Euler characteristics and geometric properties of quiver Grassmannians

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Abstract

Let k be an algebraically closed field, Q a finite quiver and M a finite-dimensional Q-representation. The quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ is the projective variety of subrepresentations of M with dimension vector $\mathbf{d}$.

Quiver Grassmannians occur naturally in different contexts. Fomin and Zelevinsky introduced cluster algebras in 2000. Caldero and Keller used Euler characteristics of quiver Grassmannians for the categorification of acyclic cluster algebras. This was generalized to arbitrary antisymmetric cluster algebras by Derksen, Weyman and Zelevinsky. The quiver Grassmannians play a crucial role in the construction of Ringel-Hall algebras. Moreover, they arise in the study of general representations of quivers by Schofield and in the theory of local models of Shimura varieties. Motivated by this, we study the geometric properties of quiver Grassmannians, their Euler characteristics and Ringel-Hall algebras. This work is divided into three parts.

In the first part of this thesis, we study geometric properties of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$. In some cases we compute the dimension of this variety, we detect smooth points and we prove semicontinuity of the rank functions and of the dimensions of homomorphism spaces. Moreover, we compare the geometry of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ with the geometry of the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ and we develop tools to decompose $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components.

In the following we consider some special classes of quiver representations, called string, tree and band modules. There is an important family of finite-dimensional k-algebras, called string algebras, such that each indecomposable module is either a string or a band module.

In the second part, for $k = \mathbb{C}$ we compute the Euler characteristics of quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$ and of quiver flag varieties $\mathcal{F}_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M)$ in the case that M is a direct sum of string, tree and band modules. We prove that these Euler characteristics are positive if the corresponding variety is non-empty. This generalizes some results of Cerulli Irelli.

In the third part, we consider the Ringel-Hall algebra $\mathcal{H}(A)$ of a string algebra A over $\mathbb{C}$. We give a complete combinatorial description of the product of the subalgebra $\mathcal{C}(A)$ of the Ringel-Hall algebra $\mathcal{H}(A)$.

In covering theory we obtain the following results, which resemble the results of the last two parts. Let $\hat{Q}$ be a locally finite quiver with a free action of a free or free abelian group and $\pi\colon \hat{Q} \to Q$ the corresponding projection on the orbit space Q. Thus for each finite-dimensional $\hat{Q}$-representation V we get a Q-representation $\pi_*(V)$ and π induces a map $\boldsymbol{\pi}\colon \mathbb{N}^{\hat{Q}_0} \to \mathbb{N}^{Q_0}$ of dimension vectors. We show that the Euler characteristic of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(\pi_*(V))$ is the sum of the Euler characteristics of $\mathrm{Gr}_{\mathbf{t}}(V)$, where $\mathbf{t}$ runs over all dimension vectors in $\boldsymbol{\pi}^{-1}(\mathbf{d})$. Moreover, the morphism $\pi\colon \hat{Q} \to Q$ of quivers induces a morphism $\mathcal{C}(\pi)\colon \mathcal{C}(\mathbb{C}Q) \to \hat{\mathcal{C}}(\mathbb{C}\hat{Q})$ of the Ringel-Hall algebras.

Contents

1 Introduction

Let k be an algebraically closed field, $Q = (Q_0, Q_1)$ a locally finite quiver, $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ a finite-dimensional Q-representation and $\mathbf{d} = (d_i)_{i\in Q_0}$ a dimension vector. A subrepresentation of M with dimension vector $\mathbf{d}$ is a tuple $(U_i)_{i\in Q_0}$ of d_i-dimensional subspaces U_i of the k-vector space M_i such that $(U_i, M_\alpha|_{U_{s(\alpha)}})_{i\in Q_0,\alpha\in Q_1}$ is again a Q-representation. The *quiver Grassmannian* $\mathrm{Gr}_{\mathbf{d}}(M)$ is the projective variety over k of all these subrepresentations of M with dimension vector $\mathbf{d}$. This is a closed subvariety of a product of classical Grassmannians (see Lemma 2.3.7).

Following [46] quiver Grassmannians appear in the study of general representations of quivers (see Crawley-Boevey [19] and Schofield [48]) and their *Euler characteristics* in the theory of *cluster algebras* (see Caldero and Chapoton [9], Caldero and Keller [11] and Derksen, Weyman and Zelevinsky [21]). Cluster algebras were introduced by Fomin and Zelevinsky [24, 25, 26] in 2000. For instance, Caldero and Keller [10, 11] showed that the Euler characteristic plays a central role for the categorification of cluster algebras. In this context the positivity of these Euler characteristics is essential. The Euler characteristic of such a projective variety is a much studied, but very rough invariant (see Caldero and Zelevinsky [13] and Cerulli Irelli [14]). The representation theoretic properties of these quiver Grassmannians are studied for instance by Fedotov [23], Lusztig [39] and Reineke [42]. Moreover, Görtz [30, Section 4] showed that they appear in the theory of local models of Shimura varieties.

It is easy to see that an *ideal* I of a quiver Q does not affect our results. Let M be a (Q, I)-representation. So M is also a Q-representation. Each subrepresentation of the Q-representation M is also a subrepresentation of the (Q, I)-representation M. Thus the variety $\mathrm{Gr}_{\mathbf{d}}(M)$ for a finite-dimensional (Q, I)-representation M equals the variety $\mathrm{Gr}_{\mathbf{d}}(M)$ for the Q-representation M.

This thesis is organized as follows: After this introduction we state the necessary basic notions in Chapter 2. Most of these definitions and results are well-known. In the remaining three chapters we present our own results. In Chapter 3 we study the geometry of the quiver Grassmannian $Gr_{\mathbf{d}}(M)$ as a scheme. In Chapter 4 we compute the Euler characteristics of some quiver Grassmannians. These results are applied to Ringel-Hall algebras in Chapter 5. Some results of the last two chapters are already published in [32].

1.1 Geometric properties of quiver Grassmannians

We study basic geometric properties of quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$ building on work of Caldero and Reineke [12] (see also Cerulli Irelli and Esposito [15], Schofield [48] and

Wolf [54]). For this it is convenient to consider a k-scheme $Gr_{\mathbf{d}}(M)$ such that its k-rational points form the variety $\mathrm{Gr}_{\mathbf{d}}(M)$.

The *module variety* $\mathrm{rep}_{\mathbf{d}}(Q) = \prod_{\alpha\in Q_1} \mathrm{Mat}\left(d_{t(\alpha)} \times d_{s(\alpha)}, k\right)$ is very well-known in representation theory. This affine variety parametrizes in some sense all Q-representations with dimension vector $\mathbf{d}$. The algebraic group $\mathrm{GL}_{\mathbf{d}}(k) = \prod_{i\in Q_0} \mathrm{GL}_{d_i}(k)$ acts by conjugation on the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$. The orbits under this action are in bijection with the isomorphism classes of Q-representations with dimension vector $\mathbf{d}$. They are irreducible, locally closed, smooth and their dimensions are well-known (see Proposition 2.3.3). The closure of such an orbit is the union of orbits. This defines the *degeneration order* on the set of orbits. We say an orbit $\mathcal{O}(U)$ is bigger than another orbit $\mathcal{O}(V)$ if and only if the orbit $\mathcal{O}(V)$ is contained in the closure of the orbit $\mathcal{O}(U)$ (see e.g. [43]). For each (semi-)admissible ideal I there is a closed subvariety $\mathrm{rep}_{\mathbf{d}}(Q, I)$ of $\mathrm{rep}_{\mathbf{d}}(Q)$, which parametrizes all (Q, I)-representations with dimension vector $\mathbf{d}$. Since this variety is not irreducible in general it is natural to decompose it into irreducible components (see e.g. [3, 20, 41, 45, 50]).

This suggest to decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components. The isomorphism classes $\mathcal{C}_U(k)$ of subrepresentations U of a Q-representation M in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ are in general not orbits of some natural action (see Remark 3.3.6). Nevertheless, these locally closed subschemes $\mathcal{C}_U$ of the scheme $Gr_{\mathbf{d}}(M)$ are irreducible, smooth and have dimension $\dim_k \mathrm{Hom}_Q(U, M) - \dim_k \mathrm{End}_Q(U)$ for each $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ by Theorem 3.1.1. In Proposition 3.1.7 we give a homological condition on $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ such that $\overline{\mathcal{C}_U(k)}$ is an irreducible component of the variety $\mathrm{Gr}_{\mathbf{d}}(M)$. Moreover, in Corollary 3.1.8 we use this criterion to construct a lot of examples of irreducible components. In these cases all points in $\mathcal{C}_U(k)$ are smooth in the quiver Grassmannian $Gr_{\mathbf{d}}(M)$.

Each homomorphism $f\colon M \to N$ of Q-representations induces an isomorphism of closed subschemes of the quiver Grassmannians $Gr_{\mathbf{d}}(M)$ and $Gr_{\mathbf{d}-\mathbf{dim}\,\mathrm{Ker}\,f}(N)$ (see Proposition 3.2.1). Rank functions on the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ are lower semicontinuous and dimensions of homomorphism spaces of Q-representations are upper semicontinuous. We show the analogous statements for the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ (see Proposition 3.3.1 and 3.3.3).

Comparing the degeneration order defined by the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ and the topology of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ we get the following result (see Theorem 3.4.1). Let $U, V \in \mathrm{Gr}_{\mathbf{d}}(M)$. If $U \in \overline{\mathcal{C}_V(k)}$ in the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$, then $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$ in the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$. Using Example 3.4.4 or the example in Section 3.6.4 we see that the converse of this theorem is not true. Nevertheless, this gives us some irreducible components of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ if there are only finitely many isomorphism classes of subrepresentations of the Q-representation M with dimension vector $\mathbf{d}$ (see Proposition 3.5.5).

Let M be an *exceptional* Q-representation, i.e. $\mathrm{Ext}^1_Q(M, M) = 0$. By Caldero and Reineke [12, Corollary 4] the corresponding quiver Grassmannian $Gr_{\mathbf{d}}(M)$ is empty or smooth (see Proposition 2.3.12). Moreover, we show in Proposition 3.5.7 the following. If $\mathrm{Gr}_{\mathbf{d}}(M)$ is non-empty and there are only finitely many isomorphism classes in $\mathrm{Gr}_{\mathbf{d}}(M)$, there is an exceptional Q-representation U such that the isomorphism class $\mathcal{C}_U(k)$ of U is dense in $\mathrm{Gr}_{\mathbf{d}}(M)$.

Of course, there are dual versions of all these results by replacing sub- by factor representations.

In Section 3.6 we consider the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$ in some examples. We try to decompose it into irreducible components and detect smooth points. The following examples are studied:

- *Linearly oriented quivers of type A:* We consider the quiver $1 \to 2$ for each Q-representation M and each dimension vector $\mathbf{d}$, the quiver $1 \to 2 \to 3$ for Q-representations M with dimension vectors of the form (n, n, n) and dimension vectors $\mathbf{d} = (d, d, d)$ and the quiver $1 \to 2 \to \cdots \to N$ for a projective or injective Q-representation M and each dimension vector $\mathbf{d}$.
- *Cyclically oriented quivers of type* $\tilde{A}$: We study the one-loop-quiver (see Figure 3.6.4) for each representation and each dimension vector and one example for the oriented two-cycle-quiver (see Figure 3.6.9). Finally, we consider the oriented N-cycle-quiver Q (see Figure 3.6.11) with each projective-injective (Q, α^N)-representation and a dimension vector for $N \in \mathbb{N}$ and $N \geq 2$. Görtz [30, Section 4] studied this example in the context of local models of Shimura varieties (see Remark 3.6.17 and also Pappas, Rapoport and Smithling [40, Section 7]).

1.2 Euler characteristics of quiver Grassmannians

Let k be the field of complex numbers $\mathbb{C}$. We use and improve a technique of Cerulli Irelli [14] to compute Euler characteristics $\chi_{\mathbf{d}}(M)$ of quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$. In general it is hard to compute the Euler characteristic of such projective varieties, but in the case of a direct sum of tree and band modules we show that this is only a simple combinatorial task.

Some special morphisms of quivers $F\colon S \to Q$ are called *windings of quivers* (see Section 2.2). Each winding induces a functor $F_*\colon \mathrm{rep}(S) \to \mathrm{rep}(Q)$ of categories of finite-dimensional quiver representations and a map $\mathbf{F}\colon \mathbb{N}^{S_0} \to \mathbb{N}^{Q_0}$ of dimension vectors of the corresponding quivers. Let S be a finite tree and $\mathbb{1}_S$ the S-representation such that every vector space of this representation is one-dimensional and every linear map is non-zero. This representation $\mathbb{1}_S$ is up to isomorphism uniquely determined and its image under the functor F_* is called a *tree module.* Let $n \in \mathbb{Z}_{>0}$, S be a quiver of type $\tilde{A}_{n-1}$ and $\mathcal{I}_S^n$ the set of indecomposable S-representations $V = (V_i, V_a)_{i \in S_0, a \in S_1}$ with V_a is an isomorphism for each $a \in S_1$ and $\dim_{\mathbb{C}} V_i = n$ for some $i \in S_0$. The Q-representation $F_*(V)$ is called a *band module* if $V \in \mathcal{I}_S^n$ and $F_*(V)$ is indecomposable.

In Theorem 4.3.1 we compute the Euler characteristics of quiver Grassmannians of all tree and band modules. Let $F_*(\mathbb{1}_S)$ be a tree module. By Part 1 of Theorem 4.3.1 the Euler characteristic of $\mathrm{Gr}_{\mathbf{d}}(F_*(\mathbb{1}_S))$ is the sum of the Euler characteristics of the quiver Grassmannians $\mathrm{Gr}_{\mathbf{t}}(\mathbb{1}_S)$, where $\mathbf{t}$ runs over all dimension vectors in $\mathbf{F}^{-1}(\mathbf{d})$. By definition of $\mathbb{1}_S$ it is very easy to compute the Euler characteristic $\chi_{\mathbf{t}}(\mathbb{1}_S)$ in this case, namely $\mathrm{Gr}_{\mathbf{t}}(\mathbb{1}_S)$ contains at most one point (see Corollary 4.4.1). For each indecomposable band module M we give an explicit formula for the Euler characteristic $\chi_{\mathbf{d}}(M)$ in Part 2 of Theorem 4.3.1. Moreover, we prove the positivity of each Euler characteristic $\chi_{\mathbf{d}}(M)$ in the case that M is a direct sum of tree and band modules (see Corollary 4.4.2).

In the proof of Theorem 4.3.1 we use the following result of Bialynicki-Birula [5, Corollary 2]. For a quasi-projective variety with a $\mathbb{C}^*$-action its Euler characteristic equals the Euler characteristic of the fixed points under this action (see Theorem 4.2.1). To construct $\mathbb{C}^*$-actions on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ we introduce the notion of *gradings* in Section 4.1.

The projective variety $\mathcal{F}_{\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}}(M)$ of flags of subrepresentations of a Q-representation M with dimension vectors $\mathbf{d}^{(1)},\ldots,\mathbf{d}^{(r)}$ is called *quiver flag variety*. The results for the Euler characteristics of quiver Grassmannians can be generalized to analogous statements for such quiver flag varieties (see Corollary 4.5.3).

Let $\hat{Q}$ be a locally finite quiver and G a free or free abelian group. An *action of the group G on $\hat{Q}$* is a pair of maps $G\times\hat{Q}_0\to\hat{Q}_0, (g,i)\mapsto gi$ and $G\times\hat{Q}_1\to\hat{Q}_1, (g,a)\mapsto ga$ such that $gs(a)=s(ga)$ and $gt(a)=t(ga)$ for all $g\in G$ and $a\in\hat{Q}_1$. We say, the group G acts *freely* on the quiver $\hat{Q}$ if for all $i\in\hat{Q}_0$ and all $a\in\hat{Q}_1$ the stabilizers are trivial. Let $Q=\hat{Q}/G$ be the orbit quiver of such an action and $\pi\colon\hat{Q}\to Q$ the canonical projection. If G acts freely on the quiver $\hat{Q}$, then π is a winding.

Let V be a finite-dimensional $\hat{Q}$-representation. In Part 3 of Theorem 4.3.1 we show that the Euler characteristic of a quiver Grassmannian of the Q-representation $\pi_*(V)$ is determined by the Euler characteristics of the quiver Grassmannians of V. More precisely, for each dimension vector $\mathbf{d}$ the Euler characteristic of $\mathrm{Gr}_{\mathbf{d}}(\pi_*(V))$ is the sum of all Euler characteristics of $\mathrm{Gr}_{\mathbf{t}}(V)$, where $\mathbf{t}$ runs over all dimension vectors in $\boldsymbol{\pi}^{-1}(\mathbf{d})$.

1.3 Ringel-Hall algebras

Let Q be a locally finite quiver, I an admissible ideal and $A=\mathbb{C}Q/I$ the corresponding $\mathbb{C}$-algebra. We associate to the algebra A the *Ringel-Hall algebra* $\mathcal{H}(A)$, its subalgebra $\mathcal{C}(A)$ and its completions $\hat{\mathcal{H}}(A)$ and $\hat{\mathcal{C}}(A)$ (see Section 2.4). We assume one of the following cases.

1. Let $\varphi\colon S\to Q$ be a tree or a band and $A=\mathbb{C}Q/I$ and $B=\mathbb{C}S/J$ finite-dimensional algebras such that φ induces a functor $\varphi_*\colon\ \mathrm{mod}(B)\to\mathrm{mod}(A)$.
2. Let $\hat{Q}$ be a locally finite quiver and G a free or free abelian group, which acts freely on $\hat{Q}$. Let $Q=\hat{Q}/G$ be the orbit quiver, $A=\mathbb{C}Q/I$ and $B=\mathbb{C}\hat{Q}/J$ algebras and $\varphi\colon\hat{Q}\to Q$ the canonical projection such that φ induces a functor $\varphi_*\colon\ \mathrm{mod}(B)\to\mathrm{mod}(A)$.

Then the winding of quivers φ induces a functorial homomorphism

$$\mathcal{C}(\varphi)\colon\mathcal{C}(A)\to\hat{\mathcal{C}}(B), f\mapsto f\circ\varphi_*$$

of Hopf algebras (see Theorem 5.1.1). Moreover, this map $\mathcal{C}(\varphi)$ can be extended to the Ringel-Hall algebras $\mathcal{H}(\varphi)\colon\mathcal{H}(A)\to\hat{\mathcal{H}}(B)$, but this map is in general not an algebra homomorphism.

Let $\mathbf{F} = \big(F^{(1)}, \dots, F^{(r)}\big)$ with $F^{(i)}\colon S^{(i)} \to Q$ be a tuple of trees, $\mathbf{B} = \big(B^{(1)}, \dots, B^{(s)}\big)$ with $B^{(i)}\colon T^{(i)} \to Q$ a tuple of bands and $\mathbf{n} = (n_1, \dots, n_s)$ a tuple of positive integers. Let

$$\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(M) = \begin{cases} 1 & \text{if } \exists V_i \in \mathcal{I}^{n_i}_{T^{(i)}} : M \cong \bigoplus_{i=1}^{r} F^{(i)}_{*}\,(\mathbb{1}_{S^{(i)}}) \oplus \bigoplus_{i=1}^{s} B^{(i)}_{*}(V_i), \\ 0 & \text{otherwise.} \end{cases}$$

for each Q-representation M. This defines constructible functions $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$, which are not necessarily in $\mathcal{C}(A)$. We compute the image of such a function $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ under the map $\mathcal{H}(\varphi)\colon \mathcal{H}(A) \to \hat{\mathcal{H}}(B)$ in Theorem 5.2.1. Roughly speaking, this is given by the sum of all maps $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}$, where $\widetilde{\mathbf{F}}$ and $\widetilde{\mathbf{B}}$ runs over all liftings of $\mathbf{F}$ and $\mathbf{B}$ by the winding φ (see Figure 1.3.1).

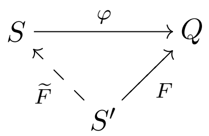

Figure 1.3.1: Lifting $\widetilde{F}\colon S' \to S$ of $F\colon S' \to Q$ by $\varphi\colon S \to Q$.

Using this we can study the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$. This gives us a combinatorial description of $(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'})(M)$ for each $\mathbb{C}$-algebra A and each direct sum of tree and band modules M such that M is an A-module (see Corollary 5.4.7).

Actually for a *string algebra* $A = \mathbb{C}Q/I$ each function in $\mathcal{C}(A)$ is a linear combination of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ and the computation of arbitrary products in $\mathcal{C}(A)$ is reduced to a purely combinatorial task (see Definition 2.2.11 and Corollary 5.5.3). If A is representation finite and Q has no loops and cyclically oriented two-cycles, then $\mathcal{C}(A) = \mathcal{H}(A)$ (see Theorem 5.5.4). Moreover, in this case the functions $\mathbb{1}_{\mathbf{F}}$ with some tuple $\mathbf{F}$ of strings form a vector space basis.

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2 Preliminaries

Let k be an algebraically closed field. We denote by $\mathbb{N}$ the natural numbers including 0. Each algebra is an associative k-algebra with a unit. For a ring R and $d, n \in \mathbb{N}$ let $\mathrm{Mat}(d \times n, R)$ be the free R-module of matrices with d rows, n columns and entries in R. Moreover, let $\mathrm{GL}_n(R)$ be the group of invertible elements in the R-algebra $\mathrm{Mat}(n \times n, R)$. The identity matrix in $\mathrm{Mat}(n \times n, R)$ is denoted I_d for each ring R.

Let $\mathcal{S}$ be a set and $\mathbf{d} = (d_i)_{i \in \mathcal{S}}, \mathbf{n} = (n_i)_{i \in \mathcal{S}} \in \mathbb{N}^{\mathcal{S}}$ some tuples. In most cases we assume that at most finitely many entries of such a tuple are non-zero. If $d_i \leq n_i$ for each $i \in \mathcal{S}$ we write $\mathbf{d} \leq \mathbf{n}$. Moreover, if $\mathbf{d} \leq \mathbf{n}$ and $\mathbf{d} \neq \mathbf{n}$, we write $\mathbf{d} < \mathbf{n}$. For a ring R we denote the product $\prod_{i \in \mathcal{S}} \mathrm{Mat}(d_i \times n_i, R)$ by $\mathrm{Mat}(\mathbf{d} \times \mathbf{n}, R)$ and the same for $\mathrm{GL}_{\mathbf{n}}(R)$.

2.1 Quivers and quiver representations

In this section we give a short introduction to the representation theory of quivers and we explain the relations to finite-dimensional k-algebras. Most of these definitions and results can be found in several books (see e.g. [1]).

Let $Q = (Q_0, Q_1, s, t)$ be a *locally finite quiver* (or $Q = (Q_0, Q_1)$ and *quiver* for short), i.e. an oriented graph with vertex set Q_0, arrow set Q_1 and maps $s, t \colon Q_1 \to Q_0$ indicating the start and terminal point of each arrow such that in each vertex only finitely many arrows start and end. A *finite-dimensional representation* $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ *of the quiver* Q (or Q*-representation* for short) is a tuple of finite-dimensional k-vector spaces $\{M_i | i \in Q_0\}$ and a tuple of k-linear maps $\{M_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)} | \alpha \in Q_1\}$ such that only finitely many of the vector spaces are non-zero. A *homomorphism* $f = (f_i)_{i \in Q_0} \colon M \to N$ of Q-representations is a tuple of k-linear maps $\{f_i \colon M_i \to N_i | i \in Q_0\}$ such that $f_{t(\alpha)} M_\alpha = N_\alpha f_{s(\alpha)}$ for all $\alpha \in Q_1$ (see Figure 2.1.1). The vector space of homomorphisms

$$\begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{f_{s(\alpha)}} & N_{s(\alpha)} \\ {\scriptstyle M_\alpha}\downarrow & & \downarrow{\scriptstyle N_\alpha} \\ M_{t(\alpha)} & \xrightarrow[f_{t(\alpha)}]{} & N_{t(\alpha)} \end{array}$$

Figure 2.1.1: The condition for a homomorphism $f \colon M \to N$ of Q-representations.

$f \colon M \to N$ of Q-representations is denoted by $\mathrm{Hom}_Q(M, N)$. Let $\mathrm{rep}(Q)$ denote the *category of finite-dimensional Q-representations.*

Let Q be a quiver and $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ a Q-representation. A *subrepresentation* $N = (N_i)_{i\in Q_0}$ of the Q-representation M is a tuple of subspaces $\{N_i \subseteq M_i | i \in Q_0\}$ such that $M_\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)}$ for all $\alpha \in Q_1$. So every subrepresentation $N = (N_i)_{i\in Q_0}$ of a Q-representation $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ is again a Q-representation $(N_i, M_\alpha|_{N_{s(\alpha)}})_{i\in Q_0,\alpha\in Q_1}$ and there is a canonical injective homomorphism $\iota\colon N \to M$ of Q-representations, which is called the *canonical embedding*. In this case we write $N \subseteq M$. Let M be a Q-representation and $\mathcal{S}$ a subset of M. Then $\langle m | m \in \mathcal{S}\rangle_Q$ denotes the minimal subrepresentation of M containing $\mathcal{S}$. A *factor representation* and the *canonical projection* are defined dually. Let M be a Q-representation. Then there is a unique largest semisimple subrepresentation of Q. This is called *socle* and denoted by $\operatorname{soc} M$. Dually the largest semisimple quotient of M is called *top* and denoted by $\operatorname{top} M$.

The *dimension* $\dim_k M$ of a Q-representation M is the dimension of the corresponding vector space $\bigoplus_{i\in Q_0} M_i$. Thus $\dim_k M = \sum_{i\in Q_0} \dim_k M_i$. The *dimension vector* of M is the tuple $\mathbf{dim}\, M = (\dim_k M_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$. So a dimension vector of Q is a tuple $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ with at most finitely many non-zero entries. This means for a dimension vector $\mathbf{d} = (d_i)_{i\in Q_0} \in \mathbb{N}^{Q_0}$ holds $|\mathbf{d}| := \sum_{i\in Q_0} d_i < \infty$. The *support* of a Q-representation $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ is the full subquiver of Q with vertices $\{i \in Q_0 | M_i \neq 0\}$.

Let $n \in \mathbb{N}$. An *oriented path* $\alpha_1 \dots \alpha_n$ of the quiver Q of length n is the concatenation of some arrows $\alpha_1, \dots, \alpha_n \in Q_1$ such that $t(\alpha_{i+1}) = s(\alpha_i)$ for all $1 \leq i < n$. Additionally we introduce a path e_i of length zero for each vertex $i \in Q_0$. The *path algebra* kQ of a quiver Q is the following k-algebra. The underlying k-vector space has a basis given by the set of oriented paths of Q. The product of basis vectors is given by the concatenation of paths if possible or by zero otherwise, e.g. $e_{t(\alpha)} \cdot \alpha \cdot e_{s(\alpha)} = \alpha$ for all $\alpha \in Q_1$.

Let Q be a locally finite quiver and kQ^+ the ideal of the path algebra kQ generated by all arrows. An ideal of the path algebra kQ contained in $(kQ^+)^2$ is called *semiadmissible ideal* and each semiadmissible ideal containing $(kQ^+)^n$ for some $n \in \mathbb{N}$ is called *admissible*. Thus the zero ideal is always semiadmissible, but not admissible in general. Based on the following observation we call also an ideal I of the path algebra kQ an *ideal of the quiver* Q.

Let Q be a quiver, I a semiadmissible ideal of Q and $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ a Q-representation. Let $\sum_{i=1}^{n} \lambda_i \alpha_{i1} \dots \alpha_{in_i}$ be a linear combination of oriented paths in the ideal I with $n, n_i \in \mathbb{N}$, $n_i \geq 2$, $\lambda_i \in k$ and $\alpha_{ij} \in Q_1$ for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_i\}$ such that there are $i_0, j_0 \in Q_0$ with $s(\alpha_{in_i}) = i_0$ and $t(\alpha_{i1}) = j_0$ for all $i \in \{1, \dots, n\}$. Then the Q-representation M is called a *(Q, I)-representation* if for each linear combination in I the linear map

$$\sum\nolimits_{i=1}^{n} \lambda_i M_{\alpha_{i1}} \circ \dots \circ M_{\alpha_{in_i}} \colon M_{i_0} \to M_{j_0}$$

vanishes. Moreover, the full subcategory of (Q, I)-representations of $\operatorname{rep}(Q)$ is denoted by $\operatorname{rep}(Q, I)$.

An additive category is called *Krull-Remak-Schmidt* if each object is isomorphic to a direct sum of indecomposable objects and this decomposition is unique. It is well-known that the category $\operatorname{rep}(Q, I)$ (especially $\operatorname{rep}(Q)$) is abelian and Krull-Remak-Schmidt for each semiadmissible ideal I. The category $\mathrm{rep}(Q, I)$ is called *representation finite* if the set of isomorphism classes of indecomposable (Q, I)-representations is finite. By a theorem of Gabriel [29, Satz 1.2] the category $\mathrm{rep}(Q)$ is representation finite if and only if the underlying graph of the quiver Q is a disjoint union of Dynkin graphs of type A, D or E.

For a quiver Q and a semiadmissible ideal I it is well-known that the *category of finite-dimensional kQ/I-modules* $\mathrm{mod}(kQ/I)$ is equivalent to the category $\mathrm{rep}(Q, I)$. So we think of (Q, I)-representations as kQ/I-modules and vice versa. Especially the categories $\mathrm{rep}(Q)$ and $\mathrm{mod}(kQ)$ are equivalent. Moreover, for each finite-dimensional k-algebra A exists a finite quiver Q and an admissible ideal I such that $\mathrm{rep}(Q, I)$ and $\mathrm{mod}(A)$ are equivalent (see e.g. [1, Corollary I 6.10., Theorem II 3.7.]).

A Q-representation M is called *nilpotent* if there is a $n \in \mathbb{N}$ such that M is a $(Q,(kQ^+)^n)$-representation. Moreover, let $\mathrm{nil}(Q)$ be the full *subcategory of nilpotent representations* of $\mathrm{rep}(Q)$. This category $\mathrm{nil}(Q)$ is an abelian and extension closed subcategory of the category $\mathrm{rep}(Q)$.

For $i \in Q_0$ we denote the Q-representation $(M_j, M_\alpha)_{j \in Q_0, \alpha \in Q_1}$ with $M_i = k$, $M_j = 0$ for each other $j \in Q_0$ and $M_\alpha = 0$ for all $\alpha \in Q_1$ by $S(i)$. These are up to isomorphism all simple Q-representations in $\mathrm{nil}(Q)$ and in $\mathrm{rep}(Q, I)$ for each admissible ideal I. Thus for a finite quiver Q and an admissible ideal I the k-algebra kQ/I is finite-dimensional and the isomorphism classes of simple representations are given by $\{S(i) | i \in Q_0\}$. Moreover, we denote the semisimple Q-representation $(M_j, M_\alpha)_{j \in Q_0, \alpha \in Q_1}$ with dimension vector $\mathbf{d}$ and $M_\alpha = 0$ for all $\alpha \in Q_1$ by $S(\mathbf{d})$.

Let $i \in \mathbb{N}$, Q a quiver and I a semiadmissible ideal. The *i-th cohomology group of extensions* of (Q, I)-representations M and N in the category $\mathrm{rep}(Q, I)$ is denoted by $\mathrm{Ext}^i_{(Q,I)}(M, N)$. If $I = 0$, we write $\mathrm{Ext}^i_Q(M, N)$ for short. Let M be a (Q, I)-representation. If the functors $\mathrm{Ext}^i_{(Q,I)}(M, -)$ are vanish for all $i \in \mathbb{N}$ with $i \geq 1$, we call it *projective in the category* $\mathrm{rep}(Q, I)$ (or a *projective (Q, I)-representation* for short). Dually the (Q, I)-representation M is called *injective in the category* $\mathrm{rep}(Q, I)$ (or an *injective (Q, I)-representation* for short) if $\mathrm{Ext}^i_{(Q,I)}(-, M) = 0$ for all $i \in \mathbb{N}$ with $i \geq 1$. Moreover, if a (Q, I)-representation is both projective and injective, we call it *projective-injective.* If $I = 0$, the category $\mathrm{rep}(Q)$ is hereditary, i.e. $\mathrm{Ext}^i_Q(M, N) = 0$ for all Q-representations M and N and $i \in \mathbb{N}$ with $i \geq 2$. Let $\mathbf{d} = (d_i)_{i \in Q_0}$ and $\mathbf{n} = (n_i)_{i \in Q_0}$ be dimension vectors. Since $\mathrm{rep}(Q)$ is hereditary the *Euler form*

$$\langle \mathbf{d}, \mathbf{n} \rangle = \sum\nolimits_{i \in Q_0} d_i n_i - \sum\nolimits_{\alpha \in Q_1} d_{s(\alpha)} n_{t(\alpha)} \tag{2.1.1}$$

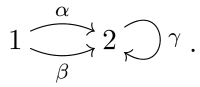
of the quiver Q behalves very well, e.g.

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \dim_k \mathrm{Hom}_Q(M, N) - \dim_k \mathrm{Ext}^1_Q(M, N) \tag{2.1.2}$$

for each Q-representations M and N.

Let Q be a quiver and M a Q-representation. If $\mathrm{Ext}^1_Q(M, M) = 0$, we call the Q-representation M *exceptional.* Thus for instance projective and injective Q-representations are exceptional.

Example 2.1.1. Let Q be the following quiver



Let $M = (M_1, M_2, M_\alpha, M_\beta, M_\gamma)$ be the Q-representation with $M_1 = k$, $M_2 = k^2$, $M_\alpha = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $M_\beta = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $M_\gamma = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$. The Q-representations $S(1)$, $S(2)$ and M are illustrated in the pictures in Figure 2.1.2.

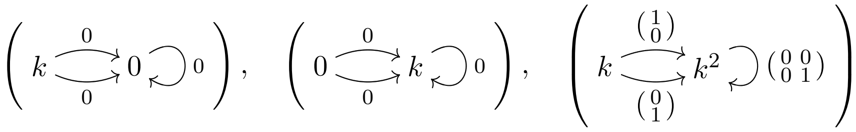


Figure 2.1.2: The Q-representations $S(1)$, $S(2)$ and M.

Thus $S(1)$ is a factor and $S(2)$ a subrepresentation of M. Moreover,

$$\mathrm{Hom}_Q(S(2), M) = \left\{ \left(0, \left(\begin{smallmatrix}\lambda\\0\end{smallmatrix}\right)\right) \,\middle|\, \lambda \in k \right\} \cong k.$$

The dimension vector of the Q-representation M is $(1, 2)$ and the support of $S(2)$ is the quiver $(\{2\}, \{\gamma\})$. Let $I = (\gamma\alpha)$ be the ideal generated by $\gamma\alpha$. Thus I is a semiadmissible ideal, which is not admissible, and M is an indecomposable (Q, I)-representation, which is not nilpotent. Using the Euler form of Q we can compute

$$\begin{aligned}
\dim_k \mathrm{Ext}^1_Q(S(2), M) &= \dim_k \mathrm{Hom}_Q(S(2), M) - \langle (0, 1), (1, 2) \rangle = 1 - (2 - 2) = 1, \\
\dim_k \mathrm{Ext}^1_Q(N, S(1)) &= \dim_k \mathrm{Hom}_k(N_1, k) - \langle (d_1, d_2), (1, 0) \rangle = d_1 - d_1 = 0
\end{aligned}$$

for each Q-representation $N = (N_1, N_2, N_\alpha, N_\beta, N_\gamma)$ with dimension vector $\mathbf{d} = (d_1, d_2)$. Thus the Q-representation $S(1)$ is injective.

2.2 Tree and band modules

Let $Q = (Q_0, Q_1, s, t)$ and $S = (S_0, S_1, s', t')$ be two quivers. A *winding of quivers* $F\colon S \to Q$ (or *winding* for short) is a pair of maps $F_0\colon S_0 \to Q_0$ and $F_1\colon S_1 \to Q_1$ such that the following hold:

1. F is a morphism of quivers, i.e. $sF_1 = F_0 s'$ and $tF_1 = F_0 t'$.
2. If $a, b \in S_1$ with $a \neq b$ and $s'(a) = s'(b)$, then $F_1(a) \neq F_1(b)$.
3. If $a, b \in S_1$ with $a \neq b$ and $t'(a) = t'(b)$, then $F_1(a) \neq F_1(b)$.

This generalizes Krause's definition of a winding [37]. Let V be a S-representation. For $i \in Q_0$ and $a \in Q_1$ set

$$(F_*(V))_i = \bigoplus\nolimits_{j \in F_0^{-1}(i)} V_j \qquad \text{and} \qquad (F_*(V))_a = \bigoplus\nolimits_{b \in F_1^{-1}(a)} V_b.$$

This induces a functor $F_*\colon \operatorname{rep}(S) \to \operatorname{rep}(Q)$ and a map of dimension vectors $\mathbf{F}\colon \mathbb{N}^{S_0} \to \mathbb{N}^{Q_0}$. The concatenation of windings behaves very well: Let $F\colon S \to Q$ and $G\colon T \to S$ be windings then $FG\colon T \to Q$ is again a winding and the functors $(FG)_*$ and F_*G_* are naturally isomorphic.

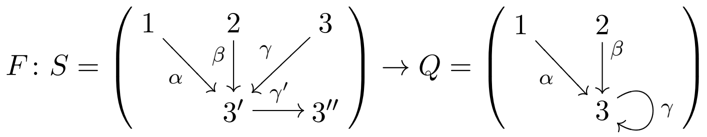
Let Q be a finite quiver. Then the Q-representation $(M_i, M_a)_{i \in Q_0, a \in Q_1}$ with $M_i = k$ for all $i \in Q_0$ and $M_a = \mathrm{id}_k$ for all $a \in Q_1$ is denoted $\mathbb{1}_Q$. For $n \in \mathbb{N}$ let $\mathcal{I}_Q^n$ be the set of all indecomposable Q-representations $(M_i, M_a)_{i \in Q_0, a \in Q_1}$ with $\dim_k M_i = n$ for all $i \in Q_0$ and M_a is an isomorphism for all $a \in Q_1$.

A simply connected and finite quiver S is called a *tree*, i.e. for two vertices in the quiver S exists a unique not necessarily oriented path from one vertex to the other.

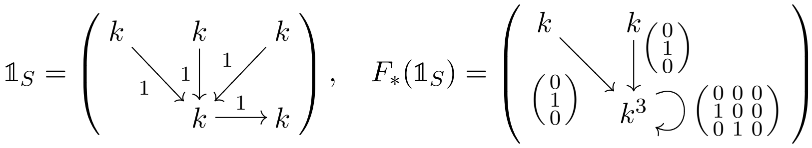
Definition 2.2.1. Let Q and S be quivers and $F\colon S \to Q$ a winding. If S is a tree, then the representation $F_*(\mathbb{1}_S)$ is called a *tree module*. We call such a winding F a *tree*, too.

By [28, Lemma 3.5] all tree modules are indecomposable.

Example 2.2.2. Let Q, S and F be described by the following picture.



This means for the morphism $F\colon S \to Q$ of quivers holds $F_0(1) = 1$, $F_0(2) = 2$, $F_0(3) = F_0(3') = F_0(3'') = 3$, $F_1(\alpha) = \alpha$, $F_1(\beta) = \beta$ and $F_1(\gamma) = F_1(\gamma') = \gamma$. Then $\mathbb{1}_S$ and $F_*(\mathbb{1}_S)$ are described by the following pictures, $F\colon S \to Q$ is a tree and $F_*(\mathbb{1}_S)$ a tree module.



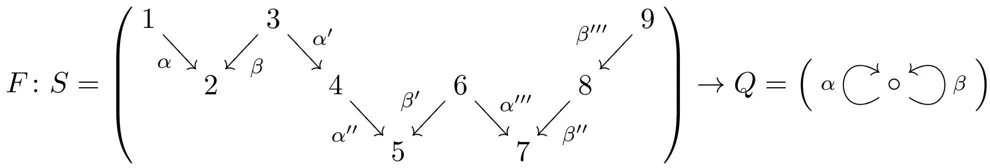
A quiver $S = (S_0, S_1, s, t)$ is called *of type* A_l for some $l \in \mathbb{Z}_{>0}$ (or *of type* A for short) if $S_0 = \{1, \dots, l\}$ and $S_1 = \{s_1, \dots, s_{l-1}\}$ such that for all $i \in S_0$ with $i \neq l$ there exists a $\varepsilon_i \in \{-1, 1\}$ with $s(s_i^{\varepsilon_i}) = i+1$ and $t(s_i^{\varepsilon_i}) = i$. We use here the convention $s(a^{-1}) = t(a)$ and $t(a^{-1}) = s(a)$ for all $a \in S_1$. A quiver S of type A_l is called *linearly oriented* if $\varepsilon_i = -1$ for all $i \in \{1, \dots, l-1\}$. Figure 2.2.1 visualizes a linearly oriented quiver of type A_l.

$$1 \xrightarrow{s_1} 2 \xrightarrow{s_2} 3 \longrightarrow \cdots \longrightarrow l-1 \xrightarrow{s_{l-1}} l$$

Figure 2.2.1: A linearly oriented quiver of type A_l.

Definition 2.2.3. Let Q and S be quivers, S of type A_l, $F\colon S \to Q$ a winding and $F_*(\mathbb{1}_S)$ a tree module. Then F is called a *string* and $F_*(\mathbb{1}_S)$ a *string module*.

Example 2.2.4. Let $Q = (\{\circ\}, \{\alpha, \beta\})$ a quiver and F the string described by the following picture.



In this case the string module $F_*(\mathbb{1}_S)$ has a basis $\{e_i | i \in S_0\}$ and is visualized by the picture in Figure 2.2.2. This means the vertices of this quiver correspond to the basis

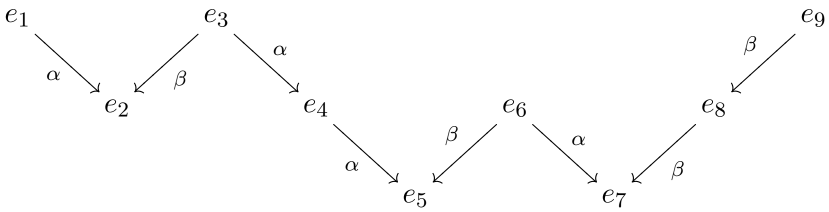


Figure 2.2.2: A string module $F_*(\mathbb{1}_S)$.

vectors $\{e_i | i \in S_0\}$ of $F_*(\mathbb{1}_S)$ and the arrows describe the linear maps corresponding to the arrows of Q. For example the basis vector e_1 is mapped to e_2 by the linear map $F_*(\mathbb{1}_S)_\alpha$ and to zero by $F_*(\mathbb{1}_S)_\beta$.

A quiver S is called *of type* $\tilde{A}_{l-1}$ for some $l \in \mathbb{Z}_{>0}$ (or *of type* $\tilde{A}$ for short) if $S_0 = \{1, \ldots, l\}$ and $S_1 = \{s_1, \ldots, s_l\}$ such that for all $i \in S_0$ a $\varepsilon_i \in \{-1, 1\}$ exists with $s(s_i^{\varepsilon_i}) = i + 1$ and $t(s_i^{\varepsilon_i}) = i$. We set $l + i := i$ in S_0, $s_{l+i} := s_i$ in S_1 and $\varepsilon_{l+i} := \varepsilon_i$ for all $i \in S_0$. A quiver S of type $\tilde{A}_{l-1}$ is called *cyclically oriented* if $\varepsilon_i = -1$ for all $i \in S_0$. We draw pictures of quivers of type $\tilde{A}_{l-1}$ in Figure 3.6.11 and 4.4.1.

Definition 2.2.5. Let Q and S be quivers, $B\colon S \to Q$ a winding and $V \in \mathcal{I}_S^n$. If S is of type $\tilde{A}_{l-1}$ and $B_*(V)$ is indecomposable, then $B_*(V)$ is called a *band module*. The winding B is called a *band* if S is of type $\tilde{A}_{l-1}$ and $B_*(\mathbb{1}_S)$ is indecomposable.

Let S be a quiver of type $\tilde{A}_{l-1}$ and $B\colon S \to Q$ a winding. The module $B_*(\mathbb{1}_S)$ is not necessarily indecomposable. This well-known feature is explained in the following example.

Example 2.2.6. Let Q and S be quivers, S of type $\tilde{A}_{l-1}$, $B\colon S \to Q$ a winding and $V \in \mathcal{I}_S^n$.

1. If there is no integer r with $1 \leq r < l$, $B_1(s_i) = B_1(s_{i+r})$ and $\varepsilon_i = \varepsilon_{i+r}$ for all $1 \leq i \leq l$ and the Jordan normal form of the linear map $V_{s_1}^{\varepsilon_1} \ldots V_{s_l}^{\varepsilon_l}$ is an indecomposable Jordan matrix, then $B_*(V)$ is indecomposable.
2. If there is an integer r with $r > 0$ as above, then $B_*(V) \cong \bigoplus_{i=1}^{s} M^{(i)}$ with $s = \frac{l}{\gcd(r,l)}$ and Q-representations $M^{(i)}$ of dimension $n \gcd(r, l)$.

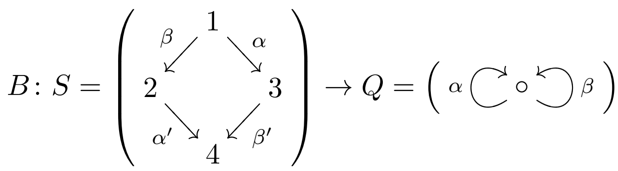
Remark 2.2.7. Using the Jordan normal form, the indecomposable modules of the polynomial ring $k[T, T^{-1}]$ of dimension r are canonically parametrized by k^* for each $r \in \mathbb{Z}_{>0}$. Let $\varphi_r : k^* \to \mathrm{mod}(k[T, T^{-1}])$ describe this parametrization and $\varphi\colon k^* \times \mathbb{Z}_{>0} \to \mathrm{mod}(k[T, T^{-1}])$ with $\varphi(\lambda, r) = \varphi_r(\lambda)$.

Let $B\colon S \to Q$ be a band and $\mathrm{mod}(k[T, T^{-1}])$ the category of finite-dimensional $k[T, T^{-1}]$-modules. There exists a full and faithful functor $F\colon \mathrm{mod}(k[T, T^{-1}]) \to \mathrm{rep}(S)$ such that $F(V) \in \mathcal{I}_S^{\dim_k V}$ for each indecomposable $V \in \mathrm{mod}(k[T, T^{-1}])$.

The map $k^* \times \mathbb{Z}_{>0} \xrightarrow{\varphi} \mathrm{mod}(k[T, T^{-1}]) \xrightarrow{F} \mathrm{rep}(S) \xrightarrow{B_*} \mathrm{rep}(Q)$ is a parametrization of all band modules of the form $B_*(V)$. The image of $(\lambda, r) \in k^* \times \mathbb{Z}_{>0}$ under this map is denoted $B_*(\lambda, r)$. Additional we define $B_*(\lambda, 0) = 0$ for all $\lambda \in k^*$. We remark that neither the functor F nor our parametrization of band modules of the form $B_*(V)$ is unique.

Let $\lambda \in k^*$ and $r, s \in \mathbb{N}$ with $r \geq s$. Then a surjective homomorphism $B_*(\lambda, r) \twoheadrightarrow B_*(\lambda, s)$ and an injective homomorphism $B_*(\lambda, s) \hookrightarrow B_*(\lambda, r)$ of Q-representations exists. Let $\varphi\colon B_*(\lambda, r) \to B_*(\lambda, s)$ be such a homomorphism. Then the kernel and the image of φ are independent of φ. So for all $r, s \in \mathbb{N}$ with $r \geq s$ exists a unique sub- and a unique factor module of $B_*(\lambda, r)$ isomorphic to $B_*(\lambda, s)$.

Example 2.2.8. Let $Q = (\{\circ\}, \{\alpha, \beta\})$ be as in Example 2.2.4, $\lambda \in k^*$ and B the band described by the following picture.



In this case we can assume that the band module $B_*(\lambda, 3)$ has a basis $\{e_{ij} | i \in S_0, j \in \{1, 2, 3\}\}$ and is visualized in Figure 2.2.3. In this case there are written some scalar multiples. This means for example $B_*(\lambda, 3)_\beta(e_{31}) = \lambda e_{41}$ and $B_*(\lambda, 3)_\beta(e_{33}) = e_{42} + \lambda e_{43}$.

Crawley-Boevey [18] and Krause [37] constructed a basis of the homomorphism spaces of tree and band modules. This description yields the following lemma.

Lemma 2.2.9. *Let Q, S, T be connected quivers, $F\colon S \to Q$ and $G\colon T \to Q$ trees or bands, $V \in \mathcal{I}_S^n$ and $W \in \mathcal{I}_T^m$. If $F_*(V) \cong G_*(W)$, then a unique bijective winding $H\colon S \to T$ exists such that $F = GH$ and $H_*(V) \cong W$.*

Proof. Since $F_*(V)$ is indecomposable the endomorphism ring $\mathrm{End}_Q(F_*(V))$ is local. Thus by [18] and [37] such a winding H exists. Since F and G are trees or bands the modules $H_*(V)$ and W are isomorphic. The winding H is unique since there is no non-trivial automorphism $H'\colon S \to S$ with $F = FH'$ for a connected tree or band $F\colon S \to Q$. □

Remark 2.2.10. Let Q be a quiver of type $\tilde{A}_{l-1}$. The category $\mathrm{rep}(Q)$ is well-known and described in [52]. The indecomposable Q-representations are divided into three

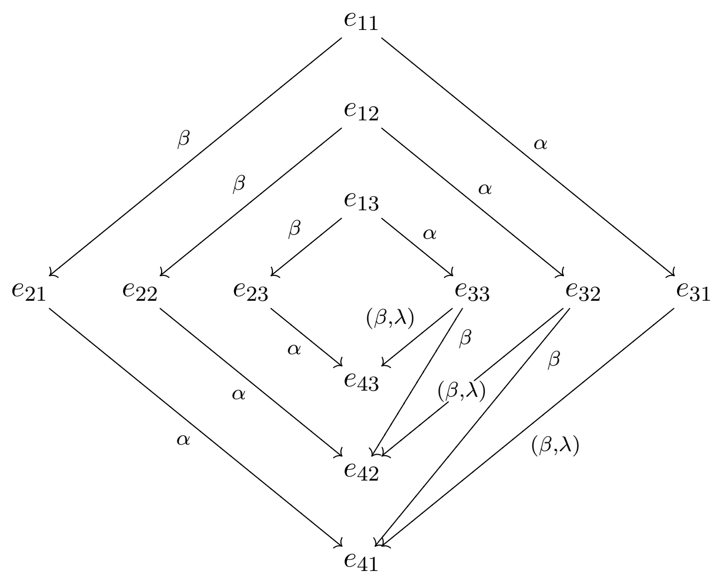


Figure 2.2.3: A band module $B_*(\lambda, 3)$.

classes: The classes of *preprojective*, *regular* and *preinjective* representations. Let M be a band module and N a string module of Q. Then the band module M is regular and the following hold:

- Let N be preprojective. Then N is exceptional, determined up to isomorphism by its dimension vector, $\mathrm{Hom}_Q(N, M) \neq 0$ and $\mathrm{Hom}_Q(M, N) = 0$. If $\dim_k N \leq \dim_k M$, then an injective homomorphism $N \hookrightarrow M$ of Q-representations and an indecomposable preinjective representation with dimension vector $\mathbf{dim}\, M - \mathbf{dim}\, N$ exists. All short exact sequences $0 \to M \to L \to N \to 0$ with a Q-representation L split.
- If N is regular, then $\mathrm{Hom}_Q(N, M) = 0$ and $\mathrm{Hom}_Q(M, N) = 0$.
- Let N be preinjective. Then N is again exceptional, determined up to isomorphism by its dimension vector, $\mathrm{Hom}_Q(N, M) = 0$ and $\mathrm{Hom}_Q(M, N) \neq 0$. All short exact sequences $0 \to N \to L \to M \to 0$ with a Q-representation L split.

Let M and N be indecomposable preprojective Q-representations such that $\dim_k M \geq \dim_k N$. Then $\mathrm{Hom}_Q(M, N) = 0$ if $M \ncong N$ and all short exact sequences $0 \to M \to L \to N \to 0$ with a Q-representation L split.

These notions of string and band modules were introduced to study the following class of finite-dimensional k-algebras.

Definition 2.2.11. Let Q be a finite quiver and I an admissible ideal. Then $A = kQ/I$ is called a *string algebra* if the following hold:

1. At most two arrows start and at most two arrows end in each vertex of Q.
2. Let $\alpha, \beta, \gamma \in Q_1$. If $\alpha \neq \beta$, then $\alpha\gamma \in I$ or $\beta\gamma \in I$. If $\beta \neq \gamma$, then $\alpha\beta \in I$ or $\alpha\gamma \in I$.

3. The ideal I is generated by oriented paths of Q.

Example 2.2.12. Let $Q = (\{\circ\}, \{\alpha, \beta\})$ be as in Example 2.2.4 and 2.2.8 and $I = (\alpha^2, \beta^2, \alpha\beta\alpha)$ the admissible ideal of kQ generated by α^2, β^2 and $\alpha\beta\alpha$. Then $A = kQ/I$ is a string algebra and the set $\{e_\circ, \alpha, \beta, \alpha\beta, \beta\alpha, \beta\alpha\beta\}$ of paths is a basis of the vector space A.

Let A be a string algebra. By [53, Proposition 2.3.] it is well-known that every indecomposable A-module is a string or a band module.

2.3 Algebraic geometry

Algebraic varieties and schemes are basic objects in algebraic geometry (see [8, 22, 31, 51]). In our studies each variety is a reduced quasi-projective variety, which is not necessarily irreducible. For example *module varieties* and *quiver Grassmannians* are studied in representation theory. Roughly speaking the quiver Grassmannian of a quiver representation is the collection of subrepresentations of this quiver representation with a fixed dimension vector. These “collections“ turn out to be algebraic varieties (see Section 2.3.2) and actually algebraic schemes (see Section 2.3.3).

In representation theory module varieties are more common. These affine varieties parametrize in some sense all quiver representations for a fixed quiver and a fixed dimension vector. Again these varieties turn out to be algebraic schemes (see [8, Section 3.1] and Section 2.3.3). Since we compare module varieties and quiver Grassmannians in Section 3.4, we have to introduce the module variety in detail (see Section 2.3.1). First of all we repeat some notions from algebraic geometry.

Let X be a topological space (e.g. an algebraic variety). A map $\varphi\colon X \to \mathbb{Z}$ is called *upper semicontinuous* if for all $n \in \mathbb{Z}$ the set $\{x \in X | \varphi(x) \geq n\}$ is closed in X. Dually it is called *lower semicontinuous* if $\{x \in X | \varphi(x) \leq n\}$ is closed for all $n \in \mathbb{Z}$.

Example 2.3.1. Let $d, n \in \mathbb{N}$ and $\operatorname{rk}\colon \operatorname{Mat}(n \times d, k) \to \mathbb{Z}$ the usual rank function. We consider $\operatorname{Mat}(n \times d, k)$ as an affine variety. Let $r \in \mathbb{N}$ and B an r-minor of the matrices in $\operatorname{Mat}(n \times d, k)$. The induced algebraic morphism $B\colon \operatorname{Mat}(n \times d, k) \to k$ is continuous and thus $B^{-1}(0)$ is closed in $\operatorname{Mat}(n \times d, k)$. The map rk is lower semicontinuous, since

$$\left\{M \in \operatorname{Mat}(n \times d, k) \,\middle|\, \operatorname{rk}(M) \leq r\right\} = \bigcap\nolimits_{B,\ (r+1)\text{-minor}} B^{-1}(0).$$

Let X be an algebraic variety and $x \in X$. We denote the maximum of the dimensions of irreducible components of X containing x as *the dimension* $\dim_x X$ *of* X *at* x.

Let $k[\varepsilon] = k[X]/(X^2)$ be the dual numbers of k, X a k-scheme and x a k-valued point of X. Using [22, VI.1.3] the *tangent space* $T_x X$ of the k-scheme X is the set of $k[\varepsilon]$-valued points, which lift the point x. If the k-scheme X is affine, i.e. $X = \operatorname{Spec} A$ for a commutative k-algebra A, we get for each k-valued point $x \in \operatorname{Hom}_{k\text{-Alg}}(A, k)$

$$T_x X \cong \left\{f \in \operatorname{Hom}_{k\text{-Alg}}(A, k[\varepsilon]) \,\middle|\, \pi f = x\right\}$$

as vector spaces with the canonical projection $\pi\colon k[\varepsilon] \to k$. In general the dimension of the vector space T_xX is at least the dimension of $X(k)$ in x. We call a k-valued point x of a k-scheme X *smooth* if $\dim_k T_xX = \dim_x X$.

Let X be a complex algebraic variety. We denote the *Euler characteristic* of the topological space X with the analytical topology by $\chi(X)$ (see e.g. [27, Section 4.5]). Then for example $\chi(X) = \chi(U) + \chi(X\backslash U)$ holds for each constructible subset U of X. Moreover, for a morphism $f\colon X \to Y$ of complex algebraic varieties holds $\chi(X) = \chi(Y)\chi(f^{-1}(y))$ for each $y \in Y$ if the map $Y \to \mathbb{Z}, y \mapsto \chi(f^{-1}(y))$ is constant.

2.3.1 The module variety

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\mathbf{d}$ a dimension vector. Each tuple $(M_\alpha)_{\alpha\in Q_1}$ of matrices in $\prod_{\alpha\in Q_1} \operatorname{Mat}\left(d_{t(\alpha)} \times d_{s(\alpha)}, k\right)$ is a Q-representation $(M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ with $M_i = k^{d_i}$ for all $i \in Q_0$. Moreover, for a semiadmissible ideal I some of these tuples $(M_\alpha)_{\alpha\in Q_1}$ are (Q, I)-representations.

Definition 2.3.2. Let Q be a quiver and $\mathbf{d}$ a dimension vector. The affine variety

$$\prod\nolimits_{\alpha\in Q_1} \operatorname{Mat}\left(d_{t(\alpha)} \times d_{s(\alpha)}, k\right)$$

is called *module variety* and denoted by $\operatorname{rep}_{\mathbf{d}}(Q)$. Each semiadmissible ideal I yields a closed subvariety

$$\operatorname{rep}_{\mathbf{d}}(Q, I) = \left\{ (M_\alpha)_{\alpha\in Q_1} \in \operatorname{rep}_{\mathbf{d}}(Q) \,\middle|\, \left(k^{d_i}, M_\alpha\right)_{i\in Q_0,\alpha\in Q_1} \in \operatorname{rep}(Q, I) \right\}$$

of (Q, I)-representations of the affine variety $\operatorname{rep}_{\mathbf{d}}(Q)$.

The dimension of the variety $\operatorname{rep}_{\mathbf{d}}(Q)$ is $\sum_{\alpha\in Q_1} d_{s(\alpha)}d_{t(\alpha)}$. In general the algebraic group $\operatorname{GL}_{\mathbf{d}}(k)$ acts by conjugation on the variety $\operatorname{rep}_{\mathbf{d}}(Q, I)$. For $U \in \operatorname{rep}_{\mathbf{d}}(Q, I)$ the orbit under this action is denoted by $\mathcal{O}(U)$. These $\operatorname{GL}_{\mathbf{d}}(k)$-orbits in $\operatorname{rep}_{\mathbf{d}}(Q, I)$ are in bijection with the isomorphism classes of (Q, I)-representations with dimension vector $\mathbf{d}$. Thus we can associate to each Q-representation N with dimension vector $\mathbf{d}$ the corresponding orbit. This is denoted by $\mathcal{O}(N)$ although N is in general not a point of the variety $\operatorname{rep}_{\mathbf{d}}(Q)$. The geometry of these orbits is very well-known by the following proposition (see [8, Proposition 2.1.7]).

Proposition 2.3.3. *Let Q be a quiver, I a semiadmissible ideal, $\mathbf{d}$ a dimension vector and $U \in \operatorname{rep}_{\mathbf{d}}(Q, I)$. Then the subset $\mathcal{O}(U)$ of $\operatorname{rep}_{\mathbf{d}}(Q, I)$ is irreducible, locally closed, smooth and has dimension $\dim \operatorname{GL}_{\mathbf{d}}(k) - \dim_k \operatorname{End}_Q(U)$.*

Since the closure of an orbit in the variety $\operatorname{rep}_{\mathbf{d}}(Q, I)$ is a union of orbits we get a partial order on the isomorphism classes of (Q, I)-representations with dimension vector $\mathbf{d}$. This much studied partial order is called *degeneration order* (see for example works of Bongartz, Kraft, Riedtmann and Zwara). It is well-known that if there is a short exact sequence of the form $0 \to U \to M \to U' \to 0$ with Q-representations U, M and U', then $\mathcal{O}(U \oplus U') \subseteq \overline{\mathcal{O}(M)}$ holds in the variety $\operatorname{rep}_{\mathbf{dim}\, M}(Q)$.

The following proposition is well-known (see Example 2.3.1 and [43, Proposition 2.1]).

Proposition 2.3.4. *Let Q be a quiver and $\mathbf{d}$ a dimension vector. For $\beta \in Q_1$ the map*

$$\mathrm{rk}_\beta\colon\ \mathrm{rep}_{\mathbf{d}}(Q) \to \mathbb{Z}, (U_\alpha)_{\alpha\in Q_1} \mapsto \mathrm{rk}\,(U_\beta)$$

is lower semicontinuous. For each Q-representation N the maps

$$\dim_k \mathrm{Hom}_Q(-,N)\colon\ \mathrm{rep}_{\mathbf{d}}(Q) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(U,N),$$
$$\dim_k \mathrm{Hom}_Q(N,-)\colon\ \mathrm{rep}_{\mathbf{d}}(Q) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(N,U)$$

are both upper semicontinuous.

Using [8, Section 3.1] we can generalize the definition of a module variety $\mathrm{rep}_{\mathbf{d}}(Q,I)$ and define a corresponding k-scheme such that its k-valued points are $\mathrm{rep}_{\mathbf{d}}(Q,I)$ (see also Section 2.3.3). Moreover, by [36, Section 2.7] or [8, Corollary 3.2.3] the normal space of the orbit $\mathcal{O}(U)$ of a k-valued point U in this scheme is isomorphic to $\mathrm{Ext}^1_{(Q,I)}(U,U)$. Thus for the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ we get the following well-known and unsurprising result.

Lemma 2.3.5. *Let Q be a quiver and $\mathbf{d}$ a dimension vector. There is a Q-representation M with $\mathrm{rep}_{\mathbf{d}}(Q) = \overline{\mathcal{O}(M)}$ if and only if there is an exceptional Q-representation with dimension vector $\mathbf{d}$.*

Proof. Using Proposition 2.3.3 and the Euler form we get for each $M \in \mathrm{rep}_{\mathbf{d}}(Q)$

$$\dim \mathrm{rep}_{\mathbf{d}}(Q) - \dim \mathcal{O}(M) = \dim_k \mathrm{End}_Q(M) - \langle \mathbf{d}, \mathbf{d}\rangle = \dim_k \mathrm{Ext}^1_Q(M,M).$$

Thus M is exceptional if and only if $\mathrm{rep}_{\mathbf{d}}(Q) = \overline{\mathcal{O}(M)}$. □

Corollary 2.3.6. *Let Q be a quiver. Then there exists up to isomorphism at most one exceptional Q-representation to each dimension vector.*

2.3.2 Grassmannians as varieties

The classical Grassmannians $\mathrm{Gr}\binom{n}{d}$ are well-known and much studied geometric objects. We use them to define the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$. It turns out that we need a schematic version of this. Nevertheless, first of all we will define it as an algebraic variety.

Let $d, n \in \mathbb{N}$ with $d \le n$. The algebraic group $\mathrm{GL}_d(k)$ acts freely by right multiplication on the open subset of matrices of rank d in $\mathrm{Mat}(n \times d, k)$. The quotient of this set is called *(classical) Grassmannian* $\mathrm{Gr}\binom{n}{d}$. Using the Plücker embedding this is a projective variety (see e.g. [51, Chapter I, Section 4]). Let

$$\pi\colon \left\{A \in \mathrm{Mat}(n\times d, k) \,\middle|\, \mathrm{rk}(A) = d\right\} \twoheadrightarrow \mathrm{Gr}\tbinom{n}{d} \qquad (2.3.1)$$

be the induced canonical projection. Thus the Grassmannian $\mathrm{Gr}\binom{n}{d}$ parametrizes all d-dimensional subspaces of a fixed n-dimensional k-vector space.

Now we define an open affine covering of the Grassmannian $\mathrm{Gr}\binom{n}{d}$. This means the Grassmannian is the glueing of these subvarieties. For each d-minor B of matrices in $\mathrm{Mat}(n \times d, k)$ we define

$$U_B = \{\pi(A) \mid A \in \mathrm{Mat}(n \times d, k), \det(B(A)) \neq 0\}$$

an open subset of $\mathrm{Gr}\binom{n}{d}$. This variety is isomorphic to the affine variety $\mathrm{Mat}((n-d) \times d, k)$. Let B be the d-minor of the first d rows. Then

$$\pi_B \colon \mathrm{Mat}((n-d) \times d, k) \to U_B,\ C \mapsto \pi\begin{pmatrix} I_d \\ C \end{pmatrix} \tag{2.3.2}$$

is a well-defined isomorphism of algebraic varieties.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\mathbf{n} = (n_i)_{i \in Q_0}$ and $\mathbf{d} = (d_i)_{i \in Q_0}$ dimension vectors. We consider the products of classical Grassmannians $\mathrm{Gr}\binom{\mathbf{n}}{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{Gr}\binom{n_i}{d_i}$. Let $\mathbf{B} = (B_i)_{i \in Q_0}$ be a tuple such that B_i is a d_i-minor of the matrices in $\mathrm{Mat}(n_i \times d_i, k)$ for each $i \in Q_0$ we call this tuple a $\mathbf{d}$-*minor*. For a $\mathbf{d}$-minor we define the following product $U_{\mathbf{B}} = \prod_{i \in Q_0} U_{B_i}$ with the affine subsets U_{B_i} of $\mathrm{Gr}\binom{n_i}{d_i}$ for all $i \in Q_0$. This forms a covering of open affine sets of $\mathrm{Gr}\binom{\mathbf{n}}{\mathbf{d}}$ indexed by the $\mathbf{d}$-minors. Let $\mathbf{B}$ be the $\mathbf{d}$-minor of the first rows. Then

$$\pi_{\mathbf{B}} \colon \mathrm{Mat}((\mathbf{n} - \mathbf{d}) \times \mathbf{d}, k) \to U_{\mathbf{B}},\ (C_i)_{i \in Q_0} \mapsto \left(\pi\begin{pmatrix} I_{d_i} \\ C_i \end{pmatrix}\right)_{i \in Q_0} \tag{2.3.3}$$

is again a well-defined isomorphism of algebraic varieties.

Lemma 2.3.7. *Let Q be a quiver, M a Q-representation with dimension vector* $\mathbf{n}$ *and* $\mathbf{d}$ *another dimension vector. Then the subset*

$$\mathrm{Gr}_{\mathbf{d}}(M) = \{U \subseteq M \mid \mathbf{dim}\, U = \mathbf{d}\}$$

of $\mathrm{Gr}\binom{\mathbf{n}}{\mathbf{d}}$ *is closed. So this subvariety* $\mathrm{Gr}_{\mathbf{d}}(M)$ *of the product of classical Grassmannians* $\mathrm{Gr}\binom{\mathbf{n}}{\mathbf{d}}$ *is called* quiver Grassmannian*.*

Hence this is a projective k-variety. In general the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ is not connected and not equidimensional, i.e. the dimensions of the irreducible components differ (see Example 2.3.9). By the following example it is neither smooth nor irreducible in general.

If k is the field of complex numbers $\mathbb{C}$, we denote the Euler characteristic $\chi(\mathrm{Gr}_{\mathbf{d}}(M))$ of a quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ by $\chi_{\mathbf{d}}(M)$ for short.

Example 2.3.8. Let $Q = 1 \xrightarrow{\alpha} 2$, $M_1 = M_2 = k^2$ and $M_\alpha \colon M_1 \to M_2$ a linear map with $\mathrm{rk}(M_\alpha) = 1$. Then $M = (M_1, M_2, M_\alpha)$ is a Q-representation such that $\mathrm{Gr}_{(1,1)}(M)$ can be described as $(\{*\} \times \mathbb{P}^1_k) \cup (\mathbb{P}^1_k \times \{*\}) \subseteq \mathbb{P}^1_k \times \mathbb{P}^1_k$. This projective variety is neither smooth nor irreducible. By the way, if $k = \mathbb{C}$, then $\chi_{(1,1)}(M) = 3$.

Proof of Lemma 2.3.7. We have to prove that the quiver Grassmannian is really described by some closed condition as a subvariety of the product of classical Grassmannians. Without loss of generality, let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be the Q-representation with

$M_i = k^{n_i}$ for all $i \in Q_0$ and $M_\alpha \in \operatorname{Mat}(n_{t(\alpha)} \times n_{s(\alpha)}, k)$ for all $\alpha \in Q_1$. We use the affine sets $U_{\mathbf{B}}$ with $\mathbf{d}$-minors. Let $\mathbf{B}$ be the $\mathbf{d}$-minor of the first rows. Thus Equation (2.3.3) holds and it is enough to show that the set

$$\left\{ (C_i)_{i \in Q_0} \in \operatorname{Mat}((\mathbf{n}-\mathbf{d}) \times \mathbf{d}, k) \,\middle|\, \exists (X_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \operatorname{Mat}(d_{t(\alpha)} \times d_{s(\alpha)}, k) : M_\alpha \begin{pmatrix} I_{d_{s(\alpha)}} \\ C_{s(\alpha)} \end{pmatrix} = \begin{pmatrix} I_{d_{t(\alpha)}} \\ C_{t(\alpha)} \end{pmatrix} X_\alpha \ \forall \alpha \in Q_1 \right\} \tag{2.3.4}$$

is closed in $\operatorname{Mat}((\mathbf{n}-\mathbf{d}) \times \mathbf{d}, k)$. If we write for each $\alpha \in Q_1$ the matrix M_α as a block matrix such that

$$\begin{pmatrix} M_{1,\alpha} & M_{2,\alpha} \\ M_{3,\alpha} & M_{4,\alpha} \end{pmatrix} \in \operatorname{Mat}\Big(\big((d_{t(\alpha)}) + (n_{t(\alpha)} - d_{t(\alpha)})\big) \times \big((d_{s(\alpha)}) + (n_{s(\alpha)} - d_{s(\alpha)})\big), k \Big),$$

then Equation (2.3.4) yields the following two equations:

$$\begin{aligned} M_{1,\alpha} + M_{2,\alpha} C_{s(\alpha)} &= X_\alpha, \\ M_{3,\alpha} + M_{4,\alpha} C_{s(\alpha)} &= C_{t(\alpha)} X_\alpha. \end{aligned}$$

Thus we have to prove that the set

$$\left\{ (C_i)_{i \in Q_0} \,\middle|\, M_{3,\alpha} + M_{4,\alpha} C_{s(\alpha)} = C_{t(\alpha)} M_{1,\alpha} + C_{t(\alpha)} M_{2,\alpha} C_{s(\alpha)} \ \forall \alpha \in Q_1 \right\}$$

is closed in $\operatorname{Mat}((\mathbf{n}-\mathbf{d}) \times \mathbf{d}, k)$. This set is obviously given by polynomials. □

Example 2.3.9. Let Q be the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\ ,$$

$M = (M_1, M_2, M_\alpha, M_\beta)$ the Q-representation with $M_1 = M_2 = k$, $M_\alpha = id_k$ and $M_\beta = 0$ and $N = (N_1, N_2, N_\alpha, N_\beta)$ with $N_1 = N_2 = k$, $N_\alpha = 0$ and $N_\beta = id_k$. Then $\operatorname{Gr}_{(1,1)}(M \oplus N) = \{M, N\}$ and in general for $i, j \in \mathbb{Z}$ with $i \geq 1$ and $j \geq 1$ the quiver Grassmannian $\operatorname{Gr}_{(1,1)}(M^i \oplus N^j)$ is isomorphic to the disjoint union of $\mathbb{P}_k^{i-1}$ and $\mathbb{P}_k^{j-1}$. Thus this variety is not connected and in general not equidimensional.

The following lemma allows us to dualize most statements.

Lemma 2.3.10. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver, $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ a Q-representation and $\mathbf{d}$ a dimension vector. Let $Q^{op} = (Q_0, \{\alpha \colon t(\alpha) \to s(\alpha) | \alpha \in Q_1\})$ be the* opposite quiver *and $M^* = (M_i^*, M_\alpha^*)_{i \in Q_0, \alpha \in Q_1^{op}}$ the dual Q^{op}-representation. Then*

$$(-)^* \colon \operatorname{Gr}_{\mathbf{d}}(M) \to \operatorname{Gr}_{\mathbf{dim}\, M - \mathbf{d}}(M^*), U \mapsto (M/U)^*$$

is an isomorphism of algebraic varieties.

Proof. For each diagram of vector spaces with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & M_1 & \longrightarrow & M_1/U_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\alpha|_{U_1}} & & \downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle \overline{f_\alpha}} & & \\ 0 & \longrightarrow & U_2 & \longrightarrow & M_2 & \longrightarrow & M_2/U_2 & \longrightarrow & 0 \end{array}$$

the dual is

$$\begin{array}{ccccccccc} 0 & \longleftarrow & U_1^* & \longleftarrow & M_1^* & \longleftarrow & (M_1/U_1)^* & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle (f_\alpha|_{U_1})^*} & & \uparrow{\scriptstyle f_\alpha^*} & & \uparrow{\scriptstyle (\overline{f_\alpha})^*} & & \\ 0 & \longleftarrow & U_2^* & \longleftarrow & M_2^* & \longleftarrow & (M_2/U_2)^* & \longleftarrow & 0 \end{array}$$

and has again exact rows. Thus the morphism $(-)^* \colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathrm{Gr}_{\mathbf{dim}\, M-\mathbf{d}}(M^*)$ is a well-defined morphism of algebraic varieties. Using the duality $M^{**} \cong M$ the morphism $\mathrm{Gr}_{\mathbf{dim}\, M-\mathbf{d}}(M^*) \to \mathrm{Gr}_{\mathbf{d}}(M^{**})$ is an inverse map of $(-)^*$. □

2.3.3 Grassmannians as schemes

We define also a k-scheme, i.e. a representable functor from the category of commutative k-algebras to sets, called *quiver Grassmannian* $\mathit{Gr}_{\mathbf{d}}(M)$, which is the schematic version of the variety $\mathrm{Gr}_{\mathbf{d}}(M)$. For this we use some observations of Caldero and Reineke [12, Section 4].

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $\mathbf{d} = (d_i)_{i\in Q_0}$ and $\mathbf{n} = (n_i)_{i\in Q_0}$ dimension vectors of Q. First we define the schematic version of the *module variety* as in [8, Section 3.1]. Let $\mathit{rep}_{\mathbf{n}}(Q)$ be the affine k-scheme defined by $\mathit{rep}_{\mathbf{n}}(Q) = \mathrm{Spec}\,(R(Q,\mathbf{n}))$ with $R(Q,\mathbf{n})$ the polynomial algebra over the field k with coefficients in $X_{ij}^{(\alpha)}$ for $\alpha \in Q_1$, $1 \le i \le n_{t(\alpha)}$ and $1 \le j \le n_{s(\alpha)}$. Let A be a commutative k-algebra. Then the A-valued points of $\mathit{rep}_{\mathbf{n}}(Q)$ are

$$\mathit{rep}_{\mathbf{n}}(Q)(A) = \mathrm{Hom}_{k\text{-Alg}}(R(Q,\mathbf{n}), A) \cong \prod\nolimits_{\alpha\in Q_1} \mathrm{Mat}(n_{t(\alpha)} \times n_{s(\alpha)}, A).$$

Analogously we define an affine k-scheme $\mathit{Hom}(\mathbf{d},\mathbf{n})$ by $\mathit{Hom}(\mathbf{d},\mathbf{n}) = \mathrm{Spec}\,(H(\mathbf{d},\mathbf{n}))$ with the polynomial ring $H(\mathbf{d},\mathbf{n})$ over k with coefficients in $Y_{ij}^{(l)}$ for $l \in Q_0$, $1 \le i \le n_l$ and $1 \le j \le d_l$. Thus $\mathit{Hom}(\mathbf{d},\mathbf{n})(A) \cong \mathrm{Mat}(\mathbf{n} \times \mathbf{d}, A)$. Moreover, we define the open subscheme $\mathit{Hom}^0(\mathbf{d},\mathbf{n})$ of $\mathit{Hom}(\mathbf{d},\mathbf{n})$ by

$$\mathit{Hom}^0(\mathbf{d},\mathbf{n}) = \bigcup\nolimits_{\mathbf{B},\ \mathbf{d}\text{-minor}} \mathrm{Spec}\left(H(\mathbf{d},\mathbf{n})_{(\mathbf{B})}\right).$$

A monomorphism $f \colon V \to W$ of free modules is called a *split monomorphism* if there is a homomorphism $g \colon W \to V$ of free modules such that $gf = \mathrm{id}_V$. Using this notation the scheme $\mathit{Hom}^0(\mathbf{d},\mathbf{n})$ parametrizes the split monomorphisms from a tuple of free modules with dimension vector $\mathbf{d}$ to a tuple of free modules with dimension vector $\mathbf{n}$.

These schemes are used to define the quiver Grassmannian. The most important case of the scheme $\mathit{Hom}^0(\mathbf{d},\mathbf{n})$ is the scheme $\mathit{GL}_{\mathbf{d}} = \mathit{Hom}^0(\mathbf{d},\mathbf{d})$ of tuples of invertible matrices. Then for the k-valued points holds $\mathit{GL}_{\mathbf{d}}(k) \cong \mathrm{GL}_{\mathbf{d}}(k)$. Let $M = (M_i, M_\alpha)_{i\in Q_0, \alpha\in Q_1}$

be a Q-representation with $M_i = k^{n_i}$ for all $i \in Q_0$ and $M_\alpha \in \mathrm{Mat}(n_{t(\alpha)} \times n_{s(\alpha)}, k)$ for all $\alpha \in Q_1$. Then we define the closed subscheme $\mathit{Hom}^0_Q(\mathbf{d}, M)$ of the scheme $\mathit{rep}_{\mathbf{d}}(Q) \times \mathit{Hom}^0(\mathbf{d}, \mathbf{n})$ by

$$\mathit{Hom}^0_Q(\mathbf{d}, M) = \bigcup\nolimits_{\mathbf{B},\ \mathbf{d}\text{-minor}} \mathrm{Spec}\left(H^0(Q, \mathbf{d}, M, \mathbf{B})\right)$$

where $H^0(Q, \mathbf{d}, M, \mathbf{B})$ is the quotient of the algebra $R(Q, \mathbf{d}) \otimes_k H(\mathbf{d}, \mathbf{n})_{(\mathbf{B})}$ by the ideal generated by the relations induced by the matrix multiplications

$$\left(Y^{(t(\alpha))}_{ij}\right)_{1\le i\le n_{t(\alpha)}, 1\le j\le d_{t(\alpha)}} \left(X^{(\alpha)}_{ij}\right)_{1\le i\le d_{t(\alpha)}, 1\le j\le d_{s(\alpha)}} - M_\alpha \left(Y^{(s(\alpha))}_{ij}\right)_{1\le i\le n_{s(\alpha)}, 1\le j\le d_{s(\alpha)}}$$

for all $\alpha \in Q_1$. Thus the A-valued points $\mathit{Hom}^0_Q(\mathbf{d}, M)(A)$ of the scheme $\mathit{Hom}^0_Q(\mathbf{d}, M)$ are

$$\Big\{\big((U_\alpha)_{\alpha\in Q_1}, (f_l)_{l\in Q_0}\big) \in \prod\nolimits_{\alpha\in Q_1} \mathrm{Mat}\, d_{t(\alpha)} \times d_{s(\alpha)}, A) \times \mathrm{Mat}(\mathbf{n} \times \mathbf{d}, A)\Big| \\ \exists \mathbf{B} = (B_l)_{l\in Q_0} : B_l(f_l) \in A^* \ \forall l \in Q_0,\ f_{t(\alpha)} U_\alpha = M_\alpha f_{s(\alpha)} \ \forall \alpha \in Q_1\Big\}$$

for each commutative k-algebra A. Thus this scheme $\mathit{Hom}^0_Q(\mathbf{d}, M)$ parametrizes the Q-representations of dimension vector $\mathbf{d}$ together with a homomorphism of Q-representations to M, which is a split homomorphism of free modules.

The A-linear morphisms

$$GL_{\mathbf{d}}(A) \times \mathit{rep}_{\mathbf{d}}(Q)(A) \to \mathit{rep}_{\mathbf{d}}(Q)(A), \big((g_l)_{l\in Q_0}, (M_\alpha)_{\alpha\in Q_1}\big) \mapsto \left(g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)}\right)_{\alpha\in Q_1}$$

define a natural transformation of the functors $A \mapsto GL_{\mathbf{d}}(A) \times \mathit{rep}_{\mathbf{d}}(Q)(A)$ and $A \mapsto \mathit{rep}_{\mathbf{d}}(Q)(A)$ from the category of commutative k-algebras to the category of sets. Thus this induces a morphism of k-schemes and an action of the scheme $GL_{\mathbf{d}}$ on the scheme $\mathit{rep}_{\mathbf{d}}(Q)$. Using

$$GL_{\mathbf{d}}(A) \times \mathit{Hom}^0(\mathbf{d}, \mathbf{n})(A) \to \mathit{Hom}^0(\mathbf{d}, \mathbf{n})(A), \big((g_l)_{l\in Q_0}, (f_l)_{l\in Q_0}\big) \mapsto \left(f_l g_l^{-1}\right)_{l\in Q_0}$$

for a commutative k-algebra A we define an algebraic action of $GL_{\mathbf{d}}$ on the scheme $\mathit{Hom}^0(\mathbf{d}, \mathbf{n})$. This action is free. These both actions induce a free action of $GL_{\mathbf{d}}$ on the scheme $\mathit{Hom}^0_Q(\mathbf{d}, M)$.

Now we review the well-known definition of the classical Grassmannian as a scheme. For the quiver $Q = (\{\circ\}, \varnothing)$ with one point and no arrow and $d, n \in \mathbb{N}$ the quotient of the scheme $\mathit{Hom}^0((d), (n))$ by $GL_{(d)}$ is again a scheme since these action is free. This scheme is called *(classical) Grassmannian* $Gr\binom{n}{d}$. For each d-minor B of $\mathrm{Mat}(n \times d, k)$ we get an open affine subscheme $\mathcal{U}_B$ of $Gr\binom{n}{d}$ defined as in Section 2.3.2. Thus the scheme $\mathcal{U}_B$ is isomorphic to the affine k-scheme $\mathit{Hom}((d), (n-d))$.

Let Q be an arbitrary quiver. For a commutative k-algebra the morphism

$$\mathit{Hom}^0_Q(\mathbf{d}, M)(A) \to \mathit{Hom}^0(\mathbf{d}, \mathbf{n})(A), \big((U_\alpha)_{\alpha\in Q_1}, (f_l)_{l\in Q_0}\big) \mapsto (f_l)_{l\in Q_0}$$

is injective and thus the induced morphism $\mathit{Hom}^0_Q(\mathbf{d}, M) \to \mathit{Hom}^0(\mathbf{d}, \mathbf{n})$ of schemes is an embedding. Generalizing the proof of Lemma 2.3.7, the image of this morphism turns out to be closed.

Using the free action of $GL_{\mathbf{d}}$ on the schemes $\mathit{Hom}^0(\mathbf{d}, \mathbf{n})$ and $\mathit{Hom}^0_Q(\mathbf{d}, M)$ we can define the quiver Grassmannian in the following way (see also [12, Lemma 2]). The quotient of the scheme $\mathit{Hom}^0(\mathbf{d}, \mathbf{n})$ by $GL_{\mathbf{d}}$ is again a k-scheme since these action is free. This is the product of Grassmannians $\mathit{Gr}\binom{\mathbf{n}}{\mathbf{d}} = \prod_{i \in Q_0} \mathit{Gr}\binom{n_i}{d_i}$.

Definition 2.3.11. Let Q be a quiver, M a Q-representation with dimension vector $\mathbf{n}$ and $\mathbf{d}$ another dimension vector. Then the quotient of the scheme $\mathit{Hom}^0_Q(\mathbf{d}, M)$ by $GL_{\mathbf{d}}$ is a closed subscheme of the product of Grassmannians $\mathit{Gr}\binom{\mathbf{n}}{\mathbf{d}}$. This subscheme is called *quiver Grassmannian* $\mathit{Gr}_{\mathbf{d}}(M)$.

The induced projection is denoted by

$$\pi\colon\ \mathit{Hom}^0_Q(\mathbf{d}, M) \twoheadrightarrow \mathit{Gr}_{\mathbf{d}}(M). \tag{2.3.5}$$

Since $\mathit{Gr}_{\mathbf{d}}(M)(k) \cong \mathrm{Gr}_{\mathbf{d}}(M)$ the scheme $\mathit{Gr}_{\mathbf{d}}(M)$ is the schematic version of the variety $\mathrm{Gr}_{\mathbf{d}}(M)$. In general $\mathrm{Gr}_{\mathbf{d}}(M)$ is not reduced (see Example 3.1.10) - in particular not smooth. The scheme $\mathit{Gr}_{\mathbf{d}}(M)$ carries more information and is more natural in some sense than the variety $\mathrm{Gr}_{\mathbf{d}}(M)$. Thus we study the geometry of the quiver Grassmannian $\mathit{Gr}_{\mathbf{d}}(M)$ instead of the geometry of the corresponding variety $\mathrm{Gr}_{\mathbf{d}}(M)$.

For example Schofield [48, Lemma 3.2] and Caldero and Reineke [12, Proposition 6 and Corollary 4] computed the tangent space of the scheme $\mathit{Gr}_{\mathbf{d}}(M)$. This result was generalized by [21, Proposition 3.5] to a so called *general representation* in a module variety.

Proposition 2.3.12. *Let Q be a quiver, M a Q-representation with dimension vector $\mathbf{n}$, $\mathbf{d}$ another dimension vector and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$. Then $T_U(\mathit{Gr}_{\mathbf{d}}(M)) \cong \mathrm{Hom}_Q(U, M/U)$ and $\langle \mathbf{d}, \mathbf{n} - \mathbf{d} \rangle \le \dim \mathrm{Gr}_{\mathbf{d}}(M) \le \langle \mathbf{d}, \mathbf{n} - \mathbf{d} \rangle + \dim_k \mathrm{Ext}^1_Q(M, M)$.*

If M is exceptional, then $\mathit{Gr}_{\mathbf{d}}(M)$ is empty or smooth with dimension $\langle \mathbf{d}, \mathbf{n} - \mathbf{d} \rangle$.

In general we get again an open affine covering $\mathcal{U}_{\mathbf{B}}$ of the product of Grassmannians $\mathit{Gr}\binom{\mathbf{n}}{\mathbf{d}}$ indexed by the $\mathbf{d}$-minors $\mathbf{B}$ such that there is an isomorphism

$$\pi_{\mathbf{B}}\colon\ \mathit{Hom}(\mathbf{d}, \mathbf{n} - \mathbf{d}) \to \mathcal{U}_{\mathbf{B}} \tag{2.3.6}$$

of k-schemes and $\mathcal{U}_{\mathbf{B}}(k) \cong U_{\mathbf{B}}$ for each $\mathbf{d}$-minor $\mathbf{B}$. Moreover, for each $\mathbf{d}$-minor $\mathbf{B} = (B_l)_{l \in Q_0}$ the A-valued points of $\mathcal{U}_{\mathbf{B}} \cap \mathit{Gr}_{\mathbf{d}}(M)$ are

$$\pi\Big(\Big\{ \big((U_\alpha)_{\alpha \in Q_1}, (f_l)_{l \in Q_0}\big) \in \prod\nolimits_{\alpha \in Q_1} \mathrm{Mat}(d_{t(\alpha)} \times d_{s(\alpha)}, A) \times \mathrm{Mat}(\mathbf{n} \times \mathbf{d}, A) \Big| \\ B_l(f_l) \in A^* \ \forall l \in Q_0,\ f_{t(\alpha)} U_\alpha = M_\alpha f_{s(\alpha)} \ \forall \alpha \in Q_1 \Big\}\Big).$$

2.4 Ringel-Hall algebras

The Ringel-Hall algebras of finite-dimensional hereditary algebras over finite fields are well-known objects (see [47] for an introduction). In this section let k be the field of complex numbers $\mathbb{C}$. We consider the *Ringel-Hall algebra* $\mathcal{H}(A)$ of constructible functions over the module varieties $\mathrm{rep}_{\mathbf{d}}(Q, I)$ of a $\mathbb{C}$-algebra $A = \mathbb{C}Q/I$ with a locally finite quiver Q and an admissible ideal I. This is an idea due to Schofield [49], which also appears in works of Lusztig [38] and Riedtmann [44]. An introduction to the construction of Kapranov and Vasserot [34] and Joyce [33], which we are using here, can be found in [7, Section 4]. For completeness we review the definition.

Let $A = \mathbb{C}Q/I$ be a path algebra of a locally finite quiver Q and an admissible ideal I. A function $f\colon X \to \mathbb{C}$ on a variety X is called *constructible* if the image is finite and every fibre is constructible. A constructible function $f\colon \mathrm{rep}_{\mathbf{d}}(Q, I) \to \mathbb{C}$ on the module variety $\mathrm{rep}_{\mathbf{d}}(Q, I)$ is called $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-*stable* (or $\mathrm{GL}(\mathbb{C})$-*stable* for short) if the fibres are $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-stable sets.

Let $\mathcal{H}_{\mathbf{d}}(A)$ be the vector space of constructible and $\mathrm{GL}_{\mathbf{d}}(\mathbb{C})$-stable functions on the variety $\mathrm{rep}_{\mathbf{d}}(Q, I)$. For a constructible and $\mathrm{GL}(\mathbb{C})$-stable subset $X \subseteq \mathrm{rep}_{\mathbf{d}}(Q, I)$ let $\mathbb{1}_X$ be the characteristic function of X in $\mathcal{H}_{\mathbf{d}}(A)$. Let $\mathcal{H}(A) = \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathcal{H}_{\mathbf{d}}(A)$ and $*\colon \mathcal{H}(A) \otimes \mathcal{H}(A) \to \mathcal{H}(A)$ with

$$(\mathbb{1}_X * \mathbb{1}_Y)(M) = \chi\Big(\Big\{N \in \mathrm{Gr}_{\mathbf{d}}(M) \Big| N \in X, M/N \in Y\Big\}\Big)$$

for all $M \in \mathrm{rep}_{\mathbf{c}+\mathbf{d}}(Q, I)$ and all constructible and $\mathrm{GL}(\mathbb{C})$-stable subsets $X \subseteq \mathrm{rep}_{\mathbf{d}}(Q, I)$ and $Y \subseteq \mathrm{rep}_{\mathbf{c}}(Q, I)$. For a dimension vector $\mathbf{d}$ let $\mathbb{1}_{\mathbf{d}}$ be the characteristic function of all representations with dimension vector $\mathbf{d}$ and $\mathbb{1}_{S(\mathbf{d})}$ the characteristic function of the semisimple representations with dimension vector $\mathbf{d}$ in $\mathcal{H}_{\mathbf{d}}(A)$. For an A-module M let $\mathbb{1}_M$ be the characteristic function of the orbit of the module M in $\mathcal{H}_{\mathbf{dim}\, M}(A)$.

Proposition 2.4.1. *Let $A = \mathbb{C}Q/I$ be a path algebra of a locally finite quiver Q and an admissible ideal I. The vector space $\mathcal{H}(A)$ with the product $*$ is a $\mathbb{N}^{Q_0}$-graded algebra with unit $\mathbb{1}_0$.*

This algebra $\mathcal{H}(A)$ is called *Ringel-Hall algebra*. Let $\mathcal{C}(A)$ be the subalgebra of $\mathcal{H}(A)$ generated by the set $\{\mathbb{1}_{\mathbf{d}} | \mathbf{d} \in \mathbb{N}^{Q_0}\}$. This algebra turns out to be a Hopf algebra although $\mathcal{H}(A)$ is not a Hopf algebra in general. Let $\hat{\mathcal{H}}(A) = \prod_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathcal{H}_{\mathbf{d}}(A)$ be the completion of the Ringel-Hall algebra $\mathcal{H}(A)$ and $\hat{\mathcal{C}}(A)$ the one of $\mathcal{C}(A)$.

Lemma 2.4.2. *Let $A = \mathbb{C}Q/I$ be a path algebra of a locally finite quiver Q and an admissible ideal I. The algebra $\mathcal{C}(A)$ is a cocommutative Hopf algebra with the coproduct $\Delta\colon \mathcal{C}(A) \to \mathcal{C}(A)\otimes\mathcal{C}(A)$ defined by $\Delta(f)(M, N) = f(M\oplus N)$ and the counit $\eta\colon \mathcal{C}(A) \to \mathbb{C}$ defined by $\eta(f) = f(0)$ for all $f \in \mathcal{C}(A)$ and all Q-representations M and N. For each dimension vector $\mathbf{d}$ and the antipode S of $\mathcal{C}(A)$, $S(\mathbb{1}_{\mathbf{d}}) = (-1)^{|\mathbf{d}|}\mathbb{1}_{S(\mathbf{d})}$ and $S(\mathbb{1}_{S(\mathbf{d})}) = (-1)^{|\mathbf{d}|}\mathbb{1}_{\mathbf{d}}$.*

Moreover, the algebra $\mathcal{C}(A)$ is also generated by $\{\mathbb{1}_{S(\mathbf{d})} | \mathbf{d} \in \mathbb{N}^{Q_0}\}$ since the subalgebra of $\mathcal{C}(A)$ generated by this set is a Hopf algebra. The first part of this lemma is known by Joyce [33] and also stated in [7, Section 4.2].

Proof of the second part. Since $\mathcal{C}(A)$ is a Hopf algebra, $*(S \otimes 1)\Delta = \eta \mathbb{1}_0$ holds. Using $\Delta(\mathbb{1}_{\mathbf{d}}) = \sum_{\mathbf{c} \in \mathbb{N}^{Q_0},\ \mathbf{c} \leq \mathbf{d}} \mathbb{1}_{\mathbf{c}} \otimes \mathbb{1}_{\mathbf{d}-\mathbf{c}}$, $S(\mathbb{1}_0) = \mathbb{1}_0$ and $S(\mathbb{1}_{\mathbf{d}}) = -\sum_{\mathbf{c} \in \mathbb{N}^{Q_0},\ \mathbf{c} < \mathbf{d}} S(\mathbb{1}_{\mathbf{c}}) * \mathbb{1}_{\mathbf{d}-\mathbf{c}}$ for $\mathbf{d} \in \mathbb{N}^{Q_0}$ with $\mathbf{d} \neq 0$. By induction we get the following result for each $M \in \operatorname{rep}_{\mathbf{d}}(Q, I)$ with $\mathbf{d} \neq 0$. Let $\mathbf{d}' = (d'_i)_{i \in Q_0}$ be the dimension vector of $\operatorname{soc} M$. Thus $0 < \mathbf{d}' \leq \mathbf{d}$ and

$$\begin{aligned}
\left((-1)^{|\mathbf{d}|}\mathbb{1}_{S(\mathbf{d})} - S(\mathbb{1}_{\mathbf{d}})\right)(M) &= \sum\nolimits_{\mathbf{c} \in \mathbb{N}^{Q_0}} \left((-1)^{|\mathbf{c}|}\mathbb{1}_{S(\mathbf{c})} * \mathbb{1}_{\mathbf{d}-\mathbf{c}}\right)(M) \\
&= \sum\nolimits_{\mathbf{c} \in \mathbb{N}^{Q_0}} \left(\prod\nolimits_{i \in Q_0} (-1)^{c_i} \binom{d'_i}{c_i}\right) \\
&= \prod\nolimits_{i \in Q_0} \left(\sum\nolimits_{j=0}^{d'_i} (-1)^j \binom{d'_i}{j}\right) = \prod\nolimits_{i \in Q_0} (1-1)^{d'_i} = 0.
\end{aligned}$$

The other equation follows analogously. □

3 Geometric properties of quiver Grassmannians

In this chapter we can assume for simplicity that all occurring quivers are finite. We study geometric properties of the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$. After defining the subschemes $\mathcal{C}_U$ of subrepresentations of M isomorphic to U, we define another class of closed subschemes of $\mathrm{Gr}_{\mathbf{d}}(M)$ and construct morphisms between them. In the last section of this chapter we decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components in some examples.

3.1 Isomorphism classes

The $\mathrm{GL}_{\mathbf{d}}(k)$-orbits in the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ are in bijection to the isomorphism classes of Q-representations with dimension vector $\mathbf{d}$. For these orbits Proposition 2.3.3 is well-known, i.e. they are irreducible, locally closed and smooth. The isomorphism classes of subrepresentations of a Q-representation are in general not orbits of some algebraic action (see Remark 3.3.6). Nevertheless, these subschemes of the scheme $Gr_{\mathbf{d}}(M)$ are by the following theorem locally closed, irreducible and smooth. The intention of this section is to prove this theorem and some corollaries.

Let Q be a quiver, M a Q-representation with dimension vector $\mathbf{n} = (n_i)_{i\in Q_0}$, $\mathbf{d} = (d_i)_{i\in Q_0}$ another dimension vector and $N = (N_i, N_\alpha)_{i\in Q_0,\alpha\in Q_1}$ a Q-representation with $N_i = k^{d_i}$ for all $i \in Q_0$ and $N_\alpha \in \mathrm{Mat}(d_{t(\alpha)} \times d_{s(\alpha)}, k)$ for all $\alpha \in Q_1$. Then the A-linear maps

$$GL_{\mathbf{d}}(A) \to rep_{\mathbf{d}}(Q)(A), (g_i)_{i\in Q_0} \mapsto \left(g_{t(\alpha)} N_\alpha g_{s(\alpha)}^{-1}\right)_{\alpha\in Q_1}$$

with a commutative k-algebra A induce a morphism $GL_{\mathbf{d}} \to rep_{\mathbf{d}}(Q)$ of schemes. The image is denoted by $\mathcal{O}(N)$ and

$$\mathcal{O}^0(N, M) = (\mathcal{O}(N) \times Hom^0(\mathbf{d}, \mathbf{n})) \cap Hom^0_Q(\mathbf{d}, M). \tag{3.1.1}$$

By Proposition 2.3.3 the subscheme $\mathcal{O}(N)$ of the scheme $rep_{\mathbf{d}}(Q)$ is locally closed. By definition $Hom^0_Q(\mathbf{d}, M)$ is closed in $rep_{\mathbf{d}}(Q) \times Hom^0(\mathbf{d}, \mathbf{n})$. Thus the k-scheme $\mathcal{O}^0(N, M)$ is a locally closed subscheme of $Hom^0_Q(\mathbf{d}, M)$. This parametrizes the Q-representations isomorphic to N with an injective homomorphism of Q-representations to M.

This subscheme $\mathcal{O}^0(N, M)$ is also $GL_{\mathbf{d}}$-stable and thus the quotient

$$\mathcal{C}_N = \mathcal{O}^0(N, M)/\, GL_{\mathbf{d}} \tag{3.1.2}$$

is a well-defined locally closed subscheme of $Gr_{\mathbf{d}}(M) = Hom^0_Q(\mathbf{d}, M)/\, GL_{\mathbf{d}}$. Thus $\mathcal{C}_N$ is the subscheme of $Gr_{\mathbf{d}}(M)$ of subrepresentations of the Q-representation M isomorphic to N. Dually we define the subscheme $\mathcal{C}'_N$ of $Gr_{\mathbf{d}}(M)$ of subrepresentations of M such that M/N is isomorphic to N.

Theorem 3.1.1. *Let Q be a quiver, M, N Q-representations and $\mathbf{d}$ a dimension vector. If $\mathcal{C}_N(k)$ is non-empty in $\mathrm{Gr}_{\mathbf{d}}(M)$, then it is locally closed, irreducible, has dimension*

$$\dim_k \mathrm{Hom}_Q(N, M) - \dim_k \mathrm{End}_Q(N)$$

and the scheme $\mathcal{C}_N$ is smooth. Moreover,

$$T_U(\mathcal{C}_N) \cong \mathrm{Hom}_Q(N, M)/\, \mathrm{End}_Q(N)$$

for $U \in \mathcal{C}_N(k)$. If $\mathcal{C}'_N(k)$ is non-empty, then it is also locally closed, irreducible, smooth and has dimension $\dim_k \mathrm{Hom}_Q(M, N) - \dim_k \mathrm{End}_Q(N)$.

In general we do not get that the locally closed subscheme $\mathcal{C}_{N,N'} = \mathcal{C}_N \cap \mathcal{C}'_{N'}$ is irreducible. The subvarieties $\mathcal{C}_N(k)$ of $\mathrm{Gr}_{\mathbf{d}}(M)$ are defined similarly to the orbits $\mathcal{O}(N)$ in the variety $\mathrm{rep}_{\mathbf{d}}(Q)$. Nevertheless, $U \in \overline{\mathcal{C}_V(k)}$ does not imply $\mathcal{C}_U(k) \subseteq \overline{\mathcal{C}_V(k)}$, since the set $\mathcal{C}_V(k)$ is in general not an orbit of some action and thus $\overline{\mathcal{C}_V(k)}$ is in general not a union of $C_U(k)$'s (see Example 3.1.3). However, in general for each quiver Q, dimension vector $\mathbf{d}$ and Q-representation M,

$$\begin{aligned} \mathrm{Gr}_{\mathbf{d}}(M) &= \bigcup\nolimits_{N \in \mathrm{rep}_{\mathbf{d}}(Q)} \mathcal{C}_N(k) = \bigcup\nolimits_{N' \in \mathrm{rep}_{\mathbf{dim}\, M - \mathbf{d}}(Q)} \mathcal{C}'_{N'}(k) \\ &= \bigcup\nolimits_{(N,N') \in \mathrm{rep}_{\mathbf{d}}(Q) \times \mathrm{rep}_{\mathbf{dim}\, M - \mathbf{d}}(Q)} \mathcal{C}_{N,N'}(k). \end{aligned} \tag{3.1.3}$$

We remark that these unions are not necessarily finite and they do not hold for the corresponding schemes.

Corollary 3.1.2. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$. Then the normal space $N_U(\mathcal{C}_U/\, Gr_{\mathbf{d}}(M))$ is isomorphic to the image of the first connecting morphism $\mathrm{Hom}_Q(U, \delta)$ of the long exact sequence of the short exact sequence $0 \to U \to M \to M/U \to 0$ and the functor $\mathrm{Hom}_Q(U, -)$, i.e.*

$$N_U(\mathcal{C}_U/\, Gr_{\mathbf{d}}(M)) \cong \mathrm{Im}\, \mathrm{Hom}_Q(U, \delta).$$

Moreover, this normal space is a subspace of $\mathrm{Ext}^1_{(Q,I)}(U, U)$ for each semiadmissible ideal I such that M is a (Q, I)-representation. Dually $N_U(\mathcal{C}'_U/\, Gr_{\mathbf{d}}(M)) \cong \mathrm{Im}\, \mathrm{Hom}_Q(\delta, U)$.

Before proving the theorem and this corollary we consider some useful lemmas. By the corollary or by Proposition 2.3.12 we get $\dim \mathcal{C}_N \leq \dim_k \mathrm{Hom}_Q(N, M/N)$ and $\dim \mathcal{C}'_{M/N} \leq \dim_k \mathrm{Hom}_Q(N, M/N)$. Using Lemma 2.3.10 both statements of the theorem and of the corollary are dual. Thus it is enough to consider the scheme $\mathcal{C}_N$ in the following lemmas. If $\mathcal{C}_N(k)$ is empty, there is nothing to prove.

Example 3.1.3. Let $Q = 1 \xrightarrow{\alpha} 2$ and $M = (M_1, M_2, M_\alpha)$ the Q-representation such that $M_1 = k^2$ with basis $\{e_1, e_2\}$, $M_2 = k^2$ with basis $\{f_1, f_2\}$ and $M_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $U = \langle e_1 \rangle_Q, V = \langle e_2, f_1 \rangle_Q, V' = \langle e_2, f_2 \rangle_Q \in \mathrm{Gr}_{(1,1)}(M)$. Then $V \cong V'$,

$$\mathcal{C}_U(k) = \{\langle e_1 + \lambda e_2 \rangle_Q | \lambda \in k\}, \quad \mathcal{C}_V(k) = \{\langle e_2, f \rangle_Q | f \in M_2, f \neq 0\} \subseteq \mathrm{Gr}_{(1,1)}(M),$$

$V \in \overline{\mathcal{C}_U(k)}$, $V' \notin \overline{\mathcal{C}_U(k)}$ and thus $\mathcal{C}_V(k) \cap \overline{\mathcal{C}_U(k)} \neq \varnothing$ and $\mathcal{C}_V(k) \nsubseteq \overline{\mathcal{C}_U(k)}$.

Lemma 3.1.4. *Let Q be a quiver, M, N Q-representations and $\mathbf{d}$ a dimension vector. Then $\mathcal{C}_N(k)$ is irreducible in $\mathrm{Gr}_{\mathbf{d}}(M)$.*

Proof. Let $U, V \in \mathcal{C}_N(k)$ and $\varphi \colon U \to V$ an isomorphism of Q-representations. We consider the homomorphisms

$$\varphi(t) = t\iota_U + (1-t)\iota_V \varphi \colon U \to M$$

of Q-representations with $\iota_U \colon U \to M, \iota_V \colon V \to M$ are the canonical embeddings and $t \in k$. Let $W(t)$ be the image of $\varphi(t)$. Thus $W(t)$ is a subrepresentation of M, $W(0) = V$, $W(1) = U$ and if $\mathbf{dim}\, W(t) = \mathbf{d}$, then $W(t) \in \mathcal{C}_N(k)$ for all $t \in k$.

Let $\{u_1, \ldots, u_d\}$ be a basis of the vector space U. Then $\{\varphi(t)(u_1), \ldots, \varphi(t)(u_d)\}$ generates the vector space $W(t)$. Let

$$M(t) = (\varphi(t)(u_1) \ldots \varphi(t)(u_d)) \in \mathrm{Mat}(d \times n, k)$$

be the matrix with columns $\varphi(t)(u_1), \ldots, \varphi(t)(u_d)$ for all $t \in k$. Now we consider the following maps $M \colon k \to \mathrm{Mat}(n \times d, k), t \mapsto M(t)$ and $\mathrm{rk} \colon \mathrm{Mat}(n \times d, k) \to \mathbb{Z}, M \mapsto \mathrm{rk}(M)$. Since rk is lower semicontinuous by Example 2.3.1 and M is continuous, also $\mathrm{rk} \circ M$ is lower semicontinuous. Thus $X = \{t \in k | \, \mathrm{rk}\, M(t) = d\} = \{t \in k | W(t) \in \mathcal{C}_N(k)\}$ is open in k. Since $0 \in X$ this set X is dense in k. Moreover, $W \colon X \to \mathcal{C}_N(k), t \mapsto W(t)$ is a well-defined morphism of algebraic varieties. Since X is irreducible, $\mathcal{C}_N(k)$ is also irreducible. □

In the dual case for $\mathcal{C}'_N$ let $\psi \colon M/V \to M/U$ be an isomorphism. Then the kernel of $\psi(t) = t\pi_U + (1-t)\psi\pi_V \colon M \to M/U$ gives a family of points in $\mathcal{C}'_N(k)$.

Lemma 3.1.5. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector and $N \in \mathrm{rep}_{\mathbf{d}}(Q)$ with $\mathcal{C}_N(k)$ is non-empty in $\mathrm{Gr}_{\mathbf{d}}(M)$. Then*

$$\dim \mathcal{C}_N = \dim_k \mathrm{Hom}_Q(N, M) - \dim_k \mathrm{End}_Q(N).$$

Proof. By Proposition 2.3.3 $\dim \mathcal{O}(N) = \dim \mathrm{GL}_{\mathbf{d}}(k) - \dim_k \mathrm{End}_Q(N)$. The projection on the first component

$$\pi_1 \colon \mathcal{O}^0(N, M)(k) \to \mathcal{O}(N)(k), \big((U_\alpha)_{\alpha \in Q_1}, (f_i)_{i \in Q_0}\big) \mapsto (U_\alpha)_{\alpha \in Q_1}$$

has fibres isomorphic to the subset of injective homomorphisms in $\mathrm{Hom}_Q(N, M)$. This set is open and non-empty in the affine space $\mathrm{Hom}_Q(N, M)$. Thus

$$\dim \mathcal{C}_N = \dim \mathcal{O}^0(N, M) - \dim \mathrm{GL}_{\mathbf{d}}(k)$$

$$= \dim \mathcal{O}(N) + \dim_k \operatorname{Hom}_Q(N, M) - \dim \operatorname{GL}_{\mathbf{d}}(k)$$
$$= \dim_k \operatorname{Hom}_Q(N, M) - \dim_k \operatorname{End}_Q(N).$$

This proves Lemma 3.1.5. □

Lemma 3.1.6. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector, $N \in \operatorname{Gr}_{\mathbf{d}}(M)$ and $U \in \mathcal{C}_N(k)$. Then $T_U(\mathcal{C}_N) \cong \operatorname{Hom}_Q(N, M)/\operatorname{End}_Q(N)$ and moreover $\mathcal{C}_N$ is smooth.*

Proof. As in the proof of Proposition 2.3.12 of this thesis in [12, Proposition 6] we compute the tangent space $T_U(\mathcal{C}_N)$ by fixing a point $(U, f) = ((U_\alpha)_{\alpha\in Q_1}, (f_i)_{i\in Q_0}) \in Hom^0_Q(\mathbf{d}, M)(k)$ in the fibre $\pi^{-1}(U)$ with π defined in Equation (2.3.5), computing the tangent space T of $\mathcal{O}^0(N, M)$ at this point and factoring it by the image of the differential of the action of $GL_{\mathbf{d}}$.

Let $\mathbf{n} = (n_i)_{i\in Q_0}$ be the dimension vector of M. Without loss of generality we assume $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ with $M_i = k^{n_i}$ for all $i \in Q_0$ and $M_\alpha \in \operatorname{Mat}(n_{t(\alpha)} \times n_{s(\alpha)}, k)$ for all $\alpha \in Q_1$. To compute T, we perform a calculation with dual numbers. Since $rep_{\mathbf{d}}(Q)\times Hom(\mathbf{d}, \mathbf{n})$ is just an affine scheme, an element of the tangent space at the point (U, f) looks like $((U_\alpha + \varepsilon V_\alpha)_{\alpha\in Q_1}, (f_i + \varepsilon g_i)_{i\in Q_0})$, with $((V_\alpha)_{\alpha\in Q_1}, (g_i)_{i\in Q_0}) \in \operatorname{rep}_{\mathbf{d}}(Q) \times \operatorname{Mat}(\mathbf{n} \times \mathbf{d}, k)$. The conditions for this to belong to the tangent space T are:

$$M_\alpha(f_{s(\alpha)} + \varepsilon g_{s(\alpha)}) = (f_{t(\alpha)} + \varepsilon g_{t(\alpha)})(U_\alpha + \varepsilon V_\alpha),$$
$$\exists (h_i)_{i\in Q_0} \in \operatorname{Mat}(\mathbf{d} \times \mathbf{d}, k) : (U_\alpha + \varepsilon V_\alpha)(1 + \varepsilon h_{s(\alpha)}) = (1 + \varepsilon h_{t(\alpha)})U_\alpha$$

for all $\alpha \in Q_1$, which yields the conditions

$$M_\alpha g_{s(\alpha)} = f_{t(\alpha)} V_\alpha + g_{t(\alpha)} U_\alpha, \tag{3.1.4}$$
$$\exists (h_i)_{i\in Q_0} \in \operatorname{Mat}(\mathbf{d} \times \mathbf{d}, k) : U_\alpha h_{s(\alpha)} + V_\alpha = h_{t(\alpha)} U_\alpha$$

for all $\alpha \in Q_1$. The differential of the action of $GL_{\mathbf{d}}$ is computed by applying the definition of the action to a point $(1 + \varepsilon x_i)_{i\in Q_0}$ of $T_1\, GL_{\mathbf{d}}$:

$$(1 + \varepsilon x_i)_{i\in Q_0}((U_\alpha + \varepsilon V_\alpha)_{\alpha\in Q_1}, (f_i + \varepsilon g_i)_{i\in Q_0})$$
$$=((U_\alpha + \varepsilon(V_\alpha + x_{t(\alpha)}U_\alpha - U_\alpha x_{s(\alpha)}))_{\alpha\in Q_1}, (f_i + \varepsilon(g_i - f_i x_i))_{i\in Q_0}).$$

By the calculation above, we get the following formula for the tangent space $T_U(\mathcal{C}_N)$:

$$\frac{\left\{ ((V_\alpha)_{\alpha\in Q_1}, (g_i)_{i\in Q_0}) \middle| \forall \alpha \in Q_1 : \begin{array}{c} M_\alpha g_{s(\alpha)} = f_{t(\alpha)} V_\alpha + g_{t(\alpha)} U_\alpha, \\ V_\alpha = h_{t(\alpha)} U_\alpha - U_\alpha h_{s(\alpha)} \end{array} \right\}}{\left\{ ((x_{t(\alpha)} U_\alpha - U_\alpha x_{s(\alpha)})_{\alpha\in Q_1}, (-f_i x_i)_{i\in Q_0}) \right\}} \tag{3.1.5}$$

with $(V_\alpha)_{\alpha\in Q_1} \in \operatorname{rep}_{\mathbf{d}}(Q)$, $(g_i)_{i\in Q_0} \in \operatorname{Mat}(\mathbf{n} \times \mathbf{d}, k)$, $(h_i)_{i\in Q_0} \in \operatorname{Mat}(\mathbf{d} \times \mathbf{d}, k)$ and $(x_i)_{i\in Q_0} \in T_1\, GL_{\mathbf{d}}$.

To understand this conditions better, we can assume without loss of generality the following:

$$M_\alpha = \begin{pmatrix} U_\alpha & U'_\alpha \\ 0 & W_\alpha \end{pmatrix}, \qquad f_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad g_i = \begin{pmatrix} g'_i \\ g''_i \end{pmatrix}$$

for all $\alpha \in Q_1$ and $i \in Q_0$. Then Condition (3.1.4) reads

$$\begin{pmatrix} U_\alpha & U'_\alpha \\ 0 & W_\alpha \end{pmatrix} \begin{pmatrix} g'_{s(\alpha)} \\ g''_{s(\alpha)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} V_\alpha + \begin{pmatrix} g'_{t(\alpha)} \\ g''_{t(\alpha)} \end{pmatrix} U_\alpha,$$

yielding the two conditions

$$U_\alpha g'_{s(\alpha)} + U'_\alpha g''_{s(\alpha)} = V_\alpha + g'_{t(\alpha)} U_\alpha \tag{3.1.6}$$

$$W_\alpha g''_{s(\alpha)} = g''_{t(\alpha)} U_\alpha \tag{3.1.7}$$

for all $\alpha \in Q_1$. The subspace to be factored out reads

$$\left\{ \left((x_{t(\alpha)} U_\alpha - U_\alpha x_{s(\alpha)})_{\alpha \in Q_1}, \begin{pmatrix} -x_i \\ 0 \end{pmatrix}_{i \in Q_0} \right) \right\}.$$

Let

$$\left(\left(h_{t(\alpha)} U_\alpha - U_\alpha h_{s(\alpha)} \right)_{\alpha \in Q_1}, \begin{pmatrix} g'_i \\ g''_i \end{pmatrix}_{i \in Q_0} \right) \in T_U(\mathcal{C}_N)$$

with $(h_i)_{i \in Q_0} \in \mathrm{Mat}(\mathbf{d} \times \mathbf{d}, k)$, $(g'_i)_{i \in Q_0} \in \mathrm{Mat}(\mathbf{d} \times \mathbf{d}, k)$ and $(g''_i)_{i \in Q_0} \in \mathrm{Mat}((\mathbf{n} - \mathbf{d}) \times \mathbf{d}, k)$. Using Equation (3.1.6) and (3.1.7) holds $\begin{pmatrix} g'_i + h_i \\ g''_i \end{pmatrix}_{i \in Q_0} \in \mathrm{Hom}_Q(U, M)$. Thus the Formula (3.1.5) yields the following short exact sequence

$$0 \to \mathrm{End}_Q(U) \to \mathrm{Hom}_Q(U, M) \to T_U(\mathcal{C}_N) \to 0$$

defined by

$$(h_i)_{i \in Q_0} \mapsto \begin{pmatrix} h_i \\ 0 \end{pmatrix}_{i \in Q_0}, \qquad \begin{pmatrix} h_i \\ g''_i \end{pmatrix}_{i \in Q_0} \mapsto \left(\left(h_{t(\alpha)} U_\alpha - U_\alpha h_{s(\alpha)} \right)_{\alpha \in Q_1}, \begin{pmatrix} 0 \\ g''_i \end{pmatrix}_{i \in Q_0} \right)$$

This implies together with Lemma 3.1.5 the statement. □

Now we are able to prove Corollary 3.1.2.

Proof of Corollary 3.1.2. Let $0 \to U \to M \xrightarrow{\pi} M/U \to 0$ be a short exact sequence and I a semiadmissible ideal such that M is a (Q, I)-representation. Then we use the following part of the corresponding long exact sequence.

$$0 \longrightarrow \mathrm{End}_Q(U) \longrightarrow \mathrm{Hom}_Q(U, M) \xrightarrow{\mathrm{Hom}_Q(U,\pi)} \mathrm{Hom}_Q(U, M/U) \xrightarrow{\mathrm{Hom}_Q(U,\delta)} \mathrm{Ext}^1_{(Q,I)}(U, U). \tag{3.1.8}$$

By the proof of Lemma 3.1.6,

$$N_U(\mathcal{C}_U /\, Gr_{\mathbf{d}}(M)) = \frac{T_U(Gr_{\mathbf{d}}(M))}{T_U(\mathcal{C}_U)} \cong \frac{\mathrm{Hom}_Q(U, M/U)}{\mathrm{Im}\,\mathrm{Hom}_Q(U, \pi)} \cong \mathrm{Im}\,\mathrm{Hom}_Q(U, \delta).$$

□

The following proposition gives a homological condition on a smooth point U in $Gr_{\mathbf{d}}(M)$ such that the set $\overline{\mathcal{C}_U(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.

Proposition 3.1.7. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$.*

1. *The linear map $\mathrm{Hom}_Q(U,\pi)\colon \mathrm{Hom}_Q(U,M) \to \mathrm{Hom}_Q(U,M/U)$ with the canonical projection $\pi\colon M \to M/U$ is surjective if and only if U is a smooth point in $Gr_{\mathbf{d}}(M)$ and $\overline{\mathcal{C}_U(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.*
2. *The linear map $\mathrm{Hom}_Q(\iota, M/U)\colon \mathrm{Hom}_Q(M,M/U) \to \mathrm{Hom}_Q(U,M/U)$ with the canonical embedding $\iota\colon U \to M$ is surjective if and only if U is a smooth point in $\mathrm{Gr}_{\mathbf{d}}(M)$ and $\overline{\mathcal{C}'_{M/U}(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.*

Moreover, these irreducible components have dimension $\dim_k \mathrm{Hom}_Q(U,M/U)$.

By Example 3.1.10 for a non-smooth point U the set $\overline{\mathcal{C}_U(k)}$ can be also an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$. Before proving this proposition we give a corollary.

Corollary 3.1.8. *Let Q, M, $\mathbf{d}$, U, $\iota\colon U \to M$ and $\pi\colon M \to M/U$ as in Proposition 3.1.7.*

1. *If $Gr_{\mathbf{d}}(M)$ is smooth, then $\mathrm{Hom}_Q(U,\pi)$ is surjective if and only if $\overline{\mathcal{C}_U(k)}$ is an irreducible component and $\mathrm{Hom}_Q(\iota,M/U)$ is surjective if and only if $\overline{\mathcal{C}'_{M/U}(k)}$ is an irreducible component.*
2. *Let I be a semiadmissible ideal such that M is a (Q,I)-representation. If $\mathrm{Ext}^1_{(Q,I)}(U,U) = 0$, then $\overline{\mathcal{C}_U(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$. If $\mathrm{Ext}^1_{(Q,I)}(M/U,M/U) = 0$, then $\overline{\mathcal{C}'_{M/U}(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.*
3. *If the maps $\mathrm{Hom}_Q(U,\pi)$ and $\mathrm{Hom}_Q(\iota,M/U)$ are both surjective, then $\overline{\mathcal{C}_{U,M/U}(k)}$ is an irreducible component.*
4. *Especially if $M \cong U \oplus V$ for Q-representations U and V, then $\overline{\mathcal{C}_{U,V}(k)}$ is an irreducible component with dimension $\dim_k \mathrm{Hom}_Q(U,V)$.*

Moreover, all points in these sets $\mathcal{C}_{U,M/U}(k)$ are smooth in $Gr_{\mathbf{d}}(M)$.

By Proposition 2.3.12 the quiver Grassmannian $Gr_{\mathbf{d}}(M)$ is empty or smooth if M is exceptional. This case is discussed in Section 3.5 in more detail. After proving the proposition we show Part 3 and 4 of this corollary. The other parts follow immediately.

Proof of Proposition 3.1.7. Since Part 2 is dual to Part 1 by Lemma 2.3.10, it is enough to prove Part 1. We use the again the exact sequence in (3.1.8). By Theorem 3.1.1 holds

$$\begin{aligned}\dim_k \mathrm{Hom}_Q(U,M/U) &= \dim_k T_U(Gr_{\mathbf{d}}(M)) \geq \dim_U \mathrm{Gr}_{\mathbf{d}}(M) \\ &\geq \dim \mathcal{C}_U(k) = \dim_k \mathrm{Hom}_Q(U,M) - \dim_k \mathrm{End}_Q(U).\end{aligned} \tag{3.1.9}$$

Thus if $\mathrm{Hom}_Q(U,\pi)$ is surjective, we get $\dim \mathcal{C}_U(k) = \dim_U \mathrm{Gr}_{\mathbf{d}}(M)$ and U is smooth in $Gr_{\mathbf{d}}(M)$. Moreover, $\mathcal{C}_U(k)$ is dense in one irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ and thus $\overline{\mathcal{C}_U(k)}$ is one irreducible component.

If the point U is smooth in $Gr_{\mathbf{d}}(M)$ and $\overline{\mathcal{C}_U(k)}$ is an irreducible component, Equation (3.1.9) yields $\dim_k \mathrm{Hom}_Q(U,M/U) = \dim_k \mathrm{Hom}_Q(U,M) - \dim_k \mathrm{End}_Q(U)$ and thus $\mathrm{Hom}_Q(U,\pi)$ is surjective. □

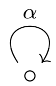
Since each regular local ring is an integral domain (see [2, 11 Dimension Theory]) we get the following lemma.

Lemma 3.1.9. *Let U be a smooth point of a projective variety X. Then U lies in a unique irreducible component of X.*

Proof of Corollary 3.1.8. Using Lemma 3.1.9 we get in Part 3 that the sets $\overline{\mathcal{C}_U(k)}$ and $\overline{\mathcal{C}_{M/U}(k)}$ coincide. Since the sets $\mathcal{C}_U(k)$ and $\mathcal{C}_{M/U}(k)$ are locally closed, the set $\mathcal{C}_{U,M/U}(k)$ is dense in this irreducible component (see [31, II, Exercise 3.18]).

For Part 4 choose $\iota\colon U \to M$ and $\pi\colon M \to V$ to be the canonical embedding and projection induced by the isomorphism $M \cong U \oplus V$. Then the exact sequence $0 \to U \xrightarrow{\iota} M \xrightarrow{\pi} V \to 0$ splits and we apply Part 3. □

Example 3.1.10. Let Q be the following quiver



and $M = (M_\circ, M_\alpha)$ the Q-representation with $M_\circ = k^2$, basis $\{e_1, e_2\}$ and $M_\alpha = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Let $U = \langle e_2 \rangle_Q \in \mathrm{Gr}_1(M)$. The exact sequence $0 \to U \to M \xrightarrow{\pi} M/U \to 0$ is described in Figure 3.1.1. Then $\mathrm{Hom}_Q(U, \pi)\colon \mathrm{Hom}_Q(U, M) \to \mathrm{Hom}_Q(U, M/U)$ is not surjective and

$$0 \longrightarrow \begin{pmatrix} e_2 \end{pmatrix} \longrightarrow \begin{pmatrix} e_1 \\ \downarrow\alpha \\ e_2 \end{pmatrix} \xrightarrow{\ \pi\ } \begin{pmatrix} e_1 \end{pmatrix} \longrightarrow 0$$

Figure 3.1.1: The exact sequence $0 \to U \to M \to M/U \to 0$.

$\mathrm{Gr}_1(M) = \{U\} = \mathcal{C}_{U,M/U}(k)$ as varieties. Since $\mathcal{C}_U$ is smooth and $Gr_1(M)$ is not smooth this equation holds only for varieties and not for schemes.

3.2 Morphisms induced by homomorphisms

In this section we construct to each homomorphism of Q-representations an isomorphism of closed subschemes of the corresponding quiver Grassmannians.

Let Q be a quiver, $\mathbf{n}$ and $\mathbf{d}$ dimension vectors and $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ a Q-representation with $M_i = k^{n_i}$ and the standard basis $\{e_1, \ldots, e_{n_i}\}$ for all $i \in Q_0$. Let $V = (V_i)_{i \in Q_0}$ be a subrepresentation of M with basis $\{e_1, \ldots, e_{t_i}\}$ of V_i for all $i \in Q_0$. The maps

$$\begin{aligned} \psi_A\colon \mathrm{Mat}((\mathbf{n}-\mathbf{t}) \times (\mathbf{d}-\mathbf{t}), A) \times GL_{\mathbf{d}}(A) &\to \mathrm{Mat}(\mathbf{n} \times \mathbf{d}, A), \\ \big((C_i)_{i \in Q_0}, (g_i)_{i \in Q_0}\big) &\mapsto \left(\left(\begin{smallmatrix} I_{t_i} & 0 \\ 0 & C_i \end{smallmatrix}\right) g_i^{-1} \right)_{i \in Q_0} \end{aligned}$$

for a commutative k-algebra A induce a morphism $\psi\colon\ \mathit{Hom}(\mathbf{d}-\mathbf{t},\mathbf{n}-\mathbf{t})\times GL_{\mathbf{d}}\to \mathit{Hom}(\mathbf{d},\mathbf{n})$ of schemes. Let

$$\mathcal{U}^0_{\mathbf{d}}(V,M) = (\mathit{rep}_{\mathbf{d}}(Q)\times \operatorname{Im}\psi)\cap \mathit{Hom}^0_Q(\mathbf{d},M). \tag{3.2.1}$$

This is $GL_{\mathbf{d}}$-stable. For each $\mathbf{d}$-minor $\mathbf{B}$ the scheme $\pi(\mathcal{U}^0_{\mathbf{d}}(V,M))\cap U_{\mathbf{B}}$ with the projection π defined in Equation (2.3.5) is by the isomorphism $\pi_{\mathbf{B}}$ defined in Equation (2.3.6) isomorphic to a closed subscheme of $\mathit{Hom}(\mathbf{d},\mathbf{n}-\mathbf{d})$. Thus this scheme $\mathcal{U}^0_{\mathbf{d}}(V,M)$ is a closed subscheme of $\mathit{Hom}^0_Q(\mathbf{d},M)$ and parametrizes subrepresentations of the Q-representation M with an injective homomorphism of Q-representations to M such that the image contains the subrepresentation V of M. The geometric quotient $\mathcal{U}_{\mathbf{d}}(V,M) = \mathcal{U}^0_{\mathbf{d}}(V,M)/\,GL_{\mathbf{d}}$ of the scheme $\mathcal{U}^0_{\mathbf{d}}(V,M)$ by the group $GL_{\mathbf{d}}$ is a well-defined closed subscheme of $\mathit{Gr}_{\mathbf{d}}(M)=\mathit{Hom}^0_Q(\mathbf{d},M)/\,GL_{\mathbf{d}}$.

This can be generalized to each pair V,M of Q-representations with an embedding $V\hookrightarrow M$. Moreover, by the canonical embedding the scheme $\mathit{Gr}_{\mathbf{d}}(V)$ is a closed subscheme of $\mathit{Gr}_{\mathbf{d}}(M)$.

In Corollary 3.2.2 we study the tangent space of the scheme $\mathcal{U}_{\mathbf{d}}(V,M)$. First we show that each homomorphism $f\colon M\to N$ of Q-representations induces an isomorphism of closed subschemes of this form of the corresponding quiver Grassmannians.

Proposition 3.2.1. *Let Q be a quiver, $\mathbf{d}$ a dimension vector and $f\colon M\to N$ a homomorphism of Q-representations. Then*

$$\begin{aligned} &f_*\colon \mathcal{U}_{\mathbf{d}}(\operatorname{Ker} f,M)\to \mathit{Gr}_{\mathbf{d}-\mathbf{dim}\operatorname{Ker} f}(\operatorname{Im} f),\ U\mapsto f(U),\\ &f^*\colon\ \mathit{Gr}_{\mathbf{d}-\mathbf{dim}\operatorname{Ker} f}(\operatorname{Im} f)\to \mathcal{U}_{\mathbf{d}}(\operatorname{Ker} f,M),\ U\mapsto f^{-1}(U)\end{aligned}$$

are both well-defined morphisms of schemes, which are inverse to each other.

Proof. Let $\mathbf{n}=(n_i)_{i\in Q_0}\in\mathbb{N}^{Q_0}$. Without loss of generality let $M=(M_i,M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ and $N=(N_i,N_\alpha)_{i\in Q_0,\alpha\in Q_1}$ with $M_i=k^{n_i}$ and $N_i=k^{n'_i}$ for all $i\in Q_0$, $M_\alpha\in \operatorname{Mat}(n_{t(\alpha)}\times n_{s(\alpha)},k)$ and $N_\alpha\in\operatorname{Mat}(n'_{t(\alpha)}\times n'_{s(\alpha)},k)$ for all $\alpha\in Q_1$, $\mathbf{t}=\mathbf{dim}\operatorname{Ker} f$, $\operatorname{Ker} f=(V_i,V_\alpha)_{i\in Q_0,\alpha\in Q_1}$ and the linear maps $f_i\colon M_i\to N_i$ are described by the matrices $\begin{pmatrix}0 & I_{n_i-t_i}\\ 0 & 0\end{pmatrix}$ for all $i\in Q_0$. Thus let $M_\alpha=\begin{pmatrix}V_\alpha & \overline{M}_\alpha\\ 0 & M'_\alpha\end{pmatrix}$ for all $\alpha\in Q_1$. Let A be a commutative k-algebra.

For each $((U_\alpha)_{\alpha\in Q_1},(C_i)_{i\in Q_0})\in \mathit{Hom}^0_Q(\mathbf{d}-\mathbf{t},\operatorname{Im} f)(A)$ we get

$$\left(\begin{pmatrix}V_\alpha & \overline{M}_\alpha C_{s(\alpha)}\\ 0 & U_\alpha\end{pmatrix}_{\alpha\in Q_1},\begin{pmatrix}I_{t_i} & 0\\ 0 & C_i\end{pmatrix}_{i\in Q_0}\right)\in \mathcal{U}^0_{\mathbf{d}}(\operatorname{Ker} f,M)(A)$$

with $f^{-1}(\operatorname{Im} C_i)=\operatorname{Im}\begin{pmatrix}I_{t_i} & 0\\ 0 & C_i\end{pmatrix}$ for all $i\in Q_0$. Thus this induces a morphism of schemes

$$\tilde{f}^*\colon\ \mathit{Hom}^0_Q(\mathbf{d}-\mathbf{t},\operatorname{Im} f)\to \mathcal{U}^0_{\mathbf{d}}(\operatorname{Ker} f,M),$$

which factors to the morphism $f^*\colon\ \mathit{Gr}_{\mathbf{d}-\mathbf{t}}(\operatorname{Im} f)\to \mathcal{U}_{\mathbf{d}}(\operatorname{Ker} f,M)$ with

$$(f^*)_A\colon\ \mathit{Gr}_{\mathbf{d}-\mathbf{t}}(\operatorname{Im} f)(A)\to \mathcal{U}_{\mathbf{d}}(\operatorname{Ker} f,M)(A),\ U\mapsto f^{-1}(U)$$

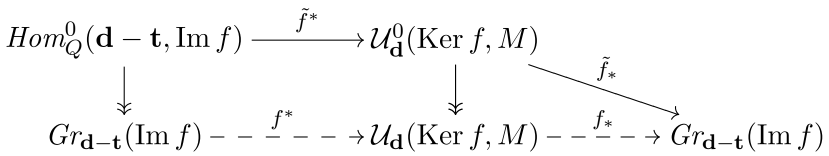


Figure 3.2.1: A commutative diagram for the proof of Proposition 3.2.1.

as in the commutative diagram in Figure 3.2.1.

Now we construct an inverse map of the morphism f^*. For a commutative k-algebra A let $((U_\alpha)_{\alpha\in Q_1}, (h_i)_{i\in Q_0}) \in \mathcal{U}^0_{\mathbf{d}}(\operatorname{Ker} f, M)(A)$. Thus there is a tuple $(g_i)_{i\in Q_0} \in GL_{\mathbf{d}}(A)$ of invertible matrices, a representation $(U'_\alpha)_{\alpha\in Q_1} \in rep_{\mathbf{d}-\mathbf{t}}(Q)(A)$ and a tuple $(C_i)_{i\in Q_0} \in Hom^0(\mathbf{d}-\mathbf{t}, \mathbf{n}-\mathbf{t})(A)$ of matrices such that $U_\alpha = g_{t(\alpha)} \begin{pmatrix} V_\alpha & \overline{M}_\alpha C_{s(\alpha)} \\ 0 & U'_\alpha \end{pmatrix} g^{-1}_{s(\alpha)}$ for all $\alpha \in Q_1$ and $h_i = \begin{pmatrix} I_{t_i} & 0 \\ 0 & C_i \end{pmatrix} g_i^{-1}$ for all $i \in Q_0$. These matrices g_i and C_i are not unique for each $i \in Q_0$. Nevertheless, the image of C_i is uniquely determined by h_i for all $i \in Q_0$. And so this defines a well-defined morphism $(\tilde{f}_*)_A \colon \mathcal{U}^0_{\mathbf{d}}(\operatorname{Ker} f, M)(A) \to Gr_{\mathbf{d}-\mathbf{t}}(\operatorname{Im} f)(A)$,

$$\left(\left(g_{t(\alpha)} \begin{pmatrix} V_\alpha & \overline{M}_\alpha C_{s(\alpha)} \\ 0 & U'_\alpha \end{pmatrix} g^{-1}_{s(\alpha)}\right)_{\alpha\in Q_1}, \left(\begin{pmatrix} I_{t_i} & 0 \\ 0 & C_i \end{pmatrix} g_i^{-1}\right)_{i\in Q_0}\right) \mapsto (\operatorname{Im} C_i)_{i\in Q_0}.$$

This morphism factors again to the morphism

$$f_* \colon \mathcal{U}_{\mathbf{d}}(\operatorname{Ker} f, M) \to Gr_{\mathbf{d}-\mathbf{t}}(\operatorname{Im} f),\ U \mapsto f(U).$$

Moreover, these morphisms f^* and f_* are inverse to each other. □

If $f\colon M \to N$ is an injective homomorphism of Q-representations, then this induces a closed embedding $f_*\colon Gr_{\mathbf{d}}(M) \to Gr_{\mathbf{d}}(N), U \mapsto f(U)$. If $f\colon M \to N$ is surjective, then $f^*\colon Gr_{\mathbf{d}-\mathbf{dim}\operatorname{Ker} f}(N) \to Gr_{\mathbf{d}}(M), V \mapsto f^{-1}(V)$ is again a closed embedding. In Example 3.2.3 we study an example.

Corollary 3.2.2. *Let Q be a quiver and M, U and V Q-representations such that $\mathbf{dim}\, U = \mathbf{d}$ and $V \subseteq U \subseteq M$. Then*

$$T_U(\mathcal{U}_{\mathbf{d}}(V, M)) \cong \operatorname{Hom}_Q(U/V, M/U).$$

By Proposition 3.2.1 the scheme $\mathcal{U}_{\mathbf{d}}(V, M)$ is isomorphic to the quiver Grassmannian $Gr_{\mathbf{d}-\mathbf{t}}(M/V)$ with $\mathbf{dim}\, V = \mathbf{t}$ and thus this corollary is clear by Proposition 2.3.12. Nevertheless, we give an independent proof. Using the canonical projection $U \to U/V$ of Q-representations the normal space $N_U(\mathcal{U}_{\mathbf{d}}(V, M)/\, Gr_{\mathbf{d}}(M))$ is isomorphic to the quotient space

$$\operatorname{Hom}_Q(U, M/U)/\operatorname{Hom}_Q(U/V, M/U).$$

Moreover, $N_V(Gr_{\mathbf{t}}(U)/\, Gr_{\mathbf{t}}(M)) \cong \operatorname{Hom}_Q(V, M/V)/\operatorname{Hom}_Q(V, U/V)$ holds by the embedding $U/V \to M/V$.

Proof. Without loss of generality let $\mathbf{dim}\, M = \mathbf{n} = (n_i)_{i\in Q_0}$, $\mathbf{dim}\, U = \mathbf{d} = (d_i)_{i\in Q_0}$, $\mathbf{dim}\, V = \mathbf{t} = (t_i)_{i\in Q_0}$ and $M = (M_i, M_\alpha)_{i\in Q_0,\alpha\in Q_1}$ a Q-representation with $M_i = k^{n_i}$ and basis $\{e_1, \dots, e_{n_i}\}$ for all $i \in Q_0$ and

$$M_\alpha = \begin{pmatrix} U_\alpha & U'_\alpha \\ 0 & W_\alpha \end{pmatrix} \in \operatorname{Mat}\left((\mathbf{d} + (\mathbf{n} - \mathbf{d}))_{t(\alpha)} \times (\mathbf{d} + (\mathbf{n} - \mathbf{d}))_{s(\alpha)}, k\right),$$
$$U_\alpha = \begin{pmatrix} V_\alpha & V'_\alpha \\ 0 & X_\alpha \end{pmatrix} \in \operatorname{Mat}\left((\mathbf{t} + (\mathbf{d} - \mathbf{t}))_{t(\alpha)} \times (\mathbf{t} + (\mathbf{d} - \mathbf{t}))_{s(\alpha)}, k\right)$$

for all $\alpha \in Q_1$. Moreover, we assume that $U = (U_i)_{i\in Q_0}$ and $V = (V_i)_{i\in Q_0}$ are the subrepresentations of M with the basis $\{e_1, \dots, e_{d_i}\}$ of U_i and $\{e_1, \dots, e_{t_i}\}$ of V_i for all $i \in Q_0$.

We use the proof of Lemma 3.1.6. Thus $\left((U_\alpha)_{\alpha\in Q_1}, \begin{pmatrix} I_{d_i} \\ 0 \end{pmatrix}_{i\in Q_0}\right) \in \pi^{-1}(U)$ with π defined in Equation (2.3.5). Let $((Y_\alpha)_{\alpha\in Q_1}, (g_i)_{i\in Q_0}) \in \operatorname{rep}_{\mathbf{d}}(Q) \times \operatorname{Hom}_k(\mathbf{d}, M)$. The conditions for

$$\left((U_\alpha + \varepsilon Y_\alpha)_{\alpha\in Q_1}, \left(\begin{pmatrix} I_{d_i} \\ 0 \end{pmatrix} + g_i\right)_{i\in Q_0}\right)$$

belonging to the tangent space T of $\mathcal{U}^0_{\mathbf{d}}(V, M)$ are (see Equation (3.1.4)):

$$M_\alpha g_{s(\alpha)} = \begin{pmatrix} I_{d_{t(\alpha)}} \\ 0 \end{pmatrix} Y_\alpha + g_{t(\alpha)} U_\alpha$$

and there exist $h_i \in \operatorname{Mat}(d_i \times d_i, k)$, $D'_i \in \operatorname{Mat}((d_i - t_i) \times (d_i - t_i), k)$ and $D_i \in \operatorname{Mat}((n_i - d_i) \times (d_i - t_i), k)$ for all $i \in Q_0$ such that

$$\begin{pmatrix} I_{d_i} \\ 0 \end{pmatrix} + \varepsilon g_i = \left(\begin{pmatrix} I_{d_i} \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} I_{t_i} & 0 \\ 0 & D'_i \\ 0 & D_i \end{pmatrix}\right)(I_{d_i} + \varepsilon h_i)$$

for all $i \in Q_0$. With $g_i = \begin{pmatrix} g'_i \\ g''_i \end{pmatrix}$ for all $i \in Q_0$ this yields:

$$U_\alpha g'_{s(\alpha)} + U'_\alpha g''_{s(\alpha)} = Y_\alpha + g'_{t(\alpha)} U_\alpha,\ W_\alpha g''_{s(\alpha)} = g''_{t(\alpha)} U_\alpha,\ g'_i = \begin{pmatrix} I_{t_i} & 0 \\ 0 & D'_i \end{pmatrix} + h_i,\ g''_i = \begin{pmatrix} 0 & D_i \end{pmatrix}$$

with $h_i \in \operatorname{Mat}(d_i \times d_i, k)$, $D'_i \in \operatorname{Mat}((d_i - t_i) \times (d_i - t_i), k)$ and $D_i \in \operatorname{Mat}((n_i - d_i) \times (d_i - t_i), k)$ for all $i \in Q_0$. For each $i \in Q_0$, g'_i and D'_i we can choose some $h_i \in \operatorname{Mat}(d_i \times d_i, k)$. So we can drop the third condition.

As in the proof of Lemma 3.1.6 the vector space $T_U(\mathcal{U}_{\mathbf{d}}(V, M))$ is isomorphic to

$$\frac{\left\{\left(\left(U_\alpha g'_{s(\alpha)} - g'_{t(\alpha)} U_\alpha + U'_\alpha g''_{s(\alpha)}\right)_{\alpha\in Q_1}, \begin{pmatrix} g'_i \\ g''_i \end{pmatrix}_{i\in Q_0}\right) \middle| \begin{array}{c} W_\alpha g''_{s(\alpha)} = g''_{t(\alpha)} U_\alpha, \\ g''_i = \begin{pmatrix} 0 & D_i \end{pmatrix} \end{array}\right\}}{\left\{\left(\left(x_{t(\alpha)} U_\alpha - U_\alpha x_{s(\alpha)}\right)_{\alpha\in Q_1}, \begin{pmatrix} -x_i \\ 0 \end{pmatrix}_{i\in Q_0}\right)\right\}}$$

with $(x_i)_{i\in Q_0} \in T_1 \operatorname{GL}_{\mathbf{d}}(k)$. Setting $x_i = g'_i$ for all $i \in Q_0$ this vector space is isomorphic to

$$\{(D_i)_{i\in Q_0} \in \operatorname{Mat}((\mathbf{n} - \mathbf{d}) \times (\mathbf{d} - \mathbf{t}), k) | W_\alpha \begin{pmatrix} 0 & D_{s(\alpha)} \end{pmatrix} = \begin{pmatrix} 0 & D_{t(\alpha)} \end{pmatrix} U_\alpha\}.$$

Using the definition of U_α for all $\alpha \in Q_1$ we get

$$T_U(\mathcal{U}_{\mathbf{d}}(V,M)) \cong \left\{(D_i)_{i\in Q_0} \middle| W_\alpha D_{s(\alpha)} = D_{t(\alpha)} X_\alpha\right\} \cong \mathrm{Hom}_Q(U/V, M/U).$$

□

Example 3.2.3. Let $Q = (\{\circ\},\{\alpha\})$ be the quiver defined in Example 3.1.10 (see also Section 3.6.3). For each $n \in \mathbb{N}$ let $M(n) = (M(n)_\circ, M(n)_\alpha)$ be a Q-representation with $M(n)_\circ = k^n$ and $M(n)_\alpha$ the nilpotent Jordan block of size n. Then for $n > 0$ the Q-representation $M(n)$ is indecomposable, nilpotent and up to isomorphism unique. For $d, n \in \mathbb{N}$ with $d \le n$ exists an injective homomorphism $\iota\colon M(d) \to M(n)$. For such an embedding the image is unique and $M(n)/M(d) \cong M(n-d)$.

For $d, n \in \mathbb{N}$ with $d \le n$ the variety $\mathrm{Gr}_d(M(n))$ contains a unique element, but the schemes $Gr_d(M(n))$ are in general pairwise non-isomorphic.

Let $n, m \in \mathbb{N}$ and $f\colon M(n) \to M(m)$ a homomorphism of Q-representations with $\operatorname{Ker} f \cong M(t)$. Then by Proposition 3.2.1 the induced morphism $f_*\colon \mathcal{U}_d(M(t), M(n)) \to Gr_{d-t}(M(n-t))$ is an isomorphism of schemes. Moreover

$$\begin{aligned} T_{M(d)}(\mathcal{U}_d(M(t),M(n))) &\cong \mathrm{Hom}_Q(M(d-t), M(n-d)) \\ \cong \mathrm{Hom}_Q(M(d-t), M(n-t)/M(d-t)) &\cong T_{M(d-t)}(Gr_{d-t}(M(n-t))). \end{aligned}$$

Let $t, n \in \mathbb{N}$ with $t \le n$ and $g\colon M(n) \to M(n-t)$ a surjective homomorphism of Q-representations. Then the induced morphism $g^*\colon\ Gr_{d-t}(M(n-t)) \to Gr_d(M(n))$ is a closed embedding of schemes. Moreover, for all $d, d', n, n' \in \mathbb{N}$ with $d' \le d$ and $n' - d' \le n - d$ there is a closed embedding of schemes $Gr_{d'}(M(n')) \to Gr_{d'}(M(n-d+d')) \to Gr_d(M(n))$.

3.3 Semicontinuity and group action

There are some well-known results for the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$. By Proposition 2.3.4 rank functions on the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ are lower semicontinuous and dimensions of homomorphism spaces of Q-representations are upper semicontinuous. In this section it turns out that this is also true for the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$. In the last part we study a canonical operation of the automorphism group $\mathrm{Aut}_Q(M)$ of the Q-representation M on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$. This is useful for the computations in Section 3.6.

Proposition 3.3.1. *Let Q be a quiver, $M = (M_i, M_\alpha)_{i\in Q_0, \alpha\in Q_1}$ a Q-representation, $\mathbf{d}$ a dimension vector and $\beta \in Q_1$. Then the maps*

$$\begin{aligned} \mathrm{rk}_\beta\colon\ \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, (U_i)_{i\in Q_0} &\mapsto \mathrm{rk}\left(M_\beta|_{U_{s(\beta)}}\colon U_{s(\beta)} \to U_{t(\beta)}\right) \\ \overline{\mathrm{rk}}_\beta\colon\ \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, (U_i)_{i\in Q_0} &\mapsto \mathrm{rk}\left((M/U)_\beta\colon M_{s(\beta)}/U_{s(\beta)} \to M_{t(\beta)}/U_{t(\beta)}\right) \end{aligned}$$

are lower semicontinuous.

Using Lemma 2.3.10 both statements are dual. Thus it is enough to consider the case of the map rk_β. In the proof of the proposition we use the following lemma.

Lemma 3.3.2. *Let $d, m, n \in \mathbb{N}$ with $d \leq n$ and $M \in \mathrm{Mat}(m \times n, k)$. For $v_1, \dots, v_d \in k^n$ let $(v_1 \dots v_d) \in \mathrm{Mat}(n \times d, k)$ be the matrix with columns $v_1, \dots, v_d$. Then the map*

$$\mathrm{rk}_M \colon \mathrm{Gr}\binom{n}{d} \to \mathbb{Z}, \langle v_1, \dots, v_d \rangle \to \mathrm{rk}(M \cdot (v_1 \dots v_d))$$

is well-defined and lower semicontinuous.

Proof of Proposition 3.3.1. Let $\mathbf{n} = (n_i)_{i \in Q_0}$ be the dimension vector of M. Then the map $\mathrm{rk}_\beta \colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}$ factorizes in the following way

$$\mathrm{Gr}_{\mathbf{d}}(M) \hookrightarrow \mathrm{Gr}\binom{\mathbf{n}}{\mathbf{d}} \xrightarrow{\pi_{s(\beta)}} \mathrm{Gr}\binom{n_{s(\beta)}}{d_{s(\beta)}} \xrightarrow{\mathrm{rk}_{M_\beta}} \mathbb{Z}$$

with the canonical projection $\pi_{s(\beta)}$. These maps are all continuous or lower semicontinuous. □

Proof of Lemma 3.3.2. Let $U \in \mathrm{Gr}\binom{n}{d}$ and $v_1, \dots, v_d$ and $v_1', \dots, v_d'$ bases of U. Then there exists $g \in \mathrm{GL}_d(k)$ with $(v_1 \dots v_d) \cdot g = (v_1' \dots v_d')$. Thus

$$\mathrm{rk}(M \cdot (v_1 \dots v_d)) = \mathrm{rk}(M \cdot (v_1 \dots v_d) \cdot g) = \mathrm{rk}(M \cdot (v_1' \dots v_d'))$$

and the map rk is well-defined.

Now we use the open affine covering $\{U_B\}_B$ of $\mathrm{Gr}\binom{n}{d}$ defined in Section 2.3.2. Let B be the d-minor of the first d rows and π_B the isomorphism defined in Equation (2.3.2). Moreover, we define the following linear map

$$\chi \colon \mathrm{Mat}((n-d) \times d, k) \to \mathrm{Mat}(m \times d, k), C \mapsto M \begin{pmatrix} I_d \\ C \end{pmatrix}.$$

We consider the commutative diagram in Figure 3.3.1.

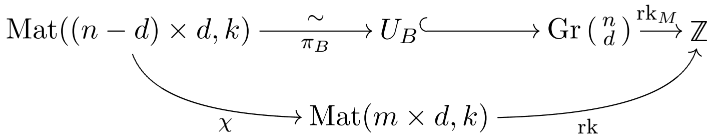


Figure 3.3.1: A commutative diagram for the proof of Lemma 3.3.2.

To prove that rk_M is lower semicontinuous it is enough to show this for the map $\mathrm{rk}_M|_{U_B} \colon U_B \to \mathbb{Z}$. This map is the concatenation of the continuous map $\chi\pi_B^{-1}$ and the lower semicontinuous map $\mathrm{rk} \colon \mathrm{Mat}(m \times d, k) \to \mathbb{Z}$ (see Example 2.3.1). □

Proposition 3.3.3. *Let Q be a quiver, M, N Q-representations and $\mathbf{d}$ a dimension vector. Then the maps*

$$\begin{aligned}
\dim_k \mathrm{Hom}_Q(-, N) &\colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(U, N)\\
\dim_k \mathrm{Hom}_Q(N, -) &\colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(N, U)\\
\dim_k \overline{\mathrm{Hom}_Q(-, N)} &\colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(M/U, N)\\
\dim_k \overline{\mathrm{Hom}_Q(N, -)} &\colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathbb{Z}, U \mapsto \dim_k \mathrm{Hom}_Q(N, M/U)
\end{aligned}$$

are upper semicontinuous.

Some of the maps defined in the following proof are used again in the proof of Theorem 3.4.1.

Proof. We use the following result of Crawley-Boevey [17, Section 3, Special Case]: Let X be a variety, V a vector space and for all $x \in X$ let V_x be a subspace of V such that the set $\{(x, v) | v \in V_x\}$ is locally closed in $X \times V$. Then the map $X \to \mathbb{Z}, x \mapsto \dim_k V_x$ is upper semicontinuous.

Let $\mathbf{m} = \mathbf{dim}\, M$, $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ with $M_i = k^{m_i}$ for all $i \in Q_0$ and $M_\alpha \in \operatorname{Mat}(m_{t(\alpha)} \times m_{s(\alpha)}, k)$ for all $\alpha \in Q_1$, $\mathbf{n} = \mathbf{dim}\, N$, $N = (N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1}$ with $N_i = k^{n_i}$ for all $i \in Q_0$ and $N_\alpha \in \operatorname{Mat}(n_{t(\alpha)} \times n_{s(\alpha)}, k)$ for all $\alpha \in Q_1$, $\iota\colon \operatorname{Gr}_{\mathbf{d}}(M) \hookrightarrow \operatorname{Gr}\binom{\mathbf{m}}{\mathbf{d}}$ the canonical inclusion and $\{U_{\mathbf{B}}\}_{\mathbf{B}}$ the affine open covering of $\operatorname{Gr}\binom{\mathbf{m}}{\mathbf{d}}$ defined in Section 2.3.2.

For each $U \in \operatorname{Gr}_{\mathbf{d}}(M)$ let V_U be the subspace $\{f \in \operatorname{Mat}(\mathbf{n} \times \mathbf{m}, k) | f|_U \in \operatorname{Hom}_Q(U, N)\}$ of $\operatorname{Mat}(\mathbf{n} \times \mathbf{m}, k)$. We show that the set

$$\big\{(U, f) \in (\operatorname{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}) \times \operatorname{Mat}(\mathbf{n} \times \mathbf{m}, k) \big| f \in V_U\big\} \tag{3.3.1}$$

is closed in $(\operatorname{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}) \times \operatorname{Mat}(\mathbf{n} \times \mathbf{m}, k)$ for each $\mathbf{B}$. The maps used for this are summarized in the commutative diagrams in Figure 3.3.2 and 3.3.3.

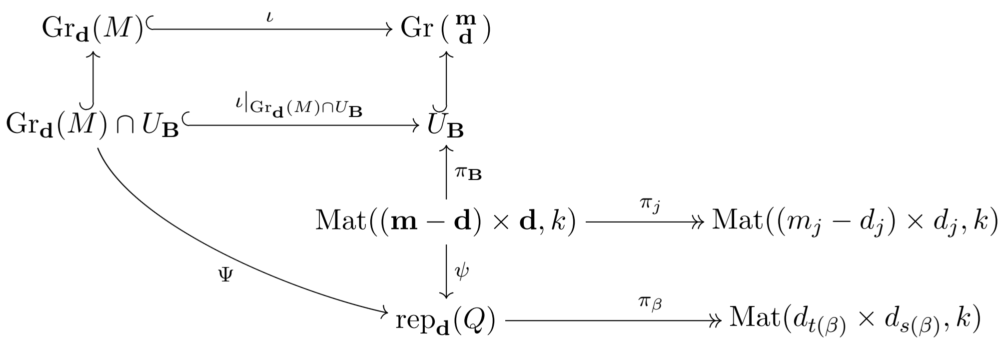


Figure 3.3.2: A commutative diagram for the proof of Proposition 3.3.3 and Theorem 3.4.1 with $j \in Q_0$ and $\beta \in Q_1$.

Without loss of generality we assume $\mathbf{B} = (B_i)_{i \in Q_0}$ is the $\mathbf{d}$-minor such that B_i is the d_i-minor of the first d_i rows for all $i \in Q_0$. Thus the isomorphism $\pi_{\mathbf{B}}\colon \operatorname{Mat}((\mathbf{m} - \mathbf{d}) \times \mathbf{d}, k) \xrightarrow{\sim} U_{\mathbf{B}}$ is given by Equation (2.3.3). We define the following linear maps

$$\begin{aligned}
\psi\colon\ & \operatorname{Mat}((\mathbf{m} - \mathbf{d}) \times \mathbf{d}, k) \to \operatorname{rep}_{\mathbf{d}}(Q),\ (C_i)_{i \in Q_0} \mapsto \left(\begin{pmatrix} I_{d_{t(\alpha)}} & 0 \end{pmatrix} \cdot M_\alpha \cdot \begin{pmatrix} I_{d_{s(\alpha)}} \\ C_{s(\alpha)} \end{pmatrix}\right)_{\alpha \in Q_1}, \\
\pi_\beta\colon\ & \operatorname{rep}_{\mathbf{d}}(Q) \to \operatorname{Mat}(d_{t(\beta)} \times d_{s(\beta)}, k), (L_\alpha)_{\alpha \in Q_1} \mapsto L_\beta, \\
\pi_j\colon\ & \operatorname{Mat}((\mathbf{m} - \mathbf{d}) \times \mathbf{d}, k) \to \operatorname{Mat}((m_j - d_j) \times d_j, k), (C_i)_{i \in Q_0} \mapsto C_j, \\
\varphi_\beta\colon\ & \operatorname{Mat}((\mathbf{m} - \mathbf{d}) \times \mathbf{d}, k) \times \operatorname{Mat}(\mathbf{n} \times \mathbf{m}, k) \to \operatorname{Mat}(n_{t(\beta)} \times d_{s(\beta)}, k), \\
& ((C_i)_{i \in Q_0}, (f_i)_{i \in Q_0}) \mapsto N_\beta f_{s(\beta)} \begin{pmatrix} I_{d_{s(\beta)}} \\ C_{s(\beta)} \end{pmatrix} - f_{t(\beta)} \begin{pmatrix} I_{d_{t(\beta)}} \\ C_{t(\beta)} \end{pmatrix} \pi_\beta \psi((C_i)_{i \in Q_0})
\end{aligned}$$

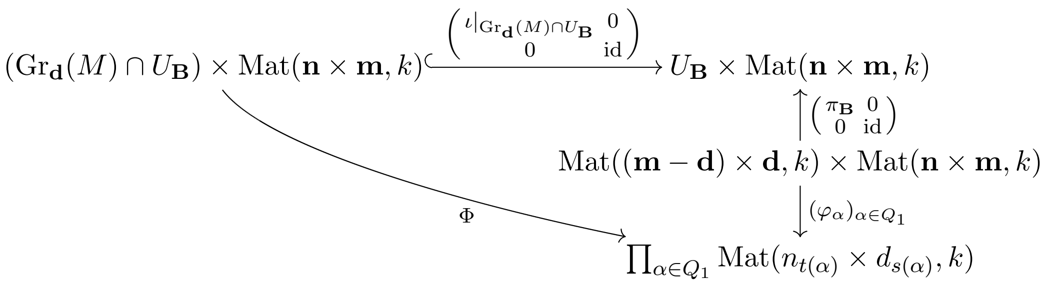


Figure 3.3.3: A commutative diagram for the proof of Proposition 3.3.3.

for $j \in Q_0$ and $\beta \in Q_1$. Moreover, we get the following morphisms of varieties

$$\Psi\colon \ \mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}} \to \mathrm{rep}_{\mathbf{d}}(Q), U \mapsto \psi \pi_{\mathbf{B}}^{-1} \iota(U),$$
$$\Phi\colon (\mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}) \times \mathrm{Mat}(\mathbf{n} \times \mathbf{m}, k) \to \prod_{\alpha \in Q_1} \mathrm{Mat}(n_{t(\alpha)} \times d_{s(\alpha)}, k),$$
$$(U, f) \mapsto \big(\varphi_\alpha(\pi_{\mathbf{B}}^{-1} \iota(U), f)\big)_{\alpha \in Q_1}.$$

Let $(U, f) \in (\mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}) \times \mathrm{Mat}(\mathbf{n} \times \mathbf{m}, k)$. Since U is a subrepresentation of M, there is some tuple $(g_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \mathrm{Mat}(d_{t(\alpha)} \times d_{s(\alpha)}, k)$ such that

$$M_\beta \begin{pmatrix} I_{d_{s(\beta)}} \\ \pi_{s(\beta)} \pi_{\mathbf{B}}^{-1} \iota(U) \end{pmatrix} = \begin{pmatrix} I_{d_{t(\beta)}} \\ \pi_{t(\beta)} \pi_{\mathbf{B}}^{-1} \iota(U) \end{pmatrix} g_\beta$$

for each $\beta \in Q_1$. This equation shows $g_\beta = \pi_\beta \Psi(U)$ for each $\beta \in Q_1$. Thus the left hand side of the diagram in Figure 3.3.4 is commutative. Using the hole diagram in Figure 3.3.4 we get $f \in V_U$ if and only if $\varphi_\beta(\pi_{\mathbf{B}}^{-1} \iota(U), f) = 0$ for all $\beta \in Q_1$.

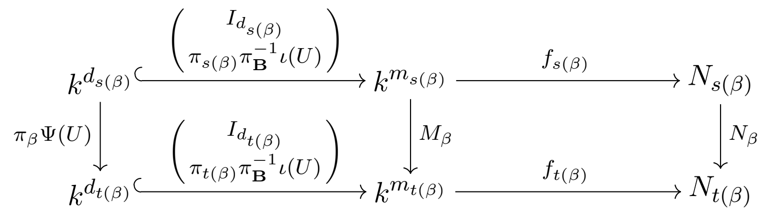


Figure 3.3.4: A not necessarily commutative diagram for $f \in V_U$ with $\beta \in Q_1$.

Thus the set defined in Equation (3.3.1) is $\Phi^{-1}(\{0\})$ and closed in $(\mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}) \times \mathrm{Mat}(\mathbf{n} \times \mathbf{m}, k)$. Since $\dim_k V_U = \dim_k \mathrm{Hom}_Q(U, N) + \sum_{i \in Q_0} (m_i - d_i) n_i$ for all $U \in \mathrm{Gr}_{\mathbf{d}}(M)$, the map $\dim_k \mathrm{Hom}_Q(-, N)$ is upper semicontinuous.

Using the subset $\{(U, f) \in \mathrm{Gr}_{\mathbf{d}}(M) \times \mathrm{Mat}(\mathbf{m} \times \mathbf{n}, k) | f \in \mathrm{Hom}_Q(N, U)\}$ of $\mathrm{Gr}_{\mathbf{d}}(M) \times \mathrm{Mat}(\mathbf{m} \times \mathbf{n}, k)$ and Lemma 2.3.10 the same holds in the other cases. □

The group $\mathrm{GL}_{\mathbf{d}}(k)$ operates on the variety $\mathrm{rep}_{\mathbf{d}}(Q)$. The orbits under this action are the isomorphism classes of Q-representations in $\mathrm{rep}_{\mathbf{d}}(Q)$. By Remark 3.3.6 there is no such natural action for $\mathrm{Gr}_{\mathbf{d}}(M)$, but by the following proposition the group $\mathrm{Aut}_Q(M)$ operates on $\mathrm{Gr}_{\mathbf{d}}(M)$.

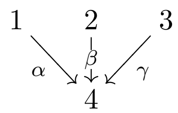
Proposition 3.3.4. *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then the group $\mathrm{Aut}_Q(M)$ operates on $\mathrm{Gr}_{\mathbf{d}}(M)$. This operation stabilizes $\mathcal{C}_N(k)$, $\overline{\mathcal{C}_N(k)}$, $\mathcal{C}'_N(k)$ and $\overline{\mathcal{C}'_N(k)}$ for all Q-representations N.*

Proof. The operation of $\mathrm{Aut}_Q(M)$ on M induces one on $\mathrm{Gr}_{\mathbf{d}}(M)$. If $U \cong N$ with $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ and $g \in \mathrm{Aut}_Q(M)$, then $gU \cong N$.

Let $g \in \mathrm{Aut}_Q(M)$. Since $\varphi_g \colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathrm{Gr}_{\mathbf{d}}(M)$, $U \mapsto g^{-1}U$ is continuous and $\overline{\mathcal{C}_N(k)}$ is closed, $\varphi_g^{-1}\big(\overline{\mathcal{C}_N(k)}\big) = g\overline{\mathcal{C}_N(k)}$ is also closed. So by $\mathcal{C}_N(k) = g\mathcal{C}_N(k) \subseteq g\overline{\mathcal{C}_N(k)}$ holds $\overline{\mathcal{C}_N(k)} \subseteq g\overline{\mathcal{C}_N(k)}$. Thus $\overline{\mathcal{C}_N(k)} = g\overline{\mathcal{C}_N(k)}$ by symmetry. □

Using the following example the number of $\mathrm{Aut}_Q(M)$-orbits of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ is not necessarily finite and do not describe the subschemes of isomorphism classes.

Example 3.3.5. Let Q be the following quiver



and $M = (M_1, M_2, M_3, M_4, M_\alpha, M_\beta, M_\gamma)$ the Q-representation with $M_1 = M_2 = M_3 = k$, $M_4 = k^2$, $M_\alpha = \binom{1}{0}$, $M_\beta = \binom{1}{1}$ and $M_\gamma = \binom{0}{1}$. Then $\mathrm{Aut}_Q(M) \cong k$ and $\mathrm{Gr}_{(0,0,0,1)}(M) \cong \mathbb{P}^1_k$, although $\mathcal{C}_{S(4)}(k) = \mathrm{Gr}_{(0,0,0,1)}(M)$.

Remark 3.3.6. In general there is no algebraic group G with an action on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ such that the orbits are the subsets $\mathcal{C}_U(k)$.

We assume there is such a continuous action in general. In this case the closure of an orbit is the union of some orbits (see proof of Proposition 3.3.4). However, this is not true in Example 3.1.3.

3.4 Connections to degenerations of representations

Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then we study in this section the relations of the topology of the module variety $\mathrm{rep}_{\mathbf{d}}(Q)$ and the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$. For example we can guess

$$\mathcal{C}_U(k) \subseteq \overline{\mathcal{C}_V(k)} \Leftrightarrow \mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)} \tag{3.4.1}$$

for $U, V \in \mathrm{Gr}_{\mathbf{d}}(M)$ or even

$$\mathcal{C}_{U,M/U}(k) \subseteq \overline{\mathcal{C}_{V,M/V}(k)} \Leftrightarrow \mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)} \wedge \mathcal{O}(M/U) \subseteq \overline{\mathcal{O}(M/V)} \tag{3.4.2}$$

for $U, V \in \mathrm{Gr}_{\mathbf{d}}(M)$. Only one of the implications in Equivalence (3.4.2) is true by the following theorem, the other is wrong by Example 3.4.4.

Theorem 3.4.1. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector and $U, V \in \mathrm{Gr}_{\mathbf{d}}(M)$ with $U \in \overline{\mathcal{C}_V(k)}$. Then $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$ in the variety $\mathrm{rep}_{\mathbf{d}}(Q)$.*

This is even stronger than the first part of Equivalence (3.4.1). However by using the dual, adding trivial implications, taking the negation and using the definitions we get the following two corollaries. Thereafter we prove the theorem.

Corollary 3.4.2. *Let Q be a quiver, M a Q-representation, $\mathbf{d}$ a dimension vector and $U, V \in \mathrm{Gr}_{\mathbf{d}}(M)$.*

- *If $U \in \overline{\mathcal{C}'_{M/V}(k)}$, then $\mathcal{O}(M/U) \subseteq \overline{\mathcal{O}(M/V)}$.*
- *If $\mathcal{C}_U(k) \subseteq \overline{\mathcal{C}_V(k)}$, then $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$ and if $\mathcal{C}'_{M/U}(k) \subseteq \overline{\mathcal{C}'_{M/V}(k)}$, then $\mathcal{O}(M/U) \subseteq \overline{\mathcal{O}(M/V)}$.*
- *Let $U \ncong V$. If $U \in \overline{\mathcal{C}_V(k)}$, then $\mathcal{O}(V) \nsubseteq \overline{\mathcal{O}(U)}$ and if $U \in \overline{\mathcal{C}'_{M/V}(k)}$, then $\mathcal{O}(M/V) \nsubseteq \overline{\mathcal{O}(M/U)}$.*
- *Let $U \ncong V$. If $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$, then $\mathcal{C}_V(k) \nsubseteq \overline{\mathcal{C}_U(k)}$ and if $\mathcal{O}(M/U) \subseteq \overline{\mathcal{O}(M/V)}$, then $\mathcal{C}'_{M/V}(k) \nsubseteq \overline{\mathcal{C}'_{M/U}(k)}$.*

Corollary 3.4.3. *Let Q be a quiver, $\mathbf{d}$ and $\mathbf{n}$ dimension vectors, M a Q-representation with dimension vector $\mathbf{n}$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$.*

1. *If the orbit $\mathcal{O}(U)$ is closed in the topological space $\bigcup_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(V) \subseteq \mathrm{rep}_{\mathbf{d}}(Q)$, then $\mathcal{C}_U(k)$ is also closed in $\mathrm{Gr}_{\mathbf{d}}(M)$.*
2. *If the orbit $\mathcal{O}(M/U)$ is closed in the topological space $\bigcup_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(M/V) \subseteq \mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q)$, then $\mathcal{C}'_{M/U}(k)$ is also closed in $\mathrm{Gr}_{\mathbf{d}}(M)$.*

This corollary does not mean that all closed orbits are of this form.

Proof of Theorem 3.4.1. We use the notations as in the proof of Proposition 3.3.3. The maps used in this proof are again summarized in the commutative diagram in Figure 3.3.2. Let $\mathbf{B}$ be again the $\mathbf{d}$-minor of the first rows. Without loss of generality we assume that $U \in U_{\mathbf{B}}$.

Let $W = (W_i)_{i \in Q_0} \in \mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}$. Then W is a subrepresentation of M and thus by $(W_i, M_\alpha|_{W_{s(\alpha)}} \colon W_{s(\alpha)} \to W_{t(\alpha)})_{i \in Q_0, \alpha \in Q_1}$ canonically a Q-representation. The tuple $\Psi(W)$ is a point in $\mathrm{rep}_{\mathbf{d}}(Q)$ and thus by $(k^{d_i}, \pi_\alpha \Psi(W))_{i \in Q_0, \alpha \in Q_1}$ again a Q-representation. Using the left hand side of the diagram in Figure 3.3.4 with $W = U$, we get that this Q-representations W and $\Psi(W)$ are isomorphic and $\Psi(W) \in \mathcal{O}(W)$ for each $W \in \mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}$.

Since Ψ is continuous the set $\Psi^{-1}\left(\overline{\mathcal{O}(V)}\right)$ is a closed subset of $\mathrm{Gr}_{\mathbf{d}}(M) \cap U_{\mathbf{B}}$ and $\Psi^{-1}\left(\overline{\mathcal{O}(V)}\right) \cup (\mathrm{Gr}_{\mathbf{d}}(M) \backslash U_{\mathbf{B}})$ a closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$ containing $\mathcal{C}_V(k)$. This means $\left(\overline{\mathcal{C}_V(k)} \cap U_{\mathbf{B}}\right) \subseteq \Psi^{-1}\left(\overline{\mathcal{O}(V)}\right)$. Thus $\Psi(U) \in \overline{\mathcal{O}(V)}$ and $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$. □

In the following example we give a quiver Q, Q-representations M, N and N', a dimension vector $\mathbf{d}$ and subrepresentations U, U' and V of M such that there are exact sequences

$$0 \to U \to V \to U' \to 0 \text{ and } 0 \to N \to M/V \to N' \to 0, \tag{3.4.3}$$

$M/(U+U') \cong N \oplus N'$ and $\mathcal{C}_{U\oplus U',N\oplus N'}(k) \nsubseteq \overline{\mathcal{C}_V(k)}$. In this case $\mathcal{O}(U \oplus U') \subseteq \overline{\mathcal{O}(V)}$ in $\mathrm{rep}_{\mathbf{d}}(Q)$ and $\mathcal{O}(N \oplus N') \subseteq \overline{\mathcal{O}(M/V)}$ in $\mathrm{rep}_{\mathbf{dim}\,M-\mathbf{d}}(Q)$.

Example 3.4.4. Let Q be the following quiver

$$\beta \circlearrowright 1 \xleftarrow{\ \alpha\ } 2$$

and $M = (M_1, M_2, M_\alpha, M_\beta)$ the Q-representation such that $M_1 = k^4$, $M_2 = k^2$, $M_\alpha = \begin{pmatrix} 00 \\ 00 \\ 10 \\ 01 \end{pmatrix}$ and $M_\beta = \begin{pmatrix} 0000 \\ 1000 \\ 0100 \\ 0000 \end{pmatrix}$. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of M_1 and $\{f_1, f_2\}$ of M_2. This representation M is described by one picture in Figure 3.4.1. We define the following sub- and factor representations of M. Let $U = \langle f_2 \rangle_Q \in \mathrm{Gr}_{(1,1)}(M)$, $U' = \langle e_2 \rangle_Q \in \mathrm{Gr}_{(2,0)}(M)$, $V = \langle e_1, f_1 \rangle_Q \in \mathrm{Gr}_{(3,1)}(M)$, $W = \langle e_2, e_4, f_1 \rangle_Q \in \mathrm{Gr}_{(3,1)}(M)$, $N = M/\langle e_2, f_1, f_2 \rangle_Q$ and $N' = M/\langle e_1, f_2 \rangle_Q$. Then these representations are described

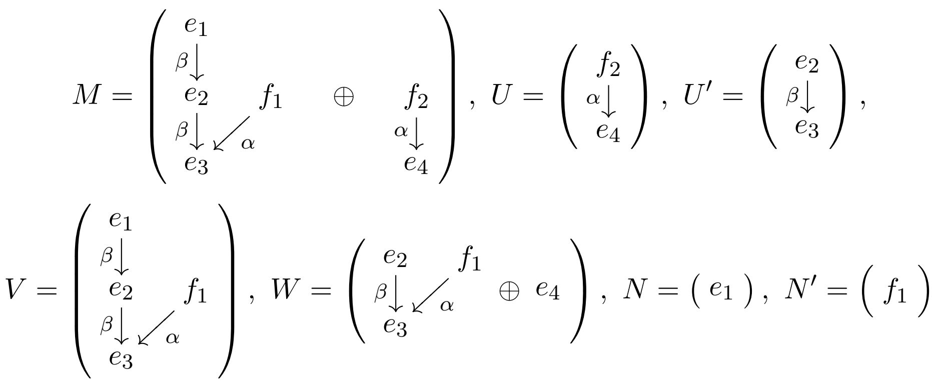


Figure 3.4.1: The Q-representations M, U, U', V, W, N and N'.

by the pictures in Figure 3.4.1. There are exact sequences as in Equation (3.4.3). Using Theorem 3.1.1 for the variety $\mathrm{Gr}_{(3,1)}(M)$ holds

$$\begin{aligned} \mathcal{C}_{U\oplus U',N\oplus N'}(k) &= \{\langle e_2, f_2 + \mu f_1 \rangle_Q | \mu \in k\}, \\ \mathcal{C}_{V,M/V}(k) &= \{\langle e_1 + \lambda e_4, f_1 \rangle_Q | \lambda \in k\}, \\ \mathcal{C}_{W,M/W}(k) &= \{W\} = \{\langle e_2, e_4, f_1 \rangle_Q\} \end{aligned}$$

and $\mathrm{Gr}_{(3,1)}(M) = \mathcal{C}_{U\oplus U',N\oplus N'}(k) \cup \mathcal{C}_{V,M/V}(k) \cup \mathcal{C}_{W,M/W}(k)$. Since $\dim \mathcal{C}_{U\oplus U'} = 1 = \dim \mathcal{C}_V$ the irreducible components of $\mathrm{Gr}_{(3,1)}(M)$ are $\overline{\mathcal{C}_{U\oplus U',N\oplus N'}(k)}$ and $\overline{\mathcal{C}_{V,M/V}(k)}$.

Especially $\mathcal{C}_{U\oplus U',N\oplus N'}(k) \nsubseteq \overline{\mathcal{C}_V \cup \mathcal{C}_{M/V}(k)}$, $\mathcal{O}(U \oplus U')(k) \subseteq \overline{\mathcal{O}(V)(k)}$ in the variety $\mathrm{rep}_{(3,1)}(Q)$ and $\mathcal{O}(N \oplus N')(k) \subseteq \overline{\mathcal{O}(M/V)(k)}$ in $\mathrm{rep}_{(1,1)}(Q)$.

By some calculations $\dim_k \mathrm{Hom}_Q(U \oplus U', N \oplus N') = 2$, $\dim_k \mathrm{Hom}_Q(V, M/V) = 1$ and $\dim_k \mathrm{Hom}_Q(W, M/W) = 3$. Thus the smooth part of $Gr_{(3,1)}(M)$ is $\mathcal{C}_{V,M/V}(k)$.

3.5 Representation finite case

Let Q be a quiver, $\mathbf{d}$ a dimension vector and M a Q-representation. The representation M is called *sub-$\mathbf{d}$-finite* if the set of isomorphism classes of subrepresentations of M with dimension vector $\mathbf{d}$ is finite. M is called *factor-$\mathbf{d}$-finite* if the set of isomorphism classes of factor representations of M with dimension vector $\mathbf{dim}\, M - \mathbf{d}$ is finite.

Example 3.5.1. If I is an admissible ideal such that $\mathrm{rep}(Q, I)$ is representation finite, then M is sub- and factor-$\mathbf{d}$-finite for each (Q, I)-representation M and each dimension vector $\mathbf{d}$. In this case for each dimension vector $\mathbf{d}$ the decomposition of the module variety $\mathrm{rep}_{\mathbf{d}}(Q, I)$ into irreducible components is given by $\mathrm{rep}_{\mathbf{d}}(Q, I) = \bigcup_{M \in \mathcal{M}_{\mathbf{d}}} \overline{\mathcal{O}(M)}$ with $\mathcal{M}_{\mathbf{d}}$ the set of elements $M \in \mathrm{rep}_{\mathbf{d}}(Q, I)$ such that $\mathcal{O}(M)$ is maximal under the degeneration order.

Using Equation (3.1.3), Theorem 3.1.1 yields the following corollary.

Corollary 3.5.2. *Let Q be a quiver, $\mathbf{d}$ a dimension vector and M a Q-representation.*
1. *If M is sub-$\mathbf{d}$-finite, then each irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ is of the form $\overline{\mathcal{C}_N(k)}$ with some Q-representation N.*
2. *If M is factor-$\mathbf{d}$-finite, then each irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ is of the form $\overline{\mathcal{C}'_N(k)}$ with some Q-representation N.*
3. *If M is sub- and factor-$\mathbf{d}$-finite, then each irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ is of the form $\overline{\mathcal{C}_{N,N'}(k)}$ with some Q-representations N and N'. Especially $\overline{\mathcal{C}_{N,N'}(k)}$ is irreducible in this case.*

Using Proposition 3.1.7, we get the following proposition.

Proposition 3.5.3. *Let Q be a quiver, $\mathbf{d}$ and $\mathbf{n}$ dimension vectors, M a Q-representation with dimension vector $\mathbf{n}$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$.*
1. *If M is sub-$\mathbf{d}$-finite, then the set*
$$\bigcup_{U \in \mathrm{Gr}_{\mathbf{d}}(M),\ \mathrm{Hom}_Q(U,\pi)\ \textit{surjective}} \mathcal{C}_U(k)$$
with the canonical projection $\pi\colon M \to M/U$ is dense in the smooth part of $\mathrm{Gr}_{\mathbf{d}}(M)$.
2. *If M is factor-$\mathbf{d}$-finite, then the set*
$$\bigcup_{U \in \mathrm{Gr}_{\mathbf{d}}(M),\ \mathrm{Hom}_Q(\iota,M/U)\ \textit{surjective}} \mathcal{C}'_{M/U}(k)$$
with the canonical embedding $\iota\colon U \to M$ is dense in the smooth part of $\mathrm{Gr}_{\mathbf{d}}(M)$.
3. *If M is sub- and factor-$\mathbf{d}$-finite, then the set*
$$\bigcup_{U \in \mathrm{Gr}_{\mathbf{d}}(M),\ \mathrm{Hom}_Q(U,\pi),\ \mathrm{Hom}_Q(\iota,M/U)\ \textit{both surjective}} \mathcal{C}_{U,M/U}(k)$$
is dense in the smooth part of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$.

Example 3.5.4. Let $Q = 1 \overset{\alpha}{\to} 2$, $\mathbf{d} = (1,1)$ and $M = (M_1, M_2, M_\alpha)$ a Q-representation such that $M_1 = k^2$ with basis $\{e_1, e_2\}$, $M_2 = k$ with basis $\{f\}$ and $M_\alpha = \left(\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right)$. Let $U = \langle e_1 \rangle_Q, V = \langle e_2, f\rangle_Q \in \mathrm{Gr}_{(1,1)}(M)$. Then

$$\mathrm{Gr}_{(1,1)}(M) = \mathcal{C}_U(k) \cup \mathcal{C}_V(k) = \overline{\mathcal{C}_U(k)} = \mathcal{C}'_{S(1)}(k).$$

Moreover, the quiver Grassmannian $\mathrm{Gr}_{(1,1)}(M)$ is smooth. Let $0 \to U \overset{\iota_U}{\to} M \overset{\pi_U}{\to} M/U \to 0$ and $0 \to V \overset{\iota_V}{\to} M \overset{\pi_V}{\to} M/V \to 0$ be the canonical short exact sequences. Since the first one splits, $\mathrm{Hom}_Q(U, \pi_U)$ and $\mathrm{Hom}_Q(\iota_U, U)$ are surjective, but $\mathrm{Hom}_Q(V, M/V)$ is not surjective since $\mathrm{Hom}_Q(V, \pi_V)$ vanishes and $\mathrm{Hom}_Q(V, M/V)$ is one dimensional. Thus the set $\mathcal{C}_U(k)$ for Part 1 of Proposition 3.5.3 is a proper subset of the smooth part of $\mathrm{Gr}_{\mathbf{d}}(M)$.

Using Theorem 3.4.1 and Corollary 3.4.2, we get the following proposition.

Proposition 3.5.5. *Let Q be a quiver, $\mathbf{d}$ and $\mathbf{n}$ dimension vectors, M a Q-representation with dimension vector $\mathbf{n}$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$.*

1. *If M is sub-$\mathbf{d}$-finite and the orbit $\mathcal{O}(U)$ is maximal in the variety*

$$\bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(V) \subseteq \mathrm{rep}_{\mathbf{d}}(Q),$$

then $\overline{\mathcal{C}_U(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.

2. *If M is factor-$\mathbf{d}$-finite and the orbit $\mathcal{O}(M/U)$ is maximal in the variety*

$$\bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(M/V) \subseteq \mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q),$$

then $\overline{\mathcal{C}'_{M/U}(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$.

3. *If M is sub- and factor-$\mathbf{d}$-finite and the $(\mathrm{GL}_{\mathbf{d}}(k) \times \mathrm{GL}_{\mathbf{n}-\mathbf{d}}(k))$-orbit $(\mathcal{O}(U) \times \mathcal{O}(W))$ is maximal in the variety*

$$\bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} (\mathcal{O}(V) \times \mathcal{O}(M/V)) \subseteq \mathrm{rep}_{\mathbf{d}}(Q) \times \mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q),$$

then $\overline{\mathcal{C}_{U,W}(k)}$ is an irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ with dimension

$$\begin{aligned} \dim \overline{\mathcal{C}_{U,W}(k)} &= \dim_k \mathrm{Hom}_Q(U, M) - \dim_k \mathrm{End}_Q(U) \\ &= \dim_k \mathrm{Hom}_Q(M, W) - \dim_k \mathrm{End}_Q(W). \end{aligned}$$

Example 3.5.6. Let $Q = 1 \overset{\alpha}{\to} 2$, $\mathbf{d} = (1,1)$ and $M = (M_1, M_2, M_\alpha)$ a Q-representation with $M_1 = M_2 = k^2$ and $\mathrm{rk}(M_\alpha) = 1$. Then

$$\bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(V) = \bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} \mathcal{O}(M/V) = \mathrm{rep}_{\mathbf{d}}(Q) \cong k$$

and with the identification $\mathrm{rep}_{\mathbf{d}}(Q)^2 \cong k^2$ holds

$$\bigcup\nolimits_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} (\mathcal{O}(V) \times \mathcal{O}(M/V)) \cong \{(a,b) \in k^2 | ab = 0\} \subsetneq k^2.$$

Example 3.4.4 and Section 3.6.4 shows that in general these are not all irreducible components of $\mathrm{Gr}_{\mathbf{d}}(M)$ for a sub- and factor-**d**-finite Q-representation M. Nevertheless, in Section 3.6 we prove that in some special cases all irreducible components are of this form.

In both cases the smooth points are not dense in the quiver Grassmannian. We look for some example such that there exists a $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ with $\overline{\mathcal{C}_{U,M/U}(k)}$ is an irreducible component, each point of $\mathcal{C}_{U,M/U}(k)$ is smooth and $\mathcal{O}(U) \times \mathcal{O}(M/U)$ is not maximal in the variety described in Part 3 of Proposition 3.5.5. By the way, in this case the linear map $\mathrm{Hom}_Q(U,\pi)\colon \mathrm{Hom}_Q(U,M) \to \mathrm{Hom}_Q(U,M/U)$ is surjective by Proposition 3.1.7.

If a Q-representation M is exceptional we get the following simple result for the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$.

Proposition 3.5.7. *Let Q be a quiver,* **d** *a dimension vector and M an exceptional Q-representation such that $\mathrm{Gr}_{\mathbf{d}}(M)$ is non-empty.*

1. *If M is sub-**d**-finite, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}_U(k)}$ with some exceptional Q-representation U.*
2. *If M is factor-**d**-finite, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}'_U(k)}$ with some exceptional Q-representation U.*
3. *If M is sub- and factor-**d**-finite, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}_{U,V}(k)}$ with some exceptional Q-representations U and V.*

We remind to Part 2 of Corollary 3.1.8. Using this each exceptional subrepresentation U with dimension vector **d** of an arbitrary Q-representation M provides an irreducible component $\overline{\mathcal{C}_U(k)}$ of $\mathrm{Gr}_{\mathbf{d}}(M)$. By this proposition the irreducible component $\overline{\mathcal{C}_U(k)}$ is the hole quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ if the Q-representation M is also exceptional.

Proof. Again it is enough to prove the first part. By Corollary 3.5.2 each irreducible component of $\mathrm{Gr}_{\mathbf{d}}(M)$ is of the form $\overline{\mathcal{C}_U(k)}$ with a $U \in \mathrm{Gr}_{\mathbf{d}}(M)$. Let U be such a subrepresentation of M and $0 \to U \to M \xrightarrow{\pi} M/U \to 0$ the corresponding short exact sequence. Using a result of [12, Proof of Corollary 3] we get $\mathrm{Ext}^1_Q(U,M) = 0$ for the subrepresentation U of the exceptional Q-representation M. We use the following part of the corresponding long exact sequence:

$$0 \longrightarrow \mathrm{End}_Q(U) \longrightarrow \mathrm{Hom}_Q(U,M) \xrightarrow{\mathrm{Hom}_Q(U,\pi)} \mathrm{Hom}_Q(U,M/U) \xrightarrow{\mathrm{Hom}_Q(U,\delta)} \mathrm{Ext}^1_Q(U,U) \longrightarrow \mathrm{Ext}^1_Q(U,M) \longrightarrow \cdots$$

Using Proposition 2.3.12 the scheme $Gr_{\mathbf{d}}(M)$ is smooth, since M is exceptional. Thus Proposition 3.1.7 yields that the map $\mathrm{Hom}_Q(U,\pi)\colon \mathrm{Hom}_Q(U,M) \to \mathrm{Hom}_Q(U,M/U)$ is surjective. This means $\mathrm{Ext}^1_Q(U,U) = 0$.

By Corollary 2.3.6 there exists up to isomorphism at most one exceptional Q-representation with dimension vector **d**. Thus U is unique up to isomorphism. □

Example 3.5.8. Let Q be the following quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \,.$$

Since Q is of type $\tilde{A}_1$, we use Remark 2.2.10. For $n \in \mathbb{N}$ let $M(n)$ be an indecomposable preprojective Q-representation with dimension vector $(n, n+1)$ (see Figure 3.5.1).

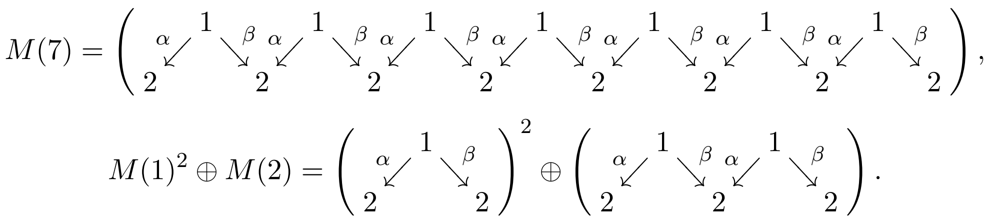


Figure 3.5.1: The Q-representations $M(7)$ and $M(1)^2 \oplus M(2)$.

Let $m \in \mathbb{N}$ and $\mathbf{d} = (d_1, d_2)$ a dimension vector. Thus $M(m)$ is a sub-$\mathbf{d}$-finite and exceptional string module and the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M(m))$ is non-empty if and only if $0 \leq d_1 < d_2 \leq m+1$. By Proposition 3.5.7, there is an exceptional Q-representation U with $\mathrm{Gr}_{\mathbf{d}}(M(m)) = \overline{\mathcal{C}_U(k)}$ in this case. Using again Remark 2.2.10, there are unique numbers $p, r, s \in \mathbb{N}$ with $r > 0$ and $U \cong M(p)^r \oplus M(p+1)^s$. Thus $r + s = d_2 - d_1$, $0 \leq s < d_2 - d_1$ and $(d_2 - d_1)p + s = d_1$. Summing up, we get $p, s \in \mathbb{N}$ by division algorithm with divisor $d_2 - d_1$ and dividend d_1.

For $m = 7$ and $\mathbf{d} = (4, 7)$ we get $p = 1$, $r = 2$ and $s = 1$. We illustrate this example in Figure 3.5.1). For an indecomposable preinjective Q-representation we get a dual result. But for a regular Q-representation we need a new strategy.

Example 3.5.9. Let Q be a quiver without oriented cycles. A Q-representation M, which is projective or injective in the abelian category $\mathrm{rep}(Q)$, is exceptional.

Let $i \in Q_0$. The *projective cover* (resp. *injective hull*) of the simple Q-representation $S(i)$ is a projective (resp. injective) Q-representation M with $\mathrm{top}\, M \cong S(i)$ (resp. $\mathrm{soc}\, M \cong S(i)$). We denote it by $P(i)$ (resp. $I(i)$). For each projective (resp. injective) Q-representation M exists a unique tuple $\mathbf{k} = (k_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ such that this representation is isomorphic to

$$P(\mathbf{k}) = \bigoplus_{i \in Q_0} P(i)^{k_i} \ \left(\text{resp. } I(\mathbf{k}) = \bigoplus_{i \in Q_0} I(i)^{k_i}\right). \tag{3.5.1}$$

Let Q be the quiver described in the picture in Figure 3.5.2. In Figure 3.5.3 we give pictures for projective Q-representations.

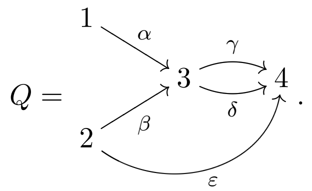


Figure 3.5.2: The quiver Q for Example 3.5.9.

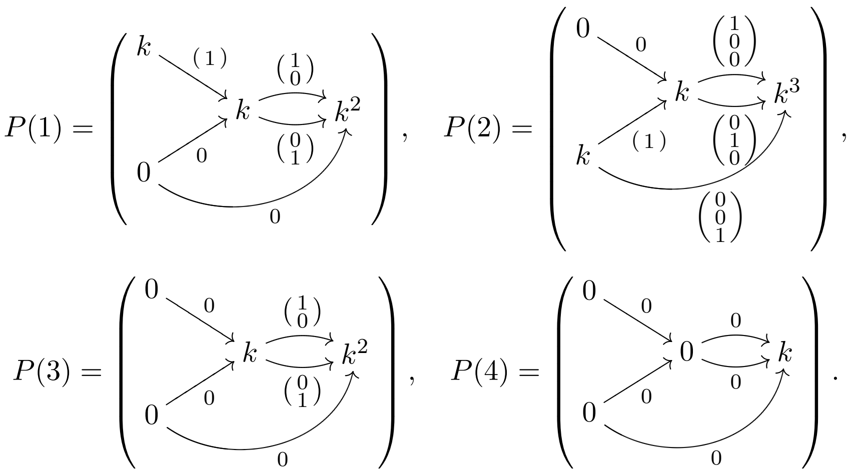


Figure 3.5.3: The projective Q-representations $P(i)$ with $i \in Q_0$.

Corollary 3.5.10. *Let Q be a quiver without oriented cycles, $\mathbf{d}$ a dimension vector and M a Q-representation such that $\mathrm{Gr}_{\mathbf{d}}(M)$ is non-empty.*

1. *If M is a projective Q-representation, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}_U(k)}$ with some projective Q-representation U.*
2. *If M is injective, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}'_V(k)}$ with some injective Q-representation V.*
3. *If M is a projective-injective Q-representation, then $\mathrm{Gr}_{\mathbf{d}}(M) = \overline{\mathcal{C}_{U,V}(k)}$ with some projective Q-representation U and some injective Q-representation V.*

We consider an example to this corollary in Section 3.6.6.

Proof. Since the category $\mathrm{rep}(Q)$ is hereditary each subrepresentation of a projective Q-representation is again projective. By Equation (3.5.1) all projective Q-representations are sub-$\mathbf{d}$-finite for each dimension vector $\mathbf{d}$. □

3.6 Examples

In this section we decompose for some examples the quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$ into its irreducible components. The considered Q-representations M are both sub- and factor-$\mathbf{d}$-finite in all cases. Thus by Part 3 of Proposition 3.5.5 we get an irreducible component $\overline{\mathcal{C}_{U,M/U}(k)}$ of $\mathrm{Gr}_{\mathbf{d}}(M)$ for each $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ such that the $(\mathrm{GL}_{\mathbf{d}}(k) \times \mathrm{GL}_{\mathbf{n}-\mathbf{d}}(k))$-orbit $(\mathcal{O}(U) \times \mathcal{O}(M/U))$ is maximal in the variety

$$\bigcup_{V \in \mathrm{Gr}_{\mathbf{d}}(M)} (\mathcal{O}(V) \times \mathcal{O}(M/V)) \subseteq \mathrm{rep}_{\mathbf{d}}(Q) \times \mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q). \tag{3.6.1}$$

Using Example 3.4.4, in general these are not all irreducible components of $\mathrm{Gr}_{\mathbf{d}}(M)$. Nevertheless, if this is the case, we use the following strategy.

Setting. Let Q be a quiver, $\mathbf{d}$ a dimension vector and M a sub- and factor-$\mathbf{d}$-finite Q-representation with dimension vector $\mathbf{n}$. Let $\mathrm{rk}\colon \mathrm{Gr}_{\mathbf{d}}(M) \to \mathcal{N}_{\mathbf{d}}(M)$ be a surjective map such that each subrepresentation U in $\mathrm{Gr}_{\mathbf{d}}(M)$ is mapped to its $\mathrm{GL}_{\mathbf{d}}(k)$-orbit $\mathcal{O}(U)$ in $\mathrm{rep}_{\mathbf{d}}(Q)$. Thus the set $\mathcal{N}_{\mathbf{d}}(M)$ is finite and $\mathrm{Gr}_{\mathbf{d}}(M) = \bigcup_{\mathcal{O}(U) \in \mathcal{N}_{\mathbf{d}}(M)} \mathcal{C}_U(k)$.

Strategy. Now we define for each example some subset $\mathcal{M}_{\mathbf{d}}(M)$ of $\mathcal{N}_{\mathbf{d}}(M)$ and associate to each $\mathcal{O}(U) \in \mathcal{M}_{\mathbf{d}}(M)$ an orbit in $\mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q)$, denoted by $\mathcal{O}(U^\star)$. We claim that the set $\{(\mathcal{O}(U) \times \mathcal{O}(U^\star)) | \mathcal{O}(U) \in \mathcal{M}_{\mathbf{d}}(M)\}$ is the set of maximal orbits in the variety described in Equation (3.6.1). Moreover, we request that the decomposition of $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components is given by

$$\mathrm{Gr}_{\mathbf{d}}(M) = \bigcup_{\mathcal{O}(U) \in \mathcal{M}_{\mathbf{d}}(M)} \overline{\mathcal{C}_{U,U^\star}(k)}. \tag{3.6.2}$$

We determine the cases such that the points in $\mathcal{C}_{U,U^\star}(k)$ are smooth points of $Gr_{\mathbf{d}}(M)$ and the cases such that the subset of smooth points is dense.

Idea of proof. We show Equation (3.6.2) in the following way. Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)$. If $\mathrm{rk}(U) \in \mathcal{M}_{\mathbf{d}}(M)$, we prove $U \in \overline{\mathcal{C}_{U,U^\star}(k)}$. Otherwise we construct some $V \in \mathrm{Gr}_{\mathbf{d}}(M)$ with $V \ncong U$ and $U \in \overline{\mathcal{C}_V(k)}$.

Now we prove that for each $\mathcal{O}(U) \in \mathcal{M}_{\mathbf{d}}(M)$ the orbit $(\mathcal{O}(U) \times \mathcal{O}(U^\star))$ is maximal in the variety described in Equation (3.6.1). Using the first part and Proposition 3.5.5 we know that the converse is true. So it is enough to prove $\mathcal{O}(U^\star) \nsubseteq \overline{\mathcal{O}(V^\star)}$ in $\mathrm{rep}_{\mathbf{n}-\mathbf{d}}(Q)$ for each $\mathcal{O}(U), \mathcal{O}(V) \in \mathcal{M}_{\mathbf{d}}(M)$ with $U \ncong V$ and $\mathcal{O}(U) \subseteq \overline{\mathcal{O}(V)}$ in $\mathrm{rep}_{\mathbf{d}}(Q)$. □

We observe the following examples:

1. For $Q = 1 \to 2$ our strategy succeeds for each Q-representation M and each dimension vector $\mathbf{d}$. In this case the subset of smooth points is dense.
2. For $Q = 1 \to 2 \to 3$ we decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components only for Q-representations M with dimension vectors of the form (n, n, n) and dimension vectors $\mathbf{d} = (d, d, d)$. In this case the subset of smooth points is not dense in general.
3. For the one-loop-quiver Q (see Figure 3.6.4) our strategy succeeds again for each Q-representation M and each dimension vector $\mathbf{d}$. But in this case the subset of smooth points is not dense in general.
4. For the cyclically oriented two-cycle-quiver Q (see Figure 3.6.9) our strategy fails since in general Part 3 of Proposition 3.5.5 do not describe all irreducible components of $\mathrm{Gr}_{\mathbf{d}}(M)$.

5. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the cyclically oriented N-cycle-quiver (see Figure 3.6.11) and M a projective-injective (Q, α^N)-representation. Using our strategy we decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ into irreducible components for each dimension vector $\mathbf{d}$. In this case the subset of smooth points is again dense.
   By Görtz [30, Section 4] and Pappas, Rapoport and Smithling [40, Section 7] these quiver Grassmannians $\mathrm{Gr}_{\mathbf{d}}(M)$ with dimension vectors $\mathbf{d}$ of the form $(d, \ldots, d)$ occur in the context of local models of Shimura varieties.
6. For $Q = 1 \to 2 \to \cdots \to N$ we can use the previous results for each projective or injective Q-representation M and each dimension vector $\mathbf{d}$. In this case the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ is smooth.

3.6.1 Quiver of type A_2

Let $Q = 1 \overset{\alpha}{\to} 2$, $\mathbf{n} = (n_1, n_2)$ and $\mathbf{d} = (d_1, d_2)$ dimension vectors and $k \in \mathbb{N}$. If $k \leq n_1$ and $k \leq n_2$ let $M(\mathbf{n}, k)$ be a Q-representation with $\mathbf{dim}\, M(\mathbf{n}, k) = \mathbf{n}$ and $\mathrm{rk}_\alpha(M(\mathbf{n}, k)) = k$. This parametrizes up to isomorphism all Q-representations.

Proposition 3.6.1. *Let $Q = 1 \overset{\alpha}{\to} 2$, $\mathbf{n} = (n_1, n_2)$ and $\mathbf{d} = (d_1, d_2)$ dimension vectors and $k \in \mathbb{N}$ with $k \leq n_1$ and $k \leq n_2$. Let* $\mathrm{rk}\colon \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{n}, k)) \to \mathbb{Z}, U \mapsto \mathrm{rk}_\alpha(U)$ *and $\mathcal{N}_{\mathbf{d}}(\mathbf{n}, k)$ the image of this map. Define a subset of $\mathcal{N}_{\mathbf{d}}(\mathbf{n}, k)$ by $\mathcal{M}_{\mathbf{d}}(\mathbf{n}, k) = \{d_1\}$ if $d_1 + n_2 - d_2 \leq k$ and*

$$\mathcal{M}_{\mathbf{d}}(\mathbf{n}, k) = \Big\{ r \in \mathbb{N} \,\Big|\, \max\{0, k + d_1 - n_1, k + d_2 - n_2\} \leq r \leq \min\{k, d_1, d_2\} \Big\}$$

otherwise and set

$$r^\star = \begin{cases} n_2 - d_2 & \text{if } d_1 + n_2 - d_2 \leq k, \\ k - r & \text{otherwise} \end{cases}$$

for each $r \in \mathcal{M}_{\mathbf{d}}(\mathbf{n}, k)$. Then the decomposition of $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{n}, k))$ into irreducible components is given by

$$\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{n}, k)) = \bigcup\nolimits_{r \in \mathcal{M}_{\mathbf{d}}(\mathbf{n}, k)} \overline{\mathcal{C}_{r, r^\star}(k)}$$

with $\mathcal{C}_{r,r^\star} = \mathcal{C}_{M(\mathbf{d}, r), M(\mathbf{n}-\mathbf{d}, r^\star)}$ for short. Moreover, for $r \in \mathcal{M}_{\mathbf{d}}(\mathbf{n}, k)$ all points in $\mathcal{C}_{r,r^\star}(k)$ are smooth points of $Gr_{\mathbf{d}}(M(\mathbf{n}, k))$.

Thus the subset of smooth points is dense.

Proof. For all $r \in \mathcal{M}_{\mathbf{d}}(\mathbf{n}, k)$ the tuple $(r, r^\star)$ is maximal in the order on $\mathbb{Z}$ induced by degenerations. Let $U \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{n}, k))$. By Proposition 3.3.4 we can decompose the Q-representation $M(\mathbf{n}, k)$ such that $M(\mathbf{n}, k) = A \oplus A' \oplus \overline{A} \oplus B \oplus B' \oplus \overline{B}$ and the following holds. This decomposition is illustrated in the picture in Figure 3.6.1.

- Let $A \oplus A' \oplus \overline{A} \cong M((k, k), k)$, $B_2 = 0$, $B'_1 = 0$ and $\overline{B}_\alpha = 0$ with $B = (B_1, B_2, B_\alpha)$, $B = (B'_1, B'_2, B'_\alpha)$ and $\overline{B} = (\overline{B}_1, \overline{B}_2, \overline{B}_\alpha)$.

$$\begin{pmatrix} \circ \cdots \circ \\ \downarrow \quad \downarrow \\ \circ \cdots \circ \end{pmatrix} \oplus \begin{pmatrix} \\ \\ a_1 \cdots a_r \end{pmatrix} \oplus 0 \oplus \begin{pmatrix} b_1 \cdots b_s \\ \\ \\ \end{pmatrix} \oplus \begin{pmatrix} \\ \\ \circ \cdots \circ \end{pmatrix} \oplus 0$$

$$\begin{pmatrix} \circ \cdots \circ \\ \downarrow \quad \downarrow \\ \circ \cdots \circ \end{pmatrix} \oplus \begin{pmatrix} \circ \cdots \circ \\ \downarrow \quad \downarrow \\ a_1 \cdots a_r \end{pmatrix} \oplus \begin{pmatrix} \circ \cdots \circ \\ \downarrow \quad \downarrow \\ \circ \cdots \circ \end{pmatrix} \oplus \begin{pmatrix} b_1 \cdots b_s \\ \\ \\ \end{pmatrix} \oplus \begin{pmatrix} \\ \\ \circ \cdots \circ \end{pmatrix} \oplus \begin{pmatrix} \circ \cdots \circ \\ \\ \circ \cdots \circ \end{pmatrix}$$

Figure 3.6.1: The Q-representations U and $M(\mathbf{n}, k)$.

- Let $U = A \oplus \tilde{A} \oplus B \oplus B'$ with $\tilde{A}_1 = 0$ and $\tilde{A}_2 = A'_2$ with $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_\alpha)$ and $A' = (A'_1, A'_2, A'_\alpha)$.

If $\tilde{A} = 0$, then $\mathrm{rk}_\alpha(U) + \overline{\mathrm{rk}}_\alpha(U) = k$ and thus $U \in \mathcal{C}_{\mathrm{rk}(U),\mathrm{rk}(U)^\star}(k)$. Also if $B = 0$ and $\overline{B}_2 = 0$, then $U \in \mathcal{C}_{d_1,n_2-d_2}(k)$ and $d_1 + n_2 - d_2 \leq k$. Let $\{a_1, \ldots, a_r\}$ be a basis of $\tilde{A}$ and $\{b_1, \ldots, b_s\}$ one of B_1. If $\tilde{A} \neq 0$ and $B \neq 0$, then

$$U(\lambda) = A \oplus \langle a_1, \ldots, a_r, \lambda a_1 + b_1, b_2, \ldots, b_s \rangle_k \oplus B' \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{n}, k))$$

for $\lambda \in k$. Moreover, $U(0) = U$ and $\mathrm{rk}(U(\lambda)) = \mathrm{rk}(U) + (1, 0)$ for each $\lambda \in k^*$. Thus $U \in \overline{\mathcal{C}_{\mathrm{rk}_\alpha(U)+1,\overline{\mathrm{rk}}_\alpha(U)}(k)}$. And if $\tilde{A} \neq 0$ and $\overline{B}_2 \neq 0$, then $U \in \overline{\mathcal{C}_{\mathrm{rk}_\alpha(U),\overline{\mathrm{rk}}_\alpha(U)+1}(k)}$.

This calculation shows the following. If $n_2 - d_2 + d_1 \leq k$, then for all $U \in \mathcal{C}_{d_1,n_2-d_2}(k)$ holds

$$\dim \mathcal{C}_{d_1,n_2-d_2}(k) = d_1(n_1 - d_1) + (d_2 - d_1)(n_2 - d_2) = \dim_k T_U(Gr_{\mathbf{d}}(M(\mathbf{d}, k))).$$

Otherwise by Corollary 3.1.8 all points of $\mathcal{C}_{r,k-r}(k)$ are smooth in $Gr_{\mathbf{d}}(M(\mathbf{d}, k))$ for $r \in \mathcal{M}_{\mathbf{d}}(\mathbf{n}, k)$. $\square$

3.6.2 Quiver of type A_3

Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. For $i \in \{1, 2, 3, 4\}$ let

$$M(i) = (M_1(i), M_2(i), M_3(i), M_\alpha(i), M_\beta(i))$$

be the Q-representation such that $M_1(i) = M_2(i) = M_3(i) = k$ with a basis $\{e_1, e_2, e_3\}$ and $e_j \in M_j(i)$ for all $j \in Q_0$, $M_\alpha(1) = M_\beta(1) = M_\alpha(2) = M_\beta(3) = \mathrm{id}_k$ and $M_\beta(2) = M_\alpha(3) = M_\alpha(4) = M_\beta(4) = 0$. Thus these Q-representations are described by the pictures in Figure 3.6.2. Let $\mathbf{n} = (n, n, n)$ be a dimension vector. Thus each Q-representation with dimension vector $\mathbf{n}$ is isomorphic to some $M(\mathbf{l}) = M(1)^{l_1} \oplus M(2)^{l_2} \oplus M(3)^{l_3} \oplus M(4)^{l_4}$ with $\mathbf{l} \in \mathbb{N}^4$ and $l_1 + l_2 + l_3 + l_4 = n$.

For the dimension vector $\mathbf{d} = (d, d, d)$ and each $\mathbf{k} \in \mathbb{N}^4$ we decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ into irreducible components. First we consider one important example.

Example 3.6.2. Let $\{e_{j1}, e_{j2} | j \in Q_0\}$ be a basis of $M = M(1) \oplus M(4)$ such that $e_{j1} \in M_j(1)$ and $e_{j2} \in M_j(4)$ for all $j \in Q_0$. Let $U = \langle e_{11} + e_{12} \rangle_Q$, $V = \langle e_{12}, e_{21} + e_{22} \rangle_Q$

$$M(1)=\begin{pmatrix} e_1\\ \downarrow\alpha \\ e_2 \\ \downarrow\beta \\ e_3\end{pmatrix},\ M(2)=\begin{pmatrix} e_1\\ \downarrow\alpha \\ e_2 \\ \\ e_3\end{pmatrix},\ M(3)=\begin{pmatrix} e_1\\ \\ e_2 \\ \downarrow\beta \\ e_3\end{pmatrix},\ M(4)=\begin{pmatrix} e_1\\ \\ e_2 \\ \\ e_3\end{pmatrix}$$

Figure 3.6.2: The Q-representations $M(i)$ with $i \in \{1,2,3,4\}$.

$$M=\begin{pmatrix} e_{11} & e_{12}\\ \downarrow\alpha & \\ e_{21} & e_{22} \\ \downarrow\beta & \\ e_{31} & e_{32}\end{pmatrix}, U=\begin{pmatrix} e_{11}+e_{12}\\ \downarrow\alpha \\ e_{21} \\ \downarrow\beta \\ e_{31}\end{pmatrix},\ V=\begin{pmatrix} e_{12}\\ \\ e_{21}+e_{22} \\ \downarrow\beta \\ e_{31}\end{pmatrix},\ W=\begin{pmatrix} e_{12}\\ \\ e_{22} \\ \\ e_{31}+e_{32}\end{pmatrix}$$

Figure 3.6.3: The Q-representations M, U, V and W.

and $W = \langle e_{12}, e_{22}, e_{31} + e_{32}\rangle_Q$ in $\mathrm{Gr}_{(1,1,1)}(M)$. These Q-representations are described by the picture in Figure 3.6.3.

Then holds $\mathrm{Gr}_{(1,1,1)} = \mathcal{C}_U(k) \cup \mathcal{C}_V(k) \cup \mathcal{C}_W(k)$ and the irreducible components are $\overline{\mathcal{C}_U(k)}$, $\overline{\mathcal{C}_V(k)}$ and $\overline{\mathcal{C}_W(k)}$. Moreover, the set $\mathcal{C}_U(k)$ is open, the set $\mathcal{C}_V(k)$ is locally closed and $\mathcal{C}_W(k)$ is closed. Since $\dim_k \mathrm{Hom}_Q(U, M/U) = \dim_k \mathrm{Hom}_Q(V, M/V) = \dim_k \mathrm{Hom}_Q(W, M/W) = 1$ the smooth part of $\mathrm{Gr}_{\mathbf{d}}(M)$ is $\mathcal{C}_{U,M/U}(k) \cup \mathcal{C}_{V,M/V}(k) \cup \mathcal{C}_{W,M/W}(k)$.

Proposition 3.6.3. *Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, $\mathbf{d} = (d,d,d)$ a dimension vector and $\mathbf{k} \in \mathbb{N}^4$. Let $\mathrm{rk}\colon \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) \to \mathbb{Z}^4, U \mapsto \mathbf{l}$ with $U \cong M(\mathbf{l})$. Define a subset of the image $\mathcal{N}_d(\mathbf{k})$ of this map* rk *by*

$$\begin{aligned}\mathcal{M}_d(\mathbf{k}) = \big\{\mathbf{l} \in \mathbb{N}^4 \big| &l_1 + l_2 + l_3 + l_4 = d, l_1 \le k_1, l_2 \le k_2,\\ &l_1 + l_3 \le k_1 + k_3, l_3 + l_4 \le k_3 + k_4, l_4 \le k_4\big\}\end{aligned}$$

and the tuple

$$\mathbf{l}^\star = \begin{cases} \mathbf{k} - \mathbf{l} & \text{if } l_3 \le k_3,\\ (k_1 - l_1 + k_3 - l_3, k_2 - l_2 - k_3 + l_3, 0, k_3 - l_3 + k_4 - l_4) & \text{if } l_3 > k_3\end{cases}$$

for each $\mathbf{l} \in \mathcal{M}_d(\mathbf{k})$. Then the decomposition of $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ into irreducible components is given by

$$\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \bigcup\nolimits_{\mathbf{l}\in\mathcal{M}_d(\mathbf{k})} \overline{\mathcal{C}_{M(\mathbf{l}),M(\mathbf{l}^\star)}(k)}.$$

For $\mathbf{l} \in \mathcal{M}_d(\mathbf{k})$ each point in $\mathcal{C}_{M(\mathbf{l}),M(\mathbf{l}^\star)}(k)$ is smooth in $Gr_{\mathbf{d}}(M(\mathbf{k}))$ if and only if $l_3 \le k_3$ or $l_2 = 0$.

Thus the subset of smooth points is dense if and only if $d \in \{0,1,n\}$ or $k_1k_2k_4 = 0$.

Proof. For all $\mathbf{l} \in \mathcal{M}_d(\mathbf{k})$ the tuple $(\mathbf{l},\mathbf{l}^\star)$ is maximal in the order induced by degenerations. Let $U \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$. We consider an injective homomorphism $f\colon M(\mathbf{l}) \to M(\mathbf{k})$ of Q-representations such that $U = \operatorname{Im} f$. For example instead of the injective homomorphism $M(4) \to M(2) \oplus M(3)$ we can consider without loss of generality the injective homomorphism $M(2) \to M(2) \oplus M(3)$. Thus without loss of generality we can assume $l_1 \le k_1$, $l_2 \le k_2$, $l_4 \le k_4$ and that the homomorphism f is given by a matrix of the form

$$\left(\begin{array}{c|c|c|c} I_{l_1} & 0 & 0 & 0 \\ 0 & 0 & f_1 & 0 \\ \hline 0 & I_{l_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_2 & 0 \\ \hline 0 & 0 & 0 & I_{l_4} \\ 0 & 0 & f_3 & 0 \end{array}\right)$$

with some smaller homomorphisms $f_1\colon M(3)^{l_3} \to M(1)^{k_1-l_1}$, $f_2\colon M(3)^{l_3} \to M(3)^{k_3}$ and $f_3\colon M(3)^{l_3} \to M(4)^{k_4-l_4}$. Moreover, if $l_3 \le k_3$ we can assume that f_2 is given by the matrix $\left(\begin{smallmatrix} I_{l_3} \\ 0 \end{smallmatrix}\right)$, $f_1 = 0$ and $f_3 = 0$, otherwise we assume $f_2 = \left(\begin{smallmatrix} I_{k_3} & 0 \end{smallmatrix}\right)$, $f_1 = \left(\begin{smallmatrix} 0 & I_{l_3-k_3}g_1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f_3 = \left(\begin{smallmatrix} 0 & I_{l_3-k_3}g_2 \\ 0 & 0 \end{smallmatrix}\right)$ such that $g_1\colon M(3) \to M(1)$ and $g_2\colon M(3) \to M(4)$ are homomorphisms of Q-representations with maximal image. Thus there is some exact sequence $0 \to M(\mathbf{l}) \xrightarrow{f} M(\mathbf{k}) \to M(\mathbf{l}^\star) \to 0$ with $\mathbf{l}^\star$ defined in the proposition.

By Proposition 2.3.12 and Theorem 3.1.1 holds for $U \in \mathcal{C}_{M(\mathbf{l}),M(\mathbf{l}^\star)}(k)$

$$\begin{aligned} &\dim_k T_U(Gr_{\mathbf{d}}(M(d,\mathbf{k}))) - \dim \mathcal{C}_{M(\mathbf{l}),M(\mathbf{l}^\star)}(k) \\ =&\dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(\mathbf{l}^\star)\big) - \big(\dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(\mathbf{k})\big) - \dim_k \mathrm{End}_Q\big(M(\mathbf{l})\big)\big) \\ =&\dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(\mathbf{l}^\star + \mathbf{l})\big) - \dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(\mathbf{k})\big). \end{aligned}$$

This is zero if $l_3 \le k_3$ (see also Corollary 3.1.8) and otherwise this equals

$$\begin{aligned} &(l_3 - k_3)\big(\dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(2) \oplus M(3)\big) - \dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(1) \oplus M(4)\big)\big) \\ =&l_2(l_3 - k_3). \end{aligned}$$

This yields the proposition. □

3.6.3 The one-loop-quiver

Let $Q = (\{\circ\}, \{\alpha\})$ be the one-loop-quiver. This is described by the picture in Figure 3.6.4.

A Q-representation $M = (M_\circ, M_\alpha)$ is a finite-dimensional vector space $M_\circ$ together with an endomorphism M_α of $M_\circ$. By the Jordan decomposition of M we decompose the abelian category $\mathrm{rep}(Q)$ of finite-dimensional Q-representations into direct summands.

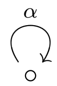


Figure 3.6.4: The one-loop-quiver Q.

Each direct summand is equivalent to the abelian category $\mathrm{nil}(Q)$ of finite-dimensional nilpotent Q-representations. Thus it is enough to consider nilpotent Q-representations.

Let $n \in \mathbb{N}$. A *partition* $\mathbf{n} = (n_1, \ldots, n_r)$ of n is a tuple of integers such that $r \in \mathbb{N}$, $n_1 \geq n_2 \geq \ldots \geq n_r \geq 0$ and $\sum_{i=1}^{r} n_i = n$. We denote such a partition by $\mathbf{n} \vdash n$. In literature often $n_r > 0$ is required, but we allow also $n_r = 0$ for simplicity. To compensate this we identify each partition $(n_1, \ldots, n_r)$ with the partition $(n_1, \ldots, n_r, 0)$. Thus we can assume for each partition without loss of generality that the last entry vanishes.

For $l \in \mathbb{N}$ we define an indecomposable nilpotent Q-representation $M = (M_\circ, M_\alpha)$ by $M_\circ = k^l$ and M_α to be the nilpotent Jordan block of size l. Following this we define for each partition $\mathbf{l} = (l_1, \ldots, l_r)$ a Q-representation as a direct sum of the Q-representation associated to the natural numbers $l_1, \ldots, l_r$. This Q-representation is unique up to isomorphism. Moreover, for $d \in \mathbb{N}$ the isomorphism classes of finite-dimensional nilpotent Q-representations of dimension d can be canonically parametrized by partitions $\mathbf{l}$ of d.

Using this we identify each partition with the associated Q-representation. Let $\mathbf{l}$ and $\mathbf{m}$ be partitions. Then $\mathbf{l} \oplus \mathbf{m}$ is well-defined as a direct sum of Q-representations. And the number of indecomposable direct summands of the Q-representation $\mathbf{l}$ is called the *length* of the partition $\mathbf{l}$, denoted by $l(\mathbf{l})$, i.e. $l(\mathbf{l}) = \max(i | l_i > 0)$ for a partition $\mathbf{l} = (l_1, \ldots, l_r)$.

Now we introduce some notions of tuples of integers. For $d \in \mathbb{Z}$, $r \in \mathbb{N}$ and $\mathbf{d}, \mathbf{n} \in \mathbb{Z}^r$ we define $r\mathbf{d}$ and $\mathbf{d} + \mathbf{n}$ componentwise. Since all partitions are tuples of integers we can use this also for partitions with the same number of integers. For $d, n \in \mathbb{Z}$ with $d \leq n$ let $[d, n] = \{d, d+1, \ldots, n-1, n\}$. For a subset $I \subseteq \mathbb{Z}$ and $d \in \mathbb{Z}$ let $I + d = \{i + d | i \in I\}$. And for a partition $\mathbf{l} = (l_1, \ldots, l_r)$, $d \in [1, r]$ and $I = \{i_1 < i_2 < \ldots < i_d\} \subseteq [1, r]$ we define the partition $\mathbf{l}_I = (l_{i_1}, l_{i_2}, \ldots, l_{i_d})$.

Example 3.6.4. The partitions of 7 with length 3 are $(5, 1, 1)$, $(4, 2, 1)$, $(3, 3, 1)$ and $(3, 2, 2)$. For the direct sum holds $(5, 4, 3, 3, 2) \oplus (4, 3, 1, 1) \cong (5, 4, 4, 3, 3, 3, 2, 1, 1)$.

With some tuples we get $3(1, 3) - (7, 0) = (-4, 9)$ and $(1, 2, 1, 1) - (4, 4, 3, 2) + 2(4, 3, 2, 1) = (5, 4, 2, 1)$. For the subset $I = \{-7, 0, 2, 3, 4\}$ of $\mathbb{Z}$ we get $I + 5 = \{-2, 5, 7, 8, 9\}$. Moreover, $(8, 5, 4, 2)_{\{2<4\}} = (5, 2)$, $(8, 5, 4, 2)_{\{2<4\}-1} = (8, 4)$ and finally $(4, 4, 4, 4, 3, 3, 3, 2, 2, 1)_{\{3<4<5<10\}} = (4, 4, 3, 1)$.

Let $\mathbf{n}$ be a partition and $d \in \mathbb{N}$. The aim of this section is to decompose the variety $\mathrm{Gr}_d(\mathbf{n})$ into irreducible components. For this we define subvarieties. For $i \in \mathbb{N}$ let

$$\mathrm{Gr}_d^i(\mathbf{n}) := \{U \in \mathrm{Gr}_d(\mathbf{n}) | U \cong (k_1, \ldots, k_i)\}.$$

We will see this is a closed subvariety of $\mathrm{Gr}_d(\mathbf{n})$. Moreover, $\mathrm{Gr}_d^i(\mathbf{n}) \subseteq \mathrm{Gr}_d^{i+1}(\mathbf{n})$ for each $i \in \mathbb{N}$ and $\mathrm{Gr}_d(\mathbf{n}) = \mathrm{Gr}_d^{l(\mathbf{n})}(\mathbf{n})$.

We define a partial order on the partitions. Let $d \in \mathbb{N}$ and $\mathbf{l} = (l_1, \ldots, l_r)$ and $\mathbf{m} = (m_1, \ldots, m_s)$ be partitions of d. Then let $\mathbf{l} \leq \mathbf{m}$ if and only if $\sum_{i=1}^{k} l_i \leq \sum_{j=1}^{k} m_j$ for each $k \in \mathbb{N}$ with $l_i = 0$ for $i > r$ and $m_j = 0$ for $j > s$. For $d \in \mathbb{N}$ it is well-known that this order of partitions of d is equivalent to the degeneration order induced by the variety $\operatorname{rep}_d(Q, \alpha^d)$.

Proposition 3.6.5. *Let Q be the one-loop-quiver, $d \in \mathbb{N}$ and $\mathbf{l} = (l_1, \ldots, l_r), \mathbf{m} = (m_1, \ldots, m_s) \vdash d$. Then $\mathcal{O}(\mathbf{l}) \subseteq \overline{\mathcal{O}(\mathbf{m})}$ in $\operatorname{rep}_d(Q)$ if and only if $\mathbf{l} \leq \mathbf{m}$.*

For the quiver Grassmannian $\operatorname{Gr}_d(\mathbf{n})$ we get the following decomposition into irreducible components.

Proposition 3.6.6. *Let Q be the one-loop-quiver, $d, n \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_r) \vdash n$ with $n_r = 0$. Let $\operatorname{rk}\colon \operatorname{Gr}_d(\mathbf{n}) \to \{\mathbf{l} | \mathbf{l} \vdash d\}, U \mapsto \mathbf{l}$ with $U \cong \mathbf{l}$. Define a subset of the image $\mathcal{N}_d(\mathbf{n})$ of this map rk by*

$$\begin{aligned} \mathcal{M}_d(\mathbf{n}) = \Big\{\mathbf{l} \Big| \mathbf{l} \vdash d, \exists f\colon [1, l(\mathbf{l})] \to [1, r] \text{ injective}, I \dot\cup J = [1, l(\mathbf{l})] : \\ \mathbf{l}_I = \mathbf{n}_{f(I)}, f(j) \neq 1, f(j) - 1 \notin \operatorname{Im} f, n_{f(j)} < l_j < n_{f(j)-1} \forall j \in J\Big\}. \end{aligned}$$

and a partition $\mathbf{l}^\star$ of $n - d$ by $\mathbf{n}_{[1,r] \setminus (\operatorname{Im} f \cup (f(J)-1))} \oplus (\mathbf{n}_{f(J)} + \mathbf{n}_{f(J)-1} - \mathbf{l}_J)$ for each $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ with f and J as above. This is well-defined and the decomposition of $\operatorname{Gr}_d(\mathbf{n})$ into irreducible components is given by

$$\operatorname{Gr}_d(\mathbf{n}) = \bigcup\nolimits_{\mathbf{l} \in \mathcal{M}_d(\mathbf{n})} \overline{\mathcal{C}_{\mathbf{l}, \mathbf{l}^\star}(k)}. \tag{3.6.3}$$

For $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ each point in $\mathcal{C}_{\mathbf{l},\mathbf{l}^\star}(k)$ is smooth in $Gr_d(\mathbf{n})$ if and only if $\mathbf{l} \oplus \mathbf{l}^\star \cong \mathbf{n}$.

Thus, for example the subset of smooth points is dense if $\{j \in \mathbb{N} | 1 \leq j \leq \min\{n_1, d\}\}$ is contained in $\{n_i | i\}$.

Proof. In Lemma 3.6.9 we show Equation (3.6.3). Now we prove $(\mathbf{l}, \mathbf{l}^\star)$ is maximal for all $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$. Let $\mathbf{l}, \mathbf{m} \in \mathcal{M}_d(\mathbf{n})$ with $\mathbf{l} < \mathbf{m}$.

Let $f\colon [1, l(\mathbf{l})] \to [1, r]$ and $g\colon [1, l(\mathbf{m})] \to [1, r]$ be the corresponding maps with $I \dot\cup J = [1, l(\mathbf{l})]$ and $I' \dot\cup J' = [1, l(\mathbf{m})]$. Let $t \in [1, l(\mathbf{l})]$ minimal such that $l_t < m_t$. Then $f(t) \geq g(t)$ and without loss of generality $f(i) = g(i)$ for each $i \in [1, t-1]$. Thus without loss of generality we assume that $t = 1$.

If $1 \in I'$, we assume without loss of generality $g(1) = 1$. Let $n_1 = \ldots = n_s = m_1$ for the maximal $s \in [1, r]$. Since $l_1 < m_1 = n_s$, we get $f(1) > s$. Thus $m_i^\star \leq n_{i+1} = m_1 = l_i^\star$ for all $i \in [1, s-1]$ and $m_s^\star \leq n_{s+1} < l_s^\star$.

If $1 \in J'$, we assume without loss of generality $g(1) = 2$. Then $n_2 < m_1 < n_1$. Thus if $l_1 > n_2$ we get $m_1^\star = n_1 + n_2 - m_1 < n_1 + n_2 - l_1 = l_1^\star$ and $m_1^\star = n_1 + n_2 - m_1 < n_1 = l_1^\star$ otherwise.

By Proposition 3.6.6 a point $U \in \mathcal{C}_{\mathbf{l},\mathbf{l}^\star}(k)$ is smooth in the scheme $Gr_d(\mathbf{n})$ if and only if $\dim_k T_U(Gr_d(\mathbf{n})) = \dim \mathcal{C}_{\mathbf{l}}(k)$. Using Proposition 2.3.12 and Theorem 3.1.1 we consider

$$\begin{aligned} &\dim_k T_U(Gr_d(\mathbf{n})) - \dim \mathcal{C}_{\mathbf{l}}(k) \\ =& \dim_k \operatorname{Hom}_Q(\mathbf{l}, \mathbf{l}^\star) - (\dim_k \operatorname{Hom}_Q(\mathbf{l}, \mathbf{n}) - \dim_k \operatorname{End}_Q(\mathbf{l})) \end{aligned}$$

$$
\begin{aligned}
&= \dim_k \operatorname{Hom}_Q(\mathbf{l}, (\mathbf{n}_{f(J)} + \mathbf{n}_{f(J)-1} - \mathbf{l}_J) \oplus \mathbf{l}_J) - \dim_k \operatorname{Hom}_Q(\mathbf{l}, \mathbf{n}_{f(J)} \oplus \mathbf{n}_{f(J)-1}) \\
&= \sum_{i=1}^{l(\mathbf{l})} \sum_{j \in J} \underbrace{\dim_k \operatorname{Hom}_Q(l_i, (n_{f(j)} + n_{f(j)-1} - l_j, l_j)) - \dim_k \operatorname{Hom}_Q(l_i, (n_{f(j)-1}, n_{f(j)}))}_{=0, \text{ if } i \neq j} \\
&= \sum_{j \in J} (\min(l_j, n_{f(j)} + n_{f(j)-1} - l_j) + l_j) - (l_j + n_{f(j)}) \\
&= \sum_{j \in J} \min(l_j - n_{f(j)}, n_{f(j)-1} - l_j)
\end{aligned}
$$

with some $f\colon [1, l(\mathbf{l})] \to [1, r]$ such that $\mathbf{l}^\star \cong \mathbf{n}_{[1,r]\setminus(\operatorname{Im} f \cup (f(J)-1))} \oplus (\mathbf{n}_{f(J)} + \mathbf{n}_{f(J)-1} - \mathbf{l}_J)$. Thus $n_1|J| \geq \dim_k T_U(Gr_d(\mathbf{n})) - \dim_U Gr_d(\mathbf{n}) \geq |J|$. This yields the proposition. □

Let $\mathbf{n} = (n_1, \ldots, n_r)$ be a partition. Then $\{e_{ij} | 1 \leq i \leq r, 1 \leq j \leq n_i\}$ is a vector space basis of the Q-representation $\mathbf{n}$ described by the picture in Figure 3.6.5.

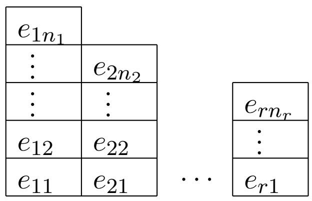


Figure 3.6.5: Basis of the Q-representation $\mathbf{n}$.

Using this basis we can describe the endomorphism ring $\operatorname{End}_Q(\mathbf{n})$ for a partition $\mathbf{n} = (n_1, \ldots, n_r)$. It is easy to see that

$$
\begin{aligned}
\operatorname{End}_Q((n)) &\xrightarrow{\sim} k[T]/T^n, (e_{1n} \mapsto e_{1k}) \mapsto T^{n-k}, \\
\operatorname{Aut}_Q((n)) &\xrightarrow{\sim} (k[T]/T^n)^* = \{p \in k[T]/T^n | p(0) \neq 0\}
\end{aligned}
$$

It is well-known that $\prod_{i=1}^n \operatorname{Aut}_Q((n_i))$ is a subgroup of $\operatorname{Aut}_Q((\mathbf{n}))$. Let $1 \leq i_0 \leq r$ and $g_i \in \operatorname{Hom}_Q((n_{i_0}), (n_i))$ for $i \neq i_0$. Then these defines a $g \in \operatorname{Aut}_Q((\mathbf{n}))$ by $g(e_{i_0 j_0}) = e_{i_0 j_0} + \sum_{i=1, i \neq i_0}^r g_i(e_{i_0 j_0})$ and $g(e_{ij}) = e_{ij}$ for each $1 \leq j_0 \leq n_{i_0}$, $i \neq i_0$ and $1 \leq j \leq n_i$.

Lemma 3.6.7. *Let Q be the one-loop-quiver, $d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_r)$ a partition with $d \leq n_1$ and $n_r = 0$. Then*

$$
\operatorname{Gr}_d^1(\mathbf{n}) = \begin{cases} \overline{\mathcal{C}_{(d),(n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_r)}(k)} & \text{if } \exists i \in [1,r] : n_i = d, \\ \overline{\mathcal{C}_{(d),(n_1,\ldots,n_{i-2},n_{i-1}+n_i-d,n_{i+1},\ldots,n_r)}(k)} & \text{if } \exists i \in [1,r] : n_i < d < n_{i-1}. \end{cases}
$$

Proof. Let $U \in \operatorname{Gr}_d^1(\mathbf{n})$. Then

$$
U = \left\langle v = \sum\nolimits_{i=1}^r \sum\nolimits_{j=1}^{n_i} \lambda_{ij} e_{ij} \right\rangle_Q
$$

with $\lambda_{ij} \in k$ and $d \leq n_1$.

- If $d \leq n_{i_0}$ for some $1 \leq i_0 \leq r$, then define $U_\lambda = \langle v + \lambda e_{i_0 d} \rangle_Q \in \mathrm{Gr}_d(\mathbf{n})$ for each $\lambda \in k$. Thus $U_0 = U$ and by some automorphism $g \in \mathrm{Aut}_Q((n_{i_0}))$ we get $gU_\lambda = \langle e_{i_0 d} + \sum_{i=1, i \neq i_0}^{r} \sum_{j=1}^{n_i} \lambda_{ij} e_{ij} \rangle_Q$ for each $\lambda \neq -\lambda_{i_0 d}$. Using again some automorphism $g' \in \mathrm{Aut}_Q((\mathbf{n}))$ defined as above we get $g'gU_\lambda = \langle e_{i_0 d} + \sum_{i=1, i > i_0}^{r} \sum_{j=1}^{n_i} \lambda_{ij} e_{ij} \rangle_Q$.
- If $d \geq n_{i_0}$ for some $1 \leq i_0 \leq r$, then define $U_\lambda = \langle v + \lambda e_{i_0 n_{i_0}} \rangle_Q \in \mathrm{Gr}_d(\mathbf{n})$ for each $\lambda \in k$. Thus $U_0 = U$ and by some $g \in \mathrm{Aut}_Q((n_{i_0}))$ we get $gU_\lambda = \langle e_{i_0 n_{i_0}} + \sum_{i=1, i \neq i_0}^{r} \sum_{j=1}^{n_i} \lambda_{ij} e_{ij} \rangle_Q$ for each $\lambda \neq -\lambda_{i_0 n_{i_0}}$. Again holds $g'gU_\lambda = \langle e_{i_0 n_{i_0}} + \sum_{i=1, i < i_0}^{r} \sum_{j=1}^{n_i} \lambda_{ij} e_{ij} \rangle_Q$ with some g'.

Using the proof of Proposition 3.6.6 this yields the lemma. □

Lemma 3.6.8. *Let Q be the one-loop-quiver, $d \in \mathbb{N}$ and $\mathbf{n} = (n_1, \ldots, n_r)$ a partition with $n_r = 0$. Then*

$$\mathrm{Gr}_d^2(\mathbf{n}) = \bigcup\nolimits_{\mathbf{l} \in \mathcal{M}_d(\mathbf{n}),\ l(\mathbf{l}) \leq 2} \overline{\mathcal{C}_{\mathbf{l}, \mathbf{l}^\star}(k)}.$$

Proof. Let $U \in \mathrm{Gr}_d^2(\mathbf{n}) \cap \mathcal{C}_{\mathbf{l}, \mathbf{m}}(k)$. Then

$$U = \left\langle v_1 = \sum\nolimits_{i=1}^{r} \sum\nolimits_{j=1}^{n_i} \lambda_{ij} e_{ij}, v_2 = \sum\nolimits_{i=1}^{r} \sum\nolimits_{j=1}^{n_i} \mu_{ij} e_{ij} \right\rangle_Q$$

with $\lambda_{ij}, \mu_{ij} \in k$. By the proof of Lemma 3.6.7 we can assume that one of the following cases holds. Some of these are illustrated in Figure 3.6.6 and 3.6.7.

1. If $U \in \mathrm{Gr}_d^1(\mathbf{n})$, we are done by Lemma 3.6.7.
2. If there is some $i \in [1, r]$ with $n_i = l_1$ and $n_{i+1} < l_2$, then
$$U = \langle e_{i n_i}, e_{i-1, l_2} + e_{i+1, n_{i+1}} \rangle_Q.$$
If $n_{i-1} = n_i$, then $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ and
$$\mathbf{m} = (n_1, \ldots, n_{i-2}, n_{i-1} + n_{i+1} - l_2, n_{i+2}, \ldots, n_r) = \mathbf{l}^\star.$$
3. If there are some $i, j \in [1, r]$ with $i < j$, $n_i = l_1$ and $n_j = l_2$, then $U = \langle e_{i n_i}, e_{j n_j} \rangle_Q$. Thus $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ and $\mathbf{m} = \mathbf{n} - \mathbf{l} = \mathbf{l}^\star$.
4. If there are some $i, j \in [1, r]$ with $i < j - 1$, $n_i = l_1$ and $n_j < l_2 < n_{j-1}$, then $U = \langle e_{i n_i}, e_{j-1, l_2} + e_{j n_j} \rangle_Q$. Thus $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ and
$$\mathbf{m} = (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_{j-2}, n_{j-1} + n_j - l_2, n_{j+1} \ldots, n_r) = \mathbf{l}^\star.$$
5. If $n_2 < l_1 < n_1$ and $n_3 < l_2$, then
$$U = \langle e_{1 l_1} + e_{2 n_2}, e_{2 l_2} + x + e_{3, n_3} \rangle_Q$$
with $x = \sum_{j=1}^{l_2 - 1} \mu_j e_{2j}$ and $\mu_j \in k$ for all $j \in [1, l_2 - 1]$.
6. If there is some $i \in [3, r]$ with $n_i < l_1 < n_{i-1}$ and $n_{i+1} < l_2$, then
$$U = \left\langle e_{i-1, l_1} + e_{i n_i}, e_{i-2, l_2} + \mu_{l_2} e_{i l_2} + x + e_{i+1, n_{i+1}} \right\rangle_Q$$
with $x = \sum_{j=1}^{l_2 - 1} \mu_j e_{ij}$, $\mu_j \in k$ for all $j \in [1, l_2 - 1]$ and $\mu_j = 0$ for all $j \in \mathbb{N}$ with $j > n_i$.

7. If there are some $i, j \in [2, r]$ with $i < j$, $n_i < l_1 < n_{i-1}$ and $n_j = l_2$, then $U = \langle e_{i-1,l_1} + e_{in_i}, e_{jn_j} \rangle_Q$. Thus $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ and

$$\mathbf{m} = (n_1, \ldots, n_{i-2}, n_{i-1} + n_i - l_1, n_{i+1}, \ldots, n_{j-1}, n_{j+1} \ldots, n_r) = \mathbf{l}^{\star}.$$

8. If there are some $i, j \in [2, r]$ with $i < j - 1$, $n_i < l_1 < n_{i-1}$ and $n_j < l_2 < n_{j-1}$, then $U = \left\langle e_{i-1,l_1} + e_{in_i}, e_{j-1,l_2} + e_{jn_j} \right\rangle_Q$. Thus $\mathbf{l} \in \mathcal{M}_d(\mathbf{n})$ and

$$\mathbf{m} = (n_1, \ldots, n_{i-2}, n_{i-1} + n_i - l_1, n_{i+1}, \ldots, n_{j-2}, n_{j-1} + n_j - l_2, n_{j+1} \ldots, n_r) = \mathbf{l}^{\star}.$$

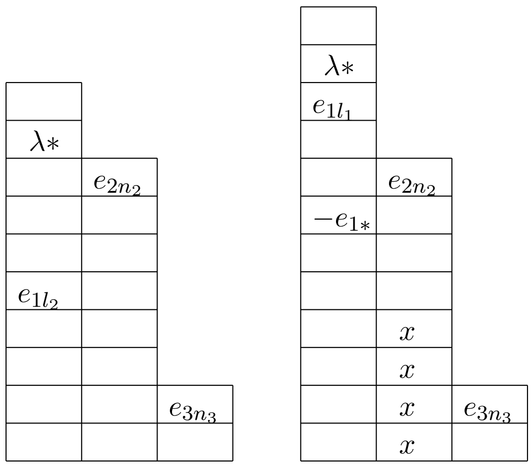


Figure 3.6.6: subrepresentations U_λ occurring in Case 2 with $i = 2$ and in Case 5.

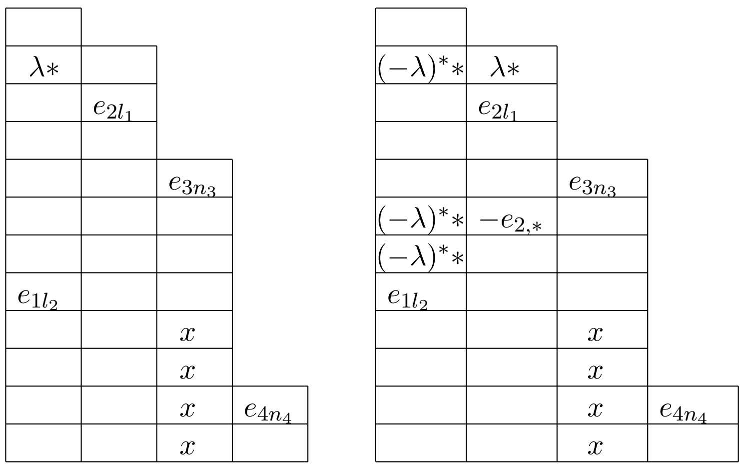


Figure 3.6.7: subrepresentations U_λ occurring in Case 6 with $i = 3$ such that $\mu_{l_2} = 0$ or $\mu_{l_2} = 1$.

By considering these cases only Cases 2, 5 and 6 are left.

- If we are in Case 2 with $n_{i-1} > n_i$, then define $U_\lambda = \langle v_1, v_2\rangle_Q$ for each $\lambda \in k$ with

$$v_1 = \lambda e_{i-1,l_1+1} + e_{in_i},$$
$$v_2 = e_{i-1,l_2} + e_{i+1,n_{i+1}}.$$

Thus $U_0 = U$ and $\dim_k U_\lambda \leq d$, since $\alpha^{l_2-1}v_2 = e_{i-1,1} = \lambda^{-1}\alpha^{l_1}v_1$ for all $\lambda \in k^*$. Using that rk is the lower semicontinuous, we get $U_\lambda \in \mathrm{Gr}_d(\mathbf{n})$ for most $\lambda \in k$ and $U_\lambda \cong (l_1+1, l_2-1) > \mathbf{l}$ for $\lambda \in k^*$.
- If we are in Case 5, then define $U_\lambda = \langle v_1, v_2\rangle_Q$ for each $\lambda \in k$ with

$$v_1 = \lambda e_{1,l_1+1} + e_{1l_1} + e_{2n_2},$$
$$v_2 = -e_{1,l_1+l_2-n_2} + x + e_{3n_3}.$$

Thus $U_0 = U$, $U_\lambda \in \mathrm{Gr}_d(\mathbf{n})$ for most $\lambda \in k$ and $U_\lambda \cong (l_1+1, l_2-1) > \mathbf{l}$ for $\lambda \neq 0$, since $-(\lambda+\alpha)\alpha^{l_2-1}v_2 = \lambda e_{1,l_1-n_2+1} + e_{1,l_1-n_2} = \alpha^{n_2}v_1$ and $n_2 + (l_2+l_1-n_2) = d$.
- Now assume we are in Case 6. If $\mu_{l_2} = 0$, then we can treat this case very similar to Case 2. Otherwise if $\mu_{l_2} \neq 0$, then we use Case 5. In detail we get the following. Define $U_\lambda = \langle v_1, v_2\rangle_Q$ for each $\lambda \in k$ with

$$v_1 = \mu_{l_2}^{-1}(-\lambda)^{l_1-n_i+1}e_{i-2,l_1+1} + \lambda e_{i-1,l_1+1} + e_{i-1,l_1} + e_{in_i},$$
$$v_2 = \sum\nolimits_{j=0}^{l_1-n_i}(-\lambda)^j e_{i-2,l_2+j} - \mu_{l_2}e_{i-1,l_1+l_2-n_i} + x + e_{i+1,n_{i+1}}.$$

Thus $U_0 = U$, $U_\lambda \in \mathrm{Gr}_d(\mathbf{n})$ for most $\lambda \in k$ and again $U_\lambda \cong (l_1+1, l_2-1) > \mathbf{l}$ for $\lambda \neq 0$, since again $-\mu_{l_2}^{-1}(\lambda+\alpha)\alpha^{l_2-1}v_2 = \alpha^{n_i}v_1$.

Using again the proof of Proposition 3.6.6 this yields the lemma. □

Lemma 3.6.9. *Let Q be the one-loop-quiver, $d \in \mathbb{N}$ and $\mathbf{n}$ a partition. Then Equation* (3.6.3) *holds.*

This completes the proof of Proposition 3.6.6.

Proof. Let $U \in \mathcal{C}_{\mathbf{l},\mathbf{m}}(k)$. By induction over $s = l(\mathbf{l})$ there is an injective map $f\colon [1, s-1] \to [1, r]$ and $I \dot\cup J = [1, s-1]$ such that

$$U = \left\langle e_{f(i),l_i}, e_{f(j)-1,l_j} + e_{f(j),n_{f(j)}}, v = \sum\nolimits_{t=1}^{r}\lambda_t e_{tl_s} + x \,\middle|\, i \in I, j \in J\right\rangle_Q$$

with $\lambda_t \in k$ for each $t \in [1, r]$, $x = \sum_{i=1}^{r}\sum_{j=1}^{l_s-1}\mu_{ij}e_{ij}$ and $\mu_{ij} \in k$ for $i \in [1, r]$ and $j \in [1, l_s - 1]$. An example of such a representation is illustrated in the picture in Figure 3.6.8. Without loss of generality we assume $\lambda_{f(i)} = 0$ if $i \in I$ and $\lambda_{f(j)-1} = 0$ if $j \in I$. If $n_{f(s-1)+1} \geq l_s$, we can assume without loss of generality $v = e_{f(s-1)+1,l_s} + x$ and we are done.

Thus we assume $v = \sum_{t=1}^{f(s-1)}\lambda_t e_{tl_s} + x$. Let t_0 be maximal in $[1, f(s-1)]$ with $\lambda_{t_0} \neq 0$. Such a t_0 exists since $\langle v\rangle_Q \cong (l_s)$. If $t_0 \notin f(J)$, we can assume without loss of generality $v = \lambda_{t_0}e_{t_0 l_s} + x$. This case can be treated very similar to Case 2 of the proof of Lemma 3.6.8.

	$e_{2,12}$										
		$e_{3,11}$									
			$e_{4,10}$								
				e_{59}	e_{69}						
$\pm\lambda^*\ast$				$\pm\lambda^*\ast$		λe_{78}					
						e_{77}					
							e_{85}				
$\pm\lambda^*\ast$				$\pm\lambda^*\ast$		$-e_{74}$		e_{94}			
$\pm\lambda^*\ast$				$\pm\lambda^*\ast$					$e_{10,3}$	$e_{11,3}$	
$\lambda_1 e_{12}$				$\lambda_5 e_{52}$							
		x		x			x		x		x

Figure 3.6.8: Example for a subrepresentation U_λ occurring in the proof of Lemma 3.6.9 with $U = \langle e_{69}, e_{11,3}, e_{2,12} + e_{3,13}, e_{4,10} + e_{59}, e_{77} + e_{85}, e_{94} + e_{10,3}, \lambda_1 e_{12} + \lambda_5 e_{52} + e_{82} + x\rangle_Q$, $j_0 = 3$ and $t_0 = f(j_0) = 8$.

Thus we assume $t_0 \in f(J)$ and $v = \sum_{t=1}^{t_0-2} \lambda_t e_{tl_s} + e_{t_0 l_s} + x$. Let $j_0 \in J$ with $f(j_0) = t_0$. Then define similar to Case 5 of the proof of Lemma 3.6.8 the subrepresentations

$$U_\lambda = \left\langle e_{f(i),l_i}, e_{f(j)-1,l_j} + e_{f(j),n_{f(j)}}, v_1, v_2 \middle| i \in I, j \in J, j \neq j_0 \right\rangle_Q$$

for each $\lambda \in k$ with

$$v_1 = \sum_{t=1}^{t_0-2} (-\lambda)^{l_{j_0} - n_{t_0} + 1} \lambda_t e_{t,l_{j_0}+1} + \lambda e_{t_0-1,l_{j_0}+1} + e_{t_0-1,l_{j_0}} + e_{t_0 n_{t_0}},$$
$$v_2 = \sum_{t=1}^{t_0-2} \sum_{j=0}^{l_{j_0}-n_{t_0}} (-\lambda)^j \lambda_t e_{t,l_s+j} - e_{t_0-1,l_{j_0}+l_s-n_{t_0}} + x.$$

Thus $U_0 = U$, $U_\lambda \in \mathrm{Gr}_d(\mathbf{n})$ for most $\lambda \in k$ and now $U_\lambda \cong \mathbf{l}_{[1,s-1]\setminus\{t_0\}} \oplus (l_{t_0}+1, l_s-1) > \mathbf{l}$ for $\lambda \neq 0$, since again

$$-(\lambda+\alpha)\alpha^{l_s-1} v_2 = -(\lambda+\alpha)\left(\sum_{t=1}^{t_0-2} \sum_{j=0}^{l_{j_0}-n_{t_0}} (-\lambda)^j \lambda_t e_{t,j+1} - e_{t_0-1,l_{j_0}-n_{t_0}+1}\right)$$
$$= \sum_{t=1}^{t_0-2} (-\lambda)^{l_{j_0}-n_{t_0}+1} \lambda_t e_{t,l_{j_0}-n_{t_0}+1} + \lambda e_{t_0-1,l_{j_0}-n_{t_0}+1} + e_{t_0-1,l_{j_0}-n_{t_0}} = \alpha^{n_{t_0}} v_1.$$

□

3.6.4 The two-cycle-quiver

Example 3.4.4 shows that the converse of Part 3 of Proposition 3.5.5 is not true. This section gives another example. Our strategy for computing all irreducible components of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ fails, since they are in general not parametrized by the maximal elements in $\mathcal{N}_{\mathbf{d}}(M)$. In the following we give an example. Therefore, we do not force to decompose all Grassmannians for all representations of the two-cycle-quiver.

Let $Q = (Q_0, Q_1)$ be the (cyclically oriented) two-cycle-quiver with $Q_0 = \mathbb{Z}/2\mathbb{Z}$ and $Q_1 = \{\alpha_i \colon i \to i+1 | i \in Q_0\}$. This is described by the picture in Figure 3.6.9. We study

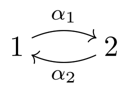


Figure 3.6.9: The cyclically oriented two-cycle-quiver Q.

again the abelian category of finite-dimensional nilpotent Q-representations nil(Q). For $i \in \mathbb{N}_{>0}$ let $M(i)$ be a Q-representation with dimension i and $\operatorname{soc} M(i) \cong S(1)$. This is unique up to isomorphism and has a basis $\{e_{ij} | 1 \le j \le i\}$ with $e_{ij} \in M(i)_j$ and $M(i)_{\alpha_j}(e_{ij}) = e_{i,j-1}$ for all $1 < j \le i$. The pictures in Figure 3.6.10 describe these representations. Let $M = M(4) \oplus M(3) \oplus M(1)$. Then $\{e_{44}, e_{43}, e_{42}, e_{41}, e_{33}, e_{32}, e_{31}, e_{11}\}$

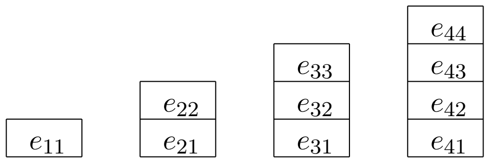


Figure 3.6.10: The Q-representations $M(1)$, $M(2)$, $M(3)$ and $M(4)$.

is a basis of M. Let $U = \langle e_{44}, e_{11} \rangle_Q$ and $U' = \langle e_{33}, e_{42} \rangle_Q$.

Then $U, U' \in \mathrm{Gr}_{(3,2)}(M)$, $U \cong M(4) \oplus M(1)$, $U' \cong M(3) \oplus M(2)$, $M/U \cong M(3)$ and $M/U' \cong M(1) \oplus M(2)$. Thus there are exact sequences $0 \to M(3) \to U \to M(2) \to 0$ and $0 \to M(2) \to M/U \to M(1) \to 0$. Moreover,

$$\mathcal{C}_{U,M/U}(k) = \{\langle e_{44} + \alpha e_{32}, \beta e_{31} + e_{11} \rangle_Q | \alpha, \beta \in k\},$$
$$\mathcal{C}_{U',M/U'}(k) = \{\langle \alpha e_{43} + e_{33} + \beta e_{11}, e_{42} \rangle_Q | \alpha, \beta \in k\}$$

and $\mathrm{Gr}_{(3,2)}(M) = \overline{\mathcal{C}_{U',M/U'}(k)} \cup \overline{\mathcal{C}_{U,M/U}(k)}$, but $U' \notin \overline{\mathcal{C}_{U,M/U}(k)}$, since for all $V \in \mathcal{C}_{U,M/U}(k)$ holds $V \subseteq \langle e_{44}, e_{32}, e_{11} \rangle_Q$ and $e_{33} \in U'$.

Nevertheless, the quiver Grassmannian $\mathrm{Gr}_{(3,2)}(M)$ has two irreducible components the set $\mathcal{N}_{(3,2)}(M)$ contains a unique maximal element. This is induced by the subrepresentation U of M. Thus there is no generalization of Proposition 3.6.6 for this case. Moreover, $\dim \mathcal{C}_{U,M/U} = 2 = \dim_k T_U(Gr_{\mathbf{d}}(M))$ and $\dim \mathcal{C}_{U',M/U'} = 2 < 3 = \dim_k T_{U'}(Gr_{\mathbf{d}}(M))$. Thus the smooth part of $Gr_{(3,2)}(M)$ is $\mathcal{C}_{U,M/U}$.

3.6.5 The N-cycle-quiver

Let $N \in \mathbb{N}$ with $N \ge 2$ and $Q = (Q_0, Q_1)$ the (cyclically oriented) N-cycle-quiver with $Q_0 = \mathbb{Z}/N\mathbb{Z}$ and $Q_1 = \{\alpha_i \colon i \to i+1 | i \in Q_0\}$. This is described by the picture in Figure 3.6.11. Let α^N be the admissible ideal generated by all paths of length N. Then

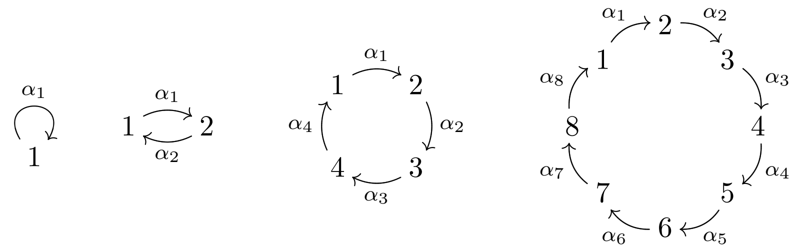


Figure 3.6.11: The cyclically oriented N-cycle-quiver Q for $N \in \{1, 2, 4, 8\}$.

we study the abelian category $\operatorname{rep}(Q, \alpha^N)$ of finite-dimensional (Q, α^N)-representations. For $i \in Q_0$ and $j \in \{1, \ldots, N\}$ let $M(i, j)$ be an indecomposable (Q, α^N)-representation with dimension j, $\operatorname{soc} M(i, j) \cong S(i)$ and $\operatorname{top} M(i, j) \cong S(i - j + 1)$. This is unique up to isomorphism and has a basis $\{e_k | 1 \leq k \leq j\}$ with $e_k \in M(i, j)_{i-k+1}$ for all $1 \leq k \leq j$ and $M(i, j)_{\alpha_{i-k+1}}(e_k) = e_{k-1}$ for all $1 < k \leq j$. Moreover, we set $M(i, 0) = 0$ for all $i \in Q_0$. The pictures in Figure 3.6.12 describe some of these (Q, α^N)-representations.

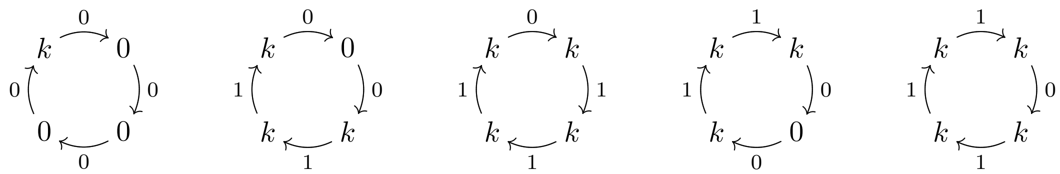


Figure 3.6.12: The indecomposable, nilpotent (Q, α^N)-representations $M(1, 1)$, $M(1, 3)$, $M(1, 4)$, $M(2, 3)$ and $M(2, 4)$ for $N = 4$.

Let M be a (Q, α^N)-representation. Then there exists a unique tuple

$$\mathbf{k} = (k_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1, \ldots, N\}}$$

such that

$$M \cong \bigoplus_{i \in Q_0} \bigoplus_{j=1}^{N} M(i, j)^{k_{ij}}.$$

All (Q, α^N)-representations are of this form and we call this (Q, α^N)-representation $M(\mathbf{k})$. In the following way we can consider $\mathbf{k} = (k_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1, \ldots, N\}}$ as a matrix with N lines, N columns and entries in $\mathbb{N}$:

$$\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & \vdots & & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{NN} \end{pmatrix}.$$

A (Q, α^N)-representation $M(\mathbf{k})$ is projective-injective in the category $\operatorname{rep}(Q, \alpha^N)$ if and only if all $k_{ij} = 0$ for all $i \in Q_0$ and $j \in \{1, \ldots, N - 1\}$, i.e. the only non-vanishing entries in the matrix $\mathbf{k}$ are in the last column.

Example 3.6.10. Let $N = 4$. The matrices for a projective-injective (Q, α^4)-representation $M(\mathbf{k})$ and the (Q, α^4)-representation $M(\mathbf{l}) = M(1,1) \oplus M(1,3) \oplus M(2,3)^3 \oplus M(2,4)^2$ with $\mathbf{k}, \mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\ldots,4\}}$ are the following ones:

$$\begin{pmatrix} 0 & 0 & 0 & k_{14} \\ 0 & 0 & 0 & k_{24} \\ 0 & 0 & 0 & k_{34} \\ 0 & 0 & 0 & k_{44} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 3.6.11. *Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver and $\mathbf{d}$ a dimension vector. For $t \in \mathbb{N}$ define*

$$\mathcal{K}_t^p = \Big\{ \mathbf{l} \Big| l_{ij} = 0 \ \forall i \in Q_0, j \in \{1, \ldots, N-1\}, \sum\nolimits_{i \in Q_0} l_{iN} = t \Big\},$$
$$\mathcal{K}_{\mathbf{d}}^0 = \Big\{ \mathbf{l} \Big| \, \mathbf{dim}\, M(\mathbf{l}) = \mathbf{d}, l_{iN} = 0 \ \forall i \in Q_0,$$
$$l_{ij} \neq 0 \Rightarrow l_{i+k,j'} = 0 \ \forall i \in Q_0, j, j', k \in \{1, \ldots, N-1\}, k \leq j' < j+k \leq N \Big\}$$

be subsets of $\mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$. Then the decomposition of $\operatorname{rep}_{\mathbf{d}}(Q, \alpha^N)$ into irreducible components is given by

$$\operatorname{rep}_{\mathbf{d}}(Q, \alpha^N) = \bigcup\nolimits_{t \in \mathbb{N},\ \mathbf{l}^p \in \mathcal{K}_t^p,\ \mathbf{l}^0 \in \mathcal{K}^0_{\mathbf{d}-(t,\ldots,t)}} \overline{\mathcal{O}(M(\mathbf{l}^p + \mathbf{l}^0))}. \tag{3.6.4}$$

Moreover, if $\mathbf{d} = (d, \ldots, d)$, then the decomposition of $\operatorname{rep}_{\mathbf{d}}(Q, \alpha^N)$ into irreducible components is given by

$$\operatorname{rep}_{\mathbf{d}}(Q, \alpha^N) = \bigcup\nolimits_{\mathbf{l} \in \mathcal{K}_d^p} \overline{\mathcal{O}(M(\mathbf{l}))}, \tag{3.6.5}$$

there are $\sum_{(l_1,\ldots,l_{d+1}) \vdash d} \prod_{j=1}^{d} \binom{N-l_{j+1}}{N-l_j}$ irreducible components and each has dimension $d^2(N-1)$.

For each $t \in \mathbb{N}$ it is very easy to list all elements of $\mathcal{K}_t^p$. This is not true for $\mathcal{K}_{\mathbf{d}}^0$ with an arbitrary $\mathbf{d}$. Since it is not really hard to check if some $\mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ is in $\mathcal{K}_{\mathbf{d}}^0$ we have to reduce the number of candidates, which can be in $\mathcal{K}_{\mathbf{d}}^0$. This is done in Proposition 3.6.13.

Remark 3.6.12. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector and $t \in \mathbb{N}$.

1. For $\mathbf{l} \in \mathcal{K}_t^p$ the (Q, α^N)-representation $M(\mathbf{l})$ is projective-injective with dimension vector $(t, \ldots, t)$ and for each $i \in Q_0$ holds
$$\operatorname{rk}_{\alpha_i}\big(M(\mathbf{l})\big) = t - l_{iN}. \tag{3.6.6}$$
Each projective-injective (Q, α^N)-representation is of this form.
2. For $\mathbf{l} \in \mathcal{K}_{\mathbf{d}}^0$ the (Q, α^N)-representation $M(\mathbf{l})$ has no projective-injective direct summand in the category $\operatorname{rep}(Q, \alpha^N)$ and by definition holds for all $i \in Q_0$ and $j, j', k \in \{1, \ldots, N-1\}$ with $k \leq j' < j+k \leq N$
$$l_{ij} \neq 0 \Rightarrow l_{i+k,j'} = 0. \tag{3.6.7}$$

And dually for all $i \in Q_0$ and $j, j', k \in \{1, \dots, N-1\}$ with $k \leq j < j' + k \leq N$

$$l_{ij} \neq 0 \Rightarrow l_{i-k,j'} = 0.$$

Moreover, for $i \in Q_0$ and $j \in \{1, \dots, N-1\}$ holds

$$l_{ij} \neq 0 \Rightarrow l_{i-j,j'} = 0 \text{ and } l_{i+j',j'} = 0 \ \forall j' \in \{1, \dots, N-1\}.$$

If $l_{Nj'} \neq 0$ for some $j' \in \{1, \dots, N-1\}$ the matrix $\mathbf{l}$ is given in Figure 3.6.13.

$$\begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & l_{1,j'+1} & \cdots & \cdots & l_{1,N-1} & 0 \\
l_{21} & \ddots & & & & \ddots & \ddots & & \vdots & \vdots \\
\vdots & \ddots & \ddots & & & & \ddots & \ddots & \vdots & \vdots \\
l_{N-j'-1,1} & \cdots & l_{N-j'-1,N-j'-2} & \ddots & & & & \ddots & l_{N-j'-1,N-1} & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & 0 & \vdots \\
l_{N-j'+1,1} & \ddots & & & & \ddots & l_{N-j'+1,N-j'+2} & \cdots & l_{N-j'+1,N-1} & \vdots \\
\vdots & \ddots & \ddots & & & & \ddots & \ddots & \vdots & \vdots \\
\vdots & & \ddots & \ddots & & & & \ddots & l_{N-2,N-1} & \vdots \\
\vdots & & & \ddots & 0 & \cdots & \cdots & \cdots & 0 & \vdots \\
l_{N1} & \cdots & \cdots & \cdots & l_{N,j'} & \cdots & \cdots & \cdots & l_{N,N-1} & 0
\end{pmatrix}$$

Figure 3.6.13: The matrix $\mathbf{l} = (l_{ij})_{i,j} \in \mathcal{K}^0_{\mathbf{d}}$ with $l_{Nj'} \neq 0$ for some $j' \in \{1, \dots, N-1\}$.

Since $\operatorname{soc} M(i-j, j') \cong S(i-j)$ and $\operatorname{top} M(i+j', j') \cong S(i+1)$ for all $i \in Q_0$ and $j, j' \in \{1, \dots, N-1\}$ we get

$$l_{ij} \neq 0 \Rightarrow d_{i-j} < d_{i-j+1} \text{ and } d_{i+1} < d_i$$

for $i \in Q_0$ and $j \in \{1, \dots, N-1\}$. Since $M(\mathbf{l})$ has no projective-injective direct summand we get for $i \in Q_0$ that $d_i \leq d_{i+1}$ if and only if $M(\mathbf{l})_{\alpha_i}$ is injective and $d_i \geq d_{i+1}$ if and only if $M(\mathbf{l})_{\alpha_i}$ is surjective. Moreover, for all $i \in Q_0$ holds

$$\operatorname{rk}_{\alpha_i}\big(M(\mathbf{l})\big) = \min(d_i, d_{i+1}). \tag{3.6.8}$$

3. Let $\mathbf{l} \in \mathcal{K}^0_{\mathbf{d}}$, $i_0 \in Q_0$, $j_0, j_0' \in \{1, \dots, N-1\}$ and $k_0 \in \{0, \dots, N-1\}$ with $l_{i_0 j_0} \neq 0$ and $l_{i_0+k_0, j_0'} \neq 0$. The (Q, α^N)-representation $M(i_0, j_0)$ and $M(i_0 + k_0, j_0')$ are illustrated by the picture on the left side of Figure 3.6.14. Then by Equation (3.6.7) holds $k_0 > j_0'$, $j_0' \geq j_0 + k_0$, $j_0 + k_0 > N$ or $k_0 = 0$.
   - If $j_0' \geq j_0 + k_0$, then $\mathbf{dim}\, M(i_0, j_0) \leq \mathbf{dim}\, M(i_0 + k_0, j_0')$.
   - If $k_0 = 0$, we can assume by the previous case that $j_0 > j_0'$. Thus in this case $\mathbf{dim}\, M(i_0, j_0) > \mathbf{dim}\, M(i_0 + k_0, j_0')$.
   - If $k_0 > j_0'$ and $j_0 + k_0 < N$ we get

   $$\big(M(i_0, j_0) \oplus M(i_0 + k_0, j_0')\big)_{i_0+1} = 0,$$
   $$\big(M(i_0, j_0) \oplus M(i_0 + k_0, j_0')\big)_{i_0+j_0} = 0$$

   and thus the support of $M(i_0, j_0) \oplus M(i_0 + k_0, j_0')$ is not connected.

- If $k_0 > j'_0$ and $N \le j_0 + k_0 < N + j'_0$ we get $l_{i_0+k_0,j'_0} \neq 0$, $l_{(i_0+k_0)+(N-k_0),j_0} = l_{i_0 j_0} \neq 0$ and $N - k_0 \le j_0 < j'_0 + (N - k_0) \le N$. This is a contradiction to Equation (3.6.7).
- If $k_0 > j'_0$ and $N + j'_0 \le j_0 + k_0$ we get for each $k \in \{k_0 - j'_0 + 1, k_0 - j'_0 + 2, \ldots, k_0\}$ the equation $\dim_k M(i_0, j_0)_{i_0+k} = 1 = \dim_k M(i_0 + k_0, j'_0)_{i_0+k}$ since $N - j_0 + 1 \le k_0 - j'_0 + 1 \le k \le k_0 \le N$. Thus $\mathbf{dim}\, M(i_0, j_0) \ge \mathbf{dim}\, M(i_0 + k_0, j'_0)$.
- If $j_0 + k_0 > N$ and $k_0 = j'_0$ we get a contradiction to Part 2.
- Thus if $j_0 + k_0 > N$, we can assume by the previous cases that $k_0 < j'_0$. This yields

$$\begin{aligned}
&\dim_k \big((M(i_0, j_0) \oplus M(i_0 + k_0, j'_0))_{i_0+1}\big) = 1,\\
&\dim_k \big((M(i_0, j_0) \oplus M(i_0 + k_0, j'_0))_{i_0}\big) = 2,\\
&\dim_k \big((M(i_0, j_0) \oplus M(i_0 + k_0, j'_0))_{i_0+k_0+1}\big) = 1,\\
&\dim_k \big((M(i_0, j_0) \oplus M(i_0 + k_0, j'_0))_{i_0+k_0}\big) = 2.
\end{aligned}$$

  Moreover, $\mathbf{dim}\,\big(M(i_0, j_0) \oplus M(i_0 + k_0, j'_0)\big) > (1, \ldots, 1)$.

Thus we get always one of the following cases:

- $\mathbf{dim}\, M(i_0, j_0) \le \mathbf{dim}\, M(i_0 + k_0, j'_0)$.
- $\mathbf{dim}\, M(i_0, j_0) \ge \mathbf{dim}\, M(i_0 + k_0, j'_0)$.
- The support of $M(i_0, j_0) \oplus M(i_0 + k_0, j'_0)$ is not connected.
- $\mathbf{dim}\,\big(M(i_0, j_0) \oplus M(i_0 + k_0, j'_0)\big) > (1, \ldots, 1)$.

Proof of Proposition 3.6.11. Let $U \in \mathrm{rep}_{\mathbf{d}}(Q, \alpha^N)$ with a tuple $\mathbf{l} = (l_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ such that $U \cong M(\mathbf{l})$. Let $\mathbf{l}^p, \mathbf{l}^0 \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ with $l^p_{iN} = l_{iN}$, $l^p_{ij} = 0$, $l^0_{iN} = 0$ and $l^0_{ij} = l_{ij}$ for all $i \in Q_0$ and $j \in \{1, \ldots, N-1\}$. This means $\mathbf{l}^p + \mathbf{l}^0 = \mathbf{l}$, $\mathbf{l}^p \in \mathcal{K}^p_t$ with $t = \sum_{i \in Q_0} l_{iN}$. Moreover, $\mathbf{dim}\, M(\mathbf{l}^0) = \mathbf{d} - (t, \ldots, t)$.

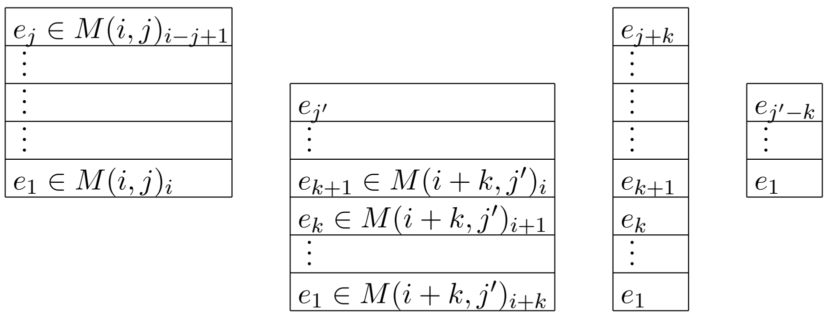


Figure 3.6.14: The (Q, α^N)-representations $M(i, j)$, $M(i + k, j')$, $M(i + k, j + k)$ and $M(i, j' - k)$.

Let $i \in Q_0$, $j, j', k \in \{1, \ldots, N - 1\}$ with $k \le j' < j + k \le N$, $l_{ij} \neq 0$ and $l_{i+k,j'} \neq 0$. Using the pictures in Figure 3.6.14 we get a short exact sequence

$$0 \to M(i + k, j') \to M(i + k, j + k) \oplus M(i, j' - k) \to M(i, j) \to 0.$$

Notice for this that $k \leq j'$ and $j + k \leq N$. Thus

$$M(i+k, j') \oplus M(i,j) \in \overline{\mathcal{O}\big(M(i+k, j+k) \oplus M(i, j'-k)\big)}.$$

Moreover, without loss of generality $\mathbf{l}^0 \in \mathcal{K}^0_{\mathbf{d}-(t,\dots,t)}$ and Equation (3.6.4) holds.

Let $t, t' \in \mathbb{N}$, $\mathbf{l}^p \in \mathcal{K}^p_t$, $\mathbf{l}'^p \in \mathcal{K}^p_{t'}$, $\mathbf{l}^0 \in \mathcal{K}^0_{\mathbf{d}-(t,\dots,t)}$ and $\mathbf{l}'^0 \in \mathcal{K}^0_{\mathbf{d}-(t',\dots,t')}$ with $M(\mathbf{l}^p + \mathbf{l}^0) \subseteq \overline{\mathcal{O}(M(\mathbf{l}'^p + \mathbf{l}'^0))}$. Now we show that $t = t'$, $\mathbf{l}^p = \mathbf{l}'^p$ and $\mathbf{l}^0 = \mathbf{l}'^0$. Using Proposition 2.3.4 and Equation (3.6.6) and (3.6.8) we get

$$l'^p_{iN} = \min(d_i, d_{i+1}) - \mathrm{rk}_{\alpha_i}\left(M(\mathbf{l}'^p + \mathbf{l}'^0)\right) \leq \min(d_i, d_{i+1}) - \mathrm{rk}_{\alpha_i}\left(M(\mathbf{l}^p + \mathbf{l}^0)\right) = l^p_{iN}$$

for all $i \in Q_0$. Using Proposition 2.3.4 again we get

$$\begin{aligned} l^p_{iN} &= d_i - \dim_k \mathrm{Hom}_Q\left(M(\mathbf{l}^p + \mathbf{l}^0), M(i, N-1)\right) \\ &\leq d_i - \dim_k \mathrm{Hom}_Q\left(M(\mathbf{l}'^p + \mathbf{l}'^0), M(i, N-1)\right) = l'^p_{iN} \end{aligned}$$

for all $i \in Q_0$. Thus $\mathbf{l}^p = \mathbf{l}'^p$ and $t = t'$.

We assume $\mathbf{l}^0 \neq \mathbf{l}'^0$. Using Part 2 of Remark 3.6.12 we get

$$\sum\nolimits_{j=1}^{N} l^0_{i'j} = \max\{d_{i'} - d_{i'+1}, 0\} = \sum\nolimits_{j=1}^{N} l'^0_{i'j}, \tag{3.6.9}$$

$$\sum\nolimits_{j=1}^{N} l^0_{i'+j-1,j} = \max\{d_{i'} - d_{i'-1}, 0\} = \sum\nolimits_{j=1}^{N} l'^0_{i'+j-1,j} \tag{3.6.10}$$

for all $i' \in Q_0$. Thus there is some $i_0 \in Q_0$ and $j_0 \in \{1, \dots, N\}$ with $l^0_{i_0 j_0} > l'^0_{i_0 j_0}$. Without loss of generality we assume $i_0 = N$ and that one of the following two cases holds:

1. $j_0 \leq \frac{N}{2}$ and $l^0_{ij} \leq l'^0_{ij}$ for all $i \in Q_0$ and $j \in \{1, \dots, j_0 - 1\} \cup \{N - j_0 + 1, \dots, N\}$.
2. $j_0 > \frac{N}{2}$ and $l^0_{ij} \leq l'^0_{ij}$ for all $i \in Q_0$ and $j \in \{1, \dots, N - j_0 - 1\} \cup \{j_0 + 1, \dots, N\}$.

Since $l^0_{Nj_0} > l'^0_{Nj_0} \geq 0$ the matrix $\mathbf{l}^0$ looks like in Figure 3.6.13.

In the first case for $i \in \{N - j_0, \dots, N-1\}$ holds

$$\begin{aligned} l^0_{ij} &\leq l'^0_{ij} \text{ for } j \in \{1, \dots, j_0 - 1\}, \\ l^0_{ij} &= 0 \text{ for } j \in \{j_0, \dots, N - j_0\} \text{ and} \\ l^0_{ij} &\leq l'^0_{ij} \text{ for } j \in \{N - j_0 + 1, \dots, N\}. \end{aligned}$$

Thus by Equation (3.6.9) we get $l^0_{ij} = l'^0_{ij}$ for all $i \in \{N - j_0, \dots, N-1\}$ and $j \in \{1, \dots, N\}$. For $i \in \{N - j_0 + 2, \dots, N\}$ holds

$$\begin{aligned} l^0_{i+j-1,j} &\leq l'^0_{i+j-1,j} \text{ for } j \in \{1, \dots, j_0 - 1\}, \\ l^0_{i+j-1,j} &= 0 \text{ for } j \in \{j_0, \dots, N - j_0\} \text{ and} \\ l^0_{i+j-1,j} &\leq l'^0_{i+j-1,j} \text{ for } j \in \{N - j_0 + 1, \dots, N\}. \end{aligned}$$

Thus by Equation (3.6.10) we get $l^0_{i+j-1,j} = l'^0_{i+j-1,j}$ for all $i \in \{N - j_0 + 2, \ldots, N\}$ and $j \in \{1, \ldots, N\}$. Using Proposition 2.3.4 again we get by Equation (3.6.9)

$$\sum_{i=N-j_0}^{N}\sum_{j=1}^{i-N+j_0} l^0_{ij} = \sum_{i=N-j_0}^{N}\sum_{j=1}^{N} l^0_{ij} - \dim_k \operatorname{Hom}_Q\left(M(N, j_0+1), M(\mathbf{l}^0)\right)$$
$$\leq \sum_{i=N-j_0}^{N}\sum_{j=1}^{N} l'^0_{ij} - \dim_k \operatorname{Hom}_Q\left(M(N, j_0+1), M(\mathbf{l}'^0)\right) = \sum_{i=N-j_0}^{N}\sum_{j=1}^{i-N+j_0} l'^0_{ij}.$$

This means

$$\begin{aligned} 0 &\leq \sum_{i=N-j_0}^{N}\sum_{j=1}^{i-N+j_0}\left(l'^0_{ij} - l^0_{ij}\right) \\ &= \sum_{i=N-j_0}^{N-1}\sum_{j=1}^{i-N+j_0}\left(l'^0_{ij} - l^0_{ij}\right) + \sum_{j=1}^{j_0-1}\left(l'^0_{(N-j+1)+j-1,j} - l^0_{(N-j+1)+j-1,j}\right) \\ &+ \left(l'^0_{Nj_0} - l^0_{Nj_0}\right) = l'^0_{Nj_0} - l^0_{Nj_0}. \end{aligned}$$

This is a contradiction to $l^0_{Nj_0} > l'^0_{Nj_0}$. In the second case we get a contradiction by using

$$d_N - \dim_k \operatorname{Hom}_Q\left(M(\mathbf{l}^0), M(N, j_0 - 1)\right) \leq d_N - \dim_k \operatorname{Hom}_Q\left(M(\mathbf{l}'^0), M(N, j_0 - 1)\right).$$

Let $\mathbf{d} = (d, \ldots, d)$, $t \in \mathbb{N}$ and $\mathbf{l} \in \mathcal{K}^0_{\mathbf{d}-(t,\ldots,t)}$. By Part 2 of Remark 3.6.12 we get $M(\mathbf{l})_{\alpha_i}$ is an isomorphism for all $i \in Q_0$. Thus $\mathcal{K}^0_{\mathbf{d}-(t,\ldots,t)} = \{0\}$ if $d = t$ and $\mathcal{K}^0_{\mathbf{d}-(t,\ldots,t)} = \varnothing$ otherwise. This yields Equation (3.6.5). In this case the irreducible components are in bijection with the set $\left\{(l_{iN})_{i\in Q_0} \in \mathbb{N}^{Q_0} \middle| \sum_{i\in Q_0} l_{iN} = d\right\}$. By reordering this tuples we get for each such tuple in this set a unique partition of d. Thus there is a map ψ from this set to the partitions of d by taking each tuple to the corresponding dual partition (see e.g. [50, Section 5] for the definition of a dual partition). The preimage of a partition $(l_1, \ldots, l_{d+1})$ of d under the map ψ contains $\prod_{j=1}^{d}\binom{N-l_{j+1}}{l_j-l_{j+1}}$ tuples. Moreover, $\dim_k \operatorname{End}_Q\left(M(\mathbf{l})\right) = d^2$ in this case. □

Now we describe the elements in $\mathcal{K}^0_{\mathbf{d}}$ in more detail.

Proposition 3.6.13. *Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver and $\mathbf{d} = (d_i)_{i\in Q_0}$ a dimension vector. For $i \in Q_0$ define*

$$\begin{aligned} \mathcal{L}_{\mathbf{d}}(i) = \Big\{\mathbf{l}\,\Big|\, &\dim_k M(\mathbf{l})_i = d_i = \sum_{i'\in Q_0}\sum_{j=1}^{N} l_{i'j}, \mathbf{dim}\, M(\mathbf{l}) \leq \mathbf{d}, \\ &\dim_k(\operatorname{top} M(\mathbf{l}))_{i'} \leq \max\{0, d_{i'} - d_{i'-1}\}, \\ &\dim_k(\operatorname{soc} M(\mathbf{l})_{i'} \leq \max\{0, d_{i'} - d_{i'+1}\}\ \forall i' \in Q_0\Big\}. \end{aligned}$$

Let $i_0 \in Q_0$, $\mathbf{l}' \in \mathcal{L}_{\mathbf{d}}(i_0)$ and $\mathbf{d}' = \mathbf{d} - \mathbf{dim}\, M(\mathbf{l}')$. We define inductively the tuple $\mathbf{l}(\mathbf{l}') = (l(\mathbf{l}')_{ij})_{i,j} \in \mathcal{K}^0_{\mathbf{d}'}$ by

$$l(\mathbf{l}')_{ij} = \min\left\{d'_{i'} \middle| i' \in Q_0, M(i,j)_{i'} \neq 0\right\} - \sum_{(i',j')\in\mathcal{S}_{ij}} l(\mathbf{l}')_{i'j'} \qquad (3.6.11)$$

with $\mathcal{S}_{ij} = \{(i',j') \in Q_0 \times \{1,\dots,N\} \,|\, \mathbf{dim}\, M(i,j) < \mathbf{dim}\, M(i',j')\}$ *for all* $i \in Q_0$ *and* $j \in \{1,\dots,N\}$. *Then holds for each* $i_0 \in Q_0$

$$\mathcal{K}^0_{\mathbf{d}} \subseteq \{\mathbf{l}' + \mathbf{l}(\mathbf{l}') \,|\, \mathbf{l}' \in \mathcal{L}_{\mathbf{d}}(i_0)\} \subseteq \mathbb{N}^{Q_0 \times \{1,\dots,N\}}.$$

In general there is no reason for $\mathbf{l}' + \mathbf{l}(\mathbf{l}')$ to be in $\mathcal{K}^0_{\mathbf{d}}$, but for small d_{i_0} there are only a few cases to check. So we should require $d_{i_0} = \min\{d_i | i \in Q_0\}$ for simplicity.

Remark 3.6.14. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector, $i_0 \in Q_0$ and $\mathbf{l}' \in \mathcal{L}_{\mathbf{d}}(i_0)$.

1. For $j \in \{1,\dots,N\}$, $k \in \{0,\dots,N-1\}$ with $l'_{i_0+k,j} \neq 0$ holds

$$d_{i_0+k+1} < d_{i_0+k} \wedge d_{i_0+k-j} < d_{i_0+k-j+1} \wedge k < j \wedge \dim_k M(i_0+k,j)_{i_0} = 1.$$

2. For $i \in Q_0$ and $j \in \{1,\dots,N\}$ holds

$$\begin{aligned} l(\mathbf{l}')_{ij} &= d - \sum\nolimits_{(i',j') \in \mathcal{S}_{ij}} l(\mathbf{l}')_{i'j'} \\ &= d - l(\mathbf{l}')_{i,j+1} - l(\mathbf{l}')_{i+1,j+1} + l(\mathbf{l}')_{i+1,j+2} - \sum\nolimits_{(i',j') \in \mathcal{S}_{i,j+1}} l(\mathbf{l}')_{i'j'} \\ &\quad - \sum\nolimits_{(i',j') \in \mathcal{S}_{i+1,j+1}} l(\mathbf{l}')_{i'j'} + \sum\nolimits_{(i',j') \in \mathcal{S}_{i+1,j+2}} l(\mathbf{l}')_{i'j'} \\ &= d - \min\{d'_{i-j}, d\} - \min\{d, d'_{i+1}\} + \min\{d'_{i-j}, d, d'_{i+1}\} \geq 0 \end{aligned}$$

with $d = \min\{d'_{i'} \,|\, i' \in Q_0, M(i,j)_{i'} \neq 0\}$ since $\mathcal{S}_{i,j+1} \cup \mathcal{S}_{i+1,j+1} \cup \{(i,j+1),(i+1,j+1)\} = \mathcal{S}_{ij}$ and $\mathcal{S}_{i,j+1} \cap \mathcal{S}_{i+1,j+1} = \mathcal{S}_{i+1,j+2} \cup \{(i+1,j+2)\}$. Thus $\mathbf{l}(\mathbf{l}') \geq 0$.

3. If $l(\mathbf{l}')_{ij_0} \neq 0$ and $l(\mathbf{l}')_{i+k_0,j'_0} \neq 0$ for some $i \in Q_0$ and $j_0, j'_0, k_0 \in \{1,\dots,N-1\}$ with $k_0 \leq j'_0 < j_0 + k_0 \leq N$, then

$$\begin{aligned} 0 < l(\mathbf{l}')_{i,j_0} &= \min\{d'_{i-j_0+1},\dots,d'_i\} - \sum\nolimits_{(i',j') \in \mathcal{S}_{ij_0}} l(\mathbf{l}')_{i'j'}, \\ 0 < l(\mathbf{l}')_{i+k_0,j'_0} &= \min\{d'_{i+k_0-j'_0+1},\dots,d'_{i+k_0}\} - \sum\nolimits_{(i',j') \in \mathcal{S}_{i+k_0,j'_0}} l(\mathbf{l}')_{i'j'}. \end{aligned}$$

Combining this two equations we get for all $k' \in \{-j_0+1, -j_0+2, \dots, k_0\}$

$$d'_{i+k'} > \left(\sum\nolimits_{(i',j') \in \mathcal{S}_{i+k_0,j_0+k_0}} l(\mathbf{l}')_{i'j'}\right) + l(\mathbf{l}')_{i+k_0,j_0+k_0}.$$

since $\mathcal{S}_{i+k_0,j_0+k_0} \cup \{(i+k_0, j_0+k_0)\} \subseteq \mathcal{S}_{ij_0} \cap \mathcal{S}_{i+k_0,j'_0}$. Thus

$$\begin{aligned} l(\mathbf{l}')_{i+k_0,j_0+k_0} &= \min\{d'_{i-j_0+1},\dots,d'_{i+k_0}\} - \sum\nolimits_{(i',j') \in \mathcal{S}_{i+k_0,j_0+k_0}} l(\mathbf{l}')_{i'j'} \\ &> l(\mathbf{l}')_{i+k_0,j_0+k_0}. \end{aligned}$$

This is a contradiction and therefore $\mathbf{l}(\mathbf{l}') \in \mathcal{K}^0_{\mathbf{d}'}$. Moreover, by Proposition 3.6.13 we get $\mathcal{K}^0_{\mathbf{d}} \subseteq \{\mathbf{l}' + \mathbf{l}(\mathbf{l}') \,|\, \mathbf{l}' \in \mathcal{L}_{\mathbf{d}}(i_0), \mathbf{l}' \in \mathcal{K}^0_{\mathbf{dim}\, M(\mathbf{l}')}\}$ for each $i_0 \in Q_0$.

Example 3.6.15. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $i_0 \in Q_0$, $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector with $d_{i_0} = 0$ and $t \in \mathbb{N}$. Let $\mathbf{l}^p \in \mathcal{K}_t^p$ and $\mathbf{l} \in \mathcal{K}^0_{\mathbf{d}-(t,\ldots,t)}$. Then $t = 0$, $\mathbf{l}^p = 0$ and $\mathcal{L}_{\mathbf{d}}(i_0) = \{0\}$.

By Part 3 of Remark 3.6.12 we get always one of the following cases for $i_0 \in Q_0$, $j_0, j_0' \in \{1, \ldots, N-1\}$ and $k_0 \in \{0, \ldots, N-1\}$ with $l_{i_0 j_0} \neq 0$ and $l_{i_0+k_0, j_0'} \neq 0$:

- $\mathbf{dim}\, M(i_0, j_0) \leq \mathbf{dim}\, M(i_0 + k_0, j_0')$.
- $\mathbf{dim}\, M(i_0, j_0) \geq \mathbf{dim}\, M(i_0 + k_0, j_0')$.
- The support of $M(i_0, j_0) \oplus M(i_0 + k_0, j_0')$ is not connected.

It is clear that this rule defines $\mathbf{l}$ uniquely. Moreover, we can use Equation (3.6.11) in Proposition 3.6.13 to define $\mathbf{l}$ as $\mathbf{l}(0)$. This is illustrated in the picture in Figure 3.6.15.

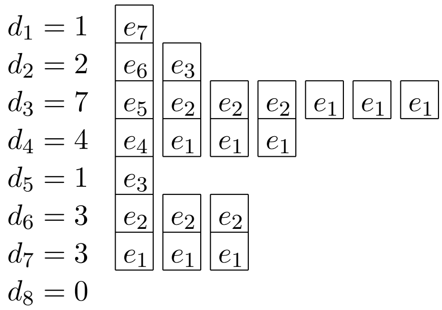


Figure 3.6.15: The (Q, α^8)-representation $M(\mathbf{l})$ for the unique element $\mathbf{l}$ in the set $\mathcal{K}^0_{(1,2,7,4,1,3,3,0)}$.

Using Proposition 3.6.11 the orbit $\mathcal{O}(M(\mathbf{l}))$ is dense in $\mathrm{rep}_{\mathbf{d}}(Q, \alpha^N)$. Moreover, by Proposition 2.3.3 holds

$$\dim \mathrm{rep}_{\mathbf{d}}(Q, \alpha^N) = \sum_{i \in Q_0} \left(d_i^2 - \tfrac{1}{2}(d_i - d_{i+1})^2\right) = \sum_{i \in Q_0} d_i d_{i+1}.$$

This is well-known since the variety $\mathrm{rep}_{\mathbf{d}}(Q, \alpha^N)$ is isomorphic to the affine variety $\prod_{i \in Q_0} \mathrm{Mat}(d_{i+1} \times d_i, k)$ in this case.

Proof of Proposition 3.6.13. Let $\mathbf{l} \in \mathcal{K}^0_{\mathbf{d}}$ and $i_0 \in Q_0$. We define a tuple $\mathbf{l}' = (l'_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ by

$$l'_{ij} = \begin{cases} l_{ij} & \text{if } M(i,j)_{i_0} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we test if $\mathbf{l}' \in \mathcal{L}_{\mathbf{d}}(i_0)$ and $\mathbf{l}(\mathbf{l}') = \mathbf{l} - \mathbf{l}'$. The first part is clear by its definition and Part 2 of Remark 3.6.12. For the second part we recognize that $\mathbf{l}(\mathbf{l}'), \mathbf{l} - \mathbf{l}' \in \mathcal{K}^0_{\mathbf{d}'}$ with $\mathbf{d}' = \mathbf{d} - \mathbf{dim}\, M(\mathbf{l}')$ by Part 3 of Remark 3.6.14. Moreover, since $d'_{i_0} = 0$ by Example 3.6.15 the set $\mathcal{K}^0_{\mathbf{d}'}$ has exactly one element. $\square$

Now we decompose the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ for each projective-injective (Q, α^N)-representation with an arbitrary dimension vector $\mathbf{d}$.

Proposition 3.6.16. *Let $N, n \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $\mathbf{d}$ a dimension vector and $\mathbf{k} \in \mathcal{K}_n^p$, i.e. $M(\mathbf{k})$ is a projective-injective (Q, α^N)-representation with dimension vector $\mathbf{n} = (n, \dots, n)$. Let* $\mathrm{rk}\colon \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) \to \mathbb{N}^{Q_0 \times \{1,\dots,N\}}, U \mapsto \mathbf{l}$ *with $U \cong M(\mathbf{l})$ and $\mathcal{N}_{\mathbf{d}}(\mathbf{k})$ the image of this map. Define for $t \in \{0, \dots, n\}$ the subsets*

$$\mathcal{M}_t^p(\mathbf{k}) = \Big\{\mathbf{l} \in \mathcal{K}_t^p \Big| l_{iN} \leq k_{iN} \ \forall i \in Q_0\Big\},$$
$$\mathcal{M}_{\mathbf{d}}^0(\mathbf{k}) = \Big\{\mathbf{l} \in \mathcal{K}_{\mathbf{d}}^0 \Big| \sum\nolimits_{j=1}^N l_{ij} \leq k_{iN} \ \forall i \in Q_0\Big\}$$

of $\mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ and for each tuple $\mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ a tuple $\mathbf{l}^\star = (l_{ij}^\star)_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ by

$$l_{ij}^\star = \begin{cases} l'_{i+j,N-j} & \text{if } 1 \leq j < N, \\ k_{iN} - \sum_{j'=1}^N l'_{ij'} & \text{if } j = N \end{cases} \tag{3.6.12}$$

for all $i \in Q_0$ and $j \in \{1, \dots, N\}$. Then the decomposition of $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ into irreducible components is given by

$$\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \bigcup\nolimits_{t \in \{0,\dots,n\},\ \mathbf{l}^p \in \mathcal{M}_t^p(\mathbf{k}),\ \mathbf{l}^0 \in \mathcal{M}_{\mathbf{d}-(t,\dots,t)}^0(\mathbf{k}-\mathbf{l}^p)} \overline{\mathcal{C}_{M(\mathbf{l}^p+\mathbf{l}^0), M((\mathbf{l}^p+\mathbf{l}^0)^\star)}(k)} \tag{3.6.13}$$

and the points of $\mathcal{C}_{M(\mathbf{l}^p+\mathbf{l}^0), M((\mathbf{l}^p+\mathbf{l}^0)^\star)}(k)$ are smooth in $Gr_{\mathbf{d}}(M(\mathbf{k}))$ for all $\mathbf{l}^p \in \mathcal{M}_t^p(\mathbf{k})$ and $\mathbf{l}^0 \in \mathcal{M}_{\mathbf{d}-(t,\dots,t)}^0(\mathbf{k} - \mathbf{l}^p)$ with $t \in \{1, \dots, n\}$. Moreover, if $\mathbf{d} = (d, \dots, d)$, then the decomposition of $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ into irreducible components is given by

$$\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \bigcup\nolimits_{\mathbf{l} \in \mathcal{M}_d^p(\mathbf{k})} \overline{\mathcal{C}_{M(\mathbf{l}), M(\mathbf{k}-\mathbf{l})}(k)}, \tag{3.6.14}$$

there are $\sum_{(l_1,\dots,l_{d+1}) \vdash d} \prod_{j=1}^d \binom{|\{i \in Q_0 | k_{iN} \geq j\}| - l_{j+1}}{l_j - l_{j+1}}$ irreducible components and each has dimension $d(n-d)$.

Thus the subset of smooth points is dense.

Remark 3.6.17. For $\mathbf{d} = (d, \dots, d)$ Görtz [30, Section 4.3] describes also the decomposition into irreducible components by Schubert cells. Their index sets are quotients of the Weyl group. To study this see also [35].

Example 3.6.18. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $i_0 \in Q_0$, $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector with $d_{i_0} = 0$, $\mathbf{k} \in \mathcal{K}_n^p$, $\mathbf{n} = (n, \dots, n)$ and $t \in \{0, \dots, n\}$. Let $\mathbf{l}^p \in \mathcal{M}_t^p(\mathbf{k})$ and $\mathbf{l} \in \mathcal{M}_{\mathbf{d}-(t,\dots,t)}^0(\mathbf{k} - \mathbf{l}^p)$. Then as in Example 3.6.15 holds $t = 0$, $\mathbf{l}^p = 0$ and $\mathcal{L}_{\mathbf{d}}(i_0) = \{0\}$.

Using Proposition 3.6.16 we get $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \overline{\mathcal{C}_{M(\mathbf{l}), M(\mathbf{l}^\star)}(k)}$. Moreover, by Theorem 3.1.1 holds

$$\dim \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \sum\nolimits_{i \in Q_0} d_i k_{iN} - \tfrac{1}{2}(d_i - d_{i+1})^2 = \sum\nolimits_{i \in Q_0} d_i \left(k_{iN} - d_i + d_{i+1}\right).$$

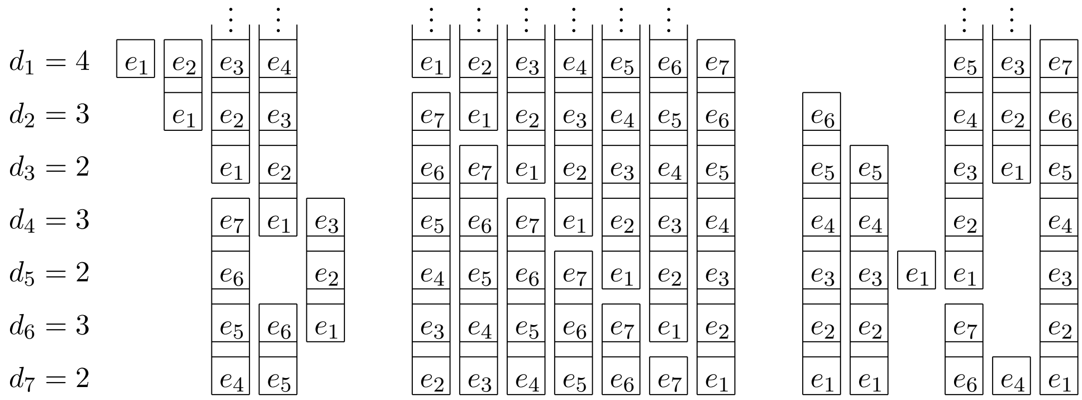


Figure 3.6.16: The (Q, α^7)-representations $M(\mathbf{l}^p + \mathbf{l}^0)$, $M(\mathbf{k})$ and $M((\mathbf{l}^p + \mathbf{l}^0)^\star)$.

Example 3.6.19. Let Q be the 7-cycle-quiver, $\mathbf{d} = (4, 3, 2, 3, 2, 3, 2)$ a dimension vector, $\mathbf{k} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ with $k_{i7} = 1$ and $k_{ij} = 0$ for all $i \in Q_0$ and $j \in \{1, \dots, 6\}$, $\mathbf{l}^p \in \mathcal{M}_1^p(\mathbf{k})$ with $l^p_{37} = 1$ and all other $l^p_{ij} = 0$ for $i \in Q_0$ and $j \in \{1, \dots, 7\}$ and $\mathbf{l}^0 \in \mathcal{M}^0_{\mathbf{d}-(1,\dots,1)}(\mathbf{k}-\mathbf{l}^p)$ with $l^0_{11} = l^0_{22} = l^0_{46} = l^0_{63} = 1$ and all other $l^0_{ij} = 0$ for $i \in Q_0$ and $j \in \{1, \dots, 7\}$. The (Q, α^7)-representations $M(\mathbf{l}^p+\mathbf{l}^0)$, $M(\mathbf{k})$ and $M((\mathbf{l}^p+\mathbf{l}^0)^\star)$ are illustrated in the pictures in Figure 3.6.16. Thus holds

$$\begin{aligned}
\dim_k \mathrm{End}_Q \big(M(\mathbf{l}^p + \mathbf{l}^0)\big) &= 1 + 2 + 3 + 3 + 2 = 11,\\
\dim_k \mathrm{Hom}_Q \big(M(\mathbf{l}^p + \mathbf{l}^0), M(\mathbf{k})\big) &= 1 + 2 + 7 + 6 + 3 = 19,\\
\dim_k \mathrm{Hom}_Q \big(M(\mathbf{l}^p + \mathbf{l}^0), M((\mathbf{l}^p + \mathbf{l}^0)^\star)\big) &= 0 + 0 + 4 + 3 + 1 = 8.
\end{aligned}$$

Moreover, the points in $\mathcal{C}_{M(\mathbf{l}^p+\mathbf{l}^0),M((\mathbf{l}^p+\mathbf{l}^0)^\star)}(k)$ are smooth in $Gr_{\mathbf{d}}(M(\mathbf{k}))$.

Remark 3.6.20. Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $\mathbf{d}$ a dimension vector, $\mathbf{k} \in \mathcal{K}^p_n$, $\mathbf{n} = (n, \dots, n)$ and $t \in \{0, \dots, n\}$. For $\mathbf{l}^p \in \mathcal{M}^p_t(\mathbf{k})$ and $\mathbf{l}^0 \in \mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k} - \mathbf{l}^p)$ there exists a $U \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ such that $\mathrm{rk}(U) = \mathbf{l}^p + \mathbf{l}^0$. Thus $\mathbf{l}^p + \mathbf{l}^0 \in \mathcal{N}_{\mathbf{d}}(\mathbf{k})$, $\mathcal{C}_{M(\mathbf{l}^p+\mathbf{l}^0)}(k)$ is non empty and the following lemma yields $\mathcal{C}_{M(\mathbf{l})}(k) = \mathcal{C}_{M(\mathbf{l}),M(\mathbf{l}^\star)}(k)$ for each $\mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$.

Before proving Proposition 3.6.16 we consider some useful lemma. For this it is important that the (Q, α^N)-representation $M(\mathbf{k})$ is projective-injective.

Lemma 3.6.21. *Let $N \in \mathbb{N}$ with $N \geq 2$, Q the N-cycle-quiver, $\mathbf{d} = (d_1, \dots, d_N)$ a dimension vector, $\mathbf{k} \in \mathcal{K}^p_n$ and $\mathbf{n} = (n, \dots, n)$. Let $U \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ with $U \cong M(\mathbf{l})$ and for each $\mathbf{m} \in Q_0 \times \{1, \dots, N\}$ define*

$$S(\mathbf{m}) = \big\{(i, j, k) \in Q_0 \times \{1, \dots, N\} \times \mathbb{N} \big| 1 \leq k \leq m_{ij}\big\}.$$

Then there is a basis $\big\{f_{i,N,k,l} \big| (i, N, k) \in S(\mathbf{k}), l \in \{1, \dots, N\}\big\}$ of $M(\mathbf{k})$ such that the following holds

- *For each tuple $(i, N, k) \in S(\mathbf{k})$ and $l \in \{1, \dots, N\}$ is $f_{i,N,k,l}$ in $M(\mathbf{k})_{i+l-1}$.*

- *For each tuple* $(i,N,k) \in S(\mathbf{k})$ *holds* $M(\mathbf{k})_{\alpha_{i+l-1}}(f_{i,N,k,l}) = f_{i,N,k,l-1}$ *for all* $l \in \{2,\dots,N\}$ *and* $M(\mathbf{k})_{\alpha_i}(f_{i,N,k,1}) = 0$.
- *There is some injective map* $S(U)\colon S(\mathbf{l}) \to S(\mathbf{k})$ *such that*

$$U = \left\langle f_{S(U)(i,j,k),j} \middle| (i,j,k) \in S(\mathbf{l}) \right\rangle_Q.$$

Moreover, $M/U \cong M(\mathbf{l}^\star)$ *with* $\mathbf{l}^\star \in \mathbb{N}^{Q_0\times\{1,\dots,N\}}$ *defined in Equation* (3.6.12) *in Proposition 3.6.16.*

Proof. For each $\mathbf{m} \in \mathbb{N}^{Q_0\times\{1,\dots,N\}}$ define a basis $\left\{e_{i,j,k,l} \middle| (i,j,k) \in S(\mathbf{m}), l \in \{1,\dots,j\}\right\}$ of $M(\mathbf{m}) = \bigoplus_{i\in Q_0}\bigoplus_{j=1}^N M(i,j)^{m_{ij}}$ such that $e_{i,j,k,l}$ is in the k-th copy of $M(i,j)$ and in $M(\mathbf{m})_{i+l-1}$ for all $(i,j,k) \in S(\mathbf{m})$ and $l \in \{1,\dots,j\}$.

Since $M(\mathbf{k})$ is projective-injective $k_{ij} = 0$ for all $i \in Q_0$ and $j \in \{1,\dots,N-1\}$. Let $\iota\colon M(\mathbf{l}) \to M(\mathbf{k})$ be an embedding with image U. Thus $\{\iota(e_{i,j,k,j}) | (i,j,k) \in S(\mathbf{l})\}$ is a minimal set generating the Q-representation U. Let $\lambda_{i,j,k,i',k',l'} \in k$ be the coefficients such that

$$\iota(e_{i,j,k,j}) = \sum\nolimits_{(i',N,k')\in S(\mathbf{k})}\sum\nolimits_{l'=1}^{j} \lambda_{i,j,k,i',k',l'} e_{i',N,k',l'}$$

for all $(i,j,k) \in S(\mathbf{l})$. Since $\iota\colon M(\mathbf{l}) \to M(\mathbf{k})$ is an injective homomorphism of Q-representations we get that

$$\left\{\sum\nolimits_{(i',N,k')\in S(\mathbf{k})}\sum\nolimits_{l'=1}^{j} \lambda_{i,j,k,i',k',l'} e_{i',N,k',N+l'-j} \middle| (i,j,k) \in S(\mathbf{l})\right\}$$

generates the minimal direct summand of the Q-representation M containing the subrepresentation U. This direct summand is isomorphic to $M(\mathbf{k}')$ with $k'_{iN} = \sum_{j=1}^N l_{ij}$ for each $i \in Q_0$. And let

$$S(\iota_1)\colon S(\mathbf{l}) \to S(\mathbf{k}'), (i,j,k) \mapsto \left(i,N,k+\sum\nolimits_{j'=1}^{j-1} l_{ij'}\right)$$

be a bijective map and $\iota_1\colon M(\mathbf{l}) \to M(\mathbf{k}'), e_{i,j,k,j} \mapsto e_{S(\iota_1)(i,j,k),j}$ the induced homomorphism of Q-representations. Now we consider the injective homomorphism

$$f\colon M(\mathbf{k}') \to M(\mathbf{k}), e_{i,N,k,N} \mapsto \sum\nolimits_{(i',N,k')\in S(\mathbf{k})}\sum\nolimits_{l'=1}^{j} \lambda_{i,j,k'',i',k',l'} e_{i',N,k',N+l'-j}$$

for $S(\iota_1)^{-1}(i,N,k) = (i,j,k'')$. Since the image of this homomorphism f is the direct summand of $M(\mathbf{k})$ considered above, this homomorphism splits and we get an isomorphism

$$\left(\begin{smallmatrix} f & g \end{smallmatrix}\right)\colon M(\mathbf{k}') \oplus M(\mathbf{k}-\mathbf{k}') \to M(\mathbf{k}).$$

Since the diagram in Figure 3.6.17 is commutative it is enough to complete the set

$$\left\{f_{i,N,k,l} = f(e_{i,N,k,l}) \middle| (i,N,k) \in S(\mathbf{k}'), l \in \{1,\dots,N\}\right\}$$

to a basis of $M(\mathbf{k})$ by $f_{i,N,k'_{iN}+k,l} = g(e_{i,N,k,l})$ for all $(i,N,k) \in S(\mathbf{k}-\mathbf{k}')$ and $l \in \{1,\dots,N\}$. □

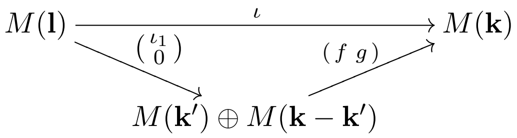


Figure 3.6.17: A commutative diagram for the proof of Lemma 3.6.21.

Now we are ready to prove Proposition 3.6.16.

Proof of Proposition 3.6.16. By Proposition 3.6.11 Equation (3.6.13) is left to show. Let $\mathbf{l} = (l_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ with $\mathrm{rk}(U) = \mathbf{l}$. Let $S(U)\colon S(\mathbf{l}) \to S(\mathbf{k})$ and $\big\{ f_{i,N,k,l} \big| (i,N,k) \in S(\mathbf{k}), l \in \{1,\dots,N\} \big\}$ a basis of $M(\mathbf{k})$ as in Lemma 3.6.21.

Let $i_0 \in Q_0$ and $j_0, j_0', k_0 \in \{1,\dots,N-1\}$ with $l_{i_0 j_0} \neq 0$, $l_{i_0+k_0, j_0'} \neq 0$ and $k_0 \leq j_0' < j_0 + k_0 \leq N$. Using again the pictures in Figure 3.6.14 we define

$$U_\lambda = \big\langle f_{S(U)(i_0,j_0,1),j_0} + \lambda f_{S(U)(i_0+k_0,j_0',1),j_0+k_0}, f_{S(U)(i,j,k),j} \big| (i,j,k) \in S(\mathbf{l}) - \{(i_0,j_0,1)\} \big\rangle_Q$$

for $\lambda \in k$. Then $U_\lambda \in \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ for all $\lambda \in k$, $U_0 = U$ and for all $\lambda \in k^*$ holds $\mathrm{rk}(U_\lambda) \neq \mathbf{l}$. Thus we can assume without loss of generality for $\mathbf{l} = (l_{ij})_{i,j}$ holds

$$l_{ij} \neq 0 \Rightarrow l_{i+k,j'} = 0 \ \forall i \in Q_0, j, j', k \in \{1,\dots,N-1\}, k \leq j' < j+k \leq N.$$

Let $\mathbf{l}^p = (\mathbf{l}^p_{ij})_{i,j}, \mathbf{l}^0 = (\mathbf{l}^0_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\dots,N\}}$ with $l^p_{iN} = l_{iN}$, $l^p_{ij} = 0$, $l^0_{iN} = 0$ and $l^0_{ij} = l_{ij}$ for all $i \in Q_0$ and $j \in \{1,\dots,N-1\}$. This means $\mathbf{l}^p + \mathbf{l}^0 = \mathbf{l}$, $\mathbf{l}^p \in \mathcal{M}^p_t(\mathbf{k})$ and $\mathbf{l}^0 \in \mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k}-\mathbf{l}^p)$ with $t = \sum_{i \in Q_0} l_{iN}$. Thus Equation (3.6.13) holds.

By some straightforward generalization of Example 3.6.19 we get that each point of $\mathcal{C}_{M(\mathbf{l}^p+\mathbf{l}^0)\oplus M((\mathbf{l}^p+\mathbf{l}^0)^\star)}(k)$ is smooth in $Gr_{\mathbf{d}}(M(\mathbf{k}))$ for $t \in \{0,\dots,n\}$, $\mathbf{l}^p \in \mathcal{M}^p_t(\mathbf{k})$ and $\mathbf{l}^0 \in \mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k}-\mathbf{l}^p)$.

Let $\mathbf{d} = (d,\dots,d)$, $t \in \{0,\dots,n\}$, $\mathbf{l} \in \mathcal{M}^p_t(\mathbf{k})$ and $\mathbf{l}^0 \in \mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k}-\mathbf{l})$. As in the proof of Proposition 3.6.11 we get $\mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k}-\mathbf{l}) = \{0\}$ if $d = t$ and $\mathcal{M}^0_{\mathbf{d}-(t,\dots,t)}(\mathbf{k}-\mathbf{l}) = \varnothing$ otherwise. This yields Equation (3.6.14). As in the proof of Proposition 3.6.11 the irreducible components are in bijection with the set

$$\Big\{ (l_{iN})_{i \in Q_0} \in \mathbb{Z}^{Q_0} \Big| l_{iN} \in \{0,1,\dots,k_{iN}\}, \sum\nolimits_{i \in Q_0} l_{iN} = d \Big\}.$$

By reordering this tuples we get again for each such tuple in this set a unique partition of d. Thus there is the map ψ taking each tuple to the corresponding dual partition. The preimage of a partition $(l_1,\dots,l_{d+1})$ of d under the map ψ contains in this case $\prod_{j=1}^{d} \binom{|\{i \in Q_0 | k_{iN} \geq j\}| - l_{j+1}}{l_j - l_{j+1}}$ tuples. Moreover, $\dim_k \mathrm{End}_Q\big(M(\mathbf{l})\big) = d^2$ in this case. Since $\dim_k \mathrm{Hom}_Q\big(M(\mathbf{l}), M(\mathbf{k}-\mathbf{l})\big) = d(n-d)$ this quiver Grassmannian has dimension $d(n-d)$. ☐

Before closing this subsection we give an explicit list of the irreducible components for some representation of the 6-cycle-quiver.

Example 3.6.22. Let Q be the 6-cycle-quiver, $r, s \in \mathbb{N}$ with $r \geq s$, $\mathbf{d} = (r, s, r, s, r, s)$ a dimension vector and $\mathbf{k} \in \mathcal{K}^p_{6r}$ such that $k_{i6} = r$ for each $i \in Q_0$. Thus $M(\mathbf{k})$ is a projective-injective (Q, α^6)-representation with dimension vector $(6r, \ldots, 6r)$. For $i \in Q_0$ and $j \in \{1, \ldots, 6\}$ let $\mathbf{e}_{ij} = ((e_{ij})_{i',j'})_{i',j'} \in \mathbb{N}^{Q_0 \times \{1,\ldots,6\}}$ with

$$(e_{ij})_{i'j'} = \begin{cases} 1 & \text{if } i = i', j = j', \\ 0 & \text{otherwise,} \end{cases}$$

for all $i' \in Q_0$ and $j' \in \{1, \ldots, 6\}$. Then

$$\mathcal{M}^p_t(\mathbf{k}) = \left\{ \sum\nolimits_{i \in Q_0} \lambda_i \mathbf{e}_{i6} \middle| \lambda_i \in \{0, 1, \ldots, s\}, \sum\nolimits_{i \in Q_0} \lambda_i = t \right\}$$

for $t \in \mathbb{N}$. Let $t \in \mathbb{N}$ with $t \leq s$ and $\mathbf{l}^p = \sum_{i \in Q_0} \lambda_i \mathbf{e}_{i6} \in \mathcal{M}^p_t(\mathbf{k})$. Then

$$\begin{aligned} \mathcal{M}^0_{\mathbf{d}-(t,\ldots,t)}(\mathbf{k} - \mathbf{l}^p) = \Big\{ & (j - 2s + r + t)(\mathbf{e}_{11} + \mathbf{e}_{31} + \mathbf{e}_{51}) \\ & + (s - t - 2j)(\mathbf{e}_{i1} + \mathbf{e}_{i+4,3} + \mathbf{e}_{i+2,5}) + j(\mathbf{e}_{15} + \mathbf{e}_{35} + \mathbf{e}_{55}) \Big| \\ & j \in \mathbb{N}, 2j \leq s - t, 2s - r - t \leq j, i \in \{1, 3, 5\} \Big\}. \end{aligned}$$

Thus by Proposition 3.6.16 each irreducible component of the variety $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ is of the form $\overline{\mathcal{C}_{N(t,\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,j,i)}(k)}$ such that $t, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \{0, 1, \ldots, s\}$ with $3s - 2r \leq t$ and $\sum_{i \in Q_0} \lambda_i = t$, $j \in \mathbb{N}$ with $2j \leq s - t$ and $2s - r - t \leq j$, $i \in \{1, 3, 5\}$ and

$$\begin{aligned} N(t, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, j, i) = & \bigoplus\nolimits_{i \in Q_0} M(\mathbf{e}_{i6})^{\lambda_i} \oplus (S(1) \oplus S(3) \oplus S(5))^{j-2s+r+t} \\ & \oplus M(\mathbf{e}_{i1} + \mathbf{e}_{i+4,3} + \mathbf{e}_{i+2,5})^{s-t-2j} \oplus M(\mathbf{e}_{15} + \mathbf{e}_{35} + \mathbf{e}_{55})^j. \end{aligned}$$

This Q-representation is described in the picture in Figure 3.6.18. If $2j = s - t$, it does not depend on $i \in \{1, 3, 5\}$.

3.6.6 Quiver of type A_N

Let $N \in \mathbb{N}$ and $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{N-1}} N$. Again we describe the Q-representations by tuples in $\mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$. Let $i \in Q_0$ and $j \in \{1, \ldots, i\}$. Then the Q-representation $M(i, j)$ is defined as in Section 3.6.5 with $\operatorname{soc} M(i, j) \cong S(i)$ and $\dim_k M(i, j) = j$. Thus for each tuple $\mathbf{l} = (l_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ with $l_{ij} = 0$ for all $i \in Q_0$ and $j \in \{i+1, \ldots, N\}$ we define a Q-representation

$$M(\mathbf{l}) = \bigoplus\nolimits_{i \in Q_0} \bigoplus\nolimits_{j=1}^{N} M(i, j)^{l_{ij}}.$$

This parametrizes all isomorphism classes of Q-representations.

A Q-representation $M(\mathbf{l})$ with $\mathbf{l} = (l_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ is injective in the abelian category $\mathrm{rep}(Q)$, if and only if the linear map M_β is surjective for each $\beta \in Q_1$, i.e.

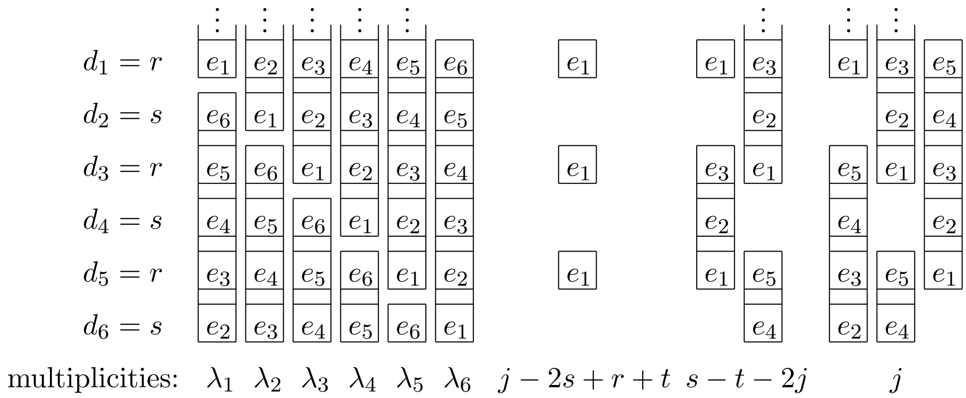


Figure 3.6.18: The (Q, α^6)-representation $N(t, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, j, 1)$.

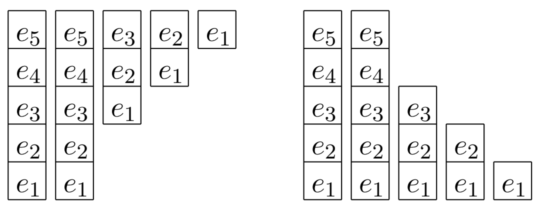


Figure 3.6.19: Some injective and projective Q-representations with $N = 5$.

$l_{ij} = 0$ for each $i \in Q_0$ and $j \in \{1, \ldots, i-1\}$. Dually a Q-representation $M(\mathbf{l})$ is projective, if and only if M_β is injective for each $\beta \in Q_1$, i.e. $l_{ij} = 0$ for each $i \in Q_0$ and $j \in \{1, \ldots, N\}$ with $i \neq N$. In Figure 3.6.19 we give pictures for injective and projective Q-representations.

The first corollary is well-known. Nevertheless it follows by Example 3.6.15.

Corollary 3.6.23. *Let $N \in \mathbb{N}$, $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{N-1}} N$ and $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector. Then $\operatorname{rep}_{\mathbf{d}}(Q) = \overline{\mathcal{O}(M(\mathbf{l}))}$ with $\mathbf{l} = (l_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ defined inductively by*

$$l_{ij} = \min\left\{d_{i'} \middle| i' \in Q_0, M(i,j)_{i'} \neq 0\right\} - \sum\nolimits_{(i',j') \in \mathcal{S}_{ij}} l_{i'j'} \tag{3.6.15}$$

and $\mathcal{S}_{ij} = \left\{(i',j') \in Q_0 \times \{1, \ldots, N\} \middle| \, \mathbf{dim}\, M(i,j) < \mathbf{dim}\, M(i',j')\right\}$ for all $i \in Q_0$ and $j \in \{1, \ldots, N\}$.

Equation (3.6.15) is some special case of Equation (3.6.11). The following result is a corollary of Corollary 3.5.10 and the last section.

Corollary 3.6.24. *Let $N \in \mathbb{N}$, $Q = 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{N-1}} N$, $\mathbf{d} = (d_i)_{i \in Q_0}$ a dimension vector and $M(\mathbf{k})$ a Q-representation with $\mathbf{k} = (k_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ and dimension vector $\mathbf{n}$.*

- *If the Q-representation $M(\mathbf{k})$ is injective, then holds* $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \overline{\mathcal{C}_{M(\mathbf{l}),M(\mathbf{m})}(k)}$ *with the tuple* $\mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ *given by Equation* (3.6.15) *and the tuple* $\mathbf{m} = (m_{ij})_{i,j} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ *is defined by*

$$m_{ij} = \begin{cases} k_{ii} - d_i + d_{i+1} & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases} \tag{3.6.16}$$

  with $d_{N+1} = 0$ *for all* $i \in Q_0$ *and* $j \in \{1, \ldots, N\}$.
- *If* $M(\mathbf{k})$ *is projective, then holds* $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) = \overline{\mathcal{C}_{M(\mathbf{l}),M(\mathbf{m})}(k)}$ *with the tuple* $\mathbf{l} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ *given by a dual version of Equation* (3.6.16) *and* $\mathbf{m} \in \mathbb{N}^{Q_0 \times \{1,\ldots,N\}}$ *given by Equation* (3.6.15) *with* $\mathbf{n} - \mathbf{d}$ *instead of* $\mathbf{d}$.

Moreover, $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}))$ *is smooth in these cases.*

Proof. We use Proposition 3.6.13 and 3.6.16. In the first case we consider the $(N+1)$-cycle-quiver Q' with $\mathbf{d}' = (d'_i)_{i \in Q_0}$ such that $d'_i = d_i$ for each $i \in \{1, \ldots, N\}$ and $d'_{N+1} = 0$. Let $\mathbf{k}' = (k'_{ij})_{i \in Q'_0, j \in \{1,\ldots,N+1\}} \in \mathbb{N}^{Q'_0 \times \{1,\ldots,N+1\}}$ with $k'_{iN} = k_{ii}$ for each $i \in \{1, \ldots, N\}$ and $k_{ij} = 0$ otherwise. Thus we can consider $M(\mathbf{k})$ as a subrepresentation of the projective-injective (Q', α^{N+1})-representation $M(\mathbf{k}')$. Moreover, each subrepresentation $(U_i)_{i \in Q'_0}$ of the Q'-representation $M(\mathbf{k}')$ with $U_{N+1} = 0$ is a subrepresentation of the Q'-representation $M(\mathbf{k})$ and thus also a subrepresentation of the Q-representation $M(\mathbf{k})$. Using Proposition 3.2.1 we get an isomorphism of varieties $\mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k})) \to \mathrm{Gr}_{\mathbf{d}}(M(\mathbf{k}'))$. Since $d_{N+1} = 0$ we can use Example 3.6.18. By Equation (3.6.11) holds the formula for $\mathbf{l}$. For $\mathbf{m}$ Equation (3.6.12) yields

$$m_{ii} = k_{ii} + \sum\nolimits_{j'=1}^{N-i} l_{i+j',j'} - \sum\nolimits_{j'=1}^{i} l_{ij'} = k_{ii} - d_i + d_{i+1}$$

for $i \in Q_0$. Since $M(\mathbf{k})$ is exceptional Proposition 2.3.12 yields the smoothness of $Gr_{\mathbf{d}}(M(\mathbf{k}))$. The second case is the dual of the first one. □

4 Euler characteristics of quiver Grassmannians

In this chapter let k be the field of complex numbers $\mathbb{C}$. We compute now the Euler characteristic $\chi_{\mathbf{d}}(M)$ of a quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ associated to a tree or band module M. Moreover, we formulate very similar results for covering theory.

4.1 Gradings

To compute the Euler characteristic of a quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ we define some algebraic actions of the one-dimensional torus $\mathbb{C}^*$ on the projective variety $\mathrm{Gr}_{\mathbf{d}}(M)$ in this section.

4.1.1 Definitions

Let Q be a quiver and $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$ a Q-representation. Let I be the set $\{1, 2, \ldots, \dim_k M\}$ and $E = \{e_j | j \in I\}$ be a basis of $\bigoplus_{i\in Q_0} M_i$ such that $E \subseteq \bigcup_{i\in Q_0} M_i$.

Definition 4.1.1. A map $\partial\colon E \to \mathbb{Z}$ is called a *grading* of M.

So every grading depends on the choice of a basis E. It is useful to change the basis during calculations. A vector $m = \sum_{j\in I} m_j e_j \in M$ with $m_j \in \mathbb{C}$ is called ∂-*homogeneous* of degree $n \in \mathbb{Z}$ if $\partial(e_j) = n$ for all $j \in I$ with $m_j \neq 0$. In this case we set $\partial(m) = n$.

The following grading was studied by Riedtmann [44, Lemma 2.2]: Let $M = \bigoplus_{k=1}^{r} N_k$, where N_k is a subrepresentation of the Q-representation M for all k and $E \subseteq \bigcup_{k=1}^{r} N_k$. Then the grading $\partial\colon E \to \mathbb{Z}$ with $\partial(e_j) = k$ for $e_j \in N_k$ is called *Riedtmann grading* (or *R-grading* for short).

Definition 4.1.2. Let ∂ and $\partial_1, \ldots, \partial_r$ be gradings of M and $\Delta(\mathbf{y}, \mathbf{z}, a) \in \mathbb{Z}$ for all $\mathbf{y}, \mathbf{z} \in \mathbb{Z}^r$ and $a \in Q_1$ such that

$$\Delta\Big((\partial_m(e_j))_{1\leq m\leq r}, (\partial_m(e_i))_{1\leq m\leq r}, a\Big) = \partial(e_i) - \partial(e_j) \tag{4.1.1}$$

for all $i, j \in I$ and $a \in Q_1$ with $e_i \in M_{t(a)}$, $e_j \in M_{s(a)}$ and $m_i \neq 0$ for $M_a(e_j) = \sum_{k\in I} m_k e_k$. Then ∂ is called a *nice* $\partial_1, \ldots, \partial_r$-grading.

The definition of nice $\partial_1, \ldots, \partial_r$-gradings generalizes the gradings introduced by Cerulli Irelli [14, Theorem 1]. He only considers the nice $\varnothing$-gradings, i.e. $r = 0$. We say *nice grading* for short. Now we can successively apply these gradings. For example each R-grading is a nice grading.

Example 4.1.3. Let Q be the following quiver

$$1 \underset{b}{\overset{a}{\rightrightarrows}} 2$$

and $M = (M_1, M_2, M_a, M_b)$ the Q-representation with $M_1 = M_2 = \mathbb{C}^2$, $M_a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $M_b = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\{e_i = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), f_i = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$ be the canonical basis of M_i for each $i \in Q_0$ and $E = \{e_1, f_1, e_2, f_2\}$. Let $\partial, \partial_1 \colon E \to \mathbb{Z}$ be gradings with $\partial(e_1) = \partial(f_1) = 0$, $\partial(e_2) = 3$, $\partial(f_2) = 5$, $\partial_1(e_i) = 0$ and $\partial_1(f_i) = 1$ for each $i \in Q_0$. Then ∂ is a nice ∂_1-grading, since $\Delta(0,0,b) = \partial(e_2) - \partial(e_1) = 3$ and $\Delta(1,1,b) = \partial(f_2) - \partial(f_1) = 5$. Moreover, ∂ is a not a nice grading.

Example 4.1.4. In this example we state two extreme cases of gradings.

- Let ∂ and ∂' be gradings such that $\partial' \colon E \to \mathbb{Z}$ is an injective map. Then ∂ is a nice ∂'-grading.
- Let ∂ be a grading such that $\partial(e_i) = \partial(e_j)$ for all $i, j \in I$. Then ∂ is a nice grading.

By the following remark, we describe a way to visualize a nice $\partial_1, \dots, \partial_r$-grading ∂ of a Q-representation of the form $F_*(\mathbb{1}_S)$.

Remark 4.1.5. Let Q and S be quivers and $F \colon S \to Q$ a winding. Let $M = F_*(\mathbb{1}_S)$ and $\{f_i \in (\mathbb{1}_S)_i | i \in S_0\}$ be a basis of $\bigoplus_{i \in S_0} (\mathbb{1}_S)_i$. Then $E := \{F_*(f_i) | i \in S_0\}$ is a basis of $\bigoplus_{i \in Q_0} M_i$.

- Now we illustrate each grading $\partial \colon E \to \mathbb{Z}$ of M by a labelling of the quiver S. For this we identify the set E and S_0. Thus $\partial \colon S_0 \to \mathbb{Z}, i \mapsto \partial(F_*(f_i))$.
- For each nice $\partial_1, \dots, \partial_r$-grading ∂ we further extend ∂ in a meaningful way to $S_0 \cup S_1$ by

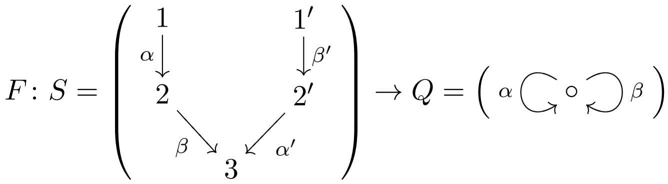
$$\partial(a) = \Delta\Big(\big(\partial_m(s(a))\big)_{1 \le m \le r}, \big(\partial_m(t(a))\big)_{1 \le m \le r}, F_1(a)\Big)$$

  for all $a \in S_1$. Then by Equation (4.1.1)

$$\partial(a) = \partial(t(a)) - \partial(s(a)) \tag{4.1.2}$$

  holds for all $a \in S_1$.
- Let $\partial \colon S_0 \cup S_1 \to \mathbb{Z}$ be a map with the following conditions:
  (S1) Equation (4.1.2) holds for all $a \in S_1$.
  (S2) $\partial(a) = \partial(b)$ for all $a, b \in S_1$ with $F_1(a) = F_1(b)$, $\partial_m(s(a)) = \partial_m(s(b))$ and $\partial_m(t(a)) = \partial_m(t(b))$ for all m.
  Then the map ∂ induces a nice $\partial_1, \dots, \partial_r$-grading $\partial \colon E \to \mathbb{Z}$ on M.
- Let $\partial \colon S_1 \to \mathbb{Z}$ be a map. If S is a tree and condition (S2) holds, then the map ∂ induces a nice $\partial_1, \dots, \partial_r$-grading $\partial \colon E \to \mathbb{Z}$ on M. If S is connected, such an induced grading ∂ is unique up to some shift.

Example 4.1.6. Let $F_*(\mathbb{1}_S)$ be the tree module described by the following picture.



Then $F_*(\mathbb{1}_S)$ has a basis $E = \{F_*(f_1), F_*(f_{1'}), F_*(f_2), F_*(f_{2'}), F_*(f_3)\}$ as above.

Let $\partial_1\colon S_1 \to \mathbb{Z}, \gamma \mapsto 1$ for all $\gamma \in S_1$ and $\partial_1(F_*(f_1)) = 0$. This induces by the previous remark a unique nice grading ∂_1 of $F_*(\mathbb{1}_S)$. In particular $\partial_1(1) = \partial_1(1') = 0$, $\partial_1(2) = \partial_1(2') = 1$ and $\partial_1(3) = 2$. Let $\partial_2\colon S_1 \to \mathbb{Z}, \beta \mapsto 1, \gamma \mapsto 0$ for all $\beta \neq \gamma \in S_1$ and $\partial_2(F_*(f_1)) = 0$. This induces a unique nice ∂_1-grading ∂_2 of $F_*(\mathbb{1}_S)$. So in particular $\partial_2(1) = \partial_2(2) = 0$, $\partial_2(1') = \partial_2(2') = \partial_2(3) = 1$.

4.1.2 Stable gradings

Let Q be a quiver, M a Q-representation and ∂ a grading. The algebraic group $\mathbb{C}^*$ acts by

$$\varphi_\partial\colon \mathbb{C}^* \to \mathrm{End}_{\mathbb{C}}(M),\ \varphi_\partial(\lambda)(e_j) := \lambda^{\partial(e_j)} e_j \tag{4.1.3}$$

on the vector space M. This defines in some cases a $\mathbb{C}^*$-action on the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$.

Definition 4.1.7. Let X be a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$ and ∂ a grading of M. If for all $U \in X$ and $\lambda \in \mathbb{C}^*$ the vector space $\varphi_\partial(\lambda)U$ is in X, then the grading ∂ is called *stable on* X.

For a locally closed subset X of $\mathrm{Gr}_{\mathbf{d}}(M)$ and gradings $\partial_1, \ldots, \partial_r$ let

$$X^{\partial_1,\ldots,\partial_r} := \Big\{U \in X \Big| U \text{ has a basis, which is } \partial_i\text{-homogeneous for each } i\Big\}. \tag{4.1.4}$$

By definition, each stable grading on X is also a stable grading on $X^{\partial_1,\ldots,\partial_r}$.

Lemma 4.1.8. *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ and $\partial_1, \ldots, \partial_r$ gradings. Then $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ if and only if $\varphi_{\partial_i}(\lambda)U = U$ as vector spaces for all i and $\lambda \in \mathbb{C}^*$.*

Proof. If $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ has a basis, which is ∂_i-homogeneous for each i, we get $\varphi_{\partial_i}(\lambda)U = U$ for each i and $\lambda \in \mathbb{C}^*$.

Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ such that $\varphi_{\partial_i}(\lambda)U = U$ for all i and $\lambda \in \mathbb{C}^*$. Our aim is to find a basis for U, which is ∂_i-homogeneous for each i. Let $s \in \mathbb{N}$ with $1 \leq s \leq r$ and $\{m_1, \ldots, m_t\}$ be a basis of U, which is ∂_i-homogeneous for each i with $1 \leq i < s$. For each j with $1 \leq j \leq t$ let $m_j = \sum_{i \in I} \lambda_{ij} e_i$ with $\lambda_{ij} \in \mathbb{C}$. For each $z \in \mathbb{Z}$ and $j \in \mathbb{N}$ with $1 \leq j \leq t$ define $m_{j,z} := \sum_{i \in I, \partial(e_i) = z} \lambda_{ij} e_i \in M$. Then $m_{j,z}$ is ∂_i-homogeneous for each i with $1 \leq i \leq s$, $\varphi_{\partial_s}(\lambda)(m_{j,z}) = \lambda^z m_{j,z}$ for all $\lambda \in \mathbb{C}^*$ and $m_j = \sum_{z\in\mathbb{Z}} m_{j,z}$. Then $\varphi_{\partial_s}(\lambda)(m_j) = \sum_{z\in\mathbb{Z}} \lambda^z m_{j,z} \in U$ for all $\lambda \in \mathbb{C}^*$ and so $m_{j,z} \in U$ for all $z \in \mathbb{Z}$ and all j. Since $\{m_{j,z} | 1 \leq j \leq t, z \in \mathbb{Z}\}$ generates U, a subset of this set is a basis of the vector space U, which is ∂_i-homogeneous for each i with $1 \leq i \leq s$. The statement follows by an induction argument. □

We will show that each nice grading is stable on $\mathrm{Gr}_{\mathbf{d}}(M)$.

Lemma 4.1.9. *Let $\partial_1, \ldots, \partial_r$ and ∂ be gradings of M. Then ∂ is stable on the variety $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ for all $\mathbf{d} \in \mathbb{N}^{Q_0}$ if and only if for all $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and $\partial_1, \ldots, \partial_r$-homogeneous elements $u \in M$ we have*

$$M_a\left(\varphi_\partial(\lambda)u\right) \in \varphi_\partial(\lambda) U_{\partial_1,\ldots,\partial_r}(u), \tag{4.1.5}$$

where $U_{\partial_1,\ldots,\partial_r}(u)$ is the minimal subrepresentation of M such that $u \in U_{\partial_1,\ldots,\partial_r}(u)$ and $U_{\partial_1,\ldots,\partial_r}(u) \in \mathrm{Gr}_{\mathbf{c}}(M)^{\partial_1,\ldots,\partial_r}$ for some $\mathbf{c} \in \mathbb{N}^{Q_0}$.

If $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ and $V \in \mathrm{Gr}_{\mathbf{c}}(M)^{\partial_1,\ldots,\partial_r}$, then Lemma 4.1.8 implies $U \cap V \in \mathrm{Gr}_{\mathbf{dim}(U\cap V)}(M)^{\partial_1,\ldots,\partial_r}$. So the submodule $U_{\partial_1,\ldots,\partial_r}(u)$ is well-defined and unique.

Proof. If ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$ for all $\mathbf{d} \in \mathbb{N}^{Q_0}$, then $\varphi_\partial(\lambda) U_{\partial_1,\ldots,\partial_r}(u)$ is a subrepresentation of M for all $\lambda \in \mathbb{C}^*$ and $u \in M$.

Let $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$. If Equation (4.1.5) holds for all $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and $\partial_1, \ldots, \partial_r$-homogeneous $u \in M$, then $M_a\left(\varphi_\partial(\lambda) U_{s(a)}\right) \subseteq \varphi_\partial(\lambda) U_{t(a)}$ for all $\lambda \in \mathbb{C}^*$ and $a \in Q_1$, since U is generated by $\partial_1, \ldots, \partial_r$-homogeneous elements. Thus $\varphi_\partial(\lambda) U \in \mathrm{Gr}_{\mathbf{d}}(M)$. □

Lemma 4.1.10. *Let Q be a quiver, M a Q-representation and $\mathbf{d}$ a dimension vector. Then every nice $\partial_1, \ldots, \partial_r$-grading ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}$.*

Proof. By Lemma 4.1.9, it is enough to consider $\lambda \in \mathbb{C}^*$, $a \in Q_1$ and a homogeneous $u \in M$. We write $u = \sum_{k\in I} u_k e_k$ with $u_k \in \mathbb{C}$, $M_a(e_k) = \sum_{j\in I} m_{jk} e_j$ with $m_{jk} \in \mathbb{C}$ for all $k \in I$ and $M_a(u) = \sum_{\mathbf{z}\in\mathbb{Z}^r} m_{\mathbf{z}}$ with $(\partial_m(m_{\mathbf{z}}))_m = \mathbf{z}$. So $m_{\mathbf{z}} = \sum_{k,j\in I, (\partial_m(e_j))_m = \mathbf{z}} u_k m_{jk} e_j$ and

$$\begin{aligned} M_a\left(\varphi_\partial(\lambda)u\right) &= \sum\nolimits_{k\in I} u_k M_a\left(\lambda^{\partial(e_k)} e_k\right) = \sum\nolimits_{k,j\in I} u_k \lambda^{\partial(e_k)} m_{jk} e_j \\ &= \sum\nolimits_{k,j\in I} \lambda^{\partial(e_k)-\partial(e_j)} u_k m_{jk} \varphi_\partial(\lambda) e_j \\ &= \varphi_\partial(\lambda)\left(\sum\nolimits_{k,j\in I} \lambda^{\Delta((\partial_m(u))_m, (\partial_m(e_j))_m, a)} u_k m_{jk} e_j\right) \\ &= \varphi_\partial(\lambda)\left(\sum\nolimits_{\mathbf{z}\in\mathbb{Z}^r} \lambda^{\Delta((\partial_m(u))_m, \mathbf{z}, a)} m_{\mathbf{z}}\right) \in \varphi_\partial(\lambda) U_{\partial_1,\ldots,\partial_r}(u). \end{aligned}$$

This gives the lemma. □

4.2 Preliminaries

By the following theorem it is enough to compute the Euler characteristic of the subset of fixed points under the induced $\mathbb{C}^*$-action on the Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$. This generalizes some result of Cerulli Irelli [14, Theorem 1].

Theorem 4.2.1. *Let Q be a locally finite quiver, M a finite-dimensional representation of Q, $X \subseteq \mathrm{Gr}_{\mathbf{d}}(M)$ a locally closed subset and ∂ a stable grading on X. Then X^∂ is a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$ and the Euler characteristic of X equals the Euler characteristic of X^∂. If the subset X is non-empty and closed in $\mathrm{Gr}_{\mathbf{d}}(M)$, then X^∂ is also non-empty and closed in $\mathrm{Gr}_{\mathbf{d}}(M)$.*

The following well-known proposition is used to prove this theorem.

Proposition 4.2.2 (Bialynicki-Birula [5, Corollary 2])**.** *Let $\mathbb{C}^*$ act on a locally closed subset X of a projective variety Y. Then the subset of fixed points $X^{\mathbb{C}^*}$ under this action is a locally closed subset of Y and $\chi(X) = \chi(X^{\mathbb{C}^*})$. If the subset X is non-empty and closed in Y, then $X^{\mathbb{C}^*}$ is also non-empty and closed in Y.*

Proof. The subset of fixed points $X^{\mathbb{C}^*}$ is closed in X. By [6], this is non-empty if X is non-empty and closed in Y (see also [16, Corollary 2.4.2.]).

So we decompose X into the locally closed subset of fixed points $X^{\mathbb{C}^*}$ and its complement $U = X \backslash X^{\mathbb{C}^*}$ in X. So $\chi(X) = \chi\left(X^{\mathbb{C}^*}\right) + \chi(U)$. Since U is the union of the non trivial orbits in X, the projection $U \to U/\mathbb{C}^*$ is an algebraic morphism. Since $\chi(\mathbb{C}^*) = 0$ the Euler characteristic of U is also zero. □

Proof of Theorem 4.2.1. The action φ_∂ of the algebraic group $\mathbb{C}^*$ on the projective variety X is well-defined. Thus Proposition 4.2.2 yields the equality of the Euler characteristic of X and the Euler characteristic of the set of fixed points under this action. By Lemma 4.1.8, a subrepresentation U of M in X is a fixed point of φ_∂ if and only if U has a basis of ∂-homogeneous elements. □

Theorem 4.2.1 yields directly the following corollary, since different $\mathbb{C}^*$-actions commute. So we can use more than one grading at the same time.

Corollary 4.2.3. *Let Q be a quiver, M a Q-representation and $\partial_1, \ldots, \partial_r$ gradings of M such that for all $1 \leq i \leq r$ the grading ∂_i is a stable grading on $\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_{i-1}}$. Then $\chi_{\mathbf{d}}(M) = \chi\left(\mathrm{Gr}_{\mathbf{d}}(M)^{\partial_1,\ldots,\partial_r}\right)$.*

The following result is well-known by Riedtmann [44]. Nevertheless, we prove it using our notation.

Proposition 4.2.4 (Riedtmann)**.** *Let Q be a quiver, $\mathbf{d}$ a dimension vector and M and N Q-representations. Then*

$$\chi_{\mathbf{d}}(M \oplus N) = \sum\nolimits_{0 \leq \mathbf{c} \leq \mathbf{d}} \chi_{\mathbf{c}}(M)\chi_{\mathbf{d}-\mathbf{c}}(N).$$

Using this proposition it is enough to consider the Euler characteristic of quiver Grassmannians associated to indecomposable representations.

Proof. We choose any basis of M and any basis of N. So the union induces a basis of $M \oplus N$. Using a R-grading ∂, we have to compute the Euler characteristic of the set of fixed points. This variety $\mathrm{Gr}_{\mathbf{d}}(M \oplus N)^{\partial}$ can be decomposed into a union of locally closed sets $X_{\mathbf{c}}$, where the subrepresentation of M has dimension vector $\mathbf{c}$ and the subrepresentation of N has dimension vector $\mathbf{d}-\mathbf{c}$. Then $\chi_{\mathbf{d}}(M) = \sum_{0\leq\mathbf{c}\leq\mathbf{d}} \chi(X_{\mathbf{c}}) = \sum_{0\leq\mathbf{c}\leq\mathbf{d}} \chi_{\mathbf{c}}(M)\chi_{\mathbf{d}-\mathbf{c}}(N)$. □

4.3 Main Theorem

The main result of this chapter is the following result. It is proven in Section 4.7.

Theorem 4.3.1.

1. *Let Q and S be finite quivers, $F\colon S \to Q$ a tree or a band, $\mathbf{d}$ a dimension vector of Q and V a finite-dimensional S-representation. Then*

$$\chi_{\mathbf{d}}(F_*(V)) = \sum\nolimits_{\mathbf{t}\in\mathbf{F}^{-1}(\mathbf{d})} \chi_{\mathbf{t}}(V). \tag{4.3.1}$$

2. *Let S be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{t} = (t_i)_{i\in S_0}$ a dimension vector of S and $V \in \mathcal{I}_S^n$, i.e. V is a band module of S and $\dim_{\mathbb{C}} V_i = n$ for some $i \in S_0$. Then*

$$\chi_{\mathbf{t}}(V) = \left(\prod_{\substack{i\in S_0\\ \text{source}}} \frac{(n-t_i)!}{t_i!}\right)\left(\prod_{\substack{i\in S_0\\ \text{sink}}} \frac{t_i!}{(n-t_i)!}\right)\left(\prod_{a\in S_1} \frac{1}{(t_{t(a)}-t_{s(a)})!}\right) \tag{4.3.2}$$

   with $0! = 1$, $s! = 0$ and $\frac{1}{s!} = 0$ for all negative $s \in \mathbb{Z}$.

3. *Let $\hat{Q}$ be a locally finite quiver, G a free (abelian) group, which acts freely on $\hat{Q}$, and $\pi\colon \hat{Q} \to Q$ the induced projection. Let $\mathbf{d}$ be a dimension vector of $Q = \hat{Q}/G$ and V a finite-dimensional $\hat{Q}$-representation. Then*

$$\chi_{\mathbf{d}}(\pi_*(V)) = \sum\nolimits_{\mathbf{t}\in\boldsymbol{\pi}^{-1}(\mathbf{d})} \chi_{\mathbf{t}}(V). \tag{4.3.3}$$

Since Part 3 of this theorem holds for free and free abelian groups, we write “free (abelian) group”. Corollaries, examples and further explanations of this theorem are given in the following sections.

4.4 Tree and band modules

All the corollaries and examples of this section are strictly related to Part 1 and 2 of Theorem 4.3.1.

Corollary 4.4.1. *Let $F\colon S \to Q$ be a tree or a band and* $\mathbf{d}$ *a dimension vector of Q. Then we have to count successor closed subquivers of S with dimension vectors in* $\mathbf{F}^{-1}(\mathbf{d})$ *to compute $\chi_{\mathbf{d}}(F_*(\mathbb{1}_S))$.*

This corollary follows immediately from Part 1 of Theorem 4.3.1.

Corollary 4.4.2. *Let Q be a quiver, M a tree or band module and* $\mathbf{d}$ *a dimension vector of Q such that the variety* $\mathrm{Gr}_{\mathbf{d}}(M)$ *is non-empty. Then $\chi_{\mathbf{d}}(M) > 0$.*

Proof. The inequality $\chi_{\mathbf{d}}(M) \geq 0$ is clear by Theorem 4.3.1. We prove the statement of Part 1 of Theorem 4.3.1 by applying Theorem 4.2.1 several times. So also the stronger inequality $\chi_{\mathbf{d}}(M) > 0$ follows. □

If the quiver S is an oriented cycle, each band module $B_*(V)$ has a unique filtration with $n = \dim_{\mathbb{C}} V_i$ pairwise isomorphic simple factors of dimension $|S_0|$. In this case Part 2 of Theorem 4.3.1 holds (see Example 4.4.3).

Therefore we assume without loss of generality that S is not an oriented cycle. Let $\{i_1, \ldots, i_r\}$ be the sources of S and $\{i'_1 \ldots, i'_r\}$ the sinks. We assume that $r > 0$ and $1 \leq i_1 < i'_1 < i_2 < i'_2 \ldots < i_r < i'_r \leq l$. Then the quiver S is visualized in Figure 4.4.1.

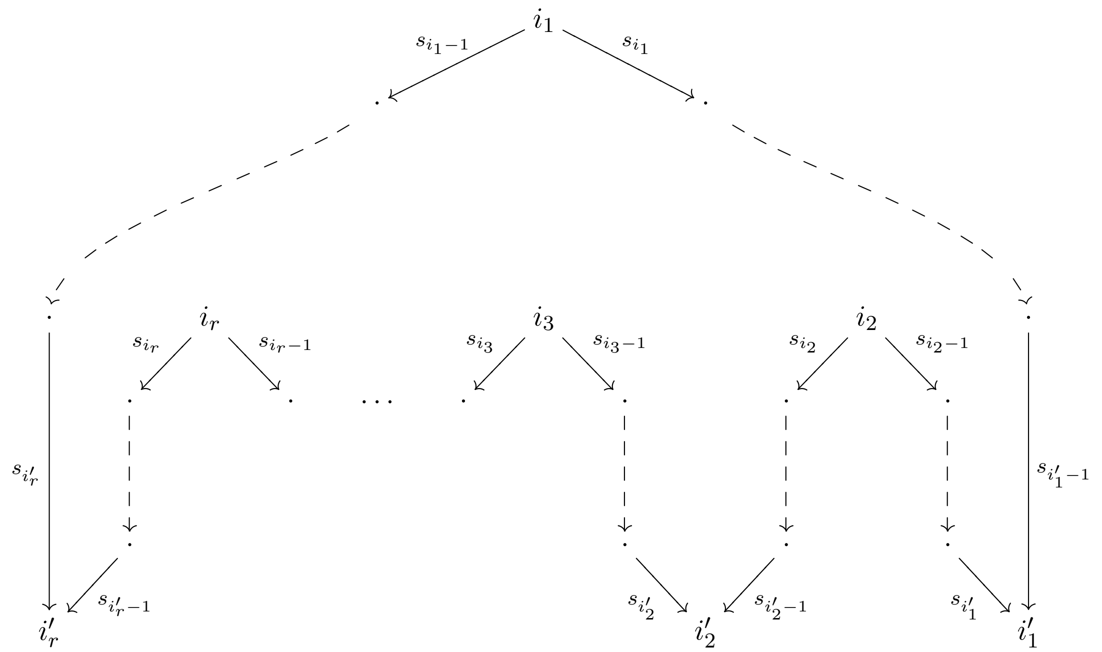


Figure 4.4.1: A quiver of type $\tilde{A}_{l-1}$.

Example 4.4.3. Let S, V and $\mathbf{t}$ be as in Part 2 of Theorem 4.3.1. Let $t_1 = t_2 = \ldots = t_l \leq n$. Then $\chi_{\mathbf{t}}(V) = 1$.

The next example shows one result of [14, Proposition 3] as a special case of Part 2 of Theorem 4.3.1.

Example 4.4.4. Let S, V and $\mathbf{t}$ be as in Part 2 of Theorem 4.3.1. Let $r = 1$, $i_1 = 1$ and $i'_1 = l$. Then

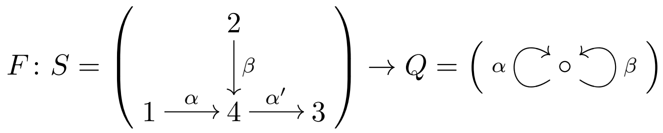
$$\chi_{\mathbf{t}}(V) = \binom{t_l}{t_1}\binom{n-t_1}{n-t_l}\frac{(t_l - t_1)!}{\prod_{i=1}^{l-1}(t_{i+1}-t_i)!}.$$

Example 4.4.5. Let Q, B be as in Example 2.2.8 and $V \in \mathcal{I}_S^2$. Using Theorem 4.3.1, it is easy to calculate the Euler characteristics $\chi_d(B_*(V))$. For instance,

$$\begin{aligned}\chi_4(B_*(V)) &= \chi_{(0,0,2,2)}(V) + \chi_{(0,2,0,2)}(V) + \chi_{(0,1,1,2)}(V) + \chi_{(1,1,1,1)}(V)\\ &= 1+1+4+1 = 7\end{aligned}$$

since $\mathbf{F}^{-1}(4) = \{\mathbf{t} = (t_1, t_2, t_3, t_4) \in \mathbb{N}^{S_0} | t_1 + t_2 + t_3 + t_4 = 4\}$ and $\mathrm{Gr}_{\mathbf{t}}(V) = \varnothing$ if $s(a) > t(a)$ for some $a \in S_1$ or $t_i > 2$ for some $i \in S_0$.

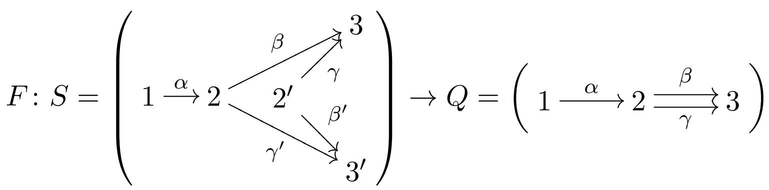
Example 4.4.6. If F is a tree or a band, Part 1 of Theorem 4.3.1 holds for each S-representation V. Let F be the tree described by the following picture.



Let V be an indecomposable S-representation with dimension vector $(1, 1, 1, 2)$. Then the dimension vector of a subrepresentation U of the S-representation V with $\dim_k U = 3$ is in $\{(1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 1, 2)\}$. Thus

$$\chi_3(F_*(V)) = \chi_{(1,0,1,1)}(V) + \chi_{(0,1,1,1)}(V) + \chi_{(0,0,1,2)}(V) = 3.$$

Example 4.4.7. If S is not a tree and not a band, Equation (4.3.1) does not hold in general. To see this we consider the winding F described by the following picture.



Then $F_*(\mathbb{1}_S)$ is indecomposable and

$$\chi_{(0,1,1)}(F_*(\mathbb{1}_S)) = 2 \neq 0 = \sum\nolimits_{\mathbf{t}\in\mathbf{F}^{-1}((0,1,1))}\chi_{\mathbf{t}}(\mathbb{1}_S).$$

It is easy to see that there exists no quiver S and no winding F such that a formula similar to Equation (4.3.1) holds. So it is not possible to describe these Euler characteristics purely combinatorial using our techniques.

4.5 Quiver flag varieties

Definition 4.5.1. Let Q be a quiver, M a Q-representation and $\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(r)}$ dimension vectors. Then the closed subvariety

$$\mathcal{F}_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M) := \left\{ 0 \subseteq U^{(1)} \subseteq \dots \subseteq U^{(r)} \subseteq M \,\middle|\, U^{(i)} \in \mathrm{Gr}_{\mathbf{d}^{(i)}}(M) \ \forall i \right\}$$

of the classical partial flag variety is called the *quiver flag variety.*

We denote the Euler characteristic of $\mathcal{F}_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M)$ by $\chi_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M)$. The following corollaries of Part 1 of Theorem 4.3.1 follow immediately from the analogous statements for the quiver Grassmannians.

Corollary 4.5.2 (Riedtmann)**.** *Let Q be a quiver, $\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(r)}$ dimension vectors and M and N Q-representations. Then*

$$\chi_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M \oplus N) = \sum\nolimits_{0 \le \mathbf{c}^{(i)} \le \mathbf{d}^{(i)}} \chi_{\mathbf{c}^{(1)},\dots,\mathbf{c}^{(r)}}(M) \chi_{\mathbf{d}^{(1)}-\mathbf{c}^{(1)},\dots,\mathbf{d}^{(r)}-\mathbf{c}^{(r)}}(N).$$

Corollary 4.5.3. *Let Q and S be quivers, $\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(r)}$ dimension vectors of Q and V a S-representation. If $F\colon S \to Q$ is a tree or a band, then*

$$\chi_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(F_*(V)) = \sum\nolimits_{\mathbf{t}^{(i)} \in \mathbf{F}^{-1}(\mathbf{d}^{(i)})} \chi_{\mathbf{t}^{(1)},\dots,\mathbf{t}^{(r)}}(V).$$

In particular we have to count flags of successor closed subquivers of S with dimension vectors in $\mathbf{F}^{-1}(\mathbf{d}^{(i)})$ to compute $\chi_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(F_*(\mathbb{1}_S))$.

Corollary 4.5.4. *Let Q be a quiver, M a tree module and $\mathbf{d}^{(1)}, \dots, \mathbf{d}^{(r)}$ dimension vectors of Q such that $\mathcal{F}_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M)$ is non-empty. Then $\chi_{\mathbf{d}^{(1)},\dots,\mathbf{d}^{(r)}}(M) > 0$.*

Example 4.5.5. Let $Q = (1 \rightrightarrows 2)$, $n \in \mathbb{N}$ with $n \ge 3$ and M an indecomposable Q-representation with dimension vector (n, n). We show $\chi_{(1,2),(2,3)}(M) = 8(n-2)$.

Let $B\colon Q \to Q$ be the identity winding. For each $\mu \in \mathbb{C}$ there is an automorphism of the algebra $\mathbb{C}Q$ such that $B_*(\lambda, n)$ is mapped to $B_*(\lambda - \mu, n)$. This is not necessarily a band module. So we assume without loss of generality that M is a string module. Let

$$T^{(n)} = \left(\begin{array}{ccccccccccc} 1^{(1)} & & & & 1^{(2)} & & & & & 1^{(n)} & & \\ & \searrow & & \swarrow & & \searrow & & & \swarrow & & \searrow & \\ & & 2^{(1)} & & & & 2^{(2)} & \cdots & 2^{(n-1)} & & & 2^{(n)} \end{array} \right),$$

$$U^{(n)} = \left(\begin{array}{cccccccccc} 1^{(1)} & & & & 1^{(2)} & & & & & 1^{(n)} \\ & \searrow & & \swarrow & & \searrow & & & \swarrow & \\ & & 2^{(1)} & & & & 2^{(2)} & \cdots & 2^{(n-1)} & \end{array} \right).$$

Thus to prove the equation above we have to count flags of successor closed subquivers of the following quiver $T^{(n)}$ associated to the dimension vectors $(1, 2)$ and $(2, 3)$. For each subquiver V of $T^{(n)}$ let

$$\mathbf{dim}\, V = \Big(\big|\{ i | 1^{(i)} \in V_0 \}\big|, \big|\{ i | 2^{(i)} \in V_0 \}\big| \Big).$$

For dimension vectors $\mathbf{c}$ and $\mathbf{d}$ of Q we set

$$X_{\mathbf{c}}(V) = \{R \subseteq V \mid R \text{ is a successor closed subquiver of } V, \mathbf{dim}\, R = \mathbf{c}\},$$
$$X_{\mathbf{c},\mathbf{d}}(V) = \{R \subseteq S \subseteq V \mid R \in X_{\mathbf{c}}(S), S \in X_{\mathbf{d}}(V)\}.$$

Using Corollary 4.5.3, it is enough to show the following equality.

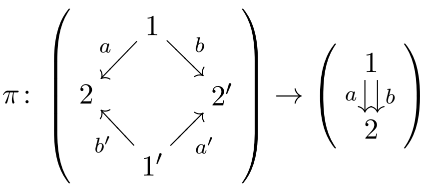
$$\begin{aligned}
\left|X_{(1,2),(2,3)}(T^{(n)})\right| =& \sum\nolimits_{i=1}^{n} \left|\{(R \subseteq S) \in X_{(1,2),(2,3)}(T^{(n)}) \mid 1^{(i)} \in R_0\}\right| \\
=& \left|X_{(0,1),(1,2)}(T^{(n-1)})\right| + \sum\nolimits_{i=2}^{n} \left|X_{(1,1)}(U^{(i-1)} \dot\cup T^{(n-i)})\right| \\
=& \tbinom{2}{1} \left|X_{(1,2)}(T^{(n-1)})\right| + \Big(\left|X_{(0,1)}(T^{(n-2)})\right| + \left|X_{(1,1)}(T^{(n-2)})\right|\Big) \\
&+ \sum\nolimits_{i=3}^{n-1} \Big(\left|X_{(1,1)}(U^{(i-1)})\right| + \left|X_{(1,1)}(T^{(n-i)})\right|\Big) + \left|X_{(1,1)}(U^{(n-1)})\right| \\
=& 2\left(\tbinom{n-2}{1} + \tbinom{n-2}{1}\right) + \left(\tbinom{n-2}{1} + 1\right) + (n-3)(2+1) + 2 = 8(n-2).
\end{aligned}$$

4.6 Coverings of quivers

We give two examples of coverings. In one case the formula in Part 3 of Theorem 4.3.1 holds and in the other it fails. This shows for this statement G has to be (abelian) free and to act freely on $\hat{Q}$.

Example 4.6.1. Let $\hat{Q} = (\mathbb{Z}, \mathbb{Z})$ and $G = \mathbb{Z}$ with $s(n) = n$, $t(n) = n+1$ and $gk = g+k$ for all $k \in \hat{Q}_0 \dot\cup \hat{Q}_1$ and $g \in G$. Let $\hat{I}$ be an ideal of $\mathbb{C}\hat{Q}$ generated by the paths of $\hat{Q}$ of length m and $I = \hat{I}/G$. Then $Q = \hat{Q}/G$ is the one loop quiver and $\mathbb{C}Q/I$ is isomorphic to $\mathbb{C}[T]/(T^m)$. Let $l \leq m$. For each indecomposable $\mathbb{C}Q/I$-module M of length l there is an indecomposable $\mathbb{C}\hat{Q}/\hat{I}$-module N with $\pi_*(N) \cong M$. Then for $0 \leq k \leq l$ holds $\chi_k(M) = \chi(\{U \subseteq N \mid \dim_k U = k\}) = 1$.

Example 4.6.2. Let $\pi\colon \hat{Q} \to Q$ be the winding described by the following picture:



Then $\mathbb{1}_{\hat{Q}}$ is indecomposable and has only one two-dimensional subrepresentation, but $\pi_*(\mathbb{1}_{\hat{Q}})$ is decomposable and has three two-dimensional subrepresentations. Thus

$$\chi_{(1,1)}(\pi_*(\mathbb{1}_{\hat{Q}})) = 2 \neq 0 = \sum\nolimits_{\mathbf{t} \in \boldsymbol{\pi}^{-1}((1,1))} \chi_{\mathbf{t}}(\mathbb{1}_{\hat{Q}}).$$

4.7 Proof of the main Theorem

4.7.1 Proof of Part 1 of Theorem 4.3.1

If F is a tree, Part 3 of Theorem 4.3.1 yields this theorem by the following property. If $F\colon S \to Q$ is a tree and $\pi\colon \hat{Q} \to Q$ a universal covering, then a factorization $F = \pi\iota$ exists such that $\iota\colon S \to \hat{Q}$ is injective. Nevertheless, we give in this section an independent prove.

For this it is enough to consider $V = \mathbb{1}_S$. By Remark 4.1.5 the set $E = \{F_*(f_i) | i \in S_0\}$ is a basis of $F_*(\mathbb{1}_S)$. We write $\partial(i)$ instead of $\partial(F_*(f_i))$ for all $i \in S_0$. To prove Part 1 of Theorem 4.3.1 we will use the following proposition inductively. This proposition holds in general and not only for trees and bands, but in the case of trees and bands there exist enough nice gradings such that Part 1 of Theorem 4.3.1 follows (see Lemma 4.7.3 and 4.7.4).

Proposition 4.7.1. *Let Q and S be locally finite quivers, T a finite subquiver of S, $F\colon S \to Q$ a winding of quivers and $\mathbf{d}$ a dimension vector of Q. Let ∂ be a nice grading of $F_*(\mathbb{1}_T)$. Define a quiver Q' by*

$$\begin{aligned} Q'_0 =& \{(F_0(i), \partial(i)) | i \in S_0\} \\ Q'_1 =& \{(\partial(s(a)), \partial(t(a)), F_1(a)) | a \in S_1\} \\ s'(\partial(s(a)), \partial(t(a)), F_1(a)) =& (s(F_1(a)), \partial(s(a))) \\ t'(\partial(s(a)), \partial(t(a)), F_1(a)) =& (t(F_1(a)), \partial(t(a))) \text{ for all } a \in Q_1. \end{aligned}$$

Define windings $F'\colon S \to Q'$ by $i \mapsto (F_0(i), \partial(i))$, $a \mapsto (\partial(s(a)), \partial(t(a)), F_1(a))$ and $G\colon Q' \to Q$ by $(F_0(i), \partial(i)) \mapsto F_0(i)$, $(\partial(s(a)), \partial(t(a)), F_1(a)) \mapsto F_1(a)$. Then

$$\chi_{\mathbf{d}}(F_*(\mathbb{1}_T)) = \sum\nolimits_{\mathbf{t} \in \mathbf{G}^{-1}(\mathbf{d})} \chi_{\mathbf{t}}({F'}_*(\mathbb{1}_T)).$$

Proof. By definition of Q', F' and G holds $F = GF'$ and $\mathrm{Gr}_{\mathbf{d}}(F_*(\mathbb{1}_T))^{\partial} =$

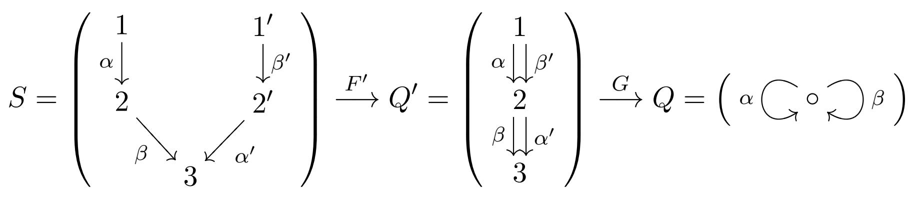
$$\left\{ U \subseteq F_*(\mathbb{1}_T) \middle| \, \mathbf{dim}\, U = \mathbf{d},\ U \text{ has a } \partial\text{-homogeneous vector space basis.} \right\}$$
$$= \dot{\bigcup}_{\mathbf{t} \in \mathbf{G}^{-1}(\mathbf{d})} \mathrm{Gr}_{\mathbf{t}}(F'_*(\mathbb{1}_T)).$$

Thus Theorem 4.2.1 implies

$$\chi_{\mathbf{d}}(F_*(\mathbb{1}_T)) = \chi\left(\mathrm{Gr}_{\mathbf{d}}(F_*(\mathbb{1}_T))^{\partial}\right) = \sum\nolimits_{\mathbf{t} \in \mathbf{G}^{-1}(\mathbf{d})} \chi_{\mathbf{t}}(F'_*(\mathbb{1}_T)).$$

□

Example 4.7.2. We have a look at Example 4.1.6. Let Q' and F' be described by the following picture.



Using the nice grading ∂_1, it is enough to observe $F'_*(\mathbb{1}_S)$ and $\chi_{\mathbf{t}}\,(F'_*(\mathbb{1}_S))$ to compute $\chi_{\mathbf{d}}(F_*(\mathbb{1}_S))$. So the nice ∂_1-grading ∂_2 induces a nice grading of $F'_*(\mathbb{1}_S)$.

Lemma 4.7.3. *Equation* (4.3.1) *holds for each tree module* $F_*(\mathbb{1}_S)$.

Proof. By Proposition 4.7.1 it is enough to treat the cases when $F_0\colon S_0 \to Q_0$ is surjective and not injective. If $i, j \in S_0$ exist with $F_0(i) = F_0(j)$ and $i \neq j$, we construct a nice grading ∂ of $F_*(\mathbb{1}_S)$ such that $\partial(i) \neq \partial(j)$. So we do an induction over $|S_0| - |Q_0|$.

Let S' be a minimal connected subquiver of S such that there exist $i, j \in S'_0$ with $F_0(i) = F_0(j)$ and $i \neq j$. Then S' is of type A_l. Let $F'\colon S' \to Q$ be the winding induced by F. Since S is a tree, every nice grading of $F'_*(\mathbb{1}_{S'})$ can be extended to a nice grading of $F_*(\mathbb{1}_S)$.

So without loss of generality let S' be equal to S. So $S_0 = \{1, \ldots, l\}$ and $S_1 = \{s_1, s_2, \ldots, s_{l-1}\}$ as in Section 2.2 and $F_0(1) = F_0(l)$ and $1 < l$. So $\partial\colon S_0 \to \mathbb{Z}, i \mapsto \delta_{i1}$ defines a grading of $F_*(\mathbb{1}_S)$ with $\partial(1) = 1 \neq 0 = \partial(l)$. Since $F_0(2) \neq F_0(l-1)$, we have $F_1(s_1)^{-\varepsilon_1} \neq F_1(s_{l-1})^{\varepsilon_{l-1}}$ and so for all $1 < k < l$ the equation $F_1(s_1) \neq F_1(s_k)$ holds by the minimality of S. Therefore ∂ is a nice grading. □

Lemma 4.7.4. *Equation* (4.3.1) *holds for each band module* $F_*(\mathbb{1}_S)$.

Proof. Let $i, j \in S_0$ with $F_0(i) = F_0(j)$, $i < j$ and $j - i$ minimal (i.e. $F_0(k) \neq F_0(m)$ for all $k, m \in S_0$ with $i \leq k < m \leq j$ and $(i, j) \neq (k, m)$). If no such tuple $(i, j) \in S_0 \times S_0$ exists, we are done. By Proposition 4.7.1 it is again enough to construct a nice grading ∂ of $F_*(\mathbb{1}_S)$ such that $\partial(i) \neq \partial(j)$.

For each $a \in Q_1$ let $\rho(a) := \sum_{i=1}^{l} \varepsilon_i \delta_{a,F_1(s_i)}$ and $\partial^{(a)}\colon Q_1 \to \mathbb{Z}, b \mapsto \delta_{ab}$ a map.

- If $a \in Q_1$ with $\rho(a) = 0$, then $\partial^{(a)}$ induces a nice grading $\partial^{(a)}$ of $F_*(\mathbb{1}_S)$ such that
$$\partial^{(a)}(i) - \partial^{(a)}(j) = \sum\nolimits_{k=i}^{j-1} \varepsilon_k \delta_{a,F_1(s_k)}.$$
- If $a, b \in Q_1$, then $\partial^{(a,b)} := \rho(a)\partial^{(b)} - \rho(b)\partial^{(a)}$ induces similarly a nice grading $\partial^{(a,b)}$ of $F_*(\mathbb{1}_S)$.

If $\rho(F_1(s_i)) = 0$, then $\partial^{(F_1(s_i))}(i) - \partial^{(F_1(s_i))}(j) = \varepsilon_i$ since $j - i$ is minimal. Thus $F_1(s_i) \neq F_1(s_k)$ for all $k \in S_0$ with $i < k < j$.

If $\rho(F_1(s_i)) \neq 0$, we should have a look at the grading $\partial^{(F_1(s_i),F_1(s_k))}$ for all $k \in S_0$. If $\partial^{(F_1(s_i),F_1(s_k))}(i) - \partial^{(F_1(s_i),F_1(s_k))}(j) \neq 0$ for some $k \in S_0$, we are done. So let us assume $\partial^{(F_1(s_i),F_1(s_k))}(i) - \partial^{(F_1(s_i),F_1(s_k))}(j) = 0$ for all $k \in S_0$ and for all tuples $(i, j) \in S_0 \times S_0$ with $0 < j - i$ minimal. If $F_1(s_i) \neq F_1(s_k)$, then

$$\begin{aligned} 0 &= \partial^{(F_1(s_i),F_1(s_k))}(i) - \partial^{(F_1(s_i),F_1(s_k))}(j) \\ &= \rho(F_1(s_i)) \left(\sum\nolimits_{m=i+1}^{j-1} \varepsilon_m \delta_{F_1(s_k),F_1(s_m)} \right) - \rho(F_1(s_k))\varepsilon_i \\ &= \rho(F_1(s_i))\varepsilon_{k'} - \rho(F_1(s_k))\varepsilon_i \end{aligned}$$

for some $k' \in S_0$ with $i < k' < j$ and $F_1(s_k) = F_1(s_{k'})$. So $\varepsilon_k \rho(F_1(s_k)) = \varepsilon_m \rho(F_1(s_m))$ for all $k, m \in S_0$. In other words, $\rho(F_1(s_k)) \neq 0$ for all $k \in S_1$ and $\varepsilon_k = \varepsilon_m$ for all $k, m \in S_0$ with $F_1(s_k) = F_1(s_m)$. So some $r \in \mathbb{Z}_{>0}$ exists such that $F_1(s_k) = F_1(s_{k+r})$ for all $k \in S_0$. By Example 2.2.6, the representation $F_*(\mathbb{1}_S)$ is decomposable if $r < l$. This is a contradiction. □

4.7.2 Proof of Part 2 of Theorem 4.3.1

Let S be a quiver of type $\widetilde{A}_{l-1}$ and $\{i_1, \dots, i_r\}$ be the sources and $\{i'_1, \dots, i'_r\}$ be the sinks of S. It is visualized in Figure 4.4.1. For all $i, j \in S_0$ with $i \le j$ let S^{ij} be the full subquiver of S with $S_0^{ij} = \{i, i+1, \dots, j\}$.

Lemma 4.7.5. *Let S be a quiver as above, $V \in \mathcal{I}_S^n$ and $\mathbf{t} = (t_1, \dots, t_l)$ be a dimension vector of S. For all $s, t \in S_0$ and $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ we define $X^{(s,t)}_{\alpha,\beta,\gamma,\delta}(\mathbf{t})$ to be*

$$\chi_{\mathbf{t}}\left(M(S^{i'_s+1,i'_r-1})^{\alpha} \oplus M(S^{i'_s+1,i_1-1})^{\beta} \oplus M(S^{i_t+1,i'_r-1})^{\gamma} \oplus M(S^{i_t+1,i_1-1})^{\delta}\right),$$

where $M(S^{ij})$ is an indecomposable S^{ij}-representation with dimension $j-i+1$ for all $i, j \in S_0$ with $i \le j$. Then

$$\chi_{\mathbf{t}}(V) = \sum\nolimits_{k\in\mathbb{Z}} \binom{t_{i_1}}{k}\binom{n-t_{i_1}}{k} X^{(1,1)}_{t_{i_1}-k,k,k,n-t_{i_1}-k}(\mathbf{t}') \tag{4.7.1}$$

with

$$\mathbf{t}' = \begin{cases} (0, t_{i_1+1}-t_{i_1}, \dots, t_{i'_1-1}-t_{i_1}, t_{i'_1}-t_{i_1}-k, t_{i'_1+1}-t_{i_1}, \dots, t_{i_1-1}-t_{i_1}) & \text{if } r = 1, \\ (0, t_{i_1+1}-t_{i_1}, \dots, t_{i'_1}-t_{i_1}, t_{i'_1+1}, \dots, t_{i'_r-1}, t_{i'_r}-t_{i_1}, \dots, t_{i_1-1}-t_{i_1}) & \text{if } r > 1. \end{cases}$$

We use here the convention $\binom{r}{s} = 0$ for all $r, s \in \mathbb{Z}$ if $s < 0$ or $s > r$. We visualize the S-representations $M(S^{i'_s+1,i'_r-1})$, $M(S^{i'_s+1,i_1-1})$, $M(S^{i_t+1,i'_r-1})$ and $M(S^{i_t+1,i_1-1})$ in Figure 4.7.1.

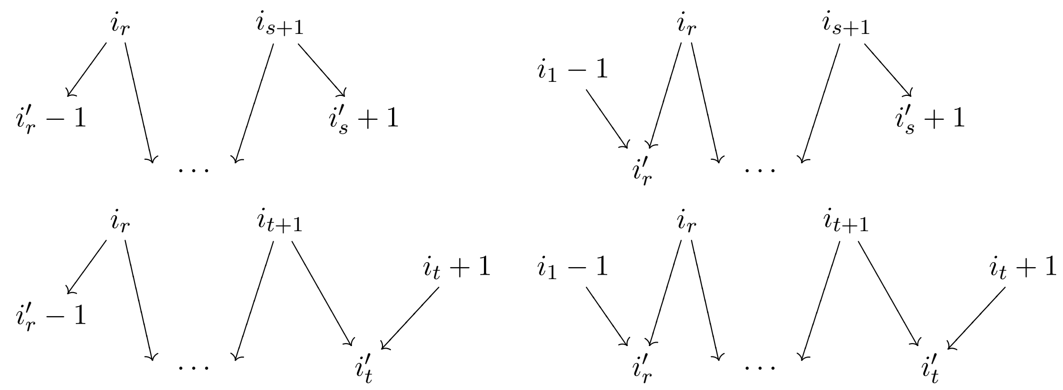


Figure 4.7.1: Modules occurring in the definition of $X^{(s,t)}_{\alpha,\beta,\gamma,\delta}(\mathbf{t})$.

Proof. Using Remark 2.2.7 we get a basis $\{e_{ik} | i \in S_0, 1 \le k \le n\}$ of $V = (V_i, V_{s_i})_{i\in S_0}$ such that the following hold.

1. For all $1 \le m \le n$, the vector space $V^{(m)} := \langle e_{i,k} | i \in S_0, 1 \le k \le m\rangle$ is a subrepresentation of V and a band module.

2. There exists a nilpotent endomorphism ψ of V such that $\psi(e_{i1}) = 0$ and $\psi(e_{ik}) = e_{i,k-1}$ for all $1 < k \leq n$ and all $i \in S_0$.

Let $U = (U_i, V_{s_i}|_{U_i})_{i \in S_0} \in \mathrm{Gr}_{\mathbf{t}}(V)$. Using the Gauß algorithm, a unique tuple

$$\mathbf{j}(U) := (1 \leq j_1 < j_2 < \ldots < j_{t_{i_1}} \leq n) \tag{4.7.2}$$

and unique $\lambda_{kj}(U) \in \mathbb{C}$ exist such that

$$\left\{ e_{i_1 j_m} + \sum\nolimits_{j=1, j \neq j_k \forall k}^{j_m - 1} \lambda_{mj}(U) e_{i_1 j} \middle| 1 \leq m \leq t_{i_1} \right\}$$

is a basis of the vector space U_{i_1}. The variety $\mathrm{Gr}_{\mathbf{t}}(V)$ is decomposed into a disjoint union of locally closed subsets

$$\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} := \left\{ U \subseteq V \middle| \operatorname{\mathbf{dim}} U = \mathbf{t}, \mathbf{j}(U) = \mathbf{j} \right\},$$

where $\mathbf{j} \in \mathbb{N}^{t_{i_1}}$. For each such tuple $\mathbf{j}$ let

$$\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0 := \left\{ U \subseteq V \middle| \operatorname{\mathbf{dim}} U = \mathbf{t}, \mathbf{j}(U) = \mathbf{j}, \lambda_{1j}(U) = 0 \forall j \right\}.$$

These are locally closed subsets of $\mathrm{Gr}_{\mathbf{t}}(V)$. The projection $\pi\colon \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} \to \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0$ with

$$U \mapsto \prod\nolimits_{j=1}^{j_1 - 1} \left(1 + \lambda_{1j}(U) \psi^{j_1 - j}\right)^{-1} (U) \tag{4.7.3}$$

is an algebraic morphism with affine fibres, since the map $\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}} \to \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}$ with $U \mapsto \left(1 + \lambda_{1j}(U)\psi^{j_1 - j}\right)(U)$ for each $1 \leq j < j_1$ can be described by polynomials and for each $U \in \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0$ holds:

$$\pi^{-1}(U) = \left\{ \prod\nolimits_{j=1}^{j_1 - 1} \left(1 + \mu_j \psi^{j_1 - j}\right)(U) \middle| \mu_1, \ldots, \mu_{j_1 - 1} \in \mathbb{C} \right\} \cong \mathbb{C}^{j_1 - 1}.$$

Thus $\chi(\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}) = \chi\left(\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0\right)$.

For $U \in \mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0$ let $U_{\mathbf{j}}$ be the subrepresentation of V generated by $e_{i_1 j_1}$. Let $V_{\mathbf{j}}$ be the subrepresentation of V with vector space basis

$$\{e_{i_1 k} | j_1 \leq k \leq n\} \cup \{e_{ik} | i_1 \neq i \in S_0, 1 \leq k \leq n\}.$$

Then $U_{\mathbf{j}} \subseteq U \subseteq V_{\mathbf{j}} \subseteq V$ and thus $\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0 \cong \mathrm{Gr}_{\mathbf{t} - \operatorname{\mathbf{dim}} U_{\mathbf{j}}}(V_{\mathbf{j}}/U_{\mathbf{j}})$. This implies

$$\chi_{\mathbf{t}}(V) = \sum\nolimits_{\mathbf{j} \in \mathbb{N}^{t_{i_1}}} \chi\left(\mathrm{Gr}_{\mathbf{t}}(V)_{\mathbf{j}}^0\right) = \sum\nolimits_{\mathbf{j} \in \mathbb{N}^{t_{i_1}}} \chi_{\mathbf{t} - \operatorname{\mathbf{dim}} U_{\mathbf{j}}}\left(V_{\mathbf{j}}/U_{\mathbf{j}}\right). \tag{4.7.4}$$

Using the representation theory of S, we get

$$(\operatorname{\mathbf{dim}} U_{\mathbf{j}})_i = \begin{cases} 0 & \text{if } i_1' < i < i_r', \\ 2 & \text{if } r = 1, i_1' = i \text{ and } j_1 > 1, \\ 1 & \text{otherwise.} \end{cases}$$

So if $j_1 = 1$ we get

$$V_{\mathbf{j}}/U_{\mathbf{j}} \cong V^{(n-1)} \oplus M\left(S^{i_1'+1,i_r'-1}\right), \tag{4.7.5}$$

and if $j_1 > 1$ we get

$$V_{\mathbf{j}}/U_{\mathbf{j}} \cong V^{(n-j_1)} \oplus M\left(S^{i_1'+1,i_1-1}\right) \oplus M\left(S^{i_1+1,i_r'-1}\right) \oplus M\left(S^{i_1+1,i_1-1}\right)^{j_1-2}. \tag{4.7.6}$$

Let $n_{\mathbf{j}} := |\{1 \le i \le n | i \neq j_m \forall m, \exists m : i+1 = j_m\}|$. A simple calculation shows

$$|\{\mathbf{j}|n_{\mathbf{j}} = k\}| = \binom{t_{i_1}}{k}\binom{n-t_{i_1}}{k}.$$

We do an induction over t_{i_1}. Then Equation (4.7.6) occurs $n_{\mathbf{j}}$-times, Equation (4.7.5) occurs $(t_{i_1} - n_{\mathbf{j}})$-times and so Equation (4.7.1) holds in general by an inductive version of Equation (4.7.4). □

The rest of the proof of Part 2 of Theorem 4.3.1 is done in the next two combinatorial lemmas.

Lemma 4.7.6. *Let $a, b, c, d, e, f \in \mathbb{N}$. Then*

$$\binom{a}{b}\binom{b}{c} = \binom{a-c}{b-c}\binom{a}{c}$$

and

$$\sum\nolimits_{m\in\mathbb{Z}} \binom{a}{b-m}\binom{b-m}{c}\binom{d}{e+m}\binom{e+m}{f} = \binom{a}{c}\binom{d}{f}\binom{a+d-c-f}{a+d-b-e}.$$

Proof. The first equation can be shown using the definition. The second equation is a consequence of the first one. □

Lemma 4.7.7. *Let S, V, $\mathbf{t}$, n as above and $1 \le m \le r$. Then*

$$\chi_{\mathbf{t}}(V) = \Lambda_m \Gamma_{i_1 i_m} \sum\nolimits_{k\in\mathbb{Z}} \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}')$$

with

$$\Lambda_m = \prod\nolimits_{k=1}^{m-1} \frac{(n-t_{i_{k+1}})!}{t_{i_k}!} \frac{t_{i_k'}!}{(n-t_{i_k'})!}, \qquad \Gamma_{ij} = \prod\nolimits_{k=i}^{j-1} \frac{1}{(\varepsilon_k(t_k - t_{k+1}))!}$$

and

$$\mathbf{t}' = \begin{cases} (0,\ldots,0,t_{i_r+1}-t_{i_r},\ldots,t_{i_r'-1}-t_{i_r}, \\ \qquad t_{i_r'}-t_{i_1}-k, t_{i_r'+1}-t_{i_1},\ldots,t_{i_1-1}-t_{i_1}) & \text{if } m = r, \\ (0,\ldots,0,t_{i_m+1}-t_{i_m},\ldots,t_{i_m'}-t_{i_m},t_{i_m'+1},\ldots,t_{i_r'-1}, \\ \qquad t_{i_r'}-t_{i_1},\ldots,t_{i_1-1}-t_{i_1}) & \text{if } m < r. \end{cases}$$

Proof. For $m = 1$ this is the statement of Lemma 4.7.5. We prove the lemma by induction. Let $1 < m \leq r$. Then

$$\begin{aligned}\chi_{\mathbf{t}}(V) =& \Lambda_{m-1}\Gamma_{i_1 i_{m-1}} \sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k} X^{(m-1,m-1)}_{t_{i_{m-1}}-k,k,t_{i_1}-t_{i_{m-1}}+k,n-t_{i_1}-k}(\mathbf{t}')\\ =& \Lambda_{m-1}\Gamma_{i_1 i_{m-1}} \sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k} \sum_p \binom{t_{i_1}-t_{i_{m-1}}+k}{t_{i'_{m-1}}-t_{i_{m-1}}-p}\binom{n-t_{i_1}-k}{p}\\ & (t_{i'_{m-1}}-t_{i_{m-1}})!\Gamma_{i_{m-1}i'_{m-1}} X^{(m-1,m)}_{t_{i'_{m-1}}-k-p,k+p,t_{i_1}-t_{i'_{m-1}}+k+p,n-t_{i_1}-k-p}(\mathbf{t}'')\\ =& \Lambda_{m-1}\Gamma_{i_1 i'_{m-1}} \sum_p \left(\sum_k \binom{t_{i_1}}{t_{i_{m-1}}-k}\binom{n-t_{i_1}}{k}\binom{t_{i_1}-t_{i_{m-1}}+k}{t_{i_1}-t_{i'_{m-1}}+p}\binom{n-t_{i_1}-k}{n-t_{i_1}-p}\right)\\ & (t_{i'_{m-1}}-t_{i_{m-1}})! X^{(m-1,m)}_{t_{i'_{m-1}}-p,p,t_{i_1}-t_{i'_{m-1}}+p,n-t_{i_1}-p}(\mathbf{t}'')\end{aligned}$$

with $\mathbf{t}'' = (0,\ldots,0,t_{i'_{m-1}+1},\ldots,t_{i'_r+1}-t_{i_1},\ldots,t_{i_1-1}-t_{i_1})$. Lemma 4.7.6 yields

$$\begin{aligned}\chi_{\mathbf{t}}(V) =& \Lambda_{m-1}\Gamma_{i_1 i'_{m-1}} \sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p}\binom{t_{i'_{m-1}}}{t_{i_{m-1}}}\\ & (t_{i'_{m-1}}-t_{i_{m-1}})! X^{(m-1,m)}_{t_{i'_{m-1}}-p,p,t_{i_1}-t_{i'_{m-1}}+p,n-t_{i_1}-p}(\mathbf{t}'')\\ =& \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i'_{m-1}} \sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p} \sum_k \binom{t_{i'_{m-1}}-p}{t_{i_m}-k}\binom{p}{k}\\ & (t_{i'_{m-1}}-t_{i_m})!\Gamma_{i'_{m-1}i_m} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}')\\ =& \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i_m} \sum_k \left(\sum_p \binom{n-t_{i_1}}{p}\binom{t_{i_1}}{t_{i'_{m-1}}-p}\binom{t_{i'_{m-1}}-p}{t_{i_m}-k}\binom{p}{k}\right)\\ & (t_{i'_{m-1}}-t_{i_m})! X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}').\end{aligned}$$

Using Lemma 4.7.6 again, we get

$$\begin{aligned}\chi_{\mathbf{t}}(V) =& \Lambda_{m-1}\frac{t_{i'_{m-1}}!}{t_{i_{m-1}}!}\Gamma_{i_1 i_m} \sum_k \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_m}}{n-t_{i'_{m-1}}}\\ & (t_{i'_{m-1}}-t_{i_m})! X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}')\\ =& \Lambda_m \Gamma_{i_1 i_m} \sum_k \binom{t_{i_1}}{t_{i_m}-k}\binom{n-t_{i_1}}{k} X^{(m,m)}_{t_{i_m}-k,k,t_{i_1}-t_{i_m}+k,n-t_{i_1}-k}(\mathbf{t}').\end{aligned}$$

□

Corollary 4.7.8. *Let S, V, $\mathbf{t}$ and n as above. Then Equation* (4.3.2) *holds.*

Proof. We have to show $\chi_{\mathbf{t}}(V) = \Lambda_{r+1}\Gamma_{1,l+1}$ with $t_{i_{r+1}} = t_{i_1}$. Lemma 4.7.7 implies

$$\chi_{\mathbf{t}}(V) = \Lambda_r \Gamma_{i_1 i_r} \sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k} X^{(r,r)}_{t_{i_r}-k,k,t_{i_1}-t_{i_r}+k,n-t_{i_1}-k}(\mathbf{t}')$$

with $\mathbf{t}' = (0, \dots, 0, t_{i_r+1} - t_{i_r}, \dots, t_{i'_r-1} - t_{i_r}, t_{i'_r} - t_{i_1} - k, t_{i'_r+1} - t_{i_1}, \dots, t_{i_1-1} - t_{i_1})$. So we have

$$\begin{aligned}\chi_{\mathbf{t}}(V) =& \Lambda_r \Gamma_{i_1 i_r} \sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_1}-k}{t_{i'_r}-t_{i_1}-k} X^{(r,r)}_{t_{i_r}-k, t_{i'_r}-t_{i_1}, t_{i'_r}-t_{i_r}, n-t_{i'_r}}(\mathbf{t}''') \\ =& \Lambda_r \Gamma_{i_1 i_r} \sum_k \binom{t_{i_1}}{t_{i_r}-k}\binom{n-t_{i_1}}{k}\binom{n-t_{i_1}-k}{t_{i'_r}-t_{i_1}-k} (t_{i'_r} - t_{i_1})! \Gamma_{i'_r i_1} (t_{i'_r} - t_{i_r})! \Gamma_{i_r i'_r}\end{aligned}$$

with $\mathbf{t}''' = (0, \dots, 0, t_{i_r+1} - t_{i_r}, \dots, t_{i'_r-1} - t_{i_r}, 0, t_{i'_r+1} - t_{i_1}, \dots, t_{i_1-1} - t_{i_1})$. Using again Lemma 4.7.6, we obtain

$$\chi_{\mathbf{t}}(V) = \Lambda_r \Gamma_{1l} \binom{n-t_{i_1}}{n-t_{i'_r}}\binom{t_{i'_r}}{t_{i_r}} (t_{i'_r} - t_{i_1})!(t_{i'_r} - t_{i_r})! = \Lambda_{r+1}\Gamma_{1,l+1}.$$

□

4.7.3 Proof of Part 3 of Theorem 4.3.1

This proof is divided into two parts.

First we assume that G is a finite abelian group. For this it is enough to consider the representation $V = \mathbb{1}_T$ with some finite subquiver $T = (T_0, T_1)$ of the quiver $\hat{Q}$. So without loss of generality we assume that G is of finite rank, e.g. $G = \mathbb{Z}^n$ for some $n \in \mathbb{N}$. By induction and Proposition 4.7.1 it is enough to assume $G = \mathbb{Z}$. Let R_0 be a set of representatives of the $\mathbb{Z}$-orbits in $\hat{Q}_0$. Since $\mathbb{Z}$ acts freely on $\hat{Q}$, there is a unique $z_i \in \mathbb{Z}$ and $r_i \in R_0$ for each $i \in \hat{Q}_0$ with $i = z_i r_i$. The Q-representation $\pi_*(\mathbb{1}_T)$ has a basis $\{f_i | i \in T_0\}$. We define a grading of $\pi_*(\mathbb{1}_T)$ by $\partial(f_i) = z_i$. This grading is well-defined and so it is enough to show that ∂ is a nice grading. Let $a, b \in T_1$ such that they are lying in the same $\mathbb{Z}$-orbit, i.e. it exists $z \in \mathbb{Z}$ with $za = b$. Thus $zs(a) = s(za) = s(b)$ and $z_{s(a)} + z = z_{s(b)}$. So

$$\partial\left(f_{t(b)}\right) - \partial\left(f_{s(b)}\right) = z_{t(b)} - z_{s(b)} = (z_{t(a)} + z) - (z_{s(a)} + z) = \partial\left(f_{t(a)}\right) - \partial\left(f_{s(a)}\right).$$

This proofs Equation (4.3.3) for a free abelian group. □

Now we assume that G is a free group. In this case we use again some induction and Part 3 of Theorem 4.3.1 with $G = \mathbb{Z}$. To illustrate the following construction we give an example afterwards. Let G be generated by $\{g_t | t \in I\}$ with some set I as a free group. Thus

$$G = \left\{g_{t_1}^{\varepsilon_1} \dots g_{t_n}^{\varepsilon_n} \middle| n \in \mathbb{N}, \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}, t_1, \dots, t_n \in I\right\}.$$

For $t_0 \in I$ we define a normal subgroup G^{t_0} of G by

$$\left\{g_{t_1}^{\varepsilon_1} \dots g_{t_n}^{\varepsilon_n} \in G \middle| \sum_{j=1}^{n} \delta_{t_0 t_j} \varepsilon_j = 0\right\}.$$

The quotient G/G^{t_0} is isomorphic to $\mathbb{Z}$ and this group G^{t_0} is isomorphic to the free group generated by

$$\left\{g_t(j) := g_{t_0}^j g_t g_{t_0}^{-j} \middle| (t, j) \in (I - \{t_0\}) \times \mathbb{Z}\right\}.$$

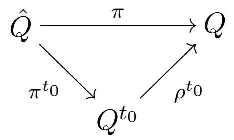


Figure 4.7.2: A commutative diagram for the proof of Part 3 of Theorem 4.3.1.

Let $Q^{t_0} = \hat{Q}/G^{t_0}$, $\pi^{t_0} \colon \hat{Q} \to Q^{t_0}$ the canonical projection and $\rho^{t_0} \colon Q^{t_0} \to Q$ the projection induced by the action of G/G^{t_0} (see Figure 4.7.2). By the first part of this proof Equation (4.3.3) holds for the winding $\rho^{t_0} \colon Q^{t_0} \to Q$. Thus it is enough to consider the action of the free group G^{t_0} on $\hat{Q}$.

Let $V = (V_i, V_\alpha)_{i \in \hat{Q}_0, \alpha \in \hat{Q}_1}$ be a $\hat{Q}$-representation. If there is some $j \in Q_0$ with $\left|\left\{i \in \pi_0^{-1}(j) \middle| \dim_k V_i \neq 0\right\}\right| \geq 2$, let i_1 and i_2 be such two diverse elements. Let $g = g_{t_1}^{\varepsilon_1} \dots g_{t_n}^{\varepsilon_n} \in G$ with $gi_1 = i_2$. Since $n > 0$, we can apply the previous induction step for t_1. If $g \notin G^{t_1}$, we are done, otherwise write $g = g_{t'_1}(j_1)^{\varepsilon'_1} \dots g_{t'_{n'}}(j_{n'})^{\varepsilon'_{n'}}$ with $n' \in \mathbb{N}$, $\varepsilon'_1, \dots, \varepsilon'_{n'} \in \{-1, 1\}$, $(t'_1, j_1), \dots, (t'_{n'}, j_{n'}) \in (I - \{t_1\}) \times \mathbb{Z}$. In this case $0 < n' < n$.

Thus by induction we assume $\left|\left\{i \in \pi_0^{-1}(j) \middle| \dim_k V_i \neq 0\right\}\right| = 1$ for all $j \in Q_0$. In this case Equation (4.3.3) is trivial. Thus it holds for a free group in general. □

Example 4.7.9. Let $Q = (\{\circ\}, \{\alpha, \beta\})$ be as in Example 2.2.4 and $\pi \colon \hat{Q} \to Q$ the universal covering. The fundamental group of Q is a free group with two generators, called a and b, such that e.g. $as(\alpha') = t(\alpha')$ for each $\alpha' \in \pi_1^{-1}(\alpha)$.

Let $g = ababa^{-1}b^{-1}a^{-1} \in G$. The quiver Q^a and the canonical projection $\pi^a \colon Q^a \to Q$ are described by the picture in Figure 4.7.3.

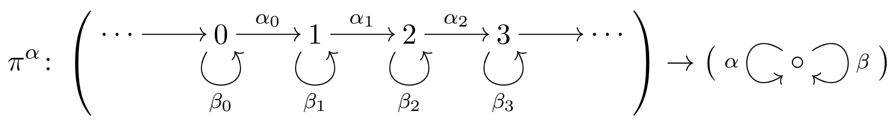


Figure 4.7.3: The covering $\pi^a \colon Q^a \to Q$.

Since $g \in G^a$ we get $g = b_1 b_2 b_1^{-1}$ with $b_i = b(i)$ for $i \in \mathbb{Z}$. For $(Q^a)^{b_1}$ see the picture in Figure 4.7.4. In $(G^a)^{b_1}$ holds $g = b_{(2,1)}$ with $b_{(j,i)} = b_j(i)$ for $(j, i) \in (\mathbb{Z} - \{1\}) \times \mathbb{Z}$. The corresponding quiver $\left((Q^a)^{b_1}\right)^{b_{(2,1)}}$ is described by the picture in Figure 4.7.5.

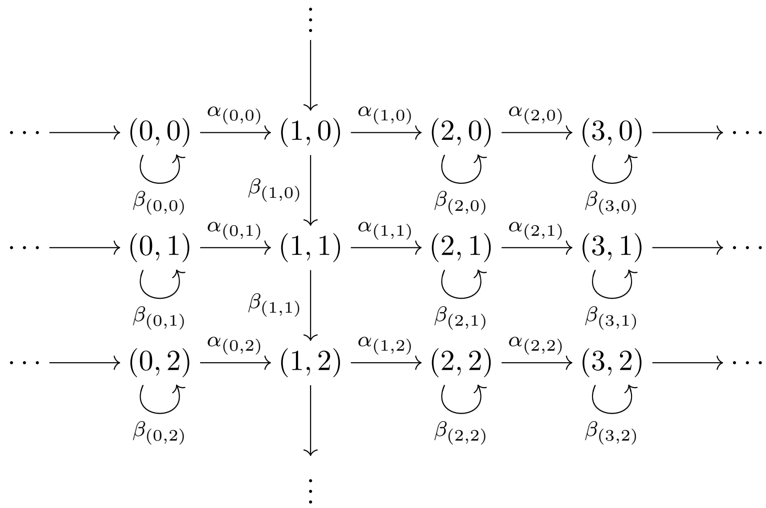


Figure 4.7.4: The quiver $(Q^a)^{b_1}$.

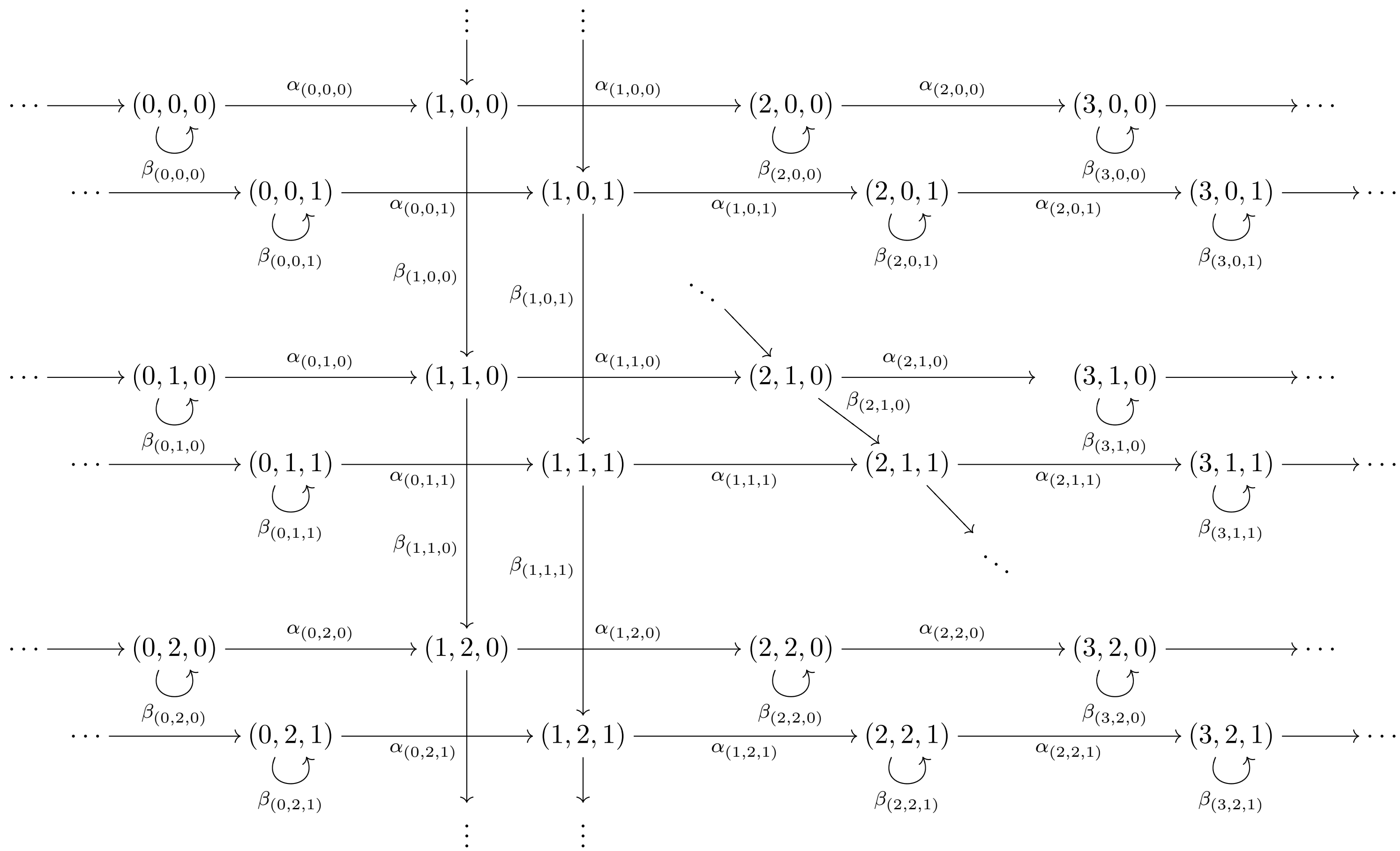


Figure 4.7.5: The quiver $\left((Q^a)^{b_1}\right)^{b_{(2,1)}}$.

5 Ringel-Hall algebras

In this chapter let k be again the field of complex numbers $\mathbb{C}$. We apply the results of the last chapter to the study of Ringel-Hall algebras.

In the first section of this chapter we construct a morphism of the Ringel-Hall algebras for some windings. Thereafter we study the images of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$. Together with the notion of gradings this simplifies the computations of products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$. In the last section we apply these results to string algebras.

5.1 Morphisms of Ringel-Hall algebras

Each winding $\varphi\colon S \to Q$ of locally finite quivers induces a functor $\varphi_*\colon \operatorname{mod}(\mathbb{C}S) \to \operatorname{mod}(\mathbb{C}Q)$ and moreover a map of constructible and $\mathrm{GL}(\mathbb{C})$-stable functions

$$\mathcal{H}(\varphi)\colon \mathcal{H}(\mathbb{C}Q) \to \hat{\mathcal{H}}(\mathbb{C}S), f \mapsto f \circ \varphi_*.$$

This is in general not an algebra homomorphism, but functorial, since $\mathcal{H}(\mathrm{id}_{\mathbb{C}Q}) = \mathrm{id}_{\mathcal{H}(\mathbb{C}Q)}$ and $\mathcal{H}(F \circ G) = \mathcal{H}(G) \circ \mathcal{H}(F)$ for windings F and G. For trees, bands, and coverings we get the following statement.

Theorem 5.1.1.

1. *Let $F\colon S \to Q$ be a tree or a band and $A = \mathbb{C}Q/I$ and $B = \mathbb{C}S/J$ finite-dimensional algebras. If F induces a functor $F_*\colon \operatorname{mod}(B) \to \operatorname{mod}(A)$, then the map*

$$\mathcal{C}(F)\colon \mathcal{C}(A) \to \mathcal{C}(B), f \mapsto f \circ F_*$$

   is a Hopf algebra homomorphism. If F is injective, this homomorphism $\mathcal{C}(F)$ is surjective. If each A-module can be lifted to a B-module, i.e. F_* is dense, the homomorphism $\mathcal{C}(F)$ is injective.

2. *Let $\hat{Q}$ be a locally finite quiver and G a free (abelian) group, which acts freely on $\hat{Q}$. Let $Q = \hat{Q}/G$, $A = \mathbb{C}Q/I$ and $B = \mathbb{C}\hat{Q}/J$ be algebras and $\pi\colon \hat{Q} \to Q$ the canonical projection. If π induces a functor $\pi_*\colon \operatorname{mod}(B) \to \operatorname{mod}(A)$, then the map*

$$\mathcal{C}(\pi)\colon \mathcal{C}(A) \to \hat{\mathcal{C}}(B), f \mapsto f \circ \pi_*$$

   is a Hopf algebra homomorphism. If each A-module can be lifted to a B-module, this homomorphism is injective.

Proof. For $V \in \operatorname{mod}(B)$ and dimension vectors $\mathbf{d}^{(i)} \in \mathbb{N}^{Q_0}$ holds:

$$\begin{aligned}&\mathcal{C}(F)\left(\prod\nolimits_{i=1}^{n} \mathbb{1}_{\mathbf{d}^{(i)}}\right)(V)\\ =\ &\chi\Big(\Big\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = F_*(V) \Big| \operatorname{\mathbf{dim}}\left(U^{(i)}/U^{(i-1)}\right) = \mathbf{d}^{(i)} \forall i\Big\}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \chi\Big(\Big\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = F_*(V) \Big| \, \mathbf{dim}\left(U^{(i)}/U^{(i-1)}\right) = \mathbf{d}^{(i)} \forall i\Big\}^{\partial_1,\ldots,\partial_n}\Big) \\
&= \chi\Big(\Big\{0 = U^{(0)} \subseteq \ldots \subseteq U^{(n)} = V \Big| \, \mathbf{dim}\left(F_*\left(U^{(i)}/U^{(i-1)}\right)\right) = \mathbf{d}^{(i)} \forall i\Big\}\Big) \\
&= \left(\prod\nolimits_{i=1}^{n} \mathcal{C}(F)\left(\mathbb{1}_{\mathbf{d}^{(i)}}\right)\right)(V)
\end{aligned}
$$

with some gradings $\partial_1, \ldots, \partial_n$ as in the proofs of Theorem 4.3.1, Lemma 4.7.3 and 4.7.4. For $\mathbf{d} \in \mathbb{N}^{Q_0}$ holds

$$
\mathcal{C}(F)(\mathbb{1}_{\mathbf{d}}) = \sum\nolimits_{\mathbf{t} \in \mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{\mathbf{t}}, \quad \mathcal{C}(F)(\mathbb{1}_{S(\mathbf{d})}) = \sum\nolimits_{\mathbf{t} \in \mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{S(\mathbf{t})}.
$$

Thus $\mathcal{C}(F)$ is a well-defined algebra homomorphism. For each $f \in \mathcal{C}(A)$ and A-modules V and W holds

$$
\begin{aligned}
&(\mathcal{C}(F) \otimes \mathcal{C}(F))(\Delta(f))(V, W) = (\Delta(f) \circ (F_*, F_*))(V, W) = \Delta(f)(F_*(V), F_*(W)) \\
&= f(F_*(V) \oplus F_*(W)) = f(F_*(V \oplus W)) = \mathcal{C}(F)(f)(V \oplus W) = \Delta(\mathcal{C}(F)(f))(V, W)
\end{aligned}
$$

and using Lemma 2.4.2 we get for $\mathbf{d} \in \mathbb{N}^{Q_0}$

$$
S(\mathcal{C}(F)(\mathbb{1}_{\mathbf{d}})) = S\left(\sum\nolimits_{\mathbf{t} \in \mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{\mathbf{t}}\right) = (-1)^{|\mathbf{d}|} \sum\nolimits_{\mathbf{t} \in \mathbf{F}^{-1}(\mathbf{d})} \mathbb{1}_{S(\mathbf{t})} = \mathcal{C}(F)(S(\mathbb{1}_{\mathbf{d}})).
$$

By this $\mathcal{C}(F)$ is actually a Hopf algebra homomorphism.

If $F \colon S \to Q$ is injective, $F_* \colon \operatorname{mod}(B) \to \operatorname{mod}(A)$ is injective and $\mathcal{C}(F)(\mathbb{1}_{\mathbf{F}(\mathbf{d})}) = \mathbb{1}_{\mathbf{d}}$ holds for each dimension vector $\mathbf{d} \in \mathbb{N}^{S_0}$. The functor F_* induces an embedding of varieties $\operatorname{mod}(B, \mathbf{d}) \to \operatorname{mod}(A, \mathbf{F}(\mathbf{d}))$. Thus $\mathcal{C}(F)$ is surjective.

Let $f \in \operatorname{Ker} \mathcal{C}(F)$ and $W \in \operatorname{mod}(A)$. If $F_* \colon \operatorname{mod}(B) \to \operatorname{mod}(A)$ is dense, a B-module V with $F_*(V) \cong W$ exists. By $f(W) = f(F_*(V)) = \mathcal{C}(F)(f)(V) = 0$ is $\operatorname{Ker} \mathcal{C}(F) = 0$ and $\mathcal{C}(F)$ injective.

The second part can be proven in a very similar way. □

5.2 Liftings

Let $F \colon S \to Q$ be a winding and $\mathbf{F} = \left(F^{(1)}, \ldots, F^{(r)}\right)$ with $F^{(i)} \colon S^{(i)} \to Q$ be a tuple of windings. In this section we study the image $\mathcal{H}(F)(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}})$. For this we define the following set of tuples: Let $\mathcal{G}_F(\mathbf{F})$ be a set of representatives of the equivalence classes of the set

$$
\left\{\widetilde{\mathbf{F}} = \left(\widetilde{F}^{(1)}, \ldots, \widetilde{F}^{(r)}\right) \Big| \, \widetilde{F}^{(i)} \colon S^{(i)} \to Q \text{ winding, } F\widetilde{F}^{(i)} = F^{(i)} \ \forall i\right\}
$$

with the equivalence relation $\sim$ defined by $\widetilde{\mathbf{F}} \sim \widetilde{\mathbf{F}}'$ if and only if $\mathbb{1}_{\widetilde{\mathbf{F}}} = \mathbb{1}_{\widetilde{\mathbf{F}}'}$ in $\mathcal{H}(A)$. Thus for all i the diagram in Figure 5.2.1 commutes. If $r = 0$, the set $\mathcal{G}_F(\mathbf{F})$ consists by convention of one trivial element.

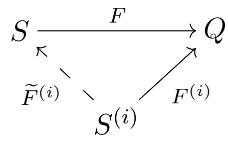


Figure 5.2.1: The lifting property.

Theorem 5.2.1. *Let* $\mathbf{F}$ *be a tuple of trees,* $\mathbf{B}$ *a tuple of bands and* $\mathbf{n}$ *a tuple of positive integers.*

1. *Let* Q *be a finite quiver and* $F\colon S \to Q$ *a tree or a band. Then*

$$\mathcal{H}(F)\,(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}) = \sum\nolimits_{\widetilde{\mathbf{F}}\in\mathcal{G}_F(\mathbf{F}),\widetilde{\mathbf{B}}\in\mathcal{G}_F(\mathbf{B})} \mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}.$$

2. *Let* $\hat{Q}$ *be a locally finite quiver,* G *a free (abelian) group, which acts freely on* $\hat{Q}$*,* $Q = \hat{Q}/G$ *and* $\pi\colon \hat{Q} \to Q$ *the canonical projection. Then*

$$\mathcal{H}(\pi)\,(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}) = \sum\nolimits_{\widetilde{\mathbf{F}}\in\mathcal{G}_\pi(\mathbf{F}),\widetilde{\mathbf{B}}\in\mathcal{G}_\pi(\mathbf{B})} \mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}.$$

This theorem is a direct consequence of the following lifting property.

Lemma 5.2.2 (Lifting property)**.** *Let* $\mathbf{F}$ *be a tuple of trees,* $\mathbf{B}$ *a tuple of bands and* $\mathbf{n}$ *a tuple of positive integers.*

1. *Let* $F\colon S \to Q$ *be a tree or a band and* $V \in \operatorname{rep}(S)$ *such that* $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(V)) = 1$*. Then there exists a tuple* $(\widetilde{\mathbf{F}},\widetilde{\mathbf{B}}) \in \mathcal{G}_F(\mathbf{F}) \times \mathcal{G}_F(\mathbf{B})$ *with* $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}(V) = 1$*.*
2. *Let* $\hat{Q}$ *be a locally finite quiver,* G *a free (abelian) group, which acts freely on* $\hat{Q}$*,* $Q = \hat{Q}/G$ *and* $\pi\colon \hat{Q} \to Q$ *the canonical projection. Let* $V \in \operatorname{rep}(\hat{Q})$ *such that* $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(\pi_*(V)) = 1$*. Then there is a tuple* $(\widetilde{\mathbf{F}},\widetilde{\mathbf{B}}) \in \mathcal{G}_\pi(\mathbf{F}) \times \mathcal{G}_\pi(\mathbf{B})$ *with* $\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}}(V) = 1$*.*

Proof of Part 1. If F is a tree, a lifting $F = \pi\iota$ with the universal covering $\pi\colon \hat{Q} \twoheadrightarrow Q$ and an embedding $\iota\colon S \hookrightarrow \hat{Q}$ exists (see Figure 5.2.2, left hand side). By the additivity of F_* and [28, Lemma 3.5] we assume without loss of generality that V, $\iota_*(V)$ and $F_*(V)$ are indecomposable. The module $F_*(V)$ can be lifted to a $\hat{Q}$-module. By $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(V)) = 1$ the module $F_*(V)$ is a tree or a band module. If $F_*(V)$ is a band module, it cannot be lifted to a $\hat{Q}$-module since $\hat{Q}$ is a tree. This is a contradiction.

So $F_*(V) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$ is a tree module. Since $F^{(1)}$ is a tree, there exists another lifting $F^{(1)} = \pi\iota'$ with an embedding $\iota'\colon S^{(1)} \hookrightarrow \hat{Q}$ (see Figure 5.2.2, left hand side). Using the proof of [28, Theorem 3.6(c)] we get $\iota_*(V)$ is (up to shift by some group element) isomorphic to $\iota'_*(\mathbb{1}_{S^{(1)}})$. So we can modify ι' such that $\iota_*(V) \cong \iota'_*(\mathbb{1}_{S^{(1)}})$ and a winding $\widetilde{F}^{(1)}\colon S^{(1)} \to S$ exists such that the diagram in Figure 5.2.2 commutes. In particular V is a tree module.

If F is a band, then V is a direct sum of some string and band modules. Since F_* is additive we assume again without loss of generality that V is indecomposable.

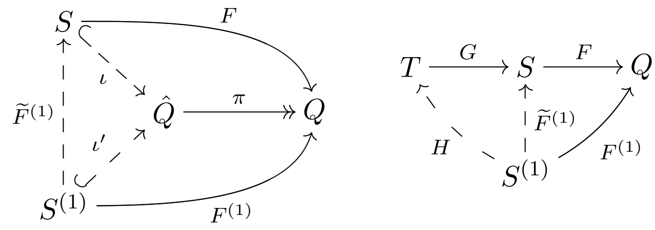


Figure 5.2.2: Windings occurring in the proof of Part 1 of Lemma 5.2.2.

Thus V is a tree or a band module and it exists a winding $G\colon T \to S$ and $W \in \mathcal{I}_T^{n'}$ with $G_*(W) \cong V$. Since $FG\colon T \to Q$ is a winding and $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}((FG)_*(W)) = 1$ we get $l(\mathbf{F}) + l(\mathbf{B}) = 1$.

If $l(\mathbf{F}) = 1$, then $(FG)_*(W) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$. By Lemma 2.2.9 exists an isomorphism of quivers $H\colon S^{(1)} \to T$ such that $(FG)H = F^{(1)}$. By setting $\widetilde{F}^{(1)} = GH$ the statement follows (see Figure 5.2.2, right hand side). For $l(\mathbf{B}) = 1$ the result follows analogously. □

Proof of Part 2. By the additivity of F_* and [28, Lemma 3.5] we assume without loss of generality that V and $\pi_*(V)$ are indecomposable. And by $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(\pi_*(V)) = 1$ the module $\pi_*(V)$ is a tree or a band module.

If $\pi_*(V)$ is a tree module, we get $\pi_*(V) \cong F_*^{(1)}(\mathbb{1}_{S^{(1)}})$. Since G acts on $\hat{Q}$ and $Q = \hat{Q}/G$ the tree $F^{(1)}$ factors through π. Let $\widetilde{F}^{(1)}$ be the lifting, e.g. $\pi\widetilde{F}^{(1)} = F^{(1)}$ (see Figure 5.2.3, left hand side). Then $\pi_*\big(\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}})\big) = F_*^{(1)}(\mathbb{1}_{S^{(1)}}) \cong \pi_*(V)$ and by the proof of [28, Theorem 3.6(c)] we get $\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}})$ is (up to shift by some group element) isomorphic to V. So we can modify again $\widetilde{F}^{(1)}$ such that $\widetilde{F}_*^{(1)}(\mathbb{1}_{S^{(1)}}) \cong V$ and still $\pi\widetilde{F}^{(1)} = F^{(1)}$.

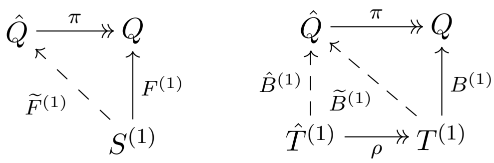


Figure 5.2.3: Windings occurring in the proof of Part 2 of Lemma 5.2.2.

If $\pi_*(V)$ is a band module, we get $\pi_*(V) \cong B_*^{(1)}(V_1)$ for some $V_1 \in \mathcal{I}_{T^{(1)}}^{n_1}$. Let $\rho\colon \hat{T}^{(1)} \to T^{(1)}$ be the universal covering of $T^{(1)}$ (see Figure 5.2.3, right hand side). Since $\pi_*(V) \cong B_*^{(1)}(V_1)$ and G is a free (abelian) group, which acts freely on $\hat{Q}$, we get not only a lifting $\hat{B}^{(1)}$ of $B^{(1)}\rho$ but also a lifting $\widetilde{B}^{(1)}$ of $B^{(1)}$. Then the result follows as above. □

5.3 Gradings

To consider the multiplication of the Ringel-Hall algebra $\mathcal{H}(A)$ we have to compute the Euler characteristics of the constructible subsets

$$\left\{N \in \mathrm{Gr}_{\mathbf{d}}(M) \middle| N \in X, M/N \in Y\right\}$$

of the quiver Grassmannian $\mathrm{Gr}_{\mathbf{d}}(M)$ with some constructible and $\mathrm{GL}(\mathbb{C})$-stable subsets $X \subseteq \mathrm{rep}_{\mathbf{d}}(A)$ and $Y \subseteq \mathrm{rep}_{\mathbf{c}}(A)$. We use the gradings to simplify the calculations of these Euler characteristics.

In the following section we study the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$. It turns out that it is enough to study the following cases and simple generalizations. For a Q-representation M and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$ let

$$\mathrm{Gr}_{\mathbf{d}}(M)^U = \left\{V \in \mathrm{Gr}_{\mathbf{d}}(M) \middle| V \cong U, M/V \cong M/U\right\}.$$

Lemma 5.3.1. *Let Q be a quiver, M be a Q-representation and $U \in \mathrm{Gr}_{\mathbf{d}}(M)$. Then every R-grading ∂ is stable on $\mathrm{Gr}_{\mathbf{d}}(M)^U$.*

Proof. The linear map $\varphi_\partial(\lambda)\colon M \to M$ is an automorphism of Q-representations for all $\lambda \in \mathbb{C}^*$. □

Lemma 5.3.2. *Let Q be a quiver, M a Q-representation and $F_*(\mathbb{1}_S) \subseteq M$ with $F\colon S \to Q$ a tree such that $M/F_*(\mathbb{1}_S)$ is a tree module, too. Let ∂ be a nice grading on $\mathrm{Gr}_{\mathbf{d}}(M)$. Then ∂ is also stable on $\mathrm{Gr}_{\mathbf{d}}(M)^{F_*(\mathbb{1}_S)}$.*

Proof. Let $a \in Q_1$, $\lambda \in \mathbb{C}^*$ and $U \in \mathrm{Gr}_{\mathbf{d}}(M)^{F_*(\mathbb{1}_S)}$. Since ∂ is a nice grading we know $M_a\varphi_\partial(\lambda) = \lambda^{\partial(a)}\varphi_\partial(\lambda)M_a$ by the proof of Lemma 4.1.10. Let $i \in S_0$ and ρ_j the unique not necessarily oriented path in S from i to some $j \in S_0$. Then we associate an integer $\partial(\rho_j)$ to each path ρ_j such that $f_j \mapsto \lambda^{\partial(\rho_j)}f_j$ induces an isomorphism $U \to \varphi_\partial(\lambda)(U)$ of quiver representations. The same holds for the quotient. □

Lemma 5.3.3. *Let Q and S be quivers, $B\colon S \to Q$ a winding, M a Q-representation and ∂ a nice grading on $\mathrm{Gr}_{\mathbf{d}}(M)$. Let*

$$X = \left\{U \in \mathrm{Gr}_{\mathbf{d}}(M) \middle| \exists B_*(V) \text{ band module} : U \cong B_*(V)\right\},$$

a locally closed subset of $\mathrm{Gr}_{\mathbf{d}}(M)$. Then ∂ is also stable on X.

Proof. We use the proof of Lemma 5.3.2. In this case the representations U and $\varphi_\partial(\lambda)(U)$ are in general non-isomorphic, but they are both band modules for the same quiver S and the same winding $B\colon S \to Q$. □

The next example shows that this lemma is not true if we restrict the action to one orbit of a band module.

Example 5.3.4. Let $F_*(\mathbb{1}_S)$ be the tree module described by the following picture.

$$F\colon \left(\begin{array}{ccc} 1 & & 1' \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & 2 & \end{array}\right) \to \left(\begin{array}{c} 1 \\ {\scriptstyle\alpha}\downarrow\downarrow{\scriptstyle\beta} \\ 2 \end{array}\right)$$

Let U be the subrepresentation of $F_*(\mathbb{1}_S)$ generated by $F_*(f_1 + f_{1'})$. Let $\lambda \in \mathbb{C}^*$ with $\lambda \neq 1$ and ∂ a nice grading of $F_*(\mathbb{1}_S)$ with $\partial(\alpha) = 1$ and $\partial(\beta) = 0$ (see Remark 4.1.5). Then $\varphi_\partial(\lambda)U$ is generated by $F_*(f_1 + \lambda f_{1'})$, and U and $\varphi_\partial(\lambda)U$ are non-isomorphic band modules.

5.4 Product in the Ringel-Hall algebra

Now we study the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in the Ringel-Hall algebra $\mathcal{H}(A)$. Using the following lemma and following example, it is enough to consider the images of indecomposable A-modules of the products of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(A)$.

Lemma 5.4.1. *Let $A = \mathbb{C}Q/I$ be an algebra, $f, g \in \mathcal{C}(A)$ and M and N be A-modules. Then*

$$(f * g)(M \oplus N) = \sum\nolimits_{i,j} \left(f_i^{(1)} * g_j^{(1)}\right)(M) \left(f_i^{(2)} * g_j^{(2)}\right)(N),$$

where $\Delta(f) = \sum_i f_i^{(1)} \otimes f_i^{(2)}$ and $\Delta(g) = \sum_j g_j^{(1)} \otimes g_j^{(2)}$.

Proof. By definition $\Delta(f)(M, N) = f(M \oplus N)$ for each $f \in \mathcal{C}(A)$. Since $\mathcal{C}(A)$ is a bialgebra the comultiplication Δ is an algebra homomorphism. □

Example 5.4.2. Let $\mathbf{F}$ be a tuple of trees, $\mathbf{B}$ a tuple of bands and $\mathbf{n}$ a tuple of positive integers. Then

$$\Delta(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}) = \sum_{\mathbf{F}^{(1)}\dot\cup\mathbf{F}^{(2)}=\mathbf{F},\mathbf{B}^{(1)}\dot\cup\mathbf{B}^{(2)}=\mathbf{B},\mathbf{n}^{(1)}\dot\cup\mathbf{n}^{(2)}=\mathbf{n}} \mathbb{1}_{\mathbf{F}^{(1)},\mathbf{B}^{(1)},\mathbf{n}^{(1)}} \otimes \mathbb{1}_{\mathbf{F}^{(2)},\mathbf{B}^{(2)},\mathbf{n}^{(2)}}.$$

In this example we have been a little bit lazy: $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ is not necessarily in $\mathcal{C}(A)$, but we can extend the comultiplication in a natural way to all functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$.

Combining Theorem 5.1.1 and 5.2.1 we get useful corollaries to compute the products of these functions $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ in $\mathcal{H}(\mathbb{C}Q)$. For this again it is not important if we compute the images in $\mathcal{H}(\mathbb{C}Q)$ resp. $\hat{\mathcal{H}}(\mathbb{C}Q)$ or in $\mathcal{H}(A)$. It is only essential that F resp. π induces a well-defined functor $\operatorname{mod}(B) \to \operatorname{mod}(A)$.

Corollary 5.4.3. *Let $\mathbf{F}$ and $\mathbf{F}'$ be tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers.*

1. *Let Q be a finite quiver and $F\colon S \to Q$ a tree or a band. Then*

$$\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right) = \mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) * \mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right).$$

2. *Let $\hat{Q}$ be a locally finite quiver, G a free (abelian) group, which acts freely on $\hat{Q}$, $Q = \hat{Q}/G$ and $\pi\colon \hat{Q} \to Q$ the canonical projection. Then*

$$\mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right) = \mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) * \mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right).$$

The functions $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$, $\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}$ and the corresponding products are in $\mathcal{H}(\mathbb{C}Q)$. The functions $\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right)$, $\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)$ are in $\mathcal{H}(\mathbb{C}S)$ and the functions $\mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right)$, $\mathcal{H}(\pi)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)$ are in $\hat{\mathcal{H}}(\mathbb{C}\hat{Q})$. So this corollary shows: To calculate

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V))$$

for a tree or band $F\colon S \to Q$ it is enough to consider some combinatorics and S-representations, where S is a tree or a quiver of type $\tilde{A}_{l-1}$.

Proof. Let $F\colon S \to Q$ be a tree or band and $V \in \operatorname{rep}(S)$. Then we have to show

$$\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V)) = \sum \left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V),$$

where the sum is over $(\widetilde{\mathbf{F}}, \widetilde{\mathbf{F}}', \widetilde{\mathbf{B}}, \widetilde{\mathbf{B}}') \in \mathcal{G}_F(\mathbf{F}) \times \mathcal{G}_F(\mathbf{F}') \times \mathcal{G}_F(\mathbf{B}) \times \mathcal{G}_F(\mathbf{B}')$. By the proof of Part 1 of Theorem 5.1.1, Lemmas 5.3.2 and 5.3.3 and Theorem 5.2.1 we get

$$\begin{aligned}
&\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V))\\
&= \chi\Big(\Big\{U \subseteq F_*(V) \,\Big|\, \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(U) = 1, \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(F_*(V)/U) = 1\Big\}^{\partial_1,\dots,\partial_n}\Big)\\
&= \chi\Big(\Big\{U \subseteq V \,\Big|\, \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(F_*(U)) = 1, \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(F_*(V/U)) = 1\Big\}\Big)\\
&= \Big(\mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}\right) * \mathcal{H}(F)\left(\mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)\Big)(V)\\
&= \sum \left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V).
\end{aligned}$$

So we only have to use the representation theory of trees and quivers of type $\widetilde{A}_{l-1}$ to calculate the Euler characteristics of the occurring varieties.

The second case can be proven similarly. □

Proposition 5.4.4. *Let A be a finite-dimensional algebra, $\mathbf{F}$ and $\mathbf{F}'$ be tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers.*

1. *Let $F_*(\mathbb{1}_S)$ be a tree module of A such that $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(\mathbb{1}_S)) \neq 0$. Then $l(\mathbf{B}) = l(\mathbf{B}') = 0$.*
2. *Let $B_*(V)$ be a band module of A such that $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(B_*(V)) \neq 0$. Then $\mathbf{B}, \mathbf{B}' \in \{0, (B)\}$, $\mathbf{F}$ and $\mathbf{F}'$ are tuples of strings and $l(\mathbf{F}) = l(\mathbf{F}')$, where $l(\mathbf{F})$ denotes the length of the tuple $\mathbf{F}$.*

Proof. Let $A = \mathbb{C}Q/I$. Corollary 5.4.3 shows, to compute $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(F_*(V))$ with a tree or band $F\colon S \to Q$ we have only to consider the products $\left(\mathbb{1}_{\widetilde{\mathbf{F}},\widetilde{\mathbf{B}},\mathbf{n}} * \mathbb{1}_{\widetilde{\mathbf{F}}',\widetilde{\mathbf{B}}',\mathbf{n}'}\right)(V)$, where S is a tree or a quiver of type $\tilde{A}_{l-1}$, and some combinatorics.

Thus for Part 1 we assume without loss of generality that Q is a tree and F is the identity winding. So we have to compute $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(\mathbb{1}_Q)$ in $\mathcal{H}(\mathbb{C}Q)$. All sub- and factor modules of the tree module $\mathbb{1}_Q$ are again tree modules. If $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(\mathbb{1}_Q) \neq 0$, then $l(\mathbf{B}) = l(\mathbf{B}') = 0$.

For Part 2 we assume without loss of generality that Q is a quiver of type $\tilde{A}_{l-1}$. All Q-modules are string or band modules $B'_*(V')$ such that $B' \colon Q \to Q$ is the identity winding. If $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(V) \neq 0$, then $l(\mathbf{B}), l(\mathbf{B}') \leq 1$ holds by Remark 2.2.7.

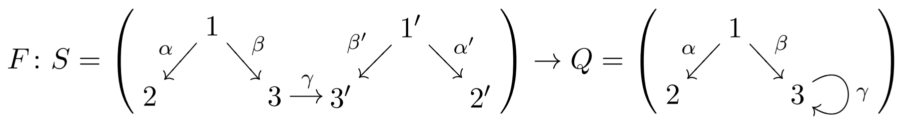
The equality $l(\mathbf{F}) = l(\mathbf{F}')$ is shown by induction. Let V be a band module and U a submodule, which is isomorphic to a string module. It is enough to show that for the representation $V/U = (W_i, W_a)_{i \in Q_0, a \in Q_1}$ the equality

$$\dim_k(V/U) - 1 = \sum\nolimits_{a \in Q_1} \mathrm{rk}(W_a)$$

holds, where $\mathrm{rk}(W_a)$ is the rank of the linear map W_a. This is clear since V is a band and U a string module with $\mathbf{dim}\, U \notin \mathbb{Z}(1, \ldots, 1)$. □

The calculation of the image of a tree module under a product $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}$ is now a purely combinatorial task. Using this proposition it is enough to consider $\left(\mathbb{1}_{(F^{(1)},\ldots,F^{(r)})} * \mathbb{1}_{(F'^{(1)},\ldots,F'^{(s)})}\right)(F_*(\mathbb{1}_S))$. By Corollary 5.4.3 it is even enough to count successor closed subquivers T of the quiver S with $F_*(\mathbb{1}_T) \cong \bigoplus_{i=1}^r F^{(i)}(\mathbb{1}_{S^{(i)}})$ and $F_*(\mathbb{1}_S/\mathbb{1}_T) \cong \bigoplus_{i=1}^s F'^{(i)}(\mathbb{1}_{S'^{(i)}})$.

Example 5.4.5. Let F be the string described by the following picture.



Let $\mathbf{F} = (2 \to Q, (3 \xrightarrow{\gamma} 3') \to Q)$ and $\mathbf{F}' = (1 \to Q, (1 \xrightarrow{\alpha} 2) \to Q)$. We compute $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(F_*(\mathbb{1}_S))$ with Corollary 5.4.3. Then

$$\begin{aligned}
\mathcal{G}_F(\mathbf{F}) &= \{(2 \to S, (3 \xrightarrow{\gamma} 3') \to S), (2' \to S, (3 \xrightarrow{\gamma} 3') \to S)\}, \\
\mathcal{G}_F(\mathbf{F}') &= \{(1 \to S, (1 \xrightarrow{\alpha} 2) \to S), (1' \to S, (1 \xrightarrow{\alpha} 2) \to S), \\
&\qquad (1 \to S, (1' \xrightarrow{\alpha'} 2') \to S), (1' \to S, (1' \xrightarrow{\alpha'} 2') \to S)\}
\end{aligned}$$

and thus $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(F_*(\mathbb{1}_S)) = 2$ by counting these subquivers.

Proposition 5.4.6. *Let Q be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{F}$ and $\mathbf{F}'$ be tuples of strings, $B \colon Q \to Q$ the identity winding, $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$.*

1. Let $n, n' \in \mathbb{N}$ with $n + n' \leq m$. Then

$$\left(\mathbb{1}_{\mathbf{F},B,n} * \mathbb{1}_{\mathbf{F}',B,n'}\right)(B_*(\lambda, m)) = (\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(B_*(\lambda, m - n - n')).$$

2. *Let $n \in \mathbb{N}$, F a string and $\mathbf{F}(n) = (F, \ldots, F)$ with $l(\mathbf{F}(n)) = n$ such that $F_*(\mathbb{1}_S)$ and $F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ are preprojective, $\dim_k F_*(\mathbb{1}_S) \geq \dim_k F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ and $F_*(\mathbb{1}_S) \ncong F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ for all i. Then $\left(\mathbb{1}_{\mathbf{F}(n)\dot\cup\mathbf{F}} * \mathbb{1}_{\mathbf{F}'}\right)(B_*(\lambda, m)) =$*
$$\sum\nolimits_{k_1,\ldots,k_n \in \mathbb{N}} (\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'}) \left(B_* \left(\lambda, m - \sum\nolimits_{i=1}^{n} k_i\right) \oplus \bigoplus\nolimits_{i=1}^{n} I_{k_i}\right)$$
with I_{k_i} is an indecomposable representation with dimension vector $(k_i, \ldots, k_i) - \mathbf{dim}\, F_*(\mathbb{1}_S)$ for all i.

If $\mathbf{dim}\, B_*(\lambda, k_i) > \mathbf{dim}\, F_*(\mathbb{1}_S)$, the module I_k exists, is preinjective and determined up to isomorphism uniquely by Remark 2.2.10.

Let Q be a quiver of type $\tilde{A}_{l-1}$, $\mathbf{F}''$ and $\mathbf{F}'$ be tuples of strings and $V \in \mathcal{I}_Q^n$ such that $(\mathbb{1}_{\mathbf{F}''} * \mathbb{1}_{\mathbf{F}'})(V) \neq 0$. Without loss of generality we assume that $\dim_k F_*''^{(1)}(\mathbb{1}_S) \geq \dim_k F_*''^{(i)}(\mathbb{1}_S)$ for all i. Then $F_*''^{(i)}(\mathbb{1}_S)$ is preprojective for all i and we apply Part 2 of Proposition 5.4.6 with $F = F''^{(1)}$ and $\mathbf{F} = \{F''^{(i)} | F''^{(i)}(\mathbb{1}_S) \ncong F(\mathbb{1}_S)\}$. Thus before proving Proposition 5.4.6 we get the following corollary.

Corollary 5.4.7. *Let $A = \mathbb{C}Q/I$ be an algebra, M a direct sum of tree and band modules of Q such that M is an A-module. Let $\mathbf{F}$ and $\mathbf{F}'$ be tuples of trees, $\mathbf{B}$ and $\mathbf{B}'$ tuples of bands and $\mathbf{n}$ and $\mathbf{n}'$ tuples of positive integers. Then $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}(M)$ is given by a combinatorial description.*

Proof of Proposition 5.4.6. First we prove Part 1. Let $M := B_*(\lambda, m)$ be an A-module, $\pi\colon M \to B_*(\lambda, m-n)$ a projection, $K := \bigoplus_i F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ and $K' := \bigoplus_i F_*'^{(i)}(\mathbb{1}_{S'^{(i)}})$. By Remark 2.2.7, there exists a unique $U \subseteq B_*(\lambda, m)$ with $U \cong B_*(\lambda, n)$, so we can assume $B_*(\lambda, n) \subseteq B_*(\lambda, m - n') \subseteq B_*(\lambda, m)$. Define the varieties
$$X := \left\{U \subseteq M \,\middle|\, U \cong B_*(\lambda, n) \oplus K, M/U \cong B_*(\lambda, n') \oplus K'\right\}$$
$$\overline{X} := \left\{V \subseteq \overline{M} \,\middle|\, V \cong K, \overline{M}/V \cong K'\right\}$$
with $\overline{U} := \big(U \cap B_*(\lambda, m - n')\big)/B_*(\lambda, n)$ for all $B_*(\lambda, n) \subseteq U \subseteq M$ and an algebraic morphism $\phi\colon X \to \overline{X}$ by $U \mapsto \overline{U}$. Using Remark 2.2.7 again, $B_*(\lambda, n) \subseteq U \subseteq B_*(\lambda, m - n')$ for all $U \in X$. So ϕ is well-defined and injective.

Let $V \in \overline{X}$. Since $V \cong K$ and $\overline{M}/V \cong K'$ we have $B_*(\lambda, m - n')/\pi^{-1}(V) \cong K'$ and $M/B_*(\lambda, m - n') \cong B_*(\lambda, n')$. There exist two short exact sequences
$$0 \to B_*(\lambda, n) \to \pi^{-1}(V) \to K \to 0$$
$$0 \to K' \to M/\pi^{-1}(V) \to B_*(\lambda, n') \to 0$$
Using Remark 2.2.10, we assume without loss of generality that the direct summands of K are preprojective Q-representations and the direct summands of K' are preinjective ones. So both sequences split and this means that $\pi^{-1}(V) \cong B_*(\lambda, n) \oplus K$ and $M/\pi^{-1}(V) \cong K' \oplus B_*(\lambda, n')$. Thus $\pi^{-1}(V) \in X$ and $\overline{\pi^{-1}(V)} = V$. This shows that the Euler characteristics of both varieties are equal.

Part 2 of Proposition 5.4.6 follows inductively by the following lemma. □

Lemma 5.4.8. *Let Q, m, $B\colon Q \to Q$, $F\colon S \to Q$, $\mathbf{F}(n)$, $\mathbf{F}$ and $\mathbf{F}'$ as in Part 2 of Proposition 5.4.6. Let $M = B_*(\lambda, m)$ and $n \in \mathbb{Z}_{>0}$. Then*

$$\left(\mathbb{1}_{\mathbf{F}(n)\dot\cup\mathbf{F}} * \mathbb{1}_{\mathbf{F}'}\right)(M) = \sum\nolimits_{k\in\mathbb{N}} \left(\left(\mathbb{1}_{\mathbf{F}(n-1)} \otimes 1\right) * \Delta\left(\mathbb{1}_{\mathbf{F}}\right) * \Delta\left(\mathbb{1}_{\mathbf{F}'}\right)\right)\left(B_*(\lambda, m-k), I_k\right)$$

with I_k is an indecomposable module and $\mathbf{dim}\, I_k = \mathbf{dim}\, B_*(\lambda, k) - \mathbf{dim}\, F_*(\mathbb{1}_S)$.

Proof. Let $M = (M_i, M_a)_{i\in Q_0, a\in Q_1}$, $\mathbf{c} = \mathbf{dim}\, F_*(\mathbb{1}_S)$, $\mathbf{d}^{(i)} = \mathbf{dim}\, F_*^{(i)}(\mathbb{1}_{S^{(i)}})$ for all i and $\mathbf{d} = n\mathbf{c} + \sum_i \mathbf{d}^{(i)}$. By Remark 2.2.10, we know $\mathbb{1}_{\mathbf{F}(n)} * \mathbb{1}_{\mathbf{F}} = \mathbb{1}_{\mathbf{F}(n)\dot\cup\mathbf{F}}$. So we have to calculate the Euler characteristic of

$$X = \left\{(0 \subseteq U \subseteq W \subseteq M) \in \mathcal{F}_{n\mathbf{c},\mathbf{d}}(M) \middle| \mathbb{1}_{\mathbf{F}(n)}(U) = \mathbb{1}_{\mathbf{F}}(W/U) = \mathbb{1}_{\mathbf{F}'}(M/W) = 1\right\}.$$

We use now the arguments of the proof of Lemma 4.7.5 in Section 4.7.2. Let $\{e_{ik} | i \in Q_0, 1 \le k \le m\}$ be a basis of M such that the following hold.

1. For all $1 \le p \le m$, the vector space $M^{(p)} := \langle e_{i,k} | i \in Q_0, 1 \le k \le p\rangle$ is a subrepresentation of M isomorphic to $B_*(\lambda, p)$.
2. There exists a nilpotent endomorphism ψ of M such that $\psi(e_{i1}) = 0$ and $\psi(e_{ik}) = e_{i,k-1}$ for all $1 < k \le m$ and all $i \in Q_0$.

The quiver S is of type $A_{|\mathbf{c}|}$ such that $S_0 = \{1, \ldots, |\mathbf{c}|\}$ and $S_1 = \{s_1, \ldots, s_{|\mathbf{c}|-1}\}$.

Let $(0 \subseteq U \subseteq W \subseteq M) \in X$. Then $U \cong F_*(\mathbb{1}_S)^n$. Using the Gauß algorithm, there exists a unique tuple $\mathbf{j}(U) = (1 \le j_1 < j_2 < \ldots < j_n \le m)$ as in Equation (4.7.2) and unique $\lambda_{kj}(U) \in \mathbb{C}$ such that the vector space U is spanned by

$$\left(M_{F_1(s_1)}^{\varepsilon_1} \ldots M_{F_1(s_q)}^{\varepsilon_q}\right)^{-1}\left(e_{F_0(1),j_p} + \sum\nolimits_{j=1, j\ne j_k \forall k}^{j_p-1} \lambda_{pj}(U) e_{F_0(1),j}\right)$$

with $1 \le p \le n$ and $0 \le q < |\mathbf{c}|$. This is well-defined since all linear maps M_i are isomorphisms. The variety X can be decomposed into a disjoint union of locally closed subsets $X_k := \{(U \subseteq W) \in X | (\mathbf{j}(U))_1 = k\}$. Define a locally closed subset of X for each k by $X_k^0 := \{(U \subseteq W) \in X_k | \lambda_{1j}(U) = 0\ \forall j\}$. Equation (4.7.3) defines again an algebraic morphism $\pi\colon X_k \to X_k^0$ with affine fibres.

For each k there exists a $U_k \subseteq M$ such that $U_k \cong F_*(\mathbb{1}_S)$, $U_k \subseteq U$ for all $(U \subseteq W) \in X_k^0$ and $M/U_k \cong M^{(m-k)} \oplus I_k$ with an indecomposable module I_k as in the lemma. Since $|\mathbf{c}| \ge |\mathbf{d}^{(i)}|$ for all i and $F_*(\mathbb{1}_S)$ is preprojective, all sequences of the form

$$0 \to F_*(\mathbb{1}_S) \to \pi^{-1}(W) \to F_*(\mathbb{1}_S)^{n-1} \oplus \bigoplus\nolimits_i F_*^{(i)}(\mathbb{1}_{S^{(i)}}) \to 0$$

with a projection $\pi\colon M \to M/F_*(\mathbb{1}_S)$ and a submodule $W \subseteq M/F_*(\mathbb{1}_S)$ split. Let

$$\overline{X_k^0} := \left\{\left(U \subseteq W \subseteq M^{(m-k)}\right), \left(W' \subseteq I_k\right) \middle| \right.$$
$$\left. \mathbb{1}_{\mathbf{F}(n-1)}(U) = \mathbb{1}_{\mathbf{F}}(W/U \oplus W') = \mathbb{1}_{\mathbf{F}'}\left(M^{(m-k)}/W \oplus I_k/W'\right) = 1\right\}.$$

Using an R-grading, we conclude, as in the proof of Part 1 of Proposition 5.4.6, that

$$\chi\left(X_k^0\right) = \chi\left(\overline{X_k^0}\right) = \left(\left(\mathbb{1}_{\mathbf{F}(n-1)} \otimes 1\right) * \Delta\left(\mathbb{1}_{\mathbf{F}}\right) * \Delta\left(\mathbb{1}_{\mathbf{F}'}\right)\right)\left(M^{(m-k)}, I_k\right)$$

and by $\chi(X) = \sum_{k\in\mathbb{N}} \chi\left(X_k^0\right)$ the lemma. □

5.5 String algebras

In this section we consider the Ringel-Hall algebras of string algebras. We apply the results of the previous sections to this special case.

Corollary 5.5.1. *Let A be a string algebra. Let $\mathbf{F}$ be a tuple of strings, $\mathbf{B}$ a tuple of bands and $\mathbf{n}$ a tuple of positive integers. Then $\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{B},\mathbf{n}} = \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} = \mathbb{1}_{\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}}$.*

Proof. If A is a string algebra, then every indecomposable A-module is a string or a band module. So this corollary follows directly from Lemma 5.4.1, Example 5.4.2 and Proposition 5.4.4. □

Example 5.5.2. Let $Q = (1 \rightrightarrows 2)$, $\mathbf{F}$ and $\mathbf{F}'$ tuples of strings, $m \in \mathbb{N}$ and $V \in \mathcal{I}_Q^m$ such that $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(V) \neq 0$. Then

$$(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(V) = \frac{l(\mathbf{F})!}{\prod_{\{F^{(i)}|i\}/\cong} |[F^{(i)}]|!} \frac{l(\mathbf{F}')!}{\prod_{\{F'^{(i)}|i\}/\cong} |[F'^{(i)}]|!}, \tag{5.5.1}$$

where $\{F^{(i)}|i\}/\cong$ is the set of isomorphism classes and $|[F^{(i)}]|$ is the number of elements in the isomorphism class of $F^{(i)}$. For instance,

$$\left(\mathbb{1}_{S(2)^{m-r} \oplus P(1)^r} * \mathbb{1}_{S(1)^{m-s} \oplus I(2)^s}\right)(V) = \binom{m}{r}\binom{m}{s}$$

for each $V \in \mathcal{I}_Q^{m+r+s}$ with $m, r, s \in \mathbb{N}$, $S(i) \in \operatorname{rep}(Q)$ is the simple representation associated to the vertex $i \in Q_0$ and $P(i) \in \operatorname{rep}(Q)$ (resp. $I(i)$) is the projective cover (resp. injective hull) of $S(i)$ for each $i \in Q_0$ (see Example 3.5.9).

Equation (5.5.1) can be proven by iterated use of Part 2 of Proposition 5.4.6. By Example 4.5.5, alternatively it is enough to show Equation (5.5.1) for a string module with dimension vector (m, m). Using Theorem 5.1.1, this can be computed by counting all listings of the strings in $\mathbf{F}$ and in $\mathbf{F}'$.

In general it is much harder to give an explicit formula for $(\mathbb{1}_{\mathbf{F}} * \mathbb{1}_{\mathbf{F}'})(V)$.

Corollary 5.5.3. *Let A be a string algebra. Then every function in $\mathcal{C}(A)$ is a linear combination of functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ with some tuple $\mathbf{F}$ of strings, some tuple $\mathbf{B}$ of bands and some tuple $\mathbf{n}$ of positive integers.*

Proof. We use an induction over dimension vectors of Q. Let $\mathbf{d}$ be a dimension vector. Then the set

$$H_{\mathbf{d}} := \left\{\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} \middle| \exists M \in \operatorname{rep}_{\mathbf{d}}(Q) : \mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}(M) \neq 0\right\}$$

is finite and the function $\mathbb{1}_{\mathbf{d}}$ is the sum of all functions in $H_{\mathbf{d}}$.

It remains to show that each product $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'} \in \mathcal{H}_{\mathbf{d}}(A)$ is a linear combination of functions in $H_{\mathbf{d}}$. Using Lemma 5.4.1 and Example 5.4.2, we have to check that for all bands B and $m \in \mathbb{N}$ the integer $\left(\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}} * \mathbb{1}_{\mathbf{F}',\mathbf{B}',\mathbf{n}'}\right)(B_*(\lambda, m))$ is independent of $\lambda \in \mathbb{C}^*$. This is clear by Part 2 of Proposition 5.4.6 and an induction argument. □

In general the functions of the form $\mathbb{1}_{\mathbf{F},\mathbf{B},\mathbf{n}}$ do not belong to $\mathcal{C}(A)$ for a string algebra A. Nevertheless they are linearly independent. So we get the following result. Unfortunately it is not clear how a basis looks like in general or if $\mathcal{C}(A) = \mathcal{H}(A)$ holds for each representation finite (string) algebra A.

Theorem 5.5.4. *Let $A = \mathbb{C}Q/I$ be a string algebra, such that Q has no loops and cyclically oriented two-cycles. Then $\mathcal{C}(A) = \mathcal{H}(A)$ if and only if A is representation finite. Moreover, in this case the set of functions of the form $\mathbb{1}_{\mathbf{F}}$ with some tuple $\mathbf{F}$ of strings is a vector space basis of $\mathcal{C}(A)$.*

For each finite-dimensional representation finite algebra A the set of characteristic functions $\mathbb{1}_M$ of the orbits of A-modules M is a basis of the vector space $\mathcal{H}(A)$. To prove the theorem we obtain the following two lemmas.

Lemma 5.5.5. *Let A be a finite-dimensional algebra.*
1. *If A is representation finite and $\mathbb{1}_M \in \mathcal{C}(A)$ for each indecomposable A-module M, then $\mathcal{C}(A) = \mathcal{H}(A)$.*
2. *If A is representation infinite, then $\mathcal{C}(A) \neq \mathcal{H}(A)$.*

Proof. For the first part we consider an A-module N and a decomposition $N = \bigoplus_{i=1}^m N_i$ of N in indecomposable A-modules. We use a result of Riedtmann [44, Lemma 2.2] and some induction over the number of indecomposable direct summands m. Since A is representation finite, we get

$$\mathbb{1}_{N_1} * \ldots * \mathbb{1}_{N_m} = \lambda \mathbb{1}_N + \sum\nolimits_{i=1}^{r} \lambda_i \mathbb{1}_{M_i}$$

with $r \in \mathbb{N}$, $\lambda, \lambda_1, \ldots, \lambda_r \in \mathbb{C}^*$ and A-modules M_i such that the number of indecomposable direct summands of M_i is smaller than m. Thus $\mathbb{1}_M \in \mathcal{C}(A)$ for each A-module M.

Otherwise, if A is representation infinite, there exists some dimension vector $\mathbf{d}$ with infinitely many isoclasses of A-modules with dimension vector $\mathbf{d}$ (see Bautista [4, Theorem 2.4]). Since $\{\mathbb{1}_M | M \in \mathrm{mod}(A)\}$ is a basis, the vector space $\mathcal{H}_{\mathbf{d}}(A)$ is - in contrast to $\mathcal{C}_{\mathbf{d}}(A)$ - not finite-dimensional. This yields $\mathcal{C}(A) \neq \mathcal{H}(A)$. □

Lemma 5.5.6. *Let $N \in \mathbb{N}$ and $Q = (Q_0, Q_1)$ be the cyclically oriented quiver of type $\tilde{A}_{N-1}$, i.e. $Q_0 = \{1, \ldots, N\}$ and $Q_1 = \{\alpha_i \colon i \to i+1 | i \in Q_0\}$. For each admissible ideal I holds $\mathcal{C}(\mathbb{C}Q/I) = \mathcal{H}(\mathbb{C}Q/I)$.*

Proof of Theorem 5.5.4. Let A be representation finite. Thus each indecomposable A-module is a string module. By Lemma 5.5.5 it is enough to show $\mathbb{1}_F \in \mathcal{C}(A)$ for each string $F \colon S \to Q$. We use some induction over the dimension d of $F_*(\mathbb{1}_S)$. If $d = 1$, we are done. Thus we assume $d \geq 2$.

Let $G \colon T \to Q$ be a string and $i \in Q_0$. Then by Corollary 5.4.3 we get

$$\mathbb{1}_{S(i)} * \mathbb{1}_G - \mathbb{1}_G * \mathbb{1}_{S(i)} = \sum\nolimits_{j=1}^{r} \varepsilon_j \mathbb{1}_{F^{(j)}} \qquad (5.5.2)$$

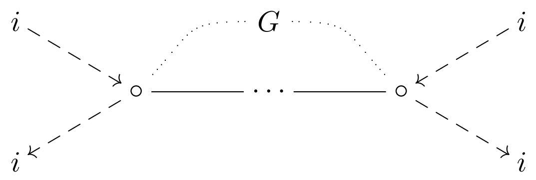


Figure 5.5.1: Possible strings $F^{(j)}\colon T^{(j)} \to Q$.

with $r \in \mathbb{N}$, $\varepsilon_j \in \{-1, 1\}$ and strings $F^{(j)}\colon T^{(j)} \to Q$ of the representation finite algebra A. Since Q has no loops and cyclically oriented two-cycles, we get $0 \leq r \leq 2$ by the picture in Figure 5.5.1.

Moreover, if $r = 2$, we obtain the following string. Let t be the dimension of $G_*(\mathbb{1}_T)$. Then without loss of generality we get a quiver $T' = (T'_0, T'_1)$ of type A_{t+2} with $T'_0 = \{0, 1, \ldots, t, t{+}1\}$ and a string $G'\colon T' \to Q$ of Q such that the following holds. For $r, s \in \mathbb{N}$ with $0 \leq r \leq s \leq t{+}1$ let $T^{r,s}$ be the full subquiver of T' with the points $\{r, r{+}1, \ldots, s\}$. The quiver $T^{1,t} = T$, $T^{0,t} = T^{(2)}$, $T^{1,t+1} = T^{(1)}$, $G'|_{T^{1,t}} = G$, $G'|_{T^{0,t}} = F^{(2)}$ and $G'|_{T^{1,t+1}} = F^{(1)}$ (see Figure 5.5.2). We remark that $G'_*(\mathbb{1}_{T'})$ is not necessarily an A-module.

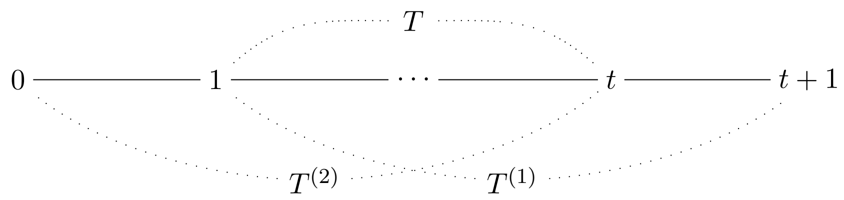


Figure 5.5.2: The quiver T' with subquivers T, $T^{(1)}$ and $T^{(2)}$.

Let $F\colon S \to Q$ be the string with $S_0 = \{1, \ldots, d\}$ and $d \geq 2$ as above. For $r, s \in \mathbb{N}$ with $1 \leq r \leq s \leq d$ let $S^{r,s}$ be the full subquiver of S with the points $\{r, \ldots, s\}$. Then we set $G = F|_{S^{1,d-1}}\colon S^{1,d-1} \to Q$ and $i = F_0(d)$. Without loss of generality $F^{(1)} = F$ in Equation (5.5.2). If $r = 1$, we are done. Otherwise we get a quiver T' and a $G'\colon T' \to Q$ as above with $d = t{+}1$ and $T^{1,t+1} = S$. Moreover, we obtain a quiver $\tilde{S}$ of type $\tilde{A}_{d-1}$ with $\tilde{S}_0 = S_0$ by identifying the points 0 and $t + 1$ in T' and an induced winding $\tilde{G}\colon \tilde{S} \to Q$ with $\tilde{G}|_S = F$ and $\tilde{G}_1 = G'_1$. Now we continue with the string $\hat{G} = G'|_{T^{0,d-2}}\colon T^{0,d-2} \to Q$ and $\hat{i} = F_0(d{-}1)$ and so on. If this construction stops sometime, we are done. Otherwise, without loss of generality this quiver $\tilde{S}$ is cyclically oriented, since the algebra $A = \mathbb{C}Q/I$ is representation finite. This means $\tilde{S}_1 = \{\alpha_i\colon i \to i + 1 | i \in \tilde{S}_0\}$ with $d + i := i$ in $\tilde{S}_0$.

Furthermore, we assume that the winding $\tilde{G}\colon \tilde{S} \to Q$ is surjective. For $r, s \in \{1, \ldots, d\}$

we define a subquiver $\tilde{S}^{r,s} = (\tilde{S}_0^{r,s}, \tilde{S}_1^{r,s})$ of $\tilde{S}$ of type A by

$$\tilde{S}_0^{r,s} = \begin{cases} \{r, \ldots, s\} & \text{if } r \leq s, \\ \{r, \ldots, d+s\} & \text{otherwise,} \end{cases}$$
$$\tilde{S}_1^{r,s} = \{\alpha_i | i \in \tilde{S}_0^{r,s}, i \neq s\}.$$

We remark again that $\tilde{G}_*(\mathbb{1}_{\tilde{S}})$ is not an A-module, but $\tilde{G}_*(\mathbb{1}_{\tilde{S}^{r,s}})$ is an A-module for all $r, s \in \{1, \ldots, d\}$.

Now we prove $\tilde{G}_1(\alpha_r) = \tilde{G}_1(\alpha_s)$ for all $r, s \in \tilde{S}_0$ with $\tilde{G}_0(r) = \tilde{G}_0(s)$. This yields that Q is a cyclically oriented quiver of type $\tilde{A}$ (see Figure 3.6.11). This case is covered in Lemma 5.5.6.

Let $\tilde{G}$ be not injective and $r, s \in \mathbb{N}$ with $1 \leq r < s \leq d$ and $\tilde{G}_0(r) = \tilde{G}_0(s)$. Now we construct a quiver S' of type $\tilde{A}_{d-1}$ by taking the disjoint union of $\tilde{S}^{r,s}$ and $\tilde{S}^{s,r}$ and identifying the points r and s crosswise (see Figure 5.5.3).

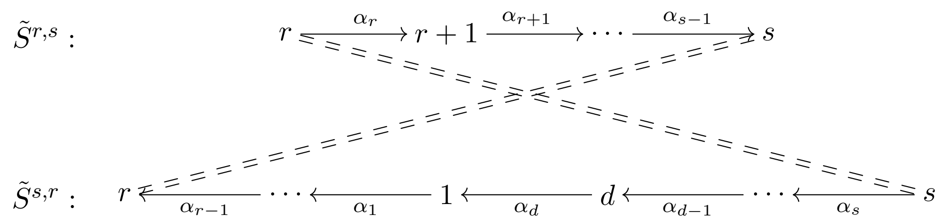


Figure 5.5.3: The construction of the quiver S'.

This quiver is not cyclically oriented. If the induced morphism $S' \to Q$ of quivers is a winding, we get a band of A. However, A is representation finite. Thus $\tilde{G}_1(\alpha_{r-1}) = \tilde{G}_1(\alpha_{s-1})$ or $\tilde{G}_1(\alpha_r) = \tilde{G}_1(\alpha_s)$. We assume the first case. Since A is a string algebra we get $\tilde{G}_1(\alpha_{r-i}) = \tilde{G}_1(\alpha_{s-i})$ for all $i \in \mathbb{N}$ with $i > 0$, $i \leq s - r$ and $i \leq r - s + d$. Without loss of generality we assume $r - s + d < s - r$. In this case $\tilde{G}_1(\alpha_s) = \tilde{G}_1(\alpha_{2s-r-d})$ and $r < 2s - r - d < s$. Thus $\tilde{G}_0(s) = \tilde{G}_0(2s - r - d)$. We construct another quiver S'' of type $\tilde{A}_{s-r-1}$ by taking the disjoint union of $\tilde{S}^{r,2s-r-d}$ and $\tilde{S}^{s,r}$ and identifying the points r and $2s - r - d$ in $\tilde{S}^{r,2s-r-d}$ and the points r and s in $\tilde{S}^{s,r}$ crosswise (see Figure 5.5.4).

This quiver induces again a morphism $S'' \to Q$ of quivers. This is again not a band. Thus by an induction over the number of points in S' and S'' we get $\tilde{G}_1(\alpha_r) = \tilde{G}_1(\alpha_s)$. Moreover, there is a $r \in \mathbb{N}$ with $1 \leq r < l$, $\tilde{G}_1(\alpha_i) = \tilde{G}_1(\alpha_{i+r})$ as in Example 2.2.6. □

Proof of Lemma 5.5.6. For $r \in \mathbb{N}$ with $r > 0$ let M_r be an indecomposable, nilpotent Q-representation with top $M_r \cong S(1)$ and $\dim M_r = r$ (see Figure 5.5.5). Using the first part of the proof of Theorem 5.5.4 $\mathbb{1}_{M_r} \in \mathcal{C}(A)$ for all $r \in \mathbb{N} \backslash N\mathbb{N}$ (see Equation (5.5.2)).

If for all $r \in \mathbb{N}$ holds $\mathbb{1}_{M_N^r} \in \mathcal{C}(A)$, we get $\mathbb{1}_{M_{rN}} \in \mathcal{C}(A)$ by induction over r and the following equation

$$\mathbb{1}_{M_{rN}} = \sum\nolimits_{i=1}^{r-1} (-1)^{i-1} \mathbb{1}_{M_{(r-i)N}} * \mathbb{1}_{M_N^i} + (-1)^{r-1} r \mathbb{1}_{M_N^r}.$$

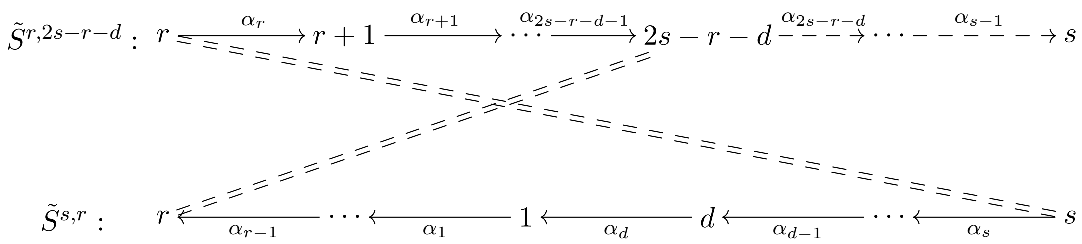


Figure 5.5.4: The construction of the quiver S''.

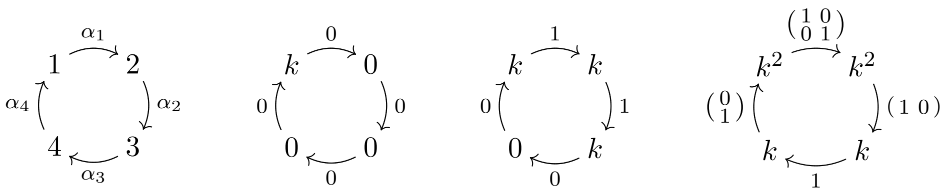


Figure 5.5.5: The quiver Q and the Q-representations M_1, M_3 and M_6 for $N = 4$.

Thus it is enough to show $\mathbb{1}_{M_N^r} \in \mathcal{C}(A)$ for all $r \in \mathbb{N}$. Let (P_s) be the following property for $s \in \{1, \dots, N\}$:

$$\mathbb{1}_{S(\mathbf{d}) \oplus M_s^r} \in \mathcal{C}(A)\ \forall r \in \mathbb{N}, \mathbf{d} \in \mathbb{N}^{Q_0}.$$

Property (P_1) holds by Lemma 2.4.2 and Property (P_N) yields the lemma. Let $\mathbf{d} = (d_i)_{i \in Q_0}$ be a dimension vector. If $N \geq 2$, we get by induction over the dimension for all $l \in \{1, \dots, r\}$

$$\mathbb{1}_{S(\mathbf{d}+(r-l)e_1+re_2)} * \mathbb{1}_{S(1)^l} = \sum_{i=0}^{l} \binom{d_1+r-i}{l-i} g_i \in \mathcal{C}(A) \tag{5.5.3}$$

with $g_i := \mathbb{1}_{S(\mathbf{d}+(r-i)e_1+(r-i)e_2) \oplus M_2^i}$ for all $i \in \{0, \dots, r\}$. Thus Property (P_1) yields $g_0 \in \mathcal{C}(A)$. By Equation (5.5.3) and another induction over l we get $g_i \in \mathcal{C}(A)$ for all $i \in \{0, \dots, r\}$. Thus $g_r = \mathbb{1}_{S(\mathbf{d}) \oplus M_2^r} \in \mathcal{C}(A)$ and Property (P_2) holds.

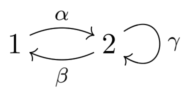
Now we assume that Property (P_s) holds for some $s \in \{2, \dots, N-1\}$. Again we get for all $l \in \{1, \dots, r\}$

$$\mathbb{1}_{S(\mathbf{d}+re_{s+1}) \oplus M_s^{r-l}} * \mathbb{1}_{M_s^l} = \sum_{i=0}^{l} \binom{r-i}{l-i} g_i \in \mathcal{C}(A) \tag{5.5.4}$$

with $g_i := \mathbb{1}_{S(\mathbf{d}+(r-i)e_{s+1}) \oplus M_s^{r-i} \oplus M_{s+1}^i}$ for all $i \in \{0, \dots, r\}$. Thus Property (P_s) yields $g_0 \in \mathcal{C}(A)$. By Equation (5.5.4) and another induction over l we get again $g_i \in \mathcal{C}(A)$ for all $i \in \{0, \dots, r\}$. Thus $g_r = \mathbb{1}_{S(\mathbf{d}) \oplus M_{s+1}^r} \in \mathcal{C}(A)$ and Property (P_{s+1}) holds. □

By the following example this proof cannot be generalized to each representation finite string algebra.

Example 5.5.7. Let Q be the following quiver



and $I = \langle \alpha\beta, \beta\alpha, \gamma^2 \rangle$ an admissible ideal. Thus $A = \mathbb{C}Q/I$ is a string algebra. This algebra is representation finite since up to an isomorphism all strings are described by the pictures in Figure 5.5.6.

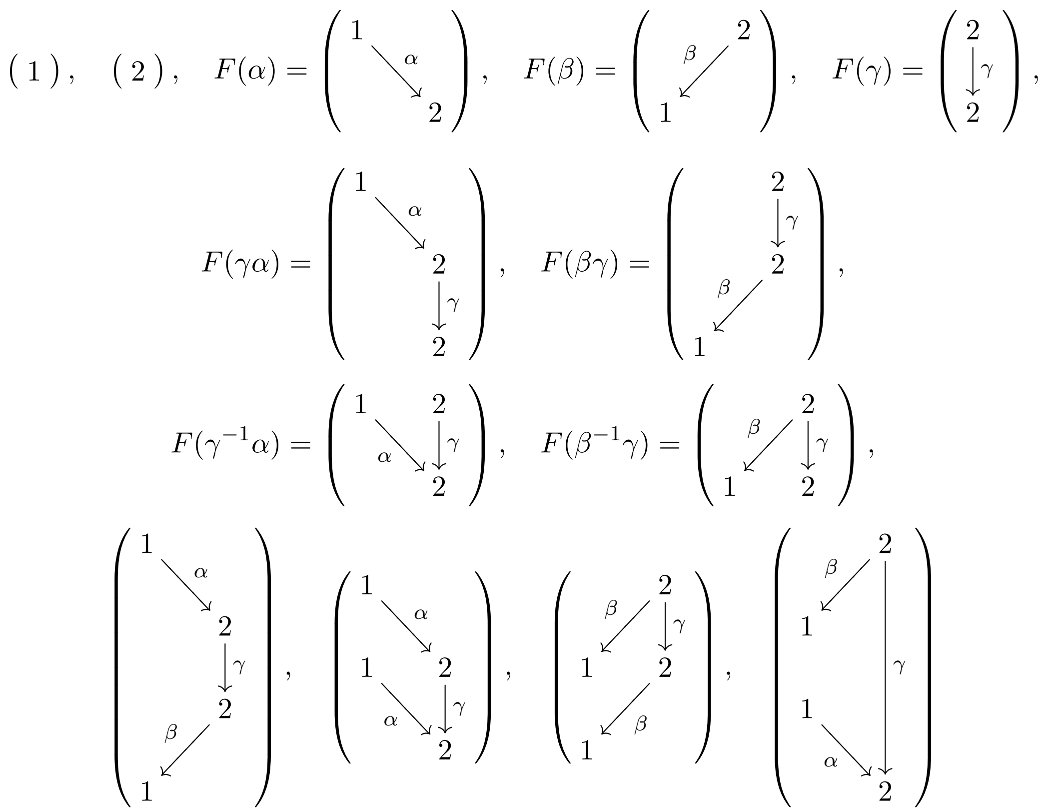


Figure 5.5.6: Strings of the string algebra A.

Then holds

$$\begin{aligned}
\mathbb{1}_{S(2)} * \mathbb{1}_{F(\alpha)} - \mathbb{1}_{F(\alpha)} * \mathbb{1}_{S(2)} &= \mathbb{1}_{F(\gamma\alpha)} - \mathbb{1}_{F(\gamma^{-1}\alpha)}, \\
\mathbb{1}_{S(2)} * \mathbb{1}_{F(\beta)} - \mathbb{1}_{F(\beta)} * \mathbb{1}_{S(2)} &= - \mathbb{1}_{F(\beta\gamma)} + \mathbb{1}_{F(\beta^{-1}\gamma)}, \\
\mathbb{1}_{S(1)} * \mathbb{1}_{F(\gamma)} - \mathbb{1}_{F(\gamma)} * \mathbb{1}_{S(1)} &= - \mathbb{1}_{F(\gamma\alpha)} + \mathbb{1}_{F(\beta\gamma)} - \mathbb{1}_{F(\gamma^{-1}\alpha)} + \mathbb{1}_{F(\beta^{-1}\gamma)}.
\end{aligned}$$

Thus the proof of Theorem 5.5.4 fails for this example, but $\mathcal{C}(A) = \mathcal{H}(A)$ by some straightforward calculation.

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