Obstruction theory for operadic algebras

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Obstruction theory for operadic algebras*

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Abstract

We study obstruction theory for formality of chain operadic algebras. We construct a canonical class $\gamma_A \in H^{2,-1}_{\mathrm{P}_\infty}(H_*A, H_*A)$ for an algebra A over a cofibrant chain operad P_∞. This class takes values in the Gamma cohomology groups of the homology algebra H_*A and depends only on the homotopy type of the algebra. If the canonical class vanishes then there is a successive obstruction. Further, we give a criterion for when two P_∞-algebras are of the same homotopy type.

Contents

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Introduction

The notion of an operad has its origin in Stasheff's work "Homotopy associativity of H-spaces" [Sta] which appeared in the early sixties. There Stasheff gives the definition of an A_n-structure and an A_n-space. It turns out that a connected space has the weak homotopy type of a loop space if and only if it is an A_∞-space. Only few years later Boardman and Vogt [BoVo] gave the definition of a "homotopy everything H-space". They proved that a CW complex X with $\pi_0(X)$ a group is weakly equivalent to an infinite loop space if and only if it is a homotopy everything H-space. In search of a general recognition principle for n-fold loop spaces May [May] gave the first explicit definition of an operad and an operadic algebra. The notions of an A_∞- and E_∞-operad are due to him. His "little cubes operads" $\mathfrak{C}_n$ meet the requirements of detecting n-fold loop spaces up to weak equivalence. As a logical generalization of the previous mathematical problems Boardman and Vogt investigated the question about homotopy invariant algebraic structures in topology [BV]. Their W-construction assigns to an operad P a new operad $W(\mathrm{P})$ with nice homotopical properties. Algebras over P are also $W(\mathrm{P})$-algebras, and if P has for example a free action of the symmetric groups, then the operad $W(\mathrm{P})$ has the "homotopy invariance property". This means that given a weak equivalence of CW complexes

$$X \underset{\sim}{\xrightarrow{f}} Y$$

and a $W(\mathrm{P})$-algebra structure on Y one can equip X with a $W(\mathrm{P})$-algebra structure, such that the map f can be extended to a $W(\mathrm{P})$-equivalence. The corresponding notion of a $W(\mathrm{P})$-equivalence was also defined by Boardman and Vogt. In summary, $W(\mathrm{P})$-structures are homotopy invariant.

Although operads arose in topology, they soon became of interest in other fields of mathematics. The concept of an operad can be applied in every symmetric monoidal category. Actually, it seems to be the perfect formal framework for encoding different types of algebras - associative, commutative, Lie, Leibniz, differential graded associative, differential graded commutative, and many more. For the past twenty-five years algebraists have been exploring the field. Some names to be mentioned in this context are Fresse, Kadeishvili, Loday, Markl, Vallette. The question about homotopy invariant structures in the algebraic setting has drawn the attention. Kadeishvili was the first to manage the breakthrough:

Theorem ([Kad])**.** *Let A be a differential graded algebra with homology free over the ground ring. There is a transferred A_∞-structure on H_*A, such that H_*A is connected to A via an A_∞-map that is a weak equivalence.*

The homology H_*A together with the transferred A_∞-structure is called a *minimal* model for A. If the structure on H_*A can be chosen to be the trivial one, i.e., just graded associative induced by A, then A is said to be a *formal* algebra. An A_∞-algebra is in particular the homotopy invariant generalization of a differential graded associative algebra.

In this context Benson, Krause and Schwede [BKS] defined a canonical class of a differential graded algebra

$$\gamma_A \in \mathbf{HH}^{3,-1}(H_*A, H_*A)$$

that takes values in the $(3,-1)^{\mathrm{st}}$ Hochschild cohomology group of H_*A and depends only on the homotopy type of the algebra. It can help to distinguish non-quasi-isomorphic algebras and it is an obstruction to formality. Another result in this direction is due to Kadeishvili [Kad88]. It states that the vanishing of the total second Hochschild cohomology group of a given graded algebra H implies that every algebra with homology isomorphic to H is formal.

The goal of this thesis is to develop a general obstruction theory for formality of algebras over a differential graded operad P with trivial differentials. Our interest is directed mainly at the commutative operad, since E_∞-algebras arise naturally for example as cochains of spaces, but the theory we introduce is valid in a more general setting. At first glance the commutative operad seems to be very similar to the associative one, there is one big difference, though: The action of the symmetric groups is not free. This fact makes the commutative case in particular in positive characteristic much more complicated. Beside the obstruction theory for formality we want to give a criterion that enables one to decide whether two given P_∞-algebras have the same homotopy type.

We proceed similarly to the associative case. We take a particular *cofibrant replacement* P_∞ of the operad P. The important thing about the replacement is that it has the "homotopy invariance property". Given a P- or more generally P_∞-algebra A over a field k, we equip H_*A with a quasi-isomorphic P_∞-structure. This structure gives us the necessary information in order to construct obstructions

$$\gamma_A^{[t]} \quad \text{for} \quad t \geq 2.$$

A major difficulty in our setting is the fact that the cofibrant replacements we have to deal with are huge and not that handy. In contrast to the associative setting where Koszul theory gives a small cofibrant replacement of the associative operad, we are forced to work with the Cobar-Bar resolution of a Σ_*-cofibrant replacement of the original operad P. Another question that appears is, which is the corresponding cohomology theory where the formality obstructions should live for a given operad P. The answer here is *Gamma cohomology.* This is the appropriate generalization of Hochschild cohomology.

Our first main result is the following:

Theorem 4.2. *Let* P *be a graded operad in* $\mathcal{C}h$ *and* A *a* P_∞*-algebra. There is a canonical class* $\gamma_A^{[2]} \in H^1_{\mathrm{P}_\infty}(H_*A)$ *in the Gamma cohomology of the strict* P_∞*-algebra* H_*A*, such that*

(i) if f is a map of P_∞*-algebras from A to B, and we denote by f^* respectively f_* the induced maps on cohomology groups as depicted below:*

$$\begin{tikzcd} H^1_{\mathrm{P}_\infty}(H_*A, H_*A) \arrow[r, "f_*"] & H^1_{\mathrm{P}_\infty}(H_*A, H_*B) \\ & H^1_{\mathrm{P}_\infty}(H_*B, H_*B) \arrow[u, "f^*"] \end{tikzcd}$$

then we have $f_*(\gamma_A^{[2]}) = f^*(\gamma_B^{[2]})$. In particular, if f is a weak equivalence then via the induced isomorphisms f_* and f^* on operadic cohomology the classes $\gamma_A^{[2]}$ and $\gamma_B^{[2]}$ coincide.

(ii) if g^c is the map on cofibrant operads

$$g^c\colon \mathcal{B}^c\mathcal{B}(\mathrm{Q} \otimes \mathrm{E}\Sigma_*) \to \mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$$

induced by a map $g\colon \mathrm{Q} \to \mathrm{P}$ of graded operads then A is also a Q_∞*-algebra, and under the map induced by g^c on operadic cohomology groups the canonical class of A as a* P_∞*-algebra is mapped to the canonical class of A as a* Q_∞*-algebra.*

If the first canonical class vanishes then one can construct a higher obstruction. The latter is not just a single class but a set in the corresponding Gamma cohomology group of the algebra. We introduce an equivalence relation in order to define a quotient group

$$\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A) \text{ for } t \geq 2$$

where the t-th obstruction takes values. We can then restate the theorem for the higher obstructions:

Theorem 4.6. *Let* P *be a graded operad in* $\mathcal{C}h$*, and* A' *and* A'' *two* P_∞*-algebras. Suppose that there are transferred* P_∞*-structures* γ' *and* γ'' *on* H_*A' *and* H_*A''*, respectively, with the property* $\gamma'^{[s]} = \gamma''^{[s]} = 0$ *for* $s = 2, \ldots, t-1$*. Then there are successive obstructions* $\gamma^{[t]}_{A'} \in \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A')$ *and* $\gamma^{[t]}_{A''} \in \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A'')$ *for which the following holds:*

(i) if f *is a map of* P_∞*-algebras from* A' *to* A''*, and we denote by* f^* *respectively* f_* *the induced maps on the quotients of cohomology groups as depicted below:*

$$\begin{tikzcd} \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A', H_*A') \arrow[r, "f_*"] & \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A', H_*A'') \\ & \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A'', H_*A'') \arrow[u, "f^*"'] \end{tikzcd}$$

then we have $f_*(\gamma^{[t]}_{A'}) = f^*(\gamma^{[t]}_{A''})$*. In particular, if* f *is a weak equivalence then via the induced isomorphisms* f_* *and* f^* *the obstructions* $\gamma^{[t]}_{A'}$ *and* $\gamma^{[t]}_{A''}$ *coincide.*

(ii) if g^c *is the map on cofibrant operads*

$$g^c \colon \mathcal{B}^c\mathcal{B}(\mathrm{Q} \otimes \mathrm{E}\Sigma_*) \to \mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$$

induced by a map $g \colon \mathrm{Q} \to \mathrm{P}$ *of graded operads then* A' *is also a* Q_∞*-algebra, and under the map induced by* g^c *on operadic cohomology the obstruction of* A' *as a* P_∞*-algebra is mapped to the obstruction of* A' *as a* Q_∞*-algebra.*

In particular, as a corollary of both theorems we can conclude that the vanishing of the first Gamma cohomology group of H_*A implies the formality of A. In other words, if the first Gamma cohomology group of a given graded algebra V is zero, then there is only one homotopy type of P_∞-algebras with homology isomorphic to V.

As we mentioned, the obstruction theory can also help to distinguish algebras of different homotopy type. However, it can happen that two algebras have the same (non-trivial) canonical class or more general higher obstruction, but are not quasi-isomorphic. To solve this case we give a criterion in terms of maps of quasi-cofree coalgebras.

Organization

The first three sections give the prerequisites for the obstruction theory developed in Section 4. In Section 1 we recall the necessary parts of operadic algebra, in particular operads, cooperads, tree representations and important constructions. In Section 2 we use the model category language to state a conceptual version of the "homotopy invariance property". Then, we pursue the question of explicit cofibrant replacements of operads as well as of operadic algebras over so called Σ_*-cofibrant operads. We give a short introduction to operadic cohomology and Gamma cohomology in Section 3. After that, in Section 4.1, we state and prove the main results. In Section 4.2 we compare these results with the ones that are already in the literature, before we come to the last section where we illustrate the theory on some examples.

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0 Conventions

Before we start let us fix some notation and conventions.

When nothing else specified we will denote by $\mathcal{C}$ a symmetric monoidal category. Usually, however, we are working in a particular symmetric monoidal category, namely in the category of unbounded chain complexes over a commutative ring k. We are going to use the notation $\mathcal{C}h$ for it. In the main part of this thesis, we in addition have the assumption that k is a field (of possibly positive characteristic). By abuse of notation we also write k for the chain complex concentrated in degree 0.

Recall that the category of chain complexes has an internal Hom object, which we denote by $\mathbf{Hom}(-,-)$. For two chain complexes A and B, the *internal Hom complex* in level s consists of k-linear maps raising the degree by s, i.e., $f \in \mathbf{Hom}(A,B)_s$ iff f is k-linear and $f(A_*) \subset B_{*+s}$. For example the differential of A is an element of degree -1. The differential on $\mathbf{Hom}(A,B)_s$ is given by $\delta(f) = \delta_B \circ f - (-1)^s f \circ \delta_A$. An element of $\mathbf{Hom}(A,B)$ will be called a *homomorphism*. The term "morphism" should be reserved for maps commuting with differentials.

For the sake of our needs we remind the reader that the *s-fold shift* of a chain complex is given by $(\Sigma^s A)_* = A_{*-s}$ with differential $(-1)^s \delta_A$. There is a canonical isomorphism

$$\begin{array}{rcl} \Sigma^1 C \otimes \Sigma^1 C & \cong & \Sigma^2 (C \otimes C) \\ \Sigma c_1 \otimes \Sigma c_2 & \mapsto & (-1)^{|c_1|} \Sigma^2 (c_1 \otimes c_2). \end{array}$$

Let A be a given chain complex with differential δ. We can alter this "internal" differential by an element of the internal Hom complex $\partial \in \mathbf{Hom}(A,A)_{-1}$, of degree -1. In order to obtain a new differential $\delta + \partial$ for A, ∂ should satisfy the equality

$$\delta(\partial) + \partial^2 = 0.$$

A map ∂ that fulfills this condition is called a *twisting differential* or a *twisting homomorphism*. We denote "internal" differentials by δ and twisting differentials by ∂. We hope that the reader is not going to be confused by the overuse of these letters. It should always be clear from the context the differential of which particular chain complex is currently meant.

Recall that a *symmetric sequence* or *Σ_*-sequence* in a category $\mathcal{C}$ is a sequence of objects $X(n)$ in $\mathcal{C}$ with a right action of the symmetric group Σ_n on $X(n)$ for every $n \geq 0$. A morphism of symmetric sequences is a collection of maps $\{f(n)\}_{n \geq 0}$ commuting with the respective group actions. We denote the category of symmetric sequences in $\mathcal{C}$ by $\mathcal{C}^{\mathbf{\Sigma}_*}$. If we are in the situation $\mathcal{C} = \mathcal{C}h$ then by "the differential" of a symmetric sequence we mean the collection of differentials on the individual levels. Similar, by a twisting differential we mean a collection of twisting differentials.

1 Recollection on operadic algebra

In this first rather long section we want to give the necessary prerequisites about operads, cooperads, algebras, coalgebras and tree representations of these. Further, we are going to recall the notions of (co)free and quasi-(co)free (co)operads and (co)algebras, as well as remind the reader of the Cobar-Bar resolution.

We start with definitions and examples of operads and operadic algebras, followed by a short presentation of the dual notions. After that we want to make the reader familiar with the tree representations we are going to use later on in this thesis. In the last section we give relevant constructions and notions such as quasi-free objects, Cobar-Bar resolution of operads and operadic twisting morphisms.

1.1 Operads

In their present form operads appeared for the first time in the early seventies in the monograph of May “The Geometry of Iterated Loop Spaces” [May]. Similar ideas could be found to that time also in Boardman and Vogt’s “Homotopy invariant algebraic structures on topological spaces” [BV] and Stasheff’s “Homotopy associativity of H-spaces” [Sta]. Initially invented for topological purposes, operads soon became of interest for algebraists since they give a conceptual way of encoding algebraic structure. A lot of literature can be found on this topic, among others by Fresse [Fre09b], Markl, Shnider and Stasheff [MSS] and Loday-Vallette [LV].

We concentrate on the parts that are necessary for our further work and give an exposition of these here. At the beginning we fix a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I})$ that has all small colimits and finite limits and such that the monoidal product preserves colimits in both variables. First we give the definition of an operad as in [May] but generalized for an arbitrary $\mathcal{C}$. May’s original definition was for topological spaces.

Definition 1.1. (Operad) An operad in $\mathcal{C}$ is a symmetric sequence $\mathrm{P} = \{\mathrm{P}(n)\}_{n\geq 0}$ in $\mathcal{C}$ together with operadic composition maps

$$\gamma\colon \mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \to \mathrm{P}(i_1 + \cdots + i_n)$$

for every set of indices $i_1, i_2, ..., i_n \in \mathbb{N}$ and $n \in \mathbb{N}$, and a unit map

$$\iota\colon \mathbb{I} \to \mathrm{P}(1)$$

such that the following conditions hold:

(i) (Associativity) The following commutes

$$\begin{tikzcd}
\Big(\mathrm{P}(n) \otimes \bigotimes_{k=1}^{n} \mathrm{P}(i_k)\Big) \otimes \bigotimes_{l=1}^{s} \mathrm{P}(j_l) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[dd] & \mathrm{P}(s) \otimes \bigotimes_{l=1}^{s} \mathrm{P}(j_l) \arrow[d, "\gamma"] \\
 & \mathrm{P}(t) \\
\mathrm{P}(n) \otimes \bigotimes_{k=1}^{n} \Big(\mathrm{P}(i_k) \otimes \bigotimes_{r=1}^{i_k} \mathrm{P}(j_{p_r^k})\Big) \arrow[r, "\mathrm{id}\otimes(\otimes_{k=1}^{n}\gamma)"] & \mathrm{P}(n) \otimes \bigotimes_{k=1}^{n} \mathrm{P}(q_k) \arrow[u, "\gamma"']
\end{tikzcd}$$

for every possible choice of indices n, s, i_k and j_l with $i_1 + \cdots + i_n = s$. The left vertical arrow is obtained from the symmetry isomorphism of the monoidal structure. Further, p_r^k stands for the sum $i_1 + \cdots + i_{k-1} + r$ and q_k for $j_{p_1^k} + \cdots + j_{p_{i_k}^k}$.

(ii) (Unitality) The following commute

$$\begin{tikzcd}
\mathbb{I} \otimes \mathrm{P}(n) \arrow[r, "\cong"] \arrow[d, "\iota\otimes\mathrm{id}"'] & \mathrm{P}(n) \\
\mathrm{P}(1) \otimes \mathrm{P}(n) \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{P}(n) \otimes \mathbb{I}^{\otimes n} \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes\iota^{\otimes n}"'] & \mathrm{P}(n) \\
\mathrm{P}(n) \otimes \mathrm{P}(1)^{\otimes n} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

(iii) (Equivariance) For every permutation $\sigma \in \Sigma_n$ the following commutes

$$\begin{tikzcd}
\mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \arrow[r, "\mathrm{id}\otimes\sigma"] \arrow[dd, "\sigma\otimes\mathrm{id}"'] & \mathrm{P}(n) \otimes \mathrm{P}(i_{\sigma^{-1}(1)}) \otimes \cdots \otimes \mathrm{P}(i_{\sigma^{-1}(n)}) \arrow[d, "\gamma"] \\
& \mathrm{P}(i_{\sigma^{-1}(1)} + \cdots + i_{\sigma^{-1}(n)}) \arrow[d, "\sigma"] \\
\mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \arrow[r, "\gamma"] & \mathrm{P}(i_1 + \cdots + i_n)
\end{tikzcd}$$

where in the top row σ acts on the left on $\mathrm{P}(i_1)\otimes\cdots\otimes\mathrm{P}(i_n)$ via the symmetry isomorphism and in the right column via a block permutation on $\mathrm{P}(i_{\sigma^{-1}(1)} + \cdots + i_{\sigma^{-1}(n)})$.
Further, the operadic composition maps

$$\gamma\colon \mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \to \mathrm{P}(i_1 + \cdots + i_n)$$

are $\Sigma_{i_1} \times \cdots \times \Sigma_{i_n}$-equivariant.

In the literature the operads we have just defined are often called "symmetric". Since we are always working with these we are not going to make the distinction. The n-th level of an operad is sometimes referred to as *arity* n of the operad.

If we restrict the operadic composition to terms of the form $\mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n)$ where only one i_j for $j = 1, \ldots, n$ is different from 1, then we are going to talk about the *partial composition product*.

Remark 1.2. The above definition is the most explicit but not the most compact one. Every symmetric sequence $\mathrm{P} = \{\mathrm{P}(n)\}_{n\geq 0}$ defines a so called *Schur functor* $S(\mathrm{P})\colon \mathcal{C} \to \mathcal{C}$ via

$$S(\mathrm{P})(C) := \bigoplus_{n\geq 0} \mathrm{P}(n) \otimes_{\Sigma_n} C^{\otimes n}.$$

We can reformulate Definition 1.1 as follows: an operad in $\mathcal{C}$ is a *triple* $(S(\mathrm{P}), \gamma, \iota)$ where P is a symmetric sequence in $\mathcal{C}$ and $S(\mathrm{P})$ the corresponding Schur functor. We are often going to use the notation $\mathrm{P} \circ C$ for the value of the Schur functor on C. This is justified by the next remark when regarding C as a symmetric sequence concentrated in degree zero.

Remark 1.3. Another reformulation uses the composition product in the category of symmetric sequences. If P and Q are two symmetric sequences then their composition product is defined by the formula

$$\mathrm{P} \circ \mathrm{Q} := \bigoplus_{n\geq 0} \mathrm{P}(n) \otimes_{\Sigma_n} \mathrm{Q}^{\otimes n},$$

where the tensor product of symmetric sequences is given by

$$(\mathrm{Q}_1 \otimes \mathrm{Q}_2)(k) := \bigoplus_{p+q=k} \mathrm{Q}_1(p) \otimes \mathrm{Q}_2(q) \otimes_{\Sigma_p\times\Sigma_q} \Sigma_k.$$

Here we denote by $\mathrm{Q}_1(p) \otimes \mathrm{Q}_2(q) \otimes \Sigma_k$ the tensor product of $\mathrm{Q}_1(p) \otimes \mathrm{Q}_2(q)$ with the set Σ_k given by $\bigoplus_{\sigma\in\Sigma_k} \mathrm{Q}_1(p) \otimes \mathrm{Q}_2(q)$, and we coequalize the left action of $\Sigma_p \times \Sigma_q$ on Σ_k with the right action on $\mathrm{Q}_1(p) \otimes \mathrm{Q}_2(q)$ in the usual way. If $\mathrm{Q}_1 = \mathrm{Q}_2 = \mathrm{Q}$ we have to say how Σ_2 acts on $\mathrm{Q}^{\otimes 2}$. The generalization for higher n is done in the same manner. If τ is the transposition in Σ_2 then τ sends the factor $\mathrm{Q}(p) \otimes \mathrm{Q}(q) \otimes \sigma$ to the factor $\mathrm{Q}(q) \otimes \mathrm{Q}(p) \otimes \tilde{\tau}\sigma$, where $\tilde{\tau}$ denotes the corresponding (p,q)-block transposition. Note that the left action of Σ_n on, say, $V^{\otimes n}$ is given by $\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$, and this is the one we are using on $\mathrm{Q}^{\otimes n}$ together with the left action of Σ_n by block permutations.

This composition product makes the category of symmetric sequences $\mathcal{C}^{\boldsymbol{\Sigma}_*}$ into a monoidal category with unit the sequence I defined by $\mathrm{I}(1) = \mathbb{I}$ and the initial object in arity n different than 1. With this notation, an operad in $\mathcal{C}$ is equivalently given by a *monoid* in $(\mathcal{C}^{\boldsymbol{\Sigma}_*}, \circ, \mathrm{I})$. If (P, μ, η) is such a monoid then the operadic composition maps γ are given by the restrictions of the product μ onto factors of the form $\mathrm{P}(n) \otimes_{\Sigma_n} (\mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \otimes \mathrm{id})$. The other way round, note that since μ is a Σ_*-equivariant map it is uniquely defined by a given operadic composition γ.

Let us mention some well-known examples of operads.

Example 1.4. (Endomorphism operad) If X is an object of a closed symmetric monoidal category $\mathcal{C}$ then using the internal Hom functor we define by $\mathrm{End_X}(n) = \mathbf{Hom}(X^{\otimes n}, X)$ the endomorphism operad of X.

Example 1.5. (Permutation operad) The symmetric groups form an operad in the category of sets with n-th level given by the symmetric group Σ_n and 0-th level the empty set. The composition product is uniquely determined by the equivariance conditions and the assignment $\gamma(\mathrm{id}_n \times \mathrm{id}_{i_1} \times \cdots \times \mathrm{id}_{i_n}) = \mathrm{id}_{i_1 + \cdots + i_n}$. More precisely, we have

$$\gamma(\sigma \times \sigma_1 \times \cdots \times \sigma_n) = \sigma(i_1, \ldots, i_n) \circ (\sigma_1 \oplus \cdots \oplus \sigma_n),$$

where $\sigma(i_1, \ldots, i_n)$ is the permutation on $(i_1 + \cdots + i_n)$ letters, which is built out of σ acting on blocks of size $i_1, i_2, \ldots, i_n$.

Example 1.6. (Associative operad) Let k be a field. We can transfer the above operad into the category of k-vector spaces. The operad Ass, given by $\mathrm{Ass}(n) = k[\Sigma_n]$ for $n \geq 1$ and $\mathrm{Ass}(0) = 0$ with a free Σ_*-action and structure maps $\iota = \mathrm{id}$ and γ uniquely determined by the assignment $\gamma(\mathrm{id}_n \otimes \mathrm{id}_{i_1} \otimes \cdots \otimes \mathrm{id}_{i_n}) = \mathrm{id}_{i_1 + \cdots + i_n}$, is the operad of non-unital associative k-algebras.

Example 1.7. (Commutative operad) Again in the context of k-vector spaces, let us set $\mathrm{Com}(n) = k$ for $n \geq 1$ and $\mathrm{Com}(0) = 0$ with the trivial Σ_*-actions and obvious structure maps. This is the operad of non-unital commutative k-algebras.

Remark 1.8. The last two examples can be generalized to the category of chain complexes over a commutative ring k. Then $\mathrm{Ass}(n)$ and $\mathrm{Com}(n)$ are chain complexes concentrated in degree 0. The rest remains the same. Further, if we want to get the operads of unital associative respectively commutative algebras, then we have to take $\mathrm{Ass}(0) = k$ respectively $\mathrm{Com}(0) = k$. The same holds in the graded case.

Example 1.9. (Barratt-Eccles operad) The simplicial version of this operad was introduced, as the name indicates, by Barratt and Eccles [BE] for the study of infinite loop spaces. Here we are interested in the corresponding operad in $\mathcal{C}h$. The n-th level of the chain Barratt-Eccles operad $\mathrm{E\Sigma_n}$ is given by $\mathcal{N}_*(E\Sigma_n)$, the normalized chains of the free contractible Σ_n-space $E\Sigma_n$. Explicitly, $(\mathrm{E\Sigma_n})_d$ is freely generated as a k-module by elements of the form $(\sigma_0, \ldots, \sigma_d)$ with $\sigma_0, \ldots, \sigma_d \in \Sigma_n$ and σ_i different from σ_{i+1} for all i between 0 and $d-1$. For $n = 0$ we set $\mathrm{E\Sigma_0} = 0$. The symmetric group is acting diagonally by

$$(\sigma_0, \ldots, \sigma_d) \cdot \sigma = (\sigma_0 \cdot \sigma, \ldots, \sigma_d \cdot \sigma),$$

and the differential is given by the alternating sum of the face maps

$$\delta(\sigma_0, \ldots, \sigma_d) = \sum_{i=0}^{d} (-1)^i (\sigma_0, \ldots, \hat{\sigma_i}, \ldots, \sigma_d),$$

where for $\sigma_{i-1} = \sigma_{i+1}$ one should set

$$(\sigma_0, \dots, \hat{\sigma_i}, \dots, \sigma_d) = 0.$$

The operadic composition is uniquely determined by the partial composites, i.e., evaluation of γ on factors of the form $\mathrm{E\Sigma_n} \otimes \mathrm{E\Sigma_1} \otimes \cdots \otimes \mathrm{E\Sigma_k} \otimes \cdots \otimes \mathrm{E\Sigma_1}$. Let us write $\sigma \circ_i \tau$ for the element

$$(\sigma_0, \dots, \sigma_d) \otimes \mathrm{id} \otimes \cdots \otimes \underset{i}{(\tau_0, \dots, \tau_f)} \otimes \cdots \otimes \mathrm{id} \in (\mathrm{E\Sigma_n})_d \otimes (\mathrm{E\Sigma_1})_0 \otimes \cdots \otimes (\mathrm{E\Sigma_k})_f \otimes \cdots \otimes (\mathrm{E\Sigma_1})_0.$$

The set $\{0, \dots, d\} \times \{0, \dots, f\}$ has a partial order given by $(n_1, n_2) \leq (m_1, m_2)$ iff $n_1 \leq m_1$ and $n_2 \leq m_2$. A maximal chain in this poset has the length $d + f + 1$. We denote by (p_*, q_*) such a longest ascending chain $\big((p_0, q_0), (p_1, q_1), \dots, (p_{d+f}, q_{d+f})\big)$. Now, the partial composition product is given by

$$\gamma(\sigma \circ_i \tau) = \sum_{(p_*, q_*)} (-1)^{\mathrm{sign}(p_*, q_*)} (\sigma_{p_0} \circ_i \tau_{q_0}, \dots, \sigma_{p_{d+f}} \circ_i \tau_{q_{d+f}}),$$

where the sum runs over the set of maximal ascending chains in $\{0, \dots, d\} \times \{0, \dots, f\}$. Further, $\sigma_{p_j} \circ_i \tau_{q_j}$ denotes the partial composition product in the permutation operad of Example 1.5 and $\mathrm{sign}(p_*, q_*)$ is the signum of the $(d+1, f)$-shuffle permutation ν that permutes the elements of (p_*, q_*) such that for $\nu(p_j, q_j) = (p_l, q_l)$ holds $p_l > p_{l-1}$ whenever $1 \leq j \leq d$, and $\nu(p_0, q_0) = (p_0, q_0)$. Respectively, for $j = d+1, \dots, d+f$ we have $\nu(p_j, q_j) = (p_l, q_l)$ with $q_l > q_{l-1}$. In plain words, we sort the chain by a shuffle permutation in a way that first exactly those elements appear that compared to the predecessor element in the chain increase their first coordinate by one. It is a well-known fact that the Barratt-Eccles operad is a so called E_∞-operad. We are going to come back to this later. For the moment let us mention that for every $n \geq 0$ there is a quasi-isomorphism $\mathrm{E\Sigma_n} \xrightarrow{\sim} \mathrm{Com}(n)$ that is induced by the classical augmentation map $\mathrm{E\Sigma_n} \to k$.

Example 1.10. There is a filtration of the chain Barratt-Eccles operad by certain suboperads $\mathrm{E_n}$. Here, we give a description following Fresse [Fre09c]. The original idea for these operads goes back to Smith [Smi] and a proof of the fact that they are indeed E_n-operads can be found for example in [Ber]. To start with we fix some notation. We can specify a given permutation σ by writing down its values $(\sigma(1), \sigma(2), \dots, \sigma(n))$. We want to keep track (of the order) of the appearance of a fixed pair $\{i, j\}$ in the value sequence of σ. We write $(\sigma(1), \dots, \sigma(n))_{ij}$ for (i, j) or (j, i), depending on the order of appearance. For example $(1, 3, 2)_{1,3} = (1, 3)$ and $(3, 2, 1)_{1,3} = (3, 1)$. Now, given a sequence $\underline{\sigma} = (\sigma_0, \sigma_1, \dots, \sigma_d)$ of permutations we construct for a pair $\{i, j\}$ a new sequence out of the order in which $\{i, j\}$ appears in every single permutation σ_k. For example let $(\sigma_0, \sigma_1, \sigma_2)$ be given by the following table in which the k-th row depicts the permutation σ_k:

(1,3,2,4)
(1,2,4,3)
(2,4,3,1)

Then our new sequence for $\{i, j\} = \{1, 4\}$, for which we write $(\sigma_0, \sigma_1, \sigma_2)_{1,4}$ or $\underline{\sigma}_{1,4}$, is given by $((1, 4), (1, 4), (4, 1))$. Denote by $\mu_{ij}(\sigma_0, \dots, \sigma_d) = \mu_{i,j}(\underline{\sigma})$ the variation of $\{i, j\}$ in $\underline{\sigma}_{i,j}$, i.e., how often we swich from (i, j) to (j, i) or the other way round. In the example above we get $\mu_{1,4}(\sigma_0, \sigma_1, \sigma_2) = 1$.

Now we are ready to define the sequence of suboperads $\mathrm{E_n}$. For every n bigger than 0, $\mathrm{E_n}(r)$ is spanned in chain degree d by those permutation sequences $\underline{\sigma} = (\sigma_0, \dots, \sigma_d) \in (\mathrm{E\Sigma_r})_d$ that satisfy the property $\mu_{i,j}(\underline{\sigma}) < n$ for all possible pairs $\{i, j\} \subset \{1, \dots, r\}$. Operadic composition is inherited from the one on $\mathrm{E}\Sigma_*$. We get in particular $\mathrm{E}_1 = \mathrm{Ass}$, and an ascending sequence of operads

$$\mathrm{Ass} = \mathrm{E}_1 \subset \mathrm{E}_2 \subset \mathrm{E}_3 \subset \cdots \subset \mathrm{E}_\mathrm{n} \subset \cdots \subset \mathrm{E}\Sigma_*.$$

Example 1.11. (Homology operad) If P is a chain operad over a field that in addition satisfies the condition $\mathrm{P}(0) = 0$, then there is a natural isomorphism $H_*\mathrm{P} \circ H_*\mathrm{P} \cong H_*(\mathrm{P} \circ \mathrm{P})$ (cf. Lemma 1.3.9 of [Fre04]) and therefore, $H_*\mathrm{P}$ is equipped with the structure of a graded operad.

Definition 1.12. (Morphism of operads) A morphism of operads is a map of symmetric sequences that commutes with the structure maps. In the language of monoids in the category of symmetric sequences this is simply a morphism of monoids.

We denote the category of operads in $\mathcal{C}$ together with the above morphisms by $\mathcal{O}p(\mathcal{C})$.

Example 1.13. There is an operadic morphism from the associative operad to the commutative operad that factors over the chain Barratt-Eccles operad

$$\mathrm{Ass} \to \mathrm{E}\Sigma_* \to \mathrm{Com}.$$

Further, the inclusions $\mathrm{E}_\mathrm{n} \hookrightarrow \mathrm{E}_{\mathrm{n}+1}$ are of course maps of operads.

Often we are going to deal with so called *connected operads*.

Definition 1.14. (Connected operad, connected symmetric sequence) An operad P is called connected if $\mathrm{P}(0)$ is the initial object of $\mathcal{C}$. More generally, a symmetric sequence M is called connected if $\mathrm{M}(0)$ is the initial object of $\mathcal{C}$. If $\mathcal{C}$ is the category $\mathcal{C}h$ of chain complexes over k we, in addition, assume that a connected operad satisfies the equality $\mathrm{P}(1) = \mathrm{I}(1) = k$.

Observe that the unit symmetric sequence I is canonically equipped with the structure of an operad. Extending established terminology we set:

Definition 1.15. (Augmented operad, augmentation ideal) An augmented operad is an operad P together with a map of operads $\epsilon\colon \mathrm{P} \to \mathrm{I}$. The kernel of the augmentation map is called the augmentation ideal and is denoted by $\tilde{\mathrm{P}}$.

The notion of an operadic ideal generalizes the well-known notion of a ring ideal. The augmentation map ϵ is indeed a retraction of the unit map of P since a map of operads is supposed to preserve units. Thus, we get a splitting $\mathrm{P} = \mathrm{I} \oplus \tilde{\mathrm{P}}$. If P is a connected operad in $\mathcal{C}h$ then it is canonically augmented and the augmentation ideal is given by

$$\tilde{\mathrm{P}}(n) = \begin{cases} 0 & \text{for } n = 0, 1\,. \\ \mathrm{P}(n) & \text{else} \end{cases}$$

We now come to the definition of an operadic algebra:

Definition 1.16. (Operadic algebra) An algebra over an operad P in a category $\mathcal{C}$ is an object A of $\mathcal{C}$ together with structure maps

$$\gamma_A\colon \mathrm{P}(n) \otimes A^{\otimes n} \longrightarrow A$$

for every $n \in \mathbb{N}$, such that the following diagrams commute:

$$\begin{tikzcd}
\mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes \cdots \otimes \mathrm{P}(i_n) \otimes A^{\otimes(i_1+\cdots+i_n)} \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[dd, "\text{shuffle}"'] & \mathrm{P}(i_1+\cdots+i_n) \otimes A^{\otimes(i_1+\cdots+i_n)} \arrow[d, "\gamma_A"] \\
 & A \\
\mathrm{P}(n) \otimes \mathrm{P}(i_1) \otimes A^{\otimes i_1} \otimes \cdots \otimes \mathrm{P}(i_n) \otimes A^{\otimes i_n} \arrow[r, "\mathrm{id}\otimes(\bigotimes \gamma_A)"] & \mathrm{P}(n) \otimes A^{\otimes n} \arrow[u, "\gamma_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I} \otimes A \arrow[r, "\cong"] \arrow[d, "\iota\otimes\mathrm{id}"'] & A \\
\mathrm{P}(1) \otimes A \arrow[ur, "\gamma_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{P}(n) \otimes A^{\otimes n} \arrow[r, "\gamma_A"] \arrow[d] & A \\
\mathrm{P}(n) \otimes_{\Sigma_n} A^{\otimes n} \arrow[ur] &
\end{tikzcd}$$

In the lower right diagram, we more precisely ask for such a factorization to exist.

Remark 1.17. In the language of triples, an algebra over an operad is an algebra over the triple corresponding to the operad.

Example 1.18. An algebra over the operad Ass or Com is precisely a non-unital associative respectively commutative algebra. In the chain versions of these operads we get differential graded and differential graded commutative algebras. There are also unital versions of these operads that yield algebras with units. Algebras over the operad $\mathrm{E}\Sigma_*$ are so called E_∞-algebras.

Remark 1.19. In the situation of Example 1.11, given an algebra A over P, we get an algebra H_*A over the operad $H_*\mathrm{P}$.

Definition 1.20. (Morphism of operadic algebras) A morphism $f\colon A \to B$ from a P-algebra A to a P-algebra B is a map in the underlying category $\mathcal{C}$ such that for every n the diagram

$$\begin{tikzcd}
\mathrm{P}(n) \otimes A^{\otimes n} \arrow[r, "\mathrm{id}\otimes f^{\otimes n}"] \arrow[d, "\gamma_A"] & \mathrm{P}(n) \otimes B^{\otimes n} \arrow[d, "\gamma_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

commutes.

We denote the category of P-algebras with the above morphisms by P-alg.

1.2 Cooperads

In the following we assume that the symmetric monoidal category $\mathcal{C}$ is pointed. We denote the zero object by 0.

This is a short chapter about cooperads and coalgebras over cooperads. The reader should be aware that we are not using standard terminology. Possibly the most natural way to define a "co-operad" is as an operad in the opposite category, or equivalently, to reverse all the arrows in Definition 1.1 and require coassociativity, counitality and the dual equivariance conditions. This is *not* what we are going to understand by a cooperad. What we will mean by this term is the following

Definition 1.21. (Cooperad) A cooperad in $\mathcal{C}$ is a comonoid in the monoidal category of symmetric sequences $(\mathcal{C}^{\mathbf{\Sigma}_*}, \circ, \mathrm{I})$ with the composition product $\circ$ defined as in Remark 1.3.

Note that the unit I has a canonical structure of a cooperad.

Definition 1.22. (Morphism of cooperads) A morphism of cooperads is a map of comonoids.

The category of cooperads in $\mathcal{C}$ is denoted by $\mathcal{C}oOp(\mathcal{C})$.

Definition 1.23. (Connected cooperad) A cooperad T is called connected if $\mathrm{T}(0)$ is the zero object of $\mathcal{C}$. If $\mathcal{C}$ is the category of chain complexes we require in addition that $\mathrm{T}(1)$ equals k.

We are always going to consider connected cooperads. Dual to the notion of an augmentation and an augmentation ideal we have

Definition 1.24. (Coaugmented cooperad, coaugmentation coideal) A cooperad T is coaugmented if it is equipped with a map of cooperads $\eta\colon \mathrm{I} \to \mathrm{T}$. The cokernel of the coaugmentation map is called the coaugmentation coideal and is denoted by $\tilde{\mathrm{T}}$.

Every connected chain cooperad is canonically coaugmented and the coaugmentation coideal is given by

$$\tilde{\mathrm{T}}(n) = \begin{cases} 0 & \text{for } n = 0, 1\,. \\ \mathrm{T}(n) & \text{else} \end{cases}$$

We again have a splitting $\mathrm{T} = \mathrm{I} \oplus \tilde{\mathrm{T}}$.

Example 1.25. Let P be a connected chain operad such that every level consists of finitely generated projective k-modules, and let $\mathrm{P}^\vee$ denote the (k-linear) dual symmetric sequence. Then there is a natural map $\mathrm{P}^\vee \circ \mathrm{P}^\vee \to (\mathrm{P} \circ \mathrm{P})^\vee$, and it is an isomorphism. A detailed account of this can be found in [Fre04], Lemma 1.2.19. The short explanation of the fact that the map is indeed an isomorphism is the presentation of the composition product of a *connected* operad in the bottom of p.18. Thus, $\mathrm{P}^\vee$ gives us an example of a cooperad.

Definition 1.26. (Coalgebra over a cooperad) A coalgebra A over a cooperad T is an object A of $\mathcal{C}$, regarded as a symmetric sequence concentrated in degree zero, together with a left coaction of the comonoid T.

In plain words, we have a coaction

$$A \to \bigoplus_{n \geq 1} \mathrm{T}(n) \otimes_{\Sigma_n} A^{\otimes n}$$

that is coassociative and counital.

Definition 1.27. (Morphism of coalgebras) A morphism of coalgebras is a map in the underlying category that commutes with the structure maps.

Later on we are going to see examples of coalgebras and morphisms between them.

1.3 The language of trees

To ease notation in this section we restrict ourselves to $\mathcal{C} = \mathcal{C}h$. We want to introduce a way to represent elements of an operad by trees as well as to show how the composition product can be described in this setting. We rely our exposition mainly on the notation introduced by Fresse in [Fre09] and [Fre04].

Symmetric sequences as contravariant functors: First observe that a symmetric sequence in $\mathcal{C}h$ can be viewed as a contravariant functor from the category of finite sets with bijections to the category $\mathcal{C}h$. These functors are also known under the term "linear species" (see for example [Joy]). Let us denote by $\mathcal{B}ij$ the category of finite sets (including the empty set) as objects, and bijections as morphisms. A symmetric sequence M defines a contravariant functor

$$\mathrm{M}\colon \mathcal{B}ij \to \mathcal{C}h$$

as follows. For the set $\underline{\mathrm{n}} = \{1, \ldots, n\}$ we have $\mathrm{M}(\underline{\mathrm{n}}) = \mathrm{M}(n)$ and for an arbitrary set I with n elements we define

$$\mathrm{M}(I) := \Big(\bigoplus_{f\colon \underline{\mathrm{n}} \to I} \mathrm{M}(n) \Big)_{\Sigma_n},$$

where the action of Σ_n on an element (f, m) for $f\colon \underline{\mathrm{n}} \to I$ and $m \in \mathrm{M}$ is given diagonally by $(f, m) \cdot \sigma = (f \circ \sigma, m\sigma)$. Conversely, to a contravariant functor as above we associate a symmetric sequence in $\mathcal{C}h$ by evaluating it on sets of the form $\underline{\mathrm{n}}$ and the empty set. Therefore, we are often not going to distinguish between a symmetric sequence and its "coordinate-free representation" as a contravariant functor from $\mathcal{B}ij$ to $\mathcal{C}h$.

An element of $\mathrm{M}(I)$ can be represented by a rooted *non-planar* tree with one vertex, indexed by an element of $\mathrm{M}(n)$, and n number of leaves labeled by the elements of I. By "non-planar" we mean that for every vertex v of the tree, there is no preferred order on the set J_v of those vertices whose outgoing edge ends in v. In the example above this translates into $m \in \mathrm{M}(I)$ being represented by a corolla without an ordering of the leaves.

We prefer to use the following *planar* tree representation, though. For this we choose an ordering of the leaves. An element $[(f, m)]$ of $\mathrm{M}(I)$ is going to be depicted as a tree in the form

$$\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown \quad \diagup \\ m \\ | \end{array}$$

where $f(j) = i_j$. The same element is also represented by the tree

$$\begin{array}{c} i_{\sigma(1)} \cdots\cdots i_{\sigma(n)} \\ \diagdown \quad \diagup \\ m\sigma \\ | \end{array}$$

for every $\sigma \in \Sigma_n$. Or reformulating the above we have:

$$\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown \quad \diagup \\ m\sigma \\ | \end{array} \qquad \sim \qquad \begin{array}{c} i_{\sigma^{-1}(1)} \cdots\cdots i_{\sigma^{-1}(n)} \\ \diagdown \quad \diagup \\ m \\ | \end{array}$$

This relation should remind the reader of the first equivariance property for operads. If I is the standard set with n elements we of course have a preferred choice for f, namely the identity map.

Composition product in the setting of contravariant functors: We want to give formulas for the tensor and composition products of symmetric sequences in the context of functors. If M and N are two contravariant functors then the tensor product mentioned in Remark 1.3 can be written as

$$(\mathrm{M} \otimes \mathrm{N})(n) = \bigoplus_{(J_1,J_2)} \mathrm{M}(J_1) \otimes \mathrm{N}(J_2)$$

where the sum ranges over all ordered pairs (J_1, J_2) with $J_1 \sqcup J_2 = \underline{n}$. The action of the symmetric group Σ_2 on $\mathrm{N} \otimes \mathrm{N}$ is given by the symmetry isomorphism and the permutation of factors. For higher tensor powers we get

$$\mathrm{N}^{\otimes r}(n) = \bigoplus_{(J_1,\dots,J_r)} \mathrm{N}(J_1) \otimes \cdots \otimes \mathrm{N}(J_r)$$

with $(J_1, \dots, J_r)$ ordered tuples that decompose $\underline{n}$, i.e., $J_1 \sqcup \cdots \sqcup J_r = \underline{n}$. The left action of Σ_r on $\mathrm{N}^{\otimes r}$ is given by permuting factors: an element $n_{i_1} \otimes \cdots \otimes n_{i_r}$ is sent up to a sign to

$$n_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes n_{i_{\sigma^{-1}(r)}}.$$

The signs come from the symmetry isomorphism and can be calculated via the Koszul sign rule. The right Σ_n-action on $\mathrm{N}^{\otimes r}(n)$ is obtained in the following way: For a given permutation $\sigma \in \Sigma_n$ we can form for every tuple $(J_1, \dots, J_r)$ the restrictions $\sigma_i \colon J_i' \to J_i$ with $J_i' := \sigma^{-1}(J_i)$ and $i = 1, \dots, r$. These induce a map

$$\sigma_1 \otimes \cdots \otimes \sigma_r \colon \mathrm{N}(J_1) \otimes \cdots \otimes \mathrm{N}(J_r) \longrightarrow \mathrm{N}(J_1') \otimes \cdots \otimes \mathrm{N}(J_r')$$

with domain the summand labeled by $(J_1, \dots, J_r)$. In this way we get a map from every summand of $\mathrm{N}^{\otimes r}(\underline{n})$, and in particular a map on $\mathrm{N}^{\otimes r}(\underline{n})$ itself, that yields the required group action.

With the formulas from above we get the following description of the composition product

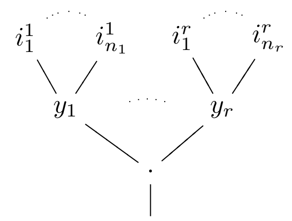
$$(\mathrm{M} \circ \mathrm{N})(n) = \bigoplus_{r \geq 0} \mathrm{M}(r) \otimes_{\Sigma_r} \Big(\bigoplus_{(J_1,\dots,J_r)} \mathrm{N}(J_1) \otimes \cdots \otimes \mathrm{N}(J_r) \Big)$$

where $(J_1, \dots, J_r)$ runs through all possible ordered decompositions of $\underline{n}$. Further, by an observation of Fresse (Lemma 1.3.9 of [Fre04]) if N is connected (cf. Definition 1.14) we can simplify the expression to

$$(\mathrm{M} \circ \mathrm{N})(n) = \bigoplus_{r \geq 0} \mathrm{M}(r) \otimes \Big(\bigoplus_{(J_1,\dots,J_r)'} \mathrm{N}(J_1) \otimes \cdots \otimes \mathrm{N}(J_r) \Big)$$

with J_k non-empty sets of the form $J_k = \{j_{k1} < \cdots < j_{kn_k}\}$, $k = 1, \dots, r$ such that $j_{11} < j_{21} < \cdots < j_{r1}$. The reason for this is that under the assumption $J_1, \dots, J_r \neq \emptyset$ the symmetric group acts freely on the set of partitions $J_1 \sqcup \cdots \sqcup J_r = \underline{n}$.

Composition product representation via trees: Now we are going to represent the composition of symmetric sequences by trees. For a set $I \in \mathcal{B}ij$ we define an I-tree with two levels to be an oriented (non-planar) tree with two levels of vertices, $|I|$ number of leaves labeled by the elements of I, and one root:



The structure of an I-tree is fully determined by the set of vertices at level 2, which we denote by I_0, and a partition of I: $I = \bigsqcup_{v\in I_0} J_v$. Note that we allow J_v to be the empty set. An isomorphism of I-trees with two levels is a bijection between the sets of vertices at level 2 that preserves the decomposition of I (sloppy speaking an isomorphism just renames the vertices at level 2). To a given I-tree τ and symmetric sequences M and N we associate the chain complex

$$\tau(\mathrm{M},\mathrm{N}) = \mathrm{M}(I_0) \otimes \Big(\bigotimes_{v\in I_0} \mathrm{N}(J_v)\Big).$$

There are some explanations about this tensor product to be given. (For a formal description compare Remark 1.28 below.) By definition an element of $\mathrm{M}(I_0)$ is represented by a pair (f, m) where, if k is the cardinality of I_0, $f : \underline{k} \to I_0$ is a bijection and $m \in \mathrm{M}(k)$. Thus, a representative (f, m) gives an order on the set I_0 and in this way defines the order of appearance of the single factors $\mathrm{N}(J_v)$. For example

$$(f, m) \otimes n_1 \otimes \cdots \otimes n_k$$

with $(f, m) \in \mathrm{M}(I_0)$ and $n_j \in \mathrm{N}(J_{f(j)})$ is an element of $\tau(\mathrm{M},\mathrm{N})$. We can represent the same element also by

$$\pm(f\sigma, m\sigma) \otimes n_{\sigma(1)} \otimes \cdots \otimes n_{\sigma(k)}$$

for any permutation $\sigma \in \Sigma_k$. One can think of the elements of $\tau(\mathrm{M},\mathrm{N})$ as (sums of) I-trees with two levels, where the first level is labeled by an element of $\mathrm{M}(I_0)$, and the second by elements of $\mathrm{N}(J_v)$. Since choosing representatives of these elements automatically gives an order on the edges entering the corresponding vertex, we can use this to think of planar trees instead. Note that this "treewise tensor product" represents a direct summand of $(\mathrm{M} \circ \mathrm{N})(I)$ up to isomorphism (cf. the representation in the previous paragraph). In particular, the identification along the symmetric group action is already encoded. In order to get the "whole" composition product, we have to take "different" I-trees into account, i.e., we have to take care of different decompositions of the set I. Thus, a first candidate for $(\mathrm{M} \circ \mathrm{N})(I)$ is the direct sum

$$\bigoplus_{\tau\in\Theta_2(I)} \tau(\mathrm{M},\mathrm{N})$$

that runs over all I-trees with two levels (we neglect for a moment set theoretical inaccuracy). Here and later on we denote by $\Theta_2(I)$ the category with objects I-trees with two levels and morphisms, isomorphisms of I-trees. However, we are not quite done yet since this sum contains too many factors. An isomorphism of I-trees $\nu : \tau \to \tau'$ defines an isomorphism of chain complexes

$$\begin{array}{rcl} \nu_* : \tau(\mathrm{M},\mathrm{N}) & \longrightarrow & \tau'(\mathrm{M},\mathrm{N}). \\ {[f, m]} \otimes n_1 \otimes \cdots \otimes n_k & \longmapsto & [\nu f, m] \otimes n_1 \otimes \cdots \otimes n_k \end{array}$$

One can now verify that

$$(\mathrm{M} \circ \mathrm{N})(I) = \bigoplus_{\tau\in\Theta_2(I)} \tau(\mathrm{M},\mathrm{N})/\sim\ ,$$

where $\sim$ is the equivalence relation induced by the isomorphisms of I-trees.

Remark 1.28. We were not really formally precise about the tensor product

$$\tau(\mathrm{M},\mathrm{N}) = \mathrm{M}(I_0) \otimes \Big(\bigotimes_{v\in I_0} \mathrm{N}(J_v)\Big).$$

The correct way would be to define an "unordered tensor product" first, so that we can make sense of the term

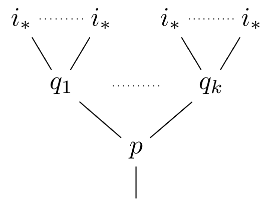
$$\bigotimes_{v \in I_0} \mathrm{N}(J_v).$$

The unordered tensor product of factors J_v for $v \in I_0$ is given by the coinvariants

$$\Big(\bigoplus_{f \colon \underline{\mathrm{k}} \to I_0} \mathrm{N}(J_{f(1)}) \otimes \cdots \otimes \mathrm{N}(J_{f(k)}) \Big)_{\Sigma_k},$$

where the sum runs over all bijections from $\underline{\mathrm{k}}$ to I_0 and the action of Σ_k on the right is given diagonally by precomposition on f and permuting the factors of the tensor product. In this way we get a tensor product without a distinguished order of the factors. Now $\tau(\mathrm{M}, \mathrm{N})$ is formally well-defined. There is a canonical isomorphism from our description of the tensor product to this formal one. We leave this verification to the reader.

Operadic composition product via trees: With the above description of the composition product of symmetric sequences and the definition of an operad as a monoid with respect to this composition product we see that the operadic product is fully determined by its values on trees of the form

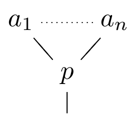


where the inputs i_* are shared out by permutations $\sigma \in \Sigma_n$, and $p, q_1 \dots, q_k$ are elements of the given operad P. If the operad is connected then we can restrict to shuffle permutations. Note that in this last case (when dealing with connected symmetric sequences) by choosing representatives of the isomorphism classes of trees together with an ordering of the set I_0 we can reduce the presentation of the composition product $(\mathrm{M} \circ \mathrm{N})(I)$ from the last paragraph to a direct sum without identifications.

Operadic algebra structure via trees: As we mentioned earlier, when we consider an object A of $\mathcal{C}$ as a symmetric sequence concentrated in degree 0 then the structure of a P-algebra on A is given by a map of symmetric sequences

$$\mathrm{P} \circ A \to A$$

that is associative and unital in the appropriate sense. Using this and the language of trees, the operadic algebra structure is determined by its values on trees of the form



with $p \in \mathrm{P}(n)$ and $a_1, \dots, a_n \in A$.

Cooperadic coproduct via trees: Let T be a cooperad and let us denote by ν the cooperadic coproduct of T. Later on we are going to use the following notation for ν:

$$\nu\left(\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu(\wp)} \left(\begin{array}{ccc} i_* \cdots\cdots i_* & & i_* \cdots\cdots i_* \\ \diagdown\ \diagup & & \diagdown\ \diagup \\ \wp''_* & \cdots\cdots & \wp''_* \\ & \diagdown\ \ \diagup & \\ & \wp' & \\ & | & \end{array}\right),$$

where $\wp \in \mathrm{T}(\{i_1, \ldots, i_n\})$ is the element whose coproduct we are taking, $\wp'$ and $\wp''$ are factors in the coproduct, and the notation i_* indicates that there is also a permutation (that we do not explicitly write down) that shares out the correct indices. We are a bit sloppy in the notation in the sense that we use the letters $\wp'$ and $\wp''$ multiple times to denote different factors of the coproduct. If we wanted to be extremely precise then we should have made the dependence on the sum factor visible in the notation. Further, we write $\wp''_*$ to avoid the overflow of indices.

The so called *quadratic coproduct* is the projection of the coproduct onto factors of the form

$$\mathrm{T}(n) \otimes \mathrm{T}(1) \otimes \cdots \otimes \mathrm{T}(1) \otimes \mathrm{T}(k) \otimes \mathrm{T}(1) \otimes \cdots \otimes \mathrm{T}(1)$$

for all possibilities of n and k. We will denote this by ν_2 and depict it as follows:

$$\nu_2\left(\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu_2(\wp)} \left(\begin{array}{ccc} & i_* \cdots\cdots i_* & \\ & \diagdown\ \diagup & \\ i_* \cdots & \wp'' & \cdots i_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array}\right).$$

Sometimes we will need a kind of "reduced" quadratic coproducts. By ν_2' we will denote the quadratic coproduct where on the right hand side we leave out the factor with $\wp' = \wp$ and $\wp'' = 1$. By ν_2'' we will denote the quadratic coproduct where we omit in addition also the factor for $\wp'' = \wp$ and $\wp' = 1$.

Cooperadic coalgebra structure via trees: For a coalgebra A with coproduct ρ and an element $a \in A$ we write

$$\rho(a) = \sum_{\rho(a)} \left(\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right).$$

Of course, $\wp$ takes again different values and a_* represents different elements of A.

1.4 Constructions

In this section we present some constructions on operads, cooperads, algebras and coalgebras. Many of them can be carried out in more general symmetric monoidal categories, nevertheless we will focus on the case $\mathcal{C} = \mathcal{C}h$ since this is the one important for this thesis, and we need explicit descriptions of the constructions.

For the rest of this section we are working in the category of chain complexes $\mathcal{C}h$.

1.4.1 Free operads and free algebras

The forgetful functor from operads to symmetric sequences $\mathcal{U}\colon \mathcal{O}p(\mathcal{C}h) \to \mathcal{C}h^{\Sigma_*}$ has a left adjoint functor, the free functor

$$\mathcal{F}\colon \mathcal{C}h^{\Sigma_*} \to \mathcal{O}p(\mathcal{C}h).$$

Let us denote the unit of this adjunction by η. Because of the adjointness relation we have that for any symmetric sequence M and any map $f\colon \mathrm{M} \to \mathrm{P}$ towards an operad P there exists a unique morphism of operads ϕ_f making the diagram

$$\begin{tikzcd}
\mathrm{M} \arrow[r, "f"] \arrow[d, "\eta"'] & \mathrm{P} \\
\mathcal{F}(\mathrm{M}) \arrow[ur, dotted, "\exists!\phi_f"'] &
\end{tikzcd}$$

commute.

Remark 1.29. Another possibility to define an operad is as an algebra over the triple induced by the above adjunction $(\mathcal{F},\mathcal{U})$ on the category of symmetric sequences.

In the following we want to recall the construction of $\mathcal{F}(\mathrm{M})$ in the language of trees. For a detailed treatment of the topic we refer the reader to §1.1.9, §3.1.1 and §3.4 of [Fre04]. The construction and proof of the properties of the free functor can also be found in §1.4 of [GJ]. Loosely speaking, the free operad is generated by formal compositions of elements of M. These last ones we are going to organize on the structure of a tree. In order to do that we have to generalize the approach from Section 1.3. There we used the formalism of an I-tree with two levels. We need to take care of composites of arbitrary finite length (not just two factors). Therefore, we define the notion of a (general) I-tree. This is a non-planar oriented tree with leaves labeled by the set I and one root. The set of vertices of an I-tree τ is denoted by $V(\tau)$.

$$\begin{tikzcd}
 & & & i_5 \arrow[dr] & i_4 \arrow[d] & i_5 \arrow[dl] \\
 & i_1 \arrow[dr] & & i_2 \arrow[dl] & v_5 \arrow[d] & \\
v_2 \arrow[drr] & & v_3 \arrow[d] & & v_4 \arrow[dll] & \\
 & & v_1 \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

For example in this picture we have $I = \{i_1, \dots, i_5\}$, $V(\tau) = \{v_1, \dots, v_5\}$ and the root is denoted by 0. Note that for the general construction we allow vertices with no incoming edges (as v_2 above) since our symmetric sequence is possibly not connected, i.e., $\mathrm{M}(0) \neq 0$. Later on we are mainly going to deal with connected objects where such trees do not contribute.

Analogous to the case of I-trees with two levels, the structure of an I-tree is fully determined by a partition

$$V(\tau) \sqcup I = \bigsqcup_{v \in (V(\tau) \cup \{0\})} I_v$$

that reflects which vertices or leaves are connected to a given vertex v. An isomorphism of I-trees is a bijection on the set of vertices that respects the tree structure and fixes the leaves (i.e. just relabeling the vertices). For a complete formal definition of an I-tree and an I-tree isomorphism we refer the reader to §1.2.1 of [Fre09]. We proceed as earlier and associate to a tree τ and a symmetric sequence M the chain complex given by the tree tensor

$$\tau(\mathrm{M}) = \bigotimes_{v \in V(\tau)} \mathrm{M}(I_v),$$

which we regard as an unordered tensor product as defined in Remark 1.28. One can think of the elements of this tensor product as sums of non-planar trees labeled by elements of M with the corresponding valence. If τ is the tree with no vertices (one leaf, one edge, one root) we have $\tau(\mathrm{M}) = k$. As before, an isomorphism of I-trees $\nu\colon \tau \to \tau'$ induces an isomorphism between the corresponding chain complexes

$$\nu_*\colon \tau(\mathrm{M}) \to \tau'(\mathrm{M}).$$

Now, the free operad generated by M is defined by

$$\mathcal{F}(\mathrm{M})(I) = \bigoplus_{\tau \in \Theta(I)} \tau(\mathrm{M})/\sim\ ,$$

where $\Theta(I)$ is the groupoid of I-trees and the equivalence relation $\sim$ is induced by the isomorphisms of I-trees. A bijection $I_1 \to I_2$ gives an isomorphism

$$\mathcal{F}(\mathrm{M})(I_2) \to \mathcal{F}(\mathrm{M})(I_1)$$

that consists of reindexing the leaves of the I_2-trees via the inverse bijection. Thus, we indeed get a symmetric sequence.

The operadic composition for $\mathcal{F}(\mathrm{M})$ is given by grafting of trees. The unit map

$$\iota\colon \mathrm{I} \to \mathcal{F}(\mathrm{M})$$

identifies $\mathrm{I}(1) = \mathbb{I} = k$ with the summand of $\mathcal{F}(\mathrm{M})$ corresponding to the tree with no vertices. The construction is obviously functorial. The adjunction unit $\eta_{\mathrm{M}}\colon \mathrm{M} \to \mathcal{F}(\mathrm{M})$ is defined by the canonical isomorphism $\mathrm{M}(\{i_1, \dots, i_n\}) \cong \tau(\mathrm{M})$ where τ is the I-tree with one vertex:

$$\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown \quad \diagup \\ \cdot \\ | \end{array}$$

The unique morphism $\phi_f\colon \mathcal{F}(\mathrm{M}) \to \mathrm{P}$ associated to a map $f\colon \mathrm{M} \to \mathrm{P}$ towards an operad P is given on generators by first applying f on every single tensor factor, and then performing operadic composition in P. The projection from $\mathcal{F}(\mathrm{M})$ onto the summand indexed by the tree with no vertices equips the free operad with an augmentation. The augmentation ideal $\tilde{\mathcal{F}}(\mathrm{M})$ is spanned by trees with a non-empty set of vertices. Observe further that the free operad is connected if and only if M is trivial in arity 0 and 1 (e.g. (co)augmentation (co)ideal of an (co)augmented (co)operad). The condition $\mathrm{M}(0) = 0$ is equivalent to the free operad being trivial in level zero, and $\mathrm{M}(1) = 0$ is needed for $\mathcal{F}(\mathrm{M})$ to be "not more" than k in arity 1.

Remark 1.30. Later on we are going to represent elements of $\mathcal{F}(\mathrm{M})$ for a connected symmetric sequence M as sums of trees. To legitimate this observe that when M is connected all the trees that appear in the construction of the free operad are "non-singular". By this we mean that every vertex has incoming edges or leaves. Such trees have no non-trivial automorphisms. Thus, if we choose a representative for every isomorphism class of I-trees together with an order of the vertices, then we can represent $\mathcal{F}(\mathrm{M})(I)$ as a direct sum of trees labeled by elements of M.

We come to the notion of a *free algebra*. Let P be an operad. We have another adjunction, this time between the categories of P-algebras and chain complexes

$$\mathcal{F}\colon \mathcal{C}h \to \text{P-alg} :\mathcal{U}.$$

Given a chain complex C, the free P-algebra associated to C is just the Schur functor applied to C:

$$\mathcal{F}(C) = S(\mathrm{P})(C) = \mathrm{P} \circ C$$

The algebra structure map is induced by the operadic composition of P.

1.4.2 Cofree cooperads and cofree coalgebras

The free functor $\mathcal{F}$ generates not only free operads but also cofree cooperads. To be precise: If we consider $\mathcal{F}$ as a functor on the category of symmetric sequences, then $\mathcal{F}$ carries the structure of a cotriple with coproduct $\Delta\colon \mathcal{F} \to \mathcal{F}\circ\mathcal{F}$ given by "cutting branches" (a tree is sent to the sum of all possible partitions into smaller trees organized on the structure of a two level tree), and a counit given by the projection onto the tree with no vertices. Coalgebras over this cotriple are precisely the cooperads, and we have an adjunction

$$\mathcal{U}\colon \mathcal{C}o\mathcal{O}p(\mathcal{C}h) \rightleftarrows \mathcal{C}h^{\Sigma_*} :\mathcal{F}^c$$

with left adjoint the forgetful functor and right adjoint the functor generating cofree cooperads that we will denote by $\mathcal{F}^c$. More details can be found in §1.7 of [GJ].

The following will be our notion of a cofree coalgebra over a cooperad. Given a cooperad T and a chain complex C, the cofree T-coalgebra generated by C is the chain complex $S(\mathrm{T}, C) = \mathrm{T}\circ C$ with coaction induced by the comultiplication of T. In the language of trees we use for the coaction ρ the representation

$$\rho\left(\begin{array}{c} c_1 \cdots\cdots c_n \\ \diagdown \;\; \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu(\wp)} \left(\begin{array}{ccc} \begin{array}{c} c_* \cdots\cdots c_* \\ \diagdown \;\; \diagup \\ \wp''_* \end{array} & \cdots\cdots & \begin{array}{c} c_* \cdots\cdots c_* \\ \diagdown \;\; \diagup \\ \wp''_* \end{array} \\ & \diagdown \quad \diagup & \\ & \wp' & \\ & | & \end{array}\right)$$

(cf. the representation of the comultiplication of a cooperad at the end of Section 1.3).

The *quadratic coaction* of the coalgebra is given by

$$\rho_2\left(\begin{array}{c} c_1 \cdots\cdots c_n \\ \diagdown \;\; \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu_2(\wp)} \left(\begin{array}{ccc} & \begin{array}{c} c_* \cdots\cdots c_* \\ \diagdown \;\; \diagup \end{array} & \\ c_* \cdots & \wp'' & \cdots c_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array}\right).$$

Further, we are going to use the summation index $\nu_2'(\wp)$ when we want to leave out the factor with $\wp' = \wp$ and $\wp'' = 1$, respectively $\nu_2''(\wp)$ if we in addition omit the factor with $\wp'' = \wp$ and $\wp' = 1$ (cf. the end of Section 1.3).

1.4.3 Quasi-free operads

Quasi-free operads are a generalization of free operads. The reason why we are interested in such operads is that they provide "cofibrant" operads under reasonable conditions. We will be more concrete later when we have discussed the model category context for operads.

In this section we use the letters M and N to denote symmetric sequences or operads. If we are regarding M or N as an operad then it is explicitly stated.

Definition 1.31. (Quasi-free operad) A quasi-free operad is an operad obtained from a free operad $\mathcal{F}(\mathrm{M})$ by altering its natural internal differential by a twisting differential ∂_α. We will denote the so resulting quasi-free operad by $(\mathcal{F}(\mathrm{M}), \partial_\alpha)$.

Note that ∂_α is a family of twisting differentials on the individual levels of the operad. In the above definition we tacitly assume that the twisting differential ∂_α not only provides $\mathcal{F}(\mathrm{M})$ with a new differential but is also compatible with the operadic composition product of $\mathcal{F}(\mathrm{M})$. This precisely means that ∂_α is an operadic derivation:

Definition 1.32. (Operadic derivation) An operadic derivation for an operad M is a family of homomorphisms $f = \{f_n\}_{n \geq 0}$

$$f_n \in \mathbf{Hom}(\mathrm{M}(n), \mathrm{M}(n))_k$$

for some fixed integer k, that are Σ_n-equivariant and satisfy the equality

$$f_*(\gamma(m_0 \otimes m_1 \otimes \cdots \otimes m_n)) = \sum_{i=0}^{n} \pm\gamma(m_0 \otimes \cdots \otimes f_*(m_i) \otimes \cdots \otimes m_n)$$

for all elements $m_0, \ldots, m_n \in \mathrm{M}$. Here γ denotes the composition product of M and f_* takes the appropriate values, depending on the input. The sign is built out of the degrees of f_* and $m_0, \ldots, m_n$ by the Koszul sign rule.

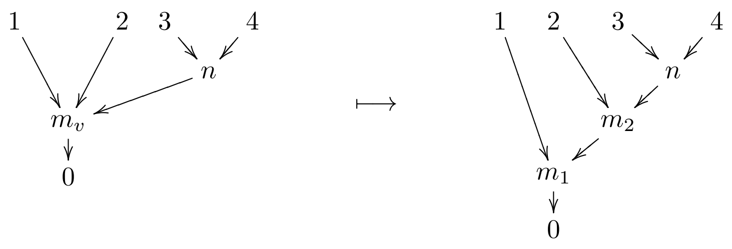
Since the elements of a free operad are formal composites of elements of M, we see that an operadic derivation of the free operad $\mathcal{F}(\mathrm{M})$ is determined by its restriction to M. In the other direction, a family of equivariant maps $\tilde{f} = \{\tilde{f}_n\}_{n \geq 0}$

$$\tilde{f}_n \in \mathbf{Hom}(\mathrm{M}(n), \mathcal{F}(\mathrm{M})(n))_k$$

defines an operadic derivation f on $\mathcal{F}(\mathrm{M})$. We sketch a construction of this derivation. More details can be found in §1.4.3 of [Fre09]. The evaluation of f on a tree τ marked by elements of M is given by a sum indexed over the set of vertices of τ. For every vertex v we apply on the element $m_v \in \mathrm{M}$ decorating v the map $\tilde{f}$, and obtain in this way a new tree (respectively sum of trees) by "blowing up the vertex" into the tree $\tilde{f}_*(m_v)$. If for example $\tilde{f}_*(m_v)$ is given by

$$\begin{tikzcd}
1 \arrow[ddr] & 2 \arrow[dr] & & 3 \arrow[dl] \\
& & m_2 \arrow[dl] & \\
& m_1 \arrow[d] & & \\
& 0 & &
\end{tikzcd}$$

then by blowing up the vertex v in the tree below we get:



We do this for each vertex of τ. Signs are occurring depending on the degree of $\tilde{f}$. One can check that this definition indeed provides an operadic derivation. An explicit calculation shows

Proposition 1.33. *An operadic derivation ∂_α on $\mathcal{F}(\mathrm{M})$ with $\partial_\alpha|_{\mathrm{M}} = \alpha$ is a twisting homomorphism if and only if α satisfies the equality*

$$\delta(\alpha) + \partial_\alpha \circ \alpha = 0,$$

where δ denotes the differential of $\mathbf{Hom}(\mathrm{M}, \mathcal{F}(\mathrm{M}))$. □

We want to draw the attention to morphisms of quasi-free operads for a moment. Given two Σ_*-sequences M und N and a collection of homomorphisms $f_n \in \mathbf{Hom}(\mathrm{M}(n), \mathrm{N}(n))_0$ we have an induced map

$$\mathcal{F}(f)\colon \mathcal{F}(\mathrm{M}) \to \mathcal{F}(\mathrm{N})$$

on the free operads generated by M and N. Assuming N is an operad, we get a map of operads

$$\tilde{f}\colon \mathcal{F}(\mathrm{M}) \to \mathrm{N}$$

by composing $\mathcal{F}(f)$ with the evaluation (operadic composition) $\mathcal{F}(\mathrm{N}) \to \mathrm{N}$ of the operad N. On the other hand, given such $\tilde{f}$ we can of course associate a family of maps $f = \{f_n\}_{n\geq 0}$ of degree 0 to it by precomposing with $\mathrm{M} \to \mathcal{F}(\mathrm{M})$. This is a kind of extension of the universal property of the free operad to maps of degree 0 that not necessarily commute with differentials. The following result is an easy calculation and can also be found in [Fre09] Proposition 1.4.7:

Proposition 1.34. *Suppose* M *is a symmetric sequence and* N *an operad. There is a bijection between operad morphisms*

$$\tilde{f}\colon (\mathcal{F}(\mathrm{M}), \partial_\alpha) \to \mathrm{N}$$

and families $f \in \mathbf{Hom}(\mathrm{M}, \mathrm{N})_0$ such that $\delta(f) = \tilde{f} \circ \alpha$. We denote here by δ the differential in $\mathbf{Hom}(\mathrm{M}, \mathrm{N})$. □

Further we have:

Proposition 1.35. *Let $f\colon \mathrm{M} \to \mathrm{N}$ be a morphism of Σ_*-modules, in particular it commutes with the differentials. The induced map $\mathcal{F}(f)$ yields a map*

$$\mathcal{F}(f)\colon (\mathcal{F}(\mathrm{M}), \partial_\alpha) \to (\mathcal{F}(\mathrm{N}), \partial_\beta)$$

of quasi-free operads if and only if it satisfies the relation $\beta \circ f = \mathcal{F}(f) \circ \alpha$ in $\mathbf{Hom}(\mathrm{M}, \mathcal{F}(\mathrm{N}))$. □

1.4.4 Quasi-free algebras

All we have said about quasi-free operads can be transferred to the setting of algebras.

Definition 1.36. (Quasi-free P-algebra) A quasi-free P-algebra is a P-algebra obtained from a free algebra $\mathrm{P} \circ C$ by the addition of a twisting homomorphism ∂_α. We use the notation $(\mathrm{P} \circ C, \partial_\alpha)$ for an algebra produced in this way.

A twisting homomorphism should respect the algebra structure in some sense if we want the resulting twisted complex to be again an algebra. To be precise, it should be a derivation:

Definition 1.37. (Derivation) A homomorphism $f \in \mathbf{Hom}(A, A)_k$ is a derivation of the P-algebra A if

$$f(\gamma_A(p \otimes a_1 \otimes \cdots \otimes a_n)) = \sum_{i=1}^{n} \pm \gamma_A(p \otimes a_1 \otimes \cdots \otimes f(a_i) \otimes \cdots \otimes a_n)$$

for all $p \in \mathrm{P}(n)$ and $a_1, \ldots, a_n \in A$.

If we now take A to be a free algebra, say $\mathrm{P} \circ C$, then a derivation f is obviously determined by its restriction on C

$$\tilde{f} \colon C \cong \mathrm{I} \circ C \xrightarrow{\eta \otimes \mathrm{id}} \mathrm{P} \circ C \xrightarrow{f} \mathrm{P} \circ C.$$

Conversely, given a homomorphism $\tilde{f} \colon C \to \mathrm{P} \circ C$ we define a derivation by the formula

$$f(p \otimes c_1 \otimes \cdots \otimes c_n) = \sum_{i=1}^{n} \pm \gamma_{\mathrm{P} \circ C}(p \otimes c_1 \otimes \cdots \otimes \tilde{f}(c_i) \otimes \cdots \otimes c_n).$$

Proposition 1.38. *A derivation ∂_α on $\mathrm{P} \circ C$ with $\partial_\alpha|_C = \alpha$ is a twisting homomorphism if and only if α satisfies the equality*

$$\delta(\alpha) + \partial_\alpha \circ \alpha = 0.$$

Proof. Since ∂_α should fulfill the twisting differential condition $\delta(\partial) + \partial^2 = 0$, the restriction on C translates into the above equation. The other direction is an explicit calculation. □

1.4.5 Quasi-cofree cooperads

Definition 1.39. (Quasi-cofree cooperad) A quasi-cofree cooperad $(\mathcal{F}^c(\mathrm{M}), \partial_\alpha)$ is a cofree cooperad altered by a twisting coderivation, i.e., a cooperad arising from a cofree one by the addition of a twisting homomorphism.

Dually to the case of quasi-free operads one can show that the coderivation ∂_α is uniquely determined by

$$\alpha \colon \mathcal{F}^c(\mathrm{M}) \xrightarrow{\partial_\alpha} \mathcal{F}^c(\mathrm{M}) \xrightarrow{pr} \mathrm{M},$$

and conversely, such a morphism α defines a unique coderivation.

1.4.6 Cobar and Bar constructions

In the topological setting Boardman and Vogt [BV] defined the so called W-construction of a topological operad. This gives a replacement of a given operad by a "homotopically well-behaved", or in other words, a cofibrant one. A generalization of the W-construction to categories with "suitable interval" is done in [BM06] by Berger and Moerdijk. In the setting of chain complexes the first functorial cofibrant constructions go back to Ginzburg/Kapranov [GK] and Getzler/Johnes [GJ], and are known under the name Cobar-Bar resolution. Since we have just defined what a quasi-free operad and a quasi-cofree cooperad is, we are now able to recall the Bar functor and the Cobar functor. The composition of these two functors applied to an operad P gives its Cobar-Bar resolution. In Section 2.3 we will come back to the question when the Cobar-Bar resolution of an operad indeed yields a cofibrant replacement.

We start with the Cobar functor

$$\mathcal{B}^c\colon \mathcal{C}o\mathcal{O}p(\mathcal{C}h)_{con} \longrightarrow \mathcal{O}p(\mathcal{C}h)_{con}$$

where the subscript *con* indicates that we are in the categories of connected operads and connected cooperads, respectively (cf. Definitions 1.14 and 1.23). The cobar operad associated to a cooperad T is a quasi-free operad $(\mathcal{F}(\Sigma^{-1}\tilde{\mathrm{T}}), \partial_{\nu_2''})$ generated by the desuspension of the coaugmentation coideal $\tilde{\mathrm{T}}$ of T (cf. Definition 1.24). The twisting differential is induced by the comultiplication of T. More precisely, the derivation is determined by the map

$$\nu_2''\colon \Sigma^{-1}\tilde{\mathrm{T}} \longrightarrow \mathcal{F}(\Sigma^{-1}\tilde{\mathrm{T}})$$

which up to addition of signs to the individual factors coincides with the reduced quadratic coproduct ν_2'' of T. The new signs come from the fact that elements are shifted by one degree. In particular, as needed for a twisting differential, the above map is of degree -1. The Cobar construction is easily seen to be functorial. More details can be found in §3.6 of [Fre09].

A few words about the dual Bar construction: The Bar functor goes in the opposite direction

$$\mathcal{B}\colon \mathcal{O}p(\mathcal{C}h)_{con} \longrightarrow \mathcal{C}o\mathcal{O}p(\mathcal{C}h)_{con}.$$

To a connected operad P we associate a quasi-cofree cooperad $(\mathcal{F}^c(\Sigma\tilde{\mathrm{P}}), \partial_{\mu_2})$ where the coderivation is determined by a map

$$\mu_2\colon \mathcal{F}^c(\Sigma\tilde{\mathrm{P}}) \xrightarrow{pr} \mathcal{F}^c_2(\Sigma\tilde{\mathrm{P}}) \xrightarrow{\mu} \Sigma\tilde{\mathrm{P}}$$

that up to signs corresponds to the quadratic product (also known as partial composition) of the operad P. By $\mathcal{F}^c_2(\Sigma\tilde{\mathrm{P}})$ we have denoted the part of the cofree cooperad generated by trees with two vertices. A complete treatment of the topic can be found in §3.5 of [Fre04].

Remark 1.40. The Cobar functor and the Bar functor form an adjoint pair. The first references on this matter are [GJ] and [GK].

1.4.7 Operadic twisting morphisms

A map $\tilde{f}$ from the Cobar construction $\mathcal{B}^c(\mathrm{T})$ to an operad P is by Proposition 1.34 determined by $f\colon \tilde{\mathrm{T}} \to \mathrm{P}$ of degree -1 such that $\delta(f) = \tilde{f} \circ \nu_2''$ holds. If we write the last condition explicitly applying the definitions of $\tilde{f}$ and ν_2'', we get

$$\delta(f)\left(\begin{array}{c} i_1 \cdots\cdots i_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu_2''(\wp)} \pm\mu_{\mathrm{P}}\left(\begin{array}{ccc} & i_* \cdots\cdots\cdots i_* & \\ & \diagdown\ \ \diagup & \\ i_* \cdots\cdots & f(\wp'') & \cdots\cdots i_* \\ \diagdown & | & \diagup \\ & f(\wp') & \\ & | & \end{array}\right)$$

for every $\wp \in \tilde{\mathrm{T}}$. The sign comes from moving a map of degree -1 past $\wp'$, and μ_{P} denotes the quadratic product of P. We can extend f on T(1) trivially. A map satisfying the above equality is called an *operadic twisting morphism*. The set of all these maps from a given connected cooperad T to an operad P is denoted by $\mathcal{T}w(\mathrm{T}, \mathrm{P})$, and, as discussed, there is a bijection

$$\mathcal{O}p(\mathcal{B}^c(\mathrm{T}), \mathrm{P}) \cong \mathcal{T}w(\mathrm{T}, \mathrm{P}).$$

For the sake of completeness let us mention that similarly one can show a bijection between the set of twisting morphisms and the maps of cooperads $\mathcal{C}o\mathcal{O}p(\mathrm{T}, \mathcal{B}(\mathrm{P}))$. All together we get for a connected operad P and a connected cooperad T

$$\mathcal{O}p(\mathcal{B}^c(\mathrm{T}), \mathrm{P}) \cong \mathcal{T}w(\mathrm{T}, \mathrm{P}) \cong \mathcal{C}o\mathcal{O}p(\mathrm{T}, \mathcal{B}(\mathrm{P})).$$

1.4.8 Quasi-cofree coalgebras

We complete our discussion on quasi-(co)free objects with some definitions and facts about quasi-cofree coalgebras.

Definition 1.41. (Quasi-cofree coalgebra) A quasi-cofree coalgebra over a cooperad T is a coalgebra constructed from a free coalgebra $\mathrm{T} \circ C$ by altering the natural differential by a twisting differential ∂_α. We will use the notation $(\mathrm{T} \circ C, \partial_\alpha)$.

In order for a twisting differential ∂_α to respect the structure map of the cofree coalgebra it should be a coderivation:

Definition 1.42. (Coderivation) A homomorphism $f\colon A \to A$ of a coalgebra A is a coderivation if it satisfies

$$\rho(f(a)) = \sum_{\rho(a)} \left(\begin{array}{ccccc} a_* & \cdots & f(a_*) & \cdots & a_* \\ & \diagdown & | & \diagup & \\ & & \wp & & \\ & & | & & \end{array} \right)$$

for every $a \in A$.

There is a slight abuse of notation in the above formula. On the right hand side we have first applied the comultiplication of A, and then we use f on every factor of the coproduct not only once but we have one summand for every element a_* in the factor. We want to keep the notation as simple as possible and abandon the use of a double sum, hoping that the reader will not be mislead.

Dually to the case of algebras, a coderivation $\tilde{f}$ on a cofree coalgebra is uniquely determined by its projection onto cogenerators:

$$f\colon \mathrm{T} \circ C \xrightarrow{\tilde{f}} \mathrm{T} \circ C \xrightarrow{pr} \mathbb{I} \circ C \cong C$$

More generally, we have the following result (Proposition 4.1.3 of [Fre09], Proposition 2.14 of [GJ]):

Proposition 1.43. *There is a bijection between coderivations $\partial_\alpha\colon \mathrm{T} \circ C \to \mathrm{T} \circ C$ and maps $\alpha \in \mathbf{Hom}(\mathrm{T} \circ C, C)$. The map α associated to a coderivation ∂_α is given by the projection onto* *the summand $C \cong \mathbb{I} \otimes C$. The coderivation associated to a given map α is defined by the formula*

$$\partial_\alpha \left(\begin{array}{ccc} c_1 & \cdots & c_n \\ \diagdown & & \diagup \\ & \wp & \\ & | & \end{array} \right) = \sum_{i=1}^{n} \pm \left(\begin{array}{ccc} c_1 \cdots & \alpha(c_i) & \cdots c_n \\ \diagdown & | & \diagup \\ & \wp & \\ & | & \end{array} \right) + \sum_{\nu_2'(\wp)} \pm \left(\begin{array}{ccc} & \alpha \left[\begin{array}{ccc} c_* & \cdots & c_* \\ \diagdown & & \diagup \\ & \wp'' & \end{array} \right] & \\ c_* \cdots & & \cdots c_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array} \right)$$

for every $\wp \in \mathrm{T}$ and $c_1, \ldots, c_n \in C$. The signs are determined by the commutation of α with elements of the tree tensor. The summation index ν_2' stands for the quadratic coproduct without the factor $\mathrm{T}(n) \otimes \mathrm{T}(1) \otimes \cdots \otimes \mathrm{T}(1)$ (cf. the end of Section 1.4.2).

Proof. The proof is a calculation. □

We need a few more observations about quasi-cofree coalgebras and their morphisms. All of these can be found in §4.1 of [Fre09]. We list the statements and add a few comments where we regard this as helpful for the reader. The next proposition describes coderivations of cofree coalgebras that are at the same time twisting homomorphisms.

Proposition 1.44. *Let $\alpha\colon \mathrm{T} \circ C \to C$ be a map of degree -1 such that the restriction $\alpha|_C$ is zero. The induced coderivation ∂_α on the coalgebra $\mathrm{T} \circ C$ is a twisting homomorphism if and only if the equality*

$$\delta(\alpha) \left(\begin{array}{ccc} c_1 & \cdots & c_n \\ \diagdown & & \diagup \\ & \wp & \\ & | & \end{array} \right) + \sum_{\nu_2'(\wp)} \pm \alpha \left(\begin{array}{ccc} & \alpha \left[\begin{array}{ccc} c_* & \cdots & c_* \\ \diagdown & & \diagup \\ & \wp'' & \end{array} \right] & \\ c_* \cdots & & \cdots c_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array} \right) = 0$$

holds for all elements of $\mathrm{T} \circ C$. Here, δ denotes the differential of the chain complex $\mathbf{Hom}(\mathrm{T} \circ C, C)$.

Proof. The "only if" direction is trivial. The other one is a calculation using the explicit construction of ∂_α from the previous proposition. □

Note that we can exchange the summation index ν_2' by ν_2'' since α vanishes on C. We call a map α that satisfies the conditions of the previous proposition a *twisting cochain*.

Proposition 1.45. *The structure of a $\mathcal{B}^c(\mathrm{T})$-algebra on a chain complex A is given by a map $\alpha\colon \mathrm{T} \circ A \to A$ that satisfies the assumptions and the equality of Proposition 1.44.*

Proof. A $\mathcal{B}^c(\mathrm{T})$-algebra structure on A is equivalent to an operadic morphism $\tilde{f}$ from $\mathcal{B}^c(\mathrm{T})$ into the endomorphism operad of A. Using this equivalence and the assignment

$$\alpha \left(\begin{array}{ccc} c_1 & \cdots & c_n \\ \diagdown & & \diagup \\ & \wp & \\ & | & \end{array} \right) = -f(\wp)(c_1 \otimes \cdots \otimes c_n)$$

the equation of an operadic twisting morphism (cf. Section 1.4.7) translates into the conditions of the previous proposition. □

Therefore, for a $\mathcal{B}^c(\mathrm{T})$-algebra A we have a naturally associated quasi-cofree coalgebra $(\mathrm{T} \circ A, \partial_\alpha)$. The last statements are concerning maps between $\mathcal{B}^c(\mathrm{T})$-algebras, and more generally maps between quasi-cofree coalgebras. A calculation shows:

Proposition 1.46. *Let A and B be two $\mathcal{B}^c(\mathrm{T})$-algebras and $f\colon A \to B$ a chain map between them. Then f defines a map of $\mathcal{B}^c(\mathrm{T})$-algebras if and only if $\mathrm{T}(f)\colon \mathrm{T} \circ A \to \mathrm{T} \circ B$ defines a map between the associated (by Proposition 1.45) quasi-cofree coalgebras $(\mathrm{T} \circ A, \partial_\alpha)$ and $(\mathrm{T} \circ B, \partial_\beta)$.* □

This motivates the study of maps between quasi-cofree coalgebras. Let us first take a look at maps between cofree coalgebras. A coalgebra map $\tilde{f}\colon \mathrm{T} \circ A \to \mathrm{T} \circ B$ of degree 0 is determined by the projection

$$f\colon \mathrm{T} \circ A \xrightarrow{\tilde{f}} \mathrm{T} \circ B \xrightarrow{pr} B$$

on the factor B. The other way round, given a map f of degree 0 we define a coalgebra map via

$$\tilde{f}\left(\begin{matrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{matrix}\right) = \sum_{\nu(\wp)} \left(\begin{matrix} f\left[\begin{matrix} a_* \cdots a_* \\ \diagdown \ \diagup \\ \wp''_* \end{matrix}\right] \cdots\cdots f\left[\begin{matrix} a_* \cdots a_* \\ \diagdown \ \diagup \\ \wp''_* \end{matrix}\right] \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{matrix}\right).$$

An easy calculation shows:

Proposition 1.47. *In the situation above, f defines a map of quasi-cofree coalgebras $(\mathrm{T} \circ A, \partial_\alpha)$ and $(\mathrm{T} \circ B, \partial_\beta)$ if and only if it satisfies the equality*

$$\delta(f)\left(\begin{matrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{matrix}\right) - \sum_{\nu'_2(\wp)} \pm f \left(\begin{matrix} \alpha\left[\begin{matrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{matrix}\right] \\ a_* \cdots \qquad | \qquad \cdots a_* \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{matrix}\right)$$

$$+ \sum_{\nu(\wp)} \beta \left(\begin{matrix} f\left[\begin{matrix} a_* \cdots a_* \\ \diagdown \ \diagup \\ \wp''_* \end{matrix}\right] \cdots\cdots f\left[\begin{matrix} a_* \cdots a_* \\ \diagdown \ \diagup \\ \wp''_* \end{matrix}\right] \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{matrix}\right) = 0$$

Again, δ denotes the differential in the internal Hom complex. □

Note that in this last proposition we still assume $\partial_\alpha|_A = 0$. Else we would have one more sum term in the equation above.

If we start with $\mathcal{B}^c(\mathrm{T})$-algebras A and B, then not every morphism of the quasi-cofree coalgebras $(\mathrm{T} \circ A, \partial_\alpha)$ and $(\mathrm{T} \circ B, \partial_\beta)$ corresponds to a morphism of $\mathcal{B}^c(\mathrm{T})$-algebras. We are going to see, however, that when T is Σ_*-cofibrant it corresponds to a morphism in the homotopy category, or in other words, it is a morphism up to homotopy.

2 Model structures and homotopy invariance property

In this section we are going to recall the semi-model structures on the categories of operads and operadic algebras in a monoidal model category. After that we will state a version of the homotopy invariance property of cofibrant operads, that is going to be needed for the construction of the canonical class later on. In the last section we give a construction of particular cofibrant replacements for operads and operadic algebras over Σ_*-cofibrant chain operads.

2.1 Semi-model structure for operads

A lot of work has been done on model structures for operads. Some of the names to be mentioned here are Berger and Moerdijk [BM], Fresse [Fre09b], Harper [Har], Hinich [Hin], Markl [Mar96] and Spitzweck [Spi]. We are going to collect some statements in this context that justify our approach later on. Our presentation of the results leans heavily on Chapter 12 of Fresse's book "Modules over operads and functors" [Fre09b].

For the remainder of this section $\mathcal{C}$ is a monoidal model category in the sense of Definition 3.1 of [SS00]. We denote by $\mathcal{C}^{\mathbf{\Sigma}_*}$ the category of symmetric sequences in $\mathcal{C}$. If $\mathcal{C}$ is cofibrantly generated and G is a discrete group then the category $\mathcal{C}^{\mathbf{G}}$ of objects of $\mathcal{C}$ with right G action is again a model category with weak equivalences and fibrations given by these in the underlying category $\mathcal{C}$. The category $\mathcal{C}^{\mathbf{\Sigma}_*}$ is a product of categories of the kind $\mathcal{C}^{\mathbf{G}}$ and inherits a (cofibrantly generated) model category structure with pointwise fibrations and weak equivalences. The general theory says that under some assumptions, using an adjunction

$$\mathcal{F}\colon \mathcal{A} \rightleftarrows \mathcal{B} :\mathcal{U},$$

one can transfer a cofibrantly generated model category structure from $\mathcal{A}$ to $\mathcal{B}$ such that the right adjoint $\mathcal{U}$ creates fibrations and weak equivalences. Therefore, one can hope to equip the category of operads with a model structure via the free-forgetful adjunction

$$\mathcal{F}\colon \mathcal{C}^{\mathbf{\Sigma}_*} \rightleftarrows \mathcal{O}p(\mathcal{C}) :\mathcal{U}.$$

In general, this does not work completely. One only gets a so called semi-model category structure. However, the latter is good enough for doing homotopical algebra. Most of the usual working tools for model categories can be adapted to the setting of semi-model categories. For an overview of the topic the reader may want to take a look at §2 of [Spi] or Chapter 12 of [Fre09b]. Let us mention the main difference between a model and a semi-model category: A semi-model category fulfills only weaken versions of the axioms M4 and M5. The factorization axiom M5 must hold for maps with cofibrant domain and the lifting axiom M4 is restricted to squares where the cofibration resp. acyclic cofibration has a cofibrant domain. In addition, complementary properties are required in two new axioms.

Now we come to

Theorem 2.1 (cf. Theorem 3 of [Spi])**.** *The category of operads $\mathcal{O}p(\mathcal{C})$ inherits a semi-model category structure such that the forgetful functor $\mathcal{U}$ creates fibrations and weak equivalences.*

Note that a fibration and a weak equivalence of operads is given by levelwise fibrations respectively weak equivalences in the underlying category $\mathcal{C}$. Cofibrations are defined by the left lifting property. Moreover, $\mathcal{U}$ maps cofibrations with domain cofibrant in $\mathcal{C}^{\mathbf{\Sigma}_*}$ to cofibrations in $\mathcal{C}^{\mathbf{\Sigma}_*}$. Thus, cofibrant operads are cofibrant as symmetric sequences. For a cofibrant replacement of an operad P we will often write P_∞.

Remark 2.2. Berger and Moerdijk give criteria for obtaining a full model structure on the subcategory of so called "reduced" operads (Theorem 3.1 of [BM]). These are operads that in arity zero equal the monoidal unit. Under the conditions of the theorem the semi-model structure on $\mathcal{O}p(\mathcal{C})$ restricts to a full model structure when regarding only reduced operads. This result can be adapted to the full subcategory of connected operads. Under the assumptions of the theorem the category $\mathcal{O}p(\mathcal{C})_{\mathbf{con}}$ inherits a full model structure. In particular, cofibrant connected operads are also cofibrant in the semi-model structure, and therefore also cofibrant as symmetric sequences.

We come to the question when the category of algebras over a given operad admits good homotopical properties. In general, if we fix an operad P that is cofibrant as a symmetric sequence then we can again use the free-forgetful adjunction between $\mathcal{C}$ and P-algebras

$$\mathcal{F} \colon \mathcal{C} \rightleftarrows \text{P-alg} : \mathcal{U}$$

to prove:

Theorem 2.3 (cf. Theorem 12.3.A of [Fre09b] and Theorem 4.3 of [Spi]). *If* P *is cofibrant as a symmetric sequence then the category of* P*-algebras inherits a semi-model category structure such that the forgetful functor* $\mathcal{U}$ *creates fibrations and weak equivalences. If in addition* P *is cofibrant as an operad and* $\mathcal{C}$ *satisfies the monoid axiom (see [SS00]) then the category of* P*-algebras inherits a full model structure.*

We will use the terminology "Σ_*-cofibrant" for operads whose underlying symmetric sequence is cofibrant. A Σ_*-cofibrant replacement of the commutative operad (in the algebraic or in the topological setting) is called an E_∞-operad. Such a replacement of the associative operad is called an A_∞-operad. These notations go back to May [May].

Further, Fresse proves:

Theorem 2.4 (cf. Theorem 12.5.A of [Fre09b]). *If* P *and* Q *are operads that are* Σ_**-cofibrant and* $f \colon \mathrm{P} \to \mathrm{Q}$ *a map between them then the induced adjoint pair of functors*

$$f_! \colon \text{P-}alg \rightleftarrows \text{Q-}alg : f^*$$

defines a Quillen functor pair of semi-model categories. If f *is a weak equivalence then* $(f_!, f^*)$ *is a Quillen equivalence.*

The construction of $f_!$ is given by a certain reflexive coequalizer in the category of Q-algebras. For the details we refer to §3.3 of [Fre09b].

We want to apply the above theory in the special case of $\mathcal{C}$ being the category of (unbounded) chain complexes $\mathcal{C}h$. This category admits a cofibrantly generated monoidal model category structure with weak equivalences the quasi-isomorphisms of chain complexes and fibrations the levelwise surjective maps. Details and proofs about this model structure can be found in Section 2.3 of [Ho]. We describe here the *cofibrant chain complexes*: A chain complex C is cofibrant if there is an exhaustive filtration of C

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots$$

such that all quotient complexes $\mathcal{F}_{k+1}/\mathcal{F}_k$ have trivial differentials and are levelwise projective. By §3 of [Sch] this gives a cofibrant chain complex in the model structure on $\mathcal{C}h$. Further, the category of chain complexes with the above model structure satisfies the conditions of [BM], Theorem 3.1 (cf. 3.3.3 of [BM]), and therefore the category $\mathcal{O}p(\mathcal{C}h)_{con}$ inherits a full model structure.

For the sake of completeness we want to make some remarks on model structures for operads.

Remark 2.5. As we already mentioned, Berger and Moerdijk have shown in [BM], that if one restricts the semi-model structure on $\mathcal{O}p(\mathcal{C})$ onto the full subcategory of reduced operads then one gets a full model structure under mild conditions on $\mathcal{C}$. These conditions are satisfied for instance when $\mathcal{C}$ is the category of simplicial sets, compactly generated spaces or chain complexes over a commutative ring. Further, Kro [Kro] and Gutiérrez-Vogt [GV] generalize the theory of Berger and Moerdijk to enriched categories. In this way they prove that also the categories of orthogonal and symmetric spectra with the positive model structures provide their categories of reduced operads (resp. all operads in the case of symmetric spectra) with a model structure. Kro also establishes the existence of a full model structure for the category of connected operads (in [Kro] referred to as "positive" operads).

2.2 Homotopy invariance property of cofibrant operads

Categories of algebras over cofibrant operads are homotopically well-behaved. They form semi-model categories so that we can talk about their homotopy categories and do homotopy theory in a good formal framework. This is, however, not an exclusive property of cofibrant operads. As we stated in the previous section, there are many other operads that allow a semi-model structure for their category of algebras - it is enough to have a Σ_*-cofibrant operad. Most of the commonly used E_∞-operads are not cofibrant as operads. For many purposes Σ_*-cofibrant operads are just as good as cofibrant ones.

The *homotopy invariance property* we are going to state next is a priority of cofibrant operads, though. It was first formulated by Boardman and Vogt in the seventies. Their work is situated in the topological context (cf. [BV]). In the eighties Kadeishvili achieved the breakthrough in the algebraic setting with his homotopy transfer theorem for A_∞-structure (Theorem 1 of [Kad]). What followed where different versions in various contexts among others by Markl, and Chuang and Lazarev. More recently, Berger and Moerdijk [BM] as well as Fresse in [Fre10] gave a model category formulation of the homotopy invariance property that forms a conceptional generalization of the previous results. We are going to restate the result of Berger and Moerdijk to the extent of our needs. For the convenience of the reader we will also give a proof.

Theorem 2.6 (Theorem 3.5 [BM]). *Let $f\colon A \to B$ be a morphism in a closed monoidal model category for which the category of connected operads admits a model structure. Assume further that B is a P_∞-algebra for some cofibrant operad P_∞.*

(i) If A is cofibrant and f is an acyclic fibration then A can be equipped with the structure of a P_∞-algebra in such a way that f is a P_∞-algebra map.

(ii) If both A and B are cofibrant-fibrant and f is an acyclic cofibration then A can be equipped with the structure of a P_∞-algebra in such a way that f preserves the P_∞-algebra structure up to homotopy.

In particular, if A is cofibrant-fibrant, B is fibrant and f is a weak equivalence then A can be equipped with a P_∞-structure such that f preserves the latter up to homotopy.

Proof. We denote by $\mathrm{End_A}$ respectively $\mathrm{End_B}$ the connected endomorphism operad of A respectively B. We define a symmetric sequence $\mathrm{Hom}_{A,B}$ in the ground category given by $\mathbf{Hom}(A^{\otimes n}, B)$ for every n in $\mathbb{N}$. We can form the following pull back in the category of symmetric sequences:

$$\begin{tikzcd} \mathrm{End_f} \arrow[r, "i_A"] \arrow[d, "i_B"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{End_A} \arrow[d, "f_*"] \\ \mathrm{End_B} \arrow[r, "f^*"] & \mathrm{Hom}_{A,B} \end{tikzcd}$$

It is an easy observation that $\mathrm{End_f}$ inherits an operad structure from $\mathrm{End_A}$ and $\mathrm{End_B}$.

We start with the proof of (i). The map f is an acyclic fibration and A is cofibrant. Thus, using the adjunction between **Hom** and $\otimes$, and the definition of acyclic fibrations via the right lifting property, we conclude that f_* is an acyclic fibration (levelwise, and thus, of symmetric sequences). Therefore, i_B is an acyclic fibration of symmetric sequences, and hence of operads, and we have a lift in the diagram

$$\begin{tikzcd}
 & \mathrm{End_f} \arrow[r, "i_A"] \arrow[d, "i_B"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{End_A} \arrow[d, "f_*"] \\
\mathrm{P}_\infty \arrow[ur, dashed, "\phi"] \arrow[r, "\phi_B"] & \mathrm{End_B} \arrow[r, "f^*"] & \mathrm{Hom}_{A,B}
\end{tikzcd}$$

where ϕ_B denotes the P_∞-algebra structure on B and $i_A \circ \phi$ gives us the desired P_∞-structure on A.

For (ii) note first that for A and B cofibrant-fibrant the endomorphism operads $\mathrm{End_A}$ and $\mathrm{End_B}$ are fibrant (adjointness and right lifting property). Again by adjointness and push out-product axiom we conclude that f^* is an acyclic fibration whenever f is an acyclic cofibration and B is fibrant. On the other hand, Ken Brown's Lemma ([Ho] Lemma 1.1.12) allows us to deduce that f_* is a weak equivalence not only when f is an acyclic fibration but also when it is just a weak equivalence between fibrant objects. Hence, in our situation f_*, f^* and i_A are weak equivalences. By the two out of three property so is i_B. As we mentioned, the operads $\mathrm{End_A}$ and $\mathrm{End_B}$ are fibrant. Since i_A is a fibration $\mathrm{End_f}$ is fibrant, too and i_B is a weak equivalence between fibrant operads. In particular, it induces a bijection on homotopy classes of operadic maps out of P_∞. We have a diagram where the triangle commutes up to homotopy:

$$\begin{tikzcd}
 & \mathrm{End_f} \arrow[r, two heads, "\sim"] \arrow[d, "\sim"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{End_A} \arrow[d, "\sim"] \\
\mathrm{P}_\infty \arrow[ur, dashed, "\phi"] \arrow[r, "\phi_B"] \arrow[ur, phantom, "\circlearrowright h", shift right=3] & \mathrm{End_B} \arrow[r, two heads, "\sim"] & \mathrm{Hom}_{A,B}
\end{tikzcd}$$

The map $i_A \circ \phi$ makes A into a P_∞-algebra and f becomes a P_∞-algebra map from A endowed with this structure to B endowed with the structure $i_B \circ \phi$. The last one is homotopic to the original P_∞-structure on B given by ϕ_B. $\square$

Remark 2.7. The original statement and proof of [BM] is about cofibrant operads in the model category of reduced operads. It can be adapted to the setting of connected operads without further changes.

We want to apply the above theorem in the case of the category $\mathcal{C}h$ of chain complexes over a field (of possibly positive characteristic). If A is a chain complex then the homology of A is a cofibrant chain complex. Indeed, it is very easy to give an exhaustive filtration of H_*A by subcomplexes

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots \subset H_*A$$

such that $\mathcal{F}_{k+1}/\mathcal{F}_k$ has trivial differentials and is levelwise projective. Take for example

$$\mathcal{F}_k = (\cdots 0 \to \cdots \to 0 \to H_kA \to H_{k-1}A \to \cdots \to H_{-k+1}A \to H_{-k}A \to 0 \to \cdots \to 0 \cdots)\,.$$

Further, by choosing cycle representatives we can build a map

$$H_*A \to A.$$

The latter is surely a weak equivalence, and since all chain complexes are fibrant we can apply the above theorem to transfer a P_∞-structure from A to H_*A. Later on in the next section we are going to specify what it means to have homotopic P_∞-algebra structures in the chain setting. Then we will see that this transferred P_∞-structure makes H_*A weakly equivalent to A as a P_∞-algebra.

2.3 Cofibrant replacements

In this section we are working in the category $\mathcal{C}h$.

Now we have had an overview about model structures for operads and have discussed the advantages of cofibrant operads, we want to come back to the question about cofibrant replacements of operads. After that we are going to give a construction of a cofibrant replacement of an algebra over a Σ_*-cofibrant operad.

As mentioned earlier, a candidate for a cofibrant resolution of a chain operad is the Cobar-Bar resolution. The following result goes back to Ginzburg/Kapranov [GK]:

Proposition 2.8. *Let* P *be a connected chain operad. The natural morphism*

$$\mathcal{B}^c\mathcal{B}(\mathrm{P}) \to \mathrm{P}$$

adjoint to the identity on $\mathcal{B}(\mathrm{P})$ *is a weak equivalence of operads.* □

Further, Fresse proves:

Theorem 2.9 (cf. Theorem 1.4.12 and §3.14 of [Fre09]). *If* P *is in addition cofibrant as a symmetric sequence then* $\mathcal{B}^c\mathcal{B}(\mathrm{P})$ *is a cofibrant operad.*

All together, if P is Σ_*-cofibrant and connected then the Cobar-Bar resolution gives us a cofibrant replacement of P in the semi-model category of operads and in the model category of connected operads. If P happens not to be Σ_*-cofibrant then we can replace it by its *Hadamard product* with the Barratt-Eccles operad (Example 1.9). Explicitly, this means

$$(\mathrm{P} \otimes \mathrm{E}\Sigma_*)(n) := \mathrm{P}(n) \otimes \mathrm{E}\Sigma_\mathrm{n}.$$

The augmentation ϵ of the Barratt-Eccles operad gives a weak equivalence

$$\mathrm{P} \otimes \mathrm{E}\Sigma_* \xrightarrow{\mathrm{id}\otimes\epsilon} \mathrm{P} \otimes \mathrm{Com} \cong \mathrm{P}.$$

The left hand side is Σ_*-cofibrant, and therefore for any (connected) chain operad P we get a cofibrant replacement $\mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \to \mathrm{P}$.

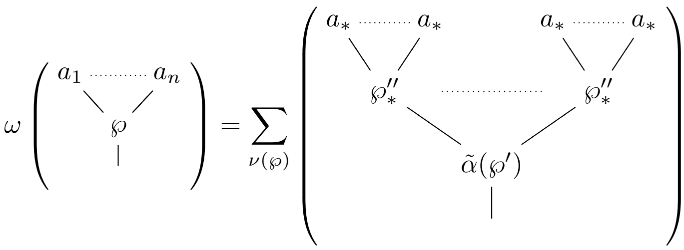
In the following we are tacitly assuming that our operads and cooperads are connected.

Now let us fix a connected Σ_*-cofibrant operad P, a Σ_*-cofibrant cooperad T together with a weak equivalence $\mathtt{A}\colon \mathcal{B}^c(\mathrm{T}) \to \mathrm{P}$, and a P-algebra A. One possibility for this constellation is to take $\mathrm{T} = \mathcal{B}(\mathrm{P})$ when P is Σ_*-cofibrant. Another one is to see A as a $\mathrm{P} = \mathcal{B}^c\mathcal{B}(\mathrm{Q} \otimes \mathrm{E}\Sigma_*)$-algebra for some operad Q, and to take $\mathtt{A} = \mathrm{id}$. We are looking for a cofibrant replacement of A in the category of P-algebras. The results we present originate from §4.2 of [Fre09]. Proofs of the statements can also be found there.

We want to construct a certain quasi-free P-algebra that we denote for the moment by $R_\mathrm{P}\,(\mathrm{T} \circ A, \partial_\alpha)$ where ∂_α is the twisting coderivation on the cofree coalgebra $\mathrm{T} \circ A$ given by Proposition 1.44 and Proposition 1.45 (note that via restriction of structure A is also a $\mathcal{B}^c(\mathrm{T})$-algebra). The underlying object of $R_\mathrm{P}\,(\mathrm{T} \circ A, \partial_\alpha)$ is the free P-algebra $\mathrm{P} \circ \mathrm{T} \circ A$, generated by the quasi-cofree coalgebra $(\mathrm{T} \circ A, \partial_\alpha)$. The twisting derivation on $R_{\mathrm{P}}\,(\mathrm{T} \circ A, \partial_\alpha)$, which we denote by ∂_ω, is determined by the map

$$\omega\colon \mathrm{T} \circ A \to \mathrm{P} \circ \mathrm{T} \circ A$$

given by



for every $\wp \in \mathrm{T}$ and $a_1, \ldots, a_n \in A$. In the above formula, we denote by $\tilde{\alpha}$ the twisting morphism from T to P corresponding to the map $\mathtt{A}$ (see 1.4.7). The close relation between this twisting morphism and the coderivation on $\mathrm{T} \circ A$ is uncovered by our choice of notation.

Proposition 2.10 (cf. §4.2.1 of [Fre09])**.** *The map ω defines a twisting morphism.* □

We want to take a closer look at the quasi-free algebra $R_{\mathrm{P}}\,(\mathrm{T} \circ A, \partial_\alpha)$. Its differential consists of several parts. It is the sum of the "internal differential" with the derivation ∂_ω. Then again the internal differential decomposes into two parts – the one coming from the internal differentials of $\mathrm{P} \circ \mathrm{T} \circ A$, and the other reflecting the twisting coderivation ∂_α. All together, we write $\partial_\omega + \partial_\alpha + \delta$ for the differential of $R_{\mathrm{P}}\,(\mathrm{T} \circ A, \partial_\alpha)$.

The natural morphism $\mathrm{P} \circ \mathrm{T} \circ A \to A$ can be seen to give a map of P-algebras

$$\chi\colon R_{\mathrm{P}}(\mathrm{T} \circ A, \partial_\alpha) \to A$$

(Proposition 4.2.3 of [Fre09]). And we have:

Theorem 2.11 (Theorem 4.2.4 of [Fre09])**.** *Suppose in the above situation that A is in addition cofibrant as a chain complex. Then*

$$\chi\colon R_{\mathrm{P}}(\mathrm{T} \circ A, \partial_\alpha) \to A$$

defines a cofibrant replacement of A in the category of P*-algebras.* □

Since the construction R_{P} is functorial not only with respect to P-algebra maps but also with respect to maps of T-coalgebras, we get the following corollaries of the theorem:

Corollary 2.12 (Proposition 4.2.7 of [Fre09])**.** *Let $R_A = R_{\mathrm{P}}(\mathrm{T} \circ A, \partial_\alpha)$ and $R_B = R_{\mathrm{P}}(\mathrm{T} \circ B, \partial_\beta)$ be the cofibrant replacements of the* P*-algebras A and B given by Theorem 2.11. A map of quasi-cofree coalgebras*

$$f\colon (\mathrm{T} \circ A, \partial_\alpha) \to (\mathrm{T} \circ B, \partial_\beta)$$

induces a map on cofibrant replacements, and in particular represents a map in the homotopy category of P*-algebras.* □

Corollary 2.13 (Proposition 4.2.8 of [Fre09])**.** *In the situation of the previous corollary, if we assume that the composition*

$$A \hookrightarrow \mathrm{T} \circ A \xrightarrow{f} \mathrm{T} \circ B \xrightarrow{pr} B$$

is a weak equivalence then f induces a weak equivalence of P*-algebras $R_A \xrightarrow{\sim} R_B$.* □

Remark 2.14. Let T be $\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$ or, if P is Σ_*-cofibrant, just $\mathcal{B}(\mathrm{P})$, and A and B two $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$- or resp. $\mathcal{B}^c\mathcal{B}(\mathrm{P})$-algebras (in particular, A and B may be P-algebras). The algebras R_A and R_B constructed from $\mathtt{A} = \mathrm{id}$ are cofibrant replacements of A and B, provided A and B are cofibrant chain complexes. Morphisms between these cofibrant replacements induced by maps of quasi-cofree coalgebras as in Corollary 2.12 are a generalization of the so called ∞-morphisms. For P = Ass see for example §3 of [Kel].

In Theorem 2.6 we have used the terminology of "homotopic" algebra structures on an object A. Let us now come to the question what precisely this means in the context of chain operads. We again fix a Σ_*-cofibrant operad P and denote by A_1 and A_2 two P-algebras, both with underlying chain complex A, and with homotopic algebra structures, i.e., the maps

$$\phi_1, \phi_2 \colon \mathrm{P} \to \mathrm{End_A}$$

giving the different P-algebra structures on A are homotopic in the model category of connected operads. In §5 of [Fre09], Fresse defines an explicit cylinder object for the operad $\mathcal{B}^c(\mathrm{T})$ so that he can obtain:

Theorem 2.15 (Theorem 5.2.1 of [Fre09])**.** *Suppose there is a weak equivalence*

$$\mathtt{A}\colon \mathcal{B}^c(\mathrm{T}) \xrightarrow{\sim} \mathrm{P}$$

of operads with P *and* T *being cofibrant symmetric sequences. Assume further that the chain complex A is equipped with two different* P*-algebra structures. We denote the resulting* P*-algebras by A_1 and A_2. We can view A_1 and A_2 as $\mathcal{B}^c(\mathrm{T})$-algebras via the map* $\mathtt{A}$*.*

In this situation, left homotopies between the $\mathcal{B}^c(\mathrm{T})$-algebras A_1 and A_2 correspond bijectively to morphisms of quasi-cofree coalgebras

$$f\colon (\mathrm{T}\circ A_1, \partial_{\alpha_1}) \to (\mathrm{T}\circ A_2, \partial_{\alpha_2})$$

which reduce to the identity on A.

Let A be a cofibrant chain complex. By Corollaries 2.12 and 2.13 such a map f of quasi-cofree coalgebras induces a weak equivalence between the cofibrant replacements R_{A_1} and R_{A_2}, and in particular, an isomorphism between A_1 and A_2 in the homotopy category of P*-algebras.* □

Therefore, homotopic algebra structures on a cofibrant chain complex indeed give us homotopic algebras.

We can now combine the above result with Theorem 2.6 of the previous section to obtain:

Theorem 2.16. *Let $f\colon A \to B$ be a weak equivalence of chain complexes, A a cofibrant chain complex and B a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra. Then A inherits a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure such that f can be extended to a map ϕ_f between the (by Proposition 1.45) corresponding quasi-cofree coalgebras $(\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha)$ and $(\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ B, \partial_\beta)$.*

Proof. We factorize f as an acyclic cofibration followed by an acyclic fibration:

$$\begin{tikzcd} A \arrow[rr, hook, "f", "\sim"'] \arrow[dr, hook, "f_1"', "\sim"] & & B \\ & C \arrow[ur, two heads, "f_2"', "\sim"] & \end{tikzcd}$$

By part (i) of Theorem 2.6 we know that there is a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure γ on C such that the map f_2 is a map of $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebras. In particular by Proposition 1.46

$$\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)(f_2)\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ C, \partial_\gamma) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ B, \partial_\beta)$$

gives a map of quasi-cofree coalgebras. By the second part of the same theorem we get a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure α on A and a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure γ' on C such that

$$\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)(f_1)\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ C, \partial_{\gamma'})$$

is a map of quasi-cofree coalgebras and the structures γ and γ' on C are left homotopic. Further, Theorem 2.15 provides us with a map of quasi-cofree coalgebras

$$h\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ C, \partial_{\gamma'}) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ C, \partial_{\gamma})$$

that reduces to the identity on C. Composing all three maps yields the required morphism. □

Let us go back to the very special case of a chain complex A over a field, and its homology H_*A. As mentioned in the previous section we can use a cycle choosing map

$$i\colon H_*A \to A$$

to transport a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-structure from A to H_*A. The map i can be then extended to a map ϕ_i between quasi-cofree coalgebras.

Remark 2.17. In this light, the statement of the last theorem can be seen as a wide generalization of the well-known result of Kadeishvili [Kad] that the homology of a differential graded algebra A can be equipped with an A_∞-structure such that there is an A_∞-morphism from H_*A to A extending a given cycle choosing map $i\colon H_*A \to A$.

Our next theorem states that there is also a map of quasi-cofree coalgebras in the opposite direction, i.e., it induces a map $R_A \to R_{H*A}$, and its restriction from A to H_*A is a weak equivalence. A similar statement for the special case char $k = 0$ can be found in §10.2 of [LV]. The methods used there are very explicit, though, and can not be applied in the general situation.

Theorem 2.18. *In the above situation, we can construct a map of quasi-cofree coalgebras*

$$\phi_p\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_{\gamma})$$

that reduces to a weak equivalence $p\colon A \to H_*A$. Here, ∂_α and ∂_γ denote the twisting homomorphisms corresponding to the $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structures on A and H_*A.

Proof. Since we are working over a field we can define a map of chain complexes, and in particular a weak equivalence, $p\colon A \to H_*A$. Obviously, the composition of this map with the inclusion $i\colon H_*A \to A$ is the identity on H_*A.

We claim that we can transfer the $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure of A to H_*A along p in a way that the latter extends to a map of the corresponding quasi-cofree coalgebras,

$$(\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha) \quad \text{and} \quad (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_{\gamma'}).$$

Note that γ' do not have to coincide with γ. Nevertheless, considering the claim to be true, they are seen to be homotopic by Theorem 2.16 and Theorem 2.15. Then, we can postcompose to get the desired map ϕ_p. Therefore, it suffices to prove the claim.

As in the proof of Theorem 2.6 we have a pull back diagram of symmetric sequences:

$$\begin{tikzcd}
\mathrm{End_p} \arrow[r, "i_A", "\sim"'] \arrow[d, "i_B"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{End_A} \arrow[d, "p_*"] \\
\mathrm{End}_{H_*\mathrm{A}} \arrow[r, "p^*", "\sim"'] & \mathrm{Hom}_{A,H_*A}
\end{tikzcd}$$

The vertical maps are levelwise surjections and thus, fibrations. The lower horizontal map is a levelwise weak equivalence (explicit calculation, H_*A is a deformation retract of A). The same holds for the upper one by right properness of $\mathcal{C}h$. Since $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$ is a cofibrant operad and End_p is fibrant (all chain complexes are fibrant), we can therefore conclude that there is a map

$$\phi\colon \mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*) \to \mathrm{End}_\mathrm{p}$$

that induces a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-structure on A homotopic to the original one. Via the map ϕ we can equip H_*A with a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure, and p can be extended to a map between the involved quasi-cofree coalgebras by similar arguments as in the previous theorem. This completes the proof of the claim. □

Finally, we want to mention a non-surprising fact. Assume that P is a graded operad. In particular, it is (isomorphic to) the homology operad of $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$, and is therefore acting on H_*A. We keep the notation α for the $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra structure of A, and γ for the structure on H_*A transferred by the map i. Then the P-algebra structures on H_*A induced by α on H_*A and γ on $H_*H_*A = H_*A$ coincide. To see this, note that the map ϕ_i given by Theorem 2.16 induces on homology the map $H_*(\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)(\phi_i)) = H_*(i) = \mathrm{id}$ and this gives us the isomorphism between H_*A with the different actions.

This is the right time to recall the following definition:

Definition 2.19. (Formal algebra) Let A be a P- or more generally a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra. The homology H_*A has an induced P-algebra structure through which we see H_*A as a $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebra. The algebra A is called formal, if in the homotopy category of $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-algebras it is isomorphic to H_*A (with the above $\mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$-structure).

3 Operadic cohomology

For a Σ_*-cofibrant operad P and a P-algebra A we can define *operadic cohomology groups* of A with coefficients in an A-representation M. For particular operads, such as Ass, Com or Lie the operadic cohomology has been known for a long time. There we have Hochschild, Harrison (André-Quillen) and Eilenberg-Chevalley cohomology (the latter two in characteristic 0 since the operads Com and Lie are not Σ_*-cofibrant in positive characteristic). In the topological setting we have further examples such as Topological Hochschild and Topological André-Quillen cohomology for ring spectra respectively commutative ring spectra. First references of the cohomology groups in the more general, operadic setting, go back to Balavoine and Hinich (cf. [Bal] and [Hin]). A more recent reference is Fresse's "Modules over operads and functors" [Fre09b]. The obstruction theory we will develop in Section 4.1 is taking values in the so called Gamma cohomology of A. This is the operadic cohomology of A viewed as a P_∞-algebra.

The aim of this section is to define Gamma cohomology. First we collect the necessary terminology in order to be able to define operadic cohomology, and do so in the second section. There we also list some examples of operadic cohomology. Finally, we come to the precise definition of Gamma cohomology.

Unless stated otherwise we are working with operads in the category of chain complexes.

3.1 Representations, derivations and enveloping algebras

In 1.37 we defined what a derivation of a P-algebra is. This definition is just a special case of a derivation from a P-algebra A to an A-representation M.

Let M be a chain complex. We denote by $(A;M)_n$ the chain complex

$$\bigoplus_{k=1}^{n} A \otimes \cdots \otimes \underset{k}{M} \otimes \cdots \otimes A,$$

where in every summand we have $n-1$ copies of the algebra A and one copy of the given chain complex M. The symmetric group Σ_n acts from the left by permutation of factors. Further, write $S(\mathrm{P})(A;M)$ for the expression

$$\bigoplus_{n=0}^{\infty} \mathrm{P}(n) \otimes_{\Sigma_n} (A;M)_n.$$

Definition 3.1. (Representation) A chain complex M is a representation of a P-algebra A if it is equipped with a map of chain complexes $\mu_M\colon\ S(\mathrm{P})(A;M) \to M$ that makes the following two diagrams commute:

$$\begin{tikzcd}
S(\mathrm{P}\circ\mathrm{P})(A;M) \arrow[rr, "S(\mu_{\mathrm{P}})(\mathrm{id};\mathrm{id})"] \arrow[d, "\cong"'] & & S(\mathrm{P})(A;M) \arrow[dd, "\mu_M"] \\
S(\mathrm{P})(\mathrm{P}(A);S(\mathrm{P})(A;M)) \arrow[d, "S(\mathrm{P})(\mu_A;\mu_M)"'] & & \\
S(\mathrm{P})(A;M) \arrow[rr, "\mu_M"] & & M
\end{tikzcd}$$

$$\begin{tikzcd} S(\mathrm{I})(A;M) \arrow[r, "S(\eta)(\mathrm{id};\mathrm{id})"] \arrow[dr, "\cong"'] & S(\mathrm{P})(A;M) \arrow[d, "\mu_M"] \\ & M \end{tikzcd}$$

In the literature representations are also known as "operadic modules". If we take P to be the operad Com and A a differential graded commutative algebra, then the representations of A indeed coincide with A-modules. If, however, A is just a differential graded associative algebra then A-representations are A-bimodules. In general, A-representations are modules over the so called *enveloping algebra* of A, which we are going to define below. The simplest example of an A-representation is A itself.

Now we come back to the definition of a derivation with values in a representation.

Definition 3.2. (Derivation) Let A be a P-algebra and M an A-representation. A map

$$f \in \mathbf{Hom}(A, M)$$

is called a derivation if it satisfies

$$f(\mu_A(p \otimes a_1 \otimes \cdots \otimes a_n)) = \sum_{i=1}^{n} \pm \mu_M(p \otimes a_1 \otimes \cdots \otimes f(a_i) \otimes \cdots \otimes a_n)$$

for all $a_1, \ldots, a_n \in A$ and $p \in \mathrm{P}$.

We denote the set of derivations from A to M by $\mathbf{Der}(A, M)$. One can check that this set is actually a subcomplex of $\mathbf{Hom}(A, M)$. We are going to view $\mathbf{Der}(A, M)$ as a *cochain* complex by changing the grading by a sign. Observe that for a free or quasi-free algebra a derivation is determined by its values on the generators. In the cases of the operads Com and Ass we recover the standard notion of a derivation. As in the classical theory the functor $\mathbf{Der}(A, -)$ is corepresentable. The *module of Kähler differentials* $\Omega^1_{\mathrm{P}}(A)$ is an A-representation such that there is a natural isomorphism

$$\mathcal{R}ep^A_{\mathrm{P}}(\Omega^1_{\mathrm{P}}(A), M) \cong \mathbf{Der}(A, M).$$

Here $\mathcal{R}ep^A_{\mathrm{P}}$ denotes the category of representations of A and a map of representations is a map of chain complexes commuting with the additional structure.

For the sake of completeness we give the definition of the enveloping algebra of a P-algebra A:

Definition 3.3. (Enveloping algebra) The enveloping algebra of A, denoted A^{en}, is the coequalizer

$$S(\mathrm{P})(\mathrm{P}(A); \mathbb{I}) \rightrightarrows S(\mathrm{P})(A; \mathbb{I}) \to A^{en}$$

in the category $\mathcal{C}h$, where the maps in the coequalizer are induced by μ_A and μ_{P}, respectively.

Remark 3.4. This object can be provided with an associative unital multiplication

$$A^{\mathrm{en}} \otimes A^{\mathrm{en}} \to A^{\mathrm{en}}.$$

The modules over A^{en} are precisely the A-representations. For more details and proofs of the statements, see Section 1 of [GH].

We want to mention that in the case $\mathrm{P} = \mathrm{Com}$ or Ass we get back the classical definition of an enveloping algebra. Let A_+ denote the algebra formed from the (non-unital) algebra A by the addition of a unit. Then the enveloping algebra of a commutative algebra A is given by A_+. If A is an associative algebra then A^{en} is given by $A_+ \otimes A_+^{\mathrm{op}}$.

3.2 Operadic cohomology and Gamma cohomology

In this section P is a Σ_*-cofibrant operad. We need this assumption in order to ensure that the category of P-algebras has a structure of a semi-model category (cf. Theorem 2.3). Further, $co\mathcal{C}h$ denotes the category of cochain complexes.

Recall that for a model category $\mathcal{M}$ and an $A \in \mathcal{M}$ the overcategory of objects over A, which we denote by $\mathcal{M}_{/A}$, inherits a model category structure with fibrations, cofibrations and weak equivalences created by the forgetful functor to $\mathcal{M}$. It is an easy verification that the same statement holds if we exchange "model" by "semi-model". Consequently, if A is a P-algebra then with P-alg also the category of P-algebras over A is a semi-model category. We denote this category by P-alg$_{/A}$, and write (B, f) for the object given by a P-algebra B together with a map f to A.

Proposition 3.5 (13.1.2 of [Fre09b])**.** *The functor* $\mathbf{Der}(-, M)$: P-*alg*$_{/A} \to co\mathcal{C}h$ *given by*

$$(B, f) \mapsto \mathbf{Der}(B, M)$$

preserves weak equivalences between cofibrant objects for every A-representation M. Here, M is viewed as a B-representation via the map

$$B \xrightarrow{f} A.$$

In particular, $\mathbf{Der}(-, M)$ *induces a functor* $\mathbf{Der}^h(-, M)$ *on the homotopy categories.*

The operadic cohomology groups of A are defined as the cohomology groups of the derived functor $\mathbf{Der}^{\mathrm{h}}(-, M)$ evaluated at the object (A, id).

Definition 3.6. (Operadic cohomology) For a P-algebra A we define the operadic cohomology groups $H^*_{\mathrm{P}}(A; M)$ with coefficients in an A-representation M as the cohomology groups of the cochain complex $\mathbf{Der}(Q_A, M)$, where

$$Q_A \xrightarrow{i_A} A$$

is a cofibrant replacement of A in the category of P-algebras, and thus, a cofibrant replacement of (A, id) in the overcategory.

Remark 3.7. If we have another Σ_*-cofibrant operad Q and a weak equivalence $\mathrm{Q} \xrightarrow{\sim} \mathrm{P}$ then A can be seen as a Q-algebra. By Theorem 2.4 the restriction and extension functors induce a Quillen equivalence between the categories of Q- and P-algebras. In particular, it is not hard to see that the operadic cohomology groups of A as a Q-algebra are isomorphic to the cohomology groups of A as a P-algebra.

Definition 3.8. (Gamma cohomology) Let A be an algebra over a not necessarily Σ_*-cofibrant operad Q. The Gamma cohomology groups of A with coefficients in an A-representation M are defined by

$$H\Gamma^*(A; M) := H^*_{\tilde{\mathrm{Q}}}(A; M),$$

where $\tilde{\mathrm{Q}}$ is some Σ_*-cofibrant replacement of the operad Q, and A is viewed as a $\tilde{\mathrm{Q}}$-algebra.

By the above remark this definition gives a well-defined notion of Γ-cohomology. We will use the notation $H^*_{\mathrm{P}}(A)$ respectively $H\Gamma^*(A)$ to denote the cohomology groups with coefficients in the A-representation A.

The notion "Gamma (co)homology" was established by Alan Robinson to denote the (co)homology theory for graded commutative and more generally E_∞-algebras, developed by him in the early nineties (cf. [RW] and [Rob]). Robinson defines his (co)homology groups by giving an explicit chain complex for calculating them. In [Hoff10] it is shown that these chain complexes indeed calculate the operadic Gamma homology and cohomology for the operad Com. The definition of operadic homology is not recalled here since we do not need it in this thesis, but it can be found for example in [Fre09b] §13.1.3.

Remark 3.9. There are other well-known theories for commutative algebras such as André-Quillen (cf. [And] and [Qui]) and Harrison (cf. [Harr]). One could ask what their relation to Gamma cohomology is. A result of Whitehouse [Whi] states that Gamma cohomology is isomorphic to Harrison and André-Quillen cohomology when k is a commutative ring containing $\mathbb{Q}$. We should mention that in this case André-Quillen and Harrison cohomology groups coincide up to a degree shift. In positive characteristic Gamma cohomology is different in general. It can be seen as the "correct" algebraic version of Topological André-Quillen cohomology. A nice overview on this topic can be found in [BR].

Let us take a short look at the associative operad. It is Σ_*-cofibrant and therefore we can associate to a differential graded algebra A and some A-representation M its operadic cohomology, which coincides by definition with its Gamma cohomology. There is a particular cofibrant replacement for A arising from its *Koszul* complex that identifies up to degree shift Hochschild and operadic cohomology of A. Similarly, when the characteristic of k is zero and therefore the operads Com and Lie are Σ_*-cofibrant, the operadic cohomology recovers Harrison respectively Eilenberg-Chevalley cohomology up to degree shift.

At the end we want to mention that the approach of operadic cohomology groups that we recalled in the dg setting has also been developed in the topological framework. There, Basterra [Bas] first defined Topological André-Quillen (co)homology for commutative $\mathbb{S}$-algebras mimicking the definitions in the algebraic setting. For (not necessarily commutative) $\mathbb{S}$-algebras Lazarev [Laz] shows a close relationship between Topological Hochschild (co)homology and a version of operadic (co)homology for $\mathbb{S}$-algebras that he calls "topological derivations". More recently, Basterra and Mandell [BasMan] gave a general definition for operadic cohomology of operadic algebras in the category of $\mathbb{S}$-modules (under some appropriate conditions on the operad).

4 Universal class for operadic algebras

This is the main part of this thesis. We are going to develop an obstruction theory for formality of algebras over a cofibrant chain operad over a field (of possibly positive characteristic). In the first section we state and prove the main theorems as well as give some straightforward implications of these. Further, we discuss a criterion for detecting if two given operadic algebras are connected by a zig-zag of weak equivalences. In the second section we compare our results with earlier ones. In the last section we illustrate the theory on examples and discuss some applications.

4.1 Obstruction theory

Let k be a field of an arbitrary characteristic and P a differential graded operad with trivial differentials. This implies that for each algebra A over P, the homology H_*A is also an algebra over P. We are in particular interested in the case of the commutative operad Com but the theory we develop works also in this higher generality. The first aim of this section is to construct a canonical class

$$\gamma_A^{[2]} \in H\Gamma^1(H_*A; H_*A) = H^1_{\mathrm{P}_\infty}(H_*A)$$

for every P-algebra or more generally P_∞-algebra A. The class takes values in the first Gamma cohomology group of H_*A and depends only on the weak equivalence type of the algebra. We are working with a particular cofibrant replacement of P given by

$$\mathrm{P}_\infty = \mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*).$$

Let A be a P_∞-algebra. Recall that the P_∞-structure on A is given by a twisting cochain, i.e., a homomorphism

$$\alpha\colon \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ A \to A$$

of degree -1 such that the following equality holds (cf. Propositions 1.44 and 1.45):

$$(\star)\qquad \delta(\alpha)\left(\begin{array}{c} a_1 \quad\cdots\cdots\quad a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) + \sum_{\nu_2'(\wp)} \pm\alpha\left(\begin{array}{c} \alpha\left[\begin{array}{c} a_* \quad\cdots\cdots\quad a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots\cdots \qquad\qquad \cdots\cdots a_* \\ \diagdown \qquad | \qquad \diagup \\ \wp' \\ | \end{array}\right) = 0$$

By Theorem 2.6, H_*A can also be equipped with a P_∞-algebra structure given by a twisting cochain

$$\gamma\colon \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A \to H_*A,$$

such that H_*A together with γ is isomorphic to A in the homotopy category of P_∞-algebras.

The coalgebra $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$ can be written as a direct sum (not respecting the differential)

$$\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A = \bigoplus_{s \geq 0} \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A$$

where $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A$ is the part generated by trees with the sum of the number of vertices and the degrees of the elements of $\mathrm{E}\Sigma_*$ equal to s. For example, $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A$

for $s = 0, 1, 2$ is generated by trees of the form

$$\underbrace{\left\{ \begin{array}{c} a \\ | \\ | \end{array} \right\}}_{s=0} \qquad \underbrace{\left\{ \begin{array}{c} a_* \cdots\cdots\cdots a_* \\ \diagdown \qquad \diagup \\ \Sigma(p \otimes e) \\ | \end{array} , \quad e \in (\mathrm{E}\Sigma_*)_0 \right\}}_{s=1}$$

$$\underbrace{\left\{ \begin{array}{c} a_* \cdots\cdots\cdots a_* \\ \diagdown \qquad \diagup \\ \Sigma(p \otimes e) \\ | \end{array} , \quad \begin{array}{c} a_* \cdots a_* \\ \diagdown \quad \diagup \\ a_* \cdots \Sigma(p_2 \otimes e_2) \cdots a_* \\ \diagdown \qquad | \qquad \diagup \\ \Sigma(p_1 \otimes e_1) \\ | \end{array} , \quad e \in (\mathrm{E}\Sigma_*)_1 \, , \, e_1, e_2 \in (\mathrm{E}\Sigma_*)_0 \right\}}_{s=2} ,$$

respectively. We have a decomposition

$$\gamma = \bigoplus \gamma^{[s]} \colon \bigoplus_{s \geq 0} \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A \to H_*A.$$

On the other hand, the homology H_*A of A is itself a $H_*\mathrm{P}_\infty$-algebra, i.e., a P-algebra. Hence, we have another P_∞-algebra structure on H_*A given by

$$\mathrm{P}_\infty = \mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \xrightarrow{\sim} \mathrm{P} \xrightarrow{H_*(\alpha)} \mathbf{End}(H_*A),$$

where the second map $H_*(\alpha)$ denotes the P-algebra structure on H_*A induced by the twisting cochain α (cf. Example 1.11 and Remark 1.19). As we know a map from the Cobar construction is uniquely determined by a map on the generators (that gives us the corresponding twisting cochain). In the case above, if we review the definition of the weak equivalence we get the following on generators:

$$\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \xrightarrow{pr} \mathrm{P} \otimes \mathrm{E}\Sigma_* \xrightarrow{\mathrm{id} \otimes \epsilon} \mathrm{P} \otimes k \xrightarrow{\cong} \mathrm{P} \xrightarrow{H_*(\alpha)} \mathbf{End}(H_*A)$$

At the end of Section 2.3 we argued that both P_∞-structures $H_*(\alpha)$ and $H_*(\gamma)$ on H_*A coincide. In other words, restricted to $\mathrm{P} \otimes (\mathrm{E}\Sigma_*)_0$ the above map equals $\gamma^{[1]}$, and elsewhere it is 0.

By abuse of notation we are going to denote the twisting cochain corresponding to this P_∞-structure on H_*A just by $\gamma^{[1]}$. Further, we will refer to it as the "strict" P_∞-structure on the homology. The reader can keep in mind the example of the commutative operad. The homology of an E_∞-algebra is a priori a graded commutative algebra, that of course can be viewed as an E_∞-algebra again. We call this structure the strict one and denote the corresponding twisting cochain by $\gamma^{[1]}$. The E_∞-structure given by Theorem 2.6 is denoted by γ.

Now we are ready to construct the canonical class:

Construction 4.1. As above, the P_∞-algebra structure on A is given as a cochain

$$\alpha \colon \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ A \to A,$$

and denote the transferred P_∞-algebra structure on H_*A by

$$\gamma = \bigoplus_{s \geq 0} \gamma^{[s]}.$$

We claim that $\gamma^{[2]}$ defines a cohomology class in $H^1_{\mathrm{P}_\infty}(H_*A)$, the Gamma cohomology of the strict P_∞-algebra H_*A with coefficients in itself.

Recall that operadic cohomology $H^*_{\mathrm{P}_\infty}(A; M)$ is given by the cohomology groups of the cochain complex $\mathbf{Der}_{\mathrm{P}_\infty}(Q(A), M)$ where $Q(A)$ is a cofibrant replacement of A in the category of P_∞-algebras. As Theorem 2.11 states, we can take

$$R_{H_*A} := R_{\mathrm{P}_\infty}\left(\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A, \partial_{\gamma^{[1]}}\right)$$

as a cofibrant replacement of the homology algebra. Since R_{H_*A} is quasi-free, a derivation from it to H_*A is determined by a homomorphism

$$\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A \to H_*A.$$

A priori the domain of $\gamma^{[2]}$ is $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[2]} \circ H_*A$ but we extend it trivially on the remaining summands. In the cochain grading $\gamma^{[2]}$ becomes an element of degree 1. It is to verify that the differential of the cochain complex of derivations sends $\gamma^{[2]}$ to zero.

The total differential on R_{H_*A} is of the form

$$\partial_{R_{H_*A}} = \delta + \partial_{\gamma^{[1]}} + \partial_\omega,$$

where δ is induced by the (internal) differentials of P_∞, $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$ and H_*A, $\partial_{\gamma^{[1]}}$ comes from the twisting differential defined by $\gamma^{[1]}$ on $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$ and $\partial_\omega|_{\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A}$ is given by the coproduct of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$ followed by the inclusion of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$ into P_∞. The differential of H_*A is trivial, and therefore calculating the differential of $\gamma^{[2]}$ in the derivation complex restricts to precomposing with the differential of R_{H_*A}. Note that we only need to take care about the restriction of $\partial_{R_{H_*A}}$ to $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$, since the latter defines the resulting derivation uniquely.

Let us separately look at the three summands of $\partial_{R_{H_*A}}$ applied to $\gamma^{[2]}$ and evaluated on an element of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$.

$$\delta(\gamma^{[2]})\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) = \gamma^{[2]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{array}\right),$$

where $\partial(\wp)$ denotes the differential in $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$. We do not have further summands since H_*A is a chain complex with trivial differentials.

$$\partial_{\gamma^{[1]}}(\gamma^{[2]})\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right)$$

$$= \sum_{i=1}^{n} \pm\gamma^{[2]}\left(\begin{array}{c} a_1 \cdots \gamma^{[1]}(a_i) \cdots a_n \\ \diagdown \quad | \quad \diagup \\ \wp \\ | \end{array}\right) + \sum_{\nu_2'(\wp)} \pm\gamma^{[2]}\left(\begin{array}{ccc} & \gamma^{[1]}\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & & \diagup \\ & \wp' & \\ & | & \end{array}\right)$$

$$= \sum_{\nu_2'(\wp)} \pm\gamma^{[2]}\left(\begin{array}{ccc} & \gamma^{[1]}\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & & \diagup \\ & \wp' & \\ & | & \end{array}\right)$$

Here, the second equality holds since $\gamma^{[1]}$ is trivial on H_*A. For the last term we get

$$\partial_\omega(\gamma^{[2]})\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu(\wp)} \tilde{\gamma}^{[2]}\left(\begin{array}{ccc} a_* \cdots a_* & & a_* \cdots a_* \\ \diagdown \ \diagup & & \diagdown \ \diagup \\ \wp''_* & \cdots\cdots & \wp''_* \\ \diagdown & & \diagup \\ & \iota(\wp') & \\ & | & \end{array}\right),$$

where $\tilde{\gamma}^{[2]}$ denotes the derivation induced by $\gamma^{[2]}$ and ι the inclusion of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$ into P_∞. To calculate the above term we have to take into account the R_{H_*A}-representation structure of H_*A. It is given by the projection

$$R_{H_*A} \to H_*A$$

and the strict P_∞-algebra structure on H_*A. Combining this with the observation that $\gamma^{[2]}$ acts trivially on H_*A we see that the sum over ν reduces to:

$$\partial_\omega(\gamma^{[2]})\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) = (-1)^2 \sum_{\nu_2'(\wp)} \pm\gamma^{[1]}\left(\begin{array}{ccc} & \gamma^{[2]}\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & & \diagup \\ & \wp' & \\ & | & \end{array}\right)$$

Note the two additional minus signs on the right hand side. The first one comes from the fact that ι contains a dimension shift on $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$, and we have to pass $\gamma^{[2]}$ by $\iota(\wp')$. The second one is the minus sign occurring in the passage from operadic twisting morphisms to twisting homomorphisms of coalgebras (cf. the proof of Proposition 1.45).

Let us first assume that $\wp$ is in $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]}$ for some s different from 3. Then we have

$$\big(\delta + \partial_{\gamma^{[1]}} + \partial_\omega\big)(\gamma^{[2]}) \left(\begin{array}{ccc} a_1 & \cdots\cdots & a_n \\ & \diagdown \quad \diagup & \\ & \wp & \\ & | & \end{array} \right) = 0$$

since $\gamma^{[2]}$ vanishes on

$$\partial(\wp) \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s-1]}$$

for $s - 1 \neq 2$. Further, for the factors

$$\wp' \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[p]} \text{ and } \wp'' \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[q]}$$

in the quadratic coproduct ν_2' we have $p+q = s$. Hence, at least one of $\gamma_{[1]}$ and $\gamma_{[2]}$ acts trivially and thus, the whole sum is zero.

For $\wp \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[3]}$ we get exactly the left hand side of the twisting cochain equality $(\star)$ on p.45 for γ in place of α. Hence, $\gamma^{[2]}$ defines a cohomology class in the operadic cohomology groups as claimed.

Now we come to the first main result of this section:

Theorem 4.2. *Let* P *be a graded operad in* $\mathcal{C}h$ *and* A *a* P_∞*-algebra. There is a canonical class* $\gamma_A^{[2]} \in H^1_{\mathrm{P}_\infty}(H_*A)$ *in the Gamma cohomology of the strict* P_∞*-algebra* H_*A*, such that*

(i) if f *is a map of* P_∞*-algebras from* A *to* B*, and we denote by* f^* *respectively* f_* *the induced maps on cohomology groups as depicted below:*

$$\begin{array}{ccc} H^1_{\mathrm{P}_\infty}(H_*A, H_*A) & \xrightarrow{f_*} & H^1_{\mathrm{P}_\infty}(H_*A, H_*B) \\ & & \uparrow{\scriptstyle f^*} \\ & & H^1_{\mathrm{P}_\infty}(H_*B, H_*B) \end{array}$$

then we have $f_*(\gamma_A^{[2]}) = f^*(\gamma_B^{[2]})$*. In particular, if* f *is a weak equivalence then via the induced isomorphisms* f_* *and* f^* *on operadic cohomology the classes* $\gamma_A^{[2]}$ *and* $\gamma_B^{[2]}$ *coincide.*

(ii) if g^c *is the map on cofibrant operads*

$$g^c \colon \mathcal{B}^c\mathcal{B}(\mathrm{Q} \otimes \mathrm{E}\Sigma_*) \to \mathcal{B}^c\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$$

induced by a map $g \colon \mathrm{Q} \to \mathrm{P}$ *of graded operads then* A *is also a* Q_∞*-algebra, and under the map induced by* g^c *on operadic cohomology groups the canonical class of* A *as a* P_∞*-algebra is mapped to the canonical class of* A *as a* Q_∞*-algebra.*

Proof. We first have to check that the class of Construction 4.1 is well-defined. We keep the notation used there. Let us suppose that we are given another transferred P_∞-algebra structure on the homology algebra, and let us denote this by

$$\gamma' = \bigoplus_{s \geq 0} \gamma'^{[s]}.$$

Since γ and γ' are homotopic, by Theorem 2.15 there is a morphism between quasi-cofree coalgebras

$$(\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_\gamma) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_{\gamma'})$$

induced by a homomorphism

$$f = \bigoplus_{s\geq 0} f^{[s]} \colon \bigoplus_{s\geq 0} \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A \to H_*A$$

with $f^{[0]}$ the identity map. As a morphism between quasi-cofree coalgebras, f satisfies the formula

$$(\star\star) \qquad \delta(f)\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) - \sum_{\nu_2'(\wp)} \pm f\left(\begin{array}{c} \gamma\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots\cdots \qquad | \qquad \cdots\cdots a_* \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{array}\right)$$

$$+ \sum_{\nu(\wp)} \gamma' \left(\begin{array}{c} f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \cdots\cdots f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{array}\right) = 0$$

by Proposition 1.47.

If we write down this equation for elements $\wp \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[2]}$ we get the following, after rearranging summands and exploiting the facts that $f^{[0]}$ is the identity and $\gamma^{[0]}$ and $\gamma'^{[0]}$ are zero:

$$-(-1)^{|f|} f^{[1]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{array}\right) - \sum_{\nu_2''(\wp)} \pm f^{[1]}\left(\begin{array}{c} \gamma^{[1]}\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots\cdots \qquad | \qquad \cdots\cdots a_* \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{array}\right)$$

$$+ \sum_{\substack{\nu(\wp) \\ \wp' \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[1]}}} \gamma'^{[1]} \left(\begin{array}{c} f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \cdots\cdots f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{array}\right) =$$

$$\gamma^{[2]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \;\; \diagup \\ \wp \\ | \end{array}\right) - \gamma'^{[2]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \;\; \diagup \\ \wp \\ | \end{array}\right)$$

The left hand side is quickly seen to be the differential of $f^{[1]}$ in $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$ when extending $f^{[1]}$ trivially on $\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s]}$ for s different from 1. Therefore $\gamma^{[2]}$ and $\gamma'^{[2]}$ define the same class in the operadic cohomology of H_*A.

We come to the proof of part (i). Suppose f is a morphism

$$f\colon A \to B$$

of P_∞-algebras. This means that f is a morphism of chain complexes that commutes with all the operations of P_∞ (cf. Definition 1.20). It induces a map of P-algebras

$$H_*f\colon H_*A \to H_*B.$$

By abuse of notation we write f_* respectively f^* for the maps induced by H_*f on the derivation complexes and on the operadic cohomology. We have to compare the homology classes of

$$f_*(\gamma^{[2]}) = H_*f \circ \gamma^{[2]} \quad \text{and} \quad f^*(\gamma'^{[2]}) = \gamma'^{[2]} \circ \phi_{H_*f}$$

in the cochain complex of derivations

$$\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*B),$$

where ϕ_{H_*f} denotes the map on cofibrant replacements induced by H_*f. Here γ and γ' denote the transferred P_∞-structures on H_*A and H_*B, respectively. By Theorems 2.16 and 2.18, and Proposition 1.46 we have maps of quasi-cofree coalgebras

$$\phi_i\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_\gamma) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha)$$
$$\phi_p\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ B, \partial_\beta) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*B, \partial_{\gamma'})$$
$$\phi_f\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ A, \partial_\alpha) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ B, \partial_\beta).$$

After composing these maps we get a map of quasi-cofree coalgebras

$$f'\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A, \partial_\gamma) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*B, \partial_{\gamma'}).$$

The zeroth component $f'^{[0]}$ of f' coincides with H_*f. As in the proof of the fact that the canonical class is well-defined, one can calculate that $f'^{[1]}$ defines a derivation of R_{H_*A} with coefficients in H_*B that bounds

$$f'^{[0]} \circ \gamma^{[2]} - \gamma'^{[2]} \circ \phi_{f'^{[0]}} = f_*(\gamma^{[2]}) - f^*(\gamma'^{[2]}).$$

Clearly, if f is a weak equivalence, f_* and f^* induce isomorphisms on cohomology groups through which we can identify the canonical classes of A and B.

For part (ii) of the theorem we are looking at the map

$$\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A) \xrightarrow{g^*} \mathbf{Der}_{\mathrm{Q}_\infty}(R^Q_{H_*A}, H_*A),$$

where the upper index of $R^Q_{H_*A}$ indicates that we are taking the (chosen) cofibrant replacement of H_*A as a Q_∞-algebra, and g^* denotes the precomposition with the map induced by g on cofibrant replacements. If γ denotes the transferred twisting cochain of the homology H_*A as a P_∞-algebra then

$$\gamma\big(\mathcal{B}(g\otimes \mathrm{id})\circ \mathrm{id}\big)$$

gives the structure of H_*A as a Q_∞-algebra and hence, we are done.

$\square$

A direct consequence of the theorem is that if A is quasi-isomorphic as a P_∞-algebra to a P_∞-algebra with trivial canonical class then $\gamma_A^{[2]}$ is also trivial.

Remark 4.3. The cochain complex $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$ is in fact a double cochain complex. First we can introduce a direct sum decomposition

$$\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A = \bigoplus_{t \in \mathbb{Z}} (\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A)_t$$

by taking into account only the degrees of the elements of H_*A and P. For instance a tree of the form

$$\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}$$

where $\wp \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[s]}$ is in the t-th summand if and only if the sum of degrees

$$|a_1| + \cdots + |a_n| + |\wp| - s$$

equals t. Recall that the Bar construction contains a shift on cogenerators. Therefore, the degree of a tree is given by the sum of the degrees of the elements on the tree *plus* the number of vertices. In the example, $|\wp| - s$ gives exactly the sum of the degrees of the elements of P. We view the above direct sum as a chain complex with trivial differentials. We get a (chain) grading on the set of derivations given by the degree of maps with respect to the new decomposition of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A$ defined above. We switch to cochain grading in order to get a cochain complex (with trivial differentials). A simple verification shows that the differential of $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$ respects this new grading (the differential coincides with the one on the Hom complex $\mathbf{Hom}(R_{H_*A}, H_*A)$). In particular, we get a double cochain complex

$$\mathbf{Der}^{p,q}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$$

where $p + q$ gives the total degree of a derivation, and q gives the degree with respect to the splitting introduced at the beginning of this remark. Since one of the differentials is trivial, we also get a grading on the cohomology groups:

$$H^*_{\mathrm{P}_\infty}(H_*A) = \bigoplus_{s \in \mathbb{Z}} H^{s,*-s}_{\mathrm{P}_\infty}(H_*A)$$

Now we can be more precise about where the canonical class lives: Since it is represented by a map

$$\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[2]} \circ H_*A \to H_*A$$

of degree -1, it lives in (chain) degree $(2-1) = 1$ with respect to the second grading. This means that in the double cochain complex we end up in degree $(2, -1)$. All together, we conclude

$$\gamma_A^{[2]} \in H^{2,-1}_{\mathrm{P}_\infty}(H_*A).$$

The canonical class of Theorem 4.2 is the first obstruction to formality. If it happens to vanish then we have a successive obstruction. The latter is a *set* of cohomology classes rather than a *single* class. We are going to define an equivalence relation on the first Gamma cohomology group. The higher obstructions take values in the resulting quotient group.

Construction 4.4. Let $H^1_{\mathrm{P}_\infty}(H_*A) = \bigoplus_{s\in\mathbb{Z}} H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$ be the splitting of the first Gamma cohomology group of H_*A given by the previous remark. We define an equivalence relation on every sum factor for $s \geq 2$. Roughly speaking, we set classes to be equivalent if they can be represented by cochains that coincide on a certain type of trees. Precisely, two classes η and η' in $H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$ are equivalent if there exist representatives $\tilde{\eta}$ and $\tilde{\eta}'$, respectively,

$$\tilde{\eta}, \tilde{\eta}' \colon \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A \to H_*A$$

such that

$$\tilde{\eta}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right) = \tilde{\eta}'\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right)$$

for those $\wp \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s]}$, for which in the coproduct

$$\nu\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown\ \diagup \\ \wp \\ | \end{array}\right) = \sum_{\nu(\wp)} \left(\begin{array}{c} a_* \cdots\cdots a_* \qquad a_* \cdots\cdots a_* \\ \diagdown\ \diagup \qquad\qquad \diagdown\ \diagup \\ \wp''_* \qquad \cdots\cdots \qquad \wp''_* \\ \diagdown \qquad\quad \diagup \\ \wp' \\ | \end{array}\right),$$

holds: if $\wp'$ is an element of $\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[1]}$ then for the corresponding tensor factors $\wp''_*$ we have $\wp''_* \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s-1]}$ or $\wp''_* \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[0]}$. In plain words the representing cochains coincide on trees of the form

$$\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown\ \diagup \\ a_* \cdots \qquad \wp'' \qquad \cdots a_* \\ \diagdown \qquad | \qquad \diagup \\ \wp' \\ | \end{array}$$

where $\wp' \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[1]}$, as well as arbitrary trees whose bottom vertex is labeled by an element $\Sigma(p\otimes e)$ with $|e| > 0$. We obtain an equivalence relation on every group $H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$ for s greater or equal to 2. For $s = 2$ the condition on $\wp$ is empty, i.e., the resulting quotient group is $H^{2,-1}_{\mathrm{P}_\infty}(H_*A)$ itself. For higher s we can not assume this a priori. We write $\tilde{H}^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$ for the quotient.

Similarly, we use the notation $\tilde{H}^{s,1-s}_{\mathrm{P}_\infty}(H_*A, H_*B)$ for the above construction applied to the direct sum factors of the group $H^1_{\mathrm{P}_\infty}(H_*A, H_*B)$ appearing in Theorem 4.2 (i).

Note that the elements of $\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*) \circ H_*A$ on which representing cocycles are supposed to be equal (cobounded) do not form a subcomplex. Otherwise, we would have been able to rewrite the above quotient groups as Gamma cohomology groups of a certain P_∞-algebra.

In the previous theorem we constructed for a given P_∞-algebra A a canonical class

$$\gamma^{[2]}_A \in \tilde{H}^{2,-1}_{\mathrm{P}_\infty}(H_*A).$$

Now we want to define inductively for $t \geq 2$ higher obstructions

$$\gamma^{[t]}_A \in \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A)$$

whenever all lower ones vanish (in a strong sense).

Construction 4.5. We proceed by induction. Again, let

$$\gamma = \bigoplus_{s \geq 0} \gamma^{[s]}$$

denote the transferred P_∞-structure on the homology of A. The idea is that the "higher" pieces of algebra structure $\gamma^{[s]}$ should correspond to the higher obstructions. They should define cohomology classes in $H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$. There is going to be some indeterminacy, though, which is why the actual obstruction lives in

$$\tilde{H}^{s,1-s}_{\mathrm{P}_\infty}(H_*A).$$

If all classes $\gamma_A^{[s]}$ for s less than or equal to some $t-1$ have been defined and can be chosen to be trivial as cohomology classes in $H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$, then we claim that there is a homotopic P_∞-algebra structure on H_*A, say γ', with $\gamma'^{[s]} = 0$ for $s = 2, \ldots, t-1$ and we can define the t-th canonical class of A to be represented by the cohomology class of the cocycle $\gamma'^{[t]}$.

There are two statements hidden in our claim. First, if the transferred P_∞-structure has the property that for every $s = 2, \ldots, t-1$ the map $\gamma^{[s]}$ is zero (i.e., already trivial as a cocycle), then $\gamma^{[t]}$ defines a cocycle in $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$ (and hence we can indeed define a cohomology class in $H^{t,1-t}_{\mathrm{P}_\infty}(H_*A)$). The proof of this is the same as the proof in 4.1 where we constructed the (first) canonical class of A. Second, assume $\gamma^{[s]}$ is zero for $s = 2, \ldots, t-1$ and $\gamma^{[t]}$ is different from zero but is a coboundary in the derivation complex, i.e., vanishes in Gamma cohomology. Then we have to show that there is a homotopic P_∞-structure γ' on H_*A with the property $\gamma'^{[s]} = 0$ for $s = 2, \ldots, t$. We are going to spend some time arguing why this last statement is true:

We have to give a homomorphism f that induces a map of quasi-cofree coalgebras

$$(\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A, \partial_\gamma) \to (\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*) \circ H_*A, \partial_{\gamma'}),$$

where $\gamma'^{[s]}$ equals to zero for $s = 2, \ldots, t$.

Since $\gamma^{[t]}$ is a coboundary in $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$ there is a map

$$f' \colon \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[t-1]} \circ H_*A \to H_*A$$

with

$$-(-1)^{|f'|} f' \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{pmatrix} - \sum_{\nu_2''(\wp)} \pm f' \begin{pmatrix} & \gamma^{[1]} \left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array} \right] & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{pmatrix}$$

$$+ \sum_{\substack{\nu_2' \\ \wp' \in \mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)^{[1]}}} \gamma^{[1]} \begin{pmatrix} & f' \left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array} \right] & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{pmatrix} = \gamma^{[t]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{pmatrix}$$

for elements $\wp$ in $\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[t]}$. Set f to be the map given by:

$$f^{[s]} = \begin{cases} \mathrm{id} & \text{for } s=0 \\ f'^{[s]} & \text{for } s=t-1 \\ 0 & \text{else} \end{cases}$$

We want to give H_*A a new P_∞-algebra structure, such that f is a map between the corresponding quasi-cofree coalgebras. By induction on d for $\wp \in \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[d]}$, using $(\star\star)$ on p.50, we define an isomorphic P_∞-structure γ' on H_*A. For $d=1$ we have

$$-\mathrm{id}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{array}\right) - \gamma^{[1]}\left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \; \diagup \\ \wp \\ | \end{array}\right) + \gamma'^{[1]}\left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \; \diagup \\ \wp \\ | \end{array}\right) = 0,$$

which reads as $\gamma^{[1]} = \gamma'^{[1]}$ since $\partial(\wp)$ is zero. Because of $f^{[s]}=0$ when s varies from 1 to $t-2$, for $d=2,\dots,t-1$ the equation $(\star\star)$ simplifies to $\gamma^{[d]} = \gamma'^{[d]}$ (and thus $=0$). Now set $\gamma'^{[t]}=0$ and observe that under this assignment the "border" case $d=t$ reduces to the equality we got from the condition $\partial f' = \gamma^{[t]}$ in $\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*A}, H_*A)$. For higher d we rewrite $(\star\star)$ so that we can inductively define $\gamma'^{[d]}$ by:

$$-\gamma'^{[d]}\left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \; \diagup \\ \wp \\ | \end{array}\right) = \delta(f)\left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \; \diagup \\ \wp \\ | \end{array}\right) - \sum_{\nu_2'(\wp)} \pm f\left(\begin{array}{c} \gamma\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots \qquad | \qquad \cdots a_* \\ \diagdown \quad\quad \diagup \\ \wp' \\ | \end{array}\right)$$

$$+ \sum_{\substack{\nu(\wp) \\ \wp' \notin \mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)^{[d]}}} \gamma'\left(\begin{array}{c} f\left[\begin{array}{c} a_* \cdots a_* \\ \diagdown \; \diagup \\ \wp''_* \end{array}\right] \cdots f\left[\begin{array}{c} a_* \cdots a_* \\ \diagdown \; \diagup \\ \wp''_* \end{array}\right] \\ \diagdown \quad\quad \diagup \\ \wp' \\ | \end{array}\right)$$

$$= -\gamma^{[d]}\left(\begin{array}{c} a_1 \cdots a_n \\ \diagdown \; \diagup \\ \wp \\ | \end{array}\right) - \sum_{\nu_2''(\wp)} \pm f\left(\begin{array}{c} \gamma\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots \qquad | \qquad \cdots a_* \\ \diagdown \quad\quad \diagup \\ \wp' \\ | \end{array}\right) +$$

$$+\sum_{\substack{\nu(\wp)\\ \wp'\notin\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)^{[d]}}} \gamma'\left(\begin{array}{c} f\begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown\ \diagup \\ \wp''_* \end{bmatrix} \cdots\cdots f\begin{bmatrix} a_*\cdots\cdots a_* \\ \diagdown\ \diagup \\ \wp''_*\end{bmatrix} \\ \diagdown\qquad\diagup \\ \wp' \\ | \end{array}\right).$$

We have to argue why γ' does define a twisting cochain on $\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A$. Remember that the twisting cochain condition says nothing more than that γ' defines a twisting differential $\partial_{\gamma'}$ on $\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A$. Since γ does so and ϕ_f is a morphism making the diagram

$$\begin{tikzcd}
\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A \arrow[r, "\phi_f"] \arrow[d, "\delta+\partial_\gamma"] \arrow[dd, bend right=60, "0"'] & \mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A \arrow[d, "\delta+\partial_{\gamma'}"] \\
\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A \arrow[r, "\phi_f"] \arrow[d, "\delta+\partial_\gamma"] & \mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A \arrow[d, "\delta+\partial_{\gamma'}"] \\
\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A \arrow[r, "\phi_f"] & \mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)\circ H_*A
\end{tikzcd}$$

commute, it suffices to note that ϕ_f is surjective. To see the surjectivity one uses the construction of ϕ_f out of f given after Proposition 1.46, and the fact that $f^{[0]}$ is the identity.

Finally, we have to say a few words about the indeterminacy of the cohomology class we defined. If there are two different (but homotopic) P_∞-algebra structures γ' and γ'' on H_*A then they are connected by a map f, and respectively by a morphism of quasi-cofree coalgebras ϕ_f with $f^{[0]}=\mathrm{id}$. If both γ' and γ'' satisfy the condition $\gamma'^{[s]}=0=\gamma''^{[s]}$ for $s=2,\dots,t-1$ the equality $(\star\star)$ for $\wp\in\mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)^{[s]}$ reduces to the statement that in the derivation complex $f^{[t-1]}$ bounds the difference of $\gamma'^{[t]}$ and $\gamma''^{[t]}$ on exactly those kind of trees that we used for the definition of $\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A)$. Therefore, we get a well-defined element

$$\gamma^{[t]}_A\in\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A).$$

The reason why we do not know if $\gamma'^{[t]}$ and $\gamma''^{[t]}$ define the same class in $H^{t,1-t}_{\mathrm{P}_\infty}(H_*A)$ is that we do not know if we can choose f in a way that $f^{[s]}=0$ for $s=1,\dots,t-2$.

Theorem 4.6. *Let* P *be a graded operad in* $\mathcal{C}h$, *and* A' *and* A'' *two* P_∞*-algebras. Suppose that there are transferred* P_∞*-structures* γ' *and* γ'' *on* H_*A' *and* H_*A''*, respectively, with the property* $\gamma'^{[s]}=\gamma''^{[s]}=0$ *for* $s=2,\dots,t-1$. *Then there are successive obstructions* $\gamma^{[t]}_{A'}\in\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A')$ *and* $\gamma^{[t]}_{A''}\in\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A'')$ *for which the following holds:*

(i) if f *is a map of* P_∞*-algebras from* A' *to* A''*, and we denote by* f^* *respectively* f_* *the induced maps on the quotients of cohomology groups as depicted below:*

$$\begin{tikzcd}
\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A',H_*A') \arrow[r, "f_*"] & \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A',H_*A'') \\
 & \tilde{H}^{t,1-t}_{\mathrm{P}_\infty}(H_*A'',H_*A'') \arrow[u, "f^*"]
\end{tikzcd}$$

then we have $f_*(\gamma^{[t]}_{A'})=f^*(\gamma^{[t]}_{A''})$. *In particular, if* f *is a weak equivalence then via the induced isomorphisms* f_* *and* f^* *the obstructions* $\gamma^{[t]}_{A'}$ *and* $\gamma^{[t]}_{A''}$ *coincide.*

(ii) if g^c is the map on cofibrant operads

$$g^c\colon \mathcal{B}^c\mathcal{B}(\mathrm{Q}\otimes \mathrm{E}\Sigma_*) \to \mathcal{B}^c\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)$$

induced by a map $g\colon \mathrm{Q}\to \mathrm{P}$ of graded operads then A' is also a Q_∞-algebra, and under the map induced by g^c on operadic cohomology the obstruction of A' as a P_∞-algebra is mapped to the obstruction of A' as a Q_∞-algebra.

Proof. Using the definition of Construction 4.5 one only has to adapt the proof of the previous theorem to this setting. Note, that the passage from $H^{t,1-t}_{\mathrm{P}_\infty}$ to $\tilde{H}^{t,1-t}_{\mathrm{P}_\infty}$ is compatible with the maps induced by f and g on cohomology. □

Immediate consequences of both theorems are the following:

Corollary 4.7. *Let A be a P_∞-algebra such that the cohomology group $H^1_{\mathrm{P}_\infty}(H_*A)$ of the strict P_∞-algebra H_*A vanishes. Then A is quasi-isomorphic as a P_∞-algebra to its homology equipped with the strict P_∞-structure. In other words, A is formal.*

Corollary 4.8. *Let V be a P-algebra with trivial differentials. If the first operadic cohomology of V viewed as a P_∞-algebra vanishes then there is only one homotopy type of P_∞-algebras with homology isomorphic to V, i.e., every P_∞-algebra with homology given by V is isomorphic to V in the homotopy category of P_∞-algebras.*

The last two results are mainly of theoretical value. Gamma cohomology groups of graded algebras are difficult to calculate and usually huge. If one wants to do calculations and compare explicit obstructions, then it can be reasonable to examine the question if there is a coboundary between two cochains rather than trying to calculate the corresponding cohomology groups. We are going to see some examples later on.

We want to note that

Remark 4.9. If the operad P is Σ_*-cofibrant then we do not need to take the tensor product of P with the Barratt-Eccles operad but can also work just with the cofibrant replacement $\mathcal{B}^c\mathcal{B}(\mathrm{P})$. Then the grading on $\mathcal{B}(\mathrm{P})$,

$$\mathcal{B}(\mathrm{P}) = \bigoplus_{s\geq 0} \mathcal{B}(\mathrm{P})^{[s]},$$

that one should use to define the individual classes, only takes the number of vertices of a given tree into account. Similar, if we take a Σ_*-cofibrant replacement of P different than $\mathrm{P}\otimes \mathrm{E}\Sigma_*$ but with *non-trivial* differentials, then we have to take the degrees of elements of the replacement into account.

Until now we were concerned with the question about formality of P_∞-algebras. Of course the canonical class and the higher obstructions can help to show that two given (possibly non-formal) P_∞-algebras are not isomorphic in the homotopy category of P_∞-algebras. It is, however, also possible that two “homotopically different” P_∞-algebras have the same canonical class, and we are not able to distinguish them by our obstruction theory. The next result should help to manage this problem.

Theorem 4.10. *Let A' and A'' be two P_∞-algebras with isomorphic cohomology. Further, let γ' and γ'' denote the transferred (by Theorem 2.6) P_∞-algebra structures on H_*A' and H_*A'', respectively. There is a zig-zag of weak equivalences of P_∞-algebras from A' to A'' if and only if there is a homomorphism*

$$f = \bigoplus_{s\geq 0} f^{[s]} \colon \bigoplus_{s\geq 0} \mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)^{[s]} \circ H_*A' \to H_*A''$$

where $f^{[0]}$ is an isomorphism and the condition

$$\delta(f)\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) - \sum_{\nu_2'(\wp)} \pm f \left(\begin{array}{c} \gamma'\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{array}\right] \\ a_* \cdots\cdots \quad | \quad \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp' \\ | \end{array}\right)$$

$$+ \sum_{\nu(\wp)} \gamma'' \left(\begin{array}{c} f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \cdots\cdots f\left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{array}\right] \\ \diagdown \quad \diagup \\ \wp' \\ | \end{array}\right) = 0.$$

holds.

Proof. One direction is clear: if such a homomorphism f is given then it induces a weak equivalence of cofibrant replacements (Corollary 2.12 and 2.13). Therefore, A' and A'' are isomorphic in the homotopy category of P_∞-algebras.

If A' and A'' are connected by a zig-zag of weak equivalences then the same holds also for (H_*A',γ') and (H_*A'',γ''). Since every P_∞-algebra is fibrant, there is a map of P_∞-algebras between the cofibrant replacements of (H_*A',γ') and (H_*A'',γ'') that is a weak equivalence:

$$\begin{array}{ccc} R_{(H_*A',\gamma')} & \xrightarrow{\sim} & R_{(H_*A'',\gamma'')} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ (H_*A',\gamma') & & (H_*A'',\gamma'') \end{array}$$

The horizontal map represents an isomorphism between (H_*A',γ') and (H_*A'',γ'') in the homotopy category. Therefore it is a weak equivalence. All together we have the following diagram of P_∞-algebras and P_∞-maps:

$$(H_*A',\gamma') \xleftarrow[\sim]{p_1} R_{(H_*A',\gamma')} \xrightarrow[\sim]{p_2} (H_*A'',\gamma'')$$

We can transfer the P_∞-algebra structure of $R_{(H_*A',\gamma')}$ to H_*A' via some cycle choosing map. Denote the twisting cochain representing the new structure on H_*A' by γ'''. By Theorem 2.16 there is a morphism of quasi-cofree coalgebras

$$(\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A', \partial_{\gamma'''}) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ R_{(H_*A',\gamma')}, \partial_{R_{(H_*A',\gamma')}}).$$

Since p_1 also induces a map of quasi-cofree coalgebras, we conclude by Theorem 2.15 that γ''' and γ' are homotopic. Hence, by the same theorem there is a morphism of quasi-cofree coalgebras in the other direction. Composing this with the map of quasi-cofree coalgebras induced by p_2 we get a map

$$\tilde{f}\colon (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A', \partial_{\gamma'}) \to (\mathcal{B}(\mathrm{P}\otimes \mathrm{E}\Sigma_*)\circ H_*A'', \partial_{\gamma''}).$$

The projection to H_*A'' gives the desired map f. □

The above result is based on the fact (which we now implicitly proved) that two P_∞-structures on a cofibrant chain complex are isomorphic in the homotopy category if and only if they are homotopic in the model category of operads. In a simplicial setting a similar statement can be found in Rezk [Re], Theorem 1.1.5.

4.2 Comparison to earlier results

We want to put our obstruction theory in the context of known results, and show in which way it is a generalization of already existing statements.

An obstruction theory in our sense has been developed for differential graded algebras over a field, or in other words for P the associative operad. As we mentioned earlier Kadeishvili [Kad] proved a homotopy transfer theorem in this case. He also indicated that the pieces of the transferred A_∞-structure can be successively viewed as Hochschild cohomology classes provided the lower ones are trivial. He further proved that the vanishing of the second Hochschild cohomology group of the homology H_*A of a given differential graded algebra A implies the formality of A.

Again in the case $\mathrm{P} = \mathrm{Ass}$ a canonical class with the properties of Theorem 4.2 and 4.6 have been considered by Benson, Krause and Schwede in [BKS]. The results of the previous section can be seen as an extension of their work on the topic to a wider class of operads.

Let us take a closer look at the particular case $\mathrm{P} = \mathrm{Ass}$. The methods used in [Kad] and [BKS] are very explicit. In their sense, an A_∞-structure on a chain complex A is given by a sequence of homogeneous maps

$$m_n\colon A^{\otimes n} \to A$$

of degree $n-2$ for n larger or equal to 2. These maps should satisfy some relations, see for example §3 of [Kel] (be aware of the fact that Keller uses cochain notation and denotes the differential of A by m_1). An alternative way of organizing this A_∞-structure is as a map of degree -1 from the reduced tensor coalgebra $\bar{T}\Sigma A$ to ΣA such that the induced coderivation on $\bar{T}\Sigma A$ defines a twisting differential. Here, $\bar{T}A$ is given by

$$\bar{T}A = \bigoplus_{n\geq 1} A^{\otimes n}$$

with comultiplication determined by

$$\Delta(a_1,\dots,a_n) = \sum_{1\leq i\leq n-1} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)$$

for every $(a_1,\dots,a_n)\in A^{\otimes n}$. To be able to switch back and forth between the two definitions one only has to note that a map

$$\bar{T}\Sigma A \to \Sigma A$$

is uniquely defined by its restrictions onto the summands of the form $(\Sigma A)^{\otimes n}$. Up to shifts the latter give the maps m_n.

Further, given two A_∞-algebras A and B, Kadeishvili defines an A_∞-morphism as a collection of maps

$$f_n\colon A^{\otimes n} \to B$$

of degree $n-1$ for $n>0$ that interact in an appropriate way with the A_∞-structures on A and B. Equivalently, an A_∞-morphism is given by a map of differential graded coalgebras

$$f\colon \left(\bar{T}\Sigma A, m_A\right) \to \left(\bar{T}\Sigma B, m_B\right),$$

where m_A and m_B are the twisting differentials induced by the A_∞-structures on A respectively B. For proofs and more details on this topic we again refer the reader to §3.6 of [Kel].

Kadeishvili's proof of the transfer theorem inductively constructs an A_∞-structure on the homology H_*A together with an A_∞-quasi-isomorphism to A via explicit calculations based on the defining equalities of these.

Hidden in the definitions of an A_∞-structure on a chain complex A and an A_∞-morphism between A_∞-algebras is a particular cofibrant replacement of the associative operad. It arises as an application of the "operadic Koszul theory". For a nice introduction to the field and some of the main results see [LV] and [Fre04]. Koszul theory assigns to a so called quadratic operad P a *Koszul dual cooperad* P^{i} together with an operadic twisting morphism

$$\kappa\colon \mathrm{P}^{\mathrm{i}} \to \mathrm{P}.$$

The operad P is called *Koszul* if the corresponding morphism of operads

$$\mathcal{B}^c\mathrm{P}^{\mathrm{i}} \to \mathrm{P}$$

is a weak equivalence.

It is well-known that the operads Ass, Com and Lie are Koszul. In characteristic 0 the Koszul dual cooperads of these form cofibrant Σ_*-sequences. In particular in this situation $\mathcal{B}^c\mathrm{P}^{\mathrm{i}}$ is a cofibrant replacement in the category of operads. For the associative operad even more is true: since it has free Σ_*-action its Koszul dual cooperad is Σ_*-cofibrant even in positive characteristic. One can use the "smaller" resolutions above to obtain an obstruction theory parallel to the one in the previous section using the methods exploited there. By Proposition 1.45, a $\mathcal{B}^c\mathrm{P}^{\mathrm{i}}$-algebra structure on A is equivalently given by a twisting cochain

$$\mathrm{P}^{\mathrm{i}} \circ A \to A.$$

Further, there is a cofibrant replacement of A in the category of $\mathcal{B}^c\mathrm{P}^{\mathrm{i}}$-algebras of the form $R_{\mathcal{B}^c\mathrm{P}^{\mathrm{i}}}\left(\mathrm{P}^{\mathrm{i}} \circ A, \partial_\alpha\right)$ that allows one to use the same arguments for defining and proving the existence of a canonical class with the properties of Theorem 4.2. The reason why we did not follow this approach is that in general it does not work in positive characteristic. Nevertheless, in the case of the associative operad it enables us to compare "our" class with the class of Benson, Krause and Schwede in [BKS]:

Proposition 4.11. *Consider the morphism of cofibrant operads*

$$g\colon \mathcal{B}^c\mathrm{Ass}^{\mathrm{i}} \to \mathcal{B}^c\mathcal{B}(\mathrm{Ass})$$

and the map induced by it on Gamma cohomology

$$g^*\colon H^{2,-1}_{\mathcal{B}^c\mathcal{B}(\mathrm{Ass})}(H_*A) \to H^{2,-1}_{\mathcal{B}^c\mathrm{Ass}^{\mathrm{i}}}(H_*A),$$

where A is a differential graded associative algebra. Then the canonical class of Theorem 4.2 is sent to the canonical class examined in [BKS]. Further, provided a higher obstruction is defined, then it is also sent to its analogue on the right hand side.

Proof. We first describe the Koszul dual cooperad $\mathrm{Ass}^{\mathsf{i}}$ of the associative operad. It is, up to shifts and signs, given by the cooperad arising from Ass by taking levelwise the k-linear dual, as in Example 1.25. More precisely, the following holds (cf. [GK], and [GJ] Theorem 3.1):

$$\mathrm{Ass}^{\mathsf{i}}(k) = \Sigma^{k-1}\mathrm{Ass}^{\vee}(k) \otimes \mathrm{sgn}_k,$$

where sgn_k is the one dimensional representation of Σ_k with an action given by multiplication with the signature of a permutation. As a chain complex $\mathrm{Ass}^{\mathsf{i}}(k)$ is concentrated in degree $k-1$ where it has one generator (as a $k[\Sigma_k]$-module) denoted by $\mu_k^{\vee}$. The latter is dual to the generating operation $\mu_k \in \mathrm{Ass}(k)$. The weak equivalence

$$\epsilon\colon \mathcal{B}^c\mathrm{Ass}^{\mathsf{i}} \xrightarrow{\sim} \mathrm{Ass}$$

is determined by the fact that $\mu_2^{\vee}$ is mapped to μ_2, and the generators of the form $\mu_l^{\vee}$ for l greater than 2 are sent to zero. Further, the Koszul dual cooperad of a connected operad P is a subcooperad of the Bar construction $\mathcal{B}(\mathrm{P})$ (see Lemma 5.2.4 of [Fre04]). In particular, we have a map of cooperads

$$\mathrm{Ass}^{\mathsf{i}} \to \mathcal{B}(\mathrm{Ass}),$$

such that the cooperad $\mathrm{Ass}^{\mathsf{i}}$ inherits a grading

$$\mathrm{Ass}^{\mathsf{i}} = \bigoplus_{s\geq 0} (\mathrm{Ass}^{\mathsf{i}})^{[s]}$$

compatible with the one of $\mathcal{B}(\mathrm{Ass})$ needed for the construction of our universal class. Spelling out the definitions in §5.2.3 of [Fre04], we get

$$(\mathrm{Ass}^{\mathsf{i}})^{[s]} = \mathrm{Ass}^{\mathsf{i}}(s+1).$$

An action of the cofibrant operad $\mathcal{B}^c\mathrm{Ass}^{\mathsf{i}}$ on a chain complex A is by Proposition 1.45 given by a map

$$\tilde{\alpha}\colon \mathrm{Ass}^{\mathsf{i}} \circ A \to A$$

that satisfies the assumptions and equality of Proposition 1.44, or in other words by a coderivation that is also a twisting differential. Note that this data is equivalent to a twisting coderivation on the reduced tensor coalgebra $\bar{T}\Sigma A$, i.e., to an A_∞-structure on A. Furthermore, an A_∞-morphism corresponds exactly to a map of coalgebras.

By Theorem 2.6, we have a transferred $\mathcal{B}^c\mathrm{Ass}^{\mathsf{i}}$-algebra structure ($A_\infty$-structure) on H_*A given by

$$\tilde{\gamma}\colon \bigoplus_{s\geq 0} (\mathrm{Ass}^{\mathsf{i}})^{[s]} \circ H_*A \to H_*A.$$

Using the cofibrant replacement of the form

$$R_{\mathcal{B}^c\mathrm{Ass}^{\mathsf{i}}}\left(\mathrm{Ass}^{\mathsf{i}} \circ H_*A, \partial_{\tilde{\gamma}^{[1]}}\right) =: R_{H_*A}$$

and the methods of the previous section, one can show that $\tilde{\gamma}^{[2]}$ defines a cohomology class in $H^{2,-1}_{\mathcal{B}^c\mathrm{Ass}^{\mathsf{i}}}(H_*A)$. The cochain complex arising from this particular cofibrant replacement "almost" coincides with the shift of the total cochain complex of the Hochschild cochain bicomplex (as reviewed in §4 of [BKS]). Let us quickly remind of this notion and say what we mean by "almost". In few words, the Hochschild cochain bicomplex is given by the "standard" Hochschild cochain complex where a second grading comes from the one on $\mathbf{Hom}(A^{\otimes n}, A)$. Now, take the sub-bicomplex that does not contain the factors coming from $\mathbf{Hom}(k, A)$, form the total complex and shift it by 1. From the description of the Koszul dual $\mathrm{Ass}^{\mathrm{i}}$ one easily sees that $\mathbf{Der}_{\mathcal{B}^c\mathrm{Ass}^{\mathrm{i}}}(R_{H_*A}, H_*A)$ is isomorphic as a graded space to the aforesaid complex. One has to verify that we also get the correct differentials. This can be done by an explicit calculation of the differential of R_{H_*A}, where one has to take into account the comultiplication of $\mathrm{Ass}^{\mathrm{i}}$ as well as the "strict" $\mathcal{B}^c\mathrm{Ass}^{\mathrm{i}}$-algebra and H_*A-representation structures on H_*A.

The canonical class of Benson, Krause and Schwede is represented in the Hochschild bicomplex by the m_3 part of the transferred A_∞-structure of H_*A. But the latter corresponds exactly to $\tilde{\gamma}^{[2]}$. Note the shift of gradings: the class in [BKS] is in $\mathbf{HH}^{3,-1}(H_*A)$, this means in total degree 2, whereas ours lives in $H^{2,-1}_{\mathrm{P}_\infty}(H_*A)$, i.e., in degree 1.

Finally, observe that the inclusion

$$\mathrm{Ass}^{\mathrm{i}} \to \mathcal{B}(\mathrm{Ass})$$

defines a map on Gamma cohomology groups that sends "our" class to the class of [BKS].

The arguments for the higher obstructions are analogous.

□

Remark 4.12. One can generalize the statement of the proposition to $\mathcal{B}^c\mathcal{B}(\mathrm{Ass})$-algebras, in the sense that the two discussed constructions of a canonical class – by the Cobar-Bar resolution and by the Cobar-Koszul resolution – coincide.

An the end we want to mention that there are other works in the literature considering the question of transferring P_∞-structures in the case where the characteristic of the ground ring is zero, e.g., Loday and Vallette [LV], or Huebschmann [Hue] for Lie algebras. In [Kad88] Kadeishvili establishes results corresponding to Corollary 4.7 and Corollary 4.8 for the commutative operad in characteristic 0.

Therefore, our results generalize the already existing ones, and are in particular new in positive characteristic.

4.3 Examples

In this section we want to illustrate the previously developed theory on some examples. We are going to examine two different E_∞-algebras that have the same homology algebra. Our first goal is to determine the associated (first) canonical classes. Then, we will compare these with the trivial class in order to conclude that neither of the examples is a formal E_∞-algebra. At the end, we will use Theorem 4.10 to argue why the two E_∞-algebras are not isomorphic in the homotopy category of E_∞-algebras.

Let us set up the general framework. We are working at the prime $p = 2$. The ground field k can be any field of characteristic 2, for instance $\mathbb{F}_2$. We are interested in E_∞-algebras. The underlying graded operad is the commutative operad $\mathrm{P} = \mathrm{Com}$. Our examples are motivated by the fact that there is an E_∞-structure on the normalized cochains of a topological space (with integer or field coefficients). Independently, McClure-Smith [MS] and Berger-Fresse [BF] constructed an operad $\mathcal{S}$ acting naturally on the cochains of spaces. This operad is a Σ_*-cofibrant replacement of the commutative operad (in particular, in the notation of May it is an E_∞-operad). We denote by $S_*(X)$ the normalized integer chains of a space X. Further, $S^*(X) := \mathbf{Hom}_{\mathbb{Z}}(S_*(X), k)$ are the normalized cochains of X with k coefficients. Since we have been using chain grading on our algebras so far, we introduce the notation $\bar{S}_*(X) := S^{-*}(X)$. The homology of the chain complex $\bar{S}_*(X)$ calculates the cohomology of X with coefficients in k and negative grading.

Before we start with the examples we shortly introduce the surjection operad $\mathcal{S}$. We follow the notation and presentation of [MS], since it is topologically motivated. More details and proofs for the statements can be found there. Note that in their work, McClure and Smith refer to $\mathcal{S}$ as the "sequential operad".

The surjection operad: Recall that we write $\underline{n}$ for the set $\{1, \ldots, n\}$. A function $f\colon \underline{m} \to \underline{n}$ is called a ***non-degenerate surjection*** if it is surjective and in addition $f(i) \neq f(i+1)$ for every $i = 1, \ldots, m-1$. Explicit surjections are written in the form

$$f(1)f(2)\cdots f(m),$$

e.g., the sequence 1212 gives a non-degenerate surjection $f\colon \underline{4} \to \underline{2}$. The degree of such an f is defined to be $m-n$. As a graded abelian group the surjection operad $\mathcal{S}(n)$ in arity n is freely generated (over k) by the set of non-degenerate surjections $f\colon \underline{m} \to \underline{n}$. The differential on $f = f(1)\cdots f(m)$ is given by

$$\delta(f(1)\cdots f(m)) = \sum_{i=1}^{m} f(1)\cdots \widehat{f(i)} \cdots f(m),$$

where $\widehat{f(i)}$ means that we leave out the i'th entry. Some of the terms in the sum are not representing surjections onto $\underline{n}$ any more, or are degenerate. Those terms are set to be zero. For instance we get

$$\delta(13123) = (3123) + (1323) + (1312).$$

If we are not working over a field of characteristic 2, then there are signs occurring in the formula for the differential.

Let us first define an action of a non-degenerate surjection on $\bar{S}_*(X)$. For an r-simplex $\sigma\colon \Delta^r \to X$ and $a_0, \ldots, a_q \in \{0, \ldots, r\}$ we denote by

$$\sigma(a_0, \ldots, a_q)\colon \Delta^q \to X$$

the q-simplex resulting by precomposing with the map $\Delta^q \to \Delta^r$ that sends the i-th vertex of Δ^q to the vertex a_i of Δ^r. Further, we define an *overlapping partition* of $\{0, \ldots, q\}$ *with* m *pieces* as a collection of subsets $A_1, \ldots, A_m$ of $\{0, \ldots, q\}$ such that:

(i) For $j < j'$ each element of A_j is smaller or equal to each element of $A_{j'}$.

(ii) For every j smaller than m, $A_j \cap A_{j+1}$ has exactly one element.

One overlapping partition of $\{0, 1, 2, \ldots, 5\}$ with 7 pieces is given for instance by

$$\{\{0\}, \{01\}, \{1234\}, \{4\}, \{45\}, \{5\}, \{5\}\}.$$

In particular, an overlapping sequence with m pieces is uniquely defined by the $m-1$ *overlap points*. In the example above these are $0, 1, 4, 4, 5, 5$. Now we can say how a non-degenerate surjection $f\colon \underline{m} \to \underline{n}$ defines a natural transformation

$$\langle f \rangle \colon \bar{S}_*(X)^{\otimes n} \to \bar{S}_*(X).$$

For a q-simplex $\sigma\colon \Delta^q \to X$ and elements $x_1, \ldots, x_n$ of $\bar{S}_*(X)$ we set

$$\langle f \rangle (x_1 \otimes \cdots \otimes x_n)(\sigma) = \sum_{\mathcal{A}} x_1(\sigma(\coprod_{f(j)=1} A_j)) \cdots x_i(\sigma(\coprod_{f(j)=i} A_j)) \cdots x_n(\sigma(\coprod_{f(j)=n} A_j)),$$

where $\mathcal{A}$ denotes the set of all overlapping partitions of q with m pieces. For example,

$$\langle 13132 \rangle (x_1 \otimes x_2 \otimes x_3)(\sigma) = \sum_{\mathcal{A}} x_1(\sigma(A_1 \coprod A_3)) \cdot x_2(\sigma(A_5)) \cdot x_3(\sigma(A_2 \coprod A_4)).$$

We remark, that the above formula has to be completed by signs when working in characteristic different than 2. Even though the formula might look slightly unmanageable at first glance, it is a generalization of the well-known formulas for $\cup_i$ products on cochains of spaces due to Steenrod [Ste]. If we take f to be given by the sequence 12 then

$$\langle f\rangle \colon \bar{S}_*(X)^{\otimes 2} \to \bar{S}_*(X)$$

is exactly the cup product: for $x_1 \in \bar{S}_{-m}(X)$, $x_2 \in \bar{S}_{-n}(X)$ and $\sigma\colon \Delta^{m+n} \to X$

$$\begin{aligned} \langle 12\rangle(x_1 \otimes x_2)(\sigma) &= \sum_{l=0}^{m+n} x_1(\sigma(0,\dots,l)) \cdot x_2(\sigma(l,\dots,m+n)) \\ &= x_1(\sigma(0,\dots,m)) \cdot x_2(\sigma(m,\dots,m+n)). \end{aligned}$$

Note that for every $l = 0,\dots,m+n$ we have one partition of $\{0,\dots,m+n\}$ with 2 pieces. There is, however, only one non-trivial summand – the one for $l = m$. If σ is a simplex of dimension other than $m+n$ then there are of course overlapping partitions with 2 pieces, but the evaluation on x_1 or x_2 is zero for dimensional reasons. For $f = 121$, $x_1 \in \bar{S}_{-m}(X)$, $x_2 \in \bar{S}_{-n}(X)$ and $\sigma\colon \Delta^{m+n-1} \to X$ we get

$$\begin{aligned} \langle 121\rangle(x_1 \otimes x_2)(\sigma) &= \sum_{l\le k} x_1(\sigma(0,\dots,l,k,\dots m+n-1)) \cdot x_2(\sigma(l,\dots,k)) \\ &= \sum_{l=0}^{m-1} x_1(\sigma(0,\dots,l,l+n,\dots,m+n-1)) \cdot x_2(\sigma(l,\dots,l+n)), \end{aligned}$$

which corresponds exactly to the $\cup_1$ product. Again for dimensional reasons, for σ a q-simplex with $q \neq m+n-1$ the sum is zero.

Finally, we have to say a few words about the operadic structure and the Σ_*-action of $\mathcal{S}$. They are inherited from another operad, denoted by $\mathcal{N}$, of which $\mathcal{S}$ is a suboperad. More precisely, $\mathcal{N}(n)$ is freely generated as a graded group by the natural transformations

$$\bar{S}_*(X)^{\otimes n} \to \bar{S}_*(X)$$

where a degree m transformation raises the chain degree by m. A differential is induced by the differentials of $\bar{S}_*(X)^{\otimes n}$ and $\bar{S}_*(X)$. The operadic composition and the symmetric action are the obvious ones. Theorem 2.15 of [MS] states that with the above definitions $\mathcal{S}$ becomes an E_∞-operad. We refer the reader to the original (very well-written) reference [MS] for more details.

We come back to our original goal: giving examples for canonical classes of E_∞-algebras. The surjection operad acts on normalized cochains of spaces. We are going to use small simplicial models for the wedge $S^1 \vee S^2$ and the mod 4 Moore space $M(4)$, respectively, in order to obtain an E_∞-action on the chain complexes $\bar{S}_*(S^1 \vee S^2)$ and $\bar{S}_*(M(4))$. These are the E_∞-algebras we are going to use for our examples.

The canonical class of $\bar{S}_*(S^1 \vee S^2)$: The chain complex $\bar{S}_*(S^1 \vee S^2)$ is an algebra over the surjection operad. In order to apply the obstruction theory from the last sections we consider $\bar{S}_*(S^1 \vee S^2)$ as an algebra over a cofibrant replacement of the commutative operad. Since $\mathcal{S}$ is Σ_*-cofibrant we have a cofibrant replacement of Com given by $\mathcal{B}^c\mathcal{B}(\mathcal{S})$, and via the canonical map

$$\mathcal{B}^c\mathcal{B}(\mathcal{S}) \to \mathcal{S}$$

$\bar{S}_*(S^1 \vee S^2)$ becomes a $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-algebra. We denote the twisting cochain giving the action of $\mathcal{B}^c\mathcal{B}(\mathcal{S})$ on $\bar{S}_*(S^1 \vee S^2)$ by

$$\alpha\colon \mathcal{B}(\mathcal{S}) \circ \bar{S}_*(S^1 \vee S^2) \to \bar{S}_*(S^1 \vee S^2).$$

The grading of α takes account of the number of vertices of a tree and the degrees of the elements of $\mathcal{S}$, i.e., the surjection operad takes the place of the Barratt-Eccles operad in our original construction. Since $\bar{S}_*(S^1 \vee S^2)$ is even an $\mathcal{S}$-algebra, the evaluation of α on trees with more than one vertex is trivial. We refer the reader to the adjunction in Section 1.4.7 to trace this fact: the map $\mathcal{B}^c\mathcal{B}(\mathcal{S}) \to \mathcal{S}$ corresponds to the identity on $\mathcal{B}(\mathcal{S})$ under the Cobar-Bar adjunction.

The space $S^1 \vee S^2$ has a small simplicial model with non-degenerate simplices e_0, e_1 and e_2 in dimensions 0, 1 and 2, respectively. The differential of the normalized chains $S_*(S^1 \vee S^2)$ is trivial everywhere. After dualizing we obtain for $\bar{S}_*(S^1 \vee S^2)$

$$\cdots \to 0 \to 0 \to \underbrace{k.c_0}_{0} \xrightarrow{0} \underbrace{k.c_1}_{-1} \xrightarrow{0} \underbrace{k.c_2}_{-2} \to 0 \to 0 \to \cdots$$

where c_0, c_1 and c_2 denote generators. Since this chain complex already has trivial differentials the canonical class of $\bar{S}_*(S^1 \vee S^2)$ is given by the class of the map $\alpha^{[2]}$. Therefore our task reduces to calculating $\alpha^{[2]}$. As we noticed α is zero on trees with more than one vertex. This means that in order to determine the canonical class we only have to calculate the action of elements $f \in \mathcal{S}(k)_1$ of degree 1. Before we start with the calculation let us mention that $\alpha^{[1]}$ gives a graded commutative structure on $\bar{S}_*(S^1 \vee S^2)$ that coincides with the cup product on the cohomology of $S^1 \vee S^2$. It is easy to see that the only non-trivial multiplication is by elements of degree zero. Non-degenerate surjections of degree 1 are maps $\underline{\mathrm{n{+}1}} \to \underline{\mathrm{n}}$ that raise the degree by one:

$$\bar{S}_*(S^1 \vee S^2)^{\otimes n} \to \bar{S}_{*+1}(S^1 \vee S^2)$$

Hence, for dimensional reasons there could be non-trivial operations only on $c_1 \otimes c_1$, $c_1 \otimes c_2$, $c_2 \otimes c_1$, $c_1 \otimes c_1 \otimes c_1$ as well as combinations including c_0. Suppose there is a non-degenerate surjection f that applied to $c_1 \otimes c_2$, $c_2 \otimes c_1$ or $c_1 \otimes c_1 \otimes c_1$ is non-trivial. For instance, let us take $c_1 \otimes c_2$. In order for $\langle f \rangle(c_1 \otimes c_2)$ to be non-zero, its evaluation on the 2-simplex e_2 should be non-trivial. However, this is not possible for any f since $e_2(i,j)$ is different from e_1, and therefore, for any f and any partition of $\{0,1,2\}$ with 3 pieces $c_1(e_2(i,j))$ is zero. For similar reasons $\alpha^{[2]}$ vanishes on $\Sigma f \otimes c_2 \otimes c_1$ and $\Sigma f \otimes c_1 \otimes c_1 \otimes c_1$. The short explanation for the vanishing of the above operations lies in the fact that S^1 and S^2 are simplicial subsets of $S^1 \vee S^2$ that have only a non-degenerate zero simplex in common.

We now consider the case $c_1 \otimes c_1$. There are two non-degenerate surjections generating $\mathcal{S}(2)_1$. These are 121 and 212. It suffices to look at the case 121 since 212 is obtained from 121 by applying the unique transposition in Σ_2. We have:

$$\alpha^{[2]} \left(\begin{array}{ccc} c_1 & & c_1 \\ & \diagdown \quad \diagup & \\ & \Sigma(121) & \\ & | & \end{array} \right) = \langle 121 \rangle (c_1 \otimes c_1) \in \bar{S}_{-1}(S^1 \vee S^2)$$

There is only one overlapping partition of $\{0,1\}$ with 3 pieces for which we have to evaluate c_1 on non-degenerate simplices. This is the partition $\{0\} \cup \{0,1\} \cup \{1\}$:

$$\langle 121 \rangle (c_1 \otimes c_1)(e_1) = c_1(e_1(\{0\} \cup \{1\}))c_1(e_1(\{0,1\})) = 1$$

Therefore, we conclude

$$\langle 121\rangle(c_1 \otimes c_1) = c_1.$$

Remark 4.13. In this way we explicitly calculated that the Steenrod operation Sq^0 performs the identity on $H^1(S^1 \vee S^2)$ since $Sq^0(c_1) = c_1 \cup_1 c_1 = c_1$. The evaluation $Sq^0(c_2) = c_2 \cup_2 c_2 = c_2$ appears in the $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-algebra structure as a part of $\alpha^{[3]}$.

Note that the element $12 \in \mathcal{S}(2)_0$ acts non-trivially

$$\bar{S}_*(S^1 \vee S^2)^{\otimes 2} \xrightarrow{\langle 12\rangle} \bar{S}_*(S^1 \vee S^2)$$

(multiplication with c_0 performs the identity). Hence, partial operadic compositions of 12 with 121 give elements in degree 1 that act non-trivially on $\bar{S}_*(S^1 \vee S^2)$, e.g.,

$$\alpha^{[2]}\left(\begin{array}{ccc} c_1 & c_1 & c_0 \\ \diagdown & | & \diagup \\ & \Sigma(1213) & \\ & | & \end{array}\right) = c_1 \in \bar{S}_{-1}(S^1 \vee S^2).$$

Loosely speaking, the "valuable information of $\alpha^{[2]}$" is concentrated in the evaluation on $\Sigma(121) \otimes c_1 \otimes c_1$.

The class of $\alpha^{[2]}$ in the derivation complex

$$\mathbf{Der}_{\mathrm{P}_\infty}(R_{\bar{S}_*(S^1 \vee S^2)}, \bar{S}_*(S^1 \vee S^2))$$

represents the canonical class of $\bar{S}_*(S^1 \vee S^2)$. Our next goal is to show that it is not cohomologous to zero. Here P_∞ stands for $\mathcal{B}^c\mathcal{B}(\mathcal{S})$. If we assume that $\alpha^{[2]}$ is a coboundary then there is a homomorphism $f\colon \mathcal{B}(\mathcal{S})^{[1]} \circ \bar{S}_*(S^1 \vee S^2) \to \bar{S}_*(S^1 \vee S^2)$ such that for every $\wp \in \mathcal{B}(\mathcal{S})^{[2]}$ the equation

$$f\left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & \partial(\wp) & \\ & | & \end{array}\right) + \sum_{\nu_2''(\wp)} f\left(\begin{array}{ccc} & \alpha^{[1]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \wp'' & \end{array}\right] & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array}\right)$$

$$+ \sum_{\substack{\nu_2' \\ \wp' \in \mathcal{B}(\mathrm{P}\otimes\mathrm{E}\Sigma_*)^{[1]}}} \alpha^{[1]}\left(\begin{array}{ccc} & f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \wp'' & \end{array}\right] & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{array}\right) = \alpha^{[2]}\left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & \wp & \\ & | & \end{array}\right)$$

holds. For $\wp = \Sigma(121)$ (one vertex tree) we are left with

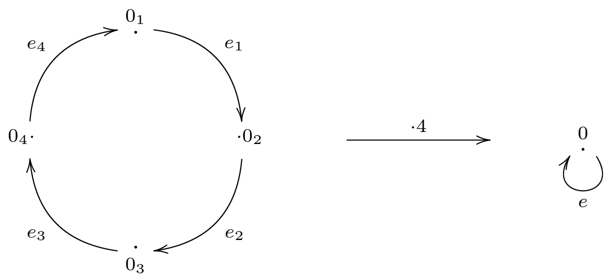
$$f\left(\begin{array}{ccc} c_1 & & c_1 \\ \diagdown & & \diagup \\ & \partial\Sigma(121) & \\ & | & \end{array}\right) = \alpha^{[2]}\left(\begin{array}{ccc} c_1 & & c_1 \\ \diagdown & & \diagup \\ & \Sigma(121) & \\ & | & \end{array}\right),$$

which reads as

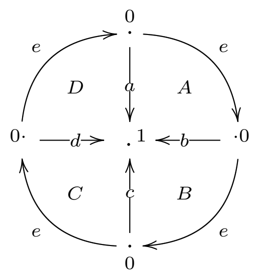
$$f\left(\begin{array}{ccc} c_1 & & c_1 \\ & \diagdown \quad \diagup & \\ & \Sigma(12) & \\ & | & \end{array}\right) + f\left(\begin{array}{ccc} c_1 & & c_1 \\ & \diagdown \quad \diagup & \\ & \Sigma(21) & \\ & | & \end{array}\right) = \alpha^{[2]}\left(\begin{array}{ccc} c_1 & & c_1 \\ & \diagdown \quad \diagup & \\ & \Sigma(121) & \\ & | & \end{array}\right).$$

The left hand side of this equation is zero (apply a transposition to one of the trees and use char $k = 2$). As we calculated earlier, the right hand side is c_1. Hence, such an f can not exist and $\alpha^{[2]}$ is not a coboundary in the derivation complex. In particular, by Theorem 4.2 we can conclude that $\bar{S}_*(S^1 \vee S^2)$ is not a formal $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-algebra.

The canonical class of $\bar{S}_*(M(4))$*:* We come to our second example. The cochains of the mod 4 Moore space with k coefficients. $M(4)$ is constructed as the mapping cone of the map "multiplication by 4" on S^1. First we have to model the map $(\cdot 4)$ simplicially. For this we take two different simplicial models of S^1.



The domain of the map $(\cdot 4)$ has a model with four non-degenerate 0-simplices, which we denote by 0_1, 0_2, 0_3 and 0_4, and four non-degenerate 1-simplices labeled by e_1, e_2, e_3 and e_4 in the picture. The codomain is the usual model of S^1, namely $\Delta^1/\partial\Delta^1$. The map in between sends e_1, e_2, e_3, e_4 to e and $0_1, 0_2, 0_3, 0_4$ to 0. We use as a simplicial model for the mapping cone the simplicial set depicted in the following way:



We have two non-degenerate 0-simplices 0 and 1, five non-degenerate 1-simplices a, b, c, d, e and four non-degenerate 2-simplices, denoted in the picture by A, B, C, D. We first have to determine the normalized chains of $M(4)$. An easy calculation shows that $S_*(M(4))$ is given by:

$$\cdots 0 \to 0 \to \mathbb{Z}\{A, B, C, D\} \xrightarrow{\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}} \mathbb{Z}\{a, b, c, d, e\} \xrightarrow{\begin{pmatrix} -1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}\{0, 1\} \to 0$$

After dualizing we obtain

$$0 \to k\{0^*, 1^*\} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}} k\{a^*, b^*, c^*, d^*, e^*\} \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}} k\{A^*, B^*, C^*, D^*\} \to 0 \to 0 \cdots .$$

We view this as an unbounded chain complex with 0^* and 1^* in degree 0. The surjection operad acts on $\bar{S}_*(M(4))$. In order to determine the canonical class we have to transport (at least a piece of) the $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-structure of $\bar{S}_*(M(4))$ to its homology. In the following we denote the homology just by H_*M. In degrees $0, -1$ and -2 it is one dimensional generated by the classes $[0^* + 1^*]$, $[a^* + c^* + e^*]$ and $[A^*]$, respectively.

We denote by β the $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-structure on $\bar{S}_*(M(4))$, and the transferred structure on H_*M by γ. To simplify the notatition, we omit shifts when labeling trees. Again, P_∞ is an abbreviation for $\mathcal{B}^c\mathcal{B}(\mathcal{S})$. We first need to calculate pieces of the structure β. We start with non-degenerate surjections of degree 0. These determine $\beta^{[1]}$. In $\mathcal{S}(2)_0$ there are two generators: 12 and 21. Using these, in arity different than 2 the action of the surjection operad is fully defined by the operadic composition product. There are many non-trivial evaluations of $\beta^{[1]}$ induced by

$$\langle 12\rangle(0\text{-cell} \otimes n\text{-cell})$$

$$\langle 21\rangle(0\text{-cell} \otimes n\text{-cell})$$

for appropriate 0- and n-cells, for $n \in \{0, 1, 2\}$. This information, however, is not going to be essential later on. It only covers the fact that $[0^* + 1^*]$ acts like a unit in the cohomology ring. The only other possibility for non-trivial evaluations comes from elements of the form

$$\langle 12\rangle(1\text{-cell} \otimes 1\text{-cell}).$$

For example we get

$$\beta^{[1]}\left(\begin{array}{ccc} e^* & & a^* \\ & \diagdown \quad \diagup & \\ & 12 & \\ & | & \end{array}\right) = \langle 12\rangle(e^* \otimes a^*) = D^*. \tag{4.1}$$

To see this, note that the only overlapping partition of $\{0, 1, 2\}$ with two pieces that has to be considered is $\{01\} \cup \{12\}$. In particular, the only 2-simplex σ on which we get a non-trivial evaluation $\langle 12\rangle(e^* \otimes a^*)(\sigma)$ is the one with $\sigma(01) = e$ and $\sigma(12) = a$. The rest is clear. We write down the other evaluations of this kind:

$$\langle 12\rangle(e^* \otimes b^*) = \quad A^* \quad = \langle 21\rangle(b^* \otimes e^*) \tag{4.2}$$

$$\langle 12\rangle(e^* \otimes c^*) = \quad B^* \quad = \langle 21\rangle(c^* \otimes e^*) \tag{4.3}$$

$$\langle 12\rangle(e^* \otimes d^*) = \quad C^* \quad = \langle 21\rangle(d^* \otimes e^*) \tag{4.4}$$

Remark 4.14. The reader might want to keep in mind that the simplicial set $M(4)$ has an automorphism κ of order 4, i.e., $\kappa^4 = \mathrm{id}$. This morphism fixes the zero simplices as well as the one simplex e, and performs a cyclic rotation on a, b, c, d and A, B, C, D in the expected way. It induces an isomorphism on $\bar{S}_*(M(4))$ which one can use in order to obtain (4.2), (4.3) and (4.4) from the evaluation $\langle 12\rangle(e^* \otimes a^*)$. In the upcoming calculations κ can be used in the same manner.

We now come to the action of non-degenerate surjection of degree 1, i.e., the one relevant for $\beta^{[2]}$. Again, for dimensional reasons there are just three kinds of interesting evaluations:

(1)	$\langle f\rangle(\text{1-cell} \otimes \text{1-cell})$
(2)	$\langle f\rangle(\text{1-cell} \otimes \text{1-cell} \otimes \text{1-cell})$
(3)	$\langle f\rangle(\text{1-cell} \otimes \text{2-cell})$

Note that evaluations of the form $\langle f\rangle(\text{2-cell} \otimes \text{1-cell})$ can be reduced to the third type via the action of a transposition. Our next assignment is to give the non-trivial evaluations in the above three cases. We start with (1): We already made this calculation in the first example, so we just list the results:

$$\langle 121\rangle(a^* \otimes a^*) = a^* = \langle 212\rangle(a^* \otimes a^*) \tag{4.5}$$
$$\langle 121\rangle(b^* \otimes b^*) = b^* = \langle 212\rangle(b^* \otimes b^*) \tag{4.6}$$
$$\langle 121\rangle(c^* \otimes c^*) = c^* = \langle 212\rangle(c^* \otimes c^*) \tag{4.7}$$
$$\langle 121\rangle(d^* \otimes d^*) = d^* = \langle 212\rangle(d^* \otimes d^*) \tag{4.8}$$
$$\langle 121\rangle(e^* \otimes e^*) = e^* = \langle 212\rangle(e^* \otimes e^*) \tag{4.9}$$

These are all non-zero evaluations in the case (1). We proceed to (2). All the non-degenerate surjections $f\colon \underline{4} \to \underline{3}$ can be obtained from $1213, 1231$ and 2131 by the action of a permutation. Hence, it is enough to consider $f_1 = 1213$, $f_2 = 1231$ and $f_3 = 2131$. For every $i = 1, 2, 3$ there is exactly one overlapping partition $\mathcal{A}_{f_i}$ of $\{0, 1, 2\}$ with 4 pieces which is relevant for f_i. This is

$$\mathcal{A}_{1213} = \{0\} \cup \{01\} \cup \{1\} \cup \{12\}$$
$$\mathcal{A}_{1231} = \{0\} \cup \{01\} \cup \{12\} \cup \{2\}$$
$$\mathcal{A}_{2131} = \{01\} \cup \{1\} \cup \{12\} \cup \{2\}$$

respectively. Using this one calculates:

$$\langle 1213\rangle(e^* \otimes e^* \otimes a^*) = D^* \tag{4.10}$$
$$\langle 1213\rangle(e^* \otimes e^* \otimes b^*) = A^* \tag{4.11}$$
$$\langle 1213\rangle(e^* \otimes e^* \otimes c^*) = B^* \tag{4.12}$$
$$\langle 1213\rangle(e^* \otimes e^* \otimes d^*) = C^* \tag{4.13}$$

$$\langle 1231\rangle(a^* \otimes e^* \otimes b^*) = A^* \tag{4.14}$$
$$\langle 1231\rangle(b^* \otimes e^* \otimes c^*) = B^* \tag{4.15}$$
$$\langle 1231\rangle(c^* \otimes e^* \otimes d^*) = C^* \tag{4.16}$$
$$\langle 1231\rangle(d^* \otimes e^* \otimes a^*) = D^* \tag{4.17}$$

$$\langle 2131\rangle(e^* \otimes a^* \otimes a^*) = D^* \tag{4.18}$$
$$\langle 2131\rangle(e^* \otimes b^* \otimes b^*) = A^* \tag{4.19}$$
$$\langle 2131\rangle(e^* \otimes c^* \otimes c^*) = B^* \tag{4.20}$$
$$\langle 2131\rangle(e^* \otimes d^* \otimes d^*) = C^* \tag{4.21}$$

In case (3) we have to consider $f_1 = 121$ and $f_2 = 212$ as well as the corresponding relevant overlapping partitions:

$$\mathcal{A}_{121} = \{0\} \cup \{012\} \cup \{2\}$$
$$\mathcal{A}^1_{212} = \{0\} \cup \{01\} \cup \{12\}$$
$$\mathcal{A}^2_{212} = \{01\} \cup \{12\} \cup \{2\}$$

We get the following tables:

$$\langle 121\rangle(a^* \otimes A^*) = A^* \quad (4.22)$$
$$\langle 121\rangle(b^* \otimes B^*) = B^* \quad (4.23)$$
$$\langle 121\rangle(c^* \otimes C^*) = C^* \quad (4.24)$$
$$\langle 121\rangle(d^* \otimes D^*) = D^* \quad (4.25)$$

$$\langle 212\rangle(e^* \otimes A^*) = A^* \quad (4.26)$$
$$\langle 212\rangle(e^* \otimes B^*) = B^* \quad (4.27)$$
$$\langle 212\rangle(e^* \otimes C^*) = C^* \quad (4.28)$$
$$\langle 212\rangle(e^* \otimes D^*) = D^* \quad (4.29)$$

$$\langle 212\rangle(b^* \otimes A^*) = A^* \quad (4.30)$$
$$\langle 212\rangle(c^* \otimes B^*) = B^* \quad (4.31)$$
$$\langle 212\rangle(d^* \otimes C^*) = C^* \quad (4.32)$$
$$\langle 212\rangle(a^* \otimes D^*) = D^* \quad (4.33)$$

Our next goal is to transport the P_∞-structure to the homology of $\bar{S}_*(M(4))$. It is an easy calculation that as a graded commutative ring the chain complex H_*M is of the form

$$\cdots \to 0 \to 0 \to \underbrace{k.c_0}_{0} \xrightarrow{0} \underbrace{k.c_1}_{-1} \xrightarrow{0} \underbrace{k.c_2}_{-2} \to 0 \to 0 \to \cdots$$

with c_0 unit in degree 0 and $c_1^2 = 0$. We write down the calculation for $c_1 \cdot c_1 = 0$ explicitly: Since c_1 is represented by $(a^* + c^* + e^*)$ we have to determine the homology class of

$$\langle 12\rangle((a^* + c^* + e^*) \otimes (a^* + c^* + e^*)) = \langle 12\rangle((e^*) \otimes (a^* + c^*)) = D^* + B^*. \quad (4.34)$$

Here we use (4.1) and (4.3) as well as the fact that elsewhere the evaluation is trivial. The sum

$$D^* + B^* = \delta(c^* + d^*) \quad (4.35)$$

is a boundary, and therefore trivial in homology. Note that as graded commutative algebras (and even as unstable algebras over the Steenrod algebra) H_*M and $\bar{S}_*(S^1 \vee S^2)$ are isomorphic.

We want to transport the P_∞-structure to H_*M via the cycle choosing map

$$\iota\colon H_*M \to \bar{S}_*(M(4))$$

that sends c_0 to $(0^* + 1^*)$, c_1 to $(a^* + c^* + e^*)$ and c_2 to A^*. Theorem 2.6 gives a theoretical proof of the existence of a P_∞-structure on H_*M. Here we need an explicit construction. Recall that the new structure γ comes together with a homomorphism

$$f = \bigoplus_{s\geq 0} f^{[s]}\colon \bigoplus_{s\geq 0} \mathcal{B}(\mathcal{S})^{[s]} \circ H_*M \to \bar{S}_*(M(4))$$

defining a map of quasi-cofree coalgebras from $(\mathcal{B}(\mathcal{S}) \circ H_*M, \partial_\gamma)$ to $(\mathcal{B}(\mathcal{S}) \circ \bar{S}_*(M(4)), \partial_\beta)$. If we reformulate the condition on f (Proposition 1.47) for $\wp \in \mathcal{B}(\mathcal{S})^{[s]}$ we obtain:

$$
\delta_{\bar{S}_*(M(4))} f^{[s]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{pmatrix} = \sum_{\nu_2'(\wp)} \pm f \begin{pmatrix} & \gamma \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{bmatrix} & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{pmatrix}
$$

$$
- \sum_{\nu(\wp)} \beta \begin{pmatrix} f \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{bmatrix} \cdots\cdots f \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{bmatrix} \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{pmatrix} + (-1)^{|f|} f^{[s-1]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{pmatrix} =
$$

$$
= \underbrace{f^{[0]} \gamma^{[s]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{pmatrix}}_{S_1^s} + \underbrace{\sum_{\nu_2''(\wp)} \pm f \begin{pmatrix} & \gamma \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp'' \end{bmatrix} & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \wp' & \\ & | & \end{pmatrix}}_{S_2^s}
$$

$$
\underbrace{- \sum_{\nu(\wp)} \beta \begin{pmatrix} f \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{bmatrix} \cdots\cdots f \begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \wp''_* \end{bmatrix} \\ \diagdown \qquad \diagup \\ \wp' \\ | \end{pmatrix}}_{S_3^s} + \underbrace{(-1)^{|f|} f^{[s-1]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \partial(\wp) \\ | \end{pmatrix}}_{S_4^s}
$$

One can use this formula to define $f^{[s]}$ and $\gamma^{[s]}$ inductively. Suppose that $f^{[s-1]}$ and $\gamma^{[s-1]}$ have been successfully defined in a way that the above equation is satisfied for $\wp \in \mathcal{B}(\mathcal{S})^{[t]}$ with $t \leq s-1$. Then it is a manageable (though not pretty) calculation that $(S_2^s + S_3^s + S_4^s)$ defines a cycle in $\bar{S}_*(M(4))$. One defines

$$
\gamma^{[s]} \begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{pmatrix}
$$

to be the class represented by this cycle. Hence, $(S_1^s + S_2^s + S_3^s + S_4^s)$ is a boundary and one can choose

$$f^{[s]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right)$$

to be an element of its preimage under the differential $\delta_{\bar{S}_*(M(4))}$. It is a similar argument as at the end of Construction 4.5 why γ indeed defines a $\mathcal{B}^c\mathcal{B}(\mathcal{S})$-structure on H_*M.

We come back to our explicit example. We have already defined $f^{[0]} = \iota$ and the first part of the transferred structure, $\gamma^{[1]}$, that coincides with the graded commutative structure on H_*M. Before we can calculate $\gamma^{[2]}$ we have to determine $f^{[1]}$. In the case $s = 1$ the defining equation gets a very simple form:

$$\delta_{\bar{S}_*(M(4))} f^{[1]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right) = \underbrace{f^{[0]}\gamma^{[1]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right)}_{S_1^1} + \underbrace{\beta^{[1]}\left(\begin{array}{c} f^{[0]}(a_1) \cdots\cdots f^{[0]}(a_n) \\ \diagdown \quad \diagup \\ \wp \\ | \end{array}\right)}_{S_3^1}$$

Assume first $a_i \neq c_0$ for every $i = 1, \ldots, n$. Then S_1^1 is zero. For dimensional reasons there is only one generating tree at which the evaluation of S_3^1 can be non-trivial. We made this calculation in (4.34) and (4.35):

$$\beta^{[1]}\left(\begin{array}{c} f^{[0]}(c_1) \qquad f^{[0]}(c_1) \\ \diagdown \quad \diagup \\ 12 \\ | \end{array}\right) = \beta^{[1]}\left(\begin{array}{c} a^*+c^*+e^* \qquad a^*+c^*+e^* \\ \diagdown \quad \diagup \\ 12 \\ | \end{array}\right)$$

$$= D^* + B^* = \delta_{\bar{S}_*(M(4))}(c^* + d^*)$$

Therefore, we set

$$f^{[1]}\left(\begin{array}{c} c_1 \qquad c_1 \\ \diagdown \quad \diagup \\ 12 \\ | \end{array}\right) = c^* + d^*. \tag{4.36}$$

It is an easy generalization that

$$f^{[1]}\left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ 12..n \\ | \end{array}\right) = c^* + d^*$$

when there are two entries $a_i = c_1 = a_j$ where $i \neq j$, and all others are c_0. The remaining values of $f^{[1]}$ are easily seen to be zero.

We proceed to $\gamma^{[2]}$. The big difference to the example of $\bar{S}_*(S^1 \vee S^2)$ is that now there is no immediate reason why γ should vanish on trees with more than one vertex. We first discuss the values of $\gamma^{[2]}$ on one-vertex trees. It suffices to look at the following trees:

$$\begin{array}{c} c_1 \quad c_1 \\ \diagdown \diagup \\ 121 \\ | \end{array} \qquad \begin{array}{c} c_1 \quad c_2 \\ \diagdown \diagup \\ 121 \\ | \end{array} \qquad \begin{array}{c} c_1 \quad c_2 \\ \diagdown \diagup \\ 212 \\ | \end{array} \qquad \begin{array}{c} c_1 \; c_1 \; c_1 \\ \diagdown | \diagup \\ 1213 \\ | \end{array} \qquad \begin{array}{c} c_1 \; c_1 \; c_1 \\ \diagdown | \diagup \\ 1231 \\ | \end{array} \qquad \begin{array}{c} c_1 \; c_1 \; c_1 \\ \diagdown | \diagup \\ 2131 \\ | \end{array}$$

We perform the calculation for the first tree and list the results for the rest. In the defining equation we get $S_2^2 = 0 = S_4^2$. Thus, $\gamma^{[2]}$ evaluated on the first tree is given by the homology class of

$$\beta^{[2]}\left(\begin{array}{ccc} f^{[0]}(c_1) & & f^{[0]}(c_1) \\ & \diagdown \quad \diagup & \\ & 121 & \\ & | & \end{array}\right) = \beta^{[2]}\left(\begin{array}{ccc} a^*+c^*+e^* & & a^*+c^*+e^* \\ & \diagdown \quad \diagup & \\ & 121 & \\ & | & \end{array}\right) = a^*+c^*+e^*, \qquad (4.37)$$

i.e., by c_1. Here we use equalities (4.5), (4.7) and (4.9). Further, the value of $f^{[2]}$ on the same tree is zero. Similar, for the other five cases we get

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & 121 & \\ & | & \end{smallmatrix}\right) \overset{(4.22)}{=} c_2 \qquad f^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & 121 & \\ & | & \end{smallmatrix}\right) = 0 \qquad (4.38)$$

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & 212 & \\ & | & \end{smallmatrix}\right) \overset{(4.26)}{=} c_2 \qquad f^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & 212 & \\ & | & \end{smallmatrix}\right) = 0 \qquad (4.39)$$

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 1213 & \\ & | & \end{smallmatrix}\right) \overset{(4.10),(4.12)}{=} 0 \qquad f^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 1213 & \\ & | & \end{smallmatrix}\right) = c^* + d^* \qquad (4.40)$$

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 1231 & \\ & | & \end{smallmatrix}\right) = 0 \qquad f^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 1231 & \\ & | & \end{smallmatrix}\right) = 0 \qquad (4.41)$$

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 2131 & \\ & | & \end{smallmatrix}\right) \overset{(4.18),(4.20)}{=} 0 \qquad f^{[2]}\left(\begin{smallmatrix} c_1 & c_1 & c_1 \\ & 2131 & \\ & | & \end{smallmatrix}\right) = c^* + d^*. \qquad (4.42)$$

We come to trees with 2 vertices. The new information here is concentrated in

$$\begin{array}{cccc} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 12 & & \\ & | & & \end{array} \quad \text{and} \quad \begin{array}{cccc} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 21 & & \\ & | & & \end{array} .$$

By the defining equation the evaluation of $\gamma^{[2]}$ on the first tree is given by the homology class of

$$\beta^{[1]}\left(\begin{array}{ccc} & f^{[1]}\left(\begin{smallmatrix} c_1 & & c_1 \\ & 12 & \end{smallmatrix}\right) & \\ f^{[0]}(c_1) & & \\ & \diagdown \quad \diagup & \\ & 12 & \\ & | & \end{array}\right) = \beta^{[1]}\left(\begin{array}{ccc} a^*+c^*+e^* & & c^*+d^* \\ & \diagdown \quad \diagup & \\ & 12 & \\ & | & \end{array}\right) \overset{(4.3),(4.4)}{=} B^* + C^*.$$

Hence,

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 12 & & \\ & | & & \end{smallmatrix}\right) = 0 \quad \text{and} \quad f^{[2]}\left(\begin{smallmatrix} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 12 & & \\ & | & & \end{smallmatrix}\right) = c^*. \qquad (4.43)$$

Similar, one sees

$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 21 & & \\ & | & & \end{smallmatrix}\right) = 0 \quad \text{and} \quad f^{[2]}\left(\begin{smallmatrix} c_1 & & c_1 & c_1 \\ & & \multicolumn{2}{c}{12} \\ & 21 & & \\ & | & & \end{smallmatrix}\right) = 0. \qquad (4.44)$$

Now we have determined "the information encoded in $\gamma^{[2]}$", we can compare the canonical class of $\bar{S}_*(M(4))$ with both the one of $\bar{S}_*(S^1 \vee S^2)$ and the trivial one. Using (4.37) and the same argument as in the first example, $\bar{S}_*(S^1 \vee S^2)$, we see that $\gamma^{[2]}$ is not representing the trivial class in Gamma cohomology, and thus, $\bar{S}_*(M(4))$ is not formal.

Unfortunately, we can show that

$$\gamma^{[2]}_{\bar{S}_*(S^1 \vee S^2)} = \gamma^{[2]}_{\bar{S}_*(M(4))}.$$

If one compares $\alpha^{[2]}$ and $\gamma^{[2]}$, the former represents $\gamma^{[2]}_{\bar{S}_*(S^1 \vee S^2)}$ and the latter $\gamma^{[2]}_{\bar{S}_*(M(4))}$, then one finds out that they differ only on

$$\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 121 & \\ & | & \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 212 & \\ & | & \end{smallmatrix},$$

as well as on degenerates of these, i.e., where additional entries c_0 appear (cf. (4.38) and (4.39)). We define a map

$$g \colon \mathcal{B}(\mathcal{S})^{[1]} \circ H_*M \to H_*M$$

by

$$g\left(\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 12 & \\ & | & \end{smallmatrix}\right) = c_2,$$

and similarly for degenerates of this tree. Elsewhere the map is zero. We claim that in the derivation complex, g is sent to the difference $(\gamma^{[2]} - \alpha^{[2]})$. The differential of g in

$$\mathbf{Der}_{\mathrm{P}_\infty}(R_{H_*M}, H_*M) \cong \mathbf{Der}_{\mathrm{P}_\infty}(R_{\bar{S}_*(S^1 \vee S^2)}, \bar{S}_*(S^1 \vee S^2))$$

is given by

$$g \circ \partial_{R_{H_*M}} = g \circ \delta + g \circ \partial_{\gamma^{[1]}} + g \circ \partial_\omega,$$

where we keep the notation of Section 2.3, used also in Construction 4.1. Because of

$$\partial_{\gamma^{[1]}}\left(\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 121 & \\ & | & \end{smallmatrix}\right) = \partial_\omega\left(\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 121 & \\ & | & \end{smallmatrix}\right) = 0$$

we get:

$$g \circ \partial_{R_{H_*M}}\begin{pmatrix} c_1 & & c_2 \\ & \diagdown\;\diagup & \\ & 121 & \\ & | & \end{pmatrix} = g \circ \delta\begin{pmatrix} c_1 & & c_2 \\ & \diagdown\;\diagup & \\ & 121 & \\ & | & \end{pmatrix} =$$

$$g\begin{pmatrix} c_1 & & c_2 \\ & \diagdown\;\diagup & \\ & \partial(121) & \\ & | & \end{pmatrix} = g\begin{pmatrix} c_1 & & c_2 \\ & \diagdown\;\diagup & \\ & 12 & \\ & | & \end{pmatrix} + g\begin{pmatrix} c_1 & & c_2 \\ & \diagdown\;\diagup & \\ & 21 & \\ & | & \end{pmatrix} = c_2$$

Similar one calculates that $g \circ \partial_{R_{H_*M}}$ has the correct value on $\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 212 & \\ & | & \end{smallmatrix}$, as well. The rest of the calculation is verifying that $g \circ \partial_{R_{H_*M}}$ vanishes on trees with two vertices. We leave this to the interested reader.

Remark 4.15. Note that the equality

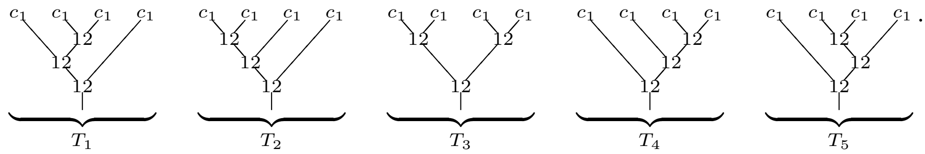
$$\gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 121 & \\ & | & \end{smallmatrix}\right) = \gamma^{[2]}\left(\begin{smallmatrix} c_1 & & c_2 \\ & \diagdown\,\diagup & \\ & 212 & \\ & | & \end{smallmatrix}\right)$$

is the reason why the map g can be constructed. This equality follows by formal arguments not depending on our particular example $M(4)$. It can be concluded from the fact

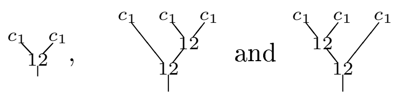
$$\delta(1212) = 212 + 121.$$

Comparing $\bar{S}_*(S^1 \vee S^2)$ and $\bar{S}_*(M(4))$: We want to use Theorem 4.10 to show that our two examples have different homotopy type. Therefore, we have to compare the corresponding transferred P_∞-structures on homology, and show that there can not be any map of quasi-cofree coalgebras. The structure map α vanishes on trees with more than one vertex. This does not have to be the case for γ. We are going to determine a non-trivial operation of γ on a particular tree with 3 vertices. This will lead to a contradiction under the assumption of the existence of a map as in Theorem 4.10.

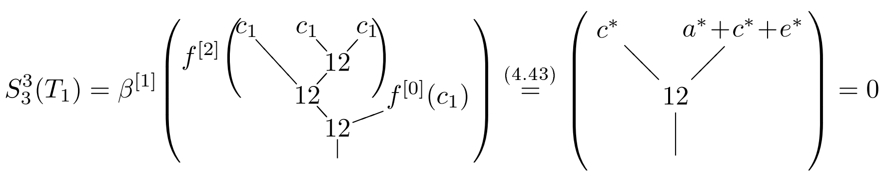
We proceed in the following way: First we calculate the evaluation of $\gamma^{[3]}$ on the trees

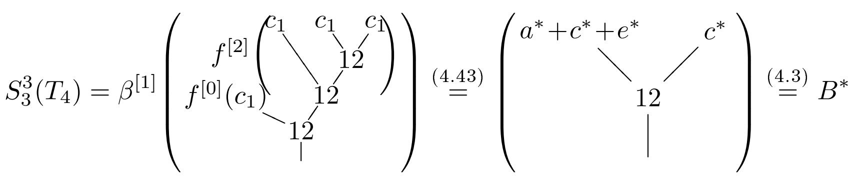


We show $\gamma^{[3]}(T_4) = c_2$ and $\gamma^{[3]}(T_1) = \gamma^{[3]}(T_2) = \gamma^{[3]}(T_3) = \gamma^{[3]}(T_5) = 0$. After that we analyze the equation that have to be satisfied if a map of quasi-cofree coalgebras between $(\bar{S}_*(S^1 \vee S^2), \alpha)$ and (H_*M, γ) exists (Proposition 1.47) for the element $(T_1 + T_2 + T_3 + T_4 + T_5)$. We use the notation from p.71 for the calculation of $\gamma^{[3]}$. Recall that $\gamma^{[3]}(T_i)$ is represented by the class of $S_2^3(T_i) + S_3^3(T_i) + S_4^3(T_i)$. Since γ vanishes on



by (4.34), (4.43) and (4.44), the term $S_2^3(T_i)$ is trivial for every $i = 1, \ldots, 5$. The term S_4 also vanishes for every one of the trees: For dimensional reasons one can set $f^{[2]}$ to be zero on every tree that results from T_i by the application of the differential (collapsing edges). We come to the factor S_3^3. The only summands that can survive are the ones where $\beta^{[1]}$ appears. We perform the computation for T_1 and T_2.





All together we conclude

$$\gamma^{[3]}(T_1) = 0 \tag{4.45}$$
$$\gamma^{[3]}(T_4) = c_2. \tag{4.46}$$

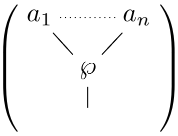
For the other trees one similarly computes that

$$\gamma^{[3]}(T_2) = \gamma^{[3]}(T_3) = \gamma^{[3]}(T_5) = 0. \tag{4.47}$$

Suppose now that there is a homomorphism

$$h\colon \mathcal{B}(\mathcal{S}) \circ H_*M \to \bar{S}_*(S^1 \vee S^2)$$

inducing a map of quasi-cofree coalgebras. In particular, f satisfies the condition of Theorem 4.10 with γ in place of γ', α in place of γ'' and



$$\in \{T_1, T_2, T_3, T_4, T_5\}.$$

Let us denote the three terms on the left hand side of the equation for h by $U_1(T)$, $U_2(T)$ and $U_3(T)$. They depend on the tree T on which they are evaluated. Since α is non-trivial on trees with only one vertex, and since in addition $\alpha^{[1]}$ is non-zero only when it performs multiplication with a unit (which can not be the case for dimensional reasons), we can conclude that

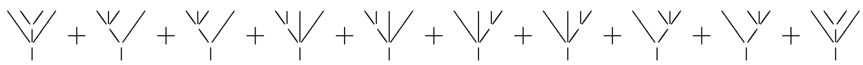
$$U_3(T_i) = 0 \text{ for } i \in \{1, 2, \ldots, 5\}.$$

There is a priori no reason for $U_1(T_i)$ to be trivial but the sum

$$\sum_{i=1}^{5} U_1(T_i)$$

is zero since

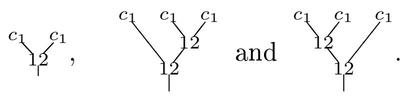
$$\partial(T_1 + T_2 + T_3 + T_4 + T_5) =$$



where all vertices are labeled by the identity and all leaves by c_1. Further,

$$\sum_{i=1}^{5} U_2(T_i) = \sum_{i=1}^{5} f^{[0]}(\gamma^{[3]}(T_i)) \overset{(4.45)-(4.47)}{=} c_2.$$

Here, we again used the fact that γ vanishes on



All together we get $c_2 = 0$ which is the desired contradiction.

Remark 4.16. What we effectively did in this last paragraph is to calculate that c_2 is an element of the 4-fold Massey product

$$c_2 \in \langle c_1, c_1, c_1, c_1 \rangle$$

of H_*M, and this is the reason why H_*M is not quasi-isomorphic to $\bar{S}_*(S^1 \vee S^2)$ as an E_∞-algebra. In particular, they are not quasi-isomorphic also as A_∞-algebras.

In conclusion we want to briefly mention the main result of Mandell [Man, Main Theorem]. It relates the homotopy category of connected p-complete nilpotent spaces of finite p-type to the homotopy category of E_∞ $\bar{\mathbb{F}}_p$-algebras via a contravariant full inclusion of categories. Mandell's Theorem shows the potential to distinguish this particular kind of spaces via the E_∞-structure on their normalized cochains. In particular our theory can be used therefor.

References

[And] M. André *Homologie des Algébres Commutatives* Grundlehren der mathematischen Wissenschaften **206** Springer-Verlag, 1974.

[Bal] D. Balavoine *Homology and cohomology with coefficients, of an algebra over a quadratic operad* J. Pure Appl. Algebra **132** (1998), 221-258.

[Bas] M. Basterra *Andre-Quillen Cohomology of Commutative S-algebras* Journal of Pure and Applied Algebra **144** (1999), 111-143.

[BasMan] M. Basterra, M. Mandell *Homology and Cohomology of E-infinity Ring Spectra* Math. Zeitschrift **249** (2005), no. 4, 903-944.

[BR] M. Basterra, B. Richter *(Co)-Homology Theories for Commutative S-algebras, in Structured Ring Spectra* 115-131, London Math. Society, Lecture Notes Series **315**, Cambridge Univ. Press, Cambridge, 2004.

[BE] M. Barratt, P. Eccles *On Γ_+-structures I. A free group functor for stable homotopy theory* Topology **13** (1974), 25-45.

[BKS] D. Benson, H. Krause, S. Schwede *Realizability of modules over Tate cohomology* Trans. Amer. Math. Soc. **356** (2004), 3621-3668.

[Ber] C. Berger *Opérades cellulaires et espaces de lacets itérés* Ann. Inst. Fourier **46** (1996), 1125-1157.

[BF] C. Berger, B. Fresse *Combinatorial operad actions on cochains* Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 135-174.

[BM] C. Berger, I. Moerdijk *Axiomatic homotopy theory for operads* Comment. Math. Helv. **78** (2003), 805-831.

[BM06] C. Berger, I. Moerdijk *The Boardman-Vogt resolution of operads in monoidal model categories* Topology **45** (2006), no. 5, 807-849.

[BoVo] J. M. Boardman, R. M. Vogt *Homotopy-everything H-spaces* Bull. Amer. Math. Soc. **74** (1968), 1117-1122.

[BV] J. M. Boardman, R. M. Vogt *Homotopy invariant algebraic structures on topological spaces* Lect. Notes Math. **347** (1973).

[Fre04] B. Fresse *Koszul duality of operads and homology of partition posets* in "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory", 115-215, Contemp. Math. **346**, Amer. Math. Soc. (2004).

[Fre09] B. Fresse *Operadic cobar constructions, cylinder objects and homotopy morphisms of algebras over operads* in "Alpine perspectives on algebraic topology" Contemp. Math. **504**, Amer. Math. Soc. (2009), 125-189.

[Fre09b] B. Fresse *Modules over operads and functors.* Lect. Notes Math. Springer Verlag (2009).

[Fre09c] B. Fresse *Koszul duality of E_n-operads* arXiv:0904.3123v6

[Fre10] B. Fresse *Props in model categories and homotopy invariance of structures* Georgian Math. J. **17** (2010), 79-160.

[GH] P. Goerss, M. Hopkins *André-Quillen (co)-homology for simplicial algebras over simplicial operads* Contemp. Math. **265** Amer. Math. Soc., Providence, RI, 2000

[GJ] E. Getzler, J. Jones *Operads, homotopy algebra, and iterated integrals for double loop spaces* preprint: http://arxiv.org/pdf/hep-th/9403055

[GK] V. Ginzburg, M. Kapranov *Koszul duality for operads* Duke Math. J. **76** (1995), 203-272.

[GV] J. J. Gutiérrez, R. M Vogt *A Model Structure for Coloured Operads in Symmetric Spectra* arXiv:1006.2316v2

[Har] J. Harper *Homotopy theory of modules over operads in symmetric spectra* Algebr. Geom. Topol. **9** (2009), no. 3, 1637-1680.

[Harr] D. Harrison *Commutative algebras and cohomology* Trans. Amer. Math. Soc. **104** (1962), 191-204.

[Hin] V. Hinich *Homological algebra of homotopy algebras* Comm. Algebra **25** (1997), 3291-3323.

[Hoff10] E. Hoffbeck *Γ-homology of algebras over an operad* Algebr. Geom. Topol. **10** (2010), no. 3, 1781-1806.

[Ho] M. Hovey *Model categories* Mathematical Surveys and Monographs of the Amer. Math. Soc. **63** (1999).

[Hue] J. Huebschmann *The Lie algebra perturbation lemma* Higher structures in geometry and physics, 159-179, Progr. Math. **287**, 2011.

[Joy] A. Joyal *Foncteurs analytiques et espèces de structures* Combinatoire enumerative (Montreal, Que., 1985/Quebec, Que., 1985), Lecture Notes in Math., vol. 1234, Springer, Berlin, 1986, pp. 126-159.

[Kad] T. Kadeishvili *On the homology theory of fibre spaces* International Topology Conference (Moscow State Univ., Moscow, 1979) Uspekhi Mat. Nauk **35** (1980), no. 3(213), 183-188.

[Kad88] T. Kadeishvili *The structure of the $A(\infty)$-algebra, and the Hochschild and Harrison cohomologies* Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR **91**, (1988), 19-27.

[Kel] B. Keller *Introduction to A-infinity algebras and modules* Homology, Homotopy Appl. **3** (2001), no.1, 1-35.

[Kro] T. A. Kro *Model structure on operads in orthogonal spectra* Homology, Homotopy Appl. **9** (2007), no. 2, 397-412.

[Laz] A. Lazarev *Cohomology theories for highly structured ring spectra* Structured ring spectra, 201-231, London Math. Soc., Lecture Note Series **315**, Cambridge Univ. Press, Cambridge, 2004.

[LV] J.-L. Loday, B. Vallette *Algebraic Operads* http://math.unice.fr/∼brunov/Operads.pdf

[MS] J. McClure, J. Smith *Multivariable cochain operations and little n-cubes* J. Amer. Math. Soc. **16** (2003), no. 3, 681-704.

[Man] M. Mandell E_∞*-algebras and p-adic homotopy theory* Topology **40** (2001), no. 1, 43-94.

[Mar96] M. Markl *Models for operads* Comm. Algebra **24** (1996), no. 4, 1471-1500.

[Mar04] M. Markl *Homotopy algebras are homotopy algebras* Forum Math. **16** (2004), no. 1, 129-160.

[MSS] M. Markl, S. Shnider, J. Stasheff *Operads in algebra, topology and physics* Mathematical Surveys and Monographs **96**, Amer. Math. Soc., Providence, Rhode Island, 2002.

[May] J. P. May *The Geometry of Iterated Loop Spaces* Lectures Notes in Mathematics **271**, Springer Verlag, 1972.

[Qui] D. Quillen *Homology of commutative rings* Mimeographed notes, MIT 1968.

[Re] C. Rezk *Spaces of Algebra Structures and Cohomology of Operads* PhD Thesis, MIT, 1996

[Rob] A. Robinson *Gamma homology, Lie representations and E_∞ multiplications* Invent. Math. **152** (2003), 331-348.

[RW] A. Robinson, S. Whitehouse *Operads and Γ-homology of commutative rings* Math. Proc. Cambridge Philos. Soc. **132** (2002), 197-234.

[Sch] S. Schwede *Morita theory in abelian, derived and stable model categories* in Structured Ring Spectra, 33-86, London Mathematical Society Lecture Notes **315.**

[Smi] J. H. Smith *Simplicial group models for $\Omega^n\Sigma^n X$* Israel J. Math. **66** (1989), 330-350.

[SS00] S. Schwede, B. Shipley *Algebras and modules in monoidal model categories* Proc. London Math. Soc. (3) **80** (2000), no. 2, 491-511.

[Spi] M. Spitzweck *Operads, algebras and modules in general model categories* PhD Thesis, Bonn (2001).

[Sta] J. Stasheff *Homotopy associativity of H-spaces I, II* Trans. Amer. Math. Soc. **108** (1963), 275-312.

[Ste] N. E. Steenrod *Products of cocycles and extensions of mappings* Ann. of Math. (2) **48** (1947), 290-320.

[Whi] S. A. Whitehouse *Gamma (co)homology of commutative algebras and some related representations of the symmetric group* PhD Thesis, University of Warwick

Notation

$\mathcal{C}$	symmetric monoidal category with small colimits and finite limits
k	commutative ring, often a field; also chain complex with k in degree 0
$\mathcal{C}h$	category of unbounded chain complexes over k
$\mathbf{Hom}(-,-)$	internal Hom object in $\mathcal{C}h$
Σ_n	symmetric group on n letters
Σ_*	refers to the sequence of symmetric groups
Σ_*-sequence	symmetric sequence
$\mathcal{B}ij$	category of finite sets and empty set as objects, morphisms are bijections
$\underline{\mathrm{n}}$	$\{1, 2, \ldots, n\}$
$\mathcal{N}$	Normalization functor
$\mathbb{I}$	monoidal unit in $\mathcal{C}$
$\otimes$	the symmetric monoidal product in $\mathcal{C}$
$\underline{\sigma}$	denotes a sequence of permutations $(\sigma_0, \ldots, \sigma_d)$
$\mu_{i,j}(\underline{\sigma})$	definition in Example 1.10
P, Q	symmetric sequence, usually (connected) operad
T	symmetric sequence, usually a (connected) cooperad
$\mathcal{O}p(\mathcal{C})$	category of operads in $\mathcal{C}$
$\mathcal{C}o\mathcal{O}p(\mathcal{C})$	category of cooperads in $\mathcal{C}$
M, N	symmetric sequence
$S(\mathrm{P})$	Schur functor associated to P
$\mathrm{M} \circ \mathrm{N}$	composition product of symmetric sequences
$\mathrm{P} \circ C$	is $S(\mathrm{P})(C)$ if C is an object of $\mathcal{C}$
$\mathrm{M}(I)$	symmetric sequence as a contravariant functor evaluated at the set I
$S(\mathrm{M})$	Schur functor applied to M
$\mathcal{C}^{\mathbf{\Sigma}_*}$	category of symmetric sequences in the category $\mathcal{C}$
I	monoidal unit in the category of symmetric sequences
γ, γ_P	operadic composition product
μ, μ_{P}	operadic composition product
ν	comultiplication of a cooperad
ν_2	quadratic coproduct of a cooperad
ν_2'	"reduced" quadratic coproduct, cf. 1.3
ν_2''	"reduced" quadratic coproduct, cf. 1.3
ι	unit map of an operad
η	unit map of an operad or coaugmentation of a cooperad
ϵ	augmentation of an operad
γ_A	composition product of an operadic algebra
μ_A	composition product of an operadic algebra
ρ	composition coproduct of a cooperad coalgebra
ρ_2	quadratic coaction (composition coproduct) of a coalgebra
$\mathrm{End_A}$	endomorphism operad of A
$E\Sigma_n$	classifying simplicial space of Σ_n
$\mathrm{E}\Sigma_*$	chain Barratt-Eccles operad as defined in Example 1.9
$\mathrm{E_n}$	refers usually to the concrete $\mathrm{E_n}$-operad constructed in Example 1.10
Ass	non-unital chain associative operad
Com	non-unital commutative operad

$\tilde{\mathrm{P}}$	augmentation ideal/coaugmentation coideal of an operad/cooperad P
P-alg, Q-alg	category of algebras over the operad P resp. Q
$\Theta_2(I)$	category of I-trees with 2 levels, morphisms are isomorphisms of I-trees
$\Theta(I)$	category of I-trees and isomorphisms of I-trees as morphisms
$V(\tau)$	vertex set of a tree τ
$(f_!, f^*)$	pair of adjoint functors induced by f
$\mathcal{F}(\mathrm{M})$	free operad generated by M
$\tilde{\mathcal{F}}(\mathrm{M})$	augmentation ideal of a free operad
$\mathcal{F}^c(\mathrm{M})$	cofree cooperad cogenerated by M
$\mathrm{M}(\emptyset)$	corresponds to $\mathrm{M}(0)$
$(\mathcal{F}(\mathrm{M}), \partial_\alpha)$	quasi-free operad with twisting differential ∂_α
$(\mathcal{F}^c(\mathrm{M}), \partial_\alpha)$	quasi-cofree cooperad with twisting differential ∂_α
$(\mathrm{P} \circ C, \partial_\alpha)$	quasi-(co)free (co)algebra with twisting (co)differential ∂_α
$\mathcal{B}^c$	Cobar construction
$\mathcal{B}$	Bar construction
$\mathcal{C}o\mathcal{O}p(\mathcal{C}h)_{con}$	category of connected cooperads
$\mathcal{O}p(\mathcal{C}h)_{con}$	category of connected operads
$\mathcal{T}w(\mathrm{T}, \mathrm{P})$	set of twisting morphisms
μ_M	structure map of an algebra representation M
$\mathbf{Der}(A, M)$	set of derivations from A to M
$\Omega^1_{\mathrm{P}}(A)$	module of Kähler differentials
$\mathcal{R}ep^A_{\mathrm{P}}$	category of representations of a P-algebra A
A^{en}	enveloping algebra of A
$\text{P-alg}_{/A}$	category of P-algebras over A
$co\mathcal{C}h$	category of cochain complexes
$H^*_{\mathrm{P}}(A; M)$	operadic cohomology of A
$H^*_{\mathrm{P}}(A)$	operadic cohomology of A with coefficients in A
$H\Gamma^*(A; M)$	Gamma cohomology of A
$H\Gamma^*(A)$	Gamma cohomology with coefficients in A
G	discrete group
$\mathcal{F}$	free functor
$\mathcal{U}$	forgetful functor
Σ_*-cofibrant	cofibrant symmetric sequence
P_∞, Q_∞	cofibrant operads; sometimes replacement for given operads P and Q
H_*A	homology of A
$R_{\mathrm{P}}\,(\mathrm{T} \circ A, \partial_\alpha)$	particular quasi-free P-algebra, c.f. §2.3
$\gamma^{[2]}_A$, $\gamma^{[s]}_A$	universal class of the algebra A resp. higher universal classes of A
$H^1_{\mathrm{P}_\infty}(H_*A)$	first Gamma cohomology group of H_*A with coefficients in H_*A
$H_*\mathrm{P}$	homology operad associated to P
R_{H_*A}	particular cofibrant replacement of H_*A
$\wp, \wp'$ and $\wp''$	elements of $\mathcal{B}(\mathrm{P} \otimes \mathrm{E}\Sigma_*)$, or more general - elements of a cooperad
$H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$	direct sum factor of $H^1_{\mathrm{P}_\infty}(H_*A)$
$\tilde{H}^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$	certain quotient of $H^{s,1-s}_{\mathrm{P}_\infty}(H_*A)$
$\mathrm{P}^{¡}$	Koszul dual cooperad of an operad P
$\bar{T}A$	reduced tensor coalgebra of A
$S_*(X)$	normalized integer chains
$S^*(X)$	normalized integer cochains
$\tilde{S}_*(X)$	normalized cochains with coefficients in k and chain grading
$\langle f \rangle$	natural transformation induced by a non-degenerate surjection

$\mathcal{S}$	surjection operad
Δ^r	standard r-simplex
$\sigma(a_0, \ldots, a_n)$	see for definition p.63
$M(4)$	mapping cone of the map $(\cdot 4)$ on S^1