Topological string theory, modularity and non-perturbative physics

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Marco Rauch
aus
Memmingen

Bonn 2011

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

1. Gutachter:	Prof. Dr. Albrecht Klemm
2. Gutachter:	PD Dr. Stefan Förste
Tag der Promotion:	19. September 2011
Erscheinungsjahr:	2011

ABSTRACT

In this thesis the holomorphic anomaly of correlators in topological string theory, matrix models and supersymmetric gauge theories is investigated. In the first part it is shown how the techniques of direct integration known from topological string theory can be used to solve the closed amplitudes of Hermitian multi-cut matrix models with polynomial potentials. In the case of the cubic matrix model, explicit expressions for the ring of non-holomorphic modular forms that are needed to express all closed matrix model amplitudes are given. This allows to integrate the holomorphic anomaly equation up to holomorphic modular terms that are fixed by the gap condition up to genus four. There is an one-dimensional submanifold of the moduli space in which the spectral curve becomes the Seiberg–Witten curve and the ring reduces to the non-holomorphic modular ring of the group $\Gamma(2)$. On that submanifold, the gap conditions completely fix the holomorphic ambiguity and the model can be solved explicitly to very high genus. Using these results it is possible to make precision tests of the connection between the large order behavior of the $1/N$ expansion and non-perturbative effects due to instantons. Finally, it is argued that a full understanding of the large genus asymptotics in the multi-cut case requires a new class of non-perturbative sectors in the matrix model. In the second part a holomorphic anomaly equation for the modified elliptic genus of two M5-branes wrapping a rigid divisor inside a Calabi-Yau manifold is derived using wall-crossing formulae and the theory of mock modular forms. The anomaly originates from restoring modularity of an indefinite theta-function capturing the wall-crossing of BPS invariants associated to D4-D2-D0 brane systems. The compatibility of this equation with anomaly equations previously observed in the context of $\mathcal{N}=4$ topological Yang-Mills theory on $\mathbb{P}^2$ and E-strings obtained from wrapping M5-branes on a del Pezzo surface which in turn is related to topological string theory is shown. The non-holomorphic part is related to the contribution originating from bound-states of singly wrapped M5-branes on the divisor. In examples it is shown that the information provided by the anomaly is enough to compute the BPS degeneracies for certain charges.

ACKNOWLEDGEMENTS

I am indebted to my "Doktorvater" Prof. Dr. Albrecht Klemm for giving me the opportunity to work, learn and study with him on this exciting subject. I am grateful for his support and that he accompanied me over the last couple of years.

Not the least less importantly I have to thank my other collaborators Prof. Dr. Marcos Mariño, Dr. Murad Alim, Dr. Babak Haghighat, Michael Hecht and Thomas Wotschke for sharing their ideas with me and for countless brilliant discussions. It was challenging, but still fun to work with and learn from them.

Further, I would like to thank the other members of our group Dr. Denis Klevers, Daniel Viera Lopes, Maximilian Poretschkin, Jose Miguel Zapata Rolon and Marc Schiereck as well as the former group members Dr. Thomas Grimm, Dr. Tae-Won Ha and Dr. Piotr Sułkowski for patiently answering numerous questions and for a joyful time and atmosphere in the offices of room 104.

Last, this work was partially supported by the Bonn-Cologne Graduate School of Physics and Astronomy.

Contents

Chapter 1

Introduction

The study of mathematical structures underlying physical models has always been a rich source of valuable insights for both physics as well as mathematics. The central theme of this thesis is to investigate the properties of certain correlators or amplitudes in topological string theory and its by large N-duality and geometric engineering related descriptions. The main emphasis is laid on an dichotomy of holomorphicity and modularity whose far-reaching physical, computational and structural implications are studied in the following.

Starting point are the various guises of superstring theory. Despite its drawbacks like the problem of featuring extremely many vacua, the lack of a non-perturbative description or the difficulty of background independence, string theory has lead to many new insights in physics and mathematics. To name a few, it offers a theoretical ground to study quantum gravity effects such as the microscopic origin of the Bekenstein-Hawking entropy of black holes. Furthermore, it provides a geometric description of supersymmetric gauge theory which led, for instance, to a geometric understanding of S-duality. Moreover, it gives a unifying description of gauge theory with gravity and has sparked many applications surrounding the AdS/CFT correspondence.

Often the study of non-perturbative aspects of string theory is most exciting. However, this has to be achieved indirectly by considering for instance BPS states of the theory. These are states that are protected by supersymmetry and are present at weak and strong coupling. Another avenue to study non-perturbative effects is to examine the large order behavior of perturbation theory. In both cases, the investigation of toy models of string theory has been highly promising and led to a deeper understanding of string theory. One such toy model is topological string theory introduced in [162, 163]. Following [139], the topological string can be understood as a localized version of the full superstring in the sense that the path integral only receives special contributions from the classical configurations. The importance lies in the fact that certain BPS observables of the physical string are localizations of the same special contributions. Thus, these physical observables can be computed in the simpler framework of topological string theory but are then valid for the physical string as well.

Beside the direct applications of topological string theory to the physical string it is interesting to study it on its own. Topological string theory is based on a non-linear sigma model with $\mathcal{N} = (2, 2)$ world-sheet supersymmetry whose fields take values in a six-dimensional target space X. Supersymmetry implies that the target space is a Kähler manifold and by imposing conformal invariance X is shown to be Calabi-Yau. By a redefinition of charge and spin of the fields the field theory is twisted into a topological theory. This twist can be achieved in two different ways giving rise to the so-called A-model and the B-model. Upon coupling to gravity one obtains two versions of topological string theory. Compared to the physical string the A-model is a restriction of type IIA theory and the B-model receives only special contributions of the type IIB string. Supersymmetric localization implies that the A-model reduces to holomorphic maps from the world-sheet onto two-cycles of the Calabi-Yau manifold and that in the B-model only constant maps contribute to the path integral. The moduli of topological string theory are given by the couplings of the non-linear sigma model and can be interpreted geometrically as the complexified Kähler parameters and complex structure parameters of the Calabi-Yau manifold X for the A-model and the B-model, respectively.

Features of topological string theory

Topological string theory is mainly concerned with the computation of its partition function $Z(g_s, t)$, where t are the moduli and g_s is the string coupling constant. Depending on the type of world-sheet, one distinguishes between open and closed topological strings. Open topological strings are built from a Riemann surface $\Sigma_{g,h}$ of genus g with h holes, whereas closed topological strings stem from a compact Riemann surface Σ_g of genus g. The latter enjoys a genus expansion of the form

$$Z(g_s, t) = \exp\left(\sum_{g=0}^{\infty} F_g(t)\, g_s^{2g-2}\right),$$

where $F_g(t)$ are the genus g free energies. Having computed $F_g(t)$ globally on the moduli space it is straightforward to derive all closed topological string correlators solving the theory perturbatively. In the following we collect some interesting features of these free energies.

Mirror symmetry

Mirror symmetry is an equivalence of A-model topological strings on a Calabi-Yau three-fold X with B-model topological strings on a different Calabi-Yau three-fold Y and can be thought of as a generalization of T-duality [152]. In particular, all correlation functions of the A-model get identified with the same correlation functions on the B-model side by providing a map between the complexified Kähler moduli t of X and the complex structure parameter z of Y. The map $t = t(z)$ is called the mirror map and in order to make sense the mirror pair (X, Y) has to fulfill $h^{p,q}(X) = h^{3-p,q}(Y)$. The tremendous benefit of mirror symmetry is, that it opens a possibility to study the prepotential $F_0(t)$, the higher genus free energies $F_g(t)$ and their interpretation in terms of enumerative geometry on the A-model side through an easier computation on the B-model side.

Holomorphic anomaly, modularity and direct integration

From the non-linear sigma model point of view there exists an anti-A-model and an anti-B-model which are related to the A- and B-model by exchanging left-movers with right-movers. Due to the topological property anti-A(B)-model and A(B)-model decouple such that $F_g(t)$ should only depend holomorphically on the moduli t. However, upon coupling to gravity this decoupling is broken and leads to a dependence of $F_g(t)$ on $\bar{t}$. This is called the holomorphic anomaly and it can be summarized in a set of differential equations governing the $F_g(t)$ that are recursive in the genus [15]. Another variant of the holomorphic anomaly is given by studying the symmetry properties of the topological string amplitudes. On the B-model side one is interested in the group Γ of large, Ω-preserving diffeomorphisms, where Ω is the holomorphic three-form on the Calabi-Yau manifold. Γ is a discrete subgroup of $\mathrm{Sp}(b_3, \mathbb{Z})$ generated by the monodromies of the periods of Ω. Then, following [1], the F_g's are either holomorphic, quasi-modular forms or almost-holomorphic modular forms of weight zero under the group Γ acting on t. We therefore conclude, that the F_g cannot be holomorphic and modular invariant simultaneously.

At first sight, the holomorphic anomaly makes the system more complicated. Fortunately, the modular symmetry in combination with the recursive holomorphic anomaly equations allows to write down solutions to the equations as polynomials of fixed degree in the ring of the corresponding modular forms. This ring is finitely generated by holomorphic and non-holomorphic modular forms and closes under differentiation. Therefore, the integration of the differential holomorphic anomaly equations is basically with respect to a finite number of non-holomorphic generators. This technique is therefore referred to as direct integration and was developed in [6, 77, 92, 169]. In fact, by integration there is a family of solutions parameterized by finitely many unknowns. In principle, one can recover the physical solution F_g by imposing enough independent boundary conditions. In practice, these are given by the behavior of the holomorphic expansions of F_g near boundary points of the moduli space. In the case of non-compact Calabi-Yau spaces the procedure is known to supply enough constraints to fix F_g completely [82].

Background independence and choice of polarization

In topological string theory background dependence refers to an explicit dependence of the correlators on a reference point in the moduli space of the theories. From the point of view of the two-dimensional field theories background dependence reflects the different values of couplings. The precise dependence is captured by the holomorphic anomaly equations and seems to be an obvious obstruction to background independence. However, in [166] it was argued that the holomorphic anomaly equations can be interpreted as actually giving the partition function $Z(g_s, t)$ a background independent meaning. The idea is to view $Z(g_s, t)$ as a wave function in an auxiliary Hilbert space obtained by quantizing $H_3(X)$, where g_s^2 plays the role of $\hbar$. The partition function being holomorphic or modular is traced back to a choice of polarization [1]. This is similar to the quantum-mechanical case where the wave functions depend either on the coordinates or momenta and the two choices of polarization are related by a Fourier transform. The choice of polarization in the B-model is given by a choice of complex structure. Once the complex structures changes, the wave functions change by a Bogoliubov transformation. The latter transformation property can be shown to coincide with the holomorphic anomaly equations [166].

Topological string theory, matrix models and geometric transition

Open topological string theory is obtained by studying world-sheets with boundaries $\Sigma_{g,h}$. This gives rise to open topological string amplitudes $F_{g,h}$. In a series of papers [45–47] Dijkgraaf and Vafa showed that the spacetime description of particular open topological string setups in terms of string field theory reduces to a matrix model. These are certain non-compact Calabi-Yau manifolds which originate from a singular geometry of the form $y^2 = (W'(x))^2$, where $W(x)$ is a polynomial of degree $n+1$. The singularities are at the critical points of $W(x)$. As in the conifold case there are two ways to smooth out the singularity either by resolving or deforming the singular Calabi-Yau manifold. We call the two manifolds X_{def} and X_{res}. The resolved geometry X_{res} with branes wrapping the blown-up $\mathbb{P}^1$'s gives rise to a multi-cut matrix model of a single $N \times N$ matrix M with polynomial potential $W(M)$. By introducing the 't Hooft parameter $t = g_s N$ and formally resuming the $F_{g,h}$ as $F_g(t) = \sum_h F_{g,h} t^h$, one obtains a closed string theory interpretation. Indeed, as explained in [23, 70], there is a geometric or large N transition relating open string Calabi-Yau backgrounds to closed string Calabi-Yau backgrounds. This allows to transfer methods from matrix models described in terms of their spectral curve to open topological string theory on the resolved Calabi-Yau manifold X_{res} or closed topological string theory on the deformed Calabi-Yau manifold X_{def}.

Gauge-String dualities

The geometric or large N transition we encountered above can in fact be embedded in the much more general framework of gauge/string theory duality. The original duality between $\mathcal{N} = 4$ Super-Yang-Mills theory and type IIB closed string theory on $\text{AdS}_5 \times S^5$ was discovered in [115] and motivated by the open-closed string dualities for D-branes. The Super-Yang-Mills theory lives on the boundary of AdS_5 and originates from a stack of D3-branes in type IIB theory. The general approach of gauge-string dualities associates world-sheet Riemann surfaces to the Feynman diagrams in 't Hooft's double line notation of the large N gauge theory. One of the simplest and oldest examples of the gauge-string duality is given by the Kontsevich matrix model [105] which agrees at large N with two-dimensional topological gravity. Another example is given by three-dimensional Chern-Simons theory on S^3 which stems from open topological string theory on T^*S^3 and is related by a large N duality to closed topological string theory on the resolved conifold [124].

Physical applications of topological string theory

It was pointed out before, that topological string theory computes certain observables of the physical string. Following [139], the perhaps best studied physical examples are the prepotential of $\mathcal{N} = 2$ gauge theories in four dimensions, the superpotential of $\mathcal{N} = 1$ gauge theories in four dimensions and black hole entropy in four and five dimensions. In the following each of these setups is described.

$\mathcal{N} = 2$ gauge theories in four dimensions

We begin with an application to $\mathcal{N} = 2$ gauge theories in four dimensions. Compactifying type IIA (or IIB) string theory on a Calabi-Yau threefold yields a $\mathcal{N} = 2$ effective gauge theory with $h^{1,1}$ ($h^{2,1}$) vector multiplets. Then topological string theory computes certain F-terms in the effective action involving the vector multiplets [12, 15]. More precisely, the free energies are the gauge kinetic couplings of graviphotons to the curvature tensor in the effective action. Of particular phenomenological interest is the genus zero contribution which gives the prepotential of the $\mathcal{N} = 2$ effective theory. Interesting gauge theories are geometrically engineered as follows [96]. First of all, one would like to decouple gravity which is achieved by considering non-compact Calabi-Yau manifolds. These can be realized by fibrations of so-called ALE spaces $\mathbb{C}^2/G$ over a Riemann surface Σ. Here, G is a finite subgroup of SU(2) and the singularities of the ALE space are zero-size two-spheres which correspond to the simple roots of a Lie algebra $\mathfrak{g}$. D2-branes which wrap these two-spheres are then identified with massless gauge bosons in four dimensions. By resolving the singularities the W bosons get massive and the volumes or Kähler parameters t_i are related to their masses. The volume of the base Σ is identified with the gauge theory coupling constant by $\text{vol}(\Sigma) = 1/g^2$. In the limit in which g^2 and the Kähler parameters t_i are sent to zero while keeping the masses of the gauge bosons constant the prepotential F_0 of topological string theory is counting the gauge theory instantons as a function of the Coulomb branch moduli which then solves the infra-red dynamics of the theory as in the case of Seiberg and Witten [147, 148].

$\mathcal{N} = 1$ gauge theories in four dimensions

The second example is obtained by compactifying type II string theory on a Calabi-Yau manifold X together with internal fluxes or space-time filling D-branes which break supersymmetry down to $\mathcal{N} = 1$. Here, the gauge symmetry is not obtained from the geometric singularity as in the previous example but originates from the stack of D-branes. In particular, N space-time filling D-branes give rise to $\mathrm{U}(N)$ gauge theory in four dimensions. The physical observable of interest is the superpotential W of $\mathcal{N} = 1$ theory. Its importance lies in the fact that the vacuum of the $\mathcal{N} = 1$ effective theory is at field configurations which extremize W. The full quantum superpotential can be obtained from the free energies of topological string theory on world-sheets with boundaries [15]. In the case of type IIB theory with D5-branes wrapping a cycle inside X the superpotential can also be directly computed in the gauge theory. Due to the relation to topological string theory the complicated Yang-Mills theory can be truncated and, as outlined before, a matrix model emerges [45–47]. More precisely, one considers the partition function of a single $N \times N$ matrix M with action $W(M)/g_s$. Then, the large N or planar limit of the matrix model determines the superpotential.

Spinning black holes in five dimensions

The third observables are the degeneracies of BPS states contributing to the entropy of spinning black holes in five dimensions obtained by compactifying M-theory on a Calabi-Yau threefold X. More precisely, we are interested in the number of BPS states of a given charge Q and spin j_L. The BPS states in five dimensions are given by M2-branes wrapping two-cycles of X where the charge Q is the $\mathrm{U}(1)$ charge of the M-theory three-form dimensionally reduced on the two-cycle. The spin j_L denotes the quantum number of the little group of massive particles in five dimensions which is $\mathrm{SO}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. A convenient way to package the information over the BPS states is provided by the $\mathcal{N} = 2$ elliptic genus $\mathrm{Tr}(-1)^{J_R} q^{J_L} e^{-\beta H}$ in five dimensions. It turns out that the elliptic genus is rigid against complex structure deformations but depends continuously on the Kähler moduli t of X. Now, the crucial observation is that the topological string partition function $Z(g_s, t)$ is precisely the elliptic genus if we identify $q = e^{-g_s}$ with the string coupling constant g_s and the spin-dependence gets related to the genus g of the world-sheet of the topological string [68, 69]. Thus, topological string theory encodes integer-valued BPS invariants n_g^Q in its perturbative expansion. This allows to make contact with the entropy S of black holes. S can be determined classically and it turns out to be proportional to the area of the event horizon. Quantum mechanically the entropy should coincide with the logarithm of the degeneracy of the quantum states of the black hole in the limit of many states. Indeed, it was shown in [91] that the asymptotic growth of the so-called Gopakumar-Vafa invariants n_g^Q agrees with the scaling of the entropy of spinning black holes in five dimensions for various examples.

Charged black holes in four dimensions

The fourth and last example is given by the degeneracies of BPS states contributing to the entropy of four-dimensional black holes. In the following consider type IIB theory on a Calabi-Yau manifold X and D3-branes wrapping three-cycles in $H_3(X, \mathbb{Z})$. These D3-branes give rise to charged BPS particles in four dimensions. The splitting of electric charges Q and magnetic charges P is obtained by making a choice of symplectic basis of $H_3(X,\mathbb{Z})$. At large charges, the number of BPS states is related to the classical Bekenstein-Hawking entropy S which in our setup is proportional to the holomorphic volume of the Calabi-Yau manifold. More precisely, denoting by $C=(P,Q)$ the three-cycle of electric and magnetic charges and by Ω the holomorphic three-form, the entropy is given by $S=\frac{i\pi}{4}\int_X \Omega\wedge\bar{\Omega}$. In this formula, Ω is fixed by the condition that $\mathrm{Re}(\Omega)$ is the Poincaré dual of C. We denote by X the vector multiplet scalars and introduce $\Phi = X-\bar{X}$. Then, a Legendre transform from Φ to Q shows that the transformed imaginary part of the prepotential $\mathrm{Im}(F_0(X))$ is in fact equal to the entropy S for large charges. The fact that the Legendre transformation is the leading approximation to the Fourier transform leads to the famous OSV conjecture [143]

$$\sum_Q \Omega(P,Q)e^{-Q\cdot\Phi} \sim |Z(P+i\Phi)|^2.$$

On the left hand side, $\Omega(P,Q)$ denotes[1] again the index of BPS black holes of charge (P,Q), while the right hand side is the square of the B-model partition function evaluated at a particular value of the coupling constant g_s and moduli t. There are at least two subtleties with this formula. First, the degeneracies $\Omega(P,Q)$ in fact depend on a choice of boundary condition of the scalar fields. The dynamics of the BPS quantum states leads to sudden jumps of $\Omega(P,Q)$ as one crosses so-called walls of marginal stability in the moduli space of scalar fields [37]. This raises the question for which choice of background value the conjecture is supposed to hold. Second, the topological string partition function Z is divergent and understood as an asymptotic series. Thus, it is unclear how the proportional sign $\sim$ is interpreted.

Outline of this thesis

This thesis is organized as follows.

- Chapter two starts with a discussion about topological string theory where most emphasis is laid on the holomorphic anomaly equations and the techniques to solve them. Thereafter matrix models and their connection to topological string theory are investigated. Some emphasis is laid on the relation between loop equations and the holomorphic anomaly as well as on the relation of large order behavior of perturbation theory and non-perturbative aspects. At last $\mathcal{N}=2$ BPS states and the wall-crossing phenomenon are studied culminating in the Kontsevich-Soibelman formula.

- In chapter three direct integration is proposed as a new method to solve the closed amplitudes for multi-cut matrix models with polynomial potentials. As an example the cubic matrix model is examined and the direct integration technique is presented explicitly. These results are then used to study large order behavior of the perturbative expansion of the partition function which is then interpreted by non-perturbative effects. This chapter is published in [102].

- In chapter four a recursive holomorphic anomaly equation for the elliptic genus of multiply wrapped M5-branes on a rigid divisor inside a Calabi-Yau threefold is derived.

[1] Do not confuse the BPS index $\Omega(P,Q)$ with the holomorphic three-form Ω.

In this context wall-crossing and non-holomorphicity are related by writing the elliptic genus in terms of mock modular forms. It is shown that this anomaly equation is the equation which was found in the context of $\mathcal{N} = 4$ SYM [156] and E-strings [131, 132]. Most parts of this chapter are published in [5].

- In chapter five we summarize and conclude. Further, open problems and future directions of research are pointed out.

Publications

This thesis is based on the following publications of the author:

- Babak Haghighat, Albrecht Klemm, Marco Rauch, "Integrability of the holomorphic anomaly equations," JHEP **0810**, 097 (2008).
- Albrecht Klemm, Marcos Mariño and Marco Rauch, "Direct Integration and Non-Perturbative Effects in Matrix Models," JHEP **1010**, 004 (2010).
- Murad Alim, Babak Haghighat, Michael Hecht, Albrecht Klemm, Marco Rauch and Thomas Wotschke, "Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes," [arXiv:1012.1608 [hep-th]].

Chapter 2

Topological string theory, matrix models and BPS states

In the introduction the tight connection between topological string theory, matrix models and BPS states was described. The following three sections concentrate on each of these setups separately, but also deepen their interrelation. Some emphasis is laid on the holomorphic anomaly equations in topological string theory and matrix models as well as on the wall-crossing phenomenon of $\mathcal{N} = 2$ BPS states in four dimensions.

2.1 Topological string theory

In the following a brief recapitulation of topological string theory is presented. Many reviews and notes have appeared about this subject by now. Some of these include [14, 15, 76, 84, 100, 124, 139, 159].

We start by recalling some properties of $\mathcal{N} = 2$ superconformal field theories in two dimensions. We note a chiral ring structure, study the marginal deformations and take a look at non-linear sigma models. Thereafter, the twisting to a topological field theory as well as the coupling to gravity is presented. The review of mirror symmetry is followed by a derivation and solution of the famous holomorphic anomaly equations. We end this section with a re-interpretation of the holomorphic anomaly equations as encoding quantum background independence of topological string theory.

2.1.1 $\mathcal{N} = (2,2)$ superconformal field theory

Topological string theory is based on an $\mathcal{N} = (2,2)$ superconformal field theory (SCFT) in two dimensions, where the two copies of $\mathcal{N} = 2$ refer to right-moving and left-moving versions of the $\mathcal{N} = 2$ superconformal algebra which are related by complex conjugation.

The $\mathcal{N} = 2$ algebra is an extension of the Virasoro algebra of the energy-momentum tensor $T(z)$ by two anti-commuting currents $G^{\pm}(z)$ and a U(1) current $J(z)$. Here $G^{\pm}(z)$ carries charge ± 1 under this U(1). The conformal weights are summarized as follows

$$\begin{array}{cccc} & & & \text{weight } h \\ & T(z) & & 2 \\ G^{+}(z) & & G^{-}(z) & 3/2 \\ & J(z) & & 1 \end{array} \tag{2.1}$$

and the central charge is denoted by c. In addition, the following boundary conditions are imposed

$$G^{\pm}(e^{2\pi i} z) = -e^{\mp 2\pi i a} G^{\pm}(z), \tag{2.2}$$

with a continuous, real parameter a. We say that integral and half-integral a corresponds to the Ramond and Neveu-Schwarz sector, respectively. Let us summarize the expansions of the fields in Fourier modes. According to the conformal weights they read

$$\begin{aligned} T(z) &= \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}}, \\ G^{\pm}(z) &= \sum_{m\in\mathbb{Z}} \frac{G^{\pm}_{m\pm a}}{z^{m\pm a+\frac{3}{2}}}, \\ J(z) &= \sum_{m\in\mathbb{Z}} \frac{J_m}{z^{m+1}}. \end{aligned} \tag{2.3}$$

It is possible to express the $\mathcal{N}=2$ superconformal algebra in terms of operator product expansions of the currents or by the commutation relations of its modes. We refrain from writing down any of them and refer the reader to the literature [76]. For different continuous parameters a the algebra is isomorphic and thus there exists a so-called spectral flow symmetry which is an operation on the states of the theory. In particular Ramond and Neveu-Schwarz sectors get related to each other which gives rise to supersymmetry in space-time.

Chiral ring

Furthermore, we have to point out an additional structure of the representation theory of the $\mathcal{N}=2$ superconformal algebra. The operators creating the highest weight state form a finite sub-sector and are endowed with a ring structure. This is important since topological string theory can be seen as a truncation of the $\mathcal{N}=(2,2)$ superconformal field theory to states which only belong to this so-called chiral ring. States in the Neveu-Schwarz sector of the theory are labelled by their eigenvalues under L_0 and J_0, which are denoted h_ϕ and q_ϕ, respectively. In the Ramond sector the $G_0^{\pm}$ eigenvalue is also needed. A highest weight state $|\phi\rangle$ is defined by

$$L_n|\phi\rangle = 0, \quad G_s^{\pm}|\phi\rangle = 0, \quad J_m|\phi\rangle = 0, \quad n,s,m>0. \tag{2.4}$$

A highest weight state is created by a primary field ϕ out of the vacuum $|0\rangle$, where the vacuum is the state whose eigenvalue labels are all zero. We will be interested in a subset of primary fields called (anti-)chiral primary fields. The states associated to chiral primary fields are further annihilated by $G^{+}_{-1/2}$, whereas states associated to anti-chiral primary fields are further annihilated by $G^{-}_{-1/2}$. Thus combining left moving and right moving sectors of the $\mathcal{N}=(2,2)$ SCFT one is lead to the notion of $(c,c),(a,c),(c,a)$ and (a,a) primary fields, where c stands for chiral and a for anti-chiral. For a chiral primary field $h_\phi=\frac{q_\phi}{2}$ and for an anti-chiral field $h_\phi=-\frac{q_\phi}{2}$. Further, the conformal weight of a chiral primary is bounded by $c/6$ and a general state $|\psi\rangle$ satisfies the inequality $h_\psi \geq \frac{q_\psi}{2}$. We denote by ϕ_i the set of all chiral primary fields and state that the before mentioned ring structure is given by[1]

$$\phi_i\phi_j = C_{ij}^k\phi_k, \tag{2.5}$$

where C_{ij}^k stems from the three-point function $C_{ijk}=\langle\phi_i\phi_j\phi_k\rangle_0$ on the sphere and the index is raised with respect to the topological metric $\eta_{ij}=\langle\phi_i\phi_j\rangle_0$. We further note, that the (a,a) ring is the complex conjugate of the (c,c) ring, and so is the (c,a) ring of the (a,c) ring.

[1] This equation is understood to hold as a relation between correlation functions.

Deformations

In order to obtain an interesting theory we are going to study the deformation space of the $\mathcal{N}=(2,2)$ SCFT. In general conformal field theories are perturbed or deformed by adding so-called marginal operators to the action. Marginal operators are fields having conformal weight $h+\bar{h}=2$. Under a marginal deformation a conformal field theory flows to a "nearby" conformal field theory in the infra-red with the same central charge. This will thus generate a continuously connected family of conformal field theories. In the following we will concentrate on the $(c,c)/(a,a)$ ring and spinless marginal deformations, i.e. $h=\bar{h}=1$. The perturbation of the action is then given by

$$\delta S = z^i \int_{\Sigma_g} \phi_i^{(2)} + z^{\bar{\imath}} \int_{\Sigma_g} \bar{\phi}_{\bar{\imath}}^{(2)}, \qquad i=1,\dots,n, \tag{2.6}$$

where $\phi_i^{(2)}$ and $\bar{\phi}_{\bar{\imath}}^{(2)}$ originate from (anti-)chiral fields $\phi_i^{(0)}$ and $\bar{\phi}_{\bar{\imath}}^{(0)}$ with conformal weights $(h_\phi,\bar{h}_\phi)=(h_{\bar{\phi}},\bar{h}_{\bar{\phi}})=(1/2,1/2)$ and charges $(q_\phi,\bar{q}_\phi)=(-q_{\bar{\phi}},-\bar{q}_{\bar{\phi}})=(1,1)$ as follows

$$\begin{aligned}\phi_i^{(2)}(w,\bar{w}) &= \{G^-_{-1/2},[\bar{G}^-_{-1/2},\phi_i^{(0)}(w,\bar{w})]\} = \oint dz\, G^-(z) \oint d\bar{z}\, \bar{G}^-(\bar{z})\, \phi_i^{(0)}(w,\bar{w}),\\ \bar{\phi}_{\bar{\imath}}^{(2)}(w,\bar{w}) &= \{G^+_{-1/2},[\bar{G}^+_{-1/2},\bar{\phi}_{\bar{\imath}}^{(0)}(w,\bar{w})]\} = \oint dz\, G^+(z) \oint d\bar{z}\, \bar{G}^+(\bar{z})\, \bar{\phi}_{\bar{\imath}}^{(0)}(w,\bar{w}).\end{aligned} \tag{2.7}$$

Further, n denotes the dimension of the subspace of the Hilbert space spanned by the charge $(1,1)$ operators. The deformation space spanned by this operators is called the moduli space of the SCFT and denoted $\mathcal{M}$. We skip the similar construction for the (a,c) ring which can be found e.g. in [76].

Non-linear sigma-models

Geometric realizations of the $\mathcal{N}=(2,2)$ SCFT are given by non-linear sigma models. This allows to identify the (a,c) and (c,c) rings with geometric quantities. The discussion follows mainly [164]. Consider a field theory with bosons ϕ^i and fermions χ^i,ψ^i on a Riemann surface Σ_g of genus g. Both fields are related by supersymmetry and the bosonic field is understood as the coordinate of some three-dimensional target space X, i.e. $\phi:\Sigma_g\to X$. A consequence of $\mathcal{N}=(2,2)$ supersymmetry on the worldsheet is that X is Kähler. The action of the non-linear sigma model is given by [164]

$$\begin{aligned}S_0 = \int_{\Sigma_g} d^2z\, (&\frac{1}{2} g_{i\bar{\jmath}}\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{\jmath}} + \frac{i}{2} g_{i\bar{\jmath}}\psi^i D_z\psi^{\bar{\jmath}}\\ &+ \frac{i}{2} g_{i\bar{\jmath}}\chi^i D_{\bar{z}}\chi^{\bar{\jmath}} + R_{i\bar{k}j\bar{l}}\psi^i\psi^{\bar{k}}\chi^j\chi^{\bar{l}}) + \int_{\Sigma_g}\phi^*(B),\end{aligned} \tag{2.8}$$

where B is the B-field. Since the beta-function at one-loop is proportional to the Ricci tensor this action is conformally invariant, if the Ricci tensor of the target space X vanishes. Thus, recalling that X is Kähler this shows that X is in fact a Calabi-Yau manifold. It is shown e.g. in [76] that the (a,c) and (c,c) rings stem from the representation of the zero-mode algebra of the fermions. In fact, the following correspondence holds

$$\begin{aligned}\mathcal{R}^{(a,c)} &\simeq \bigoplus_{p,q} H^{(p,q)}_{\bar{\partial}}(X),\\ \mathcal{R}^{(c,c)} &\simeq \bigoplus_{p,q} H^{(0,p)}_{\bar{\partial}}(X,\wedge^q TX).\end{aligned} \tag{2.9}$$

This will allow us to conclude that the (a, c) ring deformations parameterize the space of complexified Kähler deformations and that the (c, c) ring deformations parameterize the space of complex structure deformations of the underlying target space X.

2.1.2 Twisting and topological field theory

A topological theory of cohomological type is a quantum field theory which features a Grassmann, scalar symmetry operator $\mathcal{Q}$. $\mathcal{Q}$ obeys the following three properties:

1. $\mathcal{Q}$ is nilpotent, $\mathcal{Q}^2 = 0$.
2. The action is $\mathcal{Q}$-exact, $S = \{\mathcal{Q}, V\}$.
3. The energy momentum tensor is $\mathcal{Q}$-exact, $T_{\mu\nu} = \{\mathcal{Q}, G_{\mu\nu}\}$.

As a consequence the partition function is independent of the choice of the background metric which can be seen as follows. A variation of the partition function with respect to the background metric is equivalent to the insertion of the energy-momentum tensor into the correlator. Since this correlator is $\mathcal{Q}$-exact and $\mathcal{Q}$ is a symmetry of the theory the variation vanishes and the partition function is metric independent. Note, that $\mathcal{Q}$ is formally identical to a BRST operator and thus physical states correspond to cohomology classes of the operator $\mathcal{Q}$. In addition the $\mathcal{Q}$-exactness of the action implies that the semiclassical approximation is exact. For a derivation of these facts see [124].

Topological twist

The supersymmetric non-linear sigma models we discussed in the last section can be twisted in two different ways to produce two inequivalent topological field theories in two dimensions. Depending on the chiral ring under consideration the twisting procedure is achieved differently. First the Grassmann valued scalar BRST operators, denoted $Q_{A/B}$, are given by

$$\begin{aligned} (a,c): \quad & Q_A = G_0^- + \bar{G}_0^+ , \\ (c,c): \quad & Q_B = G_0^+ + \bar{G}_0^+ , \end{aligned} \tag{2.10}$$

such that by considering only those states as physical states, which are annihilated by the BRST operators, a restriction to the chiral ring under consideration is achieved. Second, the energy-momentum tensor is shifted which amounts to a redefinition of the spin of the fields of the SCFT. This is done by defining[2]

$$\begin{aligned} (a,c): \quad & T \to T + \frac{1}{2}\partial J, \quad \bar{T} \to \bar{T} - \frac{1}{2}\bar{\partial}\bar{J}, \\ (c,c): \quad & T \to T - \frac{1}{2}\partial J, \quad \bar{T} \to \bar{T} - \frac{1}{2}\bar{\partial}\bar{J}. \end{aligned} \tag{2.11}$$

It is customary to denote the restriction to the (a, c) ring as the A-model, and the restriction to the (c, c) as the B-model. In complete analogy, one also can construct the anti-A-model and anti-B-model by restricting to the (c, a) ring and the (a, a) ring, respectively.

[2] Notice, that this changes the conformal weights of the anti-commuting currents $G^{\pm}$ resulting in different mode expansions as compared to the untwisted theory.

2.1.3 Coupling to gravity

Topological string theory refers to one of the two inequivalent topological field theories that we discussed in the last section coupled to two-dimensional gravity which involves an integration over the space of all possible two-dimensional metrics of the Riemann surfaces Σ_g of genus g of the non-linear sigma model. In particular, this implies that the operators $Q_{A/B}$ have to be globally defined on every such Σ_g. This is guaranteed by the topological twist, especially by the redefinition of the energy-momentum tensor. By construction this yields an A-model topological string theory as well as a B-model topological string theory.

By denoting by m^a, $a = 1, \ldots, 3g-3$, the complex structure parameters[3] of Σ_g, the first order deformation of the worldsheet metric h modulo Weyl and diffeomorphism invariance is given by

$$\int_{\Sigma_g} d^2z\,\sqrt{h}\delta h^{\mu\nu}T_{\mu\nu} = \int_{\Sigma_g} d^2z\left(\mu_{\bar{z}}^{az}\delta m^a T_{zz} + \bar{\mu}_z^{a\bar{z}}\delta\bar{m}^a T_{\bar{z}\bar{z}}\right), \tag{2.12}$$

where $\mu^a \in H^{0,1}(\Sigma_g, T\Sigma_g)$ are the so-called Beltrami differentials. Since the notion of physical states in the bosonic string theory is exactly the same as that of chiral states in the twisted theories, the Beltrami differentials get contracted with the fields $G^{\pm}$ which play the role of ghosts. Therefore, the higher genus amplitudes or free energies of topological string theory are defined by [15]

$$F_g = \int_{\mathcal{M}_g}[dm\,d\bar{m}]\,\langle\prod_{a=0}^{3g-3}(\int_{\Sigma_g}\mu_a G^-)(\int_{\Sigma_g}\mu_{\bar{a}}\bar{G}^-)\rangle, \quad g \geq 2, \tag{2.13}$$

and the n-point functions are defined by

$$C^g_{i_1\ldots i_n} = \int_{\mathcal{M}_g}[dm\,d\bar{m}]\,\langle\prod_{r=1}^{n}\int_{\Sigma_g}\phi_{i_r}^{(2)}\prod_{a=0}^{3g-3}(\int_{\Sigma_g}\mu_a G^-)(\int_{\Sigma_g}\mu_{\bar{a}}\bar{G}^-)\rangle, \tag{2.14}$$

where (2.14) is valid for all n if $g > 1$, for $n > 0$ if $g = 1$ and for $n \geq 3$ if $g = 0$. Furthermore, $\langle\ldots\rangle$ denotes the correlation function of the CFT on Σ_g and $[dm\,d\bar{m}]$ are dual to the Beltrami differentials. In the eq. (2.13), (2.14) the correlators are taken with respect to a action that is perturbed by marginal operators, $S = S_0 + \delta S$. Due to the existence of globally defined Killing vectors on the sphere and on the torus the genus zero and one sectors are only well-defined when a certain minimum amount of fields are inserted into the correlator of (2.14) that kill these symmetries, cf. [14, 15]. A more detailed treatment of this subject can be found in the references [84, 100, 124].

2.1.4 Mirror symmetry

Mirror symmetry is an equivalence of the A-model and B-model topological string theory and can be thought of as a generalization of T-duality [152]. More precisely, one considers the A-model on a target space X together with the space of complexified Kähler deformations which we take to be parameterized locally by t^a, $a = 1, \ldots, h^{1,1}(X)$, whereas the B-model is given by another Calabi-Yau manifold Y together with the space of complex structure deformations locally parameterized by coordinates z^i, $i = 1, \ldots, h^{2,1}(Y)$. Mirror symmetry then states that the free energies and correlators of the A-model can be identified with the free energies and

[3] The virtual dimension of the moduli space of complex structures on a genus g Riemann surface is $3g-3$.

correlators of the B-model given the so-called mirror map $t = t(z)$. Obviously, this implies that the Calabi-Yau spaces involved have to fulfill $h^{1,1}(X) = h^{2,1}(Y)$.

In the following we want to examine mirror symmetry in some detail. This means that we have to study the deformation space $\mathcal{M}$ of the twisted $\mathcal{N} = (2,2)$ SCFT more thoroughly. It turns out that the ground-states of the theory do not vary over the space of deformations. However, there exists a holomorphic vector bundle over $\mathcal{M}$, called the vacuum bundle $\mathcal{V}$, which is a sub-ring generated by the charge $(1,1)$ operators of the deformations. It possesses a split graded by the charges of the operators

$$\mathcal{V} = \mathcal{H}^{0,0} \oplus \mathcal{H}^{1,1} \oplus \mathcal{H}^{2,2} \oplus \mathcal{H}^{3,3}, \tag{2.15}$$

where $\mathcal{H}^{i,i}$ is the Hilbert subspace of states of charge (i,i). This splitting or grading is what varies over the moduli space of the SCFT.

The A-model

We first turn our attention briefly to the A-model and give further reference to [84]. Here, the variation of the splitting of the vacuum bundle is rather complicated and leads to the notion of quantum cohomology, which is only completely developed for the large radius regime of the moduli space. Recall, that we study maps $\phi : \Sigma_g \to X$, but due to supersymmetry the path integral localizes onto holomorphic maps whose images depend only on the homology classes in $H_2(X,\mathbb{Z})$. Hence, we introduce a basis β_i of $H_2(X,\mathbb{Z})$, $i = 1,\ldots,b_2(X)$, and define complexified Kähler parameters by

$$t^i = \int_{\beta_i} (J + iB), \tag{2.16}$$

where J is the Kähler class on X and B is the B-field. Then, the $F_g(t)$ can be expanded up to classical terms as

$$F_g(t) = \sum_{\beta \in H_2(X,\mathbb{Z})} N_{g,\beta}\, q^\beta, \qquad q_i = e^{-t^i}, \tag{2.17}$$

where $N_{g,\beta}$ counts in a suitable way the number of holomorphic curves of genus g in the class β and is called Gromov-Witten invariant. Although, the genus g free energies have a finite radius of convergence about the large radius point, the perturbative expansion of the full partition function of the A-model,

$$Z(g_s,t) = \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t)\right), \tag{2.18}$$

is asymptotic in the string coupling constant g_s, i.e. it has zero radius of convergence.

The B-model

In the following we will briefly outline the geometric realization of the B-model side of the twisted SCFT. Many reference on this subject have been published. Our exposition mainly follows [100,123]. In contrast to the A-model, the B-model path integral localizes on constant maps of the Riemann surface Σ_g onto a point on Y and is thus independent on the Kähler moduli of Y. Actually, the moduli space of the B-model SCFT is the moduli space of complex structures on Y. The vacuum bundle is given by the middle-dimensional cohomology $H^3(Y,\mathbb{C})$ which given a choice of complex structure on Y has a natural splitting

$$H^3(Y,\mathbb{C}) = \bigoplus_{p+q=3} H^{p,q}(Y). \tag{2.19}$$

Hence, the variation of the grading of the vacuum-bundle is geometrically understood as the variation of Hodge structures on $H^3(Y,\mathbb{C})$. This is a very well-known subject in mathematics [158]. The answer is basically provided by the periods of the holomorphic three-form Ω with respect to a symplectic basis (α_I, β^I) of $H_3(Y,\mathbb{C})$. The periods are defined by

$$\Pi = \begin{pmatrix} X^I \\ \mathcal{F}_I \end{pmatrix} = \begin{pmatrix} \int_{\alpha_I} \Omega \\ \int_{\beta^I} \Omega \end{pmatrix}, \qquad I = 0,\ldots,h^{2,1}(Y), \tag{2.20}$$

and can be shown to fulfill differential equations referred to as Picard-Fuchs equations of the form

$$\mathcal{L}_\alpha \Pi = 0, \qquad \alpha = 1,\ldots,h^{2,1}(Y). \tag{2.21}$$

Further, the X^I can be regarded as local projective coordinates on $\mathcal{M}$. Since the X^I parameterize $\mathcal{M}$, the other periods $\mathcal{F}_I$ must be dependent on X^I. This allows to define a prepotential. Introducing special coordinates by

$$z^i = \frac{X^i}{X^0}, \tag{2.22}$$

the prepotential $F_0(z)$ is given by

$$(X^0)^2 F_0(z) = \frac{1}{2} \sum_{I=0}^{h^{2,1}} X^I \mathcal{F}_I. \tag{2.23}$$

It turns out that the space of complex structure deformations is a special Kähler manifold $\mathcal{M}$. This means that $\mathcal{M}$ is a Kähler manifold endowed with a line bundle, the Hodge bundle $\mathcal{L}$. $\mathcal{L}$ stems from the freedom in rescaling the $(3,0)$-form Ω. The free energies F_g are sections in $\mathcal{L}^{2-2g}$. Furthermore, the metric $G_{\bar{\imath}j}$ on $\mathcal{M}$ originates from a Kähler potential

$$K = -\log\left(i \int_Y \Omega \wedge \overline{\Omega} \right). \tag{2.24}$$

There is a natural induced connection acting on tensors in $T^*\mathcal{M} \otimes \mathcal{L}^n$ as

$$(D_i)_j^k = \delta_j^k (\partial_i + n\partial_i K) - \Gamma_{ij}^k. \tag{2.25}$$

The Yukawa couplings C_{ijk} are then determined via the prepotential as

$$C_{ijk} = D_i D_j D_k F_0, \tag{2.26}$$

and fulfill an integrability condition

$$D_i C_{jkl} = D_j C_{ikl}. \tag{2.27}$$

The last ingredient for a special Kähler manifold is a relation for the curvature tensor which reads

$$R_{i\bar{\jmath}k}{}^l = -[\bar{\partial}_{\bar{\jmath}}, D_i]_k^l = G_{\bar{\jmath}i}\delta_k^l + G_{\bar{\jmath}k}\delta_i^l - C_{ikm}\bar{C}_{\bar{\jmath}}^{lm}, \tag{2.28}$$

where

$$\bar{C}_{\bar{\imath}}^{jk} = \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} G^{i\bar{\imath}} G^{j\bar{\jmath}} e^{2K}, \tag{2.29}$$

and $\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} = \overline{C_{ijk}}$ denotes the complex conjugate Yukawa coupling.

The mirror map to the A-model is also encoded in the periods of the holomorphic three-form as

$$t^i(z) = \frac{X^i(z)}{X^0(z)}. \tag{2.30}$$

This leads to the period vector

$$\Pi(z(t)) = \begin{pmatrix} X^0 \\ X^i \\ \mathcal{F}_i \\ \mathcal{F}_0 \end{pmatrix} = X^0 \begin{pmatrix} 1 \\ t^i \\ \partial_{t_i} F_0 \\ 2F_0 - t^i \partial_{t_i} F_0 \end{pmatrix}. \tag{2.31}$$

If interpreted on the A-model side $\Pi(z(t))$ gives the quantum volume of a point, two-cycle, four-cycle and of the full Calabi-Yau manifold, respectively. The main benefit of mirror symmetry is that it opens a way to calculate the non-trivial information provided in the genus g free energies F_g of the A-model by performing a simpler since purely geometric and classical calculation on the B-model side and then using the mirror map.

2.1.5 The holomorphic anomaly equations

In the following we discuss the holomorphic anomaly equations of Bershadsky, Cecotti, Ooguri and Vafa (BCOV) [14, 15]. These are differential equations which relate genus g free energies or correlators to free energies and correlators of lower genus. BCOV derive these set of equations by a worldsheet analysis of the underlying SCFT of topological string theory. By studying the deformations of the action by marginal operators one would naively think that the A/B-model decouples from the anti-A/B-model. This can be seen for instance in the B-model by considering the anti-holomorphic perturbation which couples to $z^{\bar{\imath}}$. It can be written as a Q_B exact quantity

$$\bar{\phi}_{\bar{\imath}}^{(2)} = \{G_0^+, [\bar{G}_0^+, \bar{\phi}_{\bar{\imath}}^{(0)}]\} = -\frac{1}{2}\{Q_B, [G_0^+ - \bar{G}_0^+, \bar{\phi}_{\bar{\imath}}^{(0)}]\}, \tag{2.32}$$

and hence does not affect the correlation functions. This, however, is only true within topological field theories. The failure of this argument in topological string theory is briefly recapitulated following [15].

The derivation of the equations

A derivative of F_g with respect to $z^{\bar{\imath}}$ is generated by an insertion of the anti-chiral field $\bar{\phi}_{\bar{\imath}}$. We have

$$\begin{aligned} \bar{\partial}_{\bar{\imath}} F_g &= \int_{\mathcal{M}_g} [dm\, d\bar{m}] \int d^2 z \, \langle \oint_{C_z} G^+ \oint_{C'_z} \bar{G}^+ \bar{\phi}_{\bar{\imath}}^{(2)}(z) \prod_{a=1}^{3g-3} \int \mu_a G^- \int \bar{\mu}_{\bar{a}} \bar{G}^- \rangle \\ &= \int_{\mathcal{M}_g} [dm\, d\bar{m}] \sum_{b,\bar{b}=1}^{3g-3} 4 \frac{\partial^2}{\partial m_b \partial \bar{m}_{\bar{b}}} \langle \int \bar{\phi}_{\bar{\imath}}^{(2)}(z) \prod_{a=1}^{3g-3} \int \mu_a G^- \int \bar{\mu}_{\bar{a}} \bar{G}^- \rangle, \end{aligned} \tag{2.33}$$

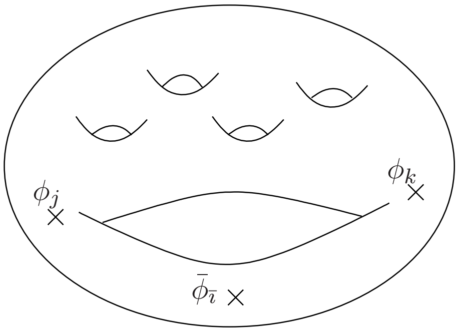

Figure 2.1: A possible contribution from the boundary of the moduli space $\mathcal{M}_g$, where a genus g curve degenerates into a curve of genus $g-1$, called A-type-sewing.

where the contours C_z and C'_z are around the insertion point z of the anti-chiral field. Moving the contours around the Riemann surface and applying commutation relations one arrives at the second line of (2.33). Applying Cauchy's theorem, the right-hand side of (2.33) is reduced to an integral over the boundary of the moduli space $\mathcal{M}_g$. It can be shown [15], that the only non-vanishing configurations from the boundary of $\mathcal{M}_g$ are given by degeneration of a curve of genus g into a curve of genus $g-1$ and from the splitting of a genus g curve into two curves of genus $g-r$ and r, respectively. In both cases, a handle is stretched into a long tube and the field $\bar{\phi}_{\bar{\imath}}$ has to be inserted on that long tube whose length goes to infinity to yield a non-vanishing contribution to $\bar{\partial}_{\bar{\imath}}F_g$ [15], cf. Fig. 2.1 and Fig. 2.2. In this limit states which propagate on the tube are projected to the ground-state. However, ground-states are generated by chiral fields and we can think of the degeneration as insertions of two chiral fields $\phi_j(z)$ and $\phi_k(z')$ at the points z and z' where the tube ends. To cut a long story short, both contributions can be summarized into a single equation. This so-called holomorphic anomaly equation of BCOV [14, 15] reads

$$\bar{\partial}_{\bar{\imath}}F_g = \frac{1}{2}\bar{C}_{\bar{\imath}}^{jk}\left(D_jD_kF_{g-1} + \sum_{r=1}^{g-1} D_jF_rD_kF_{g-r}\right), \qquad (g>1). \tag{2.34}$$

At genus one, the holomorphic anomaly is given by

$$\bar{\partial}_{\bar{\imath}}\partial_jF_1 = \frac{1}{2}\bar{C}_{\bar{\imath}}^{kl}C_{jkl} - (\frac{\chi}{24}-1)G_{\bar{\imath}j}. \tag{2.35}$$

Using the special geometry relation for the commutator $[\bar{\partial}_{\bar{\imath}}, D_j]$ and combining it with the holomorphic anomaly equations for the free energies above, the correlation functions

$$C^g_{i_1\ldots i_n} = \begin{cases} D_{i_1}\ldots D_{i_n}F_g & g\geq 1 \\ D_{i_1}\ldots D_{i_{n-3}}C_{i_{n-2}i_{n-1}i_n} & g=0 \end{cases} \tag{2.36}$$

fulfill holomorphic anomaly equations in the cases $n\geq 4$ $(g=0)$, $n\geq 1$ $(g=1)$ and for all n $(g>1)$. These equations are not reproduced here and can be found for instance in [15].

Solving the equations

In addition to the set of differential equations (2.34), BCOV proposed a method to recursively solve for the F_g. The simple idea is to write the r.h.s. of (2.34) also as an anti-holomorphic

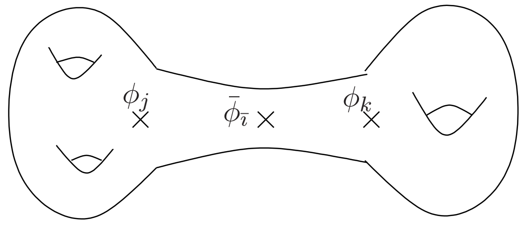

Figure 2.2: The second contribution from the boundary of the moduli space $\mathcal{M}_g$, where a genus g curve splits into two curves of genus r and $g-r$, respectively. This degeneration is called B-type-sewing.

derivative and then to integrate. Thus, one arrives at an expression for F_g up to a holomorphic function, called the holomorphic ambiguity. To do so, BCOV note, that locally the anti-holomorphic Yukawa coupling can be written as

$$\bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} = e^{-2K} D_{\bar{\imath}} D_{\bar{\jmath}} \bar{\partial}_{\bar{k}} S, \tag{2.37}$$

with a section S of $\mathcal{L}^{-2}$. Further, $S^{ij} \in \mathcal{L}^{-2} \otimes \mathrm{Sym}^2(T^*\mathcal{M})$ and $S^i \in \mathcal{L}^{-2} \otimes T^*\mathcal{M}$ are defined by

$$\bar{\partial}_{\bar{\imath}} S^{ij} = \bar{C}_{\bar{\imath}}^{ij}, \quad \bar{\partial}_{\bar{\imath}} S^i = G_{\bar{\imath}j} S^{ij}, \quad \bar{\partial}_{\bar{\imath}} S = G_{\bar{\imath}j} S^j. \tag{2.38}$$

In the solution of BCOV the holomorphic anomaly equations for F_g are reorganized in such a way that the expressions stem from Feynman diagrams in which the three sections S^{ij}, S^i and S are interpreted as propagators and the correlation functions as vertices. This gives a description to write F_g up to the holomorphic ambiguity which has to be fixed by supplying physical boundary conditions. One of the practical shortcomings is that the terms involved in the procedure proposed by BCOV grows factorial in the genus. A simpler and more sophisticated procedure is based on the idea to express F_g as a polynomial of a ring of a finite number of non-holomorphic generators on which the anti-holomorphic derivative closes. These are S^{ij}, S^i, S and K_i, the partial derivative of the Kähler potential. One then establishes that the anti-holomorphic derivative is traded for a derivative with respect to the generators. To be precise, one writes

$$\bar{\partial}_{\bar{\imath}} F_g = \bar{C}_{\bar{\imath}}^{ij} \frac{\partial F_g}{\partial S^{ij}} + G_{\bar{\imath}i}\left(\frac{\partial F_g}{\partial K_i} + S^i \frac{\partial F_g}{\partial S} + S^{ij} \frac{\partial F_g}{\partial S^j}\right). \tag{2.39}$$

By assuming linear independence of the $\bar{C}_{\bar{\imath}}^{ij}$ and $G_{\bar{\imath}i}$, the integration of the holomorphic anomaly equations is simply with respect to the generators S^{ij}, S^i and S.[4] This idea was initiated by Yamaguchi and Yau [169] who studied the mirror quintic and was developed further in [6, 77, 80, 81, 92]. Hints for such a polynomial structure were already visible in the work of refs. [87, 88]. Today, there are basically two techniques known as direct integration of the holomorphic anomaly equations. In the first framework the propagators S^{ij}, S^i and S are identified with the non-holomorphic generators, in the second setup the modular properties are used and one arrives at a modular covariant formulation of direct integration, where

[4] Note, that in the case of the free energies F_g the r.h.s of the holomorphic anomaly equations (2.34) does not depend on $G_{\bar{\imath}i}$. This allows to get rid off the non-holomorphic generator K_i by a redefinition of S^i and S. However, for correlation functions with insertions this is no longer true, as can be seen already by looking at the genus one variant of the holomorphic anomaly equation (2.35).

the propagators are expressed as (generalizations of) almost holomorphic modular forms. In this case, the modular symmetry stems from the integral monodromy of the periods around special points in the complex structure moduli space. In both direct integration procedures F_g can be written as a polynomial of degree $3g-3$ in the non-holomorphic generators. In the next chapter we briefly review both techniques in the case of topological string theory on local Calabi-Yau geometries. Here, we briefly collect the identities to express the Christoffel symbol and the covariant derivatives of the non-holomorphic generators again as polynomials in these generators

$$\begin{aligned}\Gamma^l_{ij} &= \delta^l_i K_j + \delta^l_j K_i - C_{ijk}S^{kl} + s^l_{ij},\\ D_i S^{jk} &= \delta^j_i S^k + \delta^k_i S^j - C_{imn}S^{mj}S^{nk} + h^{jk}_i,\\ D_i S^j &= 2\delta^j_i S - C_{imn}S^m S^{nj} + h^{jk}_i K_k + h^j_i,\\ D_i S &= -\frac{1}{2}C_{imn}S^m S^n + \frac{1}{2}h^{mn}_i K_m K_n + h^j_i K_j + h_i,\\ D_i K_j &= -K_i K_j - C_{ijk}S^k + C_{ijk}S^{kl}K_l + h_{ij},\end{aligned} \tag{2.40}$$

where s^l_{ij}, h^{jk}_i, h^j_i, h_i and h_{ij} are holomorphic functions that have to be fixed and therefore are called ambiguities. This completes the derivation that the non-holomorphic parts of the correlation functions and free energies can be expressed in terms of the generators.

We see, that whether one employs the old Feynman technique or the direct integration procedure leads always to a holomorphic ambiguity that one has to fix. By using the space-time interpretation of the topological string amplitudes F_g as gauge kinetic couplings of $2g-2$ graviphotons to the self-dual part of the Riemann tensor R^2_+ in the effective action which emerges by compactifying type IIA on the Calabi-Yau manifold X, it is possible to perform a Schwinger loop calculation at special points in the moduli space where the light BPS spectrum is understood. Such points are known as conifold points in the moduli space of the A-model or the B-model. Denoting by t_D the period over the vanishing three-cycle of S^3 topology,

$$t_D = \int_{S^3} \Omega, \tag{2.41}$$

the F_g feature an interesting singularity behavior known as gap condition which states that in the holomorphic limit F_g behaves like

$$F_g = \frac{B_{2g}}{2g(2g-2)}\frac{1}{t_D^{2g-2}} + \mathcal{O}(t_D^0). \tag{2.42}$$

This allows to fix part of the holomorphic ambiguity. In the case of local Calabi-Yau manifolds it was argued in [82] that the gap condition together with the leading contribution from the constant maps at large radius are enough to fix the ambiguity at every genus thus providing an integrable procedure.

2.1.6 Background independence

In topological string theory background dependence refers to an explicit dependence of the correlators on a reference point in the moduli space of the theories. From the point of view of the two-dimensional field theories background dependence reflects the different values of couplings. The precise dependence is captured by the holomorphic anomaly equations and seems to be an obvious obstruction to background independence. However, in [166] it is argued that the holomorphic anomaly can be re-interpreted as a manifestation of quantum background independence of topological string theory. The idea is to view the full partition function $Z(g_s, t)$ as a wave function in some auxiliary Hilbert space given by the geometric quantization of $H^3(X, \mathbb{R})$. Upon quantization $H^3(X, \mathbb{R})$ becomes a symplectic phase space denoted $\mathcal{W}$. However, this construction requires a choice of polarization. Given a complex structure J on X, $\mathcal{W}$ gets a complex structure as well and the Hilbert space $\mathcal{H}_J$ is constructed as the space of holomorphic sections of a line bundle over $\mathcal{W}$. We denote the wave functions in $\mathcal{W}$ by $\psi(a^i; t^i)$, where a^i are coordinates on $\mathcal{W}$ and t^i parameterize the different choices of J. Now, the idea is to identify the Hilbert spaces $\mathcal{H}_J$ using a flat connection ∇ over the space of complex structures J in such a way that as J varies, the wave functions ψ change by a Bogoliubov transformation. Background independence should then be interpreted as ψ being invariant under parallel transport by ∇. This leads to the equation

$$\left(\frac{\partial}{\partial t^i} - \frac{1}{4}\left(\frac{\partial J}{\partial t^i}\omega^{-1}\right)^{ij} \frac{D}{\partial a^i}\frac{D}{\partial a^j}\right)\psi = 0, \tag{2.43}$$

where ω is the symplectic structure on $\mathcal{W}$. Now, it is shown in [166] that the equation (2.43) is equivalent to the holomorphic anomaly equation for the full topological string partition function which is the linear equation given by

$$\left(\bar{\partial}_{\bar{\imath}} - \frac{1}{4} g_s^2 \bar{C}_{\bar{\imath}}^{ij} D_i D_j\right) Z(g_s, t) = 0. \tag{2.44}$$

2.2 Matrix models

Matrix models are toy versions of quantum field theories which share a couple of features with ordinary quantum field theories but are much easier. Our exposition follows mainly [123, 124, 128]. We start by reviewing basics of matrix models and the saddle-point analysis. Thereafter, a surprising connection of matrix models and topological string theory is developed using the connection between topological strings and $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. This is followed by presenting a technique to solve matrix models in general. This brings together the loop equations and the holomorphic anomaly of topological string theory. Combined with the above connection to topological string theory this allows for a solution of open and closed topological string theory on local Calabi-Yau manifolds. Since matrix models possess both a perturbative as well as a non-perturbative description they are a perfect playground to study the connection between non-perturbative effects and large order behavior of perturbation theory which perhaps allows to shed some light on non-perturbative topological string theory.

2.2.1 Basics of matrix models and saddle-point analysis

In this thesis we will be interested in multi-cut, Hermitian matrix models. The partition function is defined by

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \mathrm{d}M \, \mathrm{e}^{-\frac{1}{g_s} W(M)} \tag{2.45}$$

where $W(M)$ is a polynomial of degree $n+1$ in the $N \times N$ matrix M

$$\frac{1}{g_s} W(M) = \frac{1}{2g_s} \mathrm{Tr} M^2 + \frac{1}{g_s} \sum_{p=3}^{n+1} \frac{g_p}{p} \mathrm{Tr} M^p, \tag{2.46}$$

where g_p are coupling constants. The action possesses the gauge symmetry $M \to UMU^\dagger$ under a unitary transformation U. Thus the factor $\mathrm{vol}(U(N))$ in eq. (2.45) occurs. Furthermore, $\mathrm{d}M$ denotes the Haar measure. One can calculate Z by performing perturbation theory around the Gaussian point which amounts to expanding the exponential in (2.45). This yields Z as a power series in the couplings g_p. It is convenient to introduce the free energy F as

$$Z = \log F, \tag{2.47}$$

which only receives contributions from connected vacuum diagrams. Using fatgraphs the perturbative expansion can be organized into a genus expansion

$$F = \sum_{g=0}^{\infty} F_g(t)\, g_s^{2g-2}, \tag{2.48}$$

where $t = g_s N$ is the 't Hooft parameter. Eq. (2.48) can be interpreted as an expansion in g_s about $g_s = 0$ keeping t fixed or one can regard it as an expansion in $1/N$ for large N keeping t fixed. The expression on the r.h.s. of (2.48) is a formal power series in g_s – in fact it is an asymptotic series as in the case of topological string theory, where each F_g grows with $(2g)!$.

Another way of writing the matrix model partition function is by diagonalizing M. This yields an integral over the eigenvalues of M given by

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi}\, \mathrm{e}^{N^2 S_{\text{eff}}(\lambda)}, \tag{2.49}$$

where the effective action S_{eff} reads

$$S_{\text{eff}}(\lambda) = -\frac{1}{tN} \sum_{i=1}^{N} W(\lambda_i) + \frac{2}{N^2} \sum_{i<j} \log|\lambda_i - \lambda_j|. \tag{2.50}$$

The most general saddle point of this model at large N is a multi-cut solution, in which the eigenvalues of M condense along cuts

$$[a_i^-, a_i^+] \subset \mathbb{C}, \qquad i = 1, \cdots, n, \tag{2.51}$$

in the complex plane. The cuts are centered around the n critical points of $W(x)$ (cf. Fig. 2.3). One way of encoding the planar solution of the matrix model is through its resolvent

$$\omega(x) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{x - M} \right\rangle. \tag{2.52}$$

The planar limit (genus zero) of this correlator, denoted by $\omega_0(x)$, has the structure (see for example [42])

$$\omega_0(x) = \frac{1}{2t}(W'(x) - y(x)), \tag{2.53}$$

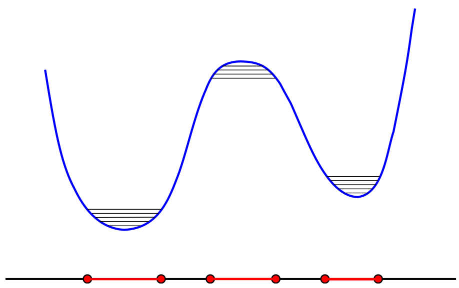

Figure 2.3: The eigenvalues condense at the critical values of the potential and define the cuts $[a_i^-, a_i^+]$ in the complex plane.

where

$$y^2(x) = \big(W'(x)\big)^2 + f(x) = c\prod_{i=1}^{n}(x-a_i^-)(x-a_i^+) \tag{2.54}$$

is called the spectral curve of the multi-cut matrix model. In the matrix model literature it is customary to write it as

$$y(x) = M(x)\sqrt{\sigma(x)}, \tag{2.55}$$

where $\sigma(x)$ is a polynomial in x,

$$\sigma(x) = \prod_{i=1}^{2s}(x-x_i), \tag{2.56}$$

and $s \leq n$. Of course, if all the roots in (2.54) are different, $s=n$ and $M(x)$ is a constant. The positions of the endpoints are fixed by the asymptotic condition

$$\omega_0(x) \sim \frac{1}{x}, \qquad x \to \infty, \tag{2.57}$$

and by the requirement that there are N_i eigenvalues in each cut,

$$\frac{N_i}{N} = \frac{1}{2}\oint_{\mathcal{C}_i}\frac{\mathrm{d}x}{2\pi\mathrm{i}}\,\omega_0(x). \tag{2.58}$$

In this equation, $\mathcal{C}_i$ is a contour encircling the cut $[a_i^-, a_i^+]$ counterclockwise. Notice, that the partial 't Hooft parameters S_i are defined by

$$S_i = g_s N_i, \tag{2.59}$$

and that obviously the following identity holds $t = \sum_{i=1}^{n} S_i$.

2.2.2 Matrix models, supersymmetric gauge theory and topological strings

In the following we are interested in a surprising connection between matrix models and topological string theory due to Dijkgraaf and Vafa [45–47]. Their aim is to study $\mathcal{N}=1$ gauge theory in four dimensions. Such theories can be obtained by compactifying type II string theory on (local) Calabi-Yau manifolds X together with space-time filling D-branes. We follow the exposition of ref. [123].

To connect to topological string theory it is necessary to study open topological strings, i.e. when the worldsheet is an open Riemann surface with boundary [142,165]. In the open B-model the appropriate boundary conditions are Dirichlet along holomorphic cycles S of X and Neumann in the remaining directions. Chan-Paton factors are given by a $\mathrm{U}(N)$ holomorphic bundle E over the holomorphic cycles S. As shown by [165] the B-model coupled to gravity can be described using the cubic string field theory of open bosonic strings [161] whose action is given by

$$S = \frac{1}{g_s} \int \mathrm{Tr} \left(\frac{1}{2} \Psi \star \mathcal{Q} \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right), \tag{2.60}$$

where $\star$ defines an associate, non-commutative product of $\mathrm{U}(N)$ string functionals Ψ, g_s is the string coupling constant and $\mathcal{Q}$ is the BRST operator of bosonic string theory. As in the case of coupling the B-model to world-sheet gravity one can employ the analogy between bosonic strings and topological strings. Under the following identifications [165]

$$\begin{aligned} \Psi \to A, \quad & \mathcal{Q} \to \bar{\partial}, \\ \star \to \wedge, \quad & \int \to \int_X \Omega \wedge, \end{aligned} \tag{2.61}$$

the action translates into an action for the open topological string theory B-model that is given by the holomorphic Chern-Simons action

$$S = \frac{1}{2g_s} \int_X \Omega \wedge \mathrm{Tr} \left(A \wedge \bar{\partial} A + \frac{2}{3} A \wedge A \wedge A \right), \tag{2.62}$$

where A is a $(0,1)$-form taking values in the endomorphisms of the holomorphic vector bundle E over S.

We are interested in the local Calabi-Yau geometry X given by

$$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1, \tag{2.63}$$

with N D-branes wrapping the two-sphere $\mathbb{P}^1$. Now, the gauge potential A splits into a $(0,1)$ gauge potential A supported on $\mathbb{P}^1$, a section of $\mathcal{O}(0)$ denoted Φ_0 and a section of $\mathcal{O}(-2)$ called Φ_1. All fields take values in the adjoint of $\mathrm{U}(N)$. It can be shown straightforwardly that (2.62) reduces to

$$S = \frac{1}{g_s} \int_{\mathbb{P}^1} \mathrm{Tr} \left(\Phi_0 \bar{D}_A \Phi_1 \right), \tag{2.64}$$

where $\bar{D}_A = \bar{\partial} + [A, \cdot]$.

Now, the idea is to modify the geometry such that one obtains n isolated $\mathbb{P}^1$s. Following [23] one introduces a polynomial potential $W(\Phi_0)$ of degree $n+1$ and makes a redefinition of the fields[5]

$$x = \Phi_0, \quad u = 2z^2 \Phi_1, \quad v = 2\Phi_1, \quad y = i(2z\Phi_1 - W'(x)), \tag{2.65}$$

where z is a local coordinate on $\mathbb{P}^1$. One then arrives at a geometry

$$uv = y^2 - W'(x)^2, \tag{2.66}$$

which is singular at each critical point of W. This geometry is smoothed out by blowing up a $\mathbb{P}^1$ at each singularity and the resolved geometry is called X_r. The idea is to distribute the

[5] See the refs. [23, 123] for the details.

N D-branes in such a way, that N_i of them wrap around the i-th $\mathbb{P}^1$. The action of this setup turns out to be [45]

$$S = \frac{1}{g_s} \int_{\mathbb{P}^1} \mathrm{Tr}\left(\Phi_0 \bar{D}_A \Phi_1 + \omega W(\Phi_0)\right), \tag{2.67}$$

where ω is the Kähler form of $\mathbb{P}^1$. It is now possible to show,[6] that the partition function using this action S reduces to a matrix model partition function where the matrix potential is precisely given by W. Thus, we have seen that the open topological string B-model amplitudes of the resolved Calabi-Yau manifold X_{r} are computed by a multi-cut matrix model. There exists another way to smooth out the singularities and this is obtained by deforming the singular geometry (2.66). This yields a deformed Calabi-Yau space X_{d} given by

$$uv = y^2 - (W'(x)^2 + f(x)), \tag{2.68}$$

where $f(x)$ is a polynomial of degree $n-1$ that splits the n double zeroes of $W'(x)^2$. In [23,70] it was argued that the closed topological string theory on X_{d} without D-branes is equivalent to the open topological string theory on X_{r} with N D-branes wrapping the blown-up $\mathbb{P}^1$'s. Further, the partial 't Hooft couplings S_i of the open string theory are identified with the periods of the closed string theory. Evidence for this geometric transition is given by the planar solution of the matrix model that is encoded by a hyperelliptic curve

$$y^2 = W'(x)^2 + f(x), \tag{2.69}$$

with the same polynomial $f(x)$ as in the deformed case.

2.2.3 Loop equations, holomorphic anomaly and their solution

Matrix models are solved completely once one knows all correlation functions. A generating function of correlation functions together with its large N expansion is given by

$$\left\langle \mathrm{Tr}\frac{dp_1}{p_1 - M} \dots \mathrm{Tr}\frac{dp_n}{p_n - M} \right\rangle = \sum_{g=0}^{\infty} N^{2-2g-n} W_n^{(g)}(p_1, \dots, p_n), \tag{2.70}$$

where the correlator is taken with respect to the measure in (2.45). The meromorphic differentials $W_n^{(g)}(p_1, \dots, p_n)$ are called genus g, n hole correlation functions. Note, that $W_1^{(0)}$ is nothing but the resolvent of the matrix model. By introducing the loop operator [9]

$$\frac{\mathrm{d}}{\mathrm{d}V}(p) = -\sum_{k=1}^{\infty} \frac{k}{p^{k+1}} \frac{\partial}{\partial g_k}, \tag{2.71}$$

it is possible to relate correlation functions to free energies

$$W_n^{(g)}(p_1, \dots, p_n) = \frac{\mathrm{d}}{\mathrm{d}V}(p_1) \dots \frac{\mathrm{d}}{\mathrm{d}V}(p_n) F_g. \tag{2.72}$$

Now, writing Ward identities for the correlators one can deduce the so-called loop equations whose simplest one can be formulated as [9]

$$\left(\widehat{K} - 2W_1^{(0)}(p)\right) W_1^{(g)}(p) = W_2^{(g-1)}(p,p) + \sum_{h=1}^{g-1} W_1^{(h)}(p) W_1^{(g-h)}(p), \tag{2.73}$$

[6] See refs. [45, 123] for a derivation.

where the operator $\widehat{K}$ acts as follows

$$\widehat{K} f(p) = \oint_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{W'(w)}{p-w} f(w), \tag{2.74}$$

where C is a contour that encloses the singularities of $f(w)$.

In virtue of the Dijkgraaf-Vafa correspondence, [56] showed that the loop equations (2.73) imply the holomorphic anomaly equations of topological string theory on the local Calabi-Yau geometries obtained from eq. (2.66) that we studied in the last section. This is achieved by promoting the genus g free energies of matrix models to modular invariant, non-holomorphic amplitudes. The idea is to write down solutions to (2.73) which are then shown to be invariants of the spectral curve, that are non-holomorphic in the partial 't Hooft couplings S_i [57]. In fact, the method of Eynard and Orantin [57] is much more general and applies to every elliptic curve independent of whether it is a spectral curve of a matrix model or not. More precisely, [57] consider an affine plane curve

$$C : \{\mathcal{E}(x,y) = 0\} \subset \mathbb{C}^2, \tag{2.75}$$

where $\mathcal{E}(x,y)$ is a polynomial in $\mathbb{C}^2$. Then, they give a recursive procedure to derive invariants F_g on the elliptic curve. In the case that C is the spectral curve of a matrix model their procedure is a purely geometric method to solve the loop equations. The benefit compared to direct integration in topological string theory is that there is no need for fixing a holomorphic ambiguity. However, in order to apply this method to topological string theory one has to modify this procedure slightly [19]. The reason is that the B-model geometry is a non-compact Calabi-Yau manifold of the form

$$uv = H(x,y), \tag{2.76}$$

where $u, v \in \mathbb{C}$ and $x, y \in \mathbb{C}^*$ [83]. All non-trivial information is encoded in the elliptic curve

$$\Sigma : \{H(x,y) = 0\} \subset \mathbb{C}^* \times \mathbb{C}^*, \tag{2.77}$$

and hence there are slightly different ingredients for the recursion. In table 2.1 we summarize the differences of these ingredients of both approaches. The ramification points q_i and the meromorphic differential are given in the table. Moreover, one needs to define the Bergmann kernel $B(p,q)$ on C and Σ, respectively. Here, $B(p,q)$ is the unique meromorphic differential with a double pole at $p = q$ with no residue and no other pole and normalized such that

$$\oint_{A_I} B(p,q) = 0, \tag{2.78}$$

for a symplectic choice (A_I, B^I) of cycles on C and Σ, respectively. Near each ramification point q_i we can define a related one-form by

$$\mathrm{d}E_q(p) = \frac{1}{2} \int_q^{\bar{q}} B(p,\xi). \tag{2.79}$$

With these ingredients the recursion is given by [19, 57]

$$W_1^{(0)}(p_1) = 0, \quad W_2^{(0)}(p_1,p_2) = B(p_1,p_2), \tag{2.80}$$

Remodeling the B-model [19]	Eynard-Orantin [57]
Ramification points	
$q_i \in \Sigma : \frac{\partial H}{\partial y}(q_i) = 0$ near each $q_i\ \exists\, q, \bar{q} \in \Sigma : x(q) = x(\bar{q})$	$q_i \in C : \frac{\partial \mathcal{E}}{\partial y}(q_i) = 0$ near each $q_i\ \exists\, q, \bar{q} \in C : x(q) = x(\bar{q})$
Meromorphic differential	
$\Theta(p) = \log y(p) \frac{\mathrm{d}x(p)}{x(p)}$ on Σ	$\Phi(p) = y(p)\mathrm{d}x(p)$ on C
Symplectic transformation	
$G_\Sigma = \mathrm{SL}(2,\mathbb{Z}) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acting as $(x,y) \mapsto (x^a y^b, x^c y^d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\Sigma$	$G_C = \mathrm{SL}(2,\mathbb{C}) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acting as $(x,y) \mapsto (ax+by, cx+dy)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_C$

Table 2.1: Differences of the ingredients needed for the recursive procedures to define/compute the symplectic invariants F_g and correlation functions $W_k^{(g)}$.

and

$$W^{(g)}_{h+1}(p,p_1,\ldots,p_h) = \sum_{q_i} \operatorname*{Res}_{q=q_i} \frac{\mathrm{d}E_q(p)}{\Phi(p)-\Phi(\bar{q})} \Big(W^{(g-1)}_{h+2}(q,\bar{q},p_1,\ldots,p_h) + \sum_{l=0}^{g} \sum_{J\subset H} W^{(g-l)}_{|J|+1}(q,p_J) W^{(l)}_{|H|-|J|+1}(\bar{q},p_{H\setminus J}) \Big), \qquad (g,h \in \mathbb{Z}^+). \tag{2.81}$$

For topological string theory we have to replace Φ by Θ in (2.81). Using $\mathrm{d}\phi(p) = \Phi(p)$ or $\mathrm{d}\theta(p) = \Theta(p)$ the invariants F_g are given by[7]

$$F_g = \frac{1}{2-2g} \sum_{q_i} \operatorname*{Res}_{q=q_i} \phi(q)\, W_1^{(g)}(q), \tag{2.82}$$

which are invariant under the symplectic transformations given in table 2.1. In the topological string case the F_g are precisely the free energies of the B-model or mirror A-model. Furthermore, the integrated correlation functions $\int W_k^{(g)}$ are equal to the open topological string amplitudes [19].

2.2.4 A digression on non-perturbative effects and large order behavior

We have quoted that the perturbative series of the partition function of matrix models and topological string theory are asymptotic as expansions in the string coupling constant g_s. In the following we will try to sharpen these statements by reviewing some facts on the large order behavior of the perturbative expansion. Our exposition on asymptotic series, Borel summability and their connection to instanton effects follows refs. [11, 109, 126–128].

[7] Again, for topological string theory one has to replace ϕ by θ in (2.82).

In field theory the number of Feynman graphs contributing to order n grows roughly as $n!$. Therefore, consider the following power series

$$f(w) = \sum_{n=0}^{\infty} a_n w^n. \tag{2.83}$$

Its coefficients are supposed to grow factorial $a_n \sim (\beta n)!$ for some real parameter β and hence the series $f(w)$ diverges. More precisely, a series is called asymptotic if there exists a bound of the form

$$|f(w) - \sum_{n=0}^{N} a_n w^n| \leq C_{N+1}|w|^{N+1}, \tag{2.84}$$

with

$$C_N = cA^{-N}(\beta N)!, \qquad (c, A \in \mathbb{R}). \tag{2.85}$$

This series has zero radius of convergence and does not uniquely define the function $f(w)$. In fact, it only determines $f(w)$ up to a non-perturbative ambiguity. An estimation argument shows that one can always add to the asymptotic expansion an analytic function that is smaller than $\varepsilon(w)$ where

$$\varepsilon(w) \sim \exp(-(A/|w|)^{\frac{1}{\beta}}). \tag{2.86}$$

Since asymptotic series occur rather naturally in quantum field theories in the perturbative expansions this immediately raises the question, how to assign a numerical value to the series and how to relate the series to the exact answer $f(w)$? The answer of course is to give an independent non-perturbative definition of $f(w)$, which however might not always be possible as is the case of string theory in general. We therefore introduce the concept of Borel resummation. We define the Borel transform of f, $B_f(z)$, as the series

$$B_f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(\beta n)!} z^n. \tag{2.87}$$

It follows that

$$\tilde{f}(w) = \int_0^{\infty} dte^{-t} B_f(t^{\beta} w) \tag{2.88}$$

defines an analytic continuation of $f(w)$ even if $f(w)$ has zero radius of convergence. Obviously, the function $\tilde{f}(w)$ is well-defined if $B_f(z)$ does not have singularities on the real axis. In this case f is said to be Borel summable. In some cases the Borel resummation is a way to define the series f non-perturbatively. But even in the case that f is not Borel summable, the Borel transform still encodes some of the large order behavior of the asymptotic series. Depending on whether $B_f(z)$ has a branch cut or poles on the real axis, it can be shown that the form of the large order behavior, i.e. basically $\varepsilon(w)$, is controlled by the singularities of $B_f(z)$. For the details reference is given to [128].

This leads us to the question what are the physical sources of these singularities? The answer can be given by a heuristic argument due to 't Hooft [109]. Consider a correlation function

$$W(\alpha) = \int \mathcal{D}\phi e^{-\frac{1}{\alpha}S(\phi)} \phi(x_1) \dots \phi(x_n), \tag{2.89}$$

and re-write it as

$$W(\alpha) = \alpha \int_0^{\infty} \mathrm{d}t F(\alpha t) e^{-t}, \tag{2.90}$$

where

$$F(z) = \int \mathcal{D}\phi\, \delta(z - S(\phi))\, \phi(x_1) \dots \phi(x_n). \tag{2.91}$$

We remark that $F(z)$ is essentially the analytic continuation of $W(\alpha)$ using the Borel transform. Thus, if the theory admits a finite action instanton ϕ^* with $z^* = S(\phi^*)$, the function $F(z)$ will be singular at $z = z^*$. The leading large order behavior in the case of complex instanton solutions is determined by the solution with smallest action in absolute value. The phase of the action results in an oscillatory character of the series.

In summary we have learned that if a quantum field theory admits an instanton configuration ϕ^* with finite action $S(\phi^*)$, the Borel transform of any correlation function possesses a singularity at $S(\phi^*)$ and the perturbative expansions have inevitably zero radius of convergence. The singularity of the Borel transform can be avoided by deforming the contour of integration. Two contour prescriptions of resummation are related purely non-perturbatively and define the so-called Stokes parameter [127, 128]. The large order behavior of the perturbative expansion encodes the instanton action, the Stokes parameter and even the coefficient of the first instanton correction. In cases where there is no clear non-perturbative definition available, the large order behavior gives useful indication to its structure. But especially for matrix models, a non-perturbative description exists and one can explicitly compute these quantities. For instance, the parameter A is given by the one-instanton action of an eigenvalue tunneling from the background $(N_1, \dots, N_n)$ to a neighboring background [126, 127], cf. Fig. 2.4. We will exploit this in more detail in chapt. 3.5.

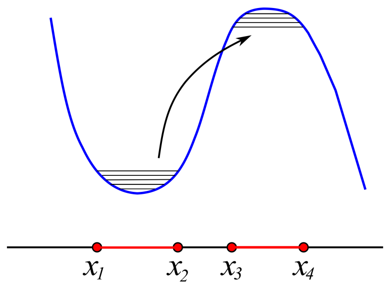

Figure 2.4: An eigenvalue tunnels from one critical point of the potential to another. The instanton action A is then given by the tunneling probability between the two cuts $A = \int_{x_2}^{x_3} y(x)\, \mathrm{d}x$.

2.3 BPS states and wall-crossing

BPS states are massive representations of the supersymmetry (SUSY) algebra whose rest mass m saturates the bound on massive states that can be derived from the algebra. In particular, in $\mathcal{N} = 2$ SUSY a BPS state fulfills $m^2 = |Z(\gamma, u)|^2$, where $Z \in \mathbb{C}$ is the central charge of the SUSY algebra, γ is the charge of the BPS particle and u denotes the collection of physical parameters of the theory. BPS states are expected to be rigid under deformations of the theory. However, it turns out that the BPS spectrum is only piecewise constant, but can undergo sudden changes when the physical parameters u of the theory vary. This endows the moduli space $\mathcal{M}$ of the theory, parameterized by u, with a chamber structure. The walls separating the chambers are real co-dimension one loci in $\mathcal{M}$ which are called walls of marginal stability. Qualitatively, BPS bound-states are formed or decay across walls of marginal stability [37]. Quantitatively, the change in the BPS spectrum is determined by the Kontsevich-Soibelman wall-crossing formula [106]. In the following, we review some of these aspects.

2.3.1 BPS black holes, the attractor mechanism and BPS indices

In the following we consider $\mathcal{N}=2$ supergravity in four dimensions coupled to Abelian vector multiplets. These theories naturally arise from compactifications of type II string theory on Calabi-Yau manifolds. Moreover, by taking local Calabi-Yau spaces gravity decouples and one can reproduce the Seiberg-Witten solution of $\mathcal{N}=2$ field theories. This section follows mainly [38].

We denote the rank of the gauge group of the vector multiplets by r. In addition, the gravity multiplet has a U(1) gauge field. This gives rise to a rank $r+1$ Abelian gauge group in four dimensions. The lattice Γ of electric and magnetic charges is hence of dimension $2r+2$ and the moduli space of the vector multiplet scalars u is of complex dimension r.[8] To state it more precisely, in type IIA string theory on a Calabi-Yau manifold X D0- and D2-branes on X can be considered as electrically charged states, while D4- and D6-branes are magnetically charges states. In type IIB theory on a Calabi-Yau space X all of the charges are realized by D3-branes wrapping three-cycles in X. Thus, a splitting into magnetic and electric charges is obtained by making a choice of symplectic basis of $H_3(X,\mathbb{Z})$.

The moduli space describing the scalars is a special Kähler manifold. In the case of type IIB theory on X we choose a symplectic basis of three-cycles (A^I, B_I), $I=1,\dots,r+1$. For a charge vector $\gamma=(p^I,q_I)$ we define the corresponding three-cycle $C=p^IB_I-q_IA^I$, then the central charge is given by

$$Z(\gamma,u)=e^{K/2}\int_C\Omega, \tag{2.92}$$

where $\Omega(u)$ is the holomorphic three-form and K denotes the Kähler potential. There exists a symplectic product of charge vectors that we write by abuse of notation as

$$\langle\gamma_1,\gamma_2\rangle=\int_X\gamma_1\wedge\gamma_2^*. \tag{2.93}$$

On the right hand side the γ_i are understood as Poincaré dual to the cycle determining the charges γ_i. Notice, that the formulae above can be translated to type IIA theory via mirror symmetry.

In the $\mathcal{N}=2$ supergravity setup BPS states are given by multi-centered, dyonic, extremal black holes whose centers are labelled by $(\vec{x}_i,\gamma_i)$. The ansatz for the metric of a stationary BPS solution is of the form

$$ds^2=-e^{2U(\vec{x})}(dt+\omega)^2+e^{-2U(\vec{x})}d\vec{x}^2, \tag{2.94}$$

where $\omega=\omega_i(\vec{x})dx^i$ and $U,\omega\to 0$ at spatial infinity. The BPS equations of motion then read [37]

$$\begin{aligned}2e^{-U}\mathrm{Im}(e^{-i\alpha}e^{K/2}\Omega)&=-H\\ *_3 d\omega&=\langle dH,H\rangle,\end{aligned} \tag{2.95}$$

[8] Note, that for type IIA compactified on X we have $r=h^{1,1}(X)$ and $\Gamma=H^{\text{even}}(X,\mathbb{Z})$. For type IIB compactified on X we have $r=h^{2,1}(X)$ and $\Gamma=H^{\text{odd}}(X,\mathbb{Z})$.

where $*_3$ is the Hodge star in flat Euclidian space $\mathbb{R}^3$ and $e^{i\alpha}$ denotes the phase of the central charge $Z(\sum_i \gamma_i, u)$. The function $H : \mathbb{R}^3 \to \Gamma \otimes \mathbb{R}$ is harmonic with poles at the centers $(\vec{x}_i, \gamma_i)$ and can be given in asymptotically flat space by

$$H(\vec{x}) = \sum_i \frac{\gamma_i}{|\vec{x} - \vec{x}_i|} - 2\mathrm{Im}(e^{-i\alpha} e^{K/2}\Omega)_{r=\infty}, \quad (r = |\vec{x}|). \tag{2.96}$$

The first equation of (2.95) can be reduced for a single center black hole to the attractor flow equation [59] whose integrated form reads

$$2e^{-U}\mathrm{Im}(e^{-i\alpha} e^{K/2}\Omega) = -\gamma r^{-1} + \mathrm{const.} \tag{2.97}$$

The attractor equation fixes the moduli in the near-horizon limit, denoted u_*, to

$$2e^{K/2}\mathrm{Im}(\overline{Z(\gamma, u_*(\gamma))}\Omega) = -\gamma, \tag{2.98}$$

and determines the classical Bekenstein-Hawking entropy of the single-centered black hole

$$S(\gamma) = \pi |Z(\gamma, u_*(\gamma))|^2. \tag{2.99}$$

For multi-center solutions eqns. (2.98), (2.99) hold for each center separately. The second equation in (2.95) has non-singular solutions if the centers $(\vec{x}_i, \gamma_i)$ satisfy the following integrability condition for all i [37]

$$\sum_{j \neq i} \frac{\langle \gamma_i, \gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\mathrm{Im}(e^{-i\alpha_{r=\infty}} Z(\gamma_i, u_{r=\infty})). \tag{2.100}$$

This is a condition on the existence of the multi-centered bound-states.

This gives a macroscopic description of the entropy of dyonic, extremal black holes. For a microscopic understanding of the entropy it is hence necessary to count the degeneracies of BPS states contributing to the entropy of the black hole. We are therefore interested in the Hilbert space of BPS states $\mathcal{H}_u^{\mathrm{BPS}}$ which is endowed with a grading originating from the charges $\gamma \in \Gamma$,

$$\mathcal{H}_u^{\mathrm{BPS}} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}^{\mathrm{BPS}}. \tag{2.101}$$

Here, the BPS Hilbert space $\mathcal{H}_{\gamma,u}^{\mathrm{BPS}}$ of charge γ is defined as the subspace of the one-particle Hilbert space satisfying $m = |Z(\gamma, u)|$. To enumerate BPS states one can define a so-called BPS index [31, 38]

$$\Omega(\gamma, u) = \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}_{\gamma,u}^{\mathrm{BPS}}}(2J_3)^2(-1)^{2J_3}, \tag{2.102}$$

where J_3 is the generator of spatial angular momentum $so(3)$. $\Omega(\gamma, u)$ is also known as second helicity supertrace. Since there can be massive states that are not BPS but whose mass saturates the bound just for particular values of u, one has to separate these "fake" representations from the "true" BPS states. The advantage of the index is that it presicely ensures this property as it vanishes on "fake" BPS representations.

2.3.2 The wall-crossing phenomenon

In the following we want to understand why the BPS index $\Omega(\gamma, u)$ can change as the moduli u vary. Reference is given to [37, 38, 63, 106].

BPS bound-states

The idea is that BPS particles of charge γ_i and mass m_i can form BPS bound-states of charge γ and mass m, but do not have to be stable at each point $u \in \mathcal{M}$ in the moduli space. First of all, conservation of charge implies that $\gamma = \sum_i \gamma_i$. For simplicity, we will concentrate on bound-states of two particles, i.e. $i = 1, 2$. Such bound-states are only stable if it is energetically favorable. Their binding energy is given by

$$E_{\text{bound}} = |Z(\gamma_1 + \gamma_2, u)| - |Z(\gamma_1, u)| - |Z(\gamma_2, u)|. \tag{2.103}$$

Because Z is linear in the charges we can use the triangle inequality to conclude that the binding energy is non-positive and therefore the bound-state exists and the two particles cannot be separated to infinity unless the central charges $Z(\gamma_1, u)$ and $Z(\gamma_2, u)$ align as complex numbers. Equivalently, a decay is possible if masses are conserved and yields the same result for marginal stability as above. It is therefore interesting to introduce walls of marginal stability which are given by

$$\text{MS}(\gamma_1, \gamma_2) = \{u \,|\, Z(\gamma_1, u)/Z(\gamma_2, u) \in \mathbb{R}^+\}. \tag{2.104}$$

These are real co-dimension one loci in the moduli space $\mathcal{M}$. It remains to answer the question, which side of the wall is the stable region? To answer it, we recall the condition on the existence of bound-states in $\mathcal{N} = 2$ supergravity, i.e. eq. (2.100). Specialized to two centers $(\vec{x}_i, \gamma_i)$, $i = 1, 2$, it reads

$$|\vec{x}_1 - \vec{x}_2| = \frac{\langle \gamma_1, \gamma_2 \rangle}{2} \frac{|Z(\gamma_1, u) + Z(\gamma_2, u)|}{\text{Im}(Z(\gamma_1, u)\overline{Z(\gamma_2, u)})}\bigg|_{r=\infty}. \tag{2.105}$$

Hence, in order to have a well-defined, that is positive, distance the necessary condition for existence in this case reads

$$\langle \gamma_1, \gamma_2 \rangle \, \text{Im}(Z(\gamma_1, u)\overline{Z(\gamma_2, u)})_{r=\infty} > 0. \tag{2.106}$$

From (2.105) it directly follows that the distance of the two centers diverges when a wall of marginal stability is approached. This is the supergravity realization of the decay of the bound-states.

Kontsevich-Soibelman wall-crossing formula

It is quantitatively understood how the BPS spectrum changes. The answer is encoded in the famous Kontsevich-Soibelman wall-crossing (KSWC) formula [106] which allows a computation of how the BPS index changes as one moves across a wall of marginal stability u_{MS}, i.e. it explicitly gives a formula to calculate

$$\Delta\Omega(\gamma, u_{\text{MS}}) = \Omega(\gamma, u_+) - \Omega(\gamma, u_-), \tag{2.107}$$

where $u_\pm$ are points infinitesimal displaced on opposite sides of the wall u_{MS}. The ingredients are the following:

– A Lie algebra defined by generators e_γ with $\gamma \in \Gamma$ and commutation relation

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle \, e_{\gamma_1 + \gamma_2}. \tag{2.108}$$

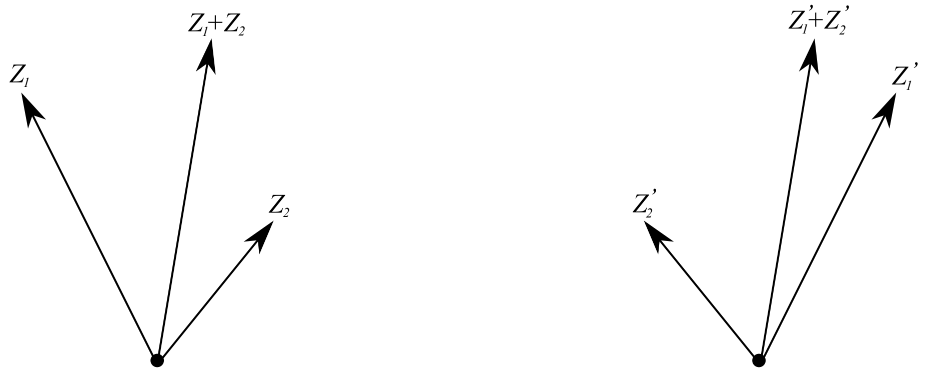

Figure 2.5: To each BPS particle of charge γ we can associate a ray in the complex plane, determined by the central charge $Z(\gamma, u)$. As we vary $u \in \mathcal{M}$ these rays start to rotate and the cyclic ordering of the rays changes precisely when u reaches a wall of marginal stability. At such a wall a subset of the BPS rays align. We depict this situation for two charges γ_i, $i = 1, 2$ with central charges denoted Z_i. The primed Z's refer to the same central charges evaluated at a different chamber in moduli space.

– A Lie group element U_γ defined for each charge $\gamma \in \Gamma$ by

$$U_\gamma = \exp\left(-\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2}\right). \tag{2.109}$$

With these ingredients the result of Kontsevich and Soibelman can be formulated by considering the following product

$$\prod_{\gamma}^{\curvearrowright} U_\gamma^{\Omega(\gamma,u)}, \tag{2.110}$$

where the product is taken over all charges γ and the factors are ordered clockwisely corresponding to the phases of the central charges $Z(\gamma, u)$. As we have argued before, when u approaches a wall of marginal stability this ordering changes and the $\Omega(\gamma, u)$ jump, Fig. 2.5. The statement of the wall-crossing formula is, that the whole product (2.110) does not change if we cross a wall. By evaluating the product on two sides of a wall u_{MS}, one can determine the change in the BPS invariant $\Delta\Omega(\gamma, u_{\mathrm{MS}})$.

In [38] a primitive and semi-primitive wall-crossing formula was derived which is of course correctly reproduced by the KSWC formula. Fixing two primitive charges γ_1 and γ_2 the change of the BPS index across a wall of marginal stability u_{MS} can be evaluated explicitly and yields

$$\Delta\Omega(\gamma_1 + \gamma_2, u_{\mathrm{MS}}) = (-1)^{\langle\gamma_1,\gamma_2\rangle-1} |\langle\gamma_1, \gamma_2\rangle|\, \Omega(\gamma_1, u_{\mathrm{MS}})\, \Omega(\gamma_2, u_{\mathrm{MS}}). \tag{2.111}$$

For these fixed charges $\gamma_{1/2}$, u_{MS} is also a wall of marginal stability for charges $N_1\gamma_1 + N_2\gamma_2$ with $N_1, N_2 > 0$. Then, there exists a semi-primitive wall crossing formula

$$\sum_{N_2>0} \Delta\Omega(\gamma_1 + N_2\gamma_2)\, q^{N_2} = \Omega(\gamma_1) \prod_{k>0} \left(1 - (-1)^{k\langle\gamma_1,\gamma_2\rangle} q^k\right)^{k|\langle\gamma_1,\gamma_2\rangle|\Omega(k\gamma_2)}, \tag{2.112}$$

where all BPS indices are evaluated at u_{MS}.

There has been a number of physical interpretations of the KSWC formula. For instance, in [10] the wall-crossing formula was derived from a notion called supersymmetric galaxies. In [63] the KSWC formula was interpreted for the BPS spectrum of Seiberg-Witten theories. There the BPS instanton corrected hyperkähler metric of the moduli space of the theory on $\mathbb{R}^3 \times S^1$ was shown to be continuous when the KSWC formula is applied to the spectrum of BPS states.

Chapter 3

Direct integration and non-perturbative effects in matrix models

This chapter has been published in ref. [102], where direct integration is proposed as a new method to solve the closed amplitudes of multi-cut matrix models with polynomial potentials.

3.1 Introduction and Results

In the following we calculate the closed partition function of multi-cut matrix models

$$Z(\underline{S}) = \exp\left(\sum_g g_s^{2g-2} F_g(\underline{S})\right) \tag{3.1}$$

perturbatively in the genus g, but exactly in the 't Hooft parameters $\underline{S}$. Exact means that the $F_g(\underline{S})$ are given in terms of period integrals of the spectral curve Σ and can be written explicitly in terms of modular forms of subgroups of $\mathrm{Sp}(2g(\Sigma), \mathbb{Z})$, where $g(\Sigma)$ denotes the genus of the spectral curve Σ.

Direct integration refers to a method of solving the holomorphic anomaly equation [15] using the modular transformation properties of the amplitudes under the monodromy group of the spectral curve. This method has been developed in the context of topological string theory in [6, 77, 80, 81, 92, 169]. The fact that the holomorphic anomaly equations govern such matrix models was suggested by the large N duality of [45]. In this duality, type B topological string amplitudes on certain local Calabi-Yau spaces turn out to be encoded in the $1/N$ expansion of matrix model partition functions. Therefore, the holomorphic anomaly of the topological string naturally carries over to these matrix models as first pointed out in [89]. It has been shown much more generally in [56] that the holomorphic anomaly equation is valid for all matrix models which are solvable by the method of [57].

The holomorphic anomaly equation relates anti-holomorphic derivatives of the closed amplitudes $F_g(\underline{S})$ at genus g to lower genus amplitudes $F_{h<g}(\underline{S})$, in a recursive way. Since only the anti-holomorphic derivative is specified by the equations, the procedure leaves a holomorphic ambiguity, i.e. $F_g(\underline{S}) = F_g^{\mathrm{nh}}(\underline{S}) + f_g(\underline{S})$ splits into a non-holomorphic term $F_g^{\mathrm{nh}}(\underline{S})$, which is determined by the holomorphic anomaly equation, and the holomorphic ambiguity $f_g(\underline{S})$, which must be fixed genus by genus by using modular properties and boundary conditions at special points in the moduli space. The modular transformation properties imply that the amplitudes are generated by a finite ring of modular forms, which have holomorphic as well as non-holomorphic generators. Modularity and the holomorphic anomaly equation imply that the total amplitude $F_g(\underline{S})$ is a polynomial in these generators whose degree grows linearly with the genus. The ambiguity $f_g(\underline{S})$ is a polynomial generated by the smaller ring of holomorphic generators. The finite number of coefficients in this polynomial must be fixed by boundary conditions.

In this paper we find that the gap conditions, which where investigated in non-compact [7, 82, 89, 90] and compact Calabi-Yau backgrounds [77, 80, 81, 92], provide enough independent boundary conditions to fix the ambiguity (and hence the amplitudes) completely. Following [82] we refer to this property as integrability of the holomorphic anomaly equation.

The large N duality relating matrix models and topological strings gives a natural geometric interpretation to the algebraic objects describing the planar limit of the matrix model [45]. The spectral curve $y(x)$ of the matrix model (which, in the case of polynomial potentials, is a hyperelliptic curve) describes the distribution of eigenvalues in the planar limit, and in the topological string dual it describes the nontrivial part of the Calabi–Yau geometry. We derive the modular ring starting from the Picard-Fuchs equations governing the periods of the form $\Omega = y(x)\mathrm{d}x$. This is a general method[1], and since we expect that the gap boundary conditions fix the ambiguity, our approach should apply to general multi-cut matrix models with polynomial potential.

Of course, the formalism of [57] gives in principle all the genus g free energies of generic multi-cut matrix model in terms of universal formulae on the spectral curve. The price to pay for such a general approach is that its detailed implementation is in practice very involved. Even in two-cut models, going beyond genus two with the methods of [57] is not very feasible. In contrast, direct integration becomes very powerful when the spectral curve and its modular group are simple.

In this paper, in order to illustrate the method of direct integration, we focus on the two-cut matrix model with a cubic potential. In this model the N eigenvalues split in two sets $N = N_1 + N_2$ and condense in sets near the two critical points of the potential. This leads to the cuts in the spectral curve shown in figure 3.1. There are two independent 't Hooft couplings $S_i = g_s N_i$, $i = 1, 2$, which correspond to the integrals of Ω over the two cuts. As shown in [45], the planar free energy of this matrix model, $F_0(S_1, S_2)$, calculates the exact superpotential W_{eff} of an $\mathcal{N} = 2$ $U(M)$ supersymmetric gauge theory broken down to an $\mathcal{N} = 1$ gauge theory $U(M_1) \times U(M_2)$, by a cubic three-level superpotential in the adjoint [45] (notice that N_i are unrelated to M_i). The higher genus amplitudes $F_g(S_1, S_2)$ in the matrix model arise as generalized couplings in a non-commutative deformation of the $\mathcal{N} = 1$ gauge theory [144].

Certain aspects of the original $\mathcal{N} = 2$ theory can be recovered from the $\mathcal{N} = 1$ theory by breaking the gauge symmetry to the Cartan subgroup and taking the limit in which the superpotential vanishes [24]. When the gauge group is $SU(2)$, a cubic superpotential is enough to go to the Coulomb branch. This implies that various quantities appearing in the Seiberg–Witten solution of pure $\mathcal{N} = 2$ super Yang–Mills theory [147] can be obtained from a matrix model calculation with a cubic potential, and on the slice $S_1 = -S_2$. These include the gauge coupling [43] and the R_+^2 gravitational coupling [44, 99]. In fact, the spectral curve of the cubic matrix model on that slice is identical to the Seiberg–Witten curve [43]. Since the modular group of this curve is particularly simple, direct integration becomes an extremely powerful method to calculate the $F_g(S_1, -S_1)$, as we show in section 3.4.

On the other hand, in the $SU(2)$, $\mathcal{N} = 2$ gauge theory there is an infinite number of couplings $F_g(a)$, $g \geq 2$, which describe the gauge-gravity couplings $F_+^{2g-2} R_+^2$ involving the graviphoton field strength F_+. These couplings appear naturally in Nekrasov's partition function [138] and they can be also obtained by using the holomorphic anomaly equations. This

[1] For example, a meromorphic modular form of weight k of $\mathrm{SL}(2, \mathbb{Z})$ or a congruence subgroup fulfills a linear differential equation of order $k + 1$ in the total modular invariant [179].

was shown for the pure gauge theory and $SU(2)$ with matter in [77, 89] and [90] respectively. However, it was noticed in [99] that these higher genus couplings $F_g(a)$ do not agree with the higher genus $F_g(S_1, S_2)$ obtained in the cubic matrix model and then restricted to the slice $S_2 = -S_1$. This disagreement is due to the fact that the Seiberg-Witten differential $\lambda_{\rm SW}$ differs from the natural differential Ω on the spectral curve of the matrix model. In contrast, τ and F_1 only depend on the spectral curve, and not on the differential, and therefore are the same in both cases. In [101, 153] matrix models are derived which encode all $\mathcal{N} = 2$ gauge theory amplitudes F_g for arbitrary g, however one has to introduce potentials involving polylogarithms and their quantum generalizations.

An interesting application of our computation of the couplings $F_g(S_1, S_2)$ at high genus is the study of non-perturbative effects in matrix models and their connection to the large order behavior of the $1/N$ expansion. It is well-known that, in many quantum systems, there is a connection between perturbation theory at large orders and instantons (see for example [109]). In matrix models, instanton configurations correspond to the tunneling of eigenvalues between different saddle points [34, 151]. A detailed analysis of these configurations for off-critical, one-cut matrix models can be found in [126], which verified the connection to the large order behavior of the $1/N$ expansion in detail in some nontrivial examples. In this paper we explore this connection in the two-cut matrix model. On the one hand, we find that the large order behavior is controlled at leading order by the action of a single eigenvalue tunneling from one saddle-point to the other, in agreement with the general ideas put forward in [34,126,151]. On the the other hand, we argue that a full understanding of this connection requires new non-perturbative sectors which have not been yet identified in the matrix model. The existence of these sectors is also suggested by a recent analysis of the asymptotic behavior of the instanton solutions of the Painlevé I equation [67]. We conjecture that these sectors might involve topological brane-antibrane systems.

3.2 Direct integration of the holomorphic anomaly equation

Below we review very briefly the generic aspects of the techniques of direct integration of the holomorphic anomaly equation of [15]

$$\bar{\partial}_{\bar{\imath}} F_g = \frac{1}{2} \bar{C}_{\bar{\imath}}^{jk} \left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_{g-r} D_k F_r \right) , \qquad (g > 1) , \tag{3.2}$$

which was derived for Calabi-Yau three-folds in [6, 77, 80, 81, 92, 169]. In our application to the spectral curve Σ of a matrix model, the D_i are covariant derivatives D_i with respect to the metric G on the moduli space of the Riemann surface Σ.

We note that $\bar{C}_{\bar{\imath}}^{jk} = \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} G^{\bar{\jmath}j} G^{\bar{k}k}$, where C_{ijk} can be derived from the holomorphic prepotential F_0 as $C_{ijk} = D_i D_j \partial_k F_0$. The prepotential F_0, the metric $G_{i\bar{\jmath}}$ and flat coordinates $\underline{S}$ can all be derived from the period integrals

$$\left(\int_{a^i} \Omega, \int_{b_i} \Omega \right), \qquad i = 1, \ldots, g(\Sigma) \tag{3.3}$$

over a symplectic basis (a^i, b_i) of $H_1(\Sigma, \mathbb{Z})$. In particular, given a point in the moduli space, one can make a choice of this symplectic basis, so that suitable flat coordinates are defined by

$$S^i = \int_{a^i} \Omega \tag{3.4}$$

while the b_i periods Π_i fulfill

$$\Pi_i = \frac{\partial F_0}{\partial S^i} \ . \tag{3.5}$$

These relations determine the prepotential F_0 up to an irrelevant constant. We define the τ matrix of the Riemann surface as

$$\tau_{ij} = \frac{\partial^2 F_0}{\partial S^i \partial S^j}. \tag{3.6}$$

The matrix $\mathrm{Im}(\tau)_{ij}$ is positive definite, and it gives the metric on the moduli space of the model. Equivalently, the metric can be obtained form the Kähler potential

$$K = \frac{1}{2\pi i}\left(\Pi_i \bar{S}^{\bar{\imath}} - \bar{\Pi}_{\bar{\imath}} S^i\right) . \tag{3.7}$$

On Riemann surfaces the period integrals can often be directly performed. Alternatively it might be useful to derive the Picard-Fuchs equations and reconstruct the periods as linear combinations of their solutions. Much of the above has been spelled out in the context of the Riemann surfaces for the B-model of topological string theory on non-compact Calabi-Yau in [82]. The relevant compact part of the geometry is given by a Riemann surface and a meromorphic differential, which comes from reducing the holomorphic $(3,0)$-form on the Riemann surface. After identification of the former with the spectral curve Σ and the later with the form Ω, we can use the formalism discussed in [82].

One property of the matrix model geometry is that the periods over the a-cycles do not fulfill the relation $\sum_{i=1}^r S^i = 0$. Usually this relation is inherited by the periods of holomorphic forms due to the homological relation of the cycles. However, in matrix models one has $\sum_{i=1}^r S^i \propto N$, because Ω has one non-vanishing residue outside the cuts. This leads to one algebraic relation between the periods in terms of the r parameters, which for the $r = 2$ case (the cubic matrix model) is expressed in eq. (3.35). The property of a non-vanishing residue is shared with Seiberg-Witten theories with matter [90] and certain non-compact Calabi-Yau geometries with more than one Kähler class [7, 82].

3.2.1 Direct integration

The so-called propagator plays a decisive role in the solutions of the B-model [15]. For the formalism on the Riemann surface Σ one needs only one type[2] of propagator S^{ij} defined by

$$\bar{\partial}_{\bar{\imath}} S^{ij} = C^{ij}_{\bar{\imath}}, \tag{3.8}$$

where $i, j = 1, \ldots, r$ and r is the number of parameters in the model. Following [15] it can be shown that the F_g can be written as

$$F_g = \sum_{|I|=0}^{3g-3} f_{g,i_1 \ldots i_{|I|}}(\underline{S})\ S^{i_1 i_2} \ldots S^{i_{|I|-1} i_{|I|}} \tag{3.9}$$

[2] In the threefold cases there are three types S^{ij}, S^i and S.

where the $f_{g,I}(\underline{S})$ are holomorphic tensors of the moduli. The most important property of the S^{ij} is that

$$\partial_{\bar{i}} F_g = C_{\bar{i}}^{ij} \frac{\partial F_g}{\partial S^{ij}}. \qquad (3.10)$$

If one assumes linear independence of the S^{ij} as functions of $\underline{S}$, it follows from this property that (3.2) can be rewritten as a set of equations

$$\frac{\partial F_g}{\partial S^{ij}} = \frac{1}{2}\left(D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_{g-r} D_k F_r \right), \qquad (g > 1)\ . \qquad (3.11)$$

These equations can be integrated algebraically, provided that the r.h.s. can be expressed in terms of the S^{ij} contracted by holomorphic tensors as in the r.h.s of (3.9). This is possible since the following closing relations are fulfilled due to special geometry [6, 15]

$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f_i^{kl}, \qquad (3.12)$$

$$\Gamma_{ij}^k = -C_{ijl} S^{kl} + \tilde{f}_{ij}^k\ , \qquad (3.13)$$

$$\partial_i F_1 = \frac{1}{2} C_{ijk} S^{jk} + A_i. \qquad (3.14)$$

Here the f_i^{kl}, $\tilde{f}_i^{kl}$ and A_i are holomorphic ambiguities, which must have the same transformation properties as the expressions on the left-hand side. These ambiguities are due to the fact that (3.8) defines S^{ij} only up to an holomorphic tensor. Different choices are possible and lead to a redefinition of the $f_{g,I}$ in (3.9). As we mentioned above the periods are not algebraically independent, see for example (3.35). As a consequence it is possible to make a choice for the above ambiguities so that for a given i one has $S^{ik} = 0$, $\forall k$, i.e. the matrix of propagators has effectively only rank $\rho = r - 1$. We call the auxiliary parameter t . There may be more auxiliary parameters stemming from the independent non-vanishing residua of Ω. If there are κ such residua, the rank is reduced to $\rho = r - \kappa$.

Whether one works with the redundant or the reduced set of propagators the equation (3.11) can easily be integrated w.r.t. S^{ij} and F_g becomes of degree $3g - 3$ in the S^{ij}. This is an efficient way to solve the recursion, but at each step one still has to determine the holomorphic ambiguity.

3.2.2 Modular covariant formulation

It is possible to relate the non-redundant set of propagators to quasimodular forms. In particular, in the holomorphic polarization, the following properties derived in [1] hold:

1. $F_g(\underline{S})$ is invariant under the monodromy group Γ of the Riemann surface Σ.

2. $F_g(\underline{S})$ is an *almost*-holomorphic modular function, i.e. its non-holomorphic dependence is encoded solely in $((\tau - \bar{\tau})^{-1})^{IJ}$, where $I, J = 1, \ldots, \rho$ and τ_{IJ} is the standard matrix valued modular parameter living in the Siegel upper half space.[3]

[3] $\rho = r - \kappa$, where κ is the number of independent non-vanishing residua of Ω.

3. The non-holomorphic dependence combines always with quasimodular forms E^{IJ} to give almost-holomorphic modular forms

$$\widehat{E}^{IJ} = E^{IJ}(\tau) + ((\tau - \bar{\tau})^{-1})^{IJ}. \tag{3.15}$$

Here we defined $E^{IJ}(\tau)$ as derivative of $\frac{\partial}{\partial \tau_{IJ}} F_1(\tau)$. The anomaly equation of [15] for $F_1(\tau)$ implies that this is a non-holomorphic modular invariant

$$F_1 = -\log\left[\det^{\frac{1}{2}}(\mathrm{Im}(\tau_{IJ}))\left(\bar{\Phi}_k(\bar{\tau})\Phi_k(\tau)\right)^a\right] \tag{3.16}$$

under the monodromy group Γ. $\Phi_k(\tau)$ is a holomorphic Siegel modular cusp form of weight k which vanishes at the discriminant Δ of the Riemann surface. It transforms as $\Phi_k(\tau_\gamma) = \det(C\tau + D)^k \Phi_k(\tau)$, where τ_γ is given by

$$\tau_\gamma = (A\tau + B)(C\tau + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2\rho, \mathbb{Z}) . \tag{3.17}$$

Such modular forms exist for all genus and can be written as products of even theta functions [103]. The exponent a will make the argument of the log invariant and the vanishing order at the discriminant $\frac{1}{12}\log(\Delta)$. For an elliptic curve $\Phi_k(\tau)$ is typically the Dedekind η-function. However, if the subgroup Γ allows for several cusp forms, $\Phi_k(\tau)$ can be a suitable multiplicative combination of them. In virtue of the definition $\widehat{E}^{IJ}$ transforms as a Siegel modular form

$$\widehat{E}^{IJ}(\tau_\gamma) = (C\tau + D)^I_K (C\tau + D)^J_L \widehat{E}^{KL}(\tau) . \tag{3.18}$$

4. $F_g(\underline{S})$ can be expanded as

$$F_g = \sum_{|I|=0}^{3g-3} \tilde{f}_{g,I_1,\ldots,I_{|I|}} \widehat{E}^{I_1 I_2} \ldots \widehat{E}^{I_{|I|-1} I_{|I|}} . \tag{3.19}$$

Note, that $\tilde{f}_{g,I}$ has to compensate for the modular transformation of τ and can in principle be expressed through holomorphic modular forms.

3.3 The two-cut cubic matrix model

As shown by Dijkgraaf and Vafa in [45], the B-model topological string theory on certain non-compact Calabi–Yau geometries is captured by a matrix model. The matrix model is the n-cut matrix model with potential $W(x)$, while the Calabi–Yau geometry is the following hypersurface in $\mathbb{C}^4$

$$uv = y^2 - (W'(x)^2 + f(x)). \tag{3.20}$$

Here, $f(x)$ is a polynomial of degree $n-1$ that splits the n double zeroes of $W'(x)^2$, see [123] for a detailed review. In the following we will combine the Dijkgraaf–Vafa correspondence with known results about the holomorphic anomaly equation in order to give a recursive solution of multi-cut matrix models.

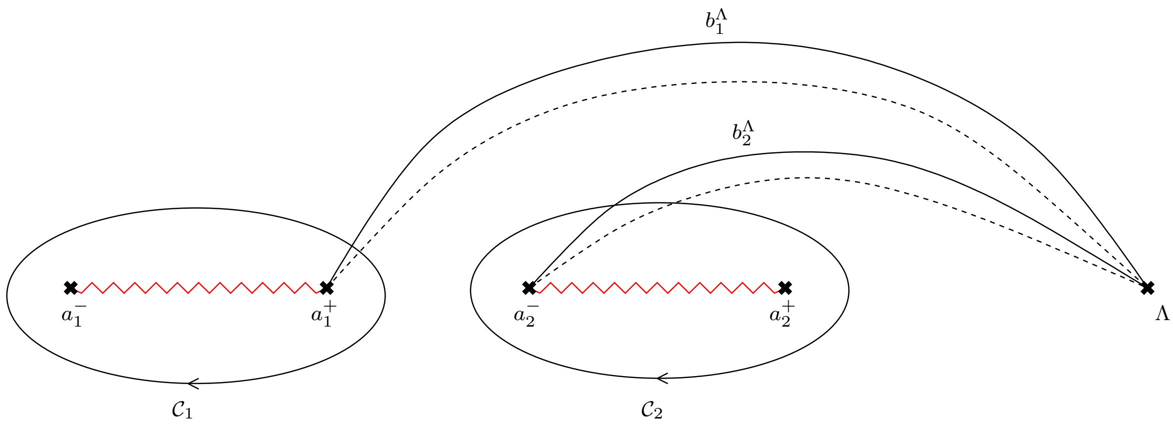

Figure 3.1: Choice of branch cuts and cycles on the elliptic geometry (3.25).

3.3.1 The geometrical setup

In the following we consider a cubic matrix model with potential W given by

$$W(x) = \frac{m}{2}x^2 + \frac{g}{3}x^3. \tag{3.21}$$

Since this model has two critical points $x = a_1$, $x = a_2$, the generic saddle will be a two-cut matrix model. If we write the matrix integral (2.45) in terms of eigenvalues, we have to distinguish two different sets $\{\mu_i\}_{i=1,\cdots,N_1}$, $\{\nu_j\}_{j=1,\cdots,N_2}$, which are expanded around a_1, a_2, respectively, and we obtain

$$Z = \frac{1}{N_1!N_2!}\int \prod_{i=1}^{N_1} \mathrm{d}\mu_i \prod_{j=1}^{N_2} \mathrm{d}\nu_j \prod_{i<j} (\mu_i - \mu_j)^2 (\nu_i - \nu_j)^2 \prod_{i,j} (\mu_i - \nu_j)^2 \, \mathrm{e}^{-\frac{1}{g_s}\left(\sum_i W(\mu_i) + \sum_j W(\nu_j)\right)}. \tag{3.22}$$

Since

$$W'(x) = mx + gx^2 = gx\Big(x + \frac{m}{g}\Big) = g(x-a_1)(x-a_2), \tag{3.23}$$

$W'(x)^2$ has two double zeroes at $x = a_1$, a_2, that are split by the degree one polynomial

$$f(x) = \lambda x + \mu \tag{3.24}$$

into four roots $a_1^\pm, a_2^\pm$. Hence, the curve for the geometry/matrix model is given by

$$y^2 = W'(x)^2 + f = g^2(x-a_1^-)(x-a_1^+)(x-a_2^-)(x-a_2^+). \tag{3.25}$$

We choose the branch cuts to be along the intervals (a_1^-, a_1^+) and (a_2^-, a_2^+), cf. Fig. 3.1. It follows from (2.58) that the 't Hooft parameters for this curve are the periods of the one-form

$$\Omega = y(x)\,\mathrm{d}x \tag{3.26}$$

around the branch cuts. Following the notation of [23], we have

$$S_i = \frac{1}{2\pi \mathrm{i}}\int_{a_i^-}^{a_i^+} \Omega, \qquad \Pi_i = \frac{1}{2\pi \mathrm{i}}\int_{b_i^\Lambda} \Omega. \tag{3.27}$$

These 't Hooft parameters are functions of the couplings in the potential m, g, and of the variables λ, μ. Equivalently, they are functions of the branch points $a_i^{\pm}$ of the quartic curve (3.25). It is convenient to define new variables given by

$$\begin{aligned} z_1 &= \frac{1}{4}(x_2 - x_1)^2, \quad z_2 = \frac{1}{4}(x_4 - x_3)^2, \\ Q &= \frac{1}{2}(x_1 + x_2 + x_3 + x_4) = -\frac{m}{g}, \\ I^2 &= \frac{1}{4}\left[(x_3 + x_4) - (x_1 + x_2)\right]^2 = \left(\frac{m}{g}\right)^2 - 2(z_1 + z_2), \end{aligned} \tag{3.28}$$

where we label the cuts more conveniently as

$$(a_1^-, a_1^+, a_2^-, a_2^+) = (x_1, x_2, x_3, x_4) \tag{3.29}$$

and we also have

$$\sigma(x) = \prod_{i=1}^{4}(x - x_i). \tag{3.30}$$

We will use this in order to expand all four periods in powers of z_1 and z_2. Notice that z_i are coordinates that parameterize the complex structure deformations of the local Calabi–Yau geometry (3.20).

Let us consider S_1. For this we change variables to $y = x - \frac{1}{2}(x_1 + x_2)$ and the integral becomes

$$S_1 = \frac{g}{2\pi}\int_{y_3}^{y_4} \sqrt{(y - y_3)(y - y_4)}\sqrt{y^2 - z_1}\mathrm{d}y.$$

Expanding the second square root for z_1 small, each term in the series can be computed explicitly and it is most easily given in terms of a generating function [23],

$$F(a) = -\pi\sqrt{(y_3 + a)(y_4 + a)} + \frac{\pi}{2}(y_3 + y_4 + 2a) \tag{3.31}$$

as follows,

$$S_1 = \frac{g}{32}(y_3 + y_4)(y_4 - y_3)^2 + \frac{g}{2\pi}\sum_{n=1}^{\infty} c_n \Delta_{21}^{2n} F^{(n)}(0)$$

where c_n are the coefficients in the expansion of $\sqrt{1-x}$ and $F^{(n)}(a)$ is the n-th derivative with respect to a.

The explicit answer has the following structure,

$$S_1 = \frac{g}{4}z_2 I - \frac{g}{2I}K(z_1, z_2, I^2), \tag{3.32}$$

where

$$K(x, y, z) = \frac{1}{4}xy\left(1 + \frac{1}{4z}(x+y) + \frac{1}{8z^2}(x+y)^2 + \frac{1}{8z^2}xy + \dots\right).$$

It is important to notice that this is symmetric in (x, y), namely, $K(x, y, z) = K(y, x, z)$. This allows us to write,

$$S_2 = -\frac{g}{4}z_1 I + \frac{g}{2I}K(z_1, z_2, I^2). \tag{3.33}$$

In the following we will simplify the expressions by putting $m = g = 1$. It will be useful to change variables to

$$t = S_1 + S_2, \qquad s = \frac{1}{2}(S_1 - S_2) \tag{3.34}$$

where t is the total 't Hooft parameter. Due to (3.32) and (3.33) one immediately obtains

$$t = \frac{1}{4}(z_2 - z_1)\sqrt{1 - 2z_1 - 2z_2}. \tag{3.35}$$

Note, that t can be regarded as a global parameter of the model. Different from t the expression of s in terms of the z_i requires a transcendental function. This more complicated function reflects the dependence of s on the choice of the symplectic basis in (3.3).

As mentioned earlier, there is another possibility to derive the periods as series in z_i which was applied in [89]. There the authors consider a set of Picard–Fuchs differential operators, $\mathcal{L}_1, \mathcal{L}_2$ associated to the spectral curve and differential Ω, which annihilate the periods. Therefore, these can be calculated as solutions to a system of ODEs. The Picard-Fuchs operators, which are given in eq. (C.1) of appendix C, have the following discriminant factors

$$\text{disc} = z_1 z_2 I^2 J = z_1 z_2 (1 - 2(z_1 + z_2))(1 - 6z_1 - 6z_2 + 9z_1^2 + 14z_1 z_2 + 9z_2^2). \tag{3.36}$$

Moreover, their solutions around $z_1 = 0$ and $z_2 = 0$ describe the periods of the elliptic geometry (3.25). Due to the fact that one can find a combination of periods such that the mirror map becomes exact (3.35), it is convenient to introduce adapted coordinates $\tilde{z}_i$, $i = 1, 2$, by

$$\tilde{z}_1 = z_1 + z_2, \quad \tilde{z}_2 = \frac{1}{4}(z_1 - z_2)\sqrt{1 - 2(z_1 + z_2)}, \tag{3.37}$$

as well as coordinates $\tilde{t}_i$, $i = 1, 2$, on the mirror by

$$\tilde{t}_1 = s = \frac{1}{2}(S_1 - S_2), \quad \tilde{t}_2 = t = S_1 + S_2. \tag{3.38}$$

The Yukawa couplings may be found in eq. (C.4) as well as the genus one free energy F_1 in eq. (C.5). Due to the special type of the mirror map

$$\tilde{z}_2 = \tilde{t}_2, \tag{3.39}$$

it is possible to derive a propagator which is of the following special form

$$S = \begin{pmatrix} S^{\tilde{z}_1 \tilde{z}_1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.40}$$

For the technical details as well as for the ambiguities that have to be computed we refer the reader to appendix C. With the help of this input it is easy to implement the direct integration procedure for the cubic matrix model as outlined in section 3.2.1. It turns out that we can recursively construct the free energies up to genus four. Moreover, we can also evaluate $F_g(S_1, S_2)$ for the cubic matrix model in perturbation theory, as was done in [89,99]. The expansions of our direct integration analysis read

$$
\begin{aligned}
F_2 &= -\frac{1}{240}\left(\frac{1}{S_1^2}+\frac{1}{S_2^2}\right)+\frac{35}{6}(S_1-S_2)+338S_1^2-1632S_1S_2+338S_2^2+\mathcal{O}(S^3)\\
F_3 &= \frac{1}{1008}\left(\frac{1}{S_1^4}+\frac{1}{S_2^4}\right)+\frac{5005}{3}(S_1-S_2)+\frac{32}{9}\left(52522S_1^2-273403S_1S_2+52522S_2^2\right)+\mathcal{O}(S^3)\\
F_4 &= -\frac{1}{1440}\left(\frac{1}{S_1^6}+\frac{1}{S_2^6}\right)+\frac{8083075}{6}(S_1-S_2)+\frac{880}{3}\left(788369S_1^2-4387436S_1S_2+788369S_2^2\right)+\mathcal{O}(S^3).
\end{aligned}
\tag{3.41}
$$

These results agree with the low-order results obtained in [89,99]. In the following we explain how to parameterize the ambiguity and how to fix the unknowns entering our ansatz.

3.3.2 Direct integration, boundary conditions and integrability

In the last section we set up the necessary ingredients to perform a direct integration of the holomorphic anomaly equations. As mentioned in section 3.2.1 the free energies F_g can be written in the following way

$$
F_g = \sum_{k=1}^{3g-3} a_k(z_1,z_2)\left(S^{\tilde{z}_1\tilde{z}_1}\right)^k + f_g(z_1,z_2), \tag{3.42}
$$

where a_k are rational functions completely determined by the recursive procedure. f_g is the holomorphic anomaly, which is not constrained by direct integration and must be fixed by supplying further boundary conditions. The amplitudes F_g should be well-defined over the whole moduli space except for points at the boundary of moduli space where the elliptic geometry (3.25) acquires a node, i.e. a cycle of $\mathbb{S}^1$-topology shrinks. Such points are known as conifold points and are given by the zero loci of the discriminant of the Picard–Fuchs system, which we also call conifold divisors.

Thus, regularity and holomorphicity imply that f_g should be a rational function of z_i, where the numerator is at most of the same degree as the denominator. The denominator is given by the discriminant factors and takes the form $(z_1z_2J^2)^{2g-2}$. This gives the following ansatz for the holomorphic ambiguity

$$
f_g(z_1,z_2) = \frac{\sum_{k,l} a_{k,l}^{(g)} z_1^k z_2^l}{(z_1z_2J^2)^{2g-2}}, \tag{3.43}
$$

where the $a_{k,l}^{(g)}$ have to be determined by the boundary conditions. Note that due to the symmetry of the model in z_1 and z_2 it is enough to restrict the numerator to a polynomial which is symmetric in z_1 and z_2. In order to be well defined as $z_i \to \infty$, the degree of this polynomial must be at most $12g-12$. It turns out that it is sufficient to truncate the degree at $9g-9$, as long as $g \leq 4$. However, this reduced ansatz may not be present at higher genus and one would have to deal with the full ansatz of degree $12g-12$.

There are two boundary conditions which we will refer to in the following as a cycle gap and b cycle gap. Let us first consider the a cycle gap. Due to the Gaussian contribution to the partition function of a multi-cut matrix model (see for example [99] for more details) it is easily seen that the holomorphic expansion of F_g at small filling fractions is of the form

$$
F_g = \frac{B_{2g}}{2g(2g-2)}\left(\frac{1}{S_1^{2g-2}}+\frac{1}{S_2^{2g-2}}\right)+\mathcal{O}(S), \qquad (g>1). \tag{3.44}
$$

Due to the absence of subleading singular terms in S_i, this property of the expansion is referred to as the gap condition. The coefficients of the subleading singular powers of S_i depend generically on the other S_j, with $j \neq i$ –in fact they are (infinite) series in the S_j. Demanding the vanishing of these series leads in principle to an over-determined system, therefore in the multi-parameter case it is not easy to count the number of independent conditions implied by (3.44).

The gap condition is also present in the expansion of genus g topological string amplitudes near a conifold divisor [7, 81, 82], where we have

$$F_g^c = \frac{B_{2g}}{2g(2g-2)\Pi^{2g-2}} + \mathcal{O}(\Pi^0), \qquad (g > 1). \tag{3.45}$$

Here, Π is a flat coordinate normal to the divisor. In view of the Dijkgraaf–Vafa correspondence, this behavior should also characterize multi-cut matrix model amplitudes near the divisors of the spectral curve geometry. Again, since the coefficients of the subleading powers of Π depend on the coordinates tangential to the conifold divisor, the counting of conditions in the multi-parameter case is not easily done.

However, it turns out that, when both constraints, (3.44) and (3.45), are taken into account, the holomorphic anomaly f_g is completely and uniquely fixed. We checked this explicitly for genus $g \leq 4$. It is then natural to conjecture that the a and b cycle gap conditions are always sufficient to fix all unknowns in the holomorphic ambiguity for general matrix models with polynomial potential. Following [82] we refer to such a property as integrability of the holomorphic anomaly equation.

3.3.3 Modular covariant formulation

In the last sections we explained how to solve the cubic matrix model with the techniques known from topological string theory. However, we used a somewhat artificial description which does not make the symmetry properties of the geometry completely explicit. Such a formulation is given by writing all quantities in a covariant modular way. Since the geometry is an elliptic curve together with meromorphic differential Ω we expect not only to parameterize the topological amplitudes F_g by the elliptic modulus τ but in addition by an auxiliary parameter. In the following we explore how this can be achieved in detail.

We start by transforming the quartic curve (3.25) to Weierstrass form, where it is easy to read off the j-function. It is given by

$$j(z_1, z_2) = \frac{16\left((1-3z_1-3z_2)^2 + 12z_1z_2\right)^3}{z_1z_2\left((1-3z_1-3z_2)^2 - 4z_1z_2\right)^2}. \tag{3.46}$$

Comparing this modular invariant to its usual Fourier expansion

$$j(\tau) = q^{-1} + 744 + 196884q + \mathcal{O}(q^2), \tag{3.47}$$

we get a relation $\tau = \tau(z_1, z_2)$. Using the definition of j in terms of modular forms yields actually a rational expression.

It is also easy to identify the auxiliary parameter which accompanies τ. Note that the periods/filling fractions are taken with respect to the differential Ω, which is meromorphic. Thus the sum of all cycles is of course homologically trivial, but the sum of the periods does not have to vanish and is rather proportional to the residue of Ω. Since this residue is related to the auxiliary parameter, it is natural to parameterize the topological amplitudes by both, τ and $t = g_s N$. Due to (3.35) we obtain a relation $t = t(z_1, z_2)$.

In principle this allows us to rewrite all quantities in terms of modular forms together with an auxiliary parameter t, by combining the rational expression $\tau = \tau(z_1, z_2)$ with $t = t(z_1, z_2)$. If we do so, we obtain

$$u = \frac{1 - 3z_1 - 3z_2}{2\sqrt{z_1 z_2}}, \qquad 4t = (z_1 - z_2)\sqrt{1 - 2z_1 - 2z_2}, \tag{3.48}$$

where u is given in terms of modular forms b, c and d defined in appendix A.3 by

$$u = \frac{c + d}{b}. \tag{3.49}$$

However, it turns out that, for the general cubic matrix model, the resulting formulae become too complicated. The reason is that the corresponding spectral curve is a generic elliptic curve. However, if we specialize the calculation to the slice $t = 0$, or $S_1 = -S_2$, the curve has $\Gamma(2)$ monodromy (it is the Seiberg–Witten curve of [147]) and it is possible to exploit the formulation in terms of modular forms, as we will see in section 3.4.

Fortunately, it is possible to give some closed and simple expressions using modular forms for the genus zero and one sectors, which will prove to be useful in due course. We start by quoting the perturbative calculation of the planar free energy [23]

$$\begin{aligned} F_0(S_1, S_2) = &\frac{1}{2} S_1^2 \log\Big(\frac{S_1}{m\Lambda^2}\Big) + \frac{1}{2} S_2^2 \log\Big(\frac{S_2}{m\Lambda^2}\Big) - \frac{3}{4}(S_1^2 + S_2^2) + 2 S_1 S_2 \log\Big(\frac{m}{\Lambda g}\Big) \\ &+ \frac{1}{g\Delta^3}\Big(\frac{2}{3} S_1^3 - 5 S_1^2 S_2 + 5 S_1 S_2^2 - \frac{2}{3} S_2^3\Big) + \mathcal{O}(S^4). \end{aligned} \tag{3.50}$$

Of course, $F_0(S_1, S_2)$ is symmetric under the exchange $S_1 \leftrightarrow -S_2$. From the prepotential we can define the tau-coupling (3.6) and also introduce

$$2\pi \mathrm{i}\tau = \frac{\partial^2 F_0}{\partial s^2}. \tag{3.51}$$

It was shown in [127] (see also [18]) that τ can be computed in terms of elliptic functions as

$$\tau = \mathrm{i}\frac{\mathcal{K}'}{\mathcal{K}} = \mathrm{i}\frac{K(k')}{K(k)}, \tag{3.52}$$

where

$$\begin{aligned} \mathcal{K} &= \int_{x_1}^{x_2} \frac{\mathrm{d}z}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}} K(k), \qquad k^2 = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}, \\ \mathcal{K}' &= \int_{x_2}^{x_3} \frac{\mathrm{d}z}{\sqrt{|\sigma(z)|}} = \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}} K(k'), \qquad k'^2 = 1 - k^2. \end{aligned} \tag{3.53}$$

This modular parameter turns out to match with our definition by the j-function mentioned above. We find, in the full theory

$$\pi \mathrm{i}\tau = \frac{1}{2}\log\Big(\frac{-S_1 S_2}{m^6}\Big) + \frac{17(S_1 - S_2)}{m^3} + \frac{2\left(83 S_1^2 - 209 S_2 S - 1 + 83 S_2^2\right)}{m^6} + \cdots \tag{3.54}$$

Let us now consider genus one. Akemann [3] gave a simple expression for F_1, that reads

$$F_1 = -\frac{1}{24}\sum_{i=1}^{4}\ln M_i - \frac{1}{2}\ln K(k) - \frac{1}{12}\ln\Delta + \frac{1}{8}\ln(a_1^- - a_2^-)^2 + \frac{1}{8}\ln(a_1^+ - a_2^+)^2, \qquad (3.55)$$

where Δ denotes the discriminant of $\sigma(x)$. Using that $M_i = g$ for the cubic matrix model as well as Thomae's formulae, cf. app. A.3, this can be written compactly as

$$F_1 = -\log\eta(\tau) - \frac{1}{24}\log\Delta, \qquad (3.56)$$

where η is the Dedekind eta-function.

3.4 The cubic model on a slice

In the following we will specialize the cubic matrix model studied in section 3.3 to the slice $S_1 = -S_2$. On this slice, $t = 0$, and the direct integration procedure simplifies. Moreover, we are able to write all quantities which are needed for direct integration in terms of simple modular forms. The underlying reason for this is that, when $t = 0$, the spectral curve of the matrix model becomes the Seiberg–Witten curve, which has simple monodromy properties. Therefore the recursive procedure will be very efficient in obtaining results at high genus.

First of all notice that, by contour deformation,

$$S_1 + S_2 = g\oint_{z=0}\frac{\mathrm{d}z}{z^4}\sqrt{1 + \frac{2m}{g}z + \frac{m^2}{g^2}z^2 + \frac{\lambda}{g^2}z^3 + \frac{\mu}{g^2}z^4} = \frac{\lambda}{2g}. \qquad (3.57)$$

Therefore, if the parameter λ in (3.24) vanishes $\lambda = 0$, we have

$$t = S_1 + S_2 = 0. \qquad (3.58)$$

In this case one also has [43]

$$\tau_{11} = \tau_{22} = -\tau_{12} = \tau, \qquad (3.59)$$

which can be seen from (3.6).

From the point of view of the original matrix model, the slice $S_1 = -S_2$ involves an analytic continuation in the space of 't Hooft parameters. This is because on this slice $S_1/S_2 = N_1/N_2 = -1$, which can not be implemented in the matrix integral (3.22), since $N_{1,2}$ are *a priori* positive integers. In terms of matrix integrals, the slice $S_1 = -S_2$ can be related to a *supermatrix model* [8,39,48,176]. A Hermitian supermatrix has the form

$$\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}, \qquad (3.60)$$

where A (C) are $N_1 \times N_1$ ($N_2 \times N_2$) Hermitian, Grassmann even matrices, and Ψ is a matrix of complex, Grassmann odd numbers. The supermatrix model is defined by the partition function

$$Z_\mathrm{s}(N_1|N_2) = \int \mathcal{D}\Phi\, \mathrm{e}^{-\frac{1}{g_s}\mathrm{Str}W(\Phi)}, \qquad (3.61)$$

where we consider a polynomial potential $W(\Phi)$ and Str denotes the supertrace. There are two types of supermatrix models with supergroup symmetry $U(N_1|N_2)$: the ordinary supermatrix model, and the physical supermatrix model [176]. The ordinary supermatrix model is obtained by requiring A, C to be real Hermitian matrices, while the physical model is obtained by requiring that, after diagonalizing Φ by a superunitary transformation, the resulting eigenvalues are real. The partition function of the physical supermatrix model reads, in terms of eigenvalues [48, 176]

$$Z_{\rm s}(N_1|N_2) = \frac{1}{N_1!N_2!}\int\prod_{i=1}^{N_1}{\rm d}\mu_i\prod_{j=1}^{N_2}{\rm d}\nu_j\frac{\prod_{i<j}\left(\mu_i-\mu_j\right)^2\left(\nu_i-\nu_j\right)^2}{\prod_{i,j}\left(\mu_i-\nu_j\right)^2}{\rm e}^{-\frac{1}{g_s}\left(\sum_i W(\mu_i)-\sum_j W(\nu_j)\right)}, \tag{3.62}$$

where the two groups of eigenvalues μ_i, ν_j are expanded around two different critical points of $W(x)$. This partition function is related to (3.22) after changing $N_2 \to -N_2$ [48], therefore it gives a physical realization of the $S_1/S_2 < 0$ slice of the moduli space. Notice, that the moduli space of the local Calabi–Yau for generic complex S_1, S_2 describes both the original matrix integral (3.22) and its supergroup extension (3.62).

3.4.1 The geometry

In the following we discuss the geometry underlying the curve with $\lambda = 0$. It is easy to see that, up to a shift in the x coordinate, it can be written as

$$y^2 = (x^2-a^2)(x^2-b^2), \qquad a > b. \tag{3.63}$$

If we compare this to the Seiberg–Witten curve [147]

$$y^2 = (x^2-u)^2 - \Lambda_{\rm SW}^4, \tag{3.64}$$

we find that they are equal once we identify the parameters as

$$u = \frac{a^2+b^2}{2}, \qquad \Lambda_{\rm SW}^2 = \frac{a^2-b^2}{2}. \tag{3.65}$$

We also want to translate these parameters in terms of the cubic matrix model variables. This was already done in [43, 99], and we have

$$\Delta = \frac{m}{g}, \qquad u = \frac{1}{4}\Delta^2. \tag{3.66}$$

We will set

$$g = 1. \tag{3.67}$$

On the other hand, we have the following relation between the Λ parameter appearing in (3.50) and the Seiberg–Witten scale

$$\Lambda = \frac{1}{\sqrt{2}}\Lambda_{\rm SW}. \tag{3.68}$$

For the simple curve (3.63) one can compute many quantities directly and relate them to modular forms or elliptic integrals. As a starting point the period integrals

$$S = S_1 = -S_2, \qquad \Pi = \partial_s F_0(S,-S) \tag{3.69}$$

can be computed in terms of simple elliptic functions, which was done for S in ref. [43]. Repeating this analysis yields

$$S = \frac{1}{2\pi \mathrm{i}} \int_b^a y(x) \mathrm{d}x = \frac{a}{6\pi} \Big[(a^2 + b^2) E(k_1) - 2b^2 K(k_1) \Big] \tag{3.70}$$

as well as

$$\Pi = \int_{-b}^{b} y(x) \mathrm{d}x = \frac{2}{3} a \Big[(a^2 + b^2) E(k_1') + (b^2 - a^2) K(k_1') \Big], \tag{3.71}$$

where the elliptic modulus k_1 and its complementary one k_1' are given by

$$k_1^2 = \frac{a^2 - b^2}{a^2}, \qquad k_1'^2 = 1 - k_1^2 = \frac{b^2}{a^2}. \tag{3.72}$$

The modulus k_1 is related to the usual cross-ratio k^2 introduced in (3.53) as

$$k_1^2 = \frac{4k}{(1+k)^2}. \tag{3.73}$$

In order to obtain expansions of the periods we introduce the parameters

$$\mu = \frac{\Lambda_{\mathrm{SW}}^2}{u}, \qquad \mu_{\mathrm{D}} = 1 - \frac{\Lambda_{\mathrm{SW}}^2}{u}. \tag{3.74}$$

Small μ corresponds to the semiclassical regime of Seiberg–Witten theory which occurs at $u \to \infty$, whereas small μ_{D} relates to the region near $u \to \Lambda_{\mathrm{SW}}^2$, where a magnetic monopole becomes massless. In these variables the periods read

$$\frac{S}{u^{\frac{3}{2}}} = \frac{\sqrt{1+\mu}}{3\pi} \Big[E\Big(\frac{2\mu}{1+\mu}\Big) + (\mu - 1) K\Big(\frac{2\mu}{1+\mu}\Big) \Big], \tag{3.75}$$

$$\frac{\Pi}{u^{\frac{3}{2}}} = \frac{4\sqrt{2 - \mu_{\mathrm{D}}}}{3} \Big[E\Big(\frac{\mu_{\mathrm{D}}}{2 - \mu_{\mathrm{D}}}\Big) + (\mu_{\mathrm{D}} - 1) K\Big(\frac{\mu_{\mathrm{D}}}{2 - \mu_{\mathrm{D}}}\Big) \Big]. \tag{3.76}$$

Note that $S/u^{3/2}$ and $\Pi/u^{3/2}$ are dimensionless. Further, we expand (3.75) around $\mu = 0$ to obtain

$$\frac{S}{u^{\frac{3}{2}}} = \frac{\mu^2}{8} + \frac{3\mu^4}{256} + \frac{35\mu^6}{8192} + \frac{1155\mu^8}{524288} + \frac{45045\mu^{10}}{33554432} + \cdots, \tag{3.77}$$

which is the expansion (4.19) of [99], after changing to the appropriate variables. The inverse expansion is given by

$$\mu^2 = 8\frac{S}{u^{\frac{3}{2}}} - 6\Big(\frac{S}{u^{3/2}}\Big)^2 - \frac{17}{2}\Big(\frac{S}{u^{3/2}}\Big)^3 - \frac{375}{16}\Big(\frac{S}{u^{3/2}}\Big)^4 - \frac{10689}{128}\Big(\frac{S}{u^{3/2}}\Big)^5 + \cdots. \tag{3.78}$$

We introduce now the following elliptic modulus τ_0 as

$$\tau_0 = \mathrm{i}\frac{K\Big(\frac{1-\mu}{1+\mu}\Big)}{K\Big(\frac{2\mu}{1+\mu}\Big)} = \mathrm{i}\frac{K(k_1')}{K(k_1)} = \frac{\mathrm{i}}{2}\frac{K(k')}{K(k)}, \tag{3.79}$$

which can be expanded in μ. By inverting this series one can derive μ as a function of τ_0. In particular we observe

$$\mu = \frac{b}{c+d}, \tag{3.80}$$

where we follow the notation[4] of [89]. In turn the expression (3.75) defines the variable μ as a function of

$$\frac{S}{u^{3/2}} = 8\frac{S}{m^3} \tag{3.81}$$

as well, and in particular the series (3.79) defines τ_0 as a function of $S/u^{3/2}$:

$$2\pi\mathrm{i}\tau_0 = \log\Big(\frac{S}{m^3}\Big) + 34\frac{S}{m^3} + 750\Big(\frac{S}{m^3}\Big)^2 + \frac{71260}{3}\Big(\frac{S}{m^3}\Big)^3 + \cdots. \tag{3.82}$$

Moreover, comparing with (3.54) yields the identity

$$\tau_0 = \frac{1}{2}\tau(S, -S), \tag{3.83}$$

which is obvious also from (3.79).

Consider now the dual elliptic modulus $\tau_{0,\mathrm{D}}$, obtained by a S-transformation on the elliptic modulus,

$$\tau_{0,\mathrm{D}} = -\frac{1}{\tau_0}. \tag{3.84}$$

Following the same lines of thought as before, this defines $\tau_{0,\mathrm{D}}$ as a series in the dual period

$$\frac{S_\mathrm{D}}{u^{3/2}} = 8\frac{S_\mathrm{D}}{m^3}. \tag{3.85}$$

In the following we will set

$$\Lambda_{\mathrm{SW}} = 1 \tag{3.86}$$

so in particular $\mu = u^{-1} = 4/m^2$. Note that (3.80) therefore defines m as a function of τ_0. Strictly speaking, m is hence a function of S, but in order to establish the relation between τ_0 and $\tau(S, -S)$, i.e. (3.83), we treated m as an independent variable. In all subsequent formulas and expansions we will do so as well.

Next, we compute the Yukawa coupling

$$C_{sss} = \frac{\partial^3 F_0}{\partial s^3}. \tag{3.87}$$

This follows from the general formula for two-cut matrix models given by [127]

$$\frac{\partial^3 F_0}{\partial s^3} = \pi^3\Big[M_1\cdots M_4\,\mathcal{K}^3\prod_{i<j}(x_i - x_j)^2\Big]^{-1}\cdot\sum_{i=1}^{4}\Big[\prod_{j\neq i}M_j\cdot\prod_{\substack{k,l\neq i,\\ k<l}}(x_k - x_l)^2\Big] \tag{3.88}$$

where $M_i = M(x_i)$, the spectral curve is written as in (2.55), and $\mathcal{K}$ is given in (3.53) When applied to the Seiberg–Witten curve (3.63) we obtain

$$C_{sss} = \frac{\partial(4\pi\mathrm{i}\tau_0)}{\partial s} = \frac{64\sqrt{2}}{m^3}\frac{(c+d)^{5/2}}{b^2cd}. \tag{3.89}$$

To see this, one has to apply Thomae's formula, which relates the branch points x_i to ϑ-functions [58] and further one has to express $\mathcal{K}$ in terms of modular forms. This is done as follows. Note that

$$\mathcal{K} = \int_b^a \frac{\mathrm{d}x}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a}K(k_1). \tag{3.90}$$

[4] For our conventions on modular forms used in this section, see appendix A.3.

The dimensionless combination $\sqrt{u}\,\mathcal{K}$ can be expanded as a series in τ_0 since

$$\sqrt{u}\,\mathcal{K} = \frac{1}{\sqrt{1+\mu}} K\left(\frac{2\mu}{1+\mu}\right). \tag{3.91}$$

This yields

$$\mathcal{K} = \frac{\pi}{m}\sqrt{\frac{c+d}{2}}. \tag{3.92}$$

We can check the formula (3.87) by calculating this quantity directly from the perturbative result. Evaluating the derivatives at $S_1 = -S_2 = S$ we obtain from (3.50)

$$\frac{\partial^3 F_0}{\partial s^3} = \frac{2}{m^3}\Big\{34 + \frac{m^3}{S} + 1500\frac{S}{m^3} + 71260\frac{S^2}{m^6} + \cdots\Big\}. \tag{3.93}$$

Using (3.82) this coincides with (3.87), if we treat m as an independent variable.

The expression (3.89) for C_{sss} is a modular form of weight -3 on the modular group $\Gamma(2)$ defined in the Appendix. We will use as generators for the ring of modular forms on $\Gamma(2)$, $M_*(\Gamma(2))$, the functions

$$K_2 = c + d, \qquad K_4 = b^2, \tag{3.94}$$

which are modular forms of weight two and four, respectively. Note that instead of considering the $\Gamma(2)$ description of the Seiberg–Witten curve (3.63) we could also use the equivalent $\Gamma_0(4)$ description, which amounts to trade τ_0 for $2\tau_0 = \tau(S, -S)$ in all expressions of this section.

3.4.2 Direct integration and higher genus amplitudes

Having discussed the genus zero sector of the cubic matrix model specialized to the slice $S_1 = -S_2$, let us now turn our attention to the higher genus free energies F_g. According to [56, 89] the matrix model free energies F_g can be promoted to modular invariant, non-holomorphic amplitudes $F_g(\tau_0, \bar\tau_0)$ which satisfy the holomorphic anomaly equations of [15] in the local limit. The matrix model F_g is recovered by formally considering the limit $\bar\tau_0 \to \infty$ while keeping τ_0 fixed.

In order to apply this, we must compute the full non-holomorphic genus one amplitude F_1 and derive the propagator S^{ss}. Using the general formula (3.55) specialized to the Seiberg–Witten curve (3.63), and by following the same argument as for the Yukawa coupling C_{sss}, we obtain

$$F_1(\tau_0, \bar\tau_0) = -\log(\sqrt{\mathrm{Im}\tau_0}\,\eta(\tau_0)\eta(-\bar\tau_0)) + \frac{1}{4}\log\left(\frac{m^2 K_2}{\sqrt{K_4}}\right). \tag{3.95}$$

Indeed, when expanded we find

$$F_1 = -\frac{1}{6}\log S + \frac{S}{3m^3} + 15\left(\frac{S}{m^3}\right)^2 + \frac{6202}{9}\left(\frac{S}{m^3}\right)^3 + 32286\left(\frac{S}{m^3}\right)^4 + \cdots, \tag{3.96}$$

which is precisely the series for F_1 obtained in [99] after setting $S_1 = -S_2 = S$.

Next we turn to the propagator S^{ss}, defined by

$$\overline{C}^{ss}_{\bar s} = \bar\partial_{\bar s} S^{ss}, \tag{3.97}$$

where $\overline{C}_{\bar s\bar s\bar s}$ is the complex conjugate of the Yukawa coupling C_{sss} and the indices are raised by means of the metric

$$G_{s\bar s} \sim \mathrm{Im}\tau_0. \tag{3.98}$$

Using the chain rule and the relation (3.87) yields

$$\partial_s F_1(\tau_0, \bar{\tau}_0) = -\frac{1}{48} C_{sss} \widehat{E}_2(\tau_0, \bar{\tau}_0) + \partial_s f_1(\tau_0), \tag{3.99}$$

where f_1 is given by

$$f_1(\tau_0) = \frac{1}{4} \log\left(\frac{m^2 K_2}{\sqrt{K_4}}\right). \tag{3.100}$$

Hence, the propagator is identified with

$$S^{ss} = -\frac{1}{24} \widehat{E}_2(\tau_0, \bar{\tau}_0). \tag{3.101}$$

Now we are prepared to apply the method of directly integrating the holomorphic anomaly equations according to [6, 77]. In the conventions of this section the holomorphic anomaly equations can be cast into the following form

$$\frac{\partial F_g}{\partial \widehat{E}_2} = -\frac{1}{192} C_{sss}^2 \left[\hat{D}_{\tau_0}^2 F_{g-1} + \frac{\hat{D}_{\tau_0} C_{sss}}{C_{sss}} \hat{D}_{\tau_0} F_{g-1} + \sum_{h=1}^{g-1} \hat{D}_{\tau_0} F_h \hat{D}_{\tau_0} F_{g-h} \right], \quad (g > 1) \tag{3.102}$$

where $\hat{D}_{\tau_0}$ denotes the Maass derivative acting on (almost-holomorphic) modular forms of weight k as

$$\hat{D}_{\tau_0} = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\tau_0} - \frac{k}{4\pi \mathrm{Im}\tau_0}. \tag{3.103}$$

Since the ring $\widehat{M_*}(\Gamma(2)) = \mathbb{C}[\widehat{E}_2, K_2, K_4]$ is closed under $\hat{D}_{\tau_0}$, and the F_g's are modular invariant forms, the holomorphic anomaly equation can be integrated with respect to $\widehat{E}_2$. We obtain the following schematic result

$$F_g(\tau_0, \bar{\tau}_0) = \widetilde{\Delta}^{2-2g} \cdot \sum_{k=1}^{3g-3} c_k^{(g)}(\tau_0) \widehat{E}_2^k(\tau_0, \bar{\tau}_0) + f_g(\tau_0), \tag{3.104}$$

where $c_k^{(g)}(\tau_0)$ are modular forms of weight $8(g-1) - 2k$, completely determined by the holomorphic anomaly equation, and $\widetilde{\Delta}$ is just the denominator of C_{sss}. In particular it is a weight eight form given by

$$\widetilde{\Delta} = m^3 (K_2^2 - K_4) K_4. \tag{3.105}$$

All the non-trivial information is encoded in the holomorphic ambiguity $f_g(\tau_0)$. It has to be derived genus by genus by supplying further boundary conditions. In the particular case of the cubic matrix model specialized to the slice $S_1 = -S_2$, we will argue in the next subsection that $f_g(\tau_0)$ can be fixed at all genera. Applying this procedure we were able to integrate the holomorphic anomaly equations and obtained the matrix model free energies to genus 52.

Let us at least present the result for the full non-holomorphic genus two amplitude

$$\begin{aligned} F_2(\tau_0, \bar{\tau}_0) = &-\frac{160 K_2^5}{81 m^6 (K_2^2 - K_4)^2 K_4^2} \widehat{E}_2^3 - \frac{16 K_2^4 (5K_2^2 - 7K_4)}{9 m^6 (K_2^2 - K_4)^2 K_4^2} \widehat{E}_2^2 \\ &- \frac{8 K_2^3 (77 K_2^4 - 132 K_2^2 K_4 + 63 K_4^2)}{27 m^6 (K_2^2 - K_4)^2 K_4^2} \widehat{E}_2 - \frac{4 K_2^4 (2051 K_2^4 - 4005 K_2^2 K_4 + 1890 K_4^2)}{405 m^6 (K_2^2 - K_4)^2 K_4^2}. \end{aligned} \tag{3.106}$$

Here we collect some low genus expansions of the free energy amplitudes of the cubic matrix model on the slice $S_1 = -S_2 = S$:

$$\begin{aligned}
m^6 F_2 &= -\frac{1}{120}\frac{m^6}{S^2} + \frac{35}{3}\frac{S}{m^3} + 2308\frac{S^2}{m^6} + \frac{1341064}{5}\frac{S^3}{m^9} + 24734074\frac{S^4}{m^{12}} + \cdots \\
m^{12} F_3 &= \frac{1}{504}\frac{m^{12}}{S^4} + \frac{10010}{3}\frac{S}{m^3} + \frac{4036768}{3}\frac{S^2}{m^6} + \frac{1883381692}{7}\frac{S^3}{m^9} + 38608040638\frac{S^4}{m^{12}} + \cdots \\
m^{18} F_4 &= -\frac{1}{720}\frac{m^{18}}{S^6} + \frac{8083075}{3}\frac{S}{m^3} + 1749491040\frac{S^2}{m^6} + \frac{4618613451580}{9}\frac{S^3}{m^9} + \cdots \\
m^{24} F_5 &= \frac{1}{528}\frac{m^{24}}{S^8} + \frac{13013750750}{3}\frac{S}{m^3} + 4038280413440\frac{S^2}{m^6} + \frac{17515677810823140}{11}\frac{S^3}{m^9} + \cdots \\
m^{30} F_6 &= -\frac{691}{163800}\frac{m^{30}}{S^{10}} + 11699361924250\frac{S}{m^3} + \frac{43710230883020800}{3}\frac{S^2}{m^6} + \cdots .
\end{aligned} \tag{3.107}$$

We can check some of these results by comparing to the perturbative calculations of [99] specialized to $S_1 = -S_2 = S$. We observe agreement for genus two and three at low order in S/m^3. All higher genus computations are new results.

The direct integration procedure outlined here is by far the most efficient method to calculate higher genus amplitudes in matrix models. It only takes a few minutes to reach e.g. genus 10 on a conventional personal computer.

3.4.3 Boundary conditions and integrability

According to [1,56] F_g is an almost-holomorphic modular invariant form under the spacetime duality group, in this case $\Gamma(2)$. Hence, F_g is regular except for some points on the boundary of moduli space.

Regularity and holomorphicity imply that f_g should be a rational function, where its denominator is given by an appropriate power of the discriminant of the curve. From the expression (3.104) we see that the denominator of f_g is given by $\widetilde{\Delta}^{2g-2}$, hence a weight $8(g-1)$ form. Modularity now implies that the numerator has to be a form of finite weight, in order to cancel the weight from the denominator. Since the space of weight k forms is finite dimensional, there are only finitely many coefficients to determine. In particular, for $\Gamma(2)$ we have

$$\dim M_k(\Gamma(2)) = \begin{cases} \frac{k+2}{2}, & k > 2, \quad k \text{ even.} \\ 0, & \text{else.} \end{cases} \tag{3.108}$$

In summary this justifies the ansatz

$$f_g(\tau_0) = \widetilde{\Delta}^{2-2g} \cdot \sum_{k=0}^{4(g-1)} a_k K_2^{2k} K_4^{4(g-1)-k}, \qquad (g > 1) \tag{3.109}$$

where $\widetilde{\Delta}$ is given in eq. (3.105). This implies that there are $4g-3$ unknown constants a_k in the ambiguity f_g. These are completely and uniquely fixed by imposing the following two boundary conditions.

First, we know that the holomorphic expansion of F_g at small S has the structure (3.44) specialized to the slice, which imposes $2g-1$ conditions on f_g and leaving $2g-2$ unknowns. Further the holomorphic expansion at conifold divisors is of the form (3.45), where Π is a suitable coordinate transverse to the divisor which vanishes at the conifold. In our case Π is the dual period. Thus, (3.45) imposes $2g-2$ further constraints on the ambiguity, and it determines it completely.

3.5 Non-perturbative aspects

In this section we address non-perturbative effects in matrix models, and its connection to the large order behavior of the $1/N$ expansion.

3.5.1 Non-perturbative effects in the one-cut matrix model

For concreteness, we will focus here on the cubic matrix model which we are analyzing in this paper.

In the one-cut cubic matrix model, the large N limit is described by a distribution of eigenvalues around the minimum of the potential at $x = 0$. The eigenvalues fill the interval $[a, b]$. It has been known for some time that instanton sectors in this model are obtained by tunneling a finite, small number of eigenvalues $\ell \ll N$ from this interval to the maximum of the effective potential, located at x_0. The structure of the partition function in the ℓ-instanton sector has been determined in [126, 127], and at one loop it has the form

$$Z^{(\ell)} = \frac{g_s^{\ell^2/2}}{(2\pi)^{\ell/2}} G_2(\ell+1)\mu_1^{\ell^2} \exp\left(-\frac{\ell A}{g_s}\right)\left\{1 + \mathcal{O}(g_s)\right\}. \tag{3.110}$$

In this equation, $G_2(z)$ is the Barnes function. A is the instanton action, and it can be computed in terms of the spectral curve of the one-cut matrix model as

$$A = \int_b^{x_0} \mathrm{d}z\, y(z). \tag{3.111}$$

Finally, μ_1 is the one-loop contribution, and it has the explicit expression

$$\mu_1 = \frac{b-a}{4} \frac{1}{\sqrt{M(x_0)[(a-x_0)(b-x_0)]^{\frac{5}{2}}}}. \tag{3.112}$$

In [126] it was argued, following standard arguments in the large order behavior of perturbation theory [109], that the free energy of the one-instanton amplitude, $F^{(1)}$, should determine the leading asymptotics at large g of the perturbative amplitudes F_g, according to the formula

$$F_g = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \frac{\mathrm{d}z}{z^{g+1}} F^{(1)}(z). \tag{3.113}$$

If we write

$$F^{(1)} = g_s^{1/2} \mathrm{e}^{-A/g_s} \sum_{\ell=1}^\infty \mu_l g_s^{\ell-1}, \tag{3.114}$$

we obtain the full $1/g$ asymptotics

$$F_g \sim_g \frac{1}{\pi} A^{-2g-b} \Gamma\Big(2g+b\Big) \sum_{\ell=1}^\infty \frac{\mu_\ell A^{\ell-1}}{\prod_{k=1}^{\ell-1}(2g+b-k)}. \tag{3.115}$$

where

$$b = -\frac{5}{2}. \tag{3.116}$$

The formula (3.115) can be regarded as a generalization of the asymptotics for formal solutions of nonlinear ODEs. The reason is as follows. In the double-scaling limit of the matrix model (see [42]), the total free energy of the matrix model becomes a function of a double-scaled variable z,

$$F(t, g_s) \to F_{\rm ds}(z), \tag{3.117}$$

and the specific heat $u = -F''_{\rm ds}(z)$ satisfies the Painlevé I equation

$$u^2 - \frac{1}{6}u'' = z. \tag{3.118}$$

In particular, the genus expansion of the cubic matrix model leads to a formal solution of Painlevé I

$$u(z) = z^{1/2} \sum_{g=0}^{\infty} u_{g,0} z^{-5g/2}. \tag{3.119}$$

On the other hand, the instanton sectors of the matrix model lead to instanton corrections of the form

$$u_\ell(z) = z^{1/2-5\ell/8} {\rm e}^{-\ell a z^{5/4}} \sum_{n=0}^{\infty} u_{n,\ell} z^{-5n/4} \tag{3.120}$$

where

$$a = \frac{8\sqrt{3}}{5}. \tag{3.121}$$

It can be shown that the coefficients of (3.119) have an asymptotic behavior at large g which is governed by the one-instanton solution $u_1(z)$ in (3.120). The precise formula is,

$$u_{g,0} \sim_g \frac{a^{-2g+\frac{1}{2}}}{\pi} \Gamma\Big(2g - \frac{1}{2}\Big) \frac{\mathcal{S}_1}{\pi {\rm i}} \Big\{ 1 + \sum_{l=1}^{\infty} \frac{u_{l,1} a^l}{\prod_{k=1}^{l} (g - 1/2 - k)} \Big\}, \tag{3.122}$$

where $\mathcal{S}_1$ is a Stokes constant. One can explicitly check [34,126] that (3.122) can be deduced from the double-scaling limit of the asymptotics (3.115). In particular, the constant a is the double-scaling limit of the instanton action.

3.5.2 Non-perturbative effects in the cubic matrix model

Non-perturbative effects in multi-cut matrix models have been studied in [18,127]. A multi-cut matrix model with a fixed choice of filling fractions must be regarded as a fixed background, and any other choice of filling fractions leads to an instanton correction to the free energy on the fixed background. To be concrete, let us consider a two-cut matrix model with a fixed background given by N_1, N_2 eigenvalues in the stable and unstable saddle points, respectively. The partial 't Hooft parameters S_1, S_2 are given as usual by $S_i = g_s N_i$. The total partition function is of the form

$$Z = Z(N_1, N_2) + \sum_{\ell \neq 0} \zeta^\ell \, Z(N_1 - \ell, N_2 + \ell). \tag{3.123}$$

The sum over ℓ corresponds to the tunneling of ℓ eigenvalues from the first cut to the second cut, and at large N, the corresponding partition functions have the form

$$Z^{(\ell)} = \zeta^\ell q^{\ell^2/2} \exp\left(-\frac{\ell A}{g_s}\right) \Big\{ 1 + \mathcal{O}(g_s) \Big\}, \qquad \ell \in \mathbb{Z}^* \tag{3.124}$$

where

$$A = \partial_s F_0 \qquad \text{and} \qquad q = \exp\left(\partial_s^2 F_0\right). \tag{3.125}$$

The variable s is given in (3.34). If the cuts of the matrix model are the intervals $[x_1, x_2]$, $[x_3, x_4]$, the instanton action A can be written as

$$A = \int_{x_2}^{x_3} y(x)\mathrm{d}x. \tag{3.126}$$

If $\mathrm{Re}(A) \neq 0$, the instanton contributions are exponentially suppressed if $\mathrm{sgn}(\mathrm{Re}(A)\ell) > 0$, and they are exponentially enhanced if $\mathrm{sgn}(\mathrm{Re}(A)\ell) < 0$. This is just reflecting the fact that the generic background is unstable and if we expand around it we will find tachyonic directions. Notice however that both corrections are non-perturbative in g_s, therefore they are invisible in the genus expansion.

It is generically expected that the existence of these non-perturbative sectors leads to the factorial divergence of the genus expansion around a fixed background. The growth of the perturbative string amplitudes at large genus (and fixed S_1, S_2) should be of the same form as in (3.115), i.e.

$$F_g(S_1, S_2) \sim_g A^{-2g-b}\,\Gamma(2g+b) + \mathcal{O}(g^{-1}) \tag{3.127}$$

where A is given by (3.126) and b is a constant.

We can test these predictions by numerical methods using our results from direct integration. We start by concentrating on the slice $S_1 = -S_2$. Note that in this case the instanton action A is given by the dual period Π, whose explicit expression is given in eq. (3.71). In order to extract the asymptotic of the sequence $\{F_g\}_{g\geq 0}$ we employ a standard numerical technique known as Richardson extrapolation. The method removes the first terms of the subleading tail and hence accelerates the convergence. Given a sequence $\{S_g\}_{g\geq 0}$ in the form

$$S_g = a_0 + \frac{a_1}{g} + \frac{a_2}{g^2} + \dots \tag{3.128}$$

its Richardson transform is defined by

$$R_S(g, N) = \sum_{k\geq 0} \frac{(-1)^{k+N}(g+k)^N}{k!\,(N-k)!} S_{g+k}, \tag{3.129}$$

such that the sub-leading terms in $\{S_g\}_{g\geq 0}$ are cancelled up to order g^{-N}. In fact, it can be shown that if $\{S_g\}_{g\geq 0}$ is a finite sequence, the Richardson transform returns exactly the leading term a_0.

Comparing (3.128) with (3.127) one can extract the instanton action by considering the sequence

$$Q_g = \frac{F_{g+1}}{4g^2 F_g} = \frac{1}{A^2}\left(1 + \frac{1+2b}{2g} + \mathcal{O}(g^{-2})\right). \tag{3.130}$$

Once A is confirmed, one can then obtain the parameter b from the new sequence

$$Q'_g = 2g\left(A^2 Q_g - 1\right) = 1 + 2b + \mathcal{O}(g^{-1}). \tag{3.131}$$

In Fig. 3.2 and Fig. 3.3 we plot the sequences Q_g, Q'_g, together with their Richardson transforms, for two values of S. It is obvious from the numerical calculation that the large genus asymptotics is controlled at leading order by the instanton action. In addition, we find numerically that

$$b = -1. \tag{3.132}$$

This value of b is different from the one characterizing the one-cut model (3.116). In fact, the value (3.116) corresponds to the universality class of pure two-dimensional gravity, while the value (3.132) corresponds rather to the universality class of the $c = 1$ string [145]. It is interesting to see that both behaviors are present in the two-cut cubic matrix model, along different submanifolds of the moduli space (the 2d gravity behavior takes place in the slice $S_2 = 0$, while the $c = 1$ behavior takes place in the slice $S_1 + S_2 = 0$).

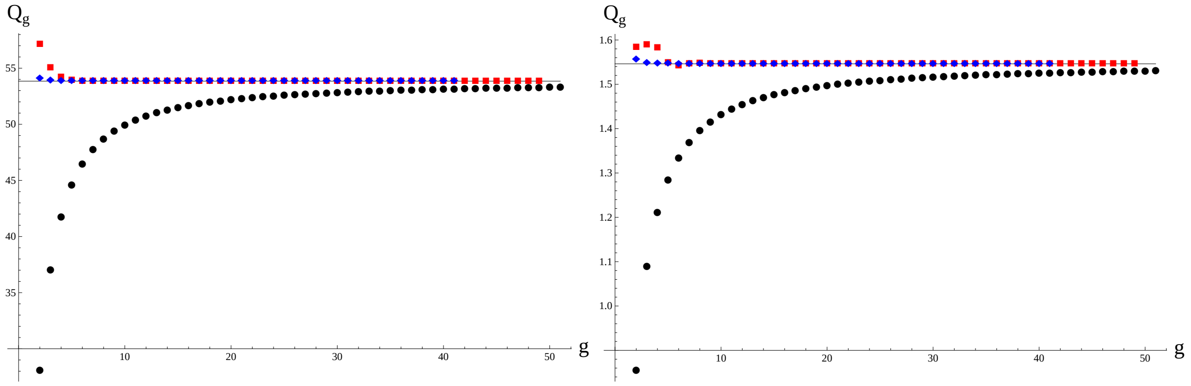

Figure 3.2: The sequence Q_g (●) and two Richardson transforms (■, ♦) at $\tau_0 = \frac{i}{2}$ (left) and $\tau_0 = \frac{2i}{3} + \frac{1}{9}$ (right) which corresponds to $S \approx 0.139$ and $S \approx 0.117{+}0.016i$, respectively. The leading asymptotics as predicted by the instanton action $|A|^{-2}$ is shown as a straight line. The error for genus 52 is about 10^{-8} % and 10^{-10} %, resp.

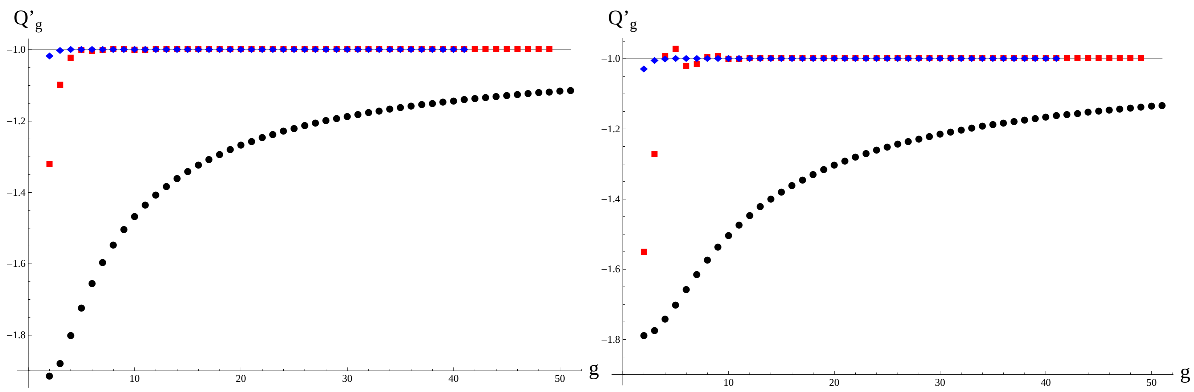

Figure 3.3: The sequence Q'_g (●) and two Richardson transforms (■, ♦) at $\tau_0 = \frac{i}{2}$ (left) and $\tau_0 = \frac{2i}{3} + \frac{1}{9}$ (right) which corresponds to $S \approx 0.139$ and $S \approx 0.117{+}0.016i$, respectively. The leading asymptotics as predicted by the parameter $b = -1$ is shown as a straight line. The error for genus 52 is about 10^{-8} % in both cases.

Turning our attention to a generic value of the filling fractions (S_1, S_2) in the cubic matrix model, we can try to test our prediction (3.127) by using the results from direct integration of section 3.3. Since we computed F_g up to genus four, we can only explore the first four elements of the sequence $\{Q_g\}_{g\geq 0}$, eq. (3.130). In order to have a better control of the error, we consider a perturbation around the submanifold $S_1+S_2=0$ where the large order behavior is well established. The instanton action (3.126) is calculated using (C.3) by

$$\begin{aligned} A &= \frac{\partial F_0}{\partial S_1} - \frac{\partial F_0}{\partial S_2} = \Pi_1 - \Pi_2 \\ &= \log(S_1)S_1 - \log(S_2)S_2 + \frac{1}{6} - S_1 + S_2 + \mathcal{O}(S^2). \end{aligned} \tag{3.133}$$

Fig. 3.4 shows Q_3, $R_Q(1,2)$ and $|A|^{-2}$ as a function of S_1 in the vicinity of the slice point $S_1=-S_2=S=0.004$, where convergence is ensured. We observe that the behavior of Q_3 and $R_Q(1,2)$ is qualitatively the same as predicted by the instanton action. Moreover, their relative errors stay roughly constant over the complete data set. This seems to indicate that the large order behavior of the genus expansion is also governed by the instanton action in the general two-cut cubic matrix model.

Unfortunately, our numerical results for the generic case are not good enough to determine the value of b reliably. It is an interesting question to know how this value changes as we move in the moduli space. We expect it to be $b=-1$ except in the one-cut slices $S_1=0$, $S_2=0$, where it takes the value (3.116).

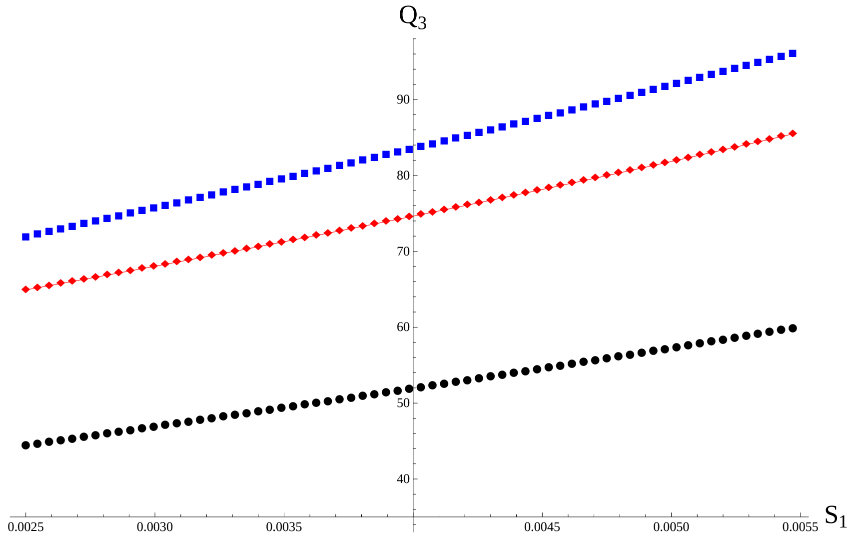

Figure 3.4: Q_3 (●), $R_Q(1,2)$ (■) and $|A|^{-2}$ (♦) are plotted for several values of S_1 around the slice point $S_1=-S_2=S=0.004$. Q_3 and $R_Q(1,2)$ have a relative error of about 30 % and 10 %, respectively, as compared to the instanton action $|A|^{-2}$ throughout the data set.

3.5.3 Asymptotics and non-perturbative sectors

In principle, one should be able to refine the asymptotic formula (3.127) and obtain a generalization of (3.115) involving the g_s expansion of instanton solutions. A natural guess is that the relevant instanton solutions are the closest ones to the given background, i.e. the instanton amplitudes (3.124) with $\ell = \pm 1$. This guess would relate the large genus behavior of $F_g(t_1, t_2)$ to an integral of the form (3.113), involving this time $F^{(1)}$ and $F^{(-1)}$. However, this expectation turns out to be too naive. Indeed, it seems that the asymptotics involves *new non-perturbative sectors* whose matrix model interpretation is yet unknown.

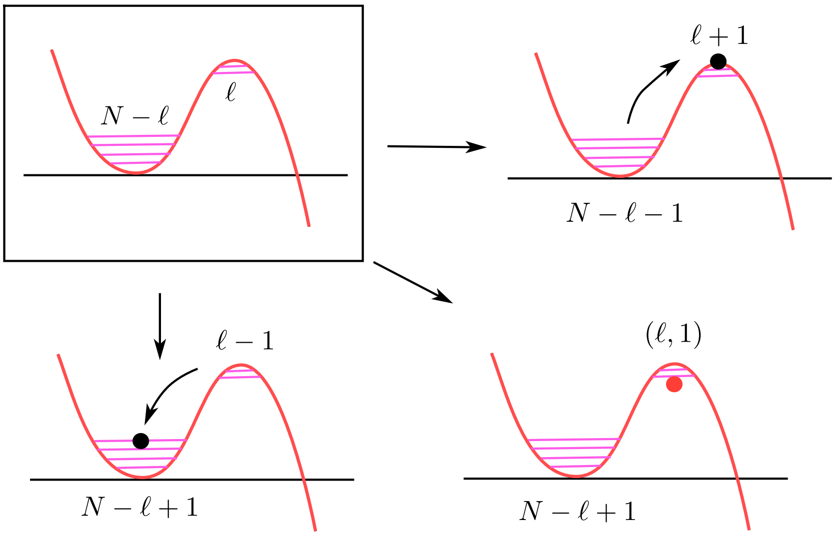

Figure 3.5: The asymptotics of the coefficients of the ℓ-th instanton solution $u_\ell(z)$ of Painlevé I is determined by the two nearest neighbor instantons, which are obtained by eigenvalue tunneling, and by the generalized instanton amplitude $u_{\ell|1}$, which is represented here by the label $(\ell, 1)$.

In order to explain this in some detail, we will come back to a simplest case where the asymptotics can be fully determined, namely the Painlevé I equation and its instanton solutions $u_\ell(z)$. It is natural to ask what is the asymptotics of the coefficients $u_{n,\ell}$ appearing in (3.120). Notice that, when ℓ is big, this instanton solution is the double-scaled limit of a two-cut solution, therefore the question of the asymptotics of this sequence is closely related to the original question concerning the asymptotics (3.127). We have seen in (3.122) that the asymptotics of the perturbative solution is governed by the one-instanton solution. In the same way, one would think that the asymptotics of the ℓ-instanton solution is governed by the $\ell \pm 1$ instanton amplitudes. It has been shown in [67] that this is not the case. In order to understand the asymptotics of a generic instanton sector, one has to consider more general amplitudes, labelled by two non-negative integers:

$$u_{n|m}(z). \tag{3.134}$$

The amplitude where $m = 0$ is the standard instanton amplitude: $u_{\ell|0}(z) = u_\ell(z)$. The other amplitudes can be obtained by requiring

$$u(z, C_1, C_2) = \sum_{n,m \geq 0} u_{n|m}(z) C_1^n C_2^m \tag{3.135}$$

to be a formal solution to the Painlevé I equation, for arbitrary C_1, C_2, and that

$$u_{n|m}(z) \sim \mathrm{e}^{-(n-m)az^{5/4}}, \qquad z \to \infty. \tag{3.136}$$

The two-parameter solution of the Painlevé I equation (3.135) is called a *trans-series solution*, and it was introduced by Jean Écalle in the context of resurgent analysis (see for example [146] for an simple introduction to resurgence). It turns out that the asymptotic behavior of the coefficients $u_{n,\ell}$ in the ℓ-th instanton u_ℓ is governed by the solutions $u_{\ell\pm1}(z)$, but also by the solution $u_{\ell|1}(z)$. This means that the asymptotics of the coefficients $u_{n,\ell}$ as $n \to \infty$ can be obtained by a relation similar to (3.113), but involving $u_{\ell\pm1}(z)$ as well as $u_{\ell|1}(z)$. For example, let us consider the one-instanton solution, and let us ask what is the asymptotics of the coefficients $u_{n,1}$ appearing in (3.120) with $\ell = 1$. An analysis based on resurgence theory, which can be verified with Riemann–Hilbert techniques, leads to the formula [67]

$$u_{n,1} \sim_n a^{-n+1/2} \frac{S_1}{2\pi \mathrm{i}} \Gamma\big(n - 1/2\big) \Big\{ 2u_{0,2} + (-1)^n \mu_{0,2} + \sum_{l=1}^{\infty} \frac{(2u_{l,2} + (-1)^{n+l} \mu_{l,2}) a^l}{\prod_{m=1}^{l} (n - 1/2 - m)} \Big\} \tag{3.137}$$

where $u_{n,2}$ are the coefficients of the two-instanton expansion ($\ell = 2$) in (3.120), and $\mu_{n,2}$ are the coefficients of the function

$$u_{1|1}(z) = z^{-3/4} \sum_{n \geq 0} \mu_{n,2} z^{-5n/4}. \tag{3.138}$$

It can be seen, by plugging (3.135) in the Painlevé I equation, that this function satisfies the linear inhomogeneous ODE

$$-\frac{1}{6} u''_{1|1} + 2u_0 u_{1|1} + 2u_1 u_{0|1} = 0. \tag{3.139}$$

There are similar, but more complicated, formulae for the asymptotic behavior of the coefficients $u_{n,\ell}$ for arbitrary ℓ, see [67]. They all involve the trans-series solutions $u_{\ell|1}(z)$.

The instanton amplitude $u_{\ell+1}(z)$ can be obtained from the solution $u_\ell(z)$ by tunneling one extra eigenvalue to the unstable saddle, while the amplitude $u_{\ell-1}(z)$ can be obtained from $u_\ell(z)$ by tunneling one eigenvalue back to the stable saddle. The amplitude $u_{\ell|1}(z)$ does not seem to have, however, an eigenvalue interpretation of this type. The different non-perturbative sectors governing the asymptotics of the ℓ-th instanton solution are depicted in Fig. 3.5.

We can now come back to the original problem of determining the large order behavior of $F_g(S_1, S_2)$, corresponding to a two-cut model with N_1, N_2 eigenvalues around the two saddles. There are two instanton configurations which are obtained by eigenvalue tunneling, with fillings $N_1 \pm 1$, $N_2 \mp 1$. It can be seen that the subleading terms in the asymptotics are not reproduced by using just these two configurations. This is not surprising, in view of the result for the instantons of Painlevé I. The above analysis, based on [67], suggests in fact that there should be two other non-perturbative configurations in the two-cut matrix model, that we denote by $(N_1, 1)$ and $(N_2, 1)$, in analogy with our notation in Fig. 3.5. These configurations are depicted at the bottom of Fig. 3.6. The instantons obtained by eigenvalue tunneling can be easily calculated from the free energy of the generic two-cut matrix model. However, we do not know how to compute the amplitudes involving these new configurations, since there is no analogue of the Painlevé I equation (or the pre-string equation) for the generic two-cut matrix model. In general, it seems that the most general saddle-point of the two-cut matrix model should be labeled by two pairs of integers, (N_1, M_1), (N_2, M_2), associated to the two critical points of the cubic potential.

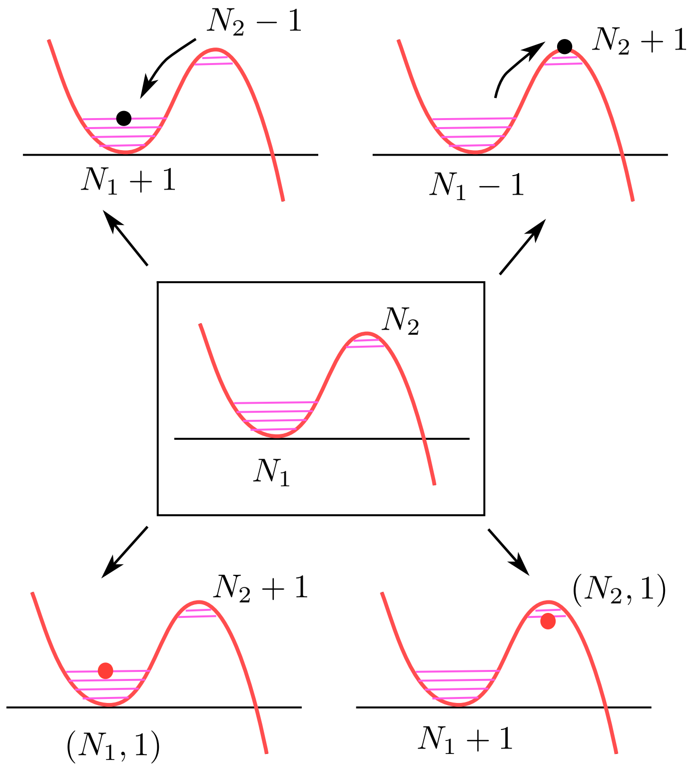

Figure 3.6: In the two-cut case, given a background with perturbative amplitudes $F_g(t_1, t_2)$, there are two instantons which are obtained by eigenvalue tunneling, and two generalized instanton amplitudes represented by $(N_1, 1)$, $(N_2, 1)$.

Chapter 4

Wall-crossing, mock modularity and multiple M5-branes

In the following we derive a recursive holomorphic anomaly equation for the elliptic genus of multiply wrapped M5-branes on a rigid divisor inside a Calabi-Yau threefold. Most results of this chapter have been published in ref. [5].

4.1 Introduction

The study of background dependence of physical theories has been a rich source of insights. Understanding the change of correlators as the background parameters are varied supplemented by boundary data can be sufficient to solve the theory. A class of theories where the question of background dependence can be sharply stated are topological field theories. Correlators in topological theories typically have holomorphic expansions near special values of the background moduli. The expansion coefficients can be given precise mathematical meaning as topological invariants of the geometrical configuration contributing to the topological non-trivial sector of the path integral. Physically the expansion often captures information of the degeneracies of BPS states of theories related to the same geometry.

An example of this is the topological A-model [162], with a Calabi-Yau three-fold (CY) X as target space, which in a large volume limit counts holomorphic maps from the world-sheet into $H_2(X, \mathbb{Z})$ and physically captures the degeneracies of BPS states coming from an M-theory compactification on X [68, 69]. Another example is the modified elliptic genus of an M5-brane wrapping a complex surface P,[1] which was related in ref. [132] to the partition function of topologically twisted $\mathcal{N} = 4$ Yang-Mills theory [156], which computes generating functions of Euler numbers of moduli spaces of instantons. This same quantity was shown in ref. [61] to capture the geometric counting of degeneracies of systems of D4-D2-D0 black holes associated to the MSW string [114].

In both cases the topological theories enjoy duality symmetries. T-duality acting on the Kähler moduli on X in the topological string case and S-duality for the $\mathcal{N} = 4$ SYM theory acting on the gauge coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. The former symmetry extends by mirror symmetry and both might extend to U-duality groups. Both symmetries can be conveniently expressed in the language of modular forms.

The holomorphic expansions of the topological string correlators are given in the moduli spaces of families of theories. Fixing a certain background corresponding to a certain point in the moduli space, the topological correlators are expected to be holomorphic expansions. In refs. [14, 15] holomorphic anomaly equations governing topological string amplitudes were derived showing that this is not the case and hence the correlators suffer from background

[1] In the following we will use the terms surface, divisor and four-cycle (of a CY) interchangeably when the context is clear.

dependence.[2] In ref. [166] a background independent meaning was given to the correlators, stating that the anomaly merely reflects the choice of polarization if the partition function is considered as a wave function only depending on half of the variables of some phase space which has a natural geometric meaning in this context.

This anomaly is also manifest in a failure of target space duality invariance of the holomorphic expansion which can only be restored at the expense of holomorphicity as shown in ref. [1].[3] A similar story showed up in $\mathcal{N}=4$ topological $U(2)$ SYM theory on $\mathbb{P}^2$ [156], where it was shown that different sectors of the partition function need a non-holomorphic completion which was found earlier in ref. [177] in order to restore S-duality invariance. An anomaly equation describing this non-holomorphicity was expected [156] in the cases where $b_2^+(P)=1$. In these cases holomorphic deformations of the canonical bundle are absent. The non-holomorphic contributions were associated with reducible connections $U(n)\to U(m)\times U(n-m)$ [132,156]. In ref. [132] this anomaly was furthermore related to an anomaly appearing in the context of E-strings [131]. These strings arise from an M5-brane wrapping a del Pezzo surface $\mathcal{B}_9$, also called $\frac{1}{2}$K3. The anomaly in this context was related to the fact that n of these strings can form bound-states of m and $(n-m)$ strings. Furthermore, the anomaly could also be related to the one appearing in topological string theory.

The anomaly thus follows from the formation of bound-states. Although the holomorphic expansion would not know about the contribution from bound-states, the restoration of duality symmetry forces one to take these contributions into account. The non-holomorphicity can be understood physically as the result of a regularization procedure. The path integral produces objects like theta-functions associated to indefinite quadratic forms which need to be regularized to avoid divergences. This regularization breaks the modular symmetry, restoring the symmetry gives non-holomorphic objects. The general mathematical framework to describe these non-holomorphic completions is the theory of mock modular forms developed by Zwegers in ref. [180].[4] A mock modular form $h(\tau)$ of weight k is a holomorphic function which becomes modular after the addition of a function $g^*(\tau)$, at the cost of losing its holomorphicity. Here, $g^*(\tau)$ is constructed from a modular form $g(\tau)$ of weight $2-k$, which is referred to as shadow.

Another manifestation of the background dependence of the holomorphic expansions of the topological theories are wall-crossing phenomena associated to the enumerative content of the expansions. Mathematically, it is known that Donaldson-Thomas invariants jump on surfaces with $b_2^+(P)=1$, see [71] and references therein, for related physical works see for example refs. [113, 135]. On the physics side wall-crossing refers to the jumping of the degeneracies of BPS states when walls of marginal stability are crossed. These phenomena were observed in the jumps of the soliton spectrum of two-dimensional theories [25] and were an essential ingredient of the work of Seiberg and Witten [148] in four-dimensional theories. Recent progress was triggered by formulae relating the degeneracies on both sides of the walls, which were given from a supergravity analysis in refs. [37, 38] and culminated in a mathematical rigorous formula of Kontsevich and Soibelman (KS) [106], which could also be derived from continuity of physical quantities in refs. [26,63] (See also refs. [27,64,65]). The fact that the holomorphic anomaly describes how to transform the counting functions when varying the

[2] The anomaly relates correlators at a given genus to lower genera thus providing a way to solve the theory. Using a polynomial algorithm [6, 77, 169] and boundary conditions [89] this can be used to compute higher genus topological string amplitudes on compact CY [92] manifolds and solve it on non-compact CY [7,82].

[3] Following the anomaly reformulation of refs. [49, 157], see also [79].

[4] See [141, 178] and sec. A.2 for an introduction and overview.

background moduli, which in turn changes the degeneracy of BPS states, suggests that non-holomorphicity and wall-crossing are closely related. In fact the failure of holomorphicity can be traced back to the boundary of the moduli space of the geometrical configuration, where the latter splits in several configurations with the same topological charges. Mock modularity was used in a physical context studying the wall-crossing of degeneracies of $\mathcal{N} = 4$ dyons[5] in ref. [33]. In the context of $\mathcal{N} = 2$ supersymmetric theories the application of ideas related to mock modularity was initiated in ref. [119] and further pursued in refs. [21, 120, 121]. These motivated parts of our work.[6]

In this paper we study the relation between wall-crossing and non-holomorphicity and relate the appearance of the two. A central role is played by a wall-crossing formula by Göttsche [73], where the Kähler moduli dependence of a generating function of Euler numbers of stable sheaves is given in terms of an indefinite theta-function due to Göttsche and Zagier [71]. We show that this formula is equivalent to wall-crossing formulae of D4-D2-D0 systems in type IIA. The latter can be related to the (modified) elliptic genus of multiple M5-branes wrapping a surface. Rigid surfaces are subject to Göttsche's wall-crossing formula. Using ideas of Zwegers [180], we translate the latter into a holomorphic anomaly equation for two M5-branes wrapping the surface/divisor. We show that this anomaly equation is the equation which was found in the context of $\mathcal{N} = 4$ SYM [156] and E-strings [131, 132]. We further propose the generalization of the anomaly equation for higher wrappings and comment on its implications for the wall-crossing of multiple D4-branes.

4.2 Effective descriptions of wrapped M5-branes

In this section we review the effective descriptions of M5-branes wrapping a complex surface P as well as previous appearances of the holomorphic anomaly which will be derived in the next section. The world-volume theory of M5-branes can have either a two-dimensional CFT description in terms of the (MSW) CFT [114] or a four-dimensional description giving the $\mathcal{N} = 4$ topologically twisted Yang-Mills theory of Vafa and Witten [156]. In the latter theory it was observed [156] that a non-holomorphicity [177] had to be introduced in order to restore S-duality, the resulting holomorphic anomaly was related in ref. [132] to an anomaly [131] appearing in the context of E-strings. The anomaly was conjectured to take into account contributions coming from reducible connections in $\mathcal{N} = 4$ SYM theory. In ref. [132] it was related to the curve counting anomaly [15] and was given the physical interpretation of taking into account the bound-state contribution of E-strings. Later we will show that the contributions from bound-states as a cause for non-holomorphicity will persist more generally for the class of surfaces we will be studying. In our work we investigate the (generalized/modified) elliptic genus which captures the content of the CFT description of the M5-branes [116, 132] and its relation to D4-D2-D0 systems [36, 38, 61, 62, 108, 117] and the associated counting of black holes which has been intensively studied (e.g. in ref. [30]). Our goal is to show that wall-crossing in D4-D2-D0 systems leads to an anomaly equation which coincides with the anomalies found before and hence our work complements in some sense this circle of ideas.

[5] See for example ref. [32] and references therein for more details.

[6] Further physical appearances of mock modularity can be found for example in refs. [29, 52–54, 154].

4.2.1 The elliptic genus and D4-D2-D0 branes

In the following we will start with the $2d$ CFT perspective of the M5-brane world-volume theory. We want to study BPS states that arise in the context of an M-theory compactification on a Calabi-Yau manifold X with r M5-branes wrapping a complex surface (or a four-cycle) P, and extended in $\mathbb{R}^{1,3}\times S^1$. Considering P to be small compared to the M-theory circle, the reduction of the world-volume theory of the M5-brane is described by a $(1+1)$-dimensional $(0,4)$ MSW CFT [114].[7] The BPS states associated to the string that remains after wrapping the M5-branes on P are captured by a further compactification on a circle. They are counted by the partition function of the world-volume theory of the M5-branes on $P\times T^2$ [132]. The effective CFT description will thus exhibit invariance under the full $\mathrm{SL}(2,\mathbb{Z})$ symmetry of the T^2. Furthermore, excitations of the M5-branes will induce M2-brane charges corresponding to the flux of the self-dual field strength of the M5-brane world-volume theory. In addition, the momentum of the M2-branes along the M-theory circle will give rise to a further quantum number. As a result BPS states of the effective two-dimensional description will be labeled by the class of the divisor the M5-branes wrap, the M2-brane charges and by the momentum along S^1. In a type IIA setup, r times the class of the divisor will correspond to D4-brane charge, the induced M2-brane charge corresponds to D2-brane charge and the momentum to D0-brane charge. Choosing a basis $\Sigma_A\,, A=1,\ldots,b_4(X)$ of $H_4(X,\mathbb{Z})$, the charge vector will be given by

$$\Gamma=(Q_6,Q_4,Q_2,Q_0)=r(0,p^A,q_A,q_0), \tag{4.1}$$

where the Q_p are the Dp-brane charges and r is the number of coincident M5-branes wrapping the divisor specified by p^A. A priori the set of all possible induced D2-brane charges, or equivalently of $U(1)$ fluxes of the world-volume of the M5-brane would be in one-to-one correspondence with $\Lambda_P=H^2(P,\mathbb{Z})$ which is generically a larger lattice than $\Lambda=i^*H^2(X,\mathbb{Z})$, where $i:P\hookrightarrow X$, however the physical BPS states are always labeled by the smaller lattice Λ. The metric d_{AB} on Λ is given by

$$d_{AB}=-\int_P \alpha_A\wedge\alpha_B, \tag{4.2}$$

where α_A is a basis of two-forms in Λ, which is the dual basis to Σ_A of $H_4(X,\mathbb{Z})$. In order to obtain a generating series of the degeneracies of those BPS states one has to sum over directions along $\Lambda^\perp$ which is the orthogonal complement to Λ in Λ_P w.r.t. d_{AB} [61]. [8]

The partition function of the MSW CFT counting the BPS states is given by the modified elliptic genus[9] [116, 132]

$$Z_P^{\prime(r)}(\tau,z)=\mathrm{Tr}_{\mathcal{H}_{\mathrm{RR}}}\,(-1)^{F_{\mathrm{R}}}\,F_{\mathrm{R}}^2\,q^{L_0'-\frac{c_{\mathrm{L}}}{24}}\,\bar{q}^{\bar{L}_0'-\frac{c_{\mathrm{R}}}{24}}e^{2\pi i z\cdot Q_2}, \tag{4.3}$$

where the trace is taken over the RR Hilbert space. Furthermore, vectors are contracted w.r.t. the metric d_{AB}, i.e. $x\cdot y=x^Ay_A=d_{AB}x^Ay^B$. For a single M5-brane it was shown in

[7] The target space sigma model description of which was given in ref. [134], for more details see ref. [78] and references therein. In the following we will be concerned with the natural extension of the analysis of the degrees of freedom to r M5-branes.

[8] In general, the lattice $\Lambda\oplus\Lambda^\perp$ is only a sublattice of $H^2(P,\mathbb{Z})$, because $\det d_{AB}\neq 1$ in general, see for example ref. [134] and ref. [38] for a more recent exposition. However, we will only be concerned with divisors P with $b_2^+(P)=1$, such that $\det d_{AB}=1$.

[9] We follow the mathematics convention of not writing out explicitly the dependence on $\overline{\tau}$ which will be clear in the context. Moreover, we denote $q=e^{2\pi i\tau}$ and $\tau=\tau_1+i\tau_2$. To avoid confusion without introducing new notation we will denote the charge vector of D2-brane charges by $\underline{q}$, its components by q_A.

ref. [36] that $Z_P'^{(1)}(\tau, z)$ transforms like a $\mathrm{SL}(2,\mathbb{Z})$ Jacobi form of bi-weight $(0,2)$ due to the insertion of F_{R}^2, we demand that the same is true for all r.

Following ref. [36] the center of mass momentum $\vec{p}_{\mathrm{cm}}$ for the system of r M5-branes can be integrated out. In this way L_0' and $\bar{L}_0'$ can be written in the form

$$L_0' = \frac{1}{2}\vec{p}_{\mathrm{cm}}^{\,2} + L_0, \quad \bar{L}_0' = \frac{1}{2}\vec{p}_{\mathrm{cm}}^{\,2} + \bar{L}_0. \tag{4.4}$$

This allows one to split up the center of mass contribution and rewrite formula (4.3) as

$$\begin{aligned} Z_P'^{(r)}(\tau, z) &= \int d^3 p_{\mathrm{cm}} (q\bar{q})^{\frac{1}{2}\vec{p}_{\mathrm{cm}}^{\,2}} Z_P^{(r)}(\tau, z) \\ &\sim (\tau_2)^{-\frac{3}{2}}\, Z_P^{(r)}(\tau, z), \end{aligned} \tag{4.5}$$

where $Z_P^{(r)}(\tau, z)$ is now a Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$ which we simply call elliptic genus for short in the following.

The decomposition of the elliptic genus

The elliptic genus $Z_P^{(r)}(\tau, z)$ and equivalently the generating function of D4-D2-D0 BPS degeneracies is subject to a theta-function decomposition, which has been studied in many places, see for example refs. [30,36,38,61,108]. This is ensured by two features of the superconformal algebra of the (0,4) CFT. One of these is that the $\overline{\tau}$ contribution entirely comes from BPS states $|\underline{q}\rangle$ satisfying

$$\left(\overline{L}_0 - \frac{c_{\mathrm{R}}}{24} - \frac{r}{2}q_{\mathrm{R}}^2\right)|\underline{q}\rangle = 0, \tag{4.6}$$

the other one is the spectral flow isomorphism of the $\mathcal{N} = (0,4)$ superconformal algebra, which we want to recall for r M5-branes here, building on refs. [36,160], see also [30]. Proposition 2.9 of ref. [160] describes the spectral flow symmetry by an isomorphism between moduli spaces of vector bundles on complex surfaces. The complex surface here is the divisor P and the vector bundle configuration describes the bound-states of D4-D2-D0 branes. Within this setup the result of [160] translates for arbitrary r to a symmetry under the transformations

$$\begin{aligned} q_0 &\mapsto q_0 - k\cdot\underline{q} - \frac{1}{2}k\cdot k, \\ \underline{q} &\mapsto \underline{q} + k, \end{aligned} \tag{4.7}$$

where $k \in \Lambda$. Physically these transformations correspond to monodromies around the large radius point in the moduli-space of the Calabi-Yau manifold [30]. Denote by Λ^* the dual lattice of Λ with respect to the metric rd_{AB}. Keeping only the holomorphic degrees of freedom one can write

$$\begin{aligned} Z_P^{(r)}(\tau, z) &= \sum_{Q_0;Q_A} d(Q, Q_0)\, e^{-2\pi i\tau Q_0}\, e^{2\pi i z\cdot Q_2} \\ &= \sum_{q_0;\underline{q}\in\Lambda^*+\frac{[P]}{2}} d(r, \underline{q}, -q_0)\, e^{-2\pi i\tau r q_0}\, e^{2\pi i r z\cdot\underline{q}}, \end{aligned} \tag{4.8}$$

where $d(r, \underline{q}, -q_0)$ are the BPS degeneracies and the shift[10] $\frac{[P]}{2}$ originates from an anomaly [60, 133]. Now, spectral flow symmetry predicts [36]

$$d(r, \underline{q}, -q_0) = (-1)^{rp\cdot k} d(r, \underline{q} + k, -q_0 + k \cdot \underline{q} + \frac{k^2}{2}). \tag{4.9}$$

Making use of this symmetry and the following definition

$$\underline{q} = k + \mu + \frac{[P]}{2}, \qquad \mu \in \Lambda^*/\Lambda, \qquad k \in \Lambda, \tag{4.10}$$

one is led to the conclusion that the elliptic genus can be decomposed in the form

$$Z_P^{(r)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} f_{\mu,J}^{(r)}(\tau) \theta_{\mu,J}^{(r)}(\tau, z), \tag{4.11}$$

$$f_{\mu,J}^{(r)}(\tau) = \sum_{r\hat{q}_0 \geq -\frac{c_L}{24}} d_\mu^{(r)}(\hat{q}_0) e^{2\pi i \tau r \hat{q}_0}, \tag{4.12}$$

$$\theta_{\mu,J}^{(r)}(\tau, z) = \sum_{k \in \Lambda + \frac{[P]}{2}} (-1)^{rp\cdot(k+\mu)} e^{2\pi i \bar{\tau} r \frac{(k+\mu)_+^2}{2}} e^{2\pi i \tau r \frac{(k+\mu)_-^2}{2}} e^{2\pi i r z \cdot (k+\mu)}, \tag{4.13}$$

where $J \in \mathcal{C}(P)$ and $\mathcal{C}(P)$ denotes the Kähler cone of P restricted to $\Lambda \otimes \mathbb{R}$ and $\hat{q}_0 = -q_0 - \frac{1}{2}\underline{q}^2$ is invariant under the spectral flow symmetry. The subscript $+$ refers to projection onto the sublattice generated by the Kähler form J and $-$ is the projection to its orthogonal complement, i.e.

$$k_+^2 = \frac{(k \cdot J)^2}{J \cdot J}, \quad k_-^2 = k^2 - k_+^2. \tag{4.14}$$

There are two issues here for the case of rigid divisors with $b_2^+(P) = 1$ on which we want to comment as this class of divisors is the focus of our work. First of all note, that q_0 contains a contribution of the form $\frac{1}{2}\int_P F \wedge F$ where $F \in \Lambda_P$. Now, F can be decomposed into $F = \underline{q} + \underline{q}_\perp$ with $\underline{q}_\perp \in \Lambda^\perp$, which allows us to write

$$\hat{q}_0 = \tilde{q}_0 + \frac{1}{2}\underline{q}_\perp^2. \tag{4.15}$$

For $b_2^+(P) = 1$ and $r = 1$, the degeneracies $d(r, \mu, \tilde{q}_0)$ are independent of the choice of $\underline{q}_\perp$ and moreover it was shown by Göttsche [72] that

$$\sum_{\tilde{q}_0} d(1, \mu, \tilde{q}_0)\, e^{2\pi i \tau \tilde{q}_0} = \frac{1}{\eta^{\chi(P)}}. \tag{4.16}$$

Then, for $r = 1$ (4.12) becomes

$$f_{\mu,J}^{(1)}(\tau) = \frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta^{\chi(P)}(\tau)}, \qquad \vartheta_{\Lambda^\perp}(\tau) = \sum_{\underline{q}_\perp \in \Lambda^\perp} e^{i\pi\tau \underline{q}_\perp^2}. \tag{4.17}$$

[10] In components, $[P]$ is given by $d_{AB}p^A$.

The second subtlety is concerned with the dependence on a Kähler class J. Due to wall-crossing phenomena we will find that $f^{(r)}_{\mu,J}(\tau)$ also depends on J. We expect that it has the following expansion ($\tilde{q}_0 = \frac{d}{r} - \frac{c_L}{24}$)

$$f^{(r)}_{\mu,J}(\tau) = (-1)^{rp\cdot\mu} \sum_{d\geq 0} \bar{\Omega}(\Gamma; J)\, q^{d-\frac{r\chi(P)}{24}}. \tag{4.18}$$

Here, the factor $(-1)^{rp\cdot\mu}$ is inserted to cancel its counterpart in the definition of $\theta^{(r)}_{\mu,J}$, which was only included to make the theta-functions transform well under modular transformations. The invariants $\bar{\Omega}(\Gamma; J)$ are rational invariants first introduced by Joyce [94,95] and are defined as follows

$$\bar{\Omega}(\Gamma; J) = \sum_{m|\Gamma} \frac{\Omega(\Gamma/m; J)}{m^2}, \tag{4.19}$$

where $\Omega(\Gamma, J)$ is an integer-valued index of BPS degeneracies, given by [31]

$$\Omega(\Gamma, J) = \frac{1}{2}\mathrm{Tr}(2J_3)^2(-1)^{2J_3}, \tag{4.20}$$

where J_3 is a generator of the rotation group Spin(3). Note, that for a single M5-brane $\bar{\Omega}$ and Ω become identical and independent of J.

4.2.2 $\mathcal{N} = 4$ SYM, E-strings and bound-states

In the following we recall the relation [132] of the elliptic genus of M5-branes to the $\mathcal{N} = 4$ topological SYM theory of Vafa and Witten [156]. Our goal is to relate the holomorphic anomaly equation which we will derive from wall-crossing in the next section to the anomalies appearing in the $\mathcal{N} = 4$ context. We review moreover the connection of the anomaly to the formation of bound-states given in ref. [132].

The $\mathcal{N} = 4$ topological SYM arises by taking a different perspective on the world-volume theory of n M5-branes on $P \times T^2$ considering the theory living on P which is the $\mathcal{N} = 4$ topologically twisted SYM theory described in ref. [156]. The gauge coupling of this theory is given by

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}, \tag{4.21}$$

and is geometrically realized by the complex structure modulus of the T^2. The partition function of this theory counts instanton configurations by computing the generating functions of the Euler numbers of moduli spaces of gauge instantons [156]. S-duality translates to the modular transformation properties of the partition function. The analogues of D4-D2-D0 charges are the rank of the gauge group, different flux sectors and the instanton number.

In ref. [132] the relation is made between this theory and the geometrical counting of BPS states of exceptional strings obtained by wrapping M5-branes around a del Pezzo surface $\mathcal{B}_9$, also called $\frac{1}{2}$K3. This string is dual to the heterotic string with an E_8 instanton of zero size [66,149] and is therefore called E-string. In F-theory this corresponds to a $\mathbb{P}^1$ shrinking to zero size [136,137,167]. The geometrical study of the BPS states of this non-critical string was initiated in ref. [98] and further pursued in refs. [110,130,131]. In ref. [132] the counting of BPS states of the exceptional string with increasing winding n was related to the $U(n)$, $\mathcal{N} = 4$ SYM partition functions.

In the following we will use the geometry of ref. [98] which is an elliptic fibration over the Hirzebruch surface $\mathbb{F}_1$, which in turn is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$.[11] We will denote by t_E, t_F and t_D the Kähler parameters of the elliptic fiber, the fiber and the base of $\mathbb{F}_1$, respectively and enumerate these by $1, 2, 3$ in this order. We further introduce $\tilde{q}_a = e^{2\pi i \tilde{t}_a}$, $a = 1, 2, 3$ the exponentiated Kähler parameters appearing in the instanton expansion of the A-model at large radius, which are also the counting parameters of the BPS states.

Within this geometry we will be interested in the elliptic genus of M5-branes wrapping two different surfaces, one is a K3 corresponding to wrapping the elliptic fiber and the fiber of $\mathbb{F}_1$, the resulting string is the heterotic string. The other possibility is to wrap the base of $\mathbb{F}_1$ and the elliptic fiber corresponding to $\frac{1}{2}$K3 and leading to the E-string studied in refs. [98, 110, 130–132]. The two possibilities are realized by taking the limits $t_D, t_F \to i\infty$, respectively. The resulting surface in both cases is still elliptically fibered which allows one to identify the D4-D0 charges n and p with counting curves wrapping n-times the base and p-times the fiber of the elliptic fibration [132]. The multiple wrapping is hence encoded in the expansion of the prepotential $F_0(\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ of the geometry. In order to get a parameterization inside the Kähler cone of the K3 in which the corresponding curves in $H_2(\mathrm{K3}, \mathbb{Z})$ intersect with the standard metric of the hyperbolic lattice $\Gamma^{1,1}$, we define $t_1 = \tilde{t}_1$, $t_2 = \tilde{t}_2 - \tilde{t}_1$ and $t_3 = \tilde{t}_3$ as well as the corresponding $q_1 = \tilde{q}_1$, $q_2 = \tilde{q}_2/\tilde{q}_1$ and $q_3 = \tilde{q}_3$. Taking q_2 or $q_3 \to 0$, the multiple wrapping of the base is expressed by

$$F_0(t_1, t_a) = \sum_{n\geq 1} Z^{(n)}(t_1) q_a^n\,, \quad a = 2 \text{ or } 3. \tag{4.22}$$

The $Z^{(n)}$ can be identified with the elliptic genus of n M5-branes wrapping the corresponding surface after taking a small elliptic fiber limit [132]. In this limit the contribution coming from the theta-functions (4.13) reduce to $\tau_2^{-3/2}\left(\tau_2^{-1/2}\right)$ for the K3($\frac{1}{2}$K3) cases, these are the contributions of 3(1) copies of the lattice $\Gamma^{1,1}$ appearing in the decomposition of the lattices of K3($\frac{1}{2}$K3). Omitting these factors gives the $Z^{(n)}$ of weight $(-2, 0)$ in both cases. The elliptic genera of wrapping n M5-branes corresponding to n strings are in both cases related recursively to the lower wrapping. The nature of the recursion depends crucially on the ability of the strings to form bound-states.

The heterotic string, no bound-states

The heterotic string is obtained from wrapping an M5-brane on the K3 by taking the $q_3 \to 0$ limit. The heterotic string does not form bound-states and the recursion giving the higher wrappings in this case is the Hecke transformation[12] of $Z^{(1)}$ as proposed in ref. [132]. The formula for the Hecke transformation in this case is given by

$$Z^{(n)}(t) = n^{w_{\mathrm{L}}-1} \sum_{a,b,d} d^{-w_L} Z^{(1)}\left(\frac{at+b}{d}\right), \tag{4.23}$$

with $ad = n$ and $b < d$ and $a, b, d \geq 0$. Which specializes for $w_{\mathrm{L}} = -2$ and $n = p$, where p is prime to

$$Z^{(p)}(t) = \frac{1}{p^3} Z^{(1)}(pt) + \frac{1}{p}\left[Z^{(1)}\left(\frac{t}{p}\right) + Z^{(1)}\left(\frac{t}{p}+\frac{1}{p}\right) + \cdots + Z^{(1)}\left(\frac{t}{p}+\frac{p-1}{p}\right)\right]. \tag{4.24}$$

[11] The toric data of this geometry is summarized in appendix B.2.

[12] For a review on Hecke transformations see Zagier's article in [179].

For example the partition functions for $n = 1, 2$ obtained from the instanton part of the prepotential of the geometry read

$$Z^{(1)} = -\frac{2E_4E_6}{\eta^{24}}, \quad Z^{(2)} = -\frac{E_4E_6\left(17E_4^3 + 7E_6^2\right)}{96\eta^{48}}, \tag{4.25}$$

and are related by the Hecke transformation. Further examples of higher wrapping are given in the appendix C.3. The fact that the partition functions of higher wrappings of the M5-brane on the K3, which correspond to multiple heterotic strings, are given by the Hecke transformation was interpreted [132] by the absence of bound-states. Geometrically, multiple M5-branes on a K3 can be holomorphically deformed off one another. This argument fails for surfaces with $b_2^+ = 1$ and in particular for $\frac{1}{2}$K3.

One reason that the higher $Z^{(n)}$ can be determined in such a simple way from $Z^{(1)}$ can be understood in topological string theory from the fact that the BPS numbers on K3 depend only on the intersection of a curve $\mathcal{C}^2 = 2g - 2$ [170], and not on their class in $H_2(\mathrm{K3}, mathdsZ)$. This allows to prove (4.23) to all orders in the limit of the topological string partition function under consideration by slightly modifying the proof in [97]. Using the Picard-Fuchs system of the elliptic fibration one shows in the limit $q_3 \to 0$ the first equality in the identity

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F_0|_{q_3\to 0} &= \frac{E_4(t_1)E_6(t_1)E_4(t_2)}{\eta(t_1)^{24}(j(t_1) - j(t_2))} \\ &= \frac{q_1}{q_1 - q_2} + E_4(t_2) - \sum_{d,l,k>0} l^3 c(kl) q_1^{kl} q_2^{ld}, \end{aligned} \tag{4.26}$$

where $j = E_4^3/\eta^{24}$ and $c(n)$ are defined as

$$-\frac{1}{2}Z^{(1)} = \sum_n c(n)q^n. \tag{4.27}$$

This equations shows two things. The BPS numbers inside the Kähler cone of K3 depend only on $\mathcal{C}^2 = kl$ and all $Z^{(n)}$ are given by one modular form. The second fact can be used as in [97] to establish that

$$\frac{1}{2}\left(\frac{\partial}{\partial t_2}\right)^3 F_0|_{q_3\to 0} = \sum_{n=0}^{\infty} F_n(t_1)q_2^n, \tag{4.28}$$

where F_n is the Hecke transform of F_1, i.e. $n^3F_n = F_1|T_n$. Using Bol's identity and restoring the n^3 factors yields (4.23).

E-strings and bound-states

The recursion relating the higher windings of the E-strings to lower winding, developed in [130–132] in contrast reads

$$\frac{\partial Z^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{s=1}^{n-1} s(n-s)Z^{(s)}\,Z^{(n-s)}, \tag{4.29}$$

which becomes an anomaly equation, when E_2 is completed into a modular object $\widehat{E}_2$ by introducing a non-holomorphic part (see appendix A.2). The anomaly reads:

$$\partial_{\bar{t}_1}\widehat{Z}^{(n)} = \frac{i(\mathrm{Im}\,t_1)^{-2}}{16\pi}\sum_{s=1}^{n-1} s(n-s)\widehat{Z}^{(s)}\widehat{Z}^{(n-s)}, \tag{4.30}$$

and was given the interpretation [132] of taking into account the contributions from bound-states. Starting from [98]

$$Z^{(1)} = \frac{E_4\sqrt{q}}{\eta^{12}}, \tag{4.31}$$

and using the vanishing of BPS states of certain charges one obtains recursively all $Z^{(n)}$ [130–132]. E.g. the $n = 2$ the contribution reads:

$$\widehat{Z}^{(2)} = \frac{qE_4E_6}{12\eta^{24}} + \frac{q\widehat{E_2}E_4^2}{24\eta^{24}}\,, \tag{4.32}$$

where the second summand has the form $\widehat{E}_2\left(Z^{(1)}\right)^2$ and takes into account the contribution from bound-states of singly wrapped M5-branes.

A relation to the anomaly equations appearing in topological string theory [15] was pointed out in ref. [132] and proposed for arbitrary genus in refs. [87, 88]. The higher genus generalization reads [87, 88]:

$$\frac{\partial Z_g^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{g_1+g_2=g}\sum_{s=1}^{n-1} s(n-s)Z_{g_1}^{(s)}\,Z_{g_2}^{(n-s)} + \frac{n(n+1)}{24}Z_{g-1}^{(n)}\,, \tag{4.33}$$

where the instanton part of the A-model free energies at genus g is denoted by $F_g(q_1, q_2, q_3)$, and $F_g(q_1, q_2 \to 0, q_3) = \sum_{n\geq 1} Z_g^{(n)} q_3^n$. The $Z_g^{(n)}$ have the form [88]

$$Z_g^{(n)} = P_g^{(n)}(E_2, E_4, E_6)\frac{q_1^{n/2}}{\eta^{12n}}\,, \tag{4.34}$$

where $P_g^{(n)}$ denotes a quasi-modular form of weight $2g + 6n - 2$.

4.2.3 Holomorphic anomaly via mirror symmetry for $\frac{1}{2}$K3

In the following we try to derive the holomorphic anomaly equation at genus zero (4.29) by adapting the proof which appeared in ref. [87] for a similar geometry. We start by studying the Picard-Fuchs operator associated to the elliptic fiber $X_6[1, 2, 3]$ only. Denoting by $\theta_x = x\partial_x$ the Picard-Fuchs operator can be written as

$$\mathcal{L} = \theta_x^2 - 12x(6\theta_x + 5)(6\theta_x + 1). \tag{4.35}$$

One can immediately write down two solutions as power series expansions around $x = 0$. They are given by

$$\phi(x) = \sum_{n\geq 0} a_n x^n, \quad \tilde{\phi}(x) = \log(x)\phi(x) + \sum_{n\geq 0} b_n x^n, \tag{4.36}$$

with

$$a_n = \frac{(6n)!}{(3n)!(2n)!n!}, \quad b_n = a_n(6\psi(1 + 6n) - 3\psi(1 + 3n) - 2\psi(1 + 2n) - \psi(1 + n)), \tag{4.37}$$

where $\psi(z)$ denotes the digamma function. The mirror map is thus given by

$$2\pi i\tau = \frac{\tilde{\phi}(x)}{\phi(x)}. \tag{4.38}$$

Using standard techniques from the Gauss-Schwarz theory for the Picard-Fuchs equation (cf. [112]) one observes

$$j(\tau) = \frac{1}{x(1-432x)}, \tag{4.39}$$

which can be inverted to yield

$$x(\tau) = \frac{1}{864}(1-\sqrt{1-1728/j(\tau)}) = q - 312q^2 + \mathcal{O}(q^3). \tag{4.40}$$

Further, the polynomial solution $\phi(x)$ can be expressed in terms of modular forms as

$$\phi(x) = {}_2F_1(\frac{5}{6}, \frac{1}{6}, 1; 432x) = \sqrt[4]{E_4(\tau)}, \tag{4.41}$$

from which one can conclude that

$$\begin{aligned} E_4(\tau) &= \phi^4(x), \\ E_6(\tau) &= \phi^6(x)(1-864x), \\ \Delta(\tau) &= \phi^{12}(x)x(1-432x), \\ \frac{1}{2\pi i}\frac{dx}{d\tau} &= \phi^2(x)x(1-432x). \end{aligned} \tag{4.42}$$

Let us now examine the periods of the mirror geometry Y in the limit that the fiber F of the Hirzebruch surface becomes small. Due to the special structure of the Picard-Fuchs system which is found in eq. (B.10) the first three period integrals in the notation of [87] read

$$\begin{aligned} w_0(x,y,0) &= \phi(x) \\ w_1^{(1)}(x,y,0) &= \tilde{\phi}(x) \\ w_2^{(1)}(x,y,0) &= \log(y)\phi(x) + \xi(x) + \sum_{m\geq 1}(\mathcal{L}_m\phi(x))y^m, \end{aligned} \tag{4.43}$$

with

$$\xi(x) = \sum_{n\geq 0} a_n(\psi(1+n)-\psi(1))x^n, \tag{4.44}$$

and

$$\mathcal{L}_m = \frac{(-)^m}{m(m!)}\prod_{k=1}^{m}(\theta_x - k + 1). \tag{4.45}$$

This can be obtained by applying the Frobenius method to derive the period integrals, see e.g. [86]. The mirror map reads

$$2\pi i t_i = \frac{w_i^{(1)}(x,y,0)}{w_0(x,y,0)}, \quad i = 1,2. \tag{4.46}$$

Comparing this with our previous discussion about the Picard-Fuchs operator of the elliptic fiber we see that for $t_1 = \tau$ there is nothing left to discuss. Hence, let's study the mirror map associated to $t_2 = t$. We observe that by formally inverting, the inverse mirror map can be determined iteratively through the relation

$$y(q,p) = p\zeta e^{-\sum_{m\geq 1} c_m(x)y^m}, \tag{4.47}$$

where $\zeta = e^{-\frac{\xi(x)}{\phi(x)}}$ and

$$c_m(x) = \frac{\mathcal{L}_m \phi(x)}{\phi(x)}. \tag{4.48}$$

Using eq. (4.42) $c_1(x)$ is given by

$$\begin{aligned} c_1(x) &= -\frac{1}{12}(f_1 - 2) - \frac{f_1}{12}\frac{E_2(\tau)}{\phi^2(x)} \\ &= -\frac{1}{\phi^6}\frac{f_1}{12}(E_2E_4 - E_6), \end{aligned} \tag{4.49}$$

where we introduced $f_1 = (1 - 432x)^{-1}$. In order to obtain the other $c_m(x)$ one uses

$$\begin{aligned} \theta_x f_1 &= f_1(f_1 - 1), \\ \theta_x \left(\frac{E_2}{\phi^2}\right) &= -\frac{1}{\phi^8}\frac{f_1}{12}\left(E_2^2 E_4 - 2E_2E_6 + E_4^2\right), \\ \theta_x \left(\frac{E_6}{\phi^6}\right) &= -\frac{1}{\phi^{12}}\frac{f_1}{12}\left(6E_4^3 - 6E_6^2\right), \end{aligned} \tag{4.50}$$

and finds the following kind of structure. One can show inductively that

$$c_m(x) = \frac{1}{\phi^{6m}}\left(\frac{f_1}{12}\right)^m Q_{6m}(E_2, E_4, E_6), \tag{4.51}$$

where Q_{6m} is a quasi-homogeneous polynomial of degree $6m$ and type $(2, 4, 6)$, i.e.

$$Q_{6m}(\lambda^2 x, \lambda^4 y, \lambda^6 z) = \lambda^{6m} Q_{6m}(x, y, z).$$

Also by induction, it follows from (4.49) and (4.50) that Q_{6m} is linear in E_2. This allows to write a second structure which is analogous to the one appearing in ref. [87] and given by

$$c_m(x) = B_m \frac{E_2}{\phi^2} + D_m, \tag{4.52}$$

where the coefficients B_m, D_m obey the following recursion relation

$$\begin{aligned} B_{m+1} &= -\frac{m}{(m+1)^2}\left[(\theta_x - m)B_m + D_1 B_m - B_1 D_m\right], \\ D_{m+1} &= -\frac{m}{(m+1)^2}\left[(\theta_x - m)D_m - D_1 D_m + B_1 B_m\right], \end{aligned} \tag{4.53}$$

with $B_1 = -\frac{f_1}{12}$ and $D_1 = -\frac{1}{12}(f_1 - 2)$. A formal solution to the recursion relation (4.53) can be given by

$$\begin{aligned} B_m &= -\frac{f_m}{12}, \\ D_m &= \frac{1}{f_1}\left[\frac{(m+1)^2}{m} f_{m+1} + (\theta_x - m - \frac{1}{12}(f_1 - 2)) f_m\right], \end{aligned} \tag{4.54}$$

where we define f_m to be

$$f_m(x) = \tilde{\phi}(x)\mathcal{L}_m \phi(x) - \phi(x)\mathcal{L}_m \tilde{\phi}(x). \tag{4.55}$$

Due to the relations (4.50) we conclude, that the f_m as well as B_m and D_m are polynomials in f_1. Since f_1 is a rational function of x, it transforms well under modular transformations. Therefore modular invariance is broken only by the E_2 term in c_m. We express this via the partial derivative of c_m

$$\frac{\partial c_m(x)}{\partial E_2} = -\frac{1}{12}\frac{f_m(x)}{\phi^2(x)}. \tag{4.56}$$

In order to prove the holomorphic anomaly equation (4.29) one first shows using the general results about the period integrals in [86] that the instanton part of the prepotential can be expressed by the functions $f_m(x)$. A tedious calculation reveals

$$\frac{1}{2\pi i}\frac{\partial}{\partial t}F_0(\tau,t) = \sum_{m\geq 1}\frac{f_m(x)}{\phi^2(x)}y^m. \tag{4.57}$$

Using the implicit function theorem and eqs. (4.56), (4.47) yields

$$\frac{\partial y}{\partial E_2} = \frac{1}{12}\left(\frac{1}{2\pi i}\frac{\partial y}{\partial t}\right)\left(\frac{1}{2\pi i}\frac{\partial F_0}{\partial t}\right). \tag{4.58}$$

Now, we have

$$\frac{\partial}{\partial E_2}\left(\frac{1}{2\pi i}\frac{\partial F_0}{\partial t}\right) = \frac{1}{12}\left(\frac{\partial^2 F_0}{\partial(2\pi i t)^2}\right)\left(\frac{1}{2\pi i}\frac{\partial F_0}{\partial t}\right), \tag{4.59}$$

which implies that up to a constant term in p one arrives at

$$\frac{\partial F_0}{\partial E_2} = \frac{1}{24}\left(\frac{1}{2\pi i}\frac{\partial F_0}{\partial t}\right)^2. \tag{4.60}$$

By definition of $Z^{(n)}$ we have $\frac{1}{2\pi i}\frac{\partial}{\partial t}F_0(\tau,t) = \sum_{m\geq 1} mZ^{(m)}p^m$ and hence obtain by resummation

$$\frac{\partial Z^{(n)}}{\partial E_2} = \frac{1}{24}\sum_{s=1}^{n-1} s(n-s)Z^{(s)}Z^{(n-s)}. \tag{4.61}$$

This almost completes the derivation of (4.29). We still need to determine the explicit structure of Z_n. To achieve this we proceed inductively. Using (4.42), (4.57) and (4.47) one obtains

$$Z^{(1)} = \frac{\zeta f_1}{\phi^2} = q^{\frac{1}{2}}\frac{E_4}{\eta^{12}}. \tag{4.62}$$

Employing the structure (4.51) one can evaluate (4.57) and calculate that

$$\begin{aligned} Z^{(n)} &= \frac{\zeta^n f_1^n}{\phi^{6n}}P_{6n-2}(E_2,E_4,E_6) \\ &= \left(\frac{\zeta f_1}{\phi^2}\right)^n \frac{1}{\phi^{4n}}P_{6n-2}(E_2,E_4,E_6) \\ &= \frac{q^{\frac{n}{2}}}{\eta^{12n}}P_{6n-2}(E_2,E_4,E_6), \end{aligned} \tag{4.63}$$

where P_{6n-2} is of weight $6n-2$ and is decomposed out of (parts of) Q_m's. This establishes a derivation of the holomorphic anomaly equation (4.29) at genus zero for the elliptic genus associated to the $\frac{1}{2}$K3 surface.

4.2.4 Generating functions from wall-crossing

In the last sections we have argued that the partition function of $\mathcal{N}=4$ $U(r)$ Super-Yang-Mills theory suffers from a holomorphic anomaly for divisors with $b_2^+(P)=1$. In fact there exists another way to see the anomaly which is also intimately related to the computation of BPS degeneracies encoded in the elliptic genus and will be the subject of this section. This method relies on wall-crossing formulas and originally goes back to Göttsche and Zagier [71, 73]. In the physics context it has also been employed in [120, 121]. It will be used in section 4.3 to derive the elliptic genus for BPS states and their anomaly rigorously. In the following presentation we will be very sketchy as we merely want to stress the main ideas. We refer to section 4.3 for details.

The starting point is the Kontsevich-Soibelman formula [106] which describes the wall-crossing of bound-states of D-branes. Specifying to the case of two M5-branes and taking the equivalent D4-D2-D0 point of view the Kontsevich-Soibelman formula reduces to the primitive wall-crossing formula

$$\Delta\Omega(\Gamma;J\to J')=\Omega(\Gamma;J')-\Omega(\Gamma;J)=(-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1,\Gamma_2\rangle\,\Omega(\Gamma_1)\,\Omega(\Gamma_2), \tag{4.64}$$

which describes the change of BPS degeneracies of a bound-state with charge vector $\Gamma=\Gamma_1+\Gamma_2$, once a wall of marginal stability specified by J_W is crossed. The symplectic charge product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle\Gamma_1,\Gamma_2\rangle=-Q_6^{(1)}Q_0^{(2)}+Q_4^{(1)}\cdot Q_2^{(2)}-Q_2^{(1)}\cdot Q_4^{(2)}+Q_0^{(1)}Q_6^{(2)}. \tag{4.65}$$

Hence, for D4-D2-D0 brane configurations $\langle\Gamma_1,\Gamma_2\rangle$ is independent of the D0-brane charge. Further, in eq. (4.64) Γ_1 and Γ_2 are primitive charge vectors such that $\Omega(\Gamma_i)$ do not depend on the moduli. Thus, the Γ_i can be thought of as charge vectors with $r=1$ whereas Γ corresponds to a charge vector with $r=2$. Assuming, that the wall of marginal stability does not depend on the D0-brane charge, formula (4.64) can be translated into a generating series $\Delta f^{(2)}_{\mu,J\to J'}$ defined by

$$\Delta f^{(2)}_{\mu,J\to J'}=\sum_{d\geq 0}\Delta\bar{\Omega}(\Gamma;J\to J')\,q^{d-\frac{\chi(P)}{12}}. \tag{4.66}$$

Assuming that there exists a reference chamber J' such that $\bar{\Omega}(\Gamma;J)=0$, this gives us directly an expression for $f^{(2)}_{\mu,J}$.

As it will turn out in the next section, $\Delta f^{(2)}_{\mu,J\to J'}$ is given in terms of an indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which contains the information about the decays due to wall-crossing as one moves from J to J'. Indefinite theta-functions were analyzed by Zwegers in his thesis [180]. One of their major properties is that they are not modular as one only sums over a bounded domain of the lattice Λ specified by J and J'. However, Zwegers showed that by adding a non-holomorphic completion the indefinite theta-functions have modular transformation behavior and fall into the class of mock modular forms.[13] Every mock modular form h of weight k has a shadow g, which is a modular form of weight $2-k$, such that the function

$$\hat{h}(\tau)=h(\tau)+g^*(\tau) \tag{4.67}$$

[13] We review some notions in appendix A.2.

transforms as a modular form of weight k but is not holomorphic. Here, g^* is a certain transformation of the function g that introduces a non-holomorphic dependence. Taking the derivative of $\hat{h}$ with respect to $\bar{\tau}$ yields a holomorphic anomaly given by the shadow

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{\partial g^*}{\partial \bar{\tau}} = \tau_2^{-k}\overline{g(\tau)}, \tag{4.68}$$

where $\tau_2 = \mathrm{Im}(\tau)$.

As described in sections 4.2.1 and 4.2.2 the (MSW) CFT and the $\mathcal{N}=4\, U(r)$ Super-Yang-Mills partition functions should behave covariantly under modular transformations of the $\mathrm{SL}(2,\mathbb{Z})$ acting on τ. Thus, the modular completion outlined above will effect the generating functions $f^{(2)}_{\mu,J}$ through their relation to the indefinite theta-function $\Theta^{J,J'}_{\Lambda,\mu}$, which needs a modular completion to transform covariantly under modular transformations, i.e.

$$\Theta^{J,J'}_{\Lambda,\mu} \mapsto \widehat{\Theta}^{J,J'}_{\Lambda,\mu} \tag{4.69}$$

and consequently $f^{(2)}_{\mu,J}$ is replaced by $\hat{f}^{(2)}_{\mu,J}$. Due to eq. (4.67) the counting function of BPS invariants $\hat{f}^{(2)}_{\mu,J}$ and thus the elliptic genus $Z^{(2)}_P$ are going to suffer from a holomorphic anomaly, to which we turn next.

4.3 Wall-crossing and mock modularity

In this section we derive an anomaly equation for two M5-branes wound on a rigid surface/divisor P with $b_2^+(P) = 1$, inside a Calabi-Yau manifold X. We begin by reviewing D4-D2-D0 bound-states in the type IIA picture and their wall-crossing in the context of the Kontsevich-Soibelman formula. Then we proceed by deriving a generating function for rank two sheaves from the Kontsevich-Soibelman formula which is equivalent to Göttsche's formula [73]. This generating function is an indefinite theta-function, which fails to be modular. As a next step we apply ideas of Zwegers to remedy this failure of modularity by introducing a non-holomorphic completion. This leads to a holomorphic anomaly equation of the elliptic genus of two M5-branes that we prove for rigid divisors P.

4.3.1 D-branes and sheaves

In order to clarify our notation we collect some facts about D-brane charges and the stability conditions for a bound-state system of D4-D2-D0 branes wrapped around a divisor $i : P \hookrightarrow X$ inside a Calabi–Yau three-fold X. See e.g. [13] for a review.

Charges of D-branes and sheaves

The D4-D2-D0 brane-system is specified by a (coherent) sheaf $\mathcal{E}$ on P. The image of the K-theory charge of the sheaf $\mathcal{E}$ in $H^{\mathrm{even}}(X,\mathbb{Q})$ is given by the Mukai vector [75, 133, 168]

$$\Gamma = \mathrm{ch}(i_*\mathcal{E})\sqrt{\mathrm{Td}(X)}, \tag{4.70}$$

where $i_*\mathcal{E}$ denotes the extension-sheaf to X. Using the Grothendieck-Riemann-Roch-theorem

$$i_*(\mathrm{ch}(\mathcal{E})\,\mathrm{Td}(P)) = \mathrm{ch}(i_*\mathcal{E})\,\mathrm{Td}(X), \tag{4.71}$$

and the expressions

$$\mathrm{Td}(Y)^a = 1 + \frac{a}{2}c_1(Y) + \frac{(3a^2-a)c_1(Y)^2 + 2a\,c_2(Y)}{24} \tag{4.72}$$

$$\mathrm{ch}(Y) = \sum_{i=0}^{3} \mathrm{ch}_i(Y) = \mathrm{rk}(Y) + c_1(Y) + \frac{1}{2}c_1(Y)^2 - c_2(Y) \tag{4.73}$$

$$\mathrm{ch}(Y^*) = \sum_{i=0}^{3} \mathrm{ch}_i(Y^*) = \mathrm{rk}(Y) - c_1(Y) + \frac{1}{2}c_1(Y)^2 - c_2(Y), \tag{4.74}$$

where Y^* denotes the dual sheaf, one obtains [41]:

$$\begin{aligned}\Gamma &= r[P] + r\, i_*\left(\frac{c_1(\mathcal{E})}{r} + \frac{c_1(P)}{2}\right)\\ &+ r i_*\left(\frac{c_1(P)^2 + c_2(P)}{12} + \frac{\frac{1}{2}(c_1(P)c_1(\mathcal{E}) + c_1(\mathcal{E})^2) - c_2(\mathcal{E})}{r}\right) - \frac{c_2(X)\cdot[P]}{24},\end{aligned} \tag{4.75}$$

where r is the rank of the sheaf $\mathcal{E}$ and one has to note, that $c_1(X) = 0$ as X is Calabi-Yau. Using the adjunction formula we arrive at

$$\Gamma = (Q_6\,,\, Q_4\,,\, Q_2\,,\, Q_0) = r\left(0\,,\, [P]\,,\, i_*F\,,\, \left[\frac{\chi(P)}{24} + \int_P \frac{1}{2}F^2 - \Delta\right]\right). \tag{4.76}$$

Here we introduce

$$F = \frac{c_1(\mathcal{E})}{r} + \frac{c_1(P)}{2}, \tag{4.77}$$

$$\mu = \frac{c_1(\mathcal{E})}{r}, \tag{4.78}$$

$$\Delta = \frac{1}{2r^2}\left(2r\, c_2(\mathcal{E}) - (r-1)\, c_1(\mathcal{E})^2\right). \tag{4.79}$$

The quantity Δ is called the discriminant.

Π-*stability*

Given the K-theory charges the expression for the central charge from mirror symmetry is

$$\begin{aligned}Z(\mathcal{E}) &= -\int e^{-(B+iJ)}\,\Gamma(\mathcal{E}) + (\text{instanton} - \text{corrections})\\ &= -\frac{r}{2}[P]\cdot t^2 + t(i_*c_1(\mathcal{E}) + \frac{r}{2}i_*c_1(P)) - \mathrm{ch}_2(\mathcal{E})\\ &\quad - \frac{1}{2}c_1(\mathcal{E})c_1(P) - \frac{r}{8}c_1(P)^2 - \frac{r}{24}c_2(P) + \mathcal{O}(e^{-t}),\end{aligned} \tag{4.80}$$

where J is the Kähler form of X and $t = B + iJ$. We now denote the phase of $Z(\mathcal{E})$ by

$$\varphi(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Arg}\, Z(\mathcal{E}) = \frac{1}{\pi}\,\mathrm{Im}\log Z(\mathcal{E}). \tag{4.81}$$

A sheaf $\mathcal{E}$ is called Π-(semi)-stable [50, 51] iff for every (well-behaved) subsheaf $\mathcal{F}$:

$$\varphi(\mathcal{F}) \leq \varphi(\mathcal{E}), \tag{4.82}$$

where the strict inequality amounts to stability. If the inequality is strictly fulfilled (a stable sheaf) a decay is impossible by charge and energy conservation. Note, that the Π-stability condition involves an infinite tower of quantum corrections at an arbitrary point in moduli space.

μ-stability

In a large volume phase ($t \to \infty$) of the Calabi-Yau the instanton-corrections are suppressed by $\mathcal{O}(e^{-t})$ and the classical expressions become exact. In this limit we are left with [40]:

$$\varphi(\mathcal{E}) = \frac{1}{\pi} \operatorname{Im} \log \left(-\frac{r}{2} J^2 \cdot [P] \right) + 2 \frac{J \cdot \mu}{J^2 \cdot [P]} + \mathcal{O}\left(\frac{1}{J^2} \right). \tag{4.83}$$

Π-stability now amounts to the definition

$$(i^* J) \cdot \frac{c_1(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} \leq (i^* J) \cdot \frac{c_1(\mathcal{E})}{\operatorname{rk}(\mathcal{E})} \quad \text{for any nice subsheaf} \quad \mathcal{F} \subseteq \mathcal{E}, \tag{4.84}$$

where $i^* J$ denotes the pullback of the Kähler form of the Calabi-Yau to P and all expressions are understood on P. The quantity appearing in the above definition is called slope and denoted by $\mu(\mathcal{E})$. The above condition is called μ-(semi-)stability and the classical notion of the stringy Π-stability. Note also, that μ-stability is not sensitive to how the lower dimensional charges are distributed among decay products. This is in contrast to Π-stability, where quantum corrections change this insensitivity.

Dimension of moduli space

On general grounds the moduli space of a D-brane modelled by a sheaf $\mathcal{E}$ is given by $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$. The elements of this group count the number of marginal open string operators in the spectrum of the BCFT describing the B-brane. We assume, that P is a rational surface and further that the sheaf $\mathcal{E}$ is μ-stable and that $(i^* J) \cdot [K_P] \leq 0$. Under these assumptions the moduli space is smooth and the following formula for its dimension holds [129]

$$\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) = 1 + r^2(2\Delta - 1). \tag{4.85}$$

A consequence is that for a slope-stable sheaf one has

$$\Delta \geq 0, \tag{4.86}$$

which is a condition on the stable bundle's Chern classes.

In the following we take on the equivalent type IIA point of view, adapting the discussion of refs. [40,120,121] to describe the relation to the Kontsevich-Soibelman wall-crossing formula [106]. We restrict our attention to the D4-D2-D0 system on the complex surface P and work in the large volume limit with vanishing B-field.

Decay of D4-D2-D0 branes

Given a choice of $J \in \mathcal{C}(P)$, a sheaf $\mathcal{E}$ is called μ-semi-stable if for every sub-sheaf $\mathcal{E}'$

$$\mu(\mathcal{E}') \cdot J \leq \mu(\mathcal{E}) \cdot J. \tag{4.87}$$

Moreover, a wall of marginal stability is a co-dimension one subspace of the Kähler cone $\mathcal{C}(P)$ where the following condition is satisfied

$$(\mu(\mathcal{E}_1) - \mu(\mathcal{E}_2)) \cdot J = 0, \tag{4.88}$$

but is non-zero away from the wall. Across such a wall of marginal stability the configuration (4.76) splits into two configurations with charge vectors

$$\begin{aligned} \Gamma_1 &= r_1 \left(0,\, [P],\, i_* F_1,\, \frac{\chi(P)}{24} + \int_P \frac{1}{2} F_1^2 - \Delta(\mathcal{E}_1)\right), \\ \Gamma_2 &= r_2 \left(0,\, [P],\, i_* F_2,\, \frac{\chi(P)}{24} + \int_P \frac{1}{2} F_2^2 - \Delta(\mathcal{E}_2)\right), \end{aligned} \tag{4.89}$$

where $r_i = \mathrm{rk}(\mathcal{E}_i)$ and $\mu_i = \mu(\mathcal{E}_i)$. By making use of the identity

$$r\Delta = r_1\Delta_1 + r_2\Delta_2 + \frac{r_1 r_2}{2r}\left(\frac{c_1(\mathcal{E}_1)}{r_1} - \frac{c_1(\mathcal{E}_2)}{r_2}\right)^2, \tag{4.90}$$

one can show that $\Gamma = \Gamma_1 + \Gamma_2$. Therefore, charge-vectors as defined in (4.76) form a vector-space which will be essential for the application of the Kontsevich-Soibelman formula.

Before we proceed, let us note, that the BPS numbers and the Euler numbers of the moduli space of sheaves are related as follows. Denote by $\mathcal{M}_J(\Gamma)$ the moduli space of semi-stable sheaves characterized by Γ. Its dimension reads [129]

$$\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma) = 2r^2 - r^2 \chi(\mathcal{O}_P) + 1. \tag{4.91}$$

The relation between BPS invariants and the Euler numbers of the moduli spaces $\mathcal{M}_J(\Gamma)$ is then given by [40]

$$\Omega(\Gamma, J) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)} \chi(\mathcal{M}(\Gamma), J)\,. \tag{4.92}$$

Moreover, for the system of charges we have specified to, the symplectic pairing of charges simplifies to [40]

$$\langle \Gamma_1, \Gamma_2 \rangle = r_1 r_2 (\mu_2 - \mu_1) \cdot [P]\,. \tag{4.93}$$

The holomorphic function $f^{(r)}_{\mu,J}(\tau)$ appearing in eq. (4.11) can now be identified with the generating function of BPS invariants of moduli spaces of semi-stable sheaves. Its wall crossing will be described in the following.

Kontsevich-Soibelman wall-crossing formula

Kontsevich and Soibelman [106] have proposed a formula which determines the jumping behavior of BPS-invariants $\Omega(\Gamma; J)$ across walls of marginal stability. The wall-crossing formula is given in terms of a Lie algebra defined by generators e_Γ and a basic commutation relation

$$[e_{\Gamma_1}, e_{\Gamma_2}] = (-1)^{\langle \Gamma_1, \Gamma_2 \rangle} \langle \Gamma_1, \Gamma_2 \rangle e_{\Gamma_1 + \Gamma_2}. \tag{4.94}$$

For every charge Γ an element U_Γ of the Lie group can be defined by

$$U_\Gamma = \exp\left(-\sum_{n\geq 1} \frac{e_{n\Gamma}}{n^2}\right). \tag{4.95}$$

The Kontsevich-Soibelman wall-crossing formula states that across a wall of marginal stability the following formula holds

$$\prod^{\curvearrowright}_{\Gamma:Z(\Gamma;J)\in V} U_\Gamma^{\Omega(\Gamma;J_+)} = \prod^{\curvearrowright}_{\Gamma:Z(\Gamma;J)\in V} U_\Gamma^{\Omega(\Gamma;J_-)}, \tag{4.96}$$

where J_+ and J_- denote Kähler classes on the two sides of the wall. Further, V is a region in $\mathbb{R}^2$ bounded by two rays starting at the origin and $\curvearrowright$ denotes a clockwise ordering of the factors in the product with respect to the phase of the central charges $Z(\Gamma; J)$, that are defined in eq. (4.80).

Restricting to the case $r = 2$ and $r_1 = r_2 = 1$, (4.96) can be truncated to

$$\prod_{Q_{0,1}} U_{\Gamma_1}^{\Omega(\Gamma_1)} \prod_{Q_0} U_\Gamma^{\Omega(\Gamma;J_+)} \prod_{Q_{0,2}} U_{\Gamma_2}^{\Omega(\Gamma_2)} = \prod_{Q_{0,2}} U_{\Gamma_2}^{\Omega(\Gamma_2)} \prod_{Q_0} U_\Gamma^{\Omega(\Gamma;J_-)} \prod_{Q_{0,1}} U_{\Gamma_1}^{\Omega(\Gamma_1)}, \tag{4.97}$$

where Q_0 is the D0-brane charge of Γ and the $Q_{0,i}$ are the D0-brane charges belonging to Γ_i, respectively. The above formula has been derived by setting all Lie algebra elements with D4-brane charge greater than two to zero. Therefore, the element e_Γ is central, using the Baker-Campbell-Hausdorff formula $e^X e^Y = e^Y e^{[X,Y]} e^X$ and the fact that the symplectic product is independent of the D0-brane charge, one finds the following change of BPS numbers across a wall of marginal stability [63, 120]

$$\Delta\Omega(\Gamma) = (-1)^{\langle\Gamma_1,\Gamma_2\rangle-1}\langle\Gamma_1,\Gamma_2\rangle \sum_{Q_{0,1}+Q_{0,2}=Q_0} \Omega(\Gamma_1)\,\Omega(\Gamma_2). \tag{4.98}$$

Moreover, one can deduce that the rank one degeneracies $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$ do not depend on the modulus J.

4.3.2 Relation of KS to Göttsche's wall-crossing formula

Göttsche has found a wall-crossing formula for the Euler numbers of moduli spaces of rank two sheaves in terms of an indefinite theta-function in ref. [73]. In this section we want to derive a modified version of this formula from the Kontsevich-Soibelman wall-crossing formula associated to D4-D2-D0 bound-states with D4-brane charge equal to two.

We use the short notation $\Gamma = (r, \mu, \Delta)$ to denote a rank r sheaf with the specified Chern classes that is associated to the D4-D2-D0 states. For rank one sheaves the generating function has no chamber dependence and we have already seen that it is given by (4.17). Following the discussion of our last section, higher rank sheaves do exhibit wall-crossing phenomena and therefore do depend on the chamber in moduli space, i.e. on $J \in \mathcal{C}(P)$.

Our aim now is to determine the generating function of the D4-D2-D0 system using the primitive wall-crossing formula derived from the KS wall-crossing formula. From now on we restrict our attention to rank two sheaves $\mathcal{E}$. They can split across walls of marginal stability into rank one sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ as outlined in section 4.2.4. Using relation (4.90) we can write

$$d = d_1 + d_2 + \xi \cdot \xi, \tag{4.99}$$

where $\xi = \mu_1 - \mu_2$ and $d = 2\Delta$. Further, a wall is given by (4.88), i.e. the set of walls given a split of charges ξ reads

$$W^\xi = \{J \in \mathcal{C}(P) \,|\, \xi \cdot J = 0\}\,. \tag{4.100}$$

Now, consider a single wall $J_W \in W^\xi$ determined by a set of vectors $\xi \in \Lambda + \mu$. Let J_+ approach J_W infinitesimally close from one side and J_- infinitesimally close from the other side. Thus, in our context the primitive wall-crossing formula (4.98) becomes

$$\bar{\Omega}(\Gamma; J_+) - \bar{\Omega}(\Gamma; J_-) = \sum_{Q_{0,1}+Q_{0,2}=Q_0} (-1)^{2\xi\cdot[P]}\, 2\,(\xi\cdot[P])\,\Omega(\Gamma_1)\,\Omega(\Gamma_2), \tag{4.101}$$

where we have used the identity (4.93). Note, that $Q_{0,i}$ and Q_0 are determined in terms of Γ and Γ_i through (4.76) and (4.89). Now, we can sum over the D0-brane charges to obtain a generating series. This yields

$$\begin{aligned} &\sum_{d\geq 0} (\bar{\Omega}(\Gamma; J_+) - \bar{\Omega}(\Gamma; J_-)) q^{d-\frac{\chi(P)}{12}} \\ = &\sum_{d_1,d_2\geq 0,\,\xi} (-1)^{2\xi\cdot[P]}\,(\xi\cdot[P])\,\Omega(\Gamma_1)\Omega(\Gamma_2) q^{d_1+d_2+\xi^2-\frac{2\chi(P)}{24}} \\ = &\ (-1)^{2\mu\cdot[P]-1}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}}\sum_\xi (\xi\cdot[P])\,q^{\xi^2}, \end{aligned} \tag{4.102}$$

where for the first equality use has been made of the identities (4.99, 4.101), and for the second equality the identity (4.17) has been used. The last line can be rewritten as

$$(-1)^{2\mu\cdot[P]-1}\frac{1}{2}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi(P)}}\mathrm{Coeff}_{2\pi i y}(\Theta^{J_+,J_-}_{\Lambda,\mu}(\tau,[P]y)), \tag{4.103}$$

where we have introduced the indefinite theta-function

$$\Theta^{J,J'}_{\Lambda,\mu}(\tau,x) := \frac{1}{2}\sum_{\xi\in\Lambda+\mu} (\mathrm{sgn}\langle J,\xi\rangle - \mathrm{sgn}\langle J',\xi\rangle)\, e^{2\pi i\langle \xi,x\rangle}\, q^{Q(\xi)}, \tag{4.104}$$

with the inner product[14] defined by $\langle x,y\rangle = 2d_{AB}x^Ay^B$ and the quadratic form $Q(\xi) = \frac{1}{2}\langle \xi,\xi\rangle$. As these theta-functions obey the cocycle condition [71]

$$\Theta^{F,G}_{\Lambda,\mu} + \Theta^{G,H}_{\Lambda,\mu} = \Theta^{F,H}_{\Lambda,\mu}, \tag{4.105}$$

we finally arrive at the beautiful relation between the BPS numbers in an arbitrary chamber J and those in a chamber J' first found by Göttsche in the case $\Lambda = H^2(P,\mathbb{Z})$:

$$f^{(2)}_{\mu,J'}(\tau) - f^{(2)}_{\mu,J}(\tau) = \frac{1}{2}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta^{2\chi(P)}(\tau)}\,\mathrm{Coeff}_{2\pi i y}(\Theta^{J,J'}_{\Lambda,\mu}(\tau,[P]y)). \tag{4.106}$$

4.3.3 Holomorphic anomaly at rank two

In this subsection we discuss the appearance of a holomorphic anomaly at rank two and give a proof of it by combing our previous results with results of Zwegers [180].

[14] Note, that this is not the symplectic product of D-brane charges defined before.

Elliptic genus at rank two and modularity

An important datum in eq. (4.106) is the choice of chambers $J, J' \in \mathcal{C}(P)$, which are any points in the Kähler cone of P. As a consequence, the indefinite theta-series does not transform well under $\mathrm{SL}(2,\mathbb{Z})$ in general. However, from the discussion of sect. 4.2.1 we expect, that the generating series $f^{(r)}_{\mu,J}(\tau)$ transforms with weight $-\frac{r(\Lambda)+2}{2}$ in a vector-representation under the full modular group, where $r(\Lambda)$ denotes the rank of the lattice Λ. Hence, there is a need to restore modularity. The idea is as follows.

Following Zwegers [180], it turns out that the indefinite theta-function can be made modular at the cost of losing its holomorphicity. From the definition (4.104) Zwegers smoothes out the sign-functions and introduces a modified function as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \frac{1}{2}\sum_{\xi\,\in\,\Lambda+\mu}\left(E\left(\frac{\langle c, \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right) - E\left(\frac{\langle c', \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c')}}\right)\right) e^{2\pi i\langle\xi,x\rangle} q^{Q(\xi)}, \tag{4.107}$$

where E denotes the incomplete error function

$$E(x) = 2\int_0^x e^{-\pi u^2} du. \tag{4.108}$$

Note, that if c or c' lie on the boundary of the Kähler cone, one does not have to smooth out the sign-function. Zwegers shows, that the non-holomorphic function $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ satisfies the correct transformation properties of a Jacobi form of weight $\frac{1}{2}r(\Lambda)$. Due to the non-holomorphic pieces it contains mock modular forms, that we want to identify in the following. In order to separate the holomorphic part of $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ from its shadow we recall the following property of the incomplete error function

$$E(x) = \mathrm{sgn}(x)(1 - \beta_{\frac{1}{2}}(x^2)), \tag{4.109}$$

which enables us to split up $\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x)$ into pieces. Here, β_k is defined by

$$\beta_k(t) = \int_t^\infty u^{-k} e^{-\pi u} du. \tag{4.110}$$

Hence, one can write eq. (4.107) as

$$\widehat{\Theta}^{c,c'}_{\Lambda,\mu}(\tau,x) = \Theta^{c,c'}_{\Lambda,\mu}(\tau,x) - \Phi^c_\mu(\tau,x) + \Phi^{c'}_\mu(\tau,x), \tag{4.111}$$

with

$$\Phi^c_\mu(\tau,x) = \frac{1}{2}\sum_{\xi\,\in\,\Lambda+\mu}\left[\mathrm{sgn}\langle\xi,c\rangle - E\left(\frac{\langle c, \xi + \frac{\mathrm{Im}\,(x)}{\tau_2}\rangle\sqrt{\tau_2}}{\sqrt{-Q(c)}}\right)\right] e^{2\pi i\langle\xi,x\rangle} q^{Q(\xi)}. \tag{4.112}$$

If c belongs to $\mathcal{C}(P) \cap \mathbb{Q}^{r(\Lambda)}$, we may write

$$\Phi^c_\mu(\tau,x) = R(\tau,x)\theta(\tau,x), \tag{4.113}$$

where we decomposed the lattice sum into contributions along the direction of c and perpendicular to c given by R and θ, respectively. Hence, θ is a usual theta-series associated to the quadratic form $Q|\langle c\rangle^\perp$, i.e. of weight $(r(\Lambda)-1)/2$. R is the part which carries the non-holomorphicity. It transforms with a weight $\frac{1}{2}$ factor and therefore $\mathrm{Coeff}_{2\pi iy}(R(\tau,[P]y))$ is of weight $\frac{3}{2}$. Following the general idea of Zagier [178] that we recapitulate in appendix A.2, we should encounter the $\beta_{\frac{3}{2}}$ function in the $2\pi iy$-coefficient of Φ. Indeed one can prove the following identity

$$\mathrm{Coeff}_{2\pi iy}\Phi^c_\mu(\tau,[P]y) = -\frac{1}{4\pi}\frac{\langle c,[P]\rangle}{\langle c,c\rangle}\sum_{\xi\,\in\,\Lambda+\mu}|\langle c,\xi\rangle|\,\beta_{\frac{3}{2}}\left(\frac{\tau_2\langle c,\xi\rangle^2}{-Q(c)}\right)q^{Q(\xi)}. \tag{4.114}$$

Taking the derivative with respect to $\bar\tau$ in order to obtain the shadow we arrive at the following final expression which evaluated at the attractor point ($c=-[P]$) simplifies as follows

$$\partial_{\bar\tau}\mathrm{Coeff}_{2\pi iy}\Phi^c_\mu(\tau,[P]y) = -\frac{\tau_2^{-\frac{3}{2}}}{8\pi i}\frac{c\cdot[P]}{\sqrt{-c^2}}(-1)^{4\mu^2}\,\theta^{(2)}_{\mu-\frac{[P]}{2},c}(\tau,0)\Big|_{c=-[P]}, \tag{4.115}$$

where we define the Siegel-Narain theta-function $\theta^{(r)}_{\mu,c}(\tau,z)$ as in eq. (4.13). For more details on the transformation properties of the indefinite theta-functions we refer the reader to appendix C.2.

Now, these results can be used to compute the elliptic genus for two M5-branes wrapping the divisor P. Consider

$$f^{(2)}_{\mu,J}(\tau) = f_{\mu,J'}(\tau) - \frac{1}{2}\frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta^{2\chi(P)}}\mathrm{Coeff}_{2\pi iy}\Theta^{J,J'}_{\Lambda,\mu}(\tau,[P]y), \tag{4.116}$$

where $f_{\mu,J'}(\tau)$ is a holomorphic ambiguity given by the generating series in a reference chamber J', which we choose to lie at the boundary of the Kähler cone $J'\in\partial\mathcal{C}(P)$. In explicit computations it may be possible to choose J' such that the BPS numbers vanish. In general, however, such a vanishing chamber might not always exist, but since J' is at the boundary of the Kähler cone, $f_{\mu,J'}(\tau)$ has no influence on the modular transformation properties, nor on the holomorphic anomaly. We write the full M5-brane elliptic genus as

$$Z^{(2)}_P(\tau,z) = \sum_{\mu\,\in\,\Lambda^*/\Lambda}\hat f^{(2)}_{\mu,J}(\tau)\theta^{(2)}_{\mu,J}(\tau,z), \tag{4.117}$$

where $\hat f^{(2)}_{\mu,J}$ denotes the modular completion as outlined above. We can show using Zwegers' results [180], that the M5-brane elliptic genus transforms like a Jacobi form of bi-weight $(-\frac{3}{2},\frac{1}{2})$. Again, we refer the reader to appendix C.2 for further details.

Proof of holomorphic anomaly at rank two

Now, we are in position to prove the holomorphic anomaly at rank two for general surfaces P with $b_2^+(P)=1$. We assume that J is evaluated at the attractor point $J=-[P]$, where we know the simple expression (4.115). The holomorphic anomaly takes the following form

$$\mathcal{D}_2 Z^{(2)}_P(\tau,z) = \tau_2^{-3/2}\frac{1}{16\pi i}\frac{J\cdot[P]}{\sqrt{-J^2}}\left(Z^{(1)}_P(\tau,z)\right)^2\Big|_{J=-[P]}, \tag{4.118}$$

where the derivative $\mathcal{D}_k$ is given as

$$\mathcal{D}_k = \partial_{\bar\tau} + \frac{i}{4\pi k}\partial^2_{z_+}, \tag{4.119}$$

and z_+ refers to the projection of z along a direction $J \in \mathcal{C}(P)$. For the proof, $\mathcal{D}_2 Z_P^{(2)}$ can be computed explicitly. Using (4.115) we obtain directly

$$\mathcal{D}_2 Z_P^{(2)}(\tau, z) = \tau_2^{-3/2} \frac{1}{16\pi i} \frac{J \cdot [P]}{\sqrt{-J^2}} \frac{\vartheta_{\Lambda^\perp}(\tau)^2}{\eta(\tau)^{2\chi}} \sum_{\mu \in \Lambda^*/\Lambda} (-1)^{4\mu^2} \theta^{(2)}_{\mu - \frac{[P]}{2}, J}(\tau, 0) \theta^{(2)}_{\mu, J}(\tau, z) \Big|_{J=-[P]}. \tag{4.120}$$

Since the following identity among the theta-functions $\theta_{\mu,J}$ holds

$$\left(\theta^{(1)}_{0,J}(\tau, z)\right)^2 = \sum_{\mu \in \Lambda^*/\Lambda} (-1)^{4\mu^2} \theta^{(2)}_{\mu - \frac{[P]}{2}, J}(\tau, 0) \theta^{(2)}_{\mu, J}(\tau, z), \tag{4.121}$$

we have proven the holomorphic anomaly equation at rank two for general surfaces P.

4.4 Application and extensions

In the following we want to apply the previous results to several selected examples. Before doing so, we explain two mathematical facts which will help to fix the ambiguity $f_{\mu,J'}(\tau)$, which are the blow-up formula and the vanishing lemma. After discussing the examples, we turn our attention to a possible extension to higher rank. This leads us to speculations about mock modularity of higher depth and wall-crossing having its origin in a meromorphic Jacobi form.

4.4.1 Blow-up formulae and vanishing chambers

There is a universal relation between the generating functions of stable sheaves on a surface P and on its blow-up $\tilde{P}$ [73, 111, 156, 172, 173]. Let P be a smooth projective surface and $\pi : \tilde{P} \to P$ the blow-up at a non-singular point with E the exceptional divisor of π. Let $J \in \mathcal{C}(P)$, r and μ such that $\gcd(r, r\mu \cdot J) = 1$. Then, the generating series $f^{(r)}_{\mu,J}(\tau; P)$ and $f^{(r)}_{\mu,J}(\tau; \tilde{P})$ are related by the blow-up formula

$$f^{(r)}_{\pi^*(\mu) - \frac{k}{r}E, \pi^*(J)}(\tau; \tilde{P}) = B_{r,k}(\tau) f^{(r)}_{\mu,J}(\tau; P), \tag{4.122}$$

with $B_{r,k}$ given by

$$B_{r,k}(\tau) = \frac{1}{\eta^r(\tau)} \sum_{a \in \mathbb{Z}^{r-1} + \frac{k}{r}} q^{\sum_{i \le j} a_i a_j}. \tag{4.123}$$

The second fact states that for a class of semi-stable sheaves on certain surfaces the moduli space of the sheaves is empty. We refer to this fact as the vanishing lemma [73]. For this let P be a rational ruled surface $\pi : P \to \mathbb{P}^1$ and J be the pullback of the class of a fiber of π. Picking a Chern class μ with $r\mu \cdot J$ odd, we have

$$\mathcal{M}((r, \mu, \Delta), J) = \emptyset \tag{4.124}$$

for all d and $r \geq 2$.

4.4.2 Applications to surfaces with $b_2^+ = 1$

The surfaces we are going to consider are $\mathbb{P}^2$, the Hirzebruch surfaces $\mathbb{F}_0$ and $\mathbb{F}_1$, the del Pezzo surfaces $\mathcal{B}_8$ and $\mathcal{B}_9$.

Projective plane $\mathbb{P}^2$

The projective plane $\mathbb{P}^2$ has been discussed quite exhaustively in the literature. The rank one result was obtained by Göttsche [72]

$$Z^{(1)}_{\mathbb{P}^2} = \frac{\vartheta_1(-\bar{\tau}, -z)}{\eta^3(\tau)}. \tag{4.125}$$

The generating functions of the moduli space of rank two sheaves or $SO(3)$ instantons of Super-Yang-Mills theory on $\mathbb{P}^2$ were written down by [156, 171, 172] and are given by

$$\begin{aligned} f_0(\tau) &= \sum_{n=0}^{\infty} \chi(\mathcal{M}((2,0,n),J))q^{n-\frac{1}{4}} = \frac{3h_0(\tau)}{\eta^6(\tau)}, \\ f_1(\tau) &= \sum_{n=0}^{\infty} \chi(\mathcal{M}((2,1,n),J))q^{n-\frac{1}{2}} = \frac{3h_1(\tau)}{\eta^6(\tau)}. \end{aligned} \tag{4.126}$$

Here, $h_j(\tau)$ are mock modular forms given by summing over Hurwitz class numbers $H(n)$

$$h_j(\tau) = \sum_{n=0}^{\infty} H(4n+3j)q^{n+\frac{3j}{4}}, \qquad (j = 0, 1). \tag{4.127}$$

Their modular completion is denoted by $\hat{h}_j(\tau)$, where the shadows are given by $\vartheta_{3-j}(2\tau)$ [177]. Explicitly, we have

$$\partial_{\bar{\tau}} \hat{h}_j(\tau) = \frac{\tau_2^{-\frac{3}{2}}}{16\pi i}\vartheta_{3-j}(-2\bar{\tau}). \tag{4.128}$$

Note, that these results are valid for all Kähler classes $J \in H^2(\mathbb{P}^2, \mathbb{Z})$ as there is no wall crossing in the Kähler moduli space of $\mathbb{P}^2$. This leads directly to the following elliptic genus of two M5-branes wrapping the $\mathbb{P}^2$ divisor

$$Z^{(2)}_{\mathbb{P}^2}(\tau, z) = \hat{f}_0(\tau)\vartheta_2(-2\bar{\tau}, -2z) - \hat{f}_1(\tau)\vartheta_3(-2\bar{\tau}, -2z). \tag{4.129}$$

Denoting by $\mathcal{D}_2 = \partial_{\bar{\tau}} + \frac{i}{8\pi}\partial_z^2$ one finds the expected holomorphic anomaly equation at rank two, given by[15]

$$\mathcal{D}_2 \, Z^{(2)}_{\mathbb{P}^2}(\tau, z) = -\frac{3}{16\pi i}\tau_2^{-\frac{3}{2}} \left(Z^{(1)}_{\mathbb{P}^2}(\tau, z)\right)^2, \tag{4.130}$$

which can be derived directly from the simple fact that

$$\vartheta_1(\tau, z)^2 = \vartheta_2(2\tau)\vartheta_3(2\tau, 2z) - \vartheta_3(2\tau)\vartheta_2(2\tau, 2z). \tag{4.131}$$

Further note, that the q-expansion of f_0, eq. (4.126), has non-integer coefficients. It was explained in [120] that this is due to the fact that the generating series involves the fractional BPS invariants $\bar{\Omega}(\Gamma)$, which we encountered before.

[15] This result has already been derived in [21].

Hirzebruch surface $\mathbb{F}_0$

Our next example is the Hirzebruch surface $P = \mathbb{F}_0$. We denote by F and B the fiber and the base $\mathbb{P}^1$'s respectively. For an embedding into a Calabi-Yau manifold one may consult app. B.2. Let us choose $J = F + B$, $J' = B$ and Chern class $\mu = F/2$. The choice $\mu = B/2$ can be treated analogously and leads to the same results. The other sectors corresponding to $\mu = 0$ and $\mu = (F + B)/2$ require a knowledge of the holomorphic ambiguity at the boundary and will not be treated here. One obtains

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi iy}(\Theta^{F+B,B}_{\Lambda,\mu}(\tau,[P]y)) \\ &= q^{-\frac{1}{3}}\left(2q + 22q^2 + 146q^3 + 742q^4 + \dots\right), \end{aligned} \tag{4.132}$$

where we denote by μ either $B/2$ or $F/2$. This exactly reproduces the numbers obtained in [107].

We want to compute the shadow of the completion given by adding Φ^{F+B}_{μ} and Φ^{B}_{μ} to the indefinite theta-series $\Theta^{F+B,B}_{\Lambda,\mu}$. Since B is chosen at the boundary, Φ^{B}_{μ} vanishes for $\mu = F/2, B/2$. The only relevant contribution has a shadow proportional to $\vartheta_2(\tau)$. Precisely, we obtain

$$\partial_{\bar\tau} f^{(2)}_{\mu,F+B}(\tau) = -\tau_2^{-3/2}\frac{1}{4\pi i\sqrt{2}}\frac{\overline{\vartheta_2(\tau)}\vartheta_2(\tau)}{\eta^8(\tau)} \qquad (\mu = \frac{F}{2},\frac{B}{2}). \tag{4.133}$$

Hirzebruch surface $\mathbb{F}_1$

The next example is the Hirzebruch surface $\mathbb{F}_1$, which is a blow-up of $\mathbb{P}^2$. Again we denote by F and B the fiber and base $\mathbb{P}^1$'s. The $\mathbb{P}^2$ hyperplane is given by the pullback of $F + B$ and B is the exceptional divisor. This example is particularly nice, since we can check our results against the blow-up formula (4.122) or use the results known from $\mathbb{P}^2$ to write generating functions in sectors which are not accessible through the vanishing lemma. Notice, that the holomorphic expansions have been already discussed in ref. [121]. From the general discussion one sees that there are four different choices for the Chern class $\mu \in \{\frac{B}{2}, \frac{F+B}{2}, \frac{F}{2}, 0\}$.

First, we choose $J = F + B$, $J' = F$ and Chern class $\mu = B/2$. We then obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi iy}(\Theta^{F+B,F}_{\Lambda,B}(\tau,[P]y)) \\ &= q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right). \end{aligned} \tag{4.134}$$

A check of this result against the blow-up formula (4.122) applied to $\mathbb{P}^2$ yields

$$\frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)} = q^{-\frac{1}{12}}\left(-\frac{1}{2} - q + \frac{15}{2}q^2 + 91q^3 + 558q^4 + \dots\right) = f^{(2)}_{\mu,F+B}(\tau). \tag{4.135}$$

Further, we calculate the shadow by differentiating $\hat f^{(2)}$ with respect to $\bar\tau$

$$\partial_{\bar\tau}\hat f^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_3(2\tau)}\vartheta_2(2\tau)}{\eta^8(\tau)}, \tag{4.136}$$

which also is in accord with the blow-up formula. Note, that (4.134) has half-integer expansion coefficients, since $J = B + F$ lies on a wall for the Chern class $\mu = B/2$.

As a second case we choose $J = F+B$, $J' = F$ and Chern class $\mu = (F+B)/2$ and obtain

$$\begin{aligned} f^{(2)}_{\mu,F+B}(\tau) &= \frac{1}{2\eta^8(\tau)}\mathrm{Coeff}_{2\pi i y}(\Theta^{F+B,F}_{\Lambda,F+B}(\tau,[P]y)) \\ &= q^{-\frac{7}{12}}\left(q+13q^2+93q^3+496q^4+\dots\right), \end{aligned} \tag{4.137}$$

which we again can check against the blow-up formula (4.122) for $\mathbb{P}^2$

$$\frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)} = q^{-\frac{7}{12}}\left(q+13q^2+93q^3+496q^4+\dots\right) = f^{(2)}_{\mu,F+B}(\tau). \tag{4.138}$$

Calculating the shadow yields

$$\partial_{\bar\tau}\hat f^{(2)}_{\mu,F+B}(\tau) = \frac{3}{16\pi i}\tau_2^{-3/2}\frac{\overline{\vartheta_2(2\tau)}\vartheta_3(2\tau)}{\eta^8(\tau)}, \tag{4.139}$$

which is also in accord with the blow-up formula.

The last two sectors $\mu = F/2, 0$ are not accessible via the vanishing lemma. However, using a blow-down to $\mathbb{P}^2$ we observe, that the above two cases reproduce correctly the two Chern classes in the cases of rank two sheaves on $\mathbb{P}^2$. Using the blow-up formulas once more we finally arrive at

$$\begin{aligned} f^{(2)}_{(0,0),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(\frac{1}{2},0),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(0,\frac{1}{2}),J}(\tau) &= \frac{3h_0(\tau)}{\eta^6(\tau)}\frac{\vartheta_2(2\tau)}{\eta^2(\tau)}, \\ f^{(2)}_{(\frac{1}{2},\frac{1}{2}),J}(\tau) &= \frac{3h_1(\tau)}{\eta^6(\tau)}\frac{\vartheta_3(2\tau)}{\eta^2(\tau)}, \end{aligned} \tag{4.140}$$

where $J = F+B$ and $\mu = (a,b) = aF + bB$. Note, that in the cases $f^{(2)}_{(0,0),J}$ and $f^{(2)}_{(0,\frac{1}{2}),J}$ the blow-up formula is not valid since we violate the gcd-condition, as $\pi_*\mu = 0$ in these cases. However, for rank two sheaves on $\mathbb{F}_1$ the blow-up formula seems to work anyway, since the generating series using the blow-up procedure and the indefinite theta-function description coincide for the Chern class $\mu = (0, \frac{1}{2})$.

Del Pezzo surface $\mathcal{B}_8$

As in [61] we embed the surface $\mathcal{B}_8$ in a certain free $\mathbb{Z}_5$ quotient[16] of the Fermat quintic $\tilde X = \{\sum_{i=1}^5 x_i^5 = 0\}$ in $\mathbb{P}^4$. The action of the group $G = \mathbb{Z}_5$ on the projective coordinates of the ambient space is given by $x_i \sim \omega^i x_i$, where $\omega = e^{2\pi i/5}$. For the hyperplane section, denoted P, we observe that $P^3 = 1$, as for the Fermat quintic the five points of intersection of three hyperplanes $\{x_i = x_j = x_k = 0\}$ are identified under the action of the group G. The Euler character of the hyperplane is given by $\chi(P) = 11$. It can be shown that the divisor P

[16] The only freely acting group actions for the quintic are a $\mathbb{Z}_5^2$ and the above $\mathbb{Z}_5$.

is rigid and has $b_2^+ = 1$. We observe that $H^2(P,\mathbb{Z}) = \mathbb{Z} \oplus (-E_8)$ as is explained in [61]. The elliptic genus of a single M5-brane is then fixed by the modular weights

$$Z_P^{(1)}(\tau, z) = \frac{E_4(\tau)}{\eta^{11}(\tau)}\,\vartheta_1(-\bar{\tau}, -z). \tag{4.141}$$

The form of $Z_P^{(2)}$ can now be calculated as for $\mathbb{P}^2$ and is given by

$$Z_P^{(2)}(\tau, z) \sim \frac{E_4(\tau)^2}{\eta(\tau)^{22}}\,(\hat{h}_0(\tau)\,\vartheta_2(-2\bar{\tau}, -2z) - \hat{h}_1(\tau)\,\vartheta_3(-2\bar{\tau}, -2z)). \tag{4.142}$$

The holomorphic anomaly equation fulfilled by $Z_P^{(2)}(\tau, z)$ can be obtained as in the $\mathbb{P}^2$ case

$$\mathcal{D}_2\, Z_P^{(2)}(\tau, z) \sim \frac{\tau_2^{-\frac{3}{2}}}{16\pi i}\left(Z_P^{(1)}(\tau, z)\right)^2. \tag{4.143}$$

Del Pezzo surface $\mathcal{B}_9$*, the* $\frac{1}{2}$*K3*

We end our examples by returning and commenting on $\frac{1}{2}$K3 or $\mathcal{B}_9$ which was the example of section (4.2.2), as M5-branes wrapping on it give rise to the multiple E-strings. The $\mathcal{B}_9$ surface can be understood as a $\mathbb{P}^2$ blown up at nine points or a rational elliptic surface. This case is interesting as one can map via T-duality along the elliptic fibration the computation of the modified elliptic genus to the computation of the partition function of topological string theory on the same surface [132]. The middle dimensional cohomology lattice of $\mathcal{B}_9$ is given by $H^2(\mathcal{B}_9,\mathbb{Z}) = \Gamma^{1,1} \oplus E_8$ and the Euler number can be computed to $\chi(\mathcal{B}_9) = 12$. Modularity then fixes the form of the elliptic genus at rank one to

$$Z_{\mathcal{B}_9}^{(1)}(\tau, z) = \frac{E_4(\tau)}{\eta(\tau)^{12}}\,\theta_{0,J}^{(1)}(\tau, z), \tag{4.144}$$

where $\theta_{0,J}^{(1)}(\tau, z)$ is the theta-function associated to the lattice $\Gamma^{1,1}$ with standard intersection form

$$(-d_{AB}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.145}$$

Choosing the Kähler form $J = (R^{-2}, 1)^T$, where $(1, 0)^T$ is the class of the elliptic fiber, one can show that

$$\theta_{0,J}^{(1)}(\tau, 0) \to \frac{R}{\sqrt{\tau_2}} \quad \text{as} \quad R \to \infty. \tag{4.146}$$

In this limit of small elliptic fiber one recovers the results of sect. 4.2.2. The factor $E_4(\tau)$ is precisely the theta-function of the E_8 lattice. The results obtained from the anomaly for higher wrappings of refs. [131, 132] were proven mathematically for double wrapping in ref. [174]. In this analysis the Weyl group of the E_8 lattice was used to perform the theta-function decomposition.

4.4.3 Extensions to higher rank and speculations

In the following sections we want to discuss the extension of our results to higher rank. Partial results for rank three can be found already in the literature [104, 107, 121, 160, 175]. Thereafter, we discuss a possible generalization of mock modularity and speculate about a contour description which stems from a relation to a meromorphic Jacobi form.

Higher rank anomaly and mock modularity of higher depth

We want to focus on the holomorphic anomaly equation at general rank as conjectured in [132]. We recall that its form is given by

$$\mathcal{D}_r Z_P^{(r)}(\tau, z) \sim \sum_{n=1}^{r-1} n(r-n) Z_P^{(n)}(\tau, z) Z_P^{(r-n)}(\tau, z), \tag{4.147}$$

where $Z_P^{(r)}(\tau, z)$ can be decomposed into Siegel-Narain theta-functions as described in section 4.2.1. One may thus ask the question what it implies for the functions $\hat{f}_{\mu,J}^{(r)}(\tau)$ for general r. In order to extract this information we want to compare the coefficients in the theta-decomposition on both sides of (4.147). For this we need a generalization of the identity (4.121). A computation shows that

$$\theta_{\nu,J}^{(n)}(\tau, z)\,\theta_{\lambda,J}^{(r-n)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} c_{\nu\lambda}^{\mu}(\tau)\,\theta_{\mu,J}^{(r)}(\tau, z), \tag{4.148}$$

where $c_{\nu\lambda}^{\mu}$ are Siegel-Narain theta-functions themselves given by

$$c_{\nu\lambda}^{\mu}(\tau) = \delta_g(\mu) \sum_{\xi \in \Lambda + \mu + \frac{g}{r}(\nu - \lambda)} \bar{q}^{-\frac{rn(r-n)}{2g^2}\xi_+^2} q^{\frac{rn(r-n)}{2g^2}\xi_-^2} \tag{4.149}$$

with $g = \gcd(n, r-n)$ and $\delta_g(\mu)$ yields one if $r\mu$ is divisible by g and vanishes otherwise. With this input one finds

$$\partial_{\bar{\tau}} \hat{f}_{\mu,J}^{(r)}(\tau) \sim \sum_{n=1}^{r-1} n(r-n) \sum_{\nu,\lambda \in \Lambda^*/\Lambda} \hat{f}_{\nu,J}^{(n)}(\tau) \hat{f}_{\lambda,J}^{(r-n)}(\tau) c_{\nu\lambda}^{\mu}(\tau), \tag{4.150}$$

which sheds some light into the question about the modular properties of generating functions at higher rank as follows.

The structure of eq. (4.150) indicates, that an appropriate description of the generating function $\hat{f}_{\mu,J}^{(r)}$ needs a generalization of the usual notion of mock modularity. This results from the fact, that on the right hand side of the anomaly equation (4.150), mock modular forms appear, such that the shadow of $\hat{f}_{\mu,J}^{(r)}$ is a mock modular form itself. Therefore, it is also subject to a holomorphic anomaly equation. This would lead to the notion of mock modularity of higher depth [181], similar to the case of almost holomorphic modular forms of higher depth. These are functions like $\widehat{E}_2(\tau)$ and powers thereof, which can be written as a polynomial in τ_2^{-1} with coefficients being holomorphic functions.

A further motivation for this comes from the observation that the generating functions $\hat{f}_{\mu,J}^{(r)}$ could be obtained from an indefinite theta-function as in the case of two M5-branes. The lattice, however, that is summed over in these higher rank indefinite theta-functions will be of higher signature. In the case of r M5-branes one would expect a signature

$$(r-1, (r-1)(r(\Lambda) - 1)) \tag{4.151}$$

due to the $r-1$ relative D2-brane charges of the possible r decay products of D4-D2-D0 bound-states [119, 120]. However, a complete discussion of the modular properties of such functions and their relation to mock modular forms of depth is beyond the scope of this work. We would like to come back to this question in future research.

The contour description

The elliptic genus of r M5-branes wrapping P is denoted by $Z_P^{(r)}(\tau, z)$, where we don't indicate any dependence of $Z_P^{(r)}$ on a Kähler class/ chamber $J \in \mathcal{C}(P)$. The basic assumption is that the elliptic genus does not depend on such a choice. We simply think about $Z_P^{(r)}$ as being a *meromorphic* Jacobi form, which has poles as a function of the elliptic variable z. We assume, that it is of bi-weight $(-\frac{3}{2}, \frac{1}{2})$. In the following we want to exploit the implications of this statement.

It is known that a Jacobi form has an expansion into theta-functions with coefficients being modular forms. Since Zwegers [180], we also know that a meromorphic Jacobi form with one elliptic variable has a similar expansion, where the coefficients are mock modular. Using our Siegel-Narain theta-function $\theta_{\mu,J}^{(r)}(\tau, z)$, eq. (4.13), we conjecture the following expansion

$$Z_P^{(r)}(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} f_{\mu,J}^{(r)}(\tau)\theta_{\mu,J}^{(r)}(\tau, z) + \text{Res}, \tag{4.152}$$

with J a point in the Kähler cone which is related to a point $z_J \in \Lambda_{\mathbb{C}}$ where the decomposition is carried out. Note, that in eq. (4.152) the term "Res" should be given as a finite sum over the residues of $Z_P^{(r)}(\tau, z)$ in the fundamental domain $z_J + e\tau + e$ with $e = [0, 1]^{r(\Lambda)}$.

Let's see how the dependence on J comes about. Doing a Fourier transform we can write

$$f_{\mu,J}^{(r)}(\tau) = (-1)^{r\mu\cdot[P]} \bar{q}^{\frac{r}{2}\mu_+^2} q^{-\frac{r}{2}\mu_-^2} \int_{\mathcal{C}_J} Z_P^{(r)}(\tau, z) e^{-2\pi i r(\mu + \frac{[P]}{2})\cdot z} dz, \tag{4.153}$$

where $\mathcal{C}_J$ is a contour which has to be specified since $Z_P^{(r)}$ is meromorphic. Due to the periodicity in the elliptic variable $\mathcal{C}_J$ can be given as $z_J + e$ for some point z_J. Now, suppose we have a parallelogram $\mathcal{P} = z_J + e z_{J'} + e$ and that there is a single pole of $Z_P^{(r)}$ inside $\mathcal{P}$, say at $z = z_0$. Then, we obtain by integrating over the boundary of $\mathcal{P}$

$$f_{\mu,J}^{(r)}(\tau) - f_{\mu,J'}^{(r)}(\tau) = 2\pi i\, \alpha_\mu(\tau) \operatorname*{Res}_{z=z_0} \left(Z_P^{(r)}(\tau, z) e^{-2\pi i r(\mu + \frac{[P]}{2})\cdot z} \right), \tag{4.154}$$

where we abbreviate

$$\alpha_\mu(\tau) = (-1)^{r\mu\cdot[P]} \bar{q}^{\frac{r}{2}\mu_+^2} q^{-\frac{r}{2}\mu_-^2}. \tag{4.155}$$

That is, the coefficients of the Laurent expansion of the elliptic genus encode the jumping of the BPS numbers across walls of marginal stability and the walls are in one-to-one correspondence with the positions of the poles of $Z_P^{(r)}$. An analogous dependence on a contour of integration for wall-crossing of $\mathcal{N} = 4$ dyons was introduced in refs. [28, 150].

Moreover, the shadow of $f_{\mu,J}^{(r)}$ should be determined in terms of the residues of $Z_P^{(r)}$, since a generalizations of the ideas of [180] should show, that it is contained in the factor "Res" of eq. (4.152). Thus, combining this result with the interpretation of eq. (4.154) one expects, that the shadow not only renders $f_{\mu,J}^{(r)}$ modular, but also encodes the decay of bound-states and hence knows about the jumping of BPS invariants across walls of marginal stability.

It is tempting to speculate even further. When comparing our results to the case of dyon state counting in $\mathcal{N} = 4$ theories [32, 33] one might suspect that there is an analog of the Igusa cusp form ϕ_{10} in our setup. In the $\mathcal{N} = 4$ dyon case there are meromorphic Jacobi forms, often denoted ψ_m, which are summed up to give ϕ_{10}. In analogy, it may be useful to introduce another parameter $\rho \in \mathcal{H}$ and to study the object

$$\phi_P^{-1}(\tau, \rho, z) = \sum_{r \geq 1} Z_P^{(r)}(\tau, z) e^{2\pi i r \rho}. \tag{4.156}$$

Chapter 5

Conclusions

In this thesis the holomorphic anomaly and its interpretations in terms of background independence and wall-crossing has been investigated. The oppositeness of holomorphy and modularity allowed to perform high precision calculations. Especially, the relation between topological string theory and matrix models has been under scrutiny. In this framework building on [89], we have shown that the direct integration technique of the holomorphic anomaly equations provides a powerful tool to calculate the $1/N$ expansion of multi-cut matrix models. We have seen that, in some circumstances, we can easily fix the holomorphic ambiguity and obtain explicit expressions for the genus g amplitudes. In general, we expect the anomaly equation to be integrable, in the sense that the gap conditions completely fix the holomorphic ambiguity. In the case of the two-cut cubic matrix model on the slice $S_1 = -S_2$, we can use this method to determine the amplitudes to very high genus.

These high genus results have allowed us to obtain quantitative evidence for the connection between large order behavior and eigenvalue tunneling in a multi-cut matrix model. However, our results indicate that the detailed large genus asymptotics of the amplitudes cannot be understood just by considering the non-perturbative sectors associated with eigenvalue tunneling. Indeed, in a similar asymptotic problem analyzed in [67], it was necessary to include new non-perturbative sectors. It is only natural to suggest that a correct understanding of the asymptotic properties, in the multi-cut case, requires also the inclusion of new non-perturbative sectors. In the one-cut case and its double-scaling limit, the amplitudes in these new sectors can be obtained algebraically, as trans-series solutions to the pre-string equation and the Painlevé I equation, respectively. In the multi-cut case there is no analogue of these equations, and therefore the corresponding generalized amplitudes can not be computed with our present tools.

One obvious question is then the following: what is the interpretation of these new non-perturbative sectors in terms of matrix models or topological strings? We will give now some hints which might help in answering this question. Let us first discuss the trans-series solutions $u_{n|m}(z)$ appearing in (3.135). It turns out that $u_{0|\ell}(z)$ can be obtained from $u_{\ell|0}(z)$, the standard instanton amplitude, by changing the sign

$$z^{5/4} \to -z^{5/4}. \tag{5.1}$$

This corresponds to changing the sign of the string coupling constant $g_s \to -g_s$. If we think about the $u_{\ell|0}(z)$ as describing a set of ℓ D-branes, then the natural interpretation of $u_{0|\ell}(z)$ is as a set of ℓ *anti-D-branes*. Indeed, it has been argued that anti-D-branes are obtained from D-branes in topological string theory just by changing the sign of the string coupling constant [155]. More generally, these should be the ghost D-branes introduced in [140], which reduce to anti-D-branes in the topological string context. It is then natural to interpret the generalized instanton amplitude $u_{n|m}(z)$ as representing a state of n D-branes and m anti-D-branes at the unstable saddle, in the background of $N-n+m$ D-branes in the stable saddle. If this interpretation is correct, the generalized amplitudes in the multi-cut matrix model, which we labeled by two pairs of integers (N_1, M_1), (N_2, M_2), should correspond to a saddle where there are N_i branes and M_i anti-D-branes at the i-th critical point, $i = 1, 2$.

One problem with this interpretation is that, as argued in [45, 155], such a configuration is described in principle by a quiver or supergroup matrix model. If this is the case, the non-perturbative configuration characterized by (N_i, M_i), $i = 1, 2$, would be equivalent to a configuration with only branes or only antibranes at the critical points. More precisely, we would get $|N_i - M_i|$ branes or $|N_i - M_i|$ anti-branes depending on the sign of $N_i - M_i$. Since explicit calculations show that the amplitude $u_{n|m}(z)$ is not equal to the amplitude $u_{n-m|0}(z)$ [67], the interpretation in terms of brane/anti-brane systems might not be completely appropriate.

We believe that the appearance of these new sectors indicates that we do not fully understand the non-perturbative structure of matrix models and of two-dimensional gravity. Therefore, it would be very important to clarify their meaning and to compute their amplitudes in the multi-cut case.

In the second part of this thesis we investigated background dependence of theories that originate from r M5-branes wrapping a smooth (semi-)rigid divisor P in a Calabi-Yau threefold background. Such divisors P have $b_2^+ = 1$ and (semi-)positive anti-canonical class. In this case the wrapped M5-brane can be studied locally in the Calabi-Yau manifold using an effective description of the M5-brane theory on $P \times T^2$ by a twisted $U(r)$ $\mathcal{N} = 4$ Super-Yang-Mills theory on P.

The main object of interest was the partition function $Z_P^{(r)}$ of the twisted gauge theory and its modular and holomorphic properties. This partition function can be related to the modified elliptic genus of the $\mathcal{N} = (0, 4)$ sigma model description of the M5-brane. Using the spectral flow symmetry one establishes for all r a decomposition of the partition function into vector-valued modular forms $\hat{f}_{\mu,J}^{(r)}(\tau)$ w.r.t. the S-duality group of $\mathcal{N} = 4$ Super-Yang-Mills and Siegel-Narain theta-functions $\theta_{\mu,J}^{(r)}(\tau, z)$.

Our main result is a rigorous proof of a holomorphic anomaly equation of the partition function valid for rank two on all P described above. The proof in section 4.3.3 relies on the large volume wall-crossing formula of Göttsche [73] for invariants associated to sheaves on P, which are related to integer BPS invariants. By summing the change of the invariants across all intermediate walls one can express the difference of the generating function of the invariants $f_{\mu,J}^{(r)}(\tau)$ in two arbitrary chambers J and J' in the Kähler cone in terms of an indefinite theta-function $\Theta_{\Lambda,\mu}^{J,J'}(\tau, z)$ [71]. This theta-function is regularized by cutting out the negative directions of the quadratic form on the homology lattice, a procedure which renders the result in general not modular. The spoiled S-duality invariance can be regained following the work of Zwegers by smoothing out the cutting procedure with the error function depending non-holomorphically on τ. The non-holomorphicity introduced by this procedure completes the mock modular forms $f_{\mu,J}^{(r)}(\tau)$ to non-holomorphic modular forms $\hat{f}_{\mu,J}^{(r)}(\tau)$. The non-holomorphicity of the Siegel-Narain theta-functions on the other hand is trivial since it is annihilated by the non-holomorphic heat operator. This allows to write a concise holomorphic anomaly equation for the partition function (4.118) when evaluated at the attractor point [122].

We check this holomorphic anomaly equation and its implications for the counting of invariants of sheaves on $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$ and $\mathcal{B}_8$ in section 4.4.2. The anomaly equation (4.118) is in particular compatible with the form of a holomorphic anomaly that has been conjectured in the context of E-strings on $\frac{1}{2}$K3 for all r and checked for certain classes using the duality to the genus zero topological string partition function [132]. Since the non-holomorphicity of the $\hat{f}^{(r)}_{\mu,J}(\tau)$ for $r > 1$ is related in an intriguing way to mock modularity and wall-crossing, we analyzed the decomposition for arbitrary rank and give a general form of the conjectured general anomaly equation at the level of the $\hat{f}^{(r)}_{\mu,J}(\tau)$ in equation (4.150), which indicates a theory of mock modular forms of higher depth [181]. The holomorphic limit of the $\hat{f}^{(r)}_{\mu,J}(\tau)$ yield generating functions for invariants associated to sheaves of rank r. However, it is in general difficult to provide boundary conditions, which fix the holomorphic ambiguity.

The wall-crossing of Göttsche, which induces in the steps described above the non-holomorphicity of the $\hat{f}^{(2)}_{\mu,J}(\tau)$, can be rederived using the Kontsevich-Soibelman wall-crossing formula, as we did in section 4.3.2. As the wall-crossing formula takes a primitive form at rank two, one can rewrite the generating function of BPS differences in terms of an indefinite theta-function. The Kontsevich-Soibelman formula can be used for arbitrary rank to determine the counting functions $f^{(r)}_{\mu,J}(\tau)$ for all sectors μ in all chambers, if it is known in one chamber for all μ, e.g. by a vanishing lemma or use of the blow-up formula. This was studied for rank three by [121], where it was also shown that the rank three wall-crossing formula is primitive. In general if the wall-crossing formula is primitive, the sum over walls induce lattice sums of signature $(r-1)(b_2^+, b_2^-)$ with similar regularization requirements as for the rank two case. It is an interesting question if the program of Zwegers to build modular objects can be extended to the higher rank situation and leads upon non-holomorphic modular completion to the conjectured form of the holomorphic anomaly equation and a precise notion of the mock modular forms of higher depth.

The problem of providing boundary conditions at least in one chamber for the del Pezzo surfaces (except for the Hirzebruch surface $\mathbb{F}_0$) can in principle be solved by using the blow-up formula in both directions in connection with the wall-crossing formula before and after the blow-up. However, the blow-up formula in the literature apply only if r and $c_1 \cdot J$ have no common divisor. This restriction forbids in general to provide boundary conditions for all sectors.

The higher genus information discussed in equation (4.33) gives finer information about the cohomology of moduli spaces of sheaves than its Euler number. Namely, an elliptic genus obtained by tracing over the right j_R^3 quantum numbers of the Lefshetz decomposition in the cohomology of the moduli space. On rigid surfaces it can be further refined to include the general Ω background parameters of Nekrasov [138], which capture the individual (j_L^3, j_R^3) quantum numbers [93]. For rank two such refined partition functions have been considered in [74] and it should be possible to extend the consideration above to the refined BPS numbers, see e.g. [121,122]. Furthermore, the relation between D6-D2-D0 brane systems as counted by topological string theory and D4-D2-D0 brane systems associated to black hole state counting is the hallmark of the OSV conjecture [143], which has been intensively studied. Wall-crossing issues in combination with this conjecture have been studied in ref. [38] and more recently from an M-theory perspective for example in ref. [2]. It would be interesting to examine the implications of the anomaly equation in these contexts.

A conceptually very interesting but at this point more speculative approach is to consider the elliptic genus as a J independent meromorphic Jacobi form, as we did in section 4.4.3. As shown by Zwegers such meromorphic Jacobi forms have an expansion in theta-functions whose coefficients are mock modular forms, just as holomorphic Jacobi forms have an expansion in theta-functions with holomorphic modular forms as coefficients. This formalism relates the changes in the BPS numbers across walls of marginal stability to the different choices of the contour in the definition of $f^{(r)}_{\mu,J}(\tau)$ as a Fourier integral of $Z^{(r)}_P$, i.e. to the poles in $Z^{(r)}_P$, like in the $\mathcal{N} = 4$ case [28, 35].

Appendix A

Modularity

A.1 Elliptic modular forms

Let us collect the definitions of various modular forms appearing in the main body text. We denote the following standard theta-functions by

$$\begin{aligned}
\vartheta_1(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_2(\tau,\nu) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_3(\tau,\nu) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi i n\nu}, \\
\vartheta_4(\tau,\nu) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n\nu}.
\end{aligned} \tag{A.1}$$

In the case that $\nu = 0$ we simply denote $\vartheta_i(\tau) = \vartheta_i(\tau, 0)$ (notice that $\vartheta_1(\tau) = 0$). Under modular transformations the theta functions $\vartheta_i(\tau)$ behave as vector-valued modular forms of weight $\frac{1}{2}$. They transform as

$$\vartheta_2(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_4(\tau), \qquad \vartheta_2(\tau+1) = e^{\frac{i\pi}{4}}\vartheta_2(\tau), \tag{A.2}$$

$$\vartheta_3(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_3(\tau), \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau), \tag{A.3}$$

$$\vartheta_4(-1/\tau) = \sqrt{\frac{\tau}{i}}\vartheta_2(\tau), \qquad \vartheta_4(\tau+1) = \vartheta_3(\tau). \tag{A.4}$$

Further, the eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \tag{A.5}$$

and transforms according to

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau). \tag{A.6}$$

The Eisenstein series are defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \frac{n^{k-1}q^n}{1-q^n}, \tag{A.7}$$

where B_k denotes the k-th Bernoulli number. E_k is a modular form of weight k for $k > 2$ and even.

A.2 Mock modular forms

Following [178], we denote the space of mock modular forms of weight k by $\mathbb{M}_k$ and the space of modular forms by M_k. Mock modular forms are holomorphic functions of τ, which is an element of the upper half plane $\mathcal{H}$, but do not transform in a modular covariant way. However, to every mock modular form h of weight k there exists a shadow $g \in M_{2-k}$ such that the function $\hat{h}$, given by

$$\hat{h}(\tau) = h(\tau) + g^*(\tau) \tag{A.8}$$

transforms as of weight k. Denoting by $g^c(z) = \overline{g(-\bar{z})}$, the completion $g^*(\tau)$ is defined by

$$g^*(\tau) = -(2i)^k \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} g^c(z)\, dz. \tag{A.9}$$

Thus, $\hat{h}$ is modular but has a non-holomorphic dependence. The corresponding space containing forms of type (A.8) is denoted by $\widehat{\mathbb{M}}_k$. Given g as the expansion $g(\tau) = \sum_{n\geq 0} b_n q^n$, the completion $g^*(\tau)$ can also be written as

$$g^*(\tau) = \sum_{n\geq 0} n^{k-1} \overline{b}_n\, \beta_k(4n\tau_2)\, q^{-n}, \tag{A.10}$$

with $\tau_2 = \mathrm{Im}\,(\tau)$ and β_k defined by

$$\beta_k(t) = \int_t^{\infty} u^{-k} e^{-\pi u} du. \tag{A.11}$$

Conversely, given $\hat{h}$, one determines the shadow g by taking the derivative of $\hat{h}$ with respect to $\bar{\tau}$. One easily sees that

$$\frac{\partial \hat{h}}{\partial \bar{\tau}} = \frac{\partial g^*}{\partial \bar{\tau}} = \tau_2^{-k} \overline{g(\tau)}. \tag{A.12}$$

This viewpoint opens another characterization of $\widehat{\mathbb{M}}_k$ as the set of real-analytic functions F that fulfill a certain differential equation. To be precise, let us define the space $\mathfrak{M}_k$ as the space of real-analytic functions F in the upper half-plane $\mathcal{H}$ transforming as a modular form under $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, i.e.

$$F(\gamma\tau) = \rho(\gamma)(c\tau + d)^k F(\tau),$$

where $\rho(\gamma)$ denotes some character of Γ and we demand exponential growth at the cusps. Hence, the space of completed mock modular forms $\widehat{\mathbb{M}}_k$ can now be characterized by

$$\widehat{\mathbb{M}}_k = \left\{ F \in \mathfrak{M}_k \,\middle|\, \frac{\partial}{\partial \tau}\left(\tau_2^k \frac{\partial F}{\partial \bar{\tau}}\right) = 0 \right\}. \tag{A.13}$$

This definition induces the following maps[1]

$$\mathfrak{M}_k = \mathfrak{M}_{k,0} \xrightarrow{\tau_2^k \partial_{\bar{\tau}}} \mathfrak{M}_{0,2-k} \xrightarrow{\tau_2^{2-k}\partial_\tau} \mathfrak{M}_{k,0} = \mathfrak{M}_k, \tag{A.14}$$

[1] A function $f \in \mathfrak{M}_{k,l}$ transforms under modular transformations $\gamma \in \Gamma$ with bi-weight (k,l) and character ρ, i.e. $f(\gamma\tau) = \rho(\gamma)(c\tau+d)^k(c\bar{\tau}+d)^l f(\tau)$.

so that the composition can be converted to the Laplace operator in weight k. Hence, mock modular forms in $\widehat{\mathbb{M}}_k$ have the special eigenvalue $\frac{k}{2}\left(1-\frac{k}{2}\right)$ and are sometimes also called harmonic weak Maass forms.

Zwegers showed in [180] that mock modular forms can be realized in three different ways, namely either as Appell-Lerch sums, indefinite theta-series or as Fourier coefficients of meromorphic Jacobi forms. Further, there is a notion of mixed mock modular forms, which are functions that transform in the tensor space of mock modular forms and modular forms. However, we will call them simply mock modular forms as well.

In the following a simple example of a mock modular form is presented.

Example: E_2 as a mock modular form

The modular completion of the holomorphic Eisenstein series E_2 has the form

$$\widehat{E_2}(\tau) = E_2(\tau) - \frac{3}{\pi\tau_2}\,.$$

From $\partial_{\overline{\tau}}\widehat{E}_2 = \tau_2^{-2}\frac{3i}{2\pi}$ we get $\overline{g} = \frac{3i}{2\pi}$, a constant shadow. Doing the integral indeed yields

$$g^*(\tau) = -(2i)^2\int_{-\overline{\tau}}^{\infty}(z+\tau)^{-2}\frac{3i}{2\pi}dz = -\frac{6i}{\pi}\left[\frac{-1}{z+\tau}\right]_{-\overline{\tau}}^{\infty} = -\frac{3}{\pi\tau_2}. \tag{A.15}$$

A.3 Elliptic integrals

We follow the conventions in [22]. The complete elliptic integral of the first kind is defined as

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{A.16}$$

The parameter k is called the elliptic modulus. Further one defines the complementary modulus as $k'^2 = 1-k^2$. The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^1 \mathrm{d}t\sqrt{\frac{1-k^2t^2}{1-t^2}}. \tag{A.17}$$

The complete elliptic integrals of the first and second kind are related to each other by derivation,

$$\frac{\mathrm{d}K}{\mathrm{d}k} = \frac{E(k)-k'^2K(k)}{kk'^2}, \qquad \frac{\mathrm{d}E}{\mathrm{d}k} = \frac{E(k)-K(k)}{k}. \tag{A.18}$$

Useful transformation formulae are

$$\begin{aligned} K\left(\frac{1-k'}{1+k'}\right) &= \frac{1+k'}{2}K(k), \\ E\left(\frac{1-k'}{1+k'}\right) &= \frac{1}{1+k'}(E(k)+k'K(k)), \\ K\left(\frac{2\sqrt{k}}{1+k}\right) &= (1+k)K(k), \end{aligned} \tag{A.19}$$

as well as the Legendre relation

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}. \tag{A.20}$$

Consider an elliptic geometry of the form

$$y^2 = \prod_{i=1}^{4}(x - x_i), \tag{A.21}$$

where $x_1 < x_2 < x_3 < x_4$ are the branch cuts. Define the half-period ratio of the elliptic geometry, τ, and the elliptic nome as $q = e^{\mathrm{i}\pi\tau}$. It can be shown that

$$\tau = \mathrm{i}\frac{K(k')}{K(k)}, \tag{A.22}$$

and moreover that

$$K(k) = \frac{\pi}{2}\vartheta_3^2, \quad k^2 = \frac{\vartheta_2^4}{\vartheta_3^4}, \quad k'^2 = \frac{\vartheta_4^4}{\vartheta_3^4}. \tag{A.23}$$

The Thomae formula relates the branch cuts of an elliptic curve to theta-functions [58]. For the geometry consider above (A.21) we obtain

$$\begin{aligned} \vartheta_2^4(\tau) &= -\mathcal{K}^2(x_1 - x_2)(x_3 - x_4) \\ \vartheta_3^4(\tau) &= -\mathcal{K}^2(x_1 - x_4)(x_2 - x_3) \\ \vartheta_4^4(\tau) &= -\mathcal{K}^2(x_1 - x_3)(x_2 - x_4), \end{aligned} \tag{A.24}$$

and thus

$$\eta^{24}(\tau) = \frac{\mathcal{K}^{12}}{256}\prod_{i<j}(x_i - x_j)^2, \tag{A.25}$$

where $\mathcal{K}$ and $\mathcal{K}'$ are given in (3.53).

It is convenient to introduce

$$b = \vartheta_2^4, \quad c = \vartheta_3^4, \quad d = \vartheta_4^4, \tag{A.26}$$

where either two of them span the ring of $\Gamma(2)$ modular forms. Here the congruence subgroup $\Gamma(2) \subset \mathrm{SL}(2,\mathbb{Z})$ is defined by

$$\Gamma(2) = \{\gamma \in \mathrm{SL}(2,\mathbb{Z}) \,|\, \gamma \equiv \mathbb{1} \bmod 2\}. \tag{A.27}$$

Appendix B

Calabi-Yau spaces and its divisors

B.1 Rigid divisors

We start with some facts about complex surfaces. The Riemann Roch formula relates the signature σ and arithmetic genus χ_0 to Chern class integrals

$$\sigma = \sum_i (b_{2i}^+ - b_{2i}^-) = \frac{1}{3}\int_P (c_1^2 - 2c_2), \qquad \chi_0 = \sum_i (-1)^i h_{i,0} = \frac{1}{12}\int_P (c_1^2 + c_2). \tag{B.1}$$

Regarding the embedding one has the distinction whether P is very ample or not, i.e. if the line bundle $\mathcal{L}_P$ is generated by its global sections or not. In the former case P has $h^0(X, \mathcal{L}_P) - 1$ deformations and there exists an embedding $j : X \to \mathbb{P}^n_P$ so that $\mathcal{L}_P = j^*(\mathcal{O}(1))$, i.e. P can be described by some polynomial. This situation has been considered in [114], where the deformations and b^+, b^- have been given. Generically one has $h^{2,0}(P) = \frac{1}{2}(b_2^+ - 1)$, which is positive in the very ample case.

In this work we consider mainly rigid smooth divisors. In this case one has no deformations and locally the Calabi-Yau manifold can be written as the total space of the canonical line bundle $\mathcal{O}(K_P) \to P$ and the latter can be globalized to a elliptic fibration over P, see section B.2, for $P = \mathbb{F}_n$. In this case $\Lambda_P = \Lambda$, compare sec. 4.2.1.

As X is a Calabi-Yau manifold and to allow no section, P has to have a positive $D^2 > 0$ anti-canonical divisor class $D = -K_P$, which is also required to be nef, i.e. $D.C \geq 0$ for any irreducible curve C. This defines a weak del Pezzo surface. If $D.C > 0$, then D is ample and P is a del Pezzo surface [4]. Del Pezzo surfaces are either $\mathcal{B}_n$, which are blow-ups of $\mathbb{P}^2$ in $n \leq 8$ points or $\mathbb{P}^1 \times \mathbb{P}^1$. We can also allow the Hirzebruch surface $\mathbb{F}_2$ which is weak del Pezzo.

As $h_{1,0} = h_{2,0} = 0$ one has $\chi_0(\mathcal{B}_n) = 1$ for all surfaces under consideration. As the Euler number $\chi(\mathcal{B}_n) = 3 + n$ one has by (B.1) that $\int_P c_1^2 = 9 - n$, which implies that $n = 9$ is the critical case for positive anti-canonical class, and $(b_2^+, b_2^-) = (1, n)$. The case $n = 9$ is called $\frac{1}{2}$K3. We include this semi-rigid situation.

In more detail the homology of $\mathcal{B}_n$ is generated by the hyperplane class h of $\mathbb{P}^2$ and the exceptional divisors of the blow-ups e_i, with the non-vanishing intersections $h^2 = 1 = -e_i^2$. The anti-canonical class is given by $-K_{\mathcal{B}_n} = 3h - \sum_{i=1}^n e_i$. Defining the lattice generated by this element in $H_2(P, \mathbb{Z})$ as $\mathbb{Z}_{K_{\mathcal{B}_n}}$ and $E_n^* = (\mathbb{Z}_{K_{\mathcal{B}_n}})^\perp$ one sees that E_1^* is trivial and E_n^* are the lattices of the Lie algebras $(A_1, A_1 \times A_2, A_4, D_5, E_6, E_7, E_8)$ for $n = 2, \ldots, 8$. The corresponding basis in terms of (h, e_i) is worked out in [4]. The homology lattice for B_9 is $\Gamma^{1,1} \oplus E_8$, where $\Gamma^{1,1}$ is the hyperbolic lattice with standard metric.

In order to study topological string theory in Calabi-Yau backgrounds realized in simple toric ambient spaces, one has to consider situations in which $\Lambda \subset \Lambda_P$, which is the case for the $\frac{1}{2}$K3 realized in the toric ambient space discussed in the next section.

B.2 An elliptically fibered Calabi-Yau space

Let X be an elliptic fibration over $\mathbb{F}_n$ for $n = 0, 1, 2$ given by a generic section of the anti-canonical bundle of the ambient spaces specified by the following vertices

$$D_0 = (0,0,0,0), \quad D_1 = (0,0,0,1), \quad D_2 = (0,0,1,0), \quad D_3 = (0,0,-2,-3)$$
$$D_4 = (0,-1,-2,-3), \quad D_5 = (0,1,-2,-3), \quad D_6 = (1,0,-2,-3), \quad D_7 = (-1,-n,-2,-3).$$

One finds large volume phases with the following Mori-vectors

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	
$l^1 =$	-6	3	2	1	0	0	0	0	C^1
$l^2 =$	0	0	0	-2	1	1	0	0	C^2
$l^3 =$	0	0	0	$n-2$	$-n$	0	1	1	C^3.

We choose a basis $\{C^A, A = 1,2,3\}$ of $H_2(X,\mathbb{Z})$. Let K_A be a Poincaré dual basis of the Chow group of linearly independent divisors of X, i.e. $\int_{C^A} K_B = \delta^A_B$. The divisors $D_i = l_i^A K_A$ have intersections with the cycles C^A given by $D_i.C^A = l_i^A$. We have the following non-vanishing intersections of the divisors given by

$$K_1.K_2.K_3 = 1, \quad K_1.K_2^2 = n, \quad K_1^2.K_2 = n+2, \quad K_1^2.K_3 = 2, \quad K_1^3 = 8. \tag{B.2}$$

The divisor giving the Hirzebruch surface inside the Calabi-Yau manifold corresponds to

$$[\mathbb{F}_n] = D_3 = K_1 - 2K_2 - (2-n)K_3. \tag{B.3}$$

Thus, the metric on $H^2(\mathbb{F}_n,\mathbb{Z})$ coming from the intersections in the Calabi-Yau manifold is

$$(K_A.K_B.[\mathbb{F}_n]) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{B.4}$$

Projecting out the direction corresponding to the elliptic fiber we reduce the problem to the Hirzebruch surface itself. We denote by $F = K_3$ and $B = K_2 - nK_3$ the class of the fiber and base, respectively. Thus, the canonical class reduces to $[\mathbb{F}_n] = -(2+n)F - 2B$. The intersection numbers are given as follows

$$\begin{pmatrix} F.F & F.B \\ B.F & B.B \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}. \tag{B.5}$$

Hence, the Kähler cone is spanned by the two vectors F and $2B + nF$, i.e.

$$\mathcal{C}(\mathbb{F}_n) = \{J \in H^2(\mathbb{F}_n,\mathbb{R}) \,|\, J = t_1 F + t_2(2B+nF),\, t_1, t_2 > 0\}. \tag{B.6}$$

For $n = 1$ the geometry admits also an embedding of a K3 and a $\mathcal{B}_9$ surface. K3 is given by the elliptical fiber E and the fiber of the Hirzebruch surface F. $\mathcal{B}_9$, sometimes also called $\frac{1}{2}$K3, is given by the elliptical fiber E and the base of the Hirzebruch surface B. The classical triple intersections, the integrals involving the second Chern class and further topological data of the Calabi-Yau threefold X for $n = 1$ are

$$\mathcal{R} = 8J_E^3 + 3J_E^2 J_F + J_E J_F^2 + 2J_E^2 J_B + J_E J_B J_F, \tag{B.7}$$

$$c_2(X)J_E = 92, \quad c_2(X)J_F = 36, \quad c_2(X)J_B = 24, \tag{B.8}$$

$$h^{1,1}(X) = 3, \quad h^{2,1}(X) = 243, \quad \chi(X) = -480. \tag{B.9}$$

The mirror geometry of X is referred to as Y. The coordinates of the complex structure moduli space are called z_i. Then, denoting by $\theta_i = z_i \partial_{z_i}$, the Picard-Fuchs system governing the periods of the mirror geometry Y reads

$$\begin{aligned}
\mathcal{L}_1 &= \theta_1(\theta_1 - 2\theta_3 - \theta_2) - 12z_1(6\theta_1 + 5)(6\theta_1 + 1) \\
\mathcal{L}_2 &= \theta_2^2 - z_2(\theta_2 - \theta_3)(2\theta_3 + \theta_2 - \theta_1) \\
\mathcal{L}_3 &= \theta_3(\theta_3 - \theta_2) - z_3(2\theta_3 + \theta_2 - \theta_1)(2\theta_3 + \theta_2 - \theta_1 + 1).
\end{aligned} \tag{B.10}$$

Appendix C

Results

C.1 Data of the two-cut example

In the following we collect the necessary data for our two-cut cubic model of the main body text. We restrict ourselves to the points in moduli space which are relevant for our discussion. For further background on e.g. the monodromy around several divisors in moduli space we refer the reader to [89].

C.1.1 Large Radius

$C_1 \cap C_2 = \{z_1 = 0\} \cap \{z_2 = 0\}$:
The Picard-Fuchs operators governing the periods of the cubic matrix model are given by

$$\begin{aligned}
\mathcal{L}_1 =&(3-2z_1-6z_2)\partial_1 - 2z_1(1-2z_1-6z_2)\partial_1^2 + (1-10z_1+12z_1^2+4z_1z_2)\partial_1\partial_2\\
&+(3-6z_1-2z_2)\partial_2 + (1-10z_2+4z_1z_2+12z_2^2)\partial_1\partial_2 - 2z_2(1-6z_1-2z_2)\partial_2^2,\\
\mathcal{L}_2 =&-3(1-12z_1+18z_1^2+14z_1z_2) + (-3z_2(1-3z_2+2z_2^2)\\
&+z_1(7+46z_1^2-18z_2+26z_2^2+z_1(-39+62z_2)))\partial_1\\
&+(-1+2z_1+2z_2)(-2z_1(1+5z_1^2-2z_1z_2-3z_2^2-4(z_1+z_2))\partial_1^2\\
&+(z_1+z_2)(1-8z_1+6z_1^2-6z_1z_2)\partial_1\partial_2)\\
&-3(1-12z_2+14z_1z_2+18z_2^2)+(-3z_1(1-3z_1+2z_1^2)\\
&+z_2(7-18z_1+26z_1^2+(-39+62z_1)z_2+46z_2^2))\partial_2\\
&+(-1+2z_1+2z_2)((z_1+z_2)(1-8z_2-6z_1z_2+6z_2^2)\partial_1\partial_2\\
&-2z_2(1-3z_1^2-2z_1z_2+5z_2^2-4(z_1+z_2))\partial_2^2).
\end{aligned} \tag{C.1}$$

Its discriminant can be determined to be

$$\text{disc} = z_1z_2I^2J = z_1z_2(1-2(z_1+z_2))(1-6z_1-6z_2+9z_1^2+14z_1z_2+9z_2^2), \tag{C.2}$$

and its solutions around $z_i = 0$, $i = 1, 2$, are given by the following expansions

$$\begin{aligned}
S_1 &= \frac{z_1}{4} - \frac{1}{8}z_1(2z_1+3z_2) + \dots\\
S_2 &= -\frac{z_1}{4} + \frac{1}{8}z_2(3z_1+2z_2) + \dots\\
\Pi_1 &= S_1 \log\left(\frac{z_1}{4}\right) + \frac{1}{12} - \frac{z_1}{4} - \frac{1}{16}(2z_1^2-10z_1z_2-5z_2^2) + \dots\\
\Pi_2 &= S_2 \log\left(-\frac{z_2}{4}\right) - \frac{1}{12} + \frac{z_2}{4} - \frac{1}{16}(5z_1^2+10z_1z_2-2z_2^2) + \dots.
\end{aligned} \tag{C.3}$$

The Yukawa couplings are given by

$$\begin{aligned}
C_{z_1z_1z_1} &= \frac{1-6z_1+9z_1^2-5z_2+9z_1z_2+6z_2^2}{16z_1I^2}\\
C_{z_1z_1z_2} &= \frac{1-3z_1-5z_2}{16I^2}\\
C_{z_1z_2z_2} &= \frac{1-5z_1-3z_2}{16I^2}\\
C_{z_2z_2z_2} &= \frac{1-5z_1+6z_1^2-6z_2+9z_1z_2+9z_2^2}{16z_2I^2},
\end{aligned} \tag{C.4}$$

where all other combinations follow by symmetry. The genus one free energy can be written as

$$F_1 = -\frac{1}{2}\log\left(\det(G_{i\bar{j}})\right) - \frac{1}{12}\log(z_1z_2) - \frac{1}{2}\log I + \frac{1}{3}\log J. \tag{C.5}$$

It is convenient to introduce new variables $\tilde{z}_i$, $i=1,2$, by

$$\tilde{z}_1 = z_1+z_2, \quad \tilde{z}_2 = \frac{1}{4}(z_1-z_2)\sqrt{1-2(z_1+z_2)}, \tag{C.6}$$

as well as coordinates $\tilde{t}_i$, $i=1,2$, on the mirror by

$$\tilde{t}_1 = s = \frac{1}{2}(S_1-S_2), \quad \tilde{t}_2 = t = S_1+S_2, \tag{C.7}$$

such that the mirror map becomes as simple as possible. E.g. we have that

$$\tilde{z}_2 = \tilde{t}_2. \tag{C.8}$$

This implies that some of the Christoffel symbols vanish:

$$\Gamma^{\tilde{z}_2}_{\tilde{z}_i\tilde{z}_j} = 0, \quad \text{for} \quad i=1,2. \tag{C.9}$$

There are only four non-vanishing ambiguities $\tilde{f}^k_{ij}$ of equation (3.13), that are given by

$$\begin{aligned}
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_1\tilde{z}_1} &= -\frac{5-28\tilde{z}_1+52\tilde{z}_1^2-32\tilde{z}_1^3-112\tilde{z}_2^2}{2(1-2\tilde{z}_1)(1-8\tilde{z}_1+20\tilde{z}_1^2-16\tilde{z}_1^3+16\tilde{z}_2^2)},\\
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_1\tilde{z}_2} &= \frac{24\tilde{z}_2}{1-8\tilde{z}_1+20\tilde{z}_1^2-16\tilde{z}_1^3+16\tilde{z}_2^2},\\
\tilde{f}^{\tilde{z}_1}_{\tilde{z}_2\tilde{z}_2} &= \frac{8-16\tilde{z}_1}{1-8\tilde{z}_1+20\tilde{z}_1^2-16\tilde{z}_1^3+16\tilde{z}_2^2}.
\end{aligned} \tag{C.10}$$

This results in a propagator that has one non-vanishing component in $\tilde{z}$-coordinates, i.e.

$$S^{\tilde{z}_1\tilde{z}_1} = 4\tilde{z}_1^2 - 64\tilde{z}_2^2 + 44\tilde{z}_1^3 - 832\tilde{z}_1\tilde{z}_2^2 + \ldots, \qquad S^{\tilde{z}_1\tilde{z}_2} = S^{\tilde{z}_2\tilde{z}_1} = S^{\tilde{z}_2\tilde{z}_2} = 0. \tag{C.11}$$

The covariant derivative closes on this propagator when one fixes yet another ambiguity f^{ij}_k, cf. eq. (3.12). The only relevant, non-vanishing component is given by

$$f^{\tilde{z}_1\tilde{z}_1}_{\tilde{z}_1} = \frac{8(1-2\tilde{z}_1)^3(\tilde{z}_1-4\tilde{z}_1^2+4\tilde{z}_1^3-64\tilde{z}_2^2+144\tilde{z}_1\tilde{z}_2^2)}{(1-8\tilde{z}_1+20\tilde{z}_1^2-16\tilde{z}_1^3+16\tilde{z}_2^2)^3}. \tag{C.12}$$

C.1.2 Conifold

Conifold $J = \{1 - 6z_1 - 6z_2 + 9z_1^2 + 14z_1z_2 + 9z_2^2 = 0\}$:
We consider the point $(z_1, z_2) = (\frac{1}{8}, \frac{1}{8}) \in J$. Convenient coordinates are given by

$$z_{c,1} = \frac{1}{\sqrt{2}}(z_1 - z_2), \qquad z_{c,2} = 1 - 4(z_1 + z_2). \tag{C.13}$$

$z_{c,1}$ parametrizes the tangential direction to the conifold divisor, whereas $z_{c,2}$ the normal one. Transforming the Picard-Fuchs system to these new coordinates the polynomial solutions are given by

$$\begin{aligned} \omega_1 &= z_{c,1}\sqrt{1 + z_{c,2}} = z_{c,1} + \frac{1}{2}z_{c,1}z_{c,2} - \frac{1}{8}z_{c,1}z_{c,2}^2 + \mathcal{O}(z_c^4), \\ \omega_2 &= z_{c,2}^2 + 8z_{c,1}^2 z_{c,2} + \mathcal{O}(z_c^4). \end{aligned} \tag{C.14}$$

We choose as flat coordinates

$$t_{c,i} = \omega_i, \quad i = 1, 2. \tag{C.15}$$

By Inverting the above relations it is easy to calculate the holomorphic limit of the metric and the Christoffel symbols in z_c coordinates. Transforming the Yukawa couplings C_{ijk} as well as the ambiguities $\tilde{f}_{ij}^k$ yields the propagator at the conifold point. This allows now to expand the free energies F_g in the holomorphic limit at the conifold point.

C.2 Modular properties of the elliptic genus

We denote by $Z_P^{(r)}(\tau, z)$ the elliptic genus of r M5-branes wrapping P as defined previously in sect. 4.2.1. The elliptic genus should transform like a Jacobi form of bi-weight $(-\frac{3}{2}, \frac{1}{2})$ and bi-index $(\frac{r}{2}(d_{AB} - \frac{J_AJ_B}{J^2}), \frac{r}{2}\frac{J_AJ_B}{J^2})$ under the full modular group. In particular, we impose

$$\begin{aligned} Z_P^{(r)}(\tau + 1, z) &= \varepsilon(T)\, Z_P^{(r)}(\tau, z), \\ Z_P^{(r)}(-\frac{1}{\tau}, \frac{z_-}{\tau} + \frac{z_+}{\bar{\tau}}) &= \varepsilon(S)\, \tau^{-\frac{3}{2}}\bar{\tau}^{\frac{1}{2}} e^{\pi i r(\frac{z_-^2}{\tau} + \frac{z_+^2}{\bar{\tau}})}\, Z_P^{(r)}(\tau, z), \end{aligned} \tag{C.16}$$

where ε are certain phases [118].

Siegel-Narain theta-function and its properties

Let us start by recalling the definition of the Siegel-Narain theta-function of eq. (4.13)

$$\theta_{\mu,J}^{(r)}(\tau, z) = \sum_{\xi \in \Lambda + \frac{[P]}{2}} (-)^{r(\xi+\mu)\cdot[P]}\bar{q}^{-\frac{r}{2}(\xi+\mu)_+^2} q^{\frac{r}{2}(\xi+\mu)_-^2} e^{2\pi i r(\xi+\mu)\cdot z}, \tag{C.17}$$

where we define

$$\xi_+^2 = \frac{(\xi \cdot J)^2}{J \cdot J}, \quad \xi_-^2 = \xi^2 - \xi_+^2. \tag{C.18}$$

Note, that $\xi_+^2 < 0$ if J lies in the Kähler cone.

If we denote by $\mathcal{D}_k = \partial_{\bar{\tau}} + \frac{i}{4\pi k}\partial_{z_+}^2$, the theta-function fulfills the heat equation

$$\mathcal{D}_r\, \theta_{\mu,J}^{(r)}(\tau, z) = 0. \tag{C.19}$$

Further, we denote by Λ^* the dual lattice to Λ w.r.t. the metric rd_{AB}. For $\mu \in \Lambda^*/\Lambda$, we can deduce the following set of transformation rules

$$\begin{aligned} \theta^{(r)}_{\mu,J}(\tau+1,z) &= (-1)^{r(\mu+\frac{[P]}{2})^2}\theta^{(r)}_{\mu,J}(\tau,z),\\ \theta^{(r)}_{\mu,J}(-\frac{1}{\tau},\frac{z_+}{\bar\tau}+\frac{z_-}{\tau}) &= \frac{(-1)^{r\frac{[P]^2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}}(-i\tau)^{\frac{r(\Lambda)-1}{2}}(i\bar\tau)^{\frac{1}{2}}e^{\pi i r(\frac{z_-^2}{\tau}+\frac{z_+^2}{\bar\tau})}\sum_{\delta\,\in\,\Lambda^*/\Lambda}e^{-2\pi i r\mu\cdot\delta}\theta^{(r)}_{\delta,J}(\tau,z). \end{aligned} \tag{C.20}$$

Rank one

At rank one we have the universal answer

$$f^{(1)}_{\mu,J}(\tau) = \frac{\vartheta_{\Lambda^\perp}(\tau)}{\eta(\tau)^\chi}. \tag{C.21}$$

The transformation rules are simply given by (A.6) for the eta-function and for $\vartheta_{\Lambda^\perp}$ we obtain (assuming $\Lambda^\perp$ even and self-dual)

$$\begin{aligned} \vartheta_{\Lambda^\perp}(\tau+1) &= \vartheta_{\Lambda^\perp}(\tau),\\ \vartheta_{\Lambda^\perp}(-\frac{1}{\tau}) &= \left(\frac{\tau}{i}\right)^{\frac{r(\Lambda^\perp)}{2}}\vartheta_{\Lambda^\perp}(\tau). \end{aligned} \tag{C.22}$$

Rank two

Using Zwegers' theta-function with characteristics $\vartheta^{c,c'}_{a,b}(\tau)$ given in def. 2.1 of his thesis [180], we can write

$$\widehat\Theta^{c,c'}_{\Lambda,\mu}(\tau,x) = q^{-\frac{1}{2}\langle a,a\rangle}e^{-2\pi i\langle a,b\rangle}\vartheta^{c,c'}_{a+\mu,b}(\tau), \tag{C.23}$$

where $x = a\tau + b$, i.e.

$$a = \frac{\mathrm{Im}(x)}{\mathrm{Im}(\tau)}, \quad b = \frac{\mathrm{Im}(\bar x\tau)}{\mathrm{Im}(\tau)}. \tag{C.24}$$

Following Corollary 2.9 of Zwegers [180], we can deduce the following set of transformations

$$\begin{aligned} \widehat\Theta^{c,c'}_{\Lambda,\mu}(\tau+1,x) &= (-1)^{\langle\mu,\mu\rangle}\widehat\Theta^{c,c'}_{\Lambda,\mu}(\tau,x),\\ \widehat\Theta^{c,c'}_{\Lambda,\mu}(-\frac{1}{\tau},\frac{x}{\tau}) &= \frac{i(-i\tau)^{r(\Lambda)/2}}{\sqrt{|\Lambda^*/\Lambda|}}e^{\pi i\frac{\langle x,x\rangle}{\tau}}\sum_{\delta\,\in\,\Lambda^*/\Lambda}e^{-2\pi i\langle\delta,\mu\rangle}\,\widehat\Theta^{c,c'}_{\Lambda,\delta}(\tau,x). \end{aligned} \tag{C.25}$$

This input enables us to write down the transformation rules for $\hat f^{(2)}_{\mu,J}$. They read

$$\begin{aligned} \hat f^{(2)}_{\mu,J}(\tau+1) &= (-1)^{\frac{\chi}{6}+2\mu^2}\hat f^{(2)}_{\mu,J}(\tau),\\ \hat f^{(2)}_{\mu,J}(-\frac{1}{\tau}) &= -\frac{(-i\tau)^{-\frac{r(\Lambda)+2}{2}}}{\sqrt{|\Lambda^*/\Lambda|}}\sum_{\delta\,\in\,\Lambda^*/\Lambda}e^{4\pi i\delta\cdot\mu}\hat f^{(2)}_{\delta,J}(\tau). \end{aligned} \tag{C.26}$$

This gives the conjectured transformation properties (C.16).

The blow-up factor

For completeness we elaborate on the transformation properties of the blow-up factor. We define

$$B_{r,k}(\tau) = \eta(\tau)^{-r} \sum_{a_i \in \mathbb{Z}+\frac{k}{r}} q^{\sum_{i\leq j\leq r-1} a_i a_j}. \tag{C.27}$$

We can deduce the following set of transformation rules

$$\begin{aligned} B_{r,k}(\tau+1) &= (-1)^{\frac{r}{12}+\frac{k^2(r-1)}{r}} B_{r,k}(\tau), \\ B_{r,k}(-\frac{1}{\tau}) &= \frac{1}{\sqrt{r}}\left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \sum_{0\leq l\leq r-1} (-1)^{\frac{2kl(r-1)}{r}} B_{r,l}(\tau). \end{aligned} \tag{C.28}$$

C.3 Elliptic genera of K3 and $\frac{1}{2}$K3

In the following we give some further examples of elliptic genera of multiple M5-branes wrapping the K3 and $\frac{1}{2}$K3 surfaces within the geometry of ref. [98]. The expressions for the elliptic genera can be read off from the instanton part of the prepotential of the geometry (see section 4.2.2) and were given in ref. [85], the $\frac{1}{2}$K3 expressions were known previously in refs. [131,132].

Elliptic genera of multiply wrapping the K3

These are obtained by setting $q_2 \to 0$ and can all be obtained from $Z^{(1)}$ by the Hecke transformation.

$$\begin{aligned} Z^{(1)} &= -\frac{2E_4E_6}{\eta^{24}} \\ Z^{(2)} &= -\frac{E_4E_6\left(17E_4^3+7E_6^2\right)}{96\eta^{48}} \\ Z^{(3)} &= -\frac{\left(9349E_4^7E_6+16630E_4^4E_6^3+1669E_4E_6^5\right)}{373248\eta^{72}} \\ Z^{(4)} &= -\frac{E_4E_6\left(11422873E_4^9+46339341E_4^6E_6^2+21978651E_4^3E_6^4+880703E_6^6\right)}{2579890176\eta^{96}} \\ Z^{(5)} &= -\frac{E_4E_6\left(27411222535E_4^{12}+198761115620E_4^9E_6^2+222886195242E_4^6E_6^4\right)}{30958682112000\eta^{120}} \\ &\quad -\frac{E_4E_6\left(45368414180E_4^3E_6^6+911966215E_6^8\right)}{30958682112000\eta^{120}} \end{aligned}$$

Elliptic genera of $\frac{1}{2}$K3, E-string bound-states

These are obtained by setting $q_3 \to 0$, the polynomials containing E_2 represent the part coming from bound-states. The polynomial appearance of E_2 at higher wrapping is an example of the appearance of mock modular forms of higher depth at higher wrapping.

$$
\begin{aligned}
Z^{(1)} &= \frac{E_4\sqrt{q}}{\eta^{12}}\\
Z^{(2)} &= \frac{E_4(E_2E_4+2E_6)q}{24\eta^{24}}\\
Z^{(3)} &= \frac{E_4\left(54E_2^2E_4^2+109E_4^3+216E_2E_4E_6+197E_6^2\right)q^{3/2}}{15552\eta^{36}}\\
Z^{(4)} &= \frac{E_4\left(24E_2^3E_4^3+109E_2E_4^4+144E_2^2E_4^2E_6+272E_4^3E_6+269E_2E_4E_6^2+154E_6^3\right)q^2}{62208\eta^{48}}\\
Z^{(5)} &= \frac{E_4\left(18750E_2^4E_4^4+150000E_2^3E_4^3E_6+1250E_2^2\left(109E_4^5+341E_4^2E_6^2\right)\right)q^{5/2}}{373248000\eta^{60}}\\
&+ \frac{E_4\left(1000E_2\left(653E_4^4E_6+505E_4E_6^3\right)+116769E_4^6+772460E_4^3E_6^2+207505E_6^4\right)q^{5/2}}{373248000\eta^{60}}
\end{aligned}
$$

Bibliography

[1] M. Aganagic, V. Bouchard and A. Klemm, "Topological Strings and (Almost) Modular Forms," Commun. Math. Phys. **277**, 771 (2008) [arXiv:hep-th/0607100].

[2] M. Aganagic, H. Ooguri, C. Vafa, and M. Yamazaki, "Wall Crossing and M-theory, " [arXiv:0908.1194 [hep-th]].

[3] G. Akemann, "Higher genus correlators for the Hermitian matrix model with multiple cuts," Nucl. Phys. B **482**, 403 (1996) [arXiv:hep-th/9606004].

[4] V. Alexeev and V. V. Nikulin, "Del Pezzo and K3 surfaces," Mathematical Society of Japan Memoirs **15** (2006) 1–164.

[5] M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch, T. Wotschke, "Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes," [arXiv:1012.1608 [hep-th]].

[6] M. Alim and J. D. Länge, "Polynomial Structure of the (Open) Topological String Partition Function," JHEP **0710**, 045 (2007) [arXiv:0708.2886 [hep-th]].

[7] M. Alim, J. D. Länge and P. Mayr, "Global Properties of Topological String Amplitudes and Orbifold Invariants," arXiv:0809.4253 [hep-th].

[8] L. Álvarez-Gaumé and J. L. Mañes, "Supermatrix models," Mod. Phys. Lett. A **6**, 2039 (1991).

[9] J. Ambjorn, L. Chekhov, C. F. Kristjansen, Y. .Makeenko, "Matrix model calculations beyond the spherical limit," Nucl. Phys. **B404**, 127-172 (1993). [hep-th/9302014].

[10] E. Andriyash, F. Denef, D. L. Jafferis, G. W. Moore, "Wall-crossing from supersymmetric galaxies," [arXiv:1008.0030 [hep-th]].

[11] I. Aniceto, R. Schiappa, M. Vonk, "The Resurgence of Instantons in String Theory," [arXiv:1106.5922 [hep-th]].

[12] I. Antoniadis, E. Gava, K. S. Narain, T. R. Taylor, "Topological amplitudes in string theory," Nucl. Phys. **B413**, 162-184 (1994). [hep-th/9307158].

[13] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," [hep-th/0403166].

[14] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," Nucl. Phys. **B405** (1993) 279–304, [hep-th/9302103].

[15] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," Commun. Math. Phys. **165**, 311 (1994) [arXiv:hep-th/9309140].

[16] D. Bessis, "A New Method In The Combinatorics Of The Topological Expansion," Commun. Math. Phys. **69**, 147 (1979).

[17] D. Bessis, C. Itzykson and J. B. Zuber, “Quantum Field Theory Techniques In Graphical Enumeration,” Adv. Appl. Math. **1**, 109 (1980).

[18] G. Bonnet, F. David and B. Eynard, “Breakdown of universality in multi-cut matrix models,” J. Phys. A **33**, 6739 (2000) [arXiv:cond-mat/0003324].

[19] V. Bouchard, A. Klemm, M. Marino, S. Pasquetti, “Remodeling the B-model,” Commun. Math. Phys. **287**, 117-178 (2009). [arXiv:0709.1453 [hep-th]].

[20] E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. **59**, 35 (1978).

[21] K. Bringmann and J. Manschot, “From sheaves on P2 to a generalization of the Rademacher expansion,” [arXiv:1006.0915 [math.NT]].

[22] P. F. Byrd, M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, 1954.

[23] F. Cachazo, K. A. Intriligator and C. Vafa, “A large N duality via a geometric transition,” Nucl. Phys. B **603**, 3 (2001) [arXiv:hep-th/0103067].

[24] F. Cachazo and C. Vafa, “N = 1 and N = 2 geometry from fluxes,” arXiv:hep-th/0206017.

[25] S. Cecotti and C. Vafa, “On classification of N=2 supersymmetric theories,” Commun. Math. Phys. **158** (1993) 569–644, [hep-th/9211097].

[26] S. Cecotti and C. Vafa, “BPS Wall Crossing and Topological Strings,” [arXiv:0910.2615 [hep-th]].

[27] S. Cecotti, A. Neitzke, and C. Vafa, “R-Twisting and 4d/2d Correspondences,” [arXiv:1006.3435 [hep-th]].

[28] M. C. N. Cheng and E. Verlinde, “Dying Dyons Don’t Count,” JHEP **09** (2007) 070, [arXiv:0706.2363 [hep-th]].

[29] M. C. N. Cheng, “K3 Surfaces, N=4 Dyons, and the Mathieu Group M24,” [arXiv:1005.5415 [hep-th]].

[30] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, “Precision counting of small black holes,” JHEP **10** (2005) 096, [hep-th/0507014].

[31] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, “Exact and asymptotic degeneracies of small black holes,” JHEP **0508** (2005) 021, [hep-th/0502157].

[32] A. Dabholkar, “Cargese lectures on black holes, dyons, and modular forms,” Nucl. Phys. Proc. Suppl. **171** (2007) 2–15.

[33] A. Dabholkar, S. Murthy, and D. Zagier, “Quantum black holes and mock modular forms.” Talks at ASC workshop on Interfaces and Wall crossing, 2009 in Munich, workshop on Automorphic Forms, Kac-Moody Algebras and Strings, 2010 in Bonn and at the conference on Topological String Theory, Modularity and Non-perturbative Physics, 2010 in Vienna.

[34] F. David, "Phases Of The Large N Matrix Model And Nonperturbative Effects In 2-D Gravity," Nucl. Phys. B **348**, 507 (1991). "Nonperturbative effects in matrix models and vacua of two-dimensional gravity," Phys. Lett. B **302**, 403 (1993) [arXiv:hep-th/9212106].

[35] J. R. David, D. P. Jatkar, and A. Sen, "Dyon spectrum in N = 4 supersymmetric type II string theories," JHEP **11** (2006) 073, [hep-th/0607155].

[36] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. Verlinde, "A farey tail for attractor black holes," JHEP **11** (2006) 024, [hep-th/0608059].

[37] F. Denef, "Supergravity flows and D-brane stability," JHEP **08** (2000) 050, [hep-th/0005049].

[38] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," [hep-th/0702146].

[39] P. Desrosiers and B. Eynard, "Supermatrix models, loop equations, and duality," arXiv:0911.1762 [math-ph].

[40] E. Diaconescu and G. W. Moore, "Crossing the Wall: Branes vs. Bundles," [arXiv:0706.3193 [hep-th]].

[41] D.-E. Diaconescu and C. Romelsberger, "D-branes and bundles on elliptic fibrations," Nucl. Phys. **B574** (2000) 245–262, [hep-th/9910172].

[42] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].

[43] R. Dijkgraaf, S. Gukov, V. A. Kazakov and C. Vafa, "Perturbative analysis of gauged matrix models," Phys. Rev. D **68**, 045007 (2003) [arXiv:hep-th/0210238].

[44] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Matrix models and gravitational corrections," Adv. Theor. Math. Phys. **7**, 1155 (2004) [arXiv:hep-th/0211241].

[45] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644**, 3 (2002) [arXiv:hep-th/0206255].

[46] R. Dijkgraaf, C. Vafa, "On geometry and matrix models," Nucl. Phys. **B644**, 21-39 (2002). [hep-th/0207106].

[47] R. Dijkgraaf, C. Vafa, "A Perturbative window into nonperturbative physics," [hep-th/0208048].

[48] R. Dijkgraaf and C. Vafa, "N = 1 supersymmetry, deconstruction, and bosonic gauge theories," arXiv:hep-th/0302011.

[49] R. Dijkgraaf, E. P. Verlinde, and M. Vonk, "On the partition sum of the NS five-brane," [hep-th/0205281].

[50] M. R. Douglas, "D-branes, categories and N = 1 supersymmetry," *J. Math. Phys.* **42** (2001) 2818–2843, [hep-th/0011017].

[51] M. R. Douglas, B. Fiol, and C. Romelsberger, "Stability and BPS branes," JHEP **09** (2005) 006, [hep-th/0002037].

[52] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions," *J. Phys.* **A42** (2009) 304010, [arXiv:0812.1151 [hep-th]].

[53] T. Eguchi and K. Hikami, "Superconformal Algebras and Mock Theta Functions 2. Rademacher Expansion for K3 Surface," [arXiv:0904.0911 [hep-th]].

[54] T. Eguchi, H. Ooguri, and Y. Tachikawa, "Notes on the K3 Surface and the Mathieu group M24," [arXiv:1004.0956 [hep-th]].

[55] B. Eynard, "Topological expansion for the 1-hermitian matrix model correlation functions," JHEP **0411**, 031 (2004) [arXiv:hep-th/0407261].

[56] B. Eynard, M. Mariño and N. Orantin, "Holomorphic anomaly and matrix models," JHEP **0706** (2007) 058 [arXiv:hep-th/0702110].

[57] B. Eynard and N. Orantin, "Invariants of algebraic curves and topological expansion," arXiv:math-ph/0702045.

[58] J.D. Fay, *Theta functions on Riemann surfaces*, Springer-Verlag, 1973.

[59] S. Ferrara, R. Kallosh, A. Strominger, "N=2 extremal black holes," Phys. Rev. **D52**, 5412-5416 (1995). [hep-th/9508072].

[60] D. S. Freed and E. Witten, "Anomalies in string theory with D-branes," [hep-th/9907189].

[61] D. Gaiotto, A. Strominger, and X. Yin, "The M5-brane elliptic genus: Modularity and BPS states," JHEP **08** (2007) 070, [hep-th/0607010].

[62] D. Gaiotto and X. Yin, "Examples of M5-brane elliptic genera," *JHEP* **11** (2007) 004, [hep-th/0702012].

[63] D. Gaiotto, G. W. Moore, and A. Neitzke, "Four-dimensional wall-crossing via three-dimensional field theory," Commun. Math. Phys. **299** (2010) 163–224, [arXiv:0807.4723 [hep-th]].

[64] D. Gaiotto, G. W. Moore, and A. Neitzke, "Wall-crossing, Hitchin Systems, and the WKB Approximation," [arXiv:0907.3987 [hep-th]].

[65] D. Gaiotto, G. W. Moore, and A. Neitzke, "Framed BPS States," [arXiv:1006.0146 [hep-th]].

[66] O. J. Ganor and A. Hanany, "Small E_8 Instantons and Tensionless Non-critical Strings," Nucl. Phys. **B474** (1996) 122–140, [hep-th/9602120].

[67] S. Garoufalidis, A. Its, A. Kapaev and M. Mariño, "Asymptotics of the instantons of Painlevé I," [arXiv:1002.3634 [math.CA]].

[68] R. Gopakumar and C. Vafa, "M-theory and topological strings. I," [hep-th/9809187].

[69] R. Gopakumar and C. Vafa, "M-theory and topological strings. II," [hep-th/9812127].

[70] R. Gopakumar, C. Vafa, "On the gauge theory / geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415-1443 (1999). [hep-th/9811131].

[71] L. Göttsche and D. Zagier, "Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_+ = 1$," Sel. math. **New ser. 4** (1998) 69–115.

[72] L. Göttsche, "The Betti numbers of the Hilbert schemes of points on a smooth projective surface," *Math. Ann. 286* (1990) 193–207.

[73] L. Göttsche, "Theta Functions and Hodge Numbers of Moduli Spaces of Sheaves on Rational Surfaces," Commun. Math. Phys. **206** (1999) 105–136.

[74] L. Göttsche, H. Nakajima, and K. Yoshioka, "K-theoretic donaldson invariants via instanton counting," [math/0611945].

[75] M. B. Green, J. A. Harvey, and G. W. Moore, "I-brane inflow and anomalous couplings on D-branes," Class. Quant. Grav. **14** (1997) 47–52, [hep-th/9605033].

[76] B. R. Greene, "String theory on Calabi-Yau manifolds," [hep-th/9702155].

[77] T. W. Grimm A. Klemm, M. Mariño and M. Weiss, "Direct integration of the topological string," JHEP **0708**, 058 (2007) [arXiv:hep-th/0702187].

[78] M. Guica and A. Strominger, "Cargese lectures on string theory with eight supercharges," Nucl. Phys. Proc. Suppl. **171** (2007) 39–68, [arXiv:0704.3295 [hep-th]].

[79] M. Gunaydin, A. Neitzke, and B. Pioline, "Topological wave functions and heat equations," JHEP **12** (2006) 070, [hep-th/0607200 [hep-th]].

[80] B. Haghighat and A. Klemm, "Topological Strings on Grassmannian Calabi-Yau manifolds," JHEP **0901**, 029 (2009) [arXiv:0802.2908 [hep-th]].

[81] B. Haghighat and A. Klemm, "Solving the Topological String on K3 Fibrations," JHEP **1001**, 009 (2010) [arXiv:0908.0336 [hep-th]].

[82] B. Haghighat, A. Klemm and M. Rauch, "Integrability of the holomorphic anomaly equations," JHEP **0810**, 097 (2008) [arXiv:0809.1674 [hep-th]].

[83] K. Hori, C. Vafa, "Mirror symmetry," [hep-th/0002222].

[84] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow, "Mirror symmetry," Providence, USA: AMS (2003) 929 p.

[85] M. Hecht, *Black Holes in M-Theory, BPS states and modularity*, Diploma thesis at the Ludwig-Maximilians University of Munich (2008).

[86] S. Hosono, A. Klemm, S. Theisen, S. -T. Yau, "Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces," Nucl. Phys. **B433**, 501-554 (1995). [hep-th/9406055].

[87] S. Hosono, M. H. Saito, and A. Takahashi, "Holomorphic anomaly equation and BPS state counting of rational elliptic surface," Adv. Theor. Math. Phys. **3** (1999) 177–208, [hep-th/9901151].

[88] S. Hosono, "Counting BPS states via holomorphic anomaly equations," [arxiv:hep-th/0206206].

[89] M. x. Huang and A. Klemm, "Holomorphic anomaly in gauge theories and matrix models," JHEP **0709**, 054 (2007) [arXiv:hep-th/0605195].

[90] M. x. Huang and A. Klemm, "Holomorphicity and Modularity in Seiberg-Witten Theories with Matter," [arXiv:0902.1325 [hep-th]].

[91] M. -x. Huang, A. Klemm, M. Marino, A. Tavanfar, "Black holes and large order quantum geometry," Phys. Rev. **D79**, 066001 (2009). [arXiv:0704.2440 [hep-th]].

[92] M. x. S. Huang, A. Klemm and S. Quackenbush, "Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions," Lect. Notes Phys. **757** (2009) 45 [arXiv:hep-th/0612125].

[93] A. Iqbal, C. Kozcaz, and C. Vafa, "The refined topological vertex," JHEP **10** (2009) 069, [arvix:hep-th/0701156].

[94] Joyce, "Holomorphic generating functions for invariants counting coherent sheaves on Calabi-Yau 3-folds," [hep-th/0607039].

[95] S. Joyce, "A theory of generalized Donaldson-Thomas invariants," [arXiv:0810.5645 [math.AG]].

[96] S. H. Katz, A. Klemm and C. Vafa, "Geometric engineering of quantum field theories," Nucl. Phys. B **497**, 173 (1997) [arXiv:hep-th/9609239].

[97] A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger, "Noether-Lefschetz theory and the Yau-Zaslow conjecture," [arXiv:0807.2477].

[98] A. Klemm, P. Mayr, and C. Vafa, "BPS states of exceptional non-critical strings," [hep-th/9607139].

[99] A. Klemm, M. Mariño and S. Theisen, "Gravitational corrections in supersymmetric gauge theory and matrix models," JHEP **0303**, 051 (2003) [arXiv:hep-th/0211216].

[100] A. Klemm, "Topological string theory on Calabi-Yau threefolds," PoS **RTN2005**, 002 (2005).

[101] A. Klemm and P. Sulkowski, "Seiberg-Witten theory and matrix models," Nucl. Phys. B **819**, 400 (2009) [arXiv:0810.4944 [hep-th]].

[102] A. Klemm, M. Marino, M. Rauch, "Direct Integration and Non-Perturbative Effects in Matrix Models," JHEP **1010**, 004 (2010). [arXiv:1002.3846 [hep-th]].

[103] H. Klingen, *Introductory lectures on Siegel modular forms,* Cambridge Univ. Press, Cambridge (1990).

[104] A. Klyachko, "Moduli of vector bundels and numbers of classes," Funct. Anal. and Appl. **25** (1991) 67–68.

[105] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function," Commun. Math. Phys. **147**, 1-23 (1992).

[106] Kontsevich and Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," [arXiv:0811.2435].

[107] M. Kool, "Euler characteristics of moduli spaces of torsion free sheaves on toric surfaces," [arXiv:0906.3393].

[108] P. Kraus and F. Larsen, "Partition functions and elliptic genera from supergravity," JHEP **01** (2007) 002, [hep-th/0607138].

[109] J.C. Le Guillou and J. Zinn–Justin (eds.), *Large Order Behavior of Perturbation Theory*, North–Holland, Amsterdam 1990.

[110] W. Lerche, P. Mayr, and N. P. Warner, "Non-critical strings, del Pezzo singularities and Seiberg-Witten curves," Nucl. Phys. **B499** [hep-th/9612085].

[111] W.-P. Li and Z. Qin, "On blowup formulae for the S-duality conjecture of Vafa and Witten," Invent. Math. **136** (1999) 451–482, [math/9805054].

[112] B. H. Lian, S. -T. Yau, "Arithmetic properties of mirror map and quantum coupling," Commun. Math. Phys. **176**, 163-192 (1996). [arvix:hep-th/9411234];
B. H. Lian, S. -T. Yau, "Mirror maps, modular relations and hypergeometric series 1," [arvix:hep-th/9507151];
B. H. Lian, S. -T. Yau, "Mirror maps, modular relations and hypergeometric series. 2.," Nucl. Phys. Proc. Suppl. **46**, 248-262 (1996). [arvix:hep-th/9507153].

[113] A. Losev, N. Nekrasov, and S. L. Shatashvili, "Issues in topological gauge theory," Nucl. Phys. **B534** (1998) 549–611, [hep-th/9711108].

[114] J. M. Maldacena, A. Strominger, and E. Witten, "Black hole entropy in M-theory," JHEP **12** (1997) 002, [hep-th/9711053].

[115] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231-252 (1998). [hep-th/9711200].

[116] J. M. Maldacena, G. W. Moore, and A. Strominger, "Counting BPS black holes in toroidal type II string theory," [hep-th/9903163].

[117] J. Manschot and G. W. Moore, "A Modern Fareytail," Commun. Num. Theor. Phys. **4** (2010) 103–159, [arXiv:0712.0573 [hep-th]].

[118] J. Manschot, "On the space of elliptic genera," Commun. Num. Theor. Phys. **2** (2008) 803–833, [arXiv:0805.4333 [hep-th]].

[119] J. Manschot, "Stability and duality in N=2 supergravity," [arXiv:0906.1767 [hep-th]].

[120] J. Manschot, "Wall-crossing of D4-branes using flow trees," [arXiv:1003.1570 [hep-th]].

[121] J. Manschot, "The Betti numbers of the moduli space of stable sheaves of rank 3 on P2," [arXiv:1009.1775 [hep-th]].

[122] J. Manschot, "BPS invariants of N=4 gauge theory on a surface," [arXiv:1103.0012 [math-ph]].

[123] M. Mariño, "Les Houches lectures on matrix models and topological strings," arXiv:hep-th/0410165.

[124] M. Marino, "Chern-Simons theory, matrix models, and topological strings," Oxford, UK: Clarendon (2005) 197 p.

[125] M. Mariño, "Nonperturbative effects and nonperturbative definitions in matrix models and topological strings," JHEP **0812**, 114 (2008) [arXiv:0805.3033 [hep-th]].

[126] M. Mariño, R. Schiappa and M. Weiss, "Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings," arXiv:0711.1954 [hep-th].

[127] M. Mariño, R. Schiappa and M. Weiss, "Multi-Instantons and Multi-Cuts," J. Math. Phys. **50**, 052301 (2009) [arXiv:0809.2619 [hep-th]].

[128] M. Mariño, "Lectures on non-perturbative effects in large N theory, matrix models and topological strings," Notes of lectures held at the conference of the "Research programm on Topological String Theory, Modularity & non-perturbative Physics" at ESI, Vienna, Notes available under `http://www.th.physik.uni-bonn.de/People/rauch/viennamarino.pdf`.

[129] M. Maruyama, "Moduli of stable sheaves. II," J. Math. Kyoto Univ. **18** (1977) 557.

[130] J. A. Minahan, D. Nemeschansky, and N. P. Warner, "Investigating the BPS spectrum of non-critical E(n) strings," Nucl. Phys. **B508** (1997) 64–106, [hep-th/9705237].

[131] J. A. Minahan, D. Nemeschansky, and N. P. Warner, "Partition functions for BPS states of the non-critical E(8) string," Adv. Theor. Math. Phys. **1** (1998) 167–183, [hep-th/9707149].

[132] J. A. Minahan, D. Nemeschansky, C. Vafa, and N. P. Warner, "E-strings and N = 4 topological Yang-Mills theories," Nucl. Phys. **B527** (1998) 581–623, [hep-th/9802168].

[133] R. Minasian and G. W. Moore, "K-theory and Ramond-Ramond charge," JHEP **11** (1997) 002, [hep-th/9710230].

[134] R. Minasian, G. W. Moore, and D. Tsimpis, "Calabi-Yau black holes and (0,4) sigma models," Commun. Math. Phys. **209** (2000) 325–352, [hep-th/9904217].

[135] G. W. Moore and E. Witten, "Integration over the u-plane in Donaldson theory," Adv. Theor. Math. Phys. **1** (1998) 298–387, [hep-th/9709193].

[136] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," Nucl. Phys. **B473** (1996) 74–92, [hep-th/9602114].

[137] D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – II," Nucl. Phys. **B476** (1996) 437–469, [hep-th/9603161].

[138] N. A. Nekrasov, "Seiberg-Witten Prepotential From Instanton Counting," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161].

[139] A. Neitzke, C. Vafa, "Topological strings and their physical applications," [arvix:hep-th/0410178].

[140] T. Okuda and T. Takayanagi, "Ghost D-branes," JHEP **0603**, 062 (2006) [arXiv:hep-th/0601024].

[141] K. Ono, "Unearthing the visions of a master: harmonic maass forms and number theory," Current developments in mathematics **2008** (2009) 347–454.

[142] H. Ooguri, Y. Oz, Z. Yin, "D-branes on Calabi-Yau spaces and their mirrors," Nucl. Phys. **B477**, 407-430 (1996). [hep-th/9606112].

[143] H. Ooguri, A. Strominger, and C. Vafa, "Black hole attractors and the topological string," *Phys. Rev.* **D70** (2004) 106007, [hep-th/0405146].

[144] H. Ooguri and C. Vafa, "Gravity induced C-deformation," Adv. Theor. Math. Phys. **7**, 405 (2004) [arXiv:hep-th/0303063], "The C-deformation of gluino and non-planar diagrams," Adv. Theor. Math. Phy **7**, 53 (2003) [arXiv:hep-th/0302109].

[145] S. Pasquetti and R. Schiappa, "Borel and Stokes Nonperturbative Phenomena in Topological String Theory and c=1 Matrix Models," arXiv:0907.4082 [hep-th].

[146] T. M. Seara and D. Sauzin, "Ressumació de Borel i teoria de la ressurgència," Butl. Soc. Catalana Mat. **18** (2003) 131.

[147] N. Seiberg and E. Witten, "Electric-Magnetic Duality, Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].

[148] N. Seiberg and E. Witten, "Monopole Condensation, And Confinement In N=2 Supersymmetric Yang-Mills Theory," Nucl. Phys. **B426** (1994) 19–52, [hep-th/9407087].

[149] N. Seiberg and E. Witten, "Comments on String Dynamics in Six Dimensions," Nucl. Phys. **B471** (1996) 121–134, [hep-th/9603003].

[150] A. Sen, "Walls of Marginal Stability and Dyon Spectrum in N=4 Supersymmetric String Theories," JHEP **05** (2007) 039, [hep-th/0702141].

[151] S.H. Shenker, "The Strength of Nonperturbative Effects in String Theory," in O. Álvarez, E. Marinari and P. Windey (eds.), *Random Surfaces and Quantum Gravity*, Plenum, New York 1992.

[152] A. Strominger, S. -T. Yau, E. Zaslow, "Mirror symmetry is T duality," Nucl. Phys. **B479**, 243-259 (1996). [hep-th/9606040].

[153] P. Sulkowski, "Matrix models for 2* theories," Phys. Rev. D **80**, 086006 (2009) [arXiv:0904.3064 [hep-th]].

[154] J. Troost, "The non-compact elliptic genus: mock or modular," JHEP **06** (2010) 104, [arXiv:1004.3649].

[155] C. Vafa, "Brane/anti-brane systems and $U(N|M)$ supergroup," arXiv:hep-th/0101218.

[156] C. Vafa and E. Witten, "A Strong coupling test of S duality," Nucl. Phys. **B431** (1994) 3–77, [hep-th/9408074].

[157] E. P. Verlinde, "Attractors and the holomorphic anomaly," [hep-th/0412139].

[158] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I & II*, Cambridge University Press, 2007

[159] M. Vonk, "A Mini-course on topological strings," [arvix:hep-th/0504147].

[160] T. Weist, "Torus fixed points of moduli spaces of stable bundles of rank three," [arXiv:0903.0723].

[161] E. Witten, "Noncommutative Geometry and String Field Theory," Nucl. Phys. **B268**, 253 (1986).

[162] E. Witten, "Topological Sigma Models," Commun. Math. Phys. **118** (1988) 411.

[163] E. Witten, "On The Structure Of The Topological Phase Of Two-dimensional Gravity," Nucl. Phys. **B340**, 281-332 (1990).

[164] E. Witten, "Mirror manifolds and topological field theory," In *Yau, S.T. (ed.): Mirror symmetry I* 121-160. [hep-th/9112056].

[165] E. Witten, "Chern-Simons gauge theory as a string theory," Prog. Math. **133**, 637-678 (1995). [hep-th/9207094].

[166] E. Witten, "Quantum background independence in string theory", [hep-th/9306122].

[167] E. Witten, "Phase Transitions In M-Theory And F-Theory," Nucl. Phys. **B471** (1996) 195–216, [hep-th/9603150].

[168] E. Witten, "D-branes and K-theory," JHEP **12** (1998) 019, [hep-th/9810188].

[169] S. Yamaguchi and S. T. Yau, "Topological string partition functions as polynomials," JHEP **0407**, 047 (2004) [arXiv:hep-th/0406078].

[170] S.-T. Yau and E. Zaslow, "BPS States, String Duality, and Nodal Curves on K3," Nucl. Phys. **B471** (1996) 503–512, [hep-th/9512121].

[171] K. Yoshioka, "The betti numbers of the moduli space of stable sheaves of rank 2 on $\mathbb{P}^2$," J. reine angew. Math. **453** (1994).

[172] K. Yoshioka, "The betti numbers of the moduli space of stable sheaves of rank 2 on a ruled surface," Math. Ann. (1995).

[173] K. Yoshioka, "The chamber structure of polarizations and the moduli of stable sheaves on a ruled surface," Int. J. of Math. **7** (1996) 411–431, [alg-geom/9409008].

[174] K. Yoshioka, "Euler characteristics of su(2) instanton moduli spaces on rational elliptic surfaces," Commun. Math. Phys. **205** (1999) 501–517.

[175] K. Yoshioka, "Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface," [alg-geom/9409008].

[176] S. A. Yost, "Supermatrix models," Int. J. Mod. Phys. A **7**, 6105 (1992) [arXiv:hep-th/9111033].

[177] D. Zagier, "Nombres de classes et formes modulaires de poids 3/2," C. R. Acad. Sci. Paris. **21** (1975) A883–A886.

[178] D. Zagier, "Ramanujan's Mock Theta Functions and their Applications d'après Zwegers and Bringmann-Ono," *Séminaire BOURBAKI* **986** (2007).

[179] D. Zagier, "Elliptic Modular Forms and Their Applications," in *The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway*, with J. H. Bruinier, G. Van Der Geer, G. Harder, Springer Heidelberg (2008).

[180] S. P. Zwegers, *Mock Theta Functions*, Proefschrift Universiteit Utrecht (2002).

[181] S. P. Zwegers, "Mock modular forms." Talk given at the conference "Partitions, q-series and modular forms", University of Florida, Gainesville, March 12-16, 2008; Talk available under `http://mathsci.ucd.ie/~zwegers/presentations/001.pdf`.