The Zinger deformation of differential equations with maximal unipotent monodromy

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Dedicated to the memory of Masoud Alimohammadi, professor of Theoretical Physics at the University of Tehran, who was assassinated on 12 Jenuary of 2010 by the explosion of a motorbike, when leaving home for the university.

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Contents

Introduction

In [16] the authors consider the hypergeometric series

$$\mathcal{F}(w,x) = \mathcal{F}_n(w,x) = \sum_{d=0}^{\infty} x^d \frac{\prod_{r=1}^{r=nd}(nw+r)}{\prod_{r=1}^{r=d}((w+r)^n - w^n)}, \quad n \in \mathbb{N} \tag{1}$$

which is a deformation of the well-known hypergeometric series

$$\mathcal{F}(x) = \mathcal{F}_n(x) = \sum_{d=0}^{\infty} \frac{(nd)!}{(d!)^n} x^d, \tag{2}$$

coming from a certain family of Calabi-Yau manifolds. In that paper they define the operator $\mathbf{M} : \mathcal{P} \to \mathcal{P}$ by

$$\mathbf{M}F(w,x) := \left(1 + \frac{x}{w}\frac{\partial}{\partial x}\right)\frac{F(w,x)}{F(0,x)},$$

where $\mathcal{P} \subset 1 + \mathbb{Q}(w)[[x]]$ is the subgroup of elements which are holomorphic at $w = 0$. Surprisingly they show that $\mathcal{F}(w,x)$ is a fixed point of $\mathbf{M}^n$, i.e. $\mathbf{M}^n\mathcal{F} = \mathcal{F}$. Moreover they give some identities among $I_p = \mathbf{M}^p\mathcal{F}(w,x)|_{w=0}$, in particular the symmetry $I_p = I_{n-p-1}$ $(0 \leq p \leq n-1)$. These I_p's play an important role in the formula given by Zinger in [17] to compute the reduced genus one Gromov-Witten invariant of Calabi-Yau projective hypersurfaces.

The first observation of this thesis is that there is nothing special about $\mathcal{F}(w,x)$. In fact let $f(x)$ be a holomorphic function with $f(0) = 1$, satisfying $L(D,x)y = 0$, a homogeneous linear differential equation with maximal unipotent monodromy, where $D = x\frac{d}{dx}$. Then we take a special deformation of $f(x)$ given by the unique holomorphic solution of

$$D_w L(D_w, x) f(w,x) = w^n f(w,x),$$

where $D_w := D + w$. We call this $f(w, x)$ the *Zinger deformation* of $f(x)$.

The first theorem in Chapter 1 says that $f(w, x)$ is a fixed point of $\mathbf{M}^n$. We prove two identities for I_p's as in [16] and we give a necessary and sufficient condition for the symmetry ($I_p = I_{n-1-p}, 0 \leq p \leq n-1$). Indeed if

$$L(D, x) = \sum_{i=0}^{r} x^i B_i(D)$$

then I_p's are symmetric if and only if $B_i(-D - i) = (-1)^{n-1} B_i(D)$ for all i.

The next chapter we study the Calabi-Yau (CY) equations of order four. We study the symmetry of I_p's. Since $n = 4$, this symmetry makes only two statements: $I_1 = I_3$ and $I_0 = I_4$. We show that I_1 is always equal to I_3 and I_0/I_4 always satisfies a first order linear differential equation. This lets us divide up the CY equations into three classes:

- Full symmetry: $I_1 = I_3$ and $I_0 = I_4$.

- Near symmetry: $I_1 = I_3$ and $(I_0/I_4)^2$ is a polynomial.

- Symmetry failure: $I_1 = I_3$ and I_0/I_4 has the form $C \prod (1 - \alpha_i)^{c_i}$ with α_i and c_i algebraic.

Surprisingly, the exceptional looking case (full symmetry) happens most of the time, and the general looking case (symmetry failure) is rare. We see experimentally among the non-hypergeometric cases (there are only 14 cases, $\#1 - 14$ in the table given in [2] which are hypergeometric and all of them are symmetric) if the leading coefficient of the differential equation reducible in $\mathbb{Q}[x]$ then $(I_0/I_4)^2$ is a polynomial (near symmetry) and if it is irreducible then $(I_0/I_4)^2$ is not a polynomial (symmetry failure). In the continuation we show that up to a constant the quotient I_2/I_1 is the Yukawa coupling.

In Chapter 3 we study the behaviour of the Zinger deformation when $w \to \infty$. In [16] the authors show that if $F(w, x) \in \mathcal{P}$ is a fixed point of $\mathbf{M}^n$ for some n then $\log F(w, x)$ has a perturbative expansion. This means that the asymptotic expansion of $\log F(w,x)$ with respect to $\hbar = \frac{1}{w}$ has at most a simple pole. We generalize this result and prove that $\log F(w,x)$ has a perturbative expansion if and only if each coefficient of $\log(\frac{\mathbf{M}^n F}{F})$ is $O_x(1)$ for some $n \geq 1$. We compute the residue and under some conditions inductively we can find each coefficient of this expansion. In the continuation we study the logarithmic derivative of the Zinger deformation. In particular we prove the conjecture which is stated in the last section of [16]. We show

$$1 + \frac{x}{w}\frac{\partial}{\partial x} \log \mathcal{F}(w,x) = L \sum_{s=0}^{\infty} \frac{P_s(n, L^n)}{(nwL)^s}, \tag{3}$$

where $L = (1 - n^n x)^{-1/n}$ and $P_s(n, L^n)(s \geq 0)$ are polynomials of n and L^n.

The second part of this thesis is devoted to study polynomials $P_s(n, L^n)$ $(s \geq 0)$. In the first two chapters of this part we give an exact formula for the first and the second top coefficient of $P_s(n, L^n)$ with respect to n. Part of these results was guessed by the authors in [16]. In the final chapter we give a recursive formula to compute the ℓth top coefficient of $P_s(n, L^n)$ where s varies and we show that these coefficients under a map (called the Euler map which is defined in Chapter 6) belong to the image of the elementary functions.

Part I

Structure properties of the Zinger deformation

Chapter 1

Structure of the Zinger deformation at $w = 0$

In this chapter first we define the Zinger deformation for a class of functions and we prove some interesting properties for this class.

1.1 Definitions

Let $W_z = W_z^{(n-2)}$ be a hypersurface in $\mathbb{CP}^{n-1}$ determined in terms of homogenous coordinates X_i by the equation

$$X_1^n + \cdots X_n^n - nzX_1 \cdots X_n = 0.$$

This defines a family of Calabi-Yau manifolds. This can be viewed as a family $\mathcal{W} \to \mathbb{A}^1$ with $\mathcal{W} \subset \mathbb{P}^{n-1} \times \mathbb{A}^1$ and z a coordinate on $\mathbb{A}^1$. We can projectivize the $\mathbb{A}^1$ and consider a family $\mathcal{W} \to \mathbb{P}^1$ with

$$\mathcal{W}_\infty = W_\infty = \{(X_1, \cdots, X_n) | X_1 X_2 \cdots X_n = 0\}.$$

The group $(\mathbb{Z}/n\mathbb{Z})^n$ acts on $\mathbb{P}^{n-1}$ as follows. Take $(a_1, \cdots, a_n) \in (\mathbb{Z}/n\mathbb{Z})^n$, then it acts on $\mathbb{P}^{n-1}$ by

$$(X_1, \cdots, X_n) \mapsto (\xi^{a_1} X_1, \cdots, \xi^{a_n} X_n),$$

where ξ is a fixed nth root of unity. On the other hand the subgroup $\mathbb{Z}/n\mathbb{Z} = \{(a, a, \cdots, a) | a \in \mathbb{Z}\}$ acts as the identity on $\mathbb{P}^{n-1}$, so in fact we have an action of $(\mathbb{Z}/n\mathbb{Z})^n/(\mathbb{Z}/n\mathbb{Z})$ on $\mathbb{P}^{n-1}$. Now the subgroup G given by

$$\{(a_1, \cdots, a_n) | a_1 + \cdots a_n = 0\},$$

acts on W_z for each z, so it makes sense to consider

$$M_z = M_z^{(n-2)} = W_z/G.$$

This M_z is quite singular. For $z^n \neq 1, \infty$ there exists a resolution of singularities $\widehat{M_z} \to M_z$ such that $\widehat{M_z}$ is also a Calabi-Yau manifold, moreover it is the mirror of W_z, i.e. $h^{1,1}(W_z) = h^{1,2}(\widehat{M_z})$ and $h^{1,2}(W_z) = h^{1,1}(\widehat{M_z})$ (cf. [6]). Its periods satisfy the following Picard-Fuchs equation

$$L(D, x)y = \Big(D^{n-1} - nx \prod_{j=1}^{n-1}(nD + j)\Big)y = 0, \tag{1.1}$$

where $D := x\frac{d}{dx}$ and $x = (nz)^{-n}$. The unique holomorphic solution of this differential equation with $y(0) = 1$ is

$$\mathcal{F}(x) = \sum_{d=0}^{\infty} \frac{(nd)!}{(d!)^n} x^d. \tag{1.2}$$

In [17] Zinger uses

$$\mathcal{F}(w, x) = \sum_{d=0}^{\infty} x^d \frac{\prod_{r=1}^{r=nd}(nw + r)}{\prod_{r=1}^{r=d}((w + r)^n - w^n)}, \tag{1.3}$$

as a deformation of $\mathcal{F}(x)$ to compute the reduced genus one Gromov-Witten for Calabi-Yau hypersurfaces. Our main task in this section is to generalize this deformation for the larger class of functions such that they satisfy some nice properties as in [16]. Since the differential equation (1.1) plays the key role as in the proofs of the statements for $\mathcal{F}(w, x)$ (cf.[16]), it is natural that our candidate might be in this space. In the following definition and lemma we introduce our candidate.

Definition 1.1.1. *Let $L(D, x) \in \mathbb{C}[x, D]$ be an operator of degree $n - 1$ for some $n > 1$. Then we say L has maximal unipotent monodromy* (**MUM**) *if $L(D, 0) = D^{n-1}$.*

Lemma 1.1.1. *The kernel of an operator L with* **MUM** *has a unique holomorphic element y with $y(0) = 1$.*

Proof. Let $y(x) = 1 + \sum_{k=1}^{\infty} a_k x^k$ be in the kernel of L. Then by assumption we have

$$D^{n-1} y = xL'(y),$$

for some $L' \in \mathbb{C}[x, D]$. Then the coefficient of x^k in both side would be

$$k^{n-1} a_k = \sum_{i=0}^{k-1} \lambda_i a_i,$$

for some $\lambda_i \in \mathbb{C}$. Therefore a_k will be uniquely determined and the solution is unique.□

Remark. From now whenever we talk about $f(x)$, the unique holomorphic solution of a differential equation with **MUM**, we mean that $f(0) = 1$.

Now we want to define a suitable deformation for this unique holomorphic solution.

Definition 1.1.2. *Let $L(D, x)$ be an operator of order $n-1$ with* **MUM** *and*

$$L(D_w, x) = \sum_{k=0}^{n-1} A_k(x) D_w^k, \tag{1.4}$$

where $A_k(x) \in x\mathbb{Q}[[x]]$, $(0 \leq k \leq n-2)$ and $D_w = D + w$. Then we call

$$(D_w L(D_w, x) - w^n) y = 0, \tag{1.5}$$

*a generalized hypergeometric differential equation (***GHD***) of order n.*

Now let $\mathcal{P} \subset 1 + x\mathbb{Q}(w)[[x]]$ be the subgroup of power series in x with constant term 1 whose coefficients are rational functions in w which are holomorphic at $w = 0$.

Lemma 1.1.2. *Let $\mathcal{L}$ be a* **GHD** *of order n. Then $\mathcal{L}$ has a unique solution in $\mathcal{P}$.*

Proof. Let $f(w, x) = \sum_{k=0}^{\infty} a_k(w) x^k$ be the holomorphic solution of $\mathcal{L} = 0$. Then from definition 1.1.2 we find $a_0(w) = 1$ and recursively

$$\Big((k + w)^n - w^n\Big) a_k(w) = \sum_{i=0}^{k-1} \lambda_i(w) a_i(w)$$

where by induction $\lambda_i(w), a_i(w) \in \mathbb{Q}(w)$ and holomorphic at $w = 0$. Hence $a_k(w)$ is so and therefore $f(w,x) \in \mathcal{P}$. □

Definition 1.1.3. *Let $f(w,x) \in \mathcal{P}$ be the unique holomorphic solution of* **GHD** (1.4). *Then we call $f(w,x)$ the Zinger deformation of $f(x) = f(0,x)$.*

The first observation is that the coefficients of $f(w,x)$ with respect to w has information about the solution of the original differential equation of $f(x)$. We see this in the following proposition.

Proposition 1.1.1. *Let*

$$f(w,x) = \sum_{i=0}^{\infty} f_i(x) w^i,$$

be the Zinger deformation of $f(x) = f_0(x)$. Set

$$F(w,x) := x^w f(w,x).$$

Then the first $n-1$ coefficients of $F(w,x)$ consist a Frobenius basis for the differential equation $L(D,x)y = 0$.

Proof. We have

$$F(w,x) = x^w.f(w,x) = (\sum_{i=0}^{\infty} w^i \frac{\log^i x}{i!})(\sum_{i=0}^{\infty} f_i(x) w^i)$$
$$= f_0 + (f_0 \log x + f_1)w + (f_2 + f_1 \log x + f_0 \frac{\log^2 x}{2})w^2 + \cdots + O(w^n).$$

We note that $D(F(w,x)) = x^w D_w f(w,x)$, so by definition 1.1.3

$$DL(D,x)F(w,x) = x^w D_w L(D_w,x) f(w,x) = x^w.w^n.f(w,x) = O(w^n).$$

Therefore

$$DL(D,x)\Big(\sum_{i=0}^{k} f_i \frac{(\log x)^{k-i}}{(k-i)!}\Big) = 0, \quad 0 \le k \le n-1.$$

But by definition we have $f_i(0) = \delta_{i0}$, hence

$$L(D,x)\Big(\sum_{i=0}^{k} f_i \frac{(\log x)^{k-i}}{(k-i)!}\Big) = 0, \quad 0 \le k \le n-1.$$

□

1.2 Periodicity

In this section we prove the main property of the Zinger deformation.

Definition 1.2.1. *Let* $\mathbf{M} : \mathcal{P} \longrightarrow \mathcal{P}$ *be the map defined by*

$$\mathbf{M}F(w,x) := \frac{1}{w} D_w\Big(\frac{F(w,x)}{F(0,x)}\Big) = \Big(1 + \frac{x}{w}\frac{\partial}{\partial x}\Big)\frac{F(w,x)}{F(0,x)}.$$

We set $F_p(w,x) := \mathbf{M}^p F(w,x)$ *and* $I_p(x) = F_p(0,x)$.

Definition 1.2.2. *We define* $\mathcal{M}_n$ *to be the set of fixed points of* $\mathbf{M}^n$, *i.e., the set of all* $F(w,x) \in \mathcal{P}$ *such that* $F_n = F$.

Theorem 1.2.1. *Let* $f(w,x)$ *be the Zinger deformation of some* $f(x)$. *Then* $f \in \mathcal{M}_n$ *for some* n.

Before giving the proof of Theorem 1.2.1 we look at some examples to check this theorem.

Example 1. Let $f(x) = \sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5} x^d$. As we have seen before in (1.1), $f(x)$ satisfies in a hypergeometric differential equation. It is the case #1 in database [2]. Then by computation one can find

$$\begin{aligned}
I_0(x) = I_4(x) = I_5(x) = f(x) &= 1 + 120\,x + 113400\,x^2 + 168168000\,x^3 + O(x^4),\\
I_1(x) = I_3(x) &= 1 + 770\,x + 1435650\,x^2 + 3225308000\,x^3 + O(x^4),\\
I_2(x) &= 1 + 1345\,x + 3296525\,x^2 + 8940963625\,x^3 + O(x^4).
\end{aligned}$$

Example 2. Let $f(x) = \sum_{d=0}^{\infty} \frac{(3d)!^2}{(d!)^6} x^d$. It satisfies the following hypergeometric differential equation

$$\Big(D^4 - 3^6 x(D + \frac{1}{3})^2(D + \frac{2}{3})^2\Big)y = 0.$$

It is the Picard -Fuchs equation for the mirror of the complete intersection of two cubics in $\mathbb{P}^5$:

$$\begin{aligned}
X_1^3 + X_2^3 + X_3^3 &= 3zX_4X_5X_6,\\
X_4^3 + X_5^3 + X_6^3 &= 3zX_1X_2X_3.
\end{aligned}$$

It is the case #4 in database [2]. Then by computation one can find

$$\begin{aligned}
&I_0(x) = I_4(x) = I_5(x) = f(x) = 1 + 36\,x + 8100\,x^2 + 2822400\,x^3 + O(x^4),\\
&I_1(x) = I_3(x) = 1 + 180\,x + 79380\,x^2 + 41920920\,x^3 + O(x^4),\\
&I_2(x) = 1 + 297\,x + 168561\,x^2 + 106224345\,x^3 + O(x^4).
\end{aligned}$$

Example 3. Let $f(x) = \sum_n A_n x^n$, where

$$A_n = \sum_{k,l} \binom{n}{k}^2 \binom{n}{l}^2 \binom{k+l}{n} \binom{2n-k}{n},$$

which satisfies in

$$\begin{aligned}
&\Big(9D^4 - 3x(173D^4 + 340D^3 + 272D^2 + 102D + 15)\\
&- 2x^2(1129D^4 + 5032D^3 + 7597D^2 + 4773D + 1083)\\
&+ 2x^3(843D^4 + 2628D^3 + 2353D^2 + 675D + 6)\\
&- x^4(295D^4 + 608D^3 + 478D^2 + 174D + 26) + x^5(D+1)^4\Big)y = 0.
\end{aligned}$$

It is the Picard-Fuchs equation for the mirror of $X(1,1,1,1,1,1,1) \subset$Grass$(2,7)$, a complete intersection of hyperplanes. It is the case #27 in the database [2].
Then we have

$$\begin{aligned}
&I_0(x) = I_5(x) = 1 + 5\,x + 109\,x^2 + 3317\,x^3 + 121501\,x^4 + O(x^5),\\
&I_1(x) = I_3(x) = 1 + 14\,x + 574\,x^2 + 26222\,x^3 + 1294286\,x^4 + O(x^5),\\
&I_2(x) = 1 + \frac{56}{3}\,x + \frac{2828}{3}\,x^2 + \frac{149408}{3}\,x^3 + \frac{8285228}{3}\,x^4 + O(x^5),\\
&I_4(x) = 1 + 6\,x + \frac{344}{3}\,x^2 + \frac{92602}{27}\,x^3 + \frac{3372103}{27}\,x^4 + O(x^5).
\end{aligned}$$

Now to prove Theorem 1.2.1 we need the following lemma:

Lemma 1.2.1. *Suppose $f(w,x) \in \mathcal{P}$ satisfies the mth order differential equation*

$$(\sum_{r=0}^{m} C_r(x) D_w^r) f(w,x) = A(w,x) \tag{1.6}$$

for some power series $C_0(x), \cdots, C_m(x) \in \mathbb{Q}[[x]]$ and $A(w,x) \in \mathbb{Q}(w)[[x]]$ with $A(0,x) \equiv 0$. Then the function $\tilde{f} = \mathbf{M}f$ satisfies the $(m-1)$st order differential equation

$$(\sum_{s=0}^{m-1} \tilde{C}_s(x) D_w^s)\tilde{f}(w,x) = \frac{1}{w} A(w,x), \tag{1.7}$$

where

$$\tilde{C}_s(x) := \sum_{r=s+1}^{m} \binom{r}{s+1} C_r(x) D^{r-1-s} f(0,x). \tag{1.8}$$

Proof. See [16].□

Now we are ready to prove Theorem 1.2.1.

Proof. The idea is to use several times Lemma 1.2.1. But we need a suitable function to apply this lemma for it. In order to do this, we construct a new function $F(w,x)$ from $f(w,x)$ which has the extra property $F(0,x) = 1$. To do this job we look at the effect of D_w on power series

$$f(w,x) = \sum_{i=0}^{\infty} p_i(w) x^i \in \mathbb{Q}(w)[[x]].$$

We see

$$D_w f(w,x) = \sum_{i=0}^{\infty} (w+i) p_i(w) x^i,$$

so D_w has an inverse operator namely, D_w^{-1} which replaces each $p_i(w)$ by $(w+i)^{-1} p_i(w)$. Therefore we can define

$$F(w,x) := w D_w^{-1} f(w,x).$$

It follows that $F(0,x) = 1$ (recall that $f(0) = 1$). Multiplying both sides of (1.5) by $w D_w^{-1}$ we find

$$\left(\sum_{s=1}^{n} \tilde{A}_s(x) D_w^s\right) F(w,x) = w^n F(w,x), \tag{1.9}$$

where $\tilde{A}_s = A_{s-1}(x)$. Now we apply Lemma 1.2.1 for equation (1.9). We note that $\mathbf{M}F(w,x) = f(w,x)$, hence we have

$$\sum_{s=0}^{n-1} C_s^{(0)}(x) D_w^s f(w,x) = w^{n-1} F(w,x), \tag{1.10}$$

where

$$C_s^{(0)}(x) = \sum_{r=s+1}^{n} \binom{r}{s+1} \tilde{A}_r(x) D^{r-1-s} F(0,x) = \tilde{A}_{s+1}(x) = A_s(x).$$

Applying Lemma 1.2.1 for equation (1.10) repeatedly, we find

$$\sum_{s=0}^{n-1-p} C_s^{(p)}(x) D_w^s f_p(w,x) = w^{n-1-p} F(w,x), \tag{1.11}$$

and $C_s^{(p)}$ for $p > 0$ is given inductively by

$$C_s^{(p)}(x) = \sum_{r=s+1}^{n-p} \binom{r}{s+1} C_r^{(p-1)}(x) D^{r-s-1} I_{p-1}(x). \tag{1.12}$$

In particular for $s = n - 1 - p$ we find that

$$\begin{aligned} C_{n-1-p}^{(p)}(x) &= C_{n-p}^{(p-1)}(x) I_{p-1}(x) \\ &= C_{n-p+1}^{(p-2)}(x) I_{p-2}(w,x) I_{p-1}(x) \\ &\vdots \\ &= C_{n-1}^{(0)}(x) \prod_{r=0}^{p-1} I_r(x) = A_{n-1}(x) \prod_{r=0}^{p-1} I_r(x). \end{aligned} \tag{1.13}$$

On the other hand from equations (1.11) for $p = n - 1$ and $s = 0$ we have

$$C_0^{(n-1)}(x) f_{n-1}(w,x) = F(w,x). \tag{1.14}$$

This equation with (1.13) for $p = n - 1$ gives

$$A_{n-1}(x) \prod_{r=0}^{n-2} I_r(x) . f_{n-1}(w,x) = F(w,x). \tag{1.15}$$

Setting $w = 0$ in this relation and using $F(0,x) = 1$ gives

$$\prod_{r=0}^{n-1} I_r(x) = A_{n-1}^{-1}(x). \tag{1.16}$$

Then substituting back into (1.15) gives

$$f_{n-1}/I_{n-1} = F(w,x). \tag{1.17}$$

Finally, applying $w^{-1}D_w$ to both sides of (1.17) implies $\mathbf{M}^n f = f_n = f$. $\square$

During the proof we found a relationship among I_p's (equation (1.16)). We state it in the next theorem and we give another identity for I_p's.

Theorem 1.2.2. *Let $f(w,x) \in \mathcal{M}_n$ be as in Theorem 1.2.1. Then we have*

$$I_0(x)\cdots I_{n-1}(x) = A_{n-1}(x)^{-1}, \tag{1.18}$$

$$I_0(x)^{n-1} I_1(x)^{n-2} \cdots I_{n-1}(x)^0 = e^{h(x)}. \tag{1.19}$$

where $h(x) = -\int_0^x \dfrac{A_{n-2}(u)}{uA_{n-1}(u)}du$.

Proof. We have already seen the first identity in equation (1.16). To prove the second identity, $p = n-2$ in (1.11) with $w = 0$, implies that

$$C_0^{(n-2)}(x) I_{n-2}(x) + C_1^{(n-2)}(x) D I_{n-2}(x) = 0. \tag{1.20}$$

From (1.13) for $p = n-2$ we get

$$C_1^{(n-2)}(x) = A_{n-1}(x) \prod_{r=0}^{n-3} I_r(x).$$

Substituting this in (1.20) we find

$$C_0^{(n-2)}(x) = -A_{n-1}(x) \frac{DI_{n-2}(x)}{I_{n-2}(x)} \prod_{r=0}^{n-3} I_r(x). \tag{1.21}$$

On the other hand from (1.12) for $s = n-p-2$

$$C_{n-p-2}^{(p)}(x) = C_{n-1-p}^{(p-1)}(x) I_{p-1}(x) + (n-p) C_{n-p}^{(p-1)}(x) D I_{p-1}(x).$$

From (1.13) we have $C_{n-p}^{(p-1)}(x) = A_{n-1}(x) \prod_{r=0}^{p-2} I_r(x)$, substituting this in above we obtain

$$C_{n-p-2}^{(p)}(x) = C_{n-1-p}^{(p-1)}(x) I_{p-1}(x) + (n-p) A_{n-1}(x) \frac{DI_{p-1}(x)}{I_{p-1}(x)} \prod_{r=0}^{p-1} I_r(x). \tag{1.22}$$

Continuing this procedure we get

$$C^{(p)}_{n-p-2}(x) = \left(C^{(p-2)}_{n-p}(x) I_{p-2}(x) + (n-p+1) A_{n-1}(x) \prod_{r=0}^{p-3} I_r(x) D I_{p-2}(x) \right) I_{p-1}(x)$$
$$+ (n-p) A_{n-1}(x) \frac{D I_{p-1}(x)}{I_{p-1}(x)} \prod_{r=0}^{p-k} I_r(x),$$

or

$$C^{(p)}_{n-p-2}(x) = C^{(p-2)}_{n-p}(x) I_p(x) I_{p-1}(x) +$$
$$\left((n-p) \frac{D I_{p-1}(x)}{I_{p-1}(x)} + (n-p+1) \frac{D I_{p-2}(x)}{I_{p-2}(x)} \right) A_{n-1}(x) \prod_{r=0}^{p-1} I_r(x),$$

and finally

$$C^{(p)}_{n-2-p}(x) = \left(A_{n-2}(x) + A_{n-1}(x) \sum_{r=0}^{p-1} (n-r-1) \frac{D I_r(x)}{I_r(x)} \right) \prod_{r=0}^{p-1} I_r(x). \quad (1.23)$$

With $p = n-2$ we find

$$C^{(n-2)}_{0}(x) = \left(A_{n-2}(x) + A_{n-1}(x) \sum_{r=0}^{n-3} (n-r-1) \frac{D I_r(x)}{I_r(x)} \right) \prod_{r=0}^{n-3} I_r(x). \quad (1.24)$$

Comparing equations (1.21) and (1.24) we find

$$\sum_{r=0}^{n-k} (n-r-k) \frac{I'_r(x)}{I_r(x)} = -\frac{A_{n-2}(x)}{x A_{n-1}(x)}. \quad (1.25)$$

Integrating this and exponentiating proves the second part.□

1.3 Symmetry

If we look at again Examples 1 and 2 in Section 1.2 we see that

$$I_0(x) = I_4(x), \quad I_1(x) = I_3(x).$$

Or in other words we have (full) symmetry in these cases. But in Example 3, instead of full symmetry we have $I_1 = I_3$ and $I_0 \neq I_4$. Hence it is natural to ask under which conditions we have symmetry for an $f \in \mathcal{M}_n$. The two identities in Theorem 1.2.2 constrain an obvious necessary condition which we state in the following lemma.

Lemma 1.3.1. *Let A_i are as in Definition 1.1.2 and $f(w,x) \in \mathcal{M}_n$ satisfy the conditions of Theorem 1.2.1. Then a necessary condition for the symmetry of I_p's is*

$$-\frac{n-1}{2}A_{n-2}(x) = D(A_{n-1}(x)). \tag{1.26}$$

Proof. If we have symmetry, namely $I_p(x) = I_{n-p-1}(x)$ for all $0 \leq p \leq n-1$, then by Theorem 1.2.2

$$\begin{aligned}
A_{n-1}(x)^{1-n} &= (\prod_{r=0}^{n-1} I_r(x))^{n-1} \\
&= \prod_{k=0}^{n-1} I_k(x)^{n-k-1} \prod_{k=0}^{n-1} I_k(x)^k \\
&= (I_0(x)^{n-1} I_1(x)^{n-2} \cdots I_{n-1}(x)^0)^2 \quad (\textit{symmetry}) \\
&= e^{-2h(x)}, \quad (\textit{from}(1.16))
\end{aligned}$$

where $h(x) = \int_0^x -\dfrac{A_{n-2}(u)}{uA_{n-1}(u)}du$, hence it is necessary

$$A_{n-1}(x)^{n-1} = e^{2\int_0^x -\frac{A_{n-2}(u)}{uA_{n-1}(u)}du}, \tag{1.27}$$

or

$$\frac{n-1}{2}\log A_{n-1}(x) = \int_0^x -\frac{A_{n-2}(u)}{uA_{n-1}(u)}du,$$

so by differentiating both sides, it turns out

$$\frac{n-1}{2}\frac{A'_{n-1}(x)}{A_{n-1}(x)} = -\frac{A_{n-2}(x)}{xA_{n-1}(x)}, \tag{1.28}$$

which proves the lemma. □

For example from (1.1) for $\mathcal{F}(x)$ we have,

$$A_{n-1}(x) = 1 - n^n x, \quad A_{n-2}(x) = -x\frac{n(n-1)}{2}, \tag{1.29}$$

and satisfies the necessary condition. Indeed I_p's are symmetric in this case and we see later in Theorem 2.1.1, for Calabi-Yau equations this necessary condition is sufficient. But for the moment we consider an arbitrary Zinger deformation and we give a necessary and sufficient condition for symmetry.

Let $W = \mathbb{C}[x, D]$ be the Weyl algebra. Then the morphism

$$* : W \to W$$

given on the basis by

$$(x^i D^k)^* = (-D)^k x^i = x^i(-D-i)^k, \tag{1.30}$$

is an anti-involution. In fact we have

$$\begin{aligned}(x^j D^\ell . x^i D^k)^* &= (x^{i+j}(D+i)^\ell D^k)^* = x^{i+j}(-D-j)^\ell(-D-i-j)^k \\ &= (-D)^k x^{i+j}(-D-j)^\ell = (x^i D^k)^*(x^j D^\ell)^*.\end{aligned}$$

Furthermore

$$(x^i D^k)^{**} = (x^i(-D-i)^k)^* = x^i\Big(-(-D-i)-i\Big)^k = x^i D^k.$$

Definition 1.3.1. *For an operator $L \in W$ we call*

$$\widehat{L} := (-1)^{deg_D(L)} L^*$$

the conjugate of L, where * is the anti-involution as in (1.30)

We note that if $L \in W$ such that $L(0) = D^{n-1}$ for some n, then L^* by definition satisfies the same property. Therefore if $f(x)$ is the unique holomorphic solution of $L(y) = 0$, then it makes sense to speak about $\widehat{f}(x)$ the unique solution of $\widehat{L}(y) = 0$.

Theorem 1.3.1. *Let $\mathcal{L}$ be an operator of order $n-1$ in the Weyl algebra W with* **MUM** *and $\widehat{\mathcal{L}}$ its conjugate. Suppose $f(w, x)$ and $\widehat{f}(w, x)$ are the Zinger deformations of the holomorphic solution of $\mathcal{L}$ and $\widehat{\mathcal{L}}$ respectively. Then*

$$I_p = \widehat{I}_{n-p-1} \quad 0 \le p \le n-1. \tag{1.31}$$

Conversely if for two Zinger deformations, identity (1.31) *holds then the corresponding operators are conjugate.*

Corollary 1.3.1. *Let $\mathcal{L}$ as in Theorem 1.3.1. We write*

$$\mathcal{L}(D,x)=\sum_{i=0}^{r}x^iB_i(D)=\sum_{i=0}^{n-1}A_k(x)D^k.$$

Then the following conditions

- $\widehat{\mathcal{L}}=\mathcal{L}$,
- $I_p(x)=I_{n-1-p}(x), \quad 0\le p\le n-1$,
- $B_i(D)=(-1)^{n-1}B_i(-D-i), \qquad 0\le i\le r$,
- $A_s(x)=\sum_{k=s}^{n-1}(-1)^{n-k+1}\binom{k}{s}D^{k-s}A_k, \qquad 0\le s\le n-1.$

are equivalent.

Proof of Corollary 1.3.1. From Theorem 1.31 immediately it follows that the first and the second identity are equivalent and by definition we see that the first and the third one are equivalent as well. Now if we set $A_k(x)=\sum_{j=0}^{r}a_{kj}x^j$, then from the third identity we have

$$\begin{aligned}\sum_{k=0}^{n-1}a_{ki}D^k &= B_i(D)=(-1)^{n-1}B_i(-D-i)\\ &=\sum_{k=0}^{n-1}(-1)^{n-1}a_{ki}(-D-i)^k=\sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^{n-1+k}a_{ki}\binom{k}{j}(i)^{k-j}D^j.\end{aligned}$$

It follows that

$$a_{si}=\sum_{k=s}^{n-1}(-1)^{n-1+k}a_{ki}\binom{k}{s}i^{k-s},$$

finally the sum over all $a_{si}x^i$ when i varies, gives the fourth identity. conversely the above equation gives the other side. We note that the fourth identity for $s=n-2$ gives the necessary condition (cf. Lemma 1.3.1).

Proof of Theorem 1.3.1. We define $\tilde{f}(w,x)\in\mathcal{P}$ as the unique holomorphic solution of

$$L(D_w,x)\tilde{f}(w,x)=w^{n-1}. \tag{1.32}$$

But on the other hand by assumption we have

$$D_w L(D_w, x)\Big(f(w,x)\Big) \pmod{w^n} = w^n + O(w^{n+1}).$$

We have seen before the effect of D_w^{-1} on the coefficients of $f(w,x)$. It turns out that

$$L(D_w, x)\Big(f(w,x)\Big) \pmod{w^n} = w^{n-1} + O(w^n).$$

Therefore the uniqueness implies that $\tilde{f}(w,x) = f(w,x) \pmod{w^n}$. Hence we have $\tilde{f}_p(0,x) = f_p(0,x) = I_p(x), (0 \le p \le n-1)$. Now applying Lemma 1.2.1 repeatedly for $\tilde{f}(w,x)$ we find

$$\sum_{s=0}^{n-p-1} \tilde{C}_s^{(p)}(x) D_w^s \tilde{f}_p(w,x) = w^{n-p-1}, \quad 0 \le p \le n-1, \tag{1.33}$$

where the coefficients $\tilde{C}_s^{(p)}(x) \in \mathbb{Q}[[x]]$ can be computed recursively. The top one is given by

$$\tilde{C}_{n-1-p}^{(p)}(x) = A_{n-1}(x) I_0(x) \cdots I_{p-1}(x) = (I_p(x) \cdots I_{n-1}(x))^{-1}.$$

Specializing to $p = n-1$ we find $\tilde{C}_0^{(n-1)}(x) = I_{n-1}(x)^{-1}$. Plugging this into (1.33) with $p = n-1$ we find

$$\tilde{f}_{n-1}(w,x) = I_{n-1}(x).$$

Now by downwards induction on p, using the equation $\tilde{f}_p = I_p w D_w^{-1} \tilde{f}_{p+1}$, we can reconstruct the all power series $\tilde{f}_p(w,x)$ $(0 \le p \le n-1)$, from their special values $I_p(x) = \tilde{f}_p(0,x)$ at $w = 0$. We obtain in particular the formula

$$w^{1-n} \tilde{f}(w,x) = I_0 D_w^{-1} I_1 D_w^{-1} \cdots I_{n-2} D_w^{-1} I_{n-1}. \tag{1.34}$$

To prove the theorem we write

$$\mathcal{L}(D_w, x) = \sum_{i=0}^{r} x^i B_i(D_w)$$

where B_i's are polynomials of degree at most $n-1$. Then by definition we have

$$\mathcal{L}^*(D_w, x) = \sum_{i=0}^{r} x^i B_i^*(D_w) = \sum_{i=0}^{r} x^i B_i(-D_w - i).$$

Let $\tilde{f}(w,x) = \sum_{d=1}^{\infty} b_d(w)x^d$, with $b_0(w) = 1$. By equation (1.32) we have

$$\sum_{i=0}^{n-1} B_i((d-i)_w)b_{d-i}(w)x^d = w^{n-1} \quad (b_{d-i}(w) = 0 \quad for \quad d-i < 0).$$

For $d = 0$ we get $B_0(w) = w^{n-1}$ and by a simple induction for $d \geq 1$

$$b_d(w) = \sum_{\substack{1 \leq i_1, \cdots, i_s \leq r \\ i_1 + \cdots i_s = d}} (-1)^s \frac{\prod_{j=1}^{s} B_{i_j}(w + d - i_1 - \cdots - i_j)}{(w+d)^{n-1} \prod_{j=1}^{s-1}(w + d - i_1 - \cdots - i_j)^{n-1}}. \quad (1.35)$$

Claim 1.3.1. $(-w-d)^{1-n} b_d(-w-d) = (-w)^{1-n} \widehat{b}_d(w)$.

Proof. The denominator of each subsum of $w^{1-n}b_d(w)$ is

$$w^{n-1}(w+d)^{n-1} \prod_{j=1}^{s-1}(w + d - i_1 - \cdots - i_j)^{n-1},$$

and under $w \to -w-d$ it is transformed to

$$\begin{aligned}
&(-w-d)^{n-1}(-w)^{n-1} \prod_{j=1}^{s-1}(-w - i_1 - \cdots - i_j)^{n-1} \\
&= (-w-d)^{n-1}(-w)^{n-1} \prod_{j=1}^{s-1}(-w - d + i_{j+1} + \cdots + i_s)^{n-1} \\
&= (-1)^{(n-1)(s+1)}(w+d)^{n-1}(w)^{n-1} \prod_{j=1}^{s-1}(w + d - i_{j+1} - \cdots - i_s)^{n-1}. \quad (1.36)
\end{aligned}$$

The numerator of each subsum of $w^{1-n}b_d(w)$ is

$$(-1)^s \prod_{j=1}^{s} B_{i_j}(w + d - i_1 - \cdots - i_j),$$

and under $w \to -w-d$ it is transformed to

$$\begin{aligned}
&(-1)^s \prod_{j=1}^{s} B_{i_j}(-w - i_1 - \cdots - i_j) \\
&= (-1)^s \prod_{j=1}^{s} (-1)^{(n-1)} \widehat{B}_{i_j}(w + i_1 + \cdots + i_{j-1}) \\
&= (-1)^{s(n-1)+s} \prod_{j=1}^{s} \widehat{B}_{i_j}(w + d - i_j - \cdots - i_s). \quad (1.37)
\end{aligned}$$

Equations (1.36) and (1.37) imply that

$$(-w-d)^{1-n}b_d(-w-d) =$$
$$(-1)^{n-1}\sum_{\substack{1\le i_1,\cdots,i_s\le r\\ i_1+\cdots i_s=d}}(-1)^s\frac{\prod_{j=1}^s \widehat{B}_{i_j}(w+d-i_j-\cdots-i_s)}{(w+d)^{n-1}(w)^{n-1}\prod_{j=1}^{s-1}(w+d-i_{j+1}-\cdots-i_s)^{n-1}}.$$

Substituting i_j with i_{s-j+1} we find

$$\begin{aligned}
&(-w-d)^{1-n}b_d(-w-d)\\
&=(-1)^{n-1}\sum_{\substack{1\le i_1,\cdots,i_s\le r\\ i_1+\cdots i_s=d}}(-1)^s\frac{\prod_{j=1}^s \widehat{B}_{i_{s-j+1}}(w+d-i_1-\cdots-i_{s-j+1})}{(w+d)^{n-1}(w)^{n-1}\prod_{j=1}^{s-1}(w+d-i_1-\cdots-i_{s-j})^{n-1}}\\
&=(-w)^{1-n}\widehat{b}_d(w),
\end{aligned}$$

which proves the claim.
Now by comparing the coefficients of x^d on both sides of (1.34), we find

$$w^{1-n}b_d(w)=\sum_{\substack{1\le i_0,\cdots,i_s\le r\\ i_0+\cdots i_{n-1}=d}}\frac{a_{i_0}(0)\cdots a_{i_{n-1}}(n-1)}{(w+i_1+\cdots+i_{n-1})(w+i_2+\cdots+i_{n-1})\cdots(w+i_{n-1})},$$

for all $d\ge 0$, where $a_i(p)$ denotes the coefficient of x^i in $I_p(x)$. Splitting up the sum on the right into the subsum over n-tuples $(i_0,\cdots,i_{n-1})$ with $\max\{i_r\}\le d-1$ and the sum over n-tuples which are permutations of $(d,0,\cdots,0)$, and using $a_0(p)=1$ for all p, we can rewrite this equation as

$$\begin{aligned}
\sum_{p=0}^{n-1}\frac{a_d(p)}{w^{n-p-1}(w+d)^p} &= w^{1-n}b_d(w)\\
&-\sum_{\substack{i_0,\cdots,i_{n-1}\ge 0\\ i_0+\cdots i_{n-1}=d}}\frac{a_{i_0}(0)\cdots a_{i_{n-1}}(n-1)}{(w+i_1+\cdots+i_{n-1})(w+i_2+\cdots+i_{n-1})\cdots(w+i_{n-1})}.
\end{aligned}$$

Now suppose by induction that $a_r(p)=\widehat{a}_r(n-p-1)$ for all $r<d$ and all $0\le p\le n-1$. The right hand side goes to its conjugate under the map $w\to -w-d$, as one sees for the second term by making the renumbering $i_r\to i_{n-r-1}$. It follows that the left-hand side has the same property, so

$a_d(p) = \widehat{a}_d(n-p-1)$ for all $0 \leq p \leq n-1$, completing the inductive proof of $\widehat{I}_{n-1-p} = I_p$.

Now for the other side if we follow the proof of the first part we see that the identity $I_p = \widehat{I}_{n-p-1}$ and Claim 1.3.1 are equivalent. Now by induction we show that

$$(-1)^{n-1} B_i(-w-i) \; = \widehat{B}_i(w),$$

and therefore it implies that $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are conjugate.
For $i=0$ we have $(-1)^{n-1}B_0(-w) = w^{n-1} = \widehat{B}_0(w)$. Now using equation (1.35) for $i=1$ we find

$$b_1(w) = -\frac{B_1(w)}{(w+1)^{n-1}}$$

$$\begin{aligned}
-\frac{\widehat{B}_1(w)}{(w+1)^{n-1}w^{n-1}} &= (-1)^{n-1}w^{1-n}\hat{b}_1(w) \\
&= (-1-w)^{1-n}b_1(-w-1) = -\frac{B_1(-w-1)}{(-w)^{n-1}(-w-1)^{n-1}}.
\end{aligned}$$

It follows that $(-1)^{n-1}B_1(-w-1) = \widehat{B}_1(w)$. Now for $d > 1$ we split the equation (1.35) into two parts

$$\begin{aligned}
\frac{B_d(w)}{w^{n-1}(w+d)^{n-1}} &= -w^{1-n}b_d(w) \\
&+ \sum_{\substack{1 \leq i_1, \cdots, i_s < d \\ i_1 + \cdots i_s = d}} (-1)^s \frac{\prod_{j=1}^{s} B_{i_j}(w+d-i_1-\cdots-i_j)}{w^{n-1}(w+d)^{n-1}\prod_{j=1}^{s-1}(w+d-i_1-\cdots-i_j)^{n-1}},
\end{aligned}$$

by induction and our assumption the right hand side goes to its conjugate under $w \to -w-d$, so the left hand side goes to its conjugate too. Therefore we find $(-1)^{n-1}B_d(-w-d) = \widehat{B}_d(w)$. $\square$

Chapter 2

Calabi-Yau equations

This chapter has two sections. In the first section, we define Calabi-Yau equations which are linear differential equations of order 4 of **MUM** with some extra properties. Consequently one can define the Zinger deformation and I_p's ($0 \leq p \leq 4$) for these. Then we prove a statement about the symmetry of I_p's.
In the next section first we explain how equation (1.2) for $n = 3, 4$ has a modular interpretation. For the case $n = 5$ which is the holomorphic solution of the Calabi-Yau equation (1.1), such interpretation has not found yet. In the continuation for Calabi-Yau equations we give a connection among I_p's and the Yukawa coupling.

2.1 Calabi-Yau equations and symmetry

In this section we study a special case, namely Calabi-Yau equations. These are differential equation of order 4 with **MUM** which satisfy some extra properties. According to [2] we list these properties in the following definition.

Definition 2.1.1. *We call the differential equation*

$$\mathcal{L} : y^{(4)} + a_3(x)y^{(3)} + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{2.1}$$

of Calabi-Yau type or shortly CY-equation, if satisfies the following conditions *i) The singular point $x = 0$ is a point of maximal unipotent monodromy, i.e. the indicial equation at $x = 0$ should have 0 as its only solution. Or equivalently if we write equation (2.1) as*

$$\sum_{i=0}^{4} A_i(x) D^i y = 0, \tag{2.2}$$

then $A_4(0) = 1$ and $A_3(0) = A_2(0) = A_1(0) = A_0(0) = 0$. We remind that $D = x\frac{d}{dx}$.

ii) The coefficients $a_i(x)$ satisfy the following condition

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3''.$$

iii) The solutions $r_1 \leq r_2 \leq r_3 \leq r_4$ of the indicial equation at $x = \infty$ are positive rational numbers satisfying $r_1 + r_4 = r_2 + r_3 - s$ for some $s \in \mathbb{Q}$. We also suppose that the eigenvalues $e^{2\pi i r}$ of the monodromy around $x = \infty$ are the zeroes of a product of cyclotomic polynomials, which can be interpreted as the characteristic polynomial of the monodromy around $x = \infty$.

iv) The power series solution near $x = 0$ has integral coefficients.

v) The genus zero instanton numbers computed by the standard recipe are integral (up to multiplication by an overall positive integer).

In [2] the authors have collected more than 300 examples of CY-equations.

Theorem 2.1.1. *Let $\mathcal{L}$ be a CY-equation defined in (2.2). Then $I_1 = I_3$ and I_0/I_4 satisfies the following first order linear differential equation*

$$(I_0/I_4)' = \frac{2xA_4'(x) - A_3(x)}{2xA_4(x)}(I_0/I_4).$$

Proof. By definition of CY-equations we have

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3'',$$

From Proposition 1 in [1] this is equivalent to

$$\frac{d^2}{dt^2}(y_3/y_0) = t\frac{d^2}{dt^2}(y_2/y_0), \tag{2.3}$$

where $y_0.y_1, y_2, y_3$ are the Frobenius basis of (2.1) with **MUM** and $t = y_1/y_0$. On the other hand we have

$$I_1 = D(y_1/y_0), \quad I_2 = D\Big(D(y_2/y_0)/I_1\Big), \quad I_3 = D(Z/I_2), \tag{2.4}$$

where $Z := D\Big(D(y_3/y_0)/I_1\Big)$. On can see $\frac{d}{dt}f(x) = D(f)/I_1$, hence we have

$$\frac{d^2}{dt^2}(y_2/y_0) = \frac{d}{dt}(D(y_2/y_0)/I_1) = I_2/I_1,$$

and similarly

$$\frac{d^2}{dt^2}(y_3/y_0) = Z/I_1.$$

We have from (2.4)

$$I_3 = D(\frac{Z}{I_1}) = \frac{DZ}{I_1} - Z\frac{DI_2}{I_2^2}. \tag{2.5}$$

But from (2.3) we have

$$\frac{y_1}{y_0}\frac{I_2}{I_1} = \frac{Z}{I_1},$$

or

$$Z = I_2\frac{y_1}{y_0}.$$

By differentiating we find

$$DZ = I_1I_2 + \frac{y_1}{y_0}DI_2 = I_1I_2 + Z\frac{DI_2}{I_2}.$$

Plugging this into (2.5) we find $I_1 = I_3$. For the second part from Theorem 1.2.2 and using the fact that $I_1 = I_3$ we have

$$\begin{aligned}\int_0^x -\frac{A_3(u)}{uA_4(u)}du &= \log(I_0^4I_1^3I_2^2I_1^3I_4^0) \\ &= 2\log(I_0/I_4) + 2\log(I_0I_1I_2I_3I_4) \\ &= 2\log(I_0/I_4) - 2\log A_4(x).\end{aligned}$$

Differentiating both sides we get the result.□

Thanks to Theorem 2.1.1 we can detect three classes among CY equations:

1) *full symmetry*: $I_1 = I_3$ and $I_0 = I_4$.
   From Theorem 2.1.1 it happens when $A_3 = 2xA_4'$. Indeed this identity is the necessary condition for symmetry which have already seen it. It is interesting because in this case this necessary condition is sufficient.

2) *near symmetry*: $I_1 = I_3$ and $(I_0/I_4)^2$ is a polynomial.
   Experimentally by checking the first 50 CY-equation in the list of CY-equations given in [2], this case happens when $A_4(x)$ is reducible in $\mathbb{Q}[x]$.

3) *symmetry failure*: $I_1 = I_3$ and I_0/I_4 has the form $C\prod(1-\alpha_i x)^{c_i}$ with α_i and c_i algebraic.
   Experimentally this case happens when $A_4(x)$ is irreducible in $\mathbb{Q}[x]$, but we do not have any proof.

Despite of the generality of third case which one can expect this is the often case, surprisingly this is rare (at least in the list of CY equations given in [2]) Table 2.1 shows the type of the first 50 CY-equations. In Table 2.2 for those which are of the second type we show the decomposition of $A_4(x)$ and the polynomial $(I_0/I_4)^2$. Table 2.3 gives the type of those CY-equations which has been already known their Calabi-Yau manifolds (according to [13]).

2.2 Modularity and the Yukawa coupling

It is a basic fact that the classic modular forms satisfy linear differential equations respect to a modular function. More precisely, let $f(z) \in M_k(\Gamma)$ be a modular form of weight k, where Γ is a subgroup of finite index in $SL_2(\mathbb{Z})$, and $t(z)$ a modular function. Then the many valued function $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a linear differential equation of order $k+1$

Table 2.1: Types of CY-equations

type 1	type 2	type 3
#1-14,15,16 18,24,25,26 28,29,30,34 36,38,41,42 45,46,47,48	#17,20,21,22 23,27,31,32 33,35,37,39 40,43,44,49	#19

with algebraic coefficients. This differential equation is the kth symmetric power of the following second order differential equation.

$$\frac{\partial^2 G}{\partial t^2} + \frac{[g,t']_1}{g\,t'^2}\frac{\partial G}{\partial t} - \frac{[g,g]_2}{2g^2\,t'^2}\,G = 0, \tag{2.6}$$

where $g := f^{1/k}$, $G := F^{1/k}$ and $[\,,\,]_n$ is the nth Rankin-Cohen bracket, defined by

$$[P,Q]_n = \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{\ell+n-1}{r} P^{(r)}Q^{(s)}, \quad (n \geq 0),$$

where P and Q are modular forms of weight k and ℓ respectively and $'$ is $\frac{d}{dz}$. The complete solution spanned by $\{f, zf, \cdots, z^k f\}$ and the monodromy group is the image of Γ in $\mathrm{sym}^k(\Gamma)$ (cf. [15]).

Since having "integral monodromy" group is a rare phenomena, one hopes a differential equation with integral monodromy might have modular property. One of the interesting question about the holomorphic solution of a Picard-Fuchs equation (Our interest are CY-equations) is to find a 'modular property' for it. We see this in some examples.

First we look at (1.2) for $n = 3$. In this case we have

$$f(x) = \sum_{d=0}^{\infty} \frac{(3d)!}{(d!)^3} x^d,$$

and it satisfies

$$\Big(D^2 - 3x(3D+1)(3D+2)\Big) f(x) = 0.$$

Table 2.2: CY-equations of second type

Database	$A_4(x)$	$(I_0/I_4)^2$
17	$(9x-5)^2(27x-1)(27x^2+1)$	$(9x-5)^3$
20	$(18x-1)^2(27x-1)^2(54x-1)$	$(18x-1)^6$
21	$(4x-1)(4x+1)(8x+5)(32x-1)$	$(8x+5)^6$
22	$(4x-7)^2(32x-1)(x^2-11x-1)$	$(4x-7)^6$
23	$(16x-1)^2(32x-3)^2(32x-1)$	$(32x-3)^6$
27	$(x-3)^2(x^3-289x^2-57x+1)$	$(x-3)^6$
31	$(1024x-1)^2$	$(1024x-1)$
32	$(27x^2+270x-1)^4$	$(27x^2+270x-1)^7$
33	$(16x-1)(128x-1)(1024x-1)$	$(128x-1)^6$
35	$(288x-1)^2(432x-1)^2(846x-1)$	$(288x-1)^6$
37	$(256x-1)^4(1024x-1)^4$	$(256x-1)^7(1024x-1)^7$
39	$(64x-1)^4(256x-1)^4$	$(64x-1)^7(256x-1)^7$
40	$(256x-1)^4$	$(256x-1)^6$
43	$(1024x-1)^4$	$(1024x-1)^6$
44	$(256x^2-544x+1)^4$	$(256x^2-544x+1)^7$
49	$(432x-1)^4$	$(432x-1)^6$

By Proposition 1.1.1 another solution is

$$g(x) = f(x) + f_1(x)\log x,$$

where

$$f_1(x) = \frac{\partial f(w,x)}{\partial w}|_{w=0}.$$

But

$$\frac{\partial f(w,x)}{\partial w} = \sum_{d=1}^{\infty} \frac{\prod_{r=1}^{3d}(3w+r)}{\prod_{r=1}^{d}((w+1)^3-w^3)} \left(\sum_{r=1}^{3d} \frac{3}{3w+r} - \sum_{r=1}^{d} 3\frac{(w+r)^2-w^2}{(w+r)^3-w^3} \right) x^d,$$

and therefore

$$f_1(x) = \sum_{d=1}^{\infty} \left(\frac{(3d)!}{(d!)^3} \sum_{i=d+1}^{3d} \frac{3}{i} \right) x^d. \tag{2.7}$$

Table 2.3: Geometric Calabi-Yau equations

Database	Description	Type
13	$X(6,6) \in \mathbb{P}^5(1,1,2,2,3,3)$	1
2	$X(10) \in \mathbb{P}^4(1,1,1,2,5)$	1
9	$X(2,12) \in \mathbb{P}^5(1,1,1,1,4,6)$	1
12	$X(3,4) \in \mathbb{P}^5(1,1,1,1,1,2)$	1
7	$X(8) \in \mathbb{P}^5(1,1,1,1,4)$	1
8,125	$X(6) \in \mathbb{P}^4(1,1,1,1,2)$	1,1
10	$X(4,4) \in \mathbb{P}^5(1,1,1,1,2,2)$	1
14,85,86	$X(2,6) \in \mathbb{P}^5(1,1,1,1,1,3)$	1,1,1
1,79,87,128	$X(5) \in \mathbb{P}^4$	1,1,1,1
11,95	$X(4,6) \in \mathbb{P}^5(1,1,1,2,2,3)$	1,1
6,75,76,96	$X(2,4) \in \mathbb{P}^6$	1,1,1,2
4	$X(3,3) \in \mathbb{P}^5$	1
51	Conj:$X \to B_5$	1
5,90,91,93	$X(2,2,3) \in \mathbb{P}^6$	1,1,1,1
99	Conj:5×5-Pfaffian $\in \mathbb{P}^6$	1
222	7×7-Pfaffian $\in \mathbb{P}^6$	2
24	$X(1,1,3) \in$ Grass$(2,5)$	1
3,72,224	$X(2,2,2,2) \in \mathbb{P}^7$	1,1,3
25	$X(1,2,2) \in$ Grass$(2,5)$	1
29	Conj:$X(1,1,1,1,1,1,2) \in X_{10}$	1
26	$X(1,1,1,1,2) \in$ Grass$(2,6)$	1
42	Conj:$X(1,1,2) \in$ LGrass$(3,6)$	1
184	Conj:$X(1,2) \in X_5$	1
27	$X(1,1,1,1,1,1,1) \in$Grass$(2,7)$	2
28	$X(1,1,1,1,1,1) \in$Grass$(3,6)$	1
247	Tjøtta's example	2

We define

$$
\begin{aligned}
q(x) &:= \exp(\frac{g(x)}{f(x)}) = x\exp(\frac{f_1(x)}{f(x)}) \\
&= x + 15\,x^2 + 279\,x^3 + 5729\,x^4 + 124554\,x^5 + O(x^6),
\end{aligned}
$$

and the *mirror map* as the inverse function

$$
x(q) = q - 15\,q^2 + 171\,q^3 - 1679\,q^4 + 15054\,q^5 + O(q^6),
$$

which is convergent for $|q| < 1$. Let $z = \frac{1}{2\pi i}\log q$ and $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ the eta- function. Then one can see

$$
X(z) = x(q) = \frac{\eta(z)^{12}}{\eta(z)^{12} + 27\eta(3z)^{12}},
$$

is a modular function for the congruence subgroup $\Gamma_0(3)$. Let $\delta = \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$. Then the derivative of a modular function is a modular form of weight 2. By definition we have $I_1(x) = 1 + x(\frac{f_1(x)}{f(x)})'$ and

$$
\delta x(q) = q(x)\frac{1}{q(x)'} = x.\exp(\frac{f_1}{f}).\frac{1}{\exp(\frac{f_1}{f})\Big(1 + (\frac{f_1}{f})'\Big)} = \frac{x(q)}{I_1(x(q))},
$$

is a modular form of weight 2 for $\Gamma_0(3)$, or $I_1(x(q)) = \frac{x(q)}{\delta(x(q))}$ is a modular form of weight -2. Thanks to Theorems 1.2.2 and 1.3.1 we have

$$
f^2(x(q)) = \frac{1}{I_1(x(q))}.\frac{1}{1 - 27x(q)},
$$

is also a modular form of weight 2 for $\Gamma_0(3)$. One can show that

$$
f(x(q)) = 1 + 6\sum_{n=1}^{\infty}\Big(\sum_{d|n}(\frac{d}{n})q^n\Big) \in M(\Gamma_0(3), (\frac{}{3})).
$$

Example (1.2) for $n = 4$ also has a modular property. Its differential equation is the symmetric square of the following one:

$$
(1 - 4^4x)D^2 - 128xD - 12x = 0.
$$

But when $n > 4$ the story is different. Here we discuss the case $n = 5$. First we see that the differential equation in this case is not a symmetric power, moreover the monodromy group is not a subgroup of $SL_2(\mathbb{Z})$, therefore the holomorphic solution can not be a modular form. But it is not the end of the story. In this case the monodromy group is still arithmetic, indeed the monodromy group is a arithmetic subgroup of $Sp(4,\mathbb{Z})$, and therefore it might there is a modular interpretation of more than one variable, e.g., Siegel modular forms (cf.[4]). In this case the Frobenius basis $\{y_i\}_{i=0}^{3}$, of the differential equation

$$\mathfrak{L} : \Big(D^4 - 5x(5D+1)(5D+2)(5D+3)(5D+4)\Big)y = 0, \tag{2.8}$$

is given by Proposition 1.1.1. The mirror map is

$$x(q) = q - 770\,q^2 + 171525\,q^3 - 81623000\,q^4 - 35423171250\,q^5 + O(q^6),$$

and

$$f(x(q)) = 1 + 120\,q + 21000\,q^2 + 14115000\,q^3 + 13414125000\,q^4 + O(q^5),$$

as we see the coefficients are too big for modularity and in fact the convergent domain of $f(x(q))$ is not the disk $|q| < 1$ (cf.[18]). From Theorem 1.3.1, I_p's are symmetric, hence from Theorem 1.2.2 we have

$$I_0^2(x(q)) = \Big(\frac{\delta x(q)}{x(q)}\Big)^3 . \frac{1}{1-5^5x(q)} . \frac{I_1(x)}{I_2(x)}. \tag{2.9}$$

We will show that the quotient $\dfrac{I_2(x(q))}{I_1(x(q))}$ up to a constant is $K(q)$ the *Yukawa coupling*, defined by

$$K(q) = N_0.\delta^2(y_2/y_0), \tag{2.10}$$

where $y_0 = f(x)$ and $y_2(x) = f_2 + f_1 \log x + f\frac{\log^2 x}{2!}$ and $N_0 = 5$ in this case. We have by definition

$$I_1 = 1 + D(\frac{F_1}{F}), \quad I_2 = \frac{1 + D(\frac{\frac{F_1}{F} + D(\frac{F_2}{F})}{I_1})}{I_1}.$$

On the other hand by chain rule

$$\begin{aligned}\delta(y_2/y_0) = q(x)\frac{(y_2/y_0)'}{q'(x)} &= \frac{D(y_2/y_0)}{I_1(x)} \\ &= \frac{D(\frac{F_2}{F}) + \log x(D(\frac{F_1}{F}) + 1) + \frac{F_1}{F}}{I_1} \\ &= \log x + \frac{D(\frac{F_2}{F}) + \frac{F_1}{F}}{I_1}.\end{aligned}$$

Therefore

$$\delta^2(y_2/y_0) = I_1^{-1} D(\log x + \frac{D(\frac{F_2}{F}) + \frac{F_1}{F}}{I_1}) = \frac{I_2}{I_1}.$$

It follows that $\dfrac{I_2(x(q))}{I_1(x(q))} = \frac{1}{5}K(q)$. We can rewrite (2.9) as

$$K(q) = I_0^{-2}(x(q))\Big(\frac{\delta x(q)}{x(q)}\Big)^3.\frac{5}{1 - 5^5 x(q)}. \tag{2.11}$$

Proposition 2.2.1. *For the CY-equation given by* (2.2), *we have*

$$K(q) = N_0.\frac{I_2(x(q))}{I_1(x(q))},$$

where $K(q)$ is the Yukawa coupling and N_0 is a constant.

Proof. See above.

Remark 2.2.1. *The main property of the Yukawa coupling when the CY-equation is the Picard-Fuchs of a family of Calabi-Yau threefold, is counting the number of rational curves of fixed degree on the mirror. For example in the quintic case, we write*

$$K(q) = 5 + \sum_{\ell=1}^{\infty} \frac{n_\ell^3 q^\ell}{1 - q^\ell} = 5 + 2875\, q + 4876875\, q^2 + O(q^3).$$

Then n_ℓ is the number of rational curves of degree ℓ on the mirror of a generic quintic threefold (for example see [9]).

Chapter 3

Structure of the Zinger deformation at $w = \infty$

3.1 Introduction

In the previous chapter we have studied the structure of the Zinger deformation at $w = 0$. In this chapter we describe some of its structures at $w = \infty$. In [16] the authors have shown that for every $f \in \mathcal{P} \cap \mathcal{M}_n$, every coefficient of the power series $\log f(w, x) \in \mathbb{Q}(w)[[x]]$ is $O_x(w)$ as $w \to \infty$. In the next theorem we generalize this result.

Theorem 3.1.1. *Let $F(w, x) \in \mathcal{P}$ such that $\log \frac{\mathbf{M}^n F}{F} = O_x(1)$, for some $n \geq 1$. Then we have*

$$\log F(w, x) = O_x(w),$$

and for every $k \geq 0$

$$\log \frac{\mathbf{M}^k F(w, x)}{F(w, x)} = O_x(1).$$

Remark 3.1.1. *This theorem gives a criteria for the set $\mathcal{P} \cap \bar{\mathcal{M}}$, where $\bar{\mathcal{M}} = \cup_{n \geq 1} \mathcal{M}_n$. Indeed if $F \in \mathcal{P}$ and $\log \frac{\mathbf{M}F(w,x)}{F(w,x)} \neq O_x(1)$, then $F \notin \bar{\mathcal{M}}$.*

Corollary 3.1.1. *Let $f(w, x) \in \mathcal{P}$ satisfies in the* **GHD** (1.5)*. Then every coefficient of the power series $\log f(w, x) \in \mathbb{Q}(w)[[x]]$ is $O_x(w)$ as $w \to \infty$.*

Proof of Corollary 3.1.1. In this case $\mathbf{M}^n f = f$ and obviously the condition of theorem 3.1.1 holds.□

Now we prove the theorem.

Proof. We set

$$H(w,x) := \log F(w,x) = \sum_{r=1}^{\infty} h_r(w)x^r,$$

and we have $\mathbf{M}(e^{H(w,x)}) = e^{H^*(w,x)}$, where

$$H^*(w,x) = H(w,x) - H(0,x) + \log(1 + \frac{DH(w,x) - DH(0,x)}{w}). \tag{3.1}$$

We have to show that $h_r(w) = O(w)$ for all $r \geq 1$. We suppose by induction, $h_1, \cdots, h_{s-1}$ are $O(w)$ and we show that $h_s(w) = O(w)$.

$$\begin{aligned} \frac{DH(w,x) - DH(0,x)}{w} &= \sum_{r=1}^{\infty} r \frac{h_r(w) - h_r(0)}{w} x^r \\ &= \sum_{r=1}^{s-1} O(1)x^r + \frac{s\, h_s(w)}{w} x^s + O(x^{s+1}). \end{aligned} \tag{3.2}$$

Hence from (3.1) and (3.2),

$$H^*(w,x) = H(w,x) + O_x(1) + \frac{s\, h_s(w)}{w} x^s + O(x^{s+1}). \tag{3.3}$$

Iterating, we find that for every $k \geq 1$

$$\log(\mathbf{M}^k F) = \log F(w,x) + O_x(1) + \frac{ks\, h_s(w)}{w} x^s + O(x^{s+1}), \tag{3.4}$$

especially when $k = n$. But by assumption $\log(\frac{\mathbf{M}^n F}{F}) = O_x(1)$, so it follows that $h_s(w) = O(w)$, which proves the first part. The second part is obvious from the first part and equation (3.4).□

3.2 Asymptotic expansion of the Zinger deformation

Now let $f(w,x)$ be the Zinger deformation of some $f(x)$. Then Theorem 3.1.1 implies that $\log f(w,x)$ has an asymptotic expansion $\sum_{j=-1}^{\infty} \mu_j(x) w^{-j}$

with $\mu_j(x) \in x\mathbb{Q}[[x]]$ for all $j \geq -1$ or equivalently,

$$f(w,x) \sim e^{\mu(x)w} \sum_{s=0}^{\infty} \Phi_s(x) w^{-s} \quad (w \to \infty) \tag{3.5}$$

Thanks to this expansion, in [16], the authors have computed $\mu(x)$ and $\Phi_0(x)$ for $\mathcal{F}(w,x)$. With a slight modification this proof works for the general case. We give it in the following proposition.

Proposition 3.2.1. *In the expansion* (3.5)

$$\mu_(x) = \int_0^x \frac{L(u)-1}{u} du, \tag{3.6}$$

where $L(x) = A_{n-1}(x)^{-1/n}$, *and* $A_{n-1}(x)$ *defined in* (1.4). *Moreover if* I_p*'s are symmetric, then* $\Phi_0(x) = L(x)$.

Proof. From Theorem 3.1.1, each $f_p(w,x) = \mathbf{M}^p f(w,x)$ has an asymptotic expansion

$$f_p(w,x) \sim e^{\mu(x)w} \sum_{s=0}^{\infty} \Phi_{p,s}(x) w^{-s} \quad (w \to \infty),$$

with the same $\mu(x)$ in the exponent. The equation $f_{p+1} = \mathbf{M} f_p$ gives

$$\Phi_{0,s} = \Phi_s, \quad \Phi_{p+1,s} = \frac{1+\mu'}{I_p}\Phi_{p,s} + \begin{cases} (\Phi_{p,s-1}/I_p)' & \text{if } s \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

The case $s = 0$ of (3.7) gives by induction

$$\Phi_{p,0} = \frac{(1+\mu')^p}{I_0 \cdots I_{p-1}} \Phi_{0,0}. \tag{3.8}$$

But $f_n = f$ or $\Phi_{n,0} = \Phi_{0,0}$, hence we obtain from (3.8)

$$(1+\mu')^n = I_0 \cdots I_{n-1} = A_{n-1}(x)^{-1},$$

which proves the first equation in (3.6) because $\mu(x)$ is a power series in x with no constant term.
For the second part, from Theorem 1.2.2 and the symmetry we have

$$I_0(x)^n I_1(x)^{n-1} \cdots I_{n-1}(x) = L(x)^{\frac{n(n+1)}{2}}. \tag{3.9}$$

Now we define

$$H_p(x) := \frac{L(x)^p}{I_0(x)\cdots I_{p-1}(x)} \quad (p \geq 0),$$

then we have

$$H_0 = 1, \quad H_p/H_{p+1} = I_p/L, \quad H_1H_2\cdots H_n = 1, \quad H_{p+n} = H_p, \quad H_{n-p} = H_p^{-1}. \tag{3.10}$$

In above the first two equations follow from definition and the third one follows from (3.9). The last two equations follow from Theorem 1.2.1 and the symmetry. We can rewrite equation (3.8) as

$$\Phi_{p,0}(x) = H_p(x)\Phi_0(x) \quad p \geq 0.$$

Now substituting this into the case $s = 1$ of (3.7) we find inductively

$$\Phi_{p,1}(x) = H_p(x)\Big(\Phi_1(x) + p\frac{\Phi_o' - L'}{L} + \frac{\Phi_0}{L}\sum_{r=1}^{p}\frac{H_r'}{H_r}\Big) \quad p \geq 0.$$

Setting $p = n$ and using the third and fourth equations (3.10) and $f_n = f$, we deduce that $\Phi_0 = L$. $\square$

Now by applying (1.5) to equation (3.5) one can compute step by step each coefficient of expresion (3.5). In [16] the authors have found a recursive differential equation for the coefficients of $\mathcal{F}(w, x)$ (equation (1.3)), and also they have computed explicitly the first four terms. Here first we do the same for general case and then we go to the main theorem of this chapter.

Let $f(w, x) \in \mathcal{M}_n$ be the Zinger deformation of $f(x)$ and $A_{n-1}(x)$ be the leading coefficient of its differential equation. We write

$$A_{n-1}(x) = \prod_{i=1}^{p}(1 - \alpha_i x),$$

for some $\alpha_i \in \mathbb{C}$. Then for $L(x) := A_{n-1}(x)^{-1/n}$ we have

$$DL = L\sum_{i=1}^{p}\frac{x_i - 1}{n}, \quad Dx_i = x_i(x_i - 1) \tag{3.11}$$

where $x_i = (1-\alpha_i x)^{-1}$ and $D = x\frac{d}{dx}$. It turns out that the ring $\mathbb{Q}[x_1, \cdots, x_p]$ is a differential ring with $D = x\frac{d}{dx}$. This property helps us to compute inductively the higher derivatives of L. For simplicity we denote

$$X = (x_1, \cdots, x_p), \quad Y = \sum_{i=1}^{p} \frac{x_i - 1}{n}. \tag{3.12}$$

We have $D_w e^{\mu w} = e^{\mu w} \widetilde{D}_w$ where $\widetilde{D}_w = D + Lw$. By induction on k

$$\widetilde{D}_w^k = \sum_{m=0}^{k} \binom{k}{m} \widetilde{D}_w^m(1) D^{k-m} = D^k + kLwD^{k-1} + \cdots \tag{3.13}$$

where

$$\widetilde{D}_w^m(1) = \sum_{k=0}^{m} \mathcal{H}_{m,k}(n, X)(Lw)^{m-k}, \tag{3.14}$$

with $\mathcal{H}_{m,k}(n, X) \in \mathbb{Q}[x_1, \cdots, x_p]$, inductively given by $\mathcal{H}_{0,k} = \delta_{0,k}$ and for $m > 1$

$$\mathcal{H}_{m,k} = \mathcal{H}_{m-1,k} + \left(\sum_{i=1}^{p} x_i(x_i - 1) \frac{d}{dx_i} + (m-k)Y \right) \mathcal{H}_{m-1,k-1}. \tag{3.15}$$

For example, for $k = 0, 1, 2$ we find

$$\mathcal{H}_{m,0} = 1, \quad \mathcal{H}_{m,1} = \binom{m}{2} Y, \quad \mathcal{H}_{m,2} = \binom{m}{3} \sum_{i=1}^{p} \frac{x_i(x_i - 1)}{n} + 3\binom{m+1}{4} Y^2. \tag{3.16}$$

In the following lemma in the case $p = 1$ we give a formula for $\mathcal{H}_{m,k}$ whose reqursive part is independent of m.

Lemma 3.2.1. *Let $\mathcal{H}_{m,k}$ $(m, k \geq 0)$ be as in* (3.15)*. Then for $p = 1$, fixed $k \geq 1$ and varying m, we have*

$$\mathcal{H}_{m,k}(n, X) = \sum_{j=1}^{k} \binom{m}{j+k} Q_{k,j}(n, X), \tag{3.17}$$

with $Q_{k,j} \in \mathbb{Z}[n^{-1}, X]$ defined inductively by

$$Q_{0,j} = \delta_{0,j}, \quad Q_{k,j} = (X-1)\left(XQ'_{k-1,j} + \frac{jQ_{k-1,j} + (k+j-1)Q_{k-1,j-1}}{n} \right) \quad k \geq 1. \tag{3.18}$$

Proof. For $k = 1$, from definition we have

$$\mathcal{H}_{m,1} = \frac{1}{n}\binom{m}{2}(X-1) = \binom{m}{2}Q_{1,1}(n,X).$$

Now let us assume the identity is true for all $k' < k$. By definition we have

$$\begin{aligned}\Delta\mathcal{H}_m = \mathcal{H}_{m,k} - \mathcal{H}_{m-1,k} &= (X-1)\left(X\frac{d}{dX} + \frac{m-k}{n}\right)\mathcal{H}_{m-1,k-1}\\ &= (X-1)\left(X\frac{d}{dX} + \frac{m-k}{n}\right)\sum_{j=1}^{k-1}\binom{m-1}{k-1+j}Q_{k-1,j}(n,X).\end{aligned}$$

Then from (3.18) we have

$$\begin{aligned}\Delta\mathcal{H}_m = &\sum_{j=1}^{k-1}\binom{m-1}{k-1+j}\left(Q_{k,j} - \frac{jQ_{k-1,j} + (k+j-1)Q_{k-1,j-1}}{n}\right)\\ &+ (X-1)\sum_{j=1}^{k-1}\frac{m-k}{n}\binom{m-1}{k-1+j}Q_{k-1,j}.\end{aligned}$$

$$\begin{aligned}\Delta\mathcal{H}_m = &\sum_{j=1}^{k-1}\binom{m-1}{k-1+j}Q_{k,j} - (X-1)\sum_{j=1}^{k-1}\frac{k+j-1}{n}\binom{m-1}{k-1+j}Q_{k-1,j-1}\\ &+ (X-1)\sum_{j=1}^{k-1}\frac{m-j-k}{n}\binom{m-1}{k-1+j}Q_{k-1,j},\end{aligned}$$

but $(m-j-k)\binom{m-1}{k-1+j} = (k+j)\binom{m-1}{k+j}$, so we find

$$\Delta\mathcal{H}_m = \sum_{j=1}^{k-1}\binom{m-1}{k-1+j}Q_{k,j} + \frac{2k-1}{n}\binom{m-1}{2k-1}(X-1)Q_{k-1,k-1},$$

but by definition (3.18) we have $Q_{k,k} = \frac{2k-1}{n}(X-1)Q_{k-1,k-1}$. Therefore

$$\begin{aligned}\mathcal{H}_{m,k} - \mathcal{H}_{m-1,k} &= \sum_{j=1}^{k}\binom{m-1}{j-1+k}Q_{k,j}\\ &= \sum_{j=1}^{k}\left(\binom{m}{k+j} - \binom{m-1}{k+j}\right)Q_{k,j}(n,X),\end{aligned}$$

which completes the induction step. □
The function $\widetilde{f}(w,x) = e^{-\mu(x)w} f(w,x)$, satisfies the differential equation $\mathcal{L}\widetilde{\mathcal{F}} = 0$, where

$$\mathcal{L} = L^n(\widetilde{D}_w L(\widetilde{D}_w, x) - w^n) = \widetilde{D}_w^n + L^n \sum_{k=0}^{n-1} (A_{k-1} + DA_k)\widetilde{D}_w^k,$$

with $A_{-1} = -w^n$.
Using (3.13) and (3.14) we can expand $\mathcal{L}$ as $\mathcal{L} = \sum_{k=1}^{n} (Lw)^{n-k}\mathcal{L}_k$, with

$$\mathcal{L}_k = \sum_{i=0}^{k} \frac{E_{k,i}(n,X)}{n^{k-i}} D^i, \tag{3.19}$$

where

$$\begin{aligned} E_{k,i}(n,x) &= \binom{n}{i} \mathcal{H}_{n-i,k-i}(n,X)\, n^{k-i} \\ &\quad - L^n \sum_{r=1}^{k-i} \binom{n-r}{i} \widetilde{A}_{n-r}(x) \mathcal{H}_{n-i-r,k-i-r}(n,X)\, n^{k-i}, \end{aligned} \tag{3.20}$$

where $\widetilde{A}_k = A_{k-1} + DA_k$ $(0 \le k \le n-1)$. We have included the factor n^{i-k} in (3.19), because then in our main example (equation (1.1)), $E_{k,i}(n,X)$ are polynomials of n and X. Finally from the differential equation $\mathcal{L}\widetilde{\mathcal{F}} = 0$ and the asymptotic expansion of $\widetilde{\mathcal{F}}$ for large w we obtain the following first order ODEs for Φ_s:

$$\mathcal{L}_1(\Phi_s) + \frac{1}{L}\mathcal{L}_2(\Phi_{s-1}) + \frac{1}{L^2}\mathcal{L}_3(\Phi_{s-2}) + \cdots + \frac{1}{L^{n-1}}\mathcal{L}_n(\Phi_{s-n+1}) = 0, \quad s \ge 0, \tag{3.21}$$

with the initial condition $\Phi_s(0) = \delta_{0,s}$.

For $\mathcal{F}(w,x)$ given by equation (1.3), $p = 1$ (the number of linear factors of $A_{n-1}(x)$). In this case we have $L = (1 - n^n x)^{-1/n}$ and with the abuse of notation we set $X = x_1 = L^n$. We obtain from the differential equation (1.1) and equation (3.20)

$$\begin{aligned} E_{k,i}(n,X) &= \binom{n}{i} \mathcal{H}_{n-i,k-i}(n,X)\, n^{k-i} \\ &\quad - (X-1) \sum_{r=1}^{k-i} \binom{n-r}{i} S_r(n)\, \mathcal{H}_{n-i-r,k-i-r}(n,X)\, n^{k-i-r}, \end{aligned} \tag{3.22}$$

where $S_r(n)$ denotes the rth elementary symmetric function of $1, 2, \cdots, n$. The table below shows few terms of $E_{k,i}$.

k	$E_{k,0}$	$E_{k,1}$	$E_{k,2}$
0	1	0	0
1	$-n(X-1)$	n	0
2	$n(n-1)\Big(\frac{(n-2)(n-11)}{24}X-1\Big)(X-1)$	$-\frac{3}{2}n^2(n-1)(X-1)$	$\frac{n(n-1)}{2}$

In this case for $s = 1$ we have $\mathcal{L}_1(\Phi_1) + \frac{1}{L}\mathcal{L}_2(\Phi_0) = 0$. From (3.19) and (3.20) we have

$$\mathcal{L}_1 = nD - (X-1) \tag{3.23}$$

$$\mathcal{L}_2 = \binom{n}{2}D^2 - \frac{3(n-1)}{2}(X-1)D + \frac{n-1}{n}\Big(\frac{(n-2)(n-11)}{24}X - 1\Big)(X-1), \tag{3.24}$$

It turns out

$$\Phi_1(x) = \frac{(n-2)(n+1)}{24n}(L(x) - L(x)^n).$$

Similarly

$$\Phi_2(x) = \frac{(n-2)^2(n+1)^2}{2(24n)^2}(L - 2L^n + L^{2n-1}),$$

and in general one can show that $\Phi_s(x)$ for fixed s and n varying is an element of $\mathbb{Q}[n, n^{-1}, L, L^{-1}, X]$, where $X = L^n$ (See.[16]).

3.3 Logarithmic derivative of the Zinger deformation

In this section we study the structure of logarithmic derivative of the Zinger deformation in special cases. The reason is the following. As we have seen in the previous section, for the asymptotic expansion of $\mathcal{F}(w, x)$, we have $\Phi_s(x) \in \mathbb{Q}[n, n^{-1}, L, L^{-1}, X]$ and this polynomial is too complicated. Part of this difficulty goes back to the recursive equation of Φ_s which is in fact a differential equation and it is not useful in practice. But the advantage of the logarithmic derivative as we will see in the next theorem is that its coefficients

up to a simple factor just depend on n, X and its recursive equation is not in the differential equation form.
Let

$$L_n = \sum_{k=0}^{n-1} A_k(n,x) D_w^k, \tag{3.25}$$

where $A_k(n,x) \in \mathbb{Q}[n,x]$, $(0 \leq k \leq n-1)$ be a family of **MUM** differential operators with parameter n and

$$\mathcal{L}_n : (D_w L_n - w^n) y = 0, \tag{3.26}$$

be the corresponding **GHD** differential equations. Then we recall that for each n, $f_n(w,x)$ the holomorphic solutions with $f_n(0,0) = 1$ is the Zinger deformation of $f_n(0,x)$. When n varies these functions form a family. In the continuation we would like to study this family.

Theorem 3.3.1. *Let $\mathcal{L}_n$ be the family of* **GHD** *as in* (3.26)*, with*

$$-\frac{n-1}{2} A_{n-2}(x) = D A_{n-1}(x), \tag{3.27}$$

$$\frac{A_{n-i}(x)}{A_{n-1}(x)} \in \frac{1}{n^i}\mathbb{Q}[n,X], \quad 1 \leq i \leq n. \tag{3.28}$$

where X as in (3.12)*.*
Suppose $f(w,x) = f_n(w,x)$ is the family of Zinger deformations corresponding to $\mathcal{L}_n$ and set $\widetilde{f}(w,x) = e^{-\mu(x)w} f(w,x)$, where $\mu(x)$ is detremined as in (3.6)*. Then there is a power series $\mathcal{P}(n,X,T) \in \mathbb{Q}[n,X][[T]]$ such that the function $x\frac{\partial}{\partial x}\log \widetilde{f}(w,x)$ has the asymptotic expansion*

$$x\frac{\partial}{\partial x}\log \widetilde{f}(w,x) \sim \frac{1}{n}\mathcal{P}(n,X,\frac{1}{nwL}) \quad w \to \infty,$$

where $L = A_{n-1}(x)^{-1/n}$. The power series $\mathcal{P}$ is characterized uniquely by the recursive equation

$$\sum_{i=0}^{\infty} \mathcal{E}_i(n,X,T)\left(\mathcal{P}(n,X,T) + x\frac{\partial}{\partial x} - nYT\frac{\partial}{\partial T}\right)^i(1) = 1, \tag{3.29}$$

where

$$\mathcal{E}_i(n,X,T) = \sum_{k=0}^{\infty} E_{k,i}(n,X) T^s, \tag{3.30}$$

with $E_{k,i}(n,X)$ as in (3.20)*.*

Remark. If I_p's are symmetric then only half of A_k's need satisfy the condition (3.28). More preciesly if $A_{n-2k-1} \in \frac{1}{n^{2k+1}L^n}\mathbb{Q}[n,X]$, then by Corollary 1.3.1 it turns out that $A_{n-k} \in \frac{1}{n^k L^n}\mathbb{Q}[n,X]$.

Corollary 3.3.1. *For $\mathcal{F}(w,x)$ as in (1.3) we have*

$$x\frac{\partial}{\partial x}\log\widetilde{\mathcal{F}}(w,x) \sim \frac{1}{n}\sum_{s=0}^{\infty}\frac{P_s(n,X)}{(nwL)^s} \tag{3.31}$$

where $X = (1-n^n x)^{-1}, L = X^{1/n}$ and each $P_s(n,X) \in \mathbb{Q}[n,X]$ is a polynomial of degree $s+1$ in X and $2s+1$ in n.

Before giving the proofs of Theorem 3.3.1 and Corollary 3.3.1, we will show how the recursive power series uniquely works. If we write $\mathcal{P}(n,X,T)$ as

$$\mathcal{P}(n,X,T) = \sum_{s=0}^{\infty}P_s(n,X)T^s,$$

then we will show that each $P_s(n,X) \in \mathbb{Q}[n,X]$. We set

$$\mathcal{P}_i(n,X,T) := \sum_{s=0}^{\infty}P_{s,i}(n,X)T^s := \left(\mathcal{P}(n,X,T) + x\frac{\partial}{\partial x} - nYT\frac{\partial}{\partial T})\right)^i(1). \tag{3.32}$$

We note that $x\frac{\partial}{\partial x} = \sum_{i=1}^{p}(x_i-1)x_i\frac{\partial}{\partial x_i}$, and from the above definition we see that $P_{s,0}(n,X) = \delta_{s,0}$, $P_{s,1}(nX) = P_s(n,X)$ and

$$\begin{aligned} P_{s,i+1}(n,X) = &\left(n\sum_{i=1}^{p}(x_i-1)x_i\frac{\partial}{\partial x_i} - snY\right)P_{s,i}(n,X) \\ &+ \sum_{r=0}^{s}P_{r,i}(n,X)P_{s-r}(n,X), \quad i = 1,2,3,... \end{aligned} \tag{3.33}$$

Also from (3.29) we have

$$\sum_{r=1}^{s}\sum_{i=0}^{r}E_{r,i}(n,X)P_{s-r,i}(n,X) = 0, \quad s = 1,2,3,... \tag{3.34}$$

From the condition (3.28) it follows that $\widetilde{A}_{n-i}(x) \in \frac{1}{n^i L^n}\mathbb{Q}[n,X]$, hence $E_k(n,x)$ is a polynomial of n and $x_1,\cdots,x_p$. It means it is an elemnt of $\mathbb{Q}[n,X]$. Now for each $s \geq 1$ with equation (3.34) and given all $P_{s',i}(n,X)$

with $s' < s-1$ we find $P_{s-1}(n,X)$. since $E_{s,i}(n,X)$ and $P_{s',i}(n,X)$ are polynomials, therefore $P_{s-1}(n,X)$ which is uniquely determined in this way will be a polynomial. With this information and equation (3.33) we find $P_{s-1,i}(n,X)$ for all $i>1$. We show this procedure in some examples. From equation (3.34) we have

$$P_0 = -E_{1,0}.$$

Now with equation (3.33)

$$P_{0,2} = n\sum_{i=1}^{p}(x_i-1)x_i\frac{d}{dx_i}P_0 + P_0^2, \quad P_{0,3} = \sum_{i=1}^{p}(x_i-1)x_i\frac{d}{dx_i}P_{0,2} + P_0P_{0,2}.$$

Now equation (3.34) for $s=2$ says

$$P_1 = -(E_{2,0} + E_{2,1}P_0 + E_{2,2}P_{0,2}),$$

so we can find P_1. Now using equation (3.33) for $s=1$,

$$P_{1,2} = n(\sum_{i=1}^{p}(x_i-1)x_i\frac{d}{dx_i} - Y)P_1 + 2P_0P_1.$$

Finally for $s=3$:

$$\begin{aligned} P_2 = -(&E_{3,0} + E_{3,1}P_0 + E_{3,2}P_{0,2} + E_{3,3}P_{0,3} \\ &+E_{2,1}P_1 + E_{2,2}P_{1,2}), \end{aligned}$$

and we can find P_2. Applying this computation to our main example, i.e. $\mathcal{F}(w,x)$ shows that

$$\begin{aligned} P_0(n,X) &= X-1, \\ P_1(n,X) &= -\frac{(n+1)(n-1)(n-2)}{24}(X-1)X, \\ P_2(n,X) &= 0. \end{aligned}$$

Proof of Corollary 3.3.1. To show equation (3.31) we just need to check the two conditions (3.27) and (3.28). Since in this case we have full symmetry for I_p's, the first condition which is in fact the necessary condition for the symmetry of I_p's (cf. Lemma 1.3.1) holds. For the second one we have by definition

$$A_{n-r} = -xn^{n-r}S_r(n) = \frac{S_r(n)}{n^r}(1-\frac{1}{L^n}), \quad 1 < r \leq n.$$

Hence $\frac{A_{n-r}(x)}{A_{n-1}(x)} = \frac{X-1}{n^r}S_r(n) \in \frac{1}{n^r}\mathbb{Q}[n,X]$ and (3.31) follows from Theorem 3.3.1.

For the degree of X, we see from the recursive equation of $\mathcal{H}_{m,j}$ by a simple induction that in this case the degree of X for $\mathcal{H}_{k,j}$ is j, so the degree of $E_{k,i}(n,X)$ will be $k-i$ and from the recursive equation (3.33), the result follows.

For the degree of n from (3.17) and (3.18) one can easily check that the degree of n in $E_{k,i}(n,X)$ is $2k-i$. From this and the recursive equation (3.34) we find the result.□

Proof of Theorem 3.3.1. We define

$$\sum_{s=0}^{\infty} \Psi_{s,i}(n,x)\,T^s := \frac{\sum_{s=0}^{\infty} D^i\,\Phi_s(n,x)\,T^s}{\sum_{s=0}^{\infty} \Phi_s(n,x)\,T^s}, \tag{3.35}$$

where $D = x\frac{d}{dx}$. We notice that

$$\sum_{s=0}^{\infty} \Psi_s w^{-s} := \sum_{s=0}^{\infty} \Psi_{s,1} w^{-s} = x\frac{\partial}{\partial x}\log\tilde{\mathcal{F}}(w,x).$$

By differentiating from equation (3.35) we have

$$\sum_{s=0}^{\infty} D\Psi_{s,i}T^s = \frac{\sum_{s=0}^{\infty} D^{i+1}\Phi_s T^s}{\sum_{s=0}^{\infty}\Phi_s T^s} - \frac{\sum_{s=0}^{\infty} D^i\Phi_s T^s}{\sum_{s=0}^{\infty}\Phi_s T^s}.\frac{\sum_{s=0}^{\infty} D\Phi_s T^s}{\sum_{s=0}^{\infty}\Phi_s T^s},$$

or

$$\sum_{s=0}^{\infty}\Psi_{s,i+1}T^s = \sum_{s=0}^{\infty} D\Psi_{s,i}T^s + (\sum_{s=0}^{\infty}\Psi_{s,i}T^s)(\sum_{s=0}^{\infty}\Psi_s T^s). \tag{3.36}$$

We have

$$\sum_{r=0}^{s}\sum_{i=0}^{r}\frac{1}{n^{r-i}L^{r-1}}E_{r,i}\Psi_{s-r,i} = 0, \quad s = 1,2,3,... \tag{3.37}$$

For the moment let us assume this is true and we show by induction on s,that

$$\Psi_{s,i} = \frac{P_{s,i}(n,X)}{n^{s+i}L^s}, \tag{3.38}$$

where $P_{s,i}(n,X) \in \mathbb{Q}[n,X]$ given by the recursive equations (3.33),(3.34). For $s=0, i=1$ we have

$$\Psi_0 = \frac{D\Phi_0}{\Phi_0} = \frac{DL}{L} = Y = \sum_{i=1}^{p} \frac{x_i - 1}{n} = \frac{P_0}{n}.$$

Now if (3.37) is true for $s' \leq s$ and $i' < i+1$, from equation (3.36) for $\Psi_{s,i+1}$ we have

$$\begin{aligned}
\Psi_{s,i+1} &= D\Psi_{s,i} + \sum_{r=0}^{s} \Psi_{r,i}\Psi_{s-r,1} \\
&= D(\frac{P_{s,i}(n,X)}{n^{s+i}L^s}) + \sum_{r=0}^{s} \frac{P_{r,i}(n,X)}{n^{r+i}L^r}.\frac{P_{s-r,i}(n,X)}{n^{s-r+i}L^{s-r}} \\
&= \frac{nDP_{s,i} - s\sum_{k=1}^{p}(x_k-1)P_{s,i}}{n^{s+i+1}L^s} + \sum_{r=0}^{s} \frac{P_{r,i}(n,X)P_{s-r,i}(n,X)}{n^{s+i+1}L^s}.
\end{aligned}$$

but $D = x\frac{d}{dx} = \sum_{k=1}^{p} x_k(x_k-1)\frac{d}{dx_k}$, hence it follows from equation (3.33)

$$\Psi_{s,i+1} = \frac{P_{s,i+1}(n,X)}{n^{s+i+1}L^s}.$$

Now coming back to equation(3.37) we get

$$\Psi_{s-1} = \frac{-1}{n^s L^{s-1}}(E_{s,0} + \sum_{r=2}^{s}\sum_{i=1}^{r} E_{r,i}P_{s-r,i}), \tag{3.39}$$

therefore from (3.34) we find

$$\Psi_{s-1} = \frac{P_{s-1}(n,X)}{n^s L^{s-1}},$$

and this completes the induction step. The only thing is to prove the identity (3.37). We show this identity by induction on s.
For $s=1$ we have to check that

$$\frac{1}{n}E_{1,0} + E_{1,1}\Psi_0 = 0,$$

but $E_{1,0} = -nY$, $E_{1,1} = n$. Therefore we have to show

$$\Psi_0 = Y.$$

But by definition (3.35) for $i = 1$ we have

$$\sum_{s=0}^{\infty} \Psi_s T^s = \frac{\sum_{s=0}^{\infty} D\Phi_s T^s}{\sum_{s=0}^{\infty} \Phi_s T^s} = Y + O(T). \tag{3.40}$$

Hence the identity is true for $s = 1$. Now suppose that the identity is true for all $s' < s$, then for s from definition (3.35)

$$D\Phi_s = \sum_{i=0}^{s} \Phi_i \Psi_{s-i} = L\Psi_s + Y\Phi_s + \sum_{i=1}^{s-1} \Phi_i \Psi_{s-i}.$$

From (3.23) we have

$$\Psi_s = \frac{1}{L}(\frac{1}{n}\mathcal{L}_1(\Phi_s) - \sum_{j=1}^{s-1} \Phi_j \Psi_{s-j}). \tag{3.41}$$

But

$$\mathcal{L}_1(\Phi_s) = -\sum_{r=2}^{s+1} \frac{1}{L^{r-1}} \mathcal{L}_r(\Phi_{s+1-r}) = -\sum_{r=2}^{s+1}\sum_{i=0}^{r} \frac{E_{r,i}}{n^{r-i-1}L^{r-1}} D^i(\Phi_{s-r+1}).$$

Plugging this into (3.41) we find

$$\Psi_s = -\sum_{r=2}^{s+1}\sum_{i=0}^{r} \frac{E_{r,i}}{n^{r-i}L^r} D^i(\Phi_{s-r+1}) - \frac{1}{L}\sum_{j=1}^{s-1} \Phi_j \Psi_{s-j}.$$

Using the induction step for $\Psi_{s-j,1}$, $1 \leq j \leq s-1$, in the last equation we find

$$\begin{aligned} \Psi_s = &-\sum_{r=2}^{s+1}\sum_{i=0}^{r} \frac{E_{r,i}}{n^{r-i}L^r} D^i(\Phi_{s-r+1}) \\ &+ \sum_{j=1}^{s-1}\sum_{r=2}^{s-j+1}\sum_{i=1}^{r} \frac{\Phi_j E_{r,i} \Psi_{s-j+1-r,i}}{n^{r-i}L^r} + \sum_{j=1}^{s-1} \frac{\Phi_j E_{s-j+1,0}}{n^{s-j+1}L^{s-j+1}} \qquad (3.42) \\ &= -\sum_{r=2}^{s+1}\sum_{i=1}^{r} \frac{E_{r,i}}{n^{r-i}L^r} D^i(\Phi_{s-r+1}) - \frac{E_{s+1,0}}{n^{s+1}L^s} \qquad (3.43) \\ &+ \sum_{j=1}^{s-1}\sum_{r=2}^{s-j+1}\sum_{i=1}^{r} \frac{\Phi_j E_{r,i} \Psi_{s-j+1-r,i}}{n^{r-i}L^r}. \qquad (3.44) \end{aligned}$$

For fixed $2 \leq r \leq s+1$ and $1 \leq i \leq r$, from the definition of $\Psi_{s,r}$ (equation (3.35)) we have

$$\Psi_{s-r+1,i} = \frac{1}{L}(D^i(\Phi_{s-r+1}) - \sum_{j=1}^{s-r+1} \Phi_j \Psi_{s-j+1-r,i}).$$

It follows

$$\Psi_s = -\frac{E_{s+1,0}}{n^{s+1}L^s} - \sum_{r=2}^{s+1} \frac{E_{r,i}\Psi_{s-r+1,i}}{n^{r-i}L^{r-1}},$$

which completes the induction step.

□

Part II

Coefficients of $P_s(n, X)$ with respect to n

In the last chapter we proved that there exist polynomials $\{P_s(n,X)\}_{s\geq 0}$, such that

$$x\frac{\partial}{\partial x}\log \tilde{\mathfrak{F}}(w,x) \sim \frac{1}{n}\sum_{s=0}^{\infty}\frac{P_s(n,X)}{(nwL)^s} \quad w\to\infty.$$

We can consider $P_s(n,X)$ as a function of n and we can write

$$P_s(n,X) = \rho_s(X)n^{2s+1} + \mu_s(X)n^{2s} + \cdots . \tag{3.45}$$

This part has four chapters. In the first two chapters we compute the first and the second top coefficients of $P_s(n,X)$, namely, $\rho_s(X)$ and $\mu_s(X)$. In Chapter 6 we give some preliminaries which is necessary for Chapter 7. We define the Euler multiplication and the Euler map and give some identities for Stirling numbers. Finally in Chapter 7 we show that the generating function of the ℓth top coefficient of $P_s(n,X)$ where s varies belongs to the image of elementary functions under the Euler map.

Chapter 4

The leading coefficient of $P_s(n, X)$

In this chapter we study the generating function of $\{\rho_s(X)\}_{s\geq 0}$, the leading coefficient of $P_s(n, X)$), and we give a complete description for it. By experiment we find

$$\rho_0(X) = \rho_2(X) = \rho_4(X) = 0,$$

and

$$\begin{aligned}
\rho_1(X) &= \frac{-1}{24}(X^2 - X), \\
\rho_3(X) &= \frac{7}{5760}(6X^4 - 12X^3 + 7X^2 - X), \\
\rho_5(X) &= -\frac{31}{967680}(120X^6 - 360X^5 + 390X^4 - 180X^3 + 31X^2 - X).
\end{aligned}$$

These results motivated the authors in [16] to guess that $\rho_s(X) = \alpha_{s+1}e_{s+1}(X)$, where α_k is the coefficient of t^k in

$$\sum_{k\geq 0} \alpha_k t^k = \frac{t/2}{\sinh t/2} = 1 - \frac{1}{24}t^2 + \frac{7}{5760}t^4 - \frac{31}{967680}t^6 + \cdots \tag{4.1}$$

and $e_k(X)$ is an Euler polynomial. The Euler polynomials are defined inductively by

$$e_1(X) = X - 1, \quad e_{k+1}(X) = X(X-1)\frac{\partial}{\partial X}e_k(X), \quad k \geq 1. \tag{4.2}$$

For example we have $e_2(X) = X^2 - X$, $e_3(X) = 2X^3 - 3X^2 + X$, and generally

$$e_k(X) = \sum_{l=1}^{k} (-1)^{k-l} (l-1)! \left\{ {k \atop l} \right\} X^l \quad \in \quad \mathbb{Z}[X], \tag{4.3}$$

where $\left\{ {k \atop l} \right\}$ is a Stirling number of the second kind. For our purpose it is convenient to set $e_0(X) = \log X$ which is compatible with inductive definition (4.2). The main task of this chapter is to prove the above guess.

Remark 4.0.1. *Let $U := 1 - \frac{1}{X}$, then $U \frac{\partial}{\partial U} = X(X-1) \frac{\partial}{\partial X}$. Therefore we have $e_k(X) = E_k(U)$, where*

$$E_1(U) = \frac{U}{1-U}, \quad E_{k+1}(U) = U \frac{\partial}{\partial U} E_k(U) \quad k \geq 1.$$

The rational function $E_k(U)$ has the power series expansion

$$E_k(U) = \sum_{d=1}^{\infty} d^{k-1} U^d. \tag{4.4}$$

Here also by extending this definition to $k = 0$, we have

$$E_0(U) = \sum_{d=1}^{\infty} \frac{U^d}{d} = -\log(1-U).$$

One of the interesting properties of Euler polynomials is that Eisenstein series can be written as a sum of Euler polynomials. Indeed if $g_k(\tau) = \sum_{n \geq 1} \sigma_{k-1}(n) q^n$ is the Eisenstein series of weight k up to a constant, then we have by definition

$$g_k(\tau) = \sum_{n \geq 1} \sigma_{k-1}(n) q^n = \sum_{\ell, m \geq 1} m^{k-1} q^{\ell m} = \sum_{\ell \geq 1} E_k(q^\ell).$$

where $\tau \in \mathcal{H}$ in the upper half plane and $q = e^{2\pi i \tau}$.

4.1 Statement and proof

Theorem 4.1.1. *Let*

$$\widehat{\mathcal{P}}(X, T) = \sum_{s=-1}^{\infty} \rho_s(X) T^s,$$

be the generating function of polynomials $\{\rho_s(X)\}_{s\geq 0}$, with an extra term $\rho_{-1}(X) = \log X$, where $\rho_s(X)$ for $s \geq 0$, is the leading coefficient of $P_s(n, X)$. Then we have

$$\widehat{\mathcal{P}}(X,T) = -\sum_{n=1}^{\infty} \frac{Uq^{n-\frac{1}{2}}}{1-Uq^{n-\frac{1}{2}}},$$

where $q = e^T, U = 1 - \frac{1}{X}$. Furthermore for each $s \geq -1$

$$\rho_s(X) = \alpha_{s+1} e_{s+1}(X), \tag{4.5}$$

with α_k as defined in (4.1).

We give the proof in some steps. First we have to find a recursive equation for $\rho_{s,i}(X)$, the leading coefficient of $P_{s,i}(n, X)$,i.e. the coefficient of n^{2s+i}. This can be done by using the recursive equations (3.34) and (3.33) for $P_s(n, X)$. But we can not use it directly, because the original definition of $\mathcal{H}_{m,k}$, i.e. equation (3.15) is not good for our porpouse. We need to free m from the recursive equation which we have done this in Lemma 3.2.1.
Now let $a_{k,i}(X)$ and $\rho_{s,i}(X)$ be the leading coefficients of $E_{k,i}(n, X)$ and $P_{s,i}(n, X)$ respectively. From recursive equations (3.34) and (3.33) we have

$$\sum_{r=1}^{s}\sum_{i=0}^{r} a_{r,i}(X)\rho_{s-r,i}(X) = 1, \tag{4.6}$$

$$\rho_{s,i+1}(X) = D\rho_{s,i}(X) + \sum_{r=0}^{s} \rho_{r,i}(X)\rho_{s-r}(X). \tag{4.7}$$

The next problem is that in the first recursive equation we have a Hadamard product of two sequences $\{a_{r,i}(X)\}_{i\geq 0}$ and $\{\rho_{s-r,i}(X)\}_{i\geq 0}$. In this form we can not separate them by means of generating function. The following lemma resolves this problem.

Lemma 4.1.1. *Let $k \geq i \geq 0$,*

$$a_{k,i}(X) = \lim_{n\to\infty} \frac{E_{k,i}(n, x)}{n^{2k-i}}, \tag{4.8}$$

be the leading term of $E_{k,i}(n, X)$. Then

$$a_{k,i}(X) = \frac{1}{i!} a_{k-i,0}(X).$$

Proof. Let $q_{k,j}(X)$ and $h_k(X)$ be the leading coefficients of $Q_{k,j}(n,X)$ and $\mathcal{H}_{n,k}(n,X)$ respectively (here k and j are fixed with $n \to \infty$).

$$Q_{k,j}(n,X) = q_{k,j}(X)\, n^{-j} + O(n^{-j-1}).$$

The first few terms are

$$\begin{aligned}
&q_{1,1}(X) = X - 1,\\
&q_{2,1}(X) = X(X-1), \quad q_{2,2}(X) = 3(X-1)^2,\\
&q_{3,1}(X) = X(X-1)(2X-1), \quad q_{3,2}(X) = 10X(x-1)^2, \quad q_{3,3}(X) = 15(X-1)^3.
\end{aligned}$$

From Lemma 3.2.1 we have

$$\mathcal{H}_{n,k}(n,X) = \sum_{j=1}^{k} \binom{n}{k+j} Q_{k,j}(n,X) = \sum_{j=1}^{k} \frac{n^{k+j}}{(k+j)!} q_{k,j}(X)\, n^{-j} + O(n^{k-1}).$$

Hence

$$h_k(X) = \sum_{j=1}^{k} \frac{q_{k,j}(X)}{(k+j)!}. \tag{4.9}$$

We note that

$$S_r(n) = \frac{1}{2^r r!}\, n^{2r} + O(n^{2r-1}),$$

Hence from equation (3.20) we find

$$\begin{aligned}
a_{k,i}(X) &= \frac{h_{k-i}(X)}{i!} - (X-1) \sum_{r=1}^{k-i} \frac{h_{k-i-r}(X)}{2^r.r!\, i!}\\
&= \sum_{j=1}^{k-i} \frac{q_{k-i,j}(X)}{i!\,(k-i+j)!} - (X-1) \sum_{r=1}^{k-i} \sum_{j=1}^{k-i-r} \frac{q_{k-i-r,j}(X)}{2^r.r!\, i!\,(k-i-r+j)!}\\
&= \frac{1}{i!} a_{k-i,0}(X).
\end{aligned}$$

□

We set

$$\mathcal{A}_0(X,T) := \sum_{k=0}^{\infty} a_{k,0}(X) T^k, \tag{4.10}$$

and

$$R(X,T,Z) := \sum_{i=0}^{\infty} \mathcal{P}_i(X,T) \frac{Z^i}{i!}, \tag{4.11}$$

where

$$\mathcal{P}_i(X,T) := \sum_{s=0}^{\infty} \rho_{s,i}(X) T^s \tag{4.12}$$

(note that $\mathcal{P}_1 = \mathcal{P}(X,T) = \sum_{s\geq 0} \rho_s(X)T^s$).
Hence we can rewrite equation (4.6) in the following compact form

$$\mathcal{A}_0(X,T)R(X,T,T) = 1. \tag{4.13}$$

Also from (4.7) we have

$$\mathcal{P}_i(X,T) = (D+\mathcal{P})^i(1) \quad i \geq 0, \tag{4.14}$$

where $D = (X-1)X\frac{\partial}{\partial X} = U\frac{\partial}{\partial U}$. Hence from equation (4.14) we have

$$\begin{aligned} R(X,T,Z) &= \sum_{i=0}^{\infty} \mathcal{P}_i(X,T)\frac{Z^i}{i!} = \sum_{i=0}^{\infty}(D+\mathcal{P})^i(1)\frac{Z^i}{i!} \\ &= \exp\Big(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}(X,T)\frac{Z^i}{i!}\Big) \\ &= \exp(\widetilde{\mathcal{P}}(Ue^Z,T) - \widetilde{\mathcal{P}}(U,T)), \end{aligned} \tag{4.15}$$

where $\widetilde{\mathcal{P}}(U,T) = \int_0^U \mathcal{P}(\frac{1}{1-z},T)\frac{dz}{z}$.

The next step is to find a closed form for $\mathcal{A}_0(X,T)$ and $R(X,T,Z)$.

Proposition 4.1.1. *Let*

$$H(X,T,Z) = \sum_{j,k=0}^{\infty} q_{k,j}(X)\frac{T^{j+k}Z^j}{(j+k)!}. \tag{4.16}$$

be the generating function of $\{q_{k,j}(X)\}_{k,j\geq 0}$, where $q_{k,j}(X)$ is the leading coefficient of $Q_{k,j}(n,X)$.

i) We have

$$H(X,T,Z) = \exp(Zh(X,T)),$$

where

$$h(X,T) = \sum_{k=1}^{\infty} e_k(X)\frac{T^{k+1}}{(k+1)!} = Li_2(U) - Li_2(Ue^T) + T\log(1-U),$$

with $U = 1 - \frac{1}{X}$ *and*

$$Li_2(z) = \int_0^z -\frac{\log(1-u)}{u} du = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is the dilogarithm function.

ii) We have

$$\mathcal{A}_0(X,T) = H(X,T,T^{-1})\Big(1-(X-1)(e^{T/2}-1)\Big).$$

Proof. From (3.18) in Lemma 3.2.1 we find

$$q_{k,j}(X) = (X-1)Xq'_{k-1,j}(X) + (X-1)(j+k-1)q_{k-1,j-1}(X). \tag{4.17}$$

Hence we have

$$\begin{aligned}\frac{\partial}{\partial T}H &= \sum_{j,k=1}^{\infty} \frac{q_{k,j}(X)}{(j+k-1)!}T^{j+k-1}Z^j \\ &= \sum_{j,k=1}^{\infty} \frac{(X-1)Xq'_{k-1,j}(X)}{(j+k-1)!}T^{j+k-1}Z^j \\ &+ (X-1)\sum_{j,k=1}^{\infty} \frac{q_{k-1,j-1}(X)}{(j+k-2)!}T^{j+k-1}Z^j \\ &= X(X-1)\frac{\partial}{\partial X}H + ZT(X-1)H.\end{aligned}$$

So $H(X,T,Z)$ satisfies the following homogenous linear differential equation

$$[\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1)]H = 0. \tag{4.18}$$

so $H = \exp(h_0(X,T,Z))$ for some h_0, which satisfies

$$\Big(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X}\Big)h_0 = ZT(X-1).$$

It follows that $h_0(X,T,Z) = Zh_1(X,T)$ and $h_1(X,T)$ satisfies

$$\Big(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X}\Big)h_1 = T(X-1). \tag{4.19}$$

Now let $h_1(X,T) = \sum_{k=2}^{\infty} \epsilon_k(X)\frac{T^k}{k!}$, then it follows from (4.19)

$$\epsilon_2(X) = X - 1, \quad \epsilon_{k+1}(X) = X(X-1)\frac{\partial}{\partial X}\epsilon_k(X) \quad k \geq 2,$$

which is exactly the definition of $e_k(X)$, and therefore $h_1(X,T) = h(X,T)$. The function $h(X,T)$ is obtained from

$$g(X,T) = \sum_{k=1}^{\infty} e_k(X)\frac{T^{k-1}}{(k-1)!},$$

by two times integrating respect to T. But by Remark 4.0.1

$$\begin{aligned} g(X,T) &= \sum_{k=1}^{\infty} E_k(U)\frac{T^{k-1}}{(k-1)!} \\ &= \sum_{k=1}^{\infty} D^{k-1}E_1(U)\frac{T^{k-1}}{(k-1)!} \\ &= \frac{Ue^T}{1-Ue^T}. \end{aligned} \tag{4.20}$$

Therefore

$$h(X,T) = \int_0^T G(X,t)dt,$$

where

$$G(X,T) = \int_0^T g(X,t)dt = \int_0^T \frac{Ue^t}{1-Ue^t}dt = \log(1-U) - \log(1-Ue^T).$$

Finally

$$\begin{aligned} h(X,T) &= \int_0^T \log(1-Ue^t)dt + \int_0^T \log(1-U) \\ &= Li_2(U) - Li_2(Ue^T) + T\log(1-U). \end{aligned} \tag{4.21}$$

For the second part we have

$$\begin{aligned} \mathcal{A}_0(X,T) &= \sum_{j,k=0}^{\infty} \frac{q_{k,j}(X)}{(j+k)!}T^j - (X-1)\sum_{r=1}^{k}\sum_{j,k=0}^{\infty} \frac{q_{k-r,j}(X)}{2^r.r!\,(j+k-r)!}T^k \\ &= \sum_{j,k=0}^{\infty} \frac{q_{k,j}(X)}{(j+k)!}T^k - (X-1)\sum_{r=1}^{\infty}\frac{T^r}{2^r.r!}\sum_{j,k=0}^{\infty} \frac{q_{k,j}(X)}{(j+k)!}T^j \\ &= H(X,T,T^{-1})\Big(1-(X-1)(e^{T/2}-1)\Big). \end{aligned}$$

$\square$

Lemma 4.1.2. *Let $S(X,T) \in Q[X][[T]]$, such that*

$$\sum_{i=1}^{\infty} D^{i-1} S(X,T) \frac{T^i}{i!} = 0, \tag{4.22}$$

where $D = X(X-1)\frac{d}{dX}$. Then $S(X,T)$ is identically zero.

Proof. We have $D = U\frac{d}{dU} = \frac{d}{d\log V}$ where $U = e^V = 1 - \frac{1}{X}$. We set $\widetilde{S}(U,T) = S(X,T)$. Differentiating once more from equation (4.22), we get

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} D^i S(X,T) \frac{T^i}{i!} \\ &= \sum_{i=1}^{\infty} (U\frac{d}{dU})^i \widetilde{S}(U,T) \frac{T^i}{i!} = \widetilde{S}(Ue^T,T) - \widetilde{S}(U,T). \end{aligned}$$

It follows that $\widetilde{S}(Ue^T,T) = \widetilde{S}(U,T)$. Now let $\widetilde{S}(U,T) = \sum_{i=0}^{\infty} \widetilde{s}_i(U)T^i$, and k be the smallest indice such that $\widetilde{s}_k(U) \neq 0$. We have

$$\begin{aligned} 0 &= \widetilde{S}(Ue^T,T) - \widetilde{S}(U,T) = T^k[\widetilde{s}_k(Ue^T) - \widetilde{s}_k(U)] + T^{k+1}O(T) + O(T^{k+2}) \\ &= T^k[\widetilde{s}_k'(U)T + O(T^2)] + O(T^{k+2}) = T^{k+1}\widetilde{s}_k'(U) + O(T^{k+2}). \end{aligned}$$

Hence this implies that $\widetilde{s}_k(U)$ is constant. Substituting this into (4.22) we get

$$[\widetilde{s}_k(U)T^k + O(T^{k+1})]T + O(T^{k+1})\frac{T^2}{2!} + ... = 0$$

Hence $\widetilde{s}_k(U) \equiv 0$, and consequently $S(X,T) \equiv 0$. $\square$

Now we are ready to proof Theorem 4.1.1.

Proof of Theorem 4.1.1. From (4.13) we have

$$\log \mathcal{A}_0(X,T) + \log R(X,T,T) = 0.$$

Hence by the second part of Proposition 4.1.1

$$\log H(X,T,T^{-1}) + \log\Big(1 - (X-1)(e^{T/2} - 1)\Big) + \log R(X,T,T) = 0. \tag{4.23}$$

We have

$$\begin{aligned}\log\Big(1-(X-1)(e^{T/2}-1)\Big) &= \log\Big(X-(X-1)e^{T/2}\Big)\\ &= \log X + \log\Big(1-\frac{(X-1)}{X}e^{T/2}\Big)\\ &= -\log(1-U) + \log\Big(1-Ue^{T/2}\Big).\end{aligned}$$

Plugging this in to (4.23) and using the first part of Proposition 4.1.1 and equation (4.15) we find

$$\widetilde{\mathcal{P}}(U,T) - \widetilde{\mathcal{P}}(Ue^T,T) = \log\Big(1-Ue^{T/2}\Big)+\frac{1}{T}(Li_2(U)-Li_2(Ue^T)). \quad (4.24)$$

Let

$$S(U,T) = -\widetilde{\mathcal{P}}(U,T) + \frac{1}{T}Li_2(U) + \sum_{k=1}^{\infty}\log\Big(1-Ue^{(k-\frac{1}{2})T}\Big). \quad (4.25)$$

It follows from (4.24), $S(Ue^T,T) = S(U,T)$. Hence by Lemma 4.1.2, we have $S(U,T) = 0$, therefore

$$\widetilde{\mathcal{P}}(U,T) = \frac{1}{T}Li_2(U) + \sum_{k=1}^{\infty}\log\Big(1-Ue^{(k-\frac{1}{2})T}\Big). \quad (4.26)$$

But by definition

$$\widetilde{\mathcal{P}}(U,T) = \int_0^U \mathcal{P}(\frac{1}{1-z},T)\frac{dz}{z}.$$

Hence Theorem 4.1.1 follows from (4.26) by derivative with respect to U and the fact that $\widehat{\mathcal{P}} = \mathcal{P} - \frac{1}{T}\log(1-U)$.
The only thing is to show that $\rho_s = \alpha_{s+1}e_{s+1}$. From the first part we have

$$\begin{aligned}\widehat{\mathcal{P}}(X,T) &= -\sum_{m=1}^{\infty}\Bigg(\sum_{\substack{n>0\\ odd}} q^{nm/2}\Bigg)U^m\\ &= \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{1}{\sinh mT/2}\right)U^m\\ &= \sum_{m=1}^{\infty}\Bigg(\sum_{k=0}^{\infty}\alpha_k(mT)^{k-1}\Bigg)U^m = \sum_{k=0}^{\infty}\alpha_k E_k(U)T^{k-1}.\end{aligned}$$

□

4.2 Elliptic property

The interesting point about Theorem 4.1.1 is that up to an elementary function and a shift $z \to z + \tau/2$, $\widehat{\mathcal{P}}(X,T)$ is quite similar to "half" of $\zeta(\tau, z) = \frac{d}{dz}\log\theta(\tau, z) + \eta(1)z$ where

$$\theta(\tau, z) = \sum_{n\in\mathbb{Z}} (\frac{-4}{n}) q^{n^2/8} y^{n/2} = q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1-q^n)(1-q^n y)(1-q^{n-1}y^{-1}),$$

is a theta function with $q = e^{2\pi i\tau}, y = e^{2\pi i z}$ and $\eta : \Lambda_\tau \to \mathbb{C}$ the quasi-period homomorphism associated to $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.
If $w \in \Lambda$ and $\frac{w}{2} \notin \Lambda$, then

$$\eta(w) = 2\,\zeta(\frac{1}{2}w;\tau). \tag{4.27}$$

Hence we have

$$\frac{1}{2\pi i}\zeta(\tau, z) = \frac{1}{2} + \frac{1}{2\pi i}\eta(1)z - \sum_{n=1}^{\infty}\left(\frac{q^n y}{1-q^n y} - \frac{q^{n-1}y^{-1}}{1-q^{n-1}y^{-1}}\right).$$

Using the above equation and (4.27) we find

$$\frac{1}{(2\pi i)^2}\eta(1) = -\frac{1}{12} + 2\sum_{n\geq 1}\frac{q^n}{(1-q^n)^2}.$$

Now by extending the recursive equation to $s \geq -1$ we can define

$$\widehat{R}(X,T,Z) = \sum_{i=0}^{\infty} \widehat{\mathcal{P}}_i(X,T)\frac{Z^i}{i!}, \tag{4.28}$$

where $\widehat{\mathcal{P}}_i(X,T) = \sum_{s=-1}^{\infty} \rho_{s,i}(X)T^s$. Then using equations (4.15) and (4.26) we find

$$\begin{aligned}
\widehat{R}(X,T,Z) &= \exp\Big(\widetilde{\mathcal{P}}(Ue^Z,T) - \widetilde{\mathcal{P}}(U,T)\Big)\\
&= \exp\Big(\sum_{n\geq 1}\log(1-Ue^Z e^{(k-\frac{1}{2})T}) - \sum_{n\geq 1}\log(1-Ue^{(k-\frac{1}{2})T})\Big)\\
&= \frac{\prod_{n\geq 1}(1-Ue^Z e^{(k-\frac{1}{2})T})}{\prod_{n\geq 1}(1-Ue^{(k-\frac{1}{2})T})} = \frac{(Uq^{\frac{-1}{2}}e^Z;q)_\infty}{(Uq^{\frac{-1}{2}};q)_\infty},
\end{aligned} \tag{4.29}$$

where $q = e^T$ and $(x;q)_\infty = \prod_{n\geq 1}(1 - xq^n)$. We notice that by (4.28) and (4.29)

$$\widehat{\mathcal{P}}(X,T) = \frac{\partial \widehat{R}}{\partial Z}|_{Z=0} = -\sum_{k\geq 1} \frac{Uq^{k-\frac{1}{2}}}{1 - Uq^{k-\frac{1}{2}}}.$$

We have also

$$U\frac{\partial}{\partial U}(\log \widehat{R}) = -\sum_{k\geq 1} \frac{Ue^Z q^{k-\frac{1}{2}}}{1 - Ue^Z q^{k-\frac{1}{2}}} + \sum_{k\geq 1} \frac{Uq^{k-\frac{1}{2}}}{1 - Uq^{k-\frac{1}{2}}}. \tag{4.30}$$

Hence up to a constant, at $Z_0 = -2\log U$,

$$U\frac{\partial}{\partial U}(\log \widehat{R})|_{Z=Z_0} = -\frac{1}{2\pi i}\zeta(\tau, z - \frac{\tau}{2}) - \frac{1}{2\pi i}\eta(1) - \frac{1}{2}.$$

Naturally one could ask whether the remaining coefficients of $P_s(n,X)$ have a similar property or not. In the next chapter we try to answer this question by computing the second coefficient.

Chapter 5

The second top coefficient of $P_s(n, X)$

In this chapter we continue our computation. We follow the same idea as before to compute the second coefficient of $P_s(n, X)$.

5.1 Statement and steps of the proof

Theorem 5.1.1. *Let $\mu_s(X)$ be the second top coefficient of $P_s(n, X)$ with respect to n, i.e.*

$$P_s(n, X) = \rho_s(X)n^{2s+1} + \mu_s(X)n^{2s} + \cdots$$

Then we have the formula

$$\mu_s(X) = \begin{cases} (1 - \frac{s}{2})\alpha_s e_{s+1}(X) & \textit{if } s \textit{ is even,} \\ \sum\limits_{\substack{j,\, h \geq 1 \\ j+h=s+1}} \beta_{j,h}\, e_j(X)e_h(X) + \alpha_{s-1}\Big(\frac{e_{s+1}(X)}{4} - \frac{e_s(X)}{6}\Big) & \textit{if } s \textit{ is odd,} \end{cases} \tag{5.1}$$

where $\sum_{s=0}^{\infty} \alpha_s T^s = \frac{T/2}{\sinh T/2} =: S(T)$ is as in (4.1)and $\beta_{j,h}(j, h \geq 0)$ are given by

$$\sum_{j,\, h \geq 0} \beta_{j,h}\, T^j Z^h = \frac{1}{2} S(T)S(Z)S(T + Z) \cosh(\frac{T - Z}{2}) := S_2(T, Z), \tag{5.2}$$

Before starting the proof of Theorem 5.1.1, we give some examples to check it. From the recursive equations (3.34),(3.33) we find

$$\begin{aligned}
\mu_0(X) &= X-1,\\
\mu_1(X) &= \frac{X(X-1)}{12},\\
\mu_2(X) &= 0,\\
\mu_3(X) &= -\frac{183}{5760}\,X^4+\frac{398}{5760}\,X^3-\frac{267}{5760}\,X^2+\frac{52}{5760}\,X,\\
\mu_4(X) &= -\frac{7}{5760}(24\,X^5-60\,X^4-50\,X^3-15\,X^2-X).
\end{aligned}\tag{5.3}$$

We have $\alpha_0=1,\alpha_2=-\frac{1}{24},\alpha_4=\frac{7}{5760}$, and

$$\sum_{j,\,h\geq 1}\beta_{j,h}\,T^jZ^h=-\frac{1}{6}\,TZ+\frac{6}{1440}\,TZ^3+\frac{6}{1440}\,T^3Z+\frac{9}{640}\,T^2Z^2+\cdots$$

Therefore we find

$$\begin{aligned}
\mu_0(X) &= e_1(X)=X-1,\\
\mu_1(X) &= -\frac{1}{6}\,e_1(X)^2+\frac{1}{4}\,e_2(X)-\frac{1}{6}\,e_1(X)=\frac{X(X-1)}{12},\\
\mu_2(X) &= 0,\\
\mu_3(X) &= \frac{12}{1440}\,e_1(X)e_3(X)+\frac{9}{640}\,e_2(X)^2-\frac{1}{96}\,e_4(X)+\frac{1}{144}\,e_3(X)\\
&= -\frac{183}{5760}\,X^4+\frac{398}{5760}\,X^3-\frac{267}{5760}\,X^2+\frac{52}{5760}\,X,\\
\mu_4(X) &= -\frac{7}{5760}\,e_5(X)=-\frac{7}{5760}(24\,X^5-60\,X^4+50\,X^3-15\,X^2+X),
\end{aligned}$$

which coincide with (5.3).
To prove this theorem we use the same idea as Chapter 4.

Step 1. Let

$$\mathcal{M}_i(X,T):=\sum_{s=0}^{\infty}\mu_{s,i}(X)T^s \tag{5.4}$$

(for simplicity set $\mathcal{M}=\mathcal{M}_1$ and $\mu_s(X)=\mu_{s,1}(X)$). Set

$$M(X,T,Z):=\sum_{i=1}^{\infty}\mathcal{M}_i(X,T)\frac{Z^i}{i!}. \tag{5.5}$$

The following proposition determines $M(X,T,Z)$ in terms of $\mathcal{P}$ and $\mathcal{M}$, where $\mathcal{P}$ is as in (4.12).

Lemma 5.1.1. *We have*

$$M(X,T,Z) = G(X,T,Z)R(X,T,Z)$$

where

$$\begin{aligned} G(X,T,Z) = &\sum_{i=1}^{\infty} D^{i-1}\mathcal{M}(X,T)\frac{Z^i}{i!} \\ &- \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i-1}\binom{i-1}{j} e_j(X) D^{i-j-1}\mathcal{P}'(X,T)\right)\frac{Z^i}{i!}, \end{aligned} \tag{5.6}$$

where $R(X,T,Z)$ is defined as in (4.11)*.*

Step 2. Let $r_{k,j}(X)$ be the second coefficient of $Q_{k,j}(n,X)$, it means

$$Q_{k,j}(n,X) = q_{k,j}(X)\, n^{-j} + r_{k,j}(X)\, n^{-j-1} + O(n^{-j-2}).$$

Set also

$$J(X,T,Z) := \sum_{j,k=0}^{\infty} \frac{r_{k,j}(X)}{(j+k)!} T^{j+k} Z^j. \tag{5.7}$$

Lemma 5.1.2. *We have*

$$J(X,T,Z) = ZH(X,T,Z)\sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\binom{k-1}{j} e_j(X) e_{k-j}(X)\frac{T^{k+1}}{(k+1)!}.$$

where H is determined as in Proposition 4.1.1

Step 3. We write

$$E_{k,i}(n,X) = a_{k,i}(X)\, n^{2k-i-1} + b_{k,i}(X)\, n^{2k-i-2} + O(n^{2k-i-3}). \tag{5.8}$$

Lemma 5.1.3. *For fixed $i \geq 0$, we have*

$$\sum_{k=1}^{\infty} b_{k,i}(X) T^k = \mathcal{B}_0(X,T)\frac{T^i}{i!} + \mathcal{B}_1(X,T)\frac{T^i}{(i-1)!} + \mathcal{B}_2(X,T)\frac{T^i}{(i-2)!},$$

where

$$\mathcal{B}_0(X,T) = f_1(X,T)J(X,T,T^{-1}) - \frac{1}{2}f_1(X,T)H''(X,T,Z)\Big|_{Z=T^{-1}}$$
$$- f_2(X,T)H(X,T,T^{-1}) + f_3(X,T)H'(X,T,Z)\Big|_{Z=T^{-1}}, \tag{5.9}$$
$$\mathcal{B}_1(X,T) = -f_1(X,T)H'(X,T,Z)\Big|_{Z=T^{-1}} + f_3(X,T)H(X,T,T^{-1}), \tag{5.10}$$
$$\mathcal{B}_2(X,T) = -\frac{1}{2}f_1(X,T)H(X,T,T^{-1}), \tag{5.11}$$

where $'$ *is* $\frac{\partial}{\partial T}$ *and* $f_1(X,T) = X-(X-1)e^{T/2}$, $f_2(X,T) = (X-1)(\frac{T}{2}-\frac{T^2}{6})e^{T/2}$, *and* $f_3(X,T) = (X-1)\frac{T}{2}e^{T/2}$.

Now we prove these steps.

Proof of Lemma 5.1.1. From the recursive equations (3.33) we have for $i \geq 1$

$$\mu_{s,i+1}(X) = (X-1)X\frac{d}{dX}\mu_{s,i}(X) - s(X-1)\rho_{s,i}(X)$$
$$+ \sum_{r=0}^{s}\rho_{r,i}(X)\mu_{s-r}(X) + \sum_{r=0}^{s}\mu_{r,i}(X)\rho_{s-r}(X). \tag{5.12}$$

It is equivalent to

$$\mathcal{M}_{i+1}(X,T) = (X-1)X\frac{d}{dX}\mathcal{M}_i(X,T) - (X-1)T\frac{d}{dT}\mathcal{P}_i(X,T)$$
$$+ \mathcal{P}_i(X,T)\mathcal{M}(X,T) + \mathcal{P}(X,T)\mathcal{M}_i(X,T), \tag{5.13}$$

or with (5.5)

$$\frac{\partial}{\partial Z}M(X,T,Z) - \mathcal{M}(X,T)$$
$$= X(X-1)\frac{\partial}{\partial X}M(X,T,Z) + \mathcal{M}(X,T)(R(X,T,Z)-1)$$
$$+ \mathcal{P}(X,T)M(X,T,Z) - (X-1)T\frac{\partial}{\partial T}R(X,T,Z), \tag{5.14}$$

or

$$(\frac{\partial}{\partial Z} - U\frac{\partial}{\partial U} - \mathcal{P})M = (\mathcal{M} - (X-1)T\frac{\partial}{\partial T})R. \tag{5.15}$$

We write

$$M(X,T,Z) = G^*(X,T,Z)R(X,T,Z),$$

for some G^*. We would like to show that $G = G^*$. Plugging this into the differential equation (5.15) and using the fact that $R = \exp(\sum_{i=1}^{\infty} D^i \mathcal{P} \frac{Z^i}{i!})$ we find

$$\begin{aligned} &R\frac{\partial}{\partial Z}G^* + G^*(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{Z^{i-1}}{(i-1)!})R \\ &- RDG^* - G^*(\sum_{i=1}^{\infty} D^{i}\mathcal{P}\frac{Z^{i}}{i!})R - \mathcal{P}.G^*R \\ &= \mathcal{M}R - e_1(X)(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}'\frac{Z^{i}}{i!})R. \end{aligned}$$

After cancelling R from both sides we have

$$\begin{aligned} &\frac{\partial}{\partial Z}G^* + G^*\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{Z^{i-1}}{(i-1)!} \\ &- DG^* - G^*\sum_{i=1}^{\infty} D^{i}\mathcal{P}\frac{Z^{i}}{i!} - \mathcal{P}.G^* \\ &= \mathcal{M} - e_1(X)\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}'\frac{Z^{i}}{i!}. \end{aligned}$$

It turns out

$$\frac{\partial}{\partial Z}G^* - DG^* = \mathcal{M} - e_1(X)(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}'\frac{Z^{i}}{i!}). \tag{5.16}$$

This equation implies that

$$G^*(X,T,Z) = \sum_{i=1}^{\infty} g_i(X,T)\frac{Z^i}{i!} + \sum_{i=1}^{\infty} D^{i-1}\mathcal{M}(X,T)\frac{Z^i}{i!}, \tag{5.17}$$

with $g_1 = 0$ and for $i \geq 1$

$$g_{i+1} - Dg_i = -e_1(X)D^{i-1}\mathcal{P}'. \tag{5.18}$$

By a simple induction one can show

$$g_i(X,T) = -\sum_{j=1}^{i-1} \binom{i-1}{j} e_j(X)D^{i-j-1}\mathcal{P}'. \tag{5.19}$$

Hence $G^* = G$. $\square$

Proof of Lemma 5.1.2. From (3.18) we have

$$r_{k,j}(X) = (X-1)X\frac{d}{dX}r_{k-1,j}(X)+(X-1)\Big((j+k-1)r_{k-1,j-1}(X)+jq_{k-1,j}(X)\Big). \tag{5.20}$$

Hence $J(X,T,Z)$ satisfies the following non homogenous linear differential equation

$$\Big(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1)\Big)J = (X-1)Z\frac{\partial}{\partial Z}H. \tag{5.21}$$

Ansatz.

$$J(X,T,Z) = Zf(X,T)H(X,T,Z),$$

for some f.
From (5.21) we have

$$\begin{aligned} Zf(X,T)\Big(\frac{\partial}{\partial T} - X(X-1)T\frac{\partial}{\partial X} - ZT(X-1)\Big)H \\ +ZH(\Big(\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X}\Big)f(X,T) = (X-1)Zh(X,T)H. \end{aligned} \tag{5.22}$$

But from (4.18)

$$[\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1)]H = 0,$$

after cancelling this identity from equation (5.22) and using the fact that

$$H(X,T,Z) = \exp\Big(Z\sum_{k=1}^{\infty} e_k(X)\frac{T^{k+1}}{(k+1)!}\Big),$$

we find that

$$\frac{\partial}{\partial T}f = X(X-1)\frac{\partial}{\partial X}f + (X-1)\sum_{k=1}^{\infty} e_k(X)\frac{T^{k+1}}{(k+1)!}. \tag{5.23}$$

We write

$$f(X,T) = \sum_{k=1}^{\infty} \epsilon_k(X)\frac{T^{k+1}}{(k+1)!}.$$

From (5.23) we have $\epsilon_2(X) = 0$, and

$$\epsilon_{k+1}(X) = X(X-1)\frac{\partial}{\partial X}\epsilon_k(X) + e_1(X)e_k(X), \quad k \geq 2.$$

By a simple induction one can show that for $k \geq 2$

$$\epsilon_{k+1}(X) = \sum_{j=1}^{k-1} \binom{k-1}{j} e_j(X) e_{k-j}(X). \tag{5.24}$$

This completes the proof.□

Proof of Lemma 5.1.3. First from the definition of $E_{k,i}$, we find $b_{k,i}$. From equation (3.20), up to the second coefficient we have

$$\begin{aligned}
E_{k,i}(n,x) &\equiv \sum_{j=1}^{k-i} \frac{n^{k+j} - S_1(k+j-1) n^{k+j-1}}{(k-i+j)!\, i!} (n^{-j} q_{k-i,j} + n^{-j-1} r_{k-i,j}) \\
&- (X-1) \sum_{j=1}^{k-i-r} \frac{n^{k+j-r}}{(k-i-r+j)!\, i!} (\sigma_r\, n^r + \tau_r\, n^{r-1})(n^{-j} q_{k-i-r,j} + n^{-j-1} r_{k-i-r,j}) \\
&+ (X-1) \sum_{j=1}^{k-i-r} \frac{n^{k+j-1} \sum_{p=r}^{k-j+1} p}{(k-i-r+j)!\, i!} (\sigma_r\, n^r + \tau_r\, n^{r-1})(n^{-j} q_{k-i-r,j} + n^{-j-1} r_{k-i-r,j}),
\end{aligned}$$

where

$$S_r(n) = \sigma_r\, n^{2r} + \tau_r\, n^{2r-1} + O(n^{2r-2}), \tag{5.25}$$

with

$$\sigma_r = \frac{1}{2^r r!}, \quad \tau_r = \frac{r(5-2r)}{3.2^r r!}.$$

Hence we have

$$\begin{aligned}
b_{k,i}(X) &= \sum_{j=1}^{k-i} \frac{r_{k-i,j}(X)}{i!\,(k-i+j)!} - \sum_{j=1}^{k-i} \frac{S_1(k+j-1)}{i!\,(k-i+j)!} q_{k-i,j}(X) \\
&- (X-1) \Bigg(\sum_{j=1}^{k-i-r} \frac{\sigma_r\, r_{k-i-r,j}(X)}{i!(k-i-r+j)!} + \sum_{j=1}^{k-i-r} \frac{\tau_r\, q_{k-i-r,j}(X)}{i!(k-i-r+j)!} \\
&- \sum_{j=1}^{k-i-r} \frac{S_1(k+j-1) - S_1(r-1)}{i!(k-i-r+j)!} \sigma_r\, q_{k-i-r,j}(X) \Bigg).
\end{aligned} \tag{5.26}$$

Now we write

$$\begin{aligned}
S_1(k+j-1) &= \frac{(k+j)(k+j-1)}{2} \\
&= \frac{(k+j-i)(k-i+j-1)}{2} + \frac{i\,(i-1)}{2} + i\,(k-i+j).
\end{aligned} \tag{5.27}$$

Similarly

$$S_1(k+j-1)-S_1(r-1) = \frac{(k-i-r+j)(k-i-r+j-1)}{2} \\ + \frac{i\,(i-1)}{2} + i\,(k-i-r+j) + r(k-i-r+j) + i\,r. \tag{5.28}$$

The lemma follows by substituting equations (5.27) and (5.28) into (5.26) and collecting the terms with the factors $\frac{1}{i!}$, $\frac{1}{(i-1)!}$, and $\frac{1}{(i-2)!}$.$\square$

Now we are ready to prove Theorem 5.1.1.

5.2 Proof and further discussion

Proof of Theorem 5.1.1. From equation (3.34) we have

$$\sum_{r=1}^{s}\sum_{i=1}^{r} a_{r,i}(X)\mu_{s-r,i}(X) + \sum_{r=1}^{s}\sum_{i=0}^{r} b_{r,i}(X)\rho_{s-r,i}(X) = 0. \tag{5.29}$$

Applying Lemma 5.1.3 we have

$$\mathcal{A}_0(X,T)\sum_{i=1}^{\infty}\mathcal{M}_i(X,T)\frac{T^i}{i!} + \mathcal{B}_0(X,T)\sum_{i=0}^{\infty}\mathcal{P}_i(X,T)\frac{T^i}{i!} \\ + \mathcal{B}_1(X,T)\sum_{i=1}^{\infty}\mathcal{P}_i(X,T)\frac{T^i}{(i-1)!} + \mathcal{B}_2(X,T)\sum_{i=2}^{\infty}\mathcal{P}_i(X,T)\frac{T^i}{(i-2)!} = 0. \tag{5.30}$$

But by definition

$$\sum_{i=1}^{\infty}\mathcal{P}_i(X,T)\frac{T^i}{(i-1)!} = T\frac{\partial R(X,T,Z)}{\partial Z}|_{Z=T} \\ = (\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})R(X,T,T), \tag{5.31}$$

and

$$\sum_{i=1}^{\infty}\mathcal{P}_i(X,T)\frac{T^i}{(i-2)!} = T^2\frac{\partial^2 R(X,T,Z)}{\partial Z^2}|_{Z=T} \\ = \left(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-2)!} + (\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})^2\right)R(X,T,T). \tag{5.32}$$

Substituting these two in equation (5.30) and using Lemma 5.1.1 we find

$$\begin{aligned}
&\mathcal{A}_0(X,T)G(X,T,T)R(X,T,T)\\
&= -\mathcal{B}_0(X,T)R(X,T,T) - \mathcal{B}_1(X,T)(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})R(X,T,T)\\
&- \mathcal{B}_2(X,T)\Bigg(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-2)!} + (\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})^2\Bigg)R(X,T,T).
\end{aligned}$$

We recall that by the second part of Proposition 4.1.1

$$\mathcal{A}_0(X,T) = f_1(X,T)H(X,T,T^{-1}),$$

using this fact and Lemma 5.1.3 we find

$$\begin{aligned}
&f_1(X,T)H(X,T,T^{-1})G(X,T,T) =\\
&- f_1(X,T)J(X,T,T^{-1}) - \frac{1}{2}f_1(X,T)H''(X,T,Z)\Big|_{Z=T^{-1}}\\
&+ f_2(X,T)H(X,T,T^{-1}) - f_3(X,T)H'(X,T,Z)\Big|_{Z=T^{-1}}\\
&\Bigg(f_1(X,T)H'(X,T,Z)\Big|_{Z=T^{-1}} - f_3(X,T)H(X,T,T^{-1}\Bigg)\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!}\\
&+ \frac{1}{2}f_1(X,T)H(X,T,T^{-1})\Bigg(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-2)!} + (\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})^2\Bigg).
\end{aligned} \tag{5.33}$$

Finally applying Lemma 5.1.2 we find

$$\begin{aligned}
&\sum_{i=1}^{\infty} D^{i-1}\mathcal{M}\frac{T^i}{i!} = \sum_{i=1}^{\infty}\sum_{j=1}^{i-1}\binom{i-1}{j}e_j(X)D^{i-j-1}\mathcal{P}'\frac{T^i}{i!}\\
&- \sum_{k=1}^{\infty}\sum_{j=1}^{k-1}\frac{e_j(X)e_{k-j}(X)}{j!(k-j)!}\frac{T^k}{2(k+1)} + \frac{1}{2}(\sum_{k=1}^{\infty} e_k(X)\frac{T^k}{k!})^2\\
&+ \frac{1}{2}\sum_{k=1}^{\infty} e_k(X)\frac{T^k}{(k-1)!} - \frac{f_3}{f_1}\sum_{k=1}^{\infty} e_k(X)\frac{T^k}{k!} + \frac{f_2}{f_1}\\
&+ \big(\sum_{k=1}^{\infty} e_k(X)\frac{T^k}{k!} - \frac{f_3}{f_1}\big)\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!}\\
&+ \frac{1}{2}\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-2)!} + \frac{1}{2}(\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^i}{(i-1)!})^2,
\end{aligned} \tag{5.34}$$

and

$$\frac{f_3}{f_1} = \frac{(X-1)\frac{T}{2}e^{T/2}}{X-(X-1)e^{T/2}} = \frac{T}{2}\frac{Ue^{T/2}}{1-Ue^{T/2}} = \sum_{k=1}^{\infty} e_k(X)\frac{T^k}{2^k(k-1)!}, \tag{5.35}$$

$$\frac{f_2}{f_1} = \frac{(X-1)(\frac{T}{2}-\frac{T^2}{6})e^{T/2}}{X-(X-1)e^{T/2}} = \sum_{k=1}^{\infty} e_k(X)\frac{T^k}{2^k(k-1)!} - \sum_{k=1}^{\infty} e_k(X)\frac{T^{k+1}}{3.2^k(k-1)!}. \tag{5.36}$$

Now by Theorem 4.1.1 we have

$$\mathcal{P}(X,T) = \sum_{k=0}^{\infty} \alpha_{k+1}e_{k+1}(X)T^k,$$

hence

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}(X,T)\frac{T^i}{(i-1)!} = \sum_{j=1}^{\infty}\sum_{k=1}^{j-1} \alpha_{k+1}e_j(X)\frac{T^j}{(j-k-1)!}. \tag{5.37}$$

Applying this equation into (5.34), implies that the coefficient of T^s in both sides is

$$\begin{aligned}
&\sum_{k=0}^{s-1}\frac{D^{s-k-1}\mu_k(X)}{(s-k)!} = \\
&\sum_{k=0}^{s-1}\sum_{j=1}^{s-k-1} k\,\alpha_{k+1}\binom{s-k-1}{j}\frac{e_j(X)\,e_{s-j}(X)}{(s-k)!} \\
&+\frac{1}{2}\sum_{k=0}^{s-1}\frac{\alpha_{k+1}\,e_s(X)}{(s-k-2)!} + \frac{1}{2}\sum_{j=1}^{s-1}\sum_{k=0}^{j-1}\sum_{i=0}^{s-j-1}\frac{\alpha_{k+1}\,\alpha_{i+1}\,e_j(X)\,e_{s-j}(X)}{(j-k-1)!\,(s-j-i-1)!} \\
&+\sum_{j=1}^{s-1}\sum_{k=0}^{j-1}\frac{\alpha_{k+1}\,e_j(X)\,e_{s-j}(X)}{(j-k-1)!(s-j)!} - \sum_{j=1}^{s-1}\sum_{k=0}^{j-1}\frac{\alpha_{k+1}\,e_j(X)\,e_{s-j}(X)}{2^{s-j}(j-k-1)!\,(s-j-1)!} \\
&-\sum_{j=1}^{s-1}\frac{\binom{s-1}{j}e_j(X)\,e_{s-j}(X)}{(s+1)!} + \frac{1}{2}\sum_{j=1}^{s-1}\frac{e_j(X)\,e_{s-j}(X)}{j!\,(s-j)!} - \sum_{j=1}^{s-1}\frac{e_j(X)\,e_{s-j}(X)}{2^{s-j}j!\,(s-j-1)!} \\
&+(\frac{1}{2}+\frac{1}{2^s})\frac{e_s(X)}{(s-1)!} - \frac{e_{s-1}(X)}{3\cdot 2^{s-1}(s-2)!}.
\end{aligned} \tag{5.38}$$

Ansatz.

$$\mu_s(X) = \sum_{j=1}^{s}\beta^*_{j,s}\,e_j(X)e_{s+1-j}(X) + \gamma_s\,e_{s+1}(X) + \delta_s\,e_s(X),$$

where $\beta^*_{j,s}$,γ_s and δ_s are values which have to be determined. For γ_s and δ_s we have from (5.38)

$$\sum_{k=0}^{s-1} \frac{\gamma_k}{(s-k)!} e_s(X) - \sum_{k=0}^{s-1} \frac{\delta_k}{(s-k)!} e_{s-1}(X) = \tag{5.39}$$

$$\frac{1}{2}\sum_{k=0}^{s-1} \frac{\alpha_{k+1}\, e_s(X)}{(s-k-2)!} + \frac{e_s(X)}{2^s(s-1)!} - \frac{e_{s-1}(X)}{3\cdot 2^{s-1}(s-2)!}. \tag{5.40}$$

It turns out

$$\sum_{k=0}^{\infty} \delta_k T^k \sum_{i=1}^{\infty} \frac{T^i}{i!} = -\sum_{s=2}^{\infty} \frac{T^s}{3\cdot 2^{s-1}(s-1)!},$$

hence

$$\sum_{k=0}^{\infty} \delta_k T^k = -\frac{T^2/6\, e^{T/2}}{(e^T-1)} = -\frac{T^2/12}{\sinh(T/2)}. \tag{5.41}$$

We have also

$$\sum_{k=0}^{\infty} \gamma_k T^k \sum_{i=1}^{\infty} \frac{T^i}{i!} = \frac{1}{2}\sum_{k=0}^{\infty} \alpha_k T^k \sum_{i=1}^{\infty} \frac{T^i}{(i-1)!} + \sum_{s=1}^{\infty} \frac{T^s}{2^s(s-1)!},$$

hence

$$\begin{aligned}\sum_{k=0}^{\infty} \gamma_k T^k &= \frac{\frac{T/2}{\sinh(T/2)}\frac{T}{2}e^T + \frac{T}{2}e^{T/2}}{e^T-1} \\ &= \frac{T/4}{\sinh(T/2)}\Big(\frac{T/2}{\sinh(T/2)}e^{T/2} + 1\Big).\end{aligned} \tag{5.42}$$

For fixed r and s the coefficient of $e_r(X)\, e_{s-r}(X)$ of both sides is

$$\begin{aligned}&\sum_{k=0}^{s-1}\sum_{j=1}^{r}\sum_{i=0}^{d} \frac{\beta^*_{j,k}\binom{s-k-1}{i}}{(s-k)!} = \\ &\sum_{k=0}^{s-r-1} k\,\alpha_{k+1}\frac{\binom{s-k-1}{r}}{(s-k)!} + \frac{1}{2}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1} \frac{\alpha_{k+1}\,\alpha_{i+1}}{(r-k-1)!\,(s-r-i-1)!} \\ &+ \sum_{k=0}^{r-1} \frac{\alpha_{k+1}}{(r-k-1)!\,(s-r)!} - \sum_{k=0}^{r} \frac{\alpha_k}{2^{s-r}(r-k)!\,(s-r-1)!} \\ &- \frac{\binom{s-1}{r}}{(s+1)!} + \frac{1}{2}\,\frac{1}{r!\,(s-r)!},\end{aligned}$$

where $d = \min\{r-1, s-k-1\}$ and $i+j=r$. For each term we write a generating function. We start with the simplest one:

$$\sum_{s=2}^{\infty}\sum_{r=1}^{s-1}\frac{T^r Z^{s-r}}{r!\,(s-r)!} = \sum_{r=1}^{\infty}\frac{T^r}{r!}\sum_{k=1}^{\infty}\frac{Z^k}{k!} = (e^T-1)(e^Z-1). \tag{5.43}$$

$$\begin{aligned}\sum_{s=1}^{\infty}\sum_{r=1}^{s-1}\frac{\binom{s-1}{r}}{(s+1)!}T^r Z^{s-r} &= Z\sum_{s=1}^{\infty}\frac{(T+Z)^{s-1}-Z^{s-1}}{(s+1)!}\\ &= Z\left(\frac{e^{T+Z}-(T+Z)-1}{(T+Z)^2}-\frac{e^Z-Z-1}{Z^2}\right).\end{aligned} \tag{5.44}$$

$$\begin{aligned}&\sum_{s=2}^{\infty}\sum_{r=1}^{s-1}\sum_{k=0}^{r}\frac{\alpha_k\,T^r Z^{s-r}}{2^{s-r}(r-k)!\,(s-r-1)!}\\ &= \frac{Z}{2}\Big(\sum_{k=0}^{\infty}\alpha_k\,T^k\sum_{j=0}^{\infty}\frac{T^j}{j!}-1\Big)\sum_{i=0}^{\infty}\frac{Z^i}{2^i i!}\\ &= \Big(\frac{T/2}{\sinh(T/2)}e^T-1\Big)\frac{Z}{2}e^{Z/2}.\end{aligned} \tag{5.45}$$

Similarly

$$\begin{aligned}&\sum_{s=2}^{\infty}\sum_{r=1}^{s-1}\sum_{k=0}^{r-1}\frac{\alpha_{k+1}T^r Z^{s-r}}{(r-k-1)!\,(s-r)!}\\ &= \sum_{k=1}^{\infty}\alpha_k\,T^k\sum_{j=0}^{\infty}\frac{T^j}{j!}\sum_{i=1}^{\infty}\frac{Z^i}{i!}\\ &= \Big(\frac{T/2}{\sinh(T/2)}-1\Big)e^T(e^Z-1).\end{aligned} \tag{5.46}$$

We have

$$\begin{aligned}
&\sum_{s=1}^{\infty}\sum_{r=1}^{s-1}\sum_{k=0}^{s-r-1} k\,\alpha_{k+1}\frac{\binom{s-k-1}{r}}{(s-k)!}T^r Z^{s-r}\\
&=\sum_{s=1}^{\infty}\sum_{k=0}^{s-1}\sum_{r=1}^{s-k-1} k\,\alpha_{k+1}\frac{\binom{s-k-1}{r}}{(s-k)!}T^r Z^{s-r}\\
&=\sum_{s=1}^{\infty}\sum_{k=0} k\,\alpha_{k+1}\,Z^{k+1}\frac{(T+Z)^{s-k-1}-Z^{s-k-1}}{(s-k)!}\\
&=\sum_{k=1}^{\infty}(k-1)\,\alpha_k\,Z^k\Big(\sum_{j=1}^{\infty}\frac{(T+Z)^{j-1}}{j!}-\sum_{j=1}^{\infty}\frac{Z^{j-1}}{j!}\Big),
\end{aligned}$$

but

$$\begin{aligned}
\sum_{k=1}^{\infty}(k-1)\,\alpha_k\,Z^k &= Z\frac{\partial}{\partial Z}\Big(\frac{Z/2}{\sinh(Z/2)}\Big)-\big(\frac{Z/2}{\sinh(Z/2)}-1\big)\\
&= 1-(\frac{Z}{2})^2\frac{\cosh Z/2}{(\sinh(Z/2))^2},
\end{aligned}$$

hence

$$\sum_{s=1}^{\infty}\sum_{r=1}^{s-1}\sum_{k=0}^{s-r-1} k\,\alpha_{k+1}\frac{\binom{s-k-1}{r}}{(s-k)!}T^r Z^{s-r} = \Big(1-(\frac{Z}{2})^2\frac{\cosh Z/2}{(\sinh(Z/2))^2}\Big)\Big(\frac{e^{T+Z}-1}{T+Z}-\frac{e^Z-1}{Z}\Big). \tag{5.47}$$

$$\begin{aligned}
&\sum_{s=2}^{\infty}\sum_{r=1}^{s-1}\sum_{k=0}^{r-1}\sum_{i=0}^{s-r-1}\frac{\alpha_{k+1}\,\alpha_{i+1}}{(r-k-1)!\,(s-r-i-1)!}T^r Z^{s-r}\\
&=\sum_{k=1}^{\infty}\alpha_k\,T^k\sum_{j=0}^{\infty}\frac{T^j}{j!}\sum_{k=1}\alpha_k\,Z^k\sum_{j=0}^{\infty}\frac{Z^j}{j!}\\
&=\Big(\frac{T/2}{\sinh(T/2)}-1\Big)\Big(\frac{Z/2}{\sinh(Z/2)}-1\Big)e^{T+Z}, \qquad (5.48)
\end{aligned}$$

and finally with $i+j=r$ we have

$$\sum_{s=2}^{\infty}\sum_{r=1}^{s-1}\sum_{k=1}^{s-1}\sum_{j=1}^{r}\sum_{i=0}^{s-k-1}\frac{\beta^*_{j,k}\binom{s-k-1}{i}}{(s-k)!}T^r Z^{s-r}=$$
$$=\sum_{s=2}^{\infty}\sum_{k=1}^{s-1}\sum_{j=1}^{k}\beta^*_{j,k}\,T^j Z^{k+1-j}\frac{(T+Z)^{s-k-1}}{(s-k)!}$$
$$=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\beta^*_{j,k}\,T^j Z^{k+1-j}\sum_{s=1}^{\infty}\frac{(T+Z)^{s-1}}{s!}$$
$$=(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\beta^*_{j,k}\,T^j Z^{k+1-j})(\frac{e^{T+Z}-1}{T+Z}). \tag{5.49}$$

We conclude from (5.43)-(5.49)

$$S^*(T,Z)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\beta^*_{j,k}\,T^j Z^{k+1-j}$$
$$=\Bigg[\frac{1}{2}(e^T-1)(e^Z-1)-Z\bigg(\frac{e^{T+Z}-(T+Z)-1}{(T+Z)^2}-\frac{e^Z-Z-1}{Z^2}\bigg)$$
$$-\bigg(\frac{T/2}{\sinh(T/2)}e^T-1\bigg)\frac{Z}{2}e^{Z/2}+\bigg(\frac{T/2}{\sinh(T/2)}-1\bigg)e^T(e^Z-1)$$
$$+\bigg(1-(\frac{Z}{2})^2\frac{\cosh Z/2}{(\sinh(Z/2))^2}\bigg)\bigg(\frac{e^{T+Z}-1}{T+Z}-\frac{e^Z-1}{Z}\bigg)$$
$$+\frac{1}{2}\bigg(\frac{T/2}{\sinh(T/2)}-1\bigg)\bigg(\frac{Z/2}{\sinh(Z/2)}-1\bigg)e^{T+Z}\Bigg]\frac{T+Z}{e^{T+Z}-1}.$$

One can check directly

$$S_2(T,Z)=\frac{1}{2}\Big(S^*(Z,T)+S^*(T,Z)\Big)-\frac{1}{2}T^2(\frac{1}{\sinh T/2})'-\frac{1}{2}Z^2(\frac{1}{\sinh Z/2})'.$$

It means that

$$\mu_s(X)=\sum_{\substack{j,h\geq 1\\ j+h=s+1}}\beta_{j,h}\,e_j(X)e_h(X)+\delta_s\,e_s(X)+\gamma_s\,e_{s+1}(X)$$

From (5.41), (5.42) with a direct calculation and the fact that $S_2(Z,T)$ is an even function one can verify the statement. $\square$

We conclude this section by relating the function $G(X,T,T)$ wich is given in Lemma 5.1.1 to the "top coefficient", namely $\widehat{\mathcal{P}}(X,T)$ as calculated in Theorem 4.1.1. Our motivation is to check a similar elliptic property for the second top coefficient. Unfortunately this attempt is failed .
Let $\zeta_k(U) = \frac{Uq^{k+1/2}}{1-Uq^{k+1/2}}$, $(k \geq 0)$ and $q = e^T$. Set

$$\mathcal{Z}(q,U) = \sum_{k\geq 0} \zeta_k(U).$$

Then by Theorem 4.1.1 we have $\widehat{\mathcal{P}}(X,T) = -\mathcal{Z}(q,U)$. With this notation we have the following corollary.

Corollary 5.2.1. *With $U = 1 - \frac{1}{X}$ and $q = e^T$ as usual we have*

$$G(X,T,T) = -\frac{T^2}{4}\Big(\mathcal{Z}(q,U)+\frac{1}{2}\zeta_0\Big)^2+\frac{9}{16}T^2\zeta_0^2-\frac{T^2}{2}U\frac{\partial}{\partial U}\mathcal{Z}(q,U)+(\frac{T}{2}+\frac{T^2}{3})\zeta_0. \tag{5.50}$$

where $G(X,T,Z) = \dfrac{M(X,T,Z)}{R(X,T,Z)} = \dfrac{\sum_{i=1}^{\infty}\mathcal{M}_i(X,T)\frac{Z^i}{i!}}{\sum_{i=1}^{\infty}\mathcal{P}_i(X,T)\frac{Z^i}{i!}}$ *as in* (5.1.1) .

Proof. From Lemma 5.1.1we have

$$\begin{aligned} G(X,T,T) &= -\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e_j(X)D^{i-j-1}\mathcal{P}'(X,T)\frac{T^i}{i!} + \sum_{i=1}^{\infty} D^{i-1}\mathcal{M}(X,T)\frac{T^i}{i!} \\ &= -\sum_{i=1}^{\infty} D^{i-1}\Big(e_0\,\mathcal{P}'\Big)\frac{T^i}{i!} + e_0\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}'\frac{T^i}{i!} + \sum_{i=1}^{\infty} D^{i-1}\mathcal{M}\frac{T^i}{i!} \\ &= \sum_{i=1}^{\infty} D^{i-1}(\mathcal{M} - e_0\,\mathcal{P}')\frac{T^i}{i!} + e_0\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}'\frac{T^i}{i!}. \end{aligned}$$

We show that

$$\mathcal{M}(X,T) - e_0\,\mathcal{P}'(X,T) = \sum_{s=0}^{\infty}\mu_s(X)T^s + \sum_{s\geq 0}\beta_{0,s+1}\,e_0(X)\,e_{s+1}(X)\,T^s$$

By Theorems 4.1.1 and 5.1.1 it is enough to show that

$$2\sum_{s\geq 0}\beta_{0,s+1}\,(mT)^sU^m = -\mathcal{P}' = \sum_{s\geq 0} s\,\alpha_{s+1}\,(mT)^sU^m,$$

equivalently

$$2\sum_{s\geq 0}\beta_{0,s}\,t^s=\sum_{s\geq 0}(s-1)\alpha_s\,t^s,$$

or

$$S(t)^2\cosh(t/2)=\frac{\partial}{\partial t}S(t)-\,S(t),$$

where $s(t)=\frac{t/2}{\sinh(t/2)}$, and one can check easily the above identity.
Now in the continuation first we note that if we have a function like $f(x)=\sum_{n\geq 0}a_n\,x^n$, then

$$\sum_{n\geq 0}a_n\,e_n(X)T^n=\sum_{n\geq 0}\sum_{d\geq 1}a_n\,d^{n-1}U^dT^n=\sum_{d\geq 1}f(dT)\frac{U^d}{d}.$$

Hence by

$$\begin{aligned}
&f(X,Y,T)=\sum_{i,j\geq 0}\beta_{i,j}\,e_i(X)e_j(Y)T^{i+j-1}\\
&=\frac{1}{2}\frac{mT/2}{\sinh\frac{mT}{2}}\,\frac{nT/2}{\sinh\frac{nT}{2}}\,\frac{(m+n)/2}{\sinh\frac{(m+n)T}{2}}\cosh(\frac{(m-n)T}{2})\frac{U^m}{m}\frac{V^n}{n}\\
&=\frac{T^2}{16}(m+n)\frac{\coth(mT/2)\coth(nT/2)}{\sinh((m+n)T/2)}U^mV^n-\frac{T^2}{8}(m+n)\frac{U^mV^n}{\sinh((m+n)T/2)}\\
&=\frac{T^2}{16}(U\frac{\partial}{\partial U}+V\frac{\partial}{\partial V})\sum_{m,n\geq 1}\frac{(q^m+1)(q^n+1)q^{\frac{m+n}{2}}}{(q^m-1)(q^n-1)(q^{m+n}-1)}U^mV^n\\
&-\frac{T^2}{16}(U\frac{\partial}{\partial U}+V\frac{\partial}{\partial V})\sum_{m,n\geq 1}\frac{q^{\frac{m+n}{2}}}{q^{m+n}-1}U^mV^n,
\end{aligned}\tag{5.51}$$

where $q=e^T, U=1-\frac{1}{X}$ and $V=1-\frac{1}{Y}$. The main part which we have to consider is

$$\phi(U,V,q)=\sum_{m,n\geq 1}\frac{(q^m+1)(q^n+1)q^{\frac{m+n}{2}}}{(q^m-1)(q^n-1)(q^{m+n}-1)}U^mV^n.$$

We write this ϕ as a sum of three terms as follows. Let

$$\phi_1 = \sum_{m,n\geq 1} \frac{q^{\frac{m+n}{2}}}{q^{m+n}-1} U^m V^n,$$

$$\phi_2 = \sum_{m,n\geq 1} \frac{q^{\frac{m+n}{2}}}{(q^m-1)(q^{m+n}-1)} U^m V^n + (m \leftrightarrow n),$$

$$\phi_3 = \sum_{m,n\geq 1} \frac{q^{\frac{m+n}{2}}}{(q^m-1)(q^n-1)(q^{m+n}-1)} U^m V^n,$$

then $\phi = \phi_1 + 2\phi_2 + 4\phi_3$. Now we have

$$\begin{aligned}\phi_1 &= \sum_{m,n\geq 1} \sum_{k\geq 0} q^{\frac{m+n}{2}} q^{(m+n)k} U^m V^n \\ &= \sum_{k\geq 0} \frac{Uq^{k+1/2}}{1-Uq^{k+1/2}} \frac{Vq^{k+1/2}}{1-Vq^{k+1/2}} \\ &= \sum_{k\geq 0} \zeta_k(U)\zeta_k(V),\end{aligned}$$

Similarly one can check that

$$\begin{aligned}\phi_2 &= \sum_{k,k'\geq 0} \frac{Uq^{k'+k+1/2}}{1-Uq^{k'+k+1/2}} \frac{Vq^{k+1/2}}{1-Vq^{k+1/2}} + (U \leftrightarrow V) \\ &= \mathcal{Z}(q, Uq^k)\zeta_k(V) + \mathcal{Z}(q, Vq^k)\zeta_k(U).\end{aligned}$$

Finally for ϕ_3 we have

$$\phi_3 = \mathcal{Z}(q, Uq^k)\mathcal{Z}(q, Vq^k).$$

Hence it follows

$$\phi(U, V, q) = \sum_{k\geq 0} \Big(2\mathcal{Z}(q, Uq^k) + \zeta_k(U)\Big)\Big(2\mathcal{Z}(q, Vq^k) + \zeta_k(V)\Big), \tag{5.52}$$

and therefore

$$f(X, X, T) = \frac{T^2}{4} U \frac{\partial}{\partial U} \sum_{k\geq 0} \mathcal{Z}(q, Uq^k)\Big(\mathcal{Z}(q, Uq^k) + \zeta_k(U)\Big). \tag{5.53}$$

Similarly one can see that

$$g(X,T) = \sum_{s\geq 0}\Big[(1-\frac{s}{2})\alpha_s + \frac{1}{4}\alpha_{s-1}\Big]e_{s+1}T^s \tag{5.54}$$

$$= \frac{T^2}{2}U\frac{\partial}{\partial U}\sum_{m\geq 1}\frac{mq^{3m/2}}{(q^m-1)^2}U^m + \frac{T}{2}U\frac{\partial}{\partial U}\sum_{m\geq 1}\frac{q^{m/2}}{q^m-1}U^m, \tag{5.55}$$

and

$$h(X,T) = \sum_{s\geq 0} -\frac{1}{6}\alpha_{s-1}e_sT^s = -\frac{T^2}{6}U\frac{\partial}{\partial U}\sum_{m\geq 1}\frac{q^{m/2}}{q^m-1}U^m, \tag{5.56}$$

Hence we have

$$G(X,T,T) = \tilde{f}(Uq) - \tilde{f}(U) + \tilde{g}(Uq) - \tilde{g}(U) + \tilde{h}(Uq) - \tilde{h}(U).$$

where $U\frac{\partial}{\partial U}\tilde{f} = f$, etc. Therefore using equations (5.53),(5.55) and (5.56) we find

$$G(X,T,T) = -\frac{T^2}{4}\Big(\mathcal{Z}^2(q,U)+\zeta_0\mathcal{Z}(q,U)\Big)-\frac{T^2}{2}U\frac{\partial}{\partial U}\Big(\mathcal{Z}(q,U)-\zeta_0\Big)+(\frac{T}{2}-\frac{T^2}{6})\zeta_0. \tag{5.57}$$

Using the identity $U\frac{\partial}{\partial U}\zeta_k = \zeta_k^2 + \zeta_k$ the result follows immediately.$\Box$

Chapter 6

The algebra of Euler polynomials and Stirling numbers

So far we have computed the first two coefficients of $P_s(n, X)$. The method which has been used, theoretically can be applied to the rest of the coefficients, but practically it is impossible because each time the computations become more and more complicated. But if we look again at the first two coefficients we see that

$$\sum_{s=1}^{\infty} \alpha_s V_1^s, \in \mathbb{Q}(V_1, e^{V_1/2}),$$
$$\sum_{j,h=1}^{\infty} \beta_{j,h} V_1^j V_2^h \in \mathbb{Q}(V_1, V_2, e^{V_1/2}, e^{V_2/2}),$$

which means that they are elementary functions. The aim of the rest of this thesis is to prove such statement for the ℓth top coefficient of $P_s(n, X)$, without giving a closed form for it. This will be done in Chapter 7. In this chapter we introduce some algebraic formalism concerning Euler polynomials and Stirling numbers which will be needed later and which seems of interest in itself.

6.1 On products of Euler Polynomials

If A is an algebra over $\mathbb{Q}$ and $x_1, x_2, \cdots$ is an additive basis of A, then each product $x_i x_j$ can be written uniquely as a finite linear combination $\sum_l c_{ijk}\, x_k$ for certain numbers $c_{ijk} \in \mathbb{Q}$ and the algebra structure on A is completely determined by specifying the "structure constants" c_{kij}. If we apply this to the algebra $A = \mathbb{Q}[X]$ and the standard basis $x_i = X^i$, then the structure constants are completely trivial, being simply 1 if $i + j = k$ and 0 otherwise. But the Euler polynomials defined in (4.3) with 1 also form a basis of $\mathbb{Q}[X]$ and we can ask what the structure constants defined by $e_i(X)e_j(X) = \sum_k c_{ijk}\, e_k(X)$ are. The surprising fact is that, up to an elementary factor, c_{ijk} is equal simply to the rth Bernoulli number.

Proposition 6.1.1. *For $r, s \geq 1$ we have*

$$\begin{aligned} e_r(X).e_s(X) &= \frac{(r-1)!(s-1)!}{(r+s-1)!} e_{r+s}(X) \\ &+ \sum_{i=1}^{r+s-1} \frac{B_i}{i} \left[(-1)^{r-1} \binom{s-1}{r+s-i-1} + (-1)^{s-1} \binom{r-1}{r+s-i-1} \right] e_{r+s-i}(X), \end{aligned}$$

where B_i is the i-th Bernoulli number.

Remark We remark that a similar result for the Bernoulli polynomials and the usual Euler polynomials (which are slightly different from our definition) is given by L. Carlitz and N. Nielsen (for example cf.[3]), but I only learned of this recently and decided to retain my original proofs in this thesis. According to this if $\mathcal{B}_n(x) = \frac{B_n(x)}{x}$, where $B_n(x)$ is the usual Bernoulli polynomial, then we have

$$\begin{aligned} \mathcal{B}_i(x)\mathcal{B}_j(x) &= (-1)^{i-1} \frac{(i-1)!(j-1)!}{(i+j)!} B_{i+j} \\ &+ \sum_{0 \leq \ell < \frac{i+j}{2}} \left[\frac{1}{i} \binom{i}{2\ell} + \frac{1}{j} \binom{j}{2\ell} \right] B_{2\ell} \mathcal{B}_{i+j-2\ell}(x). \end{aligned}$$

We give two proofs for this proposition.

First Proof. Without loss of generality we can assume $r > 1$, because the statement is correct for $r = s = 1$ and in this case we have $e_1(x)^2 = e_2(X) -$

$e_1(X)$. Now by using the alternative definition for the Euler polynomials we have

$$\begin{aligned}
E_r(u).E_s(u) &= \sum_{c=1}^{\infty} c^{r-1}u^c.\sum_{d=1}^{\infty} d^{s-1}u^d \\
&= \sum_{m=2}^{\infty}\sum_{d=1}^{m-1} d^{r-1}(m-d)^{s-1}u^m \\
&= \sum_{m=2}^{\infty}\sum_{d=1}^{m-1}\sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i d^i m^{s-1-i}d^{r-1}u^m \\
&= \sum_{m=2}^{\infty}\sum_{d=1}^{m-1}\sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i d^{i+r-1}m^{s-1-i}u^m \\
&= \sum_{m=1}^{\infty}\sum_{i=0}^{s-1}\sum_{j=0}^{r+i-1}(-1)^i\frac{1}{r+i}\binom{r+i}{j}\binom{s-1}{i}B_j m^{r+s-j-1}u^m.
\end{aligned}$$

Now if

$$\gamma_j^{(r,s)} = \sum_{i=j+1-r}^{s-1}(-1)^i\frac{1}{r+i}\binom{r+i}{j}\binom{s-1}{i},$$

then we have

$$E_r(u).E_s(u) = \sum_{j=0}^{r+s-2}\gamma_j^{(r,s)}B_jE_{r+s-j}(u).$$

We see from (4.3)

$$e_k(X) = (k-1)!X^k + O(X^{k-1}),$$

therefore $\gamma_0^{(r,s)} = \frac{(r-1)!(s-1)!}{(r+s-1)!}$.
Now for $j>0$, $\gamma_j^{(r,s)} = \alpha_j^{(r,s)} - \frac{(-1)^{j-r}}{j}\binom{s-1}{j-r}$, where

$$\alpha_j^{(r,s)} = \sum_{i=0}^{s-1}(-1)^i\frac{1}{r+i}\binom{r+i}{j}\binom{s-1}{i}.$$

We set

$$A(x) = \sum_{j=1}^{\infty}\alpha_j^{(r,s)}x^j = \sum_{i=0}^{s-1}(-1)^i\frac{1}{r+i}(1+x)^{r+i}\binom{s-1}{i},$$

differentiating both sides

$$\begin{aligned}
A'(x) &= \sum_{i=0}^{s-1}(-1)^i(1+x)^{r+i-1}\binom{s-1}{i} \\
&= (1+x)^{r-1}\Big(1-(1+x)\Big)^{s-1} = (-x)^{s-1}(1+x)^{r-1} \\
&= (-1)^{s-1}\sum_{l=0}^{r-1}\binom{r-1}{l}x^{l+s-1}.
\end{aligned}$$

Now by integrate we find

$$j\alpha_j^{(r,s)} = (-1)^{s-1}\binom{r-1}{j-s},$$

finally

$$j\gamma_j^{(r,s)} = (-1)^{s-1}\binom{r-1}{j-s} - (-1)^{j-r}\binom{s-1}{j-r}.$$

But $r>1$ and $B_j = 0$ for odd $j>1$, so we find

$$B_j\gamma_j^{(r,s)} = \frac{B_j}{j}\left[(-1)^{s-1}\binom{r-1}{j-s} + (-1)^{r-1}\binom{s-1}{j-r}\right],$$

and it completes the proof.

Second Proof. Set $\widetilde{e}_r(X) = \dfrac{e_r(X)}{(r-1)!}$ and

$$\mathcal{E}(T) = \sum_{r=1}^{\infty}\widetilde{e}_r T^{r-1} = \frac{ue^T}{1-ue^T} = -\mathfrak{B}(v+T), \tag{6.1}$$

where $\mathfrak{B}(x) = \dfrac{e^x}{e^x-1} = \sum_{j=1}^{\infty}(-1)^j\dfrac{B_j}{j!}\,x^{j-1}$ and $u=e^v$. Rewriting Proposition 6.1.1 in a new form we find

$$\widetilde{e}_r\,\widetilde{e}_s = \widetilde{e}_{r+s} + (-1)^{r-1}\sum_{j=r}^{r+s-1}\frac{B_j\,\widetilde{e}_{r+s-j}}{j\,(r-1)!\,(j-r)!} + (-1)^{s-1}\sum_{j=s}^{r+s-1}\frac{B_j\,\widetilde{e}_{r+s-j}}{j\,(s-1)!\,(j-s)!}. \tag{6.2}$$

Hence we have

$$
\begin{aligned}
&\mathcal{E}(T)\mathcal{E}(Z) = \sum_{r,s\geq 1} \widetilde{e}_r(X)\widetilde{e}_s(X)T^{r-1}Z^{s-1} = \sum_{r,s=1}^{\infty} \widetilde{e}_{r+s}(X)T^{r-1}Z^{s-1} \\
&+ \sum_{j\geq r\geq 1} (-1)^{r-1} \frac{B_j T^{r-1} Z^{j-r}}{j\,(r-1)!\,(j-r)!} \left(\sum_{q\geq 1} \widetilde{e}_q(X)\, Z^{q-1} \right) + (T \leftrightarrow Z) \\
&= \sum_{n=1}^{\infty} \widetilde{e}_n(X) \frac{T^{n-1} - Z^{n-1}}{T - Z} \\
&+ \mathcal{E}(Z) \sum_{j\geq 1} \frac{(-1)^{j-1} B_j}{j!} (T-Z)^{j-1} + \mathcal{E}(T) \sum_{j\geq 1} \frac{(-1)^{j-1} B_j}{j!} (Z-T)^{j-1} \\
&= \frac{\mathcal{E}(T) - \mathcal{E}(Z)}{T - Z} + \left(\mathfrak{B}(T-Z) - \frac{1}{Z-T} \right) \mathcal{E}(Z) + \left(\mathfrak{B}(Z-T) - \frac{1}{T-Z} \right) \mathcal{E}(T) \\
&= \mathfrak{B}(T-Z)\mathcal{E}(Z) + \mathfrak{B}(Z-T)\mathcal{E}(T).
\end{aligned}
$$

But $\mathfrak{B}(x) = \frac{1}{2}(1 + \coth(x/2))$, and $\mathfrak{B}(-x) = 1 - \mathfrak{B}(x)$. Therefore to prove the proposition one has to verify the following identity

$$
\begin{aligned}
&\frac{1}{4}\left(1 + \coth(\frac{T+v}{2})\right)\left(1 + \coth(\frac{Z+v}{2})\right) = \\
&= -\frac{1}{4}\left(1 + \coth(\frac{T-Z}{2})\right)\left(1 + \coth(\frac{Z+v}{2})\right) + (T \leftrightarrow Z),
\end{aligned}
$$

which is straightforward. □

6.2 The Euler multiplication

Theorem 6.2.1. *There is a commutative and associative action $*$ on $\mathbb{Q}[V]$, which is defined by any of the following three properties:*

- $e^{\alpha V} * e^{\beta V} = \dfrac{e^{\alpha V + \beta} - e^{\alpha + \beta V}}{e^{\alpha} - e^{\beta}}$.
  (Here we have to consider $e^V = \sum_{i=0}^{\infty} \frac{V^i}{i!}$ and $*$ acts on each monomial and comparing the coefficient of $\alpha^i \beta^j$ in both sides gives the definition for $V^i * V^j$.)
- *The map*
  $$\phi : \mathbb{Q}[V] \to (X-1)\mathbb{Q}[X] = \mathbb{Q}[e_1, e_2, \cdots],$$

sending $V^i \to e_{i+1}$ $(i \geq 0)$ is a ring isomorphism.

- *The map $*$ is the composite map $Mo\psi$, where ψ defines the isomorphism $\mathbb{Q}(V) \otimes \mathbb{Q}(V) \simeq \mathbb{Q}(V_1, V_2)$ and the map*

$$M : \mathbb{Q}[V_1, V_2] \to \mathbb{Q}[V]$$

is defiend by

$$M(P(V_1, V_2)) =$$
$$= -2P(0,0) + \int_0^V P(t, V-t)dt - \sum_{k=1}^{\infty} \Big[P(-k, V+k) + P(V+k, -k) \Big]. \tag{6.3}$$

Here the infinite summation in (6.3) *is in the sense of 'zeta summation', i.e.*

$$\sum_{k=1}^{\infty} k^{n-1} = \zeta(1-n) = \begin{cases} -\frac{B_n}{n} & n \geq 2, \\ -\frac{1}{2} & n = 1. \end{cases} \tag{6.4}$$

In the following table we give the value of $V^i * V^j$, for $0 \leq i \leq j \leq 3$ (the values for $i > j$ are ommited since $*$ is commutative):

$*$	1	V	V^2	V^3
1	$V-1$	$\frac{1}{2}V^2 - \frac{1}{2}V$	$\frac{1}{3}V^3 - \frac{1}{2}V^2 + \frac{1}{6}V$	$\frac{1}{4}V^4 - \frac{1}{2}V^3 + \frac{1}{4}V^2$
V		$\frac{1}{6}V^3 - \frac{1}{6}V$	$\frac{1}{12}V^4 - \frac{1}{12}V^2$	$\frac{1}{120}V^5 - \frac{1}{12}V^3 + \frac{1}{30}V$
V^2			$\frac{1}{30}V^5 - \frac{1}{30}V$	$\frac{1}{60}V^6 - \frac{1}{60}V^2$
V^3				$\frac{1}{140}V^7 - \frac{1}{60}V^3 - \frac{1}{42}V$

Proof. From the first definition the commutativity turns out from the fact that the right hand side of (6.2.1) is symmetric respect to α, β, which proves the commutativity. For asscociativity we have

$$
\begin{aligned}
&(e^{\alpha V} * e^{\beta V}) * e^{\gamma V} \\
&= \frac{e^{\beta}}{e^{\alpha} - e^{\beta}}(e^{\alpha V} * e^{\gamma V}) - \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}}(e^{\beta V} * e^{\gamma V}) \\
&= \frac{e^{\beta}}{e^{\alpha} - e^{\beta}}\Big(\frac{e^{\alpha V+\gamma} - e^{\alpha+\gamma V}}{e^{\alpha} - e^{\gamma}}\Big) - \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}}\Big(\frac{e^{\beta V+\gamma} - e^{\beta+\gamma V}}{e^{\beta} - e^{\gamma}}\Big) \\
&= S(\alpha, \beta, \gamma) + S(\beta, \gamma, \alpha) + S(\gamma, \alpha, \beta)
\end{aligned}
$$

where

$$S(\alpha, \beta, \gamma) = \frac{1}{(e^{\alpha-\beta} - 1)(e^{\alpha-\gamma} - 1)} e^{\alpha V}.$$

This proves the associativity.

Remark. Using the identity

$$\sum_{i=1}^{n-1} \prod_{j \neq i} \Big(\frac{e^{\alpha_n} - e^{\alpha_j}}{e^{\alpha_i} - e^{\alpha_j}}\Big) = 1, \quad n \geq 2,$$

we find in general

$$e^{\alpha_1 V} * \cdots * e^{\alpha_n V} = \sum_{i=1}^{n} \prod_{j \neq i} \Big(\frac{1}{e^{\alpha_i - \alpha_j} - 1}\Big) e^{\alpha_i V}.$$

Now for equivalency by definition of M, for $r + s \geq 1$ we have

$$
\begin{aligned}
M(V_1^{r-1} V_2^{s-1}) &= \int_0^V t^{r-1} (V - t)^{s-1}\, dt \\
&+ \sum_{k=1}^{\infty} (-k)^{r-1} (V + k)^{s-1} + \sum_{k=1}^{\infty} (-k)^{s-1} (V + k)^{r-1} \\
&= \frac{(r-1)!(s-1)!}{(r+s-1)!} V^{r+s-1} \\
&+ \sum_{j=0}^{s} (-1)^{r-1} \frac{B_{j+r}}{j+r} \binom{s}{j} V^{s-j-1} + (r \leftrightarrow s),
\end{aligned}
$$

which exactly by Proposition 6.1.1 equals the inverse image of $\phi^{-1}(e_r e_s)$. Finally from the above equation we have

$$M\bigg(\frac{(\alpha V_1)^{r-1}}{(r-1)!}\frac{(\beta V_2)^{s-1}}{(s-1)!}\bigg) = \alpha^{r-1}\beta^{s-1}\frac{V^{r+s-1}}{(r+s-1)!} + \sum_{j=r}^{r+s-1}\frac{B_j}{j!}\binom{j-1}{r-1}(-\alpha)^{r-1}\beta^{j-r}\frac{(\beta V)^{s+r-j-1}}{(s+r-j-1)!} + (\alpha \leftrightarrow \beta). \tag{6.5}$$

Summing over all $r, s \geq 1$ we find

$$\begin{aligned} M(e^{\alpha V_1+\beta V_2}) &= \frac{e^{\alpha V} - e^{\beta V}}{\alpha - \beta} + e^{\beta V}\sum_{j=1}^{\infty}\frac{B_j}{j!}(\beta-\alpha)^{j-1} + e^{\alpha V}\sum_{j=1}^{\infty}\frac{B_j}{j!}(\alpha-\beta)^{j-1} \\ &= \frac{e^{\alpha V} - e^{\beta V}}{\alpha - \beta} + e^{\beta V}\Big(\frac{1}{e^{\beta-\alpha}-1} - \frac{1}{\beta-\alpha}\Big) + e^{\alpha V}\Big(\frac{1}{e^{\alpha-\beta}-1} - \frac{1}{\alpha-\beta}\Big) \\ &= \frac{e^{\alpha V+\beta} - e^{\alpha+\beta V}}{e^{\alpha} - e^{\beta}} = e^{\alpha V} * e^{\beta V}. \end{aligned}$$

□

6.3 The Euler map

We define the Euler map

$$\Phi_d : \mathbb{Q}[V_1, \cdots, V_d] \to \mathbb{Q}[X]\,[T]$$

on the basis $\{V_1^{i_1}\cdots V_d^{i_d}\}$ by

$$\Phi_d(V_1^{i_1}\cdots V_d^{i_d}) = \begin{cases} e_{i_1}(X)\cdots e_{i_d}(X)T^{i_1+\cdots i_d} & \text{if} \quad i_1, \cdots, i_d \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6.6}$$

where $e_i(X)$'s are Euler polynomials and this map is extended by linearity. We notice that Φ_d is not injective for $d > 1$. We recall from Chapter 4 that Euler polynomials have an alternative definition given by

$$e_i(X) = E_i(U) = \sum_{m\geq 1} m^{i-1}U^m, \quad U = 1 - \frac{1}{X}.$$

Hence for every $h \in \mathbb{Q}[V_1, \cdots, V_d]$ we can represent Φ_d as follows:

$$\Phi_d\Big(V_1 \cdots V_d h(V_1, \cdots, V_d)\Big) = \sum_{m_1, \cdots, m_d \geq 1} h(m_1 T, \cdots, m_d T) T^d U^{m_1 + \cdots m_d}. \quad (6.7)$$

Proposition 6.3.1. *Let $E_d = V_1 \cdots V_d \mathbb{Q}[V_1, \cdots, V_d]$. Then there is an associative and commutative multiplication $\star$ on E_1 such that For $d > 1$, the map Φ_d on E_d is the composite of*

$$E_d \simeq E_1^{\otimes d} \xrightarrow{\star} E_1 \xrightarrow{\Phi_1} \mathbb{Q}[X][T].$$

Proof. Since multiplication by V gives an isomorphism $V\mathbb{Q}[V] \simeq \mathbb{Q}[V]$, hence we can define $\star$ on $V\mathbb{Q}[V]$ as $V^i \star V^j = V.(V^{i-1} * V^{j-1})$, where $*$ is already constructed in Theorem 6.2.1. The statement follows.$\square$

Now by linearity one can extend the map Φ_d to $\mathbb{Q}[[V_1, \cdots, V_d]]$, the completion of $\mathbb{Q}[V_1, \cdots, V_d]$. We have the following lemma.

Lemma 6.3.1. *Let $\mathbf{E}_d = V_1 \cdots V_d \mathbb{Q}[[V_1, \cdots, V_d]]$. Then the following two diagrams are commutative:*

$$\begin{array}{ccc}
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\
{\scriptstyle (V_1 + \cdots + V_d).}\big\downarrow & & \big\downarrow {\scriptstyle TX(X-1)\frac{\partial}{\partial X}} \\
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]]
\end{array}$$

and

$$\begin{array}{ccc}
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\
{\scriptstyle V_1\frac{\partial}{\partial V_1} + \cdots + V_d\frac{\partial}{\partial V_d}}\big\downarrow & & \big\downarrow {\scriptstyle T\frac{\partial}{\partial T}} \\
\mathbf{E_d} & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]]
\end{array}$$

Proof. We recall that $D = X(X-1)\frac{\partial}{\partial X}$ and $D(e_i) = e_{i+1}$, so the map $\Phi_1 : V_1\mathbb{Q}[[V_1]] \to \mathbb{Q}[X][[T]]$ satisfies

$$\Phi_1(V_1 f(V_1)) = TD\Phi_1(f(V_1)),$$

for any function $f(V_1) \in V_1\mathbb{Q}[[V_1]]$. Now by extending to d variables we find that

$$\Phi_d\Big((V_1 + \cdots + V_d) f(\mathbf{V})\Big) = TD\Phi_d(f(\mathbf{V})),$$

for any function $f(\mathbf{V}) \in V_1 \cdots V_d \mathbb{Q}[[\mathbf{V}]]$, where $\mathbf{V} = (V_1, \cdots, V_d)$.
One can also see this from (6.7). We have

$$\begin{aligned} TD\Phi_d(V_1 \cdots V_d h(\mathbf{V})) &= \sum (m_1 + \cdots m_d) h(m_1 V_1, \cdots, m_d V_d) U^{m_1 + \cdots m_d} \\ &= \Phi_d\Big((V_1 + \cdots + V_d) V_1 \cdots V_d h(\mathbf{V})\Big). \end{aligned}$$

The commutativity of the second diagram is obvious by definition. □

6.4 Review of Stirling Numbers

The number of permutations of n symbols which have exactly m cycles is called a Stirling number of the first kind and equals $\left[{n \atop m}\right]$, where $\left[{n \atop m}\right]$ given by the following generating functions:

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} x^m, \tag{6.8}$$

$$\frac{\log(1+y)^m}{m!} = \sum_{n=m}^{\infty} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} \frac{y^n}{n!}. \tag{6.9}$$

In the following table we see the values of $\left[{n \atop m}\right]$ for $0 \leq m, n \leq 5$.

$n \backslash m$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	2	3	1	0	0
4	0	6	11	6	1	0
5	0	24	50	35	10	1

With special values

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!, \quad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\,(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}).$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2}, \quad \begin{bmatrix} n \\ n-2 \end{bmatrix} = \frac{3n-1}{4}\binom{n}{3}, \quad \begin{bmatrix} n \\ n-3 \end{bmatrix} = \binom{n}{2}\binom{n}{4}.$$

The number of ways of partitioning a set of n elements into m non-empty subsets is called a Stirling number of the second kind and denoted by $\left\{ {n \atop m} \right\}$. We have the following generating functions for them:

$$x^n = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} (x)_m, \tag{6.10}$$

$$\frac{(e^y - 1)^m}{m!} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{y^n}{n!}, \tag{6.11}$$

$$\frac{z^m}{(1-z)\cdots(1-mz)} = \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} z^n, \tag{6.12}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is the Pochhammer symbol. In the following table we see the value of $\left\{ {n \atop m} \right\}$ for $0 \le m, n \le 5$.

$n \backslash m$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

With the special values

$$\left\{ {n \atop 1} \right\} = 1, \quad \left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \quad \left\{ {n \atop 3} \right\} = \frac{1}{2}(3^{n-1} + 1) - 2^{n-1}.$$

$$\left\{ {n \atop n} \right\} = 1, \quad \left\{ {n \atop n-1} \right\} = \binom{n}{2}, \quad \left\{ {n \atop n-2} \right\} = \binom{n}{2} \frac{3n^2 - 11n + 10}{12}.$$

Equations (6.8) and (6.10) say that Stirling numbers are the matrix base change of the two bases of $\mathbb{Q}[x]$, namely, $\{x^n\}_{n \ge 0}$ and $\{(x)_n\}_{n \ge 0}$. Hence we find

$$\sum_{n \ge 0} (-1)^{j-n} \left\{ {n \atop i} \right\} \left[{j \atop n} \right] = \delta_{ij}. \tag{6.13}$$

Remark 6.4.1. *Stirling numbers, like binomial coefficients, can be defined by recursive equations:*

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = n\begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}, \tag{6.14}$$

$$\begin{Bmatrix} n+1 \\ m \end{Bmatrix} = m\begin{Bmatrix} n \\ m \end{Bmatrix} + \begin{Bmatrix} n \\ m-1 \end{Bmatrix}. \tag{6.15}$$

One of the advantage of these definitions is that they hold for all integers n, m, and we have the following duality law discovered by D. Knuth

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{bmatrix} -m \\ -n \end{bmatrix}. \tag{6.16}$$

6.5 Identities for Stirling numbers

In this section we give some identities which we need later in the proof of the main theorem of Chapter 7.

Let $S_p(r,n)$ (where r is omitted if it equals 0), be the pth elementary symmetric function of $r, r+1, \cdots, n$. For $r = 0$, from equation (6.8) we have $S_p(n-1) = \left[\begin{smallmatrix} n \\ n-p \end{smallmatrix}\right]$. The following lemma gives a formula for $S_p(r,m)$ in terms of Stirling numbers.

Lemma 6.5.1. *We have for all $p, r \geq 0$,*

$$S_p(r, n-1) = \sum_{v=0}^{p} (-1)^v \begin{Bmatrix} v+r-1 \\ r-1 \end{Bmatrix} \begin{bmatrix} n \\ n-p+v \end{bmatrix}. \tag{6.17}$$

Proof. From (6.8) follows

$$(x)_{r-1}(x-r)\cdots(x-n+1) = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} x^m.$$

Now by definition of $S_p(r,n)$ and from (6.12) we find

$$\sum_{p=0}^{n-1} (-1)^p S_p(r, n-1) x^{n-p-r} = (x-r)\cdots(x-n+1)$$
$$= (x)_{r-1}^{-1} \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} x^m = \sum_{v=0}^{\infty} \begin{Bmatrix} v+r-1 \\ r-1 \end{Bmatrix} x^{-v-r} \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} x^m$$
$$= \sum_{m=0}^{n} \sum_{v=0}^{\infty} (-1)^{n-m} \begin{Bmatrix} v+r-1 \\ r-1 \end{Bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} x^{m-v-r}.$$

Setting $m = n - p + v$ in the right hand side of the above equation we get the result.

Lemma 6.5.2. *We have*

i) When m varies, for fixed p, $\begin{bmatrix} m \\ m-p \end{bmatrix}$ is a polynomial of degree $2p$ of m. More precisely we have

$$\begin{bmatrix} m \\ m-p \end{bmatrix} = \sum_{k=p}^{2p} c_{p,k}\,(m)_k, \quad p \geq 0, \tag{6.18}$$

for some rational numbers $c_{p,k}$ given by the formula

$$c_{p,k} = \sum_{j=1}^{k} \frac{(-1)^{k-j+p}}{j!\,(k-j)!} \begin{bmatrix} j \\ j-p \end{bmatrix} \tag{6.19}$$

ii) when r, m vary, for fixed p, the coefficient of m^{2r-p} in $\begin{bmatrix} m+1 \\ m+1-r \end{bmatrix}$ is $\dfrac{\sigma_p(r)}{2^r r!}$, where $\sigma_p(r)$ is a polynomial of degree $2p$.

iii) We have for all $p, r, i, t, t', t'' \geq 0$

$$\frac{\begin{bmatrix} m \\ m-p \end{bmatrix}}{i!(m-r-i)!} = \sum_{t=p}^{2p} \sum_{t'=0}^{t} \sum_{t''=0}^{t-t'} \frac{c_{p,t} \binom{t}{t',t''} (r)_{t'}}{(i-t'')!\,(m-i-r-t+t'+t'')!}, \tag{6.20}$$

Proof. We set

$$\begin{bmatrix} m \\ m-p \end{bmatrix} = \sum_{k \geq 0} c_{p,k}\,(m)_k,$$

for unknown $c_{p,k}$ and first we want to show that $c_{p,k}$ is zero for $k \notin \{p, \cdots, 2p\}$. We set $C(x,y) = e^{-y}(1+xy)^{1/x}$ and we claim that the coefficient of $x^p y^k$ in $C(x,y)$ is $(-1)^p c_{p,k}$. To show this From (6.8), (6.9) we have

$$\begin{aligned}
(1+xy)^{1/x} &= \sum_{m \geq 0} (x^{-1})_m \frac{x^m y^m}{m!} = \sum_{m \geq p \geq 0} (-1)^p \begin{bmatrix} m \\ m-p \end{bmatrix} \frac{x^p y^m}{m!} \\
&= \sum_{m,p,k \geq 0} (-1)^p k! \binom{m}{k} c_{p,k}\, x^{m-p} \frac{y^m}{m!} \\
&= \sum_{p,k \geq 0} (-1)^p c_{p,k}\, x^p \sum_{m \geq k} \frac{y^m}{(m-k)!} \\
&= e^y \sum_{p,k \geq 0} (-1)^p c_{p,k}\, x^p y^k,
\end{aligned}$$

hence we get

$$C(x,y) = \sum_{p,k\geq 0} (-1)^p c_{p,k}\, x^p y^k = e^{-\frac{xy^2}{2}u(xy)} = \sum_{d=0}^{\infty} \left(\frac{-xy^2}{2}\right)^d \frac{u(xy)^d}{d!}, \quad (6.21)$$

where

$$u(z) = -\frac{2}{z^2}(\log(z+1) - z) = 1 - \frac{2}{3}z + \frac{2}{4}z^2 - \frac{2}{5}z^3 + \cdots$$

Since the power of y in the right hand side of (6.21) is strictly bigger than the power of x (except $d = 0$), hence the left hand side is so and therefore $k \geq p+1$. Moreover a general term in the right hand side, is of the form $x^{d+r}y^{2d+r}$, so we conclude in the left hand side $2p \geq k$.
Now the coefficient of $x^p y^k$ in $C(x,y)$ in one hand is $(-1)^p c_{p,k}$ and on the other hand by definition is $\sum_{j=1}^{k} \dfrac{(-1)^{k-j}}{j!(k-j)!}\left[{j \atop j-p}\right]$, which gives the formula for $c_{p,k}$.
For the second part we write

$$\left[{m+1 \atop m-r+1}\right] = \frac{1}{2^r r!} \sum_{k=0}^{2r-1} \sigma_k(r) m^{2r-k},$$

and we want to show that for fixed p, $\sigma_p(r)$ is a polynomial of degree 2ℓ. Using the recursive equation (6.14)

$$\left[{m+1 \atop m-r+1}\right] - \left[{m \atop m-r}\right] = m\left[{m \atop m-r+1}\right], \quad (6.22)$$

the coefficient of m^{2r-p} in both sides gives us the following identity

$$\sigma_p(r) = \sum_{k=0}^{p} (-1)^{p-k} \sigma_k(r) \binom{2r-k}{p-k} + 2r \sum_{k=0}^{p-1} (-1)^{p-k-1} \sigma_k(r-1) \binom{2r-2-k}{p-k-1}, \quad (6.23)$$

or

$$\begin{aligned} 2r\,\sigma_{p-1}(r-1) - (2r-p+1)\,\sigma_{p-1}(r) = {} & \sum_{k=0}^{p-2} (-1)^{p-k} \sigma_k(r) \binom{2r-k}{p-k} \\ & + 2r \sum_{k=0}^{p-2} (-1)^{p-k-1} \sigma_k(r-1) \binom{2r-2-k}{p-k-1}. \end{aligned}$$

For $p = 1$ we find $\sigma_0(r) = 1$ and by induction we find the result.
For the third part using the identity

$$(1+x)^m = (1+x)^r(1+x)^i(1+x)^{m-i-r}, \tag{6.24}$$

the power of x^t in both sides gives the following equation

$$\binom{m}{t} = \sum_{t',t''\geq 0} \binom{r}{t'}\binom{i}{t''}\binom{m-i-r}{t-t'-t''}$$

This equation, together with the first part gives the result. □

Chapter 7

The higher coefficients of $P_s(n, X)$

In this chapter we prove our main theorem. It says that for each fixed $\ell \geq 0$ the ℓth top coefficient of $P_s(n, X)$ with respect to n is a finite sum of the image of some elementary functions under the Euler map $\Phi_d : \mathbb{Q}[[V_1, \cdots, V_d]] \longrightarrow \mathbb{Q}[X][[T]]$ defined in Chapter 6.

7.1 Statement of the main theorem

To state the theorem we introduce some notations. Let K_d be the following ring

$$K_d := \mathbb{Q}(V_1, \cdots, V_d, e^{V_1/2}, \cdots, e^{V_d/2}) \cap \mathbb{Q}[[V_1, \cdots, V_d]]. \tag{7.1}$$

Then we define

$$\mathcal{K}_d := \Phi_d(K_d) \otimes \mathbb{Q}[T, T^{-1}] \subset \mathbb{Q}[X][T^{-1}, T]]$$

and we set $\mathcal{K} = \sum_{d \geq 1} \mathcal{K}_d$.

Lemma 7.1.1. *The space $\mathcal{K} \subset \mathbb{Q}[X][[T]]$ is closed under multiplication and differentiation.*

Proof. The multiplication follows from

$$\Phi_d\Big(F(V_1, \cdots, V_d)T^i\Big).\Phi_{d'}\Big(G(V_1, \cdots, V_{d'})T^j\Big) =$$
$$= \Phi_{d+d'}\Big(F(V_1, \cdots, V_d)G(V_{d+1}, \cdots, V_{d+d'})T^{i+j}\Big).$$

The differentiation with respect to T follows from the second diagram of Lemma 6.3.1 and the fact that K_d is closed under $\sum_i V_i \frac{\partial}{\partial V_i}$. $\square$

Corollary 7.1.1. *Suppose $a, b \in \mathbb{Q}[X][[T]]$, such that the functions $\frac{a}{b}$ and $\frac{d}{dT}(\log b) = \frac{b'}{b}$ are elements of $\mathcal{K}$. Then $\frac{a^{(k)}}{b} \in \mathcal{K}$ for all $k \geq 0$.*

Proof. This follows by induction on k, since

$$\frac{a^{(k+1)}}{b} = (\frac{a^{(k)}}{b})' + \frac{a^{(k)}}{b}.\frac{b'}{b} \in \mathcal{K}.$$

$\square$

Theorem 7.1.1. *Denote by $\rho_s^{(\ell)}(X)$ the coefficient of $n^{2s+1-\ell}$ in $P_s(n, X)$ and*

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{s=0}^{\infty} \rho_s^{(\ell)}(X) T^s$$

the corresponding generating function. Then we have

$$\mathcal{P}^{(0)}(X,T) \in \mathcal{K}_1, \quad \mathcal{P}^{(\ell)}(X,T) \in \sum_{d=1}^{2\ell} \mathcal{K}_d \qquad (\ell \geq 1).$$

Remark. The proof is constructive. Indeed we show that there exist effectively computable functions

$$\Pi^{(\ell,d)}(V_1, \cdots, V_d, T) \in K_d \otimes \mathbb{Q}[T, T^{-1}],$$

such that

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{d=1}^{2\ell} \Phi_d(\Pi^{(\ell,d)}). \tag{7.2}$$

Examples. The case $\ell = 0$ is essentially the content of Chapter 4. Indeed,from Theorem 4.1.1 we have

$$\mathcal{P}^{(0)}(X,T) = \mathcal{P}(X,T) = \sum_{s=1}^{\infty} \alpha_s e_s(X) T^{s-1} = T^{-1} \Phi_1\Big(\frac{V/2}{\sinh(V/2)}\Big) \in \mathcal{K}_1.$$

Similarly the case $\ell = 1$ (the second top coefficient) follws from what we did in Chapter 5. Specifically, from Theorem 5.1.1 we have

$$\mathcal{P}^{(1)}(X,T) = \sum_{s \geq 0} \mu_s(X) T^s = \Phi_2(\Pi^{(1,2)}) + \Phi_1(\Pi^{(1,1)}),$$

where

$$\Pi^{(1,2)} = T^{-1} S_2(V_1, V_2),$$
$$\Pi^{(1,1)} = T^{-1}\Big(V + \frac{V^2}{4}\Big)S(V) - T^{-1}\frac{V^2}{2}S'(V) - \frac{V}{6}S(V).$$

(Recall that $S_2(V_1, V_2) = S(V_1)S(V_2)S(V_1 + V_2)\cosh(\frac{V_1 - V_2}{2})$, and $S(V) = \frac{V/2}{\sinh(V/2)}$).

7.2 Statements of auxiliary results

To prove Theorem 7.1.1 we will need some propositions and a theorem that we state in this section. The proofs will be given in the next section. Recall the definition of $P_{s,i}(n, X)$ from (3.32). We can write

$$P_{s,i}(n, X) = \sum_{\ell=0}^{2s+1} \rho_{s,i}^{(\ell)}(X) n^{2s+i-\ell}, \quad (s \geq 0, i \geq 1).$$

For $i \geq 1$ we set

$$\mathcal{P}_i^{(\ell)}(X, T) := \sum_{s=0}^{\infty} \rho_{s,i}^{(\ell)}(X) T^s, \tag{7.3}$$

and $\mathcal{P}_0^{(\ell)} = \delta_{0\ell}$. We define

$$R_\ell(X, T, Z) := \sum_{i=0}^{\infty} \mathcal{P}_i^{(\ell)}(X, T)\frac{Z^i}{i!}, \quad \ell \geq 0. \tag{7.4}$$

Then we have the following statements.

Proposition 7.2.1. *For $\ell \geq 1$ we have*

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X, T)\frac{T^i}{i!} \in \sum_{d=1}^{2\ell} \mathcal{K}_d.$$

Proposition 7.2.2. *Let*

$$\mathcal{H}(X, T, Z, W) = \sum_{j,k \geq 0} Q_{k,j}(\frac{1}{W}, X)\frac{T^{j+k}}{(j+k)!}(Z/W)^k,$$

where $Q_{k,j}(n, X)$ is defined as in (3.18) *and* $W = \dfrac{1}{n}$. *Then* $\mathcal{H} = \exp(Z\hbar(X, T, W))$, *where*

$$\hbar(X, T, W) = \frac{1}{W}\int_0^T \left[\left(\frac{1-U}{1-Ue^t}\right)^W - 1\right] dt. \tag{7.5}$$

Remark 7.2.1. *Define coefficients* $q_{k,j}^{(\ell)}(X)$ *by the expansions*

$$Q_{k,j}(n, X) = \sum_{\ell=0}^{j} q_{k,j}^{(\ell)}(X) n^{-j-\ell}, \quad (n \to \infty). \tag{7.6}$$

and let $H_\ell(X, T, Z)$ *be the coefficient of* W^ℓ *in* $\mathcal{H}$. *Then by definition of* $\mathcal{H}$ *we have*

$$H_\ell(X, T, Z) = \sum_{j,k\geq 0} q_{k,j}^{(\ell)}(X)\frac{T^{j+k}Z^j}{(j+k)!}. \tag{7.7}$$

We note that in the notation of the previous chapters $H_0 = H$ *and* $H_1 = J$.

Now let

$$\lambda_{p,v}(T) = \sum_{r=1}^{\infty} \frac{\sigma_p(r)\left\{ {v+r-1 \atop r-1} \right\}}{2^r r!} T^r, \qquad p, v \geq 0. \tag{7.8}$$

where $\dfrac{\sigma_p(r)}{2^r r!}$ is the coefficient of m^{2r-p} in $\left[{m+1 \atop m-r+1} \right]$ (see Lemma 6.5.2).
For example we have

$$\begin{aligned}
\lambda_{0,0}(T) &= \sum_{r\geq 1} \frac{T^r}{2^r r!} = e^{T/2} - 1, \\
\lambda_{0,1}(T) &= \sum_{r\geq 2} \frac{r(r-1)}{2^{r+1} r!} T^r = \frac{T^2}{8} e^{T/2}, \\
\lambda_{1,0}(T) &= \sum_{r\geq 1} \frac{r(5-2r)}{3\cdot 2^r r!} T^r = (\frac{T}{2} - \frac{T^2}{6}) e^{T/2}.
\end{aligned}$$

We notice that for $p+v \geq 1$, $e^{-T/2}\lambda_{p,v}(T)$ is a polynomial of degree $2(p+v)$. In fact for $p \geq 1$ $\sigma_p(r) \in r\mathbb{Q}[r]$ of degree $2p$ and $\left\{ {v+r-1 \atop r-1} \right\} \in \mathbb{Q}[r]$ is a polynomial of degree $2v$ (see Lemma 6.5.2, equation (6.23), Remark 6.4.1 and Knuth duality formula (6.16)). Therefore we can write

$$\lambda_{p,v}(T) = \sum_{i=1}^{p+v} a_i D^i(e^{T/2}) \in e^{T/2}\mathbb{Q}[T], \quad a_i \in \mathbb{Q}.$$

Now set

$$\Lambda(T,W) := \sum_{p,v\geq 0}(-1)^v\lambda_{p,v}(T)W^{v+p} = (e^{T/2}-1)+\Big(\frac{T}{2}-\frac{7}{24}T^2\Big)e^{T/2}W+\cdots. \tag{7.9}$$

Then we have the following theorem, in which Q is considered as known and we are trying to find $\mathfrak{R}$.

Theorem 7.2.1. *Let* $\mathfrak{R}(X,T,Z,W) := \sum_{\ell=0}^{\infty} R_\ell(X,T,Z)W^\ell$*, with* R_ℓ *as in* (7.4)*. Set*

$$Q(X,T,Z,W) = \mathcal{H}(X,T,Z,W)\Big(1-(X-1)\Lambda(T,W)\Big).$$

Then

$$C(W,T\frac{\partial}{\partial T})\bigg(\mathfrak{R}(X,Z_1,T,W)Q(X,T,Z_2,W)\bigg)\bigg|_{Z_1=T,\,Z_2=T^{-1}} = 1, \tag{7.10}$$

where

$$C(x,y) = e^{-y}(1+xy)^{1/x} = 1-x\frac{y^2}{2}+x^2(\frac{y^3}{3}+\frac{y^4}{8})+\cdots\in\mathbb{Q}[[x,y]]$$

(*see* (6.21)).

Remark. In applynig the formula we have to consider $(T\frac{\partial}{\partial T})^k$ as $T^k\frac{\partial^k}{\partial T^k}$.

From Proposition 7.2.2 and Theorem 7.2.1 one can compute inductively all $R_\ell(X,T,Z)$ only on the diagonal $Z=T$. We explain later how from this we can obtain $\mathcal{P}^{(\ell)}(X,T)$. In Chapters 4 and 5 we did this for $\ell=0,1$. We illustrate the theorem by verifying these cases again.

Example. For $\ell=0$ the constant term with respect to W in the left hand side of (7.10) is $R(X,T,T)H(X,T,T^{-1})(X-(X-1)e^{T/2})$ (recall that $R=R_0, H=H_0$), hence from Theorem 7.2.1 we have

$$R(X,T,T)H(X,T,T^{-1})(X-(X-1)e^{T/2}) = 1,$$

which is exactly equation (4.23) for computing the leading coefficient.
Now for $\ell=1$ we need to find the coefficient of W in the left hand side

of (7.10). Because at the end we want to compare our result with equation (5.33), it is necessary to write $\lambda_{p,v}$ $(0 \le p, v, \le 1)$, in terms of f_1, f_2 and f_3, which have been introduced in Lemma 5.1.3. We have

$$f_1(X,T) = 1-(X-1)\lambda_{0,0}(T), \quad f_2(X,T) = (X-1)\lambda_{1,0}(T).$$

We set also $f_0(X,T) = (X-1)\lambda_{0,1}(T)$, so by Theorem 7.2.1 we have

$$\Big(1-\frac{T^2}{2}W\frac{\partial^2}{\partial T^2}\Big)\Bigg(f_1HR+\Big(f_1(HR_1+RH_1)+(f_0-f_2)HR\Big)W\Bigg)\Bigg|_{Z_1=T,\,Z_2=T^{-1}} \equiv 1,$$

where $\equiv$ means the equality is valid up to $O(W^2)$.
Therefore it turns out that

$$\begin{aligned}
&f_1H(X,T,T^{-1})R_1(X,T,T) = \\
&= \Big[-f_1RH_1+(f_2-f_0)HR+\frac{T^2}{2}\frac{\partial^2}{\partial T^2}(f_1HR)\Big]\Bigg|_{Z=T^{-1}} \\
&= \Big[\Big(-f_1H_1+f_2H+\frac{T^2}{2}f_1''H-f_0H+T^2f_1'H'+\frac{T^2}{2}f_1H''\Big)R \\
&+\Big(T^2f_1H'+T^2f_1'H\Big)R'+\frac{T^2}{2}f_1HR''\Big]\Bigg|_{Z=T^{-1}},
\end{aligned}$$

where $'$ denotes $\frac{d}{dT}$. Comparing this equation with (5.33) (note that in our old notation $H_1 = J$ and $M = R_1$), and using the fact that

$$R_1 = M = RG, \quad Tf_1' = -f_3, \quad , \quad \frac{T^2}{2}f_1'' = f_0,$$

we find that the two coincide.

7.3 Proof of the main theorem

In this section we prove the propositions and main theorem of the last sections. It has been organized as follows. First we assume Proposition 7.2.1 and we prove Theorem 7.1.1. Next we prove Proposition 7.2.1 using Proposition 7.2.2 and Theorem 7.2.1 . Finally we prove Proposition 7.2.2. In the next section we give a proof for Theorem 7.2.1.

Proof of Theorem 7.1.1. Proposition 7.2.1 says that

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}\frac{T^i}{i!} = \sum_{d=1}^{2\ell}\Phi_d(\widehat{\Pi}^{(\ell,d)}(V_1,\cdots,V_d,T)),\quad \ell\geq 1$$

where $\widehat{\Pi}^{(\ell,d)} \in K_d \otimes \mathbb{Q}[T,T^{-1}]$.
Now we set $\Pi^{(\ell,d)}(\mathbf{V},T) = T^{-1}\widehat{\Pi}^{(\ell,d)}(\mathbf{V},T)\frac{V_1+\cdots+V_d}{\exp(V_1+\cdots+V_d)-1}$, and we claim that

$$\mathcal{P}^{(\ell)}(X,T) = \sum_{d=1}^{2\ell}\Phi_d(\Pi^{(\ell,d)}(V_1,\cdots,V_d,T)) \in \sum_{d=1}^{2\ell}\mathcal{K}_d.$$

First of all from Lemma 6.3.1, we have the following comutative diagram:

$$\begin{array}{ccc}
K_d & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]] \\
{\scriptstyle T\frac{\exp(V_1+\cdots+V_d)-1}{V_1+\cdots V_d}}\Big\downarrow & & \Big\downarrow {\scriptstyle \sum_{i=1}^{\infty}\frac{T^i}{i!}D^{i-1}} \\
K_d & \xrightarrow{\Phi_d} & \mathbb{Q}[X]\,[[T]]
\end{array}$$

This diagram implies that for $\mathcal{Q}(X,T) := \sum_{d\geq 1}\Phi_d(\Pi^{(\ell,d)})$, we have

$$\sum_{i=1}^{\infty} D^{i-1}\mathcal{Q}(X,T)\frac{T^i}{i!} = \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{T^i}{i!}.$$

From Lemma 4.1.2 we conclude $\mathcal{P}^{(\ell)}(X,T) = \mathcal{Q}(X,T)$ and the proof of the theorem is complete. □

Proof of Proposition 7.2.1. To prove the statement, we prove (assuming Theorem 7.2.1 and Proposition 7.2.2) the following two statements:
A. We show that $\dfrac{R_\ell(X,T,T)}{R(X,T,T)} \in \sum_{d=1}^{2\ell}\mathcal{K}_d$.
B. We prove

$$\left.\left(\frac{R_\ell(X,T,Z)}{R(X,T,Z)} - \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{Z^i}{i!}\right)\right|_{Z=T} \in \sum_{d=1}^{2\ell-1}\mathcal{K}_d.$$

These together imply the statement.
Now we start with Statement **A**. First we show that the left hand side belongs to $\mathcal{K}$ and then we give the upper bound.

From Theorem 7.2.1 and by expanding the compact form of equation (7.10), for $\ell \geq 1$ and looking for the coefficient of W^ℓ in both sides, we find

$$
\begin{aligned}
&(X-(X-1)e^{T/2})H(X,T,T^{-1})R_\ell(X,T,T) = \\
&= (X-(X-1)e^{T/2}) \sum_{t,p,i,j\geq 0} \sum_{\substack{i+j=\ell-p \\ i<\ell}} c_{p,t}\, T^t \frac{\partial^t}{\partial T^t}\Bigg(R_i(X,Z_1,T)H_j(X,T,Z_2)\Bigg)\Bigg|_{\substack{Z_1=T, \\ Z_2=T^{-1}}} \\
&- (X-1)\sum \sum_{\substack{i+j+k=\ell-p \\ i<\ell}} c_{p-v,t}\, T^t \frac{\partial^t}{\partial T^t}\Bigg(R_i(X,Z_1,T)H_j(X,T,Z_2)\lambda_{k,v}(T)\Bigg)\Bigg|_{\substack{Z_1=T, \\ Z_2=T^{-1}}},
\end{aligned}
\tag{7.11}
$$

where the first sum runs over $t,p,v,i,j,k \geq 0$.
From Lemma 7.1.1, $\mathcal{K}$ is a ring, hence from the above equation to show that $\dfrac{R_\ell(X,T,T)}{R(X,T,T)} \in \mathcal{K}$, it is enough to prove that for $i<\ell$ and $j,t,k \geq 0$

$$
\frac{R_i^{(t)}(X,Z,T)}{R(X,Z,T)}\Bigg|_{Z=T}, \quad \frac{H_j^{(t)}(X,T,Z)}{H(X,T,Z)}\Bigg|_{Z=T^{-1}}, \quad \frac{(X-1)\lambda_{k,v}^{(t)}(T)}{X-(X-1)e^{T/2}} \in \mathcal{K}. \tag{7.12}
$$

To verify (7.12), we see that $\frac{R'}{R}$, $\frac{H'}{H}$ and $\frac{(X-1)e^{T/2}}{X-(X-1)e^{T/2}}$ are elements of $\mathcal{K}_1 \subset \mathcal{K}$. In fact from (4.15) we have

$$
\begin{aligned}
\frac{R'(X,Z,T)}{R(X,Z,T)}\Bigg|_{Z=T} &= \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}\frac{T^{i-1}}{(i-1)!} = \sum_{i,s=1}^{\infty} \alpha_{s+1}\, e_{s+i}\frac{T^{s+i-1}}{(i-1)!} \\
&= T^{-1}\Phi_1\Big(\sum_{i,s\geq 1} \alpha_{s+1}\frac{V^{s+i}}{(i-1)!}\Big) \in T^{-1}\Phi_1(K_1) \subset \mathcal{K}_1.
\end{aligned}
$$

From the first part of Proposition 4.1.1 we have

$$
\begin{aligned}
\frac{H'(X,T,Z)}{H(X,T,Z)}\Bigg|_{Z=T^{-1}} &= T^{-1}h'(X,T) \\
&= T^{-1}\sum_{k\geq 1} e_k(X)\frac{T^k}{k!} = T^{-1}\Phi_1\Big(\sum_{k\geq 1}\frac{V^k}{k!}\Big) \in \mathcal{K}_1,
\end{aligned}
$$

and finally $\frac{(X-1)e^{T/2}}{X-(X-1)e^{T/2}} = \sum_{k\geq 1} e_k(X)\frac{T^{k-1}}{2^k(k-1)!} \in \mathcal{K}_1$. Hence from Corollary 7.1.1 it is enough to verify (7.12) only for $t=0$. But in that case, the first part of (7.12) is true by induction (since $i < \ell$) and the last one follows from the fact that $\lambda_{k,v}(T) \in \mathbb{Q}[T]e^{T/2}$, hence $\dfrac{(X-1)\lambda_{k,v}(T)}{X-(X-1)e^{T/2}} \in \mathcal{K}$.

For $\dfrac{H_j}{H}$ from Proposition 7.2.2 we have

$$\frac{H_j(X,T,Z)}{H(X,T,Z)}|_{Z=T^{-1}} = \frac{\partial^j \exp\Big(Z\hbar(X,T,W) - Zh(X,T)\Big)}{\partial W^j}\Bigg|_{W=0,\, Z=T^{-1}}.$$

(recall that by Proposition 4.21, $H(X,T,Z) = \exp(Zh(X,T))$, or $h(X,T) = \hbar(X,T,0)$). But we have

$$\frac{\partial^j \exp(\hbar(W)-h)}{\partial W^j}\Big|_{W=0} \in \mathbb{Q}[\partial\hbar, \partial^2\hbar, \cdots, \partial^j\hbar]\Big|_{W=0},$$

where ∂ denotes $\frac{\partial}{\partial W}$. From Theorem 7.2.2

$$\begin{aligned}
\frac{\partial^j\hbar}{\partial W^j}|_{W=0} &= \int_0^T \log^{j+1}\left(\frac{1-U}{1-Ue^t}\right)dt = \int_0^T \left(\sum_{k=1}^{\infty} e_k(X)\frac{t^k}{k!}\right)^{j+1} dt \\
&= \int_0^T \sum_{k_1,\cdots,k_{j+1}\geq 1} e_{k_1}\cdots e_{k_{j+1}} \frac{t^{k_1+\cdots k_{j+1}}}{k_1!\cdots k_{j+1}!}\,dt \\
&= T\Phi_{j+1}\Big(\sum_{k_1,\cdots,k_{j+1}\geq 1} \frac{V_1^{k_1}\cdots V_{j+1}^{k_{j+1}}}{k_1!\cdots k_{j+1}!\,(k_1+\cdots k_{j+1}+1)}\Big).
\end{aligned}$$

But

$$\begin{aligned}
&\sum_{k_1,\cdots,k_{i+1}\geq 1} \frac{V_1^{k_1}\cdots V_{j+1}^{k_{j+1}}}{k_1!\cdots k_{j+1}!\,(k_1+\cdots k_{j+1}+1)} \\
&= \int_0^1 \Big(e^{V_1T}-1\Big)\cdots\Big(e^{V_{j+1}T}-1\Big)\,dT = \sum_{\mathbf{s}} \frac{\exp(V_{\mathbf{s}})-1}{V_{\mathbf{s}}} \in K_{j+1},
\end{aligned}$$

where $\mathbf{s}$ runs over all subsets of $\{1,\cdots,j+1\}$ and $V_{\mathbf{s}} = \sum_{p\in\mathbf{s}} V_p$. As a consequence $\dfrac{H_j(X,T,Z)}{H(X,T,Z)}\Big|_{Z=T^{-1}} \in \mathcal{K}$, or more preciesly

$$\frac{H_j(X,T,Z)}{H(X,T,Z)}\Big|_{Z=T^{-1}} \in \sum_{d=1}^{j+1}\mathcal{K}_d, \quad j\geq 1. \tag{7.13}$$

Denote r_ℓ the upper bound for the sum of the right hand side of Claim **A**. We look at the equation (7.11). This maximum is obtained in the right hand side of (7.11), when $p=0$ and for $j \geq 1$ we find that

$$r_\ell = \max\{r_{\ell-j}+j+1 | j=1,\cdots,\ell\},$$

which implise that $r_\ell = 2\ell$ and the proof of the Statement **A** is complete. Now we prove Statement **B**. From the recursive equation (3.33) we have

$$\rho^{(\ell)}_{s,i+1}(X) = D\rho^{(\ell)}_{s,i}(X) - s(X-1)\rho^{(\ell-1)}_{s,i}(X) + \sum_{k=0}^{\ell}\sum_{r=0}^{s}\rho^{(k)}_{r,i}(X)\rho^{(\ell-k)}_{s-r}(X). \quad (7.14)$$

By (7.3) this is equivalent to

$$\begin{aligned}\mathcal{P}^{(\ell)}_{i+1}(X,T) &= D\mathcal{P}^{(\ell)}_{i}(X,T) - (X-1)\Theta\mathcal{P}^{(\ell-1)}_{i}(X,T)\\ &\quad + \sum_{k=0}^{\ell}\mathcal{P}^{(k)}_{i}(X,T)\mathcal{P}^{(\ell-k)}(X,T),. \end{aligned} \quad (7.15)$$

where $\Theta = T\frac{\partial}{\partial T}$. Then by (7.4) we find

$$\begin{aligned}\frac{d}{dZ}R_\ell(X,T,Z) &= DR_\ell(X,T,Z) - (X-1)T\frac{d}{dT}R_{\ell-1}(X,T,Z)\\ &\quad + \sum_{k=0}^{\ell}\mathcal{P}^{(\ell-k)}(X,T)R_k(X,T,Z).\end{aligned}$$

Hence if we set $\mathfrak{P}(X,T,W) := \sum_{\ell=0}^{\infty}\mathcal{P}^{(\ell)}(X,T)W^\ell$ we have

$$\left(\frac{d}{dZ}-D-(X-1)W\Theta\right)\mathfrak{R}(X,T,Z,W) = \mathfrak{P}(X,T,W)\mathfrak{R}(X,T,Z,W). \quad (7.16)$$

Therefore $\mathfrak{R} = \exp(F)$ for some F which satisfies

$$\left(\frac{d}{dZ}-D-(X-1)W\Theta\right)F(X,T,Z,W) = \mathfrak{P}(X,T,W),$$

with the right hand side indepent of Z. We write

$$F(X,T,Z,W) = \sum_{k=0}^{\infty}F_k(X,T,Z)W^k.$$

Then we have

$$(\frac{d}{dZ} - D)F_0 = \mathcal{P}^{(0)} = \mathcal{P}, \tag{7.17}$$

$$(\frac{d}{dZ} - D)F_k = \mathcal{P}^{(k)} + (X-1)\Theta F_{k-1} \quad (k \geq 1). \tag{7.18}$$

Equation (7.17) can easily be solved. We note that $\mathfrak{R}(X, T, 0, 0) = 1$, so $F_0(X, T, 0) = 0$, and we have

$$F_0(X, T, Z) = \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}(X, T)\frac{Z^i}{i!}, \tag{7.19}$$

and for (7.18) using the identity

$$\binom{i}{j_1, \cdots, j_s} = \binom{i-1}{j_1 - 1, j_2, \cdots, j_s} + \cdots + \binom{i-1}{j_1, \cdots, j_{s-1}, j_s - 1} + \binom{i-1}{j_1, \cdots, j_s},$$

one can check directly that the solution of (7.18) is given by

$$\begin{aligned} F_k(X, T, Z) &= \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(k)}(X, T)\frac{Z^i}{i!} \\ &+ \sum_{i=1}^{\infty}\sum_{s=1}^{k}\sum_{j_1, \cdots, j_s \geq 1} \frac{1}{s!}\binom{i-1}{j_1, \cdots, j_s} e_{j_1}\cdots e_{j_s} D^{i-\mathbf{j}-1}\Theta^s\mathcal{P}^{(k-s)}\frac{Z^i}{i!}, \end{aligned} \tag{7.20}$$

where $\mathbf{j} = j_1 + \cdots + j_s$.
Since $k < \ell$ we have by induction

$$\mathcal{P}^{(k)}(X, T, T) = \sum_{d=1}^{2k} \Phi_d(\Pi^{(k,d)}), \quad \Pi^{(k,d)} \in K_d \otimes \mathbb{Q}[T, T^{-1}]. \tag{7.21}$$

From Lemma 6.3.1 and (7.21) we have

$$\begin{aligned} &e_{j_1}\cdots e_{j_s} D^{i-\mathbf{j}-1}\Theta^s\mathcal{P}^{(k-s)}(X, T)T^{i-1} = \\ &= \sum_{d=1}^{2(k-s)}\sum_{s=1}^{k} \Phi_{d+s}\Big(V_{d+1}^{j_1}\cdots V_{d+s}^{j_s}\big(V_1 + \cdots + V_d\big)^{i-\mathbf{j}-1}\big(\mathbf{V}\frac{\partial}{\partial \mathbf{V}} + \Theta\big)^s\Pi^{(k-s,d)}\Big), \end{aligned}$$

where $\mathbf{V}\frac{\partial}{\partial \mathbf{V}} = V_1\frac{\partial}{\partial V_1} + \cdots V_d\frac{\partial}{\partial V_d}$. Hence from (7.20) and the definition of Φ_d we have

$$\begin{aligned} F_k(X, T, T) &= \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(k)}\frac{T^i}{i!} \\ &\quad + T\sum_{s=1}^{k}\sum_{d=1}^{2(k-s)} \Phi_{d+s}\Bigg(\frac{\exp(V_1 + \cdots + V_{d+s}) - 1}{s!\big(V_1 + \cdots + V_d\big)}\big(\mathbf{V}\frac{\partial}{\partial \mathbf{V}} + \Theta\big)^s\Pi^{(k-s,d)}\Bigg). \end{aligned}$$

Hence it follows for $k < \ell$, that $F_k(X,T,T) \in \mathcal{K}$ and the upper bound is $2k-1$ and consequently by Lemma 7.1.1, $\mathbb{Q}[F_1, \cdots, F_{\ell-1}] \subset \mathcal{K}$.
For $k = \ell$, the same argument gives

$$[F_\ell(X,T,Z) - \sum_{i=1}^{\infty} D^{i-1}\mathcal{P}^{(\ell)}(X,T)\frac{Z^i}{i!}]\Big|_{Z=T} \in \mathcal{K}. \tag{7.22}$$

with the upper bound $d = 2\ell - 1$. Hence Claim **B** is equivalent to

$$\frac{R_\ell(X,T,T)}{R(X,T,T)} - F_\ell(X,T,T) \in \mathcal{K}$$

with the upper bound $2\ell - 1$. But we have

$$\frac{R_\ell(X,T,T)}{R(X,T,T)} = \frac{\partial^\ell \mathfrak{R}/R}{\partial W^\ell}\Big|_{W=0} = \frac{\partial^\ell \exp(F - F_0)}{\partial W^\ell}\Big|_{W=0} = F_\ell + G,$$

where $G \in \mathbb{Q}[F_1, \cdots, F_{\ell-1}] \subset \mathcal{K}$ with upper bound $2\ell - 1$. This completes the proof of Proposition 7.2.1. $\square$

Proof of Proposition 7.2.2. From equation (3.18), $\mathcal{H}$ satisfies the following homogenous linear differential equation

$$\left[\frac{\partial}{\partial T} - X(X-1)\frac{\partial}{\partial X} - ZT(X-1) - W(X-1)Z\frac{\partial}{\partial Z}\right]\mathcal{H} = 0. \tag{7.23}$$

It follows that $\mathcal{H}(X,T,Z,W) = \exp(Z\hbar(X,T,W))$, for some $\hbar$ which satisfies the following differential equation

$$\frac{\partial}{\partial T}\hbar - X(X-1)\frac{\partial}{\partial X}\hbar - W(X-1)\hbar = T(X-1).$$

Then we write $\hbar(X,T,W) = \sum_{i=0}^{\infty} h_i(X,W)\frac{T^i}{i!}$. But by definition of $\mathcal{H}$ we have $\mathcal{H} = 1 + \frac{X-1}{2}T^2 Z + \cdots$ which follows $h_0 = h_1 = 0$ and from the above equation it turns out

$$h_2 = X - 1 = e_1, \quad h_{i+1} = (D + We_1)h_{i-1}, \quad i \geq 2,$$

or $h_i = \frac{1}{W}(D + We_1)^i(1)$, for $i \geq 1$. Finally we have

$$\begin{aligned}\hbar' &= \frac{1}{W}\sum_{i=1}^{\infty}\Big(D + We_1\Big)^i(1)\frac{T^i}{i!} = \frac{1}{W}\left[\exp\Big(W\sum_{i=1}^{\infty} e_i\frac{T^i}{i!}\Big) - 1\right] \\ &= \frac{1}{W}\left[\left(\frac{1-U}{1-Ue^T}\right)^W - 1\right].\end{aligned} \tag{7.24}$$

Therefore (7.5) follows. □

7.4 Proof of Theorem 7.2.1

In this final section we prove Theorem 7.2.1. The recursive equation (3.34), which we repeat for convenience, says

$$\sum_{k=1}^{s}\sum_{i=0}^{k} E_{k,i}(n,X)P_{s-k,i}(n,X)=0, \tag{7.25}$$

where $E_{k,i}(n,X)$ is defined as in (3.22).
We define coefficients $a_{k,j}^{(2k-\ell)}(X)$ by the expansion

$$E_{k,i}(n,X)=\sum_{\ell=0}^{2k-i-1} a_{k,i}^{(2k-\ell)}(X)\,n^{2k-i-\ell}, \quad (n\to\infty),$$

The coefficient of $n^{2s-\ell}$ in the left hand side of (7.25) is

$$\sum_{k=1}^{s}\sum_{p=0}^{\ell}\sum_{i=0}^{k} a_{k,i}^{(2k-\ell+p)}(X)\rho_{s-k,i}^{(2s-2k-p)}(X). \tag{7.26}$$

Hence by definiton of $\mathcal{P}_i^{(\ell)}$ we have

$$\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{i=0}^{\infty}\left(\sum_{k=i}^{\infty} a_{k,i}^{(2k-\ell+p)}(X)T^k\right)\mathcal{P}_i^{(p)}(X,T)=1. \tag{7.27}$$

We show that for fixed i

$$\sum_{\ell,k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^kW^\ell = C(W,T\frac{\partial}{\partial T})\left(\Big(1-(X-1)\Lambda(T,W)\Big)\mathcal{H}\,\frac{T^i}{i!}\right)\Bigg|_{Z=T^{-1}}. \tag{7.28}$$

Note that one can consider this case as an special case of Theorem 7.2.1 when all $\rho_{s,i}^{(\ell)}=1$ $(s,i,\ell\geq 0)$.
To do this we need to write $a_{k,i}^{(2k-p)}$ as a sum of terms which depend on $k-i$ and i. The function $E_{k,i}(n,X)$ in equation (3.22) depends on n through the

quantities $\binom{n-r}{i}$, $S_r(n)$ and $\mathcal{H}_{n-i-r,k-i-r}(n,X)$ $(0 \le r \le k-i)$. We expand the first of these by

$$\binom{n-r}{i} = \frac{1}{i!}\sum_{p=0}^{i}(-1)^p S_p(r, r+i-1)\, n^{i-1},$$

where coefficients can be expresed in terms of Stirling numbers of the first and second kind by (6.17). Using 3.17 and equation (7.6) we expand $\mathcal{H}_{m,k}$ in terms of $q_{k,j}(X)$. Finally $S_r(n)$ by the second part of Lemma 6.5.2 have the following expansion

$$S_r(n) = \sum_{p=0}^{2r-1} \frac{\sigma_p(r)}{2^r r!} n^{2r-p} \quad r \ge 1.$$

Therefore from (3.22) we have

$$\begin{aligned}
a_{k,i}^{(2k-\ell)}(X) &= \sum_{j=1}^{k-i}\sum_{p=0}^{\ell}(-1)^p \frac{S_p(k+j-1) q_{k-i,j}^{(\ell-p)}(X)}{i!\,(k-i+j)!} \\
&- (X-1)\sum_{r=1}^{k-i}\sum_{j=1}^{k-i-r}\sum_{p+p'+p''=\ell}(-1)^p \frac{\sigma_{p'}(r)}{2^r r!}\,\frac{S_p(r, k+j-1)}{i!\,(k-i-r+j)!} q_{k-i-r,j}^{(p'')}(X) \\
&= \sum_{j=1}^{k-i}\sum_{p=0}^{\ell}(-1)^p \begin{bmatrix} k+j \\ k+j-p \end{bmatrix} \frac{q_{k-i,j}^{(\ell-p)}(X)}{i!\,(k-i+j)!} \\
&- (X-1)\sum_{r=1}^{k-i}\sum_{j=1}^{k-i-r}\sum_{p+p'+p''=\ell}\sum_{v=0}^{p}(-1)^{v+p} \frac{\sigma_{p'}(r)\left\{ {v+r-1 \atop r-1} \right\}}{2^r r!}\,\frac{\left[{k+j \atop k+j-p+v} \right]}{i!\,(k-i-r+j)!} q_{k-i-r,j}^{(p'')}(X).
\end{aligned} \tag{7.29}$$

Expanding $\left[{k+j \atop k+j-p+v} \right]$ by (6.20) and forming a generating function we can write this as

$$
\begin{aligned}
&\sum_{k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^k = \\
&\sum_{k,j,p,t,t'\geq 0} (-1)^p \binom{t}{t'} \frac{c_{p,t}\, q_{k-i,j}^{(\ell-p)}(X)T^k}{(i-t')!\,(k+j-i-t+t')!} \\
&-(X-1) \sum_{k,j,p,p',v,t,t',t''\geq 0} (-1)^p \binom{t}{t',t''} \frac{c_{p-v,t}\, \lambda_{p',v}^{(t')}(T)\, q_{k-i,j}^{(\ell-p-p')}(X)T^{k+t'}}{(i-t'')!\,(k+j-i-t+t'+t'')!} \\
=&\sum_{p=0}^{\ell} \sum_{t,t'\geq 0} (-1)^p \binom{t}{t'} c_{p,t}\, H_{\ell-p}^{(t-t')}(X,T,Z) \frac{T^{i+t-t'}}{(i-t')!} \Bigg|_{Z=T^{-1}} \\
&-(X-1) \sum_{p+p'=\ell} \sum_{v,t,t',t''\geq 0} (-1)^p \binom{t}{t',t''} c_{p-v,t}\, \lambda_{p',v}^{(t)}(T)\, H_{\ell-p-p'}^{(t-t'-t'')}(X,T,Z) \frac{T^{i+t-t''}}{(i-t'')!} \Bigg|_{Z=T^{-1}},
\end{aligned}
$$

where $\lambda_{p,v}$ and H_ℓ are as in (7.8), (7.7) respectively.
Now multiplying by W^ℓ and summing over $\ell \geq 0$ and using the Leibniz rule we find

$$
\begin{aligned}
&\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^k W^\ell \\
&=\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p} (-1)^p c_{p,t} \frac{\partial^t}{\partial T^t}\Big(H_{\ell-p}(X,T,Z)\frac{T^i}{i!}\Big) W^\ell T^t \Bigg|_{Z=T^{-1}} \\
&-(X-1)\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{v=0}^{p}\sum_{t=0}^{2p} (-1)^p c_{p-v,t} \frac{\partial^t}{\partial T^t}\left(H_{\ell-p-p'}(X,T,Z)\lambda_{p',v}(T)\frac{T^i}{i!}\right) W^\ell T^t \Bigg|_{Z=T^{-1}}.
\end{aligned}
\tag{7.30}
$$

We note that except the term $\frac{T^i}{i!}$ the other terms are independent of i. From definitions of $\mathcal{H}$ and Λ we get

$$
\begin{aligned}
\sum_{\ell,k=0}^{\infty} a_{k,i}^{(2k-\ell)}(X)T^k W^\ell &= \sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p} (-1)^p c_{p,t}\, W^p T^t \frac{\partial^t}{\partial T^t}\Big(\mathcal{H}\,\frac{T^i}{i!}\Big)\Bigg|_{Z=T^{-1}} \\
&-(X-1)\sum_{\ell=0}^{\infty}\sum_{p=0}^{\ell}\sum_{t=0}^{2p} (-1)^p c_{p,t}\, W^p T^t \frac{\partial^t}{\partial T^t}\left(\Lambda\,\mathcal{H}\,\frac{T^i}{i!}\right)\Bigg|_{Z=T^{-1}}.
\end{aligned}
$$

Hence equation (7.28) follows from the definition of $C(x,y)$.
Now in order to prove the statement of the theorem we multiply equation (7.27) by W^{ℓ} and sum over all $\ell \geq 0$. For fixed i we replace inside the parentesis of (7.27) with the right hand side of (7.28) . Also replacing T by Z_1 in $\mathcal{P}_i(X,T)$ and passing through the parenthesis and then summing over all $i \geq 0$ the statement of theorem immediatly follows from the definition of $\mathfrak{R}$. $\square$

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