta g)_{uu}(z_0) + \iota^*(\delta g)_{u\bar{u}}(z_0) \cdot \delta z$, to obtain

$$\iota^*(\delta g)_{u\bar{u}}(z_0) = \frac{2iv^\Sigma}{\int \Omega \wedge \bar{\Omega}} \iota^*(s_{A \lrcorner} \chi_\kappa)_{\bar{u}u} (\iota^*g)^{u\bar{u}} \iota^*(\bar{s}_{\bar{B} \lrcorner} \bar{\chi}_{\bar{\kappa}})_{\bar{u}u} \mathcal{G}^{A\bar{B}} \delta z^\kappa \delta \bar{z}^{\bar{\kappa}}. \quad (3.29)$$

Here we emphasize the change in type from purely holomorphic indices δg_{uu} at z to mixed type $\delta g_{u\bar{u}}$ at z_0 . It is important to note that there are neither any metric deformations linear in the complex structure parameter δz nor any of pure type.

These calculations stress that the analysis of the open string moduli space crucially depends on the chosen background complex structure encoded by the moduli z^κ . It is hence natural that the complex structure parameters z^κ of Z_3 and the open string moduli ζ^A should be treated on an equal footing to characterize the structure of the $\mathcal{N} = 1$ field space. This has led to the introduction of the blow-up proposal in [60] that was further exploited in [81] and ultimately in [100]. It will be used in chapter 7 to unify the open-closed fields and to understand and derive the effective brane superpotential. In the next sections we derive the general four-dimensional effective D5-brane action and show that the brane superpotential is naturally encoded by the forms $s_{A \lrcorner} \Omega$ and $s_{A \lrcorner} \chi_\kappa$ that are sensible both to the complex structure on Z_3 and the open string deformations.

We conclude by noting that the analysis of $\delta(\iota^*g)$ and the induced variation (3.29) can also be derived in two alternative ways, one based on the consideration of all vector fields in $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$ and a different by analyzing the variation of $\text{vol}(\Sigma)$ under a change of complex structure. For the first way we introduce a basis s_a of the infinite dimensional Hilbert space of all sections in $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$. The natural metric $\mathcal{G}_{a\bar{b}}$ on these sections is formally identical to the integral in (3.25). Then we introduce holomorphic vielbeins e_n of the normal bundle $N_{Z_3}\Sigma$ obeying the elementary relation

$$g_{i\bar{j}} e_n^i \bar{e}_{\bar{m}}^{\bar{j}} = \eta_{n\bar{m}}, \quad g^{i\bar{j}} = \eta^{n\bar{m}} e_n^i \bar{e}_{\bar{m}}^{\bar{j}}, \quad (3.30)$$

where $g_{i\bar{j}}$ denotes the hermitian metric on $N_{Z_3}\Sigma$ and $\eta_{n\bar{m}}$ the flat hermitian metric. We readily expand the holomorphic vielbeins with respect to the basis of sections s_a as $e_n^i = \sum c_n^a s_a^i$. Upon integrating the first relation in (3.30) over Σ_+ , we obtain

$$\eta_{n\bar{m}} = \frac{1}{v_\Sigma} \int_{\Sigma_+} \eta_{n\bar{m}} \iota^*(J) = \frac{1}{v_\Sigma} \int_{\Sigma_+} g_{i\bar{j}} e_n^i \bar{e}_{\bar{m}}^{\bar{j}} \iota^*(J) = i \frac{\mathcal{V}}{v_\Sigma} \mathcal{G}_{a\bar{b}} c_n^a \bar{c}_{\bar{m}}^{\bar{b}}, \quad (3.31)$$

which is readily inserted into the second equation in (3.30) to obtain

$$g^{i\bar{j}} = \frac{v_\Sigma}{\mathcal{V}} \mathcal{G}^{a\bar{b}} s_a^i s_b^{\bar{j}}. \quad (3.32)$$

This is then again used in (3.28) to replace, to quadratic order in s_a , the metric $g^{i\bar{j}}$. By expanding in the complex structures to second order, we reproduce (3.29) as before, however, with the basis of sections s_a of $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$ instead of s_A . Nonetheless, the relation (3.29)

is still a good approximation for the purpose of deriving the effective action since most of the modes associated to the s_a are massive, cf. section 3.4, and are consistently integrated out in the effective Lagrangian maintaining only the light fields, that are precisely counted by the finite number of sections s_A in $H^0(\Sigma, N_{Z_3\Sigma})$.

For brevity, we describe the second method only in words. Since a variation $\delta(\iota^*g)$ violates the calibrations (3.8) and thus the volume minimization property of Σ , the volume of Σ will increase by $\delta \text{vol}(\Sigma)$. Analyzing this variation to appropriate order in the fields it is possible to extract (3.29) or to directly obtain the corresponding potential term in the dimensional reduction of Dirac-Born-Infeld action (3.40). Indeed we follow a similar logic in section 3.4 to determine the potential on the infinite dimensional field space of massive deformations of Σ .

3.2.3 Reduction of the D5-Brane Action

Now we are prepared to derive the four-dimensional effective action of the D5-brane in a Calabi-Yau orientifold with $O5/O9$ -planes, where we mainly follow [60]. The action is obtained by reducing the bulk supergravity action S_{IIB} as in section 2.2.2 and the effective D-brane actions using a Kaluza-Klein reduction ansatz. The string-frame Type IIB action is used in its democratic form (2.24). In addition, one includes the string-frame D5-brane action

$$S_{\text{D5}}^{\text{SF}} = -\mu_5 \int_{\mathcal{W}} d^6\xi e^{-\phi} \sqrt{-\det(\iota^*(g_{10} + B_2) - \ell F)} + \mu_5 \int_{\mathcal{W}} \sum_{q \text{ even}} \iota^*(C_q) \wedge e^{\ell F - \iota^*(B_2)}, \quad (3.33)$$

which is the specialization of the general DBI-action (3.3) and the general CS-action (3.4) to BPS D5-branes. For convenience we put $\mathcal{W} \equiv \mathcal{W}_6$. The Kaluza-Klein reduction of the bulk action (2.24) on the orientifold background has been carried out in [128] and is reviewed in chapter 2. Thus we mainly concentrate in the following on the reduction of the D5-brane action (3.33) and later include the contributions entirely due to bulk fields in the determination of the $\mathcal{N} = 1$ characteristic functions.

Dirac-Born-Infeld Action & Tadpole Cancellation

In the following we perform the Kaluza-Klein reduction of the Dirac-Born-Infeld action given in (3.33). Firstly, we expand the determinant using

$$\sqrt{\det(\mathfrak{A} + \mathfrak{B})} = \sqrt{\det\mathfrak{A}} \cdot \left[1 + \frac{1}{2} \text{Tr} \mathfrak{A}^{-1} \mathfrak{B} + \frac{1}{8} \left((\text{Tr} \mathfrak{A}^{-1} \mathfrak{B})^2 - 2 \text{Tr} (\mathfrak{A}^{-1} \mathfrak{B})^2 \right) + \dots \right]. \quad (3.34)$$

The matrix \mathfrak{A} contains the fields in the calibrated background configuration of $Z_3/\sigma \times \mathbb{R}^{1,3}$ for which the conditions (3.8) are obeyed. Additionally, \mathfrak{B} contains the fluctuation fields around this background and is small compared to \mathfrak{A} . These fluctuations are precisely the variations of the embedding ι of the two-cycle Σ_+ parametrized by the fields $\underline{\zeta}$ of (3.19), the Wilson

lines \underline{a} introduced in (3.16) as well as the perturbations (3.18) about the calibrated NS–NS B-field defined in (3.8) and about the background complex structure. The fluctuations of all fields involve the variation $\delta\iota$ of the embedding ι since they have to be pulled back to worldvolume \mathcal{W} of the D5-brane. These pullbacks are treated in the normal coordinate expansions [62, 155, 156], which is a standard technique in differential geometry to analyze families of diffeomorphisms and embeddings using the Riemann normal coordinates. In the case at hand the family of embeddings ι is parametrized by the sections $\underline{\zeta}$ of the normal bundle. Applying the normal coordinate expansion to the metric (3.10) and the NS–NS B-field (3.13) on the D5-brane world-volume as well as taking the metric variation $\delta(\iota^*g)_{u\bar{u}}$ of (3.29) into account we obtain

$$\begin{aligned} \iota^*g_{10} &= \mathcal{V}^{-1}e^{2\phi}\eta_{\mu\nu}dx^\mu \cdot dx^\nu + (\iota^*g + \delta(\iota^*g))_{u\bar{u}}du \cdot d\bar{u} + g(\partial_\mu\zeta, \partial_\nu\bar{\zeta})dx^\mu \cdot dx^\nu, \\ \iota^*B_2 - \ell\mathcal{F} &= \mathcal{B}^a\iota^*\omega_a - \ell F + \mathcal{B}^a\iota^*\omega_a(\partial_\mu\zeta, \partial_\nu\bar{\zeta})dx^\mu \wedge dx^\nu, \end{aligned} \quad (3.35)$$

where \cdot is the symmetric product and \mathcal{V} , $g_{u\bar{u}}$ are the string frame volume and the induced hermitian metric on Σ_+ . Note that the Minkowski metric η is rescaled to the four-dimensional Einstein frame². In the expansions (3.35) we readily identify the background fields and variations as

$$\begin{aligned} \mathfrak{A} &= \begin{pmatrix} \mathcal{V}^{-1}e^{2\phi}\eta_{\mu\nu} & 0 & 0 \\ 0 & 0 & g_{u\bar{u}} \\ 0 & g_{u\bar{u}} & 0 \end{pmatrix}, \\ \mathfrak{B} &= \begin{pmatrix} (2g + \mathcal{B}^a\omega_a)(\partial_\mu\zeta, \partial_\nu\bar{\zeta}) - \ell F_{\mu\nu} & -\ell\partial_\mu\bar{a}_{\bar{j}}\bar{A}_u^{\bar{j}} & -\ell\partial_\mu a_I A_{\bar{u}}^I \\ -\ell\partial_\nu\bar{a}_{\bar{j}}\bar{A}_u^{\bar{j}} & 0 & (\delta g + \frac{1}{2}\mathcal{B}^a\omega_a)_{u\bar{u}} \\ -\ell\partial_\nu a_I A_{\bar{u}}^I & (\delta g - \frac{1}{2}\mathcal{B}^a\omega_a)_{u\bar{u}} & 0 \end{pmatrix}, \end{aligned} \quad (3.36)$$

where we omitted the pullback ι^* for notational convenience. Only the terms

$$\frac{1}{2}\text{Tr } \mathfrak{A}^{-1}\mathfrak{B} - \frac{1}{4}\text{Tr } ((\mathfrak{A}^{-1}\mathfrak{B})^2) \quad (3.38)$$

of the Taylor expansion (3.34) contribute to the effective action up to quadratic order in the fields. We insert the result into the first part of (3.33) and use the calibrations (3.8) to obtain the four-dimensional action

$$S_{\text{DBI}}^{\text{EF}} = -\mu_5 \int \left[\frac{\ell^2 e^{-\phi}}{4} v^\Sigma F \wedge *F + \frac{\ell^2 e^\phi}{\mathcal{V}} \mathcal{C}^{I\bar{J}} da_I \wedge *d\bar{a}_{\bar{J}} + \frac{1}{2} e^\phi \mathcal{G}_{A\bar{B}} d\zeta^A \wedge *d\bar{\zeta}^{\bar{B}} + V_{\text{DBI}} *1 \right] \quad (3.39)$$

in the four-dimensional Einstein frame. The potential term in (3.39) is of the form

$$V_{\text{DBI}} = \frac{e^{3\phi}}{2\mathcal{V}^2} \left(v^\Sigma + \frac{2i\mathcal{G}^{A\bar{B}}}{\int \Omega \wedge \bar{\Omega}} \int_{\Sigma_+} s_{A\bar{J}} \chi_{\bar{\kappa}} \int_{\Sigma_+} \bar{s}_{\bar{B}\bar{J}} \bar{\chi}_{\bar{\kappa}} \delta z^{\bar{\kappa}} \delta \bar{z}^{\bar{\kappa}} + \frac{(\mathcal{B}^\Sigma)^2}{8v^\Sigma} \right). \quad (3.40)$$

In the following we will successively discuss the separate terms appearing in the action S_{DBI} .

²Recall that the four-dimensional metric in the Einstein frame η is related to the string frame metric η^{SF} via $\eta = e^{-2\phi}\mathcal{V}\eta^{\text{SF}}$.

The first term in (3.39) is the kinetic term for the $U(1)$ gauge boson A on the D5-brane. The gauge coupling is thus given by $1/g_{\text{D5}}^2 = \frac{1}{2}\mu_5\ell^2 e^{-\phi} v^\Sigma$, where v^Σ is the volume of the two-cycle Σ_+ evaluated via the calibration (3.8). The second term is the kinetic term for the Wilson line moduli a^I . The corresponding metric takes the form

$$\mathcal{C}^{I\bar{J}} = \frac{1}{2} \int_{\Sigma_+} A^I \wedge *_2 \bar{A}^{\bar{J}} = \frac{i}{2} \int_{\Sigma_+} A^I \wedge \bar{A}^{\bar{J}}, \quad (3.41)$$

where we have used $*_2 \bar{A}^{\bar{J}} = i\bar{A}^{\bar{J}}$ on the basis of $(1,0)$ -forms introduced in (3.16). The third term in (3.39) contains the field space metric for the deformations ζ^A which is of the form

$$\mathcal{G}_{A\bar{B}} = -\frac{i}{2\mathcal{V}} \int_{\Sigma_+} s_{A\lrcorner} \bar{s}_{\bar{B}\lrcorner} (J \wedge J) = \frac{\mathcal{K}_\alpha}{2\mathcal{V}} \mathcal{L}_{A\bar{B}}^\alpha, \quad \mathcal{L}_{A\bar{B}}^\alpha = -i \int_{\Sigma_+} s_{A\lrcorner} \bar{s}_{\bar{B}\lrcorner} \tilde{\omega}^\alpha, \quad (3.42)$$

where we have used $\mathcal{K}_\alpha = \int \omega_\alpha \wedge J \wedge J$ and $J \wedge J = \mathcal{K}_\alpha \tilde{\omega}^\alpha$.

Finally, let us comment on the potential terms V_{DBI} . In fact, the first of the three terms represents an NS–NS tadpole and takes the form of a D-term. As mentioned in section 3.1 a consistent compactification with D-branes requires cancellation of R–R as well as NS–NS tadpoles. Hence the two-cycle Σ_+ wrapped by the D5-brane has to lie in the same homology class as an $O5$ -plane arising from the fix-points of σ . Consequently, we have to add the contribution of the orientifold plane

$$S_{\text{ori}}^{\text{SF}} = \mu_5 \int_{\mathcal{W}_{\text{orie}}} d^6 \xi e^{-\phi} \sqrt{-\det(\tilde{\varphi}^*(g_{10} + B))} \quad \rightarrow \quad S_{\text{ori}}^{\text{EF}} = \mu_5 \int \frac{e^{3\phi}}{2\mathcal{V}^2} v^\Sigma * 1, \quad (3.43)$$

to the action (3.39). Here we applied the calibration condition (2.8) for orientifold planes to obtain the two-cycle volume v^Σ . Having rescaled to the Einstein frame one compares $S_{\text{ori}}^{\text{SF}}$ with (3.39) to observe a precise cancellation of the D-term potential of the D5-brane. This cancellation in V_{DBI} again confirms, from a low-energy effective action point of view, that both the $O5$ -plane and the D5-brane have to wrap volume minimizing cycles.

The last two terms in V_{DBI} describe deviations from the calibration conditions (3.8). The first potential term accounts for the metric deformations (3.29) induced by the change of the ambient complex structure and the second term describes the field fluctuation \mathcal{B}^a of the NS–NS B-field defined in (3.18). Later on, we will show that this term is actually a D-term which is consistent with the analysis of [157]. Clearly, both terms vanish at the supersymmetric ground state defined by the calibration conditions (3.8). Let us comment on the dimensional reduction yielding these two terms. The evaluation of $\text{Tr} \mathfrak{A}^{-1} \mathfrak{B}$ in the expansion (3.34) of the DBI-action yields a term given by

$$\delta \mathcal{L}_{\delta g} = \frac{ie^{3\phi} v^\Sigma \mathcal{G}^{A\bar{B}}}{\mathcal{V}^2 \int \Omega \wedge \bar{\Omega}} \delta z^\kappa \delta \bar{z}^{\bar{\kappa}} \int_{\Sigma_+} \iota^*(s_{A\lrcorner} \chi_\kappa)_{\bar{u}u} (\iota^* g)^{u\bar{u}} (\iota^* g)^{u\bar{u}} \iota^*(\bar{s}_{\bar{B}\lrcorner} \bar{\chi}_{\bar{\kappa}})_{\bar{u}u} \iota^* J. \quad (3.44)$$

originating from the variation $\delta(\iota^* g)$. Indeed, this is the only contribution to the four-dimensional effective action originating from the metric variation (3.29) that is relevant at

our low order analysis in the fields. As discussed before, cf. section 3.2.2, the $(1,1)$ -form $\iota^* J$ is essentially the only non-trivial element in the cohomology $H^{(1,1)}(\Sigma_+)$. Thus, we rewrite the pullback of any closed $(1,1)$ -form ω to Σ_+ as

$$\iota^* \omega = \frac{\int_{\Sigma_+} \omega}{v^{\Sigma}} \iota^* J \quad (3.45)$$

in cohomology, where we used again the calibration $\int_{\Sigma_+} \iota^* J = v^{\Sigma}$. In particular, we can apply this to the closed $(1,1)$ -forms $s_{A \lrcorner} \chi_{\kappa}$ in (3.44) to obtain the second term of V_{DBI} given in (3.40). Considering the fluctuation \mathcal{B}^a , the only contribution arises from the term $\text{Tr}(\mathfrak{A}^{-1} \mathfrak{B})^2$ in (3.34). Restricting to the relevant quadratic term in \mathcal{B}^a , this yields

$$\delta \mathcal{L}_{\mathcal{B}} = \frac{e^{3\phi}}{16\mathcal{V}^2} \mathcal{B}^a \mathcal{B}^b \int_{\Sigma_+} (\iota^* \omega_a)_{u\bar{u}} (\iota^* \omega_b)_{u\bar{u}} g^{u\bar{u}} g^{u\bar{u}} \iota^* J. \quad (3.46)$$

Again we use (3.45) to expand $P_- \mathcal{B}^a \iota^* \omega_a = \mathcal{B}^{\Sigma} \iota^* J / v^{\Sigma}$ in the cohomology $H^{(1,1)}(\Sigma_+)$, where P_- again denotes the sign function introduced in (3.16), to obtain the geometrical dependence of the volume v^{Σ} of the cycle as given in (3.40). Later on in section 3.3, we show explicitly that the above results of the dimensional reduction are necessary to match the F- and D-term potential arising from a brane superpotential W_{brane} and a gauging of a shift symmetry by the $U(1)$ vector A from the D5-brane gauge theory, respectively.

Chern-Simons Action

Let us now turn to the dimensional reduction of the Chern-Simons part of the D5-brane action. For this purpose we need the normal coordinate expansion of the pullback of all R-R fields (3.14) to the world-volume of the D5-brane. Here we only display the relevant terms for the reduction of the Chern-Simons action which read

$$\begin{aligned} (\iota^* C_p)_{i_1 \dots i_p} &= \frac{1}{p!} C_{i_1 \dots i_p} + \frac{1}{p!} \zeta^n \partial_n C_{i_1 \dots i_p} - \frac{1}{(p-1)!} \nabla_{i_1} \zeta^n C_{ni_2 \dots i_p} + \frac{1}{2p!} \zeta^n \partial_n (\zeta^m \partial_m C_{i_1 \dots i_p}) \\ &- \frac{1}{(p-1)!} \nabla_{i_1} \zeta^n \zeta^m \partial_m C_{ni_2 \dots i_p} + \frac{1}{2(p-2)!} \nabla_{i_1} \zeta^n \nabla_{i_2} \zeta^m C_{nmi_3 \dots i_p} + \frac{p-2}{2p!} R_{ni_1 m}^j \zeta^n \zeta^m C_{ji_2 \dots i_p}, \end{aligned} \quad (3.47)$$

where the indices i_n label the coordinates ξ^{i_n} on the D5-brane world volume. Inserting this expansion into the Chern-Simons part of (3.33), one obtains up to second order

$$\begin{aligned} S_{\text{CS}} &= \mu_5 \int \left[\frac{\ell^2}{4} c^{\Sigma} F \wedge F - \frac{\ell}{2} d(\tilde{\rho}_{(2)}^{\Sigma}) - \mathcal{C}_{(2)} \mathcal{B}^{\Sigma} \right] \wedge A + \frac{i}{4} \mathcal{L}_{A\bar{B}}^{\alpha} d\tilde{c}_{\alpha}^{(2)} \wedge (d\zeta^A \bar{\zeta}^{\bar{B}} - d\bar{\zeta}^{\bar{B}} \zeta^A) \\ &- \ell^2 \frac{i}{2} \mathcal{C}^{I\bar{J}} d\mathcal{C}_{(2)} \wedge (da_I \bar{a}_{\bar{J}} - d\bar{a}_{\bar{J}} a_I) - \frac{i}{4} \mathcal{L}_{abA\bar{B}} d(\mathcal{B}^a \tilde{\rho}_{(2)}^b) \wedge (d\zeta^A \bar{\zeta}^{\bar{B}} - d\bar{\zeta}^{\bar{B}} \zeta^A) \\ &+ \frac{1}{2} (\mathcal{N}_{AK} A_{(3)}^K + \mathcal{N}_A^K \tilde{A}_K^{(3)}) \wedge d\hat{\zeta}^A - \frac{\ell}{2} \hat{\zeta}^A (\mathcal{N}_{Ak} dV^k + \mathcal{N}_A^k dU_k) \wedge F, \end{aligned} \quad (3.48)$$

where $\mathcal{B}^{\Sigma}, \mathcal{B}^a$ are introduced in (3.18) and we similarly define $\tilde{\rho}_{(2)}^{\Sigma} = \int_{\Sigma_-} C_4$ as well as $c^{\Sigma} = \int_{\Sigma_+} C_2$. In the S_{CS} we further used the abbreviations

$$\mathcal{N}_{AK} = \int_{\Sigma_+} \hat{s}_{A \lrcorner} \alpha_K, \quad \mathcal{N}_A^K = \int_{\Sigma_+} \hat{s}_{A \lrcorner} \beta^K, \quad \mathcal{N}_{Ak} = \int_{\Sigma_-} \hat{s}_{A \lrcorner} \alpha_k, \quad \mathcal{N}_A^k = \int_{\Sigma_-} \hat{s}_{A \lrcorner} \beta^k, \quad (3.49)$$

where the forms and their orientifold parity are given in table 2.3. We emphasize that depending on the parity of the integrand under the involution σ in general both combinations Σ_+ and Σ_- occur in (3.49). However, terms involving Σ_- can be translated to Σ_+ via the sign function P_- . In addition we evaluated the coupling

$$\mathcal{L}_{abA\bar{B}} = -i \int_{\Sigma_+} s_{A\bar{B}} \bar{s}_{\bar{B}A} (\omega_a \wedge \omega_b) = \mathcal{L}_{A\bar{B}}^\alpha \mathcal{K}_{\alpha ab} , \quad (3.50)$$

where $\mathcal{L}_{A\bar{B}}^\alpha$ was introduced in (3.42) and $\mathcal{K}_{\alpha ab} = \int_Y \omega_\alpha \wedge \omega_a \wedge \omega_b$ are the only non-vanishing triple intersection numbers involving the negative $(1,1)$ -forms ω_a of Table 2.3. We note that both in the action (3.48) and in the definitions (3.49) we used the expansion $\hat{\zeta} = \hat{\zeta}^A \hat{s}_A$ into a real basis \hat{s}_A introduced in (3.19). Clearly, the expressions involving $\hat{\zeta}^A$, \hat{s}_A are readily rewritten into complex coordinates $\zeta^A, \bar{\zeta}^A$.

Let us now discuss the interpretation of the different terms appearing in the action (3.48). The first term in S_{CS} corresponds to the theta-angle term of the gauge theory on the D5-brane. Thus it is identified with the imaginary part of the gauge-kinetic function. The second term is a Green-Schwarz term which indicates the gauging of the scalar fields dual to the two-forms $\tilde{\rho}_{(2)}$ and $\mathcal{C}_{(2)}$ by the D5-brane vector field A . In fact, we show explicitly in section 3.3 that this term indeed induces a gauging of one chiral multiplet in the four-dimensional spectrum and that the corresponding D-term is precisely the one encountered in the reduction of the DBI action in section 3.2.3.

The interpretation of the remaining terms in (3.48) is of more technical nature. The third, fourth and fifth terms are mixing terms which contribute in the kinetic terms of the scalars \underline{c} , h and $\underline{\rho}$ dual to the two-forms $\tilde{c}_\alpha^{(2)}$, $\mathcal{C}_{(2)}$ and $\tilde{\rho}_{(2)}$. In section 3.3 they help us to identify the correct complex coordinates which cast the kinetic term into the standard $\mathcal{N} = 1$ form. The sixth term contains the four-dimensional three-forms $\underline{A}_{(3)}$ and $\tilde{\underline{A}}^{(3)}$. We show in the next section 3.2.4 that these terms are crucial in the calculation of the complete scalar potential of the D5-brane effective action. Finally, the last term in S_{CS} indicates a mixing of the field strength on the D5-brane with the $U(1)$ bulk vector fields $\underline{V}, \underline{U}$. The precise form of the redefined gauge-couplings is discussed in appendix A.2.

3.2.4 The Scalar Potential

In this section we compute the scalar potential of the four-dimensional effective theory. We find two types of potentials according to their origin in the $\mathcal{N} = 1$ language, i.e. an F-term potential V_{F} and D-term potential V_{D} . In the context of the D5-brane effective action the first one arises from the coupling of the three-forms $\underline{A}_{(3)}, \tilde{\underline{A}}^{(3)}$ in the Chern-Simons action and their kinetic terms from the bulk as well as from a completion term from the potential V_{DBI} in (3.40). The second potential arises entirely from the bulk and the potential V_{DBI} . We note that the potential due to background R–R and NS–NS fluxes $F_3 = \langle dC_2 \rangle$ and $H_3 = \langle dB_2 \rangle$

has already been studied in [128]. Thus, we emphasize here the additional contributions in the presence of the space-time filling D5-branes.

A first contribution to the scalar potential is induced by the couplings of the three-forms $\underline{A}_{(3)}$ and $\tilde{\underline{A}}^{(3)}$ in the Chern-Simons action (3.48). Here it is crucial to keep these forms in the spectrum despite the fact that a massless three-form has no propagating degrees of freedom in four dimensions. Moreover, if the degrees of freedom in these forms as well as the induced potential are treated quantum mechanically, as described in [158], one is able to account for the possible R–R three-form flux

$$F_3 = m^K \alpha_K - e_K \beta^K, \quad (3.51)$$

where the flux quanta (m^K, e_K) are interpreted as labeling the discrete excited states of the system and (α_K, β^K) is the real symplectic basis introduced in (2.14). This is in accord with the fact that the duality condition $G^{(3)} = (-1) *_{10} G^{(7)}$ given in (2.10) relates the three-form containing F_3 to the seven-form F_7 containing $(dA_{(3)}^K, d\tilde{A}_K^{(3)})$.

Let us collect the terms involving the non-dynamical three-forms $\underline{A}_{(3)}$ and $\tilde{\underline{A}}^{(3)}$. The first contribution arises from the effective bulk supergravity action containing the kinetic term $\frac{1}{4} \int dC_6 \wedge *dC_6$ for the R–R-form field C_6 .³ Together with the contribution from the effective Chern-Simons action (3.48) we obtain

$$S_{A_{(3)}} = \int \left[\frac{1}{4} e^{-4\phi} \mathcal{V}^2 d\vec{A}_{(3)} \wedge *E d\vec{A}_{(3)} + \frac{1}{2} \mu_5 \vec{\mathcal{N}}^T d\vec{A}_{(3)} \right], \quad (3.52)$$

where the factor $e^{-4\phi} \mathcal{V}^2$ arises due to the rescaling to the four-dimensional Einstein frame. Note that we have introduced a vector notation to keep the following equations more transparent. More precisely, we define the matrix E , the vector-valued forms $\vec{A}_{(3)}$ and the vector $\vec{\mathcal{N}}$, cf. (3.49), as

$$E = \begin{pmatrix} \int \alpha_K \wedge * \alpha_L & \int \alpha_K \wedge * \beta^L \\ \int \beta^K \wedge * \alpha_L & \int \beta^K \wedge * \beta^L \end{pmatrix}, \quad \vec{A}_{(3)} = \begin{pmatrix} \underline{A}_{(3)} \\ \tilde{\underline{A}}^{(3)} \end{pmatrix}, \quad \vec{\mathcal{N}} = \zeta^{\mathcal{A}} \begin{pmatrix} \mathcal{N}_{AK} \\ \mathcal{N}_A^K \end{pmatrix}. \quad (3.53)$$

Our aim is to integrate out the forms $dA_{(3)}^K$ and $d\tilde{A}_K^{(3)}$ similar to [158] by allowing for the discrete excited states labeled by the background fluxes (e_K, m^K) in (3.51). In fact, we can treat these as formally dualizing the three-forms $\underline{A}_{(3)}$ and $\tilde{\underline{A}}^{(3)}$ into constants (e_K, m^K) [159]. We thus add to $S_{A_{(3)}}$ the Lagrange multiplier term $\frac{1}{2} (e_K dA_{(3)}^K + m^K d\tilde{A}_K^{(3)})$ such that

$$S_{A_{(3)}} \rightarrow S'_{A_{(3)}} = \int \left[\frac{1}{4} e^{-4\phi} \mathcal{V}^2 d\vec{A}_{(3)} \wedge *E d\vec{A}_{(3)} + \frac{1}{2} (\mu_5 \vec{\mathcal{N}} + \vec{m})^T d\vec{A}_{(3)} \right], \quad (3.54)$$

where we abbreviated $\vec{m} = (e_K, m^K)^T$. Formally replacing $\vec{F}_{(4)} = d\vec{A}_{(3)}$ by its equations of motion, one finds the scalar potential

$$S'_{A_{(3)}} = \int V_{A_{(3)}} * 1, \quad V_{A_{(3)}} = \frac{e^{4\phi}}{4\mathcal{V}^2} \left(\mu_5 \vec{\mathcal{N}} + \vec{m} \right)^T E^{-1} \left(\mu_5 \vec{\mathcal{N}} + \vec{m} \right). \quad (3.55)$$

³The factor $\frac{1}{4}$ arises due to the fact that we can eliminate dC_2 contributions in this analysis by the duality condition (2.10).

Here we used the results of appendix A.1 to determine the inverse matrix given by

$$E^{-1} = \begin{pmatrix} \int \beta^K \wedge * \beta^L & - \int \beta^K \wedge * \alpha_L \\ - \int \alpha_K \wedge * \beta^L & \int \alpha_K \wedge * \alpha_L \end{pmatrix}. \quad (3.56)$$

It is convenient to rewrite the potential $V_{A(3)}$ in (3.55) in a more compact form

$$V_{A(3)} = \frac{e^{4\phi}}{4\mathcal{V}^2} \int_{Z_3} \mathcal{G} \wedge * \mathcal{G}, \quad \mathcal{G} = F_3 + \mu_5 \hat{\zeta}^A (\mathcal{N}_A^K \alpha_K - \mathcal{N}_{AK} \beta^K), \quad (3.57)$$

where we identified F_3 as given in (3.51). We note that the scalar potential contains the familiar contribution from the R–R fluxes F_3 . In addition, there are terms linear and quadratic in the D5-brane deformations $\hat{\zeta}$.

In order to prepare for the derivation of $V_{A(3)}$ from this superpotential, it is necessary to introduce the holomorphic and anti-holomorphic variables ζ^A and $\bar{\zeta}^A$. Therefore we expand the three-form \mathcal{G} in a complex basis $(\Omega, \chi_\kappa, \bar{\chi}_{\bar{\kappa}}, \bar{\Omega})$ of $H_+^3(Y) = H^{(3,0)} \oplus H_+^{(2,1)} \oplus H_+^{(1,2)} \oplus H^{(0,3)}$. Explicitly, we find the expansion

$$\mathcal{G} = (\int \Omega \wedge \bar{\Omega})^{-1} [I \bar{\Omega} + G^{\bar{\kappa}\kappa} \bar{I}_{\bar{\kappa}} \chi_\kappa - G^{\bar{\kappa}\kappa} I_\kappa \bar{\chi}_{\bar{\kappa}} - \bar{I} \Omega], \quad (3.58)$$

where the coefficient functions are given by

$$I = \int_{Z_3} \Omega \wedge \mathcal{G} = \int_{Z_3} \Omega \wedge F_3 - \mu_5 \int_{\Sigma_+} \zeta \lrcorner \Omega, \quad I_\kappa = \int_{Z_3} \chi_\kappa \wedge \mathcal{G} = \int_{Z_3} \chi_\kappa \wedge F_3 - \mu_5 \int_{\Sigma_+} \zeta \lrcorner \chi_\kappa. \quad (3.59)$$

Here we have used the relations (3.21) and (3.22) as well as the familiar metric $G_{\kappa\bar{\kappa}}$ on the space of complex structure moduli space of Z_3/σ . Using its explicit form

$$G_{\kappa\bar{\kappa}} = - \frac{\int \chi_\kappa \wedge \bar{\chi}_{\bar{\kappa}}}{\int \Omega \wedge \bar{\Omega}}, \quad (3.60)$$

the expansion (3.58) together with (3.59) is readily checked. Finally, we insert (3.58) into (3.57) and use $*\Omega = -i\Omega$, $*\chi_\kappa = i\chi_\kappa$ to cast the complete potential V_F into the form

$$V_F = \frac{ie^{4\phi}}{2\mathcal{V}^2} \int \Omega \wedge \bar{\Omega} \left[G^{\bar{\kappa}\kappa} I_\kappa \bar{I}_{\bar{\kappa}} + |I|^2 + 2\mu_5 \mathcal{G}^{A\bar{B}} e^{-\phi} \int_{\Sigma_+} s_{A\lrcorner} \chi_\kappa \int_{\Sigma_+} \bar{s}_{\bar{B}\lrcorner} \bar{\chi}_{\bar{\kappa}} \delta z^\kappa \delta \bar{z}^{\bar{\kappa}} \right]. \quad (3.61)$$

Here we have added the potential term in (3.40) originating from the reduction of the Dirac–Born–Infeld action. Once we have computed the $\mathcal{N} = 1$ Kähler metric of the effective theory in appendix A.3, we will derive in section 3.3.2 the potential V_F from a flux and brane superpotential depending on the complex structure and the D5-brane deformations. This will show that V_F is indeed an F-term potential as indicated by the notation.

Let us now turn to the remaining potential terms arising from the DBI action (3.39) and the NS–NS fluxes H_3 . For simplicity, we only discuss electric NS–NS flux such that H_3 admits the expansion

$$H_3 = -\tilde{e}_K \beta^K. \quad (3.62)$$

It was shown in [128] that the electric fluxes \tilde{e}_K result in a gauging of the scalar h dual to $\mathcal{C}_{(2)}$ in (3.14). The effect of magnetic fluxes $\tilde{m}^K \alpha_K$ is more involved since they directly gauge the two-form $\mathcal{C}_{(2)}$ as demonstrated in [128]. In order to be able to work with the scalar h , we do not allow for the additional complication and set $\tilde{m}^K = 0$. Together with the last term in (3.39) we find the potential

$$V_D = \mu_5 \frac{e^{3\phi}}{\mathcal{V}^2} \frac{(\mathcal{B}^\Sigma)^2}{16v^\Sigma} + \frac{e^{2\phi}}{4\mathcal{V}^2} \int_Y H_3 \wedge *H_3, \quad (3.63)$$

which turns out to be a D-term potential due to the gauging of two chiral multiplets⁴ as demonstrated in section 3.3.3.

3.3 D5-Brane $\mathcal{N} = 1$ Effective Couplings and Coordinates

In this section we bring the four-dimensional effective action of the D5-brane and bulk fields into the standard $\mathcal{N} = 1$ supergravity form of (2.37). First in section 3.3.1 we determine the $\mathcal{N} = 1$ chiral coordinates and the large volume expression for the Kähler potential. Then in section 3.3.2 we read off the effective superpotential by matching the F-term scalar potential in 3.2.4, cf. also appendix A.3. We conclude in section 3.3.3 with a determination of the D5-brane gauge kinetic coupling function, the gaugings of chiral multiplets and the D-term potential, that is again in perfect agreement with the results directly obtained in the reduction. The kinetic mixing between bulk and brane gauge fields can be found in appendix A.2. We note that the $\mathcal{N} = 1$ characteristic data of the D5-brane agrees with the results of chapter 2 if the D5-brane fields are frozen out.

3.3.1 The Kähler Potential and $\mathcal{N} = 1$ Coordinates

We first define the $\mathcal{N} = 1$ complex coordinates M^I which are the bosonic components of the chiral multiplets. They define those complex coordinates for which the scalar metric is Kähler. We note that the M^I consist of the D5-brane deformations ζ^A and Wilson lines a_I introduced in section 3.2.1. In addition there are the complex structure deformations z^{κ} as well as the complex fields

$$\begin{aligned} t^\alpha &= e^{-\phi} v^\alpha - i c^\alpha + \frac{1}{2} \mu_5 \mathcal{L}_{A\bar{B}}^\alpha \zeta^A \bar{\zeta}^{\bar{B}}, \\ P_a &= \Theta_{ab} \mathcal{B}^b + i \rho_a, \\ S &= e^{-\phi} \mathcal{V} + i \tilde{h} - \frac{1}{4} (\text{Re}\Theta)^{ab} P_a (P + \bar{P})_b + \mu_5 \ell^2 \mathcal{C}^{I\bar{J}} a_I \bar{a}_{\bar{J}}, \end{aligned} \quad (3.64)$$

where $\underline{v}, \underline{b}, \underline{c}, \underline{\rho}$ as well as $\underline{\mathcal{B}}$ are given in (3.13), (3.14) and (3.18) as well as $\tilde{h} = h - \frac{1}{2} \rho_a \mathcal{B}^a$. The complex symmetric tensor appearing in (3.64) is given by $\Theta_{ab} = \mathcal{K}_{ab\alpha} t^\alpha$ and $(\text{Re}\Theta)^{ab}$

⁴Recall that the first potential term in the DBI action (3.40) is canceled by the contribution (3.43) of the $O5$ -planes.

denotes the inverse of $\text{Re}\Theta_{ab}$. The function $\mathcal{L}_{A\bar{B}}^\alpha$ is defined in (3.42). Note that we recover the $\mathcal{N} = 1$ coordinates (2.39) of the bulk $O5/O9$ setup discussed in section 2.2.3 if we set $\underline{\zeta} = \underline{a} = 0$. The completion (3.64) by the open string fields is inferred from the couplings in the D5-brane action (3.39) and (3.48).

The full $\mathcal{N} = 1$ Kähler potential is determined by integrating the kinetic terms of the complex scalars $M^I = (S, \underline{t}, \underline{P}, \underline{z}, \underline{\zeta}, \underline{a})$. It takes the form

$$K = -\ln \left[-i \int \Omega \wedge \bar{\Omega} \right] + K_q, \quad K_q = -2 \ln \left[\sqrt{2} e^{-2\phi} \mathcal{V} \right], \quad (3.65)$$

where K_q has to be evaluated in terms of the coordinates (3.64). In contrast to general compactifications with $O3/O7$ -orientifold planes, cf. section 2.2.3, this can be done explicitly for $O5$ -orientifolds yielding

$$K_q = -\ln \left[\frac{1}{48} \mathcal{K}_{\alpha\beta\gamma} \Xi^\alpha \Xi^\beta \Xi^\gamma \right] - \ln \left[S + \bar{S} + \frac{1}{4} (\text{Re}\Theta)^{ab} (P + \bar{P})_a (P + \bar{P})_b - 2\mu_5 \ell^2 \mathcal{C}^{I\bar{J}} a_I \bar{a}_{\bar{J}} \right], \quad (3.66)$$

where we write

$$\Xi^\alpha = t^\alpha + \bar{t}^\alpha - \mu_5 \mathcal{L}_{A\bar{B}}^\alpha \zeta^A \bar{\zeta}^{\bar{B}}. \quad (3.67)$$

Note that the expression (3.65) for K can already be inferred from general Weyl rescaling arguments, e.g. from the factor e^K in front of the $\mathcal{N} = 1$ potential (2.38). However, the explicit form (3.66) displaying the field dependence of K has to be derived by taking derivatives of K and comparing the result with the bulk and D5-brane effective action. Let us also note that the expression (3.66) reduces to the results found in [160, 161] in the orbifold limit.

3.3.2 The Superpotential

Having defined the $\mathcal{N} = 1$ chiral coordinates as well as the Kähler potential we are prepared to deduce the effective superpotential W . Using the general supergravity formula (2.38) for the scalar potential expressed in terms of W we are able, as presented below, to deduce the superpotential W entirely by comparison to the scalar potential V_F in (3.61) as derived from dimensional reduction. This indeed identifies V_F as an F-term potential of the $\mathcal{N} = 1$ effective theory as indicated by the notation.

The superpotential W yielding V_F consists of two parts, a truncation of the familiar flux superpotential for the closed string moduli [14] and a contribution encoding the dependence on the open string moduli of the wrapped D5-brane,

$$W = \int_{Z_3} F_3 \wedge \Omega + \mu_5 \int_{\Sigma_+} \zeta \lrcorner \Omega, \quad (3.68)$$

where we introduced the R–R-flux F_3 . Now, it is a straightforward but lengthy calculation summarized in appendix A.3 to obtain the F-term contribution of the scalar potential (2.38).

The detailed calculations yield the positive definite F-term potential

$$V = \frac{i e^{4\phi}}{2\mathcal{V}^2 \int \Omega \wedge \bar{\Omega}} \left[|W|^2 + D_{z^\kappa} W D_{\bar{z}^{\bar{\kappa}}} \bar{W} G^{\kappa\bar{\kappa}} + \mu_5 \mathcal{G}^{A\bar{B}} e^{-\phi} \int_{\Sigma_+} s_{A\perp} \Omega \int_{\Sigma_+} \bar{s}_{\bar{B}\perp} \bar{\Omega} \right]. \quad (3.69)$$

Here the covariant derivatives with respect to the complex structure coordinates z^κ and the open string moduli ζ^A read

$$D_{z^\kappa} W = \int F_3 \wedge \chi_\kappa + \mu_5 \int \zeta \lrcorner \chi_\kappa, \quad D_{\zeta^A} W = \mu_5 \int s_{A\perp} \Omega + \hat{K}_{\zeta^A} W. \quad (3.70)$$

Furthermore, we have to use the first order expansion of $s_{A\perp} \Omega$ discussed in (3.21) to obtain a form of type (1, 1) that can be integrated over Σ_+ yielding a potentially non-vanishing result,

$$\int_{\Sigma_+} s_{A\perp} \Omega = \int_{\Sigma_+} s_{A\perp} \chi_\kappa \delta z^\kappa. \quad (3.71)$$

Inserting this into (3.69), the F-term potential perfectly matches the scalar potential V_F of (3.61) obtained by dimensional reduction of the D5-brane as well as the bulk supergravity action.

The superpotential (3.68) is the perturbative superpotential of the Type IIB compactification. However, in the form (3.68) it is just the leading term in the expansion of the chain integral⁵ [65, 67, 106, 107]

$$W_{\text{brane}} = \int_{\Gamma} \Omega, \quad (3.72)$$

where Γ is a three-chain with boundary given as $\partial\Gamma = \Sigma - \Sigma_0$, where Σ_0 is a fixed reference curve in the same homology class as Σ . W_{brane} depends on the closed string complex structure moduli through the holomorphic three-form Ω and on the open string fields through the deformation parameters of the curve Σ . Using the general power series expansion of a functional about a reference function, we recover our result for the superpotential (3.68) to linear order.⁶ It is one central aim of this thesis to study and exactly calculate this brane superpotential in various setups and invoking different physical and mathematical techniques.

We conclude with a discussion of the derivation and the special structure of the F-term potential. We first note that the potential (3.69) is positive definite unlike the generic F-term potential of supergravity. This is due to the no-scale structure [162–164] of the underlying $\mathcal{N} = 1$ data. Indeed, the superpotential (3.68) only depends on \underline{z} and $\underline{\zeta}$ and is independent of the chiral fields S , \underline{P} , \underline{a} and \underline{t} . Consequently, the $\mathcal{N} = 1$ covariant derivative $D_{M^I} W$ of the superpotential simplifies to $K_{M^I} W$ when applied with respect to the fields $M^I = (S, \underline{P}, \underline{a}, \underline{t})$. The Kähler potential (3.65) for these fields has the schematic form

$$K = -m \ln(t + \bar{t} + f(\zeta, \bar{\zeta})) - n \ln(S + \bar{S} + g(P + \bar{P}, t + \bar{t}) + h(a, \bar{a})) \quad (3.73)$$

⁵In this section we set $\mu_5 = 1$.

⁶The general Taylor expansion is given by $F[g] = \sum_{k=0}^{\infty} \int dx_1 \cdots dx_k \frac{1}{k!} \left. \frac{\delta^k F[g]}{\delta g(x_1) \cdots \delta g(x_k)} \right|_{g=\tilde{g}} \delta g(x_1) \cdots \delta g(x_k)$. For W as a functional of the embedding ι and $\delta\iota \equiv \zeta$ as well as $\tilde{g} = \iota$ we to first order derive the second term of (3.68).

with $m = 3$ and $n = 1$, where we concentrate on the one-modulus case for each chiral multiplet in order to clarify our exposition. The generalization to an arbitrary number of moduli is straightforward, cf. appendix A.3, where also the functions f , g and h can be found. Then the contributions of the fields $M^I = (S, P, a, t, \zeta)$ to the scalar potential V are found to take the characteristic form given by

$$K^{I\bar{J}} D_{M^I} W D_{\bar{M}^{\bar{J}}} \bar{W} = |\partial_{\zeta} W|^2 K^{\zeta\bar{\zeta}} + (n + m) |W|^2 \quad (3.74)$$

as familiar from the basic no-scale type models of supergravity.⁷ Consequently, this turns the negative term $-3|W|^2$ in (2.38) into the positive definite term $|W|^2$ of (3.69) for the case $n = 1$ and $m = 3$. A similar structure for the underlying $\mathcal{N} = 1$ data has been found for D3- and D7-branes as shown in [61–63, 165–167].⁸ In particular, this form for the scalar potential V on the complex structure and D-brane deformation space implies that a generic vacuum is de Sitter, i.e. has a positive cosmological constant, while in a supersymmetric vacuum both V and W vanish. However, the potential depends on the Kähler moduli only through an overall factor of the volume and thus drives the internal space to decompactify.

3.3.3 The Gauge-Kinetic Function, Gaugings and D-term Potential

In the following we discuss the terms of the four-dimensional effective action arising due to the $U(1)$ vector multiplets in the spectrum. Firstly, there are the kinetic terms of the D5-brane vector A and the vectors \underline{V} arising from the expansion (3.14) of the R–R form C_4 . The gauge-kinetic function is determined from the actions (3.39) and (3.48) and reads

$$f_{\Sigma\Sigma}(t^{\Sigma}) = \frac{1}{2} \mu_5 \ell^2 t^{\Sigma} \ , \quad f_{kl}(z^{\kappa}) = -\frac{i}{2} \bar{\mathcal{M}}_{kl} = -\frac{i}{2} \mathcal{F}_{kl} \Big|_{z^k=0=\bar{z}^{\bar{k}}} \ , \quad (3.75)$$

where the complex matrix \mathcal{M} is defined in appendix A.1. Here $f_{\Sigma\Sigma}$ is the gauge-coupling function for the D5-brane vector A and f_{kl} is the gauge-coupling function for the bulk vectors \underline{V} discussed in (2.51). As reviewed in section 2.2.3, we note that the latter can be expressed via $\mathcal{F}_{kl} = \partial_{z^k} \partial_{z^l} \mathcal{F}$ as the second derivative of the $\mathcal{N} = 2$ prepotential \mathcal{F} with respect to the $\mathcal{N} = 2$ coordinates z^k which have then to be set to zero in the orientifold set-up. This ensures that the gauge-coupling function is holomorphic in the coordinates z^{κ} which would not be the case for the full $\mathcal{N} = 2$ matrix $\bar{\mathcal{M}}_{KL}$ given in (A.2).

There are some remarks in order. Firstly, we note that the gauge-kinetic function encoding the mixing between the D5-brane vector and the bulk vectors is discussed in appendix A.2. Secondly, we observe that the quadratic dependence of $f_{\Sigma\Sigma}$ on the open string moduli $\underline{\zeta}$ through the coordinate \underline{t} in (3.64) is not visible on the level of the effective action. These

⁷This no-scale structure will be clarified further, extending the example of [128], in appendix A.3 using the dual description of $S + \bar{S}$ in terms of a linear multiplet L .

⁸See [168] for a similar discussion in heterotic M-theory.

corrections as well as further mixing with the open string moduli are due to one-loop corrections of the sigma model and thus not covered by our bulk supergravity approximation nor the Dirac-Born-Infeld or Chern-Simons actions of the D5-brane.

Let us now turn to the terms in the scalar potential induced by the gauging of global shift symmetries and compare to the potential V_D in (3.63). There are two sources for such gaugings. The first gauging arises due to the source term proportional to $d(\tilde{\rho}^\Sigma - \mathcal{C}_{(2)}\mathcal{B}^\Sigma) \wedge A$ in (3.48). It enforces a gauging of the scalars dual to the two-forms $\tilde{\rho}^\Sigma$ and $\mathcal{C}_{(2)}$. In fact, eliminating $d\tilde{\rho}^\Sigma$ and $d\mathcal{C}_{(2)}$ by their equations of motion, the kinetic terms of the dual scalars ρ_a and h contain the covariant derivatives

$$\mathcal{D}\rho_a = d\rho_a + \mu_5 \ell \delta_a^\Sigma A, \quad \mathcal{D}h = dh + \mu_5 \ell \mathcal{B}^\Sigma A, \quad (3.76)$$

where A is the $U(1)$ vector on the D5-brane. Rearranging this into $\mathcal{N} = 1$ coordinates we observe that the signs in the covariant derivative of h and $\underline{\rho}$ arrange⁹ to ensure that the complex scalar S defined in (3.64) remains neutral under A . However, the gaugings (3.76) imply a charge for the chiral field P_Σ . It is gauged by the D5-brane vector A . Its covariant derivative is given by

$$\mathcal{D}P_\Sigma = dP_\Sigma + i\mu_5 \ell A. \quad (3.77)$$

The second gauging arises in the presence of electric NS–NS three-form flux \tilde{e}_K introduced in (3.62). It was shown in [128], that the scalar h is gauged by the bulk $U(1)$ vectors \underline{V} arising in the expansion (3.14) of C_4 . This forces us to introduce the covariant derivative

$$\mathcal{D}S = dS - i\tilde{e}_{\tilde{K}} V^{\tilde{K}}. \quad (3.78)$$

The introduction of magnetic NS–NS three-form flux is more involved and leads to a gauged linear multiplet $(\phi, \mathcal{C}_{(2)})$ as described in [128].

Having determined the covariant derivatives (3.77) and (3.78) it is straightforward to evaluate the D-term potential. Recall the general formula for the D-term [59]

$$K_{I\bar{J}} \bar{X}_k^{\bar{J}} = i\partial_I D_k, \quad (3.79)$$

where X^I is the Killing vector of the $U(1)$ transformations defined as $\delta M^I = \Lambda_0^k X_k^J \partial_J M^I$. For the gaugings (3.77) and (3.78) we find the Killing vectors $X^{P_\Sigma} = i\mu_5 \ell$ and $X_{\tilde{K}}^S = -i\tilde{e}_{\tilde{K}}$ which are both constant. Integrating (3.79) one evaluates the D-terms using $K_{P_\Sigma} = \tilde{K}_{P_\Sigma}$ and K_S given in (A.14) respectively above (A.13) in appendix A.3 as

$$D = -\frac{1}{4}\mu_5 \ell e^\phi \mathcal{B}^\Sigma \mathcal{V}^{-1}, \quad D_{\tilde{K}} = \frac{1}{2}\tilde{e}_{\tilde{K}} e^\phi \mathcal{V}^{-1}. \quad (3.80)$$

Inserting these D-terms into the $\mathcal{N} = 1$ scalar potential (2.38) and using the gauge-kinetic functions (3.75), we precisely recover the D-term potential V_D in (3.63) found by dimensional reduction.

⁹The plus sign in the covariant derivative of h arises due to the minus sign in the duality conditions (2.10).

3.4 Extension to Infinite Degrees of Freedom

In this concluding section we slightly extend, following [100], our discussion of the D5-brane effective action to the full geometric deformation space of the D5-brane on Σ , that is given by the Hilbert space of all sections in $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$. It contains both the light fields ζ^A in $H^0(\Sigma, N_{Z_3}\Sigma)$ discussed before but also an infinite tower of massive modes¹⁰. It is the content of the following discussion to show that the massive modes already receive a mass at leading order and consequently decouple consistently from the effective action. We note a similar and mathematically more self-contained discussion in section 7.1.

In words we exploit for our effective action analysis the mathematical result of [169, 170] that the volume of a holomorphic curve Σ , $\text{Vol}(\Sigma)$, for any deformation of Σ by an infinitesimal displacement ϵs , for $s \in \mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$, increases quadratically as

$$\left. \frac{d^2}{d\epsilon^2} \text{Vol}(\Sigma_{\epsilon s}) \right|_{\epsilon=0} = \frac{1}{2} \int_{\Sigma} \|\bar{\partial}s\|^2 \text{vol}_{\Sigma}, \quad (3.81)$$

where $\Sigma_{\epsilon s}$ denotes the deformed curve. Here $\|\bar{\partial}s\|^2$ denotes a contraction of all indices, i.e. the tangential directions and the directions normal to Σ , via the metric. We prove the statement that this second variation of the volume (3.81) is a part of the F-term potential of the D5-brane effective action. We first obtain this potential by dimensional reduction of the Dirac-Born-Infeld action of the D5-brane. Then we use the D5-brane superpotential W_{brane} in (3.72) and a generalization of the Kähler metric of (3.65) [60] to the infinite dimensional space $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$ to deduce the same potential as an F-term potential when gravity is decoupled.

We start again from the Dirac-Born-Infeld action (3.33) of a single D5-brane in Z_3 . We perform the dimensional reduction for the background of a D5-brane wrapping a holomorphic curve Σ . In a background with vanishing B-field and gauge flux F the action (3.33) is just the volume of the wrapped curve. Thus, the variation of $S_{\text{DBI}}^{\text{SF}}$ under a deformation along s is captured, up to second order in the variation parameter ϵ , by (3.81) and reads

$$V_{\text{DBI}} \supset \frac{\mu_5 e^{3\phi}}{\mathcal{V}^2} \left. \frac{d^2}{d\epsilon^2} \text{Vol}(\Sigma_{\epsilon s}) \right|_{\epsilon=0} = \frac{\mu_5 e^{3\phi}}{2\mathcal{V}^2} \int_{\Sigma} \|\bar{\partial}s\|^2 \text{vol}_{\Sigma}. \quad (3.82)$$

Here we used the formula (3.81) and further a Weyl-rescaling to the four-dimensional Einstein-frame to obtain the right factors of the dilaton ϕ and the compactification volume \mathcal{V} .

In the following we deduce this potential from the $\mathcal{N} = 1$ formulation of the D5-brane effective action. Indeed, the term (3.82) is an F-term potential of the form

$$V_{\text{F}} = e^K K^{a\bar{b}} \partial_{u_a} W_{\text{brane}} \partial_{\bar{u}_{\bar{b}}} \bar{W}_{\text{brane}} \quad (3.83)$$

¹⁰This is in contrast to the usual lore of dimensional reduction, however, we refer to section 7.1.2 for a mathematical explanation of the fact, that the natural domain of the brane superpotential W_{brane} is $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$.

for the fields $u^a(x)$ associated to the expansion $s = u^a s_a$ in a basis of $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$. In order to evaluate V_F we need the Kähler metric for the modes u^a as well as a more tractable form of the brane superpotential W_{brane} . The Kähler metric for the u^a as deformations in the infinite dimensional space $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$ is a straightforward generalization of the Kähler metric of (3.65) originally considered for the modes counted by $H^0(\Sigma, N_{Z_3}\Sigma)$. It reads

$$K_{a\bar{b}} = \frac{-i\mu_5 e^\phi}{4\mathcal{V}} \int_\Sigma s_{a\perp} \bar{s}_{b\perp} (J \wedge J) = \frac{i\mu_5 e^\phi}{\int \Omega \wedge \bar{\Omega}} \int_\Sigma (\Omega_a)_{ij} (\bar{\Omega}_{\bar{b}})^{ij} \iota^*(J), \quad (3.84)$$

where we used the abbreviation $\Omega_a = s_{a\perp} \Omega$. For details of this equality we refer to appendix A.4. Firstly, we Taylor expand W_{brane} to second order in the brane deformations u^a around the holomorphic curve $\Sigma = \Sigma_h$

$$W_{\text{brane}} = \int_{\Gamma_h} \Omega + \frac{1}{2} u^a u^b \int_\Sigma s_{a\perp} ds_{b\perp} \Omega + \mathcal{O}(u^3) \quad (3.85)$$

where s_\perp denotes the interior product with s and s_a denotes a section of $N_{Z_3}\Sigma$ that is not required to be holomorphic. Γ_h is a chain ending on the holomorphic curve Σ_h , $\partial\Gamma_h = \Sigma_h - \Sigma_0$. Secondly, introducing the abbreviation $\Omega_a = s_{a\perp} \Omega$ the variation of (3.85) with respect to u_a reads

$$\partial_{u_a} W_{\text{brane}} = -\mu_5 \int_\Sigma \bar{\partial} s_\perp \Omega_a. \quad (3.86)$$

In addition we rescaled the superpotential $W_{\text{brane}} \mapsto \mu_5 W_{\text{brane}}$ to restore physical units as before [60]. In order to evaluate the contraction (3.83) we have to exploit that the Ω_a form a basis of sections of a specific bundle on Σ . Indeed, the isomorphism of $KZ_3|_\Sigma = T^*\Sigma \otimes N^*\Sigma$ which is a consequence of the normal bundle sequence of Σ tells us that the Ω_a form a basis of sections of $\Omega^{(1,0)}(\Sigma, N^*\Sigma)$ with the property that $s_{a\perp} \Omega_a = 0$. We can use this basis to represent any other section. In particular, the contraction $\bar{\partial} s_\perp J$ is a section of $\Omega^{(0,1)}(\Sigma, \overline{N^*\Sigma})$ that we can expand in the basis $\bar{\Omega}_{\bar{a}}$ as

$$\bar{\partial} s_\perp J = \frac{-i\mu_5 e^\phi}{\int \Omega \wedge \bar{\Omega}} \Omega_a K^{a\bar{b}} \int_\Sigma \bar{\partial} s_\perp \bar{\Omega}_{\bar{b}}. \quad (3.87)$$

Again we refer to appendix A.4 for the details of this calculation. Finally, we calculate the F-term potential (3.83) as

$$V_F = e^K \mu_5^2 \int_\Sigma \bar{\partial} s_\perp (\Omega_a K^{a\bar{b}} \int_\Sigma \partial \bar{s}_\perp \bar{\Omega}_{\bar{b}}) = \frac{\mu_5 e^{3\phi}}{2\mathcal{V}^2} \int_\Sigma \|\bar{\partial} s\|^2 \text{vol}_\Sigma. \quad (3.88)$$

Here we used in the second equality the identity (3.87) as well as $e^K = \frac{ie^{4\phi}}{2\mathcal{V}^2 \int \Omega \wedge \bar{\Omega}}$, cf. appendix A.4. The norm $\|\cdot\|^2$ denotes as before the contraction of all indices using the metric.

This F-term potential is in perfect agreement with contribution (3.82) to the scalar potential V_{DBI} that we obtain from the reduction of the DBI-action (3.33) using the variation (3.81) of the calibrated volume.

Part II

String Dualities

Chapter 4

Heterotic/F-Theory Duality and Five-Brane Dynamics

In this chapter we introduce heterotic/F-theory duality and its application to analyze non-perturbative heterotic string compactifications. In general, the strength of F-theory is the geometrization of physics, for the analysis of both non-perturbative Type IIB compactifications with seven-branes and of dual heterotic string compactifications with five-branes. We will mostly be interested in four-dimensional $\mathcal{N} = 1$ compactification, for which this geometrization provides strong geometrical tools to study the coupling functions of the four-dimensional effective actions. We will put particular emphasis on the geometrization of branes in F-theory, both for the Type IIB theory and for the dual heterotic string, which we will exploit explicitly in chapter 6 for concrete calculations of the corresponding effective superpotentials.

In section 4.1 we start with a discussion of the heterotic string, its strongly coupled formulation as heterotic M-theory and its compactifications. We first review heterotic M-theory compactifications on a Calabi-Yau threefold $Z_3 \times S^1/\mathbb{Z}_2$ to four-dimensions in the presence of spacetime-filling five-branes wrapped on a curve Σ . Then we briefly comment on the massless spectrum including the geometric moduli and discuss in some detail the small instanton transition of a smooth bundle into a heterotic five-brane. Then we present the full heterotic superpotential and briefly review the essential steps of the spectral cover construction of heterotic vector bundles E on elliptic Calabi-Yau threefolds Z_3 . Next in section 4.2 we review the basics of F-theory. We start with a brief discussion of Vafa's original motivation for F-theory before we proceed to a more elaborate construction of F-theory vacua from elliptic Calabi-Yau manifolds. There we readily focus on four-dimensional compactifications on Calabi-Yau fourfolds and the induced effective Gukov-Vafa-Witten flux superpotential. Finally we study heterotic/F-theory duality in section 4.3. After reviewing the underlying eight-dimensional duality between F-theory on an elliptic $K3$ and the heterotic string on T^2 , we use the adiabatic argument to construct pairs of dual lower-dimensional $\mathcal{N} = 1$ theories, in particular in four dimensions. We emphasize the duality map for the heterotic bundle E and the heterotic five-branes, that are mapped, as we discuss in detail, either to three-branes or to blow-ups in F-theory.

4.1 Heterotic String Compactifications with Five-Branes

In this section we review the construction of heterotic M-theory compactifications to four-dimensions. As detailed in section 4.1.1 an $\mathcal{N} = 1$ compactification is determined, in the absence of fluxes, by a choice of a Calabi-Yau threefold Z_3 , a holomorphic stable vector bundle E and a number of five-branes wrapped on curves Σ in Z_3 that are located at points in the M-theory interval S^1/\mathbb{Z}_2 . We note that heterotic M-theory can be used to systematically calculate corrections to these backgrounds. Furthermore, we emphasize the importance of a cancellation of tadpoles that restricts the second Chern class of E and the choice of five-brane curves Σ . Then in section 4.1.2 we present a brief account on the determination of the massless spectrum, including the geometrical and bundle moduli. We emphasize the relation of the bundle moduli and five-brane moduli via a small instanton transition of the smooth heterotic bundle E to a singular configuration, that is more appropriately described as a heterotic five-brane. Next in section 4.1.3 we discuss the perturbative heterotic superpotential, that is a sum of the heterotic flux superpotential, the holomorphic Chern-Simons functional of the bundle E and a superpotential for heterotic five-branes, that takes the form of a chain integral. The existence of the latter is inferred from the effect of the small instanton transition on the holomorphic Chern-Simons functional. We conclude in section 4.1.4 with a presentation of the essential steps in the spectral cover construction of heterotic vector bundles on elliptically fibered Calabi-Yau manifolds. There we summarize also some results for E_8 -bundles, that have to be constructed by different means.

4.1.1 Heterotic M-Theory Compactification

The formulation of the ten-dimensional $E_8 \times E_8$ heterotic string theory at strong coupling g_S is given by eleven-dimensional supergravity, which is an effective description of the low-energy interactions of the light modes of M-theory, on $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$ [171]. The heterotic string coupling is related to the radius R of S^1 in the same way the string coupling of Type IIA is related to R ,

$$g_S = R^{3/2}, \quad (4.1)$$

which is deduced in [41] by the equivalence of the corresponding effective actions. We note that at small R compared to the eleven-dimensional Planck length $\kappa^{2/9}$, the eleven-dimensional supergravity description breaks down due to large curvature effects and the perturbative heterotic string description applies. However, in order to treat non-perturbative solitonic string effects in the heterotic theory, like NS5-branes, it is appropriate to consider the M-theory formulation which is denoted heterotic M-theory.

We begin by giving a definition of M-theory on the orbifold $S^1/\mathbb{Z}_2 \times \mathbb{R}^{1,9}$. Heterotic M-theory in eleven dimensions is known by its effective action. Its Lagrangian is systematically constructed in [172] as an expansion in the gravitational coupling $\kappa^{2/3}$ from eleven-dimensional

supergravity in the bulk of S^1/\mathbb{Z}_2 coupled to two ten-dimensional Super-Yang-Mills theories on the boundaries of S^1/\mathbb{Z}_2 . Here κ and the gauge coupling λ are related as

$$\lambda^2 = 2\pi(4\pi\kappa^2)^{2/3}. \quad (4.2)$$

In this expansion the kinetic terms of the gauge bosons, proportional to $\frac{1}{\lambda^2}$, appear at order $\kappa^{2/3}$ relative to the gravitational term, that scales as $\frac{1}{\kappa^2}$. Furthermore, a consistent and gauge invariant effective theory requires the inclusion of Green-Schwarz terms [172] even in the minimal Lagrangian. This is in contrast to the weakly coupled heterotic string where the Green-Schwarz term is not present in the minimal supergravity and Super-Yang-Mills theory and only appears when quantum loops are taken into account, where it guarantees cancellation of gauge and gravitational anomalies. Thus, only to zeroth order in $\kappa^{2/3}$, i.e. without the boundary theory, heterotic M-theory exists as a consistent classical theory and at any higher order in $\kappa^{2/3}$ quantum effects, that are one-loop effects like anomalies at first order, have to be included to guarantee consistency, which is consequently gauge invariance at first order [172]. The use of $\kappa^{2/3}$ as an expansion parameter is further exploited in [173], where a strong coupling expansion is applied to calculate corrections to the perturbative heterotic string backgrounds with $\mathcal{N} = 1$ spacetime supersymmetry found in [10]. These are defined by a Calabi-Yau threefold Z_3 and a stable holomorphic vector bundle E and appear as lowest order backgrounds in the M-theory context as reviewed below¹. The correction terms to these backgrounds are organized in a double expansion in dimensionless parameters [177, 178]

$$\epsilon_s = \left(\frac{\kappa}{4\pi}\right)^{2/3} \frac{2\pi R}{\mathcal{V}^{2/3}}, \quad \epsilon_R = \frac{\mathcal{V}^{1/6}}{\pi R}, \quad (4.3)$$

which are roughly the compactification scale $\mathcal{V}^{1/6}$ and the interval size R . This expansion is valid for all types of embeddings and in particular in the presence of M5-branes. Besides its conceptual relevance the crucial point of the M-theory description and the strong coupling expansion is the supersymmetric treatment of a non-trivial Bianchi identity of the field strength G_4 of the three-form C_3 in eleven-dimensional supergravity. This is in general non-trivial, in particular in the presence of five-branes [173] that appear as singular sources of the Bianchi identity. In addition, the heterotic M-theory formalism allows to treat the backreaction of five-branes on the geometry and the background fields, in particular G_4 , cf. [178] for an analysis to first order in ϵ_s .

Let us review the construction of heterotic compactifications to four dimensions with $\mathcal{N} = 1$ supersymmetry in the M-theory context. This means that we consider the background geometry $S^1/\mathbb{Z}_2 \times Z_3 \times \mathbb{R}^{1,3}$. In order to obtain a supersymmetric background the supersymmetry variations of all fermionic fields in the theory have to vanish identically in the background. To lowest order in ϵ_s this yields a Killing spinor equation from the eleven-dimensional gravitino which is solved by vanishing background flux $G_4 = 0$ and by Z_3 being

¹We note that one already obtains corrections within the heterotic string at weak coupling for non-standard embeddings [2, 22, 174], see e.g. [175] for a review. See [176] for a detailed discussion of phenomenological applications.

a Calabi-Yau threefold [10]. Although a first order effect in ϵ_s , one usually also includes the two E_8 gauge theories on the boundary in the construction of the heterotic background². The supersymmetry variation of the ten-dimensional gauginos yields the condition on the VEV of the gauge-fields on Z_3 , which can be non-vanishing without breaking four-dimensional Lorentz invariance. The gauge-fields have to define a vector bundle $E = E_1 \oplus E_2$ with structure group being a subgroup of $E_8 \times E_8$ such that the curvature \mathcal{F} obeys

$$\mathcal{F}_{ij} = \mathcal{F}_{\bar{i}\bar{j}} = 0, \quad g^{i\bar{j}}\mathcal{F}_{i\bar{j}} = 0. \quad (4.4)$$

The first two conditions turn E into a holomorphic vector bundle and the second condition is the Donaldson-Uhlenbeck-Yau equation. The latter condition requires the class of the field strength \mathcal{F} to be primitive, i.e. $g^{i\bar{j}}\mathcal{F}_{i\bar{j}} \sim \mathcal{F} \wedge *J \sim \mathcal{F} \wedge J^2 = 0$, which is a restrictive condition rendering a direct construction of a bundle E obeying it technically challenging. A construction is possible only under particular circumstances, one of which is when Z_3 is elliptically fibered as will be discussed below in section 4.1.4. However, the existence of E obeying (4.4) was proven by Donaldson for the rank two case in [179] and by Uhlenbeck and Yau in [180] for arbitrary rank in the context of finding solutions to the Hermitian Yang-Mills equation,

$$J^2 \wedge \mathcal{F} = \frac{2\pi}{\mathcal{V}}\mu(E) \text{vol}_{Z_3} \mathbf{1}, \quad \mu(E) = \frac{1}{\text{rk}(E)} \int_{Z_3} c_1(E) \wedge J^2, \quad (4.5)$$

where $\text{rk}(E)$ and $\mu(E)$ denote the rank and the slope of E , respectively. Then the second condition in (4.4), $\mathcal{F} \wedge J^2 = 0$, implies the special case of zero slope $\mu(E) = 0$. In general, (4.5) has a unique solution if E is slope-(semi-)stable with respect to J , i.e. for any subbundle $V \subset E$ the condition

$$\mu(V) \leq \mu(E) \quad (4.6)$$

holds, or if E is a direct sum of (semi-)stable bundles³.

This setup is further constrained by the cancellation of tadpoles in heterotic M-theory. This condition is derived from the Bianchi identity of the field strength G_4 that takes the form [171, 173, 178]

$$dG_4 = 4\sqrt{2}\pi \left(\frac{\kappa}{4\pi}\right)^{\frac{2}{3}} \left[J^{(0)}\delta(x^{11}) + J^{(N+1)}\delta(x^{11} - \pi R) + \frac{1}{2} \sum_{n=1}^N \delta_{\Sigma_i}(\delta(x^{11} - x_n) + \delta(x^{11} + x_n)) \right] \quad (4.7)$$

where $\delta(x^{11} - x_n)$, δ_{Σ_i} denote delta-functions on S^1/\mathbb{Z}_2 , with coordinate x^{11} , respectively in Z_3 , that are supported on curves Σ_i wrapped by five-branes. We note that a supersymmetric five-brane preserving four-dimensional Minkowski invariance localizes to a point x_n in S^1/\mathbb{Z}_2 and to a holomorphic curve in Z_3 [36]. The sources $J^{(0)}$, $J^{(N+1)}$ contribute boundary terms

²In the weakly coupled heterotic string the $E_8 \times E_8$ gauge symmetry is inferred from the requirement of modular invariance implying anomaly cancellation in the low-energy effective theory [10].

³We note that a stable bundle requires $H^0(Z_3, E) \cong H^3(Z_3, E) = 0$ to guarantee the absence of maps $\mathcal{O} \rightarrow E$ [181].

from the two ten-dimensional gauge theories on the boundary,

$$J^{(0)} = -\frac{1}{16\pi^2} \left(\text{tr } \mathcal{F}^2 - \frac{1}{2} \text{tr } R^2 \right) \Big|_{x^{11}=0}, \quad J^{(N+1)} = -\frac{1}{16\pi^2} \left(\text{tr } \mathcal{F}^2 - \frac{1}{2} \text{tr } R^2 \right) \Big|_{x^{11}=\pi R}, \quad (4.8)$$

where R denotes the eleven-dimensional curvature. The symbol “tr” denotes the trace in the vector representation of $O(1,9)$ for R and the $\frac{1}{30}$ of the trace in the adjoint of E_8 for the gauge field \mathcal{F} , respectively. Whereas (4.7) determines the actual form G_4 pointwise it can be evaluated in the cohomology of Z_3 yielding the tadpole condition,

$$\lambda(E_1) + \lambda(E_2) + \sum_i [\Sigma_i] = c_2(Z_3), \quad (4.9)$$

where $\lambda(E)$ is the fundamental characteristic class of the vector bundle E , which, for example, is $c_2(E)$ for $SU(N)$ bundles and $c_2(E)/30$ for E_8 bundles⁴. This tadpole requires the inclusion of gauge background bundles E over Z_3 with structure group contained in the ten-dimensional heterotic gauge group [10]. Note that in many heterotic compactifications the inclusion of five-branes is not a choice, but rather a requirement for tadpole cancellation as demonstrated explicitly in the case of elliptic threefolds Z_3 in [182]. The condition (4.9) then dictates consistent choices of the cohomology classes $[\Sigma_i]$ of the curve Σ_i in the presence of a non-trivial vector bundle E to match the curvature of the threefold Z_3 as measured by the second Chern class $c_2(Z_3)$. In particular, it implies that Σ_i corresponds to an effective class in $H_2(Z_3, \mathbb{Z})$ [183]. We note that the sources in (4.7) are of order ϵ_s . Thus the solution $G_4 = 0$ above (4.4) is consistent at zeroth order. However, since the inclusion of five-branes and their backreaction is the central point in this work, an appropriate treatment of the non-trivial Bianchi identity with localized sources will be essential as demonstrated in chapter 7.

There is one further condition on the first Chern-class of E , sometimes denoted as the K-theory constraint. It can be derived from the requirement of anomaly cancellation in the sigma-model world-sheet theory [184, 185] and reads

$$c_1(E) = 0 \pmod{2} \quad \Leftrightarrow \quad c_1(E) \in H^2(Z_3, 2\mathbb{Z}). \quad (4.10)$$

Geometrically this conditions guarantees that the bundle E admits spinors, since (4.10) is equivalent to a trivial second Stiefel-Whitney class $w_2 \in H^2(Z_3, \mathbb{Z}_2)$ for holomorphic bundles.

4.1.2 Charged Matter, Moduli and Small Instanton Transitions

The low-energy effective theory obtained by compactification of the heterotic string or heterotic M-theory is very roughly an $\mathcal{N} = 1$ gauge theory with additional chiral multiplets [10]. In the perturbative string all four-dimensional fields arise from the ten-dimensional gauge

⁴Here we used the Chern characters as $\text{ch}_2(Z_3) = -c_2(Z_3)$, $\text{ch}_2(E) = -c_2(E)$ for a compactification on Calabi-Yau threefolds Z_3 assuming $c_1(E) = 0$. In addition we ignore the numerical coefficients in (4.7).

theory and the moduli of the background geometry. The four-dimensional gauge symmetry G arises from the ten-dimensional gauge symmetry as the commutant of the background bundle E with structure group H within $E_8 \times E_8$. In the group-theoretical decomposition of the adjoint of one E_8 into representations of $G \times H$,

$$248 \rightarrow (\mathbf{adj}(G), \mathbf{1}) \oplus (\mathbf{1}, \mathbf{adj}(H)) \bigoplus_i (\mathbf{R}_i, \mathbf{Q}_i), \quad (4.11)$$

this corresponds to the fields in the adjoint of G , $\mathbf{adj}(G)$. Then charged chiral multiplets occur in the representations \mathbf{R}_i under the four-dimensional gauge symmetry G . Their chirality $n(\mathbf{R}_i)$ in the four-dimensional theory is calculated by an index theorem for the Dirac-operator of the gauginos on Z_3 that take values in \mathbf{Q}_i [2, 175, 181],

$$n(\mathbf{R}_i) = \chi(E_i) = \sum_n (-1)^n h^n(Z_3, E_i) = \int_{Z_3} \mathrm{Td}(Z_3) \mathrm{ch}(E_i) = \frac{1}{2} \int_{Z_3} c_3(E_i). \quad (4.12)$$

Here we assumed $c_1(E_i) = 0$, which will be the case in the examples constructed later on, and introduced the bundle E_i associated to the representation \mathbf{Q}_i . To the index only $h^2(Z_3, E_i^*) = h^1(Z_3, E_i)$ by Serre duality contribute since $h^3(Z_3, E_i) = h^0(Z_3, E_i) = 0$ for stable bundles. Indeed, the zero modes of the ten-dimensional gauginos are given by the cohomology groups $H^1(Z_3, E_i)$, $H^1(Z_3, E_i^*)$ [2]. We note that the charged fields are interpreted as matter fields in more phenomenological applications. In addition there are neutral chiral multiplets. They correspond to the moduli of the gauge bundle E , that are related to the adjoint $\mathbf{adj}(H)$ in (4.11) and are counted by $H^1(Z_3, \mathrm{End}(E))$, and to the moduli of Z_3 , counted by $H^{(2,1)}(Z_3)$ and $H^{(1,1)}(Z_3)$. In phenomenological applications it is the issue of moduli stabilization to fix these fields in order to avoid neutral massless scalars in four dimension.

In the non-perturbative heterotic string, there are additional fields in the low-energy effective theory due to spacetime-filling five-branes. Assuming a single five-brane wrapping a holomorphic curve Σ_i of genus g in Z_3 one obtains an additional gauge group $U(1)^g$, a universal sector of a chiral multiplet with bosonic fields (x^{11}, a) and a number of chiral multiplets for the deformations of Σ_i inside Z_3 . Here the gauge theory as well as the scalar a arise from the reduction of the self-dual two-form by the $2g$ one-forms respectively the harmonic volume form on the curve Σ_i wrapped by the five-brane [186, 187], whereas x^{11} denotes the position of the five-brane in the interval S^1/\mathbb{Z}_2 . In the following we neglect additional massless states from intersections of five-branes as well as the gauge enhancement to $U(N)^g$ in the case of multiple M5-branes, see e.g. [175, 178] for a detailed discussion.

In summary, the analysis of the moduli space of the heterotic string on Z_3 requires the study of three a priori very different pieces. Firstly, we have the geometric moduli spaces of the threefold Z_3 consisting of the complex structure as well as the Kähler moduli space. Secondly, there are the moduli of the bundle E which parametrize different gauge-field backgrounds on Z_3 . Finally, if five-branes are wrapped on non-rigid curves Σ_i , the deformations of Σ_i within Z_3 of the various five-branes have to be taken into account. Thus, the global moduli

space is in general very complicated since it is a fibration of the three different individual moduli spaces. This problem, however, becomes more tractable if one focuses Calabi-Yau threefolds Z_3 admitting additional structure, like e.g. an elliptic fibration. In this case it was shown in [182] that there exist elegant constructions of stable holomorphic vector bundles E , as we review in section 4.1.4. Moreover, the moduli space of five-branes on elliptically fibered Z_3 has been discussed in great detail in [188]. Qualitatively it is generically true, however, that the heterotic moduli space on a fixed Calabi-Yau threefold Z_3 admits several different branches corresponding to the number and type of five-branes present as well as to the topology of the bundle E . There are distinguished points in this moduli space corresponding to enhanced gauge symmetry [189, 190] of the heterotic string that allow for a clear physical interpretation and that we now discuss in more detail. It will turn out that at these points an interesting transition happens where a five-brane completely dissolves into a finite size instanton of the bundle E and vice versa. Thus, this transition connects different branches of the heterotic moduli space with a different number of five-branes and in particular with different topological type of the vector bundle E as the second Chern-class $c_2(E)$ will jump in this process. Consequently this can be viewed as a heterotic extension of the familiar topology changes in Type II string compactifications that connect Calabi-Yau threefolds Z_3 of different Euler characteristic and Hodge numbers [191], see e.g. [82, 192] for a review.

Let us discuss, following [81], this small instanton transition by starting with a Calabi-Yau threefold Z_3 with $c_2(Z_3) \neq 0$ and no five-branes. Thus, the anomaly condition (4.9) forces us to turn on a background bundle E with non-trivial second Chern class $c_2(E)$ in order to cancel $c_2(Z_3)$. Then the bundle is topologically non-trivial and carries bundle instantons characterized by the topological second Chern number [22]

$$[c_2] = - \int_{Z_3} J \wedge \mathcal{F} \wedge \mathcal{F}, \quad (4.13)$$

where J denotes the Kähler form on Z_3 and \mathcal{F} the field strength of the background bundle. The heterotic gauge group G in four dimensions is generically broken and given by the commutant of the holonomy group of E in $E_8 \times E_8$. Varying the moduli of E one can ask whether it is possible to restore parts or all of the broken gauge symmetry by flattening out the bundle as much as possible [70]. To show how this can be achieved, one first decomposes $c_2(E)$ into its irreducible components each of which being dual to an irreducible curve $[\Sigma_i]$ in Z_3 modulo chains. Since the invariant $[c_2]$ has to be kept fixed, the best we can do is to consecutively split off the components of $c_2(E)$ and to localize the curvature form of E on the corresponding curves Σ_i , i.e. the curvature becomes singular with support in Σ_i . This should be contrasted with the generic situation, where the curvature form is smooth on Z_3 and meets the condition $[\Sigma_i] \subset c_2(E)$ only in cohomology, i.e. up to exact forms. In this localization limit the holonomy of E around each individual curve Σ_i becomes trivial⁵ and the gauge group G enhances accordingly. Having reached this so-called small instanton

⁵The curvature is flat away from Σ_i and only monodromy effects persist.

configuration at the boundary of the moduli space of the bundle, the dynamics of (this part of) the gauge bundle can be effectively described by a five-brane on Σ_i [189]. In particular since E is holomorphic, the corresponding curves Σ_i are holomorphic in Z_3 and the wrapped five-brane is supersymmetric.

Small instanton configurations thus allow for supersymmetric transitions between branches of the moduli space with different numbers of five-branes and topologies of E , that consequently map bundle moduli to five-brane moduli and vice versa [193]. Note that this transition is completely consistent with (4.9) since we have just shifted irreducible components between the two summands $c_2(E)$ and $[\Sigma_i]$. Thus, we are in the following allowed to think about the small instanton configuration of E as a five-brane. In particular, doing this transition for all components of $c_2(E)$ the full perturbative heterotic gauge group $E_8 \times E_8$ can be restored. Turning this argument around, a heterotic string with full perturbative $E_8 \times E_8$ gauge symmetry on a threefold Z_3 with non-trivial $c_2(Z_3)$ has to contain five-branes to cancel the anomaly according to (4.9). We mention that the small instanton transition will be of particular importance once we are working in the framework of heterotic/F-theory duality, cf. chapter 6. In particular in our concrete examples of section 6.2 we will precisely encounter a situation with full perturbative $E_8 \times E_8$ gauge symmetry, where the complete heterotic anomaly is canceled by five-branes, which will guide us to the interpretation of the F-theory flux superpotential in terms of a five-brane superpotential for a particular class of heterotic five-branes.

4.1.3 The Heterotic Superpotential

In the following we discuss the perturbative superpotential for the heterotic string. It consists of three parts, the heterotic flux-superpotential induced by a possible flux H_3 of the B-field, the holomorphic Chern-Simons superpotential for the bundle E and the superpotential induced by five-branes, which will be of particular importance for this work. The three superpotentials can formally be unified as follows. We reduce the identity (4.7) on the interval S^1/\mathbb{Z}_2 to obtain the Bianchi identity for H_3 on $Z_3 \times R^{1,3}$ as⁶

$$dH_3 = \text{Tr}(R^2) - \text{Tr}(\mathcal{F}^2) + \sum_i \delta_{\Sigma_i} . \quad (4.14)$$

Then the superpotential can be expressed in terms of H_3 and reads, motivated by [194, 195],

$$W_{\text{het}} = \int_{Z_3} \Omega \wedge H_3 = W_{\text{flux}} + W_{\text{CS}} + W_{\text{brane}} , \quad (4.15)$$

where the different terms are associated to the various contributions in H_3 in (4.14)⁷. Since a more rigorous treatment to obtain the superpotentials on the right in (4.15) will be demonstrated in section 7.1 using the notion of currents, we will just list them in the following.

⁶In addition we set the numerical constants as well as κ to unity in (4.7).

⁷According to the heterotic Bianchi identity, a term involving the Chern-Simons form $\omega_3^L(R)$ should be present. However, to our knowledge such a term is not discussed in the literature.

The superpotential W_{flux} is due to background fluxes H_3 . In general a non-trivial background flux H_3 has to be in $H^3(Z_3, \mathbb{Z})$ due to the flux quantization condition. It is expanded as $H_3 = N^K \alpha_K - M_i \beta^i$ in the integral basis α_K, β^K of $H^3(Z_3, \mathbb{Z})$ with integer flux numbers N^i, M_i . Then the flux superpotential [14] takes the form

$$W_{\text{flux}} = \int_{Z_3} \Omega \wedge H_3 = M_K X^K - N^K \mathcal{F}_K, \quad (4.16)$$

where we used the period expansion $\Omega = X^K \alpha_K - \mathcal{F}_K \beta^K$, cf. (2.16). The superpotential W_{flux} is in general a complicated function on the complex structure moduli of Z_3 . Its complete moduli dependence is encoded in the periods $(\underline{X}, \underline{\mathcal{F}})$. In order to analyze their dependence one benefits from the techniques of algebraic geometry that are applicable for a wide range of examples, see [196] and [16,29] for reviews. This is due to the fact that $(\underline{X}, \underline{\mathcal{F}})$ obey differential equations, the so-called Picard-Fuchs equations, that can be determined and solved explicitly and thus allow to fix the complete moduli dependence of W_{flux} once the flux numbers are given. We will review these techniques in section 5.2. To end our discussion of the flux superpotential, let us stress that strictly speaking we have excluded background fluxes in section 4.1.1. Indeed, there is a back-reaction of H_3 which renders Z_3 to be non-Kähler [197]. Since our main focus will be on the five-brane superpotential, we will not be concerned with this back-reaction for the moment. However, we will come back to this point in chapter 9 and present a more rigorous treatment of the flux H_3 in the context of a compactification on a manifold with $SU(3)$ -structure.

The second term in (4.15) denotes the heterotic holomorphic Chern-Simons functional [22]⁸

$$W_{\text{CS}} = \int_{Z_3} \Omega \wedge (A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A) \quad (4.17)$$

by inserting the Chern-Simons form ω_3^{YM} for H_3 in (4.15). The functional W_{CS} depends on both the complex structure moduli of Z_3 and the bundle moduli of E through the connection A . As can be readily shown its extremal points are the holomorphic vector bundles E over Z_3 of the same topological type, cf. [199].

Finally the third term comprises the five-brane superpotential. The brane superpotential has the following properties. It depends holomorphically on the complex structure moduli of Z_3 , as well as on the (obstructed) deformations corresponding to holomorphic sections of $N_{Z_3} \Sigma$.⁹ Furthermore, the F-term supersymmetry conditions of W_{brane} , i.e. its critical points, correspond to holomorphic curves. This determines W_{brane} uniquely up to a constant as done in [65] for M-theory on a Calabi-Yau threefold Z_3 with a spacetime-filling M5-brane wrapped on a curve Σ ,

$$W_{\text{brane}} = \int_{\Gamma(\underline{u})} \Omega(\underline{z}). \quad (4.18)$$

⁸The same functional was derived in [198] as the superpotential for B-branes wrapping the entire Calabi-Yau threefold Z_3 .

⁹More precisely, denoting by u an open string deformation we expect a superpotential $W_{\text{brane}} = u_a^{n+1}$ if the deformation along the direction s^a is obstructed at order n [66,67].

Here $\Gamma(\underline{u})$ denotes a three-chain bounded by the deformed curve $\Sigma_{\underline{u}}$ and a reference curve Σ_0 that is in the same homology class, i.e. $\partial\Gamma(\underline{u}) = \Sigma_{\underline{u}} - \Sigma_0$. It depends on both the deformation \underline{u} of the five-brane on Σ as well as the complex structure moduli \underline{z} of Z_3 due to the holomorphic three-form Ω . We note that the superpotential (4.18) is formally identical to the result obtained by dimensional reduction of the D5-brane action [60], as demonstrated in section 3.3.2, and can directly be obtained from the Type IIB result via dualities. A third and more rigorous way to obtain and treat the chain integral (4.18) will be presented in section 7.1.3 using the language of currents.

We conclude by investigating the consequences of the small instanton transition of section 4.1.2 for the heterotic superpotentials (4.15). The small instanton transition implies a transition between bundle and five-brane moduli [193]. However, also away from the supersymmetric configuration of a holomorphic vector bundle and a five-brane on a holomorphic curve, the transition applies. Then the corresponding obstructed deformation fields of the non-holomorphic vector bundle and curve should be identified. Since both types of deformations are generally obstructed by a superpotential, cf. section 7.1.2, also the superpotentials for bundle and five-brane have to be connected by the transition. To see how the two superpotentials (4.17) and (4.18) are mapped onto each other in the transition, let us assume a single instanton solution \mathcal{F} with $\mathcal{F} \wedge \mathcal{F}$ dual to an irreducible curve Σ . Displaying the explicit moduli dependence of the configuration \mathcal{F} [200], in the small instanton limit $\mathcal{F} \wedge \mathcal{F}$ reduces to the delta function δ_{Σ} of four real scalar parameters. They describe the position moduli of the instanton normal to the curve in the class $[\Sigma]$ on which it is localized. Inserting the gauge configuration \mathcal{F} into W_{CS} , the holomorphic Chern-Simons functional is effectively dimensionally reduced to the curve Σ , see [106] for a similar argumentation in the B-model. In the vicinity of Σ we may write the holomorphic three-form as $\Omega = d\omega$ which we insert into (4.17) in the background $\mathcal{F} \wedge \mathcal{F}$ to obtain, after a partial integration,

$$W_{\text{CS}} = \int_{\Sigma} \omega. \quad (4.19)$$

Adding a constant given by the integral of ω over the reference curve Σ_0 this precisely matches the chain integral (4.18). Applying the above discussion, we can think about the M5-brane deformations in W_{M5} as the bundle deformations describing the position of the instanton configuration \mathcal{F} .

We will verify this matching explicitly from the perspective of the F-theory dual setup later on. There we will on the one hand identify some of the fourfold complex structure moduli with the heterotic bundle moduli, on the other hand, however, show that part of the F-theory flux superpotential depending on the same complex structure moduli really calculates the superpotential of a five-brane on a curve. This way, employing heterotic/F-theory duality, we show in the case of an example the equivalence of the small instanton/five-brane picture.

4.1.4 Spectral Cover Construction

In this section we present a basic account on the construction of vector bundles on elliptic Calabi-Yau manifolds. Although the constructions we discuss are valid in any complex dimension, we directly focus to the most relevant case of Calabi-Yau threefolds. Instead of delving into the mathematical details of the construction of [182, 201, 202], we will just focus on the general idea of the spectral cover approach and on giving the formulas of the Chern classes of E . These are essential for the calculations performed in section 6.2. We will however restrict to the case of $SU(N)$ bundles and E_8 -bundles, where the construction of the latter using either parabolics or by embedding into dP_9 [182] will not be discussed for brevity¹⁰, see e.g. [181] for a review.

The basic strategy of the spectral cover is to obtain stable holomorphic bundles E on an elliptic threefold Z_3 roughly speaking by fibering the stable bundles on the fiber torus so that they globally fit into a stable bundle on the threefold Z_3 [182, 201]. This way, the topological data of the bundle E can be determined in terms of the cohomology of the two-dimensional base B_2 and the section of the elliptic fibration. More precisely one first defines a stable bundle on each elliptic fiber of Z_3 , which is, if we focus to the case of an $SU(N)$ -bundle, specified by N line bundles on T^2 , or, in the dual picture, N points Q_i on the dual torus [182] obeying $\sum_i Q_i = 0$ ¹¹. Fibering these N points over B_2 specifies a divisor in Z_3 that is an N -fold ramified cover of the base B_2 , called the spectral cover divisor or spectral cover for short. Concretely, for an elliptic Calabi-Yau threefold $Z_3 \rightarrow B_2$ with base manifold B_2 the Weierstrass form¹² reads

$$p_0 = y^2 + x^3 + fxz^4 + gz^6 \quad (4.20)$$

in an ambient space $\mathbb{P}(O_{B_2} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ for $\mathcal{L} = K_{B_2}$ ¹³. The data of the N points on each elliptic fiber is specified by the zeros of the section [103, 182]

$$C : \quad p_+ = b_0 z^N + b_2 x z^{N-2} + b_3 y z^{N-3} + \dots + \begin{cases} b_N x^{N/2} \\ b_N y x^{(N-3)/2} \end{cases}, \quad (4.21)$$

where one distinguishes the cases N even and N odd. Here the coefficients in p_+ are chosen such that it defines a section of $\mathcal{O}(\sigma)^N \otimes \mathcal{M}$, where $\sigma : B_2 \rightarrow Z_3$ denotes the section of the elliptic fibration and \mathcal{M} an arbitrary line bundle on B_2 with first Chern class $c_1(\mathcal{M}) = \eta$. Here and in the following we furthermore use the notation $\sigma = c_1(\mathcal{O}(\sigma))$ which is the Poincaré dual to the section σ . The coefficients b_i are sections of a line bundle on B_2 . In general, they depend on moduli fields u_i which encode the deformations of the spectral cover and, hence, the bundle E .

¹⁰The method of embedding into del Pezzo surfaces allows the construction of bundles with exceptional structure group $E_{6,7,8}$, whereas the construction via parabolics applies to ADE-bundles [182].

¹¹Strictly speaking, 0 should be replaced by p which is the marked point on T^2 .

¹²This projectivization of the affine Weierstrass equation corresponds to $\mathbb{P}^2(1, 2, 3)[6]$.

¹³We refer to appendix B.1 for details on elliptic Calabi-Yau manifolds.

Having defined the spectral cover divisor \mathcal{C} the vector bundle E is obtained by a sequence of formal steps, that we only mention and refer to [181, 182] for details and a review. One constructs the so-called Poincare line bundle \mathcal{P}_B over the fiber product $Z_3 \times_{B_2} Z_3$, in which the subspace $Y = \mathcal{C} \times_{B_2} Z_3$ is canonically embedded and on which \mathcal{P}_B is trivially defined by restriction. Roughly speaking the first factor Z_3 contains the moduli space of E whereas the second factor Z_3 is the space over which E will be constructed. Concretely, E is given by

$$E = (\pi_2)_*(\pi_1^*\mathcal{N} \otimes \mathcal{P}_B) \quad (4.22)$$

where \mathcal{N} is a for the moment arbitrary line bundle on \mathcal{C} and $\pi_1 : Y \rightarrow \mathcal{C}$ as well as $\pi_2 : Y \rightarrow Z_3$ denote the two canonical projections. We note that the push-forward π_2 maps a line bundle on Y at a generic point on Z_3 to a rank N vector bundle E , that degenerates on the ramification divisor of the spectral cover $\pi : \mathcal{C} \rightarrow B_2$. The requirement of vanishing first Chern class $c_1(E)$ then fixes the line bundle \mathcal{N} as

$$c_1(\mathcal{N}) = -\frac{1}{2}(c_1(\mathcal{C}) - \pi^*(c_1(B_2))) + \gamma = \frac{1}{2}(N\sigma + \eta + c_1(B_2)) + \gamma, \quad (4.23)$$

where γ is a class in $H^{(1,1)}(\mathcal{C}, \mathbb{Z})$ obeying $\pi_*(\gamma) = 0$. Thus \mathcal{N} is not completely fixed and the freedom in its definition is encoded by the class¹⁴ γ . The most important achievement of this construction for our purposes is that it allows the determination of the Chern classes of E . One readily calculates the second Chern class $c_2(E)$ using the Hirzebruch-Grothendieck-Riemann-Roch theorem to obtain [182]

$$c_2(E) = \eta\sigma - \frac{1}{24}c_1(\mathcal{L})^2(N^3 - N) - \frac{N}{8}\eta(\eta - Nc_1(\mathcal{L}) - \frac{1}{2}\pi_*(\gamma^2)), \quad (4.24)$$

where we recall from appendix B.1 that $c_1(B_2) = c_1(\mathcal{L})$. We note that nothing prevents us from setting $N = 1$, which as a bundle with structure group $SU(1)$ seems not to make sense. However, as we will discuss and exploit in section 6.2 a bundle with $SU(1)$ structure group has to be interpreted as a horizontal five-brane in the heterotic theory [103, 182].

Another case of interest is an E_8 -bundle. We omit its construction and just state its second Chern-class that reads

$$\lambda(E) = \frac{c_2(E)}{60} = \eta_i\sigma - 15\eta^2 + 135\eta c_1(B_2) - 310c_1(B_2)^2, \quad (4.25)$$

where the notation of the $SU(N)$ case still applies. Thus, we conceptually summarize the second Chern class $c_2(E)$ of the two bundles E under consideration schematically as

$$\lambda(E) = \eta\sigma + \pi^*(\omega), \quad (4.26)$$

where η and ω are classes in $H^2(B_2, \mathbb{Z})$ respectively $\sigma \cdot H^2(B_2, \mathbb{Z})$. The meaning of the class η will be physically clarified in heterotic/F-theory duality, where it can be constructed entirely from the fibration data of the base $B_3 \rightarrow B_2$ of the dual F-theory, see section 4.3.2.

¹⁴In the approach via parabolics, the class γ is set to zero by construction [181, 182].

4.2 F-Theory Compactifications

Here we present a basic account on F-theory. We start in section 4.2.1 by reviewing the original idea underlying F-theory which is the geometrization of the axio-dilaton and the $SL(2, \mathbb{Z})$ -invariance of Type IIB via an auxiliary two-torus, that is allowed to vary over spacetime defining an elliptic fibration. The dynamics of seven-branes is then encoded in the degeneration loci and type of this elliptic fibration. In section 4.2.2 we proceed to construct F-theory compactifications to lower dimension by specifying an elliptic Calabi-Yau manifold. We readily focus on four-dimensional compactifications on Calabi-Yau fourfolds and comment on consistency conditions from tadpole cancellation. Furthermore we discuss flux quantization on fourfolds and the form of allowed four-flux G_4 in F-theory. In addition we present a brief identification of the geometric F-theory moduli with the Type IIB fields and moduli. Finally in section 4.2.3 we discuss in some detail the structure of the F-theory flux superpotential as encoded, for a fixed four-flux G_4 , by the fourfold periods and comment on its splitting into flux and seven-brane superpotential of the underlying Type IIB theory.

4.2.1 Basics of F-theory Constructions

F-theory provides a geometrization of $\mathcal{N} = 1$ Type IIB backgrounds with backreacted seven-branes and a holomorphically varying axio-dilaton [72]

$$\tau = C_0 + \frac{i}{g_s} = C_0 + ie^{-\phi} \quad (4.27)$$

Due to the genuine description of non-perturbative effects like (p, q) -strings and string junctions rich gauge dynamics can be obtained allowing even for exceptional groups E_k for $k > 6$ [203, 204], that have been accessible before only in the heterotic theory.

Starting with the simplest setup of a backreacted D7-brane in flat space it is the basic question of F-theory to find a compact geometry, which is a solution to the equations of motion of Type IIB string theory, in particular the Einstein equations, and that extends the local solution of a single D7-brane. This compact solution is found in [205] in the context of stringy cosmic strings and applied to D7-branes in the seminal work [72]. The D7-brane is a magnetic source of the axio-dilaton τ through the R-R-form C_0 and a source of gravity. Consequently a solution to the Type IIB effective action is determined by a solution of τ and the metric g , which for symmetry reasons is of the form of a direct product $\mathbb{R}^{1,7} \times B_1$ where the space B_1 transverse to the D7-brane worldvolume is determined in the following. The solution of τ is given in complex coordinates on B_1 by

$$\tau(z) \sim \frac{1}{2\pi i} \log(z) \quad (4.28)$$

which is consistent with the monodromy $C_0 \mapsto C_0 + 1$ due to the C_0 -charge of the D7-brane. The holomorphicity of τ is a consequence of a BPS-condition implying that half of

the supersymmetries are preserved [72]. The metric g is found to describe a conical space centered at $z = 0$ with a deficit angle of $\pi/6$ [72, 205]. It is crucial to note that the $SL(2, \mathbb{Z})$ -symmetry of Type IIB acting on τ renders the energy of the solution (4.28) finite¹⁵. The $SL(2, \mathbb{Z})$ -action in particular implies, that $\tau(z)$ is not a well-defined function on B_1 , but a section of an $SL(2, \mathbb{Z})$ -bundle over B_1 . More invariantly, τ can be described by the modular parameter of an elliptic curve \mathcal{E} , i.e. a two-torus, that is fibered holomorphically over B_1 . To obtain a space B_1 of finite volume one considers a multicenter solution of 24 D7-branes yielding a deficit angle of 4π , i.e. B_1 curls itself up to form the compact space of $S^2 \cong \mathbb{P}^1$. Physically this is consistent since the metric is well-defined around $1/z$ and the net D7-charge on the S^2 is zero since a loop encircling all 24 D7-branes is contractible to a point yielding a trivial monodromy for C_0 . Furthermore, this already indicates that we are no more allowed to think of 24 D7-branes since the total monodromy and charge are zero. The resolution of this paradox is again the $SL(2, \mathbb{Z})$ -invariance of the solution that allows for more general seven-branes, denoted (p, q) 7-branes¹⁶, having a different monodromy and charge. In particular a seven-brane is only specified up to its conjugacy class under $SL(2, \mathbb{Z})$. We note that $\tau(z)$ varies holomorphically over \mathbb{P}^1 where g_s is not necessarily small and diverges, $\tau \rightarrow i\infty$, at the location of a seven-brane. This and the presence of non-perturbative seven-branes indicates that this eight-dimensional Type IIB vacuum is non-perturbative. It is denoted as an F-theory compactification to eight dimensions.

Geometrically the F-theory setup is described by a fibration of an elliptic curve over $B_1 = \mathbb{P}^1$ where the generic elliptic fiber \mathcal{E} degenerates at the 24 loci of the seven-branes. However, the total space of the fibration remains smooth. A smooth complex surface which is an elliptic fibration over \mathbb{P}^1 is a two-dimensional Calabi-Yau manifold, which is $K3$. This can be seen as follows. First we note that every two-torus T^2 is algebraic. Indeed by means of the Weierstrass \mathfrak{p} -function, every point u on the lattice quotient $T^2 = \mathbb{C}/L$ is mapped bijectively to the projective plane curve $\mathcal{E} = \{y^2 = 4x^3 - g_2x - g_3\}$ via $u \mapsto (\mathfrak{p}(u), \mathfrak{p}'(u)) \equiv (x, y)$, due to the algebraic differential equation obeyed by \mathfrak{p} [206]. In a fibration, the curve \mathcal{E}_z depends on the point z in \mathbb{P}^1 . Then the discriminant $\Delta = g_2^3 - 27g_3^2$ of \mathcal{E}_z has to vanish to first order at 24 points, which fixes the degree of g_2, g_3 as polynomials in the local coordinate z on \mathbb{P}^1 to be eight and twelve so that the total space of the fibration, denoted as X_2 , obeys the Calabi-Yau condition. Thus, we see that an F-theory vacuum in eight dimensions is in one-to-one correspondence with an elliptic $K3$ -surface. Analogously, lower-dimensional F-theory vacua are obtained using the adiabatic argument [207] by compactifying on n -dimensional elliptically fibered Calabi-Yau manifolds X_n , where the base B_1 is replaced by a complex $n-1$ -dimensional Fano variety B_{n-1} [72–74]. This explains the relevance of elliptically fibered Calabi-Yau manifolds as discussed next in section 4.2.2. In all these cases, the relation of the F-theory setup specified by the elliptic Calabi-Yau manifold X_n to the Type IIB physics is made precise in [75, 76] by identifying the Type IIB manifold as the double cover of the base

¹⁵The domain of integration is reduced from the complex plane to \mathcal{F} , the fundamental domain of the torus.

¹⁶A (p, q) 7-brane is an object on which a (p, q) -string can end, cf. sections 2.2.1 and 3.1.

B_{n-1} branched over the divisor wrapped by the $O7$ -plane.

We conclude this general discussion by noting the M-theory description of F-theory, see [29] for a detailed derivation. Using the adiabatic argument [207] for M-theory on an elliptically fibered Calabi-Yau n -fold X_n and for the application of fiberwise T-duality, F-theory is identified with M-theory on X_n in the limit $\text{vol}(T^2) \rightarrow 0$, where T^2 denotes the class of the generic elliptic fiber. This M-theory description in particular yields an alternative and independent explanation why F-theory vacua have $\mathcal{N} = 1$ spacetime supersymmetry.

4.2.2 Elliptic Calabi-Yau Manifolds and Seven-Branes in F-theory

As we have seen in an F-theory compactification on a Calabi-Yau n -fold X_n to $(11 - 2n, 1)$ -dimensional Minkowski space the axio-dilaton τ is described as the complex structure modulus of an elliptic curve fibered over the Type IIB target manifold B_{n-1} that is a Kähler manifold with positive curvature,

$$\text{F-theory on } X_n = \text{Type IIB on } B_{n-1} . \quad (4.29)$$

The Calabi-Yau X_n geometrizes non-perturbative seven-branes by non-trivial monodromies of τ around degeneration loci of the elliptic curve, which includes D7-branes and O7-branes as special cases. Here we systematically discuss compactifications of F-theory with a focus on four dimensional vacua and on the consistency condition imposed by tadpoles inherited from the M-theory description of F-theory [29].

Let us study the F-theory geometry of an elliptically fibered Calabi-Yau n -fold $X_n \rightarrow B_{n-1}$ with a section. This section can be used to express X_n as an analytic hypersurface in the projective bundle $\mathcal{W} = \mathbb{P}(\mathcal{O}_{B_{n-1}} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ with coordinates (z, x, y) , which is a fiber-bundle over B_{n-1} with generic fiber $\mathbb{P}^2(1, 2, 3)$. The hypersurface constraint can be brought to the Weierstrass form

$$y^2 = x^3 + g_2(\underline{u})xz^4 + g_3(\underline{u})z^6 . \quad (4.30)$$

Here $\mathcal{L} = K_{B_{n-1}}^{-1}$ for X_n being Calabi-Yau and $g_2(\underline{u})$, $g_3(\underline{u})$ are sections of \mathcal{L}^4 and \mathcal{L}^6 for (4.30) to be a well-defined constraint equation. Locally on the base B_{n-1} they are functions in local coordinates \underline{u} on B_{n-1} . We refer to appendix B.1 for details.

F-theory defined on X_n automatically takes care of a consistent inclusion of spacetime-filling seven-branes, as we have seen in section 4.2.1. These are supported on the in general reducible divisor Δ in the base B_{n-1} determined by the degeneration loci of (4.30) given by the discriminant

$$\Delta = \{\Delta := 27g_2^2 + 4g_3^3 = 0\} . \quad (4.31)$$

The degeneration type of the fibration specified by the order of vanishing of g_2 , g_3 and Δ along the irreducible components Δ_i of the discriminant have an ADE-type classification that physically specifies the four-dimensional gauge group G . It can be determined explicitly

using generalizations of the Tate formalism [101]. For the order of vanishing of Δ at most one the Calabi-Yau manifold X_n is smooth corresponding to a single seven-brane on Δ . This is the I_1 locus of the elliptic fibration [101]. However, this is not the generic and interesting situation in F-theory in general and in this work. Indeed in all our examples we will consider a singular X_n with a rich gauge symmetry generated.

The weak string coupling limit of F-theory is given by $\text{Im } \tau \rightarrow \infty$ and yields a consistent orientifold setup with D7-branes and O7-planes on a Calabi-Yau manifold [75, 76]. In general, as the axio-dilaton of Type IIB string theory τ corresponds to the complex structure of the elliptic fiber, it can be specified by the value of the classical $SL(2, \mathbb{Z})$ -modular invariant j -function which is expressed through the functions g_2 and g_3 in (4.30) as

$$j(\tau) = \frac{4(24g_2)^3}{\Delta}, \quad \Delta = 27g_2^2 + 4g_3^3. \quad (4.32)$$

The function $j(\tau)$ admits a large $\text{Im } \tau$ expansion $j(\tau) = e^{-2\pi i \tau} + 744 + \mathcal{O}(e^{2\pi i \tau})$ from which we can directly read off the monodromy (4.28) of τ around a D7-brane, for example.

There are further building blocks necessary to specify a consistent F-theory setup. This is due to the fact that a four-dimensional compactification generically has a tadpole of the form [77, 78, 208]

$$\frac{\chi(X_4)}{24} = n_3 + \frac{1}{2} \int_{X_4} G_4 \wedge G_4, \quad (4.33)$$

which can be deduced by considering the dual M-theory formulation. There a tadpole of the the three-form C_3 is induced due to the Green-Schwarz term $\int C_3 \wedge X_8$, the coupling to the M2-brane and the Chern-Simons term $\int C_3 \wedge (dC_3)^2$. In the case that the Euler characteristic $\chi(X_4)$ of X_4 is non-zero a given number n_3 of spacetime-filling three-branes on points in B_3 and a specific amount of quantized four-form flux G_4 have to be added in order to fulfill (4.33). In addition, non-trivial fluxes on the seven-brane worldvolume contribute as [102]

$$\frac{\chi(X_4)}{24} = n_3 + \frac{1}{2} \int_{X_4} G_4 \wedge G_4 + \sum_i \int_{\Delta_i} c_2(E_i), \quad (4.34)$$

where E_i denotes the corresponding gauge bundle¹⁷ localized on the discriminant component Δ_i . The cancellation of tadpoles in F-theory compactifications on Calabi-Yau fourfolds thus is restrictive for global model building since the total amount of allowed flux is bounded by the Euler characteristic of X_4 that is, in known explicit constructions, at most 1820448 [95].

The choice of G_4 is further constrained by the F-theory consistency conditions on allowed background fluxes. The first constraint comes from the quantization condition for G_4 , which depends on the second Chern class of X_4 as [209]

$$G_4 + \frac{c_2(X_4)}{2} \in H^4(X_4, \mathbb{Z}) \quad (4.35)$$

¹⁷It has been argued in [29] that also the brane fluxes should be describable by transcendental flux G_4 .

and has been deduced from anomaly freedom of the M2-brane theory. More restrictive is the condition that G_4 has to be primitive, i.e. orthogonal to the Kähler form of X_4 . In the F-theory limit of vanishing elliptic fiber this yields the constraints

$$\int_{X_4} G_4 \wedge J_i \wedge J_j = 0 . \quad (4.36)$$

for every generator J_i , $i = 1, \dots, h^{(1,1)}(X_4)$ of the Kähler cone. There is one caveat in order when working in the Kähler sector of X_4 and when evaluating topological constraints like (4.33), (4.35) and (4.36). In the case that X_4 is singular, which happens for enhanced gauge symmetry, it is not possible to directly work with the singular space since the topological quantities such as the Euler characteristic and intersection numbers are not well-defined. Thus the above constraints can be naively evaluated only in the case of a smooth X_4 , which corresponds to the physically simplest situation with single D7-branes¹⁸. To remedy the problems when working with singular X_4 we systematically blow up the singularities¹⁹ to obtain a smooth geometry [101] for which the constraints (4.33), (4.35) and (4.36) are valid. The resulting smooth geometry still contains the information about the gauge-groups on the seven-branes and allows to analyze the compactification in detail.

Let us comment on the effect of three-branes and fluxes on the F-theory gauge group. For a generic setup with three-branes and fluxes, the four-dimensional gauge symmetry as determined by the seven-brane content is not affected. However, if the three-brane happens to collide with a seven-brane, it can dissolve, by a similar transition as discussed in section 4.1.2, into a finite-size instanton on the seven-brane worldvolume that breaks the four-dimensional gauge group G . During this transition the number n_3 of three-branes jumps and a bundle E_i over the seven-brane worldvolume is generated describing the gauge instanton [29]. However, we will not encounter this since we restrict our discussion to the case that the gauge bundle on the seven-branes is trivial and no three-branes sit on top of their worldvolumes.

We conclude by a discussion of the complex structure moduli space of elliptic fourfolds X_4 , that is of central importance in this work, and the interpretation in terms of the Type IIB moduli. Consider a smooth elliptic Calabi-Yau fourfold X_4 with a number of $h^{(3,1)}(X_4)$ complex structure moduli, that may be obtained from a singular fourfold by multiple blow-ups²⁰. In order to compare to the Type IIB weak coupling picture the complex structure moduli can be split into three classes [29]:

- (1) One complex modulus that physically corresponds to the complex axio-dilaton τ and that geometrically parametrizes the complex structure of the elliptic fiber.

¹⁸Our examples in chapter 6 are significantly more complicated and admit seven-branes with rather large gauge groups. Technically this is a consequence of working with fourfolds X_4 with few complex structure moduli, which typically have in the order of thousand elements of $H^{(1,1)}(X_4)$ of which many correspond to blow-ups of singular elliptic fibres signaling the presence of enhanced gauge groups.

¹⁹In the cases considered in this work this is done using the methods of toric geometry [101, 210, 211].

²⁰This affects only the number of Kähler moduli, which we will not discuss in the following.

- (2) A number of $\sum_i h^{(2,0)}(\Delta_i)$ complex structure moduli corresponding to the deformations of the seven-branes wrapped on the discriminant loci in B_3 .
- (3) $h^{(2,1)}(Z_3)$ complex structure moduli corresponding to the deformations of the basis and its double covering Calabi-Yau threefold Z_3 obtained in the orientifold limit.

For a more detailed analysis of this and the organization in terms of the low-energy effective action of F-theory we refer to [212].

4.2.3 The Flux Superpotential

Next we discuss the perturbative F-theory superpotential, which is given by a flux superpotential. We emphasize only the physical and qualitative aspects of the superpotential and refer to chapter 5 for a presentation of the expected mathematical structure and of the tools to calculate it efficiently.

The F-theory flux superpotential is generated upon switching on four-form flux G_4 in the M-theory perspective of F-theory [29, 150]. In an M-theory compactification on X_4 one encounters the famous Gukov-Vafa-Witten superpotential [14]

$$W_{G_4}(\underline{z}) = \int_{X_4} G_4 \wedge \Omega_4 = N^a \Pi^b(\underline{z}) \eta_{ab}, \quad a, b = 1, \dots, b^4(X_4), \quad (4.37)$$

which directly applies to the F-theory setup, once the condition (4.36) is met. Here Ω_4 is the holomorphic $(4, 0)$ form on X_4 that depends on the complex structure moduli \underline{z} of X_4 counted by $h^{(3,1)}(X_4)$. For even second Chern class $c_2(X_4)$, cf. (4.35), we expanded the flux $G_4 = N^a \hat{\gamma}_a$ into a cohomology basis $\hat{\gamma}_a$ of the horizontal cohomology $H_H^4(X_4, \mathbb{Z})$, which is a subgroup of $H^4(X_4, \mathbb{Z})$ taking (4.36) into account. We define a dual basis γ^a of the integral homology group $H_4^H(X_4, \mathbb{Z})$ in order to define the flux quantum numbers $N^a = \int_{\gamma^a} G_4$ that are integral. Then the whole complex structure dependence of W_{G_4} is encoded by the complex structure dependence of the fourfold periods

$$\Omega_4(\underline{z}) = \Pi^a(\underline{z}) \hat{\gamma}_a, \quad \Pi^a(\underline{z}) = \int_{\gamma^a} \Omega_4(\underline{z}), \quad (4.38)$$

as in the Calabi-Yau threefold case. We introduce the topological metric [93–95]

$$\eta_{ab} = \int_{X_4} \hat{\gamma}_a \wedge \hat{\gamma}_b, \quad \int_{\gamma^a} \hat{\gamma}_b = \delta_b^a, \quad (4.39)$$

that parametrizes the intersection of four-cycles in X_4 . We note that in contrast to $H^3(Z_3, \mathbb{Z})$ of Calabi-Yau threefolds the fourth cohomology group of X_4 does not carry a symplectic structure which necessitates the introduction of η_{ab} . This technically complicates mirror symmetry on fourfolds compared to the threefold case, as we will review in section 5.1.

One goal of this work is to explicitly compute (4.37) for specific elliptically fibered Calabi-Yau fourfolds, see section 6.1. Exploiting the geometrization of both bulk and brane dynamics in the complex geometry of X_4 we match the results of this calculation with the superpotentials in the language of Type IIB theory, which are the flux superpotential and the seven-brane superpotential. Thus, let us present a brief review of the Type IIB superpotential in setups with seven-branes and $O7$ -planes. The superpotential is induced by three-form fluxes F_3 and H_3 of the R–R and NS–NS sectors, as well as by two-form fluxes F_2 for the field strength of the $U(1)$ gauge-potential A on the internal part of the seven-brane worldvolume. Thus F_2 defines an element of $H^2(D, \mathbb{Z})$, where D denotes the complex surface wrapped by the seven-brane. The respective superpotentials are then given by [9, 14, 29, 199]

$$W_{\text{flux}}(\underline{z}) = \int_{Z_3} (F_3 - \tau H_3) \wedge \Omega_3, \quad W_{\text{brane}}(z, \zeta) = \int_{\Gamma_5} F_2 \wedge \Omega_3, \quad (4.40)$$

where τ is the axio-dilaton and Ω is the holomorphic three-form on the Calabi-Yau threefold Z_3 . These are well-defined e.g. for the weak coupling limit with D7-branes in $O3/O7$ -orientifolds since both Ω and the fluxes F_3, H_3 have negative parity with respect to the orientifold involution σ and thus the integrals in (4.40) can be non-zero, see section 2.1.2. We emphasize that W_{flux} only depends on the complex structure deformations of Z_3 due to the appearance of Ω_3 , while W_{brane} will also depend on the deformation moduli ζ of the seven-brane. To see the latter, one notes that Γ_5 is a five-chain which ends on the divisor D , i.e. one has $D \subset \partial\Gamma_5$, and thus carries the information about the embedding of the seven-brane into Z_3 . We note that the seven-brane superpotential is related to a localized five-brane charge on the worldvolume of the seven-brane²¹ and thus has the form of the five-brane superpotential encountered in (3.72) in Type IIB and in (4.18) in heterotic string theory upon taking the Poincare dual of F_2 in Γ_5 . Thus, we can interpret the seven-brane superpotential as a special case of the five-brane superpotential, where a description of the five-brane curve Σ in terms of seven-brane flux $[F_2] = \Sigma$ is applicable [79], see also [99, 119, 120, 122, 125–127] for a similar use of the seven-brane superpotential.

It is the great advantage of the F-theory formulation, that the calculation of the superpotential (4.40) can be performed in a fully consistent string vacuum including the backreaction of seven-branes and orientifold planes. Indeed, one expects, as we will demonstrate explicitly in chapter 6, a match of the F-theory superpotential with the superpotentials (4.40) for specific flux choices G_4 and seven-branes,

$$\int_{X_4} G_4 \wedge \Omega \quad \rightarrow \quad \int_{Z_3} (F_3 - \tau H_3) \wedge \Omega_3 + \sum_m \int_{\Gamma_5^m} F_2^m \wedge \Omega_3. \quad (4.41)$$

where m labels all seven-branes on divisors Δ_m carrying two-form fluxes F_2^m .

²¹It is important to emphasize that the five-brane charge is only locally non-trivial, i.e. the class of F_2 is trivial in Z_3 . This is necessary in a setup with $O7$ -planes since the five-brane charge in the quotient geometry Z_3/σ is projected out [29].

Let us conclude by emphasizing once more that it is crucial in the context of this naive matching to consider fluxes in $H_H^4(X_4, \mathbb{Z})$. We note that already by a pure counting of the flux quanta in $H^4(X_4, \mathbb{Z})$, as well as in $F_3, H_3 \in H^3(Z_3, \mathbb{Z})$ and in $F_2^m \in H^2(\Delta_m, \mathbb{Z})$ one will generically encounter a mismatch. This can be traced back to the fact that not all fluxes G_4 are actually allowed in an F-theory compactification, since in the duality between M-theory on X_4 and F-theory on X_4 the Kähler class of the elliptic fiber is sent to infinity turning one of the dimensions of the elliptic fiber into a space-time dimension [29, 100].

4.3 Heterotic/F-Theory Duality

Let us now come to a more systematic discussion of heterotic/F-theory duality. Since the fundamental duality that underlies heterotic/F-theory duality in any dimensions is the eight-dimensional equivalence of the heterotic string compactified on T^2 and F-theory on elliptic $K3$ we begin with a summary of this duality in section 4.3.1. In this most simple setup, the duality can be readily checked by comparison of the moduli space on both sides of the duality. Then we apply the adiabatic argument in section 4.3.2 to obtain heterotic/F-theory dual setups in lower dimensions, with particular emphasis on four-dimensional setups. We present a detailed and explicit discussion of the unified description of both the heterotic Calabi-Yau threefold Z_3 and the bundle E in terms of the complex geometry of the F-theory fourfold X_4 . In this context we put special emphasis on the split of the F-theory fourfold constraint in the stable degeneration limit, that gives back the heterotic threefold Z_3 as well as the spectral cover data of E . We conclude in section 4.3.3 by considering the duality map between F-theory and heterotic moduli and of heterotic five-branes on curves Σ . We review that a horizontal five-brane maps to a blow-up in X_4 along the five-brane curve Σ , where we explicitly construct the blow-up both locally and globally as a complete intersection. We end our discussion with the details of the duality map for the moduli in both theories. It is important to note that the F-theory complex structure moduli also have to account for the deformation modes of the curve Σ for consistency if a horizontal five-brane on Σ is included.

4.3.1 Heterotic/F-theory in Eight Dimensions

The fundamental duality underlying heterotic/F-theory duality in any dimension is the eight-dimensional duality of the heterotic string on T^2 and F-theory on an elliptic $K3$ [72]. It can be checked by matching the moduli and the gauge symmetry on both sides of the duality [72, 74]. In addition heterotic/F-theory duality agrees with the duality of M-theory on $K3$ and the heterotic string on T^3 [41], when $K3$ is taken to be elliptic and one further compactifies the eight-dimensional heterotic/F-theory setup on an additional S^1 .

On the heterotic side the torus T^2 is defined by the Weierstrass equation (4.30) in

$\mathbb{P}^2(1, 2, 3)$. A gauge background on T^2 obeying (4.4) is defined by a flat connection $F_{u\bar{u}}$ for complex coordinates u on T^2 , since $g_{u\bar{u}} = \frac{v}{2\tau_2}$ for²² $\tau_2 = \text{Im}(\tau)$ and v its volume. This is solved by switching on 16 Wilson lines $A_i^a = C_i^a$, $i = 1, 2$ and $a = 1, \dots, 16$, that are defined up to periodicity on T^2 [193]. This is an $U(1)^{16}$ gauge bundle E on T^2 with bundle moduli space²³ parametrized by the real parameters C_i^a , up to permutations by S_{15} ,

$$\mathcal{M}(E) = T^{30}/S_{15} \times T^2 \cong \mathbb{P}^{15} \times T^2. \quad (4.42)$$

For this gauge background, the eight-dimensional gauge symmetry contributed from the ten-dimensional gauge bosons of the heterotic string is $U(1)^{16}$. We note that more general bundles E on T^2 with different structure groups have to be described by other means like an embedding of T^2 into dP_9 for E_8 -bundles [182].

From the CFT-point of view the heterotic string on T^2 can be analyzed directly and the global moduli space of the heterotic string on T^2 is identified as the moduli space of (complexified) metrics on the Narain-lattice $\Gamma^{18,2}$ of signature $(18, 2)$ [213, 214],

$$\mathcal{M}_{\text{het}} = O(18, 2, \mathbb{Z}) \backslash O(18, 2) / (O(18) \times O(2)) \times \mathbb{R}_+ \quad (4.43)$$

that takes into account the complex structure τ , the Kähler structure ρ of T^2 and the heterotic dilaton, that corresponds to \mathbb{R}_+ . Here the homogeneous space counts the number of metrics on the lattice $\Gamma^{18,2}$, whereas the discrete group $SO(18, 2, \mathbb{Z})$ is the physical T-duality group. Taking the reduction of the ten-dimensional metric and the B-field into account, the heterotic gauge symmetry in eight dimensions has a maximal rank of 20, which at a generic point in \mathcal{M}_{het} is $U(1)^{20}$. It enhances to non-abelian gauge groups for special symmetric points in the moduli space \mathcal{M}_{het} yielding up to $G = E_8 \times E_8 \times G'$ for $G' = SU(2) \times U(1)^3$, $SU(3) \times U(1)^3$, $SU(2)^4$, $SU(3)^2$ gauge symmetry²⁴ in eight dimensions [5, 80]. Geometrically this means that we have a trivial gauge background on T^2 , which we are allowed to have since the heterotic tadpole (4.9) is trivial on T^2 , and the moduli τ and ρ of T^2 take special symmetric values.

On the F-theory side, the geometry of the elliptic $K3$ -surface is obtained by fibering the heterotic two-torus T^2 over the base $B_1 = \mathbb{P}^1$ as in section 4.2.1. Then, a first step to check the duality is to match the moduli and gauge symmetry of the heterotic string from the F-theory perspective. It is already very promising that the complex structure moduli space of a generic, elliptic $K3$ -surface is given by the complex 18-dimensional manifold

$$\mathcal{M}_{cs} = O(18, 2, \mathbb{Z}) \backslash O(18, 2) / (O(18) \times O(2)), \quad (4.44)$$

²²If we construct T^2 as the lattice quotient by $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ with coordinates (x_1, x_2) on \mathbb{C} , we define $z = x_2 + \tau x_1$. The connection with the algebraic representation (4.30) is established via $\tau = \frac{\int_{\gamma_2} \Omega_1}{\int_{\gamma_1} \Omega_1}$, where γ_i denote the A- and B-cycle and Ω_1 the holomorphic $(1, 0)$ -form on T^2 . Locally it is $\Omega_1 = dz$.

²³We note that in the construction of $SU(n)$ -bundles on elliptic Calabi-Yau manifolds Z_n , this moduli space occurs as the fiber of the moduli space of bundles on Z_n , cf. the spectral cover construction in section 4.1.4.

²⁴Only the first two gauge groups G' appear in [80].

which is the moduli space of an algebraic $K3$ -surface with Picard-number $\rho = 2$ for the elliptic fiber and the base. This manifold is identical with the first factor in (4.43). In addition we have a factor \mathbb{R}_+ for the real Kähler volume of the base \mathbb{P}^1 that is identified with the heterotic dilaton. Since the volume of the elliptic fiber is physically irrelevant and formally sent to zero/infinity, it does not contribute a Kähler class. In addition, it is important to note that in F-theory Kähler moduli are in general not complexified which is consistent with the $\mathcal{N} = 1$ coordinates (2.42) for $O7$ -orientifolds where the v^α , measuring volumes of curves, are not complexified, but the quadratic combination \mathcal{K}_α .

The heterotic gauge symmetry is matched by the singularity type of the elliptic fibration of $K3$. For a smooth $K3$ there is an $U(1)^{20}$ gauge theory originating from 18 elements in $H^{(1,1)}(K3)$, since both the class of the elliptic fiber and the class of the base \mathbb{P}^1 do not contribute vector fields [212]²⁵, and from $H^{(2,0)}(K3) \oplus H^{(0,2)}(K3)$. In the case of enhanced gauge symmetry G the type of the degeneration of the elliptic fibration of $K3$ in F-theory has to match the heterotic gauge group G . For the example of $G = E_8 \times E_8 \times U(1)^4$ the corresponding singular F-theory geometry is given in the affine patch $z = 1$ by [74]

$$y^2 = x^3 + \alpha x s^4 + (s^5 + \beta s^6 + s^7), \quad \alpha, \beta \in \mathbb{C}, \quad (4.45)$$

which is obtained by specializing the complex structure deformations in g_2, g_3 in (4.30) accordingly. We note that (4.45) has an E_8 singularity at $s = 0, \infty$ on the \mathbb{P}^1 -base. Indeed for $s \rightarrow 0$ we obtain the constraint $y^2 = x^3 + s^5$ of an E_8 singularity, where we ignored the irrelevant deformations s^6, s^7, xs^4 . Similarly the E_8 singularity at infinity is visible in coordinates $\tilde{s} = 1/s$. For the case of the singular $K3$ (4.45) the Picard lattice is at least 18-dimensional, of which 16 are the vanishing cycles of the $E_8 \times E_8$ -singularity and of which two are the class of the base and the elliptic fiber, that is sent to zero/infinity. Further enhancement to $E_8 \times E_8 \times G'$ for $G' = SU(2) \times U(1)^3, SU(3) \times U(1)^3, SU(2)^4, SU(3)^2$ are possible²⁶.

4.3.2 Heterotic/F-theory Duality in Lower Dimensions

After establishing heterotic/F-theory duality in eight dimensions, lower dimensional versions of the duality can be constructed using the adiabatic argument [207]. The general idea is to consider a family of dual eight-dimensional theories parameterized by a manifold B_{n-2} . Upon a slow variation of the parameters, the pairs of dual theories at every point $p \in B_{n-2}$ should glue together to form globally consistent duality between the heterotic string and F-theory in

²⁵In the notation of [212] the 18 elements on $H^{(1,1)}(K3)$ are blow-up modes. In the lift from M- to F-theory the class of the elliptic fiber lifts to the eighth dimension in Minkowski space, $\mathbb{R}^{1,6} \rightarrow \mathbb{R}^{1,7}$, and the Kähler class of the base \mathbb{P}^1 yields a Kähler coordinate T_α of the Type IIB theory.

²⁶However it is not clear to the author how to realize the enhancement of more than two $U(1)$ factors of $E_8 \times E_8 \times U(1)^4$ since only 18 of the generators of $H^{(1,1)}(K3)$ are allowed to correspond to the Cartan generators of non-abelian gauge groups [212].

lower dimensions [72]. Geometrically we obtain pairs of smooth dual geometries X_n and Z_{n-1} on the F-theory, respectively, heterotic side with generic fiber of an elliptic $K3$, respectively, an elliptic curve. For technical reasons B_{n-2} is chosen to be a complex Kähler manifold and in order to preserve $\mathcal{N} = 1$ supersymmetry, X_n and also Z_{n-1} obey the Calabi-Yau condition. We summarize the fibration structure of the heterotic/F-theory dual geometries schematically as

$$\text{Heterotic } Z_{n-1} \left\{ \begin{array}{ccc} T^2 & \longleftrightarrow & \left. \begin{array}{c} T^2 \\ \downarrow \\ \mathbb{P}^1 \end{array} \right\} K_3 \\ \downarrow & & \downarrow \\ B_{n-2} & & B_{n-2} \end{array} \right\} X_n \quad \text{F-theory} \quad (4.46)$$

This fibration structure implies that the base of X_n with respect to the elliptic fibration on the F-theory side is a holomorphic \mathbb{P}^1 -fibration over B_{n-2} forming a manifold \tilde{B}_{n-1} . In other words \tilde{B}_{n-1} is the total space of the projective bundle $\mathbb{P}(\mathcal{O}_{B_{n-2}} \oplus L)$ where the line bundle $L = \mathcal{O}_{B_{n-2}}(-\Gamma)$ is associated to an effective divisor Γ in B_{n-2} . There are two distinguished classes in $H^2(\tilde{B}_{n-1}, \mathbb{Z})$, namely the class of the hyperplane of the \mathbb{P}^1 -fiber denoted by $r = c_1(\mathcal{O}(1))$ and of the line bundle L with $c_1(L) = t$. We note that, in contrast to the $K3$ -fibration of X_n , the \mathbb{P}^1 -fibration of \tilde{B}_{n-1} in (4.46) is not fixed by the geometrical data of the heterotic Calabi-Yau Z_{n-1} . As we see below, it is fixed by the topology of the heterotic vector bundle E .

In order to make connection to the heterotic gauge bundle E we define two holomorphic sections denoted C_0, C_∞ of the fibration $p : \tilde{B}_{n-1} \rightarrow B_{n-2}$ corresponding to the first and second coordinate of $\mathbb{P}(\mathcal{O}_{B_{n-2}} \oplus L)$ set to zero,

$$C_\infty = C_0 - p^*\Gamma. \quad (4.47)$$

In terms of these divisors the perturbative gauge group $G = G_1 \times G_2$, where we denote the group factors from the first E_8 as G_1 and from the second E_8 as G_2 , is realized by seven-branes over C_0 and C_∞ with singularity type G_1 and G_2 , respectively [73, 74, 102]. These sections are simply the higher dimensional analog of the points $u = 0, \infty$ on \mathbb{P}^1 , cf. (4.45), where the singularities of $K3$ dual to the heterotic gauge symmetry are located in the eight-dimensional duality of section 4.3.1. Conversely, singularities that do not descend from the eight-dimensional theory have to correspond to new physics in the lower-dimensional theory. In particular, components of the discriminant on which Δ vanishes of order greater than one that project onto codimension two subvarieties Σ_i in B_{n-2} correspond to heterotic five-branes on the same subvarieties in Z_n [73, 101, 102]. Consequently, the corresponding seven-branes induce a gauge symmetry that maps to a non-perturbative effect due to the five-branes on the heterotic side.

Applying the duality map for the case $n = 3$ with base $B_1 = \mathbb{P}^2$ we obtain the duality of the heterotic string on $Z_2 = K3$ and F-theory on an elliptic $K3$ -fibered Calabi-Yau threefold

X_3 [73, 74]. The base \tilde{B}_2 on the F-theory side is given by a Hirzebruch surface²⁷ $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$, which is a \mathbb{P}^1 -bundle over \mathbb{P}^1 . Then the F-theory Calabi-Yau threefold X_3 is constructed as a Weierstrass form of the elliptic fibration over \mathbb{F}_k . The complex structure moduli of X_3 encode both the complex structure moduli of $K3$ and the bundle moduli of E on the heterotic side [74]. Introducing the Kähler classes k_b, k_f of the base and the fiber of \mathbb{F}_k one has the matching

$$\frac{1}{e^{2\phi}} = \frac{k_b}{k_f} \quad (4.48)$$

of the heterotic dilaton ϕ forming a six-dimensional tensor multiplet and of $k_b k_f$ that is related to the overall volume $k_b(k_f - nk_b)$ of \mathbb{F}_k yielding a universal hypermultiplet in six dimensions [72, 181]. Finally, the integer k specifying the base \mathbb{F}_k is determined from the topology of the heterotic vector bundle E . We note that in a heterotic compactification on $K3$ the tadpole condition (4.9) evaluated on the single four-cycle $K3$ itself implies the condition

$$n_1 + n_2 + n_5 = 24, \quad (4.49)$$

where the right hand side denotes the Euler characteristic $24 = \int_{K3} c_2(K3)$. The integers n_1, n_2 denote the instanton numbers in the heterotic gauge-bundle E over $K3$, where we split E into $E = E_1 \oplus E_2$, and the integer n_5 takes into account possible five-branes. From the analysis of phase transitions in the heterotic string as well as in the Kähler moduli space of \mathbb{F}_k it is possible to infer the relation $n_{1/2} = 12 \pm k$ for $n_5 = 0$ [73]. Thus, once the topology of E is fixed (as well as the number of five-branes is put to zero) the complete F-theory geometry is determined and vice versa. For a more refined analysis one applies the spectral cover construction of E reviewed in section 4.1.4.

We conclude by noting that a more detailed matching of the F-theory and heterotic fields can be inferred by working out the number and type of multiplets in the $\mathcal{N} = 2$ theory in six dimensions on both sides of the duality. This analysis can be performed e.g. by compactifying on an additional T^2 for which F-theory becomes dual to Type IIA string theory on X_3 and heterotic/F-theory duality becomes the duality of the heterotic string on $K3 \times T^2$ and Type IIA on X_3 , cf. [70, 215] for a review.

Next we construct a pair of dual heterotic/F-theory setups yielding a four-dimensional theory [102]. Then heterotic/F-theory duality states the equivalence of the heterotic string on an elliptic Calabi-Yau threefold Z_3 and F-theory on an elliptic K3-fibered Calabi-Yau fourfold X_4 . According to (4.46) the three-dimensional base \tilde{B}_3 of the elliptic fibration of X_4 is ruled over the base B_2 of the heterotic threefold Z_3 . As we will make more precise below the complex structure moduli of X_4 map to the complex structure moduli of Z_3 as well as the bundle moduli of E . The Kähler sector of X_4 in general contains the class of the elliptic fiber, several Kähler blow-ups in the elliptic fibration of X_4 , corresponding to non-abelian

²⁷Here we do assume that there are now heterotic five-branes present. A five-brane at a point in B_1 corresponds to a blow-up of the same point in \mathbb{F}_k yielding new Kähler moduli and tensor multiplets in six dimensions [72–74].

gauge symmetry, and the classes of $H^{(1,1)}(\tilde{B}_3)$, which are $h^{(1,1)}(B_2)$ Kähler classes of B_2 , one additional class²⁸ of the generic \mathbb{P}^1 -fiber in \tilde{B}_3 as well as several blow-ups in \tilde{B}_3 corresponding to heterotic five-branes. We note that these qualitatively different Kähler classes have also qualitatively different meaning for the heterotic theory. On the one hand, singularities within the elliptic fibration of the $K3$ -fiber correspond to the heterotic perturbative gauge symmetry with $h^{(1,1)}(K3) - 2$ Cartan elements, which is consistent with the statement below (4.47). On the other hand, gauge symmetry that is not already visible in eight dimensions, i.e. in the $K3$ -fiber, has to correspond to non-perturbative effects on the heterotic side. Analogously to (4.48), the heterotic dilaton ϕ is obtained as

$$e^{-2\phi} = \frac{\text{vol}(B_2)}{k_f}, \quad (4.50)$$

where k_f denotes the Kähler class of the \mathbb{P}^1 -fiber in \tilde{B}_3 and $\text{vol}(B_2)$ the volume of B_2 , see [149] for a similar analysis in heterotic/M-theory duality.

It turns out that again the fibration data of \tilde{B}_3 is crucial for the construction of the stable vector bundle E on Z_3 in the dual heterotic theory. To analyze this issue in a more refined way it is necessary to use the methods developed in [182], in particular the spectral cover. For example, consider the heterotic string with an $E_8 \times E_8$ -bundle on Z_3 . Besides the required singularities of the elliptic fibration of X_4 at the divisors C_0 and C_∞ matching the perturbative heterotic gauge group G only the base B_2 of the $K3$ -fibration is fixed by duality. The threefold $\tilde{B}_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus L)$ can be freely specified by choosing the \mathbb{P}^1 -fibration over B_2 that is fixed by a line bundle L on B_2 , whose Chern-class we denote by $c_1(L) = t$, cf. the discussion above (4.47). Then, the heterotic bundle $E = E_1 \oplus E_2$ is given in terms of the cohomology of \tilde{B}_2 as [182]

$$\eta(E_1) = 6c_1 + t, \quad \eta(E_2) = 6c_1 - t, \quad (4.51)$$

which uniquely determines the η -classes, cf. section 4.1.4, of the two bundles by the choice of \mathbb{P}^1 -fibration. In particular, we note that the heterotic anomaly (4.9) is trivially fulfilled without the inclusion of any horizontal five-branes, i.e. five-branes wrapping curves in B_2 . However, a number of five-branes wrapping the elliptic fiber have to be included as deduced in [182]. We note that the duality map (4.51) remains valid also for $SU(n)$ -bundles where X_4 is singular as concluded in [216]. Also in this case five-branes on the elliptic fiber are required by the heterotic anomaly cancellation (4.9).

Again we refer to the literature for a more detailed analysis of the four-dimensional $\mathcal{N} = 1$ spectrum and the translation to the heterotic side, see [181, 217–219] and [212] for a detailed derivation of the F-theory effective action in a dimensional reduction on X_4 . Upon a further compactification on T^2 heterotic/F-theory duality becomes heterotic/Type II duality in two dimensions which can alternatively be used for an discussion of the massless spectrum and effective action [149, 220].

²⁸ This provides a universal tensor multiplet containing the dilaton [73, 74, 101, 102], see (4.50).

Finally, since we are in this work mainly interested in the complex structure moduli of X_4 and their map to the heterotic moduli, let us conclude with a more detailed discussion of this aspect of heterotic/F-theory duality. As mentioned above, in the absence of five-branes the complex structure moduli of X_4 split into complex structure moduli as well as bundle moduli of the heterotic compactification. This splitting can be made directly visible by investigation of the Weierstrass equation defining the F-theory Calabi-Yau geometry. In eight dimensions, for simplicity, an $E_8 \times E_8$ -bundle over T^2 on the heterotic side yields the Weierstrass equation for the elliptic $K3$ on the F-theory side as [74]

$$y^2 = x^3 + \sum_{n=-4}^4 f_{4-n} s^{4-n} t^{4+n} x z^4 + \sum_{n=-6}^6 g_{6-n} s_1^{6-n} s_2^{6+n} z^6 \quad (4.52)$$

which contains two deformed E_8 singularities at $s_1 = 0$, $s_2 = 0$ in projective coordinates on the \mathbb{P}^1 -base of $K3$, cf. (4.45). Based on the observation that the heterotic geometry T^2 is encoded only by the terms

$$p_0 := -y^2 + x^3 + f_0 x \tilde{z}^4 + g_0 \tilde{z}^6, \quad (4.53)$$

upon identifying the projective coordinate \tilde{z} of T^2 as $\tilde{z} = \frac{z}{s_1 s_2}$, it was concluded in [74] that the information about the heterotic bundle E is encoded in the terms in (4.52) with different powers in s_1, s_2 . This reflects the fact that precisely these terms contain the information about the singularities of the elliptic fibration, cf. (4.45) for the case of an $E_8 \times E_8$ singularity. This observation was made rigorous and adapted to the case of toric hypersurfaces in [103], which allows a systematic application to compactifications to lower dimensions.

More precisely in the duality between the heterotic string on Z_3 with bundle $E = E_1 \oplus E_2$ and F-theory on X_4 the constraint P of X_4 is shown to split as [103]

$$P = p_0 + p_+ + p_- = 0, \quad (4.54)$$

where $p_0 = 0$ specifies the threefold Z_3 as in (4.53), $p_+ := \sum_{i>0} p_{+,i} v^i = 0$ the bundle E_1 , and $p_- := \sum_{i<0} p_{-,i} v^i = 0$ the bundle E_2 . This is also referred to as the stable degeneration limit of X_4 , cf. [182]. Here the coordinate v is the coordinate on the \mathbb{P}^1 -base in the $K3$ -fibers, that is related to the projective coordinates $(s_1 : s_2)$ on \mathbb{P}^1 as $v = s/t$. We note that the description of the heterotic bundle E via (4.54) agrees with the spectral cover construction of [182] if applicable, so that p_+ is for example given by the spectral cover divisor (4.21) in the case of an $SU(N)$ -bundle. However, the split (4.54) applies even for bundles not describable in the spectral cover approach like bundles with exceptional structure group. Another advantage of (4.54) is the fact that the map of the complex structure moduli of X_4 to the heterotic bundle moduli can be directly studied.

In fact, this will allow us in our concrete examples of section 6.1 to show that some complex structure moduli of X_4 , that are also the moduli of seven-branes are precisely mapped to the coefficients of the spectral cover p_+ of an $SU(1)$ -bundle or heterotic five-brane. Consequently,

switching on appropriate four-form fluxes G_4 in F-theory generates a superpotential for these fields which we will determine explicitly and which in the dual heterotic theory corresponds to the five-brane superpotential (4.18).

4.3.3 Five-Branes in Heterotic/F-Theory Duality: Blowing Up in F-Theory

In this section we will discuss the F-theory dual of five-branes, in particular horizontal five-branes, in very much detail following [81]. This is of particular importance on the one hand side for conceptual reasons since it turns out that the F-theory dual to the heterotic string with maximal perturbative gauge symmetry $G = E_8 \times E_8$ has to be analyzed more thoroughly and then naturally yields a description of the F-theory dual of horizontal five-branes. On the other hand a complete understanding of the heterotic string requires the inclusion of five-brane dynamics, in particular in the light of the small instanton transition reviewed in section 4.1.2. Furthermore, as we will see, it will be precisely the horizontal five-branes introduced in this section that are geometrized by blow-ups into exceptional divisors E in \tilde{B}_3 for which our analysis and calculation of the superpotential will be performed in sections 6.2 and 8.4.

In general there are two qualitatively different types of five-branes on curves Σ in an elliptically fibered Calabi-Yau threefold. This is reflected in the decomposition of the class $[\Sigma]$ of the five-brane curve as

$$\Sigma = n_f F + \Sigma_B , \quad (4.55)$$

where Σ_B denotes a curve in the base B_2 of the elliptic fibration, $F = \pi^*([p])$ denotes the elliptic fiber over a point p in B_2 , and n_f is a positive integer. This is split into five-branes vertical to the projection $\pi : Z_3 \rightarrow B_2$, where the integer n_f counts the number of five-branes wrapping the elliptic fiber, and into horizontal five-branes on Σ_B in the base B_2 . Both cases lead to different effects in the F-theory dual theory. Vertical five-branes correspond to spacetime filling three-branes at a point in the base \tilde{B}_3 of the F-theory fourfold X_4 [182, 216]. Conversely, horizontal five-branes on the curve Σ_B map completely to the geometry of the F-theory side. They map to seven-branes supported on a component Δ_i of the discriminant²⁹ in the fourfold base \tilde{B}_3 which projects under $p : \tilde{B}_3 \rightarrow B_2$ onto the curve Σ_B in B_2 [73, 101, 102] and that has to be blown-up in \tilde{B}_3 into a divisor E [103–105]. Consequently, they correspond to seven-branes that induce a non-perturbative gauge symmetry.

This duality map for branes is in particular consistent with the small instanton transition of a brane, i.e. a coherent sheaf, into a smooth bundle on both sides of the duality. Indeed, the three-branes undergoing a three-brane/instanton transition in F-theory are precisely the dual of the transition of a vertical five-brane into a finite size instanton breaking the gauge group on the F-theory and the heterotic side accordingly. However, this transition is not of relevance for our considerations since our analysis does not depend on the specific four-dimensional gauge symmetry.

²⁹As we see in the following Δ vanishes of order greater than one on Δ_i [73, 101, 102].

Let us now perform a detailed analysis of the F-theory geometry dual to a horizontal five-brane. For simplicity we consider the enhanced symmetry point with $G = E_8 \times E_8$ due to small instantons/five-branes such that the heterotic bundle is trivial. For this setup the following considerations are the most comprehensible. For general vector bundles E on the heterotic side, an analysis of the local F-theory geometry near the five-brane curve Σ_B is possible [105] applying the method of stable degeneration [182, 221]. However, since the essential point in this analysis is the locally trivial heterotic gauge bundle, the results of [105] carry immediately over to our situation with a globally trivial gauge bundle.

As follows in general, using the adjunction formula, the canonical bundle of the ruled base³⁰ $B_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus L)$ can be determined from the general formula for the total Chern class of a projective \mathbb{P}^1 -bundle as $c_1(B_3) = c_1(B_2) + 2r + t$, cf. appendix B.1. Then we use (4.47) to obtain

$$K_{B_3} = -2C_0 + p^*(K_{B_2} - \Gamma) = -C_0 - C_\infty + p^*(K_{B_2}). \quad (4.56)$$

From this we deduce the classes F , G and Δ of the divisors defined by³¹ f , g and Δ as sections of $K_{B_3}^{-4}$, $K_{B_3}^{-6}$ and $K_{B_3}^{-12}$, respectively. To match the heterotic gauge symmetry $G = E_8 \times E_8$, there have to be II^* fibers over the divisors C_0, C_∞ in B_3 . Since II^* fibers require that f , g and Δ vanish to order 4, 5 and 10 over C_0 and C_∞ [101], their divisor classes split accordingly with remaining parts

$$\begin{aligned} F' &= F - 4(C_0 + C_\infty) = -4p^*(K_{B_2}), \\ G' &= G - 5(C_0 + C_\infty) = C_0 + C_\infty - 6p^*(K_{B_2}), \\ \Delta' &= \Delta - 10(C_0 + C_\infty) = 2C_0 + 2C_\infty - 12p^*(K_{B_2}). \end{aligned} \quad (4.57)$$

This generic splitting implies that the component Δ' can locally be described as a quadratic constraint in a local normal coordinate s to C_0 or C_∞ , respectively. Thus, Δ' can be understood locally as a double cover over C_0 respectively C_∞ branching over each irreducible curve Σ_i of $\Delta' \cdot C_0$ and $\Delta' \cdot C_\infty$. In fact, near one irreducible curve Σ_i intersecting say C_0 the splitting (4.58) implies that the sections f, g take the form

$$f = s^4 f', \quad g = s^5 (g_5 + s g_6) \equiv s^5 g' \quad (4.58)$$

with f' denoting a section of $K_{B_3}^{-4}$ and g_5, g_6 sections of $K_{B_3}^{-6} \otimes L, K_{B_3}^{-6}$, respectively. The discriminant then takes the form $\Delta = k^{10} \Delta'$ where Δ' is calculated from f' and g' . Thus, the intersection curve is given by $g_5 = 0$ and the degree of the discriminant Δ rises by two over Σ_i with f' and g' vanishing of order zero and one. Precisely the singular curves Σ_i in X_4 that appear in g as above are the locations of the small instantons/horizontal five-branes in Z_3 [104, 105] on the heterotic side. In the fourfold X_4 the collision of an II^* and

³⁰If we are working entirely with fourfolds X_4 we drop the superscript \tilde{B}_3 and denote the base of the elliptic fibration by B_3 .

³¹Here we follow the common convention to denote (g_2, g_3) by (f, g) in the standard Weierstrass form 4.30.

an I_1 singularity over Σ_i induces a singularity of X_4 exceeding Kodaira's classification of singularities. Thus, it requires a blow-up $\pi : \hat{B}_3 \rightarrow B_3$ in the three-dimensional base of the curves Σ_i into divisors E_i . This blow-up can be performed without violating the Calabi-Yau condition since the shift in the canonical class of the base, $K_{\hat{B}_3} = \pi^*K_{B_3} + E_i$, can be absorbed into a redefinition of the line bundle $\mathcal{L}' = \pi^*\mathcal{L} - E_i$ entering (4.30) such that $K_{X_4} = p^*(K_{B_3} + \mathcal{L}) = p^*(K_{\hat{B}_3} + \mathcal{L}') = 0$.

To describe this blow-up explicitly let us restrict to the local neighborhood of one irreducible curve Σ_i of the intersection of Δ and C_0 . We note that the curve Σ_i in B_2 is given by the two constraints

$$h'_1 := vs = 0, \quad h'_2 := g_5 = 0, \quad (4.59)$$

for s and g_5 being sections of the normal bundle $N_{B_3}C_0$ and of $KB_3^{-6} \otimes L$, respectively. Then if X_4 is given as a hypersurface $P = 0$ we obtain the blow-up as the complete intersection [154]

$$P = 0, \quad Q = l_1 h'_2 - l_2 h'_1 = 0, \quad (4.60)$$

where we have introduced coordinates (l_1, l_2) parameterizing the \mathbb{P}^1 -fiber of the exceptional divisor. For a detailed discussion of the blow-up construction we refer to section 7.2.1.

However, at least in a local description, we can introduce a local normal coordinate t to Σ_i in B_2 such that $g_5 = tg'_5$ for a section g'_5 which is non-vanishing at $t = 0$. Then by choosing a local coordinate ℓ of the \mathbb{P}^1 -fiber of the exceptional divisor we can solve the blow-up relation Q of (4.60) to obtain $s = \ell t$. This coordinate transformation can be inserted into the constraint $P = 0$ of X_4 to obtain the blown-up fourfold \hat{X}_4 as a hypersurface³². The polynomials f', g' of this hypersurface are given by

$$f' = \ell^4 f, \quad g' = \ell^5 (g_5 + \ell t g_6 + \dots) \quad (4.61)$$

In particular, calculating the discriminant Δ' of \hat{X}_4 it can be demonstrated that the I_1 singularity no longer hits the II^* singularity over C_0 [105]. This way we have one description of \hat{X}_4 as the complete intersection (4.60) and another as a hypersurface. Both are of importance for the explicit examples discussed in sections 6.2.2, 6.2.3 and in particular section 8.4.2.

To draw our conclusions of the blow-up in F-theory, we summarize what we just discussed. The F-theory counterpart of a heterotic string with full perturbative gauge group is given by a fourfold with II^* fibers over the sections C_0, C_∞ in B_3 . The component Δ' of the discriminant enhances the degree of Δ on each intersection curve Σ_i such that a blow-up in B_3 becomes necessary. On the other hand, each blow-up corresponds to a small instanton in the heterotic bundle [70, 73], that we previously identified in section 4.1.2 as a horizontal five-brane on the curve Σ_i in the heterotic threefold Z_3 or as an $SU(1)$ -bundle in section 4.1.4. Indeed, this agrees also with the observation mentioned above that a vertical component of

³²By abuse of notation since we directly construct the blown-up fourfold \hat{X}_4 as the F-theory dual of a heterotic setup with five-branes, cf. section 6.2, we denote \hat{X}_4 simply by X_4 .

the discriminant with degree greater than one corresponds to a horizontal five-brane [102] as the degree of Δ' on C_0 and C_∞ is two, as we saw explicitly.

We finish this discussion by a briefly looking at the moduli map between F-theory and its heterotic dual, where we focus on the fate of the five-brane moduli in the just mentioned blow-up process. We note that only now where we have discussed all degrees of freedom on both sides of the duality, in particular five-branes, we can expect a complete and consistent moduli map. Indeed if five-branes were excluded a possible mismatch can occur as pointed out in [104]. The first step in establishing the moduli analysis is to relate the dimensions of the various moduli spaces in both theories with each other. Then, as is argued in [104], the relation of the fourfold Hodge numbers $h^{(3,1)}(X_4)$ and $h^{(1,1)}(X_4)$ counting complex structure and Kähler deformations, respectively, to $h^{(2,1)}(Z_3)$, $h^{(1,1)}(Z_3)$ and the bundle moduli and characteristic data has to be modified in the presence of five-branes compared to the results in [216, 222] obtained before without five-branes. The extra contributions are due to deformation moduli of the curve Σ_i supporting the five-brane, counted by $h^0(\Sigma_i, N_{Z_3}\Sigma_i)$, as well as the n_5^{hor} blow-ups in B_3 that increase $h^{(1,1)}(B_3)$. Consequently, one obtains [104]

$$\begin{aligned} h^{(3,1)}(X_4) &= h^{(2,1)}(Z_3) + I(E_1) + I(E_2) + h^{(2,1)}(X_4) + 1 + \sum_i h^0(\Sigma_i, N_{Z_3}\Sigma_i), \\ h^{(2,1)}(X_4) &= n_o \\ h^{(1,1)}(X_4) &= 1 + h^{(1,1)}(Z_3) + \text{rk}(G) + n_5^{\text{hor}}, \end{aligned} \tag{4.62}$$

where now X_4 is the fourfold including the blow-ups in the base B_3 . Here the sum index i runs over all irreducible curves Σ_i , $\text{rk}(G)$ denotes the rank of the four-dimensional non-abelian gauge group and n_o is the number of chiral multiplets odd with respect to τ , which denotes the involution on Z_3 promoted from T^2 mapping $y \mapsto -y$. The index $I(E_{1,2})$ counts a topological invariant of the bundle moduli and is given by [182, 223]

$$I(E_i) = \text{rk}(E_i) + \int_{B_2} (4(\eta_i\sigma - \lambda_i) + \eta_i c_1(B_2)) . \tag{4.63}$$

In general it counts the difference $n_e - n_o$ of even and off chiral multiplets with respect to τ . It reads, by application of an index theorem,

$$I = - \sum_0^3 (-1)^i h^i(Z_3, \text{ad}(E))_e = n_e - n_o, \tag{4.64}$$

where we note that the usual index $\sum_0^3 (-1)^i h^i(Z_3, \text{ad}(E))$ vanishes by Serre duality [181, 182].

The map for $h^{(3,1)}(X_4)$ reflects the fact that the four-dimensional gauge symmetry G is on the heterotic side determined by the gauge bundle E whereas on the F-theory side G is due to the seven-brane content defined by the discriminant Δ that is sensitive to a change of complex structure. For an explicit demonstration of this map exploiting the techniques of [103] we refer to sections 6.1 and 6.2.3 and our works [79, 81].

Let us now discuss how (4.62) changes during the blow-up procedure. To actually perform the blow-up along the curve Σ_i it is necessary to first degenerate the constraint of X_4 such that X_4 develops the singularity over Σ_i described above. This requires a tuning of the coefficients entering the fourfold constraint thus restricting the complex structure of X_4 accordingly which means $h^{(3,1)}(X_4)$ is lowered. Then, we perform the actual blow-up by introducing the new Kähler class associated to the exceptional divisor E_i . Thus, we end up with a new fourfold with decreased $h^{(3,1)}$ and with $h^{(1,1)}(\hat{B}_3)$ increased by one. This is also clear from the general argument [102] that, enforcing a given gauge group G in four dimensions in F-theory, the allowed complex structure moduli of X_4 , which is then singular according to G , have to respect the form of Δ dictated by the singularity type G . Since the blow-up which is dual to the heterotic small instanton/five-brane transition enhances the gauge symmetry G , the form of the discriminant becomes more restrictive, thus fixing more complex structures. Conversely, the blow-down can be understood as allowing for new complex structure moduli to be switched on that decrease the singularity type of the elliptic fibration.

Similarly, we can understand (4.62) from the heterotic side. For every small instanton transition between a smooth bundle E and a five-brane, the bundle loses parts of its moduli since the small instanton is on the boundary of the bundle moduli space. Consequently, the index I reduces accordingly. In the same process, the five-brane in general gains position moduli counted by $h^0(\Sigma_i, N_{Z_3 \Sigma_i})$, that have to be added to (4.62).

We close the discussion of moduli by making a more refined and illustrative statement about the heterotic meaning of the Kähler modulus of the exceptional divisor E_i . To do so, we have to consider heterotic M-theory on $Z_3 \times S^1/\mathbb{Z}_2$, see section 4.1.1. In this picture the instanton/five-brane transition can be understood [224] as a spacetime-filling five-brane wrapping Σ_i and moving onto one of the end-of-the-world branes of S^1/\mathbb{Z}_2 where one perturbative E_8 gauge group is located. There, it dissolves into a finite size instanton of the heterotic bundle E . With this in mind the distance of the five-brane x^{11} on the interval S^1/\mathbb{Z}_2 away from the end-of-the-world brane complexified by the axion a of the self-dual two-form, cf. section 4.1.2, precisely maps to the Kähler modulus of the divisor E_i resolving Σ_i in B_3 .

Chapter 5

Mirror Symmetry and Five-Branes

This chapter is devoted to the study of mirror symmetry for Calabi-Yau manifolds, in particular for Calabi-Yau three- and fourfolds, and to mirror symmetry with D-branes. Since essentially all calculations of $\mathcal{N} = 1$ coupling functions performed in this thesis involve the use of mirror symmetry and the geometrical tools related to its study, this chapter provides the technical core of this work.

In its weak formulation mirror symmetry states the equivalence of the complex structure moduli space of X and the (instanton corrected) Kähler moduli space of its mirror \tilde{X} . As was pointed out in [89] this equivalence can be formulated in physical terms by considering topological field theories called the A- and B-model with target spaces (\tilde{X}, X) , respectively. These theories are consistent cohomological truncations of some particular $\mathcal{N} = (2, 2)$ superconformal field theories. Their physical observables are the vertical subspace of the de Rham groups¹ $H^{(p,p)}(\tilde{X})$, $0 \leq p \leq n$, and the horizontal subspace of $H^p(X, \wedge^q TX)$, $p + q = n$, respectively. In particular their marginal deformations coincide with the cohomology groups $H^{(1,1)}(\tilde{X})$ for the A-model and $H^1(X, TX)$ for the B-model that are, in geometrical terms, precisely the infinitesimal directions on the Kähler and complex structure moduli space of \tilde{X} and X , respectively. Therefore, the physical statement of mirror symmetry is the equivalence of the A-model constructed from \tilde{X} and the B-model constructed from X . We note that for our purposes there is a favored point in the moduli space of marginal deformations to perform the matching between the A- and B-model. This is the large radius point in the Kähler moduli space of the A-model, which is identified with a point of maximal unipotent monodromy in the complex structure moduli space of the B-model.

Our final goal is the calculation of the periods of X and special holomorphic quantities F^0 and $F^0(\gamma)$ for Calabi-Yau three- and fourfolds, respectively. They are identified with the holomorphic superpotential of $\mathcal{N} = 1$ effective actions, the flux superpotential (4.40) of a Type IIB compactification on X in the threefold case and the Gukov-Vafa-Witten superpotential (4.37) arising in M- and F-theory compactifications. At the same time F^0 and $F^0(\gamma)$ are generating functions of the genus zero Gromov-Witten invariants of the mirror three- respectively fourfold. In the case of F-theory on elliptic fourfolds the superpotential (4.37) is

¹We keep our discussion as general as possible. For concreteness set $n = 3, 4$ for Calabi-Yau three- and fourfolds.

further identified with the flux and seven-brane superpotential in the limit (4.41) to Type IIB for particular G_4 -flux choices. Thus some of the functions $F^0(\gamma)$ can be further interpreted as generating functions of disk instantons in a dual type IIA theory. However, although the A-model perspective is essential for a complete understanding of mirror symmetry, we will be rather brief and use, at large volume, the enumerative interpretation of the A-model side rather as a technical tool and for cross-checks of our B-model calculations².

Following mainly the guideline of [79] and [81] we begin in section 5.1 with the introduction of toric geometry which provides the central geometrical tools in order to construct a broad class of examples of Calabi-Yau manifolds realized as toric hypersurfaces. There we also present the toric realization of mirror symmetry both in the closed string case as well as in the open string or brane case. In section 5.2 we provide a basic account on mirror symmetry for Calabi-Yau threefolds. There we focus almost exclusively on the B-model side by reviewing the complex geometry of Z_3 as encoded mathematically by the variations of Hodge structures. This yields the concepts of special geometry and the Picard-Fuchs equations from which the Calabi-Yau threefold periods can be obtained. In this context we put special emphasis on the use of the toric GKZ-system and the structure of the periods at large volume as fixed by the A-model classical intersection data. Then we proceed to the higher dimensional case of mirror symmetry for Calabi-Yau fourfolds in section 5.3. This involves a more systematic review of the B-model observables and the Frobenius algebra underlying both the A- and B-model operator algebra. Finally we use this structure to establish the mirror map for Calabi-Yau fourfolds. There we present, following [79], novel results from analytic continuation and global monodromy analysis of the periods of the Calabi-Yau fourfold that allow us to fix the integral basis of $H_H^4(X_4, \mathbb{Z})$ and the classical terms in the periods at large volume/large complex structure. We conclude with section 5.4 where we present a basic account of the A-model perspective on the prepotentials F^0 , the disk amplitude and the generating functions $F^0(\gamma)$. In this context we emphasize the enumerative interpretation of the three- and fourfold flux superpotential and of the brane superpotential.

5.1 Toric Calabi-Yau Hypersurfaces and Toric Branes

In this section we briefly introduce a basic account on toric geometry that is an inevitable tool for the rest of this work. We review the construction of mirror pairs (\tilde{X}, X) of toric Calabi-Yau hypersurfaces in section 5.1.1, where we introduce basic notions of toric geometry. We review the general formulas to obtain the Hodge numbers of (\tilde{X}, X) . In section 5.1.2 we extend our discussion to toric branes, mainly following [106, 107, 229]. We discuss the relevance of the symplectic quotient construction to conveniently visualize toric branes in toric Calabi-Yau threefolds and present the classical open mirror map to obtain a mirror pair of toric branes.

²We refer to [225–228] for a detailed discussion of the topological string on Calabi-Yau threefolds and [93–95] for an extension to higher dimensions.

5.1.1 Toric Mirror Symmetry

A powerful tool to construct Calabi-Yau manifolds X and their mirrors \tilde{X} for an arbitrary complex dimension n is by realizing them as hypersurfaces in toric ambient spaces. These hypersurfaces are specified by reflexive polyhedra [230] that encode all information necessary to define the hypersurface. Since the cases of most interest for us are Calabi-Yau three- and fourfolds we will restrict to these cases, but note, that all formulas given below apply for a general complex dimension³ n . Threefold mirror pairs are in the following denoted as (\tilde{Z}_3, Z_3) .

We start our considerations for a Calabi-Yau threefold⁴ \tilde{Z}_3 in Type IIA which is mirror dual to Type IIB on the mirror Calabi-Yau threefold Z_3 . We realize the compact Calabi-Yau \tilde{Z}_3 as a hypersurface in a toric ambient variety $\mathbb{P}_{\tilde{\Delta}}$. The toric ambient space is constructed from a reflexive polyhedron $\Delta_4^{\tilde{Z}}$ that is encoded by a set of k vectors $\ell^{(i)}$ forming a basis of relations among the $m = k + 4$ vertices \tilde{v}_j in $\Delta_4^{\tilde{Z}}$. Then the construction of the Calabi-Yau constraint for \tilde{Z}_3 involves the pair of reflexive polyhedra $\Delta_4^{\tilde{Z}}$ and Δ_4^Z in lattices N, M , that are dual, i.e. $\Delta_4^{\tilde{Z}} = (\Delta_4^Z)^*$. In general, the dual polyhedron Δ^* of a given polyhedron Δ in a lattice M is defined as the set of points p in the real span $N_{\mathbb{R}} = N \otimes \mathbb{R}$ of the dual lattice N of M such that

$$\Delta^* = \{p \in N_{\mathbb{R}} \mid \langle q, p \rangle \geq -1 \text{ for all } q \in \Delta\}. \quad (5.1)$$

$\mathbb{P}_{\tilde{\Delta}}$ can be represented as the quotient $(\mathbb{C}^m - \text{SR})//G$ defined by dividing out the isometry or gauge group $G = (U(1))^k$ and by imposing vanishing D-terms or moment maps⁵,

$$\tilde{x}_j \mapsto e^{i\ell_j^{(i)}\phi_i} \tilde{x}_j, \quad \sum_{j=1}^{k+4} \ell_j^{(i)} |\tilde{x}_j|^2 = r^i. \quad (5.2)$$

Here, the r^i denote the real volumes of distinguished, effective curves in $\mathbb{P}_{\tilde{\Delta}}$. They form the basis of the Mori cone of curves C_i each of which is associated to one charge vector $\ell^{(i)}$. The basis of charge vectors, thus the generators of the Mori cone, are determined by the triangulation of $\Delta_4^{\tilde{Z}}$ that correspond to the different phases of $\mathbb{P}_{\tilde{\Delta}}$ [231]. This then fixes the form of the D-term constraints in (5.2) from which the Stanley-Reisner ideal SR can be read off [29]. Geometrically the toric variety can be viewed as follows. Upon solving the second condition in (5.2) and using coordinates $p_j = |\tilde{x}_j|^2$ and angles θ_j the toric variety $\mathbb{P}_{\tilde{\Delta}}$ can be visualized as a T^{m-k} -fibration over a real $(m-k)$ -dimensional base \mathbb{B}_{m-k} [106, 232]. The degeneration loci of the T^{m-k} -fibration where one or more S^1 shrink are on the boundary of \mathbb{B}_{m-k} which is determined by $p_j = 0$ or intersections thereof since $p_j \geq 0$.

³We give the caveat that mirror symmetry in even dimension relates the Type IIA/B with itself while it interchanges Type IIA and IIB with each other in odd dimensions.

⁴For convenience we denote quantities in Type IIA always with ‘ \sim ’ in order to omit them for their mirror quantities on the Type IIB side where we will mainly work.

⁵The symplectic quotient description is equivalent to the homogeneous coordinate description $\tilde{x}_j \mapsto \lambda_i^{i\ell_j^{(i)}} \tilde{x}_j$ for $\lambda_i \in \mathbb{C}$ of the toric variety $\mathbb{P}_{\tilde{\Delta}}$.

The action of G on the coordinates is generated infinitesimally by k vector fields

$$V^{(i)} = \sum_j \ell_j^i \tilde{x}_j \frac{\partial}{\partial \tilde{x}_j}. \quad (5.3)$$

Using these vector fields it is straight forward to construct forms on $\mathbb{P}_{\tilde{\Delta}}$ from invariant forms of \mathbb{C}^{k+4} . Of particular importance is the holomorphic top-form $\Delta_{\mathbb{P}_{\tilde{\Delta}}}$ on $\mathbb{P}_{\tilde{\Delta}}$ that is constructed from the holomorphic top-form $\Delta_{\mathbb{C}} = d\tilde{x}_1 \wedge \dots \wedge d\tilde{x}_{k+4}$ on \mathbb{C}^{k+4} as

$$\Delta_{\mathbb{P}_{\tilde{\Delta}}} = V^{(1)} \lrcorner \dots \lrcorner V^{(k)} \lrcorner (\Delta_{\mathbb{C}}), \quad (5.4)$$

where \lrcorner denotes the interior product defined by contracting a form with a vector. For the example of projective space \mathbb{P}^n this yield the unique holomorphic section of $\Omega_{\mathbb{P}^n}^n(n+1) := \Omega_{\mathbb{P}^n}^n \otimes \mathcal{O}(n+1)$ given by

$$\Delta_{\mathbb{P}^n} = \sum_{i=1, n+1} (-1)^{i-1} x_i dx_1 \wedge \dots \wedge \widehat{dx}_i \dots \wedge dx_n, \quad (5.5)$$

where the x_i denote the homogeneous coordinates on \mathbb{P}^n and \widehat{dx}_i indicates, that dx_i is omitted.

The Calabi-Yau threefold \tilde{Z}_3 is then given as the hypersurface $\{\tilde{P} = 0\}$ in $\mathbb{P}_{\tilde{\Delta}}$ where \tilde{P} is given as the polynomial [230]

$$\tilde{P}(\tilde{\underline{x}}, \tilde{\underline{a}}) = \sum_{q \in \Delta_4^Z \cap M} \tilde{a}_q \prod_i \tilde{x}_i^{\langle \tilde{v}_i, q \rangle + 1} \quad (5.6)$$

in the m projective coordinates $\tilde{\underline{x}}$ of $\mathbb{P}_{\tilde{\Delta}}$ associated to each vertex \tilde{v}_j . This formula provides a direct way to count the number of complex structure parameters $\tilde{\underline{a}}$ (up to automorphisms of $\mathbb{P}_{\tilde{\Delta}}$) by counting the integral points $q \in \Delta_4^Z$. Furthermore, \tilde{Z}_3 is Calabi-Yau since (5.6) contains the monomial $\tilde{x}_1 \dots \tilde{x}_m$ corresponding to the origin in Δ_4^Z so that \tilde{P} is a section of the anti-canonical bundle $K_{\mathbb{P}_{\tilde{\Delta}}}^* = \mathcal{O}(\sum_i D_i)$, where $D_i = \{\tilde{x}_i = 0\}$ denotes a toric divisor.

For the case of hypersurfaces in toric varieties, (closed string) mirror symmetry [230] is realized in a very elegant way. The mirror threefold Z_3 on the Type IIB side is obtained by simply interchanging the roles of $\Delta_4^{\tilde{Z}}$ and Δ_4^Z so that (5.6) describes Z_3 as the hypersurface in the toric variety \mathbb{P}_{Δ} associated to the polyhedron Δ_4^Z ,

$$P(\underline{x}, \underline{a}) = \sum_{p \in \Delta_4^Z \cap N} a_p \prod_i x_i^{\langle v_i, p \rangle + 1}. \quad (5.7)$$

Here, we again associated the projective coordinates x_i to each vertex v_i of Δ_4^Z . In general there can be a discrete orbifold symmetry group Γ such that Z_3 is an orbifolded hypersurface. Indeed, the necessary requirements for mirror symmetry, $h^{(1,1)}(Z_3) = h^{(2,1)}(\tilde{Z}_3)$ and $h^{(2,1)}(Z_3) = h^{(1,1)}(\tilde{Z}_3)$, are fulfilled for this construction. This is obvious from Batyrev's formula for the Hodge numbers [230] of a given n -fold X in a toric ambient space specified by

an $n + 1$ -dimensional polyhedron Δ_{n+1}^X and its dual $\Delta_{n+1}^{\tilde{X}}$

$$\begin{aligned} h^{(n-1,1)}(X_n) &= h^{(1,1)}(\tilde{X}) \\ &= l(\Delta_{n+1}^{\tilde{X}}) - (n+2) - \sum_{\dim \tilde{\theta}=n} l'(\tilde{\theta}) + \sum_{\text{codim} \tilde{\theta}_i=2} l'(\tilde{\theta}_i) l'(\theta_i), \end{aligned} \quad (5.8)$$

$$\begin{aligned} h^{(1,1)}(X_n) &= h^{(n-1,1)}(\tilde{X}_n) \\ &= l(\Delta_{n+1}^X) - (n+2) - \sum_{\dim \theta=n} l'(\theta) + \sum_{\text{codim} \theta_i=2} l'(\theta_i) l'(\tilde{\theta}_i). \end{aligned} \quad (5.9)$$

In this expression θ ($\tilde{\theta}$) denote faces of Δ_{n+1}^X ($\Delta_{n+1}^{\tilde{X}}$), while the sum is over pairs $(\theta_i, \tilde{\theta}_i)$ of dual faces. The $l(\theta)$ and $l'(\theta)$ count the total number of integral points of a face θ and the number inside the face θ , respectively. Finally, $l(\Delta)$ is the total number of integral points in the polyhedron Δ . Using these formulas one notes that polyhedra with a small number of points will correspond to Calabi-Yau manifolds with few Kähler moduli, i.e. small $h^{(1,1)}$, and many complex structure moduli $h^{(n-1,1)}$. Since $h^{(1,1)}$ and $h^{(n-1,1)}$ are exchanged by mirror symmetry Calabi-Yau manifolds with small $h^{(n-1,1)}$ are obtained as mirror manifolds of hypersurfaces specified by a small number of lattice points in the polyhedron.

For the case of Calabi-Yau fourfolds (\tilde{X}_4, X_4) the complete list of model dependent Hodge numbers is $h^{(1,1)}(X_4)$, $h^{(3,1)}(X_4)$, $h^{(2,1)}(X_4)$ and $h^{(2,2)}(X_4)$, However only three of these are independent due to the Hirzebruch-Riemann-Roch index theorem implying [95]

$$h^{(2,2)}(X_4) = 2(22 + 2h^{(1,1)}(X_4) + 2h^{(3,1)}(X_4) - h^{(2,1)}(X_4)). \quad (5.10)$$

Therefore, only $h^{(2,1)}(X_4)$ has to be calculated in addition to fix the basic topological data of (\tilde{X}_4, X_4) . Analogously to (5.8) it is readily given by the mirror symmetric expression

$$h^{(2,1)}(X_4) = h^{(2,1)}(\tilde{X}_4) = \sum_{\text{codim} \tilde{\theta}_i=3} l'(\tilde{\theta}_i) l'(\theta_i). \quad (5.11)$$

This finally enables us to calculate the Euler number of fourfolds by

$$\chi(X_4) = \chi(\tilde{X}_4) = 6(8 + h^{(3,1)} + h^{(1,1)} - h^{(2,1)}). \quad (5.12)$$

5.1.2 Toric Branes

Now we are prepared to add an open string sector to the Calabi-Yau threefold mirror pair (\tilde{Z}_3, Z_3) . We will consider supersymmetric branes and their deformations. In local toric Calabi-Yau threefolds mirror symmetry with calibrated branes, so-called Harvey-Lawson branes, has been studied intensively in [106, 107]. In this case the toric variety $\mathbb{P}_{\tilde{\Delta}}$ itself defines a non-compact Calabi-Yau threefold. To fix our notation to include this case we will work in the following with a toric space of dimension $m - k$ denoted by $\mathbb{P}_{\tilde{\Delta}}$, where $m - k = 3$ in the non-compact case, and $m - k = 4$ in the compact case.

In the type IIA theory supersymmetric branes wrap special Lagrangian cycles L . In the toric ambient space $\mathbb{P}_{\tilde{\Delta}}$ one describes such a three-cycle L by r additional, so-called brane charge vectors $\hat{\ell}^{(a)}$ restricting the $|\tilde{x}_j|^2$ and their angles θ_i as [106]

$$\sum_{j=1}^{k+4} \hat{\ell}_j^{(a)} |\tilde{x}_j|^2 = c^a, \quad \theta_i = \sum_{a=1}^r \hat{\ell}_i^{(a)} \phi_a, \quad (5.13)$$

for angular parameters ϕ_a . To fulfill the ‘special’ condition of L , which is equivalent to $\sum_i \theta_i = 0$, one demands $\sum_j \hat{\ell}_j^{(a)} = 0$. Wrapping a brane on such a cycle we obtain a so-called toric brane, in this case a toric D6-brane, a specific brane type well studied in the context of toric mirror symmetry in non-compact Calabi-Yau manifolds [106, 107]. The great advantage of the description of toric branes in terms of toric data similar to the charge vectors $\hat{\ell}^{(i)}$ is the technically straightforward construction of the mirror branes, as presented below.

However, before discussing the mirror side we comment on the non-compact case. For non-compact Calabi-Yau threefolds $\tilde{Z}_3 = \mathbb{C}^{k+3}/G$ the Calabi-Yau condition $\sum_j \ell_j^{(i)} = 0$ has to hold. Then A-branes introduced by (5.13) are graphically represented as real codimension r subspaces of the toric base \mathbb{B}_3 . The case which was considered for the non-compact examples in [106, 107] is $r = 2$ where the non-compact three-cycle L is represented by a straight line ending on a point when projected onto the base \mathbb{B}_3 . The generic fiber is a T^2 so that the topology of L is just $\mathbb{R} \times S^1 \times S^1$. However, upon tuning the moduli c^a it is most convenient to move the L_a to the boundary of \mathbb{B}_3 where two $\{p_j = 0\}$ -planes intersect. Then one of the two moduli is frozen, and one S^1 pinches such that the topology becomes $\mathbb{C} \times S^1$. These D6-branes are mirror dual to non-compact D5-branes which intersect a Riemann surface at a point. We note, that this is the local geometry we encounter in section 6.1.

Let us now turn to the mirror Type IIB description [106, 229]. To include also non-compact Calabi-Yau threefolds and D-branes into the setup it is convenient to use another representation of the mirror Calabi-Yau Z_3 ,

$$P = \sum_{j=0}^m a_j y_j, \quad \prod_{j=0}^m (a_j y_j)^{\ell_j^{(i)}} = z^i, \quad i = 1, \dots, k, \quad (5.14)$$

Here the z^i denote the complex structure moduli of Z_3 that are related to the complex numbers \underline{a} by the second relation in (5.14). Classical mirror symmetry is established by the relation $z^i = e^{2\pi i t^i}$ to the (complexified) Kähler moduli t^i of \tilde{Z}_3 . We note that we introduced a further coordinate y_0 in (5.14) for which we also have to include a zeroth component of the charge vectors as $\ell_0^{(i)} = -\sum_{j=1}^m \ell_j^{(i)}$. The relation to the constraint (5.7) is established by identifying the y_i with monomials $m_i(\underline{x})$ in the coordinates \underline{x} , $y_i \mapsto m_i(\underline{x})$, also known as the étalé map, that are associated to the integral points \tilde{v}_i in $\Delta_4^{\tilde{Z}}$ by the mirror construction of Batyrev [230, 233]. Then the two representations (5.7) and (5.14) for Z_3 agree for the case of compact Calabi-Yau mirror pairs (\tilde{Z}_3, Z_3) . In the non-compact case, Z_3 is similarly given as

$$xz = P(\underline{x}), \quad (5.15)$$

where we emphasize the use affine coordinates x, z of \mathbb{C} and $x_i \in \mathbb{C}^*$, respectively.

Finally, toric holomorphic submanifolds Σ in \mathbb{P}_Δ , that support calibrated B-branes mirror dual to the A-branes in (5.13), are specified by the constraints

$$\prod_{j=0}^m (a_j y_j)^{\tilde{\ell}_j^{(a)}} = u^a, \quad a = 1, \dots, r. \quad (5.16)$$

The classical open mirror map takes the form $u^a = \epsilon^a e^{-c^a}$, where the phases ϵ^a are dual to the Wilson line background of the flat $U(1)$ -connection on the special Lagrangian L and complexify the moduli c^a in (5.13) to the open moduli u^a [88]. This can also be re-expressed in terms of the coordinates \underline{x} using again the étalé map $y_i \mapsto m_i(x_j)$. Intersected with the Calabi-Yau constraint $P = 0$ the constraint (5.16) describes families of submanifolds of codimension r in Z_3 . For the configuration $r = 2$, Σ is a curve in Z_3 that we call a toric curve and which is precisely the geometry we are interested in for the study of five-branes. Other cases can be considered as well leading to mirrors given by D7-branes on divisors ($r = 1$) or D3-branes on points ($r = 3$).

5.2 Mirror Symmetry for Calabi-Yau Threefolds

In this section we discuss, in the light of mirror symmetry, the geometric structure underlying the complex structure moduli space of a Calabi-Yau threefold Z_3 . Here we restrict almost entirely to the B-model side which is the main focus in this work in general. We refer for example to [225–228] for a more detailed discussion including the A-model and symplectic geometry side. We begin in section 5.2.1 by reviewing the description of deformations of complex structures as found in standard textbooks like [13, 234] or in the original works [235, 236]. Then in section 5.2.2 we discuss how this analysis is captured by variations of Hodge structures applied to the Calabi-Yau case. This yields differential relations among the periods of a Calabi-Yau threefold, that lead to special geometry, reviewed in section 5.2.3, and finally to Picard-Fuchs equations that govern the complex structure dependence of the periods. We present the basic techniques in order to obtain the Picard-Fuchs equations both in the generic hypersurface case as well as in the toric case in section 5.2.4. We conclude with the structure of the solutions to the Picard-Fuchs system at the point of maximal unipotent monodromy [237] and the identification of those linear combinations of solutions that correspond to the periods of Ω with respect to an integral symplectic basis of $H^3(Z_3, \mathbb{Z})$. As we demonstrate this is the only occasion where the comparison with the A-model is needed.

5.2.1 Deformations of Complex Structures

We consider a complex manifold Z_3 as a real manifold equipped with a background complex structure $I : TZ_3 \rightarrow TZ_3$ with $I^2 = -1$ for which the Nijenhuis-tensor N vanishes

ensuring integrability⁶ $[T^{(0,1)}Z_3, T^{(0,1)}Z_3] \subset T^{(0,1)}Z_3$. The complex structure determines the Dolbeault operator $\bar{\partial}$ and vice versa. Finite deformations of I are described by elements $A(\underline{t})$ in $\Omega^{(0,1)}(Z_3, TZ_3)$ that are $(0,1)$ -forms taking values in TZ_3 and depend on parameters \underline{t} denoting coordinates on a parameter manifold⁷ M . In fact, if we perturb $\bar{\partial}$ by A and demand $(\bar{\partial} + A)^2 = 0$, the deformation A has to obey the Maurer-Cartan equation

$$\bar{\partial}A + \frac{1}{2}[A, A] = 0. \quad (5.17)$$

If we write A as a formal power series in t , $A = A_1(\underline{t}) + A_2(\underline{t}) + \dots$ we obtain

$$\bar{\partial}A_1 = 0, \quad \bar{\partial}A_n + \frac{1}{2} \sum_{i=1}^{n-1} [A_i, A_{n-i}] = 0, \quad n > 1, \quad (5.18)$$

where $A_n(\underline{t})$ denotes a homogeneous polynomial in t of degree n . Taking coordinate transformations into account that trivially change the complex structure I , we learn that first order or infinitesimal deformations of I for which $t \sim 0$ are in one-to-one correspondence with classes $[v] = [A_1]$ in the cohomology group $H^1(Z_3, TZ_3)$, called the Kodaira-Spencer class of A_1 . However, deformation classes A_1 lift to finite deformations only if we can recursively solve (5.18) for the A_n at every finite order in n . We immediately identify the necessary condition for $[A_1] = [v]$ being integrable with the integrability condition

$$\bar{\partial}A_2 = -\frac{1}{2}[A_1, A_1], \quad (5.19)$$

which means that $[A_1, A_1]$ has to be $\bar{\partial}$ -exact in order to find a solution. Thus, one associates to every class v in $H^1(Z_3, TZ_3)$ the class $[v, v]$ in $H^2(Z_3, TZ_3)$ on the right hand side of (5.19) called the obstruction class. It necessarily has to vanish in order to define a finite $A(\underline{t})$ obeying (5.18) with $[A_1] = [v]$. This is in particular the case if $H^2(Z_3, TZ_3) = 0$. However, this is not a necessary condition for the existence of $A(\underline{t})$ since the obstruction classes for integrating an infinitesimal A_1 can be zero even for $H^2(Z_3, TZ_3) \neq 0$. Indeed, this is the generic case for Calabi-Yau manifolds, where $h^2(Z_3, TZ_3) = h^{(1,1)}(Z_3)$. It is the content of the classical theorem by Tian and Todorov that for every Calabi-Yau manifold all commutators $\sum_i [A_i, A_{n-i}]$ in the recursive equations (5.18) are exact so that a finite $A(\underline{t})$ exists for every infinitesimal deformation $[A_1] = [v]$. Thus, complex structure deformations of a Calabi-Yau manifold are generically unobstructed such that there is a global moduli space of complex structures of complex dimension⁸ $h^{(2,1)}$. In physics, this is reflected by the fact that, as long as background fluxes are absent, there is no scalar potential in the effective theory of a Calabi-Yau compactification for the fields $t(x)$ associated to complex structure deformations.

⁶In the following we denote the (anti-)holomorphic tangent bundle just by $(\overline{TZ_3}) TZ_3$.

⁷We choose M so that the Kodaira-Spencer map $T_0M \rightarrow H^1(Z_3, TZ_3)$ is bijective at a point $0 \in M$.

⁸Here we use the isomorphism $H^1(Z_3, TZ_3) = H^{(2,1)}(Z_3)$ by contraction with the $(3,0)$ -form Ω .

5.2.2 Hodge Structures for Picard-Fuchs Equations

For Calabi-Yau manifolds the infinitesimal study of the complex structure moduli space can be carried out by the study of the variation of the Hodge structure on its cohomology groups⁹. This analysis is particularly convenient since there is an unique non-vanishing holomorphic three-form Ω on a Calabi-Yau manifold. On the one hand, this enables us to map the infinitesimal deformations in $H^1(Z_3, TZ_3)$ simply to forms in $H^{(2,1)}(Z_3)$ which are identified with distinguished periods X^i . On the other hand, the change of the Hodge type of Ω under a complex structure deformation allows us to study the variation of the Hodge structure explicitly. We note that we will later apply the variations of Hodge structures in section 7.3 to also study the complex structure deformations of projective Kähler threefolds with $h^{(3,0)} = 1$, that are a blow-up of a Calabi-Yau threefold along a curve Σ .

If we consider $H^3(Z_3)$ over every point of the complex structure moduli space \mathcal{M}^{cs} , we obtain a holomorphic vector bundle over \mathcal{M}^{cs} which we denote by $\mathcal{H}^3(Z_3)$. It is endowed with a locally constant frame given by $H^3(Z_3, \mathbb{Z})$. We define a decreasing filtration on $H^3(Z_3)$, the Hodge filtration, which equips $H^3(Z_3)$ with a (pure) Hodge structure

$$\begin{aligned} F^3 H^3(Z_3) &= H^{(3,0)}(Z_3) , \\ F^2 H^3(Z_3) &= H^{(3,0)}(Z_3) \oplus H^{(2,1)}(Z_3) , \\ F^1 H^3(Z_3) &= H^{(3,0)}(Z_3) \oplus H^{(2,1)}(Z_3) \oplus H^{(1,2)}(Z_3) , \\ F^0 H^3(Z_3) &= H^{(3,0)}(Z_3) \oplus H^{(2,1)}(Z_3) \oplus H^{(1,2)}(Z_3) \oplus H^{(0,3)}(Z_3) = H^3(Z_3) , \end{aligned} \quad (5.20)$$

where we recover the familiar decomposition of the de Rham group $H^3(Z_3)$ into (p, q) -forms for Kähler manifolds. This filtration is decreasing since $F^m H^3(Z_3)$ is contained in $F^{m-1} H^3(Z_3)$ for all m . It is convenient to study the filtration $F^m H^3(Z_3)$ instead of the individual groups $H^{(p,q)}(Z_3)$ because the $F^m H^3(Z_3)$ form a holomorphic subbundle \mathcal{F}^m of $\mathcal{H}^3(Z_3)$ [241–244], but $H^{(p,q)}(Z_3)$ do not. The bundle $\mathcal{H}^3(Z_3)$ has a flat connection ∇ which is called the Gauß-Manin connection [238]. It has the so-called Griffiths transversality property [241, 242]

$$\nabla \mathcal{F}^m \subset \mathcal{F}^{m-1} \otimes \Omega_{\mathcal{M}^{cs}}^1 . \quad (5.21)$$

This together with $h^{(3,0)} = 1$ is one of the main ingredients for the formulation of the $\mathcal{N} = 2$ special geometry for Calabi-Yau manifolds, as we will see below. Starting in $F^3 H^3(Z_3) = H^{(3,0)} Z_3$ generated by Ω , we can study the variation of the complex structure by analyzing how Ω changes under complex structure deformations δz in the diagram

$$H^{(3,0)}(Z_3) = \mathcal{F}^3 \xrightarrow{\nabla} \mathcal{F}^2 \xrightarrow{\nabla} \mathcal{F}^1 \xrightarrow{\nabla} \mathcal{F}^0 = H^3(Z_3) . \quad (5.22)$$

This diagram encodes the variations of Hodge structures [240] with respect to complex structure deformations δz . The form Ω and its derivatives $\nabla^k \Omega$, $k \leq 3$, span the spaces $F^3 H^{3-k}(Z_3)$

⁹See e.g. [238–240] for a review and a discussion of the local Torelli theorem underlying this statement.

and thus the complete space $H^3(Z_3) = F^0 H^3(Z_3)$. Consequently a fourth order derivative of any element of $H^3(Z_3)$ is expressible as a linear combination of $\nabla^k \Omega$ for $k \leq 3$. These linear relations between the derivatives of Ω up to fourth order yield the Picard-Fuchs equations which are for this very reason of maximal order four, see e.g. [237] for a mathematically thorough presentation of one-modulus Calabi-Yau hypersurfaces.

5.2.3 Calabi-Yau Threefold Special Geometry

Next we exploit the special geometrical properties of \mathcal{M}_{cs} which reflect the structure of the variations of Hodge structures in terms of differential relations among the periods of the Calabi-Yau threefold Z_3 . In particular it is special to the threefold case [12] that there exists a holomorphic prepotential F^0 governing the structure of all periods. This is denoted as special geometry or special Kähler geometry. From the physics point of view this is a consequence of $\mathcal{N} = 2$ supersymmetry in the four dimensional effective theory, since in general the vector multiplet moduli space of any $\mathcal{N} = 2$ theory is encoded by a prepotential [245–247].

We start by exploiting Griffiths transversality (5.21) to obtain $\frac{\partial}{\partial z_i} \Omega = c_i(z) \Omega + \chi_i$ for a function $c_i(z)$ of the complex structure moduli and a basis χ_i of $H^{(2,1)}(Z_3)$. This implies

$$0 = \int_{Z_3} \Omega \wedge \frac{\partial}{\partial z_i} \Omega. \quad (5.23)$$

Recalling the period expansion $\Omega = X^K \alpha_K - \mathcal{F}_K \beta^K$ we note that the basis (α_K, β^K) of $H^3(Z_3, \mathbb{Z})$ can be chosen such that the X^K form projective coordinates on \mathcal{M}_{cs} , where we use that Ω is only defined up to multiplication with e^f for a function f on the complex structure moduli space¹⁰. Mathematically the existence of these coordinates reflects the isomorphism $H^1(Z_3, T Z_3) \cong H^{(2,1)}(Z_3)$ and the local Torelli theorem. Then since the dimension of \mathcal{M}_{cs} is $h^{(2,1)}$ the periods $\underline{\mathcal{F}}$ have to be functions of the \underline{X} and can be written, again by using Griffiths transversality (5.23), as derivatives of a single function

$$\mathcal{F}_L = \frac{1}{2} \frac{\partial}{\partial X^L} (X^K \mathcal{F}_K) \equiv \frac{\partial}{\partial X^L} \mathcal{F}. \quad (5.24)$$

This is the projective prepotential \mathcal{F} which is a homogeneous (local) function of degree two in the \underline{X} and consequently a section of the square of the vacuum line bundle \mathcal{L}^2 over \mathcal{M}_{cs} ¹¹. We divide by one coordinate, say X^0 , to define affine coordinates $t^a = X^a/X^0$, $a \neq 0$, and introduce the prepotential

$$F^0(\underline{t}) = (X^0)^{-2} \mathcal{F}(\underline{X}). \quad (5.25)$$

It will be the special coordinates \underline{t} that are relevant in the context of mirror symmetry as we will discuss below and more generally in section 5.3.1. According to (5.21) the first non-trivial

¹⁰More precisely Ω is a section of the vacuum line bundle \mathcal{L} over \mathcal{M}_{cs} , see [82].

¹¹The integrability condition $\partial_K \mathcal{F}_L - \partial_L \mathcal{F}_K = 0$ follows from the special structure of the Riemann tensor that is characteristic for special Kähler geometry, see e.g. [227].

pairing are the Yukawa couplings [2, 4], that take in the normalization $\Omega \rightarrow \frac{1}{X^0}\Omega$ the form

$$C_{ijk}(\underline{t}) = \int_{Z_3} \Omega \wedge \frac{\partial^3}{\partial t^i \partial t^j \partial t^k} = \frac{\partial^3}{\partial t^i \partial t^j \partial t^k} F^0(\underline{t}) \quad (5.26)$$

for $i, j, k = 1, \dots, h^{(2,1)}$, that form sections of the bundle $Sym^3 T\mathcal{M}_{cs} \otimes \mathcal{L}^2$.

Next we study the hermitian geometry of \mathcal{M}_{cs} and \mathcal{L} , emphasizing aspects of special geometry. Firstly, we note that there is a natural hermitian metric on the vacuum line bundle \mathcal{L} spanned by Ω [82],

$$h = i \int \Omega \wedge \bar{\Omega}. \quad (5.27)$$

Here we used a standard construction in mathematics to obtain the hermitian metric of a line bundle \mathcal{L} once sections s_i generating \mathcal{L} everywhere are specified, cf. [13, p. 166]. As we have noted before in (2.30) the metric h also defines the Kähler potential on the complex structure moduli space \mathcal{M}_{cs} itself, that reads

$$K_{cs} = -\log(i \int \Omega \wedge \bar{\Omega}) = -\log i(\bar{X}^K \mathcal{F}_K - X^I \bar{\mathcal{F}}_I) = -\log i \|X^0\|^2 ((t - \bar{t})^i (F + \bar{F})_i + 2\bar{F} - 2F), \quad (5.28)$$

where we used the period expansion of Ω . Similarly h is used to construct the hermitian/Chern connection and the Chern curvature form R on the holomorphic line bundle \mathcal{L} , cf. [13, p. 177, p. 186], that takes the form

$$A_i = \partial_i \log(h) = -\partial_i K_{cs} = \frac{i}{h} (\bar{X}^K \partial_i \mathcal{F}_K - \bar{\mathcal{F}}_i), \quad R_{i\bar{j}} = (\bar{\partial} \partial \log(h))_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K_{cs}. \quad (5.29)$$

The existence of the bundle \mathcal{L} , whose curvature form is simultaneously the Kähler form on its base manifold is one central property of $\mathcal{N} = 2$ special geometry¹². With these definitions at hand we can determine the constant $c_i(z)$ in $\frac{\partial}{\partial z_i} \Omega$ as

$$\frac{\partial}{\partial X^i} \Omega = \chi_i - \frac{\partial}{\partial X^i} K_{cs} \Omega, \quad c_i = -\partial_i K_{cs}. \quad (5.30)$$

We conclude by presenting the consequences of the existence of the prepotential F^0 for the structure of the period vector $\Pi := (\underline{X}, \underline{\mathcal{F}})$. Upon normalizing Ω by X_0 the period vector can be expressed as

$$\Pi = X^0 \begin{pmatrix} 1 \\ t^i \\ \partial_i F^0 \\ 2F^0 - t^i \partial_i F^0 \end{pmatrix}. \quad (5.31)$$

We note that the period vector is only defined up to symplectic transformations in the fundamental representation of $Sp(b^3, \mathbb{Z})$ that leave the pairing (5.27) invariant. Geometrically

¹²A very simple example is projective space \mathbb{P}^n with the Fubini-Study metric that is simultaneously the curvature tensor on $\mathcal{O}_{\mathbb{P}^n}(1)$.

this corresponds to the freedom in the choice of symplectic basis of $H^3(Z_3, \mathbb{Z})$. Thus Π forms a section of a symplectic vector bundle over \mathcal{M}_{cs} of rank b^3 . Since this bundle is flat the remaining gauge freedom will lead to a monodromy action $\Pi \mapsto M\Pi$, $M \in Sp(b^3, \mathbb{Z})$ around singularities in \mathcal{M}_{cs} , which are the regular singular points of the Picard-Fuchs equations [238] introduced next.

5.2.4 Picard-Fuchs Equations, Toric GKZ-Systems and the Mirror Map

In the context of toric hypersurface the structure implied by the variations of Hodge structures (5.22) can be explicitly studied. In fact in the case of an algebraic representation $P(\underline{x}; \underline{a}) = 0$ of a family of Calabi-Yau hypersurfaces residue integral expressions for the three-form Ω and its periods are known. These can be used to find Picard-Fuchs equations governing their complex structure dependence. We start by presenting a very brief review of the general techniques.

The explicit analysis of the holomorphic three-form Ω is based on the algebraic representation of Ω and its periods Π^k by the residue integral expressions [154, 238, 243, 244]

$$\Omega(\underline{z}) = \int_{S_P^1} \frac{\Delta_{\mathbb{P}_\Delta}}{P(\underline{x}, \underline{z})}, \quad \Pi^k(\underline{z}) = \int_{\Gamma^k \times S_P^1} \frac{\Delta_{\mathbb{P}_\Delta}}{P(\underline{x}, \underline{z})}, \quad (5.32)$$

where Z_3 is given as the zero locus of a polynomial constraint $P(\underline{x}, \underline{z})$ in coordinates \underline{x} in a toric ambient space \mathbb{P}_Δ and the holomorphic top-form $\Delta_{\mathbb{P}_\Delta}$ was introduced in (5.4). Here Γ^k , $k = 1, \dots, b_3$ with $b_3 = 2h^{(2,1)} + 2$, denote a basis of $H_3(Z_3, \mathbb{Z})$ and S_P^1 denotes a small S^1 surrounding the zero locus of $P(\underline{x}, \underline{z})$ in the normal direction. The \underline{z} are the complex structure moduli of Z_3 . Then one performs the so-called Griffiths-Dwork reduction method¹³, also referred to as reduction of pole order [243], to obtain differential operators \mathcal{D}_a obeying

$$\mathcal{D}_a(\underline{z})\Omega(\underline{z}) = d\alpha_a \quad (5.33)$$

for a two-form α_a . Integration over Γ^k yields homogeneous linear differential equations for the periods $\Pi^k(\underline{z})$, the Picard-Fuchs equations, that directly reflect the diagram (5.22).

In the toric context the determination of the Picard-Fuchs equations¹⁴ simplifies significantly due to the toric symmetries of \mathbb{P}_Δ and the requirement of invariance of the residue integrals (5.32). For this analysis it is necessary to use instead of the coordinates \underline{z} on \mathcal{M}_{cs} first the redundant parameterization of the complex structure variables of Z_3 by the coefficients \underline{a} on which the toric symmetries of \mathbb{P}_Δ act canonically. The infinitesimal version of these symmetries acting on the \underline{a} give rise to very simple differential operators $\mathcal{L}_l(\underline{a})$ and $\mathcal{Z}_k(\underline{a})$ called the Gelfand-Kapranov-Zelevinski or short GKZ differential system of the toric

¹³See e.g. [237, 248] for a review and [249] for an extension and algorithm for complete intersections.

¹⁴There is one caveat in order since the toric means, in contrast to the Griffiths-Dwork reduction method, not necessarily yield differential equations of minimal order.

complete intersection [250, 251]. The operators either annihilate $\Omega(\underline{a})$ or, if the toric symmetries are broken in a specific way, annihilate $\Omega(\underline{a})$ up to an exact form as in (5.33). The latter operators, the $\mathcal{Z}_k(\underline{a})$, express the fact that $\Omega(\underline{a})$ depends only on specific combinations of the \underline{a} , which are the genuine complex structure deformations \underline{z} defined in (5.14). From the operators $\mathcal{L}_k(\underline{a})$ it is possible in simple situations to obtain a complete system of Picard-Fuchs operators $\mathcal{D}_a(\underline{z})$ [233].

Let us give the explicit representation of the operators forming the GKZ-system. Both the operators $\mathcal{L}_l(\underline{a})$ and $\mathcal{Z}_k(\underline{a})$ are completely determined by the toric data encoded in the points \tilde{v}_j and relations $\ell^{(a)}$ of $\Delta_4^{\tilde{Z}}$.¹⁵ The first set of operators is given by

$$\mathcal{L}_i = \prod_{\ell_j^{(i)} > 0} \left(\frac{\partial}{\partial a_j} \right)^{\ell_j^{(i)}} - \prod_{\ell_j^{(i)} < 0} \left(\frac{\partial}{\partial a_j} \right)^{-\ell_j^{(i)}}, \quad i = 1, \dots, k. \quad (5.34)$$

These operators annihilate $\Omega(\underline{a})$ in (5.32) and its periods $\Pi^k(\underline{a})$ identically as can be checked as a simple consequence of the second relation in (5.14). In other words, the differential operators \mathcal{L}_i express the trivial algebraic relations between the monomials entering $P(\underline{x}; \underline{a})$. The second differential system of operators encoding the automorphisms of \mathbb{P}_Δ is given by

$$\mathcal{Z}_j = \sum_n (\bar{v}_n)^j \vartheta_n - \beta_j, \quad j = 0, \dots, 4. \quad (5.35)$$

Here $\beta = (-1, 0, 0, 0, 0)$ is the so-called exponent of the GKZ-system and $\vartheta_n = a_n \frac{\partial}{\partial a_n}$ denote logarithmic derivatives. We have embedded the points \tilde{v}_n into a hypersurface at distance 1 away from the origin by defining $\bar{v}_n = (1, \tilde{v}_n)$ so that all zeroth components are $(\bar{v}_n)^0 = 1$. The operators \mathcal{Z}_j simply represent the torus symmetries of \mathbb{P}_Δ on the periods $\Pi^k(\underline{a})$ that are functions of the parameters \underline{a} . In detail, the operator \mathcal{Z}_0 expresses the effect of a rescaling of the constraint $P \mapsto \lambda P$ by the homogeneity property $\Pi^k(\lambda \underline{a}) = \lambda \Pi^k(\underline{a})$. It is often convenient to absorb this scaling freedom by performing the redefinition $\Omega(\underline{a}) \mapsto a_0 \Omega(\underline{a})$ and $\Pi^k(\underline{a}) \mapsto a_0 \Pi^k(\underline{a})$ so that the periods become invariant functions under the overall rescaling $\underline{a} \mapsto \lambda \underline{a}$. Accordingly, we obtain a shift $\vartheta_0 \mapsto \vartheta_0 - 1$ in all the above operators, such that the $\mathcal{Z}_0 = \sum_n \vartheta_n$ in particular. The operators \mathcal{Z}_j , $j \neq 0$, express rescalings of the coordinates like e. g. $x_m \mapsto \lambda x_m$, $x_n \mapsto \lambda^{-1} x_n$ that can be compensated by rescalings of the monomial coefficients \underline{a} . In particular these coordinate rescalings leave the measure $\Delta_{\mathbb{P}_\Delta}$ on \mathbb{P}_Δ invariant. Examples for the operators \mathcal{Z}_i and the corresponding action on the coordinates x_j are given in section 6.1 and chapter 8.

The coordinates \underline{z} introduced in (5.14) are solutions to the \mathcal{Z}_j -system and generally read

$$z^i = (-)^{\ell_0^{(i)}} \prod_{j=0}^{k+4} a_j^{\ell_j^{(i)}}, \quad i = 1, \dots, k. \quad (5.36)$$

¹⁵We note that the following discussion applies to any Calabi-Yau manifold of complex dimensions n by replacing $\Delta_4^{\tilde{Z}} \rightarrow \Delta_{n+1}^{\tilde{Z}}$.

They indeed turn out to be appropriate coordinates at the point of maximally unipotent monodromy in the complex structure moduli space of Z_3 , which is centered at $\underline{z} = 0$. Indeed the Picard-Fuchs operators \mathcal{D}_a extracted from the GKZ-system in the coordinates \underline{z} is solved explicitly using the Frobenius method [233, 252], cf. section 5.3.3 for more details. The obtained solutions have the well-known log-grading in terms of powers of $(\log(z^i))^k$, $k = 0, \dots, 3$ with a unique holomorphic power series solution X^0 . This is expected by the known shift symmetry of the NS-NS B-field, $b^i \mapsto b^i + 1$, in the Type IIA theory, which implies on the B-model side a maximal logarithmic degeneration of periods near the point $z_i = 0$ [35, 233].

In order to relate these solutions with the geometrical periods Π in (5.31) of Ω it is convenient to exploit mirror symmetry. It relates the complex geometry of Z_3 and $H^3(Z_3)$ to the symplectic geometry of \tilde{Z}_3 and $H^{\text{ev}}(\tilde{Z}_3) = \bigoplus_{i=0}^3 H^{(i,i)}(\tilde{Z}_3)$. In particular an integral basis of $H^3(Z_3, \mathbb{Z})$ is determined by the integral basis of $H^{\text{ev}}(\tilde{Z}_3, \mathbb{Z})$ upon demanding a matching of the A- and B-model operator algebra. This fixes the leading structure of the periods of Ω in the expansion by an integral basis of $H^3(Z_3, \mathbb{Z})$ that is mirror dual to $H^{\text{ev}}(\tilde{Z}_3, \mathbb{Z})$, see e.g. (5.66) for the general matching at large volume. It is special for the threefold case that there is a canonical basis of $H^{\text{ev}}(\tilde{Z}_3, \mathbb{Z})$ given by products of the Kähler cone generators J_i on \tilde{Z}_3 , $(1, J_i, a_k^{i_1 i_2} J_{i_1} J_{i_2}, \frac{1}{\mathbb{V}} J^3)$. The constant coefficients $a_k^{i_1 i_2}$ are determined by demanding duality between $H^{(1,1)}(\tilde{Z}_3)$ and $H^{(2,2)}(\tilde{Z}_3)$, which just involves the calculation of the classical intersections on \tilde{Z}_3 . For the marginal deformations the mirror map is then established at large volume as

$$t_A^i \equiv t^i = \frac{X^i}{X^0} = \frac{1}{2\pi i} (\log(z^i) + p(\underline{z})), \quad (5.37)$$

which identifies, in geometrical terms, the Kähler moduli t_A^i on Z_3 with the affine complex structure moduli t^i of Z_3 . On the level of the cohomology basis or in other words on the level of the A- and B-model operators this corresponds to $J_i \leftrightarrow \beta_i^{(1)} = \theta_i \Omega = \chi_i z^i - \theta^i K_{\text{cs}}$, cf. the general mirror map (5.65). We note that this maps $H^{(p,p)}(\tilde{Z}_3)$ to $F^{3-p} H^3(Z_3)$ in order to appreciate the holomorphicity of the vector bundles \mathcal{F}^m . In order to extract the more intuitive matching of $H^{(3-p,p)}(Z_3) = F^p H^{3-p}(Z_3) \cap \overline{F^p H^3(Z_3)}$ we have to perform an anholomorphic projection map.

Thus, we have seen that the knowledge of the classical terms fixes the coefficients of the leading order in $\log(z^i)$. The subleading terms are restricted further by demanding integral monodromy in $Sp(b^3, \mathbb{Z})$ under the Peccei-Quinn symmetry $t_A^i \mapsto t_A^i + 1$. This allows to unambiguously identify the solutions to the Picard-Fuchs system with the periods with respect to the integral basis of $H_3(Z_3, \mathbb{Z})$. We refer to section 5.3 for a more systematic derivation of the structure of the periods and conclude by summarizing the periods at large complex structure. The prepotential is given by [227]

$$F^0 = -\frac{1}{3!} \mathcal{K}_{ijk} t^i t^j t^k - \frac{1}{2!} \mathcal{K}_{ij} t^i t^j + \mathcal{K}_i t^i + \frac{1}{2} \mathcal{K}_0 + \sum_{\beta} n_{\beta}^0 \text{Li}_3(q^{\beta}) \quad (5.38)$$

where $q^{\beta} = e^{2\pi i d_j t^j}$ for a class $\beta = d_i \beta$ with entries $d_i \in \mathbb{Z}_{\geq 0}$, cf. section 5.4. The last term

contains the quantum corrections by string world-sheet instantons to the A-model and will further be discussed in section 5.4 on enumerative geometry. Here

$$\begin{aligned}\mathcal{K}_{ijk} &= \int_{\tilde{Z}_3} J_i \wedge J_j \wedge J_k, & \mathcal{K}_{ij} &= \frac{1}{2} \int_{\tilde{Z}_3} \iota_*(c_1(J_j)) \wedge J_i, \\ \mathcal{K}_j &= \frac{1}{2^2 3!} \int_{\tilde{Z}_3} c_2(T_{\tilde{Z}_3}) \wedge J_j, & \mathcal{K}_0 &= \frac{\zeta(3)}{(2\pi i)^3} \int_{\tilde{Z}_3} c_3(T_{\tilde{Z}_3}),\end{aligned}\tag{5.39}$$

are determined by the classical intersections of the mirror Calabi-Yau threefold \tilde{Z}_3 of Z_3 . Note that $c_1(J_j)$ denotes the first Chern class of the divisor line bundle associated to J_j and $\iota_* = P_{\tilde{Z}_3} i_* P_{J_j}^{-1}$ is the Gysin homomorphism where $P_{\tilde{Z}_3}$ (P_{J_j}) is the Poincaré-duality map on \tilde{Z}_3 (J_j) and i_* is the push-forward on the homology. Thus, $\iota_*(c_1(J_j))$ is a four-form. We insert the prepotential F^0 into the general expression of the period vector (5.31) to obtain all integral periods of Z_3 as

$$\Pi = X^0 \begin{pmatrix} 1 \\ t^i \\ -\frac{1}{2}\mathcal{K}_{ijk} t^j t^k - \mathcal{K}_{ij} t^j + \mathcal{K}_i + \sum_{\beta} n_{\beta}^0 d_i \text{Li}_2(q^{\beta}) \\ \frac{1}{3!}\mathcal{K}_{ijk} t^i t^j t^k + \mathcal{K}_i t^i + \mathcal{K}_0 + \sum_{\beta} n_{\beta}^0 (2\text{Li}_3(q^{\beta}) - d_i t^i \text{Li}_2(q^{\beta})) \end{pmatrix}.\tag{5.40}$$

Ultimately the quantum corrected Yukawa couplings $C_{ijk}(t)$ are evaluated from (5.26) as

$$C_{ijk}(t) = \partial_i \partial_j \partial_k F^0(t) = -\mathcal{K}_{ijk} + \sum_{\beta} n_{\beta}^0 d_i d_j d_k \frac{q^{\beta}}{1 - q^{\beta}},\tag{5.41}$$

where we used the elementary identity $\text{Li}_0(x) = \frac{x}{1-x}$. We immediately observe that the sub-leading terms do not affect the quantum corrections to the A-model intersections. However, the subleading terms are essential in the determination of the $\mathcal{N} = 2$ Kähler potential K_{cs} in (5.28) for the vector multiplet moduli and for the flux and brane superpotentials, as we will discuss in sections 5.2.3 and 5.4.3.

We conclude by noting that it is the aim of chapter 7 to extend the use of the Picard-Fuchs equations, in particular the GKZ-system, the coordinates at large volume as well as the concept of periods to the open string sector of a five-brane. This is achieved by formulating a GKZ-system for a threefold \hat{Z}_3 obtained as a blow-up of the curve Σ wrapped by the brane and by analyzing its solutions.

5.3 Mirror Symmetry for Calabi-Yau Fourfolds

In this section we describe mirror symmetry for Calabi-Yau fourfolds (\tilde{X}, X) . The key quantities we analyze are holomorphic functions $F^0(\gamma)$, that are the fourfold analogue of the holomorphic prepotential F^0 (5.25) on Calabi-Yau threefolds in the sense, that the (covariant) double derivatives of $F^0(\gamma)$ with respect to the moduli t^a yield the three-point correlators

$C_{ab\gamma}^{(1,1,2)}$ of the underlying topological field theory. These three-point correlators together with the two-point correlation functions are the fundamental couplings of the topological A- respectively B-model that encode all other correlators in the theory. Whereas the classical two-point coupling is exact, the three-point couplings encode, as in the Yukawa couplings (5.41) in the threefold case, to leading order the classical intersections of two divisors with an element of $H_4(\tilde{X})$ on the A-model side, that receive quantum corrections due to world-sheet instantons. As in the threefold case it is the power of mirror symmetry for Calabi-Yau fourfolds to calculate these couplings exactly by relating the classical geometric calculations in the B-model on X_4 to the quantum cohomology ring of A-model on \tilde{X}_4 .

We begin in section 5.3.1 by introducing the B-model operator ring, that is formed by the operators representing elements of $H^p(X, \bigwedge^q TX)$, together with their two-point and three-point correlators, that are given by period integrals. In the large radius limit the A-model is defined by another ring, namely the vertical part of the classical cohomology ring $H^{(*,*)}(\tilde{X})$. To prepare the analysis of mirror symmetry we note in section 5.3.2 that both rings with the two- and three-point correlators just mentioned carry a natural Frobenius structure. Mirror symmetry then identifies the quantum cohomology ring of the A-model away from the large volume limit with the B-model ring. For the precise identification one has to specify matching points in the moduli spaces. The natural candidate for our present purposes is the large radius point/large complex structure point in the A- respectively B-model. We describe in section 5.3.3 the details of this matching, namely the use of the mirror map, properties of the Picard-Fuchs system near the point of maximal unipotent monodromy and the classical intersection ring of \tilde{X} . We then exploit the precise matching to obtain enumerative predictions for the A-model and to construct an integral basis of the horizontal cohomology of $H^{(n,n)}(X)$. For the latter it is a particularly important step to determine the classical terms in the leading logarithms of the fourfold periods. We conclude our discussion in section 5.3.4 by fixing these classical terms by means of analytic continuation to other points on the fourfold complex structure moduli space, namely the universal conifold, and a monodromy analysis. Our presentation follows mainly [93–95] and [79].

Since our discussion can at several points be generalized to arbitrary complex dimensional Calabi-Yau n -folds [93] we denote a mirror pair of Calabi-Yau n -folds by (\tilde{X}, X) and identify with the fourfold case by putting $n = 4$ and writing (\tilde{X}_4, X_4) instead.

5.3.1 States and Correlation Function of the B-model

In the B-model one considers a family of n -folds X_z fibered over the complex structure moduli space \mathcal{M}_{cs} , $X_z \rightarrow \mathcal{M}_{\text{cs}}$. The states¹⁶ in the B-model are elements $B_k^{(j)}$ of the cohomology

¹⁶We use in the following the same symbol for states and operators.

groups $H^j(X_z, \wedge^j T)$. Their cubic forms are defined as

$$C(B_a^{(i)}, B_b^{(j)}, B_c^{(k)}) = \int_X \Omega_n(B_a^{(i)} \wedge B_b^{(j)} \wedge B_c^{(k)}) \wedge \Omega, \quad (5.42)$$

and vanish unless $i + j + k = n$. Here Ω_n is the unique holomorphic $(n, 0)$ -form and $\Omega_n(B_a^{(i)} \wedge B_b^{(j)} \wedge B_c^{(k)})$ is the contraction of the n upper indices of $B_a^{(i)} \wedge B_b^{(j)} \wedge B_c^{(k)} \in H^n(X_z, \wedge^n TX_z)$ with Ω_n , which produces an anti-holomorphic $(0, n)$ -form on X_z . Note that this is the standard isomorphism $H^i(X_z, \wedge^j TX_z) \cong H_H^{(n-j, i)}(X_z)$ obtained by contraction with Ω_n . It is important to emphasize that not all elements of $H^n(X_z)$ can be reached as variations of Ω_n with respect to the complex structure, in contrast to the Calabi-Yau threefold case, which makes it necessary to introduce the primary horizontal subspace $H_H^n(X_z)$ of $H^n(X_z)$. Its complement is the vertical subspace¹⁷ $H_V^n(X_z)$, that is generated by powers of the Kähler cone generators in $H^{(1,1)}(X_z)$ [93]. We denote the image of $B_k^{(i)}$ in $H_H^{(n-i, i)}(X_z)$ by $b_k^{(i)} = \Omega(B_k^{(i)})$ and the inverse¹⁸ by $\bar{B}_k^{(i)} = (b_k^{(i)})^\Omega$. We define the hermitian metric

$$G(B_c^{(i)}, \bar{B}_d^{(i)}) = \int_X b_c^{(i)} \wedge \bar{b}_d^{(i)}. \quad (5.43)$$

The image $b_a^{(i)}$ of the states $B_a^{(i)}$ lies in the horizontal subspace $H_H^{(n-j, i)}(X_z)$ and the elements $b_a^{(i)}$ form a basis of this space. For $B_c^{(1)} \in H^1(X_z, TX_z)$ the image spans all of $H^{(n-1, 1)}(X_z)$ and (5.43) is the Weil-Petersson metric on \mathcal{M}_{cs} .

The integrals (5.42) and (5.43) are calculable by period integrals of the holomorphic n -form Ω_n . It is in general hard to integrate the periods directly. They encode however, as in the Calabi-Yau threefold case in section 5.2.2, the variations of Hodge structures of the family $X_z \rightarrow \mathcal{M}_{\text{cs}}$, which leads to the Picard-Fuchs differential equations on \mathcal{M}_{cs} . The periods are therefore determined as solutions of the latter up to linear combination. The precise identification as addressed in section 5.3.3 is an important problem, in particular for physical applications like the determination of the flux superpotential (4.37) on the fourfold X .

For a given base point $z = z_0$ in the complex structure moduli space \mathcal{M}_{cs} with fiber X_{z_0} , one fixes a graded topological basis $\hat{\gamma}_a^{(p)}$ of the primary horizontal subspace $H_H^n(X_{z_0}, \mathbb{Z})$. Here $a = 1, \dots, h_H^{(n-p, p)}$ labels the basis $\hat{\gamma}_a^{(p)}$ for fixed $p = 0, \dots, n$ of each graded piece $H^{(n-p, p)}(X_z)$. In addition, these forms can be chosen to satisfy¹⁹

$$\begin{aligned} \int_{X_4} \hat{\gamma}_a^{(i)} \wedge \hat{\gamma}_b^{(4-i)} &= \eta_{ab}^{(i)}, \\ \int_{X_4} \hat{\gamma}_a^{(i)} \wedge \hat{\gamma}_b^{(j)} &= 0 \quad \text{for } i + j > 4. \end{aligned} \quad (5.44)$$

¹⁷This is completely analogous to the two-dimensional case of $K3$, where $H_V^{(1,1)}$ spans the Picard group.

¹⁸The inversion is just the contraction $(b_k^{(i)})^\Omega = \frac{1}{\|\Omega\|^2} \bar{\Omega}^{a_1 \dots a_{n-i} b_1 \dots b_i} (b_k^{(i)})_{a_1 \dots a_{n-i} \bar{b}_1 \dots \bar{b}_i}$ such that $(\Omega)^\Omega = 1$. Formally it is the multiplication with the inverse \mathcal{L}^{-1} of the Kähler line bundle $\mathcal{L} = \langle \Omega \rangle$, see e.g. [82].

¹⁹The generalization from fourfolds to n -folds is the obvious one.

Later on in section 5.3.3 we will identify this grading by p with the natural grading on the observables of the A-model which are given by the vertical subspaces $H_V^{(p,p)}(\tilde{X})$ of the mirror cohomology.

Note that the $\hat{\gamma}_a^{(p)}$ basis serves as a local frame of the vector bundle $\mathcal{H}_H^n(X) \rightarrow \mathcal{M}_{\text{cs}}$ over the moduli space whose fiber at the point $z \in \mathcal{M}_{\text{cs}}$ is $H_H^n(X_z)$. Thus, the bundle $\mathcal{H}_H^n(X)$ is flat and admits a flat connection ∇ denoted the Gauss-Manin connection. However, the individual cohomology groups $H^{(n,n-p)}(X_z)$ form no holomorphic vector bundles over \mathcal{M}_{cs} since holomorphic and anti-holomorphic coordinates are mixed under a change of complex structure z . Analogously to the threefold case, cf. section 5.2.2, only the horizontal parts of $F^k = \bigoplus_{p=0}^k H^{(n-p,p)}(X_z)$ form holomorphic vector bundles for which we introduce frames $\beta_a^{(k)}$ specified by the basis expansion

$$\beta_a^{(k)} = \hat{\gamma}_a^{(k)} + \sum_{p>k} \Pi_a^{(p,k)} \hat{\gamma}_c^{(p)}. \quad (5.45)$$

In special coordinates t^a at the point of maximal unipotent monodromy, cf. (5.49), this can be written as $\{\beta^{(0)} = \Omega_n, \beta_a^{(1)} = \partial_a \Omega_n, \dots\}$. We note that for this basis choice we fixed the overall normalization of each $\beta_a^{(k)}$ such that the coefficient of $\hat{\gamma}_a^{(k)}$ is unity. This is needed to obtain the right inhomogeneous flat coordinates on \mathcal{M}_{cs} and to make contact with enumerative predictions for the A-model, see section 5.3.3. For grade $k = 0$ it corresponds to the familiar gauge choice in the vacuum line bundle \mathcal{L} generated by Ω_n , cf. section 5.2.3 for the threefold case and (5.49) for fourfolds. We also introduce a basis of integral homology cycles $\gamma_a^{(p)}$ dual to $\hat{\gamma}_a^{(p)}$ obeying

$$\int_{\gamma_a^{(i)}} \hat{\gamma}_b^{(j)} = \delta^{ij} \delta_{ab}. \quad (5.46)$$

Then, by construction of the basis (5.45), the period matrix P defined by period integrals takes an upper triangular form in this basis,

$$P = \int_{\gamma_a^{(p)}} \beta_b^{(k)} = \begin{cases} \Pi_b^{(p,k) a}, & p > k, \\ \delta_{ab}, & p = k, \\ 0, & p < k, \end{cases} \quad (5.47)$$

where (p, k) is the bi-grade of the non-trivial periods $\Pi_b^{(p,k) a}$. Since $\hat{\gamma}_a^{(p)}$ are topological and thus are locally constant on \mathcal{M}_{cs} , the moduli dependence of (5.45) is captured by the moduli dependence of the period matrix $P \equiv P(z)$ ²⁰. For Calabi-Yau fourfolds we summarize the basis choices and the periods in Table 5.1. The $(n, 0)$ -form Ω_n can always be expanded over the topological basis $\hat{\gamma}_a^{(p)}$ of $H_H^n(X_z, \mathbb{Z})$ as

$$\Omega_n = \Pi_1^{(p,0) a} \hat{\gamma}_a^{(p)} \equiv \Pi^{(p) a} \hat{\gamma}_a^{(p)}, \quad (5.48)$$

where we introduced a simplified notation $\Pi^{(p) a}$ for the periods $\Pi_1^{(p,0) a}$ of the holomorphic n -form already given in (5.45) for $k = 0$. For an arbitrarily normalized n -form Ω_n , the periods

²⁰By means of $\nabla_a \beta_b^k = \nabla_a \Pi_b^{(k+1,k) c} \beta_c^{(k+1)}$ [94] the periods of Ω_n fix the periods P .

Dimension	1	$h^{(3,1)}(X_4)$	$h^{(2,2)}(X_4)$	$h^{(3,1)}(X_4)$	1
Basis of	$\hat{\gamma}_{a_0}^{(0)}$	$\hat{\gamma}_a^{(1)}$	$\hat{\gamma}_\beta^{(2)}$	$\hat{\gamma}_a^{(3)}$	$\hat{\gamma}_{a_0}^{(4)}$
$H_H^4(X_4, \mathbb{Z})$	$\beta_{a_0}^{(0)}$	$\beta_a^{(1)}$	$\beta_\beta^{(2)}$	$\beta_a^{(3)}$	$\beta_{a_0}^{(4)}$

Table 5.1: Topological $\hat{\gamma}$ and moduli-dependent β basis of $H_H^4(X_4, \mathbb{Z})$

$(X^0, X^a) = (\Pi^{(0)1}, \Pi^{(1)a})$ with $a = 1, \dots, h^{(n,1)}(X_{z_0})$ at the fixed reference point z_0 form for all Calabi-Yau n -folds homogeneous projective coordinates of the complex moduli space \mathcal{M}_{cs} . The choice of inhomogeneous coordinates set by

$$t^a = \frac{X^a}{X^0} = \frac{\int_{\gamma_a^{(1)}} \Omega}{\int_{\gamma_1^{(0)}} \Omega}, \quad (5.49)$$

which agrees with the basis choice in (5.45), corresponds to a gauge in the Kähler line bundle $\langle \Omega_n \rangle$. At the point of maximal unipotent monodromy X^0 and X^a are distinguished by their monodromy, as X^0 is holomorphic and single-valued and $X^a \sim \log(z^a)$ has monodromy $X^a \mapsto X^a + 1$. Below the t^a in (5.49) are identified with the complexified Kähler parameters of the mirror \tilde{X} .

The t^a defined in (5.49) are flat coordinates for the Gauss-Manin connection ∇_a , i.e. the latter becomes just the ordinary differential $\frac{\partial}{\partial t_a}$. This can be seen from the gauge choice reflecting in the basis (5.45) combined with the Griffiths transversality constraint $\nabla_a(\mathcal{F}^k) \subset \mathcal{F}^{k+1}/\mathcal{F}^k$ which together imply [93] $\nabla_a t^b = \delta_a^b$, i.e. $\nabla_a = \partial_a$. In these coordinates the three-point coupling becomes

$$C_{abc}^{(1,k,n-k-1)} = C((\beta_a^{(1)})^\Omega, (\beta_b^{(k)})^\Omega, (\beta_c^{(n-k-1)})^\Omega) = \int_X \beta_c^{(n-k-1)} \wedge \partial_a \beta_b^{(k)}. \quad (5.50)$$

Here we use (5.45), (5.47) and the fact that $(\beta_a^{(1)})^\Omega \wedge (\beta_b^{(k)})^\Omega$ can be replaced by $(\nabla_a \beta_b^{(k)})^\Omega$ [93] under the integral (5.42) to obtain (5.50). This triple coupling is a particularly important example of (5.42). Furthermore, one can show from the properties of the Frobenius algebra that all other triple couplings in (5.42) can be expressed in terms of (5.50), see section 5.3.2.

The holomorphic topological two-point couplings of (5.44) in this basis trivially read

$$\eta_{ab}^{(k)} = \int_X \beta_b^{(n-k)} \wedge \beta_a^{(k)} \quad (5.51)$$

since only the lowest $\hat{\gamma}^{(p)}$ for $p = k$ in the upper-triangular basis transformation (5.45) contributes to the integral due to the second property of (5.44). In particular $\eta_{ab}^{(k)}$ is moduli independent. From the above it is easy to see the basis expansion at grade $k + 1$,

$$\partial_a \beta_b^{(k)} = C_{abc}^{(1,k,n-k-1)} \eta_{(n-k-1)d}^{cd} \beta_d^{(k+1)}, \quad (5.52)$$

where $\eta_{(p)}^{cd}$ is the inverse of $\eta_{cd}^{(p)}$.

Let us finish this section with some comments on general properties of the periods and their implications on the $\mathcal{N} = 1$ effective action. The period integrals (5.47) obey differential and algebraic relations, which are different from the special geometry relations of Calabi-Yau threefold periods in section 5.2.3. The differential relations have however exactly the same origin namely the Griffiths transversality constraints $\int_X \Omega_n \wedge \partial_{i_1} \dots \partial_{i_k} \Omega_n = 0$ for $k < n$. However since $a \wedge b = (-1)^n b \wedge a$ for a, b real n -forms one has additional algebraic relations from $\int_X \Omega_n \wedge \Omega_n = 0$ between the periods $\Pi^{(p)a}$ for n even like $n = 4$, which are absent for n odd, in particular the threefold case. $\mathcal{N} = 2$ compactifications of Type II on Calabi-Yau threefolds to four dimensions inherit the special geometry structure induced by the above differential relations in the vector moduli space. In contrast for a generic $\mathcal{N} = 1$ supergravity theory in 4d there is no special structure beyond Kähler geometry.

5.3.2 The Frobenius Algebras

As was mentioned above the B-model operators form a Frobenius algebra. Since also the A-model classical cohomology operators as well as its quantum cohomology operators form such an algebra it is worthwhile to describe the general structure before discussing the precise matching via mirror symmetry in the next section.

A Frobenius algebra has the following structures. It is a graded vector space $\mathcal{A} = \bigoplus_{i=0}^n \mathcal{A}^{(i)}$ with $\mathcal{A}^{(0)} = \mathbb{C}$ equipped with a non-degenerate symmetric bilinear form η and a cubic form

$$C^{(i,j,k)} : \mathcal{A}^{(i)} \otimes \mathcal{A}^{(j)} \otimes \mathcal{A}^{(k)} \rightarrow \mathbb{C} , \quad (5.53)$$

$i, j, k \geq 0$ and the following properties:

(F1) Degree: $C^{(i,j,k)} = 0$ unless $i + j + k = n$

(F2) Unit: $C_{1bc}^{(0,i,j)} = \eta_{bc}^{(i)}$

(F3) Nondegeneracy: $C^{(1,i,j)}$ is non-degenerate in the second slot

(F4) Symmetry: $C_{abc}^{(i,j,k)} = C_{\sigma(abc)}^{(\sigma(i,j,k))}$ under any permutation of the indices.

(F5) Associativity: $C_{abp}^{(i,j,n-i-j)} \eta_{(n-i-j)}^{pq} C_{qef}^{((i+j),k,(n-i-j-k))} = C_{aeq}^{(i,k,n-i-k)} \eta_{(n-i-k)}^{qp} C_{pbf}^{(i+k,j,n-i-j-k)}$
where the sum over common indices is over a basis of the corresponding spaces.

The product

$$\mathcal{O}_a^{(i)} \cdot \mathcal{O}_b^{(j)} = C_{abq}^{(i,j,i+j)} \eta_{(i+j)}^{qp} \mathcal{O}_p^{(i+j)} \quad (5.54)$$

defines now the Frobenius algebra for a basis of elements $\mathcal{O}_k^{(i)}$ of $\mathcal{A}^{(i)}$ which is easily seen using (F4) to be commutative. Note that the associativity implies that n -point correlators can be factorized in various ways in the three-point functions. Also the three-point correlators are

not independent and by associativity, non-degeneracy and symmetry it can be shown²¹ that all three-correlators can be expressed in terms of the $C^{(1,r,n-r-1)}$ correlators defined in (5.50).

It is easy to see that the $B^{(i)}$ operators of the B-model with the correlators defined by (5.42) or equivalently (5.50) and (5.51) fulfill the axioms of a Frobenius algebra.

Let us consider the A-model operators. As already mentioned such an operator corresponds to an element in the vertical subspace $H_V^{(*,*)}(\tilde{X})$. It is generated by the Kähler cone generators J_i , $i = 1, \dots, h^{(1,1)}(\tilde{X})$ and is naturally graded,

$$A_\alpha^{(p)} = a_\alpha^{i_1, \dots, i_p} J_{i_1} \wedge \dots \wedge J_{i_p} \in H^{(p,p)}(\tilde{X}). \quad (5.55)$$

For the classical A-model the correlation functions are simply the intersections

$$C_{abc}^{0(i,j,k)} = C(A_a^{(i)} A_b^{(j)} A_c^{(k)}) = \int_{\tilde{X}} A_a^{(i)} \wedge A_b^{(j)} \wedge A_c^{(k)}. \quad (5.56)$$

They vanish unless $i + j + k = n$. The topological metric is similarly defined by

$$\eta_{ab}^{(k)} = \int_{\tilde{X}} A_a^{(k)} \wedge A_b^{(n-k)} \quad (5.57)$$

and together with (5.56) this defines a Frobenius algebra. Clearly the $A_k^{(p)}$ are not freely generated by the J_i . The products $J_{i_1} \wedge \dots \wedge J_{i_p}$ are set to zero if their pairings (5.57) with all other cohomology elements vanish. This is easily calculated using toric techniques and reflects geometrical properties of \tilde{X} like for instance fibration structures. Likewise we calculate the quartic intersections, i.e. the intersection numbers, on a fourfold X_4 as

$$\mathcal{K}_{abcd}^0 = \int_{\tilde{X}_4} A_a^{(1)} \wedge A_b^{(1)} \wedge A_c^{(1)} \wedge A_d^{(1)} = C_{ab\alpha}^{0(1,1,2)} \eta_{(2)}^{\alpha\beta} C_{cd\beta}^{0(1,1,2)} \quad (5.58)$$

using the algebra relation (5.54). Thus, we see that the four-point couplings are bilinears in the triple intersections.

The classical intersections are extended using mirror symmetry to the quantum cohomological intersections²²

$$C(A_a^{(i)} A_b^{(j)} A_c^{(k)}) = C_{abc}^{0(i,j,k)} + \text{instanton corrections}, \quad (5.59)$$

where the instanton corrections due to holomorphic curves with meeting conditions on the homology cycles dual to the $A_k^{(p)}$. Again the correlators (5.59) vanish unless $i + j + k = n$. Note that the $C(A_a^{(i)} A_b^{(j)} A_c^{(k)})$ depend via the instantons on the complexified Kähler parameters of \tilde{X} , while $\eta_{ab}^{(k)} = \int_{\tilde{X}} A_a^{(k)} \wedge A_b^{(n-k)}$ is still purely topological. There are no instanton corrections present because in the moduli space for the two-pointed sphere not all zero modes are saturated due to the one remaining conformal Killing field on the sphere.

²¹See [95] for an explicit inductive proof.

²²We denote both the operators of the classical algebra and of quantum cohomology algebra by $A_k^{(p)}$.

5.3.3 Matching of the A-model and B-model Frobenius Algebra

We now describe the matching of the B-model Frobenius algebra with the A model quantum Frobenius algebra. At the large radius point of the Kähler structure, the correlation functions of the classical A-model can be calculated using toric intersection theory. We will match this information with the leading logarithmic behavior of the periods at the corresponding point in the complex structure moduli space, the point of large complex structure, which is characterized by its maximal logarithmic degeneration, which leads to a maximal unipotent monodromy.

Let us now discuss the Picard-Fuchs differential operators associated to the mirror Calabi-Yau X at the large complex structure point. In general the Picard-Fuchs system can be obtained also in the fourfold case by applying the method of reduction of pole order to the residue integral for Ω_4 , that reads

$$\Omega_4 = \int_{\epsilon_1} \cdots \int_{\epsilon_s} \prod_{k=1}^s \frac{a_0^{(k)}}{P_k} \Delta, \quad (5.60)$$

for Calabi-Yau fourfold hypersurfaces and complete intersections $\{P_1 = \dots = P_s = 0\}$ with $dP_1 \wedge \dots \wedge dP_s \neq 0$ in a toric variety \mathbb{P}_Δ of dimensions $s + 4$. Here ϵ_i are paths in \mathbb{P}_Δ , which encircle $P_i = 0$ and Δ is a measure invariant under the torus actions in \mathbb{P}_Δ . The parameter $a_0^{(k)}$ denotes a distinguished coefficient in the defining constraint P_k , cf. section 5.2.4 for the analogous Calabi-Yau threefold expression. Since we are dealing throughout our paper with mirror pairs (\tilde{X}, X) that are given torically, the derivation of the Picard-Fuchs operators simplifies significantly, since they are completely encoded by the toric data. To the Mori cone generators $\ell^{(a)}$ on the A-model side one associates the canonical GKZ-system of differential operators \mathcal{L}_a on the B-model side given in (5.34), where the derivative is taken with respect to the coefficients a_i of monomials in the constraint $f = 0$ defining X . By the methods described in [233] we obtain linear Picard-Fuchs operators $\mathcal{D}_a(\underline{\theta}, \underline{z})$, which are written in terms of the logarithmic derivatives $\theta_a = z^a \frac{\partial}{\partial z^a}$ with respect to the algebraic complex variables z_a defined by (5.36) that vanish at the large complex structure point.

We extract the leading term in θ_a of the differential operators as the formal limit $\mathcal{D}_i^{\text{lim}}(\underline{\theta}) = \lim_{\underline{z} \rightarrow 0} \mathcal{D}_i(\underline{\theta}, \underline{z})$, $i = 1, \dots, r$ and consider the algebraic ring

$$\mathcal{R} = \mathbb{C}[\underline{\theta}] / (\mathcal{J} = \{\mathcal{D}_1^{\text{lim}}, \dots, \mathcal{D}_r^{\text{lim}}\}). \quad (5.61)$$

This ring is graded by the degree k in θ and we denote the ring at grade k by $\mathcal{R}^{(k)}$ which is generated a number $h_{(n-k,k)}^H(X) = h_{(k,k)}^V(\tilde{X})$ of elements for $k = 0, \dots, n$. We note that for $k = 0, 1, n-1, n$ there is no difference between the vertical (horizontal) homology and the full homology groups. Let us explain in more detail how this ring connects the A- and the B-model structure at large radius:

(A) The construction of the ring \mathcal{R} is up to a normalization equivalent to the calculation of the intersection numbers of the A-model. In particular the n -point intersections appear as coefficients of the up to a normalization unique ring element of maximal degree, $\mathcal{R}^{(n)} = \sum_{i_1 \leq \dots \leq i_n} C_{i_1, \dots, i_n}^0 \theta_{i_1} \dots \theta_{i_n}$ and similar the $\mathcal{R}^{(k)}$ are generated by $\mathcal{R}_\alpha^{(k)} = \sum_{i_1 \leq \dots \leq i_k} a_\alpha^{i_1, \dots, i_k} \theta_{i_1} \dots \theta_{i_k}$, $\alpha = 1, \dots, h_{(k,k)}^V(\tilde{X})$, where the coefficients obey $a_\alpha^{i_1, \dots, i_k} \kappa_{i_1, \dots, i_k, j_1, \dots, j_{n-k}} = C_{\alpha, j_1, \dots, j_{n-k}}^0$, cf. [94].

(B) The ring $\mathcal{R}^{(k)}$ is in one-to-one correspondence to the solutions to the Picard-Fuchs equations at large radius. As discussed before the solutions are characterized by their monodromies around this point, i.e. they are graded by their leading logarithmic structure. To a given ring element $\mathcal{R}_a^{(k)} = \sum_{|\underline{\alpha}|=k} \frac{1}{(2\pi i)^k} m_{a, \underline{\alpha}} \theta_1^{\alpha_1} \dots \theta_h^{\alpha_h}$ in $\mathcal{R}^{(k)}$ we associate a solution of the form

$$\tilde{\Pi}_a^{(k)} = X_0(\underline{z}) \left[\mathbb{L}_a^{(k)} + \mathcal{O}(\log(\underline{z})^{|\alpha|-1}) \right], \quad (5.62)$$

with leading logarithmic piece of order k ,

$$\mathbb{L}_a^{(k)} = \sum_{|\underline{\alpha}|=k} \frac{1}{(2\pi i)^k} \tilde{m}_{a, \underline{\alpha}} \log^{\alpha_1}(z^1) \dots \log^{\alpha_h}(z^h), \quad (5.63)$$

by assigning $\tilde{m}_{\underline{\alpha}}^a(\prod_i \alpha_i!) = m_{\underline{\alpha}}^a$. In particular we map the unique element 1 of $\mathcal{R}^{(0)}$ to the unique holomorphic solution $X_0(\underline{z}) = 1 + \mathcal{O}(\underline{z})$. The above map follows from the fact that all $\mathcal{D}_i^{\text{lim}}$ in the ideal \mathcal{J} annihilate the leading logarithmic term $\mathbb{L}_a^{(k)}$ for $\tilde{\Pi}_a^{(k)}$ to be a solution. But this is just the same conditions as for the element $\mathcal{R}^{(k)}$ to be normal to \mathcal{J} in (5.61).

The two facts (A) and (B) imply mirror symmetry at the level of the classical couplings. Furthermore, mirror symmetry can be proven for toric hypersurfaces by matching the toric intersection calculation with the toric derivation of the Picard-Fuchs operators as it was argued in the threefold case in [252] and in [93–95] in the fourfold case. The identification

$$\theta_i \leftrightarrow J_i \quad (5.64)$$

provides a map between $\mathcal{R}_a^{(k)}$ and the classical A-model operators $A_a^{(k)}$ defined in (5.55). This provides also the matching of the A- and B-model Frobenius algebra at the large radius limit by identifying the periods of Ω_n and the solutions of the Picard-Fuchs system as follows.

To a given element $\mathcal{R}_a^{(k)}$ we can associate an A-model operator $A_a^{(k)}$ by the replacement (5.64). Similarly we can construct the dual B-model operators $\beta_a^{(k)}$ in F^k by applying the elements of the ring $\mathcal{R}^{(k)}$ as differentials in the coordinates \underline{z} to the holomorphic form Ω_n . We obtain the mirror map

$$A_a^{(k)} \longmapsto \beta_a^{(k)}|_{\underline{z}=0} = \mathcal{R}_a^{(k)} \Omega_n|_{\underline{z}=0}. \quad (5.65)$$

This determines the Frobenius algebra of the B-model completely as we show now. However, to relate the two- and three-point functions to the period integrals of the $\beta_a^{(k)}$ along the lines of section 5.3.1 we have to specify the topological basis $\gamma_a^{(k)}$, which simultaneously fixes the structure of the periods $\Pi_a^{(k,l)b}$, in terms of the A-model operator $A_a^{(k)}$ as well.

First we select a basis of solutions denoted $\Pi^{(k)a}(\underline{z})$ of the Picard-Fuchs system with leading logarithm $\mathbb{L}^{(k)a}$ that is dual to the $A_a^{(k)}$ at large radius as

$$\Pi^{(k)a}(\underline{z}) = \mathbb{L}^{(k)b}(\underline{z}) + \mathcal{O}((\log(\underline{z}))^{k-1}), \quad \mathcal{R}_a^{(k)} \mathbb{L}^{(k)b} = \delta_a^b, \quad (5.66)$$

in the limit $\underline{z} \rightarrow 0$ [94]. Then, the basis $\gamma_a^{(k)}$ is fixed by identifying the periods in the expansion (5.48) of $\Omega_n = \Pi^{(k)a}(\underline{z}) \hat{\gamma}_a^{(k)}$ with the solutions as $\Pi_1^{(k,0)a} \equiv \Pi^{(k)a}(\underline{z})$ that associates the $\mathbb{L}^{(k)a}$ to the $\hat{\gamma}_a^{(k)}$. With these definitions the requirements (5.47) on the upper triangular form of the basis $\beta_a^{(k)}$ are trivially fulfilled since $\beta_b^{(q)} = \mathcal{R}_b^{(q)} \Omega_n = \mathcal{R}_b^{(q)} (\Pi^{(q)a} \hat{\gamma}_a^{(q)} + \dots) \rightarrow \hat{\gamma}_b^q + \dots$ where the dots indicate forms $\hat{\gamma}_a^{(k)}$ at grade $k > q$ with higher logarithms. For concrete calculations it is convenient to present the solution to (5.66) [94],

$$\mathbb{L}^{(k)b} = \sum_{\underline{a}} \mathbb{L}_{a_1 \dots a_k}^{(k)b} l_{a_1} \cdots l_{a_k}, \quad \mathbb{L}_{a_1 \dots a_k}^{(k)b} = \mathcal{K}_{a_1 \dots a_k b_{k+1} \dots b_n} \hat{a}_b^{b_{k+1} \dots b_n} \quad (5.67)$$

where we used the abbreviation $\log(z^{a_i}) = l_{a_i}$ and introduced the operator that is Poincare dual to $A_c^{(k)}$, $\hat{A}_c^{(n-k)} = \hat{a}_c^{b_1 \dots b_{n-k}} J_{b_1} \cdots J_{b_{n-k}}$ obeying $\int_{\tilde{X}} A_a^{(k)} \wedge \hat{A}_c^{(n-k)} = \delta_{ac}$. In other words, the solution $\mathbb{L}^{(k)b}$ is the Poincare dual to $A_b^{(k)}$ under the map $\log(z^{a_i}) \mapsto J_{a_i}$. For Calabi-Yau fourfolds we express (5.67) on the grade two operators $k = 2$, using the definition of the operator algebra relation (5.54) in terms of the three-point function $C_{ab\alpha}^{(1,1,2)}$, as

$$\mathbb{L}_{a_1 a_2}^{(2)\alpha} = \mathcal{K}_{a_1 a_2 a_3 a_4} \hat{a}_\alpha^{a_3 a_4} = \int_{\tilde{X}_4} J_{a_1} J_{a_2} J_{a_3} J_{a_4} = C_{a_1 a_2 \beta}^{0(1,1,2)} \eta_{(2)}^{\beta\gamma} \int_{\tilde{X}_4} A_\gamma^{(2)} \wedge \hat{A}_\alpha^{(2)} = C_{a_1 a_2 \beta}^{0(1,1,2)} \eta_{(2)}^{\beta\alpha}. \quad (5.68)$$

Thus, the coefficients of $\mathbb{L}^{(2)\alpha}$ are given by the classical intersections (5.56) and the topological metric (5.57) of the A-model, which is essential for the matching of intersections of \tilde{X}_4 at large complex structure of X_4 . We will also make extensive use of this relation in our explicit examples in chapter 6.

After this technical preparations we present explicitly that the matching (5.65) of the A- and B-model operators as implies of the complete Frobenius algebra. This is checked easily by matching by the fundamental coupling $C_{abc}^{(1,1,n-2)}$ on both sides, which determines, as mentioned above, all other correlators. We restrict our attention to the fourfold case for which we evaluate (5.50) as

$$C_{ab\gamma}^{(1,1,2)} = \partial_{t^a} \Pi_b^{(2,1)\delta} \eta_{\delta\gamma}^{(2)} = \partial_{t^a} \partial_{t^b} \Pi^{(2)\delta} \eta_{\delta\gamma}^{(2)} =: \partial_{t^a} \partial_{t^b} F^{g=0}(\gamma), \quad (5.69)$$

where $a, b = 1, \dots, h^{(3,1)}$ and $\gamma = 1, \dots, h_{(2,2)}^H$. Here we used the upper triangular form (5.45) of $\beta_a^{(k)}$ and the intersection properties (5.44) of the $\hat{\gamma}_i$ for the first equality. Then we replaced $\partial_{t^a} \beta^{(0)} = \beta_a^{(1)}$ for general dimension n as follows from (5.52) and used the property (F2) in

flat coordinates. If we now let $\underline{z} \rightarrow 0$ and use the flat coordinates (5.49), which are given by (5.62) as $t^i \sim \log(z^i) + \text{hol.} \rightarrow \log(z^i)$, we see using (5.68) that in this limit the classical intersection $C_{ab\gamma}^{0(1,1,2)}$ of (5.56) are reproduced. Once the matching is established in this large radius limit we can define the full quantum cohomological Frobenius structure by (5.69) in the coordinates (5.49). The latter can then be viewed as the classical topological intersections deformed by instanton corrections.

For the Calabi-Yau fourfold case at hand the intersections are obtained from the second derivative of the holomorphic quantities $F^0(\gamma)$ introduced in (5.69) for each basis element $\beta_\gamma^{(2)}$, $\gamma = 1, \dots, h_H^{(2,2)}(X_4)$. These are the analogues of the holomorphic prepotential F^0 familiar from the threefold case and they are obtained as the special case $k = 1$ from the general generating functionals²³ (5.88) in section 5.4. The relation (5.69) tells us that we obtain the $F^0(\gamma)$ simply from the Picard-Fuchs equation as double-logarithmic solutions, that we will identify below.

However as mentioned above the identification using the ring structure fixes the solutions of the Picard-Fuchs system so far only up to normalizations. The normalization of the unique holomorphic solution is determined by the fact that the leading term in $X_0(\underline{z}) = 1 + \mathcal{O}(\underline{z})$ has to be one. Also the dual period can be uniquely normalized by the classical n -point intersections. The single logarithmic solutions are normalized to reproduce the effect of a shift of the NS-background B-field on $t^i = \int_{\mathcal{C}} (B + iJ)$, where \mathcal{C} is a generator of $H_2(\tilde{X}_4, \mathbb{Z})$ and J is the Kähler form. The shift is then $t^i \rightarrow t^i + 1$ which corresponds to the monodromy around $z_i = 0$ and implies according to (5.49) that $\tilde{m}_{\alpha\underline{a}} = 1$ for $|\alpha| = 1$ in (5.62). All other t -dependent quantities are restricted further by the monodromy of the period vector Π of the holomorphic $(4, 0)$ -form, $\Pi = (\Pi^{(0)}, \Pi^{(1)*}, \Pi^{(2)*}, \Pi^{(3)*}, \Pi^{(4)})^T$ around $\underline{z} = 0$. Let Σ be the matrix representing the intersection $K = \int_{X_4} \Omega_4 \wedge \bar{\Omega}_4 = \Pi \Sigma \Pi^\dagger$. Using (5.48), (5.44) it is easy to see that the anti-diagonal of Σ is given by the blocks $(1, (\eta^{(1)})^T, \eta^{(2)}, \eta^{(1)}, 1)$. The monodromies act by $\Pi \rightarrow M_i \Pi$, where M_i is a $(h_4^H \times h_4^H)$ matrix. The monodromy invariance of K and $\hat{\gamma}_a^{(p)} \in H_H^4(X_4, \mathbb{Z})$ implies

$$M_i^T \Sigma M_i = \Sigma \tag{5.70}$$

with M_i an integer matrix. However the monodromy at $\underline{z} \rightarrow 0$ is not enough to fix the integral basis completely and the monodromy at other points in the moduli space and analytic continuation have to be applied to fix undetermined constants in the solutions to the Picard-Fuchs system. This analysis is performed in the next section.

We conclude with some remarks about the enumerative geometry of the prepotentials $F^0(\gamma)$ of (5.69). As in the threefold case there is an enumerative geometry or counting interpretation of mirror symmetry in higher dimensions for the A-model [97]. The results and formulas necessary for our analysis are summarized in section 5.4 to which we refer at several points. As will be discussed there, the label γ is associated to a flux choice that is necessary to

²³We note that the terms $b_{a\gamma}^0, a_\gamma^0$ are irrelevant for the quantum cohomology, but important for the large radius limit of the superpotential (4.37).

reduce to a counting of curves with the prepotential $F(\gamma)$ as a generating function, cf. (5.88). The prepotentials furthermore have a Li_2 -structure and it is possible to calculate the genus zero BPS invariants $n_\beta^0(\gamma)$ of section 5.4 for a suitable basis of $H_V^{(2,2)}(\tilde{X}_4)$ and class β in $H_2(\tilde{X}, \mathbb{Z})$. This is in particular part of the calculations performed in chapter 6.

5.3.4 Fourfold Conifold: Monodromies and Classical Terms

In general the analysis of the global properties of the complex structure moduli space of X_4 is tedious since it requires analytic continuation of the periods which is numerically challenging. In general one expects due to the flatness of the Hodge bundle only monodromy effects that occur at the singular points in the moduli space. Upon demanding integrality of the corresponding monodromy matrix M_i around these points the integral basis is fixed. In this section we explicitly present such an analytic continuation to the universal conifold of the one modulus example of the sextic Calabi-Yau fourfold.

Before delving into these technicalities of this analysis we note that useful information about some of the irrational constants that appear for example in the leading logarithmic solution follow already from the Frobenius method [252, 253]. By this method the leading logarithmic solution can be obtained by applying the operator $D^{(4)} = \frac{1}{4!(2\pi i)^4} \mathcal{K}_{i_1 i_2 i_3 i_4} \partial_{\rho_{i_1}} \partial_{\rho_{i_2}} \partial_{\rho_{i_3}} \partial_{\rho_{i_4}}$ on the fundamental solution

$$\omega_0(\underline{z}, \underline{\rho}) = \sum_{\underline{n}} c(\underline{n}, \underline{\rho}) z^{\underline{n} + \underline{\rho}} \quad (5.71)$$

with

$$c(\underline{n}, \underline{\rho}) = \frac{\Gamma(-\sum_{\alpha} l_0^{(\alpha)} (n_{\alpha} + \rho_{\alpha}) + 1)}{\prod_i \Gamma(\sum_{\alpha} l_i^{(\alpha)} (n_{\alpha} + \rho_{\alpha}) + 1)} \quad (5.72)$$

and setting $\underline{\rho} = 0$. The general leading logarithmic solution, i.e. with all possible admixtures of lower logarithmic solutions, for $X^0 := \omega_0(\underline{z})|_{\underline{\rho}=0}$ reads

$$\Pi^{(4)} = X^0 \left(\frac{1}{4!} \mathcal{K}_{ijkl} t^i t^j t^k t^l + \frac{1}{3!} a_{ijk} t^i t^j t^k + \frac{1}{2!} a_{ij} t^i t^j + a_i t^i + a_0 \right), \quad (5.73)$$

where as in the threefold case \mathcal{K}_{ijkl} is the classical top intersection. It was observed in [252] for the threefold case that the Frobenius method reproduces some of the topological constants (5.38) in the leading and subleading logarithmic terms of F^0 . In particular we deduce the relations $\int_{\tilde{Z}_3} c_2 \wedge J_i = \frac{3}{\pi^2} \mathcal{K}_{ijk} \partial_{\rho_j} \partial_{\rho_k} c(\underline{0}, \underline{\rho})|_{\underline{\rho}=0}$ and $\int_{\tilde{Z}_3} c_3 = \frac{1}{3! \zeta(3)} \mathcal{K}_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} c(\underline{0}, \underline{\rho})|_{\underline{\rho}=0}$, where we denote by c_i the i -th Chern class of the Calabi-Yau threefold Z_3 , cf. (5.39). If we generalize these to fourfolds, we obtain

$$\int_{\tilde{X}_4} \frac{3}{4} c_2^2 + c_4 = \frac{1}{4! \zeta(4)} \mathcal{K}_{ijkl} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} \partial_{\rho_l} c(\underline{0}, \underline{\rho})|_{\underline{\rho}=0}. \quad (5.74)$$

These constants are expected to appear as coefficients of the subleading logarithms in (5.73).

Similar as in the threefold case one can also use the induced K-theory charge formula [254, 255] in combination with central charge formula

$$\vec{Q} \cdot \vec{\Pi} = - \int_{\tilde{X}_4} e^{-J} \text{ch}(A) \sqrt{\text{td}(\tilde{X}_4)} = Z(A) \quad (5.75)$$

for A denoting the bundle on the brane wrapping \tilde{X}_4 and mirror symmetry to obtain information about the subleading logarithmic terms in the periods.

Let us apply, however, a more direct argument and use properties of the simplest Calabi-Yau fourfold, the sextic in \mathbb{P}^5 . The mirror sextic has the Picard-Fuchs equation, see e.g. [97],

$$\theta^5 - 6z \prod_{k=1}^5 (6\theta + k) . \quad (5.76)$$

We can easily construct solutions at $z = 0$ using the Frobenius method, but let us first give a different basis of logarithmic solutions namely

$$\hat{\Pi}_k = \frac{1}{(2\pi i)^k} \sum_{l=0}^k \binom{k}{l} \log(z)^l s_{k-l}(z) , \quad (5.77)$$

where

$$\begin{aligned} X^0 &= s_0 = 1 + 720z + 748440z^2 + \dots , & s_1 &= 6246z + 7199442z^2 + \dots , \\ s_2 &= 20160z + 327001536z^2 + \dots , & s_3 &= -60480z - 111585600z^2 + \dots \\ s_4 &= -2734663680z^2 - 57797926824960z^3 + \dots . \end{aligned} \quad (5.78)$$

The point is that under the mirror map one obtains $\hat{\Pi}_k = t^k + \mathcal{O}(q)$, so that these solutions correspond to the leading volume term of branes of real dimension $2k$. The ‘‘conifold’’ locus of the sextic is at $\Delta = 1 - 6^6 z = 0$. Near that point the Picard-Fuchs equation has the indicials $(0, 1, 2, 3, \frac{3}{2})$. It is easy to construct solutions and we choose a basis in which the solution to indicial $k \in \mathbb{Z}$ has the next power z^4 . The only unique solution possesses a branch cut and reads

$$\nu = \Delta^{\frac{3}{2}} + \frac{17}{18} \Delta^{\frac{5}{2}} + \frac{551}{648} \Delta^{\frac{7}{2}} + \dots . \quad (5.79)$$

The situation at the universal conifold is crucial for mirror symmetry in various dimensions n . At this point the non-trivial monodromy affects a cycle of topology \mathbb{T}^n that corresponds to the solution X^0 , i.e. the zero-dimensional brane in the A -model, which is uniquely defined at $z = 0$, and a cycle of topology \mathbb{S}^n that corresponds to the solution $\Pi^{(n)}$, i.e. the top dimensional brane in the A -model. The topological intersection between these cycles is 1 and their classes in homology are the fiber and the base of the SYZ-fibration respectively [88]. In odd dimensional Calabi-Yau manifolds, like threefolds Z_3 , the conifold monodromy acts on the vector $\Pi_{\text{red}} = (\Pi^{(3)}, X_0)^T$ by a matrix as

$$M_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} . \quad (5.80)$$

This corresponds to the Lefschetz formula with vanishing $\Pi^{(3)}$, i.e. the quantum volume of \tilde{Z}_3 vanishes.

In four dimensions we have a monodromy of order two and the only way to obtain an integral idempotent monodromy compatible with the intersection (5.70) is

$$M_{2 \times 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5.81)$$

It is noticeable that the zero- and the highest dimensional brane get exchanged by the conifold monodromy in even dimensions. This implies the identification $X^0 = \eta - c\nu$ and²⁴ $\Pi^{(4)} = \eta + c\nu$. Here η is a combination of solutions at $\Delta = 0$ without branch cut. We can determine the latter by analytic continuation of X^0 to the conifold. While the precise combination is easily obtained, the only constant that matters below is c , which turns out to be $c = \frac{1}{\sqrt{3}\pi^2}$.

Now we can determine the combination which corresponds to the correct integral choice of the geometric period $\Pi^{(4)}$ as

$$\Pi^{(4)} = 2c\nu + X^0 \quad (5.82)$$

from the uniquely defined periods (X^0, ν) at $z = 0$ and $\Delta = 0$. The analytic continuation of ν to $z = 0$ fits nicely with our expectation from above and fixes most of the numerical coefficients in (5.73) universally. We obtain

$$a_0 = \frac{\zeta(4)}{2^4(2\pi i)^6} \int_{\tilde{X}_4} 5c_2^2, \quad (5.83)$$

and

$$a_i = -\frac{\zeta(3)}{(2\pi i)^3} \int_{\tilde{X}_4} c_3 \wedge J_i, \quad a_{ij} = \frac{\zeta(2)}{2(2\pi i)^2} \int_{\tilde{X}_4} c_2 \wedge J_i \wedge J_j, \quad a_{ijk} = \tilde{c} \int_{\tilde{X}_4} \iota_*(c_1(J_i)) \wedge J_j \wedge J_k, \quad (5.84)$$

where as before $c_i = c_i(T_{\tilde{X}_4})$ and $c_1(J_i)$ denotes the first Chern class of the divisor associated to J_i which is mapped to a four-form via the Gysin homomorphism ι_* of the embedding map of this divisor into \tilde{X}_4 . This is the generalization of (5.39) to the case of Calabi-Yau fourfolds. To be precise, the coefficients a_{ijk} are not fixed by the sextic example, because it turns out to be zero in this case, and for the canonical choice of $\Pi^{(4)}$ (5.82). This does not mean that it is absent in general. Rather it implies that it is physically irrelevant for the sextic because the divisibility of the correctly normalized solution, which is cubic in the logarithms, allows an integral symplectic choice of the periods in which this term can be set to zero. This might not be in general the case and other hypersurface in weighted projective space indicate that $\tilde{c} = 1$. It is similarly possible to use the orbifold monodromy to fix the exact integral choice of the other periods. The principal form of the terms should again follow from the Frobenius method.

²⁴The sign is chosen so that the t^4 term in $\Pi^{(4)}$ has a positive sign.

5.4 Basics of Enumerative Geometry

In this section we describe the relevant enumerative quantities for the A-model, which are calculated in this work in the B-model using mirror symmetry. We start with a discussion of the genus zero and genus one closed string Gromov-Witten invariants in section 5.4.1 before we review the open string Gromov-Witten invariants in section 5.4.2. Then we apply the definitions and structures of sections 5.4.1 and 5.4.2 to the Calabi-Yau threefold superpotentials W_{flux} and W_{brane} in section 5.4.3. Similarly we proceed with the flux superpotential on fourfolds in section 5.4.4. It will be this enumerative meaning of the B-model calculations for the A-model that will provide important cross-checks for the computations of the superpotentials in chapter 6.

5.4.1 Closed Gromov-Witten Invariants

First we review the theory of closed Gromov-Witten invariants, i.e. the theory of holomorphic maps

$$\phi : \Sigma_g \rightarrow \tilde{X} \tag{5.85}$$

from an oriented closed curve Σ_g into a Calabi-Yau manifold \tilde{X} , where we do not consider marked points for simplicity. It can be defined mathematically rigorously in general and explicitly calculated using localization techniques if \tilde{X} is represented for example by a hypersurface in a toric variety. Here g is the genus of the domain curve Σ_g and we denote by $\beta \in H_2(\tilde{X}, \mathbb{Z})$ the homology class of the image curve. The multi-degrees of the latter are measured with respect to an ample polarization L of \tilde{X} , i.e. $\beta = \sum_{i=1}^{h(1,1)} d_i \beta_i$ for $\deg(\beta) = \int_{\beta} c_1(L) = \sum_{i=1}^{h(1,1)} d_i t_i$ with $d_i \in \mathbb{Z}_+$. In string theory and in the context of the mirror symmetry the volume of the curve β_i is complexified by an integral over the antisymmetric two-form field B . Thus, one defines the complexified closed Kähler moduli $t^i = \int_{\beta_i} (B + i c_1(L))$.

For smooth \tilde{X} the virtual (complex) dimension of the moduli space of these maps ϕ is computed by an index theorem and reads

$$\text{vir dim } \mathcal{M}_g(\tilde{X}, \beta) = \int_{\beta} c_1(\tilde{X}) + (\dim \tilde{X} - 3)(1 - g) . \tag{5.86}$$

In particular for Calabi-Yau fourfolds one obtains $\text{vir dim } \mathcal{M}_g(\tilde{X}_4, \beta) = 1 - g$. Thus in order to define genus zero Gromov-Witten invariants one requires an incidence relation of the image curve with $k = (\dim(\tilde{X}) - 3)$ surfaces to reduce the dimension to zero in order to arrive at a well-defined counting problem. For fourfolds one thus needs one incidence surface and we denote the dual cycle of the surface by $\gamma \in H^{(2,2)}(\tilde{X}_4)$. Note that for $\dim \tilde{X} \geq 4$ and $g \geq 2$ the dimension of the moduli space is negative and no holomorphic maps exist. The Calabi-Yau threefolds are critical in the sense that the dimension of the moduli space for all genera is zero. Thus, in general invariants associated to the maps are non-zero for all values of g .

We define a generating function for each genus g Gromov-Witten invariant as follows:

$$F^g(\gamma_1) = \sum_{\beta \in H_2(\tilde{X}, \mathbb{Z})} r_\beta^g(\gamma_1, \dots, \gamma_k) q^\beta . \quad (5.87)$$

They are labeled by g , β and for $\dim \tilde{X} \geq 4$ also by cycles γ_i dual to the incidence surfaces. Here q^β is a shorthand notation for $q^\beta = \prod_{i=1}^{h^{(1,1)}} e^{2\pi i t_i d_i}$. We note that this is not just a formal power series²⁵, but rather has finite region of convergence for large volumes of the curves β_i , i.e. for $\text{Im}(t^i) \gg 0$. This puts a bound on the growth of the Gromov-Witten invariants $r_\beta^g(\gamma_1)$. The contributions of the maps is divided by their automorphism groups and the associated Gromov-Witten invariants $r_\beta^g(\gamma_1, \dots, \gamma_k)$ are in general rational.

Although the discussion of (5.86) indicates that the Gromov-Witten theory on higher dimensional Calabi-Yau manifolds is less rich than in the threefold case, one has a remarkable integrality structure associated to the invariants. In particular at genus zero one can define integer invariants $n_\beta^g(\gamma_1, \dots, \gamma_k) \in \mathbb{Z}$ for arbitrary $\dim(\tilde{X}) = k + 3$ dimensional manifolds as

$$F^0(\gamma_1, \dots, \gamma_k) = \frac{1}{2} C_{ab\gamma_1 \dots \gamma_k}^{0(1,1,n-2)} t^a t^b + b_{a\gamma_1 \dots \gamma_k}^0 t^a + a_{\gamma_1 \dots \gamma_k}^0 + \sum_{\beta > 0} n_\beta^g(\gamma_1, \dots, \gamma_k) \text{Li}_{3-k}(q^\beta) , \quad (5.88)$$

where $\text{Li}_p(q) = \sum_{d=1}^{\infty} \frac{q^d}{d^p}$ and $C_{ab\gamma_1 \dots \gamma_k}^{0(1,1,n-2)}$ are the classical triple intersections. For threefolds an analogous formula, see (5.90) below, was found in [35] and the multicovering was explained in [256]. Note that $b_{a\gamma_1 \dots \gamma_k}^0$, $a_{\gamma_1 \dots \gamma_k}^0$ are irrelevant for the quantum cohomology, as the latter is defined by the second derivative of $F^0(\gamma_1, \dots, \gamma_k)$, cf. (5.69).

Genus one Gromov-Witten invariants exist on Calabi-Yau manifold of all dimensions with the need of incidence conditions as discussed above. For fourfolds the authors of [97] define the following integrality condition

$$F^1 = \sum_{\beta > 0} n_\beta^1 \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta > 0} n_\beta^0(c_2(\tilde{X}_4)) \log(1 - q^\beta) - \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1 + \beta_2}) . \quad (5.89)$$

Here the m_{β_1, β_2} are so called meeting invariants, which are likewise integers as the $n_\beta^g(\cdot)$ and the function σ is defined by $\sigma(d) = \sum_{i|d} i$.

For Calabi-Yau threefolds there is also the general counting formula for BPS-states that is obtained by evaluating a one-loop Schwinger computation for M2-branes in the M-theory lift of Type IIA [257]

$$F(\lambda, q) = \sum_{g=0}^{\infty} \lambda^{2g-2} F^g = \sum_{k=1}^{\infty} \sum_{\beta > 0, g \geq 0} n_\beta^g \frac{1}{k} \left(2 \sin \frac{k\lambda}{2}\right)^{2g-2} q^{k\beta} . \quad (5.90)$$

Here g not only corresponds to genus of the curve wrapped by the M2-branes but also to the left spin of the M2-brane in the five-dimensional theory in a fixed basis of the representation of the little group for massive particles, $SO(4) \cong SU(2) \times SU(2)$.

²⁵This is important for the interpretation of such terms in the effective action. In fact, analyticity allows to define such terms beyond the large radius limit in terms of period integrals on the mirror geometry.

5.4.2 Open Gromov-Witten Invariants

Let us come now to the open Gromov-Witten invariants on Calabi-Yau threefolds \tilde{Z}_3 . They arise in the open topological A-model on \tilde{Z}_3 . We consider a Calabi-Yau threefold \tilde{Z}_3 together with a special Lagrangian submanifold L and consider a map from an oriented open Riemann surface, i.e. a Riemann surface with boundary,

$$\psi : \Sigma_{g,h} \rightarrow (\tilde{Z}_3, L) \quad (5.91)$$

into the Calabi-Yau threefold \tilde{Z}_3 . Here the Riemann surface is mapped with a given winding number into L such that the h boundary circles B_i of $\Sigma_{g,h}$ are mapped on non-trivial elements $\vec{\alpha} = (\alpha_1, \dots, \alpha_h) \in H^1(L, \mathbb{Z})^{\oplus h}$. As in the closed case we do not consider marked points. For threefolds the moduli space $\mathcal{M}_{g,h}(\tilde{Z}_3, L, \beta, \vec{\alpha}, \mu)$ with the Maslov index μ has virtual dimension zero [258]. If $H^1(L, \mathbb{Z})$ is non-trivial, the special Lagrangian has classical geometric deformation moduli. The open string moduli \underline{u} are complexifications of the geometric moduli by the Wilson-Loop integrals of the flat $U(1)$ gauge connection on the brane.

The open BPS-state counting formula analogous to (5.90) is derived in [98] by counting degeneracies of open M2-branes ending on an M5-brane wrapping L or of D4-branes wrapping L in the Type IIA picture. It reads

$$\begin{aligned} F(\underline{t}, \underline{u}) &= \sum_{g,h} \lambda^{2g-2+h} F_h^g(\underline{t}, \underline{u}) = \sum_{g=0, h=1}^{\infty} \sum_{\alpha_n^i=0}^{\infty} \lambda^{2g-2+h} F_{n_1, \dots, n_h}^{g, \alpha_n^i}(\underline{t}) \prod_{n=1}^h \text{tr} \bigotimes_{i=1}^{b^1(L)} U_i^{\alpha_n^i} \\ &= i \sum_{n=1}^{\infty} \sum_{\mathcal{R}} \sum_{\beta > 0, r \in \mathbb{Z}/2} \frac{n_{\beta, \mathcal{R}}^g}{2n \sin\left(\frac{n\lambda}{2}\right)} q_\lambda^{nr} q^{n\beta} \text{Tr}_{\mathcal{R}} \bigotimes_{i=1}^{b^1(L)} U_i^n. \end{aligned} \quad (5.92)$$

Here $\alpha^i = (\alpha_1^i, \dots, \alpha_h^i) \in H^1(L, \mathbb{Z})^{\oplus h}$ and $q_\lambda = e^{2\pi i \lambda}$. The instanton numbers counting BPS-particles of the M2-brane ending on the M5-branes in the representation \mathcal{R} and of spin g are given by the integers $n_{\beta, \mathcal{R}}^g$. The class β corresponds to the bulk charge as in the closed string formula. The holonomies of the gauge field on the D4-brane along non-trivial one-cycles in $H_1(L)$ are given by $\text{tr} U_i$, $i = 1, \dots, b_1(L)$, where the trace is taken in the fundamental representation. The integers α_n^i are the winding numbers of the n -th boundary along the i -th element of $H^1(L, \mathbb{Z})$. The open string moduli u enter the definition of the matrices U_i . In particular the disk amplitude, which gives rise to the superpotential, is given by

$$W_{\text{brane}} = F_{h=1}^0 = \sum_{\beta, m} n_\beta^m \text{Li}_2(q^\beta Q^m) \quad (5.93)$$

with $Q^m = e^{2\pi i m_i u^i}$. Comparison with (5.88) suggest that the counting problem of specific disks amplitudes can be mapped to the counting of rational curves in fourfolds since the integrality structure is the same given by the Li_2 -double-covering.

5.4.3 Closed and Open Superpotentials on Calabi-Yau Threefolds

Let us first discuss the flux superpotential W_{flux} as it occurs in Type IIB and heterotic compactifications, cf. sections 3.3.2 respectively 4.1.3, and its meaning for mirror symmetry. As noted before it takes in both setups the same form and can be evaluated in terms of the periods (X^K, \mathcal{F}_K) of the holomorphic three-form Ω as

$$W_{\text{flux}} = \hat{N}_K X^K(\underline{z}) - \hat{M}^K \mathcal{F}_K(\underline{z}), \quad X^K = \int_{A^K} \Omega, \quad \mathcal{F}_K = \int_{B^K} \Omega, \quad (5.94)$$

where $(\hat{M}_K, \hat{N}^K) = (M_K - \tau \tilde{M}_K, N^K - \tau \tilde{N}^K)$ are complex numbers in an $\mathcal{N} = 2$ Type IIB theory formed from the flux quantum numbers (M_K, N^K) of F_3 and $(\tilde{M}_K, \tilde{N}^K)$ of H_3 . Whereas in the $O3/O7$ -orientifold setup both F_3 and H_3 fluxes contribute to W_{flux} , in the $O5/O9$ -orientifold setup of section 3.3.2 there are only fluxes F_3 due to the orientifold projection \mathcal{O} . Similarly in the heterotic setup of 4.1.3 only NS-NS fluxes are present. As before we have introduced the symplectic basis (A^K, B^K) of three-cycles in $H_3(Z_3, \mathbb{Z})$.

As we have reviewed in section 5.2 the complete complex structure moduli dependence of W_{flux} is determined, once the flux is specified, from the periods obeying the Picard-Fuchs differential system. The use of the Picard-Fuchs system makes it even possible to evaluate the flux superpotential deep inside the complex structure moduli space, where conventional supergravity breaks down due to strong curvature effects for example from singularities in Z_3 like the conifold. Thus, W_{flux} in general inherits the characteristic properties of special geometry, in particular the existence of the prepotential. In other words, even in $\mathcal{N} = 1$ effective actions, the flux superpotential enjoys remnants of an underlying $\mathcal{N} = 2$ structure.

In the context of mirror symmetry and the enumerative interpretation of the A-model a few further general observations can be made. Although the concrete form of the flux superpotential W_{flux} , will highly depend on the point at which it is evaluated on the complex structure moduli space, we can make statements about its structure at particular points in the moduli space. One particularly distinguished point is the large complex structure point which by mirror symmetry corresponds to a large volume compactification of Type IIA string theory. As we have seen in section 5.2 the solutions at this point have a characteristic grading by powers of $\log(\underline{z})$. Mirror symmetry then maps the logarithmic terms to classical large-volume contributions on the Type IIA side while the regular terms in the periods $(\underline{X}, \underline{\mathcal{F}})$ encode the closed string world-sheet instantons corrections²⁶. Inserting the form of the prepotential (5.38) into the flux superpotential (5.94) one finds the characteristic structure

$$W_{\text{flux}} = X^0 \left[\hat{N}_0 + \hat{M}^0 \mathcal{K}_0 - \hat{M}^i \mathcal{K}_i + (\hat{N}_i - \hat{M}^j \mathcal{K}_{ij} + \hat{M}^0 \mathcal{K}_i) t^i - \frac{1}{2} \hat{M}^i \mathcal{K}_{ijk} t^j t^k + \frac{1}{3!} \hat{M}^0 \mathcal{K}_{ijk} t^i t^j t^k \right. \\ \left. + (\hat{M}^i - \hat{M}^0 t^i) \sum_{\beta} d_i n_{\beta}^0 \text{Li}_2(q^{\beta}) + 2 \hat{M}^0 \sum_{\beta} n_{\beta}^0 \text{Li}_3(q^{\beta}) \right]. \quad (5.95)$$

²⁶Recall that the mirror map takes the form $z^i = e^{2\pi i t^i} + \dots$, where $t^i = X^i/X^0$ on the IIB side which is identified with the world-sheet volume complexified with the NS-NS B-field on the Type IIA side.

This expression directly shows that in addition to a cubic polynomial of classical terms, also instanton correction terms proportional to $\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ for $k = 2, 3$ are induced by non-vanishing fluxes \hat{M}^i, \hat{M}^0 . Thus, we can directly read off the Gromov-Witten invariants n_{β}^0 of (5.90) from the flux superpotential W_{flux} for particular flux choices. Conversely, given a few invariants n_{β}^g , we can determine the flux numbers for any given superpotential W_{flux} or compare to superpotentials obtained from different setups, like e.g. Calabi-Yau fourfolds, or different string theories using string dualities. Indeed, this will be our strategy to explicitly relate Calabi-Yau fourfold superpotentials to dual Type IIB or heterotic superpotentials in sections 6.1 and 6.2.

Let us now turn to the superpotential for the open string sector that is given, for a curve $\Sigma_{\underline{u}}$, by the chain integral

$$W_{\text{brane}} = \int_{\Gamma(\underline{u})} \Omega(\underline{z}), \quad \partial\Gamma(\underline{u}) \supset \Sigma_{\underline{u}}, \quad (5.96)$$

which we encountered in the Type IIB context in section 3.3.2 as a D5-brane superpotential, in section 4.1.3 in the heterotic context as a small instanton/five-brane superpotential and in F-theory setups in section 4.2.3 as a seven-brane superpotential.

Ideally one would like to explicitly compute the functional dependence of W_{brane} on the brane deformations \underline{u} of the curve $\Sigma_{\underline{u}}$ and the complex structure moduli \underline{z} e.g. by evaluating, as in the closed string case, a set of *open-closed* Picard-Fuchs equations. Indeed, one way to achieve this is to lift the setup to an F-theory compactification on a Calabi-Yau fourfold, as we will demonstrate in section 6.1. Another, more direct and mathematically canonical procedure is discussed in part III of this work [60, 100]. There a constructive method is used to directly compute the superpotential W_{brane} for five-branes on a curve $\Sigma_{\underline{u}}$ on the complex geometry side by mapping the deformation problem of the curve $\Sigma_{\underline{u}}$ in Z_3 to the deformation problem of complex structures on a non-Calabi-Yau threefold \hat{Z}_3 , that is canonically obtained by blowing up Z_3 along the curve $\Sigma_{\underline{u}}$.

Before performing any calculations we infer some crucial properties of W_{brane} by applying mirror symmetry at the large complex structure/large volume point. Recall that under mirror symmetry, a Type IIB compactification with D5- or D7-branes is mapped to a Type IIA compactification with D6-branes wrapping special Lagrangian cycles L in the mirror Calabi-Yau space \tilde{Z}_3 , cf. section 5.1. Thus, in order for mirror symmetry with branes to hold [96] the superpotentials have to agree on both sides. However, on the A-model side the moduli of L are counted by elements in $H^1(L, \mathbb{Z})$ and are generically unobstructed [170]. In contrast, the deformations of the curve $\Sigma_{\underline{u}}$ are in general obstructed, which is a basic fact in classical geometry [259], and reflected in physics by the non-trivial superpotential W_{brane} [66, 67]. Consequently, the superpotential in the A-model has to be induced entirely by quantum corrections²⁷, which are string world-sheet discs ending on L . As was reviewed in detail in

²⁷In mathematical terms this equivalence can be formulated as a matching of classical obstruction theory of Σ on the B-model side with quantum obstructions L in the A-model.

section 5.4.2 this superpotential induced by the open string world-sheets reads

$$W_{\text{brane}} = C_i t^i \hat{t} + C_{ij} t^i t^j + C \hat{t}^2 + \sum_{\beta, m} n_{\beta}^m \text{Li}_2(q^{\beta} Q^m), \quad Q^m = e^{2\pi i m_i u^i}. \quad (5.97)$$

with constants C, C_i, C_{ij} and the Gromov-Witten invariants n_{β}^m determined by the brane geometry and \tilde{Z}_3 , as well as the flux F_2 in the seven-brane context.

5.4.4 Flux Superpotentials on Calabi-Yau Fourfolds

In this section we discuss the F-theory flux superpotential and recall how mirror symmetry for Calabi-Yau fourfolds [79, 93–95] allows to relate it to the enumerative geometry of the A-model. In this route we demonstrate some general features of the flux superpotential.

Recall, that the F-theory superpotential is induced by four-form flux G_4 and given by [14]

$$W_{G_4}(\underline{z}) = \sum_k N^{(k)a} \Pi^{(k)b}(\underline{z}) \eta_{ab}^{(k)} = \mathbf{N} \Sigma \Pi, \quad (5.98)$$

where \underline{z} collectively denote the $h^{(3,1)}(X_4)$ complex structure deformations of X_4 , $\Pi_a^{(k)}$ the periods and the integers $N^{(k)a}$ the flux quanta. Both periods and flux quanta are summarized in vectors \mathbf{N}, Π , where Σ denotes a $h_H^4 \times h_H^4$ -matrix containing the topological metric²⁸ $\eta_{ab}^{(k)}$ in (5.44). Here we further used the expansions into an integral basis $\hat{\gamma}_a^{(k)}$ of $H_H^4(X_4, \mathbb{Z})$ as

$$\Omega_4 = \sum_k \Pi^{(k)a} \hat{\gamma}^{(k)a}, \quad \Pi^{(k)a} = \int_{\gamma_a^{(k)}} \Omega_4, \quad G_4 = \sum_k N^{(k)a} \hat{\gamma}_a^{(k)}, \quad N^{(k)a} = \int_{\gamma_a^{(k)}} G_4. \quad (5.99)$$

We refer to section 5.3 and in particular (5.45) for more details on the notation.

In F-theory setups the flux G_4 is restricted by the two conditions (4.35) and (4.36). The latter condition implies that G_4 is an element in the primary horizontal subgroup

$$H^{(2,2)}(X_4) = H_V^{(2,2)}(X_4) \oplus H_H^{(2,2)}(X_4). \quad (5.100)$$

A corollary of this statement is that the Chern classes are in the vertical subspace, so that half integral flux quantum numbers are not allowed if condition (4.36) is met. In general, it is an important open problem to have a description of four-flux G_4 on a generic Calabi-Yau fourfold. Formally, this can be solved by mirror symmetry established via the map (5.65). This implies that one can think of the integral basis $\hat{\gamma}_a$ in terms of their corresponding differential operators $\mathcal{R}_a^{(k)}$ acting on Ω_4 . In particular, this formalism allows us to express the flux G_4 in an integral basis in the form

$$G_4 = \sum_{k=0}^4 \sum_{p_k} N^{p_k(k)} \mathcal{R}_{p_k}^{(k)} \Omega_4 \Big|_{\underline{z}=0}. \quad (5.101)$$

²⁸Recall that in contrast to $H^3(Z_3, \mathbb{Z})$ of Calabi-Yau threefolds the fourth cohomology group of X_4 does not carry a symplectic structure which necessitates the introduction of $\eta_{ab}^{(k)}$.

The representation of the integral basis as differential operators will be particularly useful in the identification of the heterotic and F-theory superpotential, cf. chapter 6 and section 8.4.2.

Once the flux G_4 is fixed, also in the fourfold case the flux superpotential (5.98) is completely determined in terms of the periods Π . As we have discussed in section 5.3.3 the periods Π can in principle be determined from the Picard-Fuchs differential system which allows for an analytic continuation of W_{G_4} deep into the complex structure moduli space of X_4 . However, it is the most important task on the B-model side to find the fourfold periods which are evaluated with respect to an integral basis of $H_4^H(X_4, \mathbb{Z})$. Moreover, an intrinsic definition of the integral basis $\gamma_a^{(k)}$ seems to be technically impossible, due to the absence of a symplectic basis as in the threefold case, and mirror symmetry and analytic continuation, like the monodromy analysis at the conifold in section 5.3.4, have to be used in order to construct an integral basis.

In the context of mirror symmetry it is meaningful to comment on the structure of (5.98) at distinguished points in the complex structure moduli space. Again the large complex structure/large volume point is of particular importance since an interpretation as classical volumes and quantum instantons on the A-model side is possible.

For a toric hypersurface X_4 the point of maximal unipotent monodromy is the origin in the Mori cone coordinate system \underline{z} introduced in section 5.2.4 as in the threefold case. Geometrically at the point $\underline{z} = 0$ several cycles $\gamma_a^{(k)}$ hierarchically vanish which is encoded in the grading of the solutions to the Picard-Fuchs system by powers in $(\log(z^i))^k$, $k = 0, 4$, see (5.63). According to the map (5.65) and the condition (5.66) there is one analytic solution $X^0(z) = \int_{\gamma_0} \Omega_4$ corresponding to the fundamental period, $h^{(3,1)}(X_4)$ logarithmic periods $X^a(z) = \int_{\gamma_a} \Omega_4 \sim X^0(z) \log(z_a)$, $h_H^{(2,2)}(X_4)$ double logarithmic solutions, $h^{(3,1)}(X_4)$ triple logarithms and one quartic logarithms. Noting that $t^a \sim \log(z^a)$ at this point we can use these flat coordinates to write the leading logarithmic structure of the period vector as

$$\Pi^T = \left(\int_{\gamma_0} \Omega_4, \dots, \int_{\gamma_{b_H^4}^{(4)}} \Omega_4 \right) \sim X^0 \left(1, t^a, \frac{1}{2} C_{ab}^0 \delta t^a t^b, \frac{1}{3!} C_{bcd}^0 t^b t^c t^d, \frac{1}{4!} C_{abcd}^0 t^a t^b t^c t^d \right). \quad (5.102)$$

Here we have introduced the constant coefficients $C_{ab}^{0\delta} := \eta^{(2)} \delta\gamma C_{ab\gamma}^{0(1,1,2)}$, $C_{abc}^{00} = \eta^{(1)ed} \mathcal{K}_{abcd}^0$ that are related to the classical three-point function $C_{ab\gamma}^{0(1,1,2)}$ and the intersection numbers \mathcal{K}_{abcd}^0 , cf. (5.68). These couplings can be calculated in the classical cohomology ring of \tilde{X}_4 in the basis (5.55) via the integrals (5.56) and (5.58). In particular, the grading $(\{k\}) = (0, 1, 2, 3, 4)$ in powers of t^a corresponds to a grading of $\gamma_a \in H_4(X_4)$ which matches the grading of the dual cohomology group $H_H^4(X_4, \mathbb{Z})$ in the fixed complex structure given by the point \underline{z} . We note that the periods (5.102) contain instanton corrections that we suppressed for convenience, that are however crucial for the A-model.

For applications of fourfolds for example to F-theory the instanton corrections in particular help to identify the physical meaning of the periods, like e.g. the interpretation in terms of the

flux or brane superpotential of the underlying Type IIB theory in the limit (4.41). Indeed the comparison of the enumerative data of the double logarithmic periods $F^0(\gamma)$ with the Ooguri-Vafa double-covering (5.97) will allow us in section 6.1 to identify periods corresponding to W_{brane} for specific flux choices G_4 .

Chapter 6

Constructions and Calculations in String Dualities

In this chapter we present explicit calculations of the effective superpotentials in F-theory and in heterotic/F-theory dual setups, where we mainly follow [79] and [81]. First in section 6.1 we explicitly calculate the F-theory flux superpotential for four-dimensional F-theory compactifications on elliptic Calabi-Yau fourfolds and extract the Type IIB flux and seven-brane superpotential along the lines of the discussion in section 4.2.3. Before delving into the details of the calculations we first present a general strategy to obtain elliptic Calabi-Yau fourfolds X_4 with a little number of complex structure moduli. This is necessary in order to work with a technically controllable complex structure moduli space of X_4 , which we will mainly study using Picard-Fuchs equations and cross-checks from mirror symmetry. We put particular emphasis on the toric realization of examples with few moduli and the toric means to analyze possible fibration structures. Then we start with the construction of a concrete Calabi-Yau fourfold, that we realize as a toric Calabi-Yau hypersurface. We focus on one main example and refer to appendix C.2 for two further examples. For this example we perform a detailed analysis of the seven-brane dynamics as encoded by the discriminant of the elliptic fibration, comment on the Calabi-Yau threefold geometry of its heterotic dual and finally calculate, using fourfold mirror symmetry, the Type IIB flux and seven-brane superpotential. This is technically achieved by specifying appropriate four-flux G_4 , that singles out linear combination of fourfold periods, that are then identified with the Type IIB Calabi-Yau threefold periods upon matching the classical terms and the instanton corrections on the fourfold with the Calabi-Yau threefold results. The Type IIB seven-brane superpotential is similarly identified by a matching of the brane disk instanton invariants with the fourfold instanton invariants.

Then in section 6.2 we compute the heterotic superpotential for a Calabi-Yau threefold compactification on Z_3 from its F-theory dual setup. Before dealing with concrete examples we comment on the general matching strategy of the F-theory flux superpotential with the heterotic superpotential. We emphasize the duality map for heterotic five-branes and their superpotential to the F-theory side. Then we study in detail two explicit examples of heterotic string compactifications. Here we essentially take the same threefold geometries that we used in section 6.1.2 in order to obtain Calabi-Yau fourfolds X_4 with a small number of complex

structure moduli. This surprising coincidence goes back to the rich fibration structure of X_4 and its mirror \tilde{X}_4 , which is, on the one hand, a consequence of the requirement of a little number of moduli but, on the other hand, also natural for heterotic/F-theory dual setups, see the schematic diagram (6.1) and [103].

The first example of a heterotic threefold we consider, denoted by \tilde{Z}_3 , has a small number of Kähler moduli, which allows to explicitly calculate intersection numbers and topological indices as necessary to e.g. concretely construct heterotic bundles E via the methods of section 4.1.4. We equip \tilde{Z}_3 with an $E_8 \times E_8$ bundle and explicitly construct the F-theory dual geometry X_4 , once in the absence and once in the presence of horizontal heterotic five-branes. Indeed we are able to check the duality map (4.62) between F-theory and heterotic moduli explicitly for both cases. The corresponding heterotic superpotentials can however not be computed due to a huge amount of complex structure moduli of \tilde{Z}_3 and X_4 . The converse situation applies in the second example, which is basically a heterotic compactification on the mirror threefold Z_3 of \tilde{Z}_3 . For this example we are not able to explicitly demonstrate the matching of moduli, but directly study the complex geometry of Z_3 and its F-theory dual fourfold X_4 . This allows both to extract the heterotic bundle and five-brane moduli and to calculate the heterotic flux and five-brane superpotential explicitly. Finally, we specify the corresponding flux G_4 that, together with the knowledge of the periods from section 6.1, determines the dual heterotic superpotential completely.

Before we start let us for future reference summarize the fibration structures of the Calabi-Yau three- and fourfolds we consider in the following, both in sections 6.1 and 6.2:

$$\begin{array}{ccccccc}
 & & & \xrightarrow{MS} & & & \\
 & & & \curvearrowright & & & \\
 & & & MS & & & \\
 & & & \curvearrowleft & & & \\
 & & & & & & \\
 Z_3 & \xrightarrow{\mathcal{E}} & K3 & \xrightarrow{MS} & X_4 & \xrightarrow{MS} & \tilde{X}_4 & \xrightarrow{\tilde{\mathcal{E}}} & \tilde{Z}_3 . \\
 \downarrow \pi_Z & & & & \downarrow \pi_X & & \downarrow \pi_{\tilde{X}} & & \\
 B_2^Z & & & & B_2^Z & & \mathbb{P}^1 & &
 \end{array} \tag{6.1}$$

Here we used the abbreviations MS and Het/F for the action of mirror symmetry respectively heterotic/F-theory duality. In words, starting from a heterotic string compactification on an elliptic threefold $\pi_Z : Z_3 \rightarrow B_2^Z$ with generic elliptic fiber \mathcal{E} we obtain the F-theory dual elliptic $K3$ -fibered fourfold $\pi_X : X_4 \rightarrow B_2^Z$ as the fourfold mirror to the Calabi-Yau threefold fibered fourfold $\pi_{\tilde{X}} : \tilde{X}_4 \rightarrow \mathbb{P}^1$ with generic fiber \tilde{Z}_3 , which in turn is the mirror of Z_3 .

6.1 F-theory, Mirror Symmetry and Superpotentials

In this section we explicitly perform the computation of the F-theory flux superpotential (5.98). The class of Calabi-Yau fourfolds X_4 that we consider here have to have, for technical reasons, a low number of complex structure moduli. We outline in section 6.1.1 a strategy,

cf. [94, 95], to construct such examples of fourfolds X_4 as in (6.1) as the mirror dual to a fourfold \tilde{X}_4 with a small number of Kähler moduli, that itself is realized as a Calabi-Yau threefold fibration \tilde{Z}_3 over a \mathbb{P}^1 -base. In addition we discuss toric means to identify interesting fibration structures like an elliptic or $K3$ -fibration, which is of particular importance both for constructions of F-theory and heterotic/F-theory dual geometries.

Then in section 6.1.2 we fix a concrete Calabi-Yau fourfold \tilde{X}_4 by specifying the threefold fiber \tilde{Z}_3 , that is given for the example at hand as an elliptic fibration over \mathbb{P}^2 . This guarantees a small number of only four complex structure moduli in the mirror X_4 . We emphasize that \tilde{Z}_3 can be viewed as a compactification of the local geometry $K_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ which was studied in [107] in the context of mirror symmetry with D5-branes on the local mirror geometry given by a Riemann surface Σ . We exploit this fact in our analysis of the seven-brane content of the F-theory compactification on X_4 , where the local brane geometry of Σ can be made visible as an additional deformation modulus of the discriminant of the elliptic fibration of X_4 . Finally in section 6.1.3 we determine the solutions of the Picard-Fuchs system for X_4 and obtain the linear combination of solutions $F^0(\gamma)$, which depends on this distinguished deformation modulus and which we thus identify as the Type IIB seven-brane superpotential. In addition, we check this assertion further by mirror symmetry, namely a comparison of the fourfold instanton invariants of $F^0(\gamma)$ with the disk instantons¹ in the limit of the local brane geometry considered in [107]. Analogously we determine the Type IIB flux superpotential by a matching of the classical terms and the world-sheet instanton corrections from the fourfold periods. This explicitly demonstrates the split of the F-theory flux superpotential into the Type IIB flux and brane superpotential as required in (4.41) and thus confirms the unified description of Type IIB open-closed deformations and obstructions in F-theory. We conclude with an independent check via the heterotic dual on Z_3 , compare to the diagram in (6.1).

6.1.1 Constructing Elliptic Fourfolds: Strategy

In the following we discuss the construction of elliptically fibered Calabi-Yau fourfolds X_4 for which we are able to compute the F-theory flux superpotential (5.98) explicitly. As outlined before our strategy is to find examples of elliptic fourfolds X_4 that are constructed as toric hypersurfaces and that admit a small number of complex structure moduli, such that the Picard-Fuchs equations and the fourfold periods can be explicitly determined. One main aim of the following discussion is to understand the various arrows and fibrations in the the diagram (6.1).

The Calabi-Yau fourfolds X_4 studied in this paper are be obtained, according to (6.1), as mirror duals to a Calabi-Yau threefold fibration \tilde{X}_4 over \mathbb{P}^1 . Denoting by \tilde{Z}_3 the generic

¹It is already the required matching of the integral structure of closed fourfold invariants in W_{G_4} and the open disk instanton invariants in W_{brane} , which has a Li_2 -structure, that necessarily identifies the appropriate linear combination of fourfold periods with a specific prepotential $F^0(\gamma)$, cf. section 5.4.

Calabi-Yau threefold fiber we summarize this structure in a schematic diagram of the form

$$\begin{array}{ccc} \text{fiber} & \longrightarrow & \text{total space} , \\ & & \downarrow \\ & & \text{base} \end{array} , \quad \begin{array}{ccc} \tilde{Z}_3 & \longrightarrow & \tilde{X}_4 . \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array} . \quad (6.2)$$

It can now be shown using toric means, see the discussion below or the original literature [103, 260], that the mirror Calabi-Yau fourfold X_4 is elliptically fibered given that the mirror Calabi-Yau threefold Z_3 is elliptically fibered. Then, upon taking a threefold Z_3 with a little number of complex structure moduli, its mirror \tilde{Z}_3 has a small number of Kähler moduli, which is inherited by \tilde{X}_4 . In addition, it is also guaranteed that the mirror fourfold X_4 has a small number of complex structure moduli, as required in order to be able to concretely work on the complex structure moduli space of X_4 . Thus the examples we consider have the following structure,

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & Z_3 , \\ & & \downarrow \\ & & B_2^Z \end{array} , \quad \begin{array}{ccc} \mathcal{E} & \longrightarrow & X_4 , \\ & & \downarrow \\ & & B_3^X \end{array} , \quad (6.3)$$

where \mathcal{E} denotes again the generic elliptic fiber and B_2^Z and B_3^X are the two- and three-dimensional base manifolds of the elliptic fibrations, respectively.

In the concrete examples we construct in this section we in addition consider Calabi-Yau threefolds \tilde{Z}_3 which themselves admit an elliptic fibration and are obtained as compactifications of local Calabi-Yau threefolds $K_{B_2^{\tilde{Z}}} \rightarrow B_2^{\tilde{Z}}$. For these geometries the requirement of a small number $h^{(1,1)}(\tilde{Z}_3)$ of Kähler moduli can readily be realized by the choice of a trivial base $B_2^{\tilde{Z}}$ like \mathbb{P}^2 or \mathbb{F}_n , as we choose in the main example of section 6.1.2 and in appendix C. We also restrict to elliptic fibrations with generic fiber $\mathcal{E} = \mathbb{P}^2(1, 2, 3)[6]$, which has only one section σ [95] and which is both the generic elliptic fiber² of Z_3 and X_4 in (6.3). The general recipe to describe such an elliptic fibration by appropriate toric data is described in detail in [95]. We apply this to construct the Calabi-Yau fourfold \tilde{X}_4 which by construction has a small number of Kähler moduli. Then, we invoke toric mirror symmetry, as reviewed in section 5.1, to readily obtain the desired elliptic fourfold X_4 .

Before we conclude we recall a general theorem of [260] to analyze the fibration structure of a given Calabi-Yau manifold. This is of particular use if we are interested in quickly checking a desired fibration structure of a given mirror pair of Calabi-Yau manifolds, like e.g. the pair of Calabi-Yau fourfolds (\tilde{X}_4, X_4) just constructed, and in particular in order to understand the relations in (6.1) more thoroughly in concrete examples. The main statement of this theorem is the observation that it suffices to analyze the reflexive polyhedra $(\Delta_5^{\tilde{X}}, \Delta_5^X)$ of either \tilde{X}_4 or its mirror X_4 without explicitly computing the intersection numbers [260]. In fact, this is essential if the considered Calabi-Yau manifolds have a huge number of Kähler

²We note that the polyhedron of $\mathbb{P}^2(1, 2, 3)$ is self-dual.

moduli, as in our examples of the Calabi-Yau fourfolds X_4 , which makes a determination of the intersection numbers technically challenging.

The basic theorem can now be formulated as follows, where we restrict to the Calabi-Yau fourfold case for simplicity. Suppose (\tilde{X}_4, X_4) are given as hypersurfaces in the toric varieties constructed from the reflexive pair $(\Delta_5^X, \Delta_5^{\tilde{X}})$ in the pair of dual lattices (M, N) . Then, [260] gives two equivalent conditions for the existence of a Calabi-Yau fibration structure of the given fourfold X_4 once in terms of Δ_5^X and another time in terms of its dual $\Delta_5^{\tilde{X}}$. Assume there exists a $(n-k)$ -dimensional lattice hyperplane in N through the origin such that $\Delta_k^F := H \cap \Delta_5^X$ is a k -dimensional reflexive polyhedron. Then this is equivalent with the existence of a projection P to a k -dimensional sublattice of M such that $P\Delta_5^{\tilde{X}}$ is a k -dimensional reflexive polyhedron $\Delta_k^{\tilde{F}}$ which is the dual of Δ_k^F . If these conditions are satisfied, then the Calabi-Yau manifold X_4 which is obtained as a hypersurface of Δ_5^X has a Calabi-Yau fibration whose $(k-1)$ -dimensional fiber F_{k-1} is encoded by Δ_k^F . The crucial point of these two equivalent criteria is that we can turn things around and analyze X_4 by not looking at hyperplanes H in the complicated polyhedron Δ_5^X , but at projections P in $\Delta_5^{\tilde{X}}$ which is simple by construction. In both cases the base of the fibration can be found by considering the quotient polyhedron Δ_5^X / Δ_k^F [211]. Here this quotient polyhedron is obtained by first determining the quotient lattice in $M \supset \Delta_5^X$ by dividing out the lattice generated by the integral points of Δ_k^F . Then the integral points of Δ_5^X / Δ_k^F are the equivalence classes of integral points in Δ_5^X in this quotient lattice. Schematically the analysis of the fibration structure can be summarized as

Fibration structure	$(\Delta_5^{\tilde{X}}, \tilde{X}_4)$	\leftrightarrow	(Δ_5^X, X_4)	(6.4)
\tilde{X}_4 admits	Injection	\leftrightarrow	Projection	
CY $_{m-1}$ – fiber \tilde{f}_{m-1}	$\Delta_m^{\tilde{f}} = \tilde{H} \cap \Delta_5^{\tilde{X}}$	\leftrightarrow	$\Delta_m^f = P\Delta_5^X$	
X_4 admits	Projection	\leftrightarrow	Injection	
CY $_{k-1}$ – fiber F_{k-1}	$\Delta_k^{\tilde{f}} = P\Delta_5^{\tilde{X}}$	\leftrightarrow	$\Delta_k^F = H \cap \Delta_5^X$	

where the arrow ‘ \leftrightarrow ’ indicates the action of mirror symmetry interchanging projection and injection. Clearly, this analysis can be also used to determine Calabi-Yau fibers \tilde{f}_{m-1} of the mirror \tilde{X}_4 . In general, it is not the case that mirror symmetry preserves fibration structures. However, in the constructions which we will analyze in the section 6.1.2, we will find that both X_4 and \tilde{X}_4 admit an intriguingly rich fibration structure

We conclude by mentioning two appealing byproducts of the geometric structure of the discussed Calabi-Yau manifolds. One interesting consequence of the fibration structure indicated in (6.1) is the simultaneous use also for heterotic/F-theory duality, where the Calabi-Yau fourfold X_4 defines the F-theory setup of a dual heterotic compactification on Z_3 [103], see chapter 4 for more details on heterotic/F-theory duality. Indeed it can be shown explicitly using the simple rules (6.4) that X_4 also admits a $K3$ -fibration with a heterotic dual on the

elliptic threefold Z_3 . We will exploit this briefly in section 6.1.2 and come to more systematic computations in heterotic/F-theory duality in section 6.2.

Secondly, for X_4 obtained from an elliptic Calabi-Yau threefold \tilde{Z}_3 according to (6.1) it is possible to take the local limit of (\tilde{Z}_3, Z_3) , which is then also promoted to the fourfold mirror pair³ (\tilde{X}_4, X_4) . We probe the local geometry of Z_3 , being a conic over a Riemann surface Σ as reviewed in section 5.1, by placing a heterotic five-brane on the non-compact fiber that hits Σ in one point. Then the F-theory compactification on X_4 in this local limit reproduces, in an appropriate further limit, precisely the setup of the local brane geometry of [107], where the same geometry and the brane superpotential were studied in the B-model. Thus, by invoking the results of [107] from local mirror symmetry, this serves as a cross-check for our computation of the brane superpotential in section 6.1.3 and 6.2.3. In particular the compact fourfold X_4 will provide a canonical extension of the local results to the compact Calabi-Yau mirror pair (\tilde{Z}_3, Z_3) . Alternatively, we can directly take the limit of (\tilde{Z}_3, Z_3) to the non-compact geometry in a Type II description where similar results can be obtained, cf. also [119, 122].

6.1.2 Constructing Elliptic Fourfolds: Concrete Examples

In the following we exemplify in detail our construction for a main example of an elliptic fourfold X_4 . It is specified by the choice of the base manifold $B_2^{\tilde{Z}}$ of the elliptic fibration of \tilde{Z}_3 that we here take to be the most simple one, namely $B_2^{\tilde{Z}} = \mathbb{P}^2$. We refer to appendix C.2 for further examples.

We start with the discussion of the non-compact Calabi-Yau threefold $\mathcal{O}_{\mathbb{P}^2}(-3)$ and its mirror Σ and discuss in some detail mirror pairs of toric branes. Then we compactify the non-compact fiber to an elliptic curve \mathcal{E} to obtain the compact threefold \tilde{Z}_3 and its mirror Z_3 , that is elliptically fibered as well. We put strong emphasis on the Weierstrass model of Z_3 and check that, in the limit of large elliptic fiber, we recover the local B-model geometry, a conic over Σ . Finally we obtain \tilde{X}_4 by fibering \tilde{Z}_3 over \mathbb{P}^1 and determine the elliptic mirror fourfold X_4 as in diagram (6.1). We study the geometry of X_4 in great detail and identify dynamical seven-branes by analyzing the discriminant of the elliptic fibration of X_4 . These can be understood as a lift of the brane in the non-compact geometry Σ . This allows us to single out those complex structure moduli of X_4 that map to seven-brane moduli in Type IIB. Then, we briefly exploit the existence of a heterotic dual compactification on Z_3 to determine, following section 4.3.2, the splitting of the Weierstrass constraint of X_4 into the heterotic threefold constraint p_0 and the bundle data p_{\pm} . This allows an explicit identification of the complex structure moduli of X_4 with complex structure moduli and brane respectively bundle moduli in the heterotic string. A more thorough discussion of the heterotic dual and

³We note that in our explicit example in this section, also X_4 is a Z_3 -fibration over \mathbb{P}^1 , that makes the lifting of the local limit to the fourfolds even more obvious.

the dual heterotic five-brane is given in section 6.2.

A toric Calabi-Yau threefold with D-branes

In the following we discuss the local Calabi-Yau threefold over $B_2^{\tilde{Z}} = \mathbb{P}^2$, i.e. $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$, in the presence of toric branes. Then, as a next step, we consider the elliptically fibered Calabi-Yau threefold in the weighted projective space $\mathbb{P}^4(1, 1, 1, 6, 9)$ that contains the non-compact geometry in the limit of large elliptic fiber.

In [107] the non-compact $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ Calabi-Yau threefold with non-compact Harvey-Lawson branes was considered. The local Calabi-Yau is defined as the toric variety $\mathbb{P}_{\tilde{\Delta}}$ characterized by the polyhedron

$$\left(\begin{array}{c|ccc|c|c} & & \tilde{\Delta}_3 & & \ell^{(1)} & \\ \hline v_1 & 0 & 0 & 1 & -3 & X_0 \\ v_1^b & 1 & 1 & 1 & 1 & X_1 \\ v_2^b & -1 & 0 & 1 & 1 & X_2 \\ v_3^b & 0 & -1 & 1 & 1 & X_3 \\ \hline \end{array} \right), \quad (6.5)$$

where the superscript b denotes the two-dimensional basis \mathbb{P}^2 and the X_i denote homogeneous coordinates. The D-term constraint (5.2) for this geometry reads

$$-3|X_0|^2 + |X_1|^2 + |X_2|^2 + |X_3|^2 = r \quad (6.6)$$

where r denotes the Kähler modulus of \mathbb{P}^2 and $\mathbb{P}_{\tilde{\Delta}}$ can be viewed as a $(S^1)^3$ -fibration over a three-dimensional base \mathbb{B}_3 . The degeneration loci of the fiber, $|X_i| = 0$, are shown in figure 6.1. The brane is defined torically by the brane charge vectors

$$\hat{\ell}^{(1)} = (1, 0, -1, 0), \quad \hat{\ell}^{(2)} = (1, 0, 0, -1). \quad (6.7)$$

This leads to the two constraints

$$|X_0|^2 - |X_2|^2 = c^1, \quad |X_0|^2 - |X_3|^2 = c^2, \quad (6.8)$$

where the c^a denote the open string moduli. The brane geometry is $\mathbb{C} \times S^1$ and can be described by a one dimensional half line in the three real dimensional toric base geometry \mathbb{B}_3 ending on a line where two of the three \mathbb{C}^* -fibers degenerate. The A-brane has two inequivalent brane phases I and II as indicated in Figure 6.1.⁴

Mirror symmetry for this geometry was analyzed in [107] where the disk instantons of the A-model were calculated exploiting the fact that the mirror geometry of $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ effectively reduces to the Riemann surface Σ defined by $P(x_1, x_2) = 0$ in (5.15). The D6-brane

⁴Note that our phase II is precisely phase III of [107]. The phase II of [107] has been omitted since it is equivalent to phase I by symmetry of \mathbb{P}^2 .

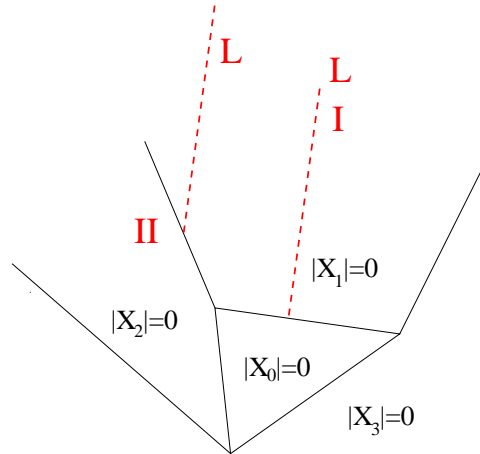


Figure 6.1: Toric base \mathbb{B}_3 and Harvey-Lawson Lagrangians for non-compact \mathbb{P}^2

is mapped under mirror symmetry to a D5-brane which intersects Σ in a point. It will be this D5-brane picture which can be reformulated as a seven-brane with flux and embedded into an F-theory compactification below.

The compact elliptic Calabi-Yau threefold

This local Calabi-Yau threefold can easily be embedded into a compact Calabi-Yau threefold \tilde{Z}_3 . The compactification can be understood as a replacement of the non-compact \mathbb{C} -fiber in $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$, that is dual to the divisor associated to v_1 in $\tilde{\Delta}_3$, by an elliptic fiber. Here we choose the generic fiber to be the elliptic curve in $\mathbb{P}^2(1, 2, 3)$ which we fiber over the \mathbb{P}^2 -base the same way as the non-compact \mathbb{C} -fiber before. Thus, the polyhedron of this compact threefold \tilde{Z}_3 , its charge vectors, the homogeneous coordinates \tilde{x}_i as well as the corresponding monomials for the mirror geometry Z_3 , cf. (6.11), are given by⁵

$$\left(\begin{array}{c|ccc|cc|c|c} & \Delta_4^{\tilde{Z}} & & & \ell^{(1)} & \ell^{(2)} & & \\ \hline v_0 & 0 & 0 & 0 & 0 & -6 & \tilde{x}_0 & zxyu_1u_2u_3 \\ v_1 & 0 & 0 & 2 & -3 & 1 & \tilde{x}_1 & z^6u_1^6u_2^6u_3^6 \\ v_1^b & 1 & 1 & 2 & 1 & 0 & \tilde{x}_2 & z^6u_3^{18} \\ v_2^b & -1 & 0 & 2 & 1 & 0 & \tilde{x}_3 & z^6u_1^{18} \\ v_3^b & 0 & -1 & 2 & 1 & 0 & \tilde{x}_4 & z^6u_2^{18} \\ v_2 & 0 & 0 & -1 & 0 & 2 & \tilde{x}_5 & x^3 \\ v_3 & 0 & 0 & 0 & 0 & 3 & \tilde{x}_6 & y^2 \end{array} \right). \quad (6.9)$$

Here the points v_1, v_2, v_3 carry the information of the elliptic fiber where we added the inner point v_1 in order to recover $\mathbb{P}^2(1, 2, 3)$, in particular its homogeneous coordinate \tilde{x}_1 with

⁵Besides the chosen $(2, 3)$, which leads to an elliptic fibration with one section, the values $(1, 2)$ and $(1, 1)$ are also admissible in the sense that these choices lead to reflexive polyhedra. The corresponding elliptic fibration has two and three sections, respectively [95].

weight one under the new \mathbb{C}^* -action $\ell^{(2)}$. Furthermore, applying the insights of (6.4), the elliptic fibration structure of \tilde{Z}_3 is obvious from the fact, that the polyhedron of $\mathbb{P}^2(1, 2, 3)$ occurs in the hyperplane $H = \{(0, 0, a, b)\}$, but also as a projection P on the (3-4)-plane is found that indicates an elliptic fibration of the mirror Z_3 , too.

The polyhedron (6.9) describes the degree 18 hypersurface in the weighted projective space $\mathbb{P}^4(1, 1, 1, 6, 9)$ considered in [261] that is blown up along the singular curve $\tilde{x}_2 = \tilde{x}_3 = \tilde{x}_4 = 0$ with exceptional divisor v_1 . Its Euler number is $\chi = -540$ whereas $h^{(1,1)} = 2$, $h^{(2,1)} = 272$. Denoting the toric divisors $\tilde{x}_i = 0$ by D_i , the two Kähler classes $J_1 = D_2$ and $J_2 = 3D_2 + D_1$ correspond to the Mori vectors $\ell^{(1)}$ and $\ell^{(2)}$ in (6.9). They represent a curve in the hyperplane class of the \mathbb{P}^2 base and a curve in the elliptic fiber, respectively. The triple intersections of the dual divisors and the intersections with the second Chern class are respectively computed to be⁶

$$\begin{aligned} \mathcal{C}_0 &= 9J_2^3 + 3J_2^2J_1 + J_2J_1^2, \\ \mathcal{C}_2 &= 102J_2 + 36J_1. \end{aligned} \quad (6.10)$$

In this notation the coefficients of the top intersection ring \mathcal{C}_0 are the cubic intersection numbers $J_i \cap J_j \cap J_k$, while the coefficients of \mathcal{C}_2 are $[c_2(T_{\tilde{Z}_3})] \cap J_i$.

Mirror symmetry for this example has been studied in [233, 261]. In order to construct the mirror pair (Z_3, \tilde{Z}_3) as well as their constraints (5.6), (5.7) we need the dual polyhedron

$$\left(\begin{array}{c|ccc|c} & \Delta_4^Z & & & \\ \hline v_1 & 0 & 0 & 1 & 1 & z \\ v_1^b & -12 & 6 & 1 & 1 & u_1 \\ v_2^b & 6 & -12 & 1 & 1 & u_2 \\ v_3^b & 6 & 6 & 1 & 1 & u_3 \\ v_2 & 0 & 0 & -2 & 1 & x \\ v_3 & 0 & 0 & 1 & -1 & y \end{array} \right), \quad (6.11)$$

where again the basis was indicated by a superscript b . Again we added the inner point v_1 to recover the polyhedron of $\mathbb{P}^2(1, 2, 3)$ as the injection with $H = \{0, 0, a, b\}$, thus confirming the elliptic fibration of the mirror Z_3 . Here we distinguish between the two-dimensional basis $B_2^Z = \mathbb{P}^2$ and the elliptic fiber by denoting the homogeneous coordinates of $\mathbb{P}^2(1, 2, 3)$ by (z, x, y) and of B_2^Z by (u_1, u_2, u_3) . The elliptic fibration structure reflects in particular in the constraint of Z_3 which takes a Weierstrass form⁷

$$p_0 := a_6y^2 + a_5x^3 + a_0zxyu_1u_2u_3 + z^6(a_3u_1^{18} + a_4u_2^{18} + a_1u_1^6u_2^6u_3^6 + a_2u_3^{18}) = 0. \quad (6.12)$$

The generic elliptic fiber can be seen by setting the coordinates \underline{u} of the basis B_2^Z to some reference point, such that p_0 takes the form of a degree six hypersurface in $\mathbb{P}^2(1, 2, 3)$. The base itself is obtained as the section $z = 0$ of the elliptic fibration over B_2^Z .

⁶In performing these toric computations we have used the Maple package `Schubert`.

⁷In order to prepare for a heterotic/F-theory duality analysis, we renamed the constraint P in (5.7) to p_0 .

The complex structure dependence of Z_3 is evident from the dependence of p_0 on the parameters \underline{a} which are coordinates on \mathbb{P}^6 . However, they redundantly parameterize the complex structure of Z_3 due to the symmetries of $\mathbb{P}^4(1, 1, 1, 6, 9)$. Indeed there is a $(\mathbb{C}^*)^6/(\mathbb{C}^*)^2$ rescaling symmetry of the coordinates that enables us to eliminate four of the \underline{a} recovering the two complex structure parameters that match $h^{(1,1)}(\tilde{Z}_3) = h^{(2,1)}(Z_3) = 2$. The appropriate coordinates \underline{z} obeying $\underline{z} = 0$ at the large complex structure/large volume point are completely determined by the phase of the A-model, i.e. the choice of charge vectors $\ell^{(i)}$ of $\Delta_4^{\tilde{Z}}$. They are given in general by (5.36) which we readily apply for the situation at hand to obtain⁸

$$z_1 = \frac{a_2 a_3 a_4}{a_1^3}, \quad z_2 = \frac{a_1 a_5^2 a_6^3}{a_0}. \quad (6.13)$$

Thus, we can use the $(\mathbb{C}^*)^4$ action and the overall scaling to set $a_i = 1, i = 2, \dots, 6$ for five parameters to obtain

$$p_0 = y^2 + x^3 + zxy m_1(\underline{u}) + z^6 m_6(\underline{u}), \quad (6.14)$$

where we have abbreviated

$$m_1(\underline{u}) = z_2^{-1/6} z_1^{-1/18} u_1 u_2 u_3, \quad m_6(\underline{u}) = u_1^{18} + u_2^{18} + u_3^{18} + z_1^{-1/3} u_1^6 u_2^6 u_3^6. \quad (6.15)$$

Alternatively, this result can be obtained more directly by the mirror construction (5.14). In this case one needs the following assignment of coordinates y_i to points of $\Delta_4^{\tilde{Z}}$ and monomials

$$\left(\begin{array}{c|c|c} y_0 & v_0 & a_0 zxy u_1 u_2 u_3 \\ y_1 & v_1 & a_1 z^6 u_1^6 u_2^6 u_3^6 \\ y_2 & v_1^b & a_2 z^6 u_3^{18} \\ y_3 & v_2^b & a_3 z^6 u_1^{18} \\ y_4 & v_3^b & a_4 z^6 u_2^{18} \\ y_5 & v_2 & a_5 x^3 \\ y_6 & v_3 & a_6 y^2 \end{array} \right). \quad (6.16)$$

This defines the étalé-map that solves the constraints of (5.14) automatically when (6.13) holds. By setting $a_0 = z_2^{-1/6} z_1^{-1/18}$, $a_1 = z_1^{-1/3}$ and $a_i = 1, i = 2, \dots, 6$, we solve (6.13) and $P = \sum_j y_j$ immediately reproduces p_0 in (6.14).

Next we show that (6.14) indeed gives back the local geometry which is a conic over a genus one Riemann surface Σ [107]. The local limit in the A-model geometry is given as a double scaling limit in which the elliptic fiber decompactifies. This corresponds to $z_2 \rightarrow 0$ in the B-model geometry. Indeed we parameterize z_2 by $\varepsilon \equiv z_2$ such that the local limit is given by $\varepsilon \rightarrow 0$. At the end we should obtain an affine equation, thus, using the two \mathbb{C}^* -action we set the coordinates z and u_3 to one. By redefining the coordinates x and y as follows [262]

$$y \rightarrow \varepsilon^{-1/2} y + k_1^{1/2}, \quad x \rightarrow \varepsilon^{-1/3} x + k_2^{2/3}, \quad (6.17)$$

⁸Compared to the general definition (5.36) we changed a superscript to a subscript for convenience.

the hypersurface equation $p_0 = 0$ becomes

$$p_0 = \frac{1}{\varepsilon} \tilde{p}_0 + k_1^2 + k_2^2 + m_6 = 0 \quad (6.18)$$

where we set $z = 1$ and $u_3 = 1$. Now we split this equation as

$$\tilde{p}_0 = \varepsilon \quad , \quad k_1^2 + k_2^2 + m_6 = -1. \quad (6.19)$$

If we now take the $\varepsilon \rightarrow 0$ limit we obtain, after appropriately redefining the k_i , the equation for the local geometry of the form

$$uv = H(x, y) = x + 1 - \phi \frac{x^3}{y} + y. \quad (6.20)$$

We observe that the Riemann surface defined by $H(x, y) = 0$ is isomorphic to the surface $m_6 = 0$ up to isogeny, i.e. the homology lattice differs only by integral multiples.

As discussed in section 4.3.2 considering heterotic string theory on the elliptically fibered Calabi-Yau threefold Z_3 is expected to be dual to F-theory on X_4 if the fourfold admits a $K3$ fibration. This is automatic in the construction in (6.2) by fibering the mirror \tilde{Z}_3 over \mathbb{P}^1 [103]. We have shown that Z_3 is indeed an elliptic fibration, and will confirm in the next section that X_4 is a $K3$ fibration. However, it is crucial to point out that there will be a large heterotic non-perturbative gauge group from the blown-up singularities of the elliptic fibration of Z_3 . Indeed by calculation of the discriminant of (6.14) one notes that the elliptic fibration not only degenerates over the curves $m_6 = 0$ and $432m_6 + m_1^6 = 0$ in the base of Z_3 , but also over many curves described by the additional coordinates corresponding to the inner points in Δ_4^Z . Let us point out that we will similarly find a large gauge group in the F-theory compactification on X_4 . However, the identification of the moduli of the heterotic gauge bundle E with the complex structure moduli of X_4 can still be performed by focusing on the heterotic perturbative gauge symmetry. This is technically achieved by extracting the spectral cover constraint p_+ of (4.21) in the splitting (4.54) of the constraint P of the fourfold X_4 , as demonstrated in section 6.2.3.

Before continuing with the construction of the Calabi-Yau fourfold, let us close with another comment on the use of the vectors $\hat{\ell}^{(1)}$ and $\hat{\ell}^{(2)}$ given in (6.7). On the compact threefold they translate to

$$\hat{\ell}^{(1)} = (0, 1, 0, -1, 0, 0, 0) \quad , \quad \hat{\ell}^{(2)} = (0, 1, 0, 0, -1, 0, 0) \quad , \quad (6.21)$$

due to the new origin in the polyhedron (6.9). In fact, applying (5.16) and using (6.16), they define the divisors

$$z_1^{-1/3} u_1^6 u_2^6 u_3^6 = \hat{z}_1 u_1^{18} \quad , \quad z_1^{-1/3} u_1^6 u_2^6 u_3^6 = \hat{z}_2 u_2^{18} \quad , \quad (6.22)$$

in the compact Z_3 . Here we introduced the moduli \hat{z}_a corresponding to the charge vector $\hat{\ell}^{(a)}$.

Note that in our F-theory compactification of the next section we will find seven-branes, that are localized on components of the discriminant of the elliptic fibration, which possess additional moduli. These additional fields correspond precisely to either \hat{z}_1 or \hat{z}_2 and parametrize deformations of the seven-brane constraint by the terms in (6.22). Hence, \hat{z}_i can be interpreted as deformations of the seven-brane divisors in X_4 , or as spectral cover moduli in the heterotic dual theory as we will see in section 6.2.3. Upon turning on a brane-flux F_2 on these divisors, the moduli \hat{z} can get obstructed by the brane superpotential (4.40). Upon lifting the brane flux F_2 to a G_4 -flux on X_4 , see e.g. [29], this is mapped to the flux superpotential (5.98) on X_4 matching the brane superpotential in the limit (4.41).

The elliptically fibered Calabi-Yau fourfold

Having discussed the threefold geometry (Z_3, \tilde{Z}_3) , we are now in the position to construct and analyze the elliptically fibered Calabi-Yau fourfold X_4 which is used as an F-theory compactification.

We start by constructing the mirror \tilde{X}_4 first. According to (6.2) it is obtained by fibering the Calabi-Yau threefold \tilde{Z}_3 over a \mathbb{P}^1 . The fibration data can be specified in such a way that one of the D-brane vectors $\tilde{\ell}^{(i)}$ of the local model (6.5) appears as a new charge vector of the polyhedron defining \tilde{X}_4 . As we demonstrate later on, this new charge vector dictates the location of the moving seven-brane, while a second additional vector not used in the construction of the fourfold controls the volume of the \mathbb{P}^1 -basis of the dual fourfold \tilde{X}_4 in (6.2). The flux superpotential and the corresponding four-flux are determined in section 6.1.3. In the following we exemplify our constructions in detail and list all toric and geometrical data necessary to reproduce our results.

The Calabi-Yau fourfolds (X_4, \tilde{X}_4) are realized as hypersurfaces in a toric ambient space described by a dual pair of reflexive polyhedra $(\Delta_5^X, \Delta_5^{\tilde{X}})$. The reflexive polyhedron $\Delta_5^{\tilde{X}}$ describes a fibration of the toric variety constructed from $\Delta_4^{\tilde{Z}}$ over \mathbb{P}^1 and is specified as

$$\Delta_5^{\tilde{X}} = \left(\begin{array}{cccc|c} \Delta_4^{\tilde{Z}} & & & & 0 \\ -1 & 0 & 2 & 3 & -1 \\ 0 & 0 & 2 & 3 & -1 \\ 0 & 0 & 2 & 3 & 1 \end{array} \right). \quad (6.23)$$

By construction, one finds $\Delta_4^{\tilde{Z}}$ by intersecting the hyperplane $\tilde{H} = (p_1, p_2, p_3, p_4, 0)$ with $\Delta_5^{\tilde{X}}$. Following (6.4) this indeed identifies \tilde{X}_4 as a \tilde{Z}_3 -fibration, and by performing the quotient $\Delta_5^{\tilde{X}}/\Delta_4^{\tilde{Z}}$ the base is readily shown to be the toric variety $((-1), (1))$, i.e. a rational curve \mathbb{P}^1 . The additional points which do not lie on \tilde{H} determine the fibration structure of the \tilde{Z}_3 -fibration. Firstly, they are chosen such that the mirror X_4 is elliptically fibered⁹ which means, that using the projection to the third and fourth coordinate one finds the polyhedron of a

⁹The fact that \tilde{X}_4 is also elliptically fibered in the example at hand is not crucial in the construction,

torus in $\mathbb{P}^2(1, 2, 3)$ just as in the threefold case in (6.9). Secondly, they can be arranged such that one charge vector of the Calabi-Yau fourfold is of the form $(\hat{\ell}^{(1)}, -, -, -)$, i.e. contains the brane charge vector $\hat{\ell}^{(1)}$. As we see below this will imply a lift of the toric brane of (6.22) to F-theory on the mirror fourfold X_4 .

Before proceeding with the concrete example let us note an alternative perspective on the construction of $\Delta_5^{\tilde{X}}$. In fact, $\Delta_5^{\tilde{X}}$ can be understood more thoroughly from the perspective of the GKZ-system obtained using the blow-up procedure [60, 100] presented in chapter 7. In this context the connection of the fourfold geometry $\Delta_5^{\tilde{X}}$ with the brane charge vectors can be understood as a consequence of heterotic/F-theory duality [81]. We note that adding this vector to form a higher-dimensional non-reflexive polyhedron was first proposed in [108, 119, 263] in the context of the B-model and then extended to the compact case in [99, 126], where a connection with heterotic/F-theory duality was exploited.

We begin by choosing the open string vector $\hat{\ell}^{(1)}$ to construct¹⁰ the \mathbb{P}^1 -fibration in (6.23). The Calabi-Yau fourfold \tilde{X}_4 is then realized as a hypersurface in the toric space described by the polyhedron $\Delta_5^{\tilde{X}}$. Its topological numbers are computed to be

$$h^{(3,1)} = 2796, \quad h^{(1,1)} = 4, \quad h^{(2,1)} = 0, \quad h^{(2,2)} = 11244, \quad \chi = 16848. \quad (6.24)$$

Here we first used (5.8), (5.9) as well as (5.11) and next applied (5.10), (5.12).

Next, we note that $\Delta_5^{\tilde{X}}$ has three triangulations, which correspond to non-singular Calabi-Yau phases which are connected by flop transitions. In the following we consider two of these phases in detail. These phases match, as we will show explicitly, the two brane phases in figure 6.1 in the local Calabi-Yau threefold geometry.

To summarize the topological data of the Calabi-Yau fourfold for the two phases of interest, we specify the generators of the Mori cone $\ell_I^{(i)}$ and $\ell_{II}^{(i)}$ for $i = 1, \dots, 4$,

$$\left(\begin{array}{c|cccc|cccc|cccc} & \Delta_5^{\tilde{X}} & & & & \ell_I^{(1)} & \ell_I^{(2)} & \ell_I^{(3)} & \ell_I^{(4)} & \ell_{II}^{(1)} & \ell_{II}^{(2)} & \ell_{II}^{(3)} & \ell_{II}^{(4)} \\ \hline v_0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -6 & 0 & 0 \\ v_1^b & 0 & 0 & 2 & 3 & -2 & 1 & -1 & -1 & -3 & 0 & 1 & -2 \\ v_2^b & 1 & 1 & 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_3^b & -1 & 0 & 2 & 3 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & 0 \\ v_4^b & 0 & -1 & 2 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ v_2 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 \\ \hat{v}_1 & -1 & 0 & 2 & 3 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ \hat{v}_2 & 0 & 0 & 2 & 3 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ \hat{v}_3 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right). \quad (6.25)$$

The charge vectors are best identified in the phase II. Here $\ell_{II}^{(1)}$ and $\ell_{II}^{(2)}$ are the extensions of the threefold charge vectors $\ell^{(1)}, \ell^{(2)}$ in (6.9) to the fourfold. The brane vector $\hat{\ell}^{(1)}$ is visible cf. (6.1). In particular, the construction also applies e.g. for the quintic hypersurface fibered over \mathbb{P}^1 , since the mirror quintic admits an elliptic fibration with generic elliptic fiber being a torus in \mathbb{P}^2 .

¹⁰We could have used also $\hat{\ell}^{(2)}$, reproducing the same local D5-brane limit.

in phases II as a subvector of $\ell_{II}^{(3)}$. The remaining vector $\ell_{II}^{(4)}$ arises since we had to complete the polyhedron such that it becomes reflexive implying that \tilde{X}_4 is a Calabi-Yau manifold. It is the class of the \mathbb{P}^1 in the base of the Calabi-Yau threefold fibration of \tilde{X}_4 . Phase I is related to phase II by a flop transition of the curve associated to $\ell_I^{(3)}$. Hence, in phase I the brane vector is identified with $-\ell_I^{(3)}$. Furthermore, in the flop transition we identify

$$\ell_{II}^{(3)} = -\ell_I^{(3)} , \quad \ell_{II}^{(1)} = \ell_I^{(1)} + \ell_I^{(3)} , \quad \ell_{II}^{(2)} = \ell_I^{(2)} + \ell_I^{(3)} , \quad \ell_{II}^{(4)} = \ell_I^{(4)} + \ell_I^{(3)} . \quad (6.26)$$

Note that the charge vectors $\ell_I^{(i)}$ and $\ell_{II}^{(i)}$ are chosen to be generators of the Mori cone of \tilde{X}_4 , that is dual to Kähler cone. The generators of the latter for phase I are then given by

$$J_1 = D_2 , \quad J_2 = D_1 + 2D_2 + D_3 + 2D_9 , \quad J_3 = D_3 + D_9 , \quad J_4 = D_9 , \quad (6.27)$$

where $D_i := \{x_i = 0\}$ are the nine toric divisors associated to the points $\Delta_5^{\tilde{X}}$ which differ from the origin. In phase II we find analogously

$$J_1 = D_2 , \quad J_2 = D_1 + 2D_2 + D_3 + 2D_9 , \quad J_3 = D_1 + 3D_2 + 2D_9 , \quad J_4 = D_9 . \quad (6.28)$$

The generators J_i provide a distinguished integral basis of $H^{(1,1)}(\tilde{X}_4)$ since in the expansion of the Kähler form J in terms of the J_i all coefficients are positive and parameterize physical volumes of cycles in \tilde{X}_4 . The J_i are also canonically used as a basis in which one determines the topological data of \tilde{X}_4 . The complete set of topological data of \tilde{X}_4 including the intersection ring as well as the non-trivial Chern classes are summarized in appendix C.1.

The polyhedron $\Delta_5^{\tilde{X}}$ has only few Kähler classes which makes it possible to identify part of the fibration structures directly from the intersection numbers. However, an analogous analysis is not possible for the mirror manifold X_4 since the dual polyhedron Δ_5^X has 2796 Kähler classes. Therefore, we apply the methods reviewed in section 6.1.1 for analyzing both \tilde{X}_4 and X_4 . As already mentioned above, $\Delta_5^{\tilde{X}}$ intersected with the two hyperplanes

$$H_1 = (0, 0, p_3, p_4, 0) , \quad H_2 = (p_1, p_2, p_3, p_4, 0) \quad (6.29)$$

yields two reflexive polyhedra corresponding to a generic elliptic fiber and the generic three-dimensional Calabi-Yau fiber \tilde{Z}_3 . The fibration structure of the mirror X_4 is studied by identifying appropriate projections to $\Delta_F^k \subset \Delta_5^{\tilde{X}}$. Three relevant projections P_i are

$$P_1(\underline{p}) = (p_3, p_4) , \quad P_2(\underline{p}) = (p_1, p_2, p_3, p_4) , \quad P_3(\underline{p}) = (p_3, p_4, p_5) , \quad (6.30)$$

where $\underline{p} = (p_1, \dots, p_5)$ denote the columns in the polyhedron $\Delta_5^{\tilde{X}}$. Invoking the theorem of section 6.1.1, we see from P_1 that X_4 is also elliptically fibered and since the polyhedron of $\mathbb{P}^2(1, 2, 3)$ is self-dual, the fibration is of $\mathbb{P}^2(1, 2, 3)$ -type. In addition, it is clear from P_2 that X_4 is Calabi-Yau threefold fibered. The fiber threefold is Z_3 , the mirror to \tilde{Z}_3 . The fact, that the threefold fibers of X_4 and \tilde{X}_4 are mirror manifolds is special to this example since the subpolyhedra obtained by H_2 and P_2 are identical. Finally, we note that X_4 is K3 fibered

as inferred from the projection P_3 . This ensures the existence of a heterotic dual theory by fiberwise applying the duality of F-theory on $K3$ to heterotic strings on T^2 , as reviewed in section 4.3.

The hypersurface constraint for X_4 depends on four complex structure moduli \underline{z} . This dependence is already captured by only introducing 12 out of the many coordinates needed to specify a non-singular X_4 . This subset of points in Δ_5^X is given by

$$\Delta_5^X \supset \left(\begin{array}{c|cccccc|c} v_1 & 0 & 0 & 1 & 1 & 0 & z \\ v_2 & -12 & 6 & 1 & 1 & 0 & u_1 \\ v_3 & 6 & -12 & 1 & 1 & 0 & u_2 \\ v_4 & 6 & 6 & 1 & 1 & 0 & u_3 \\ v_5 & 0 & 0 & -2 & 1 & 0 & x \\ v_6 & 0 & 0 & 1 & -1 & 0 & y \\ v_1^b & -12 & 6 & 1 & 1 & -6 & x_1 \\ v_2^b & -12 & 6 & 1 & 1 & 6 & x_2 \\ v_3^b & 6 & -12 & 1 & 1 & -6 & x_3 \\ v_4^b & 6 & 6 & 1 & 1 & -6 & x_4 \\ v_5^b & 0 & -6 & 1 & 1 & 6 & x_5 \\ v_6^b & 0 & 6 & 1 & 1 & 6 & x_6 \end{array} \right) \quad (6.31)$$

where we have omitted the origin. Note that we have listed in (6.31) the vertices of Δ_5^X and added the inner points v_1 and v_2 to list all points necessary to identify the polyhedron Δ_4^Z with vertices (6.11) in the hyperplane orthogonal to $(0, 0, 0, 0, 1)$. Thus we directly observe the Calabi-Yau fibration with generic fiber Z_3 . The base of this fibration is given by the points labeled by a superscript b . Note that v_1 is also needed to display the elliptic fibration. The base of the elliptic fibration is obtained by performing the quotient $\Delta_3^{\text{base}} = \Delta_5^X / (P_1 \Delta_5^{\tilde{X}})^*$ which amounts to simply dropping the third and fourth entry in the points of Δ_5^X .

In addition, one can also see the elliptic fibration directly on the defining polynomial P of X_4 which can be written in a Weierstrass form. Indeed if we apply (5.7) for the points displayed in (6.31) of Δ_5^X and all points p of $\Delta_5^{\tilde{X}}$ that are not on codimension one faces we obtain a hypersurface of the form¹¹

$$P = a_6 y^2 + a_5 x^3 + \tilde{m}_1(\underline{x}, \underline{u})xyz + \tilde{m}_6(\underline{x}, \underline{u})z^6 = 0. \quad (6.32)$$

Here $\underline{x}, \underline{u}$ are the homogeneous coordinates on the base of the elliptic fibration, while x, y , and z are the homogeneous coordinates of the $\mathbb{P}^2(1, 2, 3)$ -fiber. The polynomials \tilde{m}_1 and \tilde{m}_6 are given by

$$\tilde{m}_1(\underline{x}, \underline{u}) = a_0 u_1 u_2 u_3 x_1 x_2 x_3 x_4 x_5 x_6, \quad (6.33)$$

$$\begin{aligned} \tilde{m}_6(\underline{x}, \underline{u}) = & u_1^{18} (a_7 x_1^{24} x_2^{12} x_3^6 x_4^6 + a_3 x_1^{18} x_2^{18} x_5^6 x_6^6) + a_4 u_2^{18} x_3^{18} x_5^{12} + a_2 u_3^{18} x_4^{18} x_6^{12} \\ & + u_1^6 u_2^6 u_3^6 (a_1 x_1^6 x_2^6 x_3^6 x_4^6 x_5^6 x_6^6 + a_9 x_2^{12} x_5^{12} x_6^{12} + a_8 x_1^{12} x_3^{12} x_4^{12}), \end{aligned} \quad (6.34)$$

¹¹The polynomial P can be easily brought to the standard Weierstrass form by completing the square and the cube, i.e. $\tilde{y} = y + \frac{1}{2}\tilde{m}_1 x z$ and $\tilde{x} = x - \frac{1}{12}\tilde{m}_1^2 z^2$.

where the coefficients \underline{a} encode the complex structure deformations of X_4 . However, since $h^{(3,1)}(X_4) = h^{(1,1)}(\tilde{X}_4) = 4$ there are only four complex structure parameters rendering six of the \underline{a} redundant. It is also straightforward to recover from \tilde{m}_1, \tilde{m}_6 of the fourfold X_4 the corresponding threefold data m_1, m_6 in (6.14) and (6.15), upon fixing the coordinates of the \mathbb{P}^1 -base of the Z_3 -fibration as $\underline{x} = 1$.

For the different phases we identify the complex structure moduli in the hypersurface constraint P by using the charge vectors $\ell^{(i)}$ in (6.25) and by applying the general formula (5.36). For phase I one finds

$$z_1^I = \frac{a_2 a_4 a_7}{a_1^2 a_8}, \quad z_2^I = \frac{a_1 a_5^2 a_6^3}{a_0^6}, \quad z_3^I = \frac{a_3 a_8}{a_1 a_7}, \quad z_4^I = \frac{a_7 a_9}{a_1 a_3}, \quad (6.35)$$

while for the phase II one finds in accord with (6.26) that

$$z_1^{II} = z_1^I z_3^I, \quad z_2^{II} = z_2^I z_3^I, \quad z_3^{II} = (z_3^I)^{-1}, \quad z_4^{II} = z_4^I z_3^I. \quad (6.36)$$

In order to prepare for a comparison with the constraint p_0 in (6.14) of the threefold Z_3 we chose the gauge $a_i = 1, i = 2, \dots, 6$ and $a_8 = 1$, such that

$$a_0^6 = \frac{1}{(z_1^{II})^{1/3} z_2^{II} z_3^{II}}, \quad a_1 = \frac{1}{(z_1^{II})^{1/3}}, \quad a_7 = z_3^{II} (z_1^{II})^{1/3}, \quad a_9 = \frac{z_4^{II}}{(z_1^{II})^{2/3}}. \quad (6.37)$$

It is straightforward to find the similar expression for phase I by inserting (6.36) into this expression for a_0, a_1 and a_7, a_9 .

Having determined the defining equation P for the Calabi-Yau fourfold we readily evaluate the discriminant $\Delta(X_4)$ of the elliptic fibration. Using (4.32) for a Calabi-Yau fourfold with constraint in the Weierstrass form (4.30) we find

$$\Delta(X_4) = -\tilde{m}_6(432\tilde{m}_6 + \tilde{m}_1^6). \quad (6.38)$$

We conclude that there are seven-branes on the divisors $\tilde{m}_6 = 0$ and $432\tilde{m}_6 + \tilde{m}_1^6 = 0$ in the base B_3^X . The key observation is that in addition to a moduli independent part \tilde{m}_6^0 the full \tilde{m}_6 is shifted as

$$\tilde{m}_6 = \tilde{m}_6^0 + a_1(u_1 u_2 u_3 x_1 x_2 x_3 x_4 x_5 x_6)^6 + a_7 u_1^{18} x_1^{24} x_2^{12} x_3^6 x_4^6 + a_9 u_1^6 u_2^6 u_3^6 x_2^{12} x_5^{12} x_6^{12}. \quad (6.39)$$

The moduli dependent part is best interpreted in the phase II with a_1, a_7 and a_9 given in (6.37). In fact, when setting the fourth modulus to $z_4^{II} = 0$, one notes that the deformation of the seven-brane locus $\tilde{m}_6 = 0$ is precisely parameterized by z_3^{II} . By setting $x_i = 1$ one fixes a point in the base of X_4 viewed as fibration with generic fiber Z_3 . One is then in the position to compare the shift in (6.39) with the first constraint in (6.22) finding agreement if one identifies $\hat{z}_1 = z_3^{II} (z_1^{II})^{1/3}$.

In the next section we will exploit this further by showing that the open string BPS numbers of the local model with D5-branes of (6.5) are recovered in the z_3^{II} -direction. The shift

of the naive open modulus \hat{z}_1 by the closed complex structure modulus z_1^{II} fits then nicely with a similar redefinition made for the local models in [107]. This leaves us with the interpretation that indeed z_3^{II} deforms the seven-brane locus and matches an open string modulus in the local picture. As we will show in the next section, a z_3^{II} -dependent superpotential is induced upon switching on fluxes on the seven-brane or equivalently by specifying four-flux G_4 . The superpotential can be computed explicitly and is matched with the results for a D5-brane in the local Calabi-Yau.

A second interpretation of the shifts in the discriminant (6.39) by the monomials proportional to $z_3^{\text{II}}, z_4^{\text{II}}$ is given by the heterotic dual theory on Z_3 and the encoding of the spectral cover data of the heterotic bundle E by the fourfold constraint of X_4 as discussed in section 4.3.2. To see this, we bring P into the form (4.54) by an appropriate coordinate redefinition. Setting $v = x_1^6 x_3^6 x_4^6 x_2^{-6}$, $\tilde{u}_1 = u_1 x_1 x_2$, $\tilde{u}_2 = u_2 x_3$, $\tilde{u}_3 = u_3 x_4$, and picking the local patch $x_5 = x_6 = 1$ one rewrites (6.32) as

$$P = p_0 + v p_+ + v^{-1} p_- , \quad (6.40)$$

where $p_0(y, x, z, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = 0$ is the threefold constraint (6.12) of Z_3 , and

$$p_+ = (a_7 \tilde{u}_1^{18} + a_8 \tilde{u}_1^6 \tilde{u}_2^6 \tilde{u}_3^6) z^6 , \quad p_- = a_9 \tilde{u}_1^6 \tilde{u}_2^6 \tilde{u}_3^6 z^6 , \quad (6.41)$$

describe the two heterotic bundles $E_1 \oplus E_2$ in $E_8 \times E_8$. Hence, in the local mirror limit in which $p_- \rightarrow 0$ [103], it is natural to interpret the modulus z_3^{II} as a bundle modulus of an $E_1 = \text{SU}(1)$ in the heterotic dual theory. One might be surprised that an $\text{SU}(1)$ -bundle carries any bundle moduli due to the trivial structure group. Indeed the adequate physical interpretation of this configuration is in terms of heterotic five-branes, cf. section 4.1.4 and [103], as discussed in detail in section 6.2.3 and in [81].

Finally, as a side remark, let us note again that (6.38) with (6.33) and (6.34) is not the full answer for the discriminant since we have set many of the blow-up coordinates in X_4 to unity. However, one can use the toric methods of [101, 210, 211] to determine the full minimal gauge group in the absence of fluxes to be

$$G_{X_4} = E_8^{25} \times F_4^{69} \times G_2^{184} \times \text{SU}(2)^{276} . \quad (6.42)$$

Groups of such large rank are typical for elliptically fibered Calabi-Yau geometries with many Kähler moduli corresponding to blow-ups of singular fibers [211].

6.1.3 Mirror Symmetry Applications to F-Theory

In this section we calculate the flux and brane superpotentials (4.40) of the Type IIB theory from the perspective of F-theory on the fourfold X_4 with fluxes G_4 in the limit (4.41). We perform this analysis by applying the methods of section 5.3 to the example discussed in section 6.1.2. We use the following strategy.

First we identify the periods of the threefold fiber Z_3 of X_4 among the fourfold periods. This implies a matching of all instanton numbers as well as the classical intersections of the mirror \tilde{Z}_3 . Then we explicitly identify fourfold periods that reproduce the physics of branes on the local geometry of \tilde{Z}_3 discussed in section 6.1.2, namely all disk instantons calculated in [107]. This is equivalent to the statement that we calculate the flux superpotential (5.95) and the seven-brane superpotential (5.97) for a specific brane flux from the fourfold perspective of F-theory. For the mirror fourfold \tilde{X}_4 the closed BPS-states are encoded in $F^0(\gamma)$ so that this matching implies, in mathematical terms, a map of the integral structure of $F^0(\gamma)$ to the integral structure in (5.97) encoded by the Ooguri-Vafa invariants. Thus, we explicitly show that there is an element $\hat{\gamma} \in H_H^{(2,2)}(Z_3)$ such that the complete enumerative geometry of the threefold mirror pair (\tilde{Z}_3, Z_3) with and without branes is reproduced. We already note that the results presented below are of further importance since they provide a concrete check of heterotic/F-theory duality along the lines of section 4.3.2 as the Calabi-Yau threefold Z_3 can also be promoted to the background geometry a heterotic string compactification. The details of this analysis are found in section 6.2.

Let us now perform the concrete calculations. The Calabi-Yau fourfold X_4 introduced in (6.31) has four complex moduli. We find that the moduli dependence of its periods is determined by a complete set of six Picard-Fuchs operators which are linear differential operators \mathcal{D}_α , $\alpha = 1, \dots, 6$ of order $(3, 2, 2, 2, 3, 2)$. These are obtained from the GKZ-system (5.34) associated to the charge vectors $\ell_I^{(1)}, \ell_I^{(2)}, \ell_I^{(3)}, \ell_I^{(4)}, \ell_I^{(1)} + \ell_I^{(3)}, \ell_I^{(3)} + \ell_I^{(4)}$, by the methods described in [233]. We use logarithmic derivatives $\theta_a = z_a \frac{d}{dz_a}$ in the canonical complex variables (5.36) and present only the leading piece of the differential equations $\mathcal{D}_\alpha^{\text{lim}} = \lim_{z_a \rightarrow 0} \mathcal{D}_\alpha(\theta_a, z_a)$, $a = 1, \dots, 4$. For the case at hand we obtain

$$\begin{aligned}
 \mathcal{D}_1^{\text{lim}} &= \theta_1^2(\theta_3 - \theta_1 - \theta_4), & \mathcal{D}_2^{\text{lim}} &= \theta_2(\theta_2 - 2\theta_1 - \theta_3 - \theta_4), \\
 \mathcal{D}_3^{\text{lim}} &= (\theta_1 - \theta_3)(\theta_3 - \theta_4), & \mathcal{D}_4^{\text{lim}} &= \theta_4(\theta_1 - \theta_3 + \theta_4), \\
 \mathcal{D}_5^{\text{lim}} &= \theta_1^2(\theta_4 - \theta_3), & \mathcal{D}_6^{\text{lim}} &= \theta_4(\theta_1 - \theta_3).
 \end{aligned} \tag{6.43}$$

For the complete Picard-Fuchs system as well as the cohomology basis we extract from it by calculating the ring \mathcal{R} in (5.61) we refer to appendix C.1.

The calculation of \mathcal{R} is readily performed yielding $(1, 4, 6, 4, 1)$ generators of the ring \mathcal{R}

at grade $(0, 1, 2, 3, 4)$, which are

$$\begin{array}{l|l}
\mathcal{R}^{(0)} & 1 \\
\hline
\mathcal{R}_a^{(1)} & \theta_1, \theta_2, \theta_3, \theta_4, \\
\hline
\mathcal{R}_\alpha^{(2)} & \theta_1^2, (\theta_1 + \theta_3)\theta_4, (\theta_1 + \theta_3)\theta_3, (\theta_1 + 2\theta_2)\theta_2, (\theta_2 + \theta_4)\theta_2, (\theta_2 + \theta_3)\theta_2 \\
\hline
\mathcal{R}_\alpha^{(3)} & (\theta_3 + \theta_4)(\theta_1^2 + \theta_1\theta_3 + \theta_3^2), \theta_2(\theta_3^2 + 3\theta_2\theta_3 + 5\theta_2^2 + \theta_1(\theta_2 + \theta_3)), \\
& \theta_2(\theta_1(\theta_2 + \theta_4) + \theta_4(\theta_3 + 3\theta_2) + \theta_2(\theta_3 + 6\theta_2)), \theta_2(\theta_1^2 + 2\theta_1\theta_2 + 4\theta_2^2) \\
\hline
\mathcal{R}^{(4)} & \theta_4(\theta_1^2\theta_2 + 3\theta_1\theta_2^2 + 9\theta_2^3 + \theta_1\theta_2\theta_3 + 3\theta_2^2\theta_3 + \theta_2\theta_3^2) \\
& + \theta_2(46\theta_2^3 + 15\theta_2^2\theta_3 + 4\theta_2\theta_3^2 + \theta_3^3 + \theta_1^2(2\theta_2 + \theta_3) + \theta_1(11\theta_2^2 + 4\theta_2\theta_3 + \theta_3^2)) .
\end{array} \tag{6.44}$$

These elements can be associated to solutions of the Picard-Fuchs equations and to a choice of basis elements of the Chow ring as explained in section 5.3.3. At grade $k = 2$ the leading solutions $\mathbb{L}^{(k)\alpha}$ of the full Picard-Fuchs system (C.3), that obey the normalization $\mathcal{R}_\alpha^{(k)}\mathbb{L}^{(k)\beta} = \delta_\alpha^\beta$ of (5.66), are then given by

$$\begin{aligned}
\mathbb{L}^{(2)1} &= l_1^2, \quad \mathbb{L}^{(2)2} = \frac{1}{2}l_4(l_1 + l_3), \quad \mathbb{L}^{(2)3} = \frac{1}{2}l_3(l_1 + l_3), \\
\mathbb{L}^{(2)4} &= \frac{1}{7}l_2(3l_1 - 2(l_3 + l_4 - l_2)), \quad \mathbb{L}^{(2)5} = \frac{1}{7}l_2(-2l_1 + l_2 + 6l_4 - l_3), \\
\mathbb{L}^{(2)6} &= \frac{1}{7}l_2(-2l_1 + l_2 + 6l_3 - l_4),
\end{aligned} \tag{6.45}$$

where we used the abbreviation $\log(z_k) \equiv l_k$ and omitted the prefactor X_0 . In comparison to the complete solutions $\Pi^{(2)\alpha}$ of the Picard-Fuchs equations we focused on the leading terms only and omitted terms of order $\mathcal{O}(\underline{L})$. The full solutions with the leading logarithms (6.45) directly occur in the periods expansion of the holomorphic four-form Ω_4 . Since we are calculating the holomorphic potentials $F^0(\gamma)$ of (5.69) and the corresponding BPS-invariants we have to change the basis of solutions such that to any operator $\mathcal{R}_\alpha^{(2)}$ in (6.44) we associate a solution with leading logarithm determined by the classical intersection $C_{ab\alpha}^0(1,1,2)$ of the A-model defined in (5.56) as

$$\mathbb{L}_\alpha^{(2)} = \frac{1}{2}X_0C_{\alpha ab}^0l_al_b. \tag{6.46}$$

As we have already proven in (5.68) these solutions are related to the leading logarithms $\mathbb{L}^{(2)\alpha}$ as $\mathbb{L}_\alpha^{(2)} = \mathbb{L}^{(2)\beta}\eta_{\alpha\beta}^{(2)}$. Thus, we readily obtain the leading terms $\mathbb{L}_\alpha^{(2)}$ from the classical intersection data (6.44) in $\mathcal{R}^{(4)}$.

As in the discussion below (5.65) the choice of periods $\Pi^{(2)\alpha}$ with leading terms $\mathbb{L}^{(2)\alpha}$ corresponds to a particular choice of a basis $\hat{\gamma}_\alpha^{(2)}$ of $H_V^{(2,2)}(\tilde{X}_4)$. In fact, by construction one identifies the flux basis at grade two as

$$\hat{\gamma}_\alpha^{(2)} = \mathcal{R}_\alpha^{(k)}\Omega_4|_{z=0}. \tag{6.47}$$

However, this choice of basis for $H_V^{(2,2)}(\tilde{X}_4)$ is not necessarily a basis of integral cohomology. An integral basis can, however, be determined by an appropriate basis change. We next

identify some basis elements of the integral cohomology of \tilde{X}_4 . First we note that the Kähler generator J_4 is identified as the class of the Calabi-Yau threefold fiber \tilde{Z}_3 , where we refer to appendix C.1 for more details on this identification from the intersection numbers. Moreover, we identify the fourfold Kähler generators J_i with the threefold generators $J_k(\tilde{Z}_3)$ as

$$J_1 + J_3 \leftrightarrow J_1(\tilde{Z}_3), \quad J_2 \leftrightarrow J_2(\tilde{Z}_3), \quad (6.48)$$

by comparing the coefficient of J_4 in the intersection form $\mathcal{C}_0(\tilde{X}_4)$, given in (C.2) in appendix C.1, with the threefold intersections $\mathcal{C}_0(\tilde{Z}_3)$ in (6.10). A subset of the basis elements of the fourfold integral basis are now naturally induced from the threefold integral basis. In order to do so we identify the threefold periods $\partial_i F_{\tilde{Z}_3}$, with derivatives in the directions $J_1(\tilde{Z}_3)$ and $J_2(\tilde{Z}_3)$, with an appropriate linear combination of the fourfold periods $\Pi^{(2)\alpha}$ [94]. In other words we determine a new basis $\hat{\gamma}_i^{(2)}$ such that one fourfold prepotential equals $\partial_i F_{\tilde{Z}_3}$,

$$\partial_i F_{\tilde{Z}_3} = F^0(\hat{\gamma}_i^{(2)})|_{z_4=0} \equiv \Pi_i^{(2)}|_{z_4=0}. \quad (6.49)$$

In this matching both the classical part of the periods as well as the threefold BPS-invariants n_{d_1, d_2} and the fourfold BPS-invariants $n_{d_1, d_2, d_1, 0}(\gamma)$ have to match in the limit of large \mathbb{P}^1 -base.

The match (6.49) is most easily performed by first comparing the classical parts of the periods. In fact, using the classical intersections of \tilde{Z}_3 in (6.10) one deduces that the leading parts of the threefold periods are

$$\mathbb{L}_1(Z_3) = \frac{1}{2} X_0 \tilde{l}_2 (2\tilde{l}_1 + 3\tilde{l}_2) \quad \mathbb{L}_2(Z_3) = \frac{1}{2} X_0 (\tilde{l}_1 + 3\tilde{l}_2)^2, \quad (6.50)$$

where $\tilde{l}_i = \log \tilde{z}_i$ correspond to the two threefold directions $J_k(\tilde{Z}_3)$ in (6.48). Using (6.48) and (6.49) one then finds the appropriately normalized leading fourfold periods as

$$\mathbb{L}_2^{(2)} = \frac{1}{2} X_0 l_2 (2l_1 + 3l_2 + 2l_3) \quad \mathbb{L}_5^{(2)} = \frac{1}{2} X_0 (l_1 + 3l_2 + l_3)^2. \quad (6.51)$$

A direct computation also shows that the threefold BPS invariants $d_i n_{d_1, d_2}$ and fourfold BPS invariants $n_{d_1, d_2, d_1, 0}(\hat{\gamma}_i)$ match in the large \mathbb{P}^1 -base limit, such that (6.49) is established on the classical as well as quantum level. This match then fixes corresponding integral basis elements of $H_V^{(2,2)}(\tilde{X}_4)$ as follows.

First we determine those two ring elements $\tilde{\mathcal{R}}_\alpha^{(2)}$, $\alpha = 2, 5$, such that we obtain (6.51) from the new three-point couplings using (6.46). Then we complete them into a new basis of ring elements $\tilde{\mathcal{R}}_\alpha^{(2)}$ by choosing

$$\begin{aligned} \tilde{\mathcal{R}}_1^{(2)} &= \theta_1^2, \quad \tilde{\mathcal{R}}_2^{(2)} = \frac{1}{2} \theta_4 (\theta_1 + \theta_3), \quad \tilde{\mathcal{R}}_3^{(2)} = \frac{1}{2} \theta_3 (\theta_1 + \theta_3), \\ \tilde{\mathcal{R}}_4^{(2)} &= \frac{1}{7} \theta_2 (3\theta_1 - 2(\theta_3 + \theta_4 - \theta_2)), \quad \tilde{\mathcal{R}}_5^{(2)} = \frac{1}{7} \theta_2 (-2\theta_1 + \theta_2 + 6\theta_4 - \theta_3), \\ \tilde{\mathcal{R}}_6^{(2)} &= \frac{1}{7} \theta_2 (-2\theta_1 + \theta_2 + 6\theta_3 - \theta_4). \end{aligned} \quad (6.52)$$

Consequently, these operators fix the two integral basis elements

$$\hat{\gamma}_2^{(2)} = \tilde{\mathcal{R}}_2^{(2)} \Omega_4|_{z=0}, \quad \hat{\gamma}_5^{(2)} = \tilde{\mathcal{R}}_5^{(2)} \Omega_4|_{z=0}. \quad (6.53)$$

which reproduce the corresponding part of the flux superpotential (5.95) on Z_3 for $\hat{N}_i = \hat{M}^0 = 0$ when turning on four-form flux G_4 on X_4 in these directions,

$$W_{\text{flux}} \equiv M^1 F^0(\hat{\gamma}_2^{(2)}) + M^2 F^0(\hat{\gamma}_5^{(2)}) = \int_{X_4} \Omega_4 \wedge G_4 = M^1 \Pi_2^{(2)} + M^2 \Pi_5^{(2)} \quad (6.54)$$

for the flux

$$G_4 = M^1 \hat{\gamma}_2^{(2)} + M^2 \hat{\gamma}_5^{(2)}. \quad (6.55)$$

For the choices $M^i = \delta^{ij}$ we extract the threefold invariants $d_j n_{d_1, d_2}$ from this superpotential, i.e. from the prepotentials $F^0(\hat{\gamma}_2^{(2)})$ and $F^0(\hat{\gamma}_5^{(2)})$. We note that the above grade two basis elements (6.52) map under $\theta_i \leftrightarrow l_i$ exactly to the leading solutions of the Picard-Fuchs system (6.45). Using the same identification we find $\mathbb{L}^{(2)2} = X_0(l_1 + l_3)l_4$ and $\mathbb{L}^{(2)5} = X_0(l_2 + l_4)l_2$ as the leading behavior of corresponding periods $\int_{\gamma_\alpha^2} \Omega = \Pi^{(2)\alpha}$ of the holomorphic four-form. This agrees with the naive expectation from the large base limit that a partial factorization of the periods occurs as $t_4 \cdot t_i^{\tilde{Z}_3}$ where $t_i^{\tilde{Z}_3}$, $i = 1, 2$, denoted the two classes in \tilde{Z}_3 [94].

It is one crucial point of our whole analysis that we can extend this matching of threefold invariants even to disk invariants counting curves with boundaries on Lagrangian cycles L in \tilde{Z}_3 , cf. section 5.4.2. Having explained the F-theory origin of this fact via an analysis of the discriminant in (6.39) we now explicitly find the flux choice in $H_H^{(2,2)}(X_4)$ for which the flux superpotential (5.98) on X_4 reproduces the brane superpotential (5.97).

By construction the fourfold \tilde{X}_4 inherits information of the fiber \tilde{Z}_3 and in particular the local limit geometry $\mathcal{O}(-3) \rightarrow \mathbb{P}_2$ for which the disk invariants have been computed in [107]. As noted earlier the brane data is translated to the F-theory picture of the fourfold \tilde{X}_4 by the Mori cone generator $\ell^{(3)}$ and its dual divisor J_3 . Therefore, we expect to reproduce all classical terms as well as to extract the disk instantons of [107] from the Gromov-Witten invariants $n_{d,0,d+k,0}(\hat{\gamma}_3)$ of a fourfold period, that we construct via (6.46) from operators of (6.45) of the form $\mathcal{R}_\gamma^{(2)} = \theta_3(\theta_1 + \theta_3) + \dots$. However, the geometry is more complicated and the ring element $\mathcal{R}_\gamma^{(2)}$ with this property is not unique. It takes the form

$$\mathcal{R}_\gamma^{(2)} = -\mathcal{R}_1^{(2)} + \frac{1}{3}\mathcal{R}_2^{(2)} + \mathcal{R}_3^{(2)} = -\theta_1^2 + \frac{1}{2}\theta_3(\theta_1 + \theta_3) + \frac{1}{6}\theta_4(\theta_1 + \theta_3) \quad (6.56)$$

that is the most convenient choice by setting the arbitrary coefficients of $\mathcal{R}_\alpha^{(2)}$, $\alpha = 4, 5, 6$, to zero. We note that only the coefficient in front of $\mathcal{R}_3^{(2)}$ was fixed to unity by the requirement of reproducing the disk instanton invariants. The two further coefficients $(-1, \frac{1}{3})$ were fixed by the requirement of reproducing the Gromov-Witten invariants n_d of local \mathbb{P}^2 [264] by the fourfold invariants $n_{d,0,d,0} \equiv n_d$, i.e. for $m = 0$, as explained below. The relation between $\mathcal{R}_\gamma^{(2)}$ and the corresponding solution is established as $\hat{\gamma} = \mathcal{R}_\gamma \Omega_4|_{z=0}$ and $\Pi^{(2)\gamma} = \int_\gamma \Omega_4$, i.e. $\mathcal{R}_\gamma^{(2)} \Pi^{(2)\gamma} = 1$, so that

$$\mathbb{L}^{(2)\gamma} = -X_0 l_1^2, \quad \mathbb{L}_\gamma^{(2)} = \frac{1}{6} X_0 l_2 (8l_1 + 9l_2 + 2l_3). \quad (6.57)$$

In other words, this implies that we have explicitly calculated the seven-brane superpotential (5.97) from the fourfold superpotential (5.98) by turning on the flux $G_4 = \hat{\gamma}$,

$$W_{\text{brane}} \equiv F^0(\hat{\gamma}) = \int \Omega_4 \wedge \hat{\gamma} = \Pi_\gamma^{(2)}. \quad (6.58)$$

By listing the numbers $n_{d_1,0,d_3,0}(\hat{\gamma})$ extracted from $F^0(\hat{\gamma})$ we obtain the results of table 6.1. The BPS invariants of the holomorphic disks depend only on the relative homology class

d_1	$d_3 = 0$	$d_3 = 1$	$d_3 = 2$	$d_3 = 3$	$d_3 = 4$	$d_3 = 5$	$d_3 = 6$
0	0	1	0	0	0	0	0
1	1	n_1	-1	-1	-1	-1	-1
2	-1	-2	$2n_2$	5	7	9	12
3	1	4	12	$3n_3$	-40	-61	-93
4	-2	-10	-32	-104	$4n_4$	399	648
5	5	28	102	326	1085	$5n_5$	-4524
6	-13	-84	-344	-1160	-3708	-12660	$6n_6$

Table 6.1: BPS invariants $n_{d_1,0,d_3,0}(\hat{\gamma})$ for the disks. With the identification $d_3 - d_1 = m$ (winding) and $d_1 = d$ (\mathbb{P}^2 degree) this agrees with Tab. 5 in [107].

of the latter. In the table $d_3 - d_1 = m$ labels the winding number of the disks and $d_1 = d$ the degree with respect to canonical class of \mathbb{P}^2 , i.e. if the open string disk superpotential is defined in terms of the closed string parameter $q = e^{2\pi i t}$ and the open string string parameter $Q = e^{2\pi i \hat{t}}$ for the outer brane as

$$W_{\text{brane}} = a_{tt}t^2 + a_{t\hat{t}}t\hat{t} + a_{\hat{t}\hat{t}}\hat{t}^2 + a_t t + a_{\hat{t}}\hat{t} + a_0 + \sum_{d=1}^{\infty} \sum_{m=-d}^{\infty} n_{d,m} \text{Li}_2(q^d Q^m), \quad (6.59)$$

then $n_{d_1,0,d_3,0} = n_{d_1,d_3-d_1}$. Note that the numbers $n_{d_1,0}$ are not calculated in the framework of [107]. However it is natural and calculable in the topological vertex formalism [265] that they should be identified with dn_d , where n_d is the closed string genus zero BPS invariant, defined via the prepotential as $F^0 = \sum_{d=1}^{\infty} n_d \text{Li}_3(q^d)$. The factor of d stems from the fact that we identify $\frac{d}{dt}F^0 \subset W_{\text{brane}}$. This interpretation as $n_{d,0,d,0} = dn_d$ could be consistently imposed and yields two further conditions as mentioned above.

To obtain the open BPS invariants of phase III of [107], we use the phase II in (6.25). In this phase the fiber class is not realized as a generator of the Kähler cone. However we readily recover the classes of \tilde{Z}_3 as

$$J_1 \leftrightarrow J_1(\tilde{Z}_3) \quad J_2 + J_3 \leftrightarrow J_2(\tilde{Z}_3) \quad (6.60)$$

by comparison of the Mori cone (6.25) with the Mori cone (6.9) of \tilde{Z}_3 . Then we fix a basis $\mathcal{R}_\alpha^{(2)}$ of the ring at grade two as

$$\theta_1^2, \quad 2\theta_2(\theta_1 + 3\theta_3), \quad \theta_3(\theta_1 + 3\theta_3), \quad \theta_1\theta_4, \quad \theta_2^2, \quad (\theta_2 + \theta_3)(2\theta_3 + \theta_4), \quad (6.61)$$

from which we obtain a basis of dual solutions $\mathbb{L}^{(k)\alpha}$ to the Picard-Fuchs system (C.9)

$$\begin{aligned}\mathbb{L}^{(2)1} &= l_1^2, \quad \mathbb{L}^{(2)2} = \frac{1}{140} (l_1 (16l_2 + 9l_3) + 3 (l_2 (6l_3 - 5l_4) - l_3 (l_3 + 5l_4))), \\ \mathbb{L}^{(2)3} &= \frac{1}{70} (l_1 (9l_2 + 16l_3) - 3 (l_3 (-6l_3 + 5l_4) + l_2 (l_3 + 5l_4))), \quad \mathbb{L}^{(2)4} = l_1 l_4, \\ \mathbb{L}^{(2)5} &= l_2^2, \quad \mathbb{L}^{(2)6} = \frac{1}{14} (l_2 + l_3) (-3l_1 + l_3 + 5l_4),\end{aligned}\quad (6.62)$$

Next we construct two solutions with leading logarithms matching the two threefold periods in (6.50) for which we are able to match the threefold invariants $d_i n_{d_1, d_2}$ in the large base limit as well. The leading logarithms of the corresponding fourfold periods read

$$\begin{aligned}\mathbb{L}_4^{(2)} &= \frac{1}{2} X_0 (l_1 + 3 (l_2 + l_3))^2, \\ \mathbb{L}_6^{(2)} &= \frac{1}{2} X_0 (l_2 + l_3) (2l_1 + 3 (l_2 + l_3)).\end{aligned}\quad (6.63)$$

which is in perfect agreement with (6.50) under the identification (6.60). We fix the corresponding operators $\tilde{\mathcal{R}}_4^{(2)}, \tilde{\mathcal{R}}_6^{(2)}$ by matching the above two leading logarithms with the classical intersections $C_{\alpha ab}^0$ via (6.46). We complete them to a basis of $\tilde{\mathcal{R}}^{(2)}$ as

$$\begin{aligned}\tilde{\mathcal{R}}_1^{(2)} &= \theta_1^2, \quad \tilde{\mathcal{R}}_2^{(2)} = \frac{1}{140} (\theta_1 (16\theta_2 + 9\theta_3) + 3 (\theta_2 (6\theta_3 - 5\theta_4) - \theta_3 (\theta_3 + 5\theta_4))), \\ \tilde{\mathcal{R}}_3^{(2)} &= \frac{1}{70} (\theta_1 (9\theta_2 + 16\theta_3) - 3 (\theta_3 (-6\theta_3 + 5\theta_4) + \theta_2 (\theta_3 + 5\theta_4))), \quad \tilde{\mathcal{R}}_4^{(2)} = \theta_1 \theta_4, \\ \tilde{\mathcal{R}}_5^{(2)} &= \theta_2^2, \quad \tilde{\mathcal{R}}_6^{(2)} = \frac{1}{14} (\theta_2 + \theta_3) (-3\theta_1 + \theta_3 + 5\theta_4),\end{aligned}$$

where again this basis relates to the leading periods (6.62) by $\theta_i \leftrightarrow l_i$. The corresponding integral basis elements of $H_H^{(2,2)}(X_4)$ read

$$\hat{\gamma}_4^{(2)} = \tilde{\mathcal{R}}_4^{(2)} \Omega_4|_{z=0}, \quad \hat{\gamma}_6^{(2)} = \tilde{\mathcal{R}}_6^{(2)} \Omega_4|_{z=0}. \quad (6.64)$$

Furthermore, we determine the ring element $\mathcal{R}_\gamma^{(2)}$ that matches the open superpotential by turning on four-form flux in the direction $\hat{\gamma} = \mathcal{R}_\gamma^{(2)} \Omega_4|_{z=0}$. Again we fix

$$\mathcal{R}_\gamma^{(2)} = a_1 \mathcal{R}_2^{(2)} - \frac{1}{10} (1 + 6a_2) \mathcal{R}_3^{(2)} + \mathcal{R}_4^{(2)} + a_3 \mathcal{R}_5^{(2)} + a_2 \mathcal{R}_6^{(2)} \quad (6.65)$$

by extracting the disk invariants from the solution associated to it via (6.46) which reads

$$\begin{aligned}\mathbb{L}^{(2)\gamma} &= c(a_1) X_0 l_2 (l_1 + 3l_3), \\ \mathbb{L}_\gamma^{(2)} &= \frac{1}{6} (l_2 + l_3) (2l_1 + 3 (l_2 + l_3)) - \frac{1}{10} (l_1 + 3 (l_2 + l_3)) (3l_1 + 29l_2 + 29l_3 + 10l_4).\end{aligned}$$

Here we explicitly displayed the dependence on the three free parameters a_i for $\mathbb{L}^{(2)\gamma}$ by $c(a_1) = \frac{7}{9+140a_1}$ and evaluated $\mathbb{L}_\gamma^{(2)}$ for the convenient choice $\underline{a} = 0$. The associated disk instantons are listed in table 6.2.

6.2 Heterotic/F-Theory Duality: Moduli and Superpotentials

In this section we present calculations in heterotic/F-theory duality for two concrete four-dimensional compactifications. These calculations serve as a direct check of heterotic/F-theory duality on the level of matching the moduli as well as the holomorphic superpotentials on both sides.

d	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
0	0	n_1	$2n_2$	$3n_3$	$4n_4$	$5n_5$	$6n_6$
1	-1	2	-5	32	-286	3038	-35870
2	0	1	-4	21	-180	1885	-21952
3	0	1	-3	18	-153	1560	-17910
4	0	1	-4	20	-160	1595	-17976
5	0	1	-5	26	-196	1875	-20644
6	0	1	-7	36	-260	2403	-25812

Table 6.2: BPS invariants $n_{k,0,i,0}(\gamma)$ for the disks of the second triangulation.

Before delving into the details of the calculations we summarize in section 6.2.1, from a conceptual point of view, the matchings we perform in order to finally compute the heterotic superpotential, in particular the heterotic flux and five-brane superpotential, from its F-theory dual flux superpotential. Then in section 6.2.2 we consider an F-theory Calabi-Yau fourfold geometry, that will have few Kähler moduli and many complex structure moduli. It is constructed directly using the techniques of section 4.3.2 from its heterotic dual, which is an elliptic Calabi-Yau threefold with base \mathbb{P}^2 and an $E_8 \times E_8$ bundle with no perturbative heterotic gauge symmetry in four dimensions. In this case we can use toric geometry to compute explicitly in the Kähler sector, which allows us to evaluate both sides of the moduli map (4.62). From this we can read off the number of deformations of the five-brane curve, if present, and check the anomaly formula (4.9), from which we determine the number of vertical five-branes as well as the class of possible horizontal five-branes. A direct calculation of the superpotential is however not possible due to a large dimension of the complex structure moduli space of X_4 . The second Calabi-Yau fourfold example, introduced in section 6.2.3, will admit few complex structure moduli and many Kähler moduli. It will basically be the mirror geometry of section 6.2.2. This allows us to identify the bundle moduli and five-brane moduli under duality by studying the Weierstrass constraint. The analysis of the Kähler sector is technically challenging and omitted. We exploit the calculations of section 6.1.3 to identify the heterotic superpotential for a horizontal five-brane as the F-theory flux superpotential for a specific four-flux, that we determine.

6.2.1 Heterotic Superpotentials from Compact Calabi-Yau Fourfolds

In this section we are mainly interested in the map of the complex structure moduli of X_4 to the moduli of the heterotic compactification. In fact, not only the moduli on both sides map according to (4.62) but also their obstructions expressed by the superpotentials (4.15) on the heterotic side and (4.37) on the F-theory side. In particular we perform calculations in the complex structure moduli space of X_4 in section 6.2.3 to derive the heterotic superpotentials explicitly for a specific four-form flux G_4 , that we determine.

Let us prepare for the following explicit calculations by a more formal study of the map of the dual superpotentials. Let us recall, that the heterotic superpotential (4.15), is given by

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^o) = W_{\text{flux}}(\underline{t}^c) + W_{\text{CS}}(\underline{t}^c, \underline{t}^g) + W_{\text{brane}}(\underline{t}^c, \underline{t}^o), \quad (6.66)$$

where $\underline{t}^c, \underline{t}^g$ and \underline{t}^o denote the complex structure, bundle and five-brane moduli respectively¹². As we have discussed in section 4.1.2 the last two terms are not inequivalent, since by tuning the moduli \underline{t}^g or \underline{t}^o one can condense or evaporate five-branes and explore different branches of the heterotic moduli space. Clearly also the moduli spaces parametrized by \underline{t}^c and \underline{t}^g do not factorize globally in complex structure and bundle moduli since the notion of a holomorphic gauge bundle on Z_3 depends on the complex structure of Z_3 . Similarly, \underline{t}^c and \underline{t}^o do not factorize as well as the notion of a holomorphic curve in Z_3 does depend on the complex structure of Z_3 .

The key point of heterotic/F-theory duality is then that we can map as in (4.62) the heterotic moduli $(\underline{t}^c, \underline{t}^g, \underline{t}^o)$ to the complex structure moduli \underline{t} of X_4 , which are encoded in the fourfold period integrals. This matching involves at least an explicit counting on both sides, as performed in section 6.2.2, and involves calculations in the Kähler moduli space of Z_3 . Then, to make precise also the extension of the equivalence to the obstructions as

$$W_{\text{het}}(\underline{t}^c, \underline{t}^g, \underline{t}^o) = W_{G_4}(\underline{t}), \quad (6.67)$$

we need to establish a dictionary between the topological data on the heterotic side, which consist of the heterotic flux quanta, the topological classes of gauge bundles and the class of the curves Σ wrapped by five-branes, and the F-theory flux quanta.

The identification (6.67) implies then as a minimal check, that the periods of Ω and complex structure moduli of Z_3 arise as a subset of the periods and complex structure moduli of Ω_4 in specific directions, cf. also section 6.1.3 and [79,94]. In order to study the duality map (6.67) further, one can include additional parts of the heterotic superpotential like the five-brane moduli and superpotential. One strategy to study this explicitly in concrete examples, as in section 6.2.3, is by restricting the heterotic gauge bundle E to be of trivial $SU(1) \times SU(1)$ type, which automatically implies the inclusion of heterotic five-branes to satisfy the anomaly cancellation condition (4.9). Then, it remains to single out the four-flux in F-theory that is dual to the curve Σ supporting the heterotic five-brane.

In general, it is the challenging part of (6.67) to determine the integral F-theory flux that is dual to a particular heterotic effect. One immediate strategy to determine G_4 is to calculate either the classical terms or some of the instanton corrections in W_{het} and then to match with the fourfold periods. Then, the G_4 -flux is automatically integral and we can use the full periods of the fourfold and the map (6.67) to determine the instanton corrections and vice versa. Formally, the flux G_4 can always be expressed via differential operators acting on Ω_4

¹²Here we denote the moduli by \underline{t} in order to indicate a choice of appropriate flat $\mathcal{N} = 1$ coordinates, which have to be chosen for $\mathcal{N} = 1$ mirror symmetry and for the matching of the heterotic with the F-theory side.

as discussed in section (5.3.3). In particular, for the five-brane superpotential $W_{\text{brane}}(t^o, t^e)$, which is the case of most interest for us, one finds that the dual flux G_4^{M5} can be expressed as

$$G_4^{\text{M5}} = \sum_p N^{p(2)} \mathcal{R}_p^{(2)} \Omega_4 \Big|_{\underline{z}=0} . \quad (6.68)$$

In addition, we note that for fluxes G_4 generated by operators $\mathcal{R}^{(2)}$ the superpotential yields an integral structure of the fourfold symplectic invariants at large volume of the mirror \tilde{X}_4 of X_4 in terms of the double-covering formula (5.88) for $k = 1$ where the co-dimension two cycle $\gamma \equiv \gamma(G_4)$ is specified by the flux [79]. Thus, by the matching (6.67) a similar integrality structure is predicted for the heterotic superpotentials in geometric phases of their parameter spaces. For superpotentials from five-branes wrapped on a curve Σ this matches naturally the disk multi-covering formula (5.93) of [98], that encodes the disk instantons ending on special Lagrangians L mirror dual to Σ . It would be interesting to explore a generalization of this integral structure to the vector bundle sector of the holomorphic Chern-Simons functional of the heterotic theory.

Finally, we conclude by a geometric way to identify the flux G_4^{M5} which corresponds to the five-brane superpotential given by the chain integral $\int_{\Gamma} \Omega$. The three-chain Γ can be mapped to a three-chain Γ in the F-theory base B_3 whose boundary two-cycles Σ lie in the worldvolume of a seven-brane over which the cycles of the F-theory elliptic fiber degenerate. By fibering the one-cycle of the elliptic fiber which vanishes at the seven-brane locus over Γ , one obtains a transcendental cycle in $H_4(X_4, \mathbb{Z})$. Its dual form lies then in the horizontal part $H_H^4(X_4, \mathbb{Z})$ and therefore yields the sought-after flux G_4 , see [29] for a review on such constructions. For a recent very explicit construction of these cycles in F-theory compactifications on elliptic K3 surfaces and Calabi-Yau threefolds see refs. [266].

6.2.2 Example 1: Horizontal Five-Branes in $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18]

We begin the discussion of a first example of heterotic/F-theory dual four-dimensional theories. First we construct the setup on the heterotic side and then infer the F-theory fourfold by duality.

Following section 4.1.1 the heterotic theory is specified by an elliptic Calabi-Yau threefold with a stable holomorphic vector bundle $E = E_1 \oplus E_2$ obeying the heterotic anomaly constraint (4.9). We choose the threefold \tilde{Z}_3 as the elliptic fibration over the base $B_2 = \mathbb{P}^2$ with generic torus fiber $\mathbb{P}_{1,2,3}[6]$, which is precisely the example (6.9) considered in section 6.1.2¹³. However, since we are studying \tilde{Z}_3 in the context of heterotic/F-theory duality where we explicitly analyze the duality relation (4.62) of the moduli on both sides, in particular including heterotic horizontal five-branes, we directly study the Kähler geometry of \tilde{Z}_3 .

¹³In order to ensure consistency of our notation we thus denote the heterotic threefold by \tilde{Z}_3 .

We recall that \tilde{Z}_3 is given as the hypersurface $p_0 = 0$ in the toric ambient space given in (6.9) with the class of the hypersurface \tilde{Z}_3 given by

$$[\tilde{Z}_3] = \sum D_i = 6B + 18H. \quad (6.69)$$

Here we denoted the two independent toric divisor classes D_i by $H = D_2$ and $B = D_1$, that are the pullback of the hyperplane class of the \mathbb{P}^2 base respectively the class of the base itself. They are identified with the Kähler cone generators J_1, J_2 in section 6.1.2 as $J_1 \equiv H$ and $J_2 \equiv 3H + B$ so that the intersections of H and B can be readily read off from (6.10). From the toric data the basic topological numbers of Z_3 are obtained as

$$\chi(\tilde{Z}_3) = 540, \quad h^{(1,1)}(\tilde{Z}_3) = 2, \quad h^{(2,1)}(\tilde{Z}_3) = 272. \quad (6.70)$$

We note that this little amount of Kähler classes allows for direct calculations in the Kähler sector, whereas a direct calculation in the complex structure sector seems technically impossible. The second Chern-class of \tilde{Z}_3 is in general given in terms of the Chern classes $c_1(B_2), c_2(B_2)$ and the section $\sigma : B_2 \rightarrow \tilde{Z}_3$ of the elliptic fibration by the formula $c_2(\tilde{Z}_3) = 12c_1(B_2)\sigma + 11c_1(B_2)^2 + c_2(B_2)$. To see this we refer to section 4.1.4 and recall that the toric variety (6.9) is in the case at hand is the projective bundle $\mathbb{P}_{\tilde{\Delta}} = \mathbb{P}(\mathcal{O} \oplus K_{B_2}^{-2} \oplus K_{B_2}^{-3})$. We further identify $\sigma = B, B_2 = \mathbb{P}^2$ and thus obtain, using $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3H)$ and $c_2(\mathbb{P}^2) = 3H^2$,

$$c_2(\tilde{Z}_3) = 36H \cdot B + 102H^2. \quad (6.71)$$

Next, in order to satisfy the heterotic anomaly formula (4.9), we have to construct the heterotic vector bundle $E_1 \oplus E_2$ and to compute the characteristic classes $\lambda(E_i)$. Since \tilde{Z}_3 is elliptically fibered the classes $\lambda(E_i)$ can be constructed using the basic methods of [182] that were briefly reviewed in section 4.1.4. We restrict $E_1 \oplus E_2$ to be an $E_8 \times E_8$ bundle over Z_3 and choose the classes $\eta_1, \eta_2 \in H^2(B_2, \mathbb{Z})$ in (4.26) as $\eta_1 = \eta_2 = 6c_1(B_2)$. Then, we use formula (4.25) for the second Chern class of E_8 -bundles to obtain

$$\lambda(E_1) = \lambda(E_2) = 18H \cdot B - 360H^2. \quad (6.72)$$

The anomaly condition (4.9) then leads to conditions on the coefficients of the independent classes in $H^4(\tilde{Z}_3)$. For the class $H \cdot B$ contributed by the base via $\sigma \cdot H^2(\mathbb{P}^2, \mathbb{Z})$ the anomaly is trivially satisfied by the $\lambda(E_i)$ in (6.72). This implies that no horizontal five-branes are present. For the class of the fiber F the anomaly forces the inclusion of vertical five-branes in the class

$$[\Sigma] = c_2(B_2) + 91c_1(B_2)^2 = 822H^2 \equiv n_f F. \quad (6.73)$$

Since F is dual to the base B_2 the number n_f of vertical branes is determined by integrating Σ over \mathbb{P}^2 as

$$n_f = \int_{\mathbb{P}^2} \Sigma = \Sigma \cdot B = 822. \quad (6.74)$$

To conclude the heterotic side we compute the index $I(E_i)$ since it appears in the identification of moduli (4.62) and thus is crucial for the analysis of heterotic/F-theory duality. We readily use the formula (4.63) to obtain that $I(E_1) = I(E_2) = 8 + 4 \cdot 360 + 18 \cdot 3 = 1502$.

Next we include horizontal five-branes to the setup by shifting the classes η_i appropriately [104]. We achieve this by putting $\eta_2 = 6c_1(B) - H$. The class of the five-brane Σ can then be determined analogous to the above discussion by evaluating (4.25) and imposing the anomaly (4.9). It takes the form

$$[\Sigma] = 91c_1(B_2)^2 + c_2(B_2) - 45c_1(B_2) \cdot H + 15H^2 + H \cdot B = 702H^2 + H \cdot B, \quad (6.75)$$

which means that we have to include five-branes in the base on a curve Σ in the class H of the hyperplane of \mathbb{P}^2 . Additionally the number of five-branes on the fiber F is altered to $n_f = 702$. Accordingly, the shifting of η_2 changes the second index to $I_2 = 1019$, whereas $I_1 = 1502$ remains unchanged.

Let us now turn to the dual F-theory description. We first construct the fourfold X_4 dual to the heterotic setup with no five-branes. In this case the base B_3 of the elliptic fibration of the fourfold X_4 is $B_3 = \mathbb{P}^1 \times \mathbb{P}^2$. This can be seen from the relation (4.51) of the classes η_i and the fibration structure of $B_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus L)$ for E_8 -bundles. Since both classes η_i equal $6c_1(B_2)$ we have $t = 0$ and thus the bundle $L = \mathcal{O}_{\mathbb{P}^2}$ is trivial as well as the projective bundle $B_3 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$. Then the fourfold X_4 is constructed as the elliptic fibration over B_3 with generic fiber given by $\mathbb{P}_{1,2,3}[6]$. Again X_4 is described as a hypersurface in a five-dimensional toric ambient space \mathbb{P}_Δ^5 as described by the toric data in (6.79) if one drops the point $(3, 2, -1, 0, 1)$ and sets the divisor D to zero. The class of X_4 is then given by

$$[X_4] = \sum_i D_i = 6B + 18H + 12K, \quad (6.76)$$

where the independent divisors are the base B_3 denoted B , the pullback of the hyperplane H in \mathbb{P}^2 and of the hyperplane K in \mathbb{P}^1 . Then, the basic topological data reads

$$\chi(X_4) = 19728 \quad , \quad h^{(1,1)}(X_4) = 3 \quad , \quad h^{(3,1)}(X_4) = 3277 \quad , \quad h^{(2,1)}(X_4) = 0. \quad (6.77)$$

Now we have everything at hand to discuss heterotic/F-theory duality along the lines of section 4.3.3, in particular the map of moduli (4.62). As discussed there, the complex structure moduli of the F-theory fourfold are expected to contain the complex structure moduli of \tilde{Z}_3 on the heterotic side as well as the bundle and brane moduli of possible horizontal five-branes. Indeed we obtain a complete matching by adding up all contributions in (4.62),

$$h^{(3,1)}(X_4) = 3277 = 272 + 1502 + 1502 + 1, \quad (6.78)$$

where it is crucial that no horizontal five-branes with possible brane moduli are present.

Finally, to obtain the F-theory dual of the heterotic theory with horizontal five-branes, we apply the recipe discussed in section 4.3.3. We have to perform the described geometric

transition of first tuning the complex structure moduli of the fourfold X_4 such that it becomes singular over the five-brane curve Σ which we then blow up into a divisor E . This way we obtain a new smooth Calabi-Yau fourfold denoted by \hat{X}_4 . The toric data of this fourfold are given by

$$\Delta_5^{\hat{X}} = \left(\begin{array}{ccccc|ccc} -1 & 0 & 0 & 0 & 0 & 3D + 3B + 9H + 6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 2D + 2B + 6H + 4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & H - E & D_5 \\ 3 & 2 & 0 & -1 & 0 & H & D_5 \\ 3 & 2 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & -1 & K + E & D_8 \\ 3 & 2 & -1 & 0 & 1 & E & D_9 \end{array} \right), \quad (6.79)$$

which agrees with (6.25) up to a similarity transformation. Here we included the last point $(3, 2, -1, 0, 1)$ and a corresponding divisor $D_9 = E$ to perform the blow-up along the curve Σ .

To demonstrate that (6.79) indeed describes a blow-up of X_4 can be seen as follows. Since the curve Σ in the heterotic theory is in the class H we have to blow-up over the hyperplane class of $B_2 = \mathbb{P}^2$ in B_3 . First we project the polyhedron $\Delta_5^{\hat{X}}$ to the base B_3 which is done just by omitting the first and second column in (6.79). Then the last point maps to the point $(-1, 0, 1)$ that subdivides the two-dimensional cone spanned by $(-1, 0, 0)$ and $(0, 0, 1)$ in the polyhedron of B_3 . This two-dimensional cone, however, corresponds precisely to the curve $\Sigma = H$. Thus, upon adding the point $(-1, 0, 1)$ the curve Σ in B_2 is removed from B_3 and replaced by the divisor E corresponding to the new point. Thus, we see that the toric data (6.79) contain the blow-up of Σ in the base B_3 in the last three columns.

The fourfold \hat{X}_4 is then realized as a generic constraint $P = 0$ in the class

$$[\hat{X}_4] = 6B + 18H + 12K + 6E. \quad (6.80)$$

Note that this fourfold has now three different triangulations which correspond to the various five-brane phases on the dual heterotic side. The topological data for the new fourfold \hat{X}_4 is given by

$$\chi(\hat{X}_4) = 16848, \quad h^{(1,1)}(\hat{X}_4) = 4, \quad h^{(3,1)}(\hat{X}_4) = 2796, \quad h^{(2,1)}(\hat{X}_4) = 0, \quad (6.81)$$

where the number of complex structure moduli has reduced in the transition as expected.

If we now analyze the map (4.62) of moduli in heterotic/F-theory duality we observe that we have to put $h^0(\Sigma, N_{Z_3}\Sigma) = 2$ in order to obtain a matching. This implies, from the point of view of heterotic/F-theory duality, that the horizontal five-brane wrapped on Σ has to have two deformation moduli. Indeed, this precisely matches the fact that the hyperplane class of \mathbb{P}^2 has two deformations since a general hyperplane is given by the linear constraint

$a_1x_1 + a_2x_2 + a_3x_3 = 0$ in the three homogeneous coordinates \underline{x} of \mathbb{P}^2 . Upon the overall scaling it thus has two moduli parameterized a \mathbb{P}^2 with homogeneous coordinates \underline{a} .

This way we have found an explicit construction of an F-theory fourfold \hat{X}_4 with complex structure moduli encoding the dynamics of a horizontal heterotic five-brane. Unfortunately, we are not able to check this matching further for example by calculation of a brane superpotential from F-theory as done in section 6.1.3 for different examples. However, this is not a conceptual problem but merely a technical difficulty due to the large number of complex structure moduli of the fourfold \hat{X}_4 which makes the determination of the solutions to the Picard-Fuchs equations very hard. It would be interesting, however, to extract the F-theory superpotential for a subsector of the moduli including the two brane deformations.¹⁴ In the next section 6.2.3 we take a different route and consider an example with only a few complex structure moduli that is constructed by using mirror symmetry. However in these cases the calculation of the indices $I(E_i)$ becomes very hard due to a big number of Kähler moduli.

We conclude by noting that we will provide further evidence for the identification of the complex structure moduli space of \hat{X}_4 with the brane moduli space in section 8.4.2. Indeed, we will show there that one can also construct \hat{X}_4 as a complete intersection starting with a heterotic non-Calabi-Yau threefold obtained by blowing up along the curve Σ in \tilde{Z}_3 . This exploits already on the heterotic side the map of deformations of Σ to complex structure moduli of the blow-up threefold under the blow-up procedure discussed in chapter 7.

6.2.3 Example 2: Five-Brane Superpotential in Heterotic/F-Theory

Let us now discuss a second example for which the F-theory flux superpotential can be computed explicitly since the F-theory fourfold admits only few complex structure moduli. Clearly, using mirror symmetry such fourfolds can be obtained as mirror manifolds of examples with few Kähler moduli.

To start with, let us consider heterotic string theory on the *mirror* of the Calabi-Yau threefold \tilde{Z}_3 which we studied in the last section 6.2.2. This mirror is the heterotic manifold Z_3 that we studied already in section 6.1.2 in the context of mirror symmetry on fourfolds. As we noted there using the methods of [260] Z_3 is also elliptically fibered, such that it has an F-theory dual description and that bundles can at least in principle be constructed explicitly using the spectral cover construction of section 4.1.4. The polyhedron of Z_3 is the dual polyhedron to (6.9) presented in (6.11) and we recall its Weierstrass form,

$$p_0 = x^3 + y^2 + xy\tilde{z}a_0u_1u_2u_3 + \tilde{z}^6(a_1u_1^{18} + a_2u_2^{18} + a_3u_3^{18} + a_4u_1^6u_2^6u_3^6). \quad (6.82)$$

The coordinates \underline{u} are the homogeneous coordinates of the twofold base B_2 and (x, y, \tilde{z}) denote

¹⁴If one considers exactly the mirror of \hat{X}_4 , as we will in fact do in section 6.2.3, it might be possible to embed this reduced deformation problem into the complicated deformation problem of \hat{X}_4 constructed in this section.

the homogeneous coordinate of $\mathbb{P}^2(1, 2, 3)$. Note that one finds that the elliptic fibration of Z_3 is highly degenerate over B_2 . The threefold is nevertheless non-singular since the singularities are blown up by many divisors in the toric ambient space of Z_3 . However, in writing (6.82) many of the coordinates parameterizing these additional divisors have been set to one¹⁵. Turning to the perturbative gauge bundle $E_1 \oplus E_2$ we restrict in the following to the simplest bundle $SU(1) \times SU(1)$ which thus preserves the full perturbative $E_8 \times E_8$ gauge symmetry in four dimensions.

To nevertheless satisfy the anomaly condition (4.9) one has to include five-branes. In particular, we consider a five-brane in Z_3 given by the equations

$$h_1 := b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 = 0, \quad h_2 := \tilde{z} = 0. \quad (6.83)$$

The curve Σ wrapped by the five-brane is thus in the base B_2 of Z_3 and horizontal. Unfortunately, it is hard to check the heterotic anomaly (4.9) explicitly as in the example of section 6.2.2 since there are too many Kähler classes in Z_3 .

However, one can proceed to construct the associated Calabi-Yau fourfold X_4 which encodes a consistent completion of the setup by duality. The associated fourfold X_4 cannot be constructed as it was done in section 6.2.2. However, one can follow the strategy of [103] summarized in diagram (6.1) to construct heterotic/F-theory dual geometries by employing mirror symmetry to first obtain the mirror fourfold \tilde{X}_4 of X_4 as the Calabi-Yau threefold fibration with generic fiber \tilde{Z}_3 being the mirror of the heterotic threefold Z_3 . This then naturally leads us to identify X_4 as the mirror to the fourfold (6.79) in section 6.2.2. This fourfold is also the main example discussed in detail in section 6.1 and in [79].

In the following we check that this is indeed the correct identification by using the formalism of [103, 105]. The Weierstrass form of X_4 can be computed using the polyhedron (6.31) that is dual to (6.79) yielding

$$P = y^2 + x^3 + m_1(\underline{u}, \underline{w}, \underline{k})xyz + m_6(\underline{u}, \underline{w}, \underline{k})z^6 = 0, \quad (6.84)$$

where we abbreviated

$$\begin{aligned} m_1(\underline{w}, \underline{u}, \underline{k}) &= a_0 u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6 k_1 k_2, \\ m_6(\underline{w}, \underline{u}, \underline{k}) &= a_1 (k_1 k_2)^6 u_1^{18} w_1^{18} w_2^{18} w_5^6 w_6^6 + a_2 (k_1 k_2)^6 u_2^{18} w_3^{18} w_5^{12} \\ &\quad + a_3 (k_1 k_2)^6 u_3^{18} w_4^{18} w_6^{12} + a_4 (k_1 k_2)^6 (u_1 u_2 u_3 w_1 w_2 w_3 w_4 w_5 w_6)^6 \\ &\quad + b_1 k_2^{12} u_1^{18} w_1^{24} w_2^{12} w_3^6 w_4^6 + b_2 k_2^{12} (u_1 u_2 u_3)^6 (w_1 w_3 w_4)^{12} \\ &\quad + c_1 k_1^{12} (u_1 u_2 u_3)^6 (w_2 w_5 w_6)^{12}. \end{aligned} \quad (6.85)$$

The coordinates \underline{u} are the coordinates of the twofold base B_2 as before and \underline{w}, k_1, k_2 are the additional coordinates of the threefold base B_3 ¹⁶. Again, note that we have set many

¹⁵The blow-down of these divisors induces a large non-perturbative gauge group in the heterotic string.

¹⁶We note that we slightly changed our labeling of the coordinates compared to (6.32), (6.33).

coordinates to one to display P . The chosen coordinates correspond to divisors which include the vertices of Δ_5^X and hence determine the polyhedron completely. In particular, one finds that (k_1, k_2) are the homogeneous coordinates of the \mathbb{P}^1 -fiber over B_2 . The coefficients $\underline{a}, \underline{b}, c_1$ encode the complex structure deformations of X_4 . However, since $h^{(3,1)}(\hat{X}_4) = 4$, there are only four complex structure parameters rendering six of them redundant.

As the first check that X_4 is indeed the correct dual Calabi-Yau fourfold, we follow the discussion of section 4.3.2 and use the stable degeneration limit [182] and write P in a local patch with appropriate coordinate redefinition as [103]

$$P = p_0 + p_+ + p_- , \quad (6.86)$$

where

$$\begin{aligned} p_0 &= x^3 + y^2 + xy\tilde{z}a_0u_1u_2u_3 + \tilde{z}^6 (a_1u_1^{18} + a_2u_2^{18} + a_3u_3^{18} + a_4u_1^6u_2^6u_3^6) , \\ p_+ &= v\tilde{z}^6 (b_1u_1^{18} + b_2u_1^6u_2^6u_3^6) , \\ p_- &= v^{-1}\tilde{z}^6 c_1u_1^6u_2^6u_3^6 . \end{aligned} \quad (6.87)$$

The coordinate v is the affine coordinate of the fiber \mathbb{P}^1 . In the stable degeneration limit $\{p_0 = 0\}$ describes the Calabi-Yau threefold of the heterotic string. In this case p_0 coincides with the constraint (6.12) of the threefold geometry Z_3 as discussed in section 6.1.2 which directly identifies it as the dual heterotic Calabi-Yau threefold to X_4 . This shows in particular that the geometric moduli of the heterotic compactification on Z_3 are correctly embedded in X_4 . The polynomials p_{\pm} encode the perturbative bundles, and the explicit form (6.87) shows that we are dealing with a trivial $SU(1) \times SU(1)$ bundle, which is also directly checked by analyzing the polyhedron of \hat{X}_4 using the methods of [101, 210]. Indeed, one shows explicitly that over each divisor $k_i = 0$ in B_3 a full E_8 gauge group is realized. Since the full $E_8 \times E_8$ gauge symmetry is preserved we are precisely in the situation of section 4.3.3, where we recalled from [105] that a smooth X_4 has to contain a blow-up corresponding to a horizontal heterotic five-brane. We now check that this allows us to identify the heterotic five-brane and its moduli explicitly in the duality to F-theory.

Let us begin by making contact to the formalism of section 4.3.3. First, to make the perturbative $E_8 \times E_8$ gauge group visible in the Weierstrass equation (6.84), we have to include new coordinates $(\tilde{k}_1, \tilde{k}_2)$ replacing (k_1, k_2) . This can be again understood by analyzing the toric data using the methods of [101, 210]. We denote by $(3, 2, \vec{\mu})$ the toric coordinates of the divisor corresponding to \tilde{k}_1 in the Weierstrass model. Then the resolved E_8 singularity corresponds to the points¹⁷

$$\begin{aligned} &(3, 2, n\vec{\mu}), \quad n = 1, \dots, 6 \quad , \quad (2, 1, n\vec{\mu}), \quad n = 1, \dots, 4 \quad , \\ &(1, 1, n\vec{\mu}), \quad n = 1, 2, 3 \quad , \quad (1, 0, n\vec{\mu}), \quad n = 1, 2 \quad , \quad (0, 0, \vec{\mu}) \end{aligned} \quad (6.88)$$

¹⁷Note that we have chosen the vertices in the $\mathbb{P}_{1,2,3}[6]$ to be $(-1, 0), (0, -1), (3, 2)$ to match the discussion in refs. [101, 210]. However, if one explicitly analyses the polyhedron of X_4 one finds that one has to apply a $Gl(2, \mathbb{Z})$ transformation to find a perfect match. This is due to the fact that X_4 , in comparison to its mirror \tilde{X}_4 , actually contains the dual torus as elliptic fiber.

While $(3, 2, 6\bar{\mu})$ corresponding to k_1 is a vertex of the polyhedron, $(3, 2, \bar{\mu})$ corresponding to \tilde{k}_1 is an inner point. Using the inner point for \tilde{k}_1 , the Weierstrass form P changes slightly, while the polynomials p_0, p_+ and p_- can still be identified in the stable degeneration limit.

Next, to determine g_5 in (4.58), we compute g of the Weierstrass form in a local patch where $\tilde{k}_2 = 1$

$$g = \tilde{k}_1^5 \left(b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6 + \tilde{k}_1 (a_1 u_1^{18} + a_2 u_2^{18} + \dots) \right). \quad (6.89)$$

The dots contain only terms of order zero or higher in \tilde{k}_1 . Comparing this with (4.61), we note that the Calabi-Yau fourfold X_4 can be understood as a blow-up¹⁸ along the curve $\tilde{k}_1 = g_5 = 0$ in the base of X_4 , where g_5 is given by

$$g_5 = b_1 u_1^{18} + b_2 u_1^6 u_2^6 u_3^6. \quad (6.90)$$

This identifies $\{g_5 = 0\}$ with the curve of the five-brane Σ in the base B_2 of Z_3 , which is in perfect accord with (6.83). Thus, one concludes that X_4 is indeed a correct fourfold associated to Z_3 with the given horizontal five-brane. As we can see from (6.90), the five-brane has one modulus. If we compare g_5 with p_+ , we see that $p_+ = v\tilde{z}^6 g_5$. This nicely fits with the bundle description of a five-brane as a small instanton. In fact, in our configuration, p_+ and p_- should describe $SU(1)$ -bundle since we have the full unbroken perturbative $E_8 \times E_8$ -bundle as described above. The $SU(1)$ -bundles do not have any moduli, such that the moduli space corresponds to just one point [182]. In the explicit discussion of the Weierstrass form in our setting, p_+ has one modulus which corresponds to the modulus of the five-brane.

Finally, we consider the computation of the F-theory flux superpotential that has been carried out already in 6.1.3 but in a different context, see also [79]. Here, we do not need to recall all the details. As was discussed there, the different triangulations of \tilde{X}_4 correspond to different five-brane phases. It was proposed that the four-form flux, for the two possible five-brane phases, is given by the basis element

$$\hat{\gamma}_1^{(2)} = (-\theta_1^2 + \frac{1}{2}\theta_3(\theta_1 + \theta_3) + \frac{1}{6}\theta_4(\theta_1 + \theta_3))\Omega_4|_{\underline{z}=0}, \quad (6.91)$$

and the element (6.65) for the other phase, where the $\theta_i = z_i \frac{d}{dz_i}$ are the logarithmic derivatives as introduced in (5.65). The moduli z_1, z_2 could be identified as the deformations of the complex structure of the heterotic threefold Z_3 , while z_3 corresponded to the deformation of the heterotic five-brane.¹⁹ Indeed, a non-trivial check of this identification was provided in section 6.1.3 and in [79], where it was shown that the F-theory flux superpotential (5.98) in the directions (6.91) matches with the superpotential for a five-brane configuration in a local Calabi-Yau threefold $K_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ obtained by decompactifying Z_3 , where the non-compact five-brane is described as a point on a Riemann surface in the base B_2 of Z_3 as discussed in section 6.1.2. Using heterotic F-theory duality as in section 4.3.3 then implies that the flux (6.91) actually describes a compact heterotic five-brane setup.

¹⁸This blow-up can be equivalently described as a complete intersection as we will discuss in sections 7.2. A simple example of such a construction will be presented in section 8.4.2.

¹⁹The deformation z_4 describes the change in p_- .

Part III

Blow Up Geometries and SU(3)-Structure Manifolds

Chapter 7

Five-Branes and Blow-Up Geometries

In this chapter we provide, following [60,81,100], a novel geometric description of the dynamics of five-branes via a dual non-Calabi-Yau threefold \hat{Z}_3 . Starting with a spacetime-filling five-brane on a curve Σ in a Calabi-Yau threefold Z_3 , as it may occur in Type IIB and heterotic M-theory compactifications, the basic motivation of [60] was to find a natural description, where both the bulk and brane fields associated to deformations of Z_3 respectively Σ are treated equally in an unified framework. To obtain such a description a very canonical procedure was proposed in [60], where the pair (Z_3, Σ) is replaced by a new threefold \hat{Z}_3 with a distinguished divisor E , that is obtained by blowing up along Σ in Z_3 . It was further argued, that both the geometric deformations of (Z_3, Σ) are canonically unified as complex structure deformations of \hat{Z}_3 and it was suggested to use Picard-Fuchs equations for the complex structure deformations of \hat{Z}_3 to study the open-closed deformations of (Z_3, Σ) . Ultimately, both the flux and five-brane superpotential should be given as solutions to these Picard-Fuchs equations.

Indeed the details of this proposal and concrete examples of five-branes in Calabi-Yau threefolds Z_3 , their open-closed Picard-Fuchs equations and the superpotentials have been worked out in [100]. Furthermore it has been argued to view the geometry of the blow-up \hat{Z}_3 not only as a tool for calculations but as being dynamically generated by the five-brane backreaction. Evidence for its use to define a flux compactification of the Type IIB and heterotic string that is dual to the original five-brane setup was provided by the definition of an $SU(3)$ -structure on \hat{Z}_3 . It is the purpose of this chapter to introduce the physical and mathematical ideas that lead to the blow-up threefold \hat{Z}_3 as the natural geometry to analyze five-brane dynamics. Simultaneously this serves as a preparation both for the calculations of the Picard-Fuchs system on \hat{Z}_3 as well as for the physical interpretation of the blow-up. The details of the calculations of the open-closed superpotential for two concrete examples and the definition of the $SU(3)$ -structure are presented in chapter 8 respectively in chapter 9.

We begin our discussion of five-brane dynamics in $\mathcal{N} = 1$ Calabi-Yau compactifications in section 7.1. We emphasize that the presence of five-brane sources for the flux is more thoroughly treated using the notion of currents and naturally leads to the consideration of the open manifold $Z_3 - \Sigma$. In addition we discuss geometric deformations of the five-brane

curve Σ , where we distinguish between deformations leading, even at first order, to massive or light fields in the four dimensional effective action. We comment on the use of the five-brane superpotential for determining higher order obstructions of the light fields. Next in section 7.2 we are naturally led to the blow-up \hat{Z}_3 , that we construct explicitly both locally and globally as a complete intersection. We give a detailed explanation for the unification of the bulk and brane geometrical deformations of Z_3 respectively Σ and their obstructions as pure complex structure deformations on \hat{Z}_3 and a specific flux element. Then in section 7.3 we analyze the open-closed deformation space by studying the variation of the pullback $\hat{\Omega}$ of the Calabi-Yau three-form Ω of Z_3 to \hat{Z}_3 under a change of complex structure on \hat{Z}_3 . There we also discuss the general structure of the open-closed Picard-Fuchs equations as obtained from a residue integral for $\hat{\Omega}$. Finally in section 7.4 we present the lift of the flux and five-brane superpotential of (Z_3, Σ) to the blow-up \hat{Z}_3 and argue that both are obtained as solutions to the same open-closed Picard-Fuchs equations on \hat{Z}_3 .

7.1 Five-Brane $\mathcal{N} = 1$ Effective Dynamics

In this section we discuss basic aspects of five-brane dynamics. Our point of view will be geometrical and appropriate to formulate the blow-up proposal in section 7.2. We begin our discussion in section 7.1.1 with a brief summary of heterotic and Type IIB string compactifications with five-branes focusing on global consistency conditions and the use of currents to describe the localized brane sources. We are naturally led to work on the open manifold $Z_3 - \Sigma$, where all fields in the theory are well-behaved, even in the presence of the singular brane source. The geometrical deformations of the five-brane around the supersymmetric configuration specified by a holomorphic curve Σ are discussed in section 7.1.2, where we present a physically motivated discussion of both light and massive fields and their behavior under complex structure deformations of Z_3 . A purely mathematical analysis of analytic families of holomorphic curves provides the necessary background to appreciate the use of the brane superpotential to determine higher order obstructions, as discussed in section 7.1.3.

7.1.1 Five-Branes, Currents and Open Manifolds

In the following we consider four-dimensional $\mathcal{N} = 1$ compactifications of Type IIB string theory and the heterotic string, where we are particularly interested in including spacetime-filling five-branes. The Type IIB setups are orientifold compactifications with D5-branes and $O5$ -planes on a three-dimensional Calabi-Yau manifold Z_3 that is modded out by the orientifold involution, cf. chapter 2. In the heterotic string we consider NS5-branes and vector bundles on a smooth Calabi-Yau threefold Z_3 as reviewed in section 4.1. In both compactifications global consistency conditions restrict the choice of valid configurations and link the discrete data counting branes and fluxes via tadpole cancellation conditions.

We briefly recall how tadpole cancellation arises both in the Type IIB string and the heterotic string. In the presence of magnetic sources like smooth heterotic bundles or localized branes, the Bianchi identity of the R–R-form F_3 and the NS–NS B-field take the form

$$dF_3 = \delta_\Sigma + \sum_i \delta_{\Sigma_i} - 2 \sum_\alpha \delta_{\tilde{\Sigma}_\alpha}, \quad (7.1)$$

$$dH_3 = \delta_\Sigma + \text{tr}\mathcal{R} \wedge \mathcal{R} - \frac{1}{30} \text{Tr}\mathcal{F} \wedge \mathcal{F} \quad (7.2)$$

Note that the $O5$ -planes carry -2 times the charge of a D5-brane. In both theories these equations imply, on the level of cohomology, global tadpole cancellation conditions. Read as equations for actual forms they imply in both theories that the wavefunction of the five-brane is sharply peaked¹ at the curve Σ which is reflected by the delta-function δ_Σ in the Bianchi identity. In contrast to the global tadpole condition that fixes only the cohomology classes, the Bianchi identity fixes, up to gauge transformations, the actual forms F_3 and H_3 pointwise and implies that globally defined forms C_2 and B_2 with F_3 and H_3 do not exist². However, in a local patch we can solve the Bianchi identity for the field strength and the potential explicitly as we present in the following. Let us note here that one crucial point that leads to the blow-up proposal below is the appropriate mathematical treatment of the actual forms H_3 or F_3 that become singular near the brane.

In the vicinity of a single brane source the Bianchi identity reads

$$d\sigma_3 = \delta_\Sigma, \quad (7.3)$$

where σ_3 is identified with the singular part in the field strength H_3 , F_3 in both theories. For the setups we will consider the other localized sources do not interfere with the following local analysis and can be treated similarly. Furthermore, we will ignore the smooth part of the heterotic bundle $c_2(E)$. Then equation (7.3) is best treated in the theory of currents, see e.g. [154]. In this context σ_3 can be understood as the Poincaré dual of a chain Γ in the following way. First we associate a functional T_Γ to every three-chain Γ with boundary $\partial\Gamma = \Sigma$ by two defining properties. For any smooth three- and two-form η_3 , φ_2 we have

$$T_\Gamma(\eta_3) = \int_\Gamma \eta_3, \quad dT_\Gamma(\varphi_2) = \int_\Gamma d\varphi_2 = \int_\Sigma \varphi_2 = T_\Sigma(\varphi_2). \quad (7.4)$$

Such a map from smooth forms to complex numbers is usually denoted as a current and is a generalization of distributions to forms. In this language (7.4) is usually written as $dT_\Gamma = T_\Sigma$. This is precisely the dual of the expression (7.3) on the level of currents. Indeed we can use σ_3 to define a current T_{σ_3} that also enjoys $dT_{\sigma_3} = T_\Sigma$ as follows

$$\begin{aligned} T_{\sigma_3}(\eta_3) &= \int_{Z_3} \sigma_3 \wedge \eta_3, \\ dT_{\sigma_3}(\varphi_2) &= \int_{Z_3} \sigma_3 \wedge d\varphi_2 = \int_{Z_3} \delta_\Sigma \wedge \varphi_2 = \int_\Sigma \varphi_2 = T_\Sigma(\varphi_2). \end{aligned} \quad (7.5)$$

¹The delta-form δ_Σ is the wavefunction of a brane in eigenstate of the position space operator.

²For a discussion of the global structure of C_2 , B_2 in terms of Čech de Rham complexes we refer to [267].

Thus we identify σ_3 and δ_Σ as the Poincare dual of Γ and Σ respectively. However, both σ_3 and δ_Σ are not forms in the usual sense. δ_Σ fails to be a form similar to the fact that the delta-function fails to be a function. σ_3 is not a form on Z_3 . Though, it is a smooth form on the open space $Z_3 - \Sigma$. This can be seen directly in a local analysis in the fiber of the normal bundle $N_{Z_3}\Sigma$, that is isomorphic to \mathbb{C}^2 . Let us summarize the essential results.

On $N_{Z_3}\Sigma|_p \cong \mathbb{R}^4$ the form σ_3 is the unique rotationally invariant form on $\mathbb{R}^4 - \{0\}$ that is orthogonal to dr and integrates to 1 over a three-sphere S_r^3 of any radius r . In hyperspherical coordinates we obtain

$$\sigma_3 = \frac{1}{2\pi^2} \text{vol}_{S^3}, \quad \int_{S_r^3} \text{vol}_{S^3} = 1. \quad (7.6)$$

Thus, σ_3 is ill-defined at $r = 0$ where the three-sphere S_r^3 degenerates. Consequently, we can deal with σ_3 rigorously by working on the open manifold $\mathbb{R}^4 - \{0\}$ where σ_3 is a smooth form and by taking boundary contributions into account in the following way.

Whenever we have a bulk integral over Z_3 we replace it by an integral over the open manifold $Z_3 - \Sigma$ as [267]

$$\int_{Z_3} \mathcal{L} := \lim_{\epsilon \rightarrow 0} \int_{Z_3 - \mathcal{U}_\epsilon^{(4)}(\Sigma)} \mathcal{L}. \quad (7.7)$$

where we subtract a tubular neighborhood $\mathcal{U}_\epsilon^{(4)}(\Sigma)$ of radius ϵ over Σ . All integrands are regular when evaluated on this open manifold, even those including the singular form σ_3 in H_3, F_3 . One has to consider two cases, either \mathcal{L} is well-behaved in Σ and thus the limit $\epsilon \rightarrow 0$ in (7.7) just gives back the integral over Z_3 . In the other case \mathcal{L} contains the form σ_3 and a boundary term is produced by partial integration as follows

$$\lim_{\epsilon \rightarrow 0} \int_{Z_3 - \mathcal{U}_\epsilon^{(4)}(\Sigma)} \sigma_3 \wedge d\varphi_2 = \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon^3(\Sigma)} \sigma_3 \wedge \varphi_2 = \int_\Sigma \varphi_2, \quad (7.8)$$

where we used in the second equality that σ_3 is locally exact, $\sigma_3 = \frac{1}{2\pi^2} \text{vol}_{S^3}$, and integrates σ_3 to 1 in each S_ϵ^3 -fiber of the sphere bundle $S_\epsilon^3(\Sigma) = \partial\mathcal{U}_\epsilon^{(4)}(\Sigma)$ over Σ .

We conclude by discussing the global structure of this construction. Since the normal bundle $N_{Z_3}\Sigma$ is in general non-trivial, we have to take into account the effects of a non-trivial connection. As worked out in [267] the adequate globalization of σ_3 is related to the Thom-class $e_3/2$ of the normal bundle, see e.g. [268] for a reference. The Thom class is the unique closed form $de_3 = 0$, that is gauge invariant under the $\text{SO}(4)$ structure on $N_{Z_3}\Sigma$ and that integrates to 1 over any fiber S_r^3 . The basic idea is to smooth out the localized source of the five-brane (7.3) using a smooth bump form $d\rho$ normalized to integral 1 with $\rho(r) = -1$ around $r \sim 0$ and $\rho(r) = 0$ for $r > 2\epsilon$ such that the support $\text{supp}(d\rho) \subset]\epsilon, 2\epsilon[$. Then

$$d\sigma_3 = d\rho \wedge e_3/2 \quad (7.9)$$

approaches δ_Σ when taking the limit $\epsilon \rightarrow 0$. Thus, we identify the contribution of the five-brane as $\sigma_3 = -d\rho \wedge e_2^{(0)}/2$ with $e_3 = de_2^{(0)}$ locally, where a possible term ρe_3 has been

discarded since e_3 is not well-defined at the position $r = 0$ of the brane. We note further that $e_2^{(0)}$ is not a global form since it is not gauge invariant under the $\text{SO}(4)$ action on the normal bundle. Then, we obtain the global expressions for the field strength F_3 and H_3 as

$$F_3 = \langle F_3 \rangle + dC_2 - d\rho \wedge e_2^{(0)}/2, \quad H_3 = \langle H_3 \rangle + dB_2 - d\rho \wedge e_2^{(0)}/2 + \omega_3, \quad (7.10)$$

where $\omega_3 = \omega_3^L - \omega_3^G$ denotes the Chern-Simons form for $\text{tr}R^2 - \frac{1}{30}\text{Tr}F^2$ and $\langle F_3 \rangle, \langle H_3 \rangle$ are background fluxes in $H^3(Z_3, \mathbb{Z})$. With these formulas at hand we immediately check that the reasoning of (7.7) and the localization (7.8) to the boundary of the open manifold $Z_3 - \Sigma$ applies globally. Furthermore, the expansion (7.10) formally unifies the superpotentials as we will discuss in detail below in section 7.1.3.

Finally, we note that (7.10) implies that C_2 respectively B_2 have an anomalous transformation under the $\text{SO}(4)$ gauge transformations of $N_{Z_3}\Sigma$. This is necessary to compensate the anomalous transformation $\delta e_2^{(0)}$ so that F_3 respectively H_3 are gauge invariant. This anomalous transformation plays a crucial role for anomaly cancellation in the presence of five-branes [267].

7.1.2 Deformations and Supersymmetry Conditions

In this section we discuss the massive and light fields associated to geometric deformations of a five-brane. In many situations there is a superpotential for the light fields that obstructs deforming the brane at higher order. As a preparation of the following discussion, we refer the reader to the unobstructed case of the complex structure moduli of Calabi-Yau threefolds Z_3 as reviewed in section 5.2. This serves as an introduction to the necessary concepts that similarly apply in the more complicated case of brane deformations and superpotentials presented here.

Brane Deformations I: Infinitesimal Deformations of Holomorphic Curves

In the following we consider a five-brane wrapped on a curve Σ in a given Calabi-Yau background Z_3 . The five-brane preserves $\mathcal{N} = 1$ supersymmetry if Σ is a holomorphic curve, cf. section 3.1 for the Type II analysis. A holomorphic curve can be specified as curve of minimal volume in its homology class. In the language of calibrations this condition reads

$$\text{vol}_\Sigma = J|_\Sigma \quad (7.11)$$

using the calibration by the Kähler form J on Z_3 . In the effective four-dimensional theory the volume of the wrapped curve contributes terms to the scalar potential. However, the leading term for holomorphic curves is canceled, in a consistent compactification obeying (7.1) or (7.2), by contributions from the supersymmetric $O5$ -planes in Type IIB, as demonstrated explicitly for the example of a D5-brane in section 3.2, or by bundle and curvature contributions in the

heterotic string. Thus, this part of the vacuum energy cancels which is a necessary condition for supersymmetry. Indeed, this is easily seen, for example, in orientifold setups. As we have seen in chapter 2 the orientifold compactification preserve $\mathcal{N} = 1$ supersymmetry in the effective theory if the geometric part of the orientifold projection is a holomorphic and isometric involution σ acting on Z_3 . Hence, the $O5$ -planes, being the fix-point set of σ , wrap holomorphic curves inside Z_3 and are also calibrated with respect to J . Thus, they contribute the same potential in the vacuum but with opposite sign as argued in section 3.2 [60].

Let us now consider a general fluctuation of the supersymmetric $\Sigma \equiv \Sigma_0$ to a nearby curve Σ_s . From the above one expects the generation of a positive potential when deforming Σ non-holomorphically. A deformation is described by a complex section s of the normal bundle $N_{Z_3}\Sigma \equiv N_{Z_3}^{1,0}\Sigma$. The split of the complexified normal bundle has been performed in a background complex structure of Z_3 . Clearly, the space of such sections is infinite dimensional as is the space of all Σ_s . To make the identification between Σ_s and s more explicit, one recalls that in a sufficiently small neighborhood of Σ_0 the exponential map exp_s is a diffeomorphism of Σ_0 onto Σ_s . Roughly speaking, one has to consider geodesics through each point p on Σ_0 with tangent $s(p)$ and move this point along the geodesic for a distance of $\|s\|$ to obtain the nearby curve Σ_s as depicted in figure 7.1. That the holomorphic curve Σ_0 is of minimal volume

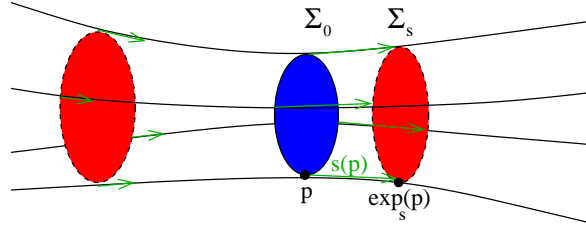


Figure 7.1: Deformations of Σ_0 along geodesics in Z_3 . The tangent vector $s(p)$ along the geodesic passing p is a normal vector of Σ_0 at p .

can now be seen infinitesimally. For any normal deformation $\Sigma_{\epsilon s}$ to Σ_0 , i.e. a deformation with infinitesimal displacement ϵs , the volume increases quadratically [169, 170] as

$$\left. \frac{d^2}{d\epsilon^2} \text{Vol}(\Sigma_{\epsilon s}) \right|_{\epsilon=0} = \frac{1}{2} \int_{\Sigma} \|\bar{\partial}s\|^2 \text{vol}_{\Sigma}, \quad (7.12)$$

where $\|\bar{\partial}s\|^2$ denotes the contraction of indices, both on the curve as well as on its normal bundle $N_{Z_3}\Sigma$ via the metric. In the effective action, (7.12) is the leading F-term potential when deforming the D5-brane curve non-holomorphically as we derived explicitly in section 3.4. A quadratic term of the form (7.12) in the scalar potential implies that the four-dimensional fields corresponding to these non-holomorphic deformations s acquire masses given by the value of the integral (7.12)³. When integrating out the massive deformations with $\bar{\partial}s \neq 0$ the

³The first variation of $\text{vol}(\Sigma_u)$ vanishes by the First Cousin Principle [170].

remaining sections are elements of

$$H^0(\Sigma, N_{Z_3}\Sigma) \equiv H^0_{\bar{\partial}}(\Sigma, N_{Z_3}^{1,0}\Sigma) \subset C^\infty(N_{Z_3}\Sigma). \quad (7.13)$$

Reversely only holomorphic sections $s \in H^0(\Sigma, N_{Z_3}\Sigma)$ deforming Σ into a nearby curve Σ_s can lead to massless or light fields in the effective theory. It is crucial to note that even for an $s \in H^0(\Sigma, N_{Z_3}\Sigma)$ the deformation might be obstructed at higher order and hence not yield a massless deformation. The higher order mass terms for these deformations can be studied by computing the superpotential as we will discuss throughout the next sections.

Before delving into the discussion of these holomorphic deformations, let us conclude with a discussion of the effect of complex structure deformations on the F-term potential (7.12) and the hierarchy of masses of fields associated to brane deformations. Deformations generated by non-holomorphic vector fields s do not obey the classical equations of motion⁴. The main complication is that there are infinitely many such off-shell deformations and it would be very hard to compute their full scalar potential. In contrast to $C^\infty(N_{Z_3}\Sigma)$, the space $H^0(\Sigma, N_{Z_3}\Sigma)$ is finite dimensional. However, there is a distinguished finite dimensional subset of $C^\infty(N_{Z_3}\Sigma)$ that should not be integrated out in the effective action. This is related to the fact that the dimension of $H^0(\Sigma, N_{Z_3}\Sigma)$ is not a topological quantity and will generically jump when varying the complex structure of Z_3 . For example, this can lift some of the holomorphic deformations $s \in H^0(\Sigma, N_{Z_3}\Sigma)$ since the notion of a holomorphic section is changed. Indeed, by deforming $\bar{\partial}$ by A in $H^1(Z_3, TZ_3)$ we obtain

$$\left. \frac{d^2}{d\epsilon^2} \text{Vol}(\Sigma_{\epsilon s}) \right|_{\epsilon=0} = \frac{1}{2} \int_{\Sigma} \|As\|^2 \text{vol}_{\Sigma} = \frac{1}{2} |t|^2 \int_{\Sigma} \|A_1 s\|^2 \text{vol}_{\Sigma} + \mathcal{O}(t^4), \quad (7.14)$$

where A_1 is the first order complex structure deformation of Z_3 as introduced above. Here we used that the complex structure on Σ is induced from Z_3 and s is in $H^0(N_{Z_3}\Sigma)$ in the unperturbed complex structure on Z_3 , $\bar{\partial}s = 0$. This result is clear from the point of view of the new complex structure $\bar{\partial}' = \bar{\partial} + A$, since $\bar{\partial}'s = As \neq 0$. Thus s is a section in $C^\infty(N_{Z_3}\Sigma)$ in the new complex structure unless s is in the kernel of A . Similarly, the corresponding field acquires a mass given by the integral (7.14). However, the main difference to a generic massive mode in $C^\infty(N_{Z_3}\Sigma)$ with mass at the compactification scale, cf. eq. (7.12) and (3.88), is the proportionality to the square of the VEV of t . Consequently the mass of this field can be made parametrically small tuning the value of t . Thus, we can summarize our approach to identify the light fields as follows: (1) drop an infinite set of deformations s which are massive via (7.12) at each point in the complex structure moduli space, (2) include any brane deformation that has vanishing (7.12) at some point in the closed string moduli space. These remaining deformations are not necessarily massless at higher orders in the complex structure deformations, or at higher ϵ order when expanding $\text{Vol}(\Sigma_{\epsilon s})$. This induces a five-brane superpotential W which can be computed using the blow-up proposal as we will show for a number of examples in section 8.⁵

⁴This can be seen readily by varying (7.12) with respect to s .

⁵The critical locus of W will either set the VEV t back to zero promoting s to a unobstructed deformation

Brane Deformations II: Analytic Families of Holomorphic Curves

Let us now present the standard account on deformations of holomorphic curves [259]. The basic question in this context is, as in the case of complex structure deformations, whether a given infinitesimal deformation can be integrated. Mathematically, finite deformations are described by the existence of an analytic family of compact submanifolds, in our context of curves. An analytic family of curves is a fiber bundle over a complex base or parameter manifold M with fibers of holomorphic curves Σ_u in Z_3 over each point $u \in M$.

Given a single curve Σ in Z_3 one can ask the reverse question, namely under which conditions does an analytic family of curves exist? The answer to this question was formulated by Kodaira [259]. In general an analytic family of holomorphic curves⁶ exists if the obstructions ψ , that are elements in $H^1(\Sigma, N_{Z_3\Sigma})$, vanish at every order m , which is of course trivially the case if $H^1(\Sigma, N_{Z_3\Sigma}) = 0$. Then \underline{u} are coordinates of points u in M and a basis of holomorphic sections in $H^0(\Sigma_u, N_{Z_3\Sigma_u})$ is given by the tangent space of TM_u via the isomorphism⁷

$$\varphi_*^z : \frac{\partial}{\partial u^a} \longmapsto \frac{\partial \varphi^i(z^i; u)}{\partial u^a} \quad (7.15)$$

at every point u in M . Here, φ^i , $i = 1, 2$, are local normal coordinates to Σ_u , cf. eqn. (7.16). In other words, in this case every deformation $H^0(\Sigma, N_{Z_3\Sigma})$ corresponds to a finite direction u^a in the complex parameter manifold M of the analytic family of curves.

This theorem can be understood locally [259] but is somewhat technical. Starting with the single holomorphic curve Σ we introduce patches U_i on Z_3 covering Σ with coordinates y_1^i, y_2^i, z^i . Then Σ is described as $y_1^i = y_2^i = 0$ and z^i is tangential to Σ . A deformation $\Sigma_{\underline{u}}$ of $\Sigma = \Sigma_0$ is described by finding functions $\varphi_l^i(z^i; \underline{u})$, $l = 1, 2$, with the boundary condition $\varphi_l^i(z^i; 0) = 0$ such that $\Sigma_{\underline{u}}$ reads

$$y_1^i = \varphi_1^i(z^i; \underline{u}), \quad y_2^i = \varphi_2^i(z^i; \underline{u}) \quad (7.16)$$

upon introducing parameters \underline{u} for convenience chosen in polycylinders $\|\underline{u}\| < \epsilon$. Furthermore, the first derivatives $\frac{\partial}{\partial u^a} \varphi_k^i|_{\underline{u}=0}$ should form a basis s^a of $H^0(\Sigma, N_{Z_3\Sigma})$. In addition, these functions have to obey specific consistency conditions, that we now discuss. As in the complex structure case, these functions are explicitly constructed as a power series

$$\varphi^i(z^i; \underline{u}) = \varphi^i(0) + \varphi_1^i(\underline{u}) + \varphi_2^i(\underline{u}) + \dots, \quad \|\underline{u}\| < \epsilon, \quad (7.17)$$

where we suppress the dependence on z^i and further denote a homogeneous polynomial in u of degree n by $\varphi_n^i(\underline{u})$. The first order deformation is defined as

$$\varphi_1^i(\underline{u}) = \sum_a u^a s_a^{(i)}(z^i), \quad (7.18)$$

or will leave a discrete set of holomorphic curves.

⁶Kodaira considered the general case of a compact complex submanifold in an arbitrary complex manifold.

⁷This map is called the infinitesimal displacement of Σ_u along $\frac{\partial}{\partial u^a}$ [259].

where $a = 1, \dots, h^0(N\Sigma)$ in the basis s_a of $H^0(\Sigma, N_{Z_3}\Sigma)$ so that (7.15) is obviously an isomorphism.

Then the m^{th} obstructions $\psi^{ik}(z^k; \underline{u})$ are homogeneous polynomials of order $m+1$ taking values in Čech 1-cocycles on the intersection $U_i \cap U_j \cap U_k$ of the open covering of Σ with coefficients in $N_{Z_3}\Sigma$. This means that the collection of local section $\psi^{ik}(z^k; \underline{u})$ defines an element in the Čech-cohomology $H^1(\Sigma, N_{Z_3}\Sigma)$. It expresses the possible mismatch in gluing together the $\varphi^i(z^i; \underline{u})$ defined on open patches $U_i \cap \Sigma_{\underline{u}}$ consistently to a global section on $\Sigma_{\underline{u}}$ at order $m+1$ in \underline{u} . In other words if the obstruction at m^{th} order is trivial and we consider (7.16) on U_i and U_k ,

$$U_i : y_l^i = \varphi_l^i(z^i; \underline{u}), \quad U_k : y_l^k = \varphi_l^k(z^k; \underline{u}), \quad (l = 1, 2), \quad (7.19)$$

then there exist functions f^{ik} and g^{ik} with $y_l^i = f_l^{ik}(\underline{y}^k, z^k)$, $z^i = g^{ik}(\underline{y}^k, z^k)$ so that

$$\begin{aligned} y^i &= \varphi^i(z^i; \underline{u}) = \varphi^i(g^{ik}(\underline{y}^k, z^k); \underline{u}) = \varphi^i(g^{ik}(\varphi^k(z^k; \underline{u}), z^k); \underline{u}) \\ y^i &= f^{ik}(\underline{y}^k, z^k) = f^{ik}(\varphi^k(z^k; \underline{u}), z^k) \\ \Rightarrow \varphi^i(g^{ik}(\varphi^k(z^k; \underline{u}), z^k); \underline{u}) &= f^{ik}(\varphi^k(z^k; \underline{u}), z^k) \end{aligned} \quad (7.20)$$

holds at order $m+1$ in \underline{u} . Here we suppressed the index l labeling the coordinates y_1^i, y_2^i . Then $\psi^{ik}(z; \underline{u})$ is the homogeneous polynomial of degree $m+1$

$$\psi^{ik}(z^k; \underline{u}) := [\varphi^i(g^{ik}(\varphi^k(z^k; \underline{u}), z^k); \underline{u}) - f^{ik}(\varphi^k(z^k; \underline{u}), z^k)]_{m+1} \quad (7.21)$$

where we expand φ^i, φ^k to order m in \underline{u} . It can be shown to have the transformation

$$\psi^{ik}(z^k; \underline{u}) = \psi^{ij}(z^j; \underline{u}) + F^{ij}(z^j) \cdot \psi^{jk}(z^k; \underline{u}), \quad (7.22)$$

where $F^{ij}(z^j)$ is the complex 2×2 transition matrices on $N_{Z_3}\Sigma$ at a point z^j in $\Sigma_{\underline{u}}$ that acts on the two-component vector $\psi^{jk} \equiv (\psi_1^{jk}, \psi_2^{jk})$. This equation identifies the ψ^{ik} as elements in $H^1(N_{Z_3}\Sigma)$ which can be identified by the Dolbeault theorem with $\bar{\partial}$ -closed $(0, 1)$ -forms taking values in $N_{Z_3}\Sigma$, $H^1(N_{Z_3}\Sigma) = H_{\bar{\partial}}^{(0,1)}(N_{Z_3})$. Assuming that all ψ^{ik} are trivial in cohomology and further proving the convergence of the power series (7.17), the analytic family of holomorphic curves is constructed.

In principle one can calculate the obstructions ψ^{ik} according to this construction at any order m . However, the obstructions are precisely encoded in the superpotential of the effective theory of a five-brane on Σ . This superpotential is in general a complicated function of both the brane and bulk deformations. Thus, determining the superpotential is equivalent to solving the deformation theory of a pair given by the curve Σ and the Calabi-Yau threefold Z_3 containing it. It is this physical ansatz that we will take in the following.

7.1.3 Formalizing Five-Brane Superpotentials

In this section we discuss, in the light of currents and brane deformations, the perturbative superpotentials both in the type IIB as well as in the heterotic theory. Using the expansion

(7.10) the superpotential can conveniently be written as

$$W_{\text{IIB}} = \int_{Z_3} \Omega \wedge F_3 \quad W_{\text{het}} = \int_{Z_3} \Omega \wedge H_3, \quad (7.23)$$

where both F_3 and H_3 have to be considered appropriately as currents obeying (7.1). According to the expansion (7.10) this contains now both the conventional flux superpotential, the heterotic Chern-Simons functional and the five-brane superpotential. Let us put special emphasis on the part contributed by the brane W_{brane} .

The brane superpotential has the following properties. It depends holomorphically on the complex structure moduli of Z_3 , as well as on the first order deformations corresponding to holomorphic sections of $N_{Z_3}\Sigma$. More precisely, we expect a superpotential $W_{\text{brane}} = (u^a)^{n+1}$ if the deformation along the direction s_a is obstructed at order n [66, 67]. Furthermore, the F-term supersymmetry conditions or critical values of W_{brane} in mathematical terms precisely correspond to holomorphic curves. Demanding these basic properties of W_{brane} was sufficient to determine the brane superpotential⁸ in [65] in the context of M-theory on a Calabi-Yau threefold Z_3 with a spacetime-filling M5-brane supported on a curve Σ ,

$$W_{\text{brane}} = \int_{\Gamma(\underline{u})} \Omega(z). \quad (7.24)$$

Here $\Gamma(\underline{u})$ denotes a three-chain bounded by the deformed curve $\Sigma_{\underline{u}}$ and the reference curve Σ_0 that is in the same homology class. Indeed, W_{brane} depends on both the deformations \underline{u} of the five-brane on Σ as well as the complex structure moduli \underline{z} of Z_3 due to the holomorphic three-form Ω . In the language of currents the chain integral can be rewritten in the form (7.23) using the expansion (7.10) as

$$W_{\text{brane}} = \int_{Z_3 - \mathcal{U}_\epsilon^{(4)}(\Sigma)} \Omega \wedge \rho e_3. \quad (7.25)$$

We note that this is gauge invariant under $\text{SO}(4)$ gauge transformations on $N_{Z_3}\Sigma$ and has only support on a small neighborhood of Σ so that it localizes on Σ as expected.

In the next section 7.2 we present the lift of the superpotential (7.23) to the blow-up threefold \hat{Z}_3 , with a focus on the flux and brane superpotential. We employ the geometrization of bulk-brane deformations on \hat{Z}_3 and the extension of the geometrical tools familiar from the study of the complex structure moduli space of Calabi-Yau threefolds, cf. section 5.2, to the non-Calabi-Yau threefold \hat{Z}_3 to determine the superpotential explicitly. Finally in chapter 9, we even understand both the closed superpotential W_{flux} as well as the brane superpotential W_{brane} as part of the generalized flux superpotential on the $SU(3)$ -structure manifold \hat{Z}_3 .

⁸Alternatively, W_{brane} can be directly deduced by dimensional reduction of the D5-brane action as presented in section 3.3 [60].

7.2 Five-Brane Blow-Ups and Unification of Deformations

In section 7.1.1 we started from the Bianchi identities for F_3 and H_3 and explained how five-brane sources on curves Σ are properly described by delta-currents. Using the language of currents it was further explained, how to relate period and chain integrals on Z_3 to regularized integrals on the open manifold $Z_3 - \Sigma$. The crucial point of the blow-up proposal [60] presented in this section is to replace the open manifold $Z_3 - \Sigma$ by a physically equivalent geometry \hat{Z}_3 with a distinguished divisor E . In addition the blow-up proposal naturally yields a flux $F_2 = [\Sigma]$ on E which can be understood as the partially dissolved five-brane charge. Furthermore, this implies an embedding of the open and closed deformations of the geometry Z_3 and the brane on Σ into pure complex structure deformations on \hat{Z}_3 .

In the first part of this section, section 7.2.1, we construct the manifold \hat{Z}_3 by blowing up a \mathbb{P}^1 -bundle along Σ . This introduces a new divisor E in \hat{Z}_3 , the exceptional divisor, that is by construction a ruled surface over Σ . Away from the exceptional divisor the open manifolds $Z_3 - \Sigma$ and $\hat{Z}_3 - E$ are biholomorphic. Using this fact and the formalism of section 7.1.1 we can evaluate the open integrals on $\hat{Z}_3 - E$. Furthermore it becomes possible to extend all open integrals, in particular the brane superpotential W_{brane} , the forms H_3 and F_3 as well as the closed periods⁹ $\Pi^k(\underline{z})$ from $Z_3 - \Sigma = \hat{Z}_3 - E$ to \hat{Z}_3 by constructing local completions of these quantities in the vicinity of the divisor E . As explained in section 7.2.2 for the case that Σ is given as a complete intersection, the blow-up proposal unifies the description of the closed and open deformations of (Z_3, Σ) , which both become complex structure deformations on \hat{Z}_3 . Of particular importance for the superpotential (7.23) and deformation theory on \hat{Z}_3 is the pullback $\hat{\Omega}$ of the Calabi-Yau three-form Ω of Z_3 to \hat{Z}_3 , which we construct in section 7.3. In section 7.4 we describe how the superpotentials concretely map to the blow-up \hat{Z}_3 .

Ultimately as explained in chapter 9, the flux and the brane superpotentials on \hat{Z}_3 of section 7.1.3 are unified to a flux superpotential on \hat{Z}_3 . The latter structure requires in addition to the extension of Ω also the extension of the Kähler form J and the flux H_3 from Z_3 to \hat{Z}_3 . Our formalism as presented in this section can be understood as the first step in the full geometrization of the five-brane and prepares the approach of chapter 9 to consider the flux-geometry \hat{Z}_3 , F_2 as a string background with $SU(3)$ -structure.

7.2.1 Geometric Properties of the Blow-Up along Σ

Given a k -dimensional complex submanifold Σ_k in an n -dimensional complex manifold Z_n , it is a standard technology in algebraic geometry [154] to blow-up along Σ_k to obtain a new n -dimensional complex manifold \hat{Z}_n . This directly applies to a supersymmetric five-brane on a holomorphic curve¹⁰ $\Sigma = \Sigma_1$ inside a Calabi-Yau threefold Z_3 .

⁹See section 5.2 for a review of the definition of periods on a Calabi-Yau threefold Z_3 .

¹⁰For the use of the blow-up proposal to analyze non-holomorphic deformations of Σ cf. section 7.2.2.

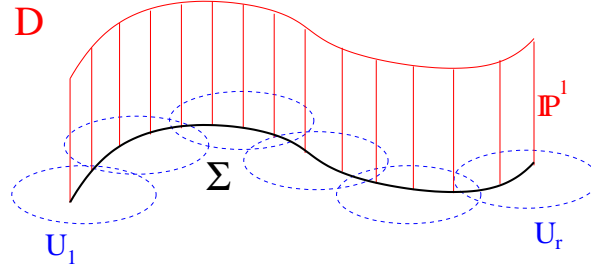


Figure 7.2: Blow up of the curve Σ to the ruled surface E

Since the blow-up is a local operation, it can be described near the complex submanifold within neighborhoods U_α with the topology of a disk, which cover Σ . Let $y_{\alpha,i}$, $i = 1, \dots, 3$ be local coordinates¹¹ on U_α in which Σ is specified by the intersection of two divisors $D_{\alpha,1} \cap D_{\alpha,2}$ i.e. by $y_{\alpha,i} = 0$, $i = 1, 2$.

The local blow-up can be described as a hypersurface constraint [154]

$$\hat{U}_\alpha = \{(y_{\alpha,1}, y_{\alpha,2}, y_{\alpha,3}, (l_{\alpha,1} : l_{\alpha,2})) \in U_\alpha \times \mathbb{P}^1 : y_{\alpha,2}l_{\alpha,1} - y_{\alpha,1}l_{\alpha,2} = 0\}. \quad (7.26)$$

Here $(l_{\alpha,1} : l_{\alpha,2})$ are projective coordinates of the \mathbb{P}^1 over the local patch U_α . We define a projection map $\pi_\alpha : \hat{U}_\alpha \rightarrow U_\alpha$ by discarding the direction of the \mathbb{P}^1 ,

$$\pi_\alpha(y_{\alpha,1}, y_{\alpha,2}, y_{\alpha,3}, (l_{\alpha,1} : l_{\alpha,2})) = (y_{\alpha,1}, y_{\alpha,2}, y_{\alpha,3}). \quad (7.27)$$

Obviously, $\hat{U}_\alpha - \pi_\alpha^{-1}(\Sigma)$ is biholomorphic to $U_\alpha - \Sigma$, as we can eliminate the $l_{\alpha,i}$ in the hypersurface (7.26) outside of the locus $y_{\alpha,1} = y_{\alpha,2} = 0$ that defines Σ . Conversely, the set $E_\alpha := \pi_\alpha^{-1}(0, 0, y_{\alpha,3}) = \pi_\alpha^{-1}(\Sigma)$ is described as follows. Over a point $(0, 0, y_{\alpha,3}) \in \Sigma$ in U_α the fibres of the projection π_α are canonically, i.e. independently of the coordinate system, identified with lines in the projectivized normal bundle $\mathbb{P}(N_{U_\alpha}\Sigma) = \mathbb{P}(\mathcal{O}(D_{\alpha,1}) \oplus \mathcal{O}(D_{\alpha,2}))$,

$$(0, 0, y_{\alpha,3}, (l_{\alpha,1} : l_{\alpha,2})) \mapsto l_{\alpha,1} \frac{\partial}{\partial y_{\alpha,1}} + l_{\alpha,2} \frac{\partial}{\partial y_{\alpha,2}}. \quad (7.28)$$

This allows to glue the open sets \hat{U}_α to obtain \hat{Z}_3 and the E_α to obtain a divisor $E = \pi^{-1}(\Sigma)$ which is identified with

$$E = \mathbb{P}(N_{Z_3}\Sigma). \quad (7.29)$$

Similarly we obtain a unique global projection map π that is trivially extended to all open sets \hat{U}_α and thus to \hat{Z}_3 as the identity map on $\hat{Z}_3 - E = Z_3 - \Sigma$. The divisor E is the exceptional divisor and it is a \mathbb{P}^1 -ruled surface over Σ by (7.29).

¹¹Given the submanifold locally by $h_i(\underline{x}) = 0$, $i = 1, 2$ in generic x_1, \dots, x_3 coordinates of U_α this choice is fixed by the inverse function theorem stating that for every point $x_0 \in \mathbb{C}^3$ with $(\partial_k h_1 \partial_l h_2 - \partial_l h_1 \partial_k h_2)|_{x_0} \neq 0$ for $k, l \neq j$, there exists a local parameterization of \mathcal{C} near x_0 as a graph over x_j . In particular, the blow-up is independent of the coordinates used, cf. p. 603 [154].

Essential facts that are intensively used in this paper are the biholomorphism

$$\pi : (\hat{Z}_3 - E) \rightarrow (Z_3 - \Sigma) \quad (7.30)$$

and the statement, that all relevant aspects of the blow-up can be analyzed locally in patches near Σ , except for the non-triviality of $N_{Z_3}\Sigma$, which is captured by the Thom class $\frac{e_3}{2}$.

The relation between the local and global construction is particularly easy if Z_3 is given by a family hypersurface $P(\underline{x}, \underline{z}) = 0$ and Σ is constructed as a complete intersection of $P(\underline{x}, \underline{z}) = 0$ and divisors D_i given by $h_i(\underline{x}, \underline{u}) = 0$, $i = 1, 2$ in \mathbb{P}_Δ . Here the ambient space is in general a toric variety \mathbb{P}_Δ with homogeneous coordinates \underline{x} and the variables $\underline{z}, \underline{u}$ parameterize the complex structure and brane moduli respectively. Then we can choose in any patch U_α coordinates so that $y_{\alpha,1} = h_1(\underline{x})|_{U_\alpha}$, $y_{\alpha,2} = h_2(\underline{x})|_{U_\alpha}$ and $y_{\alpha,3}$ is a coordinate along Σ . Now $N_{U_\alpha}\Sigma$ is globally given as the sum of two line bundles, $N_{Z_3}\Sigma = \mathcal{O}(D_1) \oplus \mathcal{O}(D_2)$, and \hat{Z}_3 is given globally as the complete intersection in the total space of the projective bundle

$$\mathcal{W} = \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2)). \quad (7.31)$$

Indeed, using the projective coordinates $(l_1, l_2) \sim \lambda(l_1, l_2)$ on the \mathbb{P}^1 -fiber of the blow-up, \hat{Z}_3 can be written as

$$P(\underline{x}, \underline{z}) = 0, \quad Q \equiv l_1 h_2(\underline{x}, \underline{u}) - l_2 h_1(\underline{x}, \underline{u}) = 0, \quad (7.32)$$

in the projective bundle \mathcal{W} .

We conclude by mentioning selected geometrical properties of the blow-up threefold \hat{Z}_3 and refer to appendix B.3 for details. First of all we note that \hat{Z}_3 does not meet the Calabi-Yau condition. This is a consequence of the general formula for the Chern classes of the blow-up of a complex manifold along a curve [154], $c_1(\hat{Z}_3) = -K_{\hat{Z}_3} = \pi^*(c_1(Z_3)) - [E] = -[E] \neq 0$ where we used that Z_3 is a Calabi-Yau manifold in the last equality. In addition, \hat{Z}_3 is not a Fano variety. One important consequence from this is that complex structure deformations in $H^1(\hat{Z}_3, T\hat{Z}_3)$ are not necessarily isomorphic to $H^{(2,1)}(\hat{Z}_3)$, since the holomorphic three-form $\hat{\Omega}$ vanishes along E . Secondly, it is crucial to emphasize that the blow-up is the minimal description of the deformations of the pair (Z_3, Σ) in the sense that in the blow-up procedure no *new* degrees of freedom associated to deformations of E are introduced. Indeed by construction the normal bundle to E in \hat{Z}_3 is the tautological bundle T [154], see appendix B.3 for details, which is a negative bundle on E . Thus E has no deformations, $H^0(E, N_{\hat{Z}_3}E) = \emptyset$, i.e. E is isolated. Furthermore, it can be shown mathematically rigorously that all deformations of Z_3 and the curve Σ , that are deformations of complex structures of Z_3 and deformations of Σ in Z_3 , map to complex structure deformations of \hat{Z}_3 .¹² The Kähler sector of Z_3 maps to that of \hat{Z}_3 that contains one additional class of the exceptional divisor E .

¹²We thank Daniel Huybrechts for a detailed explanation of the equivalence of the two deformation theories.

7.2.2 Unification of Open and Closed Deformations on Blow-Up Threefolds

In the first part of this section we present the key points of the blow-up proposal suggested in [60], namely the unification of open and closed deformations, and its use to analyze the geometrical dynamics of five-branes. Then, in a second part we study the deformation space of branes wrapping rational curves via the complete intersection curves (7.32) and the blow-up of the latter.

Matching Deformations and Obstructions: a Mathematical Proposal

We have described in section 7.1.2 that the infinitesimal elements $\varphi_*^z(\frac{\partial}{\partial u_a}) = \partial_{u_a}\varphi(z; \underline{u})$ span the tangent space to the open deformations space and live in $H^0(\Sigma, N_{Z_3}\Sigma)$, while it was reviewed in section 5.2 that deformation of the closed complex structure deformations of a manifold M live¹³ in $H^1(M, T_M)$. Up to global automorphisms of the toric ambient space \mathbb{P}_Δ , which are compatible with the torus action, this cohomology can be represented by the infinitesimal deformations δ_z of the parameters \underline{z} multiplying monomials in $P(\underline{x}, \underline{z}) = 0$ of the hypersurface. Likewise for the complete intersection (7.32) elements in $H^1(M, T_M)$ can be represented by infinitesimal deformations $\delta_{(z,u)} =: \delta_z$ of the parameters in (7.32), modulo global automorphisms of \mathcal{W} . Using these facts, it is easy to check for the complete intersections description (7.32) that the moduli \underline{u} of Σ described by the coefficients of the monomials in $h_i(\underline{x}, \underline{u})$, $i = 1, 2$ turn into complex structure moduli $\hat{\underline{z}}$ of \hat{Z}_3 since $h_i(\underline{x}, \underline{u})$, $i = 1, 2$ enter the defining equations of \hat{Z}_3 via Q in (7.32).

As noted below (B.26), the divisor E is isolated in \hat{Z}_3 , i.e. on \hat{Z}_3 there are no deformations associated to E . From this follows more illustratively, that blowing up along Σ for different values of \underline{u} yields diffeomorphic blow-ups \hat{Z}_3 which just differ by a choice of complex structure. The situation is visualized in figure 7.3. Mathematically, the equivalence even of the full deformation theory of (Z_3, Σ) and \hat{Z}_3 are expected in general. This means that not only the deformations of (Z_3, Σ) , counted by elements in $H^1(Z_3, TZ_3)$ and $H^0(\Sigma, N_{Z_3}\Sigma)$, agree with the complex structure deformations of \hat{Z}_3 , that are in $H^1(\hat{Z}_3, T\hat{Z}_3)$, but also the obstruction problems of both geometries. In particular, this implies an order-by-order match of the obstructions on both sides of the correspondence.

As detailed in the section 5.2 and 7.1.2 possible obstructions to the closed deformation space live in $H^2(M, T_M)$ while possible obstructions to the open deformations space live in $H^1(\Sigma, N_{Z_3}\Sigma)$. While one can conclude from the vanishing of these homology groups that the corresponding deformations are unobstructed, it is not necessarily true that the deformation problems are obstructed, if these homology groups do not vanish. In particular the complex structure deformations of Calabi-Yau spaces, such as Z_3 , are unobstructed despite the fact

¹³Most statements about the complex structure deformations apply to Z_3 and \hat{Z}_3 . We denote both complex manifolds by M in the following.

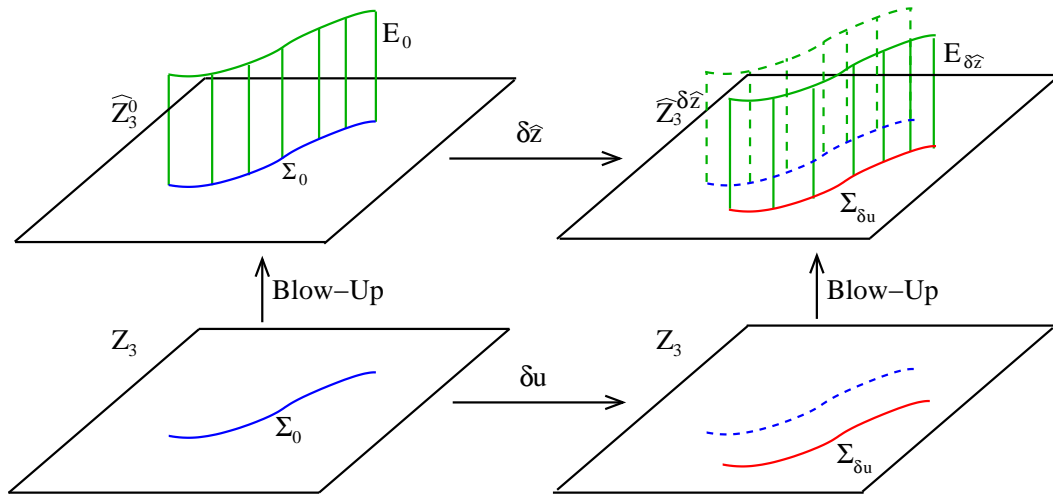


Figure 7.3: Map of brane to complex structure deformations. A displacement $\Sigma_{\delta u}$ of a holomorphic curve Σ_0 yields a new blow-up $\hat{Z}_3^{\delta \hat{z}}$ with induced complex structure deformation $\delta \hat{z}$ and divisor $E_{\delta \hat{z}}$. The divisor E_0 , which is holomorphic in \hat{Z}_3^0 , is not holomorphic in $\hat{Z}_3^{\delta \hat{z}}$.

that $H^2(Z_3, T_{Z_3}) \neq 0$. However the deformations of the curve Σ can in general be obstructed by elements $H^1(\Sigma, N_{Z_3}\Sigma)$ at some order. Given the equivalence of the obstruction problems we expect that these are precisely matched by the obstructions to complex structure deformations on \hat{Z}_3 in $H^2(\hat{Z}_3, T\hat{Z}_3)$. In physical terms, the obstruction problem is in general expressed by a superpotential, which in the case of the obstruction problem of (Z_3, Σ) is given by the superpotential (7.23). Our strategy to investigate the possible obstructions of (Z_3, Σ) will be to match the calculation of the superpotential (7.23) before and after the blow-up. Here it will be crucial to understand the lift of the brane superpotential W_{brane} of (7.24) under the blow-up, that will be replaced by a specific flux superpotential on \hat{Z}_3 . This flux superpotential induces obstructions to complex structure deformations on \hat{Z}_3 that are equivalent to the original obstructions on moving the brane on Σ expressed by W_{brane} . We will discuss the match of the superpotentials in a two step procedure in section 7.4 and in chapter 9. In particular the blow-up \hat{Z}_3 will yield an easy calculational scheme of the superpotential as explained and applied to specific examples in section 8.

Matching Deformations and Obstructions: Concrete Examples

Before we proceed we have to explain how we use the blow-up \hat{Z}_3 constructed as the complete intersection (7.32) to calculate the superpotentials W_{brane} for five-branes on rational curves. This is crucial since the families of holomorphic curves themselves defined by the complete

intersection of complex equations $h_1 = h_2 = 0$ are unobstructed. Similarly on \hat{Z}_3 the corresponding complex structure deformations are unobstructed and the deformation problem and the corresponding superpotentials are trivial.

The general statement for the moduli space of holomorphic curves¹⁴ on Calabi-Yau threefolds is that its virtual deformation space is zero-dimensional [66, 67, 82, 269]. Naively this could be interpreted as the statement, that generically holomorphic curves in a Calabi-Yau threefold never occur in families. However, this conclusion is not true as one can learn already from the case of rational curves¹⁵ in the quintic as explained in [269]. Rational curves in a generic Calabi-Yau manifold Z_3 , like the quintic with a constraint $P = 0$ including 101 complex structure parameters \underline{z} at generic values, are isolated and have a moduli space consisting of points, which we denote by $\mathcal{M}^{\underline{z}}(\mathbb{P}^1) = pts$. However, at special loci \underline{z}_0 in the complex structure moduli space, which correspond to specially symmetric Calabi-Yau constraints $P = 0$ like the Fermat point

$$P = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \quad (7.33)$$

of the quintic, a family of curves parametrized by a finite dimensional moduli space $\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1)$ can appear. Physically this means that the open superpotential becomes a constant of the brane moduli and the scalar potential has a flat direction along $\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1)$. However it can be generally argued [66] that in the vicinity of the special loci in the closed string deformation space \underline{z}_0 a superpotential develops for the rational curves. In agreement with (7.14) the superpotential starts linear in the closed string deformation $t \propto \delta_z$ away from \underline{z}_0 and is of arbitrary order in the open string moduli so that it has $(-1)^{\dim(\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1))} \chi(\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1))$ minima¹⁶. We note that this is precisely the most interesting physical situation, as t can be made parametrically small against the compactification scale, as explained below equation (7.14).

There is one important caveat in order when working with concrete algebraical curves. A given family of holomorphic curves in a specific algebraic representation $P = 0$ of Z_3 can become obstructed due to the presence of non-algebraic complex structure deformations, i.e. those \underline{z} that are not contained in $P = 0$. For example, this situation occurs in the Calabi-Yau hypersurfaces Z_3 of degree $2 + 2(n_1 + n_2 + n_3)$ in weighted projective spaces of the type $\mathbb{P}^4(1, 1, 2n_1, 2n_2, 2n_3)$ with $n_i \in \mathbb{Z}$ as discussed in [66]. This realization of the Calabi-Yau manifold Z_3 contains always a ruled surface, i.e. an \mathbb{P}^1 fibered over a (higher genus) Riemann surface¹⁷. The embedding of Z_3 in this particular ambient space is such that the generic obstructed situation, which corresponds to a non-vanishing superpotential, is not accessible using the algebraic deformations, i.e. upon tuning the parameters in the Calabi-Yau constraint $P = 0$. The absence of these deformations as algebraic deformations in $P = 0$ happens since

¹⁴In the following we will use the term of a 'moduli space' of holomorphic curves in Z_3 to denote an analytic family of holomorphic curves as introduced in section 7.1.2.

¹⁵A rational curve is birationally equivalent to a line i.e. a curve of genus zero which is a \mathbb{P}^1 .

¹⁶This formula follows from complex deformation invariance of the BPS numbers associated to holomorphic curves [270].

¹⁷The same Riemann surface is identified with the moduli space $\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1)$ in this case.

the corresponding monomials are not compatible with the symmetries of the ambient space.

Let us next describe the obstructed deformation problem of rational curves and the relation to the complete intersection curves and the blow-up (7.32). The basic idea is to map the obstructed deformations of the rational curves to the algebraic moduli space of the complete intersection in the following way. As mentioned in (7.32) the algebraic deformations parametrized by the closed moduli \underline{z} and the open moduli \underline{u} are unobstructed. Let us denote the corresponding open and closed moduli space of the complete intersection Σ , defined by $P(\underline{z}) = 0$ and $h_i(\underline{x}, \underline{u}) = 0$, $i = 1, 2$, by $\mathcal{M}(\Sigma)$ and the open moduli space of Σ for fixed closed moduli \underline{z} by $\mathcal{M}^z(\Sigma)$. The generic dimension of this open moduli space $h^0(N\Sigma)$ is positive. The idea is to consider a representation of Z_3 which is compatible with a discrete symmetry group G . This symmetry group allows us to identify lower degree and genus curves with the complete intersection Σ at a special sublocus of the moduli space $\mathcal{M}(\Sigma)$. In our main examples in section 8.1 and 8.3 these are rational curves, i.e. curves of degree one. Let us denote this sublocus by $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$. This sublocus is determined by the requirement that the algebraic constraints P , h_i degenerate so that they can be trivially factorized as powers of linear constraints,

$$\mathcal{M}_{\mathbb{P}^1}(\Sigma) : \quad P(\underline{z}) = h_i(\underline{u}) = 0 \quad \Leftrightarrow \quad \prod_k \sum_l a_{lk}^{(s)} x_l = 0, \quad s = 1, 2, 3, \quad (7.34)$$

where the different linear factors $L_k^{(s)} = \sum_l a_{lk}^{(s)} x_l$ are identified by the discrete group G , $L_{k_1}^{(s)} \leftrightarrow L_{k_2}^{(s)}$. Then the right hand side of this identification describes rational curves

$$L_{k_1}^{(1)} = L_{k_1}^{(2)} = L_{k_1}^{(3)} = 0 \quad (7.35)$$

modulo G in the ambient space and in Z_3 since $P = 0$ is trivially fulfilled. For a concrete situation we refer to section 8.1. In particular this identification embeds the moduli space $\mathcal{M}^{\mathbb{Z}_0}(\mathbb{P}^1)$ into $\mathcal{M}^{\mathbb{Z}_0}(\Sigma)$ and more trivially the (discrete) $\mathcal{M}^z(\mathbb{P}^1)$ into $\mathcal{M}^z(\Sigma)$.

More generally, i.e. away from the sublocus $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ defined by (7.34), this embedding implies that the *obstructed* deformation space of the rational curves (7.35) is identified with the *unobstructed* moduli space $\mathcal{M}(\Sigma)$. This can be compared to the method presented in [114, 119, 120, 122] where the obstructed deformations of a curve are identified with the unobstructed moduli of an appropriate divisor. For the curves we consider we depict the embedding of the deformation spaces of the rational curves into the moduli space $\mathcal{M}(\Sigma)$ of complete intersection Σ in figure 7.4, where we introduce new open moduli \hat{z}^1, \hat{z}^2 that are functions of the u^i . The point is that away from $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ the identification (7.34) of the complete intersection curve Σ with the holomorphic rational curves in the locus $P = 0$ fails. We can understand this failure in two different ways emphasizing different aspects of the identification of the deformation space of rational curves with the true moduli space $\mathcal{M}(\Sigma)$. If we analyze the identification of Σ with holomorphic rational curves infinitesimally close to $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ one can either keep the linear constraints (7.35) and relax the condition that these rational curves lie identically on

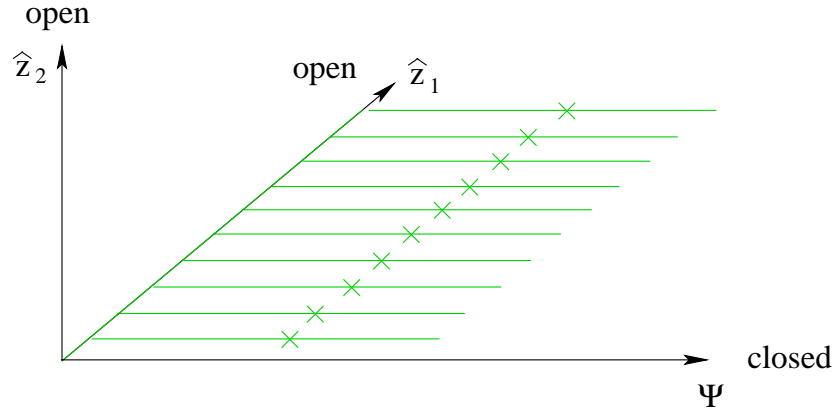


Figure 7.4: Picture of the moduli space of $\mathcal{M}(\Sigma)$ given by the quintic modulo \mathbb{Z}_5^3 and the loci $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ where Σ degenerates to (holomorphic) rational curves. All lines in the $\hat{z}_2 = 0$ plane correspond to the embedding of the moduli space of rational curves $\mathcal{M}_{\mathbb{P}^1}(\Sigma) \subset \mathcal{M}(\Sigma)$, cf. (8.4). At $\Psi = 0$ the \hat{z}_1 direction opens up as a modulus of a family of \mathbb{P}^1 's over the genus 6 curve $\mathcal{M}^{\Psi=0}(\mathbb{P}^1)$. For generic values of Ψ only $(-1)^{\dim(\mathcal{M}^{\Psi=0}(\mathbb{P}^1))} \chi(\mathcal{M}^{\Psi=0}(\mathbb{P}^1)) = 10$ points belong to $\mathcal{M}^{\Psi}(\mathbb{P}^1)$. Away from $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ the holomorphic configuration in Z_3 is an irreducible higher genus curve, which corresponds to non-holomorphic S^2 's in Z_3 .

the $P = 0$ locus or we linearize the equations $P = h_i = 0$ such that the rational curves keep lying in the $P = 0$ locus. However, since the trivial factorization (7.34) and the identification of the linear factors $L_k^{(s)}$ modulo G fails, the latter possibility introduces non-holomorphic equations with nontrivial branching. This implies that the rational curve away from $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ inside $\mathcal{M}(\Sigma)$ is not holomorphic in Z_3 . In particular, the process of turning on $t = \delta z$ around \underline{z}_0 for values of the open moduli, that are only close to $\mathcal{M}^{\underline{z}_0 + \delta z}(\mathbb{P}^1)$, can be understood as deforming the analytic family $\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1)$ of rational curves inside the $P = 0$ locus to rational curves, which are non-holomorphic. For concreteness, for the later example (8.4) of the quintic, we identify $\delta z = \Psi$ with the complex structure of the mirror quintic deforming the Fermat locus (7.33) as

$$P = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\Psi x_1 x_2 x_3 x_4 x_5. \tag{7.36}$$

Then, $\delta z \neq 0$ deforms away from the one-dimensional moduli space of rational curves $\mathcal{M}^{\underline{z}_0}(\mathbb{P}^1)$ in $P = 0$, that exists at the Fermat locus $\underline{z}_0 = 0$.

As noted before, a brane on a non-holomorphic curve is not supersymmetric and thus violates the F-term supersymmetry condition that is expressed by the superpotential W_{brane} . In the following, in particular in the examples of sections 8.1 and 8.3, we will consider five-branes on rational curves in a given Calabi-Yau threefold Z_3 and their excitation about

the supersymmetric minimum that correspond, in geometric terms, to possibly obstructed deformations about holomorphic curves. The brane excitations we consider correspond on the one hand to light fields that are generically obstructed and that become, as discussed in section 7.1.2, massless at some point in the complex structure moduli space. On the other hand we include fields that parameterize non-holomorphic deformations for all values of the closed moduli. In figure 7.4 the first type of fields corresponds to \hat{z}_1 , which becomes massless at $\Psi = 0$, and the second type of fields corresponds to \hat{z}_2 . However, the crucial point for the consideration of this deformation space, as noted above, is the identification with the moduli space $\mathcal{M}(\Sigma)$ of complete intersection curves Σ . This identification and the unification of the open-closed moduli space of (Z_3, Σ) in the blow-up (7.32) of the complete intersection curve Σ will enable us to calculate the superpotential W_{brane} for branes wrapping rational curves. We determine the periods on the complex structure moduli space of the blow-up \hat{Z}_3 that physically describe the closed and open superpotential W_{brane} upon turning on an appropriate flux on \hat{Z}_3 . For this purpose we will describe W_{brane} explicitly by chain and flux integrals on the blow-up \hat{Z}_3 respectively in section 7.4 and chapter 9. Finally, we note that the periods on \hat{Z}_3 can equivalently be understood as a definition of the concept of periods on the *brane moduli space* $\mathcal{M}(\Sigma)$, extending the familiar notion of periods on the complex structure moduli space of a Calabi-Yau manifold.

7.3 Probing the Open-Closed Deformation Space

The key in describing the deformations of complex structures on \hat{Z}_3 are the construction and the properties of the pull-back $\hat{\Omega} = \pi^*(\Omega)$ of the holomorphic three-form Ω from the Calabi-Yau threefold Z_3 to \hat{Z}_3 . Since the blow-up is a local procedure $\hat{\Omega} := \pi^*(\Omega)$ will first be constructed in the local patches \hat{U}_α and then be globalized as a residue integral for the complete intersection (7.32). From this we obtain differential equations, the Picard-Fuchs equations, which determine the full complex structure dependence of $\hat{\Omega}$ and its periods.

Let us summarize the results of the actual calculation, which is done in appendix B.2. As in section 7.2.1 we assume that Σ is represented as a complete intersection of divisors D_i , $i = 1, 2$, in Z_3 given by constraints $h_i(\underline{x}, \underline{u}) = 0$ for coordinates \underline{x} . If we consider a patch U_α on Z_3 near Σ , then the holomorphic three-form Ω is locally given by

$$\Omega = dx_1 \wedge dx_2 \wedge dx_3 = \det J^{-1} dy_{\alpha,1} \wedge dy_{\alpha,2} \wedge dy_{\alpha,3}, \quad (7.37)$$

where J is the Jacobian matrix for choosing coordinates $y_{\alpha,i} = h_i(\underline{x})$, $i = 1, 2$, and $y_{\alpha,3} = x_3$, starting with generic coordinates \underline{x} . This expression is pulled back via the projection map $\pi : \hat{Z}_3 \rightarrow Z_3$ to the patch \hat{U}_α defined in (7.26). We introduce coordinates

$$l_1 \neq 0 : \quad z_{\alpha,1}^{(1)} = y_{\alpha,1}, \quad z_{\alpha,2}^{(1)} = \frac{l_2}{l_1} = \frac{y_{\alpha,2}}{y_{\alpha,1}}, \quad z_{\alpha,3}^{(1)} = y_{\alpha,3} \quad (7.38)$$

on \hat{U}_α for $l_1 \neq 0$ to obtain

$$\hat{\Omega} = \pi^*(\Omega) = z_{\alpha,1}^{(1)} \det J^{-1} dz_{\alpha,1}^{(1)} \wedge dz_{\alpha,2}^{(1)} \wedge dz_{\alpha,3}^{(1)}. \quad (7.39)$$

Here the subscript $*_\alpha$ and the superscript $*^{(1)}$ label the patches U_α on Z_3 as well as the patch $l_1 \neq 0$ on the exceptional \mathbb{P}^1 with projective coordinates $(l_1 : l_2)^{18}$. We obtain a similar expression on the second patch $l_2 \neq 0$ of \mathbb{P}^1 using local coordinates $z_{\alpha,1}^{(2)} = \frac{l_1}{l_2}$, $z_{\alpha,2}^{(2)} = y_{\alpha,2}$ and $z_{\alpha,3}^{(2)} = y_{\alpha,3}$.

Now one can show that the pull-back map π^* on Ω can be written as the residue

$$\hat{\Omega} = \int_{S_Q^1} \frac{h_i}{l_i} \frac{\Delta_{\mathbb{P}^1}}{Q} \wedge \Omega, \quad i = 1, 2. \quad (7.40)$$

One can easily check that this is globally well-defined on both patches $l_i \neq 0$, $i = 1, 2$, covering the \mathbb{P}^1 using the blow-up constraint Q in (7.26) respectively (7.32). Here we insert the local expression (7.37) for Ω and

$$\Delta_{\mathbb{P}^1} = l_1 dl_2 - l_2 dl_1, \quad (7.41)$$

which is the invariant top-form on \mathbb{P}^1 . In fact, the residuum (7.40) specializes correctly to the local expressions (7.39) of $\hat{\Omega}$ in every chart. This ensures that the residuum expression on \hat{U}_α can be globalized to \hat{Z}_3 . We use the standard residuum expression for the holomorphic three-form Ω given in (5.32) to replace the local expression (7.37) by

$$\hat{\Omega} = \int_{S_P^1} \int_{S_Q^1} \frac{h_1}{l_1} \frac{\Delta}{PQ} = \int_{S_P^1} \int_{S_Q^1} \frac{h_2}{l_2} \frac{\Delta}{PQ}, \quad (7.42)$$

where P, Q are the two constraints of (7.32). The five-form Δ denotes an invariant holomorphic top-form on the five-dimensional ambient space \mathcal{W} defined in (7.31) and S_P^1, S_Q^1 are small loops around $\{P = 0\}, \{Q = 0\}$ encircling only the corresponding poles. The measure Δ is given explicitly in section 5.1. For the example of a trivial fibration it takes the schematic form

$$\Delta = \Delta_{\mathbb{P}_\Delta} \wedge \Delta_{\mathbb{P}^1}. \quad (7.43)$$

where $\Delta_{\mathbb{P}_\Delta}$ denotes the invariant top-form on the toric basis \mathbb{P}_Δ .

Let us now discuss the essential properties of $\hat{\Omega}$ and of the residue integral expression (7.42). By construction of \hat{Z}_3 , in particular by the isomorphism $H^{(3,0)}(\hat{Z}_3) \cong H^{(3,0)}(Z_3)$, $\hat{\Omega}$ is the unique generator of $H^{(3,0)}(\hat{Z}_3)$ [60]. In general $\hat{\Omega}$ varies under a deformation of the complex structure on \hat{Z}_3 . This is due to the fact that the notion of holomorphic and anti-holomorphic coordinates changes when changing the complex structure. More rigorously, this is described by the variation of Hodge structures, where the split

$$H^3(\hat{Z}_3) = \bigoplus_{i=0}^3 H^{(3-i,i)}(\hat{Z}_3) \quad (7.44)$$

¹⁸We drop the label α on the coordinates l_i in order to shorten our formulas.

by the Hodge type (p, q) is analyzed over the complex structure moduli space $\mathcal{M}(\hat{Z}_3)$ of \hat{Z}_3 . Then $\hat{\Omega}$ is a holomorphic section of the locally constant vector bundle $H^3(\hat{Z}_3)$ over $\mathcal{M}(\hat{Z}_3)$ and is of type $(3, 0)$ at a fixed point \hat{z}_0 in $\mathcal{M}(\hat{Z}_3)$. When moving away from \hat{z}_0 by an infinitesimal displacement δ_z the Hodge type of $\hat{\Omega}$ changes according to the diagram

$$H^{(3,0)}(\hat{Z}_3) = \mathcal{F}^3 \xrightarrow{\delta_z} \mathcal{F}^2 \xrightarrow{\delta_z} \mathcal{F}^1 \xrightarrow{\delta_z} \mathcal{F}^0 = H^3(\hat{Z}_3) \quad (7.45)$$

where we define the holomorphic vector bundles \mathcal{F}^p over \hat{z} in $\mathcal{M}(\hat{Z}_3)$ by

$$\mathcal{F}^p|_{\hat{z}} = \bigoplus_{i \geq p} H^{(i, 3-i)}(\hat{Z}_3)|_{\hat{z}}. \quad (7.46)$$

Most importantly, this implies the existence of differential equations, the Picard-Fuchs equations, on $\mathcal{M}(\hat{Z}_3)$ since the diagram (7.45) terminates at fourth order in δ_z . These can be explicitly obtained from the residue integral representation (7.42) of $\hat{\Omega}$ by applying the Griffiths-Dwork reduction method. The Picard-Fuchs system in turn determines the full moduli dependence of $\hat{\Omega}$ and its periods.

Our general strategy will be to calculate all integrals relevant for the evaluation of the superpotential, discussed in section 7.1.3, on \hat{Z}_3 using this Picard-Fuchs system. In fact, in the toric examples of sections 8.1, 8.3 we obtain a GKZ-system whose solutions are the periods of Ω and the brane superpotential W_{brane} . This, as explained in 7.2.2, unifies the closed and open deformations of (Z_3, Σ) , but, as we will see later in more detail, also the expression for individual pieces of the superpotential into a flux superpotential on \hat{Z}_3 . Indeed, the unification of open-closed deformations, as mentioned before, as complex structure deformations on \hat{Z}_3 guarantees, that the study of variations of pure Hodge structures (7.45) is sufficient [60]. This is true despite the fact that $\hat{\Omega}$ vanishes¹⁹ as a section of $K\hat{Z}_3 = E$ along the exceptional divisor E . Consequently $\hat{\Omega}$ naturally defines an element in open cohomology $H^3(\hat{Z}_3 - E) \cong H^3(\hat{Z}_3, E)$, which in general carries a mixed Hodge structure [240]. However, the analysis of variations of this mixed Hodge structure reduces to the variation of the pure Hodge structure (7.45) on the graded weight $Gr_3^W H^3(\hat{Z}_3 - E) = H^3(\hat{Z}_3)$ since E has no deformation²⁰. To derive the superpotential as a solution of the Picard-Fuchs equations we also have to use an appropriate chain integral or current as explained below.

We conclude by discussing the expected structure of the Picard-Fuchs equations on $\mathcal{M}(\hat{Z}_3)$ from the residue (7.42). For a detailed discussion along the lines of concrete examples we refer to sections 8.1 and 8.3. In general all periods of $\hat{\Omega}$ over closed three-cycles in \hat{Z}_3 are solutions to this Picard-Fuchs system. First, we note that the Calabi-Yau constraint $P(\underline{x}, \underline{a})$ appears both in the residues (7.42) on \hat{Z}_3 as in well as in (5.32). The parameters \underline{a} multiplying monomials in P are identified, modulo the symmetries of the toric ambient space \mathbb{P}_Δ , with complex

¹⁹This can be directly seen, using the fact that E is given in local coordinates by $z_{\alpha,1}^{(1)} = 0$ respectively $z_{\alpha,2}^{(2)} = 0$, from the local expression (7.40) and its global counterpart (7.42).

²⁰In the diagram of variations of mixed Hodge structures, the downward arrows corresponding to deforming E do not exist, cf. equation (4.40) of [60].

structure moduli \underline{z} . Since monomials in $Q(\underline{x}, \underline{b})$ are multiplied by independent parameters \underline{b} , the Picard-Fuchs operators $\mathcal{L}_k(\underline{a})$ annihilating Ω , expressed by derivatives w.r.t. \underline{a} , annihilate $\hat{\Omega}$ as well. Second, since \mathbb{P}_Δ is the basis of the \mathbb{P}^1 -fibration \mathcal{W} the toric symmetries of \mathbb{P}_Δ are contained in those of \mathcal{W} . They act on $P(\underline{x}, \underline{a})$ in the same way in \mathcal{W} as in \mathbb{P}_Δ , but also act on the parameters \underline{b} in the constraint Q . Therefore, the operators $\mathcal{Z}_i(\underline{a})$, expressing the toric symmetries of \mathbb{P}_Δ on Z_3 , are lifted to \hat{Z}_3 as

$$\hat{\mathcal{Z}}_i(\underline{a}, \underline{b}) = \mathcal{Z}_i(\underline{a}) + \mathcal{Z}'_i(\underline{b}), \quad (7.47)$$

where the first operator is as before on Z_3 and the second operator just contains differentials of the \underline{b} . Then, (7.47) is easily solved by choosing the coordinates \underline{z} as on Z_3 . The new torus symmetries of \mathcal{W} , that correspond to its \mathbb{P}^1 -fiber, do not involve the \underline{a} , but only the \underline{b} . They merely fix the variables \underline{u} in terms of the parameters \underline{a} and \underline{b} . Consequently, dividing out all toric symmetries of \mathcal{W} , the form $\hat{\Omega}$ depends on $\hat{\underline{z}} \equiv (\underline{z}, \underline{u})$ only,

$$\hat{\Omega}(\underline{a}, \underline{b}) \equiv \hat{\Omega}(\underline{z}, \underline{u}) = \hat{\Omega}(\hat{\underline{z}}) \quad (7.48)$$

and the differential operators $\mathcal{L}_k(\underline{a})$ take, possibly after a factorization to operators $\mathcal{D}_k(\hat{\underline{z}})$ of lower degree, the schematic form

$$\mathcal{D}_k(\hat{\underline{z}}) = \mathcal{D}_k^{Z_3}(\underline{z}) + \mathcal{D}'_k(\underline{z}, \underline{u}) \quad (7.49)$$

in these coordinates. Here $\mathcal{D}_k^{Z_3}(\underline{z})$ are the Picard-Fuchs operators of Z_3 in the coordinates \underline{z} and $\mathcal{D}'_k(\underline{z}, \underline{u})$ are at least linear in derivatives w.r.t. \underline{u} . It follows immediately that the Picard-Fuchs system for $\hat{\Omega}(\hat{\underline{z}})$ contains the Picard-Fuchs system for $\Omega(\underline{z})$, as determined by the $\mathcal{D}_k(\hat{\underline{z}})$, as a closed subsystem. Consequently the periods $\Pi^l(\underline{z})$ of Z_3 over closed three-cycles, that fulfill the differential equations $\mathcal{L}_k^{Z_3}(\underline{z})$, are also solutions to (7.49). We note that there are new operators $\mathcal{L}_m(\hat{\underline{z}})$ due to the constraint Q that do not have a counterpart on Z_3 . They form, together with (7.49), a complete differential system on $\mathcal{M}(\hat{Z}_3)$. However, the $\mathcal{L}_m(\hat{\underline{z}})$ are again at least linear in differentials of \underline{u} , thus act trivially on functions independent of \underline{u} , such as $\Pi^l(\underline{z})$.

Geometrically the lift of the periods of Ω to \hat{Z}_3 is a consequence of the isomorphism of $\pi : Z_3 - \Sigma \rightarrow \hat{Z}_3 - E$ and the fact that $\hat{\Omega}$ vanishes on E . The analog lift of the more interesting open periods on Z_3 , in particular W_{brane} is discussed in the next section, section 7.4, where we provide the corresponding expressions for the lifted superpotential on \hat{Z}_3 .

We conclude by mentioning that the structure (7.49) allows to directly determine the inhomogeneous Picard-Fuchs equations obeyed by the domain wall tension $\mathcal{T}(\underline{z})$ between two five-branes,

$$\mathcal{D}_k^{Z_3}(\underline{z})\mathcal{T}(\underline{z}) = f_k(\underline{z}). \quad (7.50)$$

The tension \mathcal{T} is obtained as $\mathcal{T}(\underline{z}) = W_{\text{brane}}(\underline{z}, \underline{u}^{rmc}) - W_{\text{brane}}(\underline{z}, \underline{u}_0^{rmc})$ and the inhomogeneity $f(\underline{z})$ upon evaluating $f_k(\underline{z}) = \mathcal{D}'_k(\underline{z}, \underline{u})W_{\text{brane}}(\underline{z}, \underline{u})|_{\underline{u}^c}$, where \underline{u}^c , \underline{u}_0^c are critical points of W_{brane} . This inhomogeneous system was obtained from residues in [111, 113, 115, 121].

7.4 Lift of the Superpotentials

We have seen at the end of the previous section from the discussion of the Picard-Fuchs equations on \hat{Z}_3 that the periods $\Pi^l(\underline{z})$ over closed three-cycles lift from Z_3 to \hat{Z}_3 . Illustratively this is clear because these integrals depend only on the geometry of Z_3 with the five-brane removed, cf. section 7.1.1. Thus, in order to lift the flux superpotential in (4.16), (7.23), which is just a linear combination of the periods $\Pi^l(\underline{z})$, we just need to lift the flux data. This is straightforward as the third cohomology of \hat{Z}_3 is given by [60], cf. appendix B.3,

$$H^3(\hat{Z}_3) = \pi^*H^3(Z_3) \oplus H^3(E) \quad (7.51)$$

so that any flux G_3 on Z_3 has a counterpart $\hat{G}_3 = \pi^*G_3$ on \hat{Z}_3 . Thus we readily obtain the lift of the flux superpotential to \hat{Z}_3 as

$$W_{\text{flux}} = \int_{\hat{Z}_3} \hat{\Omega} \wedge \hat{G}_3. \quad (7.52)$$

Again the precise integral basis of cycles on \hat{Z}_3 for which this integral can be expanded in terms of periods of $\hat{\Omega}$ with integral coefficients has to be obtained by matching the classical terms at large radius and by assuring that the monodromy acts by integral transformations. Obviously, on \hat{Z}_3 flux configurations associated to the new three-cycles in E , which are not promoted from fluxes on Z_3 can be considered.

In order to lift the five-brane superpotential W_{brane} let us first make a local heuristic analysis, which casts already much of the general structure. Locally on a patch U_α we can write $\Omega = d\omega$ and evaluate the integral over the chain $\Gamma(u)$ leading to the five brane superpotential localized to the boundary $\partial\Gamma(u) = \Sigma - \Sigma_0$,

$$W_{\text{brane}} = \int_{\Sigma} \omega. \quad (7.53)$$

Here we suppress the integral over the fixed reference curve Σ_0 in the same homology class as Σ , as it gives only rise to an irrelevant constant in W_{brane} . Then we use the fact that the original curve Σ is contained in the second cohomology $H^2(E)$ by its Poincare dual class [60]. Thus we can write

$$W_{\text{brane}} = \int_E \pi^*(\omega) \wedge F_2 \quad (7.54)$$

where $[F_2]$ is the class of Σ in the exceptional divisor E , i.e.

$$[F_2] = \Sigma \quad \text{in } E \quad (7.55)$$

We note that at the supersymmetric minimum F_2 is equal to the Kähler form of the Fubini study metric on the \mathbb{P}^1 , i.e. $F_2 = \omega_{FS}$. Locally (7.54) can be written as an integral on \hat{Z}_3

$$W_{\text{brane}} = \int_{\Gamma_5} \hat{\Omega} \wedge F_2 \quad (7.56)$$

over a five-chain Γ_5 with $\partial\Gamma_5 = E - E_0$ on which we extend F_2 . E_0 denotes a reference divisor in the same homology as E , e.g. the blow-up of Σ_0 , to match the constant contributions.

To prove this more rigorously it is instructive to consider the lift of the Bianchi identity to \hat{Z}_3 . The formalism is equal for the three-form R-R field strength F_3 in type IIB and the three-form NS-NS field strength H_3 in the heterotic string. For the following analysis of the Bianchi identities, which is local along the curve Σ , one can focus on the source term δ_Σ of one five-brane neglecting the other terms in (7.1) respectively (4.14). The only aspect of the geometry which cannot be seen locally in a patch near a point in Σ is the non triviality of the bundle $N_{\hat{Z}_3}E$, which is captured by its Thom class $\frac{e_1}{2}$. As can be calculated explicitly by evaluating the pull-back π^* to the blow-up \hat{Z}_3 , the form H_3 in (4.14) is replaced by²¹

$$d\hat{H}_3 = \delta_E \wedge F_2 = d\rho \wedge \frac{e_1}{2} \wedge F_2 , \quad (7.57)$$

where the limit $\epsilon \rightarrow 0$ is implicit and we used

$$\lim_{\epsilon \rightarrow 0} d\rho \wedge \frac{e_1}{2} = \delta_E . \quad (7.58)$$

Formally (7.57) can be integrated in the language of currents to

$$\hat{H}_3 = d\rho \wedge \frac{e_0^{(0)}}{2} F_2 + dB_2 , \quad (7.59)$$

where $e_1 = de_0^{(0)}$ and a possible term $\rho \frac{e_1}{2} \wedge F_2$ is neglected by the requirement of regularity of \hat{H}_3 . Thus, by pulling back both H_3 and Ω to \hat{Z}_3 we lift the superpotential (7.23) as

$$W_{\text{brane}} = \lim_{\epsilon \rightarrow 0} \int_{\hat{Z}_3} \hat{\Omega} \wedge \hat{H}_3 = \lim_{\epsilon \rightarrow 0} \int_{\hat{Z}_3} \hat{\Omega} \wedge F_2 \wedge \rho e_1 \quad (7.60)$$

where we restrict to the singular part (7.59) of \hat{H}_3 only. By (7.58) and the identity (7.59) for \hat{H}_3 we see that this is equivalent to (7.54) and hence to (7.56). We note that we can easily switch between the open manifold $\hat{Z}_3 - E$ and \hat{Z}_3 in (7.60) since $\hat{\Omega}|_E = 0$. Mathematically, this match of (7.60) and the original superpotential W_{brane} in (7.25) follows more geometrically by the canonical identification $\hat{\Omega}|_{\hat{Z}_3 - E} = \Omega|_{Z_3 - \Sigma}$ under the biholomorphism $Z_3 - \Sigma \cong \hat{Z}_3 - E$, by $\hat{\Omega}|_E = 0$ and the lift of the Thom classes, $\rho e_3 \cong \rho e_1 \wedge F_2$.

Finally, we conclude by arguing that W_{brane} is a solution to the Picard-Fuchs system on \hat{Z}_3 . In fact, this is confirmed for the examples of sections 8.1, 8.3 using the corresponding open-closed GKZ-system on \hat{Z}_3 . All we have to ensure is that integration of $\hat{\Omega}$ over \hat{H}_3 given in (7.59), which is not a closed form, commutes with the application of the Picard-Fuchs operators, that annihilate $\hat{\Omega}$. In addition, the whole integral has to be annihilated as well. Since \hat{H}_3 does not depend on the complex structure on \hat{Z}_3 , all differential operators indeed commute with integration. Furthermore, for the GKZ-system of the form discussed below

²¹Note that mathematically $\delta_E \wedge F_2 \equiv \mathcal{T}(F_2)$ is given by the Thom isomorphism $\mathcal{T} : H^\bullet(E) \rightarrow H^{\bullet+2}(Y)_{\text{cpt}}$ of the normal bundle $N_{\hat{Z}_3}E$ in \hat{Z}_3 . \mathcal{T} maps cohomology classes on E to compactly supported classes in \hat{Z}_3 .

(7.47), (7.49), the operators $\mathcal{L}_k(\hat{\underline{z}})$, $\hat{\mathcal{L}}_m(\hat{\underline{z}})$ annihilate $\hat{\Omega}$ identically and consequently also W_{brane} in (7.60). However, the operators $\hat{\mathcal{Z}}_i(\underline{a}, \underline{b})$ expressing the toric symmetries of \mathcal{W} obey in general [230, 248]

$$\hat{\mathcal{Z}}_i(\underline{a}, \underline{b})\hat{\Omega}(\underline{a}, \underline{b}) = d\hat{\alpha} \quad (7.61)$$

for a two-form $\hat{\alpha}$. This can potentially lead to a non-zero result in W_{brane} by partial integration since $d\hat{H}_3 \neq 0$. In fact we exploit (7.57) to rewrite this as an integral over E ,

$$\hat{\mathcal{Z}}_i(\underline{a}, \underline{b})W_{\text{brane}} = - \int_E \iota^*(\hat{\alpha}) \wedge F_2, \quad (7.62)$$

where $\iota : E \hookrightarrow \hat{Z}_3$ denotes the embedding of E . Fortunately, the pull-back $\iota^*(\hat{\alpha})$ vanishes on E by the fact that we are dealing with variations of pure Hodge structures (7.45) and thus can not reach the cohomology of E ²². Thus we have argued that W_{brane} is a solution of the GKZ-system on \hat{Z}_3 . This will be further confirmed for the examples in section 8.

²²As noted below (7.46), the cohomology of E is included in the variations of mixed Hodge structures on $H^3(\hat{Z}_3 - E)$ and can only be reached by deformations of E . These do not exist in our case since E is rigid.

Chapter 8

Superpotentials from Brane Blow-Ups

In this chapter we apply the blow-up proposal to a selection of examples of five-branes in compact Calabi-Yau threefolds. We consider two different Calabi-Yau threefolds, the one parameter example of the quintic and the two parameter Calabi-Yau in $\mathbb{P}^4(1, 1, 1, 6, 9)$, where the latter was the main example in chapter 6 [79]. The type of branes we consider have the geometric interpretation of five-branes on rational curves, i.e. holomorphic curves $\Sigma = \mathbb{P}^1$, at special loci in the open-closed deformation space. The number of open deformations is two. It is the common feature of both geometries that there is for generic values of the closed string moduli only a discrete number of such lines [269], however, for special values of the moduli, at the Fermat point, an one parameter family of holomorphic curves.

We start, mainly following [100], our discussion in section 8.1 for the case of the quintic Calabi-Yau threefold. We show that five-branes on rational curves in the quintic can be described as toric branes Σ at a special sublocus $\mathcal{M}(\mathbb{P}^1)$ of their moduli space $\mathcal{M}(\Sigma)$. This establishes the mapping of the moduli space $\mathcal{M}(\Sigma)$ with the obstructed deformations space of rational curves $\tilde{\mathbb{P}}^1$ as discussed in section 7.2.2. This is crucial since it enables us to work with a well-defined complex moduli space of (Z_3, Σ) to describe the space of *anholomorphic*, obstructed deformations of $\tilde{\mathbb{P}}^1$. From this description of the rational curve as a toric brane Σ we readily construct the blow-up \hat{Z}_3 . We study the complex structure moduli space of \hat{Z}_3 in section 8.1.2 by exploiting that \hat{Z}_3 is also governed by new toric data that is canonically related to the toric data of (Z_3, Σ) . First we construct the pull-back form $\hat{\Omega}$ on \hat{Z}_3 along the lines of section 7.3. Then we read of a toric GKZ-system that is associated to the underlying toric data of the blow-up \hat{Z}_3 . From the GKZ-system we derive a complete Picard-Fuchs-system that we solve in section 8.1.3 at various loci in the moduli space. These are the large radius point of Z_3 and the five-brane and selected discriminant loci of the Picard-Fuchs system for \hat{Z}_3 . Since the periods of $\hat{\Omega}$ are also understood as periods on $\mathcal{M}(\Sigma)$, cf. section 7.2.2, we finally obtain the brane superpotential W_{brane} for the line $\tilde{\mathbb{P}}^1$ as a linear combination of solutions fixed by an appropriate flux on \hat{Z}_3 that encodes the information about the five-brane. In particular we use open mirror symmetry to obtain predictions for the disk invariants at large volume, that match and extend independent results in the literature. Similarly, the disk instantons are obtained for a different brane phase in section 8.1.4. These are the first available results for branes with *two* open string deformations.

In section 8.2 we discuss the toric structure of \hat{Z}_3 obtained by blowing up toric curves Σ . We give a general recipe in section 8.2 to obtain a toric polyhedron $\Delta_{\hat{Z}_3}$ from the charge vectors of the geometry Z_3 and the curve Σ , that can be efficiently used to obtain the open-closed GKZ-system for the brane deformation problem associated to arbitrary toric curves Σ . We comment on a connection to Calabi-Yau fourfolds which applies to special choices of toric branes, but *not* in general. Next in section 8.3 we consider another two parameter example of the elliptically fibered Calabi-Yau threefold in $\mathbb{P}^6(1, 1, 1, 6, 9)$ with a five-brane supported on a rational curve. There we make similar use of the toric GKZ-system on \hat{Z}_3 that we solve completely and from which we construct the five-brane superpotential. Under toric mirror symmetry we obtain the disk instanton invariants of the dual A-model geometry, cf. appendix C.3. In addition, we comment on the connection to heterotic/F-theory duality.

Finally in section 8.4 we connect, following [81], the construction of the blow-up threefold with heterotic/F-theory duality as reviewed in section 4.3 and applied in section 6.2. We demonstrate that the blow-up \hat{Z}_3 for a horizontal five-brane on a curve Σ in the heterotic theory can be understood as the heterotic dual to the blow-up in the F-theory fourfold X_4 due to the presence of the horizontal heterotic five-brane. More concretely we start in section 8.4.1 by blowing up the heterotic threefold Z_3 along the five-brane curve Σ into \hat{Z}_3 [60] which we realize as a complete intersection. Then, according to the discussion of section 7.2, the non-Calabi-Yau threefold \hat{Z}_3 contains the five-brane moduli as a subsector of its complex structure moduli. We propose, and explicitly demonstrate for examples, that the F-theory geometry X_4 can in turn be entirely constructed from \hat{Z}_3 . In particular, this identification becomes apparent when also realizing X_4 as a complete intersection. Moreover, in this way the complex structure moduli of \hat{Z}_3 naturally form a subsector of the complex structure moduli of X_4 . Finally in section 8.4.2 we exploit the matching of the geometrical data of \hat{Z}_3 and X_4 , together with the lift of the heterotic superpotential to the blow-up \hat{Z}_3 as in section 7.4, to match the heterotic superpotential with the F-theory flux superpotential. We conclude with a formal matching of the fluxes. In summary, the general idea of this discussion is to reformulate and slightly extend the heterotic/F-theory duality map schematically as:

$$\begin{array}{ccc}
 \text{Heterotic string on CY threefold } Z_3, & \longleftrightarrow & \text{F-theory on CY fourfold } X_4 \\
 \text{vector bundle } E, \text{ five-brane on } \Sigma & & \text{blown up along } \Sigma, G_4\text{-flux} \\
 & \searrow \quad \swarrow & \\
 & \text{non-Calabi-Yau } \hat{Z}_3 \text{ blown up along } \Sigma, & \\
 & \text{vector bundle } \hat{E}, \text{ flux } F_2 = [\Sigma]_E &
 \end{array}$$

where the horizontal arrow indicates the action of heterotic/F-theory duality.

Before we start, let us summarize some other approaches to find open-closed Picard-Fuchs equations for the brane superpotential. The most thoroughly studied cases are non-compact toric Calabi-Yau manifolds with Harvey-Lawson type branes [106, 107]. It was first proposed and demonstrated for non-compact toric Calabi-Yau threefolds in [108–110, 263] to use open-

closed Picard-Fuchs systems to obtain the brane superpotential. For D5-branes in compact geometries the superpotential was calculated in [111,113,115–117,121] by deriving and solving inhomogeneous Picard-Fuchs equations. Finally the methods of [109,110] have been extended to compact geometries in [114] and further applied in [119,120,122–127].

8.1 Open-Closed Picard-Fuchs Systems on the Quintic

We now apply the blow-up proposal of section 7.2 to the case of toric curves¹ Σ on the one parameter quintic. As explained in section 7.2.2 the moduli space $\mathcal{M}(\Sigma)$ of Σ is identified with the obstructed deformation space of rational curves $\tilde{\mathbb{P}}^1$, on which we wrap a five-brane. The holomorphic \mathbb{P}^1 is directly visible from the complete intersection description of Σ at a special sublocus $\mathcal{M}(\mathbb{P}^1)$ of $\mathcal{M}(\Sigma)$ where Σ degenerates appropriately modulo the action of the quintic orbifold $G = (\mathbb{Z}_5)^3$. This way, we understand the five-brane on a rational curve \mathbb{P}^1 as a special case of a toric brane and consequently apply open mirror symmetry along the lines of section 5.1. Thus, we start in section 8.1.1 with the toric curve Σ , then determine the sublocus $\mathcal{M}(\mathbb{P}^1)$ and identify the wrapped rational curve \mathbb{P}^1 that we represent via the standard Veronese embedding in \mathbb{P}^4 . After this definition of the deformation problem, we determine the Picard-Fuchs system on the deformation space of the rational curves $\tilde{\mathbb{P}}^1$ from the GKZ-system on the complex structure moduli space of the blow-up \hat{Z}_3 along Σ in section 8.1.2. The solutions of this system including the large volume expressions for the flux and brane superpotential, the open-closed mirror map and the disk instantons are summarized in section 8.1.3. Furthermore, we find the solutions of the Picard-Fuchs system at its discriminant loci, namely in the vicinity of the sublocus $\mathcal{M}(\mathbb{P}^1)$ and at the involution brane, where we also identify the superpotential W_{brane} . Finally in section 8.1.4 we obtain the disk invariants for a different brane geometry at large volume.

8.1.1 Branes on Lines in the Quintic and the Blow-Up

The quintic Calabi-Yau \tilde{Z}_3 is given as the general quintic hypersurface \tilde{P} in \mathbb{P}^4 . It has 101 complex structure moduli corresponding to the independent coefficients of the monomials entering \tilde{P} . Its Kähler moduli space is one-dimensional generated by the unique Kähler class of the ambient \mathbb{P}^4 . Toric Lagrangian submanifolds of this geometry were discussed in [106] along the lines of section 5.1.

The mirror quintic threefold Z_3 is given as the hypersurface

$$Z_3 : \quad P = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\Psi x_1 x_2 x_3 x_4 x_5, \quad (8.1)$$

¹We emphasize that the five-brane does *not* wrap Σ for generic values of the closed and open moduli.

where Ψ denotes its complex structure modulus. It is obtained via (5.14) from the toric data

$$\begin{array}{cccccc} \ell^{(1)} = (-5, & 1, & 1, & 1, & 1, & 1) \\ \hline y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 x_2 x_3 x_4 x_5 & x_1^5 & x_2^5 & x_3^5 & x_4^5 & x_5^5 \end{array} \quad (8.2)$$

where the y_i corresponding to the entries $\ell_i^{(1)}$ of the charge vector are given as monomials in the projective coordinates x_i constructed using the formula (5.7). In addition we divide by an orbifold group $G = (\mathbb{Z}_5)^3$ that acts on the coordinates so that $x_1 x_2 x_3 x_4 x_5$ is invariant. A convenient basis of generators $g^{(i)}$ is given by $v^{(i)} = (1, -1, 0, 0, 0) \pmod{5}$ and all permutations of its entries where we use

$$g^{(i)} : x_k \mapsto e^{2\pi i v_k^{(i)}/5} x_k. \quad (8.3)$$

We note that the Fermat point $\Psi = 0$ is a point of enhanced symmetry where G enhances to $(\mathbb{Z}_5)^4$. As required by mirror symmetry we have $h^{(2,1)}(Z_3) = 1$, $h^{(1,1)}(Z_3) = 101$.

Next we introduce an open string sector by putting a five-brane on a line \mathbb{P}^1 in the quintic Z_3 . Following the above logic we first construct a toric curve Σ to define the deformation space $\tilde{\mathbb{P}}^1$ of the rational curve. The holomorphic \mathbb{P}^1 is then obtained at the sublocus $\mathcal{M}(\mathbb{P}^1)$ where Σ degenerates accordingly. Up to a relabeling of the projective coordinates x_i of \mathbb{P}^4 we consider the toric curves Σ given by

$$\begin{aligned} \Sigma : P = 0, \quad h_1 \equiv \beta^5 x_3^5 - \alpha^5 x_4^5 = 0, \quad h_2 \equiv \gamma^5 x_3^5 - \alpha^5 x_5^5 = 0, \\ \hat{\ell}^{(1)} = (0, 0, 0, 1, -1, 0), \quad \hat{\ell}^{(2)} = (0, 0, 0, 1, 0, -1), \end{aligned} \quad (8.4)$$

where the brane charge vectors $\hat{\ell}^{(i)}$ correspond to the constraints h_i using (5.16) and the toric data (8.2). The complete intersection (8.4) describes for all values of the parameters α , β and γ , that take values in \mathbb{P}^2 , an analytic family of holomorphic curves in the quintic. Consequently, α , β and γ parameterize the unobstructed moduli space of Σ on which we introduce coordinates $u^1 = \frac{\beta^5}{\alpha^5}$ and $u^2 = \frac{\gamma^5}{\alpha^5}$.

The obstructed deformation problem is defined by the definition of a non-holomorphic family $\tilde{\mathbb{P}}^1$ and the identification of the locus $\mathcal{M}(\mathbb{P}^1)$. As discussed before in section 7.2.2 the obstructed deformation space of lines is identified with the moduli space $\mathcal{M}(\Sigma)$. For generic values of the moduli in (8.4) the curve Σ is an irreducible higher genus Riemann surface. However, we can always linearize (8.4) for generic values of the moduli,

$$\tilde{\mathbb{P}}^1 : \quad \eta_1 x_1 + \sqrt[5]{x_2^5 + x_3^3 m(x_1, x_2, x_3)} = 0, \quad \eta_2 \beta x_3 - \alpha x_4 = 0, \quad \eta_3 \gamma x_3 - \alpha x_5 = 0, \quad (8.5)$$

Here we inserted h_1 and h_2 into P , introduced fifths roots of unity $\eta_i^5 = 1$ and the polynomial

$$m(x_1, x_2, x_3) = \frac{\alpha^5 + \beta^5 + \gamma^5}{\alpha^5} x_3^2 - 5\Psi \frac{\beta\gamma}{\alpha^2} x_1 x_2. \quad (8.6)$$

This equation (8.5) is evidently non-holomorphic due to the non-trivial branching of the fifth root, in other word $\tilde{\mathbb{P}}^1$ is a non-holomorphic family of rational curves in the quintic. However

at special loci Σ degenerates as follows. We rewrite (8.4) as

$$\Sigma : \quad x_1^5 + x_2^5 + x_3^3 m(x_1, x_2, x_3) = 0, \quad \alpha^5 x_4^5 - \beta^5 x_3^5 = 0, \quad \alpha^5 x_5^5 - \gamma^5 x_3^5 = 0. \quad (8.7)$$

Whereas h_1, h_2 can be linearized for generic values of the moduli, $m(x_1, x_2, x_3)$ forbids a holomorphic linearization of (8.4) and accordingly to take the fifth root in (8.5). However, at the sublocus

$$\mathcal{M}_{\mathbb{P}^1}(\Sigma) : \quad \alpha^5 + \beta^5 + \gamma^5 = 0, \quad \Psi \alpha \beta \gamma = 0 \quad (8.8)$$

the polynomial m vanishes identically and the Riemann surface Σ in (8.4) degenerates to

$$\Sigma : \quad h_0 \equiv x_1^5 + x_2^5, \quad h_2 = \beta^5 x_3^5 - \alpha^5 x_4^5 = 0, \quad h_2 = \gamma^5 x_3^5 - \alpha^5 x_5^5 = 0. \quad (8.9)$$

This can be trivially factorized as in the general discussion (7.34) in linear factors that differ only by fifth roots of unity η_i , that are the 125 solutions to (8.5). In other words, at the locus $\mathcal{M}(\mathbb{P}^1)$ the curve Σ degenerates to 125 lines corresponding to each choice of η_i in the three constraints h_i . However, in contrast to (8.4) which is invariant under the orbifold G , the linearized equations do transform under G . In fact, all the 125 different lines are identified modulo the action of $G = (\mathbb{Z}_5)^3$ so that (8.9) describes a single line on the quotient by G ,

$$\mathcal{M}(\mathbb{P}^1) : \quad \eta x_1 + x_2 = 0, \quad \alpha x_4 - \beta x_3 = 0, \quad \alpha x_5 - \gamma x_3 = 0. \quad (8.10)$$

Equivalently, these lines are given parametrically in \mathbb{P}^4 in terms of homogeneous coordinates U, V on \mathbb{P}^1 as the Veronese mapping

$$(U, V) \mapsto (U, -\eta U, \alpha V, \beta V, \gamma V), \quad \eta^5 = 1. \quad (8.11)$$

This way, the family Σ contains the holomorphic lines (8.11) at the sublocus $\mathcal{M}(\mathbb{P}^1)$ of (8.8) defined by the vanishing of $m(x_1, x_2, x_3)$. In summary this shows that a five-brane wrapping the line (8.11) falls in the class of toric branes at the sublocus $\mathcal{M}(\mathbb{P}^1)$ of their moduli space. We emphasize again that (8.11) is not invariant under the orbifold group G and that the identification of the 125 distinct solutions to (8.5) under G is essential to match an in general higher genus Riemann surface Σ with a rational curve of genus $g = 0$.

This picture is further confirmed from the perspective of the rational curve (8.11) since the constraint (8.8) defining $\mathcal{M}(\mathbb{P}^1)$ is precisely the condition for the line (8.11) to lie holomorphically in the quintic constraint P . Thus, the sublocus $\mathcal{M}(\mathbb{P}^1)$ defined in (8.8) is precisely the moduli space of the five-brane wrapping the holomorphic lines in the quintic. For generic $\Psi \neq 0$ this moduli space is only a number of discrete points whereas at the Fermat point $\Psi = 0$ there is a one-dimensional moduli space of lines in the quintic parametrized by a Riemann surface² of genus $g = 6$, cf. figure 7.4. In the language of superpotentials, we understand $\mathcal{M}(\mathbb{P}^1)$ as the critical locus of W_{brane} at which the five-brane on the rational curve is supersymmetric. Conversely, deforming away from the critical locus $\mathcal{M}(\mathbb{P}^1)$ in $\mathcal{M}(\Sigma)$ is

²The first constraint in (8.8) is a quintic constraint in \mathbb{P}^2 describing a Riemann surface of genus $g = 6$.

obstructed, inducing a non-trivial superpotential. Thus, we consider in the following the deformation space defined by $\tilde{\mathbb{P}}^1$, more precisely by the coefficients of m ,

$$\hat{z}^1 = \frac{\beta\gamma}{\alpha^2} = (u^1 u^2)^{\frac{1}{5}}, \quad \hat{z}^2 = \frac{\alpha^5 + \beta^5 + \gamma^5}{\alpha^5} = 1 + u^1 + u^2, \quad (8.12)$$

which agrees with the choice of variables used in figure 7.4. As noted before, we can canonically identify this deformation space with the moduli space $\mathcal{M}(\Sigma)$ of Σ in Z_3 by dividing out the orbifold group G and working with the holomorphic constraint (8.4) instead of (8.5).

Most importantly for the blow-up procedure, the description of the toric curve Σ of (8.4) is precisely in the form used in section 7.2.1 to construct the blow-up geometry \hat{Z}_3 . In particular, we can easily read off the normal bundle $N_{Z_3\Sigma}$ of Σ in the quintic which is $N_{Z_3\Sigma} = \mathcal{O}(5) \oplus \mathcal{O}(5)$ by simply noting the degree of the divisors $h_1 = 0$, $h_2 = 0$. Then the blow-up \hat{Z}_3 is given by the complete intersection (7.32), which in the case at hand reads

$$\hat{Z}_3 : \quad P = 0, \quad Q = l_1(u^2 x_3^5 - x_5^5) - l_2(u^1 x_3^5 - x_4^5) = 0. \quad (8.13)$$

Since both the closed modulus Ψ as well as the open moduli u^1 , u^2 enter (8.13), we formally obtain the embedding of the open-closed moduli space of (Z_3, Σ) , and equivalently the obstructed deformation space of $(Z_3, \tilde{\mathbb{P}}^1)$, into the complex structure moduli space of \hat{Z}_3 . In particular this trivially embeds the moduli space of the rational curves (8.11) by restricting to the critical locus (8.8).

On the blow-up \hat{Z}_3 this embedding as well as the obstructions can be understood purely geometrically. First of all we note that the action of the quintic orbifold G directly lifts to \hat{Z}_3 . Then by deforming away from the critical values (8.8) we change the topology of the blow-up divisor E from a ruled surface over \mathbb{P}^1 to a ruled surface E over a Riemann-surface Σ of higher genus. The one-cycles of the Riemann-surface in the base lift to new three-cycles on the blow-up \hat{Z}_3 that correspond to new non-algebraic complex structure deformations³, compare to the similar discussion of [66, 67]. Upon switching on flux on these three-cycles turns on higher order obstructions for the complex structure of \hat{Z}_3 destroying the ruled surface E and thus driving us back to the critical locus where Σ degenerates to \mathbb{P}^1 . This way the flux obstructs the complex structure in (8.13) which is expressed by a flux superpotential on \hat{Z}_3 that is the sought for superpotential W_{brane} .

In the following we will use the complete intersection (8.13) to analyze the open-closed deformation space $(Z_3, \tilde{\mathbb{P}}^1)$. The crucial point is that we are working with a well-defined complex moduli space of (Z_3, Σ) respectively of complex structures on \hat{Z}_3 to describe the space of *anholomorphic* deformations $\tilde{\mathbb{P}}^1$. In this context this is another reason for the effectiveness of the blow-up \hat{Z}_3 for the description of the obstructed brane deformations $\tilde{\mathbb{P}}^1$.

³Although related these new complex structure deformations should not be confused with the parameters entering Q since these are algebraic by definition.

8.1.2 Toric Branes on the Quintic: the Open-Closed GKZ-System

In the following we analyze the open-closed deformation space of $(\tilde{\mathbb{P}}^1, Z_3)$ embedded in the complex structure moduli of \hat{Z}_3 augmented by appropriate flux data. We perform this analysis by toric means, i.e. the GKZ-system. Thus we supplement the polyhedron $\Delta_4^{\tilde{Z}}$ and the charge vectors $\ell^{(1)}$, $\hat{\ell}^{(1)}$, $\hat{\ell}^{(2)}$ of the quintic Calabi-Yau (8.1) and the toric brane (8.4),

$$\left(\begin{array}{c|cccc|c|c} & & \Delta_4^{\tilde{Z}} & & & \ell^{(1)} & & \hat{\ell}^{(1)} & \hat{\ell}^{(2)} \\ \hline \tilde{v}_0 & 0 & 0 & 0 & 0 & -5 & y_0 = x_1 x_2 x_3 x_4 x_5 & 0 & 0 \\ \tilde{v}_1 & -1 & -1 & -1 & -1 & 1 & y_1 = x_1^5 & 0 & 0 \\ \tilde{v}_2 & 1 & 0 & 0 & 0 & 1 & y_2 = x_2^5 & 0 & 0 \\ \tilde{v}_3 & 0 & 1 & 0 & 0 & 1 & y_3 = x_3^5 & 1 & 1 \\ \tilde{v}_4 & 0 & 0 & 1 & 0 & 1 & y_4 = x_4^5 & -1 & 0 \\ \tilde{v}_5 & 0 & 0 & 0 & 1 & 1 & y_5 = x_5^5 & 0 & -1 \end{array} \right). \quad (8.14)$$

The points of the dual polyhedron $\Delta_4^{\tilde{Z}}$ are given by $v_1 = (-1, -1, -1, -1)$, $v_2 = (4, -1, -1, -1)$, $v_3 = (-1, 4, -1, -1)$, $v_4 = (-1, -1, 4, -1)$ and $v_5 = (-1, -1, -1, 4)$. These monomials both enter the constraints P and h_i according to (5.7) and (5.16) yielding

$$Z_3 : \quad P = \sum_{i=1}^5 a_i x_i^5 + a_0 x_1 x_2 x_3 x_4 x_5, \quad \Sigma : \quad h_1 = a_6 x_3^5 + a_7 x_4^5, \quad h_2 = a_8 x_3^5 + a_9 x_5^5, \quad (8.15)$$

where we introduced free complex-valued coefficients \underline{a} .⁴ From the polyhedron (8.14) we readily obtain the standard toric GKZ-system for Z_3 along the lines of eqs. (5.34) and (5.35),

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}_i = \vartheta_{i+1} - \vartheta_1 \quad (i = 1, \dots, 4), \\ \mathcal{L}_1 &= \prod_{i=1}^5 \frac{\partial}{\partial a_i} - \frac{\partial^5}{\partial a_0^5}, \end{aligned} \quad (8.16)$$

where we use the logarithmic derivative $\vartheta_i = a_i \frac{\partial}{\partial a_i}$. The \mathcal{Z}_i express the coordinate rescalings leaving the measure Δ and the monomial $x_1 x_2 x_3 x_4 x_5$ in (5.32) invariant. They express infinitesimal rescalings of the parameters \underline{a} and the coordinates \underline{x} entering P . For example the rescaling $(x_1, x_2) \mapsto (\lambda^{1/5} x_1, \lambda^{-1/5} x_2)$ combined with $(a_1, a_2) \mapsto (\lambda^{-1} a_1, \lambda a_2)$ leaves P invariant and consequently the periods have the symmetry $\Pi^k(a_0, \lambda^{-1} a_1, \lambda a_2, a_3, a_4, a_5) = \Pi^k(\underline{a})$. The corresponding generator of this symmetry is \mathcal{Z}_1 . These homogeneity properties of the $\Pi^k(a_i)$ imply that they are functions of only a specific combination of the \underline{a} , which in the case of the quintic takes the form

$$z^1 = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}. \quad (8.17)$$

This is perfectly consistent with (5.36) and the charge vector $\ell^{(1)}$.

⁴Conversely to the conventions in (7.47), we denote the parameters b_i by a_{5+i} for convenience.

The analysis of the combined system of the quintic and the curve (8.15) is performed by replacing (Z_3, Σ) by the blow-up (\hat{Z}_3, E) given by the family of complete intersections in $\mathcal{W} = \mathbb{P}(\mathcal{O}(5) \oplus \mathcal{O}(5)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$,

$$\hat{Z}_3 : \quad P = 0, \quad Q = \ell_1(a_8x_3^5 + a_9x_5^5) - \ell_2(a_6x_3^5 + a_7x_4^5). \quad (8.18)$$

Then the holomorphic three-form $\hat{\Omega}$ is constructed using the residue (7.42). Using this explicit residue integral expression, it is straightforward to find the Picard-Fuchs system on the blow-up \hat{Z}_3 that encodes the complex structure dependence of $\hat{\Omega}$. As it can be directly checked the GKZ-system is given by \mathcal{L}_1 as in (8.16) complemented to the system

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^5 \vartheta_i + 1, & \mathcal{Z}_1 &= \sum_{i=6}^9 \vartheta_i, & \mathcal{Z}_2 &= \vartheta_2 - \vartheta_1, & \mathcal{Z}_3 &= \vartheta_3 - \vartheta_1 + \vartheta_6 + \vartheta_8, \\ \mathcal{Z}_4 &= \vartheta_4 - \vartheta_1 + \vartheta_7, & \mathcal{Z}_5 &= \vartheta_5 - \vartheta_1 + \vartheta_9, & \mathcal{Z}_6 &= \vartheta_8 + \vartheta_9 - \vartheta_6 - \vartheta_7, \end{aligned} \quad (8.19)$$

encoding the torus symmetries of \mathcal{W} and the actual Picard-Fuchs operators

$$\mathcal{L}_1 = \prod_{i=1}^5 \frac{\partial}{\partial a_i} - \frac{\partial^5}{\partial a_0^5}, \quad \mathcal{L}_2 = \frac{\partial^2}{\partial a_3 \partial a_7} - \frac{\partial^2}{\partial a_4 \partial a_6}, \quad \mathcal{L}_3 = \frac{\partial^2}{\partial a_3 \partial a_9} - \frac{\partial^2}{\partial a_5 \partial a_8}. \quad (8.20)$$

We emphasize that there are two new second order differential operators $\mathcal{L}_2, \mathcal{L}_3$ that annihilate $\hat{\Omega}$ identically and that incorporate the deformations a_i , $i = 6, 7, 8, 9$ associated to the curve Σ . It is clear from the appearance of the \mathbb{P}^1 -coordinates (l_1, l_2) in the constraint Q that there are no further operators \mathcal{L}_a on \hat{Z}_3 of minimal degree. Let us briefly explain the origin of the operators \mathcal{Z}_k . The first two are simply associated to an overall rescaling of the two constraints $P \mapsto \lambda P$, $Q \mapsto \lambda' Q$ which acts on $\hat{\Omega}(\underline{a})$ as $\hat{\Omega}(\lambda a_0, \dots, \lambda a_6, a_7, \dots, a_{10}) = \lambda \hat{\Omega}(\underline{a})$ and $\hat{\Omega}(a_0, \dots, a_6, \lambda' a_7, \dots, \lambda' a_{10}) = \hat{\Omega}(\underline{a})$. For the rescaling of Q the factor λ' is compensated by the non-trivial prefactor h_i/ℓ_i in (7.42). The third to sixth operators are associated to the torus symmetries of the \mathbb{P}^4 as before, $(x_1, x_j) \mapsto (\lambda_j x_1, \lambda_j^{-1} x_j)$, $j = 2, \dots, 5$, and the last operator \mathcal{Z}_6 is related to the torus symmetry $(l_1, l_2) \mapsto (\lambda l_1, \lambda^{-1} l_2)$ of the exceptional \mathbb{P}^1 . It is important to note that the operators \mathcal{Z}_i of the \mathbb{P}^4 are altered due to the blow-up \hat{Z}_3 , i. e. due to the presence of the five-brane, as compared to the closed string case of (8.16).

Before delving into the determination of the solutions to this differential system let us reconsider the operators we just found from a slightly different perspective. This will in particular allow for a straightforward systematization of the constructions of GKZ-system. Comparing (8.19) to the closed GKZ-system (8.16) associated to Z_3 we recover a very similar structure. Indeed the above differential system governing the complex structure on \hat{Z}_3 defines a new GKZ-system with exponent β . To obtain the set of integral points \hat{v}_i associated to this

GKZ-system we apply the general definition of the \mathcal{Z}_i in (5.35) backwards to obtain

$$\left(\begin{array}{c|cccccc|ccc|c} & & & \Delta_7^{\hat{Z}} & & & & \hat{\ell}^{(1)} & \hat{\ell}^{(2)} & \hat{\ell}^{(3)} & \\ \hline \hat{v}_0 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & \hat{y}_0 = x_1 x_2 x_3 x_4 x_5 \\ \hat{v}_1 & 1 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & \hat{y}_1 = x_1^5 \\ \hat{v}_2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \hat{y}_2 = x_2^5 \\ \hat{v}_3 & 1 & 0 & 0 & 1 & 0 & 0 & 3 & -1 & -1 & \hat{y}_3 = x_3^5 \\ \hat{v}_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \hat{y}_4 = x_4^5 \\ \hat{v}_5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \hat{y}_5 = x_5^5 \\ \hat{v}_6 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & \hat{y}_6 = l_1 \hat{y}_3 \\ \hat{v}_7 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & \hat{y}_7 = l_1 \hat{y}_4 \\ \hat{v}_8 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & \hat{y}_8 = l_2 \hat{y}_3 \\ \hat{v}_9 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & \hat{y}_9 = l_2 \hat{y}_5 \end{array} \right). \quad (8.21)$$

Here we have displayed the points \hat{v}_i , the corresponding monomials \hat{y}_i and a basis of relations $\hat{\ell}^{(i)}$, that we obtain as a Mori cone of a triangulation of the polyhedron $\Delta_7^{\hat{Z}}$. We emphasize that besides the closed string charge vectors of Z_3 embedded as $\ell^{(1)} = \hat{\ell}^{(1)} + \hat{\ell}^{(2)} + \hat{\ell}^{(3)}$, the brane charge vectors $\hat{\ell}^{(a)}$ are among the charge vectors $\hat{\ell}^{(j)}$ of $\Delta_7^{\hat{Z}}$ as well. Furthermore, for the above triangulation of $\Delta_7^{\hat{Z}}$ we immediately obtain the full GKZ differential system \mathcal{L}_a , \mathcal{Z}_i of (8.19) by the standard formulas for the *standard* GKZ-system in (5.34), (5.35) using the points \hat{v}_i and relations $\hat{\ell}^{(j)}$ from $\Delta_7^{\hat{Z}}$ with exponent $\beta = (-1, 0, 0, 0, 0, 0, 0)$.

This GKZ-system defines coordinates \hat{z}^a on the complex structure moduli space of \hat{Z}_3 as before. We apply the closed string formula (5.36) for the charge vectors $\hat{\ell}^{(a)}$ of $\Delta_7^{\hat{Z}}$ to obtain the three coordinates

$$\hat{z}^1 = -\frac{a_1 a_2 a_3^3 a_7 a_9}{a_0^5 a_6 a_8}, \quad \hat{z}^2 = \frac{a_4 a_6}{a_3 a_7}, \quad \hat{z}^3 = \frac{a_5 a_8}{a_3 a_9}. \quad (8.22)$$

We obtain a complete system of differential operators \mathcal{D}_a , the Picard-Fuchs operators, by adding to (8.19) further operators \mathcal{L}_a associated to scaling symmetries specified by integer positive linear combinations of the charge vectors $\hat{\ell}^{(a)}$ in (8.21). By factorizing these operators \mathcal{L}_a expressed in the coordinates (8.22) we obtain the differential system generated by

$$\begin{aligned} \mathcal{D}_1 &= \theta_1 \theta_2 \theta_3 (3\theta_1 - \theta_2 - \theta_3) - 5 \prod_{i=1}^4 (5\theta_1 - i) \hat{z}^1 \hat{z}^2 \hat{z}^3, \\ \mathcal{D}_i &= (\theta_1 - \theta_i) \theta_i + (1 + \theta_1 - \theta_2) (1 + 3\theta_1 - \theta_2 - \theta_3) \hat{z}^i, \quad i = 2, 3, \end{aligned} \quad (8.23)$$

where we introduced $\theta_i = \hat{z}^i \frac{\partial}{\partial \hat{z}^i}$ and further rescaled the holomorphic three-form $\hat{\Omega}$ by a_0 . Each of these three operator \mathcal{D}_a corresponds to a linear combination of the charge vectors $\hat{\ell}^{(i)}$, whose integer coefficients can be read off from the powers of \hat{z}^i in the last term of \mathcal{D}_a . Obviously the deformation problem is symmetric under exchange of \hat{z}^2 and \hat{z}^3 . While \mathcal{D}_1 is symmetric under that symmetry, \mathcal{D}_2 and \mathcal{D}_3 map onto each other under $\hat{z}^2 \leftrightarrow \hat{z}^3$.

This Picard-Fuchs system is perfectly consistent with the expected structure from section (7.49), that in particular implies that the periods of Ω directly lift to the blow-up \hat{Z}_3 . Upon the identification of the coordinate $z^1 = \hat{z}^1 \hat{z}^2 \hat{z}^3$ on the complex structure moduli space of the quintic⁵, and keeping \hat{z}^2, \hat{z}^3 unchanged, we rewrite the operators (8.23) as

$$\begin{aligned} \mathcal{D}_1 &= \mathcal{D}_1^{Z_3} + [\theta_1(\theta_1 + \theta_3)(\theta_1 - \theta_2 - \theta_3)\theta_2 + (\theta_2 \leftrightarrow \theta_3)], \\ \mathcal{D}_i &= -[\theta_1 + \theta_i - z_i(\theta_1 - \theta_2 - \theta_3)] \cdot \theta_i, \end{aligned} \quad (8.24)$$

where we write $\theta_1 = z^1 \frac{\partial}{\partial z^1}$ by abuse of notation. The first operator \mathcal{D}_1 splits into the well-known fourth order quintic operator $\mathcal{D}_1^{Z_3} = \theta_1^4 - 5 \prod_{i=1}^4 (5\theta_1 - i) z^1$ and a term linear in the derivatives θ_2, θ_3 . The other operators $\mathcal{D}_2, \mathcal{D}_3$ are proportional to θ_2, θ_3 . Consequently, it is ensured that the solutions to (8.23) contain the closed string periods $\Pi^k(z^1)$ of the quintic as the unique solutions independent of the open string parameters \hat{z}^2, \hat{z}^3 .

Thus, we summarize by emphasizing that the complete information for the study of complex structure variations in the family \hat{Z}_3 of complete intersection threefolds $P = Q = 0$ just reduces to the determination of the toric data $\Delta_{\hat{Z}_3}^{\hat{Z}}$ and the associated GKZ-system.

8.1.3 Superpotentials from Blow-Up Threefolds

The complex structure moduli space of the blown up quintic orbifold \hat{Z}_3 described above is the model for our open/closed deformation space and (8.23) is the Picard-Fuchs system annihilating its periods. We will analyze and interpret the global properties of the deformations space and the solutions at special points in the deformation space. First we analyze the solutions at the locus $\hat{z}_i = 0$. Different than for systems that can be embedded in a Calabi-Yau fourfold, as the one in sections 8.3, we find at $\hat{z}_i = 0$ no maximal unipotent monodromy. Rather the indicial equations of the system (8.23) have the solutions $(0, 0, 0)^{12}, (\frac{1}{3}, 0, 0), (\frac{2}{3}, 0, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2})$. So in total we find 16 solutions. The twelve-times degenerate solution $((0, 0, 0)^{12}$ gives rise to one power series solution

$$X_1^{(0)} = 1 + 120z + 113400z^2 + 168168000z^3 + \mathcal{O}(z^4), \quad (8.25)$$

where $z = \hat{z}_1 \hat{z}_2 \hat{z}_3$ is the quintic complex structure parameter near the point of maximal unipotent monodromy in its moduli space. This solution is identified with the fundamental period X_0 of the quintic. Denoting $\hat{l}_i := \log(\hat{z}_i)$ we get additional eleven logarithmic solutions

$$\begin{aligned} X_i^{(1)} &: \hat{l}_1, \hat{l}_2, \hat{l}_3, \\ X_\alpha^{(2)} &: \frac{1}{2} \hat{l}_1^2, \hat{l}_2(\frac{1}{2} \hat{l}_2 + \hat{l}_1), \hat{l}_3(\frac{1}{2} \hat{l}_3 + \hat{l}_2), \hat{l}_2 \hat{l}_3 \\ X_\beta^{(3)} &: \frac{1}{6} \hat{l}_1^3, \frac{1}{6} \hat{l}_2^3 + \frac{1}{2} \hat{l}_1^2 \hat{l}_2 + \frac{1}{2} \hat{l}_2^2 \hat{l}_1, \frac{1}{6} \hat{l}_3^3 + \frac{1}{2} \hat{l}_1^2 \hat{l}_3 + \frac{1}{2} \hat{l}_3^2 \hat{l}_1, \frac{1}{2} \hat{l}_2^2 \hat{l}_3 + \frac{1}{2} \hat{l}_3^2 \hat{l}_2 + \hat{l}_1 \hat{l}_2 \hat{l}_3. \end{aligned} \quad (8.26)$$

⁵This is perfectly consistent with the embedding of the quintic charge vector as $\ell^{(1)} = \hat{\ell}^{(1)} + \hat{\ell}^{(2)} + \hat{\ell}^{(3)}$.

The single logarithmic solutions are

$$\begin{aligned} X_1^{(1)} &= X^{(0)} \log(\hat{z}_1) - 60\hat{z}_1(\hat{z}_2 + \hat{z}_3) + 770z + 9450\hat{z}_1^2(\hat{z}_2^2 + \hat{z}_3^2) + 60\hat{z}_1(\hat{z}_2^2\hat{z}_3 + \hat{z}_2\hat{z}_3^2) + \mathcal{O}(\hat{z}^5), \\ X_2^{(1)} &= X^{(0)} \log(\hat{z}_2) + 60\hat{z}_1\hat{z}_3 - 9450\hat{z}_1^2\hat{z}_3^2 - 60\hat{z}_1\hat{z}_2^2\hat{z}_3 + \mathcal{O}(\hat{z}^5), \\ X_3^{(1)} &= X^{(0)} \log(\hat{z}_3) + 60\hat{z}_1\hat{z}_2 - 9450\hat{z}_1^2\hat{z}_2^2 - 60\hat{z}_1\hat{z}_3^2\hat{z}_2 + \mathcal{O}(\hat{z}^5). \end{aligned} \quad (8.27)$$

It is easy to check that the single logarithmic period of the mirror quintic is obtained as $\sum_i X_i^{(1)}$. Similarly we have chosen the normalization of (8.26) so that $\sum_\alpha X_\alpha^{(2)}$ and $\sum_\beta X_\beta^{(3)}$ are double and triple logarithmic solutions of the Picard-Fuchs equation of the mirror quintic Z_3 . Using the information about the classical terms of the mirror quintic [35, 252] one can identify the precise combination of periods corresponding to a basis of $H_3(Z_3, \mathbb{Z})$.

Notable are the four fractional power series solutions to the remaining indices,

$$\begin{aligned} X_2^{(0)} &= \hat{z}_1^{\frac{1}{3}} + \hat{z}_1^{\frac{1}{3}}\left(\frac{1}{2}\hat{z}_2 + \frac{1}{2}\hat{z}_3 + \frac{6545}{2592}\hat{z}_1\right) + \mathcal{O}(\hat{z}^{\frac{7}{3}}), \\ X_3^{(0)} &= \hat{z}_1^{\frac{2}{3}} + \hat{z}_1^{\frac{2}{3}}(4\hat{z}_2 + 4\hat{z}_3 + \frac{86944}{10125}\hat{z}_1) + \mathcal{O}(\hat{z}^{\frac{7}{3}}), \\ X_4^{(0)} &= \sqrt{\hat{z}_1\hat{z}_2} + \sqrt{\hat{z}_1\hat{z}_2\hat{z}_3} - \frac{5005}{72}(\hat{z}_1\hat{z}_2)^{\frac{3}{2}} + \mathcal{O}(\hat{z}^4), \\ X_5^{(0)} &= \sqrt{\hat{z}_1\hat{z}_3} + \sqrt{\hat{z}_1\hat{z}_3\hat{z}_2} - \frac{5005}{72}(\hat{z}_1\hat{z}_3)^{\frac{3}{2}} + \mathcal{O}(\hat{z}^4). \end{aligned} \quad (8.28)$$

Let us discuss now the global properties of the moduli space of the branes on the quintic orbifold defined by (8.4). As discussed in section 8.1.1 there are critical points, where the unobstructed deformation problem of the complete intersection (8.4) gives rise to superpotentials associated to obstructed deformation problems such as the lines in the quintic orbifold. Clearly these loci must occur at the discriminant of the Picard-Fuchs equation determined by $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ described in the last section. We find

$$\begin{aligned} \Delta &= (1 + \hat{z}_2)(1 - \hat{z}_2)(1 + \hat{z}_3)(1 - \hat{z}_3)(1 - \hat{z}_2 - \hat{z}_3)(1 + 2\hat{z}_2 - \hat{z}_3)(1 + 2\hat{z}_3 - \hat{z}_2) \\ &\quad \times (4 + 5^5\hat{z}_1\hat{z}_2(1 - \hat{z}_3)^2)(4 + 5^5\hat{z}_1\hat{z}_3(1 - \hat{z}_2)^2)(1 - 5^5\hat{z}_1\hat{z}_2\hat{z}_3)(27 + 5^5\hat{z}_1(1 - \hat{z}_2 - \hat{z}_3)^3). \end{aligned} \quad (8.29)$$

We expect to get a degeneration of the holomorphic curve Σ of (8.4) at the discriminant locus and thus obstructed deformation problems that can be characterized by appropriate flux quantum numbers. Let us consider two discriminant loci of particular interest.

At the locus $\hat{z}_2 = -1$ and $\hat{z}_3 = -1^6$ the complete intersection becomes holomorphic in the quintic and in fact the toric A-brane, which is mirror to the holomorphic constraint, becomes compatible with the involution brane, i.e. the fixpoint locus of the involution

$$(x_1, x_2, x_3, x_4, x_5) \rightarrow (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5). \quad (8.30)$$

More precisely the toric A-branes is given by the constraints (5.13) defined by the charge vectors $\hat{\ell}^{(a)}$ in (8.15) with vanishing relative Kähler/Wilson line parameters $c^a = 0$. Comparing the solutions at that locus we obtain a two open parameter deformation of the brane

⁶We note that $\hat{z}_2 = \hat{z}_3 = -1$ agrees with $u^1 = u^2 = 1$ in the notation of (5.16) since $\hat{z}_a = -u^a$.

discussed in [111]. The relevant periods at the involution brane point are trivially obtained from the solutions at the large complex structure point by analytic continuation. In particular the solutions at large complex structure, which are at most linear in the logarithms of the \hat{z}_a converge in the variables $(v_1 = z_1, v_2 = (1 + \hat{z}_2), v_3 = (1 + \hat{z}_3))$. The solutions with the square root cuts $X_4^{(0)}$ and $X_5^{(0)}$ are expected to specialize to the superpotential for the involution brane, if the open moduli v_2 and v_3 are set to zero. Indeed, if we symmetrize in the two square root solutions, we find up to a normalization worked out in [111] the series

$$W^{quant} = \frac{30}{4\pi^2} \left(v_1^{1/2} + \frac{5005}{9} v_1^{3/2} + \frac{52055003}{75} v_1^{5/2} + v_1^{1/2} \left(\frac{1}{2} (v_2 + v_3) - \frac{1}{16} (v_2^2 - 4v_1v_2 + v_3^2) \right) + \frac{5005}{6} v_1^{3/2} (v_2 + v_3) + \mathcal{O}(v^{7/2}) \right). \quad (8.31)$$

In particular we note that for $v_2 = v_3 = 0$ this superpotential is exactly the one for the involution brane obtained in [111]. Using the mirror map of the quintic it is possible to obtain from (8.31) at $v_2 = v_3 = 0$ the disk instantons for the involution brane. We expect that the scalar potential induced by (8.31) has a minimum along the $v_2 = v_3 = 0$ direction. However to see this minimalization explicitly requires the construction of the Kähler potential, a choice of flat coordinates and a choice of the gauging of the superpotential as a section in the Kähler line bundle. We note that the above discussion of the involution brane is similar to the one of [119] in the context of a one open parameter family of a toric brane on the quintic.

A similarly interesting locus is the $(1 - \hat{z}_2 - \hat{z}_3) = 0$ and $\frac{1}{\hat{z}_1} = 0$. According to the discussion in section 8.1.1 this is the locus $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ of (8.8), $\frac{(\alpha^5 + \beta^5 + \gamma^5)}{\alpha^5} = 0$ and $\frac{\psi\beta\gamma}{\alpha^2} = 0$, where the constraints (8.4) factorize and the holomorphic lines occur. We expect the superpotential to vanish at this locus. Indeed if we expand in $(w_1 = \frac{1}{\hat{z}_1}, w_2 = (1 - \hat{z}_2 - \hat{z}_3), w_3 = \hat{z}_2 - \hat{z}_3)$ we find 16 solutions having the indicials $(\frac{k}{5}, i, j)$, where $k = 1, \dots, 4$ and $(i, j) = (0, 0), (1, 0), (0, 1), (1, 1)$. Thus, the solutions vanish with $\hat{z}_1^{-\frac{1}{5}}, \hat{z}_1^{-\frac{2}{5}}, \hat{z}_1^{-\frac{3}{5}}, \hat{z}_1^{-\frac{4}{5}}$ for $\hat{z}_1^{-\frac{1}{5}} = \frac{\psi\beta\gamma}{\alpha^2}$. This is compatible with the vanishing of the superpotential at the locus of the holomorphic lines. Again one would need the flat coordinates and the gauge choice in order to perform a detailed local analysis of the orbifold superpotential.

In summary, from the two examples above it is clear that the discriminant of the Picard-Fuchs equation contains the expected information about the degeneration of the two open parameter brane system are special loci, where the problem can be related to obstructed deformation problems. We expect this also to be true at the other loci of the discriminant, where a brane interpretation is not yet available.

8.1.4 Brane Superpotential at Large Volume: Disk Instantons

In this section we apply the blow-up \hat{Z}_3 to a different five-brane on the quintic. The following analysis is focused on the determination of the disk instanton invariants at large radius of the

A-model and thus brief at several points for the sake of brevity.

The Calabi-Yau geometry of the B-model is given by the one parameter mirror quintic with the constraint P as in (8.15). We add an open string sector of a five-brane, that we describe as before under the identification of the moduli space $\mathcal{M}(\Sigma)$ with the deformation space \tilde{P}^1 , by the toric curve Σ specified by brane charge vectors $\hat{\ell}^{(a)}$ as

$$\begin{aligned} \Sigma : \quad P = 0, \quad h_1 \equiv a_6 x_1 x_2 x_3 x_4 x_5 + a_7 x_1^5 = 0, \quad h_2 \equiv a_8 x_1^5 + a_9 x_2^5 = 0, \\ \hat{\ell}^{(1)} = (-1, 1, 0, 0, 0, 0), \quad \hat{\ell}^{(2)} = (0, -1, 1, 0, 0, 0). \end{aligned} \quad (8.32)$$

For this geometry we readily construct the blow-up \hat{Z}_3 as the complete intersection in the toric variety $\mathcal{W} = \mathbb{P}(\mathcal{O}(5) \oplus \mathcal{O}(5)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$,

$$\hat{Z}_3 : \quad P = 0, \quad Q = l_1(a_8 x_1^5 + a_9 x_2^5) - l_2(a_6 x_1 x_2 x_3 x_4 x_5 + a_7 x_1^5). \quad (8.33)$$

From these constraints we construct the holomorphic three-form $\hat{\Omega}$ as the residue (7.42) from which we read off the GKZ-system for \hat{Z}_3 as

$$\begin{aligned} \mathcal{Z}_0 = \sum_{i=0}^5 \vartheta_i + 1, \quad \mathcal{Z}_1 = \sum_{i=6}^9 \vartheta_i, \quad \mathcal{Z}_2 = \vartheta_2 - \vartheta_1 - \vartheta_7 - \vartheta_8 + \vartheta_9, \\ \mathcal{Z}_i = \vartheta_i - \vartheta_1 - \vartheta_7 - \vartheta_8, \quad i = 3, 4, 5, \quad \mathcal{Z}_6 = \vartheta_8 + \vartheta_9 - \vartheta_6 - \vartheta_7, \\ \mathcal{L}_1 = \prod_{i=1}^5 \frac{\partial}{\partial a_i} - \frac{\partial^5}{\partial a_0^5}, \quad \mathcal{L}_2 = \frac{\partial^2}{\partial a_1 \partial a_6} - \frac{\partial^2}{\partial a_0 \partial a_7}, \quad \mathcal{L}_3 = \frac{\partial^2}{\partial a_2 \partial a_8} - \frac{\partial^2}{\partial a_1 \partial a_9}, \end{aligned} \quad (8.34)$$

for the logarithmic derivative $\vartheta_i = a_i \frac{\partial}{\partial a_i}$. Again there are two second order differential operators $\mathcal{L}_2, \mathcal{L}_3$ that include the curve moduli $a_i, i = 6, 7, 8, 9$, and one fifth order operator \mathcal{L}_1 which is lifted from the quintic Calabi-Yau to the blow-up. There are no further operators of minimal degree. We obtain the GKZ-system (8.34) from the following toric data of \hat{Z}_3

$$\left(\begin{array}{c|cccccc|ccc|c} & & & & \Delta_{\hat{Z}_3} & & & & \hat{\ell}^{(1)} & \hat{\ell}^{(2)} & \hat{\ell}^{(3)} & \\ \hline \hat{v}_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 0 & \hat{y}_0 = x_1 x_2 x_3 x_4 x_5 \\ \hat{v}_1 & 1 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & \hat{y}_1 = x_1^5 \\ \hat{v}_2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \hat{y}_2 = x_2^5 \\ \hat{v}_3 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \hat{y}_3 = x_3^5 \\ \hat{v}_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \hat{y}_4 = x_4^5 \\ \hat{v}_5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \hat{y}_5 = x_5^5 \\ \hat{v}_6 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -2 & 1 & 0 & \hat{y}_6 = l_1 \hat{y}_0 \\ \hat{v}_7 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 2 & -1 & 0 & \hat{y}_7 = l_1 \hat{y}_1 \\ \hat{v}_8 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & \hat{y}_8 = l_2 \hat{y}_1 \\ \hat{v}_9 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & \hat{y}_9 = l_2 \hat{y}_2 \end{array} \right). \quad (8.35)$$

We note that the second and third charge vector realize the brane charge vectors (8.32) and the closed string charge vector of the quintic is embedded as $\ell^{(1)} = \hat{\ell}^{(1)} + 2\hat{\ell}^{(2)} + \hat{\ell}^{(3)}$. Here the generators of the Mori cone $\hat{\ell}^{(a)}$ are obtained as a triangulation of the polyhedron $\Delta_{\hat{Z}_3}$.

The GKZ-system (8.34) defines three local coordinates \hat{z}^a on the complex structure moduli space of \hat{Z}_3 , that are chosen according to the basis of charge vectors in (8.35),

$$\hat{z}^1 = -\frac{a_3 a_4 a_5 a_7^2 a_9}{a_0^3 a_6^2 a_8}, \quad \hat{z}^2 = \frac{a_1 a_6}{a_0 a_7}, \quad \hat{z}^3 = \frac{a_2 a_8}{a_1 a_9}. \quad (8.36)$$

The complete system of differential operators \mathcal{D}_a constituting the Picard-Fuchs system are found by linear combinations of the charge vectors $\hat{\ell}^{(a)}$ in (8.35). They are obtained by factorizing the corresponding operators \mathcal{L}_a , that are directly associated to the scaling symmetries of the charge vectors. We obtain the system

$$\begin{aligned} \mathcal{D}_1 &= \theta_1^3 (2\theta_1 - \theta_2) (\theta_1 - \theta_3) + (2\theta_1 - \theta_2 - 2) (\theta_1 - \theta_3 - 1) \prod_{i=0}^{i=2} (3\theta_1 + \theta_2 - i) \hat{z}^1, & (8.37) \\ \mathcal{D}_2 &= (2\theta_1 - \theta_2) (\theta_2 - \theta_3) + (2\theta_1 - \theta_2 + 1) (3\theta_1 + \theta_2) \hat{z}^2, \\ \mathcal{D}_3 &= (\theta_1 - \theta_3) \theta_3 - (\theta_1 - \theta_3 + 1) (-\theta_2 + \theta_3 - 1) \hat{z}^3, \\ \mathcal{D}_4 &= -\theta_1^3 (2\theta_1 - \theta_2) \theta_3 - (2\theta_1 - \theta_2 - 1) \prod_{i=0}^3 (3\theta_1 + \theta_2 - i) \hat{z}^1 \hat{z}^2 \hat{z}^3, \\ \mathcal{D}_5 &= \theta_1^3 (\theta_1 - \theta_3) (-\theta_2 + \theta_3) (1 - \theta_2 + \theta_3) + (-1 + \theta_1 - \theta_3) \prod_{i=0}^4 (3\theta_1 + \theta_2 - i) \hat{z}^1 (\hat{z}^2)^2, \\ \mathcal{D}_6 &= \theta_1^3 (-\theta_2 + \theta_3) \theta_3 + \prod_{i=0}^4 (3\theta_1 + \theta_2 - i) \hat{z}^1 (\hat{z}^2)^2 \hat{z}^3, \end{aligned}$$

where the corresponding linear combination of the $\hat{\ell}^{(a)}$ can be read off from the powers of the \hat{z}^a . We note that this system has the structure advertised in eq. (7.49) and thus the periods $\Pi^k(z^1)$ of the quintic Z_3 with $z^1 = \hat{z}^1 (\hat{z}^2)^2 \hat{z}^3$ are solutions to it.

Indeed, we identify 12 solutions of the following form at $\hat{z}^i \rightarrow 0$. There is one solution $X^{(0)}$ with a power series expansion, three single logarithmic solutions $X_i^{(1)}$, four double logarithmic solutions $X_\alpha^{(2)}$ and four triple logarithmic solutions $X_\beta^{(3)}$. The unique power series solution starts with a constant, that we normalize to 1,

$$X^{(0)} = 1 + 120z^1 + 113400(z^1)^2 + 168168000(z^1)^3 + 305540235000(z^1)^4 + \mathcal{O}((z^1)^5), \quad (8.38)$$

where we set $z^1 = \hat{z}^1 (\hat{z}^2)^2 \hat{z}^3$. Thus, we identify this as the fundamental period $\Pi^0(z^1)$ of the quintic. We recover the three other quintic periods by first noting that the leading logarithms of the solutions are given by

$$\begin{aligned} X_i^{(1)} &: \hat{l}_1, \hat{l}_2, \hat{l}_3, & (8.39) \\ X_\alpha^{(2)} &: \frac{1}{2}\hat{l}_1^2, \hat{l}_2(\hat{l}_1 - 2\hat{l}_3), \hat{l}_3(\hat{l}_1 + 2\hat{l}_2 + \frac{1}{2}\hat{l}_3), \hat{l}_2(\frac{1}{2}\hat{l}_2 + \hat{l}_3) \\ X_\beta^{(3)} &: \frac{1}{6}\hat{l}_1^3, \frac{1}{2}\hat{l}_1^2\hat{l}_2 - \frac{1}{3}\hat{l}_2^3 - 2\hat{l}_1\hat{l}_2\hat{l}_3 - 3\hat{l}_2^2\hat{l}_3 - \hat{l}_2\hat{l}_3^2, \\ & \frac{1}{2}\hat{l}_1^2\hat{l}_3 + 2\hat{l}_1\hat{l}_2\hat{l}_3 + 2\hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_1\hat{l}_3^2 + \hat{l}_2\hat{l}_3^2 + \frac{1}{6}\hat{l}_3^3, \frac{1}{2}\hat{l}_1\hat{l}_2^2 + \frac{1}{2}\hat{l}_2^3 + \hat{l}_1\hat{l}_2\hat{l}_3 + \frac{3}{2}\hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_2\hat{l}_3^2, \end{aligned}$$

where we used the abbreviation $\log(\hat{z}^i) = \hat{l}_i$. We immediately observe that all quintic periods $\Pi^k(z^1)$ with leading logarithms $l_1, \frac{1}{2}l_1^2$ and $\frac{1}{6}l_1^3$ for $l_1 = \hat{l}_1 + 2\hat{l}_2 + \hat{l}_3$ are indeed contained in the leading logarithms (8.39) of the solutions on \hat{Z}_3 . We readily check that the complete z^1 -series of the quintic periods $\Pi^k(z^1)$ are reproduced as well on the blow-up.

The remaining six logarithmic solutions are related to the open string sector. In particular, we can cross-check this statement by finding the brane superpotential W_{brane} by its A-model interpretation at large volume as a generating functional for disk instantons. First we interpret the single logarithms in (8.39) as the mirror map of the open-closed system at $z \rightarrow 0$ defining the flat coordinates via $\hat{t}_i = X_i^{(1)}/X^{(0)}$,

$$\begin{aligned} X_1^{(1)} &= X^{(0)} \log(\hat{z}_1) + 2\hat{z}_2 - \hat{z}_2^2 - 60\hat{z}_1\hat{z}_2^2 + \frac{2\hat{z}_2^3}{3} - \frac{\hat{z}_2^4}{2} + \frac{2\hat{z}_2^5}{5} - 48\hat{z}_1\hat{z}_2\hat{z}_3 + 462z_1 + \mathcal{O}(\hat{z}^6), \\ X_2^{(1)} &= X^{(0)} \log(\hat{z}_2) - \hat{z}_2 + \frac{\hat{z}_2^2}{2} - \frac{\hat{z}_2^3}{3} + \frac{\hat{z}_2^4}{4} - \frac{\hat{z}_2^5}{5} + 24\hat{z}_1\hat{z}_2\hat{z}_3 + 154z_1 - 360\hat{z}_1\hat{z}_2^3\hat{z}_3 + \mathcal{O}(\hat{z}^6), \\ X_3^{(1)} &= X^{(0)} \log(\hat{z}_3) + 60\hat{z}_1\hat{z}_2^2 - 9450\hat{z}_1^2\hat{z}_2^4 + 75600\hat{z}_1^2\hat{z}_2^4\hat{z}_3 - 60\hat{z}_1\hat{z}_2^2\hat{z}_3^2 + \mathcal{O}(\hat{z}^8). \end{aligned} \quad (8.40)$$

Here we omit a factor of $\frac{1}{2\pi i}$ in front of the logarithms for brevity⁷. This is perfectly consistent with the mirror map of the quintic that is obtained as $t = \hat{t}_1 + 2\hat{t}_2 + \hat{t}_3$ or as $\Pi^1(z_1) = X_1^{(1)} + 2X_2^{(1)} + X_3^{(1)} = X^{(0)} \log(z_1) + 770z_1 + \dots$ as required by the charge vectors $\hat{\ell}^{(a)}$ in (8.35). Upon inversion of the mirror map, we obtain the \hat{z}^i as a series of $q_a = e^{2\pi i \hat{\ell}_a}$, that we readily insert into the double logarithmic solutions in (8.39). Then we construct a linear combination of the double logarithmic solutions in (8.39) as

$$W_{\text{brane}} = (2X_1^{(2)} + 4X_2^{(2)} + aX_3^{(2)} + 4X_4^{(2)})/X^{(0)} \quad (8.41)$$

in which we insert the inverse mirror map to obtain

$$W_{\text{brane}} = 2t^2 + 2\hat{t}_2^2 + \frac{1}{2}(4-a)\hat{t}_3^2 - t\hat{t}_2 - (4-a)t\hat{t}_3 - \frac{1}{4\pi} \sum_{n_i} n_{d_1, d_2, d_3} \text{Li}_2(q_1^{d_1} q_2^{d_2} q_3^{d_3}), \quad (8.42)$$

where a denotes a free complex parameter. This has the expected integrality properties of the Ooguri-Vafa Li_2 -double cover formula, such that we obtain the disk instantons n_{d_1, d_2, d_3} . Selected invariants $n_{j, i+j, j}$ are summarized in table 8.1, where the rows and columns are labeled by i and j , respectively. We note that the parameter a does not affect these instantons, however, it does affect the classical terms⁸. It should be fixed by the determination of the symplectic basis on the blow-up \hat{Z}_3 .

8.2 Open-Closed GKZ-Systems from Blow-Up Threefolds

In the following section we present a general recipe to easily obtain the toric GKZ-system of an arbitrary toric brane Σ in an arbitrary toric Calabi-Yau hypersurface Z_3 .

⁷We also label the variables \hat{z}_i by a subscript instead of a superscript in order to shorten the expressions.

⁸The classical term $4t^2 - 2\hat{t}_2^2$ of [122] can not be reproduced by tuning the parameter a . The ‘‘closest’’ match is $2(t - \hat{t}_2)^2$ for $a = 4$, for which the only non-vanishing disk instantons are those in table 8.1.

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
0	0	-320	13280	-1088960	119783040	-15440622400
1	20	1600	-116560	12805120	-1766329640	274446919680
2	0	2040	679600	-85115360	13829775520	-2525156504560
3	0	-1460	1064180	530848000	-83363259240	16655092486480
4	0	520	-1497840	887761280	541074408000	-95968626498800
5	0	-80	1561100	-1582620980	931836819440	639660032468000
6	0	0	-1152600	2396807000	-1864913831600	1118938442641400
7	0	0	580500	-2923203580	3412016521660	-2393966418927980
8	0	0	-190760	2799233200	-5381605498560	4899971282565360
9	0	0	37180	-2078012020	7127102031000	-9026682030832180
10	0	0	-3280	1179935280	-7837064629760	14557931269209000
11	0	0	0	-502743680	7104809591780	-20307910970428360
12	0	0	0	155860160	-5277064316000	24340277955510560
13	0	0	0	-33298600	3187587322380	-24957649473175420
14	0	0	0	4400680	-1549998228000	21814546476229120
15	0	0	0	-272240	597782974040	-16191876966658500
16	0	0	0	0	-178806134240	10157784412551120
17	0	0	0	0	40049955420	-5351974901676280
18	0	0	0	0	-6332490480	2348019778753280

Table 8.1: Disk instanton invariants $n_{j,i+j,j}$ on the quintic at large volume. These results agree with [122].

Motivated by the above example it is possible in a simple manner to construct the toric data $\Delta_7^{\hat{Z}}$ right from the original polyhedron $\Delta_3^{\hat{Z}}$ and the toric curve Σ as specified by the charge vectors $\hat{\ell}^{(1)}, \hat{\ell}^{(2)}$. We denote the vertices of $\Delta_3^{\hat{Z}}$ by $\tilde{v}_i, i = 1, \dots, n$, with \tilde{v}_0 the origin, its charge vectors by $\ell^{(i)}, i = 1, \dots, n-4$ and the two brane vectors by $\hat{\ell}^{(1)}, \hat{\ell}^{(2)}$ as before. We define $n+5$ points \hat{v}_i spanning a seven-dimensional polyhedron $\Delta_7^{\hat{Z}}$ as

$$\begin{aligned}
Z_3 : \quad \hat{v}_i &= (1, 0, \tilde{v}_i, 0), & i &= 0, \dots, n, \\
\hat{\ell}^{(1)} : \quad \hat{v}_{n+1} &= (0, 1, v_1^{(-)}, -1), & \hat{v}_{n+2} &= (0, 1, v_1^{(+)}, -1), \\
\hat{\ell}^{(2)} : \quad \hat{v}_{n+3} &= (0, 1, v_2^{(-)}, 1), & \hat{v}_{n+4} &= (0, 1, v_2^{(+)}, 1),
\end{aligned} \tag{8.43}$$

where we use the abbreviations

$$v_1^{(+)} = \sum_{\hat{\ell}_i^{(1)} > 0} \hat{\ell}_i^{(1)} \tilde{v}_i, \quad v_1^{(-)} = - \sum_{\hat{\ell}_i^{(1)} < 0} \hat{\ell}_i^{(1)} \tilde{v}_i, \quad v_2^{(+)} = \sum_{\hat{\ell}_i^{(2)} > 0} \hat{\ell}_i^{(2)} \tilde{v}_i, \quad v_2^{(-)} = - \sum_{\hat{\ell}_i^{(2)} < 0} \hat{\ell}_i^{(2)} \tilde{v}_i. \tag{8.44}$$

The first line of (8.43) simply embeds the original toric data associated to Z_3 into \hat{Z}_3 , whereas the second and third line translate the brane data into geometric data of \hat{Z}_3 . The structure of the points \hat{v}_i is quite generic for the description of a toric complete intersection, cf. [252]

for the Calabi-Yau case.

In our context this structure in addition reflects the distinction between the closed and open string sector. It is encoded by the two canonical hyperplanes in the first and second row of the \hat{v}_i . Points in the hyperplane $H_1 = \{(1, 0, w_1, w_2, w_3, w_4, w_5)\}$ correspond to the closed string sector, i.e. the geometry of the Calabi-Yau encoded in the constraint P , whereas points in the hyperplane $H_2 = \{(0, 1, w_1, w_2, w_3, w_4, w_5)\}$ contribute to the open string sector as encoded in the constraints h_1, h_2 of Σ . On the blow-up \hat{Z}_3 , that we construct as a complete intersection (7.32), this translates to the rule, which monomial $y_i(\underline{x})$ contributes to which of the constraints P, Q . Points \hat{v}_i in H_1 contribute to P , whereas those in H_2 contribute to Q . We summarize this in the following table

	$\Delta_{\hat{Z}_7}^{\hat{Z}}$				$\hat{\ell}^{(1)}$...	$\hat{\ell}^{(n-4)}$	$\hat{\ell}^{(n-3)}$	$\hat{\ell}^{(n-2)}$	monomials
\hat{v}_0	1	0	\tilde{v}_0	0		...				$\hat{y}_0 = y_0$
...	\vdots	\vdots	\vdots	\vdots	$\ell^{(1)}$...	$\ell^{(n-4)}$	$\hat{\ell}^{(1)}$	$\hat{\ell}^{(2)}$	$P :$...
\hat{v}_n	1	0	\tilde{v}_n	0		...				$\hat{y}_{n-4} = y_{n-4}$
\hat{v}_{n+1}	0	1	$v_1^{(-)}$	-1	0	...	0	1	0	$\hat{y}_{n-3} = l_1 \prod_{\hat{\ell}_i^{(1)} < 0} y_i^{-\hat{\ell}_i^{(1)}}$
\hat{v}_{n+2}	0	1	$v_1^{(+)}$	-1	0	...	0	-1	0	$Q : \hat{y}_{n-2} = l_1 \prod_{\hat{\ell}_i^{(1)} > 0} y_i^{\hat{\ell}_i^{(1)}}$
\hat{v}_{n+3}	0	1	$v_2^{(-)}$	1	0	...	0	0	1	$\hat{y}_{n-1} = l_2 \prod_{\hat{\ell}_i^{(1)} < 0} y_i^{-\hat{\ell}_i^{(2)}}$
\hat{v}_{n+4}	0	1	$v_2^{(+)}$	1	0	...	0	0	-1	$\hat{y}_n = l_2 \prod_{\hat{\ell}_i^{(1)} > 0} y_i^{\hat{\ell}_i^{(2)}}$

(8.45)

Here we displayed besides the points \hat{v}_i of (8.43) also a natural choice of basis $\hat{\ell}^{(a)}$ of the lattice of relations of $\Delta_{\hat{Z}_7}^{\hat{Z}}$. In this basis the first $n - 4$ charge vectors are identical to those of $\Delta_{\hat{Z}_4}^{\hat{Z}}$ up to the extension by four further entries 0. More importantly the last two charge vectors naturally contain the two original brane vectors $\hat{\ell}^{(a)}$ extended by four further entries with $\pm 1, 0$. As before their entries sum up to zero. In the last row we associated monomials \hat{y}_i to the points \hat{v}_i where the $y_i(x_j)$ merely denote the polynomials on the original geometry of $\Delta_{\hat{Z}_4}^{\hat{Z}}$ computed by the Batyrev formula (5.7). The coordinates l_1, l_2 denote the homogeneous coordinates on \mathbb{P}^1 . We note that the form of the polynomials \hat{y}_i associated to the four new points $\hat{v}_n, \dots, \hat{v}_{n+4}$ reflects the definition of the brane constraints h_1, h_2 defined via (5.16). As mentioned before and indicated in (8.45) the constraints P and Q are given by

$$\hat{Z}_3 : \quad P = \sum_{i=0}^{n-4} a_i \hat{y}_i \equiv \sum_{i=0}^{n-4} a_i y_i, \quad Q = \sum_{i=n-3}^n a_i \hat{y}_i, \quad (8.46)$$

where the a denote free complex parameters. As can be easily checked the general toric data in (8.45) immediately reproduce the toric data of the blow-up (8.21) of the curve Σ in the quintic Z_3 . Similarly we obtain the toric data (8.65) of our second example in section 8.3

To the general form of the toric data (8.45) of \hat{Z}_3 we associate a GKZ-system on the

complex structure moduli space of \hat{Z}_3 by the standard formulas

$$\begin{aligned}\mathcal{L}_a &= \prod_{\hat{\ell}_i^{(a)} > 0} \left(\frac{\partial}{\partial a_i} \right)^{\hat{\ell}_i^{(a)}} - \prod_{\hat{\ell}_i^{(a)} < 0} \left(\frac{\partial}{\partial a_i} \right)^{-\hat{\ell}_i^{(a)}}, \quad a = 1, \dots, n-2, \\ \mathcal{Z}_j &= \sum_i (\hat{v}_i)^j \vartheta_i - \beta_j, \quad j = 1, \dots, 7.\end{aligned}\tag{8.47}$$

This immediately yields a natural choice of complex coordinates given by

$$\hat{z}^a = (-)^{\hat{\ell}_0^{(a)}} \prod_{i=0}^{n+4} a_i^{\hat{\ell}_i^{(a)}}, \quad a = 1, \dots, n-2.\tag{8.48}$$

We readily apply these formulas to reproduce the GKZ-system (8.19) and the coordinates (8.22) for the choice of charge vectors $\hat{\ell}^{(a)}$ as in (8.21). The same applies for the GKZ-system in the second example discussed in section 8.3.

Let us conclude by mentioning some remarkable properties of the geometry of \hat{Z}_3 as encoded by $\Delta_7^{\hat{Z}}$. First, the last row in (8.45) is associated to the toric symmetries of the exceptional \mathbb{P}^1 in the blow-up divisor E . In fact, this \mathbb{P}^1 can be made directly visible in $\Delta_7^{\hat{Z}}$ by projection on the ray $(0, 0, 0, 0, 0, 0, w_1)$. Second, one might be tempted to map the toric data $\Delta_7^{\hat{Z}}$ of the complete intersection (8.46) to toric data of a hypersurface defined by six-dimensional vectors obtained by adding the first and second row, $((\hat{v}_i)^1, (\hat{v}_i)^2, \dots) \mapsto ((\hat{v}_i)^1 + (\hat{v}_i)^2, \dots) \equiv (1, v'_i)$, where we note that $(\hat{v}_i)^1 + (\hat{v}_i)^2 = 1$. This defines a five-dimensional polyhedron Δ_5 with points v'_i . Clearly, this polyhedron has one further charge vector $\hat{\ell}^{(n-1)}$ so that the dimension of a corresponding toric variety is five-dimensional. Furthermore, for special choices of the brane charge vectors $\hat{\ell}^{(a)}$, but by far not for all choices⁹, this toric data defines a mirror pair of *compact* Calabi-Yau fourfolds \tilde{X}_4, X_4 with the hypersurface constraint given by the standard Batyrev formalism (5.7). In combination with the first observation about the universal presence of the \mathbb{P}^1 in $\Delta_7^{\hat{Z}}$, the geometry of \tilde{X}_4 will contain this very \mathbb{P}^1 as the basis of a Calabi-Yau threefold fibration with generic fiber \tilde{Z}_3 . This is precisely the geometric structure we encountered in the Calabi-Yau geometries used in [79] and in the context of heterotic/F-theory duality in [81]. In fact, the Calabi-Yau fourfold X_4 we obtain from $\Delta_7^{\hat{Z}}$ for the example of the next section 8.3 precisely agrees with the F-theory dual fourfold of a heterotic setup with horizontal five-branes as predicted by heterotic/F-theory. We suspect that heterotic/F-theory duality for horizontal five-branes is in general the reason for the occurrence of a Calabi-Yau fourfold geometry associated to some blow-up threefolds \hat{Z}_3 . This nicely completes the discussion of the application of the blow-up proposal for heterotic five-branes and heterotic/F-theory duality in [81]. Let us conclude by emphasizing that only parts of the fourfold geometry X_4 , if present, are intrinsically related to the original five-brane as e.g. signaled by the additional charge vector $\hat{\ell}^{(n-1)}$.¹⁰ However,

⁹The polyhedron Δ_5 defined by the v'_i is not generically reflexive. This is the case in section 8.1.4.

¹⁰On the heterotic side, the additional data of X_4 is related to the heterotic bundle.

the toric data $\Delta_7^{\hat{Z}}$ of the blow-up \hat{Z}_3 should by construction always carry the minimal amount of information in order to study the open-closed system of the five-brane in Z_3 .

A technical similar but differently motivated method to obtain toric data and a GKZ-system governing the deformations of toric D5-branes was presented in [108, 114, 119, 120, 122, 263] and formulated mathematically rigorously in [124].

8.3 Open-Closed Picard-Fuchs Systems on $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18]

As a second demonstration of the application of the blow-up proposal we consider a two-parameter Calabi-Yau threefold Z_3 . Since the discussion are similar to the case of the quintic we keep it as brief as possible. First in section 8.3.1 we consider branes on lines in $Z_3 = \mathbb{P}^4(1, 1, 1, 6, 9)$ [18] and identify the toric curve Σ , that degenerates appropriately at the critical locus $\mathcal{M}(\mathbb{P}^1)$. We construct the blow-up \hat{Z}_3 along Σ as a complete intersection and construct the Picard-Fuchs system in section 8.3.2. Finally in section 8.3.3 we determine the solutions at large volume and identify the brane superpotentials from which we extract compact disk instanton invariants, cf. appendix C.3, that match the local results of [107].

8.3.1 Branes on Lines in $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18] and the Blow-Up

The Calabi-Yau threefold Z_3 ¹¹ is defined as the mirror of the Calabi-Yau hypersurface \tilde{Z}_3 in $\mathbb{P}^4(1, 1, 1, 6, 9)$ which admits $h^{(2,1)}(Z_3) = 2$ complex structure moduli and is an elliptic fibration over \mathbb{P}^2 . In the conventions of [261] the two complex structures denoted Ψ_1, Ψ_2 enter the constraint P as

$$P = y^2 + x^3 + u_1^{18} + u_2^{18} + u_3^{18} - 3\Psi_1(u_1 u_2 u_3)^6 - 18\Psi_2 x y u_1 u_2 u_3, \quad (8.49)$$

where we introduce the homogeneous coordinates (u_i, x, y, z) , $i = 1, 2, 3$, for the \mathbb{P}^2 -base and the elliptic fiber $\mathbb{P}^2(1, 2, 3)$, respectively. Note however that we are working in an affine patch $z = 1$ of the elliptic fiber¹². This is reflected in the toric data used to obtain P ,

$$\begin{array}{l} \ell^{(1)} = (0, \quad -3, \quad 1, \quad 1, \quad 1, \quad 0 \quad 0) \\ \ell^{(2)} = (-6, \quad 1, \quad 0, \quad 0, \quad 0, \quad 2 \quad 3) \\ \hline \hline y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \\ zxyu_1u_2u_3 \quad (zu_1u_2u_3)^6 \quad z^6u_1^{18} \quad z^6u_2^{18} \quad z^6u_3^{18} \quad x^3 \quad y^2 \end{array} \quad (8.50)$$

where the y_i corresponding to the entries $\ell_i^{(j)}$ of the charge vectors are monomials in the homogeneous coordinates on $\mathbb{P}^4(1, 1, 1, 6, 9)$. This hypersurface data is augmented by the action

¹¹We will abbreviate this simply by Z_3 .

¹²Strictly speaking one has to include the divisor z to resolve a curve of Z_3 -singularity in $\mathbb{P}^4(1, 1, 1, 6, 9)$ [261].

of a discrete orbifold group G which is $G = \mathbb{Z}_6 \times \mathbb{Z}_{18}$ generated by $v^{(i)} = (0, 1, 3, 2, 0) \bmod 6$, $v'^{(j)} = (1, -1, 0, 0, 0) \bmod 18$ acting on the coordinates as

$$g^{(i)} : x_k \mapsto e^{2\pi i v_k^{(i)}/6} x_k, \quad g'^{(j)} : x_k \mapsto e^{2\pi i v_k'^{(j)}/18} x_k. \quad (8.51)$$

To this setup we add five-brane wrapping a rational curve \mathbb{P}^1 in Z_3 that will be of similar type as the lines (8.11) considered in the quintic. As before we use the moduli space $\mathcal{M}(\Sigma)$ as a model for the deformation space \tilde{P}^1 of this line, where we specify the analytic family of curves Σ in the form of toric branes

$$\begin{aligned} \Sigma : \quad & P = 0, \quad h_1 \equiv \beta^{12}(u_1 u_2 u_3)^6 - \alpha^6 \gamma^6 u_2^{18} = 0, \quad h_2 \equiv \gamma^{12}(u_1 u_2 u_3)^6 - \alpha^6 \beta^6 u_3^{18} = 0, \\ & \tilde{\ell}^{(1)} = (0, -1, 0, 1, 0, 0, 0), \quad \tilde{\ell}^{(2)} = (0, -1, 0, 0, 1, 0, 0). \end{aligned} \quad (8.52)$$

The brane charge vectors $\tilde{\ell}^{(i)}$ are used to construct the constraint h_i via (5.16). This basis of constraints and parameters might look inconvenient, is however justified by noting the convenient, equivalent form

$$\Sigma : \quad P = 0, \quad \gamma^{18} u_2^{18} - \beta^{18} u_3^{18} = 0, \quad \gamma^{18} u_1^{18} - \alpha^{18} u_3^{18} = 0 \quad (8.53)$$

upon a trivial algebraic manipulation. We introduce affine coordinates parameterizing this analytic family of curves in Z_3 that we choose to be $u^1 = \frac{\beta^{18}}{\gamma^{18}}$ and $u^2 = \frac{\alpha^{18}}{\gamma^{18}}$.

Next we proceed by linearizing (8.53) to describe a non-holomorphic deformation \tilde{P}^1 of rational curves given by

$$\tilde{P}^1 : \quad \eta_1 x + \sqrt[3]{y^2 + m(u_3, x, y)} = 0, \quad \eta_2 \gamma u_2 - \beta u_3 = 0, \quad \eta_3 \gamma u_1 - \alpha u_3 = 0. \quad (8.54)$$

Here we inserted h_1 and h_2 into P and introduced the third and eighteenth roots of unity η_1 , respectively η_2, η_3 as well as the polynomial

$$m(u_3, x, y) = \frac{\alpha^{18} + \beta^{18} + \gamma^{18}}{\gamma^{18}} - 3\Psi_1 \left(\frac{\alpha\beta}{\gamma^2} \right)^6 u_3^{18} - 18\Psi_2 \frac{\beta\alpha}{\gamma^2} x y u_3^{18}. \quad (8.55)$$

At the critical locus of the parameter space α, β, γ where the polynomial m vanishes identically, the generically higher genus Riemann surface Σ degenerates. This locus reads

$$\mathcal{M}_{\mathbb{P}^1}(\Sigma) : \quad \alpha^{18} + \beta^{18} + \gamma^{18} - 3\Psi_1 \alpha^6 \beta^6 \gamma^6 = 0, \quad \Psi_2 \alpha \beta \gamma = 0. \quad (8.56)$$

At this locus the Riemann surface Σ in (8.53) degenerates to

$$\Sigma : \quad h_0 \equiv y^2 + x^3, \quad \gamma^{18} u_2^{18} - \beta^{18} u_3^{18} = 0, \quad \gamma^{18} u_1^{18} - \alpha^{18} u_3^{18} = 0. \quad (8.57)$$

Modulo the action of G identifying the different solutions in (8.54) we can solve (8.54) holomorphically and consistent with the weights of $\mathbb{P}^2(1, 2, 3)$ at the locus $\mathcal{M}_{\mathbb{P}^1}(\Sigma)$ by the Veronese embedding of a line in $\mathbb{P}^4(1, 1, 1, 6, 9)$,

$$(U, V) \mapsto (\alpha U, \beta U, \gamma U, -\eta_1^2 V^6, V^9), \quad \eta_1^3 = 1. \quad (8.58)$$

From the perspective of this line, the constraint (8.56) on the parameters (α, β, γ) is precisely the condition for it to lie holomorphically in the Calabi-Yau constraint P of \hat{Z}_3 . This implies that at the point $\Psi_2 = 0$ there is an analytic family of lines in Z_3 and otherwise, for $\Psi_2 \neq 0$, only a discrete number of lines.

To study the open-closed system defined by the five-brane in Z_3 we construct the blow-up threefold \hat{Z}_3 . As explained above, cf. section 7.2.2, we use the holomorphic description by the toric curve Σ of the anholomorphic brane deformations \mathbb{P}^1 in order to construct the blow-up. Before we construct \hat{Z}_3 , we switch to a full toric description.

8.3.2 Toric Branes on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$: the Open-Closed GKZ-System

We begin the analysis of the open-closed moduli space using the toric means. First of all let us recall the toric construction of the Calabi-Yau Z_3 by giving its constraint as well as the curve Σ ,

$$\begin{aligned} Z_3 : \quad P &= a_6 y^2 + a_5 x^3 + a_1 u_1^6 u_2^6 u_3^6 + a_2 u_1^{18} + a_3 u_2^{18} + a_4 u_3^{18} + a_0 x y u_1 u_2 u_3, \\ \Sigma : \quad h_1 &= a_7 u_1^6 u_2^6 u_3^6 + a_8 u_2^{18}, \quad h_2 = a_9 u_1^6 u_2^6 u_3^6 + a_{10} u_3^{18}. \end{aligned} \quad (8.59)$$

The coefficients \underline{a} redundantly parameterize the complex structure of Z_3 respectively the moduli of the curve Σ in Z_3 . The information in (8.59) is directly encoded in the toric data specifying (Z_3, Σ) via the polyhedron $\Delta_4^{\hat{Z}}$ and the two brane-vectors $\hat{\ell}^{(1)}, \hat{\ell}^{(2)}$ reading

$$\left(\begin{array}{c|ccc|cc|cc|cc} & \Delta_4^{\hat{Z}} & & & \ell^{(1)} & \ell^{(2)} & & \hat{\ell}^{(1)} & \hat{\ell}^{(2)} \\ \hline \tilde{v}_0 & 0 & 0 & 0 & 0 & -6 & y_0 = zxyu_1u_2u_3 & 0 & 0 \\ \tilde{v}_1 & 0 & 0 & 2 & 3 & -3 & 1 & y_1 = z^6u_1^6u_2^6u_3^6 & -1 & -1 \\ \tilde{v}_2 & 1 & 1 & 2 & 3 & 1 & 0 & y_2 = z^6u_1^{18} & 0 & 0 \\ \tilde{v}_3 & -1 & 0 & 2 & 3 & 1 & 0 & y_3 = z^6u_2^{18} & 1 & 0 \\ \tilde{v}_4 & 0 & -1 & 2 & 3 & 1 & 0 & y_4 = z^6u_3^{18} & 0 & 1 \\ \tilde{v}_5 & 0 & 0 & -1 & 0 & 0 & 2 & y_5 = x^3 & 0 & 0 \\ \tilde{v}_6 & 0 & 0 & 0 & -1 & 0 & 3 & y_6 = y^2 & 0 & 0 \end{array} \right). \quad (8.60)$$

The points of the dual polyhedron $\Delta_4^{\hat{Z}}$ are given by $v_1 = (-12, 6, 1, 1)$, $v_2 = (6, -12, 1, 1)$, $v_3 = (6, 6, 1, 1)$, $v_4 = (0, 0, -2, 1)$ and $v_5 = (0, 0, 1, -1)$, where the point $(0, 0, 1, 1)$ corresponds to the z -coordinate on the elliptic fiber that we set to 1. The Calabi-Yau as well as the two brane constraints of (8.59) are then associated to this data via (5.14) and (5.16), respectively.

Accordingly, the variational problem of complex structures on Z_3 can be studied by exploiting the existence of the GKZ-system (5.34), (5.35) associated to $\Delta_4^{\hat{Z}}$. For the example at

hand it reads

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_i &= \vartheta_2 - \vartheta_{i+2} \quad (i = 1, 2) & \mathcal{Z}_3 &= 2 \sum_{i=1}^4 \vartheta_i - \vartheta_5, & \mathcal{Z}_4 &= 3 \sum_{i=1}^4 \vartheta_i - \vartheta_6, \\ \mathcal{L}_1 &= \prod_{i=2}^4 \frac{\partial}{\partial a_i} - \frac{\partial^3}{\partial a_1^3}, & \mathcal{L}_2 &= \frac{\partial^6}{\partial a_1 \partial a_5^2 \partial a_6^3} - \frac{\partial^6}{\partial a_0^6}, \end{aligned} \quad (8.61)$$

for $\vartheta_i = a_i \frac{\partial}{\partial a_i}$ as before. The differential system \mathcal{Z}_i then determines two algebraic coordinates z^1, z^2 on the complex structure moduli space that are given in terms of the Mori generators $\ell^{(i)}$ according to (5.36) as

$$z^1 = \frac{a_2 a_3 a_4}{a_1^3}, \quad z^2 = \frac{a_1 a_5^2 a_6^3}{a_0^6}. \quad (8.62)$$

To analyze the open-closed system (Z_3, Σ) described by (8.59) we now apply the blow-up proposal, i.e. construct the geometry (\hat{Z}_3, E) given as the family of complete intersections (7.32) in $\mathcal{W} = \mathbb{P}(\mathcal{O}(18H) \oplus \mathcal{O}(18H)) \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O})$ which now reads

$$\hat{Z}_3 : \quad P = 0, \quad Q = l_1(a_7 u_1^6 u_2^6 u_3^6 + a_8 u_2^{18}) - l_2(a_9 u_1^6 u_2^6 u_3^6 + a_{10} u_3^{18}). \quad (8.63)$$

We define the three-form $\hat{\Omega}$ by the residue expression (7.42) and determine a system of differential operators, the Picard-Fuchs operators, for the family \hat{Z}_3 . First we determine the GKZ-system on the complex structure moduli space of the blow-up Z_3 . We check that $\hat{\Omega}$ is identically annihilated by the two differential operators \mathcal{L}_1 and \mathcal{L}_2 of (8.61) that are complemented to the system

$$\begin{aligned} \mathcal{Z}_0 &= \sum_{i=0}^6 \vartheta_i + 1, & \mathcal{Z}_1 &= \sum_{i=7}^{10} \vartheta_i, & \mathcal{Z}_2 &= \vartheta_2 - \vartheta_3 - \vartheta_8, & \mathcal{Z}_3 &= \vartheta_2 - \vartheta_4 - \vartheta_{10}, \\ \mathcal{Z}_4 &= 2 \left(\sum_{i=1}^4 + \sum_{i=7}^{10} \right) \vartheta_i - \vartheta_5, & \mathcal{Z}_5 &= 3 \left(\sum_{i=1}^4 + \sum_{i=7}^{10} \right) \vartheta_i - \vartheta_6, & \mathcal{Z}_6 &= \vartheta_9 + \vartheta_{10} - \vartheta_7 - \vartheta_8. \\ \mathcal{L}_1, & \quad \mathcal{L}_2, & \mathcal{L}_3 &= \frac{\partial^2}{\partial a_3 \partial a_7} - \frac{\partial^2}{\partial a_1 \partial a_8}, & \mathcal{L}_4 &= \frac{\partial^2}{\partial a_4 \partial a_9} - \frac{\partial^2}{\partial a_1 \partial a_{10}} \end{aligned} \quad (8.64)$$

There are two additional second order differential operators \mathcal{L}_3 and \mathcal{L}_4 that annihilate $\hat{\Omega}$ and incorporate the parameters a_7, \dots, a_{10} that are associated to the moduli of Σ . Clearly, there are no further operators of low minimal degree. The operators \mathcal{Z}_k are related to the symmetries of \mathcal{W} . The first two operators are associated to an overall rescaling of the two constraints $P \mapsto \lambda P, Q \mapsto \lambda' Q$. The third and fourth operator describe the torus symmetries of the \mathbb{P}^2 -base, $(u_1, u_j) \mapsto (\lambda_j u_1, \lambda_j^{-1} u_j)$, $j = 2, 3$, the fifth and sixth operator are due to the $\mathbb{P}^2(1, 2, 3)$ -fiber symmetries, $(x, y, z) \mapsto (\lambda_1' x, y, \lambda_1'^{-1} z)$, $(x, y, z) \mapsto (x, \lambda_2' y, \lambda_2'^{-1} z)$ and the last operator \mathcal{Z}_6 is related to the torus symmetry $(l_1, l_2) \mapsto (\lambda l_1, \lambda^{-1} l_2)$ of the exceptional \mathbb{P}^1 . All operators \mathcal{Z}_i of the original system (8.61) are altered due to the lift to the blow-up \hat{Z}_3 .

As discussed in detail in section 8.2 the constructed GKZ-system can be obtained as a GKZ-system associated to toric data of the blow-up \hat{Z}_3 . The set of integral points \hat{v}_i reads

$$\left(\begin{array}{c|cccccc|cccc|c} & \Delta_7^{\hat{Z}} & \hat{\ell}^{(1)} & \hat{\ell}^{(2)} & \hat{\ell}^{(3)} & \hat{\ell}^{(4)} & & & & & \\ \hline \hat{v}_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & \hat{y}_0 = zxyu_1u_2u_3 \\ \hat{v}_1 & 1 & 0 & 0 & 0 & 2 & 3 & 0 & -2 & 0 & -1 & \hat{y}_1 = z^6u_1^6u_2^6u_3^6 \\ \hat{v}_2 & 1 & 0 & 1 & 1 & 2 & 3 & 0 & 1 & 0 & 0 & \hat{y}_2 = z^6u_1^{18} \\ \hat{v}_3 & 1 & 0 & -1 & 0 & 2 & 3 & 0 & 0 & 0 & 1 & \hat{y}_3 = z^6u_2^{18} \\ \hat{v}_4 & 1 & 0 & 0 & -1 & 2 & 3 & 0 & 1 & 1 & 0 & -1 & \hat{y}_4 = z^6u_3^{18} \\ \hat{v}_5 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & \hat{y}_5 = x^3 \\ \hat{v}_6 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & 0 & 0 & \hat{y}_6 = y^2 \\ \hat{v}_7 & 0 & 1 & 0 & 0 & 2 & 3 & -1 & -1 & 0 & 1 & 0 & \hat{y}_7 = l_1\hat{y}_1 \\ \hat{v}_8 & 0 & 1 & -1 & 0 & 2 & 3 & -1 & 1 & 0 & -1 & 0 & \hat{y}_8 = l_1\hat{y}_3 \\ \hat{v}_9 & 0 & 1 & 0 & 0 & 2 & 3 & 1 & 0 & 1 & 0 & -1 & \hat{y}_9 = l_2\hat{y}_1 \\ \hat{v}_{10} & 0 & 1 & 0 & -1 & 2 & 3 & 1 & 0 & -1 & 0 & 1 & \hat{y}_{10} = l_2\hat{y}_4 \end{array} \right). \quad (8.65)$$

Here we have displayed the points \hat{v}_i , the basis of relations $\hat{\ell}^{(i)}$ and the corresponding monomials \hat{y}_i . We emphasize that besides the closed string charge vectors of Z_3 embedded as $\ell^{(1)} = \hat{\ell}^{(1)} + \hat{\ell}^{(3)}$, $\ell^{(2)} = \hat{\ell}^{(2)} + \hat{\ell}^{(4)}$ the brane charge vectors $\hat{\ell}^{(a)}$ are among the $\hat{\ell}^{(i)}$ of $\Delta_7^{\hat{Z}}$ as well. We note that this toric data is completely consistent with the general formula (8.43) to obtain $\Delta_7^{\hat{Z}}$. Similarly the associated GKZ-system precisely reproduces (8.64) upon using the general formula of the GKZ-system (8.47). We confirm as mentioned in section 8.2 that this polyhedron can be mapped to the five-dimensional polyhedron with an associate compact Calabi-Yau fourfold by adding the first and second row. This agrees with the heterotic/F-theory dual fourfold when considering the elliptic Z_3 as a heterotic compactification [81]. Furthermore, the GKZ-system (8.64) is a closed and more restrictive subsystem of GKZ-system for the Calabi-Yau fourfold.

The GKZ-system (8.64) defines four coordinates \hat{z}^a on the complex structure moduli space of \hat{Z}_3 that we calculate, according to the triangulation (8.65) of $\Delta_7^{\hat{Z}}$, as

$$\hat{z}^1 = \frac{a_2a_4a_8}{a_1^2a_7}, \quad \hat{z}^2 = \frac{a_4a_5^2a_6^3a_9}{a_0^6a_{10}}, \quad \hat{z}^3 = \frac{a_3a_7}{a_1a_8}, \quad \hat{z}^4 = \frac{a_1a_{10}}{a_4a_9}. \quad (8.66)$$

We obtain a complete system of linear differential operators \mathcal{D}_a , the Picard-Fuchs system, by considering operators \mathcal{L}_a associated to linear combinations of the charge vectors $\hat{\ell}^{(a)}$ forming the lattice of relations of $\Delta_7^{\hat{Z}}$. By factorizing these operators as expressed in the algebraic

coordinates (8.66) we obtain the differential system

$$\begin{aligned}
\mathcal{D}_1 &= \theta_1 (\theta_1 - \theta_3) (\theta_1 + \theta_2 - \theta_4) + (\theta_1 - \theta_3 - 1) \prod_{i=1}^2 (2\theta_1 + \theta_3 - \theta_4 - i) \hat{z}^1, & (8.67) \\
\mathcal{D}_2 &= \theta_2 (\theta_2 - \theta_4) (\theta_1 + \theta_2 - \theta_4) + 12 (6\theta_2 - 5) (6\theta_2 - 1) (\theta_2 - \theta_4 - 1) \hat{z}^2, \\
\mathcal{D}_3 &= (\theta_1 - \theta_3) \theta_3 - (1 + \theta_1 - \theta_3) (2\theta_1 + \theta_3 - \theta_4 - 1) \hat{z}^3, \\
\mathcal{D}_4 &= (\theta_2 - \theta_4) (-2\theta_1 - \theta_3 + \theta_4) + (1 + \theta_2 - \theta_4) (1 + \theta_1 + \theta_2 - \theta_4) \hat{z}^4, \\
\mathcal{D}_5 &= \theta_1 \theta_3 (\theta_1 + \theta_2 - \theta_4) + \prod_{i=1}^3 (2\theta_1 + \theta_3 - \theta_4 - i) \hat{z}^1 \hat{z}^3, \\
\mathcal{D}_6 &= \theta_1 (\theta_1 - \theta_3) (\theta_2 - \theta_4) + (\theta_1 - \theta_3 - 1) (1 + \theta_2 - \theta_4) (2\theta_1 + \theta_3 - \theta_4 - 1) \hat{z}^1 \hat{z}^4, \\
\mathcal{D}_7 &= \theta_2 (2\theta_1 + \theta_3 - \theta_4) + 12 (6\theta_2 - 5) (6\theta_2 - 1) \hat{z}^2 \hat{z}^4, \\
\mathcal{D}_8 &= \theta_1 \theta_3 (\theta_2 - \theta_4) - (-\theta_2 + \theta_4 - 1) \prod_{i=1}^2 (2\theta_1 + \theta_3 - \theta_4 - i) \hat{z}^1 \hat{z}^3 \hat{z}^4.
\end{aligned}$$

The linear combination of charge vectors corresponding to each of these operators can be read off from the powers of the \hat{z}^a in the last term of the \mathcal{D}_a . We note that this system has the expected structure advertised in eq. (7.49) of section 7.3. Consequently the periods $\Pi^k(z^1, z^2)$ lift to the blow-up \hat{Z}_3 upon the identification $z^1 = \hat{z}^1 \hat{z}^3$, $z^2 = \hat{z}^2 \hat{z}^4$.

8.3.3 Brane Superpotential at Large Volume: Disk Instantons

In this section we solve the Picard-Fuchs system (8.68) at the point of maximal unipotent monodromy $\hat{z}^a \rightarrow 0$ in the complex structure moduli space of \hat{Z}_3 . In addition we exploit the local limit $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ of Z_3 , which is given by a decompactification of the elliptic fiber $t_2 \rightarrow i\infty$, to determine the compact brane superpotential W_{brane} and the compact disk instanton invariants.

We find 16 solutions at large volume, that split into one power series solution $X^{(0)}$, four single logarithmic solutions $X_i^{(1)}$, seven double logarithmic solutions $X_\alpha^{(2)}$ and four triple logarithmic solutions $X_\beta^{(3)}$. As already expected from the observation made below (8.65), that \hat{Z}_3 connects to a compact Calabi-Yau fourfold, these are the only solutions. In particular, there are now square root and third root at large volume. The unique power series solution reads in the chosen normalization as

$$X^{(0)} = 1 + 60z_2 + 13860z_2^2 + 4084080z_2^3 + 24504480z_1z_2^3 + 1338557220z_2^4 + \mathcal{O}(\hat{z}^{10}), \quad (8.68)$$

where we identify $z_1 = \hat{z}_1 \hat{z}_3$ and $z_2 = \hat{z}_2 \hat{z}_4$ as the complex structure moduli of Z_3 corresponding to $\ell^{(1)}$, $\ell^{(2)}$ in (8.60). Thus, $X^{(0)}$ is precisely the fundamental period $\Pi^0(z_1, z_2)$ of Z_3 , cf. [233, 261]. In addition we recover the other five periods of $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18] as linear

combinations of the solutions of the GKZ-system on \hat{Z}_3 with leading logarithms

$$\begin{aligned}
X_i^{(1)} &: \hat{l}_1, \hat{l}_2, \hat{l}_3, \hat{l}_4 & (8.69) \\
X_\alpha^{(2)} &: \frac{\hat{l}_1^2}{2}, \hat{l}_2(\hat{l}_1 + 2\hat{l}_4), \hat{l}_3(\hat{l}_1 + \frac{\hat{l}_3}{2}), \hat{l}_4(\hat{l}_1 + \hat{l}_4), \frac{\hat{l}_2^2}{2}, \hat{l}_2(\hat{l}_3 + \hat{l}_4), \hat{l}_4(\hat{l}_3 + \frac{\hat{l}_4}{2}) \\
X_\beta^{(3)} &: \frac{1}{2}\hat{l}_1^2\hat{l}_2 + \frac{1}{2}\hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_1^2\hat{l}_4 + 2\hat{l}_1\hat{l}_2\hat{l}_4 + \frac{1}{2}\hat{l}_2^2\hat{l}_4 + \hat{l}_1\hat{l}_4^2 + 2\hat{l}_2\hat{l}_4^2 + \frac{2\hat{l}_3^3}{3}, \hat{l}_1\hat{l}_2^2 - \frac{3}{2}\hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_2^2\hat{l}_4, \\
&\hat{l}_1\hat{l}_2\hat{l}_3 + \hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_2\hat{l}_3^2 + \hat{l}_1\hat{l}_2\hat{l}_4 + \hat{l}_2^2\hat{l}_4 + \hat{l}_1\hat{l}_3\hat{l}_4 + 3\hat{l}_2\hat{l}_3\hat{l}_4 + \frac{1}{2}\hat{l}_3^2\hat{l}_4 + \frac{1}{2}\hat{l}_1\hat{l}_4^2 + \frac{5}{2}\hat{l}_2\hat{l}_4^2 + \frac{3}{2}\hat{l}_3\hat{l}_4^2 + \frac{5\hat{l}_4^3}{6}, \\
&\frac{\hat{l}_3^3}{3} + \frac{1}{2}\hat{l}_2^2\hat{l}_3 + \frac{1}{2}\hat{l}_2^2\hat{l}_4,
\end{aligned}$$

where we used the abbreviation $\hat{l}_i = \log(\hat{z}_i)$. Indeed the threefold periods $l_1 = \hat{l}_1 + \hat{l}_3$, $l_2 = \hat{l}_2 + \hat{l}_4$, l_1^2 , $l_1l_2 + \frac{3}{2}l_2^2$ and $l_2(\frac{1}{2}l_1^2 + \frac{3}{2}l_1l_2 + \frac{3}{2}l_2^2)$ are linear combinations of the solutions in (8.69). Furthermore, we readily check that the complete (z_1, z_2) -series of the periods agree with the solutions on the blow-up.

Next we obtain the disk instanton invariants of the A-model dual to the five-brane on Z_3 from the local limit geometry $K_{\mathbb{P}^2}$. First we use the single logarithms of (8.69) as the mirror map $\hat{t}_i = \frac{X_i^{(1)}}{X^{(0)}}$ at large volume, where we have the series expansions

$$\begin{aligned}
X_1^{(1)} &= X^{(0)} \log(\hat{z}_1) - 4\hat{z}_1\hat{z}_3 + 120\hat{z}_2\hat{z}_4 + 60\hat{z}_2\hat{z}_3\hat{z}_4 + 30(\hat{z}_1^2\hat{z}_3^2 + 4\hat{z}_1\hat{z}_2\hat{z}_3\hat{z}_4 + 1386\hat{z}_2^2\hat{z}_4^2) + \mathcal{O}(\hat{z}^5), \\
X_2^{(1)} &= X^{(0)} \log(\hat{z}_2) - 60\hat{z}_2 - 3080\hat{z}_2^2(442\hat{z}_2 + 9\hat{z}_4) + 6\hat{z}_2(1155\hat{z}_2 + 62\hat{z}_4) + \mathcal{O}(\hat{z}^4), \\
X_3^{(1)} &= X^{(0)} \log(\hat{z}_3) - 2\hat{z}_1\hat{z}_3 + 60\hat{z}_2\hat{z}_4 - 60\hat{z}_2\hat{z}_3\hat{z}_4 + 15(\hat{z}_1^2\hat{z}_3^2 + 4\hat{z}_1\hat{z}_2\hat{z}_3\hat{z}_4 + 1386\hat{z}_2^2\hat{z}_4^2) + \mathcal{O}(\hat{z}^5), \\
X_4^{(1)} &= X^{(0)} \log(\hat{z}_4) + 60\hat{z}_2 - 6930\hat{z}_2^2 + 2\hat{z}_1\hat{z}_3 - 60\hat{z}_2\hat{z}_4 + 3080\hat{z}_2^2(442\hat{z}_2 + 9\hat{z}_4) + \mathcal{O}(\hat{z}^4). \quad (8.70)
\end{aligned}$$

Here we omit a factor $2\pi i$ for brevity. This confirms the consistency with the mirror of the threefold Z_3 , $t_1 = \hat{t}_1 + \hat{t}_3$ and $t_2 = \hat{t}_2 + \hat{t}_4$, since the periods agree as $\Pi^{(1)}(\underline{z}) = X_1^{(1)} + X_3^{(1)} = X^{(0)} \log(z_1) - 6z_1 + 45z_1^2 + \dots$ and $\Pi^{(2)}(\underline{z}) = X_1^{(1)} + X_3^{(1)} = X^{(0)} \log(z_2) + 2z_1 + 312z_2 + \dots$. Upon inversion of the mirror map, we obtain the \hat{z}^i as a series of $\hat{q}_a = e^{2\pi i \hat{t}_a}$, that we readily insert into the double logarithmic solutions in (8.69). Then we construct a linear combination of these solutions to match the disk instantons in [106, 107] of the local geometries. Since $q_2 = e^{2\pi i t_2} \rightarrow 0$ in the local limit this means that we match, as a first step, only the part of the q -series, that is independent of q_2 . Morally speaking, this procedure fixes part of the flux on \hat{Z}_3 specifying the five-brane. Indeed we obtain a perfect match of the disk instantons for both brane phases I/II , III considered in [107] for the choices of superpotential

$$W_{\text{brane}}^{I/II} = \left(\left(\frac{1}{2}(a_4 - a_7) + a_3 - \frac{1}{2} \right) X_1^{(2)} + \sum_{i=2}^7 a_i X_i^{(2)} \right) / X^{(0)}, \quad (8.71)$$

$$W_{\text{brane}}^{III} = \left(a_1 X_1^{(2)} + \frac{1}{3}(1 + a_4 + a_5 - 2a_6 + a_7) X_1^{(2)} + \sum_{i=3}^7 a_i X_i^{(2)} \right) / X^{(0)}, \quad (8.72)$$

where we in addition fix the parameters $a_3 = a_7 = 0$ to switch off the contribution of the closed instantons of Z_3 through its double logarithmic periods. We note that the two choices

corresponding to (8.71) are compatible with each other so that we can also find a single superpotential matching both phases of [107] simultaneously, corresponding to the parameters $a_1 = -1 + \frac{3}{2}a_2 + a_3 - \frac{1}{2}a_5 + a_6 - a_7$, $a_4 = -1 + 3a_2 - a_5 + 2a_6 - a_7$. Most notably, this match and the match in (8.71) of the disk instantons of the local geometry already predicts the parts of the disk instantons in the *compact* threefold Z_3 . The compact disk instantons according to the Ooguri-Vafa multi-covering formula are listed in the tables in appendix C.3.

8.4 Heterotic/F-Theory Duality and Blow-Up Threefolds

In this section we apply the blow-up procedure of section 7.2 in the context of heterotic/F-theory duality. This allows us to relate the geometry of the blow-up \hat{Z}_3 of an elliptic Calabi-Yau threefold Z_3 as well as the heterotic superpotential, in particular the superpotential induced by fluxes and horizontal five-branes, to the dual F-theory fourfold X_4 and flux superpotential. Following [81] we demonstrate explicitly in section 8.4.1 that the geometry of the blow-up \hat{Z}_3 , constructed as a complete intersection, can be used to obtain the dual Calabi-Yau fourfold X_4 on the F-theory side, which is also given as a complete intersection. In particular we recover the toric data defining the F-theory fourfold. Then in section 8.4.2 we use the the blow-up description \hat{Z}_3 of the heterotic string with horizontal five-branes for a direct geometric map of the heterotic flux and five-brane superpotential to the F-theory dual flux superpotential. In particular we show how to formally identify the four-form flux G_4 dual to the five-brane from the perspective of the blow-up \hat{Z}_3 . This all agrees with the observations of section 8.3, where it was noted that the GKZ-system of \hat{Z}_3 is related to the toric data of the dual F-theory fourfold X_4 .

8.4.1 F-theory Fourfolds from Heterotic Blow-Up Threefolds

In this section we discuss the example of section 6.2.2 employing the blow-up procedure of chapter 7. We find that the geometry of the F-theory Calabi-Yau fourfold X_4 is naturally obtained from \hat{Z}_3 by an additional \mathbb{P}^1 -fibration and in particular identical to the fourfold \hat{X}_4 considered in section 6.2.2, despite the fact that it is now realized as a complete intersection.

As in section 6.2.2 the starting point is the elliptic fibration \tilde{Z}_3 over $B_2 = \mathbb{P}^2$ with a five-brane wrapping the hyperplane class of the base. Let us describe the explicit construction of \hat{Z}_3 . Recall that \tilde{Z}_3 is a hypersurface $\{p_0 = 0\}$ in a toric variety $\mathbb{P}_{\tilde{\Delta}}$ and assume that the curve Σ is given as a complete intersection of two hypersurfaces in \tilde{Z}_3 that takes for a horizontal curve the form

$$h_1 \equiv \tilde{z} = 0, \quad h_2 \equiv g_5 = 0. \quad (8.73)$$

Here $\{\tilde{z} = 0\}$ restricts to the base B_2 and g_5 specifies Σ within B_2 , cf. section 4.3.3. Assume further that the charge vectors of $\mathbb{P}_{\tilde{\Delta}}$ are given by $\{\ell^{(i)}\}$ with $i = 1, \dots, k$. We are aiming

to torically construct a five-dimensional toric variety which is given by $\hat{\mathbb{P}}_{\tilde{\Delta}} = \mathbb{P}(N_{\mathbb{P}_{\tilde{\Delta}}}\Sigma)$ and in which the blow-up \hat{Z}_3 is given by the complete intersection (7.32). Let us denote the divisor classes defined by h_i as H_i and the charges of h_i by $\mu_i = (\mu_i^{(1)}, \dots, \mu_i^{(k)})$. Then, the coordinates l_i of $N_{\mathbb{P}_{\tilde{\Delta}}}H_i$ transform with charge $\mu_i^{(m)}$ under the k scaling relations. Since we have to projectivize $N_{\mathbb{P}_{\tilde{\Delta}}}\Sigma$, we have to include another \mathbb{C}^* -action with charge vector $\ell_{\hat{\mathbb{P}}_{\tilde{\Delta}}}^{(k+1)}$ acting non-trivially only on the new coordinates l_i . The new charge vectors of $\hat{\mathbb{P}}_{\tilde{\Delta}}$ are thus given by the following table:

	coordinates of $\mathbb{P}_{\tilde{\Delta}}$	l_1	l_2
$\ell_{\hat{\mathbb{P}}_{\tilde{\Delta}}}^{(1)}$	$\ell^{(1)}$	$\mu_1^{(1)}$	$\mu_2^{(1)}$
\vdots	\vdots	\vdots	\vdots
$\ell_{\hat{\mathbb{P}}_{\tilde{\Delta}}}^{(k)}$	$\ell^{(k)}$	$\mu_1^{(k)}$	$\mu_2^{(k)}$
$\ell_{\hat{\mathbb{P}}_{\tilde{\Delta}}}^{(k+1)}$	0	1	1

Then the blown-up threefold \hat{Z}_3 is given by the complete intersection (7.32) with $P \equiv p_0$ in this toric space.

To apply this to the elliptic fibration over \mathbb{P}^2 with the polyhedron (6.9), one picks the curve Σ given by $h_1 :=$ and $g_5 := x_1$. Σ is a genus zero curve and we readily infer that the exceptional divisor E is the first del Pezzo surface dP_1 in accord with the general formulas for the topology of E in appendix B.3. We construct the five-dimensional ambient manifold $\hat{\mathbb{P}}_{\tilde{\Delta}}$ as explained above,

$$\Delta_5^{\hat{Z}_3} = \left(\begin{array}{ccccc|c} -1 & 0 & 0 & 0 & 0 & 3B + 3D + 9H \\ 0 & -1 & 0 & 0 & 0 & 2B + 2D + 6H \\ 3 & 2 & 0 & 0 & 0 & B \\ 3 & 2 & 1 & 1 & 1 & H \\ 3 & 2 & -1 & 0 & 0 & H \\ 3 & 2 & 0 & -1 & 0 & H \\ 3 & 2 & 0 & 0 & -1 & E \\ 0 & 0 & 0 & 0 & -1 & H - E \end{array} \right). \tag{8.74}$$

Note that one has to include the inner point $(3, 2, 0, 0, 0)$ which corresponds to the base of the elliptic fibration \hat{Z}_3 . Furthermore, one readily shows that the point $(0, 0, 0, 0, 1)$, required for the above scalings, can be omitted since the associated divisor does not intersect the complete intersection \hat{Z}_3 . Explicitly the complete intersection \hat{Z}_3 is given by a generic constraint in the class

$$\hat{Z}_3 : (6B + 6E + 18H) \cap H, \tag{8.75}$$

where H, B, E are the divisor classes of the ambient space (8.74). The first divisor in (8.75) is the sum of the first seven divisors in (8.74) and corresponds to the original Calabi-Yau

constraint $p_0 = 0$ of Z_3 . The second divisor in (8.75) is the sum of the last two divisors and is the class of the second constraint Q in (7.32). This complete intersection threefold has $\chi(\hat{Z}_3) = -538 = \chi(\tilde{Z}_3) - \chi(\mathbb{P}^1) + \chi(dP_1)$, and one checks that the exceptional divisor E has the characteristic data of a del Pezzo surface dP_1 . This means that we have replaced the hyperplane which is isomorphic to \mathbb{P}^1 in the base by the exceptional divisor which is dP_1 . It can be readily checked that the first Chern class of \hat{Z}_3 is non-vanishing and equals $-E$.

Having described the blow-up geometry \tilde{Z}_3 , we now turn to the construction of the fourfold \hat{X}_4 for F-theory. This fourfold will also be constructed as complete intersection, but it will be the same manifold as the fourfold described in section 6.2.2 by equation (6.79). Geometrically we fiber an additional \mathbb{P}^1 over $\hat{\mathbb{P}}_{\Delta}$ which is only non-trivially fibered along the exceptional divisor. This is analogous to the construction of the dual fourfold in heterotic/F-theory duality where one also fibers a \mathbb{P}^1 over the base B_2 of the Calabi-Yau threefold to obtain the base B_3 of the F-theory fourfold. Here we proceed in a similar fashion but construct a \mathbb{P}^1 -fibration over the base of the non-Calabi-Yau manifold \hat{Z}_3 . This base is a complete intersection and thus leads to a realization of \hat{X}_4 as a complete intersection. Concretely, we have the following polyhedron

$$\Delta_6^{\hat{X}} = \left(\begin{array}{cccccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 & 3D + 3B + 9H + 6K & D_1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2D + 2B + 6H + 4K & D_2 \\ 3 & 2 & 0 & 0 & 0 & 0 & B & D_3 \\ 3 & 2 & 1 & 1 & 1 & 0 & H & D_4 \\ 3 & 2 & -1 & 0 & 0 & 0 & H & D_5 \\ 3 & 2 & 0 & -1 & 0 & 0 & H & D_6 \\ 3 & 2 & 0 & 0 & 0 & 1 & K & D_7 \\ 3 & 2 & 0 & 0 & 0 & -1 & K + E & D_8 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & E & D_9 \\ 0 & 0 & 0 & 0 & -1 & 0 & H - E & D_{10} \end{array} \right). \quad (8.76)$$

The fourfold \hat{X}_4 is then given as the complete intersection in the class

$$\hat{X}_4 : (6B + 6E + 18H + 12K) \cap H. \quad (8.77)$$

Note that this fourfold indeed obeys Calabi-Yau condition as can be checked explicitly by analyzing the toric data (8.76). For complete intersections the Calabi-Yau constraint is realized via the two partitions, so-called nef partitions, in (8.76) as in [271]. The first nef partition yields the sum of the first eight divisors $\sum_{i=1}^8 D_i$ in (8.76) and gives the first constraint in (8.77). The second nef partition yields the sum of the last two divisors $D_9 + D_{10}$ in (8.76) and determines the second constraint in (8.77). The divisors D_7 and D_8 correspond to the \mathbb{P}^1 -fiber in the base of \hat{X}_4 obtained by dropping the first two columns in (8.76). This fibration is only non-trivial over the exceptional divisors $D_9 = E$ in the second nef partition of (8.76). Note that if one simply drops K from the expression (8.77) one formally recovers the

constraint (8.75) of \hat{Z}_3 . To check that the complete intersection \hat{X}_4 is precisely the fourfold constructed in section 6.2.2, one has to compute the intersection ring and Chern classes. In particular, it is not hard to show that also (8.76) has three triangulations matching the result of section 6.2.2.

In summary, we have found that there is a natural construction of \hat{X}_4 as complete intersection with the base obtained from the heterotic non-Calabi-Yau threefold \hat{Z}_3 . Let us stress that this construction will straightforwardly generalize to dual heterotic/F-theory setups with other toric base spaces B_2 and different types of bundles. For example, to study the bundle configurations on Z_3 of section 6.2.2 with $\eta_{1,2} = 6c_1(B_2) \pm kH$, $k = 0, 1, 2$ one has to replace

$$D_4 \rightarrow (3, 2, 1, 1, k) , \quad D_4 \rightarrow (3, 2, 1, 1, 1, k) , \quad (8.78)$$

in the polyhedra (6.79) and (8.76), respectively. Moreover, also bundles which are not of the type $E_8 \times E_8$ can be included by generalizing the form of the \mathbb{P}^1 -fibration just as in the standard construction of dual F-theory fourfolds.

8.4.2 Superpotentials and G_4 -fluxes from Heterotic Blow-Ups

In this concluding section we show how the geometric matching of section 8.4.1 can be used to map the heterotic flux and brane superpotential to the F-theory side. In particular we use the geometry of \hat{Z}_3 to specify the corresponding G_4 -flux on X_4 .

For concreteness, let us again consider an elliptic Calabi-Yau threefold Z_3 described as the hypersurface $\{p_0 = 0\}$ in a projective or toric ambient space \mathbb{P}_Δ . Then, the blown-up threefold \hat{Z}_3 is given by the complete intersection (7.32) in the projective bundle \mathcal{W} . Next we use the lift of both the flux as well as the brane superpotential discussed in section 7.4, where its complete moduli dependence is entirely encoded in the complex structure dependence of the lifted three-form $\hat{\Omega}$. On \hat{Z}_3 the combined heterotic superpotential can be written as

$$W_{\text{flux}} + W_{\text{brane}} = \int_{\hat{Z}_3} \hat{H}_3 \wedge \hat{\Omega} \quad (8.79)$$

where \hat{H}_3 contains the current (7.57) and also a smooth flux part encoding W_{flux} .

Now we can in principle proceed as in section 8.1 and derive Picard-Fuchs differential equations for $\hat{\Omega}$ by studying its complex structure dependence explicitly. Then the closed periods of Z_3 as well as the superpotential W_{brane} are expected to be among the solutions to this system as argued in sections 7.3 and 7.4. However, instead of performing a direct analysis on the heterotic side we follow the route of heterotic/F-theory duality to calculate the superpotential.

In accord with the discussion of section 8.4.1 the dual fourfold X_4 can in general be realized as a complete intersection blown up along the five-brane curves, cf. also section 4.3.3 for the

general construction. We want to match this description with the heterotic theory on \hat{Z}_3 . Indeed, one can now identify the blow-up constraints on the heterotic and on the F-theory side as

$$Q_{\text{het}} = l_1 g_5(\underline{u}) - l_2 \tilde{z} \longmapsto Q_{\text{F}} = l_1 g_5(\underline{u}) - l_2 k \quad , \quad \tilde{z} \longmapsto k \quad , \quad (8.80)$$

where \underline{u} denote coordinates on the base B_2 , ($\tilde{z} = 0$) defines the base B_2 in Z_3 , which is realized in X_4 as $(z = 0) \cap (k = 0)$.¹³ The map (8.80) is possible since both Z_3 and X_4 share the twofold base B_2 with the curve Σ . The identification of \tilde{z} with k corresponds to the fact that in heterotic/F-theory duality the elliptic fibration of Z_3 is mapped to the \mathbb{P}^1 -fibration of B_3 , see section 4.3.2. Clearly, the map (8.80) identifies the deformations of Σ realized as coefficients in the constraint Q_{het} of \hat{Z}_3 with the complex structure deformations of X_4 realized as coefficient in Q_{F} .

We also have to match the remaining constraints p_0 and P of \hat{Z}_3 and X_4 , respectively. Clearly, there will not be a general match. However, as was argued in [103] for Calabi-Yau fourfold hypersurfaces, one can split $P = p_0 + V_E$ yielding a map

$$p_0 + V_E \longmapsto P \quad , \quad (8.81)$$

where V_E describes the spectral cover of the dual heterotic bundles $E = E_1 \oplus E_2$. Again, this requires an identification of \tilde{z} and k . For $SU(1)$ -bundles this map was given in (8.80), but can be generalized for non-trivial bundles. Note that the maps (8.80) and (8.81) can also be formulated in terms of the GKZ-systems of the complete intersections \hat{Z}_3 and X_4 . It implies that the charge vectors $\ell_i^{(a)}$ of X_4 contain charge vectors of Z_3 and the five-brane charge vectors $\hat{\ell}^{(a)}$, which is precisely the situation encountered for the blow-up GKZ-system constructed in section 8.2 and in [79, 119, 122–124].

To match the superpotentials W_{het} and W_{G_4} as in (6.67) one finally has to identify the flux quanta on the blow-up \hat{Z}_3 including the five-brane superpotential¹⁴ with elements of $H^4(X_4, \mathbb{Z})$ and show that the periods of $\hat{\Omega}$ as well as W_{brane} can be identified with a subset of the fourfold periods of Ω_4 . In order to do that, one compares the residue integrals (7.42) and (5.60) for both $\hat{\Omega}$ and Ω_4 in the case of complete intersections. Using the maps (8.80) and (8.81) one then shows that each Picard-Fuchs operator annihilating $\hat{\Omega}$ also annihilates Ω_4 . Hence, also a subset of the solutions to the Picard-Fuchs equations can be matched accordingly. As a minimal check, one finds employing the arguments of section 7.3 that the periods of Ω on Z_3 before the blow-up arise as a subset of the periods of Ω_4 [79]. The map between the cohomologies $H^3(\hat{Z}_3 - E, \mathbb{Z}) \hookrightarrow H^4(X_4, \mathbb{Z})$ is then best formulated in terms of operators $\mathcal{R}_p^{(i)}$ applied to the forms $\hat{\Omega}$ and Ω_4 ,

$$\mathcal{R}_p^{(i)} \tilde{\Omega}_3(\underline{t}^c, \underline{t}^o) \Big|_{\underline{t}^c = \underline{t}^o = 0} \longmapsto \mathcal{R}_p^{(i)} \Omega_4(\underline{z}) \Big|_{z=0} \quad . \quad (8.82)$$

¹³Note that the \mathbb{P}^1 -fibration $B_3 \rightarrow B_2$ has actually two sections, one of which is an isolated section and the other comes in a family of sections. As in section 4.3.3, $k = 0$ is one of the two sections, say, the zero section.

¹⁴Formally these integrals are defined by the cohomology of currents [154] or its dual the relative homology group $H_3(\hat{Z}_3, E) = H_3(\hat{Z}_3 - E)$ [108–110].

Here we used a notation from the context of mirror symmetry on fourfolds, where horizontal fluxes are described by derivatives of Ω_4 , see section 5.3, also for the blow-up \hat{Z}_3 . The map (8.82) implies the assumption that the heterotic flux \hat{H}_3 can be represented by $\mathcal{R}_p^{(i)}\tilde{\Omega}_3$. We conclude by noting that the preimage of this map will in general contain derivatives with respect to the closed and open string variables $\underline{t}^c, \underline{t}^o$. One then finds that by identifying the heterotic and F-theory moduli at the large complex structure point $\underline{z} = 0$, one obtains an embedding map of the integral basis on \hat{Z}_3 .

Chapter 9

SU(3)-Structure on Brane Blow-Ups

In the previous chapters we have argued that five-brane deformations can be equivalently described on \hat{Z}_3 with a two-form flux localized on the blow-up divisor E . The aim of this section is to delocalize the flux further to three-form flux and propose an $SU(3)$ -structure on the open manifold $\hat{Z}_3 - E$ and on \hat{Z}_3 . Let us note that this chapter is independent of any explicit toric construction and the reader only interested in the enumerative aspects of the superpotential can safely skip this section. We hope to provide concrete proposals which should only be viewed as first steps to identify a complete back-reacted vacuum.

We begin our discussion, following [100], in section 9.1 by recalling some basic facts about $SU(3)$ -structure manifolds and the generalized flux superpotential. As an intermediate step, we show next how the blow-up threefold \hat{Z}_3 can be endowed with a Kähler structure in section 9.2. However, it is well-known that a supersymmetric vacuum with background three-form fluxes requires that the internal space is non-Kähler (section 9.1). Furthermore, there exists no globally defined, nowhere vanishing holomorphic three-form on \hat{Z}_3 . We propose a resolution to these issues in two steps in section 9.3. In a first step we argue that there is a natural non-Kähler structure \hat{J} on the open manifold $\hat{Z}_3 - E$ which, in a supersymmetric vacuum, matches the flux via $i(\bar{\partial} - \partial)\hat{H}_3 = d\hat{J}$. While $(\hat{J}, \hat{\Omega}, \hat{H}_3)$ are well-defined forms on the open manifold, they have poles, as it is the case for \hat{J} and \hat{H}_3 , and zeros, as we have seen for $\hat{\Omega}$, when extended to all of \hat{Z}_3 . Hence, in a second step, we argue that the poles and zeros can be removed by an appropriate local logarithmic transformation yielding new differential forms (J', Ω', H'_3) on \hat{Z}_3 . In fact, the new global forms are defined such that the zeros and poles precisely cancel in the superpotential which can then be evaluated on \hat{Z}_3 .

9.1 Brief Review on SU(3)-Structures and the Superpotential

To begin with, let us recall some basic facts about compactifications on non-Calabi-Yau manifolds \hat{Z}_3 . In order that the four-dimensional effective theory obtained in such compactifications has $\mathcal{N} = 1$ supersymmetry one demands that \hat{Z}_3 has $SU(3)$ -structure [272]. $SU(3)$ -structure manifolds can be characterized by the existence of two no-where vanishing forms, a real two-form J' and a real three-form ρ' . Following [273] one demands that J' and ρ' are stable forms, i.e. are elements of open orbits under the action of general linear transformations $GL(6, \mathbb{R})$

at every point of the tangent bundle $T\hat{Z}_3$. Then one can set $\hat{\rho}' = *\rho'$ and show that $\hat{\rho}'$ is a function of ρ' only [273]. These forms define a reduction of the structure group from $GL(6, \mathbb{R})$ to $SU(3)$ if they satisfy $J \wedge \Omega' = 0$, with a nowhere vanishing three-form $\Omega' = \rho' + i\hat{\rho}'$ and J' .

By setting $I_m^n = J'_{mp}g^{pn}$ one defines an almost complex structure with respect to which the metric g_{mn} is hermitian. The almost complex structure allows to introduce a (p, q) grading of forms. Within this decomposition the form J' is of type $(1, 1)$ while Ω' is of type $(3, 0)$. In general, neither J' nor Ω' are closed. The non-closedness is parameterized by five torsion classes \mathcal{W}_i which transform as $SU(3)$ irreducible representations [274, 275]. One has

$$\begin{aligned} dJ' &= \frac{3}{2}\text{Im}(\bar{\mathcal{W}}_1\Omega') + \mathcal{W}_4 \wedge J' + \mathcal{W}_3 \\ d\Omega' &= \mathcal{W}_1 J' \wedge J' + \mathcal{W}_2 \wedge J' + \bar{\mathcal{W}}_5 \wedge \Omega' , \end{aligned} \quad (9.1)$$

with constraints $J' \wedge J' \wedge \mathcal{W}_2 = J' \wedge \mathcal{W}_3 = \Omega' \wedge \mathcal{W}_3 = 0$. The pattern of vanishing torsion classes defines the properties of the manifold \hat{Z}_3 . In a supersymmetric vacuum the pattern of torsion classes is constraint by the superpotential.

Let us first discuss the pure flux superpotential for heterotic and Type IIB orientifolds with $O5$ -planes. Recall from section 7.1.3 that the pure flux superpotential of these theories is of the form $W_{\text{flux}} = \int \Omega \wedge H_3$ and $W_{\text{flux}} = \int \Omega \wedge F_3$. It is easy to check that there are no supersymmetric flux vacua for Calabi-Yau compactifications. In fact, in the absence of branes W_{flux} is the only perturbative superpotential for a Calabi-Yau background. The supersymmetry conditions are

$$D_{z^k} W_{\text{flux}} = 0 , \quad W_{\text{flux}} = 0 , \quad (9.2)$$

where the latter condition arises from the fact that $D_S W_{\text{flux}} = K_S W_{\text{flux}} = 0$, for other moduli S which do not appear in W_{flux} . One easily checks that the first condition in (9.2) implies that H_3 cannot be of type $(2, 1) + (1, 2)$, while the second condition implies that it cannot be of type $(3, 0) + (0, 3)$. This implies that H_3 has to vanish and there are no flux vacua in a Calabi-Yau compactification.

The situation changes for non-Calabi-Yau compactifications since the superpotential in this case is of more general form. More precisely, denoting by \hat{Z}_3 a generic $SU(3)$ -structure manifold it takes the form [195, 276, 277]

$$W = \int_{\hat{Z}_3} \Omega' \wedge (H'_3 + idJ') . \quad (9.3)$$

It is straightforward to evaluate the supersymmetry conditions for this superpotential. Firstly, we note that in a supersymmetric background the compact manifold \hat{Z}_3 is complex, thus the torsion classes vanish, $\mathcal{W}_1 = \mathcal{W}_2 = 0$. Second the superpotential W is independent of the dilaton superfield and hence one evaluates in the vacuum that $W = 0$, which implies that the $(0, 3)$ part of $H'_3 + idJ'$ has to vanish. However, since the $(3, 0) + (0, 3)$ component of dJ' vanishes for a vanishing \mathcal{W}_1 , one concludes that also $H'_3 + idJ'$ has no $(3, 0)$ component, and

hence can be non-zero in the $(2, 1)$ and $(1, 2)$ directions. The Kähler covariant derivative of Ω' yields the condition that $H'_3 + idJ'$ has only components along the $(2, 1)$ direction. Hence, using the fact that J' is of type $(1, 1)$ one finally concludes

$$H'_3 = i(\bar{\partial} - \partial)J' . \quad (9.4)$$

This matches the long-known relation found in [197] for a supersymmetric vacuum of the heterotic string with background three-form flux. It should be stressed that there will be additional conditions which have to be respected by the heterotic vacuum. These involve a non-constant dilaton and cannot be captured by a superpotential.

9.2 The Blow-Up Space as a Kähler Manifold

Before introducing an $SU(3)$ -structure on \hat{Z}_3 , it is necessary to recall that every variety \hat{Z}_3 obtained by blowing up a holomorphic curve naturally admits a Kähler structure [154, 240]. We ‘twist’ this Kähler structure to obtain a non-Kähler $SU(3)$ -structure in section 9.3. It is crucial, however, to look first at the geometry of \hat{Z}_3 near E more closely, and introduce the Kähler structure very explicitly. As a preparation we consider in section 9.2.1 the blow-up of the origin in \mathbb{C}^2 and construct its canonical Kähler form. This can be viewed as the normal bundle $N_{Z_3\Sigma}$ at a point in Σ and thus as local version of the blow-up of Z_3 along Σ . Indeed the Kähler structure on \hat{Z}_3 is defined in section 9.2.2 in a formally very similar way.

9.2.1 Kähler Geometry on the Blow-Up: Warm-Up in Two Dimensions

To warm up for the more general discussion, let us first consider a simpler example and blow up a point in a complex surface. In a small patch U_ϵ around this point this looks like blowing up the the origin in \mathbb{C}^2 into an exceptional divisor $E = \mathbb{P}^1$. Let us denote by $B_{\mathbb{C}^2}$ the space obtained after blowing up as in section 7.2.1. Our aim is to explicitly define a Kähler form \tilde{J} on $B_{\mathbb{C}^2}$ following ref. [154].

To define \tilde{J} the key object we will study is the line bundle $\mathcal{L} \equiv \mathcal{O}_{B_{\mathbb{C}^2}}(E)$, or rather $\mathcal{L}^{-1} \equiv \mathcal{O}_{B_{\mathbb{C}^2}}(-E)$. To get a clearer picture of this bundle, we give a representation of \mathcal{L} near E . As in subsection 7.2.1 we first introduce the patch $\hat{U}_\epsilon = \pi^{-1}(U_\epsilon)$. One can embed the fibers of \mathcal{L} into $\hat{U}_{2\epsilon} = \{U_{2\epsilon} \times \mathbb{P}^1 : y_2 l_1 - y_1 l_2 = 0\}$ as

$$\mathcal{L}_{(y,l)} = \{\lambda \cdot (l_1, l_2), \lambda \in \mathbb{C}\} , \quad (9.5)$$

where (l_1, l_2) are the projective coordinates of \mathbb{P}^1 and y collectively denote the coordinates on U_ϵ . This implies that holomorphic sections σ of \mathcal{L} are locally specified by $\sigma \equiv \lambda(\underline{y}, \underline{l})$. To explicitly display the expressions we introduce local coordinates on patches $\hat{U}_{2\epsilon}^{(1)}$ and $\hat{U}_{2\epsilon}^{(2)}$,

which cover the \mathbb{P}^1 such that $l_i \neq 0$ on $\hat{U}_{2\epsilon}^{(i)}$. We set

$$\hat{U}_{2\epsilon}^{(1)} : \quad u_1 = y_1, \quad \ell_1 = \frac{l_2}{l_1}, \quad \hat{U}_{2\epsilon}^{(2)} : \quad u_2 = y_2, \quad \ell_2 = \frac{l_1}{l_2}, \quad (9.6)$$

Using the blow-up relation one finds the following coordinate transformation on the overlap

$$\hat{U}_{2\epsilon}^{(1)} \cap \hat{U}_{2\epsilon}^{(2)} : \quad (\ell_2, u_2) = (1/\ell_1, \ell_1 u_1), \quad (9.7)$$

which shows that $B_{\mathbb{C}^2}$ is identified with $\mathcal{O}_{\mathbb{P}^1}(-1)$. Let us point out that this matches the local description presented in appendix B.2 if we interpret $B_{\mathbb{C}^2}$ as a local model of $(N_{\hat{Z}_3}\Sigma)_p$ at a point p on Σ .¹

One now can introduce a metric $\|\sigma\|$ for sections σ of the line bundle \mathcal{L} as follows. Since \mathcal{L} is non-trivial one cannot simply specify $\|\sigma\|$ by using a single global holomorphic section σ . Each such global holomorphic section will have either poles or zeros. However, we can specify $\|\cdot\|$ on local holomorphic sections patchwise and glue these local expressions together. Let us define the local expression on $B_{\mathbb{C}^2} - E$ by evaluation on a global holomorphic section $\sigma_{(0)}$ with zeros along E . One defines

$$B_{\mathbb{C}^2} - E : \quad \|\sigma_{(0)}\|_1 := 1. \quad (9.8)$$

On the patches $\hat{U}_{2\epsilon}$ covering E one needs to use other sections which are non-vanishing also along E . Using the explicit realization of \mathcal{L} as in (9.5) with sections $\sigma = \lambda$ one can specify $\|\cdot\|_2$ in $\hat{U}_{2\epsilon}$ setting

$$\hat{U}_{2\epsilon} : \quad \|\sigma\|_2 := |\lambda| (|l_1|^2 + |l_2|^2)^{1/2}. \quad (9.9)$$

Note that the section $\sigma_{(0)}$ is also defined in $\hat{U}_{\epsilon} - E$ and can be given in the representation (9.5) of \mathcal{L} . In the same representation we can also define local sections $\sigma_{(i)}$ near E , such that

$$\hat{U}_{2\epsilon} : \quad \sigma_{(0)} = \frac{y_1}{l_1} = \frac{y_2}{l_2}, \quad \hat{U}_{2\epsilon}^{(i)} : \quad \sigma_{(i)} = \frac{1}{l_i}, \quad (9.10)$$

Recall that the y_i , as also introduced in subsection 7.2.1, specify the point which is blown up as $y_1 = y_2 = 0$. The metric (9.9) for these sections is simply given by

$$\hat{U}_{\epsilon}^{(i)} : \quad \|\sigma_{(i)}\|_2 = (1 + |\ell_i|^2)^{1/2}, \quad \|\sigma_{(0)}\|_2 = |u_i| \cdot (1 + |\ell_i|^2)^{1/2}, \quad (9.11)$$

where we have used the local coordinates (9.6). To give the global metric one next splits $B_{\mathbb{C}^2}$ into patches $(B_{\mathbb{C}^2} - \hat{U}_{\epsilon}, \hat{U}_{2\epsilon})$, and introduces a partition of unity (ρ_1, ρ_2) . The local expressions (9.8) and (9.9) are glued together as

$$\|\cdot\| := \rho_1 \|\cdot\|_1 + \rho_2 \|\cdot\|_2. \quad (9.12)$$

¹To avoid cluttering of indices we introduce the new notation for $z_1^{(1)} \equiv u_1$, $z_2^{(1)} \equiv \ell_1$ and $z_2^{(2)} \equiv u_2$, $z_2^{(2)} \equiv \ell_2$.

Using this metric one can now determine the Chern curvature form $\frac{i}{2}\Theta$ of the line bundle \mathcal{L} . Locally one has to evaluate

$$\frac{i}{2}\Theta = -\frac{1}{4\pi}\partial\bar{\partial}\log\|\sigma\|^2, \tag{9.13}$$

for holomorphic sections σ which have no poles or zeros in the considered patch. Using (9.8) and (9.11) one finds that \mathcal{L} is trivial on $B_{\mathbb{C}^2} - \hat{U}_{2\epsilon}$, but non-trivial in the patches $\hat{U}_\epsilon^{(i)}$:

$$\begin{aligned} B_{\mathbb{C}^2} - \hat{U}_{2\epsilon} : \quad \Theta &= 0, \\ \hat{U}_\epsilon^{(i)} : \quad \Theta &= -\frac{i}{2\pi}\partial\bar{\partial}\log(1 + |\ell_i|^2) = -\omega_{FS}, \end{aligned} \tag{9.14}$$

where ω_{FS} is the Fubini-Study metric. One notes that the form Θ is strictly negative on E , since it is given by minus the pull-back of the Fubini-Study metric under the restriction map to \mathbb{P}^1 . Hence, one finds that $-\Theta > 0$ on E , and $-\Theta \geq 0$ on \hat{U}_ϵ . In the region $\hat{U}_{2\epsilon} - \hat{U}_\epsilon$ the form Θ interpolates in a continuous way. Finally, we can give the Kähler form \tilde{J} on $B_{\mathbb{C}^2}$. Since $-\Theta \geq 0$ on \hat{U}_ϵ and $\Theta = 0$ on $B_{\mathbb{C}^2} - \hat{U}_{2\epsilon}$ continuity implies that $-\Theta$ is bounded from below. This fact can be used to define a Kähler form on $B_{\mathbb{C}^2}$ by setting

$$\tilde{J} = \pi^*J - v_{\text{bu}}\Theta, \tag{9.15}$$

where π^*J is the pull-back of the Kähler form on \mathbb{C}^2 . \tilde{J} is a closed $(1, 1)$ -form, and positive for a sufficiently small blow-up volume v_{bu} . In other words, one finds that the manifold $B_{\mathbb{C}^2}$ is naturally endowed with a Kähler structure. Since the blow-up is a local operation, one uses this construction to blow up a point in any Kähler surface identifying J in (9.15) with the Kähler form before the blow-up.

9.2.2 Blow-Up Threefolds as Kähler Manifolds

Having discussed the Kähler structure on the blow-up of a point in a Kähler surface, we can now generalize this construction to blowing up curves in Z_3 . This is again textbook knowledge [240] and we can be brief.

As in subsection 9.2.1 we study the line bundles $\mathcal{L} \equiv \mathcal{O}_{\hat{Z}_3}(E)$ and $\mathcal{L}^{-1} \equiv \mathcal{O}_{B_{\hat{Z}_3}}(-E)$. One proceeds as in the two-dimensional example and first examines the bundle near E . One shows that restricted to E one obtains

$$\mathcal{L}^{-1}|_E \cong N_{\hat{Z}_3}^*E \cong \mathcal{O}_E(1), \tag{9.16}$$

where $N_{\hat{Z}_3}^*E$ is the co-normal bundle to E in \hat{Z}_3 . The bundle $E = \mathbb{P}^1(N_{Z_3}\Sigma) \rightarrow \Sigma$ admits a natural closed $(1, 1)$ -form which is positive on the fibers. This form can be obtained from a hermitian metric induced from $N_{Z_3}\Sigma$ and is obtained from the Chern curvature of $\mathcal{O}_E(1)$. As in the previous section this curvature can be extended to \mathcal{L}^{-1} by a partition of unity as in (9.12). Let us denote the Chern curvature of \mathcal{L} again by $\frac{i}{2}\Theta$. The Kähler form on \hat{Z}_3 is then

given by $\tilde{J} = \pi^* J - v_{\text{bu}} \Theta$, for sufficiently small blow-up volume v_{bu} . While the construction of Θ depends on the explicit metric $\|\cdot\|$ on \mathcal{L} one can also evaluate the corresponding cohomology classes which are topological in nature. Clearly, one has

$$c_1(\mathcal{L}) = \frac{1}{\pi}[\Theta], \quad c_1(\mathcal{L})|_E = c_1(N_{\hat{Z}_3} E) = [E]_E, \quad (9.17)$$

where we have also displayed the restriction to the exceptional divisor.

The above discussion identifies \hat{Z}_3 as a Kähler manifold. If one directly uses this Kähler structure, however, one finds that \hat{Z}_3 cannot arise as an actual supersymmetric flux background. Recall that, for example, in heterotic compactifications with background fluxes H_3 the internal manifold has to be non-Kähler to satisfy (9.4) as shown in [197]. In the following we will show that there actually exists a natural $SU(3)$ -structure on \hat{Z}_3 which renders it to be non-Kähler and allows to identify a supersymmetric flux vacuum on \hat{Z}_3 .

9.3 Defining the $SU(3)$ -Structure: the Non-Kähler Twist

In the following we propose an $SU(3)$ -structure on the open manifold $\hat{Z}_3 - E$ and the blow-up space \hat{Z}_3 in sections 9.3.2 and 9.3.3, respectively. Before turning to the three-dimensional case we first discuss the complex two-dimensional case $B_{\mathbb{C}^2}$ in section 9.3.1.

9.3.1 The Non-Kähler Twist: Warm-Up in two complex Dimensions

To warm up for the more general discussion, let us first introduce a non-Kähler structure on the simpler two-dimensional example $B_{\mathbb{C}^2}$. Recall from (9.7) that $B_{\mathbb{C}^2}$ is the total space of the universal line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ over \mathbb{P}^1 . There are two geometries related to $B_{\mathbb{C}^2}$ which admit a non-trivial non-Kähler structure. Firstly, to render $B_{\mathbb{C}^2}$ into a compact space $\hat{B}_{\mathbb{C}^2}$ one can replace each fiber \mathbb{C} of the line bundle $B_{\mathbb{C}^2}$ by a two-torus \mathbb{C}^*/\mathbb{Z} .² Secondly, one can consider the open manifold $B_{\mathbb{C}^2} - E$, where one simply removes the origin and the attached exceptional blow-up divisor $E = \mathbb{P}^1$. On the geometries $\hat{B}_{\mathbb{C}^2}$ and $B_{\mathbb{C}^2}$ one can introduce coordinates (ℓ_i, u_i) as in (9.6). The new geometries have been modified from $B_{\mathbb{C}^2}$ such that, in particular, one has $u_i \neq 0$. For the compact space $\hat{B}_{\mathbb{C}^2}$ one further has a periodicity since $u_i \in \mathbb{C}^*/\mathbb{Z}$. Note that one inherits from the blow-up the coordinate transformation (9.7) on the overlaps $\hat{U}^{(1)} \cap \hat{U}^{(2)}$ covering the north and south pole of the \mathbb{P}^1 .

One observes that with the transformations (9.7) the surface $\hat{B}_{\mathbb{C}^2}$ can be identified as the *Hopf surface* $S^1 \times S^3$. This surface does not admit a Kähler structure since $h^{(1,1)} = 0$, while $h^{(0,0)} = h^{(1,0)} = h^{(2,1)} = h^{(2,2)} = 1$. Note however, that $\hat{B}_{\mathbb{C}^2}$ admits a natural globally defined no-where vanishing $(1,1)$ -form \hat{J} . The construction of \hat{J} was given in ref. [278]. In

²More precisely, parameterizing a fiber by $\lambda = r e^{i\theta}$ one removes the origin $r = 0$ and identifies $r \cong r + 1$.

fact, one can also introduce a three-form flux \hat{H}_3 such that $i(\bar{\partial} - \partial)\hat{J} = \hat{H}_3$, just as in the supersymmetry conditions (9.4).

Keeping this connection with the Hopf surface in mind, we aim to introduce \hat{J} and \hat{H}_3 on $B_{\mathbb{C}^2} - E$. Firstly, we introduce on $U_\epsilon^{(i)}$ the B-field

$$\begin{aligned} U_\epsilon^{(i)} - E : \quad \hat{B}_i &= \frac{1}{4\pi}(\partial \log(1 + |\ell_i|^2) \wedge \partial \log u_i + \bar{\partial} \log(1 + |\ell_i|^2) \wedge \bar{\partial} \log \bar{u}_i) , \\ &= \frac{1}{2\pi} \text{Re}(\partial \log \|\sigma_{(i)}\|^2 \wedge \partial \log \|\sigma_{(0)}\|^2) . \end{aligned} \quad (9.18)$$

Here we have inserted the definitions (9.10) of the sections $\sigma_{(0)}, \sigma_{(i)}$ and the norm $\|\cdot\|$ given in (9.11). One realizes that these \hat{B}_i are of type $(2,0) + (0,2)$. They extend continuously to $B_{\mathbb{C}^2} - E$ and one checks that $\hat{B}_i = 0$ on $B_{\mathbb{C}^2} - \hat{U}_{2\epsilon}$, since $\|\sigma_{(0)}\| = 1$ outside $\hat{U}_{2\epsilon}$. However, note that the \hat{B}_i do not patch together on the overlap $\hat{U}_\epsilon^{(1)} \cap \hat{U}_\epsilon^{(2)}$ as a form, but rather satisfy

$$\hat{U}_\epsilon^{(1)} \cap \hat{U}_\epsilon^{(2)} : \quad \hat{B}_2 - \hat{B}_1 = F_{21} , \quad (9.19)$$

where one identifies

$$F_{21} = -\frac{1}{2\pi} \text{Re}(d \log \ell_1 \wedge d \log u_1) , \quad (9.20)$$

The $(2,0) + (0,2)$ form F_{21} on $\hat{U}_\epsilon^{(1)} \cap \hat{U}_\epsilon^{(2)}$ can be used to define a Hermitian line bundle on this overlap. In mathematical terms F_{01} and \hat{B}_i define a gerbe with curvature

$$\hat{H}_3 = d\hat{B}_i = -\frac{i}{2} \omega_{FS} \wedge (\bar{\partial} - \partial) \log |u_i|^2 , \quad (9.21)$$

where we have introduced the Fubini-Study metric $\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log \|\sigma_{(i)}\|^2$ on \mathbb{P}^1 . One can now check that indeed on the Hopf surface $\hat{B}_{\mathbb{C}^2} \cong S^1 \times S^3$ one has a non-vanishing integral of \hat{H}_3 over the S^3 . Similarly, one can evaluate the integral on the open manifold $B_{\mathbb{C}^2} - E$, where the integral of \hat{H}_3 is performed on \mathbb{P}^1 and the S^1 encircling the zero section which has been removed.

It is now straightforward to read off the non-Kähler $(1,1)$ -form \hat{J} , satisfying the supersymmetry condition $i(\bar{\partial} - \partial)\hat{J} = \hat{H}_3$. There are two immediate choices. One could set

$$\hat{U}_\epsilon^{(1)} - E : \quad \hat{J} = \frac{1}{2\pi} \text{Im}(\partial \log \|\sigma_{(1)}\|^2 \wedge \bar{\partial} \log \bar{u}_1) , \quad (9.22)$$

which is the choice used on the Hopf surface in [278]. To evaluate \hat{J} in the second patch U_1 one uses the coordinate transformation (9.7) and transforms \hat{J} like a standard differential form. However, the second choice

$$\hat{U}_\epsilon^{(1)} - E : \quad \hat{J} = -\frac{1}{2} \log |u_1|^2 \omega_{FS} , \quad (9.23)$$

is more appropriate to the blow-up case. The reason is that (9.23) can be extended to $B_{\mathbb{C}^2} - E$ by replacing $\omega_{FS} \rightarrow -\Theta$, where Θ is proportional to the Chern curvature of the line bundle $\mathcal{L} = \mathcal{O}(E)$ introduced in subsection 9.2.1. As noted in (9.14) the Chern curvature of \mathcal{L} is localized near E and vanishes outside $\hat{U}_{2\epsilon}$. It will be the expression (9.23) which we will extend to the three-dimensional case in the next subsection.

Note that for the expressions in (9.18)-(9.23) to define well-behaved differential forms, one has to exclude the origin $u_i = 0$ in $\hat{U}_\epsilon^{(i)}$. This is precisely the reason why we considered the restriction of the blow-up space $B_{\mathbb{C}^2}$ to the open manifold $B_{\mathbb{C}^2} - E$. The extension of this structure to $B_{\mathbb{C}^2}$ will be discussed in subsection 9.3.3 below. However, one can extend the above constructions to \mathbb{C}^2 and $B_{\mathbb{C}^2}$ by including currents as in sections 7.1.1 and 7.4. Note that on \mathbb{C}^2 the delta-current $\delta_{\{0\}} = \bar{\partial}T_\beta$ localizing on the origin is defined using the Cauchy kernel

$$\beta = -\frac{1}{4\pi}\partial\log(|x_1|^2 + |x_2|^2) \wedge \partial\bar{\partial}\log(|x_1|^2 + |x_2|^2) , \quad (9.24)$$

where x_i are the coordinates on \mathbb{C}^2 . Here one has to use the same formalism as in section 7.1.1. The Cauchy kernel β can be lifted to the blow-up space $B_{\mathbb{C}^2}$ using the blow-down map $\pi : B_{\mathbb{C}^2} \rightarrow \mathbb{C}^2$ as

$$\pi^*\beta = i\partial\log u_i \wedge \omega_{FS} , \quad (9.25)$$

where u_i is identified with the coordinate on the fiber of $B_{\mathbb{C}^2}$ viewed as a line bundle over \mathbb{P}^1 , and ω_{FS} is the Fubini-Study metric on \mathbb{P}^1 defined below (9.21). Hence, we see that \hat{H}_3 as defined in (9.21) is not a form on the full space $B_{\mathbb{C}^2}$. Rather the $(2,1)$ -part of \hat{H}_3 is seen to be the pull-back of the Cauchy kernel (9.24), $\hat{H}_3^{(2,1)} = -i\partial\hat{J} = \pi^*\beta$. One readily evaluates

$$d\hat{H}_3 = \pi^*d\beta = \pi^*\delta_{\{0\}} = \omega_{FS} \wedge \delta_E , \quad (9.26)$$

where δ_E is the delta-current localizing on the exceptional divisor E as in section 7.4. Therefore, we can reduce the integrals involving the so-defined \hat{H}_3 to chain integrals over a one-chain ending on $\{0\}$ in \mathbb{C}^2 since

$$\int_\Gamma \gamma = \int_{\mathbb{C}^2} \gamma \wedge \bar{\beta} = \int_{B_{\mathbb{C}^2}} \pi^*\gamma \wedge \hat{H}_3^{(1,2)} = \frac{1}{2} \int_{B_{\mathbb{C}^2}} \pi^*\gamma \wedge (\hat{H}_3 + id\hat{J}) , \quad (9.27)$$

for a compactly supported $(1,0)$ -form γ . Note that this is the analog of the computation performed in section 7.4, where we have shown how the five-brane superpotential is translated to a superpotential on the blow-up space \hat{Z}_3 . We are now in the position to introduce an $SU(3)$ -structure on the open manifold $\hat{Z}_3 - E$.

9.3.2 The $SU(3)$ -Structure on the Open Manifold $\hat{Z}_3 - E$

We are now in the position to discuss the $SU(3)$ -structure on the open manifold $\hat{Z}_3 - E$ by introducing forms $\hat{\Omega}, \hat{J}$ of type $(3,0)$ and $(1,1)$, respectively. We show that these forms satisfy

$$d\hat{\Omega} = 0 , \quad d\hat{J} = \mathcal{W}_4 \wedge \hat{J} + \mathcal{W}_3 , \quad (9.28)$$

for non-trivial \mathcal{W}_4 and \mathcal{W}_3 . Note that $\hat{\Omega} = \pi^*\Omega$ has been already discussed in detail in section 7.3. It was noted that $\hat{\Omega}$ has a zero along E , but is a well-defined, nowhere vanishing form on the open manifold $\hat{Z}_3 - E$. The basic idea to introduce \hat{J} is to extend the definition (9.23) to

$\hat{Z}_3 - E$ by using the sections and metric on \hat{Z}_3 defined in section 9.2. It is further illuminating to view the construction of subsection 9.3.1 as a local model for $N_p\Sigma$.

Let us recall that on patches $\hat{U}_{2\epsilon}$ one can introduce the holomorphic sections $\sigma_{(0)}$ and $\sigma_{(1)}, \sigma_{(2)}$ of $\mathcal{L} = \mathcal{O}_{\hat{Z}_3}(E)$ as in (9.10). One recalls that $\sigma_{(0)}$ is a global section which has zeros along E just as the form $\hat{\Omega}$. We therefore work on $\hat{Z}_3 - E$. Moreover, we first consider a situation where a supersymmetric five-brane curve has been blown up, and hence the flux \hat{H}_3 on \hat{Z}_3 is supersymmetric and satisfies $i(\bar{\partial} - \partial)\hat{J} = \hat{H}_3$. Following the steps of subsection 9.3.1 we first introduce the B-fields

$$\hat{U}_{2\epsilon}^{(i)} - E : \quad \hat{B}_i = \frac{1}{2\pi} \text{Re}(\partial \log \|\sigma_{(i)}\|^2 \wedge \partial \log \|\sigma_{(0)}\|^2) , \quad (9.29)$$

where $\hat{U}_{2\epsilon}^{(i)}$ are the patches which cover Σ and the blow-up \mathbb{P}^1 's such that $\sigma_{(i)}$ is well-defined. Note that this implies that we have to refine the cover $\hat{U}_{2\epsilon}^{(i,\alpha)}$ as in section 7.2.1. For simplicity we will drop these additional indices in the following. Evaluating $\hat{H}_3 = d\hat{B}_i$ this implies

$$\begin{aligned} \hat{U}_{2\epsilon}^{(i)} - E : \quad \hat{H}_3 &= -\frac{i}{2} \Theta \wedge (\bar{\partial} - \partial) \log \left(\frac{\|\sigma_{(0)}\|^2}{\|\sigma_{(i)}\|^2} \right) \\ &= -\frac{i}{2} \Theta \wedge (\bar{\partial} - \partial) \log |z_i^{(i)}|^2 , \end{aligned} \quad (9.30)$$

where $\frac{i}{2}\Theta$ is the Chern curvature of \mathcal{L} as introduced in section 9.2, and we inserted the sections as given in (9.10) using local coordinates $z_i^{(i)} = y_i$ in $\hat{U}_{2\epsilon}^{(i)}$. Note that E is given in $\hat{U}_{2\epsilon}^{(i)}$ as $z_i^{(i)} = 0$ and has been excluded from $\hat{U}_{2\epsilon}^{(i)}$. One can further evaluate $\hat{B}_2 - \hat{B}_1 = F_{21}$ on the overlap $\hat{U}_{2\epsilon}^{(i)} \cap \hat{U}_{2\epsilon}^{(i)}$ and show in local coordinates that $dF_{21} = 0$.

With this preparation one next defines \hat{J} using the above supersymmetric flux. One finds the natural extension of (9.23) given by

$$\hat{U}_{2\epsilon}^{(1)} - E : \quad \hat{J} = \pi^* J - (1 - \log |y_1|^2) \Theta , \quad (9.31)$$

where J is the Kähler form on Z_3 and π is the blow-down map. In order to transform this \hat{J} into the other patches $\hat{U}_{2\epsilon}^{(2)}$ covering \mathbb{P}^1 one transforms \hat{J} as a differential form. Note that Θ vanishes outside $\cup \hat{U}_{2\epsilon}$ and one finds

$$\hat{Z}_3 - \cup \hat{U}_{2\epsilon} : \quad \hat{J} = \pi^* J . \quad (9.32)$$

In other words, the departure from the original Kähler structure only arises in a small neighborhood around E . Using the explicit form of \hat{J} one checks that $d\hat{J}$ has non-trivial torsion classes $\mathcal{W}_3, \mathcal{W}_4$ in (9.28). In conclusion one finds that near E we introduced a non-Kähler geometry on the open manifold.

Let us close this section by noting that the constructed structure $(\hat{J}, \hat{\Omega})$ and \hat{H}_3 are only well-defined differential forms on the open manifold $\hat{Z}_3 - E$. Furthermore, one finds on $\hat{Z}_3 - E$ that $d\hat{H}_3 = 2i\partial\bar{\partial}\hat{J} = 0$, which implies that there is no source term for the blown-up five-brane. To include such a source one has to work on the whole manifold \hat{Z}_3 using currents as in section 7.4. One then finds

$$d\hat{H}_3 = 2i\partial\bar{\partial}\hat{J} = \delta_E \wedge \Theta , \quad (9.33)$$

in accord with (7.57). Analog to (9.27) one also rewrites the chain integral on Z_3 to \hat{Z}_3 as

$$\int_{\Gamma} \chi = \int_{\hat{Z}_3} \pi^* \chi \wedge (\hat{H}_3 + id\hat{J}) , \quad (9.34)$$

where Γ is a three-chain ending on Σ . Here we have used $\hat{H}_3 = i(\bar{\partial} - \partial)\hat{J}$ for a supersymmetric configuration. Clearly, upon varying the complex structure of \hat{Z}_3 this relation will no longer hold, since $(\hat{\Omega}, \partial\hat{J})$ are still forms of type $(3,0)$ and $(2,1)$ in the new complex structure, respectively. The flux \hat{H}_3 , however, is fixed as a form but changes its type under a complex structure variation. Thus in the supersymmetric configuration the complex structure on \hat{Z}_3 is adjusted so that \hat{H}_3 is of type $(2,1) + (1,2)$.

Note that the described structure is not yet satisfying, since $(\hat{\Omega}, \hat{J}, \hat{H}_3)$ are no well-defined differential forms on all of \hat{Z}_3 . In the next subsection we resolve this issue by proposing a redefinition which allows us to work with differential forms on the full space \hat{Z}_3 .

9.3.3 The SU(3)-Structure on the Manifold \hat{Z}_3

In this subsection we will finally completely resolve the five-brane into a non-Kähler geometry. Recall that so far we had to work on the open manifold if we wanted to use forms, while on \hat{Z}_3 we had to use currents due to the singularities of \hat{H}_3 and \hat{J} along the exceptional divisor E . In the following we will introduce an $SU(3)$ -structure on \hat{Z}_3 by specifying a new $(1,1)$ -form J' , a non-singular three-form flux H'_3 and a non-vanishing $(3,0)$ -form Ω' .³

To begin with we note that there exists no holomorphic non-vanishing $(3,0)$ form on \hat{Z} since $K\hat{Z}_3$ is non-trivial. Hence, Ω' has to be non-holomorphic and we will find non-vanishing torsion classes $\mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5$ such that

$$d\Omega' = \bar{\mathcal{W}}_5 \wedge \Omega' , \quad dJ' = \mathcal{W}_4 \wedge J' + \mathcal{W}_3 . \quad (9.35)$$

The basic idea to define Ω' and J' is rather simple. We first note that \hat{H}_3 and $d\hat{J}$ have a pole of order one along E , while $\hat{\Omega}$ has a first order zero along E . Then we want to cancel the zero against the pole and introduce (J', Ω') such that

$$\int_{\hat{Z}_3 - E} (\hat{H}_3 + id\hat{J}) \wedge \hat{\Omega} = \int_{\hat{Z}_3} (H'_3 + idJ') \wedge \Omega' , \quad (9.36)$$

which is essential in the matching of the superpotentials.

In order to implement this cancellation we note that $d\hat{J}$ is proportional to $d|y_1|/|y_1|$ in $\hat{U}_\epsilon^{(1)}$. We thus define

$$\hat{U}_\epsilon^{(1)} : \quad \Omega' = \frac{1}{|y_1|} \hat{\Omega} , \quad (9.37)$$

³Recent constructions of non-Kähler geometries can be found in ref. [279,280]. The constructions of [279,280] share sensible similarities with our approach. We hope to come back to this issue in future works.

and transform Ω' to the other patches like a differential form. Note that Ω' no longer admits a zero along E , but also is no longer holomorphic. Evaluating $d\Omega'$ one finds a non-vanishing real $\mathcal{W}_5 = -d \log |y_1|$ in $\hat{U}_\epsilon^{(1)}$. Let us stress, however, that by construction of \hat{Z}_3 ,

$$\hat{Z}_3 - \cup \hat{U}_{2\epsilon} : \quad \Omega' = \pi^* \Omega, \quad \mathcal{W}_5 = 0 . \quad (9.38)$$

This implies that we need to ‘glue in’ the non-holomorphic dependence of Ω' using a partition of unity for $(\hat{Z}_3 - \cup \hat{U}_\epsilon, \hat{U}_{2\epsilon})$.

Let us comment on a more global approach to define Ω' on the blow-up \hat{Z}_3 realized as in (7.32). Instead of starting with $\hat{\Omega}$ we begin with a globally defined and no-where vanishing $\tilde{\Omega}$. However, this $\tilde{\Omega}$ is not a differential form, but rather a section of $\mathcal{L}^{-1} \otimes K \hat{Z}_3$ given by the residue expression

$$\tilde{\Omega} = \int_{\epsilon_1} \int_{\epsilon_2} \frac{\Delta}{PQ} . \quad (9.39)$$

Comparing this expression with $\hat{\Omega}$ given in (7.42) one immediately sees that $\tilde{\Omega}$ does not vanish along E . This $\tilde{\Omega}$ captures the complex structure dependence of the \hat{Z}_3 . It also satisfies the same Picard-Fuchs equations as $\hat{\Omega}$, and hence can be given in terms of the same periods. By the blow-up construction it depends on the complex structure deformations of Z_3 and the five-brane deformations. However, $\tilde{\Omega}$ transforms under \mathcal{L}^{-1} since the scaling-weight of Δ is not entirely canceled. In other words, if one insists on holomorphicity in the coordinates of \hat{Z}_3 one can either work with $\hat{\Omega}$ which has zeros along E , or with $\tilde{\Omega}$ which transforms also under \mathcal{L}^{-1} and thus is not a three-form. A natural global definition of the non-holomorphic Ω' is then given by the residue integral

$$\Omega' = \int_{\epsilon_1} \int_{\epsilon_2} \frac{\Delta}{PQ} \frac{h_i}{l_i} \frac{|l_i|}{|h_i|} . \quad (9.40)$$

Note that this expression is invariant under phase transformations of the global projective coordinates (l_1, l_2) and x_k entering the constraints h_i as in (7.32). In contrast Ω' would scale under real scalings of the global coordinates. However, these are fixed by the conditions defining the Kähler volumes. For example, the blow-up volume v_{bu} fixes the real scalings of the (l_1, l_2)

$$|l_1|^2 + |l_2|^2 = v_{\text{bu}} . \quad (9.41)$$

Similarly, the real scalings of the $h_i(x)$ are fixed by the definitions of the Kähler moduli of Z_3 . It would be very interesting to check if the conjecture (9.40) for a global Ω' can be used to derive Picard-Fuchs equations for \hat{Z}_3 , probably including anti-holomorphic derivatives.

Finally, let us turn to the definition of the non-Kähler form J' . In a patch $\hat{U}_\epsilon^{(1)}$ it is natural to identify

$$\hat{U}_\epsilon^{(1)} : \quad J' = \pi^* J - v_{\text{bu}}(1 - |y_1|)\Theta , \quad (9.42)$$

which agrees with \hat{J} given in (9.31) up to the logarithmic singularity along E . The non-trivial torsion classes from dJ' are again $\mathcal{W}_3, \mathcal{W}_4$. Since we expect the non-Kählerness to be localized

near E one has to have

$$\hat{Z}_3 - \cup \hat{U}_{2\epsilon} : \quad J' = \pi^* J . \quad (9.43)$$

This condition is indeed satisfied for any ansatz for J' which involves the curvature Θ of \mathcal{L} , since Θ vanishes outside a patch covering E . For a blow-up of a curve wrapped by a holomorphic five-brane one infers the flux H'_3 using $H'_3 = i(\bar{\partial} - \partial)J'$. H'_3 is a differential form on all of \hat{Z}_3 , and appears in the superpotential

$$W = \int_{\hat{Z}_3} (H'_3 + idJ') \wedge \Omega' , \quad (9.44)$$

which is now valid also for complex structure variation yielding a setup departing from a supersymmetric configuration. Clearly, by construction one can use (9.36) to equate the superpotential on the open manifold $\hat{Z}_3 - E$ with the expression (9.44).

Let us close this section by noting that the above construction should be considered as a first step in finding a fully back-reacted solution of the theory which dissolves the five-brane into flux. It will be interesting to extend these considerations to include the remaining supersymmetry conditions of [197] which are not encoded by a superpotential. In particular, this requires a careful treatment of the dilaton ϕ and the warp factor already in Z_3 , where e^ϕ becomes infinite near the five-brane.

Part IV

Conclusion and Appendix

Chapter 10

Conclusions and Outlook

In this work we have studied non-perturbative four-dimensional $\mathcal{N} = 1$ string compactifications with spacetime-filling branes. As a starting point of this analysis we have derived the effective action of a D5-brane in a generic Type IIB Calabi-Yau orientifold compactification. Then we have continued with a more involved study of non-perturbative string compactifications, namely heterotic string or heterotic M-theory Calabi-Yau threefold compactifications with five-branes and F-theory Calabi-Yau fourfold compactifications. We used techniques from mirror symmetry and heterotic/F-theory duality to calculate the exact flux and brane superpotentials for these compactifications. Furthermore we have presented a novel geometric description of five-brane dynamics by a dual blow-up threefold \hat{Z}_3 . We provided, by constructing an $SU(3)$ -structure on \hat{Z}_3 , first evidence for an interpretation of \hat{Z}_3 as a valid flux compactification of the Type IIB and heterotic string with flux due to the dissolved five-brane. We have verified this duality on the level of the effective flux and five-brane superpotentials, that we calculated from open-closed Picard-Fuchs equations on \hat{Z}_3 .

In chapter 2 we started with a systematic review of Calabi-Yau orientifold compactifications of Type II string theory. We readily focused on Type IIB orientifolds, that are $O3/O7$ - and $O5/O9$ -orientifolds, for which we reviewed the full $\mathcal{N} = 1$ effective action. This served as a preparation for our actual derivation of the full effective action of a spacetime-filling D5-brane wrapping an internal curve Σ in chapter 3, which contains the bulk $O5/O9$ -orientifold action in the case that all brane degrees of freedom are frozen out. After a brief discussion of D-branes in Calabi-Yau manifolds in section 3.1, we performed in section 3.2 the Kaluza-Klein reduction of the six-dimensional Dirac-Born-Infeld and Chern-Simons actions of the D5-brane coupled to the ten-dimensional bulk supergravity action in the most generic situations allowing background and brane fluxes. Then we read off the entire $\mathcal{N} = 1$ characteristic data in section 3.3. Most notably we derived the explicit corrections of the D5-brane fields to the bulk $O5/O9$ -fields to form new $\mathcal{N} = 1$ chiral coordinates, the D5-brane superpotential, a gauging of a bulk chiral multiplet by the D5-brane gauge field inducing a new D-term and the kinetic mixing of the bulk and D5-brane gauge fields, cf. appendix A for the combined bulk-brane gauge kinetic function. Of particular conceptual interest was the derivation of the $\mathcal{N} = 1$ scalar potential which was shown to consist of both F- and D-term contributions. One D-term needed to be canceled by the tension of the $O5$ -planes in order for the setup to be stable which reflects, on the level of the effective action, tadpole cancellation. The two further terms were

induced by gaugings of chiral fields, one of which encodes the BPS-calibration condition of equality of the NS–NS B-field with the D5-brane gauge flux and the other forbids non-trivial NS–NS three-form flux. In order to calculate the complete F-term potential in our purely bosonic reduction, we had to consider the couplings of four-dimensional non-dynamical three-forms. After performing a formal dualization of these fields into constants¹ we were able to derive the complete scalar potential in the presence of the D5-brane and of background R–R three-form flux F_3 . Furthermore we extended in section 3.4 the considerations of [60] to the full geometric deformation modes of the D5-brane, including an infinite set of massive modes. This led to a more complete study of the geometric $\mathcal{N} = 1$ open-closed deformation modes and to the derivation of some special relations between its light modes. In addition we showed that the massive modes are generically obstructed by a quadratic potential.

In the second part of this work on string dualities we have studied and calculated the holomorphic flux superpotential in F-theory compactifications on Calabi-Yau fourfolds using techniques from mirror symmetry and string dualities. This was used to deduce both the flux and seven-brane superpotential of the associated IIB theory as well as the superpotential of a dual heterotic compactification. More concretely we started with a systematic discussion of heterotic/F-theory duality in chapter 4. In our presentation of heterotic string and heterotic M-theory compactifications in section 4.1 we readily included non-perturbative five-branes on curves Σ . We put special emphasis on the small instanton transition of a smooth heterotic gauge bundles E to a singular, maximally localized limit, which is more appropriately described by a five-brane. This led in particular to the deduction of the heterotic five-brane superpotential from the holomorphic Chern-Simons functional, which localizes in the singular small instanton limit of E to a chain-integral over the Calabi-Yau form Ω . Then, after the introduction of the basic concepts and constructions of F-theory in section 4.2, we discussed in detail in section 4.3, that in the duality between the heterotic string on an elliptic Calabi-Yau threefold Z_3 and F-theory on a $K3$ -fibered elliptic fourfold X_4 horizontal heterotic five-branes are completely geometrized in F-theory via a blow-up along Σ . As argued further this implied a map of heterotic five-brane moduli and superpotentials, being a sum of the flux, Chern-Simons and five-brane superpotential, to complex structure moduli respectively the flux superpotential in F-theory. Computing the F-theory flux superpotential and counting F-theory complex structure moduli in comparison with the heterotic superpotentials and moduli thus served, on the one hand, as an ideal testing ground for heterotic/F-theory duality. On the other hand, the controllable geometric setup of the F-theory fourfold allowed to compute the heterotic superpotentials explicitly, as demonstrated in chapter 6 for the flux and five-brane superpotential.

The essential geometrical tools for this analysis are toric geometry, briefly reviewed in section 5.1, and mirror symmetry in combination with enumerative geometry as presented in

¹In fact, the correct interpretation of these constants being the flux quantum numbers of F_3 was given in [158] as labeling quantum mechanical states of the system.

chapter 5. Mirror symmetry on fourfolds (section 5.3) is in general less studied and understood as in the Calabi-Yau threefold case (section 5.2), which goes mainly back to the less restrictive $\mathcal{N} = 1$ structure of the fourfold complex structure moduli space, like e.g. the absence of $\mathcal{N} = 2$ special geometry, and an incomplete understanding of its global structure and monodromies. Essential progress in this direction was presented in section 5.3 where the analytic continuation to the universal conifold was performed for the example of the sextic fourfold. A novel \mathbb{Z}_2 -monodromy around the conifold divisor was observed that in turn allowed, by the requirement of integral monodromy, to determine the classical terms in the periods at the large complex structure/large volume point. This was essential for fixing an integral basis that is for example needed to specify a quantized four-flux which in turn induces the effective flux superpotential. Furthermore, we made use of the unified description of open and closed moduli in F-theory to explicitly compute in chapter 6, section 6.1, the flux and seven-brane superpotentials of the underlying Type IIB compactification from the F-theory flux superpotential for a selection of Calabi-Yau fourfold geometries, cf. also appendix C. For this matching, the enumerative meaning, both of the flux superpotentials on Calabi-Yau three- and fourfolds and of the brane superpotential, for the A-model was essential, as discussed in section 5.4. In particular we calculated both closed and open Gromov-Witten invariants of compact Calabi-Yau threefolds and compact branes that extend the well established results from local branes consistently. In addition we used in section 6.2 the physically independent explanation of heterotic/F-theory duality for the fact that the periods of the threefold Z_3 as well as the five-brane superpotential are contained in the fourfold periods on X_4 .

In the third part of this work we studied directly the dynamics of five-brane wrapped on curves Σ in a compact Calabi-Yau threefold Z_3 , where we focused on NS5-branes and D5-branes in $\mathcal{N} = 1$ heterotic and orientifold compactifications. We analyzed the geometric deformations of the five-brane curve Σ in Z_3 by replacing the geometry by a new threefold \hat{Z}_3 which is no longer Calabi-Yau and canonically obtained by blowing up along Σ in Z_3 . It was shown that this replacement naturally unifies the open-closed deformations of (Σ, Z_3) as pure complex structure deformations of \hat{Z}_3 . We used this description explicitly to derive open-closed Picard-Fuchs differential equations that are precisely solved by the effective flux and five-brane superpotential of the original compactification. In particular we showed that the blow-up threefold \hat{Z}_3 is in agreement with the familiar blow-up in F-theory, which is the dual of a heterotic five-brane in heterotic/F-theory duality. We concluded by constructing an $SU(3)$ -structure on \hat{Z}_3 , which provides first evidence to use \hat{Z}_3 as a consistent flux background, that naturally takes into account the five-brane backreaction.

In more detail we started in chapter 7 with a systematic and formal analysis of five-brane dynamics. We emphasized in section 7.1 that a rigorous treatment of the five-brane backreaction in the Bianchi identity for C_2 respectively B_2 naturally requires to work on the open manifold $Z_3 - \Sigma$ and also that the five-brane superpotential has to be treated more rigorously in the language of currents. Equipped with this mathematical observations we were

naturally led to the replacement of the original setup by the blow-up threefold \hat{Z}_3 in chapter 7.2. There we also presented an explicit construction of \hat{Z}_3 for Calabi-Yau hypersurfaces and the geometrization of closed and open deformations of (Z_3, Σ) on \hat{Z}_3 was explained, first in general and then via a concrete example of the quintic threefold with a curve Σ realized as a complete intersection. Finally in sections 7.3 and 7.4 we used the pullback $\hat{\Omega}$ of the Calabi-Yau form Ω on Z_3 to probe the complex structure of \hat{Z}_3 . This readily yielded open-closed Picard-Fuchs equations on \hat{Z}_3 that are precisely solved by both the flux superpotential as encoded by the closed periods of Ω , and the five-brane superpotential W_{brane} . In this context it was essential to note the general structure of the expected Picard-Fuchs differential equations and to rigorously formulate the lift of the five-brane superpotential from Z_3 to \hat{Z}_3 using currents.

While the proposal to study deformations of a holomorphic curve Σ in a complex variety Z_3 via the blown up manifold \hat{Z}_3 is very general, we concretely obtained in chapter 8 the corresponding open-closed Picard-Fuchs equations for the deformation problem in some generality for hypersurfaces in toric varieties in the presence of toric branes (section 8.2). We focused on two examples, namely branes in the mirror quintic in section 8.1 and in the mirror of the degree-18 hypersurface in $\mathbb{P}^4(1, 1, 1, 6, 9)$ in section 8.3. We noted that there exist maps from the unobstructed deformation problem of a complex higher genus curve Σ in Z_3 , which is realized as a complete intersection with the Calabi-Yau threefold Z_3 , to generically obstructed configurations of branes on rational curves or the involution brane studied in [111]. This map was defined around a critical locus in the moduli space of Σ where it degenerates holomorphically to e.g. the rational curves. Away from this locus one has to identify the rational curves using a non-holomorphic map which has branch cuts and the corresponding obstructions are encoded, upon wrapping these cycles by a five-brane, in a superpotential that we calculate from \hat{Z}_3 . This picture was further confirmed by noting that the discriminant components of the Picard-Fuchs system for \hat{Z}_3 factorizes into several components, each of which corresponding to the critical locus of an underlying deformation problem of a five-brane on an in general obstructed curve. We checked our results for the superpotential by extraction of integral disk instanton invariants at large volume that agreed with the available results in the literature or partially extend them. We concluded our calculations by a different cross-check in section 8.4, namely the application of the blow-up proposal to heterotic Calabi-Yau threefolds Z_3 that in addition admit an F-theory dual description. Indeed we were able to construct the dual F-theory fourfold X_4 as a toric complete intersection directly from the blow-up threefold \hat{Z}_3 of the heterotic string with five-branes, that was also realized as a toric complete intersection. Furthermore, we established a map of the heterotic flux and five-brane superpotential to the flux superpotential on X_4 by formally constructing the dual four-flux G_4 directly from \hat{Z}_3 .

In the final chapter 9 of this thesis we presented a proposal for formulating an $SU(3)$ -structure on the blow-up threefold \hat{Z}_3 , in order to view \hat{Z}_3 as a dual physical description of the backreacted five-brane. The basic idea was to first construct a no-where vanishing $(3, 0)$ -form Ω' , which was roughly obtained by canceling the first order zero of $\hat{\Omega}$ along the exceptional

divisor E and the first order pole of the exterior derivative of the three-form flux \hat{H}_3 along E with each other. This could only be achieved by introducing a non-holomorphic form Ω' , which is no longer closed near E . In addition to a local construction in sections 9.1 and 9.2 we proposed a global residuum representation for Ω' in the case that \hat{Z}_3 is given as a toric hypersurface in section 9.3. Simultaneously, this procedure yielded a smooth three-form flux H'_3 . In addition we defined a new $(1, 1)$ non-Kähler-form J' by smoothing out a logarithmic singularity in the pullback Kähler form \hat{J} , which is dictated by the supersymmetry condition $id^c \hat{J} = \hat{H}_3$. Finally, employing these definitions the complete effective superpotential takes the form $\int_{\hat{Z}_3} (H'_3 + idJ') \wedge \Omega'$ as familiar from flux compactifications on $SU(3)$ -structure manifolds.

We end our conclusion by mentioning some directions for future research. First, we note that the derived generic D5-brane effective action allows for various phenomenological applications. Examples include the study of D5-branes for mechanism of inflation, e.g. via D5-branes on the vanishing \mathbb{P}^1 of the conifold [281] or via the D5-brane Wilson line moduli [282], or for dynamical supersymmetry breaking, e.g. by extending the non-compact setups of [283] to D5-branes on vanishing two-cycles in compact Calabi-Yau orientifolds. A different application is provided by explicit GUT model building in Type IIB compactifications, where it would be of particular importance to include matter from intersecting D5-branes using similar techniques as developed for intersecting D7-branes [284, 285]. Furthermore, the effective action allows also some general conclusions about moduli stabilization. From the independence of the Type IIB flux² and brane superpotential on the dilaton multiplet S as well as from the positive definite induced F-term scalar potential due to the no-scale structure of the $\mathcal{N} = 1$ data we deduce that the dilaton can not be stabilized perturbatively. It might be interesting to break the no-scale structure due to non-perturbative corrections or by treating the backreaction of the fluxes and the D5-brane on the geometry more thoroughly, which naturally leads to more general non-Calabi-Yau backgrounds or strong warp effects.

Possible applications of the knowledge of the full F-theory flux superpotential are provided by the use to systematically calculate g_S -corrections to the underlying type IIB compactification or to directly stabilize complex structure moduli in F-theory. In the light of GUT model building in F-theory this might allow to analyze whether phenomenologically preferred settings, e.g. like in the study of local geometries initiated in [286–288], can indeed be stabilized by fluxes in a global setup. Furthermore, one can use the explicit knowledge of the F-theory flux superpotential to analyze the question of moduli stabilization in the dual heterotic string. This is of particular interest since in F-theory all complex structure moduli can generically be stabilized by fluxes which in the heterotic dual would correspond to a stabilization of complex structure, bundle and five-brane moduli. In addition one might wonder about extracting, like in [99], the heterotic Chern-Simons functional for non-trivial bundles E , e.g. to analyze its integral structure, or to deduce Kähler corrections from the elliptic fiber of Z_3 to the heterotic superpotential. Furthermore it would be very interesting to extend the calculations to

²The usual dilaton-dependent flux combination $G_3 = F_3 - \tau H_3$ reduces to F_3 in $O5/O9$ -orientifolds.

the heterotic Kähler potential, which might be a tractable task in heterotic/F-theory duality since the F-theory Kähler potential is computable as a function of the fourfold periods. More conceptually one might analyze the structure of the fourfold periods at other points of the moduli space including a general analysis of monodromies. Physical interpretations of their different structure compared to the threefold case might shed some light on additional massless states that are tightly related to singularities in the moduli space. Furthermore it would be challenging to identify and calculate the geometric quantity in F-theory that encodes the second holomorphic $\mathcal{N} = 1$ coupling, namely the seven-brane gauge kinetic function. This would be of particular interest in the context of open mirror symmetry by extending the local results of [289] and probably the use of the Bergmann kernel to compact Calabi-Yau compactifications.

Finally it would be very interesting to improve the understanding of the blow-up threefold \hat{Z}_3 as defining a string background of a backreacted five-brane. This includes a more thorough treatment of the dilaton and the warp factor, that both might no more be constant, and a better understanding of the size modulus of the blow-up \mathbb{P}^1 . This is essential in order to establish a physical duality, where in particular all new fields need to be identified with new quantum degrees of freedom. In the heterotic compactifications, for example, it is natural to identify the Kähler modulus of the \mathbb{P}^1 with the positions of the heterotic five-branes in the interval of heterotic M-theory. It would be interesting to make this map explicit and clarify the interpretation of the blow-up mode in the Type II setups. Moreover, the computation of disk instanton numbers via the superpotential is only the simplest check of this duality. One can attempt to compute amplitudes of higher genus and with more boundaries like the annulus. It would be of interest to investigate if these can be derived on the blow-up space \hat{Z}_3 as well. From a more technical point of view it would further be essential to gain a more complete picture of the open-closed field space as encoded by the complex structure moduli space of \hat{Z}_3 . This might require to find a general method to fix the integral symplectic basis of the third cohomology on \hat{Z}_3 which might allow an identification of the open-closed mirror map similar to the Calabi-Yau threefold case. The knowledge of the flat open coordinates is for example particularly relevant to determine critical values of the superpotential in the open-closed moduli space.

Appendix A

Effective Actions

A.1 The $\mathcal{N} = 2$ Gauge-Kinetic Coupling Function

In this appendix we collect some useful formulas applied in the derivation of the $\mathcal{N} = 1$ scalar potential and the $\mathcal{N} = 1$ gauge-kinetic function for the bulk vectors. Both quantities depend on the complex structure deformations of the internal Calabi-Yau manifold Z_3 . In the underlying $\mathcal{N} = 2$ theory the complex structure deformations are in vector multiplets together with vectors \underline{V} in the expansion $C_4 = V^K \alpha_K + \dots$ where $K = 0, \dots, h^{(2,1)}$ labels the symplectic basis of $H^3(Z_3, \mathbb{Z})$ and V^0 is identified with the graviphoton in the $\mathcal{N} = 2$ gravity multiplet, cf. table 2.4. The four-dimensional $\mathcal{N} = 2$ action for the vectors \underline{V} is of the form

$$S_{\underline{V}} = \int \left[\frac{1}{4} \text{Im} \mathcal{M}_{KL} dV^K \wedge *dV^L + \frac{1}{4} \text{Re} \mathcal{M}_{KL} dV^K \wedge dV^L \right]. \quad (\text{A.1})$$

The complex matrix \mathcal{M}_{KL} can be expressed in terms of the periods (X^K, \mathcal{F}_K) in the expansion $\Omega = X^K \alpha_K - \mathcal{F}_K \beta^K$ as

$$\mathcal{M}_{KL} = \bar{\mathcal{F}}_{KL} + 2i \frac{(\text{Im} \mathcal{F})_{KL} X^M (\text{Im} \mathcal{F})_{LN} X^N}{X^N (\text{Im} \mathcal{F})_{NM} X^M}, \quad (\text{A.2})$$

where $\mathcal{F}_{KL} = \partial_{X^K} \mathcal{F}_L$. To derive this expression one uses the natural scalar product on the cohomology $H^3(Z_3)$. This can be encoded in the following matrix [290]

$$E = \begin{pmatrix} \int \alpha_K \wedge * \alpha_L & \int \alpha_K \wedge * \beta^L \\ \int \beta^K \wedge * \alpha_L & \int \beta^K \wedge * \beta^L \end{pmatrix} = \begin{pmatrix} -(A + BA^{-1}B) & -BA^{-1} \\ -A^{-1}B & -A^{-1} \end{pmatrix}, \quad (\text{A.3})$$

where $A = \text{Im} \mathcal{M}$ and $B = \text{Re} \mathcal{M}$. A matrix of this form can be easily inverted where the inverse matrix reads

$$E^{-1} = \begin{pmatrix} -A^{-1} & A^{-1}B \\ BA^{-1} & -(A + BA^{-1}B) \end{pmatrix} = \begin{pmatrix} \int \beta^K \wedge * \beta^L & -\int \beta^K \wedge * \alpha_L \\ -\int \alpha_K \wedge * \beta^L & \int \alpha_K \wedge * \alpha_L \end{pmatrix}. \quad (\text{A.4})$$

These matrices will be used in the derivation of the $\mathcal{N} = 1$ scalar potential in section 3.2.4, where the indices $K = 0, \dots, h_+^{(2,1)}$ are in the positive eigenspace $H_+^3(Z_3)$. The complex matrix \mathcal{M} will also appear in the $\mathcal{N} = 1$ gauge-kinetic coupling function in section 3.3.3 where now the indices $k = 1, \dots, h_-^{(2,1)}$ are in the negative eigenspace $H_-^3(Z_3)$.

A.2 Kinetic Mixing of Bulk and Brane Gauge Fields

The reduction of the Chern-Simons action to the effective Lagrangian (3.48) contains mixing terms between the bulk vector fields \underline{V} , \underline{U} and the D5-brane U(1)-field F . Since the vectors \underline{U} are the magnetic duals to the vector \underline{V} , a dualization procedure has to be performed in order to reveal the effective action for the propagating fields. Here, we will present this dualization in detail and demonstrate how it affects the kinetic term of the D5-brane vector F such that a further intertwining between open and closed moduli appears.

First, we have to collect all terms of the effective action that are relevant for the dualization procedure. These are the kinetic terms from the bulk vectors \underline{V} , \underline{U} of the bulk supergravity action, the kinetic as well as instanton term of the D5-brane vector F given in the Dirac-Born-Infeld action (3.39) and the Chern-Simons action (3.48), respectively, and mixing terms between bulk and brane vectors of (3.48). Thus, the starting point of the dualization is the action

$$S_{\text{vec}} = - \int \left[\frac{1}{8} d\vec{V}^T \wedge *E d\vec{V} + \frac{1}{2} \mu_5 \ell^2 (v^\Sigma e^{-\phi} F \wedge *F - c^\Sigma F \wedge F) + \frac{1}{2} \mu_5 \ell \vec{\mathcal{N}}^T d\vec{V} \wedge F \right], \quad (\text{A.5})$$

where we again used the matrix E introduced in (3.53) and the convenient shorthand notation

$$\vec{V} = \begin{pmatrix} \underline{V} \\ \underline{U} \end{pmatrix}, \quad \vec{\mathcal{N}} = \hat{\zeta}^{\mathcal{A}} \begin{pmatrix} \mathcal{N}_{\mathcal{A}k} \\ \mathcal{N}_{\mathcal{A}}^l \end{pmatrix} = \begin{pmatrix} \mathcal{N}_k \\ \mathcal{N}^l \end{pmatrix}. \quad (\text{A.6})$$

Next we have to add the Lagrange multiplier term $\frac{1}{4} dV^k \wedge F_k$ to the Lagrangian (A.5) in order to integrate out the magnetic field strength $F_k = dU_k$. However, the equations of motion for the vectors \underline{V} and their duals \underline{U} are not compatible with each other after the naive addition of this term. In order to restore consistency of the equations of motion, we have to shift the field strengths dV^k , dU_k in the kinetic terms appropriately by

$$dV^k \rightarrow \tilde{F}^k := dV^k - 2\mu_5 \ell \mathcal{N}^k F, \quad dU_l \rightarrow \tilde{F}_l := dU_l - 2\mu_5 \ell \mathcal{N}_l F. \quad (\text{A.7})$$

Now, we can integrate out the magnetic dual \tilde{F}_l consistently and obtain

$$\begin{aligned} S_{\text{vec}} &= \int \left[\frac{1}{4} \text{Im} \mathcal{M}_{kl} F^k \wedge *F^l + \frac{1}{4} \text{Re} \mathcal{M}_{kl} F^k \wedge F^l \right. \\ &\quad - \frac{1}{2} \mu_5 \ell^2 \left((v^\Sigma e^{-\phi} + 2\mu_5 \text{Im} \mathcal{M}_{kl} (N^k + \bar{N}^k)(N^l + \bar{N}^l)) F \wedge *F \right. \\ &\quad + (c^\Sigma + i\mu_5 \text{Im} \mathcal{M}_{kl} (N^k N^l - \bar{N}^k \bar{N}^l)) F \wedge F \\ &\quad \left. \left. + \mu_5 \ell (\text{Im} \mathcal{M}_{kl} *F + \text{Re} \mathcal{M}_{kl} F) \wedge F^k (N^l + \bar{N}^l) \right] \right]. \quad (\text{A.8}) \end{aligned}$$

Here we introduced the complex fields

$$N^k = \int_{\Sigma_-} \zeta_{\lrcorner} \beta^k, \quad \bar{N}^k = \int_{\Sigma_-} \bar{\zeta}_{\lrcorner} \beta^k. \quad (\text{A.9})$$

The crucial point of this dualization is the change of the gauge-kinetic term in (A.8) compared to the form in (A.5) before dualization.

A.3 Derivation of the F-term Potential: Massless Modes

The calculation of the F-term contribution of the scalar potential (2.38) using the superpotential (3.68) and Kähler potential (3.65) is straightforward but tedious. To simplify this computation it is convenient to exploit one of the shift symmetries of the Kähler potential $S \rightarrow S + i\Lambda$ and dualize the chiral multiplet with bosonic scalar S into a linear multiplet with bosonic components (L, C_2) . Here L is a real scalar associated to $\text{Re}S$ while C_2 is a two-form dual to $\text{Im}S$. In the context of $O5$ -orientifolds without D5-brane moduli this dualization has been carried out in refs. [128, 130], and we refer the reader to these references for more details on the linear multiplet formalism and references. Here we will be mainly interested in the scalar potential in the new scalar variables L and $M^I = (\underline{P}, \underline{a}, \underline{t}, \underline{\zeta})$. First we express the Kähler potential (3.65) in terms of the new variables $L = -K_S = \frac{1}{2}e^\phi \mathcal{V}^{-1}$ and M^I such that

$$K = -\ln \left[-i \int \Omega \wedge \bar{\Omega} \right] - \ln \left[\frac{1}{48} \mathcal{K}_{\alpha\beta\gamma} \Xi^\alpha \Xi^\beta \Xi^\gamma \right] + \ln[L] , \quad (\text{A.10})$$

where Ξ^α is given in (3.67). The kinetic terms in the effective action with a linear multiplet are then obtained as derivatives of the kinetic potential

$$\tilde{K}(L, M^I, \bar{M}^I) = K + (S + \bar{S}) \cdot L , \quad (\text{A.11})$$

where $S + \bar{S} \equiv (S + \bar{S})(L, M^I)$ is now a function of (L, M^I) . In fact, this is just a Legendre transformation of the function K with respect to $S + \bar{S}$ to obtain the Legendre transformed \tilde{K} as a function of L . In terms of this data the scalar potential takes the general form

$$V = e^K (\tilde{K}^{IJ} D_I W D_{\bar{J}} \bar{W} - (3 - LK_L) |W|^2) , \quad (\text{A.12})$$

where $D_I W = \partial_I W + K_I W$ and $K_L = \partial_L K$. Note that in front of $|W|^2$ as well as in $D_I W$ only the derivatives of the Kähler potential (A.10) appear.

With this formalism at hand we evaluate the scalar potential. We first take derivatives of (A.10) and (A.11) such that

$$K_{t_\alpha} = -\frac{e^\phi}{4\mathcal{V}} \mathcal{K}_\alpha , \quad K_{P_a} = 0 , \quad K_{a_I} = 0 , \quad K_{\zeta^A} = \frac{1}{2} \mu_5 e^\phi \mathcal{G}_{A\bar{B}} \bar{\zeta}^{\bar{B}} , \quad (\text{A.13})$$

when $K \equiv K(L, M^I)$ is viewed as a function of L instead of $S + \bar{S}$.¹ Similarly we obtain

$$\tilde{K}_{t_\alpha} = \frac{e^\phi}{4\mathcal{V}} (\mathcal{K}_{\alpha ab} \mathcal{B}^a \mathcal{B}^b - \mathcal{K}_\alpha) , \quad \tilde{K}_{P_a} = -\frac{e^\phi}{2\mathcal{V}} \mathcal{B}^a , \quad \tilde{K}_{a_I} = \frac{\mu_5 \ell^2 e^\phi}{\mathcal{V}} \mathcal{C}^{I\bar{J}} \bar{a}_{\bar{J}} , \quad \tilde{K}_{\zeta^A} = K_{\zeta^A} \quad (\text{A.14})$$

where we made use of the solution

$$S + \bar{S} = \frac{1}{L} - \frac{1}{4} (\text{Re}\Theta)^{ab} (P + \bar{P})_a (P + \bar{P})_b + 2\mu_5 \ell^2 \mathcal{C}^{I\bar{J}} a_I \bar{a}_{\bar{J}} . \quad (\text{A.15})$$

¹It is a basic fact that a Legendre transformation leaves the derivatives with respect to the untransformed variables invariant, i.e. $\partial_{M^I} K = \partial_{M^I} \tilde{K}$ where both sides have to be evaluated in the same variables *after* differentiation.

From this we can easily determine the metric $\tilde{K}_{I\bar{J}}$ for the remaining fields which is block-diagonal with one block $\tilde{K}_{a_I\bar{a}_J} = \mu_5 \ell^2 e^\phi \mathcal{V}^{-1} C^{I\bar{J}}$ for the Wilson lines and another block of the following type

$$\tilde{K}_{I\bar{J}} = \begin{pmatrix} A + B^\dagger G B & -B^\dagger G & 0 \\ -G B & G + D^\dagger C D & D^\dagger C \\ 0 & C D & C \end{pmatrix} \quad (\text{A.16})$$

for the moduli $(\zeta, \underline{t}, \underline{P})$. Its inverse $\tilde{K}^{I\bar{J}}$ is then given by

$$\tilde{K}^{I\bar{J}} = \begin{pmatrix} A^{-1} & A^{-1} B^\dagger & -A^{-1} B^\dagger D^\dagger \\ B A^{-1} & G^{-1} + B A^{-1} B^\dagger & -(G^{-1} + B A^{-1} B^\dagger) D^\dagger \\ -D B A^{-1} & -D(G^{-1} + B A^{-1} B^\dagger) & C^{-1} + D(G^{-1} + B A^{-1} B^\dagger) D^\dagger \end{pmatrix}. \quad (\text{A.17})$$

Here, we abbreviated the various matrices as follows,

$$A = \frac{1}{2} e^\phi \mu_5 \mathcal{G}_{A\bar{B}}, \quad G = e^{2\phi} G_{\alpha\beta}, \quad B = \mu_5 \mathcal{L}_{A\bar{B}}^\alpha \bar{\zeta}^{\bar{B}}, \quad C = -\frac{e^\phi}{2\mathcal{V}} (\text{Re}\Theta)_{ab}, \quad D = \frac{1}{2} \mathcal{K}_{ab\alpha} \mathcal{B}^b, \quad (\text{A.18})$$

where the matrix \mathcal{G} is defined in (3.42) and we introduced the Kähler metric on the Kähler moduli space in (2.34),

$$G_{\alpha\beta} = \frac{1}{4\mathcal{V}} \left(\frac{\mathcal{K}_\alpha \mathcal{K}_\beta}{4\mathcal{V}} - \mathcal{K}_{\alpha\beta} \right). \quad (\text{A.19})$$

Now we use this to compute the F-term scalar potential. First we note the no-scale structure of K and W . The superpotential does not depend on the moduli $(S, \underline{a}, \underline{P})$ as well as on \underline{t} such that the covariant derivative $D_I = \partial_I + K_I$ reduces just to K_I . Moreover, for the dual linear multiplet to S we find a contribution $1 \cdot |W|^2$ to the scalar potential V which is an immediate consequence of $K_L L = 1$ in (A.12). The block matrix for the Wilson lines \underline{a} does not contribute to V since $K_{a_I} = 0$. However, the block for the moduli $(\zeta, \underline{t}, \underline{P})$ yields a contribution of the form

$$D_{(\zeta, \underline{t}, \underline{P})} W D_{(\bar{\zeta}, \bar{\underline{t}}, \bar{\underline{P}})} \bar{W} \tilde{K}^{(\zeta, \underline{t}, \underline{P})(\bar{\zeta}, \bar{\underline{t}}, \bar{\underline{P}})} = \frac{\mathcal{K}_\alpha (G_{\text{ks}})^{\alpha\beta} \mathcal{K}_\beta}{(4\mathcal{V})^2} |W|^2 + 2\mu_5 \left(\int_{\Sigma_+} s_{A\perp} \Omega \int_{\Sigma_+} \bar{s}_{\bar{B}\perp} \bar{\Omega} \right) e^{-\phi} \mathcal{G}^{A\bar{B}}. \quad (\text{A.20})$$

Using the various intersection matrices $v^\alpha = \int J \wedge \tilde{\omega}^\alpha$, \mathcal{K}_α , $\mathcal{K}_{\alpha\beta}$ and its formal inverse $\mathcal{K}^{\alpha\beta}$ as well as the inverse metric

$$G_{\text{ks}}^{\alpha\beta} = 2v^\alpha v^\beta - 4\mathcal{V} \mathcal{K}^{\alpha\beta}, \quad (\text{A.21})$$

we deduce the useful relation

$$\mathcal{K}_\alpha (G_{\text{ks}})^{\alpha\beta} \mathcal{K}_\beta = (8\mathcal{V})^2 v^\alpha (G_{\text{ks}})_{\alpha\beta} v^\beta = 3(4\mathcal{V})^2. \quad (\text{A.22})$$

Finally, we obtain the F-term contribution to the scalar potential V of the form

$$V = \frac{ie^{4\phi}}{2\mathcal{V}^2 \int \Omega \wedge \bar{\Omega}} \left[|W|^2 + D_{z^\kappa} W D_{\bar{z}^{\bar{\kappa}}} \bar{W} G^{\kappa\bar{\kappa}} + 2\mu_5 e^{-\phi} \mathcal{G}^{A\bar{B}} \int_{\Sigma_+} s_{A\perp} \Omega \int_{\Sigma_+} \bar{s}_{\bar{B}\perp} \bar{\Omega} \right] \quad (\text{A.23})$$

A.4 Derivation of the F-term Potential: Massive Modes

This section provides the necessary background to perform the calculation of the F-term potential (3.83). The following calculation extends the analysis made in chapter 3 [60] for deformations associated to holomorphic sections $H^0(\Sigma, N_{Z_3}\Sigma)$ to the case of the infinite dimensional space of deformations $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$. In particular, the Kähler metric for the open string deformations is accordingly generalized.

First we need the general form of the Kähler potential of the D5-brane action that is given by [60]

$$K = -\ln \left[-i \int \Omega \wedge \bar{\Omega} \right] + K_q, \quad K_q = -2 \ln \left[\sqrt{2} e^{-2\phi} \mathcal{V} \right], \quad (\text{A.24})$$

that immediately implies that

$$e^K = \frac{ie^{4\phi}}{2\mathcal{V}^2 \int \Omega \wedge \bar{\Omega}}. \quad (\text{A.25})$$

In order to evaluate the Kähler metric the potential K has to be expressed as a function of the $\mathcal{N} = 1$ complex coordinates. For the purpose of our discussion in section 3.4 we only need the part of the Kähler metric for the open string deformations u^a .

We straight forwardly extend the Kähler metric deduced in section 3.3.1 for the fields associated to $H^0(\Sigma, N_{Z_3}\Sigma)$ to the infinite dimensional space $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$. This is possible since the condition of holomorphicity of sections does not enter the calculations of section 3.2. Thus, we obtain the inverse Kähler metric

$$K^{a\bar{b}} = 2\mu_5^{-1} e^{-\phi} \mathcal{G}^{a\bar{b}} \quad (\text{A.26})$$

cf. appendix A.3 for the process of inversion of the full Kähler metric, where the matrix $\mathcal{G}^{a\bar{b}}$ is the inverse of

$$\mathcal{G}_{a\bar{b}} = \frac{-i}{2\mathcal{V}} \int_{\Sigma} s_{a\lrcorner} \bar{s}_{b\lrcorner} (J \wedge J). \quad (\text{A.27})$$

Here we introduce a basis s_a of $\mathcal{C}^\infty(\Sigma, N_{Z_3}\Sigma)$ so that a generic section s enjoys the expansion $s = u^a s_a$. Next we Taylor expand the superpotential (3.72) in the open string deformations u^a around the supersymmetric vacuum of the holomorphic curve $\Sigma = \Sigma_h$ as

$$W_{\text{brane}} = \int_{\Gamma_u} \Omega \equiv \int_{\Sigma_0}^{\Sigma + \delta\Sigma_u} \Omega = \int_{\Gamma_h} \Omega + \int_{\Sigma} u_{\lrcorner} \Omega + \frac{1}{2} u^a u^b \int_{\Sigma} s_{a\lrcorner} d s_{b\lrcorner} \Omega + \mathcal{O}(u^3). \quad (\text{A.28})$$

To evaluate the derivatives $\frac{\partial^k}{\partial^k u^a} W_{\text{brane}}|_{u=0}$ we use that for every $\frac{\partial}{\partial u^a}$ the Lie-derivative $\mathcal{L}_{s_a} = ds_{a\lrcorner} + s_{a\lrcorner} d$ acts on the integrand Ω , where we further denote the interior product with a vector s_a by $s_{a\lrcorner}$. In addition we use that on the holomorphic curve Σ there are no $(2,0)$ -forms such that the linear term in u^a in the Taylor expansion vanishes identically. Then the derivative with respect to u^a is obtained as

$$\partial_{u^a} W_{\text{brane}} = -\mu_5 \int_{\Sigma} \bar{\partial} s_{\lrcorner} s_{a\lrcorner} \Omega \quad (\text{A.29})$$

where we perform a partial integration on Σ . In addition, we rescale the superpotential by the D5-brane charge, $W_{\text{brane}} \mapsto \mu_5 W_{\text{brane}}$, as in [60]. Now we can calculate the F-term potential

$$V = \frac{2\mu_5}{e^\phi} e^K \int_\Sigma \bar{\partial} s_\perp s_a \lrcorner \Omega \mathcal{G}^{a\bar{b}} \int_\Sigma \partial \bar{s}_\perp \bar{s}_{\bar{b}} \lrcorner \bar{\Omega}. \quad (\text{A.30})$$

To further evaluate this we have to rewrite the matrix $\mathcal{G}_{a\bar{b}}$ as follows. Consider the integral

$$I_{a\bar{b}} := \int_\Sigma (s_a \lrcorner \Omega)_{ij} (\bar{s}_{\bar{b}} \lrcorner \bar{\Omega})^{ij} \iota^*(J). \quad (\text{A.31})$$

The contracted indices i, j denote the coordinates on Z_3 , one of which is tangential and two are normal to Σ . Then using $\bar{s}_{\bar{a}} \lrcorner \Omega = 0$, $\Omega \wedge \bar{\Omega} = \frac{\int \Omega \wedge \bar{\Omega}}{6\mathcal{V}} J^3$ where \mathcal{V} denotes the compactification volume and the rule $s_a \lrcorner (\alpha \wedge \beta) = (s_a \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (s_a \lrcorner \beta)$ for a p -form α we can rewrite this as

$$\begin{aligned} I_{a\bar{b}} &= - \int_\Sigma (s_a \lrcorner \bar{s}_{\bar{b}} \lrcorner (\Omega \wedge \bar{\Omega}))_{ij}^{ij} \iota^*(J) = - \frac{\int \Omega \wedge \bar{\Omega}}{6\mathcal{V}} \int_\Sigma (s_a \lrcorner \bar{s}_{\bar{b}} \lrcorner (J^3))_{ij}^{ij} \iota^*(J) \\ &= - \frac{\int \Omega \wedge \bar{\Omega}}{6\mathcal{V}} \int_\Sigma [3(s_a \lrcorner \bar{s}_{\bar{b}} \lrcorner J) J^2 - 6(\bar{s}_{\bar{b}} \lrcorner J) \wedge (s_a \lrcorner J) \wedge J]_{ij}^{ij} \iota^*(J) \\ &= \frac{\int \Omega \wedge \bar{\Omega}}{4\mathcal{V}} \int_\Sigma s_a \lrcorner \bar{s}_{\bar{b}} \lrcorner J^2 = \frac{i \int \Omega \wedge \bar{\Omega}}{2} \mathcal{G}_{a\bar{b}}. \end{aligned} \quad (\text{A.32})$$

Consequently, introducing the abbreviation $\Omega_a = s_a \lrcorner \Omega$ we can write the matrix $\mathcal{G}_{a\bar{b}}$ as

$$\mathcal{G}_{a\bar{b}} = \frac{-i}{2\mathcal{V}} \int_\Sigma s_a \lrcorner \bar{s}_{\bar{b}} \lrcorner (J \wedge J) = \frac{-2i}{\int \Omega \wedge \bar{\Omega}} \int_\Sigma (\Omega_a)_{ij} (\bar{\Omega}_{\bar{b}})^{ij} \iota^*(J). \quad (\text{A.33})$$

Next we use the basis Ω_a of $\Omega^{(1,0)}(\Sigma, N_{Z_3}\Sigma)$ to expand the section

$$\bar{\partial} s_\perp J = c^{\bar{b}} \bar{\Omega}_{\bar{b}}. \quad (\text{A.34})$$

The coefficients $c^{\bar{b}}$ are determined by contraction of (A.34) with $\Omega_a^{\bar{b}}$, where we have raised the form indices using the hermitian metric on Z_3 . This way we obtain a function on Σ that we can integrate over Σ using the volume form $\text{vol}_\Sigma = \iota^*(J)$ to determine the $c^{\bar{b}}$ so that

$$\bar{\partial} s_\perp J = \frac{-2i}{\int \Omega \wedge \bar{\Omega}} \mathcal{G}^{a\bar{b}} \bar{\Omega}_{\bar{b}} \int_\Sigma \bar{\partial} s_\perp \Omega_a. \quad (\text{A.35})$$

With this expansion we immediately obtain the desired form of the F-term superpotential

$$V = \frac{2e^K \mu_5}{e^\phi} \int_\Sigma \bar{\partial} s_\perp (\Omega_a \mathcal{G}^{a\bar{b}} \int_\Sigma \partial \bar{s}_\perp \bar{\Omega}_{\bar{b}}) = \frac{-\mu_5 e^{3\phi}}{2\mathcal{V}^2} \int_\Sigma \bar{\partial} s_\perp \partial \bar{s}_\perp J = \frac{\mu_5 e^{3\phi}}{2\mathcal{V}^2} \int_\Sigma \|\bar{\partial} s\|^2 \iota^*(J). \quad (\text{A.36})$$

This perfectly matches the potential (3.82) obtained from the reduction of the DBI-action of the D5-brane on Σ

Appendix B

Geometrical Background

B.1 Topology of Elliptic Calabi-Yau's and Ruled Threefolds

In this section we summarize useful relation when working with elliptically fibered Calabi-Yau manifolds X_n given by a Weierstrass constraint

$$y^2 = x^3 + g_2(\underline{u})xz^4 + g_3(\underline{u})z^6 \quad (\text{B.1})$$

in homogeneous coordinates (z, x, y) of a projective bundle $\mathbb{P}(\mathcal{O}_{B_{n-1}} \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ as in sections 4.1.4 and 4.2.2. Here \mathcal{L} denotes an a priori undetermined line bundle on B_{n-1} and $g_2(\underline{u})$, $g_3(\underline{u})$ are sections of \mathcal{L}^4 and \mathcal{L}^6 for (4.30) to be a well-defined constraint equation. Locally on the base B_{n-1} they are functions in local coordinates \underline{u} on B_{n-1} .

Next, we determine the line bundle \mathcal{L} by the requirement of X_n being Calabi-Yau. First, we note that the total Chern classes of the projective bundle \mathcal{W} and of X_n read [181, 182, 202]

$$c(\mathcal{W}) = c(B_{n-1})(1+r)(1+2r+2l)(1+3r+3l), \quad (\text{B.2})$$

$$c(X_n) = c(B_{n-1}) \frac{(1+r)(1+2r+2l)(1+3r+3l)}{1+6r+6l}, \quad (\text{B.3})$$

where we used that the class of X_n is $[X_n] = 6r + 6l$ as well as the adjunction formula [154] for X_n in \mathcal{W} . Here we used $c_1(\mathcal{L}) = l$ and $r = c_1(\mathcal{O}_{\mathbb{P}^2(1,2,3)}(1))$ to denote the first Chern class of the hyperplane in the fiber $\mathbb{P}^2(1, 2, 3)$. Then the Calabi-Yau condition on X_n implies $l = c_1(B)$ or equivalently $\mathcal{L} = K_{B_3}^{-1}$. Alternatively we draw the same conclusion by noting that the normal bundle to σ in Z_3 is given by $\mathcal{L}^{-1} = N_{Z_3}\sigma$ which implies by adjunction $K_{Z_3}|_\sigma = K_\sigma \otimes \mathcal{L}$ the condition $c_1(B_2) = l$ as well.

In order to calculate the second Chern-class of X_n we first note that $r(2r+2l)(3r+3l) = 0$ and since the class of X_n is given as $6r + 6c_1(L) = 0$ we obtain in the intersection ring of X_n the relation

$$\sigma^2 = -c_1(\mathcal{L})\sigma, \quad (\text{B.4})$$

which is generally valid for projective bundles [154]. Here we identified $r = c_1(\mathcal{O}(\sigma)) = (z = 0)$ where σ denotes the section of $\pi : X_n \rightarrow B_{n-1}$. The second Chern class of X_n is determined

by expanding (B.3) to second order using (B.4) to obtain

$$\begin{aligned} c_2(X_2) &= 12c_1(B_2)\sigma, \\ c_2(X_3) &= 12c_1(B_2)\sigma + 11c_1(B_2)^2 + c_2(B_2) \\ c_2(X_4) &= 12c_1(B_2)\sigma + 11c_1(B_2)^2 + c_2(B_2). \end{aligned} \tag{B.5}$$

We conclude by a discussion of the Chern class of the projective bundle $B_3 = \mathbb{P}(\mathcal{O}_{B_2} \oplus L)$ for some arbitrary line bundle L over B_2 with $c_1(L) = t$ that reads

$$c(\mathbb{P}(\mathcal{O}_{B_2} \oplus L)) = c(B_2)(1+r)(1+r+t). \tag{B.6}$$

This implies $c_1(B_3) = c_1(B_2) + 2r + t$ and $c_2(B_3) = c_2(B_2) + c_1(B_2)(t + 2r)$.

B.2 A Local Study of the Blow-Up Threefold \hat{Z}_3

In this appendix we study the geometry of the blow-up \hat{Z}_3 in more detail in a local analysis. The obtained results provide insights in the blow-up process that immediately apply for the global discussion of section 7.2 and allow for a derivation of the expressions used for \hat{Z}_3 as a complete intersection and in particular $\hat{\Omega}$ as a residue integral.

The following discussion bases on the general lore in algebraic geometry that the process of blowing up is local in nature, i.e. just affects the geometry near the subvariety Σ which is blown-up leaving the rest of the ambient space invariant. Thus the geometrical properties of the blow up geometry can be studied in a completely local analysis in an open neighborhood around Σ . In particular, this applies for the case at hand, the blow-up of the curve Σ in the compact Calabi-Yau threefold Z_3 into a divisor E in the threefold \hat{Z}_3 .

Starting from a given open covering of Z_3 by local patches $U_k \cong \mathbb{C}^3$ we choose a neighborhood U centered around the curve Σ . Thus, this local patch can be modeled by considering just \mathbb{C}^3 on which we introduce local coordinates x_1, x_2, x_3 . The holomorphic three-form Ω on Z_3 takes then simply the local form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. The curve Σ is accordingly described as the complete intersection

$$D_1 \cap D_2 = \{h_1(x_i) = 0\} \cap \{h_2(x_i) = 0\}, \tag{B.7}$$

for two given polynomials h_i with corresponding divisor classes D_i in \mathbb{C}^3 .

To construct the blow-up along Σ , denoted by $\hat{\mathbb{C}}^3$, we have to consider the new ambient space of the projective bundle $\mathcal{W} = \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$. This is locally of the form “ $\mathbb{C}^3 \times \mathbb{P}^1$ ” as necessary for the blow-up procedure described in standard textbooks, see e.g. p. 182 and 602 of [154]. Next we introduce homogeneous coordinates (l_1, l_2) on the \mathbb{P}^1 to obtain the blow-up $\hat{\mathbb{C}}^3$ as the hypersurface

$$Q \equiv l_1 h_2 - l_2 h_1 = 0 \tag{B.8}$$

in \mathcal{W} as before in (7.32).

In the following we construct and study the pullback form $\hat{\Omega} = \pi^*(\Omega)$ that is a section of the canonical bundle $K\hat{\mathbb{C}}^3$, cf. p. 187 of [154]. To simplify the calculations we first perform a coordinate transformation to coordinates y_i such that $y_1 = h_1(x_i)$, $y_2 = h_2(x_i)$ and $y_3 = x_j$ for appropriate¹ j . For notational convenience we relabel the coordinates such that $y_3 = x_3$. Thus, we obtain

$$Q = l_1 y_2 - l_2 y_1, \quad \Omega = \det J^{-1} dy_1 \wedge dy_2 \wedge dy_3 \tag{B.9}$$

in the coordinates y_i , where $J = \frac{\partial y_i}{\partial x_j}$ denotes the Jacobian of the coordinate transformation that is generically non-zero by assumption of a complete intersection Σ .

Now we perform the blow-up on the two local patches on the \mathbb{P}^1 -fiber of \mathcal{W} that are defined as usual by $U_i = \{l_i \neq 0\}$ for $i = 1, 2$. In the patch U_1 , for example, we introduce coordinates $z_1^{(1)}, z_2^{(1)}$ on $\hat{\mathbb{C}}^3$ as

$$z_1^{(1)} = y_1, \quad z_2^{(1)} = -\frac{l_2}{l_1} = \frac{y_2}{y_1}, \quad z_3^{(1)} = y_3 \tag{B.10}$$

which allows us to evaluate the pullback map as

$$\begin{aligned} \pi^*(\Omega) &= \det J^{-1} \pi^*(dy_1 \wedge dy_2 \wedge dy_3) = \det J^{-1} dz_1^{(1)} \wedge d(z_2^{(1)} z_1^{(1)}) \wedge dz_3^{(1)} \\ &= z_1^{(1)} \det J^{-1} dz_1^{(1)} \wedge dz_2^{(1)} \wedge dz_3^{(1)}. \end{aligned} \tag{B.11}$$

From this expression we can read off the canonical bundle $K\hat{\mathbb{C}}^3$, cf. p. 608 of [154], by determining the zero-locus of $\pi^*(\Omega)$. Indeed we obtain $K\hat{\mathbb{C}}^3 = E$ as mentioned in section 7.2.1 and in [60] since $z_1^{(1)} = 0$ describes the exceptional divisor E in U_1 and $\det J$ is non-zero by assumption. Analogously, we obtain a similar expression in the patch U_2 for local coordinates

$$z_1^{(2)} = \frac{l_1}{l_2} = \frac{y_1}{y_2}, \quad z_2^{(2)} = y_2, \quad z_3^{(2)} = y_3 \tag{B.12}$$

that reads

$$\pi^*(\Omega) = z_2^{(2)} \det J^{-1} dz_1^{(2)} \wedge dz_2^{(2)} \wedge dz_3^{(2)}, \tag{B.13}$$

which as well vanishes on E since $E = \{z_2^{(2)} = 0\}$ in U_2 . Thus we extract the transition functions g_{ij} of $\mathcal{O}(E)$ on $U_1 \cap U_2$ to be given as

$$g_{ij} = z_i^{(j)} = \frac{y_i}{y_j} = \frac{l_i}{l_j} \tag{B.14}$$

which reflects the fact that $E = \mathbb{P}(\mathcal{O}(D_1) \oplus \mathcal{O}(D_2))$ over Σ with fiber \mathbb{P}^1 . Additionally, the transition functions g_{ij} , when restricted to E , are just those of $\mathcal{O}_{\mathbb{P}^1}(-1)$ on the \mathbb{P}^1 fiber in E and thus we obtain $\mathcal{O}(E)|_E = \mathcal{O}(-1)$ as used in chapter 9 and in [60, 79, 81].

¹This choice is fixed by the inverse function theorem stating that for every point $x_0 \in \mathbb{C}^3$ with $(\partial_k h_1 \partial_l h_2 - \partial_l h_1 \partial_k h_2)|_{x_0} \neq 0$ for $k, l \neq j$, there exists a local parameterization of \mathcal{C} near x_0 as a graph over x_j . In particular, the blow-up is not independent of the coordinates used, cf. p. 603 of [154].

Let us now take a different perspective on the pullback-form $\pi^*(\Omega)$ of the blow-up $\hat{\mathbb{C}}^3$ that is more adapted for the global geometry of \hat{Z}_3 as a complete intersection $P = Q = 0$ in (7.32). The key point will be the description of the blow-up $\hat{\mathbb{C}}^3$ as the hypersurface (B.8) in \mathcal{W} that will allow for a residue integral representation of $\pi^*(\Omega)$. In particular, the advantage of this residue expression in contrast to the local expressions (B.11), (B.13) is the fact that it can straight forwardly be extended to a global expression on \hat{Z}_3 as used in (7.42).

Let us start with an ansatz $\hat{\Omega}$ for $\pi^*(\Omega)$,

$$\hat{\Omega} = \int_{S^1} A(x_i, l_j) \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge \Delta_{\mathbb{P}^1}}{Q}, \quad (\text{B.15})$$

where $\Delta_{\mathbb{P}^1} = l_1 dl_2 - l_2 dl_1$ is the measure on \mathbb{P}^1 obtained from (5.5) and S_Q^1 denotes a loop in \mathcal{W} centered around $Q = 0$. The function $A(x_i, l_j)$ of the coordinates x_i, l_j is fixed by its scaling behavior w.r.t. the \mathbb{C}^* -action $(l_1, l_2) \mapsto \lambda(l_1, l_2)$. Since $Q \mapsto \lambda Q$ and $\Delta_{\mathbb{P}^1} \mapsto \lambda^2 \Delta_{\mathbb{P}^1}$ we demand $A(x_i, \lambda l_j) = \lambda^{-1} A(x_i, l_j)$, i.e. it is a section of $\mathcal{O}(E)$. Thus we make the ansatz

$$A(x_i, l_j) = a_1 \frac{h_1}{l_1} + a_2 \frac{h_2}{l_2} = -a_1 \frac{Q}{l_1 l_2} + (a_2 + a_1) \frac{h_2}{l_2} = a_2 \frac{Q}{l_1 l_2} + (a_1 + a_2) \frac{h_1}{l_1}, \quad (\text{B.16})$$

which we insert in $\hat{\Omega}$ of (B.15) to obtain

$$\hat{\Omega} = (a_1 + a_2) \int_{S_Q^1} \frac{h_i dx_1 \wedge dx_2 \wedge dx_3 \wedge \Delta_{\mathbb{P}^1}}{l_i Q}, \quad i = 1, 2. \quad (\text{B.17})$$

Now we show that this precisely reproduces the local expressions (B.11), (B.13), this way fixing the free parameters a_i .

Let us perform the calculations in the local patch U_1 . Then the measure on \mathbb{P}^1 reduces to $\Delta_{\mathbb{P}^1} = (l_1)^2 dz_2^{(1)}$ with $z_2^{(1)} = l_2/l_1$ and we obtain, after a change of coordinates to y_i ,

$$\hat{\Omega} = (a_1 + a_2) \int_{S_Q^1} \det J^{-1} \frac{y_i l_1}{l_i} \frac{dy_1 \wedge dy_2 \wedge dy_3 \wedge dz_2^{(1)}}{-z_2^{(1)} y_1 + y_2} \quad (\text{B.18})$$

$$= -(a_1 + a_2) (\det J^{-1} \frac{y_i l_1}{l_i} dy_1 \wedge dy_3 \wedge dz_2^{(1)})|_{y_2=z_2^{(1)} y_1}. \quad (\text{B.19})$$

In the last line we indicated that the residue localizes on the locus $Q = 0$. This implies that $z_2^{(1)} = \frac{y_2}{y_1} = \frac{l_2}{l_1}$ as before in (B.10) and we put $z_1^{(1)} = y_1, z_3^{(1)} = y_3$ as well. Next, we evaluate $\hat{\Omega}$ for $i = 1$ for which the prefactor reduces to $\frac{y_1 l_1}{l_1} = z_1^{(1)}$, and for $i = 2$, for which we get $\frac{y_2 l_1}{l_2} = z_1^{(1)}$. Thus, the two expressions in (B.17) for $i = 1, 2$ yield one unique form $\hat{\Omega}$ after evaluating the residue integral,

$$\hat{\Omega} = (a_1 + a_2) \det J^{-1} z_1^{(1)} dz_1^{(1)} \wedge dz_2^{(1)} \wedge dz_3^{(1)}, \quad (\text{B.20})$$

which agrees with $\pi^*(\Omega)$ on U_1 for $a_1 + a_2 = 1$. Thus, we propose the global residue expression

$$\hat{\Omega} \equiv \pi^*(\Omega) = \int_{S_Q^1} \frac{h_1}{l_1} \frac{\Omega \wedge \Delta_{\mathbb{P}^1}}{Q} = \int_{S_Q^1} \frac{h_2}{l_2} \frac{\Omega \wedge \Delta_{\mathbb{P}^1}}{Q} \quad (\text{B.21})$$

The transition from the local chart $U \cong \mathbb{C}^3$ on Z_3 to the global threefold Z_3 is then effectively performed by replacing the three-form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ on \mathbb{C}^3 by the residue integral $\Omega = \text{Res}_P(\frac{\Delta_P \Delta}{P})$ of (5.32) on Z_3 in (B.21). This way the local analysis motivates and proves the global expression of (7.42) for the pullback form $\hat{\Omega} \equiv \pi^*(\Omega)$ of the holomorphic three-form Ω to \hat{Z}_3 and its properties mentioned there.

We conclude this analysis by briefly checking that this residue also reproduces $\pi^*(\Omega)$ on U_2 as evaluated in (B.13). We set $z_1^{(2)} = \frac{l_1}{l_2}$ for which the measure reduces as $\Delta_{\mathbb{P}^1} = -l_2^2 dz_1^{(2)}$ which we readily insert into (B.21) to obtain

$$\hat{\Omega} = \int_{S_Q^1} \det J^{-1} \frac{y_i l_2}{l_i} \frac{dy_1 \wedge dy_2 \wedge dy_3 \wedge dz_1^{(2)}}{y_1 - z_1^{(2)} y_2} \tag{B.22}$$

$$= (\det J^{-1} \frac{y_i l_2}{l_i} dy_2 \wedge dy_3 \wedge dz_1^{(2)})|_{y_1 = z_1^{(2)} y_2} = \det J^{-1} z_2^{(2)} dz_1^{(2)} \wedge dz_2^{(2)} \wedge dz_3^{(2)}. \tag{B.23}$$

Here we again use $z_1^{(2)} = \frac{y_1}{y_2} = \frac{l_1}{l_2}$ on $Q = 0$ and introduce the local $z_2^{(2)} = y_2$, $z_3^{(2)} = y_3$ as in (B.12). The result is in perfect agreement with the local expression (B.13) on U_2 .

B.3 Topology of the Blow-Up Threefold \hat{Z}_3

In this appendix we summarize the basic topological data of the blow-up threefold \hat{Z}_3 of the Calabi-Yau threefold Z_3 along the curve Σ . For more details we refer to [60] as well as a standard reference [154]. See also [291].

We start with the cohomology of E . The cohomology ring of E is generated by $\eta = c_1(T)$, the first Chern-class of the tautological bundle² on E , as an $H^\bullet(\Sigma)$ -algebra, i.e. $H^\bullet(E) = H^\bullet(\Sigma)\langle\eta\rangle$. Thus, for a single smooth curve Σ of genus g the non-vanishing Hodge numbers of E are determined to be

$$h^{(0,0)} = h^{(2,2)} = 1, \quad h^{(1,0)} = g, \quad h^{(1,1)} = 2 \tag{B.24}$$

as usual for a ruled surface E over a genus g curve. This is due to the fact the \mathbb{P}^1 -fibration of E does not degenerate and thus the Hodge numbers of E are equal to those of $\mathbb{P}^1 \times \Sigma$. Thus the elements in $H^{(1,0)}(E)$ are precisely the holomorphic Wilson lines a_i of Σ and the three-forms in $H^{(2,1)}(E)$ take the form $a_i \wedge \eta$. Since, by construction, the normal bundle to E in \hat{Z}_3 is the tautological bundle T the class η , being one of the two classes in $H^{(1,1)}(E)$, is induced from the ambient space \hat{Z}_3 and given by $\eta = c_1(N_{\hat{Z}_3} E)$. The second element spanning $H^{(1,1)}(E)$ is given by the Poincaré dual $[\Sigma]_E$ of the curve Σ in E , $[\Sigma]_E = c_1(N_E \Sigma)$. It is related to the first Chern class $c_1(\Sigma)$ and thus to the genus as

$$c_1(N_E \Sigma) = -c_1(\Sigma) - 2\eta, \tag{B.25}$$

²The tautological bundle can be defined on any projectivization of a vector bundle, like $E = \mathbb{P}(N_{Z_3} \Sigma)$, by the defining property that T restricted to each fiber agrees with the universal bundle $\mathcal{O}(1)$ on projective space [154].

by using the adjunction formula in \hat{Z}_3 . Note that as an exceptional divisor, E is rigid in \hat{Z}_3 . This is a crucial feature of the blow-up procedure, that guarantees that no *new* degrees of freedom associated to deformations of E are introduced. By construction the normal bundle to E in \hat{Z}_3 is T , which is a negative bundle on E . Thus we have

$$H^0(E, N_{\hat{Z}_3} E) = \emptyset. \quad (\text{B.26})$$

Next we note that the cohomology of the blow-up \hat{Z}_3 takes the form

$$H^\bullet(\hat{Z}_3) = \pi^* H^\bullet(Z_3) \oplus H^\bullet(E)/\pi^* H^\bullet(\Sigma), \quad (\text{B.27})$$

where $\pi : Z_3 \rightarrow Z_3$ denotes the blow-down map. This in particular implies $H^{(3,0)}(\hat{Z}_3) = \pi^* H^{(3,0)}(Z_3)$ since $H^{(3,0)}(\Sigma) = H^{(3,0)}(E)$ for dimensional reasons. Thus, $h^{(3,0)}(\hat{Z}_3) = 1$ as in the Calabi-Yau case. Then, the intersection ring on \hat{Z}_3 has the following relations on the level of intersection curves

$$E^2 = -\pi^* \Sigma - \chi(\Sigma)F, \quad E \cdot \pi^* D = (\Sigma \cdot D)F, \quad (\text{B.28})$$

where D is any divisor in Z_3 , F is the class of the \mathbb{P}^1 -fiber of E , and $\chi(\Sigma)$ is the Euler number of the blow-up curve Σ . The intersection numbers read

$$E^3 = \chi(\Sigma), \quad F \cdot E = -1, \quad F \cdot \pi^* D = 0, \quad E \cdot \pi^* \tilde{\Sigma} = F \cdot \pi^* \tilde{\Sigma} = 0, \quad (\text{B.29})$$

where $\tilde{\Sigma}$ is any curve in Z_3 . We conclude with the first and second Chern class of \hat{Z}_3 , that are in general affected by the blow-up as

$$c_1(\hat{Z}_3) = \pi^*(c_1(Z_3)) - c_1(N_{\hat{Z}_3} E), \quad (\text{B.30})$$

$$c_2(\hat{Z}_3) = \pi^*(c_2(Z_3) + [\Sigma]) - \pi^*(c_1(Z_3))E. \quad (\text{B.31})$$

Clearly, if Z_3 is a Calabi-Yau manifold one can use $c_1(Z_3) = 0$ to find

$$c_1(\hat{Z}_3) = -\eta, \quad c_2(\hat{Z}_3) = \pi^*(c_2(Z_3) + [\Sigma]), \quad (\text{B.32})$$

in particular that \hat{Z}_3 is no more Calabi-Yau. This implies that the lift of the holomorphic three-form Ω vanishes as

$$\hat{\Omega} = \pi^*(\Omega), \quad \hat{\Omega}|_E = 0. \quad (\text{B.33})$$

Appendix C

Details, Examples and Tables

C.1 Topological Data of the Main Example

Here we supply the topological data of the fourfold \tilde{X}_4 that was omitted in the main text for convenience. Besides the intersection rings we will also present the full Picard-Fuchs system at the large radius/large complex structure point. These determine as explained in section 5.3 the primary vertical subspace $H_V^{(p,p)}(\tilde{X}_4)$ of the A-model.

As was mentioned before there are four triangulations whereas only three yield non-singular varieties. Again we restrict our exposition to the two triangulations mentioned in section 6.1.2. For the following we label the points in the polyhedron $\Delta_5^{\tilde{X}}$ given in (6.25) consecutively by ν_i , $i = 0, \dots, 9$ and associated coordinates x_i to each ν_i . Then the toric divisors are given by $D_i := \{x_i = 0\}$.

Phase I: In phase I of the toric variety defined by the polyhedron $\Delta_5^{\tilde{X}}$ in (6.25) one has the following Stanley-Reisner ideal

$$SR = \{D_3D_8, D_7D_9, D_8D_9, D_1D_5D_6, D_2D_3D_4, D_2D_4D_7\} . \quad (\text{C.1})$$

From this we compute by standard methods of toric geometry the intersection numbers

$$\begin{aligned} \mathcal{C}_0 &= J_4(J_1^2J_2 + J_1J_3J_2 + J_3^2J_2 + 3J_1J_2^2 + 3J_3J_2^2 + 9J_2^3) + J_1^2J_3J_2 + J_1J_3^2J_2 + J_3^3J_2 \\ &\quad + 2J_1^2J_2^2 + 4J_1J_3J_2^2 + 4J_3^2J_2^2 + 11J_1J_2^3 + 15J_3J_2^3 + 46J_2^4, \\ \mathcal{C}_2 &= 24J_1^2 + 36J_1J_4 + 48J_1J_3 + 36J_4J_3 + 48J_3^2 + 128J_1J_2 + 102J_2J_4 \\ &\quad + 172J_2J_3 + 530J_2^2, \\ \mathcal{C}_3 &= -660J_1 - 540J_4 - 900J_3 - 2776J_2 . \end{aligned} \quad (\text{C.2})$$

Here we denoted generators of the Kähler cone of (6.27) dual to the Mori cone by J_i as before. The notation for the \mathcal{C}_k is as follows. Denoting the dual two-forms to J_i by ω_i the coefficients of the top intersection ring \mathcal{C}_0 are the quartic intersection numbers $J_i \cap J_j \cap J_k \cap J_l = \int_{\tilde{X}_4} \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_l$, while the coefficients of \mathcal{C}_2 and \mathcal{C}_3 are $[c_2(\tilde{X}_4)] \cap J_i \cap J_j = \int_{\tilde{X}_4} c_2 \wedge \omega_i \wedge \omega_j$ and $[c_3(\tilde{X}_4)] \cap J_i = \int_{\tilde{X}_4} c_3 \wedge \omega_i$ respectively.

As reviewed in section 5.2.4 in the threefold case, that readily applies to the fourfold case as well, the Picard-Fuchs operators of the mirror fourfold X_4 at the large complex structure

point are calculated by the methods described in [233]. In the appropriate coordinates z_i defined by (5.36) and evaluated in (6.35) we obtain the full Picard-Fuchs system on \tilde{X}_4 given by

$$\begin{aligned} \mathcal{D}_1^I &= -\theta_1^2(\theta_1 + \theta_4 - \theta_3) \\ &\quad -(-1 + \theta_1 - \theta_3)(-2 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_1, \\ \mathcal{D}_2^I &= \theta_2(-2\theta_1 - \theta_4 - \theta_3 + \theta_2) - 12(-5 + 6\theta_2)(-1 + 6\theta_2)z_2, \\ \mathcal{D}_3^I &= (\theta_1 - \theta_3)(-\theta_4 + \theta_3) - (1 + \theta_1 + \theta_4 - \theta_3)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_3, \\ \mathcal{D}_4^I &= \theta_4(\theta_1 + \theta_4 - \theta_3) - (-1 + \theta_4 - \theta_3)(-1 + 2\theta_1 + \theta_4 + \theta_3 - \theta_2)z_4. \end{aligned} \quad (\text{C.3})$$

Now we calculate the ring \mathcal{R} given by the orthogonal complement of the ideal of Picard-Fuchs operators defined in (5.61). Using the isomorphism $\theta_i \mapsto J_i$ discussed in section 5.3.3 we obtain the topological basis of $H_V^{(p,p)}(\tilde{X}_4)$ by identification with the graded ring $\mathcal{R}^{(p)}$. Since J_i form the trivial basis of $H^{(1,1)}(\tilde{X}_4)$ and $H^{(3,3)}(\tilde{X}_4)$ is fixed by duality to $H^{(1,1)}(\tilde{X}_4)$, the non-trivial part is the cohomology group $H_V^{(2,2)}(X_4)$. We calculate the ring $\mathcal{R}^{(2)}$ by choosing the basis

$$\begin{aligned} \mathcal{R}_1^{(2)} &= \theta_1^2, & \mathcal{R}_2^{(2)} &= \theta_4(\theta_1 + \theta_3), & \mathcal{R}_3^{(2)} &= \theta_3(\theta_1 + \theta_3), & \mathcal{R}_4^{(2)} &= \theta_2(\theta_1 + 2\theta_2), \\ \mathcal{R}_5^{(2)} &= \theta_2(\theta_4 + \theta_2), & \mathcal{R}_6^{(2)} &= \theta_2(\theta_3 + \theta_2). \end{aligned} \quad (\text{C.4})$$

Then we can use the intersection ring \mathcal{C}_0 to determine the topological metric $\eta^{(2)}$ of (5.57) given by

$$\eta_I^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 4 & 3 & 3 \\ 0 & 0 & 0 & 14 & 6 & 8 \\ 0 & 0 & 0 & 18 & 10 & 10 \\ 4 & 14 & 18 & 230 & 124 & 137 \\ 3 & 6 & 10 & 124 & 64 & 73 \\ 3 & 8 & 10 & 137 & 73 & 80 \end{pmatrix}. \quad (\text{C.5})$$

The entries are just the values of the integrals

$$\mathcal{R}_\alpha^{(2)}\mathcal{R}_\beta^{(2)} = \int_{\tilde{X}_4} (\mathcal{R}_\alpha^{(2)}\mathcal{R}_\beta^{(2)})|_{\theta_i \mapsto J_i}, \quad (\text{C.6})$$

where we think of it in terms of the Poincaré duals and the quartic intersections are given as the coefficients of monomials in \mathcal{C}_0 . The basis $\mathcal{R}_i^{(3)}$ at grade $p = 3$ is determined by requiring $\eta_{ab}^{(3)} = \delta_{a, h^{(1,1)} - b + 1}$ where $h^{(1,1)} = 4$ for the case at hand. Then the basis reads

$$\begin{aligned} \mathcal{R}_1^{(3)} &= \theta_1(-\theta_1\theta_4 - \theta_2\theta_4 + \theta_2\theta_3), & \mathcal{R}_2^{(3)} &= \theta_1(-\theta_1\theta_4 + \theta_1\theta_2 + \theta_2\theta_4 - \theta_2\theta_3), \\ \mathcal{R}_3^{(3)} &= \theta_1^2\theta_4, & \mathcal{R}_4^{(3)} &= \theta_1(-2\theta_1\theta_4 - \theta_1\theta_2 + \theta_2\theta_3). \end{aligned} \quad (\text{C.7})$$

Finally, we choose a basis of $\mathcal{R}^{(4)}$ by $\mathcal{R}^{(4)} = \frac{1}{103}\mathcal{C}_0|_{J_i \mapsto \theta_i}$ such that $\eta_{a_0, b_0}^{(4)} = 1$ for $\mathcal{R}^{(0)} = 1$.

Phase II: Turning to the phase II of (6.25) the Stanley-Reisner ideal and the intersection numbers read

$$\begin{aligned}
SR &= \{D_1D_7, D_7D_9, D_8D_9, D_1D_5D_6, D_2D_3D_4, D_2D_4D_7, D_3D_5D_6D_8\}, \\
\mathcal{C}_0 &= J_1^2J_4J_3 + 2J_1^2J_3^2 + 3J_1J_4J_3^2 + 12J_1J_3^3 + 9J_4J_3^3 + 54J_3^4 + J_1^2J_2J_4 + 2J_1^2J_3J_2 \\
&\quad + 3J_1J_2J_3J_4 + 12J_1J_3^2J_2 + 9J_2J_3^2J_4 + 54J_3^3J_2 + 2J_1^2J_2^2 + 3J_1J_4J_2^2 + 12J_1J_3J_2^2 \\
&\quad + 9J_4J_3J_2^2 + 54J_3^2J_2^2 + 11J_1J_3^2 + 9J_4J_3^2 + 51J_3J_2^3 + 46J_2^4, \\
\mathcal{C}_2 &= 24J_1^2 + 36J_1J_4 + 138J_1J_3 + 102J_4J_3 + 618J_3^2 + 128J_1J_2 + 102J_2J_4 \\
&\quad + 588J_3J_4 + 530J_4^2, \\
\mathcal{C}_3 &= 660J_1 - 540J_4 - 3078J_3 - 2776J_2,
\end{aligned} \tag{C.8}$$

where the Kähler cone generators were given in (6.28).

The complete Picard-Fuchs system consists of four operators given by

$$\begin{aligned}
\mathcal{D}_1^{II} &= -\theta_1^2(\theta_1 + \theta_2 - \theta_3) \\
&\quad - (-3 + 3\theta_1 - \theta_3 + 2\theta_4)(-2 + 3\theta_1 - \theta_3 + 2\theta_4)(-1 + 3\theta_1 - \theta_3 + 2\theta_4)z_1, \\
\mathcal{D}_2^{II} &= -\theta_2(\theta_1 + \theta_2 - \theta_3)(\theta_2 - \theta_3 + \theta_4) - 12(-5 + 6\theta_2)(-1 + 6\theta_2)(-1 + \theta_2 - \theta_3)z_2, \\
\mathcal{D}_3^{II} &= -(\theta_2 - \theta_3)(-3\theta_1 + \theta_3 - 2\theta_4) - (1 + \theta_1 + \theta_2 - \theta_3)(1 + \theta_2 - \theta_3 + \theta_4)z_3, \\
\mathcal{D}_4^{II} &= \theta_4(\theta_2 - \theta_3 + \theta_4) - (-2 + 3\theta_1 - \theta_3 + 2\theta_4)(-1 + 3\theta_1 - \theta_3 + 2\theta_4)z_4.
\end{aligned} \tag{C.9}$$

This enables us to calculate $H_V^{(p,p)}(\tilde{X}_4)$ as before. The basis at grade $p = 2$ reads

$$\begin{aligned}
\mathcal{R}_1^{(2)} &= \theta_1^2, \quad \mathcal{R}_2^{(2)} = \theta_2(2\theta_1 + 6\theta_3), \quad \mathcal{R}_3^{(2)} = \theta_3(\theta_1 + 3\theta_3), \quad \mathcal{R}_4^{(2)} = \theta_1\theta_4, \\
\mathcal{R}_5^{(2)} &= \theta_2^2, \quad \mathcal{R}_6^{(2)} = \theta_3(2\theta_2 + 2\theta_3 + \theta_4) + \theta_2\theta_4,
\end{aligned} \tag{C.10}$$

for which the topological metric $\eta^{(2)}$ is given by

$$\eta^{(2)} = \begin{pmatrix} 0 & 12 & 6 & 0 & 2 & 10 \\ 12 & 2240 & 1120 & 20 & 328 & 1512 \\ 6 & 1120 & 560 & 10 & 174 & 756 \\ 0 & 20 & 10 & 0 & 3 & 12 \\ 2 & 328 & 174 & 3 & 46 & 228 \\ 10 & 1512 & 756 & 12 & 228 & 1008 \end{pmatrix}. \tag{C.11}$$

Again the basis of $H^{(3,3)}(\tilde{X}_4)$ is fixed by $\eta_{ab}^{(3)} = \delta_{a,h(1,1)-b+1}$ to be

$$\begin{aligned}
\mathcal{R}_1^{(3)} &= -\frac{1}{91}(182\theta_1^2 + 25\theta_2^2 + \theta_1(-225\theta_2 + 85\theta_3))(\theta_1 + \theta_2 + \theta_3 + \theta_4), \\
\mathcal{R}_2^{(3)} &= \frac{1}{91}(91\theta_1^2 + 10\theta_2^2 + \theta_1(\theta_2 - 57\theta_3))(\theta_1 + \theta_2 + \theta_3 + \theta_4), \\
\mathcal{R}_3^{(3)} &= -\theta_1(\theta_2 - \theta_3)(\theta_1 + \theta_2 + \theta_3 + \theta_4), \\
\mathcal{R}_4^{(3)} &= -\frac{1}{91}(273\theta_1^2 + 23\theta_2^2 + \theta_1(-207\theta_2 + 60\theta_3))(\theta_1 + \theta_2 + \theta_3 + \theta_4).
\end{aligned} \tag{C.12}$$

We conclude with the basis of $H^{(4,4)}(\tilde{X}_4)$ fixed by $\mathcal{R}^{(0)} = 1$ as $\mathcal{R}^{(4)} = \frac{1}{359}\mathcal{C}_0|_{J_i \rightarrow \theta_i}$.

C.2 Further Examples of Fourfolds

Here we consider a broader class of Calabi-Yau fourfolds (\tilde{X}_4, X_4) that are constructed as described in section 6.1.2 by fibering Calabi-Yau threefolds \tilde{Z}_3 over \mathbb{P}^1 . The threefolds we consider here are itself elliptically fibered over the two-dimensional base of the Hirzebruch surfaces \mathbb{F}_n for $n = 0, 1$,

$$\begin{array}{ccc} \mathbb{F}_n & \rightarrow & \tilde{Z}_3 \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array} \tag{C.13}$$

Therefore, we will distinguish the constructed mirror pairs (\tilde{X}_4, X_4) by the two-dimensional base \mathbb{F}_n we used to construct the threefold \tilde{Z}_3 .

In the following we will present the toric data of the threefolds \tilde{Z}_3 and fourfolds \tilde{X}_4 including some of their topological quantities. Then we will determine the complete system of Picard-Fuchs differential operators at the large complex structure point of the mirror Calabi-Yau fourfold and calculate the holomorphic prepotential F^0 . From this we extract the invariants n_β^g which are integer in all considered cases. Furthermore we show that there exists a subsector for these invariants that reproduces the closed and open Gromov-Witten invariants of the local Calabi-Yau threefolds obtained by a suitably decompactifying the elliptic fiber of the original compact threefold. This matching allows us to determine the four-form flux G_4 for the F-theory compactification on these fourfolds such that the F-theory flux superpotential (5.98) admits the split (4.41) into flux and brane superpotential of the Type IIB theory.

C.2.1 Fourfold with \mathbb{F}_0

We start with an elliptically fibered Calabi-Yau threefold \tilde{Z}_3 with base given by the toric Fano basis of the zeroth Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Its polyhedron and charge vectors read

$$\left(\begin{array}{c|cccc|ccc} & & \Delta_{\tilde{Z}_3} & & & \ell^{(1)} & \ell^{(2)} & \ell^{(3)} \\ \hline v_0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ v_1^b & 0 & 0 & 2 & 3 & 1 & -2 & -2 \\ v_2^b & 1 & 0 & 2 & 3 & 0 & 1 & 0 \\ v_3^b & -1 & 0 & 2 & 3 & 0 & 1 & 0 \\ v_4^b & 0 & 1 & 2 & 3 & 0 & 0 & 1 \\ v_5^b & 0 & -1 & 2 & 3 & 0 & 0 & 1 \\ v^1 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ v_2 & 0 & 0 & 0 & -1 & 3 & 0 & 0 \end{array} \right), \tag{C.14}$$

where points in the base are again labeled by a superscript b . There is one triangulation for which the Stanley-Reisner ideal in terms of the toric divisors $D_i = \{x_i = 0\}$ takes the form

$$SR = \{D_2D_3, D_4D_5, D_1D_6D_7\}. \tag{C.15}$$

This threefold \tilde{Z}_3 has Euler number $\chi = -480$, $h_{1,1} = 3$ and $h_{2,1} = 243$, where the three Kähler classes correspond to the elliptic fiber and the two \mathbb{P}^1 's of the base \mathbb{F}_0 . The intersection ring for this Calabi-phase in terms of the Kähler cone generators

$$J_1 = D_1 + 2D_2 + 2D_4, \quad J_2 = D_2, \quad J_3 = D_4 \tag{C.16}$$

reads $\mathcal{C}_0 = 8J_1^3 + 2J_1^2J_3 + 2J_1^2J_2 + J_1J_2J_3$ and $\mathcal{C}_2 = 92J_1 + 24J_2 + 24J_3$.

In the local limit $K_{\mathbb{F}_0} \rightarrow \mathbb{F}_0$ Harvey-Lawson type branes described by the brane charge vectors $\hat{\ell}^{(1)} = (-1, 0, 1, 0, 0)$ and $\hat{\ell}^{(1)} = (-1, 0, 0, 1, 0)$ were studied in [107]. To construct the Calabi-Yau fourfold \tilde{X}_4 we use the construction of section 6.1.2 with the brane vector $\hat{\ell}^{(1)}$ and expand $\Delta_{\tilde{Z}_4}$ to the polyhedron $\Delta_{\tilde{X}_5}$ and determine the Mori cone generators $\ell^{(i)}$ with $i = 1, \dots, 5$ for the four different triangulations of the corresponding Calabi-Yau phases. Here we display one of the four triangulations on which we focus our following analysis:

	$\Delta_{\tilde{X}_5}$						$\ell^{(1)}$	$\ell^{(2)}$	$\ell^{(3)}$	$\ell^{(4)}$	$\ell^{(5)}$	
v_0	0	0	0	0	0	-6	0	0	0	0	0). \tag{C.17}
v_1	0	0	2	3	0	1	-1	-2	-1	-1	0	
v_2	1	0	2	3	0	0	1	0	0	0	0	
v_3	-1	0	2	3	0	0	0	0	1	-1	0	
v_4	0	1	2	3	0	0	0	1	0	0	0	
v_5	0	-1	2	3	0	0	0	1	0	0	0	
v_6	0	0	-1	0	0	2	0	0	0	0	0	
v_7	0	0	0	-1	0	3	0	0	0	0	0	
v_8	-1	0	2	3	-1	0	1	0	-1	1	0	
v_9	0	0	2	3	-1	0	-1	0	1	0	0	
v_{10}	0	0	2	3	1	0	0	0	0	1	0	

In this triangulation the Stanley-Reisner ideal takes the form

$$SR = \{D_2D_3, D_2D_8, D_3D_9, D_4D_5, D_8D_{10}, D_9D_{10}, D_1D_6D_7\}. \tag{C.18}$$

The generators of the Kähler cone of the fourfold \tilde{X}_4 in the given triangulation are

$$J_1 = D_1 + 2D_{10} + D_2 + D_3 + 2D_4, \quad J_2 = D_{10}, \quad J_3 = D_4, \quad J_4 = D_{10} + D_3, \quad J_5 = D_2, \tag{C.19}$$

for which the intersections are determined to be

$$\begin{aligned} \mathcal{C}_0 &= 42J_1^4 + 8J_1^3J_2 + 7J_1^3J_3 + 2J_1^2J_2J_3 + 12J_1^3J_4 + 2J_1^2J_2J_4 + 3J_1^2J_3J_4 + J_1J_2J_3J_4 \\ &\quad + 2J_1^2J_4^2 + J_1J_3J_4^2 + 8J_1^3J_5 + 2J_1^2J_2J_5 + 2J_1^2J_3J_5 + J_1J_2J_3J_5 + 2J_1^2J_4J_5 + J_1J_3J_4J_5, \end{aligned} \tag{C.20}$$

$$\begin{aligned} \mathcal{C}_2 &= 92J_1J_2 + 486J_1^2 + 24J_2J_3 + 82J_1J_3 + 24J_3J_5 + 92J_1J_5 + 24J_2J_5 \\ &\quad + 24J_2J_4 + 138J_1J_4 + 36J_3J_4 + 24J_4J_5 + 24J_4^2, \end{aligned} \tag{C.21}$$

$$\mathcal{C}_3 = -2534J_1 - 480J_2 - 420J_3 - 720J_4 - 480J_5.$$

We calculate the core topological quantities to be

$$\chi = 15408, \quad h_{3,1} = 2555, \quad h_{2,1} = 0, \quad h_{1,1} = 5. \quad (\text{C.22})$$

Furthermore, we note that these intersections reveal the fibration structure of \tilde{X}_4 . We recognize the Euler number of the threefold \tilde{Z}_3 as the coefficient of J_2 and J_5 in \mathcal{C}_3 and the fact that both J_2 and J_5 appear at most linear in $\mathcal{C}_0, \mathcal{C}_2$. This is consistent with the fact that the fiber F of a fibration has intersection number 0 with itself which implies $c_3(F) = c_3(\tilde{X}_4)$ using the adjunction formula as well as $c_1(F) + c_1(N_{\tilde{X}_4} F) = c_1(N_{\tilde{X}_4} F) = 0$ for \tilde{X}_4 Calabi-Yau. Thus we observe a fibration of \tilde{Z}_3 represented by the classes J_2 and J_5 over the base curves corresponding to $\ell^{(2)}, \ell^{(5)}$, respectively.

The Picard-Fuchs operators are determined as before and read

$$\begin{aligned} \mathcal{D}_1 &= \theta_1(\theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5) - 12(-5 + 6\theta_1)(-1 + 6\theta_1)z_1, \\ \mathcal{D}_2 &= \theta_2(\theta_2 - \theta_4 + \theta_5) - (-1 + \theta_2 - \theta_4)(-1 - \theta_1 + \theta_2 + 2\theta_3 + \theta_4 + \theta_5)z_2, \\ \mathcal{D}_3 &= \theta_3^2 - (1 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)(2 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)z_3, \\ \mathcal{D}_4 &= (\theta_2 - \theta_4)(\theta_4 - \theta_5) - (1 + \theta_2 - \theta_4 + \theta_5)(-1 - \theta_1 + \theta_2 + 2\theta_3 + \theta_4 + \theta_5)z_4, \\ \mathcal{D}_5 &= \theta_5(\theta_2 - \theta_4 + \theta_5) - (1 + \theta_1 - \theta_2 - 2\theta_3 - \theta_4 - \theta_5)(1 + \theta_4 - \theta_5)z_5. \end{aligned} \quad (\text{C.23})$$

Then we can proceed with fixing the basis of $H_V^{(p,p)}(\tilde{X}_4)$ at each grade p by determining the ring \mathcal{R} of (5.61). We choose a basis at grade $p = 2$ as

$$\begin{aligned} \mathcal{R}_1^{(2)} &= \theta_1(\theta_1 + \theta_5), \quad \mathcal{R}_2^{(2)} = \theta_1(\theta_1 + \theta_2), \quad \mathcal{R}_3^{(2)} = \theta_1(2\theta_1 + \theta_3), \quad \mathcal{R}_4^{(2)} = \theta_1(\theta_1 + \theta_4), \\ \mathcal{R}_5^{(2)} &= \theta_2\theta_3, \quad \mathcal{R}_6^{(2)} = (\theta_2 + \theta_4)(\theta_4 + \theta_5), \quad \mathcal{R}_7^{(2)} = \theta_3\theta_4, \quad \mathcal{R}_8^{(2)} = \theta_3\theta_5. \end{aligned} \quad (\text{C.24})$$

The basis of solution dual to this basis choice is given by

$$\begin{aligned} \mathbb{L}_1^{(2)} &= \frac{1}{8}l_1(l_1 - l_2 - 2l_3 - l_4 + 7l_5), \quad \mathbb{L}_2^{(2)} = \frac{1}{8}l_1(l_1 + 7l_2 - 2l_3 - l_4 - l_5), \\ \mathbb{L}_3^{(2)} &= \frac{1}{4}l_1(l_1 - l_2 + 2l_3 - l_4 - l_5), \quad \mathbb{L}_4^{(2)} = \frac{1}{8}l_1(l_1 - l_2 - 2l_3 + 7l_4 - l_5), \quad \mathbb{L}_5^{(2)} = l_2l_3, \\ \mathbb{L}_6^{(2)} &= \frac{1}{4}(l_2 + l_4)(l_4 + l_5), \quad \mathbb{L}_7^{(2)} = l_3l_4, \quad \mathbb{L}_8^{(2)} = l_3l_5. \end{aligned} \quad (\text{C.25})$$

The topological two-point coupling between the $\mathcal{R}_\alpha^{(2)}$ in the chosen basis reads

$$\eta^{(2)} = \begin{pmatrix} 58 & 60 & 109 & 64 & 3 & 8 & 4 & 2 \\ 60 & 58 & 109 & 64 & 2 & 8 & 4 & 3 \\ 109 & 109 & 196 & 118 & 4 & 20 & 6 & 4 \\ 64 & 64 & 118 & 68 & 3 & 8 & 4 & 3 \\ 3 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\ 8 & 8 & 20 & 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 6 & 4 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{C.26})$$

The basis of $\mathcal{R}^{(3)}$ determining $H^{(3,3)}(\tilde{X}_4)$ that is fixed by Poincaré duality to the Kähler cone generators satisfying $\eta_{ab}^{(3)} = \delta_{a,h(1,1)-b+1}$ is chosen to be

$$\begin{aligned} \mathcal{R}_1^{(3)} &= \frac{1}{4}(9\theta_1\theta_5 - 2\theta_1\theta_3 - \theta_3^2)\theta_3 + \theta_2\theta_3^2 - \theta_1\theta_2\theta_5, \\ \mathcal{R}_2^{(3)} &= \frac{1}{8}(\theta_1\theta_3 + 2\theta_3^2 - 10\theta_1\theta_5)\theta_3 - \theta_2\theta_3^2 - \theta_1\theta_2\theta_5, \quad \mathcal{R}_3^{(3)} = \theta_1(\frac{1}{2}\theta_3^2 - \theta_3\theta_5 - 2\theta_2\theta_5), \\ \mathcal{R}_4^{(3)} &= \theta_1\theta_2\theta_5, \quad \mathcal{R}_5^{(3)} = \frac{1}{8}\theta_3(2\theta_3^2 - 3\theta_1\theta_3 - 10\theta_1\theta_5 - 4\theta_2\theta_3) - \theta_1\theta_2\theta_5. \end{aligned} \quad (\text{C.27})$$

We choose the basis of $H^{(4,4)}(\tilde{X}_4)$ such that the volume is normalized as $\eta_{a_0,b_0}^{(4)} = 1$ for $\mathcal{R}^{(0)} = 1$, i.e. $\mathcal{R}^{(4)} = \frac{1}{96}\mathcal{C}_0|_{J \rightarrow \theta}$.

In order to fix the integral basis of $H_V^{(2,2)}(\tilde{X}_4)$ we again match the threefold periods from the fourfold periods via (6.49). The first step is to identify the Kähler classes of \tilde{Z}_3 . As discussed above J_5 represents the class of the Calabi-Yau fiber \tilde{Z}_3 . The intersections of \tilde{Z}_3 are obtained from (C.20) upon the identification

$$J_1 \leftrightarrow J_1(\tilde{Z}_3) \quad J_2 + J_4, \leftrightarrow J_2(\tilde{Z}_3), \quad J_3 \leftrightarrow J_3(\tilde{Z}_3). \quad (\text{C.28})$$

With this in mind we calculate the leading logarithms $\mathbb{L}_\alpha(Z_3)$ on the threefold given by

$$\mathbb{L}_1(Z_3) = \frac{1}{2}X_0(2\tilde{l}_1 + \tilde{l}_2)(2\tilde{l}_1 + \tilde{l}_3), \quad \mathbb{L}_2(Z_3) = \frac{1}{2}X_0\tilde{l}_1(\tilde{l}_1 + \tilde{l}_3), \quad \mathbb{L}_3(Z_3) = \frac{1}{2}X_0\tilde{l}_1(\tilde{l}_1 + \tilde{l}_2). \quad (\text{C.29})$$

This together with the requirement of matching the instanton numbers¹ n_{d_1,d_2,d_3} of \tilde{Z}_3 via $n_{d_1,d_2,d_3,d_2,0}$ on \tilde{X}_4 fixes unique solutions of the Picard-Fuchs system

$$\mathbb{L}_1^{(2)} = \frac{1}{2}X_0(2l_1+l_3)(2l_1+l_2+l_4), \quad \mathbb{L}_6^{(2)} = \frac{1}{2}X_0l_1(l_1+l_3), \quad \mathbb{L}_8^{(2)} = \frac{1}{2}X_0l_1(l_1+l_2+l_4), \quad (\text{C.30})$$

that upon (C.28) coincide with the threefold solutions. This fixes three ring elements $\tilde{\mathcal{R}}_\alpha^{(2)}$, $\alpha = 1, 6, 8$, by the map induced from (6.46) that we complete to a new basis

$$\begin{aligned} \tilde{\mathcal{R}}_1^{(2)} &= \frac{1}{8}\theta_1(\theta_1 - \theta_2 - 2\theta_3 - \theta_4 + 7\theta_5), \quad \tilde{\mathcal{R}}_2^{(2)} = \frac{1}{8}\theta_1(\theta_1 + 7\theta_2 - 2\theta_3 - \theta_4 - \theta_5), \\ \tilde{\mathcal{R}}_3^{(2)} &= \frac{1}{4}\theta_1(\theta_1 - \theta_2 + 2\theta_3 - \theta_4 - \theta_5), \quad \tilde{\mathcal{R}}_4^{(2)} = \frac{1}{8}\theta_1(\theta_1 - \theta_2 - 2\theta_3 + 7\theta_4 - \theta_5), \\ \tilde{\mathcal{R}}_5^{(2)} &= \theta_2\theta_3, \quad \tilde{\mathcal{R}}_6^{(2)} = \frac{1}{4}(\theta_2 + \theta_4)(\theta_4 + \theta_5), \quad \tilde{\mathcal{R}}_7^{(2)} = \theta_3\theta_4, \quad \tilde{\mathcal{R}}_8^{(2)} = \theta_3\theta_5. \end{aligned} \quad (\text{C.31})$$

Then the integral basis elements are given by

$$\hat{\gamma}_1^{(2)} = \tilde{\mathcal{R}}_1^{(2)}\Omega_4|_{z=0}, \quad \hat{\gamma}_6^{(2)} = \tilde{\mathcal{R}}_6^{(2)}\Omega_4|_{z=0}, \quad \hat{\gamma}_8^{(2)} = \tilde{\mathcal{R}}_8^{(2)}\Omega_4|_{z=0}, \quad (\text{C.32})$$

where again the new grade $p = 2$ basis is obtained by replacing $l_i \leftrightarrow \theta_i$ in the dual solutions of (C.25). We conclude by presenting the leading logarithms of the periods $\Pi^{(2)\alpha}$ when integrating Ω_4 over the duals $\gamma^{(2)\alpha}$ for $\alpha = 1, 6, 8$. They are then as well given by $\mathbb{L}^{(2)1} = X_0l_1(l_1 + l_5)$, $\mathbb{L}^{(2)6} = X_0(l_2 + l_4)(l_4 + l_5)$ and $\mathbb{L}^{(2)8} = X_0l_3l_5$.

¹We note here that by just matching the threefold instantons the solution on the fourfold could not be fixed. The two free parameters could only be determined by matching the classical terms, too.

Finally we determine a $\hat{\gamma}$ flux in $H_H^{(2,2)}(X_4)$ such that we match the disk invariants of [107] for both classes of the local geometry $K_{\mathbb{F}_0} \rightarrow \mathbb{F}_0$ with the brane class. Furthermore we reproduce the closed invariants of [264] for the two \mathbb{P}^1 -classes for zero brane winding $m = 0$. First we identify in the polyhedron (C.17) the vector $\ell^{(4)}$ as corresponding to the brane vector. Then we expect to recover the disk invariants from the fourfold invariants $n_{0,d_1,d_2,d_1+m,0}$. Then the flux $\hat{\gamma}$ deduced this way still contains a freedom of three parameters and takes the form

$$\hat{\gamma} = (-\mathcal{R}_5^{(2)} + \frac{1}{4}\mathcal{R}_6^{(2)} + \mathcal{R}_7^{(2)} + \frac{1}{2}\mathcal{R}_8^{(2)})\Omega_4|_{z=0} \quad (\text{C.33})$$

where we choose the free parameters \underline{a} in front of $\mathcal{R}_1^{(2)}$, $\mathcal{R}_2^{(2)}$, $\mathcal{R}_3^{(2)}$ and $\mathcal{R}_4^{(2)}$ to be zero. Note that $a_7 = 1$ is fixed by the requirement of matching the disk invariants. For this parameter choice the leading logarithmic structures of the corresponding period $\int_\gamma \Omega_4$ and of the solution matching the invariants are respectively given by

$$\mathbb{L}^{(2)\gamma} = X_0(l_2 + l_4)(l_4 + l_5) \ , \quad \mathbb{L}_\gamma^{(2)} = \frac{1}{2}X_0l_1(4l_1 + 3l_2 + 2l_3 + l_4) \ . \quad (\text{C.34})$$

C.2.2 Fourfold with \mathbb{F}_1

Here we consider an elliptically fibered Calabi-Yau threefold \tilde{Z}_3 with base twofold given by $\mathbb{F}_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ which is the blow-up of \mathbb{P}^2 at one point. The polyhedron and charge vectors read

$$\left(\begin{array}{c|ccc|ccc} & & \Delta_{\tilde{Z}_4} & & \ell^{(1)} & \ell^{(2)} & \ell^{(3)} \\ \hline v_0 & 0 & 0 & 0 & 0 & -6 & 0 \\ v_1^b & 0 & 0 & 2 & 3 & -1 & 0 & -2 \\ v_2^b & 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ v_3^b & -1 & 0 & 2 & 3 & 1 & 0 & 0 \\ v_4^b & 0 & 1 & 2 & 3 & -1 & 0 & 1 \\ v_5^b & 0 & -1 & 2 & 3 & 0 & 0 & 1 \\ v^1 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ v_2 & 0 & 0 & 0 & -1 & 0 & 3 & 0 \end{array} \right) \ . \quad (\text{C.35})$$

where the labels by a superscript b again denote points in the base. There are two Calabi-Yau phases and for the triangulation given above the Stanley-Reisner ideal reads

$$SR = \{D_2D_3, D_4D_5, D_1D_6D_7\}. \quad (\text{C.36})$$

This threefold has Euler number $\chi = 480$, $h_{1,1} = 3$ and $h_{2,1} = 243$, where the three Kähler classes correspond to the elliptic fiber and the two \mathbb{P}^1 's of the base \mathbb{F}_1 . The intersection ring for this Calabi-phase in terms of the Kähler cone generators

$$J_1 = D_2, \quad J_2 = D_1 + 3D_2 + 2D_4, \quad J_3 = D_2 + D_4 \quad (\text{C.37})$$

reads $\mathcal{C}_0 = 2J_1J_2^2 + 8J_2^3 + J_1J_2J_3 + 3J_2^2J_3 + J_2J_3^2$ and $\mathcal{C}_2 = 24J_1 + 92J_2 + 36J_3$.

For the second Calabi-Yau phase we have the following data:

$$\begin{aligned} & \left(\begin{array}{c|cccccc} \ell^{(1)} & -6 & 0 & 1 & 1 & -1 & 0 & 2 & 3 \\ \ell^{(2)} & 0 & -3 & 1 & 1 & 0 & 1 & 0 & 0 \\ \ell^{(3)} & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{array} \right), \\ SR &= \{D_1 \cdot D_4, D_4 \cdot D_5, D_1 \cdot D_6 \cdot D_7, D_2 \cdot D_3 \cdot D_5, D_2 \cdot D_3 \cdot D_6 \cdot D_7\}, \\ J_1 &= D_1 + 3D_2 + 2D + 4, \quad J_2 = D_2 + D_4, \quad J_3 = D_1 + 3D_2 + 3D_4, \\ \mathcal{C}_0 &= 8J_1^3 + 3J_1^2J_2 + J_1J_2^2 + 9J_1^2J_3 + 3J_1J_2J_3 + J_2^2J_3 + 9J_1J_3^2 + 3J_2J_3^2 + 9J_3^3, \\ \mathcal{C}_2 &= 92J_1 + 36J_2 + 102J_3. \end{aligned} \tag{C.38}$$

Harvey-Lawson type branes were considered in [107] for the brane charge vectors $\hat{\ell}^{(1)} = (-1, 1, 0, 0, 0)$ and $\hat{\ell}^{(1)} = (-1, 0, 0, 1, 0)$ for the non-compact model $K_{\mathbb{F}_1} \rightarrow \mathbb{F}_1$. The Calabi-Yau fourfold \tilde{X}_4 is constructed from the brane vector $\hat{\ell}^{(1)}$ for which there are eleven triangulations. Again we restrict our attention to one triangulation with the following data

$$\begin{aligned} & \left(\begin{array}{c|cccccccccc} \ell^{(1)} & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \ell^{(2)} & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \ell^{(3)} & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ \ell^{(4)} & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \ell^{(5)} & -6 & 1 & 0 & 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 \end{array} \right), \\ SR &= \{D_2 \cdot D_3, D_2 \cdot D_8, D_3 \cdot D_9, D_4 \cdot D_5, D_8 \cdot D_{10}, D_9 \cdot D_{10}, D_1 \cdot D_6 \cdot D_7\}, \\ J_1 &= D_2, \quad J_2 = D_1 + 2D_{10} + D_2 + D_3 + 2D_4, \quad J_3 = D_4, \quad J_4 = D_{10}, \quad J_5 = D_{10} + D_3 \end{aligned}$$

with intersections

$$\begin{aligned} \mathcal{C}_0 &= J_1J_2J_4J_5 + J_2^2J_4J_5 + J_1J_3J_4J_5 + J_2J_3J_4J_5 + J_1J_4^2J_5 + J_2J_4^2J_5 \\ &+ 2J_1J_2J_5^2 + 2J_2^2J_5^2 + 2J_1J_3J_5^2 + 2J_2J_3J_5^2 + 3J_1J_4J_5^2 + 4J_2J_4J_5^2 \\ &+ 2J_3J_4J_5^2 + 2J_4^2J_5^2 + 8J_1J_5^3 + 12J_2J_5^3 + 8J_3J_5^3 + 11J_4J_5^3 + 42J_5^4, \\ \mathcal{C}_2 &= 24J_1J_2 + 24J_2^2 + 24J_1J_3 + 24J_2J_3 + 36J_1J_4 + 48J_2J_4 + 24J_3J_4 + 24J_4^2 \\ &+ 92J_1J_5 + 138J_2J_5 + 92J_3J_5 + 128J_4J_5 + 486J_5^2, \\ \mathcal{C}_3 &= -480J_1 - 270J_2 - 480J_3 - 660J_4 - 2534J_5. \end{aligned} \tag{C.39}$$

Furthermore, we determine $\chi = 15408$, $h^{(3,1)} = 2555$, $h^{(2,1)} = 0$ and $h^{(1,1)} = 5$.

Again the Euler number of the threefold \tilde{Z}_3 appears in \mathcal{C}_3 in front of J_1 and J_3 confirming the fibration structure. By comparing the coefficient polynomial of J_1, J_3 with the threefold intersection rings presented in appendix C.2.1, C.2.2 we infer that J_1 is precisely $\tilde{Z}_3 = \mathcal{E} \rightarrow \mathbb{F}_1$, whereas J_3 is $\tilde{Z}'_3 = \mathcal{E} \rightarrow \mathbb{F}_0$. Since we discussed \mathbb{F}_0 in detail before we will just concentrate on the fibration structure involving \mathbb{F}_1 .

The Picard-Fuchs operators of X_4 read as

$$\begin{aligned}
\mathcal{D}_1 &= \theta_1(\theta_1 - \theta_2 + \theta_3) - (-1 + \theta_1 - \theta_2)(-1 + \theta_1 + \theta_2 + 2\theta_4 - \theta_5)z_1, \\
\mathcal{D}_2 &= (\theta_1 - \theta_2)(\theta_2 - \theta_3) - (1 + \theta_1 - \theta_2 + \theta_3)(-1 + \theta_1 + \theta_2 + 2\theta_4 - \theta_5)z_2, \\
\mathcal{D}_3 &= -\theta_3(\theta_1 - \theta_2 + \theta_3) - (1 + \theta_2 - \theta_3)(-1 + \theta_3 - \theta_4)z_3, \\
\mathcal{D}_4 &= \theta_4(-\theta_3 + \theta_4) - (-2 + \theta_1 + \theta_2 + 2\theta_4 - \theta_5)(-1 + \theta_1 + \theta_2 + 2\theta_4 - \theta_5)z_4, \\
\mathcal{D}_5 &= \theta_5(-\theta_1 - \theta_2 - 2\theta_4 + \theta_5) - 12(-5 + 6\theta_5)(-1 + 6\theta_5)z_5,
\end{aligned} \tag{C.40}$$

from which we determine the basis of $\mathcal{R}^{(2)}$ as

$$\begin{aligned}
\mathcal{R}_1^{(2)} &= (\theta_1 + \theta_2)(\theta_2 + \theta_3), & \mathcal{R}_2^{(2)} &= \theta_1\theta_4, & \mathcal{R}_3^{(2)} &= \theta_5(\theta_1 + \theta_5), & \mathcal{R}_4^{(2)} &= \theta_2\theta_4, \\
\mathcal{R}_5^{(2)} &= \theta_5(\theta_2 + \theta_5), & \mathcal{R}_6^{(2)} &= \theta_4(\theta_3 + \theta_4), & \mathcal{R}_7^{(2)} &= \theta_3\theta_5, & \mathcal{R}_8^{(2)} &= \theta_5(\theta_4 + 2\theta_5)
\end{aligned} \tag{C.41}$$

with the two-point coupling

$$\eta^{(2)} = \begin{pmatrix} 0 & 0 & 8 & 0 & 8 & 0 & 0 & 20 \\ 0 & 0 & 3 & 0 & 4 & 0 & 1 & 7 \\ 8 & 3 & 58 & 5 & 64 & 6 & 10 & 114 \\ 0 & 0 & 5 & 0 & 5 & 0 & 1 & 9 \\ 8 & 4 & 64 & 5 & 68 & 6 & 10 & 123 \\ 0 & 0 & 6 & 0 & 6 & 0 & 0 & 8 \\ 0 & 1 & 10 & 1 & 10 & 0 & 0 & 18 \\ 20 & 7 & 114 & 9 & 123 & 8 & 18 & 214 \end{pmatrix}. \tag{C.42}$$

The dual basis of solutions reads

$$\begin{aligned}
\mathbb{L}^{(2)1} &= \frac{1}{4}(l_1 + l_2)(l_2 + l_3), & \mathbb{L}^{(2)2} &= l_1l_4, & \mathbb{L}^{(2)3} &= \frac{1}{7}l_5(6l_1 - l_2 - 2l_4 + l_5), \\
\mathbb{L}^{(4)1} &= l_2l_4, & \mathbb{L}^{(2)5} &= \frac{1}{7}l_5(-l_1 + 6l_2 - 2l_4 + l_5), & \mathbb{L}^{(2)6} &= \frac{1}{2}l_4(l_3 + l_4), & \mathbb{L}^{(2)7} &= l_3l_5, \\
\mathbb{L}^{(2)8} &= \frac{1}{7}l_5(-2l_1 - 2l_2 + 3l_4 + 2l_5).
\end{aligned} \tag{C.43}$$

We determine $H^{(3,3)}(\tilde{X}_4)$ by duality to the canonical basis of $H^{(1,1)}(\tilde{X}_4)$ by the basis choice of $\mathcal{R}^{(3)}$ given as

$$\begin{aligned}
\mathcal{R}_1^{(3)} &= \theta_1\theta_2\theta_4, & \mathcal{R}_2^{(3)} &= -2\theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_5, & \mathcal{R}_3^{(3)} &= -\theta_1\theta_2\theta_5 + \theta_2\theta_4\theta_5 - \theta_3\theta_4\theta_5, \\
\mathcal{R}_4^{(3)} &= -\theta_1\theta_2\theta_4 + \theta_1\theta_4\theta_5 - \theta_2\theta_4\theta_5 + \theta_3\theta_4\theta_5, & \mathcal{R}_5^{(3)} &= -\theta_1\theta_2\theta_4 - \theta_1\theta_4\theta_5 + \theta_2\theta_4\theta_5.
\end{aligned}$$

Our choice for a basis of $H^{(4,4)}(\tilde{X}_4)$ is given by $\mathcal{R}^{(4)} = \frac{1}{106}\mathcal{C}_0|_{J_i \rightarrow \theta_i}$.

Again we fix the integral basis of $H^{(2,2)}(\tilde{X}_4)$ by the requirement of recovering the threefold periods from the fourfold ones. We readily identify the Kähler classes of the threefold \tilde{Z}_3 among the fourfold classes as

$$J_2 + J_3 \leftrightarrow J_1(\tilde{Z}_3), \quad J_5 \leftrightarrow J_2(\tilde{Z}_3), \quad J_4 \leftrightarrow J_3(\tilde{Z}_3), \tag{C.44}$$

which matches the threefold intersections by identifying $J_1 \equiv \tilde{Z}_3$ in the fourfold intersections (C.39). Then we calculate the classical terms of the threefold periods to be

$$\mathbb{L}_1(\tilde{Z}_3) = \tilde{l}_2(\tilde{l}_2 + \tilde{l}_3) , \quad \mathbb{L}_2(\tilde{Z}_3) = \frac{1}{2}(2\tilde{l}_2 + \tilde{l}_3)(2\tilde{l}_1 + 4\tilde{l}_2 + \tilde{l}_3) , \quad \mathbb{L}_3(\tilde{Z}_3) = \frac{1}{2}l_2(2l_1 + 3l_2 + 2l_3) . \quad (\text{C.45})$$

On the fourfold X_4 we determine the periods that match this leading logarithmic structure. They are given by

$$\begin{aligned} \mathbb{L}_1^{(2)} &= X_0 l_5 (l_4 + l_5) , & \mathbb{L}_2^{(2)} &= \frac{1}{2} X_0 (l_4 + 2l_5) (2(l_2 + l_3) + l_4 + 4l_5) , \\ \mathbb{L}_3^{(2)} &= \frac{1}{2} X_0 l_5 (2(l_2 + l_3) + 2l_4 + 3l_5) \end{aligned} \quad (\text{C.46})$$

and immediately coincide with the threefold result using (C.44). It can be shown explicitly that the instanton series contained in the corresponding full solution matches the series on the threefold as well. The threefold invariants n_{d_1, d_2, d_3} are obtained as $n_{0, d_1, d_1, d_3, d_2}$ from the fourfold invariants. To these solutions we associate using (6.46) ring elements $\mathcal{R}_\alpha^{(2)}$, $\alpha = 1, 3, 2$, that we complete to a new basis as

$$\begin{aligned} \tilde{\mathcal{R}}_1^{(2)} &= \frac{1}{4}(\theta_1 + \theta_2)(\theta_2 + \theta_3) , & \tilde{\mathcal{R}}_2^{(2)} &= \theta_1 \theta_4 , & \tilde{\mathcal{R}}_3^{(2)} &= \frac{1}{7} \theta_5 (6\theta_1 - \theta_2 - 2\theta_4 + \theta_5) , \\ \tilde{\mathcal{R}}_4^{(2)} &= \theta_2 \theta_4 , & \tilde{\mathcal{R}}_5^{(2)} &= \frac{1}{7} \theta_5 (-\theta_1 + 6\theta_2 - 2\theta_4 + \theta_5) , & \tilde{\mathcal{R}}_6^{(2)} &= \frac{1}{2} \theta_4 (\theta_3 + \theta_4) , \\ \tilde{\mathcal{R}}_7^{(2)} &= \theta_3 \theta_5 , & \tilde{\mathcal{R}}_8^{(2)} &= \frac{1}{7} \theta_5 (-2\theta_1 - 2\theta_2 + 3\theta_4 + 2\theta_5) , \end{aligned} \quad (\text{C.47})$$

where we again note that the basis of dual solutions and the new ring basis coincide by $l_i \leftrightarrow \theta_i$. Then the integral basis elements read

$$\hat{\gamma}_1^{(2)} = \tilde{\mathcal{R}}_1^{(2)} \Omega_4|_{z=0} , \quad \hat{\gamma}_2^{(2)} = \tilde{\mathcal{R}}_2^{(2)} \Omega_4|_{z=0} , \quad \hat{\gamma}_3^{(2)} = \tilde{\mathcal{R}}_3^{(2)} \Omega_4|_{z=0} , \quad (\text{C.48})$$

such that we obtain the full solution with the above leading parts $\mathbb{L}_\alpha^{(2)}$ as $\Pi_\alpha^{(2)} = \int \Omega_4 \wedge \hat{\gamma}_\alpha$. The leading behavior of the periods $\Pi^{(2)\alpha}$ is then given as $\mathbb{L}^{(2)1} = X_0(l_1 + l_2)(l_2 + l_3)$, $\mathbb{L}^{(2)2} = X_0 l_1 l_4$, $\mathbb{L}^{(2)3} = X_0 l_5(l_1 + l_5)$, respectively

We conclude by determining the flux element $\hat{\gamma}$ in $H_H^{(2,2)}(X_4)$ that reproduces the disk invariants in the phase II of [107], where the local geometry $K_{\mathbb{F}_1} \rightarrow \mathbb{F}_1$ is considered. First we identify $\ell^{(2)}$ of the toric data in (C.39) as the vector encoding the brane physics. Therefore, we expect the fourfold invariants $n_{0, m+d_1, d_1, d_2, 0}$ to coincide with the disk invariants what can be checked in a direct calculation. The ring element yielding this result reads $\hat{\gamma} = \mathcal{R}_4^{(2)}$ where the free coefficients in front of the other ring elements were chosen to vanish. The leading logarithmic parts of the period $\int_\gamma \Omega_4$ and of the solution $\Pi_\gamma^{(2)} = \int \Omega_4 \wedge \hat{\gamma} \equiv W_{\text{brane}}$ respectively read

$$\mathbb{L}_\gamma^{(2)} = X_0 l_5 (l_1 + l_2 + l_3 + l_4 + 2l_5) , \quad \mathbb{L}^{(2)\gamma} = X_0 l_2 l_4 . \quad (\text{C.49})$$

C.3 Compact Disk Instantons on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
0	0	1	0	0	0	0	0	0
1	2	-1	-1	-1	-1	-1	-1	-1
2	-8	5	7	9	12	15	19	23
3	54	-40	-61	-93	-140	-206	-296	-416
4	-512	399	648	1070	1750	2821	4448	6868
5	5650	-4524	-7661	-13257	-22955	-39315	-66213	-109367
6	-68256	55771	97024	173601	312704	559787	989215	1719248
7	879452	-729256	-1293185	-2371088	-4396779	-8136830	*	*
8	-11883520	9961800	17921632	33470172	*	*	*	*
9	166493394	-140747529	*	*	*	*	*	*

Table C.1: $k = 0$: Disk instanton invariants $n_{i,k,i+j,k}$ on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$ at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. These results for $k = 0$ agree with phase I, II of [107]. Entries * exceed the order of our calculation.

i	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
0	**	0	0	0	0	0
1	300	300	300	300	300	300
2	-2280	-3180	-4380	-5880	-7680	-9780
3	24900	39120	61620	95400	144060	211800
4	-315480	-526740	-892560	-1500900	-2477580	-3996780
5	4340400	7516560	13329060	23641980	41421000	71240400
6	-62932680	-111651720	-204177600	-375803820	-686849280	*
7	946242960	1707713040	3192621180	*	*	*

Table C.2: $k = 1$: Compact disk instanton invariants $n_{i,k,i+j,k}$ for brane phase I, II on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$ at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. The entry ** as well as the $j = 0$ column could not be fixed by our calculation.

i	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
0	**	0	0	0	0	0
1	-62910	-62910	-62910	-62910	-62910	-62910
2	778560	1146690	1622580	2206530	2898240	3698010
3	-12388860	-20596140	-33454530	-52626780	-80081460	-118092960
4	208471080	368615070	645132360	1103916150	1838367780	2976756210
5	-3588226470	-6587809920	-12083913000	-21840712470	*	*
6	62538887280	117754228980	*	*	*	*

Table C.3: $k = 2$: Compact disk instanton invariants $n_{i,k,i+j,k}$ for brane phase I, II on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$ at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. The entry ** as well as the $j = 0, 1$ columns could not be fixed by our calculation.

i	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
0	0	-1	0	0	0	0	0	0
1	**	2	1	1	1	1	1	1
2	**	-5	-4	-3	-4	-5	-7	-9
3	**	32	21	18	20	26	36	52
4	**	-286	-180	-153	-160	-196	-260	-365
5	**	3038	1885	1560	1595	1875	2403	3254
6	**	-35870	-21952	-17910	-17976	-20644	-25812	-34089
7	**	454880	275481	222588	220371	249120	*	*
8	**	-6073311	-3650196	-2926959	*	*	*	*
9	**	84302270	*	*	*	*	*	*

Table C.4: $k = 0$: Disk instanton invariants $n_{i,k,i,k+j}$ on $\mathbb{P}^4(1, 1, 1, 6, 9)[18]$ at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. These results for $k = 0$ agree with phase III of [107].

i	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
0	0	0	0	0	0	0	0
1	-540	-300	-300	-300	-300	-300	-300
2	2160	1620	1680	2280	3180	4380	5880
3	-18900	-12960	-12300	-15000	-21060	-31200	-47220
4	216000	140940	126240	142380	185280	261300	386160
5	-2800980	-1775520	-1535160	-1653900	-2046060	-2750280	-3896760
6	39087360	24316200	20544720	21489780	25725600	*	*
7	-572210460	-351319680	-292072920	*	*	*	*
8	8663561280	*	*	*	*	*	*

Table C.5: $k = 1$: Compact disk instanton invariants $n_{i,k,i,k+j}$ for brane phase III on $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18] at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. The $j = 0$ column could not be fixed by our calculation.

i	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
0	0	0	0	0	0	0
1	**	62910	62910	62910	62910	62910
2	-430110	-413640	-557010	-836340	-1223730	-1718880
3	5190480	3923100	4415580	6237810	9720180	15561180
4	-76785570	-52941600	-52475850	-65786040	-93752550	-143003760
5	1227227760	806981670	747944550	869842800	1154721060	*
6	-20387141100	-13027278600	-11592978930	*	*	*
7	346430247840	*	*	*	*	*

Table C.6: $k = 2$: Compact disk instanton invariants $n_{i,k,i,k+j}$ for brane phase III on $\mathbb{P}^4(1, 1, 1, 6, 9)$ [18] at large volume. i, k label the classes t_1, t_2 of Z_3 , where j labels the brane winding. The entry ** and the $j = 0$ column could not be fixed by our calculation.

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