Weak and Measure-Valued Solutions of the Incompressible Euler Equations

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Emil Wiedemann
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Erstgutachter:	Prof. Dr. László Székelyhidi Jr.
Zweitgutachter:	Prof. Dr. Herbert Koch
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Abstract

This thesis is concerned with the existence problem for weak solutions of the incompressible Euler equations in arbitrary dimension, and with the relationship between weak solutions and other "very weak" concepts of solution. In particular, measure-valued solutions as introduced by R. DiPerna and A. Majda (*Oscillations and concentrations in weak solutions of the incompressible fluid equations.* Comm. Math. Phys., 108(4):667-689, 1987) are studied.

There are three main results of this thesis: Theorem 1.1 asserts the global existence of weak solutions for the incompressible Euler equations. However, these solutions are physically not admissible since their kinetic energy increases at least at the initial time. Moreover, the solutions constructed are highly non-unique in the sense that there exist infinitely many weak solutions with the same initial data. Concerning admissible weak solutions (i.e. such whose energy never exceeds the initial energy), the second result, Theorem 1.2, shows that they exist globally in time at least for an L^2-dense subset of initial data. The last result, Theorem 1.3, elucidates the relationship between weak and measure-valued solutions: It is shown that every measure-valued solution is generated by a sequence of weak solutions and that therefore, surprisingly, weak solutions are as flexible as measure-valued solutions.

A common feature of these results is their relying on methods recently developed by C. De Lellis and L. Székelyhidi Jr. (*On admissibility criteria for weak solutions of the Euler equations.* Arch. Ration. Mech. Anal., 195(1):225-260, 2010). This thesis includes a fairly detailed presentation of these methods.

Preface

Res severa verum gaudium.

During my time as a PhD student in Bonn and Leipzig I have encountered a great deal of exciting mathematics as well as many friendly and interesting mathematicians. First and foremost, I would like to thank Professor László Székelyhidi for supervising this thesis. I have been very lucky to be the student of such a distinguished researcher, who introduced me to some deep and intriguing mathematics. Needless to say this piece of work would not have been possible without his kind and thoughtful guidance.

My studies have been supported by the Bonn International Graduate School in Mathematics at the Hausdorff Center in Bonn. Besides supporting me financially, the Graduate School evoked an extremely stimulating atmosphere to study mathematics together with many highly motivated fellow students, and provided every opportunity I could wish for. I have also enjoyed support by the German Scholarship Foundation (Studienstiftung des deutschen Volkes), to which I am very grateful.

It has been my great pleasure to attend summer schools, conferences, and the like at the following institutions: Università di Roma "La Sapienza", Scuola Normale Superiore di Pisa, University of Cambridge, ICMAT Madrid, University of Warwick, SISSA Trieste, and Carnegie Mellon University. I would like to thank all of them for their very kind hospitality. I am also deeply indebted to Christian Seis and Professor Gianluca Crippa for inviting me to the Max Planck Institute in Leipzig and the Universität Basel, respectively.

It is impossible to list all the fellow students and senior mathematicians with whom I had interesting discussions. At this point I only wish to mention that it was great to have Filip Rindler as a visitor to Bonn, and that I have benefited a lot from talking to him.

I have experienced recreation and *verum gaudium* when meeting my dear friends living in Fürth, München, Bonn, Hamburg, and Berlin. I have always enjoyed returning to my parents' place and heartily thank them for their support. Last but most, I am truly thankful to my wonderful girlfriend Caroline for all her love and support.

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Contents

Chapter 1

Introduction

1.1 The Incompressible Euler Equations

This thesis is concerned with the incompressible Euler equations of ideal fluid motion in $d \geq 2$ dimensions,

$$\begin{aligned} \partial_t v(x,t) + \operatorname{div}\left(v(x,t) \otimes v(x,t)\right) + \nabla p(x,t) &= 0 \\ \operatorname{div} v(x,t) &= 0. \end{aligned} \tag{1.1}$$

This nonlinear system of partial differential equations was derived by Leonhard Euler in 1757. It describes the motion of an incompressible fluid with velocity field $v : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ and scalar pressure $p : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ in the absence of external forces. Here, the time T can be positive or infinity, $v \otimes v$ denotes the matrix with entries $v_i v_j$, and the divergence of $v \otimes v$ is taken row-wise. The term "ideal" is synonymous with "inviscid", i.e. it refers to the absence of viscosity and thus the absence of effects of friction within the fluid.

The first equation (or rather, the first d equations) in (1.1) follows from Newton's Second Law (or, equivalently, the conservation of momentum), whereas the last equation reflects the incompressibility of the fluid. To see the latter, let $\Omega' \subset \mathbb{R}^d$ and observe that for an incompressible fluid with constant density, the total flux across $\partial\Omega'$ must be zero:

$$\int_{\partial\Omega'} v(x,t) \cdot n(x) dS(x) = 0,$$

where $n(x)$ is the outer unit normal of $\partial\Omega'$ at x. The divergence theorem and the fact that Ω' can be arbitrarily chosen then imply $\operatorname{div} v = 0$.

Regarding the first equation, let $X(x,t)$ denote the position at time t of a fluid particle that was located at point x at time 0. For this particle trajectory map we have

$$\frac{d}{dt} X(x,t) = v\left(X(x,t),t\right),$$

and Newton's Second Law for this particle reads

$$\frac{d^2}{dt^2} X(x,t) = -\nabla p\left(X(x,t),t\right)$$

since there are no external forces. Combining the last two equations and using the chain rule, we obtain

$$\begin{aligned}-\nabla p\left(X(x,t),t\right) &= \frac{d}{dt}v\left(X(x,t),t\right)\\ &= \partial_t v\left(X(x,t),t\right) + \frac{d}{dt}X(x,t)\cdot\nabla v\left(X(x,t),t\right)\\ &= \partial_t v\left(X(x,t),t\right) + v\left(X(x,t),t\right)\cdot\nabla v\left(X(x,t),t\right),\end{aligned}$$

from which the first equation in (1.1) follows since v is divergence-free and X is a bijection. Thus we have derived, in a non-rigorous way at least, the Euler equations from basic physical assumptions.

The Cauchy problem for (1.1) now consists of finding solutions v and p to (1.1) such that

$$v(\cdot,0) = v_0$$

for a given divergence-free initial velocity field v_0.

Of course one can also study these equations on a domain $\Omega \subset \mathbb{R}^d$, but then a boundary condition is required. For Euler, one usually imposes

$$v(x,t)\cdot n(x) = 0 \text{ on } \partial\Omega,$$

where $n(x)$ is the outer unit normal to $\partial\Omega$ at x. This condition ensures that the fluid cannot flow through the boundary of the domain. However, in this thesis, I consider the Euler equations only on the whole space or with periodic boundary conditions, i.e. when v_0, v, and p are assumed to be periodic in the space variable.

A fundamental quantity in the study of the Euler equations is the *kinetic energy*

$$\frac{1}{2}\int_{\mathbb{R}^d}|v(x,t)|^2dx.$$

Suppose that v and p are a smooth solution of (1.1), and that v decays sufficiently fast at spatial infinity. Then the kinetic energy is conserved in time, as can be seen by the following simple calculation:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|v|^2dx &= \int_{\mathbb{R}^d}v\cdot\partial_t v dx\\ &= -\int_{\mathbb{R}^d}v\cdot\operatorname{div}(v\otimes v)dx - \int_{\mathbb{R}^d}v\cdot\nabla p dx\\ &= -\int_{\mathbb{R}^d}\sum_{i,j}v_i v_j\partial_j v_i dx + \int_{\mathbb{R}^d}p\operatorname{div} v dx = 0.\end{aligned}$$

To see that the first integral in the last line is indeed zero, use an integration by parts and the divergence-free property of v to find that

$$\int_{\mathbb{R}^d}\sum_{i,j}v_i v_j\partial_j v_i dx = -\int_{\mathbb{R}^d}\sum_{i,j}\partial_j v_i v_j v_i dx,$$

hence it must be zero.

Another elementary property of classical (i.e. smooth) solutions is their uniqueness. More precisely: If v and u are smooth and sufficiently decaying solutions of (1.1) with smooth pressure fields p and q respectively and with the same initial data v_0, then $v = u$. Indeed, if $\nabla_{sym} v = \frac{1}{2}(\nabla v + \nabla^T v)$ denotes the symmetric gradient and $d^-(v)$ the negative part of its smallest eigenvalue, we have the estimate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|v-u|^2dx &= 2\int_{\mathbb{R}^d}(v-u)\cdot\partial_t(v-u)dx\\ &= -2\int_{\mathbb{R}^d}(v-u)\cdot(v\cdot\nabla v-u\cdot\nabla u)\,dx - 2\int_{\mathbb{R}^d}(v-u)\cdot\nabla(p-q)dx\\ &= -2\int_{\mathbb{R}^d}(v-u)\cdot(v\cdot\nabla v-u\cdot\nabla u)\,dx\\ &= -2\int_{\mathbb{R}^d}(v-u)\cdot\nabla_{sym}v(v-u)dx\\ &\leq 2\left\|d^-(v)\right\|_{L^\infty_x}\int_{\mathbb{R}^d}|v-u|^2dx.\end{aligned} \tag{1.2}$$

Grönwall's inequality then yields

$$\int_{\mathbb{R}^d}|v(t)-u(t)|^2dx \leq \exp\left(\int_0^t 2\left\|d^-(v)\right\|_{L^\infty_x}ds\right)\int_{\mathbb{R}^d}|v(0)-u(0)|^2dx = 0 \tag{1.3}$$

because the two solutions were assumed to agree at time zero. In fact, a similar argument was used by P.-L. Lions to motivate his definition of dissipative solutions, see Section 3.2 below.

On the other hand, the existence of smooth solutions is unknown even for smooth initial data. What is known, however, is *local* existence; here, "local" refers to the time variable. In other words, given smooth und sufficiently decaying initial data v_0, there exists a finite time $T > 0$ such that there exists a smooth solution on $\mathbb{R}^d \times [0, T]$, which is unique by the above argument. This was proved for the first time by L. Lichtenstein in Chapter 11 of [35]. Various proofs are available for this assertion. For instance, it is possible to use a Faedo-Galerkin approach as demonstrated in Section 2.5 of [38]. To this end, one considers a Hilbert basis (e_i) for a suitable function space in x and solves a "truncated" version of the Euler equations in the finite-dimensional space spanned by $\{e_1, \ldots, e_N\}$ for the "truncated" initial data

$$v_0^N = \sum_{i=1}^N (v_0 \cdot e_i)e_i.$$

This amounts to solving a system of ordinary differential equations in the time variable for the first N Fourier coefficients of the desired solution. One thus obtains a sequence v^N of truncated solutions, which one can show to exist for all times. The difficulty then lies in proving that the sequence (v^N) converges in an appropriate topology to a solution of the Euler equations with initial data v_0. For this one exploits various estimates for the nonlinearity and, most importantly, a uniform estimate on a higher Sobolev norm of v^N (this is where the regularity assumption for v_0 comes into play). However this bound for the Sobolev norms may become infinite in finite time, and thus one only obtains local existence of the limit.

It is, however, completely open whether this local solution can be extended to a global smooth solution or if it blows up in finite time. The same question is open also for the Navier-Stokes equations

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= \nu \Delta v \\ \operatorname{div} v &= 0, \end{aligned} \tag{1.4}$$

which model the flow of a *viscous* fluid with viscosity $\nu > 0$. In fact, the question of global well-posedness for Navier-Stokes and Euler is considered one of the most challenging open problems in the theory of partial differential equations and is among the seven Millennium Prize Problems (see [23] for a detailed problem description). At this point it also seems appropriate to refer the reader to the books and surveys [3, 10, 17, 37, 38] for more information about various aspects of the study of ideal incompressible flows.

In the next section I would like to give an overview and summary of the further content of this thesis, and in particular I wish to explain what original results I have achieved.

1.2 Results and Organisation of the Thesis

The thesis is concerned with various weaker notions of solutions, which behave completely differently with respect to uniqueness and energy conservation than the classical solutions discussed so far. The motivation for the study of such weak or even "very weak" solutions comes, at least to some extent, from the fact that no global existence result is available for classical solutions of the 3-dimensional Euler equations, or that certain turbulent effects that one observes cannot be described in the framework of classical solutions.

In the study of partial differential equations it is a common phenomenon that it is not possible to directly construct classical solutions. Instead, one weakens the concept of solution to allow a priori for non-differentiable solutions and shows that such weak solutions exist. If one is lucky, one can then show that weak solutions are unique and regular for sufficiently regular boundary and initial data, and that therefore they are in fact classical solutions. The prime example for the success of this strategy is formed by linear elliptic equations, see e.g. [28].

A natural way to define such weak solutions is to integrate the given equation against a test function and then perform an integration by parts; more concretely, in the case of Euler, suppose first that v is a smooth solution of (1.1), and let $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ be a divergence-free vector field. Multiplication of (1.1) by ϕ, integration in space and time, and an integration by parts yield

$$\int_0^T \int_{\mathbb{R}^d} (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dx dt = 0, \tag{1.5}$$

where $A : B = \sum_{i,j} A_{ij} B_{ij}$ denotes the scalar product of two matrices. Similarly, for every function $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ the incompressibility condition implies

$$\int_0^T \int_{\mathbb{R}^d} v \cdot \nabla \psi dx dt = 0. \tag{1.6}$$

One says that $v \in L^2_{loc}(\mathbb{R}^d \times (0,T))$ is a *weak solution* to (1.1) if (1.5) and (1.6) are satisfied for every divergence-free $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ and every $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$, respectively. Chapter 2 is dedicated to the study of such weak solutions.

Note that, whereas v should be differentiable for (1.1) to make sense, (1.5) and (1.6) could conceivably hold even if v is only in L^2_{loc}. One can also incorporate the initial condition in the weak formulation: $v \in L^2_{loc}(\mathbb{R}^d \times [0,T))$ is then a weak solution with divergence-free initial data $v_0 \in L^2(\mathbb{R}^d)$ if

$$\int_0^T \int_{\mathbb{R}^d} (v \cdot \partial_t \phi + v \otimes v : \nabla \phi) dx dt + \int_{\mathbb{R}^d} v_0(x)\phi(x,0)dx = 0 \tag{1.7}$$

holds for every $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ (notice we are no longer assuming $\phi(x,0) = 0$) and (1.6) holds for every $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$. It is, however, not clear a priori in what sense a weak solution assumes its initial value - since v is only in L^2, it is not defined on the nullset $\{(x,0) : x \in \mathbb{R}^d\}$. We will see however that this issue is resolvable.

Another remark concerns the pressure, which has been eliminated in the weak formulation by testing only against divergence-free vector fields. One can show that, given a weak solution v, there exists an associated pressure in a weak sense; see e.g. Theorem 2.1 in [27].

We have just seen that every classical solution is a weak solution, but the converse is not true. Indeed, we will encounter many non-differentiable weak solutions in the course of this thesis. The simplest example of a non-classical weak solution is arguably given by a shear flow, which I will discuss in Section 2.6.

Since the weak formulation enables us to look for solutions in a much larger class of functions, we expect the existence problem to become easier and the uniqueness to become harder compared with the classical formulation. It turns out that not only is uniqueness of weak solutions difficult to prove, but it is even false. This has been known since V. Scheffer's groundbreaking counterexample [44]. Scheffer, and later Shnirelman [45], constructed a weak solution in $L^2(\mathbb{R}^2 \times \mathbb{R})$ with compact support in space and time. Obviously such a solution is physically meaningless: It suggests that a fluid which is perfectly at rest at some time suddenly starts moving and then, after another period of time, comes to rest again. Certainly, such a solution violates the conservation of energy, but we will see that energy conservation and other conditions related to the energy of the fluid provide no remedy for the non-uniqueness.

The results of Scheffer and Shnirelman have been reproduced and improved in a completely different framework by C. De Lellis and L. Székelyhidi Jr. [15,16]. Their proof is based on the so-called method of *convex integration* and is thus reminiscent of the isometric imbedding theorems [33, 42] of J. Nash and N. Kuiper and of the construction of nowhere differentiable solutions to elliptic systems [40] by S. Müller and V. Šverák.

The construction of De Lellis and Székelyhidi will be explained in some detail in Sections 2.2 and 2.3. Very roughly, the idea goes like this: First one reformulates the Euler equations (1.1) as the combination of the (highly underdetermined) linear system of partial differential equations

$$\begin{aligned} \partial_t v + \operatorname{div} u + \nabla q &= 0 \\ \operatorname{div} v &= 0 \end{aligned} \tag{1.8}$$

and the nonlinear pointwise constraints

$$u = v \otimes v - \frac{1}{d}|v|^2 I_d, \qquad q = p + \frac{1}{d}|v|^2, \tag{1.9}$$

so that u is a trace-free symmetric matrix. Given a pair (v, u) that satisfies (2.2) for some pressure field q, one defines the *generalised energy density* as

$$e(v, u) = \frac{d}{2}\lambda_{max}(v \otimes v - u),$$

where λ_{max} denotes the largest eigenvalue. This definition is justified by the observation that the generalised energy density coincides with the usual energy density $\frac{1}{2}|v|^2$ if and only if v and u satisfy (1.9). Otherwise, the generalised energy density is strictly greater than the usual one.

If $\bar{e}$ is a given energy density, a *subsolution* with respect to the initial value v_0 and the energy $\bar{e}$ is defined as a pair (v, u) which solves (1.8) for some q and such that $v(\cdot, 0) = v_0$ and $e(v(x,t), u(x,t)) \leq \bar{e}(x,t)$ for almost every $x \in \mathbb{R}^d$ and $t \in (0, T)$. If X denotes the space of velocity fields which can be complemented by some u to become a subsolution, then the functional

$$I(v) = \inf_t \int_{\mathbb{R}^d} \left(\frac{1}{2}|v|^2 - \bar{e}\right) dx$$

on X is non-positive and will be zero if and only if v is a weak solution of Euler with initial data v_0 and energy density $\frac{1}{2}|v|^2 = \bar{e}$. One can say that I measures how far a subsolution is from being an exact solution. The idea is now, in order to obtain such a solution, to start with some subsolution (v, u), for which $I(v) < 0$, and to add highly oscillatory perturbations to this subsolution in order to increase the value of the functional I, which is essentially the L^2-norm. An iteration of this perturbation process should then eventually yield an element of X whose functional I is zero.

One of the main difficulties of this strategy is that one has to ensure that the perturbed subsolution is again in X, that is, it still satisfies (1.8), and its energy density is still below $\bar{e}$. It is thus desirable to find highly oscillatory solutions of the system (1.8). Fortunately, these equations admit a large number of *plane wave solutions*, i.e. solutions of the form

$$(v(x,t), u(x,t)) = h(\xi \cdot x, \tau t)(\bar{v}, \bar{u}),$$

where (ξ, τ) is the wave direction in the domain, $(\bar{v}, \bar{u})$ is the wave direction in the range, and the profile function h can be chosen arbitrarily. In particular, h can be chosen to be periodic with a very small period (so that the wave oscillates at a high frequency).

On the other hand, the perturbation must be chosen differently at different points x and t, because the subsolution and the difference $\frac{1}{2}|v|^2 - \bar{e}$ depend on space and time. Consequently, it is necessary to localise such plane waves. I will discuss this plane wave analysis in Section 2.2.

With these tools at hand, one can then embark on the actual construction. It is a delicate issue to show that the perturbations can be chosen so large that they significantly increase the L^2-norm of the subsolution yet are small enough to assure that the perturbed subsolution is still contained in X. This will be discussed in Section 2.3. Since the choice of the perturbations is highly non-unique, one obtains by this procedure not only one solution, but infinitely many. In fact, the set of exact solutions is "fat" in the Baire sense in the weak topology of L^2. Keep in mind, however, that the starting point of the construction is a subsolution with respect to the desired initial data and energy density, so in order to prove the existence of infinitely many exact solutions, one first needs to exhibit a suitable subsolution, which is not trivial.

In Section 2.4, I prove that De Lellis' and Székelyhidi's approach can be used to establish a global existence (and non-uniqueness) theorem for weak solutions of Euler for arbitrary dimension $d \geq 2$, thus justifying one of the motivations behind the definition of weak solutions. This is the first original result of this thesis (it has also been published separately, see [50]), and it is the first global existence result for weak solutions of Euler. More precisely, we have (cf. Theorem 2.16):

Theorem 1.1. *Let $v_0 \in L^2(\mathbb{T}^d)$ be periodic and divergence-free. Then there exists a periodic weak solution $v \in L^\infty([0,\infty); L^2(\mathbb{T}^d))$ (in fact, infinitely many) of the Euler equations with $v(0) = v_0$.*

The idea of the proof is to find an appropriate subsolution by solving the *fractional heat equation*

$$\begin{aligned} \partial_t v + (-\Delta)^{1/2} v &= 0 \\ \operatorname{div} v &= 0 \\ v(\cdot, 0) &= v_0, \end{aligned}$$

which can easily be done by Fourier transform. One thus gets a solution explicitly given in terms of its Fourier series. Observing that $(-\Delta)^{1/2} v = -\operatorname{div} \mathcal{R} v$, where $\mathcal{R}$ denotes the Riesz transform, one then has a solution of (1.8) with $u = -\mathcal{R} v$ and $q \equiv 0$. Since everything is explicitly given, it poses no problem to verify that the generalised energy density $e(v,u)$ is bounded in $L^\infty([0,\infty); L^1(\mathbb{T}^d))$ and that one may therefore choose a suitable energy density $\bar{e}$ for the exact solutions.

The freedom to choose the energy density $\bar{e}$ is one of the aspects of De Lellis' and Székelyhidi's method that make it so powerful. In [16] they construct specific initial data which admit subsolutions - and thus exact solutions - with energy densities having certain properties. In Section 2.5 I will review these admissibility properties. One of them is the *weak energy inequality*, which is satisfied by a solution v if

$$\frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0|^2 dx$$

for all times t. This is a minimal physical requirement for a solution of Euler. Another (stronger) admissibility requirement would be the conservation of energy. In this thesis, an *admissible weak solution* refers to a weak solution satisfying the weak energy inequality. The examples in [16] show that none of the conceivable energy criteria - not even the local conservation of energy - single out a unique weak solution. In [16], admissible weak solutions that emerge from the mentioned construction, and that are therefore highly non-unique and irregular, are dubbed "wild", and a vector field v_0 admitting wild solutions is called "wild initial data". Among the examples in [16] there are also wild solutions with decreasing energy, a phenomenon which already Shnirelman [46] had discovered.

An apparent downside of the global solutions of Theorem 1.1 is that they are not admissible, as can easily be seen from the proof. Indeed, the solutions exhibit a discontinuous increase of their kinetic energy at time zero:

$$\liminf_{t\searrow 0} \frac{1}{2}\int_{\mathbb{T}^d} |v(x,t)|^2 dx > \frac{1}{2}\int_{\mathbb{T}^d} |v_0(x)|^2 dx.$$

The solution thus assumes its initial value in the sense that $v(t)$ converges to v_0 weakly in L^2, but not strongly, as $t \to 0$.

This lack of admissibility is the reason why these solutions are not dissipative in the sense of P.-L. Lions, and the existence theorem therefore does not contradict the weak-strong uniqueness for dissipative solutions and the local existence for smooth initial data (see Section 3.2 below).

Since weak-strong uniqueness and local existence for smooth data rule out wild (admissible) solutions with smooth initial data, it is natural to ask how large the set of wild initial data is. In particular, in [16] the question is posed whether this set is dense among all solenoidal L^2-vector fields. In this thesis, the question is answered with "yes":

Theorem 1.2. *Let H be the set of vector fields in $L^2(\mathbb{R}^d)$ that are weakly divergence-free. There exists a subset $\mathcal{E} \subset H$ which is dense in the strong topology of L^2 such that for every $v_0 \in \mathcal{E}$, there exist infinitely many weak solutions of Euler with initial data v_0 and constant energy, and infinitely many weak solutions with initial data v_0 and strictly decreasing energy.*

Since the proof is a consequence of Theorem 1.3 below and the proof thereof, it is presented in Section 4.5.

One particularly interesting example for non-uniqueness of admissible weak solutions is discussed in Section 2.6. In his recent paper [47], L. Székelyhidi applies the strategy of constructing a subsolution in order to obtain non-unique exact solutions to the case of vortex sheet initial data. More precisely, the initial data considered is the 2π-periodic extension of

$$v_0(x) = \begin{cases} e_1 & \text{if } x_d \in (0,\pi), \\ -e_1 & \text{if } x_d \in (-\pi,0), \end{cases}$$

and one is looking for a periodic solution. This initial vector field is called a "vortex sheet" since the vorticity $\operatorname{curl} v_0$ is a measure supported on the hyperplane $\{x_d = 0\}$. It is easy to verify that the stationary solution, $v(\cdot,t) = v_0$ for all $t \geq 0$, is a weak solution of the Euler equations. Székelyhidi now constructs other admissible solutions, some of them with decreasing energy, by exhibiting a particular subsolution. This subsolution is obtained from the entropy solution of Burgers' equation with an initial data related to v_0. Since Székelyhidi's solutions are admissible, his result is not merely a special case of Theorem 1.1. An interesting feature of his solutions is that they are constant except in a "turbulent zone" around the initial vortex sheet, and this turbulent zone expands in the x_d direction with constant speed. This could be interpreted as a mixing effect and might suggest that wild solutions are not mere mathematical artefacts, but reflect "real" turbulent behaviour.

Chapter 3 deals with other concepts of solution for the incompressible Euler equations. Various types of "very weak solutions", i.e. solutions even weaker than the weak solutions defined above, have emerged in the literature, e.g. Y. Brenier's generalised flows [5, 6], P.-L. Lions' dissipative solutions [36], R. DiPerna's and A. Majda's measure-valued solutions [20], or the subsolutions of De Lellis and Székelyhidi that I have already discussed. In this thesis I will present, besides the subsolutions, the dissipative solutions of Lions (Section 3.2) and, more extensively, the measure-valued solutions (Section 3.3). A common feature of dissipative and measure-valued solutions - and also of subsolutions - is that important examples of such solutions arise from the vanishing viscosity limit of weak solutions for Navier-Stokes. Therefore I would like to explain the problem of the vanishing viscosity limit in Section 3.1. The idea is that formally, the Navier-Stokes equations (1.4) turn into the Euler equations (1.1) in the vanishing viscosity limit $\nu \to 0$. Now, for the Navier-Stokes equations, the global existence of admissible weak solutions with initial data in L^2 has been known since the pioneering work [34] of J. Leray. Weak solutions are defined similarly for Navier-Stokes as for Euler, and the admissibility criterion for Navier-Stokes reads

$$\frac{1}{2}\int_{\mathbb{R}^d}|v(x,t)|^2dx \leq \frac{1}{2}\int_{\mathbb{R}^d}|v_0(x)|^2dx - \nu\int_0^t\int_{\mathbb{R}^d}|\nabla v(x,s)|^2dxds \tag{1.10}$$

(for smooth solutions, (1.10) holds with equality). The second term on the right hand side represents a loss of kinetic energy due to viscous frictional effects. If (v_ν) is now a family of such Leray solutions corresponding to viscosities $\nu \to 0$, it could be naively expected that (v_ν) converges in some sense to a solution of the Euler equations. Unfortunately, this is not true, at least if one expects the limit to be a weak solution. Indeed, from the energy inequality (1.10) one can deduce the weak convergence of a subsequence in $L^2(\mathbb{R}^d \times [0,T])$, but the weak limit need not be a solution of the Euler equations. This is because $v_\nu \rightharpoonup v$ does *not* imply $v_\nu \otimes v_\nu \rightharpoonup v \otimes v$ due to possible oscillation and concentration effects in the sequence. As a simple analogy, one may consider the sequence $f_n(x) = \sin(nx)$ on $[0,1]$, which obviously converges weakly to 0; however, the squares f_n^2 do not converge weakly to 0. Instead, one can follow two strategies: Either one furnishes a concept of solution which is weak enough to ensure that the weak limit is a solution in this sense; subsolutions or dissipative solutions are examples for such concepts. Or, alternatively, one alters the notion of weak convergence and shows that the Leray solutions converge in a larger space to a limiting object which is no longer an L^2 function but a parametrised measure. In a certain sense, this measure can be viewed as a solution for Euler. The strength of these measure-valued solutions is that they retain the information about oscillations and concentrations in the generating sequence, which is lost when only the usual weak limit is considered.

I will give and explain the definition of dissipative solutions in Section 3.2 below. Here I will just mention their most important properties. Lions describes the two essential merits of dissipative solutions in Section 4.4 of [36]: First, their global existence can be proved for arbitrary initial data (the existence is obtained precisely as a weak limit of a vanishing viscosity sequence of Navier-Stokes). Second, they have the *weak-strong uniqueness* property. The latter means that given a sufficiently regular solution of Euler up to time T, every dissipative solution with the same initial data agrees with it up to time T. "Sufficiently regular" means, more precisely, that

$$\int_0^T \left\|\nabla v + \nabla^T v\right\|_{L_x^\infty} dt < \infty.$$

The weak-strong uniqueness follows immediately from the very definition of dissipative solutions: Indeed, dissipative solutions are required by definition to satisfy a Grönwall inequality similar to (1.3) for all smooth divergence-free vector fields. Inserting the regular solution into the inequality then gives the uniqueness property.

Two particularly interesting classes of functions belong to the set of dissipative solutions: The vanishing viscosity limits, as mentioned before, and the admissible weak solutions of Euler (see Proposition 8.2 in [16]). The weak-strong uniqueness thus implies that if a sufficiently smooth solution of Euler exists for some initial data, any vanishing viscosity sequence of Navier-Stokes solutions with the same initial data converges to it on the interval of existence. Cf. also [9,39] for the local strong convergence of vanishing viscosity sequences. Another implication of the weak-strong uniqueness is that a non-uniqueness result like Theorem 1.1 cannot hold for *admissible* solutions with arbitrary initial data: Indeed, for smooth initial data there exists a local smooth solution, and non-uniqueness of admissible weak solutions would therefore contradict the weak-strong uniqueness.

The discussion of measure-valued solutions will require some preparation. As mentioned before, a measure-valued solution is no longer a vector field, i.e. a map $(x,t) \mapsto v(x,t)$, but a *parametrised measure* or *Young measure*, i.e. a map $(x,t) \mapsto \nu_{x,t}$, where for almost every x and t, $\nu_{x,t}$ is a probability measure on $\mathbb{R}^d$. The intuition is that a measure-valued solution does not give the deterministic velocity of the fluid at a certain point in space-time, but only a probability distribution for the velocity. Such measures were introduced by L. C. Young [51, 52] in order to study the relaxation of certain variational problems and have since then been employed as a useful tool in the calculus of variations (see e.g. [2,29,41]) and partial differential equations (for instance in [19,49]).

In dealing with weakly precompact sequences of solutions to the Euler or Navier-Stokes equations, it is necessary to extend the classical notion of Young measure. Indeed, the problem is that from a uniform energy bound, i.e. a bound in $L_t^\infty L_x^2$, one cannot conclude that the sequence in question is equi-integrable, as would be required in order to work with the classical Young measure. Instead, the conceivable occurrence of non-equi-integrability, i.e. of concentration effects in the sequence, can be described by an appropriate modification of the Young measure, which was developed by DiPerna and Majda [20]. In this thesis, we will use the modified framework of J. Alibert and G. Bouchitté [1] (see also [31]).

In their setup, a *generalised Young measure* is a triple of measures, consisting of

- the *oscillation measure* (or classical Young measure) $\nu_{x,t}$, which is a probability measure on $\mathbb{R}^d$ for Lebesgue-a.e. x and t;
- the *concentration measure* λ, which is a non-negative measure on $\mathbb{R}^d \times [0,T]$;

- the *concentration-angle measure* $\nu^\infty_{x,t}$, which is a probability measure on the $d-1$-dimensional unit sphere S^{d-1} for λ-a.e. x and t.

To define a notion of convergence of a sequence to a Young measure, we have to introduce a suitable class of test functions: Let $\mathcal{F}_2$ be the set of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$ such that the *L^2-recession function*

$$f^\infty(\theta) := \lim_{s\to\infty} \frac{f(s\theta)}{s^2}$$

is well-defined and continuous.

Let now (v_n) be a sequence of vector fields which is (uniformly) bounded in $L^2(\mathbb{R}^d \times [0,T])$. The *Fundamental Theorem for (generalised) Young measures* then states that there exists a subsequence (not relabeled) and a generalised Young measure $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ such that for all $f \in \mathcal{F}_2$, the following limit representation holds:

$$f(v_n(x,t))dxdt \overset{*}{\rightharpoonup} \int_{\mathbb{R}^d} f(z)d\nu_{x,t}(z)dxdt + \int_{S^{d-1}} f^\infty(\theta)d\nu^\infty_{x,t}(\theta)\lambda$$

in the sense of measures. We also use the notation $\langle \nu_{x,t}, f\rangle = \int_{\mathbb{R}^d} f(z)d\nu_{x,t}(z)$. In the situation of the Fundamental Theorem, one says that the subsequence (v_n) *generates* the Young measure.

The intuition behind this concept is that the sequence (v_n) may display oscillatory and concentrating behaviour, which is encoded in the Young measure generated by this sequence. The oscillation measure then contains the information about oscillations in the sequence; the concentration measure tells where in physical space concentration occurs; and the concentration-angle measure gives the direction of the concentrations wherever they occur (this is why $\nu^\infty_{x,t}$ is defined only λ-a.e.). The concrete examples given in Subsection 3.3.2 will hopefully elucidate this concept further.

In Subsections 3.3.3 and 3.3.4 we study how Young measure theory can be applied to the Euler equations. For this one should observe that the nonlinearity $v \otimes v$, which causes the problems in the vanishing viscosity limit, is a function of v that belongs to $\mathcal{F}_2$ and has the recession function $\theta\otimes\theta$ (this is actually the motivation for the definition of $\mathcal{F}_2$, cf. [20]). Therefore, given a sequence (v_n) of weak solutions of Navier-Stokes or Euler with uniformly bounded energy, by the Fundamental Theorem we conclude the existence of a subsequence and a Young measure generated by this subsequence, such that the Young measure satisfies the following Euler-like equations:

$$\begin{aligned} \partial_t\langle \nu_{x,t}, \xi\rangle + \operatorname{div}\left(\langle \nu_{x,t}, \xi\otimes\xi\rangle + \langle \nu^\infty, \theta\otimes\theta\rangle\lambda\right) + \nabla p = 0 \\ \operatorname{div}\langle \nu_{x,t}, \xi\rangle = 0 \end{aligned} \tag{1.11}$$

in the sense of distributions. A Young measure which satisfies these equations is called a *measure-valued solution* of the Euler equations. Although it is clear that every sequence of weak solutions of the Euler equations gives rise to such a measure-valued solution, the converse assertion - that for any measure-valued solution there exists a generating sequence of exact solutions for Euler - is not at all obvious. In fact, it is not something which one would expect, given that the notion of measure-valued solution is extremely flexible. This issue will be the subject of Chapter 4, where it turns out that the converse statement is actually true.

Before describing more extensively the contents of Chapter 4, I would like to briefly mention some properties of measure-valued solutions. Measure-valued solutions are clearly a generalisation of weak solutions: Given a weak solution v, it can be identified with the measure-valued solution $\nu_{x,t} = \delta_{v(x,t)}$, $\lambda = 0$. On the other hand, there exist measure-valued solutions which are not deterministic: Consider, for instance, the stationary vortex sheet solution v_0 of Section 2.6 and set $v_n(x) := v_0(nx)$. This corresponds to a sequence of shear flows where the different layers become thinner and thinner, thus exhibiting high-frequency oscillations. It is then not difficult to check that (v_n) generates the measure-valued solution $\nu_{x,t} = \frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{-e_1}$ (which is actually independent of x and t). More sophisticated explicit examples, also for concentrations, can be found in [20].

Moreover, it is possible to define a meaningful notion of initial data and of kinetic energy for measure-valued solutions. If the energy satisfies $E(t) \leq \frac{1}{2}\int |v_0|^2 dx$ for a.e. t, the measure-valued solution is called *admissible* in analogy with admissible weak solutions. Admissible measure-valued solutions then also satisfy Lions' two requirements (if I may call them so): First, they exist for all times, because a sequence of Leray solutions to Navier-Stokes with initial data v_0 generates an admissible measure-valued solution. Second, admissible measure-valued solutions surprisingly have the weak-strong uniqueness property, as proved in [7]: If there exists a smooth solution v for Euler until time T and $(\nu, \lambda, \nu^\infty)$ is an admissible measure-valued solution with the same initial data, then $\nu_{x,t} = \delta_{v(x,t)}$ and $\lambda = 0$ for $t \leq T$.

I have already mentioned the question whether every measure-valued solution can be generated by a sequence of weak solutions. Chapter 4 is dedicated to the proof that this is indeed the case. We have the following result:

Theorem 1.3. *a) If $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ is a measure-valued solution for $t \in [0, T]$ (where $T = \infty$ is allowed) with bounded energy, then there exists a sequence (v_n) of weak solutions of the Euler equations with uniformly bounded energy generating this measure-valued solution.*

b) If in addition $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ is admissible and takes the initial value v_0, then the v_n can be chosen to be admissible and to satisfy

$$\|v_n(\cdot, 0) - v_0\|_{L^2(\mathbb{R}^d)} \to 0 \quad \textit{as } n \to \infty.$$

This is another original result of this thesis (it is also contained in a joint article with L. Székelyhidi [48]). The Theorem shows that in a sense measure-valued solutions and weak solutions are essentially the same for the incompressible Euler equations in dimension $d \geq 2$. This is surprising because a priori, the concept of measure-valued solution seems to be much weaker than the concept of weak solution. Indeed, note that the defining equations (1.11) only constrain the first two moments of the Young measure. Apart from the expectation and the variance, hence, one has complete freedom to choose the measures $\nu_{x,t}$ and $\nu^\infty_{x,t}$ at every point (x, t). In particular, measure-valued solutions merely describe the one-point statistics of the velocity field in a weakly convergent sequence, i.e.

they are not able to account for correlations of the velocities at different space-time points. Theorem 1.3 states that the same is true for weak solutions. We also see that sequences of weak solutions allow for *any* conceivable combination of oscillations and concentrations. In this aspect, Theorem 1.3 generalises the considerations in [20], where only specific examples of oscillatory and concentrating sequences were constructed.

At this point the remark is in order that generalised Young measures are of importance not only in fluid mechanics, where they emerged, but have also been recognised a useful tool in the calculus of variations. In particular, the question has been of some interest how Young measures that arise from certain constrained sequences can be characterised: The prototypic result is the theorem of Kinderlehrer and Pedregal [30] which states that a (classical) Young measure is generated by a sequence of gradients if and only if it satisfies a certain Jensen-type inequality. The result has been generalised to so-called $\mathcal{A}$-free sequences [25] and to generalised Young measures [24, 26, 31]. Theorem 1.3 also gives a characterisation of Young measures that are generated by a constrained sequence (namely a sequence of Euler solutions), but it differs from the previously known results in two important respects: First, our problem does not fit into the $\mathcal{A}$-free framework since the so-called constant rank condition is not satisfied; and second, our sequence not only satisfies a linear system of PDE's, but in addition a nonlinear pointwise constraint. More concretely, not only do we generate the Young measure with an $\mathcal{A}$-free sequence, but with a sequence of exact solutions of the Euler equations.

A few words should be said about the proof of Theorem 1.3. In a first step, the weak density of exact solutions in the space of subsolutions, as proved in [16], is exploited, so that it suffices to generate the desired Young measure by subsolutions (Section 4.1). There is a technical issue here because subsolutions, viewed as pairs (v, u), take values in a different space than exact solutions v. Therefore we have to adjust the notion of Young measure to this particular situation. This is however possible in a canonical way, see Subsection 3.3.5. In Section 4.2 we then use more or less standard Young measure techniques to reduce to the case of discrete homogeneous oscillation measures. This step does not use any specific properties of the Euler equations. The generation of such homogeneous discrete measures is then achieved via an explicit laminate construction (Section 4.3), which is possible by virtue of the large wave cone of the linear system 1.8 (see Section 2.2). This is in contrast to the Hahn-Banach argument that is usually employed to characterise homogeneous Young measures (as for instance in [25, 30]). In the laminate construction we also use cutoff techniques introduced in [15], see Section 2.2. Finally, with the construction outlined so far, we do not quite achieve the admissibility of our generating sequence. In a final step in Section 4.4 this will be fixed using a convex integration argument similar to the one in Section 5 of [16].

The thesis concludes with a brief overview of open questions related to the research presented here.

1.3 Basic Notation

Let us fix some notation that we will use throughout this paper. We will denote by $\mathcal{M}^+(X)$ and $\mathcal{M}^1(X)$ the space of positive finite measures and probability measures on a measurable space X respectively, and for an open or closed subset $U \subseteq \mathbb{R}^m$, a positive Borel measure μ on U and an open or closed subset $V \subseteq \mathbb{R}^l$ we denote by $L^\infty_w(U,\mu;\mathcal{M}^1(V))$ the space of μ-weakly*-measurable maps from U into $\mathcal{M}^1(V)$. That such a map ν is μ-weakly*-measurable means that for each bounded measurable function $f : V \to \mathbb{R}$, the map

$$x \mapsto \langle \nu_x, f\rangle := \int_V f(z) d\nu_x(z)$$

is μ-measurable. In case μ is the Lebesgue measure we omit the specification of the measure; d-dimensional Lebesgue measure will be denoted by $\mathcal{L}^d$.

We will denote by L^2_x the space $L^2(\mathbb{R}^d)$, and by $L^\infty_t L^2_x$ the space $L^\infty\left([0,T];L^2(\mathbb{R}^d)\right)$.

Another space of importance for us is the space $C\left([0,T];L^2_w(\mathbb{R}^d)\right)$ of functions that are weakly continuous in time and L^2 in space; more precisely, it is the space of maps $v : [0,T] \to L^2(\mathbb{R}^d)$ such that the map

$$t \mapsto \int_{\mathbb{R}^d} v(x,t)\phi(x) dx$$

is continuous for each test function $\phi \in L^2(\mathbb{R}^d)$. Often we will simply write CL^2_w for this space.

We shall write $A : B$ for the scalar product of two matrices in $\mathbb{R}^{d\times d}$, that is, $A : B = \sum_{i,j} A_{ij}B_{ij}$, and $v \otimes w$ for the tensor product of two vectors in $\mathbb{R}^d$, which is defined as a $(d \times d)$-matrix with entries $(v \otimes w)_{ij} = v_i w_j$. Moreover we define for $v \in \mathbb{R}^d$

$$v \bigcirc v := v \otimes v - \frac{1}{d}|v|^2 I_d,$$

where I_d is the $d \times d$ identity matrix. Note that $v \bigcirc v$ is symmetric and has zero trace. The space of symmetric $(d\times d)$-matrices will be denoted by $\mathcal{S}^d$ and the space of traceless symmetric $(d \times d)$-matrices by $\mathcal{S}^d_0$. If $\phi : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a differentiable matrix-valued function, then $\operatorname{div}\phi$ is a vector field defined by $(\operatorname{div}\phi)_i = \sum_j \partial_{x_j}\phi_{ij}$.

If $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are maps from some sets X, Y into, say, $\mathbb{R}$, then $f \otimes g$ is a map $X \times Y \to \mathbb{R}$ defined by $f \otimes g(x,y) = f(x)g(y)$, whereas for two measures μ and ν living on two measurable spaces X and Y respectively, $\mu \otimes \nu$ is a measure on $X \times Y$ defined by $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$ for measurable subsets $A \subseteq X$, $B \subseteq Y$.

If f is a map from a topological space X into $\mathbb{R}$, $\operatorname{supp} f$ will be the support of f, and $C_c(X)$ is the space of continuous functions whose support is compact in X. Finally, S^{d-1} is the $(d-1)$-dimensional unit sphere and χ_A denotes the characteristic function of a set A.

Chapter 2

Weak Solutions

2.1 Weak Continuity in Time

Recall from the introduction that a vector field $v \in L^2_{loc}(\mathbb{R}^d \times [0,T))$ is called a weak solution of the Cauchy problem for the Euler equations with initial data v_0 ($v_0 \in L^2(\mathbb{R}^d)$ weakly divergence-free) if v is weakly divergence-free in the sense of (1.6) and if for every $\phi \in C_c(\mathbb{R}^d \times [0,T))$ with zero divergence,

$$\int_0^T \int_{\mathbb{R}^d} (v \cdot \partial_t \phi + (v \otimes v) : \nabla \phi) dxdt + \int_{\mathbb{R}^d} v_0(x)\phi(x,0)dx = 0 \tag{2.1}$$

holds. As a preliminary consideration, we will show that a weak solution with essentially bounded energy has a continuity property that allows us to make sense of $v(\cdot,t)$ as an L^2 function for *every* (and not just almost every) $t \in [0,T]$. This is the content of the following Lemma. We essentially follow Appendix A in [16]. Recall that $C([0,T];L^2_w(\mathbb{R}^d))$ is the set of maps $[0,T] \to L^2(\mathbb{R}^d)$ which are continuous with respect to the weak topology in $L^2(\mathbb{R}^d)$.

Lemma 2.1. *Let v be a weak solution for Euler in $L^\infty([0,T];L^2(\mathbb{R}^d))$. Then there exists a representative $\bar{v} \in C([0,T];L^2_w(\mathbb{R}^d))$ of v, i.e. $\bar{v}(\cdot,t) = v(\cdot,t)$ as L^2 functions for a.e. $t \in [0,T]$.*

Proof. Let $\{\phi_i + \nabla p_i\}$ be a set of vector fields as in Lemma 4.12 in the appendix; that is, ϕ_i is divergence-free and $\phi_i, p_i \in C_c^\infty(\mathbb{R}^d)$. Moreover, the set $\{\phi_i + \nabla p_i\}$ is (strongly) dense in L^2. Let now $\chi \in C_c^\infty((0,T))$, and set

$$\Phi_i(t) := \int_{\mathbb{R}^d} (\phi_i(x) + \nabla p_i(x)) \cdot v(x,t)dx.$$

By definition of weak solution (where $\chi\phi_i$ is inserted as a test function),

$$\int_0^T \partial_t \chi \int_{\mathbb{R}^d} \phi_i \cdot v dxdt = -\int_0^T \chi \int_{\mathbb{R}^d} \nabla \phi_i : (v \otimes v)dxdt,$$

and because v is weakly divergence-free, it is even true that

$$\int_0^T \Phi_i \partial_t \chi dt = -\int_0^T \chi \int_{\mathbb{R}^d} \nabla \phi_i : (v \otimes v)dxdt.$$

Therefore, $\int \nabla\phi_i : (v \otimes v)dx$ is the distributional derivative of Φ_i, and since $v \otimes v \in L^\infty([0,T]; L^1(\mathbb{R}^d))$, we have

$$\int_0^T |\Phi_i'|dt \leq \int_0^T \int_{\mathbb{R}^d} |v \otimes v||\nabla\phi_i|dxdt \leq \|\nabla\phi_i\|_{L_t^1 L_x^\infty} \|v \otimes v\|_{L_t^\infty L_x^1},$$

i.e. $\Phi_i' \in L^1([0,T])$. Consequently, there exists a nullset $\tau_i \subset [0,T]$ such that Φ_i can be altered on τ_i to become (absolutely) continuous in $[0,T]$. The altered functions are still denoted by Φ_i. The union $\tau := \bigcup_i \tau_i$ is also a nullset, and it holds for every $t \in [0,T]\backslash\tau$ and every $i \in \mathbb{N}$ that $\Phi_i(t) = \int(\phi_i + \nabla p_i) \cdot vdx$. Moreover, by continuity,

$$|\Phi_i(t)| \leq \|v\|_{L_t^\infty L_x^2} \|\phi_i + \nabla p_i\|_{L_x^2} \quad \text{for every } t \in [0,T],$$

so that for each $t \in [0,T]$, the $\Phi_i(t)$ define a unique bounded linear functional L_t on L^2 through $L_t(\phi_i + \nabla p_i) = \Phi_i(t)$ (recall the $\phi_i + \nabla p_i$ are dense). By the Riesz representation theorem, we find for every t a function $\bar{v}(\cdot,t) \in L_x^2$ which coincides with v for every $t \in [0,T]\backslash\tau$ and which satisfies

$$\|\bar{v}(\cdot,t)\|_{L_x^2} \leq \|v\|_{L_t^\infty L_x^2} \quad \text{for every } t \in [0,T]$$

as well as

$$\Phi_i(t) = \int (\phi_i + \nabla p_i) \cdot \bar{v}dx \quad \text{for every } t \in [0,T].$$

It remains to show $\bar{v} \in C([0,T]; L_w^2)$. So let $\psi \in L_x^2$ be given, and let (by abuse of notation) $\phi_i + \nabla p_i \to \psi$ strongly in L^2 as $i \to \infty$. If $\Phi := \int \bar{v} \cdot \psi dx$, it follows that

$$|\Phi(t) - \Phi_i(t)| \leq \|v\|_{L_t^\infty L_x^2} \|\psi - \phi_i - \nabla p_i\|_{L_x^2},$$

so that $\Phi_i \to \Phi$ uniformly. Φ therefore inherits the continuity from the Φ_i, and by definition, this means that $\bar{v} \in C([0,T]; L_w^2)$. □

2.2 Subsolutions and Plane Waves

We recall some formalism from [15] which is needed for the convex integration method. First, observe that if v is a weak solution of Euler, then there exists a distribution p such that v and p solve (1.1) in the sense of distributions; that is, v is weakly divergence-free and

$$\int_0^T \int_{\mathbb{R}^d} (v \cdot \partial_t\psi + (v \otimes v) : \nabla\psi + p \operatorname{div}\psi)\, dxdt = 0$$

holds for every $\psi \in C_c^\infty(\mathbb{R}^d \times (0,T))$ (not necessarily divergence-free). In fact, p is a distributional solution of $-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v)$. An immediate consequence is the following reformulation of the Euler equations:

Lemma 2.2. *Let $v \in L^\infty\left([0,T]; L^2(\mathbb{R}^d;\mathbb{R}^d)\right)$, $u \in L^\infty\left([0,T]; L^1(\mathbb{R}^d;\mathcal{S}_0^d)\right)$ and q be a distribution such that*

$$\begin{aligned} \partial_t v + \operatorname{div} u + \nabla q &= 0 \\ \operatorname{div} v &= 0 \end{aligned} \tag{2.2}$$

in the sense of distributions. If it also holds that

$$u = v \bigcirc v := v \otimes v - \frac{1}{d}|v|^2 I_d \tag{2.3}$$

for almost every $(x,t) \in \mathbb{R}^d \times [0,T]$, *then* v *and* $p := q - \frac{1}{d}|v|^2$ *are a distributional solution to the Euler equations. Conversely, if* (v,p) *is a distributional solution of Euler with bounded energy, then* (v,u,q) *with* $u := v \bigcirc v$ *and* $q := p + \frac{1}{d}|v|^2$ *solve (2.2) and (2.3) in the sense of distributions.*

Recall also the following proposition from [15]:

Proposition 2.3. *a) Let* $\mathcal{M}$ *be the linear space of symmetric* $(d+1)\times(d+1)-$*matrices* U *such that* $U_{(d+1),(d+1)} = 0$. *Then the map*

$$\begin{aligned} \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} &\longrightarrow \mathcal{M} \\ (v,u,q) &\mapsto U = \begin{pmatrix} u + qI_d & v \\ v & 0 \end{pmatrix} \end{aligned} \tag{2.4}$$

is a linear isomorphism.

b) Introducing the variable $y = (x,t) \in \mathbb{R}^d \times [0,T]$, *(2.2) is equivalent to*

$$\operatorname{div}_y U = 0. \tag{2.5}$$

c) For any $v \in \mathbb{R}^d$ *and* $u \in \mathcal{S}_0^d$ *there exists* $q \in \mathbb{R}$ *such that the corresponding matrix* U *has zero determinant.*

Proof. a) and b) are obvious. For c), let $V^\perp$ be the orthogonal complement of the span of v in $\mathbb{R}^d$, and define $P_{V^\perp}$ to be the orthogonal projection from $\mathbb{R}^d$ onto $V^\perp$. Since u is self-adjoint, then so is the restriction of $P_{V^\perp}u$ to $V^\perp$. Hence there exists at least one eigenvalue of this operator, which we call $-q$, and an eigenvector $\xi \in V^\perp$, so that $P_{V^\perp}(u+qI_d)\xi = 0$. Hence there exists $\lambda \in \mathbb{R}$ such that

$$(u + qI_d)\xi = \lambda v.$$

It follows that the nonzero vector $(\xi, -\lambda)$ is in the nullspace of U, so that $\det U = 0$. □

The significance of part c) of the proposition lies in the fact that the set

$$\Lambda = \left\{ (v,u,q) \in \mathbb{R}^d \times \mathcal{S}_0^d \times \mathbb{R} : \det \begin{pmatrix} u + qI_d & v \\ v & 0 \end{pmatrix} = 0 \right\}$$

constitutes the *wave cone* for the system (2.2). This means that for $(v,u,q) \in \Lambda$, there exists a direction $\eta \in \mathbb{R}^{d+1}\backslash\{0\}$ such that for any profile function $h : \mathbb{R} \to \mathbb{R}$,

$$h(y \cdot \eta)(v,u,q)$$

is a solution of (2.2).

Such plane waves will be essential in the sequel. In order to localise these plane waves, a cutoff technique from [15] is useful:

Lemma 2.4. *Let $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{d+1})$ be functions for $i,j,k,l = 1,\dots,d+1$ so that the tensor E is skew-symmetric in ij and kl, and so that $E_{(d+1)i}^{(d+1)j} = 0$ for every i and j. Then the matrix U defined by*

$$U_{ij} = \mathcal{L}(E)_{ij} = \frac{1}{2}\sum_{k,l} \partial_{k,l}^2 (E_{kj}^{il} + E_{ki}^{jl})$$

takes values in $\mathcal{M}$ and is divergence-free, i.e. it satisfies (2.5).

Thus the differential operator $\mathcal{L}$ produces solutions of (2.5) and thus of (2.2) given any potential E. The proof of this lemma is a direct computation that we shall omit here.

When working with pairs (v,u) which solve (2.2), it is desirable to assign to them some kind of energy density. The following definition was made in [16]:

Definition 2.5. For $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, the *generalised energy* is defined by

$$e(v,u) := \frac{d}{2}\lambda_{max}(v \otimes v - u),$$

where λ_{max} denotes the largest eigenvalue.

Lemma 2.6 (Lemma 3.2 in [16]). *a) $e : \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ is convex.*

b) For every $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, $\frac{1}{2}|v|^2 \leq e(v,u)$, with equality if and only if $u = v \bigcirc v$.

c) For every $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$, $|u|_\infty \leq 2\frac{d-1}{d}e(v,u)$, $|u|_\infty$ being the operator norm of the matrix u.

Note that b) implies in particular $e(v,u) \geq 0$. We are now ready to define the notion of subsolution.

Definition 2.7. Let

$$\bar{e} \in C(\mathbb{R}^d \times (0,T)) \cap L^\infty([0,T]; L^1(\mathbb{R}^d)) \cap C([0,T]; L^1(\mathbb{R}^d)). \tag{2.6}$$

Suppose further that

$$v \in C([0,T]; L_w^2(\mathbb{R}^d)) \cap L^\infty([0,T]; L^2(\mathbb{R}^d)) \cap C^\infty(\mathbb{R}^d \times (0,T))$$

with $v(\cdot,0) = v_0$ and that

$$u \in L^\infty([0,T]; L^1(\mathbb{R}^d)) \cap C^\infty(\mathbb{R}^d \times (0,T)).$$

If there exists a function $q \in C^\infty(\mathbb{R}^d \times (0,T))$ such that (v,u,q) satisfies (2.2), and if

$$e(v(x,t), u(x,t)) < \bar{e}(x,t) \tag{2.7}$$

for every x and every $t > 0$, then (v,u) is said to be a (smooth) *subsolution* with respect to the energy density $\bar{e}$ and the initial data v_0.

Note that in this terminology a smooth subsolution need not be smooth up to time zero.

2.3 Convex Integration for Euler

The goal of this section is to construct exact weak solutions from subsolutions via a convex integration method. Actually, a variant of convex integration, the so-called Baire category method, will be used. The result to be shown is the following:

Theorem 2.8 (Proposition 3.3 in [16]). *Let $v_0 \in L^2(\mathbb{R}^d)$ be a weakly divergence-free vector field and $\bar{e}$ an energy density as in (2.6). If there exists a smooth subsolution with respect to v_0 and $\bar{e}$, then there exist infinitely many weak solutions $v \in C([0,T]; L^2_w(\mathbb{R}^d))$ of Euler with*

$$v(\cdot,0) = v_0 \quad \textit{and} \quad \frac{1}{2}|v(x,t)|^2 = \bar{e}(x,t) \quad \textit{for every } t > 0 \textit{ and a.e. } x \in \mathbb{R}^d.$$

Moreover, the given subsolution can be approximated arbitrarily closely in the topology of CL^2_w by weak solutions with these properties.

Remark 2.9. In fact, in [16] it is proved that the pressure field of the weak solutions v constructed from the subsolution can be chosen as $p = q - \frac{1}{d}|v|^2$, where q is the pressure of the subsolution. To achieve this additional assertion, in the proof below we would have to employ a more sophisticated potential that produces oscillations at constant pressure. See Proposition 4.8 in [16].

Remark 2.10. It is evident from the proof that one can replace $\mathbb{R}^d$ by a domain $\Omega \subset \mathbb{R}^d$ in the theorem in the following sense: One requires the subsolution to be compactly supported on Ω for all positive times, and one then gets a weak solution v with energy density

$$\frac{1}{2}|v(x,t)|^2 = \bar{e}(x,t)\chi_\Omega \quad \text{for every } t > 0, \text{ a.e. } x \in \mathbb{R}^d.$$

The proof can be thought of as consisting of two parts: A “soft” part, where a suitable function space is defined, whose functional analytic and measure theoretic properties are exploited; and a “hard” part, where the specific properties of the Euler equations enter. We start with the soft analysis.

2.3.1 “Soft” Analysis

Definition 2.11. Given v_0 and $\bar{e}$ as in Definition 2.7, we denote by X_0 the set of $v \in C([0,T]; L^2_w(\mathbb{R}^d))$ for which there exists a suitable matrix field u such that (v,u) is a smooth subsolution with respect to v_0 and $\bar{e}$. Moreover, the space X is defined to be the closure of X_0 in the topology of $C([0,T]; L^2_w(\mathbb{R}^d))$.

By assumption on $\bar{e}$, there is a bounded subset $B \in L^2_x$ such that $v(\cdot,t) \in B$ for all t and for all $v \in X$. Since B can be assumed to be weakly compact, the weak topology on B is metrisable by a metric d_B. Therefore, the space $C([0,T]; B) \subset C([0,T]; L^2_w)$ is metrisable by

$$d(v_1, v_2) := \sup_{t \in [0,T]} d_B(v_1(\cdot,t), v_2(\cdot,t)),$$

and this metric makes $C([0,T]; B)$ complete: Indeed, if $v_n \xrightarrow{d} v$, this means that $v_n(t) \xrightarrow{d_B} v(t)$ uniformly in t, and the continuity of v_n in t thus implies that also $v \in C([0,T]; B)$. It follows that also (X,d) is a complete metric space.

Proposition 2.12. *Any element $v \in X$ satisfies $v(\cdot, 0) = v_0$ and there exists $u \in L^\infty([0,T]; L^1_x)$ and a distribution q such that (2.2) holds. Moreover, $e(v,u) \leq \bar{e}$ a.e.*

Proof. Let $v \in X$ and $v_k \overset{d}{\to} v$ where $v_k \in X_0$. Let moreover $u_k \in L^\infty_t L^1_x$ be the corresponding matrix fields, which are bounded pointwise by $(2(d-1)/d)\bar{e}$ (recall Proposition 2.6c). Therefore, the u_k are uniformly bounded in $L^\infty_{loc}(\mathbb{R}^d \times (0,T))$. We deduce that (up to a not relabeled subsequence) $u_k \overset{*}{\rightharpoonup} u$ in L^∞_{loc}, and u is bounded pointwise a.e. by $(2(d-1)/d)\bar{e}$, so that $u \in L^\infty_t L^1_x$. The weak convergence preserves equations (2.2), and the convexity of e ensures $e(v,u) \leq \bar{e}$ a.e. □

On X, one now defines an *error functional*, which measures how far a subsolution is from being an exact solution:

Definition 2.13. Let $\Omega \in \mathbb{R}^d$ be a bounded open subset and $[t_1, t_2] \in (0, T)$ an interval with $0 < t_1 < t_2 < \infty$. The error functional is then defined on X as

$$I_{\Omega,t_1,t_2}(v) = \inf_{t_1 \leq t \leq t_2} \int_\Omega \left(\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t) \right) dx.$$

The error functional is essentially the L^2 norm of v, and it is therefore no surprise that it is lower semicontinuous with respect to convergence in the metric d. See Lemma 4.3 in [16] for a proof. Also, it follows from the assumptions on $\bar{e}$ that I_{Ω,t_1,t_2} is bounded from below.

From the definition of X, the lower semicontinuity of I_{Ω,t_1,t_2}, and Proposition 2.12, it is clear that $I_{\Omega,t_1,t_2}(v) \leq 0$ and that $I_{\Omega,t_1,t_2}(v) = 0$ for all Ω, t_1, t_2 if and only if v is a weak solution of Euler with initial data v_0 and energy density $\bar{e}$ for $t > 0$.

The following "perturbation property" (Proposition 4.5 in [16]) is the cornerstone of the convex integration method.

Proposition 2.14. *Fix Ω, t_1, and t_2 as above. For every $\alpha > 0$ there exists $\beta > 0$ such that if $v \in X_0$ with $I_{\Omega,t_1,t_2}(v) < -\alpha$, then there exists a sequence $(v_k) \subset X_0$ with $v_k \overset{d}{\to} v$ but*

$$\liminf_{k\to\infty} I_{\Omega,t_1,t_2}(v_k) \geq I_{\Omega,t_1,t_2}(v) + \beta.$$

Before embarking on the proof of the proposition, we shall see how it implies Theorem 2.8. The lower semicontinuity of I_{Ω,t_1,t_2} on the complete metric space X and the boundedness of I_{Ω,t_1,t_2} imply by standard results in general topology (see the references in [15,16]) that I_{Ω,t_1,t_2} is a Baire-1 map and hence its points of continuity are residual in X (i.e. the set of points of discontinuity is nowhere dense w.r.t. d). But if $v \in X$ is a point of continuity, then $I_{\Omega,t_1,t_2}(v) = 0$: Indeed, suppose $I_{\Omega,t_1,t_2}(v) < -\alpha$ for some $\alpha > 0$, and let $(v_k) \subset X_0$ be a sequence with $v_k \overset{d}{\to} v$. By assumption, $I_{\Omega,t_1,t_2}(v_k) \to I_{\Omega,t_1,t_2}(v)$, so that we may assume $I_{\Omega,t_1,t_2}(v_k) < -\alpha$ for all k. By Proposition 2.14 there exists, however, another sequence $(\tilde{v}_k) \subset X_0$ with $\tilde{v}_k \overset{d}{\to} v$ such that

$$\liminf_{k\to\infty} I_{\Omega,t_1,t_2}(\tilde{v}_k) \geq I_{\Omega,t_1,t_2}(v) + \beta,$$

which contradicts the continuity of I_{Ω,t_1,t_2} at v. An exhaustion argument then yields that there exists a residual subset $\Xi \subset X$ such that $v \in \Xi$ implies $I_{\Omega,t_1,t_2}(v) = 0$ *for all* Ω, t_1, t_2, and therefore v is a weak solution of Euler as desired. Note that by the assumptions of Theorem 2.8, X_0 and hence X is not empty; moreover, adding localised plane waves with sufficiently small amplitude (cf. next subsection) shows that X_0 (hence X) has infinite cardinality, hence Ξ, as a residual set, is infinite and dense in X. This proves Theorem 2.8.

2.3.2 Proof of the Perturbation Property

Since Proposition 2.14 essentially states the existence of a sequence which converges weakly but not strongly to a given function, it appears advisable to add highly oscillatory perturbations to the function in order to prove the proposition. For this we will use the tools introduced in Section 2.2. Thanks to the large wave cone, we have a sufficient choice of wave directions. The next Lemma (cf. Lemma 4.7 in [16]) ensures that the oscillations can be chosen to have a reasonably large amplitude.

Define, for $r \geq 0$, $K_r = \{(v, v \bigcirc v) : |v| = r\}$. One can then show (Lemma 3.2 in [16]) that the convex hull of this set in $\mathbb{R}^d \times \mathcal{S}_0^d$ is given by

$$K_r^{co} = \left\{ (v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d : e(v,u) \leq \frac{r^2}{2} \right\}.$$

Lemma 2.15. *There exists a constant C depending only on the dimension (but not on r) such that if $(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d$ satisfies $(v,u) \in \operatorname{int} K_r^{co}$, then there exists $(\bar{v},\bar{u}) \in \mathbb{R}^d \times \mathcal{S}_0^d$ with the following properties: The line segment*

$$[(v,u) - (\bar{v},\bar{u}), (v,u) + (\bar{v},\bar{u})]$$

is contained in $\operatorname{int} K_r^{co}$*, and*

$$|\bar{v}| \geq \frac{C}{r}(r^2 - |v|^2).$$

The proof, which relies on Carathéodory's Theorem for convex hulls, can be found in [16].

Let now the bounded domain $\Omega \in \mathbb{R}^d$ and the numbers $0 < t_1 < t_2 < \infty$ be given, and suppose (v,u) is a smooth subsolution with respect to $\bar{e}$ and v_0. We will decompose the domain $\Omega \times [t_1, t_2]$ into small cubes and discretise the subsolution and the energy density to be constant on each cube. On such a cube we then add a localised plane wave oscillating in the direction given by Lemma 2.15. Since the perturbation property contains a uniform estimate in t, we have to ensure that at every time t there are enough oscillations. The trick in achieving this is to use a "shifted grid".

More precisely, for $\zeta \in \mathbb{Z}^d$ and mesh size $h > 0$, define the families of cubes (Q_ζ) and $(\tilde{Q}_\zeta)$ in $\mathbb{R}^d_x$ by

$$Q_\zeta = h\zeta + \left[-\frac{h}{2}, \frac{h}{2}\right)^d, \qquad \tilde{Q}_\zeta = h\zeta + \left[-\frac{3h}{8}, \frac{3h}{8}\right)^d,$$

so that $\tilde{Q}_\zeta \subset Q_\zeta$. Moreover, for $(\zeta, i) \in \mathbb{Z}^{d+1}$ define space-time cubes in $\mathbb{R}^d_x \times \mathbb{R}_t$ by

$$C_{\zeta,i} = \begin{cases} Q_\zeta \times [ih, (i+1)h) & \text{if } \sum_{j=1}^d \zeta_j \text{ is even,} \\ Q_\zeta \times [(i - \frac{1}{2})h, (i + \frac{1}{2})h) & \text{if } \sum_{j=1}^d \zeta_j \text{ is odd,} \end{cases}$$

such as to obtain a “shifted grid” in space-time (see Figure 1 in [16]). Similarly, we can define space-time cubes $\tilde{C}_{\zeta,i} \subset C_{\zeta,i}$ with sidelength $\frac{3}{4}h$ by

$$\tilde{C}_{\zeta,i} = \begin{cases} \tilde{Q}_\zeta \times [(i+\frac{1}{8})h, (i+\frac{7}{8})h) & \text{if } \sum_{j=1}^d \zeta_j \text{ is even,} \\ \tilde{Q}_\zeta \times [(i-\frac{3}{8})h, (i+\frac{3}{8})h) & \text{if } \sum_{j=1}^d \zeta_j \text{ is odd.} \end{cases}$$

Next, let $0 \le \phi^h \le 1$ be a smooth cutoff function on $\mathbb{R}^{d+1}$ which equals 1 on the “small cubes”, i.e. on $\bigcup_{\zeta,i} \tilde{C}_{\zeta,i}$, and which is zero near the boundaries of the “big cubes”, e.g. on

$$\left\{ (x,t) \in \mathbb{R}^{d+1} : \operatorname{dist}\left((x,t), \bigcup_{\zeta,i} \partial C_{\zeta_i} \right) \le \frac{1}{16} \right\}.$$

Moreover, define

$$\Omega_1^h = \bigcup \left\{ \tilde{Q}_\zeta : \sum_{j=1}^d \zeta_j \;\text{ even}, \; Q_\zeta \subset \Omega \right\}$$

and

$$\Omega_2^h = \bigcup \left\{ \tilde{Q}_\zeta : \sum_{j=1}^d \zeta_j \;\text{ odd}, \; Q_\zeta \subset \Omega \right\}$$

(recall that Ω was the given bounded domain in the definition of the error functional) and observe that

$$\lim_{h\to 0} \mathcal{L}^d \left(\Omega_\nu^h\right) = \frac{1}{2}\left(\frac{3}{4}\right)^d \mathcal{L}^d (\Omega)$$

for $\nu = 1, 2$, and that, thanks to the “shift”, for every time $t \in [t_1, t_2]$ the set $\{x \in \Omega : \phi^h(x,t) = 1\}$ contains at least one of the sets Ω_ν^h (consult again Figure 1 in [16]).

Assume now that the given smooth subsolution (v, u) satisfies $I_{\Omega,t_1,t_2}(v) < -\alpha$ for an $\alpha > 0$, and let E_h be the piecewise constant approximation of the integrand on $\Omega \times [t_1, t_2]$, given by

$$E_h(x,t) = \frac{1}{2}|v(h\zeta, hi)|^2 - \bar{e}(h\zeta, hi) \quad \text{if } (x,t) \in C_{\zeta,i}.$$

By uniform continuity of v and $\bar{e}$ on $\Omega \times [t_1, t_2]$, it holds for $\nu = 1, 2$ that

$$\lim_{h\to 0} \int_{\Omega_\nu^h} E_h(x,t)dx = \frac{1}{2}\left(\frac{3}{4}\right)^d \int_\Omega \left(\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right) dx$$

uniformly in $t \in [t_1, t_2]$. Consequently, there exists a constant $c > 0$ such that $\int_\Omega \left(\frac{1}{2}|v(x,t)|^2 - \bar{e}(x,t)\right) dx \le -\frac{\alpha}{2}$ implies

$$\int_{\Omega_\nu^h} |E_h(x,t)| dx \ge c\alpha \tag{2.8}$$

for h sufficiently small. Set $z_{\zeta,i} = (v(h\zeta, hi), u(h\zeta, hi))$. Now, if, for some $\delta > 0$ sufficiently small, $C_{\zeta,i} \subset \Omega \times [t_1 - \delta, t_2 + \delta]$, Lemma 2.15 tells us that there exists $\bar{z}_{\zeta,i} = (\bar{v}_{\zeta,i}, \bar{u}_{\zeta,i}) \in \mathbb{R}^d \times \mathcal{S}_0^d$ such that all points on the line segment

$$\sigma_{\zeta,i} = [z_{\zeta,i} - \bar{z}_{\zeta,i}, z_{\zeta,i} + \bar{z}_{\zeta,i}]$$

have generalised energy density less than $\bar{e}(h\zeta, hi)$, and such that

$$|\bar{v}_{\zeta,i}|^2 \geq \frac{C}{\bar{e}(h\zeta, hi)}|E_h(h\zeta, hi)|^2 \geq \frac{C}{M}|E_h(h\zeta, hi)|^2, \tag{2.9}$$

where we set $r = \sqrt{2\bar{e}(h\zeta, hi)}$ in Lemma 2.15 and $M = \sup\{\bar{e}(x,t) : (x,t) \in \Omega \times [t_1 - \delta, t_2 + \delta]\}$. Finally, by uniform continuity of $z := (v, u)$ and $\bar{e}$, we can choose h so small that

$$e(z(x,t) + \lambda \bar{z}_{\zeta,i}) < \bar{e}(x,t) \tag{2.10}$$

for all $\lambda \in [-1, 1]$ and $(x,t) \in C_{\zeta,i}$. We fix now h so small that all the estimates obtained so far hold for this choice of h.

We can now define the perturbation. Consider a fixed (ζ, i). By Proposition 2.3c) there exists $\bar{q}_{\zeta,i}$ such that $(\bar{v}_{\zeta,i}, \bar{u}_{\zeta,i}, \bar{q}_{\zeta,i}) \in \Lambda$, the wave cone for the system (2.2) (in fact, in this case $\bar{q}_{\zeta,i} = 0$ will do, as shown in the proof of Lemma 4.3 in [15]). If $\bar{U}_{\zeta,i}$ is the matrix corresponding to $(\bar{v}_{\zeta,i}, \bar{u}_{\zeta,i}, \bar{q}_{\zeta,i})$ via (2.4), then this means that there exists $\eta_{\zeta,i} \in \mathbb{R}^{d+1}$ such that $h(y \cdot \eta_{\zeta,i})\bar{U}_{\zeta,i}$ solves (2.5) for any profile function h (recall $y = (x,t)$). Moreover, since $|\bar{v}_{\zeta,i}| > 0$, we have that $\bar{\eta}_{\zeta,i}$ is not parallel to e_{d+1} (the time direction). For the moment, let us assume $\bar{\eta}_{\zeta,i} = e_1$.

We define a tensor field E^{lm}_{jk}, $j, k, l, m = 1, \ldots, d + 1$, by

$$E^{k1}_{j1} = -E^{k1}_{1j} = -E^{1k}_{j1} = E^{1k}_{1j} = (\bar{U}_{\zeta,i})_{jk}\frac{\sin(Ny_1)}{N^2}$$

and all other entries zero. It is readily checked that this tensor field has the properties required in Lemma 2.4, and that

$$\mathcal{L}(E) = \bar{U}_{\zeta,i}\sin(Ny_1),$$

where $\mathcal{L}$ is the differential operator defined in Lemma 2.4. Let now $\chi_{\zeta,i}$ be the characteristic function of $C_{\zeta,i}$ and consider the cutoff function $\phi_{\zeta,i} := \phi^h\chi_{\zeta,i}$ (recall that it is 1 on $\tilde{C}_{\zeta,i}$ and is compactly supported in $C_{\zeta,i}$). Since $\mathcal{L}$ is a homogeneous differential operator of second order, we have

$$\begin{aligned} \|\mathcal{L}(\phi_{\zeta,i}E) - \phi_{\zeta,i}\mathcal{L}(E)\|_\infty &\leq C\,\|\phi_{\zeta,i}\|_{C^2}\,\|E\|_{C^1} \\ &\leq C\,\|\phi_{\zeta,i}\|_{C^2}\,\frac{1}{N}, \end{aligned} \tag{2.11}$$

where C is of course independent of N.

The case $\eta_{\zeta,i} \neq e_1$ can be reduced to the present case by a linear algebra argument that uses the Galilean invariance of (2.2). I refer to Lemma 3.3 and Step 2 in the proof of Proposition 3.2 in [15] (cf. also Section 4.3 below).

We now define the perturbation as

$$\tilde{U}_N := \sum_{(\zeta,i):C_{\zeta,i}\in\Omega\times[t_1-\delta,t_2+\delta]} \mathcal{L}(\phi_{\zeta,i}E)$$

and set the perturbed subsolution to be

$$(v_N, u_N) = (v, u) + (\tilde{v}_N, \tilde{u}_N),$$

where $(\tilde{v}_N, \tilde{u}_N)$ are obtained from $\tilde{U}_N$ via the isomorphism (2.4). By (2.10) and (2.11) we have $v_N \in X_0$ for $N \geq N_0$. Recall now that at each time $t \in [t_1, t_2]$ there exists $\nu \in \{1, 2\}$ such that $\phi^h(x,t) \equiv 1$ for $x \in \Omega^h_\nu$. If $\tilde{Q}_\zeta \subset \Omega^h_\nu$, therefore, we obtain

$$\begin{aligned}\lim_{N\to\infty} \int_{\tilde{Q}_\zeta} |\tilde{v}_N(x,t)|^2 dx &= \lim_{N\to\infty} \int_{\tilde{Q}_\zeta} |\bar{v}_{\zeta,i}|^2 \sin^2(N\eta_{\zeta,i} \cdot (x,t)) dx \\ &= \frac{1}{2} \int_{\tilde{Q}_\zeta} |\bar{v}_{\zeta,i}|^2 dx\end{aligned}$$

uniformly in t, because $\eta_{\zeta,i}$ is not parallel to the time direction. Here, the i is determined by the time t. From (2.9) we thus get

$$\lim_{N\to\infty} \int_{\Omega^h_\nu} \frac{1}{2} |\tilde{v}_N(x,t)|^2 dx \geq \frac{C}{M} \int_{\Omega^h_\nu} |E_h(x,t)|^2 dx \tag{2.12}$$

uniformly in t for $\nu = \nu(t)$ suitably chosen.

At last we are prepared for the concluding estimates. If $t \in [t_1, t_2]$, then by definition of v_N

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2} |v_N(x,t)|^2 - \bar{e}(x,t) \right) dx \\ &\quad = \int_\Omega \left(\frac{1}{2} |v(x,t)|^2 - \bar{e}(x,t) \right) dx + \int_\Omega \frac{1}{2} |\tilde{v}_N(x,t)|^2 dx + \int_\Omega \tilde{v}_N(x,t) \cdot v(x,t) dx.\end{aligned}$$

Since $\tilde{v}_N$ converges weakly to 0 uniformly in t, the last integral can be made arbitrarily small. Therefore, and because of (2.12), we can estimate

$$\begin{aligned}\liminf_{N\to\infty} I_{\Omega,t_1,t_2}(v_N) &\geq \inf_{t\in[t_1,t_2]} \left(\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e} \right) dx + \int_\Omega \frac{1}{2} |\tilde{v}_N|^2 dx \right) \\ &\geq \inf_{t\in[t_1,t_2]} \left[\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e} \right) dx + \frac{C}{M} \min_{\nu\in\{1,2\}} \int_{\Omega^h_\nu} |E_h|^2 dx \right] \\ &\geq \inf_{t\in[t_1,t_2]} \left[\int_\Omega \left(\frac{1}{2}|v|^2 - \bar{e} \right) dx + \frac{C}{\mathcal{L}^d(\Omega) M} \min_{\nu\in\{1,2\}} \left(\int_{\Omega^h_\nu} |E_h| dx \right)^2 \right].\end{aligned}$$

Taking into account (2.8), we conclude

$$\begin{aligned}\liminf_{N\to\infty} I_{\Omega,t_1,t_2}(v_N) &\geq \min \left\{ -\frac{\alpha}{2}, I_{\Omega,t_1,t_2}(v) + \frac{C}{\mathcal{L}^d(\Omega) M} \alpha^2 \right\} \\ &\geq I_{\Omega,t_1,t_2}(v) + \min \left\{ \frac{\alpha}{2}, \frac{C}{\mathcal{L}^d(\Omega) M} \alpha^2 \right\}\end{aligned}$$

since $I_{\Omega,t_1,t_2}(v) < -\alpha$ by assumption. This proves Proposition 2.14 with

$$\beta = \min \left\{ \frac{\alpha}{2}, \frac{C}{\mathcal{L}^d(\Omega) M} \alpha^2 \right\}.$$

□

2.4 Global Existence and Non-Uniqueness

The rest of this chapter is devoted to applications of Theorem 2.8. The first one is a global existence theorem for weak solutions of Euler with periodic boundary conditions, see also [50].

Let $\mathbb{T}^d$ be the d-dimensional torus with sidelength 2π, $d \geq 2$, so that $L^2(\mathbb{T}^d)$ can be identified with the space of 2π-periodic functions in $L^2_{loc}(\mathbb{R}^d;\mathbb{R}^d)$, i.e. $u(x+2\pi l) = u(x)$ for a.e. $x \in \mathbb{R}^d$ and every $l \in \mathbb{Z}^d$. Then, as usual when dealing with periodic boundary conditions for fluid equations (cf. for instance [12]), we define the space

$$\begin{aligned} H^m(\mathbb{T}^d) = \{ & v \in L^2(\mathbb{T}^d) : \sum_{k\in\mathbb{Z}^d} |k|^{2m}|\hat{v}(k)|^2 < \infty,\ \hat{v}(k)\cdot k = 0 \text{ for every } k \in \mathbb{Z}^d, \\ & \text{and } \hat{v}(0) = 0\}, \end{aligned}$$

where $\hat{v} : \mathbb{Z}^d \to \mathbb{C}^d$ denotes the Fourier transform of v. We shall write $H(\mathbb{T}^d)$ instead of $H^0(\mathbb{T}^d)$ and $H_w(\mathbb{T}^d)$ for the space $H(\mathbb{T}^d)$ equipped with the weak L^2 topology.

Similarly to the problem in the whole space, a vector field $v \in L^2_{loc}\left([0,\infty); H(\mathbb{T}^d)\right)$ is called a *weak solution* of these equations with periodic boundary conditions and initial data $v_0 \in H(\mathbb{T}^d)$ if

$$\int_0^\infty \int_{\mathbb{T}^d} (v\cdot\partial_t\phi + v\otimes v : \nabla\phi)\,dxdt + \int_{\mathbb{T}^d} v_0(x)\phi(x,0)\,dx = 0$$

for every divergence-free $\phi \in C^\infty\left(\mathbb{T}^d\times[0,\infty);\mathbb{R}^d\right)$ with compact support in the time variable.

We recall Theorem 1.1:

Theorem 2.16. *Let $v_0 \in H(\mathbb{T}^d)$. Then there exists a weak solution $v \in C([0,\infty); H_w(\mathbb{T}^d))$ (in fact, infinitely many) of the Euler equations with $v(0) = v_0$. Moreover, the kinetic energy*

$$E(t) := \frac{1}{2}\int_{\mathbb{T}^d} |v(x,t)|^2 dx$$

is bounded and satisfies $E(t) \to 0$ as $t \to \infty$.

Remark 2.17. Note that the condition $\hat{v}(0) = 0$ in the definition of $H(\mathbb{T}^d)$, i.e. $\int_{\mathbb{T}^d} v\,dx = 0$, is no actual constraint due to Galilean invariance of the Euler equations: Indeed, if $\int v_0 dx \neq 0$, then write $v_0 = \tilde{v}_0 + \bar{v}$ with $\int \tilde{v}_0 dx = 0$ and $\bar{v}$ constant. If $\tilde{v}$ is a solution for the initial data $\tilde{v}_0$ as in the theorem, then it is easy to check that

$$v(x,t) = \tilde{v}(x-\bar{v}t,t) + \bar{v}$$

defines a solution with initial data v_0.

As announced in the introduction, the strategy of proof is as follows: Owing to Theorem 2.8, it suffices to construct a suitable smooth subsolution with the desired initial data; we obtain such a subsolution by solving the Cauchy problem for the fractional heat equation

$$\begin{aligned} \partial_t v + (-\Delta)^{1/2} v &= 0 \\ \operatorname{div} v &= 0 \\ v(\cdot,0) &= v_0, \end{aligned}$$

which is not difficult since, owing to periodicity, we can work in Fourier space.

For convenience, Theorem 2.8 is restated here for the periodic setting. It is easy to convince oneself that the proof of Theorem 2.8 applies also to this situation with only minor modifications.

Theorem 2.18. *Let* $\bar{e} \in C\left(\mathbb{T}^d \times (0,\infty)\right) \cap C\left([0,\infty); L^1(\mathbb{T}^d)\right)$ *be such that*

$$\sup_{0\le t<\infty} \int_{\mathbb{T}^d} \bar{e}(x,t)dx < \infty,$$

and let $(\bar{v}, \bar{u}, \bar{q})$ *be a smooth, periodic (in space) solution of*

$$\begin{aligned} \partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} &= 0 \\ \operatorname{div} \bar{v} &= 0 \end{aligned}$$

in $\mathbb{T}^d \times (0,\infty)$ *such that*

$$\bar{v} \in C([0,\infty); H_w(\mathbb{T}^d)),$$

$$\bar{u}(x,t) \in \mathcal{S}_0^d$$

for every $(x,t) \in \mathbb{T}^d \times (0,\infty)$*, and*

$$e\left(\bar{v}(x,t), \bar{u}(x,t)\right) < \bar{e}(x,t)$$

for every $(x,t) \in \mathbb{T}^d \times (0,\infty)$*.*

Then there exist infinitely many weak solutions $v \in C([0,\infty); H_w(\mathbb{T}^d))$ *of the Euler equations with* $v(x,0) = \bar{v}(x,0)$ *for a.e.* $x \in \mathbb{T}^d$ *and*

$$\frac{1}{2}|v(x,t)|^2 = \bar{e}(x,t)$$

for every $t \in (0,\infty)$ *and a.e.* $x \in \mathbb{T}^d$*.*

Proof of Theorem 2.16. By Theorem 2.18, it suffices to find suitable $(\bar{v}, \bar{u}, \bar{q})$ and $\bar{e}$.

Let us define $\bar{v}$ and $\bar{u}$ by their Fourier transforms as follows:

$$\hat{\bar{v}}(k,t) = e^{-|k|t}\hat{v}_0(k), \tag{2.13}$$

$$\hat{\bar{u}}_{ij}(k,t) = -i\left(\frac{k_j}{|k|}\hat{\bar{v}}_i(k,t) + \frac{k_i}{|k|}\hat{\bar{v}}_j(k,t)\right) \tag{2.14}$$

for every $k \neq 0$, and $\hat{\bar{u}}(0,t) = 0$. Note that $\bar{u}_{ij}$ thus defined equals $-\mathcal{R}_j\bar{v}_i - \mathcal{R}_i\bar{v}_j$, where $\mathcal{R}$ denotes the Riesz transform. Clearly, for $t > 0$, $\bar{v}$ and $\bar{u}$ are smooth. Moreover, $\bar{u}$ is symmetric and trace-free. Indeed, the latter can be seen by observing

$$\sum_{i=1}^d \left(\frac{k_i}{|k|}\hat{\bar{v}}_i(k,t) + \frac{k_i}{|k|}\hat{\bar{v}}_i(k,t)\right) = \frac{2}{|k|}e^{-|k|t}k \cdot \hat{v}_0(k) = 0$$

for all $k \neq 0$ (for $k = 0$ this is obvious).

Next, we can write equations (2.2) in Fourier space as

$$\begin{aligned} \partial_t \hat{\bar{v}}_i + i\sum_{j=1}^d k_j \hat{\bar{u}}_{ij} + ik_i\hat{\bar{q}} &= 0 \\ k \cdot \hat{\bar{v}} &= 0 \end{aligned} \tag{2.15}$$

for $k \in \mathbb{Z}^d$, $i = 1, \dots, d$. It is easy to check that $(\hat{\bar{v}}, \hat{\bar{u}}, 0)$ as defined by (2.13) and (2.14) solves (2.15) and hence $(\bar{v}, \bar{u}, 0)$ satisfies (2.2).

Concerning the energy, we have the pointwise estimate $e(\bar{v}, \bar{u}) \leq C(|\bar{v}|^2 + |\bar{u}|)$, and because of

$$\int_{\mathbb{T}^d} |\bar{v}|^2 dx = \sum_{k \in \mathbb{Z}^d} |\hat{\bar{v}}|^2 = \sum_{k \in \mathbb{Z}^d} e^{-2|k|t} |\hat{v}_0|^2 \leq \|v_0\|^2_{L^2(\mathbb{T}^d)}$$

and, similarly,

$$\int_{\mathbb{T}^d} |\bar{u}| dx \leq C \int_Q |\bar{u}|^2 dx \leq C \|v_0\|^2_{L^2(\mathbb{T}^d)},$$

we conclude that $\sup_{t>0} \|e(\bar{v}(x,t), \bar{u}(x,t))\|_{L^1(\mathbb{T}^d)} < \infty$. Moreover, from the same calculation and the dominated convergence theorem we deduce

$$\|e(\bar{v}(x,t), \bar{u}(x,t))\|_{L^1(\mathbb{T}^d)} \to 0$$

as $t \to \infty$ as well as

$$\bar{v}(t) \to v_0$$

strongly in $L^2(\mathbb{T}^d)$ and

$$\bar{u}(t) \to u_0 := -(\mathcal{R}_j(v_0)_i + \mathcal{R}_i(v_0)_j)_{ij}$$

strongly in $L^1(\mathbb{T}^d)$. We claim that then

$$e(\bar{v}, \bar{u}) \in C\left([0, \infty); L^1(\mathbb{T}^d)\right).$$

The only issue is continuity at $t = 0$. First, one can easily check that the map

$$(v, u) \mapsto e\left(\frac{v}{\sqrt{|v|}}, u\right)$$

is Lipschitz continuous with Lipschitz constant, say, L; thus, using the inequality $||a|a - |b|b| \leq (|a| + |b|)|a - b|$, we have

$$\begin{aligned}\int_{\mathbb{T}^d} |e(\bar{v}, \bar{u}) - e(v_0, u_0)| &\leq L \int_{\mathbb{T}^d} (||\bar{v}|\bar{v} - |v_0|v_0| + |\bar{u} - u_0|)\, dx \\ &\leq 2L \sup_{t \geq 0} \|\bar{v}(t)\|_{L^2} \|\bar{v}(t) - v_0\|_{L^2} + L \|\bar{u} - u_0\|_{L^1} \to 0\end{aligned}$$

as $t \to 0$. This proves the claim.

Therefore, $\bar{e}$ defined by

$$\bar{e}(x,t) := e(\bar{v}(x,t), \bar{u}(x,t)) + \min\{t, \frac{1}{t}\}$$

satisfies the requirements of Theorem 2.18 and, in addition, $\int_{\mathbb{T}^d} \bar{e} dx \to 0$ as $t \to \infty$. Theorem 2.18 then yields the desired weak solutions of Euler. □

However, since in general

$$\liminf_{t \searrow 0} \int_{\mathbb{T}^d} \bar{e}(x,t) dx > \frac{1}{2} \int_{\mathbb{T}^d} |\bar{v}(x,0)|^2 dx,$$

the solutions constructed here exhibit a jump in their kinetic energy at $t = 0$. In particular, the energy is not non-increasing, thus contradicting common physical intuitions (and experiments). It is therefore reasonable to impose upon weak solutions certain assumptions concerning their energy. In the next section we will study such admissibilty properties and see that none of them are sufficient to exclude "wild" solutions.

2.5 Admissibility Criteria

The following admissibility criteria for weak solutions are listed in [16]. One could hope that (for given initial data) one of these properties is exclusively enjoyed by only one weak solution, and that this solution is the unique "physical" solution to the Cauchy problem. However this turns out not to be the case.

Definition 2.19. a) If $v \in C([0,T]; L^2_w(\mathbb{R}^d))$ is a weak solution for Euler with initial data v_0, we say it satisfies the *weak energy inequality* if for every $t > 0$,

$$\frac{1}{2}\int_{\mathbb{R}^d} |v(x,t)|^2 dx \le \frac{1}{2}\int_{\mathbb{R}^d} |v_0(x)|^2 dx. \tag{2.16}$$

b) If in addition

$$\frac{1}{2}\int_{\mathbb{R}^d} |v(x,t)|^2 dx \le \frac{1}{2}\int_{\mathbb{R}^d} |v(x,s)|^2 dx$$

for all s, t with $t > s$, then v satisfies the *strong energy inequality.*

c) If (2.16) holds with equality, then the *energy equality* is said to be satisfied.

d) Let now v be a weak solution in $L^3_{loc}(\mathbb{R}^d \times (0,T))$. Then it satisfies the *local energy inequality* if

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(\left(\frac{|v|^2}{2} + p\right) v\right) \le 0 \tag{2.17}$$

in the sense of distributions. If this holds with equality, then v satisfies the *local energy equality.*

Some words of explanation are in order for the local energy inequality: If $v \in L^3_{loc}$, then p, being a solution of $-\Delta p = \operatorname{div}\operatorname{div}(v \otimes v)$, can be assumed to be in $L^{3/2}_{loc}$. That v, p satisfy (2.17) in the sense of distributions then means that

$$\int_0^T \int_{\mathbb{R}^d} \frac{|v|^2}{2}\partial_t \phi + \left(\frac{|v|^2}{2} + p\right) v \cdot \nabla \phi \ge 0$$

for every $\phi \in C^\infty_c(\mathbb{R}^d \times (0,T))$ with $\phi \ge 0$. Note that the integral is well-defined since $v \in L^3_{loc}$ and $p \in L^{3/2}_{loc}$. The local energy inequality was proposed as an admissibility criterion by J. Duchon and R. Robert [21].

Some of these criteria allow for a possible loss of kinetic energy. Similar effects are known for conservation laws, where a decrease in the (mathematical) entropy is observed due to the formation of shocks. For scalar conservation laws the criterion of non-increasing entropy ensures uniqueness of weak solutions. The next theorem, which is one of the main results in [16], shows that the Euler equations behave differently:

Theorem 2.20. *There exists bounded and compactly supported initial data v_0 for which there are*

a) infinitely many weak solutions of Euler satisfying the energy equality and the local energy equality;

b) *infinitely many weak solutions satisfying the strong energy inequality but not the energy equality;*

c) *infinitely many weak solutions satisfying the weak energy inequality but not the strong energy inequality.*

Remark 2.21. Although I only treat part b) here, I wish to point out that one has to take into account Remark 2.9 to show the local energy equality in a).

Proof. This theorem is another application of Theorem 2.8. Here, I only present the proof of b). The main step is to find suitable subsolutions. Similarly to the proof of Theorem 2.8, we define a function space X_0 as the set of divergence-free $v \in C^\infty(\mathbb{R}^d \times (-1,1);\mathbb{R}^d)$ such that also $v \in C((-1,1;L^2_w))$ and for which there exists $u \in C^\infty(\mathbb{R}^d \times (-1,1);\mathcal{S}^d_0)$ and $q \in C^\infty(\mathbb{R}^d \times (-1,1))$ such that

$$\partial_t v + \operatorname{div} u + \nabla q = 0$$

and $e(v(x,t),u(x,t)) < 1$ for all $(x,t) \in B_1(0) \times (-1,1)$. In addition, we require

$$\operatorname{supp}(v,u,q) \subset B_1(0) \times \left(-\frac{1}{2},\frac{1}{2}\right)$$

for v to be in X_0. As before, the elements of X_0 take values in a bounded subset $B \subset L^2(\mathbb{R}^d)$, and hence there exists a metric d on $C((-1,1),B)$ which induces the topology of CL^2_w.

Similar arguments as in the proof of Proposition 2.14 yield a perturbation property for X_0:

Perturbation property. If $v \in X_0$ with associated matrix field u is such that

$$\int_{B_1} \left(\frac{1}{2}|v(x,0)|^2 - 1\right) dx < -\alpha$$

for some $\alpha > 0$, then for any $\epsilon > 0$ there exists a sequence $(v_k) \subset X_0$ with corresponding smooth matrix fields (u_k) such that

$$\operatorname{supp}(v_k - v, u_k - u) \in B_1(0) \times (-\epsilon,\epsilon),$$

$$v_k \overset{d}{\to} v,$$

and

$$\liminf_{k\to\infty} \int_{B_1} \frac{1}{2}|v_k(x,0)|^2 dx \geq \int_{B_1} \frac{1}{2}|v(x,0)|^2 dx + \min\left\{\frac{\alpha}{2}, C\alpha^2\right\}$$

for a constant C independent of ϵ, α, and v.

We will now pursue an iteration process that eventually gives us a subsolution with $\int \frac{1}{2}|v(x,0)|^2 dx = 1$. For this, let ρ be a standard mollifier in $\mathbb{R}^d$ and $\rho_\epsilon = \epsilon^{-d}\rho(\cdot\epsilon^{-1})$. Set $\eta_1 = 1/4$, $v_1 = u_1 = 0$. If (v_i,u_i) $(v_k \in X_0)$ has been defined for $i = 1,\dots,k$ and positive real numbers η_i have been defined for $i = 1,\dots,k-1$, we choose $\eta_k < 2^{-k}$ so small that

$$\|v_k(\cdot,0) - v_k(\cdot,0) * \rho_{\eta_k}\|_{L^2} < 2^{-k}.$$

We also set

$$\alpha_k = -\int_{B_1} \left(\frac{1}{2}|v_k(x,0)|^2 - 1\right) dx$$

(by definition of X_0, $\alpha_k > 0$).

The perturbation property just stated now yields $v_{k+1} \in X_0$ and u_{k+1} with

$$\operatorname{supp}(v_{k+1} - v_k, u_{k+1} - u_k) \subset B_1(0) \times (-2^{-k}, 2^{-k}),$$
$$d(v_{k+1}, v_k) < 2^{-k},$$

and

$$\int_{B_1} \frac{1}{2}|v_{k+1}(x,0)|^2 dx \geq \int_{B_1} \frac{1}{2}|v_k(x,0)|^2 dx + \frac{1}{4}\min\{\alpha_k, C\alpha_k^2\}. \tag{2.18}$$

Moreover, since d induces essentially the weak topology in L^2_x, we may assume that

$$\left\|(v_k(\cdot,0) - v_{k+1}(\cdot,0)) * \rho_{\eta_j}\right\|_{L^2} < 2^{-k} \quad \text{for all } j < k.$$

Since the v_k thus defined form a Cauchy sequence in the metric d, there exists a limit $\bar{v} \in C((-1,1), L^2_w(B_1))$. Moreover, since the members of the sequence coincide on sets of the form $B_1 \times ((-1,-\delta) \cup (\delta,1))$ for $\delta > 0$ and sufficiently large indices, $\bar{v}$ and the corresponding $\bar{u}$ form a smooth solution of (2.2) for $t \in (0,1)$. Moreover, the support of $(\bar{v}, \bar{u})$ is contained in $\overline{B_1(0)} \times (-1,1)$ and $\operatorname{supp}(\bar{v}(\cdot,t), \bar{u}(\cdot,t)) \subset B_1(0)$ for $t \neq 0$, and $e(\bar{v},\bar{u}) < 1$ for $t \neq 0$.

We want to show that $\frac{1}{2}|\bar{v}(x,0)|^2 = 1$ for almost every $x \in B_1(0)$. By (2.18),

$$\alpha_{k+1} \leq \alpha_k - \frac{1}{4}\min\{\alpha_k, C\alpha_k^2\},$$

which implies $\lim_{k\to 0} \alpha_k = 0$, which means by definition of α_k that

$$\lim_{k\to\infty} \int_{B_1} \left(\frac{1}{2}|v_k(x,0)|^2 - 1\right) dx = 0.$$

We can now estimate

$$\begin{aligned}\|v_k(\cdot,0) - \bar{v}(\cdot,0)\|_{L^2} \leq{}& \|v_k(\cdot,0) - v_k(\cdot,0) * \rho_{\eta_k}\|_{L^2} \\ &+ \|v_k(\cdot,0) * \rho_{\eta_k} - \bar{v}(\cdot,0) * \rho_{\eta_k}\|_{L^2} + \|\bar{v}(\cdot,0) * \rho_{\eta_k} - \bar{v}(\cdot,0)\|_{L^2}.\end{aligned}$$

The first and third terms on the right hand side converge to zero as $k \to \infty$. For the second term, observe that

$$\begin{aligned}\|v_k(\cdot,0) * \rho_{\eta_k} - \bar{v}(\cdot,0) * \rho_{\eta_k}\|_{L^2} &\leq \sum_{j=0}^{\infty} \|v_{k+j}(\cdot,0) * \rho_{\eta_k} - v_{k+j+1}(\cdot,0) * \rho_{\eta_k}\|_{L^2} \\ &\leq \sum_{j=0}^{\infty} 2^{-k-j} \leq 2^{-(k-1)}.\end{aligned}$$

It follows that $v_k(\cdot,0) \to \bar{v}(\cdot,0)$ strongly in L^2 and therefore

$$\frac{1}{2}|\bar{v}(x,0)|^2 = 1 \quad \text{for a.e. } x \in B_1(0).$$

This equality together with $e(\bar{v},\bar{u}) < 1$ for $t > 0$ and $(\bar{v},\bar{u}) = 0$ for $t \geq 1$ implies that there exists a function $\bar{e} : B_1 \times [0,\infty)$ satisfying the assumptions of Theorem 2.8, and such that $e(\bar{v},\bar{u}) < \bar{e} < 1$ for $t > 0$ and $\int_{B_1} \bar{e}dx$ is a strictly monotone decreasing function of t. Theorem 2.8 and Remark 2.10 then yield infinitely many weak solutions of Euler with initial data $\bar{v}(\cdot,0)$ and energy density $\bar{e}\chi_{B_1(0)}$ for $t > 0$. In particular, these solutions have strictly decreasing energy.

□

2.6 Vortex Sheet Initial Data

The preceding section exhibited an example of initial data for which there exist infinitely many admissible solutions. As mentioned in the introduction, such initial data can be referred to as "wild". In this section we will see that vortex sheet initial data is also wild. In particular, this shows that wild behaviour can arise from very simple and physically realistic initial conditions. The result presented here is due to L. Székelyhidi [47].

We consider again the case of periodic boundary conditions and define periodic initial data v_0 by

$$v_0(x) = \begin{cases} e_1 & \text{if } 0 < x_d < \pi, \\ -e_1 & \text{if } -\pi < x_d < 0. \end{cases}$$

A short computation shows that the stationary solution $v(\cdot,t) = v_0$ for all $t \geq 0$ is in fact a weak solution of the Euler equations: Indeed, let $\phi \in C^\infty(\mathbb{T}^d \times (0,\infty))$ be divergence-free and compactly supported in t. Then

$$\int_0^\infty \int_{\mathbb{T}^d} \partial_t \phi(x,t) \cdot v_0(x) dx dt = 0$$

because v_0 is time-independent, and

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{T}^d} \nabla\phi(x,t) : (v_0(x) \otimes v_0(x)) dx dt \\ &\quad = \int_0^\infty \int_{\mathbb{T}^{d-1}\times(0,\pi)} \nabla\phi : (e_1 \otimes e_1) dx dt + \int_0^\infty \int_{\mathbb{T}^{d-1}\times(-\pi,0)} \nabla\phi : (-e_1 \otimes -e_1) dx dt \\ &\quad = \int_0^\infty \int_{\mathbb{T}^d} \nabla\phi : (e_1 \otimes e_1) dx dt \\ &\quad = \int_0^\infty \int_{\mathbb{T}^d} \partial_1 \phi_1 dx dt = 0, \end{aligned}$$

hence v_0 is a weak solution (it is clearly divergence-free). That this is not the only admissible solution is stated in the next theorem (Theorem 1.1 in [47]):

Theorem 2.22. *There exist infinitely many weak solutions with vortex sheet initial data which satisfy the energy equality, and infinitely many which satisfy the strong energy inequality but not the energy equality.*

For the proof, we need yet another formulation of Theorem 2.8:

Theorem 2.23. *Suppose $v_0 \in L^2(\mathbb{T}^d)$ is weakly divergence-free. For some $T > 0$, let $\bar{e} \in C([0,T]; L^1(\mathbb{T}^d))$ such that*

$$\int \bar{e}(x,t) dx \leq \int \frac{1}{2} |v_0(x)|^2 dx \quad \text{for all } t \in [0,T].$$

Suppose there exists a subsolution $(\bar{v},\bar{u})$ with respect to $\bar{e}$ and v_0 and an open set $\Omega \subset \mathbb{T}^d \times (0,T)$ such that $(\bar{v},\bar{u})$, the corresponding pressure $\bar{q}$, and $\bar{e}$ are continuous on Ω, $\bar{v} \in C([0,T]; L^2_w(\mathbb{T}^d))$, and

$$\bar{v} \otimes \bar{v} - \bar{u} < \frac{2}{d} \bar{e} I_d \quad \text{in } \Omega,$$

$$\bar{v} \otimes \bar{v} - \bar{u} = \frac{2}{d}\bar{e}I_d \quad a.e.\ in\ \Omega^c.$$

Then there exist infinitely many weak solutions to Euler with initial data v_0 and energy density $\bar{e}$.

Proof of Theorem 2.22. For notational simplicity, we consider only the case $d = 2$. Set

$$s(\tau) = \begin{cases} 1 & \text{if } 0 < \tau < \pi, \\ -1 & \text{if } -\pi < \tau < 0, \end{cases}$$

and extend s periodically. Let $\lambda \in (0,1)$ and $\alpha = \alpha(x_2, t)$ be the entropy solution of Burgers' equation

$$\partial_t \alpha(x_2, t) + \frac{\lambda}{2}\partial_2 \left(\alpha(x_2, t)^2\right) = 0$$

with initial data $\alpha(x_2, 0) = s(x_2)$. α is explicitly given by

$$\alpha(x_2, t) = \begin{cases} -1 & \text{if } -\pi < x_2 < -\lambda t, \\ \frac{x_2}{\lambda t} & \text{if } -\lambda t < x_2 < \lambda t, \\ 1 & \text{if } \lambda t < x_2 < \pi \end{cases}$$

up to time $T = \frac{\pi}{\lambda}$.

Next, set $\beta = \beta(x_2, t) = \frac{1}{2}\alpha^2$ and $\gamma = \gamma(x_2, t) = -\frac{\lambda}{2}(1 - \alpha^2)$. Then a subsolution is given by

$$\bar{v} = (\alpha, 0), \quad \bar{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \quad \bar{q} = \beta.$$

A simple calculation then shows that for $T = \frac{\pi}{\lambda}$ and

$$\bar{e} = \frac{1}{2} - \epsilon\frac{1 - \lambda}{2}(1 - \alpha^2),$$

where $\epsilon \in [0, 1)$ is arbitrary, $(\bar{v}, \bar{u}, \bar{q})$ satisfies the assumptions of Theorem 2.23. Note that the solutions obtained are energy-conserving if $\epsilon = 0$ and energy-decreasing if $\epsilon > 0$. □

Remark 2.24. One can view the proof just presented as an alternative, and arguably simpler, proof of Theorem 2.20.

Remark 2.25. Observe that the set of (x_1, x_2) on which the solution differs from the initial vortex sheet - the *turbulent zone* - is given by $\{(x_1, x_2) : |x_2| < \lambda t\}$ and thus expands in time with constant speed λ. This behaviour is reminiscent of a "real" physical effect known as the Kelvin-Helmholtz instability.

Chapter 3

Very Weak Solutions

3.1 Motivation: The Vanishing Viscosity Limit

Consider the Navier-Stokes equations (1.4),

$$\begin{aligned}\partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= \nu \Delta v \\ \operatorname{div} v &= 0,\end{aligned}$$

with initial data v_0. Since the work [34] of J. Leray the global existence of weak solutions has been known; in addition, Leray's solutions are in $C([0,\infty); L^2_w)$ and in $L^2([0,\infty); H^1(\mathbb{R}^d))$, and they are admissible in the sense of (1.10). Uniqueness, however, is not known. Select a sequence $\nu_k \to 0$ and for each k a Leray solution v_{ν_k} of Navier-Stokes with viscosity ν_k. By weak compactness, (v_{ν_k}) converges in the weak-* topology of $L^\infty([0,\infty); L^2)$ to a limit v. The problem is that the limit v need not be a solution of the Euler equations. This is because $v_{\nu_k} \overset{*}{\rightharpoonup} v$ does *not* imply

$$v_{\nu_k} \otimes v_{\nu_k} \rightharpoonup v \otimes v.$$

Indeed, oscillations and concentrations occurring in the sequence could destroy the weak convergence of the tensor product. A way to describe this lack of compactness is provided by the *Reynolds stress tensor*. Suppose the $(v_{\nu_k} \otimes v_{\nu_k})$ converge weakly in the sense of distributions to a symmetric matrix field w. The Reynolds stress tensor is then defined as

$$R = w - v \otimes v$$

(recall v is the weak limit of the v_{ν_k}). We have then

$$\begin{aligned}\partial_t v + \operatorname{div}(v \otimes v + R) + \nabla p &= 0 \\ \operatorname{div} v &= 0.\end{aligned}$$

The Reynolds tensor being nontrivial thus indicates the persistence of high-frequency oscillations in the sequence, which are "forgotten" by the weak limit.

Since one can also write

$$R = \text{w-}\lim_{k\to\infty} (v_{\nu_k} - v) \otimes (v_{\nu_k} - v),$$

we have $R \geq 0$. Setting u to be the traceless part of $v \otimes v + R$, we have that on the one hand (v, u) satisfies (2.2), and on the other hand the positive semi-definiteness of R implies

$$v \otimes v - u \leq \frac{2}{d} \left(\text{w-}\lim_{k \to \infty} \frac{1}{2} |v_{\nu_k}|^2 \right) I_d.$$

Thus (v, u) is a subsolution with respect to the initial data v_0 and the energy density $\bar{e} := \text{w-}\lim \frac{1}{2} |v_{\nu_k}|^2$. In particular, if $R = 0$, this means that $u = v \bigcirc v$, which implies that v is a weak solution and $v_{\nu_k} \otimes v_{\nu_k} \rightharpoonup v \otimes v$.

In any case, the vanishing viscosity limit gives rise to a subsolution (for the motivation of subsolutions via the Reynolds tensor, see Section 2.2 in [17]). The convex integration method demonstrated in Section 2.3 thus adds oscillations to the subsolutions that were originally "lost" in the weak limit.

Another question concerns the energy of the vanishing viscosity limit. Suppose that $v_{\nu_k} \to v$ strongly, so that v is indeed a weak solution to Euler, and that the v_{ν_k} only lose the energy to be expected by frictional effects:

$$\frac{1}{2} \int_{\mathbb{R}^d} |v_{\nu_k}(x,t)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx - \nu_k \int_0^t \int_{\mathbb{R}^d} |\nabla v_{\nu_k}(x,s)|^2 dx ds.$$

Then the limit v will conserve energy in $[0, T]$ only if

$$\lim_{k \to \infty} \nu_k \int_0^T \int_{\mathbb{R}^d} |\nabla v_{\nu_k}(x,s)|^2 dx ds = 0.$$

If this is not the case, v could have decreasing energy. This phenomenon is sometimes called *anomalous dissipation*, see e.g. [10] for further discussion.

The fact that the weak limit of v_{ν_k} need not be a weak solution of Euler can be handled with basically in two ways: Either one endorses the weak limit as a "very weak" solution and shows that it does have properties which qualify it as a meaningful concept of solution; or one relaxes the limit problem to obtain only a weaker object as a limit, which however retains more information about the sequence than the weak limit does. The concepts of subsolution and dissipative solution follow the first strategy, respectively, while the second line of thought is exemplified by measure-valued solutions.

3.2 Dissipative Solutions

The notion of dissipative solution is due to P.-L. Lions [36]. The aim is to furnish a concept of solution which has the weak-strong uniqueness property. The definition of dissipative solutions is motivated by the following argument, which is similar to the computation (1.2): Suppose the vector field $u \in C^\infty(\mathbb{R}^d \times [0, \infty))$ has sufficient decay at infinity and is divergence-free, and let P be the Helmholtz projection onto the space of divergence-free vector fields (cf. the appendix). Let

$$E(u) = -\partial_t u - P(\operatorname{div}(u \otimes u)),$$

so that $E(u) = 0$ if and only if u solves the Euler equations. If v is a (smooth, sufficiently decaying) solution of the Euler equations, then a calculation similar to (1.2) yields

$$\frac{d}{dt} \int_{\mathbb{R}^d} |v - u|^2 dx \leq 2 \|d^-(u)\|_\infty \int_{\mathbb{R}^d} |v - u|^2 dx + 2 \int_{\mathbb{R}^d} E(u) \cdot (v - u) dx,$$

where $d^-(u)$ denotes again the negative part of the smallest eigenvalue of the symmetric gradient of u. By Grönwall's inequality this implies

$$\begin{aligned}\int_{\mathbb{R}^d} |v-u|^2 dx \le \exp\left(2\int_0^t \|d^-(u)\|_\infty\, ds\right) \int_{\mathbb{R}^d} |v(\cdot,0)-u(\cdot,0)|^2 dx \\ +2\int_0^t \int_{\mathbb{R}^d} \exp\left(2\int_s^t \|d^-(u)\|_\infty\, d\tau\right) E(u)\cdot(v-u)\,dx\,ds.\end{aligned} \tag{3.1}$$

In particular, when u is a solution of Euler with initial value $u(\cdot,0) = v(\cdot,0)$, then $v = u$ follows for all time. The idea is now to define a dissipative solution precisely as a function which satisfies (3.1) for every divergence-free u, so that the weak-strong uniqueness follows automatically. More precisely, we have

Definition 3.1 (P.-L. Lions)**.** A dissipative solution of the Euler equations with initial data v_0 is a vector field $v \in L^\infty([0,\infty); L^2(\mathbb{R}^d))$ with $v \in C([0,\infty); L^2_w(\mathbb{R}^d))$ such that $v(\cdot,0) = v_0$, $\operatorname{div} v = 0$ weakly, and (3.1) holds for every weakly divergence-free vector field $u \in C([0,\infty); L^2(\mathbb{R}^d))$ for which $d(u) \in L^1_{loc}([0,\infty); L^\infty_x)$ and $E(u) \in L^1_{loc}([0,\infty); L^2_x)$.

Of course one can also define dissipative solutions up to a finite time $T > 0$; one only needs to replace ∞ by T in the above definition.

Proposition 3.2. *a) Suppose there exists a solution u of the Euler equations with the properties stated in Definition 3.1. Then every dissipative solution with the same initial data coincides with u.*

b) If v is a dissipative solution, then it satisfies

$$\frac{1}{2}\int_{\mathbb{R}^d} |v(x,t)|^2 dx < \frac{1}{2}\int_{\mathbb{R}^d} |v(x,0)|^2 dx \quad \textit{for all } t \ge 0.$$

Proof. For a), insert the solution into the definition of dissipative solution. For b), insert $u \equiv 0$. □

Part b) of this proposition is the reason why these solutions are called "dissipative".

Theorem 3.3. *a) Let (v_{ν_k}) be a sequence of Leray solutions for Navier-Stokes with vanishing viscosity, $\nu_k \to 0$. Then the weak limit is a dissipative solution of Euler.*

b) Weak solutions of Euler satisfying the weak energy inequality are dissipative solutions.

Proof. a) is proved in Section 4.4 of [36]. A proof of b) can be found in Appendix B of [16]. □

3.3 Measure-Valued Solutions

Given a vanishing viscosity sequence of Leray solutions, one can identify the function $v_{\nu_k}(x,t)$ with the probability measure $\delta_{v_{\nu_k}(x,t)}$ on $\mathbb{R}^d$, i.e. with an x- and t-dependent probability measure on the space of velocities. Instead of considering the weak limit of v_{ν_k} as a function, one can then study the parametrised probability measure that arises as the weak*-limit of the probability distributions $\delta_{v_{\nu_k}(x,t)}$. The framework for such study is provided by the theory of (generalised) Young measures.

3.3.1 Young Measures

In this subsection we recall the notion of generalised Young measure as introduced in [20], [1]. For a more detailed and exhaustive discussion of generalised Young measures, see [31].

Let $\Omega \subseteq \mathbb{R}^m$ be a (possibly unbounded) measurable set, $p \in [1,\infty)$, and $(w_n)_{n\in\mathbb{N}}$ a sequence of maps $\Omega \to \mathbb{R}^l$ bounded in $L^p(\Omega)$. We want to study the limit behaviour of sequences of the form $(f(x, w_n(x)))_{n\in\mathbb{N}}$ for a certain class of test functions f. Let us define $\mathcal{F}_p$ as the collection of continuous functions $f : \Omega \times \mathbb{R}^l \to \mathbb{R}$ for which the limit

$$f^\infty(x,z) := \lim_{\substack{x'\to x\\ z'\to z\\ s\to\infty}} \frac{f(x', sz')}{s^p}$$

exists for all $(x,z) \in \overline{\Omega} \times \mathbb{R}^l$ and is continuous in (x,z). f^∞ is called the L^p-*recession function* of f. Note that it is p-homogeneous in the z variable, i.e. $f^\infty(x,\alpha z) = \alpha^p f^\infty(x,z)$ for all $\alpha \ge 0$.

Examples of functions in $\mathcal{F}_p$ are given by continuous functions satisfying $|f(x,z)| \le C(1+|z|^q)$ with $0 \le q < p$, in which case $f^\infty = 0$, or by continuous functions which are p-homogeneous in z, in which case $f^\infty = f$. Of course, functions in $\mathcal{F}_p$ always satisfy a bound $|f(x,z)| \le C(1+|z|^p)$ (where C may depend on x, however).

A *generalised Young measure on* $\mathbb{R}^l$ *with parameters in* Ω is now defined as a triple $(\nu, \lambda, \nu^\infty)$ such that

$$\nu \in L^\infty_w(\Omega; \mathcal{M}^1(\mathbb{R}^l)),$$
$$\lambda \in \mathcal{M}^+(\overline{\Omega}),$$

and

$$\nu^\infty \in L^\infty_w(\overline{\Omega}, \lambda; \mathcal{M}^1(S^{l-1})).$$

Note carefully that ν is only defined Lebesgue-a.e. on Ω and ν^∞ is defined only λ-a.e. on $\overline{\Omega}$. As in [31], we call ν the *oscillation measure*, λ the *concentration measure* and ν^∞ the *concentration-angle measure*. For a motivation of these terms, see Subsection 3.3.2, or [1, 31] and the examples therein.

We are now able to state the following important result of Alibert and Bouchitté, which is a refinement of the construction in [20] (for proofs, see [1], [31]):

Theorem 3.4. (Fundamental Theorem for Generalised Young Measures.)
For $p \in [1,\infty)$ let $(w_n)_{n\in\mathbb{N}}$ be a sequence of maps $\Omega \to \mathbb{R}^l$ bounded in $L^p(\Omega)$. Then there exist a subsequence (not relabeled) and a generalised Young measure $(\nu, \lambda, \nu^\infty)$ such that, for every $f \in \mathcal{F}_p$,

$$f(x, w_n(x))dx \overset{*}{\rightharpoonup} \langle \nu_x, f(x,\cdot)\rangle dx + \langle \nu_x^\infty, f^\infty(x,\cdot)\rangle \lambda$$

in the sense of measures, where, as before, $\langle \nu_x, f(x,\cdot)\rangle = \int_{\mathbb{R}^l} f(x,z)d\nu_x(z)$ and $\langle \nu_x^\infty, f^\infty(x,\cdot)\rangle = \int_{S^{l-1}} f^\infty(x,z)d\nu^\infty(z)$.

Moreover, we then have that $\int_\Omega \langle \nu_x, |\cdot|^p\rangle dx < \infty$.

In the situation of this Theorem, we say that the subsequence (w_n) *generates* the Young measure $(\nu, \lambda, \nu^\infty)$ in $L^p(\Omega)$, and occasionally we write $w_n \overset{\mathbf{Y}}{\to} (\nu, \lambda, \nu^\infty)$. We also use the shorthand notation

$$\langle\!\langle \nu, \lambda, \nu^\infty; f \rangle\!\rangle := \int_\Omega \langle \nu, f \rangle dx + \int_{\bar{\Omega}} \langle \nu^\infty, f^\infty \rangle d\lambda,$$

which emphasises the duality between the space of generalised Young measures and $\mathcal{F}_p$, see [31].

Roughly speaking, the oscillation measure generated by a sequence records its oscillatory properties, whereas the concentration and concentration-angle measures contain information about the concentration effects occurring in the sequence. Hence if we are only interested in the oscillatory behaviour of the sequence, we may use bounded test functions f (so that $f^\infty = 0$) to obtain

$$f(x, w_n(x))dx \overset{*}{\rightharpoonup} \langle \nu_x, f \rangle dx.$$

If this holds for every bounded continuous f, we say that (w_n) generates the oscillation measure (or classical Young measure) ν.

The following proposition collects some well-known properties of generalised Young measures.

Proposition 3.5. *a) For $p = 1$, there exists a countable set of functions $\{f_k\} = \{\phi_k \otimes h_k : k \in \mathbb{N}\} \subset \mathcal{F}_1$, where $\phi_k \in C_c(\Omega)$ and the h_k are Lipschitz continuous in $\mathbb{R}^l$, such that $\langle\!\langle \nu, \lambda, \nu^\infty; f_k \rangle\!\rangle = \langle\!\langle \tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty; f_k \rangle\!\rangle$ for all k implies $(\nu, \lambda, \nu^\infty) = (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$.*

b) If (u_n) and (w_n) are sequences bounded in $L^p(\Omega)$ and $u_n - w_n \to 0$ locally in measure (i.e. for each $\tilde{\Omega} \Subset \Omega$, $(u_n - w_n) \restriction_{\tilde{\Omega}} \to 0$ in measure), and if (u_n) generates an oscillation measure ν, then (w_n) generates the same oscillation measure ν.

c) If (u_n) and (w_n) are sequences bounded in $L^p(\bar{\Omega})$ and $u_n - w_n \to 0$ in $L^p_{loc}(\Omega)$, and if (u_n) generates the generalised Young measure $(\nu, \lambda, \nu^\infty)$, then (w_n) generates the same generalised Young measure.

d) If $w_n \to w$ strongly in $L^p_{loc}(\bar{\Omega})$, then (w_n) generates the Young measure $\nu_x = \delta_{w(x)}$ and $\lambda = 0$.

e) Suppose $w_n \overset{\mathbf{Y}}{\to} (\nu, \lambda, \nu^\infty)$ in $L^p(\Omega)$ and $w \in L^p(\Omega)$. Then $w_n + w \overset{\mathbf{Y}}{\to} (S_w\nu, \lambda, \nu^\infty)$ in L^p, where $S_w\nu$ is defined by duality: $\langle S_{w(x)}\nu_x, f(x, \cdot)\rangle = \langle \nu_x, f(x, \cdot + w(x))\rangle$.

Proof. The stated properties are well-known, however I give some proofs or precise references here for convenience. a) is Lemma 3 in [31]. We will prove b)-e) only for the case $p = 1$; the general case will then follow by observing that for $f \in \mathcal{F}_p$, the function $\tilde{f}(x, z) := f(x, |z|^{\frac{1}{p}-1} z)$ is in $\mathcal{F}_1$, that for (w_n) bounded in L^p the sequence $\tilde{w}_n := |w_n|^{p-1} w_n$ is bounded in L^1, and that moreover $\tilde{f}(x, \tilde{w}_n(x)) = f(x, w_n(x))$.

For b), then, choose $f_k = \phi_k \otimes h_k$ as in a) with the additional assumption that h_k is bounded in $\mathbb{R}^l$ (we are only considering the oscillation measure). Set

$\tilde{\Omega} := \operatorname{supp} \phi_k$ and let L be the Lipschitz constant for h_k. Then, for any $\epsilon > 0$,

$$\left| \int_{\tilde{\Omega}} \phi_k \left(h_k(u_n) - h_k(w_n) \right) dx \right| \le \|\phi_k\|_{L^\infty} \int_{\{|u_n - w_n| < \epsilon\} \cap \tilde{\Omega}} L \epsilon dx$$
$$+ \|\phi_k\|_{L^\infty} \int_{\{|u_n - w_n| \ge \epsilon\} \cap \tilde{\Omega}} |h_k(u_n) - h_k(w_n)| dx$$
$$\le \|\phi_k\|_{L^\infty} |\tilde{\Omega}| L\epsilon + 2 \|h_k\|_{L^\infty} \|\phi_k\|_{L^\infty} |\{|u_n - w_n| \ge \epsilon\}|,$$

and we can make this expression arbitrarily small if we choose ϵ sufficiently small and then n large enough, since $u_n - w_n \to 0$ in measure on $\tilde{\Omega}$.

For c), select again $f_k = \phi_k \otimes h_k$ as in a), and let L be the Lipschitz constant for h_k, then

$$\left| \int_{\tilde{\Omega}} \phi_k (h_k(u_n) - h_k(w_n)) dx \right| \le \|\phi_k\|_{L^\infty} L \|u_n - w_n\|_{L^1(\tilde{\Omega})}$$

which tends to zero as $n \to \infty$. d) now follows from c) if we set $u_n = w$ for all n and observe that the constant sequence (w) generates the Young measure $\nu_x = \delta_{w(x)}$, $\lambda = 0$. Finally, e) is a special case of Proposition 6 in [31]. □

3.3.2 Some Examples

Before applying Young measure theory to the Euler equations, we pause to look at some examples which display the various oscillation and concentration effects that may occur in a sequence of functions. See also [1, 31, 41]. In all the examples, $m = l = 1$, $p = 1$, and $\Omega = [-1, 1]$.

1. **(Strong convergence).** We have already seen (Proposition 3.5) that if $w_n \to w$ strongly in L^1, then the Young measure generated is given by $\nu_x = \delta_{w(x)}$, $\lambda \equiv 0$. This corresponds with the intuition that no oscillations or concentrations form in a strongly convergent sequence.

2. **(Oscillations.)** Define the function

$$v(x) = \begin{cases} \alpha & \text{if } 0 < x < \lambda, \\ \beta & \text{if } \lambda < x < 1, \end{cases}$$

and extend v periodically. Define $w_n(x) = v(nx)$. Then (w_n) generates the Young measure $\nu_x = \lambda\delta_\alpha + (1 - \lambda)\delta_\beta$, $\lambda \equiv 0$. Indeed, testing against a bounded continuous function f, we have

$$f(v(x)) = \begin{cases} f(\alpha) & \text{if } 0 < x < \lambda, \\ f(\beta) & \text{if } \lambda < x < 1, \end{cases}$$

and therefore

$$f(w_n) \rightharpoonup \lambda f(\alpha) + (1 - \lambda) f(\beta) = \langle \lambda\delta_\alpha + (1 - \lambda)\delta_\beta, f \rangle.$$

Moreover, setting now $f = |\cdot|$, it holds by the Fundamental Theorem that (up to extraction of a subsequence)

$$|w_n| \overset{*}{\rightharpoonup} \lambda|\alpha| + (1 - \lambda)|\beta| + \lambda,$$

from which $\lambda = 0$ follows. More generally, it is easily seen that L^∞-bounded sequences generate no concentration.

3. **(More general oscillations.)** If $v \in L^1([-1,1])$ is extended periodically and $w_n(x) = v(nx)$, then (w_n) generates the (x-independent) Young measure $\nu = \mathcal{L}^1 \circ v^{-1}$. Indeed, for bounded continuous f,

$$f(w_n) \rightharpoonup \int_{-1}^{1} f(v(x))dx = \int_{\mathbb{R}} f(z)\mathcal{L}^1 \circ v^{-1}(dz).$$

This example underlines the probabilistic interpretation of the oscillation measure: It can be viewed as the probability distribution of the values of the sequence when the frequency becomes higher and higher. This can be made rigorous in the following way: If w_n is any sequence generating the oscillation measure ν_x, then

$$\nu_x(B) = \lim_{\delta\to 0}\lim_{n\to\infty} \frac{\mathcal{L}^1\left(\{y \in B_\delta(x) : w_n(y) \in B\}\right)}{\mathcal{L}^1(B_\delta(x))}$$

for every Borel set $B \subset \mathbb{R}$ and a.e. $x \in [-1,1]$. This also holds for arbitrary domains and arbitrary dimensions m, l and was observed in [2].

4. **(Concentration.)** Let $w_n = n\chi_{[-\frac{1}{2n},\frac{1}{2n}]}$. Then (w_n) is bounded in L^1 and generates the Young measure $\nu_x \equiv \delta_0$, $\lambda = \delta_0$, $\nu_0^\infty = \delta_1$. To see this, first note that $f(w_n) \to f(0)$ strongly for every *bounded* f, whence $\nu_x \equiv \delta_0$. On the other hand, $|w_n|dx \overset{*}{\rightharpoonup} \delta_0$ in the sense of measures, so that $\lambda = \delta_0$. Since moreover for every $f \in \mathcal{F}_1$ we have $f(w_n) \overset{*}{\rightharpoonup} f^\infty(1)\delta_0$, it follows that $\nu_0^\infty = \delta_0$.

5. **(Concentration in various directions.)** Set now

$$w_n = n\left(\chi_{[-\frac{1}{2n},0]} - \chi_{[0,\frac{1}{2n}]}\right).$$

A similar argument as in the previous example gives $\nu_x \equiv \delta_0$, $\lambda = \delta_0$, $\nu_0^\infty = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

6. **(Diffuse concentration.)** This example demonstrates that the concentration measure need not be singular with respect to Lebesgue measure. Indeed, consider

$$w_n(x) = n\sum_{k=-n}^{n-1} \chi_{\left[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}\right]}.$$

This sequence generates the Young measure $\nu_x \equiv \delta_0$, $\lambda = \mathcal{L}^1 \restriction_{[-1,1]}$, $\nu_x^\infty \equiv \delta_1$. Indeed, $w_n \to 0$ in measure because

$$\mathcal{L}^1\left(\bigcup_{k=-n}^{n-1}\left[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}\right]\right) = \frac{2n}{n^2} \to 0,$$

and hence by Proposition 3.5 $\nu = \delta_0$. Let now $\phi \in C_c^\infty((-1,1))$. Then

$$\begin{aligned}\int_{-1}^{1} \phi(x)|w_n(x)|dx &= \int_{-1}^{1} \phi(x) n \sum_{k=-n}^{n-1} \chi_{\left[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}\right]}(x)dx \\ &= \int_{-n}^{n} \phi\left(\frac{y}{n}\right) \sum_{k=-n}^{n-1} \chi_{\left[k,k+\frac{1}{n}\right]}(y)dy \\ &= \sum_{k=-n}^{n-1} \int_{k}^{k+\frac{1}{n}} \phi\left(\frac{y}{n}\right) dy \\ &= \sum_{k=-n}^{n-1} \frac{1}{n}\phi\left(\frac{k}{n}\right) + o(1)\end{aligned}$$

as $n \to 0$, and the expression in the last line is just a Riemann sum and thus converges to $\int_{-1}^{1} \phi dx$. It follows that $\lambda = \mathcal{L}^1$ on $[-1,1]$, and it is readily seen that $\nu_x^\infty \equiv \delta_1$.

3.3.3 Measure-Valued Solutions for Euler

We recall from [7] the notion of admissible measure-valued solutions of Euler. As a preparation, we recall the following proposition from [7]:

Proposition 3.6. *Let $(v_n(x,t))$ be a sequence of functions $\mathbb{R}^d \times [0,T] \to \mathbb{R}^l$ which is bounded in $L^\infty\left([0,T];L^2\left(\mathbb{R}^d\right)\right)$ and generates a Young measure (ν,λ,ν^∞) in $L^2\left(\mathbb{R}^d \times [0,T]\right)$. Then*

$$\operatorname{esssup}_t \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle dx\right) < \infty,$$

and the concentration measure λ admits a disintegration of the form $\lambda(dx,dt) = \lambda_t(dx) \otimes dt$, where $t \mapsto \lambda_t$ is a bounded (w.r.t. the total variation norm) measurable map from $[0,T]$ into $\mathcal{M}^+\left(\mathbb{R}^d\right)$.

Proof. Define a measure on $[0,T]$ by $\mu(A) := \lambda(\mathbb{R}^d \times A)$ for a Borel subset $A \subseteq [0,T]$. By standard measure theory, there exists a measurable map $t \mapsto \tilde{\lambda}_t$ with $\tilde{\lambda}_t \in \mathcal{M}^+\left(\mathbb{R}^d\right)$ such that $\lambda(dx,dt) = \tilde{\lambda}_t(dx) \otimes \mu(dt)$. Application of the Fundamental Theorem with $f = |\cdot|^2$ and integration over x yields

$$\|v_n\|_{L_x^2}^2 (t)dt \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle dx\right) dt + \mu(dt), \tag{3.2}$$

which means that for every $\phi \in C_c([0,T])$, $\phi \geq 0$,

$$\int_0^T \phi(t) \|v_n\|_{L_x^2}^2 (t)dt \to \int_0^T \phi(t) \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle dx\right) dt + \int_0^T \phi(t)\mu(dt).$$

Hence,

$$\begin{aligned}\left|\int \phi \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle dx\right) dt\right| &\leq \sup_n \left|\int \phi \|v_n\|_{L_x^2}^2 dt\right| \\ &\leq \sup_{n,t} \|v_n\|_{L_x^2}^2 \|\phi\|_{L^1([0,T])},\end{aligned}$$

from which it follows that $\operatorname{esssup}_t \left(\int_{\mathbb{R}^d} \langle \nu_{x,t}, |\cdot|^2\rangle dx\right) < \infty$.

Similarly,

$$\left|\int \phi\mu(dt)\right| \le \sup_n \|v_n\|^2_{L^2_x} \|\phi\|_{L^1([0,T])},$$

whence by the Radon-Nikodým theorem $\mu = h(t)dt$, and again $h \in L^\infty([0,T])$. So by setting $\lambda_t = h(t)\tilde{\lambda}_t$ we obtain the desired disintegration. □

A *measure-valued solution* to the Euler equations is now a generalised Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times [0,T]$ which satisfies the Euler equations in an average sense. More precisely, we require the Young measure to satisfy

$$\begin{aligned}\partial_t\langle\nu_{x,t},\xi\rangle + \operatorname{div}\left(\langle\nu_{x,t},\xi\otimes\xi\rangle + \langle\nu^\infty_{x,t},\theta\otimes\theta\rangle\lambda\right) + \nabla p = 0\\ \operatorname{div}\langle\nu_{x,t},\xi\rangle = 0\end{aligned} \tag{3.3}$$

in the sense of distributions. Here, the quantity

$$\bar{v}(x,t) := \langle\nu_{x,t},\xi\rangle \tag{3.4}$$

is called the *barycentre* of $\nu_{x,t}$. As usual, we have written $\langle\nu,\xi\otimes\xi\rangle = \int \xi\otimes\xi\nu(d\xi)$ etc. It is well-known that if $\bar{v}(x,t)$ solves an equation like (3.3) and belongs to $L^\infty_t L^2_x$, then it can be redefined on a set of times of measure zero so that it belongs to the space CL^2_w (see Appendix A of [16] and cf. Section 2.1), and therefore, the initial average $\bar{v}(\cdot,0)$ is a well-defined L^2 function that is assumed in the sense that $\bar{v}(\cdot,t) \rightharpoonup \bar{v}(\cdot,0)$ weakly in L^2 as $t \to 0$. Later we will be interested only in Young measures generated by sequences which are bounded in $C\left([0,T];L^2_w\right)$. The concentration measure of such a Young measure admits a disintegration $\lambda = \lambda_t \otimes dt$ by Proposition 3.6. Unless otherwise stated, we will only deal with Young measures whose concentration measure has the form $\lambda_t \otimes dt$. We may therefore write the equations (3.3) as

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\partial_t\phi\cdot\langle\nu,\xi\rangle + \nabla\phi:\langle\nu,\xi\otimes\xi\rangle dxdt + \int_0^T\int_{\mathbb{R}^d}\nabla\phi:\langle\nu^\infty,\theta\otimes\theta\rangle\lambda_t(dx)dt\\ = -\int_{\mathbb{R}^d}\phi(x,0)\bar{v}(x,0)dx\end{aligned} \tag{3.5}$$

for all $\phi \in C^1_c\left(\mathbb{R}^d\times[0,T);\mathbb{R}^d\right)$ with $\operatorname{div}\phi = 0$.

Let us recall from [7] the definition of the *energy* of a Young measure for almost every time t:

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^d}\langle\nu_{x,t},|\cdot|^2\rangle dx + \frac{1}{2}\lambda_t(\mathbb{R}^d). \tag{3.6}$$

The following definition in the spirit of DiPerna-Majda [20] is made in [7] (cf. also [19] for the case of conservation laws):

Definition 3.7. (Measure-Valued Solutions.)

a) A Young measure (ν,λ,ν^∞) on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d\times[0,T]$ is called a *measure-valued solution* of the Euler equations with barycentre $\bar{v} := \langle\nu,\xi\rangle$ if it satisfies (3.3) in the sense of distributions.

b) A Young measure (ν,λ,ν^∞) on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d\times[0,T]$ and disintegration $\lambda = \lambda_t\otimes dt$ is called an *admissible* measure-valued solution of the Euler equations with initial data $v_0 \in L^2(\mathbb{R}^d)$ and barycentre $\bar{v} := \langle\nu,\xi\rangle$ if

1. $\operatorname{div} \bar{v} = 0$ in the sense of distributions;
2. $\bar{v}(\cdot, 0) = v_0$;
3. (3.5) holds;
4. $E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx$ for almost every t.

Proposition 3.8. *Let $(\nu, \lambda, \nu^\infty)$ be an admissible measure-valued solution of the Euler equations and $\bar{v}$ its barycentre. Then*

$$\bar{v}(\cdot, t) \to \bar{v}(\cdot, 0) = v_0$$

strongly in $L^2(\mathbb{R}^d)$ as $t \to 0$.

Proof. We have already seen that $\bar{v} \in CL^2_w$ and therefore

$$\liminf_{t \to 0} \|\bar{v}(t)\|_{L^2} \geq \|\bar{v}(0)\|_{L^2}.$$

On the other hand,

$$\begin{aligned} \int |\bar{v}(t)|^2 dx &= \int |\langle \nu_{x,t}, \xi \rangle|^2 \, dx \\ &\leq \int \langle \nu_{x,t}, |\xi|^2 \rangle dx + \lambda_t(\mathbb{R}^d) \\ &= 2E(t) \leq \int |\bar{v}(0)|^2 dx, \end{aligned}$$

where we used the weak energy inequality in Definition 3.7. Combining both inequalities yields $\|\bar{v}(t)\|_{L^2} \to \|\bar{v}(0)\|_{L^2}$ as $t \to 0$, and since weak convergence together with convergence of the norms implies strong convergence, we are done. □

It follows directly from the definitions that a vanishing viscosity sequence of Leray solutions generates an admissible measure-valued solution of Euler. This was already observed in [20]. We state it here for completeness:

Theorem 3.9. *If (v_{ν_k}) is a sequence of Leray solutions with viscosities $\nu_k \to 0$ and initial data v_0, then a subsequence generates an admissible measure-valued solution of Euler with the same initial data. In particular, there exists at least one global admissible measure-valued solution for every initial data.*

The main result of [7] is the weak-strong uniqueness for admissible measure-valued solutions. Recall that $d(v)$ denotes the symmetric gradient of v.

Theorem 3.10. *If $v \in C([0,T]; L^2(\mathbb{R}^d))$ is a solution of the Euler equations with $\int_0^T \|d(v)\|_{L^\infty_x} dt < \infty$, then every admissible measure-valued solution with the same initial data coincides with v in the sense that $\nu_{x,t} = \delta_{v(x,t)}$ a.e. and $\lambda \equiv 0$.*

The last two theorems show that admissible measure-valued solutions satisfy Lions' two minimal requirements for a notion of solution: Global existence and weak-strong uniqueness.

3.3.4 Measure-Valued Subsolutions

Consider sequences of Euler subsolutions of the form (v_n, u_n), where $v_n \in L^\infty\big([0,T]; L^2(\mathbb{R}^d)\big)$ and $u_n \in L^\infty\big([0,T]; L^1(\mathbb{R}^d)\big)$ and (v_n, u_n) take values in $\mathbb{R}^d \times \mathcal{S}_0^d$. We will need to determine the limit behaviour of $f(v_n, u_n)$, where $f : \mathbb{R}^d \times [0,T] \times \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ is continuous and such that the function

$$f^\infty(x,t;v,u) := \lim_{\substack{(x',t')\to(x,t)\\(v',u')\to(v,u)\\ s\to\infty}} \frac{f(x',t';sv',s^2u')}{s^2}$$

exists and is continuous. Let us denote the class of such functions by $\mathcal{F}_{2,1}$, and set $S = \{(v,u) \in \mathbb{R}^d \times \mathcal{S}_0^d : |v|^4 + |u|^2 = 1\}$. The following version of the Fundamental Theorem for Young measures is most suitable for this situation:

Theorem 3.11. *Suppose $w_n = (v_n, u_n)$ is a sequence bounded in $L^\infty([0,T]; L^2 \times L^1(\mathbb{R}^d))$, and $f \in \mathcal{F}_{2,1}$. Then there exist a subsequence (not relabeled) (w_n) and a Young measure $(\nu, \lambda, \nu^\infty)$, with $\nu \in L^\infty_w(\mathbb{R}^d \times [0,T]; \mathcal{M}^1(\mathbb{R}^d \times \mathcal{S}_0^d))$, $\lambda \in \mathcal{M}^+(\mathbb{R}^d \times [0,T])$, $\nu^\infty \in L^\infty_w\big(\mathbb{R}^d \times [0,T], \lambda; \mathcal{M}^1(S)\big)$, such that for all $f \in \mathcal{F}_{2,1}$*

$$f(x,t;w_n(x,t))dxdt \overset{*}{\rightharpoonup} \langle \nu_{x,t}, f(x,t;\cdot)\rangle dxdt + \langle \nu^\infty_{x,t}, f^\infty(x,t;\cdot)\rangle \lambda$$

in the sense of measures.

Proof. Consider the homeomorphism $\mathbb{R}^d \to \mathbb{R}^d$, $v \mapsto |v|v$, with inverse $v \mapsto \frac{v}{\sqrt{|v|}}$. For given $f \in \mathcal{F}_{2,1}$, define $g(x,t;|v|v,u) := f(x,t;v,u)$. Then g is a well-defined continuous function and, for $|v|^4 + |u|^2 = 1$, it holds that

$$\lim_{\substack{(x',t')\to(x,t)\\(v',u')\to(v,u)\\ s\to\infty}} \frac{g(x',t';s|v'|v',su')}{s} = \lim_{\substack{(x',t')\to(x,t)\\(v',u')\to(v,u)\\ s\to\infty}} \frac{f(x',t';sv',s^2u')}{s^2} = f^\infty(x,t;v,u),$$

thus $g \in \mathcal{F}_1$ and we may apply Theorem 3.4 to g, $(\tilde\nu, \tilde\lambda, \tilde\nu^\infty)$ being the Young measure generated by $(|v_n|v_n, u_n)$ in L^1 (we suppress the variables x and t):

$$\begin{aligned} f(v_n,u_n)dxdt =& g(|v_n|v_n,u_n)dxdt \\ \overset{*}{\rightharpoonup}& \langle\tilde\nu, g\rangle dxdt + \langle \tilde\nu^\infty, g^\infty\rangle\tilde\lambda \\ =& \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda, \end{aligned}$$

where $(\nu, \lambda, \nu^\infty)$ is given by

$$\int_{\mathbb{R}^d\times\mathcal{S}_0^d} f(\xi,\zeta)d\nu(\xi,\zeta) = \int_{\mathbb{R}^d\times\mathcal{S}_0^d} f\left(\frac{\xi}{\sqrt{|\xi|}}, \zeta\right) d\tilde\nu(\xi,\zeta),$$

$\lambda = \tilde\lambda$, and

$$\int_S f^\infty(\xi,\zeta)d\nu^\infty(\xi,\zeta) = \int_{\{|\xi|^2+|\zeta|^2=1\}} f^\infty\left(\frac{\xi}{\sqrt{|\xi|}}, \zeta\right) d\tilde\nu^\infty(\xi,\zeta).$$

□

The following proposition is a straightforward adaptation of Proposition 3.5 to the $\mathcal{F}_{2,1}$ framework.

Proposition 3.12. *a) There exists a countable set of functions $\{f_k\} = \{\phi_k \otimes h_k : k \in \mathbb{N}\} \subset \mathcal{F}_{2,1}$, where $\phi_k \in C_c(\mathbb{R}^d \times [0,T])$ and $h_k \in \mathcal{F}_{2,1}$ is independent of x and t, such that $\langle\langle \nu, \lambda, \nu^\infty; f_k \rangle\rangle = \langle\langle \tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty; f_k \rangle\rangle$ for all k implies $(\nu, \lambda, \nu^\infty) = (\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$.*

b) If $(w_n) = (v_n, u_n)$ and $(\tilde{w}_n) = (\tilde{v}_n, \tilde{u}_n)$ are sequences bounded in $(L^2 \times L^1)(\mathbb{R}^d \times [0,T])$ and $w_n - \tilde{w}_n \to 0$ locally in measure, and if (w_n) generates an oscillation measure ν, then $(\tilde{w}_n)$ generates the same oscillation measure ν.

c) If (w_n) and $(\tilde{w}_n)$ are sequences bounded in $(L^2 \times L^1)(\mathbb{R}^d \times [0,T])$ and $w_n - \tilde{w}_n \to 0$ in $(L^2 \times L^1)_{loc}(\mathbb{R}^d \times [0,T])$, and if (w_n) generates the generalised Young measure $(\nu, \lambda, \nu^\infty)$, then $(\tilde{w}_n)$ generates the same generalised Young measure.

d) If $w_n \to w$ strongly in $(L^2 \times L^1)_{loc}(\mathbb{R}^d \times [0,T])$, then (w_n) generates the Young measure $\nu_x = \delta_{w(x)}$ and $\lambda = 0$.

e) Suppose $w_n \overset{\mathbf{Y}}{\to} (\nu, \lambda, \nu^\infty)$ in $(L^2 \times L^1)(\mathbb{R}^d \times [0,T])$ and $w \in (L^2 \times L^1)(\mathbb{R}^d \times [0,T])$. Then $w_n + w \overset{\mathbf{Y}}{\to} (S_w \nu, \lambda, \nu^\infty)$ in $L^2 \times L^1$, where $S_w \nu$ is the shifted Young measure defined by $\langle S_w \nu, f \rangle = \langle \nu, f(\cdot + w) \rangle$.

Let $(\nu, \lambda, \nu^\infty)$ be a Young measure on $\mathbb{R}^d \times \mathcal{S}_0^d$ with parameters in $\mathbb{R}^d \times [0,T]$, then we can define its *barycentre* $\bar{w} = (\bar{v}, \bar{u})$ by

$$\bar{v}(x,t) := \langle \nu_{x,t}, \pi_1 \rangle \tag{3.7}$$

and

$$\bar{u}(x,t) := \langle \nu_{x,t}, \pi_2 \rangle dxdt + \langle \nu^\infty_{x,t}, \pi_2 \rangle \lambda \tag{3.8}$$

for a.e. x, t, where π_1 and π_2 are the canonical projections from $\mathbb{R}^d \times \mathcal{S}_0^d$ onto $\mathbb{R}^d$ and $\mathcal{S}_0^d$, respectively. Note that $\bar{u}(x,t)$ is only a measure. Such a Young measure is called a *measure-valued subsolution* if $(\bar{v}, \bar{u})$ is a subsolution in the sense of distributions, i.e. if it satisfies (2.2).

Recall the generalised energy $e : \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ from Section 2.2. The *energy* of a Young measure on $\mathbb{R}^d \times \mathcal{S}_0^d$ is defined by

$$E(t) = \int \langle \nu_{x,t}, e \rangle dx + \int \langle \nu^\infty_{x,t}, e \rangle \lambda_t(dx). \tag{3.9}$$

If for a measure-valued subsolution $E(t) \le \frac{1}{2} \int \bar{v}(x,0)dx$ for a.e. $t \ge 0$, we call it an *admissible* measure-valued subsolution.

3.3.5 Lifting

Finally, we would like to "lift" Young measures from $\mathbb{R}^d$ to $\mathbb{R}^d \times \mathcal{S}_0^d$. Define a map $Q : \mathbb{R}^d \to \mathbb{R}^d \times \mathcal{S}_0^d$ by

$$Q(\xi) = (\xi, \xi \bigcirc \xi).$$

Given now a Young measure $(\nu, \lambda, \nu^\infty)$ on $\mathbb{R}^d$, we may identify it with a Young measure $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ on $\mathbb{R}^d \times \mathcal{S}_0^d$ via

$$\langle \tilde{\nu}, f \rangle = \langle \nu, f \circ Q \rangle$$

and

$$\langle \tilde{\nu}^\infty, f^\infty \rangle \tilde{\lambda} = \langle \nu^\infty, f \circ Q \rangle \lambda$$

where $f \in \mathcal{F}_{2,1}$. Observe that $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ is uniquely defined by the above equations (cf. Proposition 3.12a)). The point of this lifting is the following:

Proposition 3.13. *Let $(\nu, \lambda, \nu^\infty)$ be a measure-valued solution with bounded energy and $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ be defined as above. Suppose a sequence (v_n, u_n), bounded in $L^\infty_t(L^2_x \times L^1_x)$, generates $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$. Then*

a) the barycentres of $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ (cf. (3.7) and (3.8)) form an Euler subsolution, i.e. they satisfy (2.2);

b) if $\tilde{E}(t)$ denotes the energy of the Young measure $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ in the sense of (3.9) and $E(t)$ the energy of (ν, λ, ν) in the sense of (3.6), then $\tilde{E}(t) = E(t)$ for a.e. t;

c) the sequence v_n generates the Young measure $(\nu, \lambda, \nu^\infty)$ in L^2;

d) $|u_n - v_n \bigcirc v_n| \to 0$ in $L^1_{loc}(\mathbb{R}^d \times [0, T])$.

Proof. a) follows straightforwardly by the definition of $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ and the fact that $(\nu, \lambda, \nu^\infty)$ is a solution to (3.3).

b) By definition of $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$, and applying Lemma 2.6,

$$\begin{aligned}
\tilde{E}(t) &= \int \langle \tilde{\nu}_{x,t}, e \rangle dx + \int \langle \tilde{\nu}^\infty_{x,t}, e \rangle \tilde{\lambda}_t(dx) \\
&= \int \langle \nu, e(\xi, \xi \bigcirc \xi) \rangle dx + \int \langle \nu^\infty, e(\xi, \xi \bigcirc \xi) \rangle \lambda_t(dx) \\
&= \frac{1}{2} \int \langle \nu, |\xi|^2 \rangle dx + \frac{1}{2} \int \langle \nu^\infty, |\xi|^2 \rangle \lambda_t(dx) \\
&= \frac{1}{2} \int \langle \nu, |\xi|^2 \rangle dx + \frac{1}{2} \lambda_t(\mathbb{R}^d) = E(t),
\end{aligned}$$

where we used that ν^∞ is supported on S^{d-1}.

c) Let $f \in \mathcal{F}_2(\mathbb{R}^d)$ and define $g := f \circ \pi_1$. Then $g \in \mathcal{F}_{2,1}$ with

$$\begin{aligned}
g^\infty(\xi, \zeta) &= \lim_{\substack{\xi' \to \xi \\ \zeta' \to \zeta \\ s \to \infty}} \frac{g(s\xi', s^2\zeta')}{s^2} \\
&= \lim_{\substack{\xi' \to \xi \\ s \to \infty}} \frac{f(s\xi')}{s^2} = f^\infty(\xi).
\end{aligned}$$

We have

$$\begin{aligned}
f(v_n) dx dt &= g(v_n, u_n) dx dt \\
&\overset{*}{\rightharpoonup} \langle \tilde{\nu}, g \rangle dx dt + \langle \tilde{\nu}^\infty, g^\infty \rangle \tilde{\lambda} \\
&= \langle \nu, f \rangle dx dt + \langle \nu^\infty, f^\infty \rangle \lambda
\end{aligned}$$

by definition of $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ and since $g \circ Q = g$.

d) Note that the function $f(\xi, \zeta) = |\zeta - \xi \bigcirc \xi|$ belongs to $\mathcal{F}_{2,1}$ with $f^\infty = f$. We can thus apply Theorem 3.11 with f to obtain

$$|u_n - v_n \bigcirc v_n| dx dt \overset{*}{\rightharpoonup} \langle \tilde{\nu}, f \rangle dx dt + \langle \tilde{\nu}^\infty, f^\infty \rangle \tilde{\lambda}_t dt = 0$$

because the set $\{(\xi, \xi \bigcirc \xi) : \xi \in \mathbb{R}^d\}$ contains the supports of $\tilde{\nu}$ and $\tilde{\nu}^\infty$, respectively, and on this set, f and f^∞ vanish. □

Chapter 4

The Relation Between Weak and Measure-Valued Solutions

In this chapter I prove that any measure-valued solution is generated by a sequence of weak solutions (Theorem 1.3). More precisely, we have the following two theorems, which will be proved simultaneously:

Theorem 4.1. *A Young measure $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times [0,T]$ is a measure-valued solution of the Euler equations with bounded energy if and only if there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of weak solutions to the Euler equations bounded in $C\big([0,T]; L^2_w(\mathbb{R}^d;\mathbb{R}^d)\big)$ which generate the Young measure $(\nu, \lambda, \nu^\infty)$ in the sense that*

$$f(v_n)dxdt \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} f d\nu\right) dxdt + \left(\int_{S^{d-1}} f^\infty d\nu^\infty\right)\lambda$$

in the sense of measures for every $f \in \mathcal{F}_2$.

Theorem 4.2. *Suppose that $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ is an* admissible *measure-valued solution with initial data $v_0 \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ $(\operatorname{div} v_0 = 0)$. Then the generating sequence (v_n) as in Theorem 4.1 may be chosen such that in addition*

$$\|v_n(t=0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$$

and

$$\sup_{t\in[0,T]} \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,t)|^2 dx \le \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,0)|^2 dx.$$

Before we begin to prove these results, we state a weaker version of Theorem 4.2 that we can prove along with Theorem 4.1. In Section 4.4 we then conclude from this weaker statement the full assertion of Theorem 4.2.

Proposition 4.3. *Suppose that $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ is an* admissible *measure-valued solution with initial data $v_0 \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ $(\operatorname{div} v_0 = 0)$. Then the generating sequence (v_n) as in Theorem 4.1 may be chosen such that in addition*

$$\|v_n(t=0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$$

and

$$\sup_{t\in[0,T]} \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,t)|^2 dx \le \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,0)|^2 dx + \frac{1}{n}.$$

We prove this Proposition in three steps: In Section 4.1 we use the result of [16] to show that it suffices to generate measure-valued subsolutions by sequences of subsolutions. Section 4.2 adapts various well-known Young measure techniques to our framework and does not use any specific properties of the Euler equations. It is shown that it suffices to construct generating sequences for discrete homogeneous oscillation measures. This is done in Section 4.3, where the plane wave analysis of the system (2.2) is exploited to give an explicit construction of the generating sequence.

4.1 From Subsolutions to Exact Solutions

First of all observe that whenever a sequence of weak Euler solutions bounded in CL^2_w generates a Young measure, then this measure will be a measure-valued solution with bounded energy. If the generating sequence consists of admissible weak solutions with initial data v_0, then the measure inherits these properties. This follows directly from the Fundamental Theorem of Young measures (see also [20], [7]).

The result of this section is the first step towards the converse statement (i.e. Theorems 4.1 and 4.2).

Proposition 4.4. *a) We can generate $(\nu, \lambda, \nu^\infty)$ as required in Theorem 4.1 provided we can generate the corresponding lifted measure (cf. Subsection 3.3.5) $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ in the sense of Theorem 3.11 by a sequence (v_n, u_n) bounded in $L^\infty_t L^2_x \times L^\infty_t L^1_x$ with the properties*

- *(v_n, u_n) are smooth in $\mathbb{R}^d \times [0,T]$;*
- *(v_n, u_n) solve (2.2).*

b) If $(\nu, \lambda, \nu^\infty)$ is admissible, then we can generate it as required in Proposition 4.3 if the sequence (v_n, u_n) additionally satisfies

- *$\limsup_n \sup_t \int e(v_n, u_n) dx \le \operatorname{esssup}_t \tilde{E}(t)$;*
- *$v_n(\cdot, 0) \to v_0 = \langle \nu_{\cdot,0}, \xi\rangle$ strongly in L^2.*

Proof. Suppose now (v_n, u_n) generates the Young measure $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ as in part a) of the proposition. We choose for each n a function $\bar{e}_n \in C\left(\mathbb{R}^d \times (0,T); \mathbb{R}\right) \cap C\left([0,T]; L^1(\mathbb{R}^d; \mathbb{R})\right)$ such that $\bar{e}_n > e_n := e(v_n, u_n)$ on $\mathbb{R}^d \times (0,T)$ and

$$\sup_t \int_{\mathbb{R}^d} (\bar{e}_n - e_n)\, dx < \frac{1}{n} \tag{4.1}$$

for all n.

By Theorem 2.8, for $n \in \mathbb{N}$ we can find a sequence $(v_n^k)_{k\in\mathbb{N}} \subset CL^2_w$ of Euler solutions with $v_n^k \to v_n$ as $k \to \infty$ in the CL^2_w-topology, i.e.

$$\sup_{t\in[0,T]} \left| \int (v_n^k - v_n) \cdot \phi dx \right| \to 0$$

for each $\phi \in L^2$. We have, for all $t \in [0,T]$ and all bounded subdomains $\Omega \subset \mathbb{R}^d$,

$$\int_\Omega |v_n^k - v_n|^2 dx = \int_\Omega |v_n^k|^2 dx - \int_\Omega |v_n|^2 dx - 2\int_\Omega v_n \cdot (v_n^k - v_n) dx,$$

and by the CL^2_w-convergence we can choose $k = k(n)$ so large that

$$2\left|\int v_n \cdot (v_n^k - v_n) dx\right| < \frac{1}{n}$$

for all t.

It remains to estimate the difference of the $L^2_{loc}(\mathbb{R}^d \times [0,T])$-norms of v_n^k and v_n. Being interested only in L^2_{loc}, we may assume for the moment that $T < \infty$. Since $\frac{1}{2}|v_n^k|^2 = \bar{e}_n$ by Theorem 2.8,

$$\left|\int |v_n^k|^2 dxdt - \int |v_n|^2 dxdt\right| \le 2\left|\int (\bar{e}_n - e_n) dxdt\right| + 2\left|\int (e_n - \frac{1}{2}|v_n^2|) dxdt\right|,$$

where by choice of $\bar{e}_n$ the first expression is less than $\frac{T}{n}$. Finally, let us examine the last term:

$$\begin{aligned}\left|\int_{\Omega\times[0,T]} (e_n - \frac{1}{2}|v_n^2|) dxdt\right| &= \left|\int_\Omega \frac{d}{2}\lambda_{max}(v_n \otimes v_n - u_n) - \frac{1}{2}|v_n|^2 dxdt\right| \\ &= \left|\int_{\Omega\times[0,T]} \frac{d}{2}\lambda_{max}\left(v_n \bigcirc v_n - u_n + \frac{1}{d}|v_n|^2 I_d\right) - \frac{1}{2}|v_n|^2 dxdt\right| \\ &= \left|\int_{\Omega\times[0,T]} \frac{d}{2}\lambda_{max}(v_n \bigcirc v_n - u_n) dxdt\right| \\ &\le C\int_{\Omega\times[0,T]} |v_n \bigcirc v_n - u_n| dxdt \to 0\end{aligned}$$

as $n \to \infty$, by Proposition 3.13. We also used in this calculation that for a matrix A, $\lambda_{max}(A + \alpha I_d) = \lambda_{max}(A) + \alpha$.

Thus we have shown that there exists a subsequence $v_n^{k(n)}$ of Euler solutions such that $v_n - v_n^{k(n)} \to 0$ in $L^2_{loc}(\mathbb{R}^d \times [0,T])$, and hence by Propositions 3.12c) and 3.13c), this yields that the sequence $(v_n^{k(n)})$ generates the Young measure $(\nu, \lambda, \nu^\infty)$ in L^2. This proves part a) of the proposition.

For part b), recall that $\frac{1}{2}|v_n^{k(n)}|^2 = \bar{e}_n$, so by (4.1), the assumption about the energy in part b) of the claim, and the fact that $\tilde{E} = E$, we have

$$\limsup_n \sup_t \frac{1}{2}\int |v_n^{k(n)}|^2 dx \le \operatorname{esssup}_t E(t) \le \frac{1}{2}\int |v_0|^2 dx. \tag{4.2}$$

Since $v_n^{k(n)}(\cdot,0) = v_n(\cdot,0)$, we also get

$$\left\|v_n^{k(n)}(\cdot,0) - v_0\right\|_{L^2_x} = \|v_n(\cdot,0) - v_0\|_{L^2_x} = o(1)$$

as $n \to \infty$, which, together with (4.2), completes the proof of the proposition. □

4.2 Approximation of Generalised Young Measures

This section contains some standard approximation techniques for generalised Young measures similar to the ones developed and employed in [31]. The goal is to reduce the problem of generating an arbitrary generalised Young measure $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ which arises from a measure-valued solution by lifting to the study of discrete homogeneous (i.e. independent of x and t) oscillation measures of the form

$$\nu = \sum_{i=1}^N \alpha_i \delta_{(v_i,u_i)}$$

with $\alpha_i > 0$, $\sum_{i=1}^N \alpha_i = 1$, and $(v_i, u_i) \in \mathbb{R}^d \times \mathcal{S}^d_0$. A first observation is the following:

Proposition 4.5. *Suppose that for each $k \in \mathbb{N}$, there exists a sequence (v^k_n, u^k_n) generating the generalised Young measure $(\nu^k, \lambda^k, \nu^{k,\infty})$ in the sense of Theorem 3.11, and such that $\sup_t \sup_{n,k}(\|v^k_n\|_{L^2} + \|u^k_n\|_{L^1}) < \infty$. Assume further that*

$$\langle \nu^k, f\rangle dxdt + \langle \nu^{k,\infty}, f^\infty\rangle \lambda^k(dxdt) \overset{*}{\rightharpoonup} \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda(dxdt) \tag{4.3}$$

in the sense of measures for all $f \in \mathcal{F}_{2,1}$ as $k \to \infty$. Then there exists a diagonal sequence $(v^{k(l)}_{n(l)}, u^{k(l)}_{n(l)})_{l\in\mathbb{N}}$ which generates the Young measure $(\nu, \lambda, \nu^\infty)$.

Proof. By Proposition 3.12a) it suffices to consider countably many $f_i \in \mathcal{F}_{2,1}$ and $\phi_i \in C_c(\mathbb{R}^d \times [0,T])$ as test functions. For each i, we can then estimate

$$\begin{aligned}
&\left|\int \phi_i f_i(v^k_n, u^k_n) dxdt - \int \phi_i \langle \nu, f_i\rangle dxdt - \int \phi_i \langle \nu^\infty, f_i^\infty\rangle d\lambda\right| \\
&\le \left|\int \phi_i f_i(v^k_n, u^k_n) dxdt - \int \phi_i \langle \nu^k, f_i\rangle dxdt - \int \phi_i \langle \nu^{k,\infty}, f_i^\infty\rangle d\lambda^k\right| \\
&+ \left|\int \phi_i \langle \nu^k, f_i\rangle dxdt + \int \phi_i \langle \nu^{k,\infty}, f_i^\infty\rangle d\lambda^k - \int \phi_i \langle \nu, f_i\rangle dxdt - \int \langle \nu^\infty, f_i^\infty\rangle d\lambda\right| \\
&= |I_1| + |I_2|.
\end{aligned}$$

Given $\epsilon > 0$ and $j \in \mathbb{N}$, by assumption we may choose a $k_0 = k_0(j)$ such that $|I_2| < \epsilon/2$ for all $k \ge k_0$ and $i \le j$. For any such k, by the definition of Young measure generation, we may find $n_0 = n_0(k,j)$ such that $|I_1| < \epsilon/2$ if $n \ge n_0(k,j)$ and $i \le j$. Setting $k(l) = l$ and $n(l) = n_0(l,l)$ yields the desired diagonal sequence. □

The next result is the goal of this section. It comprises several well-known techniques for Young measures.

Theorem 4.6. *a) Let $(\nu, \lambda, \nu^\infty)$ be a measure-valued subsolution with bounded energy. Then there exists a sequence of oscillation measures $\nu^k_{x,t}$ on $\mathbb{R}^d \times \mathcal{S}^d_0$ with zero barycentres which are piecewise constant (in x,t) on a lattice and discrete on each cube on which they are constant, and there exists a sequence of smooth subsolutions $(\bar{v}^k, \bar{u}^k)$ bounded in $L^\infty_t(L^2_x \times L^1_x)$, such that*

$$\langle \nu^k, f(x,t;\cdot + \bar{v}^k, \cdot + \bar{u}^k)\rangle dxdt \overset{*}{\rightharpoonup} \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda$$

for all $f \in \mathcal{F}_{2,1}$.

b) If $(\nu, \lambda, \nu^\infty)$ is an admissible *measure-valued subsolution with $\langle \nu_{\cdot,0}, \pi_1\rangle = v_0$, then the ν^k from a) can be chosen such that in addition*

$$\int_{\mathbb{R}^d} \langle \nu^k, e(\cdot + \bar{v}^k, \cdot + \bar{u}^k)\rangle dx \le \frac{1}{2}\int_{\mathbb{R}^d} |v_0|^2 dx + \frac{1}{k} \quad \textit{for a.e. } t \ge 0. \tag{4.4}$$

Proof. The proof is divided into several steps.

Step 1. From classical to generalised Young measures. Let $Q = [0,1]^{d+1}$ be the $d+1$-dimensional unit cube. To begin with, we consider a generalised Young measure on $\mathbb{R}^d \times \mathcal{S}_0^d$ which is homogeneous, i.e. independent of x and t, and discrete. More precisely, we assume $\nu = \sum_{i=1}^N \mu_i \delta_{(v_i,u_i)}$, $\lambda = \alpha \mathcal{L}^{d+1} \restriction Q$, and $\nu^\infty = \sum_{i=1}^M \tau_i \delta_{(v_i^\infty, u_i^\infty)}$. Here, $(v_i, u_i) \in \mathbb{R}^d \times \mathcal{S}_0^d$, $(v_i^\infty, u_i^\infty) \in S$ (recall the definition of S from Subsection 3.3.4), $\alpha \ge 0$, $\mu_i > 0$, $\sum_{i=1}^N \mu_i = 1$, $\tau_i > 0$, $\sum_{i=1}^M \tau_i = 1$, and $\mathcal{L}^{d+1} \restriction Q$ denotes $(d+1)$-dimensional Lebesgue measure restricted to Q. Moreover, we assume the barycentre to be zero, i.e.

$$\sum_{i=1}^N \mu_i v_i = 0, \qquad \sum_{i=1}^N \mu_i u_i + \alpha \sum_{i=1}^M \tau_i u_i^\infty = 0.$$

We will show that $(\nu, \lambda, \nu^\infty)$ can be approximated by a sequence of classical (oscillation) Young measures, i.e. Young measures whose concentration part is zero, with zero barycentre.

Indeed, define a sequence (ν^m) of probability measures by

$$\nu^m = \left(1 - \frac{1}{m}\right)\sum_{i=1}^N \mu_i \delta_{(v_i,u_i)} + \frac{1}{m}\sum_{i=1}^M \tau_i \delta_{(\sqrt{\alpha m} v_i^\infty, \alpha m u_i^\infty)}.$$

Then, ν^m with $\lambda^m = 0$ converges to $(\nu, \lambda, \nu^\infty)$ in the sense of (4.3): Indeed, for $f \in \mathcal{F}_{2,1}$,

$$\begin{aligned}
\langle \nu^m, f\rangle dxdt &= \left(1 - \frac{1}{m}\right)\sum_{i=1}^N \mu_i f(v_i, u_i) dxdt + \frac{1}{m}\sum_{i=1}^M \tau_i f\left(\sqrt{\alpha m} v_i^\infty, \alpha m u_i^\infty\right) dxdt \\
&= \left(1 - \frac{1}{m}\right)\sum_{i=1}^N \mu_i f(v_i, u_i) dxdt + \alpha \sum_{i=1}^M \tau_i \frac{f\left(\sqrt{\alpha m} v_i^\infty, \alpha m u_i^\infty\right)}{\alpha m} dxdt \\
&\overset{*}{\rightharpoonup} \sum_{i=1}^N \mu_i f(v_i, u_i) dxdt + \alpha \sum_{i=1}^M \tau_i f^\infty(v_i^\infty, u_i^\infty) dxdt \\
&= \langle \nu, f\rangle dxdt + \langle \nu^\infty, f\rangle \lambda
\end{aligned}$$

as $m \to \infty$, where we used the definition of f^∞.

Moreover, one readily checks that the barycentres $\bar{v}^k$, $\bar{u}^k$ of ν^k (cf. (3.7), (3.8)) converge to zero, using the assumption that $(\nu, \lambda, \nu^\infty)$ has zero barycentre. It is then obvious that also the shifted measures $S_{-(\bar{v}^k, \bar{u}^k)}\nu^k$, which have exactly zero barycentre, converge to $(\nu, \lambda, \nu^\infty)$ in the sense of (4.3).

Step 2. From discrete to general measures. Consider the case that $(\nu, \lambda, \nu^\infty)$ is a Young measure with zero barycentre where λ is still a constant multiple of Lebesgue measure and ν and ν^∞ are still homogeneous (i.e. independent of x and t) but not necessarily discrete. We also require $\langle \nu, e\rangle < \infty$ (recall that e is the generalised energy).

Assume for the moment that ν is compactly supported. By a standard result in measure theory (see e.g. [4], §30), we may find sequences of discrete measures ν_k with uniformly compact support and ν_k^∞ such that $\nu_k \overset{*}{\rightharpoonup} \nu$ and $\nu_k^\infty \overset{*}{\rightharpoonup} \nu^\infty$. Without loss of generality we can assume the corresponding generalised Young measure $(\nu_k, \lambda, \nu_k^\infty)$ to have zero barycentre. If $f \in \mathcal{F}_{2,1}$ is independent of x and t (this suffices by Proposition 3.12a)) and $\phi \in C_c(Q)$, we have by the weak*-convergence that $\langle \nu_k^\infty - \nu^\infty, f^\infty \rangle \to 0$, so that

$$\int_Q \phi \langle \nu_k^\infty - \nu^\infty, f^\infty \rangle \lambda(dxdt) \to 0$$

as $k \to \infty$. Similarly, since ν and ν_k are uniformly compactly supported, it holds that $\langle \nu_k - \nu, f \rangle \to 0$ (although f is, in general, not compactly supported). So we obtain

$$\int_Q \phi \langle \nu_k - \nu, f \rangle dxdt \to 0$$

and thus the desired convergence.

Now drop the assumption that ν has compact support and only impose $\langle \nu, e \rangle < \infty$. Using an idea from [32], we may approximate ν by compactly supported measures in the following way:

For $\rho \in \mathbb{N}$, let $r^\rho : \mathbb{R}^d \times \mathcal{S}_0^d \to \mathbb{R}$ be a smooth function which is 1 on B_ρ, zero on $\left(\mathbb{R}^d \times \mathcal{S}_0^d\right) \backslash B_{\rho+1}$ and $0 \le r \le 1$ everywhere. Define also a number s^ρ by

$$s^\rho = \langle \nu, 1 - r^\rho \rangle,$$

which measures how much mass ν carries outside of $B_\rho(0)$. We then define

$$\nu^\rho := r^\rho \nu + s^\rho \delta_0,$$

which is a probability measure with support in $B_{\rho+1}$. Heuristically, we obtain ν^ρ by cutting off ν outside of $B_{\rho+1}$ and concentrating the remaining mass at zero. Although $(\nu^\rho, \lambda, \nu^\infty)$ need no longer have zero barycentre, for sufficiently large ρ its barycentre becomes arbitrarily close to zero, and we can fix this issue by slightly shifting ν^ρ (cf. Step 1).

We have now that

$$\langle \nu^\rho, f \rangle \to \langle \nu, f \rangle$$

for all $f \in \mathcal{F}_{2,1}$ independent of x and t. Indeed,

$$\langle \nu - \nu^\rho, f \rangle = \langle (1 - r^\rho)\nu, f \rangle - s^\rho f(0).$$

Observe that, since $r^\rho \to 1$ pointwise as $\rho \to \infty$, by dominated convergence (recall $\langle \nu, e \rangle < \infty$ and therefore also $\langle \nu, |f| \rangle < \infty$) we have $\langle (1 - r^\rho)\nu, \phi \rangle \to 0$ as well as $s^\rho \to 0$ as $\rho \to \infty$.

A diagonal argument now shows that $(\nu, \lambda, \nu^\infty)$ can be approximated by discrete measures.

Step 3. From homogeneous to non-homogeneous measures. We will now study the case in which $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ is a possibly inhomogeneous Young measure with zero barycentre and bounded energy (in particular, λ is no longer assumed to be a constant multiple of Lebesgue measure). We require however that $\lambda \ll \mathcal{L}^{d+1}$. In order to approximate our given Young measure by piecewise homogeneous ones, we use the well-known technique of averaging, see also Lemma 4.22 in [41] and Proposition 7 in [31].

Consider now an arbitrary open cube $C \in \mathbb{R}^d \times [0,T]$. We define the *average Young measure* corresponding to $(\nu, \lambda, \nu^\infty)$ on C in the following way:

Definition 4.7. Suppose $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ is a Young measure, $(x,t) \in C$. We define the average Young measure $(\bar{\nu}, \bar{\lambda}, \bar{\nu}^\infty)$ by duality:

$$\langle \bar{\nu}, f\rangle = \fint_C \langle \nu_{x,t}, f\rangle dxdt$$

$$\bar{\lambda} = \frac{\lambda(C)}{\mathcal{L}^{d+1}(C)} \mathcal{L}^{d+1} \restriction_C$$

$$\langle \bar{\nu}^\infty, f^\infty\rangle = \fint_C \langle \nu^\infty_{x,t}, f^\infty\rangle d\lambda(x,t)$$

for all $f \in \mathcal{F}_{2,1}$ which are independent of x and t, where $\fint_C g d\mu := \frac{1}{\mu(C)} \int_C g d\mu$ for any measure μ and any $g \in L^1(C;\mu)$.

Note that $(\bar{\nu}, \bar{\lambda}, \bar{\nu}^\infty)$ is homogeneous on C, and note further that it has zero barycentre if $(\nu, \lambda, \nu^\infty)$ does.

The following approximation result is essentially Proposition 8 in [31]:

Proposition 4.8. *Suppose $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ is a Young measure with zero barycentre and $\lambda \ll \mathcal{L}^{d+1}$. For every $l \in \mathbb{N}$, consider the lattice $\frac{T}{l}\mathbb{Z}^{d+1}$, which partitions $\mathbb{R}^d \times [0,T]$ into cubes Q^l_i with sidelength $\frac{T}{l}$ ($\frac{1}{l}$ if $T = \infty$). On Q^l_i, define $(\nu^l_i, \lambda^l_i, \nu^{l,\infty}_i)$ to be the average Young measure corresponding to $(\nu, \lambda, \nu^\infty)$ on Q^l_i, and set*

$$(\nu^l_{x,t}, \lambda^l, \nu^{l,\infty}_{x,t}) := (\nu^l_i, \lambda^l_i, \nu^{l,\infty}_i)$$

if $(x,t) \in Q^l_i$. Then for every $f \in \mathcal{F}_{2,1}$ it holds that

$$\langle \nu^l, f\rangle dxdt + \langle \nu^{l,\infty}, f^\infty\rangle \lambda^l(x,t) \overset{*}{\rightharpoonup} \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda(x,t). \tag{4.5}$$

Since the approximations of Steps 1 and 2 can be performed on each cube Q^l_i separately, this shows that $(\nu, \lambda, \nu^\infty)$ can be approximated by oscillation measures with zero barycentres that are piecewise constant and discrete on the Q^l_i.

Concerning the energy, suppose $\bar{w} = (\bar{v}, \bar{u})$ is a smooth function in $L^\infty_t(L^2_x \times L^1_x)$ (in the next step this will be the barycentre of the Young measure in question). Denoting the approximating sequence for $(\nu^l, \lambda^l, \nu^{l,\infty})$ by $\nu^{k,l}$, we have (for fixed $l \in \mathbb{N}$ and $j = 0, \dots, l-1$)

$$\int_{\frac{Tj}{l}}^{\frac{T(j+1)}{l}} \int_{\mathbb{R}^d} \langle \nu^{k,l}, e(\cdot + \bar{w})\rangle dxdt \to$$
$$\int_{\frac{Tj}{l}}^{\frac{T(j+1)}{l}} \int_{\mathbb{R}^d} \langle \nu^l, e(\cdot + \bar{w})\rangle dxdt + \int_{\frac{Tj}{l}}^{\frac{T(j+1)}{l}} \int_{\mathbb{R}^d} \langle \nu^{l,\infty}, e\rangle \lambda^l$$

as $k \to \infty$, and since $\nu^{k,l}$ and $(\nu^l, \lambda^l, \nu^{l,\infty})$ are time-independent in the time interval considered and $\bar{w}$ is smooth, we conclude (after passing to a subsequence if necessary) that for a.e. $t \in [0,T]$

$$\int_{\mathbb{R}^d} \langle \nu^{k,l}, e(\cdot + \bar{w})\rangle dx \leq \sup_{[0,T]} \left(\int_{\mathbb{R}^d} \langle \nu^l, e(\cdot + \bar{w})\rangle dx + \int_{\mathbb{R}^d} \langle \nu^{l,\infty}, e\rangle \lambda_t^l(dx) \right) + \frac{1}{k}$$

for l and $k = k(l)$ large enough.

Moreover, the following inequality follows immediately from the definition of the average Young measure: For $f \in \mathcal{F}_{2,1}$,

$$\begin{aligned} &\sup_{[0,T]} \left(\int_{\mathbb{R}^d} \langle \nu^l, f\rangle dx + \int_{\mathbb{R}^d} \langle \nu^{l,\infty}, f^\infty\rangle \lambda_t^l(dx) \right) \\ &\leq \sup_{[0,T]} \left(\int_{\mathbb{R}^d} \langle \nu, f\rangle dx + \int_{\mathbb{R}^d} \langle \nu^\infty, f^\infty\rangle \lambda_t(dx) \right). \end{aligned} \tag{4.6}$$

Hence it even holds that

$$\int_{\mathbb{R}^d} \langle \nu^{k,l}, e(\cdot + \bar{w})\rangle dx \leq \sup_{[0,T]} \left(\int_{\mathbb{R}^d} \langle \nu, e(\cdot + \bar{w})\rangle dx + \int_{\mathbb{R}^d} \langle \nu^\infty, e\rangle \lambda_t(dx) \right) + \frac{1}{k}. \tag{4.7}$$

Step 4. From zero barycentre to general barycentres. In this step, let $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ be a Young measure with $\lambda \ll \mathcal{L}^{d+1}$ and smooth barycentre $(\bar{v}, \bar{u}) \in L_t^\infty L_x^2 \times L_t^\infty L_x^1$ which solve (2.2). Define another Young measure $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ by shifting $(\nu, \lambda, \nu^\infty)$ by its barycentre:

$$(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty) = (S_{-(\bar{v},\bar{u})}\nu, \lambda, \nu^\infty),$$

where S is the shift operator defined in Proposition 3.12e), i.e. $\langle S_{-(\bar{v},\bar{u})}\nu, f\rangle = \langle \nu, f(\cdot - \bar{v}, \cdot - \bar{u})\rangle$. Obviously $(\tilde{\nu}, \tilde{\lambda}, \tilde{\nu}^\infty)$ has zero barycentre, and so there exist ν^k as in Theorem 4.6a) with

$$\langle \nu^k, f\rangle dxdt \overset{*}{\rightharpoonup} \langle \tilde{\nu}, f\rangle dxdt + \langle \tilde{\nu}^\infty, f^\infty\rangle \tilde{\lambda} \quad \text{for every } f \in \mathcal{F}_{2,1}. \tag{4.8}$$

Therefore,

$$\begin{aligned} \langle \nu^k, f(x,t; \cdot + \bar{v}, \cdot + \bar{u})\rangle dxdt &= \langle S_{(\bar{v},\bar{u})}\nu^k, f\rangle dxdt \\ &\overset{*}{\rightharpoonup} \langle S_{(\bar{v},\bar{u})}\tilde{\nu}, f\rangle dxdt + \langle \tilde{\nu}^\infty, f^\infty\rangle \tilde{\lambda} \\ &= \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda. \end{aligned}$$

Moreover, it follows from (4.7) that

$$\int_{\mathbb{R}^d} \langle \nu^k, e(\cdot + \bar{w})\rangle dx \leq \sup_{[0,T]} \left(\int_{\mathbb{R}^d} \langle \nu, e\rangle dx + \int_{\mathbb{R}^d} \langle \nu^\infty, e\rangle \lambda_t(dx) \right) + \frac{1}{k}.$$

Step 5. From regular to general concentration measures. In the previous step, we were still assuming λ to be absolutely continuous with respect to Lebesgue measure, and the barycentre was assumed smooth. Let now $(\nu_{x,t}, \lambda, \nu_{x,t}^\infty)$ be any Young measure on $\mathbb{R}^d \times \mathcal{S}_0^d$ with parameters in $\mathbb{R}^d \times [0,T]$ whose energy is bounded, and whose barycentre solves (2.2) in the sense of distributions.

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a standard mollification kernel, that is, smooth and non-negative, supported on, say, $B_1(0)$, and $\int \psi dx = 1$. Let furthermore $\chi : \mathbb{R} \to \mathbb{R}$ be another mollification kernel with the same properties as ψ, but whose support is required to be contained in $(-1,0)$. Define now $\psi_\epsilon(x) = \frac{1}{\epsilon^d}\psi\left(\frac{x}{\epsilon}\right)$ and $\chi_\epsilon(t) = \frac{1}{\epsilon}\chi\left(\frac{t}{\epsilon}\right)$, so that the mass is still 1 and the supports are in $B_\epsilon(0)$ and $(-\epsilon, 0)$ respectively. Set $\phi_\epsilon(x,t) = \psi_\epsilon(x)\chi_\epsilon(t)$. We can now define for every $t \in [0, T-\epsilon]$ and $x \in \mathbb{R}^d$ another Young measure $(\nu_\epsilon, \lambda_\epsilon, \nu_\epsilon^\infty)$ by

$$\langle \nu_\epsilon, f\rangle = \langle \nu, f\rangle * \phi_\epsilon,$$

$$\lambda_\epsilon = \lambda * \phi_\epsilon,$$

$$\langle \nu_\epsilon^\infty, f^\infty\rangle = \frac{(\langle \nu^\infty, f^\infty\rangle \lambda) * \phi_\epsilon}{\lambda_\epsilon}$$

for all $f \in \mathcal{F}_{2,1}$ *that are independent of* x *and* t. Note that ν_ϵ^∞ only has to be defined λ_ϵ-almost everywhere and we therefore have no problems for $\lambda_\epsilon = 0$. Moreover, that this mollified Young measure is defined for all times $t \in [0, T-\epsilon]$ is precisely the reason we chose the support of χ the way we did. Since T was arbitrary in the first place, we may as well assume that the new Young measure lives on $[0,T]$. For the barycentre $(\bar{v}_\epsilon, \bar{u}_\epsilon)$ of this measure we have $\bar{v}_\epsilon = \bar{v} * \phi_\epsilon$ and $\bar{u}_\epsilon = \bar{u} * \phi_\epsilon$, so the barycentre is smooth and, by linearity, solves (2.2).

For any $f \in \mathcal{F}_{2,1}$ of the form $\phi \otimes h$, where $\phi \in C_c(\mathbb{R}^d \times [0,T])$ and $h \in \mathcal{F}_{2,1}$ is independent of x and t it holds now that

$$\langle \nu_\epsilon, f\rangle dxdt + \langle \nu_\epsilon^\infty, f^\infty\rangle \lambda_\epsilon(dxdt) \overset{*}{\rightharpoonup} \langle \nu, f\rangle dxdt + \langle \nu^\infty, f^\infty\rangle \lambda(dxdt)$$

as $\epsilon \to 0$, since for a measure μ it holds that $\mu * \phi_\epsilon \overset{*}{\rightharpoonup} \mu$ as $\epsilon \to 0$.

Let now $E_\epsilon(t)$ denote the energy of the measure $(\nu_\epsilon, \lambda_\epsilon, \nu_\epsilon^\infty)$ at time t as defined in (3.9), and $E(t)$ the energy of $(\nu, \lambda, \nu^\infty)$. Then we have

$$\sup_{t\in[0,T]} E_\epsilon(t) \le \sup_{t\in[0,T]} E(t).$$

Indeed, by definition of the mollified Young measure, we have

$$\begin{aligned}\int \langle (\nu_\epsilon)_{x,t}, e\rangle dx &= \int \langle \nu_{\cdot,\cdot}, e\rangle * (\psi_\epsilon \chi_\epsilon)(x,t) dx \\ &= \int \langle \nu_{x-y,t-s}, e\rangle \psi_\epsilon(y)\chi_\epsilon(s) dyds dx \\ &= \left(\int \langle \nu_{x,\cdot}, e\rangle dx\right) * \chi_\epsilon(t),\end{aligned}$$

where the last equality follows from Fubini's Theorem. Similarly, we obtain

$$\int \langle (\nu_\epsilon^\infty)_{x,t}, e\rangle (\lambda_\epsilon)_t(dx) = \left(\int \langle \nu_{x,\cdot}^\infty, e\rangle \lambda_\cdot(dx)\right) * \chi_\epsilon(t),$$

so that in fact $E_\epsilon(t) = E * \chi_\epsilon(t)$, and hence for every $t \in [0,T]$ we get

$$E_\epsilon(t) = \int E(t-s)\chi_\epsilon(s)ds \le \sup_t E(t) \int \chi = \sup_t E(t).$$

Hence we obtain

$$\int_{\mathbb{R}^d} \langle \nu^k, e(\cdot + (\bar{v}_\epsilon, \bar{u}_\epsilon))\rangle dx \leq \sup_{[0,T]} E(t) + \frac{1}{k}.$$

If $(\nu, \lambda, \nu^\infty)$ is admissible, this implies

$$\int_{\mathbb{R}^d} \langle \nu^k, e(\cdot + (\bar{v}_\epsilon, \bar{u}_\epsilon))\rangle dx \leq \frac{1}{2}\int_{\mathbb{R}^d} |v_0|^2 + \frac{1}{k}.$$

Taking a suitable diagonal sequence of the ϵ and k and relabeling the sequence yields ν^k and $(\bar{v}^k, \bar{u}^k)$ as desired.

□

Finally, we remark for later reference that by Proposition 3.8 the map $t \mapsto \langle \nu_{\cdot,t}, \xi\rangle$ is strongly continuous at $t = 0$ and it is easy to see that therefore

$$\langle (\nu_\epsilon)_{\cdot,0}, \xi\rangle \to \langle \nu_{\cdot,0}, \xi\rangle$$

strongly in $L^2(\mathbb{R}^d)$ as $\epsilon \to 0$.

4.3 Discrete Homogeneous Young Measures

In this section we show the following:

Proposition 4.9. *Let $Q = [0,1]^{d+1}$ be the $d+1$-dimensional unit cube and $Q_t = [0,1]^d \times \{t\}$ for $t \in [0,1]$. Let moreover ν be a discrete homogeneous oscillation Young measure with zero barycentre, i.e. a probability measure of the form*

$$\nu = \sum_{i=1}^N \mu_i \delta_{(v_i,u_i)}$$

with $\sum_{i=1}^N \mu_i(v_i,u_i) = 0$. Then there exists a sequence (v^n,u^n) bounded in $L_t^\infty L_x^2 \times L_t^\infty L_x^1$ of smooth solutions of (2.2) with compact support in Q which generates the Young measure ν on Q, and if $f \in \mathcal{F}_{2,1}$ is convex in (v,u) for every $(x,t) \in Q$, then it holds that

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \int_{Q_t} f(x,t;v^n(x,t),u^n(x,t))dx \leq \sup_{t\in[0,1]} \int_{Q_t} \langle \nu, f\rangle dx. \tag{4.9}$$

With this result, Proposition 4.3 follows immediately: Indeed, given an admissible measure-valued solution with initial data v_0, by Proposition 4.4 it suffices to generate the corresponding lifted admissible measure-valued subsolution by a suitable sequence of subsolutions. By Theorem 4.6, Proposition 4.5, and the shifting property (Proposition 3.12e)) this can be done as long as we can generate discrete homogeneous oscillation measures that are piecewise constant on a lattice. Given Proposition 4.9, this is possible: By translation and scaling, for each cube of the lattice we find a sequence of subsolutions generating the respective Young measure, and since the members of the sequences are compactly supported, they can be glued together to arrive at a sequence $(\tilde{v}_n, \tilde{u}_n)$. A suitable diagonal sequence of $(\tilde{v}_n + \bar{v}^k, \tilde{u}_n + \bar{u}^k)$, where $(\bar{v}^k, \bar{u}^k)$ are the functions from Theorem 4.6, will then generate the measure-valued subsolution, and inserting $e(\cdot + (\bar{v}^k, \bar{u}^k))$ for f in Proposition 4.9 and then applying Theorem 4.6b) yields that the energy requirement of Proposition 4.4b) is satisfied. Moreover, by the observation at the end of the previous section together with the fact that $(\tilde{v}_n(t=0), \tilde{u}_n(t=0)) = 0$, the initial values converge strongly to v_0.

Proof. The proof of Proposition 4.9 goes by induction over N (the number of atoms of ν).

Induction basis: $N = 1, 2$. For $N = 1$, $\nu = \delta_0$ and so we can simply take $(v_n, u_n) \equiv 0$. For $N = 2$, $\nu = \mu_1 \delta_{(v_1,u_1)} + (1 - \mu_1)\delta_{(v_2,u_2)}$ with $\mu_1(v_1, u_1) + (1 - \mu_1)(v_2, u_2) = 0$. If we consider the matrices U_1 and U_2 corresponding to (v_1, u_1) and (v_2, u_2), respectively, via the isomorphism (2.4), by Proposition 2.3 we may choose q_1 and q_2 in such a way that there exists a non-zero vector η with $(U_2 - U_1)\eta = 0$, so that the plane wave (with frequency n)

$$\tilde{U}^n(y) := (U_2 - U_1)h(n(y \cdot \eta)) \tag{4.10}$$

is a solution of (2.5) for any choice of h (recall that we write $y = (x, t)$). We set

$$h(s) = \begin{cases} \mu_1 & \text{if } s \in [0, 1 - \mu_1), \\ -(1 - \mu_1) & \text{if } s \in [1 - \mu_1, 1) \end{cases}$$

and extend h periodically to obtain solutions to (2.5) whose associated (v^n, u^n) clearly generate ν as $n \to \infty$.

Since we need to construct *smooth* solutions, the profile function should also be smooth. However this poses no problem since we can mollify h suitably (e.g. on a scale $1/n^2$), thus obtaining a smooth function that agrees with h except on a very small neighbourhood of the discontinuities. By abuse of notation we still call the mollified profile function h.

Our goal is now to cut off this plane wave (fix n for the moment) in such a way that the resulting map is compactly supported in Q and is equal to $\tilde{U}^n$ except on a small cutoff region.

To do this, assume first that $\eta = e_1$ (where (e_j) is the canonical basis in $\mathbb{R}^{d+1}$ and e_{d+1} is the time direction). Let $H : \mathbb{R} \to \mathbb{R}$ be a function with the property $H'' = h$. Since h has mean zero, H can be chosen bounded. Let us now define, for each n, a potential as in Lemma 2.4 by

$$E^{j1}_{i1} = -E^{j1}_{1i} = -E^{i1}_{1j} = E^{1j}_{1i} = (U_2 - U_1)_{ij} \frac{H(ny_1)}{n^2} \tag{4.11}$$

and all other entries zero. A direct computation shows that E thus defined is well-defined and has all the properties required in Lemma 2.4, and that moreover $\mathcal{L}(E) = \tilde{U}^n = (U_2 - U_1)h(ny_1)$. Let now $\phi \in C_c^\infty(\Omega)$, $0 \le \phi \le 1$ be a function with $\phi = 1$ on the set $Q \cap \{\operatorname{dist}(y, \partial Q) \ge n^{-1/2}\}$. Then, by Lemma 2.4, the map

$$U^n := \mathcal{L}(\phi E) \tag{4.12}$$

will be a smooth solution to (2.5). Note that ϕ and E depend on n. We may even control U^n in the cutoff region: Keeping in mind that $\mathcal{L}$ is a homogeneous differential operator of second order, we can estimate

$$\begin{aligned} \|\mathcal{L}(\phi E) - \phi\mathcal{L}(E)\|_{L^\infty} &\leq C\|\phi\|_{C^1}\|E\|_{C^1} + C\|\phi\|_{C^2}\|E\|_{C^0} \\ &\leq C'(n^{1/2}n^{-1} + nn^{-2}) \to 0 \end{aligned} \tag{4.13}$$

as $n \to \infty$. In particular, the cutoff region does not contribute to the Young measure, and the U^n are uniformly bounded in $L^\infty(Q)$.

Next, we have to extend our argument to the case where η is not necessarily equal to e_1. As long as η is not parallel to e_{d+1} (the time direction), this can be done using the Galilean invariance of the Euler equations as in Lemma 3.3 and Step 2 of the proof of Proposition 3.2 in [15]. Indeed, write $\bar{U} = U_1 - U_2$ and assume that $\bar{U}\eta = 0$, $\bar{U}e_{d+1} \neq 0$. We choose a basis $\{f_1, \dots, f_{d+1}\}$ of $\mathbb{R}^{d+1}$ such that $f_1 = \eta$ and $f_{d+1} = e_{d+1}$. Let A be the corresponding transformation matrix, i.e. $Ae_i = f_i$ for $i = 1, \dots, d+1$, and set $\bar{V} = A^t\bar{U}A$. Then $\bar{V}e_1 = 0$ and $\bar{V}e_{d+1} \neq 0$, so that we can apply our considerations in the case $\eta = e_1$ with Q replaced by A^tQ (indeed, the construction did not rely on Q being a cube) to obtain a sequence $V^n(y)$ of smooth solutions of (2.5) oscillating at frequency n between the values A^tU_1A and A^tU_2A and compactly supported in the set A^tQ. The functions $U^n(y) = (A^{-1})^tV^n(A^ty)A^{-1}$ then are also solutions of (2.5) by Lemma 3.3 of [15], are uniformly bounded in $L^\infty(Q)$ and compactly supported in Q and agree with a plane wave oscillating in direction η between the values U_1 and U_2 at a frequency proportional to n. In particular, the associated (v^n, u^n) generate the Young measure ν. Moreover, we may assume (by passing to a subsequence if necessary) that for all $t \in \left[n^{-1/2}, 1 - n^{-1/2}\right]$

$$\left|\mathcal{L}^d\left(\{x \in [0,1]^d : (v^n(x,t), u^n(x,t)) = (v_i, u_i)\}\right) - \mu_i\right| < n^{-1/2}, \quad i = 1, 2. \tag{4.14}$$

It is crucial here that $\eta \nparallel e_{d+1}$.

Let now $f \in \mathcal{F}_{2,1}$ be a function which is convex in the variables (v, u) for every (x,t). Observe first that for fixed x, t, v, u, the map $[0,1] \ni \phi \mapsto f(x,t;\phi v, \phi u)$ is convex, and that on the other hand $f(x,t;0,0) \leq \langle \nu, f(x,t;\cdot,\cdot)\rangle$ again by convexity. Moreover, recalling the plane waves $\tilde{U}^n(y)$ and their corresponding $(\tilde{v}^n, \tilde{u}^n)$ from (4.10), we may assume (after passing to a subsequence if necessary) that

$$\sup_{t\in[0,1]}\left|\int_{Q_t} f(x,t;\tilde{v}^n, \tilde{u}^n)dx - \int_{Q_t}\langle \nu, f\rangle dx\right| < \frac{C_f}{n},$$

where C_f is a constant depending only on f and it is again crucial that the plane wave direction is not parallel to the time direction (i.e. η and e_{d+1} are linearly independent). Combining these considerations, we arrive at

$$\sup_{t\in[0,1]}\int_{Q_t} f(\phi\tilde{v}^n, \phi\tilde{u}^n)dx \leq \sup_{t\in[0,1]}\int_{Q_t}\langle \nu, f\rangle dx + \frac{2C_f}{n}$$

where ϕ is the cutoff function from the preceding discussion. By estimate (4.13) and continuity of f we can assume that

$$\|f(v^n, u^n) - f(\phi\tilde{v}^n, \phi\tilde{u}^n)\|_{L^\infty(Q)} \leq \frac{1}{n},$$

so that

$$\sup_{t\in[0,1]}\int_{Q_t} f(v^n, u^n)dx \leq \sup_{t\in[0,1]}\int_{Q_t}\langle \nu, f\rangle dx + \frac{2C_f + 1}{n},$$

whereby (4.9) is established.

Finally, notice that the condition $\eta \nparallel e_{d+1}$ is not a major restriction for our purposes: Indeed, $(U_1 - U_2)e_{d+1} = 0$ means that $v_1 = v_2$, hence by slightly perturbing v_2 we can arrange for the condition to be satisfied.

Induction hypothesis: *Suppose that for any discrete probability measure with N atoms and zero barycentre there exists a sequence (v^n, u^n) with the properties stated in Proposition 4.9, and such that for every $i = 1, \ldots, N$*

$$\left|\mathcal{L}^{d+1}\left(\{(x,t) \in Q : (v^n(x,t), u^n(x,t)) = (v_i, u_i)\}\right) - \mu_i\right| < n^{-1/2}. \tag{4.15}$$

We have just seen that the induction hypothesis is satisfied for $N = 1, 2$ (in particular, (4.15) follows from (4.14)).

Induction step: Let $\nu = \sum_{i=1}^{N+1} \mu_i \delta_{(v_i,u_i)}$. Define

$$(\bar{v}_N, \bar{u}_N) = \sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} (v_i, u_i)$$

and observe that the measure

$$\bar{\nu} := \mu_{N+1}\delta_{(v_{N+1},u_{N+1})} + (1 - \mu_{N+1})\delta_{(\bar{v}_N,\bar{u}_N)}$$

has zero barycentre. Therefore, by virtue of the induction basis ($N = 2$), there exists a sequence $(\bar{v}^n, \bar{u}^n)$ which generates the Young measure $\bar{\nu}$ and satisfies for every $t \in [n^{-1/2}, 1 - n^{-1/2}]$

$$\left|\mathcal{L}^{d}\left(\left\{x \in [0,1]^d : (\bar{v}^n(x,t), \bar{u}^n(x,t)) = (\bar{v}_N, \bar{u}_N)\right\}\right) - (1 - \mu_{N+1})\right| < n^{-1/2} \tag{4.16}$$

as well as

$$\left|\mathcal{L}^{d+1}\left(\{(x,t) \in Q : (\bar{v}^n(x,t), \bar{u}^n(x,t)) = (\bar{v}_N, \bar{u}_N)\}\right) - (1 - \mu_{N+1})\right| < n^{-1/2} \tag{4.17}$$

and

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} \int_{Q_t} f(x,t; \bar{v}^n, \bar{u}^n) dx \leq \sup_{t\in[0,1]} \int_{Q_t} \langle \bar{\nu}, f \rangle dx \tag{4.18}$$

for all convex $f \in \mathcal{F}_{2,1}$.

Next we set

$$V^n := \{(x,t) \in Q : (\bar{v}^n(x,t), \bar{u}^n(x,t)) = (\bar{v}_N, \bar{u}_N)\} \cap \left\{t \in [n^{-1/2}, 1 - n^{-1/2}]\right\}$$

and for every $k \in \mathbb{N}$ we find a finite family of disjoint cubes,

$$V_k^n = \bigcup_{j=1}^{M_k} (y_{n,k}^j + \alpha_{n,k}^j Q), \quad y_{n,k}^j \in Q, \quad \alpha_{n,k}^j > 0,$$

such that $V_k^n \subset V^n$ and

$$\mathcal{L}^{d+1}(V^n \backslash V_k^n) < \frac{1}{k}. \tag{4.19}$$

By the induction hypothesis and the shift property (Proposition 3.12e)), there exists a sequence $(v_N^n, u_N^n)_{n\in\mathbb{N}}$ of smooth solutions to (2.2) with compact support in Q such that $(v_N^n + \bar{v}_N, u_N^n + \bar{u}_N)$ generate the Young measure

$$\sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} \delta_{(v_i,u_i)}$$

and such that

$$\left|\mathcal{L}^{d+1}\left(\{(x,t)\in Q : (v_N^n + \bar{v}_N, u_N^n + \bar{u}_N) = (v_i, u_i)\}\right) - \mu_i\right| < n^{-1/2} \tag{4.20}$$

for $i = 1, \dots, N$. We now set

$$(v_n^k(y), u_n^k(y)) = (\bar{v}_n(y), \bar{u}_n(y)) + \sum_{j=1}^{M_k} \left(v_N^n \left(\frac{y - y_{n,k}^j}{\alpha_{n,k}^j} \right), u_N^n \left(\frac{y - y_{n,k}^j}{\alpha_{n,k}^j} \right) \right).$$

Clearly, the functions (v_n^k, u_n^k) are smooth and compactly supported in Q, and solve equations (2.2).

Let $f \in \mathcal{F}_{2,1}$ be convex in (v,u) for every (x,t). If $t < n^{-1/2}$ or $t > 1 - n^{-1/2}$, then (4.9) follows directly from the induction hypothesis, because for these t we have $(v_n^k(y), u_n^k(y)) = (\bar{v}_n(y), \bar{u}_n(y))$, whence by (4.18) and convexity of f

$$\begin{aligned}\limsup_{n\to\infty} \sup_{t\in[0,1]\setminus(n^{-1/2},1-n^{-1/2})} &\int_{Q_t} f(x,t; v_n^k(y), u_n^k(y))dx \\ &\le \sup_{t\in[0,1]} \int_{Q_t} \langle \bar{\nu}, f\rangle dx \le \sup_{t\in[0,1]} \int_{Q_t} \langle \nu, f\rangle dx.\end{aligned}$$

For $t \in [n^{-1/2}, 1 - n^{-1/2}]$, define the sets

$$\begin{aligned} A_t^{n,k} &= \{x : (x,t) \in V_k^n\}, \\ B_t^{n,k} &= \{x : (x,t) \in V^n \setminus V_k^n\}, \\ C_t^{n,k} &= \{x : (v_n^k(x,t), u_n^k(x,t)) = (v_{N+1}, u_{N+1})\}. \end{aligned}$$

By estimate (4.16), we have

$$\left|\mathcal{L}^d(A_t^{n,k} \cup B_t^{n,k}) - (1 - \mu_{N+1})\right| < n^{-1/2} \tag{4.21}$$

and

$$\left|\mathcal{L}^d(C_t^{n,k}) - \mu_{N+1}\right| < n^{-1/2}. \tag{4.22}$$

We can therefore estimate

$$\begin{aligned} &\int_{Q_t} f(v_n^k, u_n^k)dx \\ &\le \int_{A_t^{n,k}} f(v_n^k, u_n^k)dx + \int_{B_t^{n,k}} f(v_n^k, u_n^k)dx + \int_{C_t^{n,k}} f(v_n^k, u_n^k)dx + o(1) \\ &=: I_1 + I_2 + I_3 + o(1) \end{aligned}$$

as $n \to \infty$. By the induction hypothesis, we obtain

$$I_1 \le \int_{A_t^{n,k}} \langle \sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} \delta_{(v_i,u_i)}, f\rangle dx + o(1)$$

as $n \to \infty$. By convexity of f, we also have

$$I_2 = \int_{B_t^{n,k}} f(\bar{v}_N, \bar{u}_N) \le \int_{B_t^{n,k}} \langle \sum_{i=1}^{N} \frac{\mu_i}{1 - \mu_{N+1}} \delta_{(v_i,u_i)}, f\rangle dx,$$

and

$$I_3 = \int_{C_t^{n,k}} \langle \delta_{(v_{N+1},u_{N+1})}, f\rangle dx.$$

By construction, in particular by (4.21) and (4.22), we obtain

$$\begin{aligned} I_1 + I_2 &\leq \int_{A_t^{n,k}\cup B_t^{n,k}} \langle \sum_{i=1}^{N} \frac{\mu_i}{1-\mu_{N+1}} \delta_{(v_i,u_i)}, f\rangle dx + o(1) \\ &= \int_{Q_t} \langle \sum_{i=1}^{N} \mu_i \delta_{(v_i,u_i)}, f\rangle dx + o(1) \end{aligned}$$

and

$$I_3 \leq \int_{Q_t} \langle \mu_{N+1}\delta_{(v_{N+1},u_{N+1})}, f\rangle dx + o(1)$$

as $n \to \infty$. Putting the last two inequalities together we conclude that the (v_n^k, u_n^k) satisfy (4.9).

Finally, from (4.17), (4.19), and (4.20), we deduce

$$\left|\mathcal{L}^{d+1}\left(\left\{(x,t) \in Q : (v_n^k(x,t), u_n^k(x,t)) = (v_i,u_i)\right\}\right) - \mu_i\right| < 2n^{-1/2} + k^{-1}$$

for $i = 1,\dots,N+1$, and it follows easily from the definition of Young measure generation (Theorem 3.11) that a suitably chosen subsequence, e.g. $(v_n^n(x,t), u_n^n(x,t))$, generates the Young measure ν. Moreover, by passing to a subsequence if necessary, we can assure that (4.15) holds for all $i = 1,\dots,N+1$.

This completes the proof of Proposition 4.9 and hence of Proposition 4.3 and Theorem 4.1.

□

4.4 Conclusion of the Proof of Theorem 1.3

Finally, let (ν,λ,ν^∞) be an admissible measure-valued solution. By abuse of notation, we will denote by (ν,λ,ν^∞) also the corresponding lifted Young measure on $\mathbb{R}^d \times \mathcal{S}_0^d$ as in Subsection 3.3.5. In the preceding discussion, we were able to prove for every $\epsilon > 0$ the existence of a pair $(v^\epsilon, u^\epsilon) \in C^\infty(\mathbb{R}^d \times [0,T])$ solving (2.2), with the properties

$$\left|\frac{1}{2}\int |v^\epsilon(0)|^2 dx - \frac{1}{2}\int |v_0|^2 dx\right| < \epsilon \tag{4.23}$$

and

$$\int e(v^\epsilon, u^\epsilon) dx < \frac{1}{2}\int |v_0|^2 dx + \epsilon \tag{4.24}$$

for all $t \in [0,T]$. Moreover, we saw that (v^ϵ, u^ϵ) generate (ν,λ,ν^∞) as a Young measure when $\epsilon \to 0$. In this final step, we want to deduce from this the full statement of Theorem 4.2. We will do so by using an argument from [16].

To this end, we need subsolutions as above, however not only defined for $t \in [0,T]$, but also in a small neighbourhood of 0, say in $[-\delta, T]$. An obvious way to obtain such subsolutions is to shift the given ones in time:

$$S_{-\delta}(v^\epsilon, u^\epsilon)(x,t) := (v^\epsilon, u^\epsilon)(x, t+\delta),$$

so that $S_{-\delta}(v^\epsilon, u^\epsilon) \in C^\infty(\mathbb{R}^d \times [-\delta, T-\delta])$ (as T was chosen arbitrarily, we may as well assume that $S_{-\delta}(v^\epsilon, u^\epsilon)$ is defined up to time T). Clearly (2.2) remains unaffected by the time shift, and so does inequality (4.24). To verify (4.23) also for the shifted sequence, observe that on the one hand,

$$\frac{1}{2}\int |v^\epsilon(\delta)|^2 dx \le \int e(v^\epsilon(\delta), u^\epsilon(\delta))dx \le \frac{1}{2}\int |v_0|^2 dx + \epsilon$$

by (4.24), and on the other hand

$$\frac{1}{2}\int |v^\epsilon(\delta)|^2 dx \ge \frac{1}{2}\int |v^\epsilon(0)|^2 dx - \epsilon \ge \frac{1}{2}\int |v_0|^2 dx - 2\epsilon$$

for sufficiently small δ, by weak continuity of v^ϵ. Hence, without loss of generality, we may assume that there exists $\delta > 0$ such that (v^ϵ, u^ϵ) are smoothly defined for all $t \in [-\delta, T]$. This enables us to prove

Proposition 4.10. *Suppose $\bar{e} \in C(\mathbb{R}^d \times (-\delta, T)) \cap C([-\delta, T]; L^1(\mathbb{R}^d))$ satisfies $e(v^\epsilon(x,t), u^\epsilon(x,t)) < \bar{e}(x,t)$ for all $x \in \mathbb{R}^d$, $t \in (-\delta, T)$. Then there exists a triple $(\bar{v}^\epsilon, \bar{u}^\epsilon, \bar{q}^\epsilon)$ satisfying (2.2) with the properties*

$$(\bar{v}^\epsilon, \bar{u}^\epsilon) \in C^\infty(\mathbb{R}^d \times ([-\delta, T]\setminus\{0\})) \text{ and } \bar{v}^\epsilon \in C([-\delta, T]; L^2_w(\mathbb{R}^d)),$$

$$e(\bar{v}^\epsilon(x,t), \bar{u}^\epsilon(x,t)) < \bar{e}(x,t)$$

for all $(x,t) \in \mathbb{R}^d \times ([-\delta, T]\setminus\{0\})$, and

$$\frac{1}{2}|\bar{v}^\epsilon(x,0)|^2 = \bar{e}(x,0)$$

for all $x \in \mathbb{R}^d$.

We omit the proof, since it is virtually identical to the proof of Proposition 5.1 in [16] (which was demonstrated in Section 2.5 above). But with this assertion at hand, we are done: Indeed, if we choose $\bar{e}$ such that $\sup_t \int (\bar{e} - e(v^\epsilon, u^\epsilon))dx$ is sufficiently small, we can argue as in Section 4.1 to find exact Euler solutions with energy density $\bar{e}$ and initial data close to v_0 in L^2; moreover we can choose such a solution to be close in $L^\infty_t L^2_x$ to v^ϵ, and by choosing $\bar{e}$ such that $\int \bar{e} dx$ is non-increasing in t, we can ensure that the solution satisfies the weak energy inequality. Theorem 4.2 is thus proven. □

4.5 Proof of Theorem 1.2

We recall Theorem 1.2 from the introduction:

Theorem 4.11. *Let H be the set of vector fields in $L^2(\mathbb{R}^d)$ that are weakly divergence-free. There exists a subset $\mathcal{E} \subset H$ which is dense in the strong topology of L^2 such that for every $v_0 \in \mathcal{E}$, there exist infinitely many weak solutions of Euler with initial data v_0 and constant energy, and infinitely many weak solutions with initial v_0 and strictly decreasing energy.*

Proof. Let $v_0 \in L^2(\mathbb{R}^d)$ be weakly divergence-free. As discussed before, a vanishing viscosity sequence of Leray solutions for Navier-Stokes with initial data v_0 generates an admissible measure-valued solution of Euler. Let $\epsilon > 0$ and

(v^ϵ, u^ϵ) be the subsolution of Section 4.4, which satisfies (4.23) and (4.24). Let $\bar{e}$ be an energy density with $e(v^\epsilon, u^\epsilon) < \bar{e}$, and such that $\int \bar{e} dx$ is constant and $\left\| \bar{e}(x,0) - \frac{1}{2}|v_0|^2 \right\|_{L^1_x} < 2\epsilon$ for all t. We obtain $(\bar{v}^\epsilon, \bar{u}^\epsilon)$ as in Proposition 4.10, and these yield infinitely many weak solutions for Euler with energy density $\bar{e}$ and an initial data $\tilde{v}_0$ with

$$\left| \frac{1}{2} \int |\tilde{v}_0|^2 dx - \frac{1}{2} \int |v_0|^2 dx \right| < 3\epsilon.$$

Similarly, one obtains energy-decreasing solutions by choosing $\bar{e}$ such that $\int \bar{e} dx$ is strictly decreasing. □

Outlook: Open Problems

In the realm of the topics discussed in this thesis there exist of course innumerable open questions. I would like to briefly mention only some of them. A related discussion of open problems in this field can also be found in Section 7 in [17].

Wild initial data. We have seen in the course of this thesis that there exists wild initial data, i.e. initial data for which there exist infinitely many admissible weak solutions with regularity no better than CL^2_w. Moreover, we have shown that the set of wild initial data is dense in the set of solenoidal vector fields in the (strong) L^2 topology (Theorem 1.2). On the other hand, by local existence and weak-strong uniqueness, smooth initial data can not be wild. The third piece of information available about wild initial data is that vortex sheet data belongs to it (Section 2.6). It would be interesting to characterise further the wild set. In particular, is there a critical regularity below which initial data is wild, and above which admissible weak solutions are unique (and regular)? This problem, however, seems to be out of reach presently: The global well-posedness for Euler is considered one of the most challenging problems in the field (cf. [23]).

Onsager's conjecture. A related question concerns the critical regularity that a solution of Euler must have in order to conserve energy. Similarly to the case of conservation laws, where energy is lost due to the formation of shocks (i.e. discontinuities), it is plausible to believe that there is a connection between regularity and energy dissipation in solutions of the Euler equations. L. Onsager [43] conjectured that Hölder continuous (in space) solutions conserve energy if the Hölder exponent is greater than 1/3, and that less regularity may imply energy dissipation. Whereas it is known that the mentioned regularity indeed suffices to ensure energy conservation [11, 22], it is still an open question whether for every $\alpha < 1/3$ there exist energy-dissipating solutions. In fact, Hölder continuous energy-dissipating solutions are unknown for any exponent (although very recently, continuous energy-dissipating solutions have been constructed [18]). If methods like the ones demonstrated in Chapter 2 are to be employed for the construction of Hölder continuous solutions, then obviously the convex integration method would have to be refined. In the context of isometric embeddings of manifolds, such a refinement has been developed in [13]. Convex integration techniques which yield better regularity may be powerful tools also in the study of other partial differential equations, like conservation laws or continuity equations.

I should remark that of course lack of regularity does not imply energy dissipation: Above we have come across many examples of energy conserving, irregular (and non-unique) solutions for the Euler equations.

Restoration of uniqueness. From a physical viewpoint at least, it is very unsatisfactory to describe a deterministic mechanical system with a mathematical model that admits non-unique solutions. In Chapter 2 we have seen that admissibility criteria on conservation or dissipation of energy are not sufficient to guarantee uniqueness for the Cauchy problem. One might wonder if there are other criteria which ensure uniqueness. Two candidates are natural and have proved successful in the context of conservation laws: Vanishing viscosity limits and the entropy rate admissibility criterion. The former criterion would suggest that a solution arising from a vanishing viscosity limit of Navier-Stokes solutions is uniquely determined by this property, whereas the latter criterion (introduced by C. Dafermos [14] for hyperbolic conservation laws) would claim uniqueness of the solution whose energy decreases as fast as possible. However, as pointed out in [17], the example of vortex sheet initial data shows that even if each of these two criteria guaranteed uniqueness, the respective solutions would be distinct. Indeed, the vanishing viscosity limit is given by the stationary (hence energy-conserving) solution, whereas there also exist energy-dissipating solutions.

Compressible Euler. The compressible Euler equations are

$$\begin{aligned} \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla p(\rho) &= 0 \\ \partial_t \rho + \operatorname{div}(\rho v) &= 0, \end{aligned} \tag{4.25}$$

where the density ρ and the velocity v are sought for and the pressure p is now a given function of ρ. As usual, (4.25) is also equipped with the additional *entropy condition*:

$$\partial_t \left(\frac{\rho|v|^2}{2} + \rho\epsilon(\rho) \right) + \operatorname{div}\left(\left(\frac{\rho|v|^2}{2} + \rho\epsilon(\rho) + p(\rho) \right) v \right) \leq 0,$$

where the internal energy ϵ is another given function of ρ. The question is now whether various results for incompressible Euler carry over to the compressible case. Non-uniqueness of entropy solutions has been shown in [16] for a particular density and in [8] for arbitrary densities.

The question which measure-valued solutions can be recovered by entropy solutions appears to be harder than in the incompressible case (Chapter 4). Recall that our construction of generating sequences in the incompressible case relied significantly on the fact that the wave cone of the associated linear system is very large, which in turn was due to the freedom to choose the pressure q. Since for the compressible Euler equations the pressure is completely determined by the density, we can no longer "play" with the pressure and thus only have a smaller wave cone. It is therefore conceivable that not every measure-valued solution can be generated by a sequence of weak solutions, but that further conditions are required (like a Jensen-type inequality in the case of gradient Young measures, see [30]). Unfortunately, the well-known theory about characterisation of Young measures generated by constrained sequences (see e.g. [25]) does not apply here because the constant rank property is not satisfied.

Appendix: Separability and the Helmholtz Decomposition

Here I wish to prove a technical result which is used in Section 2.1. It is certainly not new, but I could not find it explicitly stated in the literature and therefore I give the proof here for convenience.

Lemma 4.12. *There exist countably many divergence-free vector fields* $\phi_i \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ *and scalar fields* $p_i \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ *such that the set*

$$\{\phi_i + \nabla p_i : i \in \mathbb{N}\}$$

is dense in $L^2(\mathbb{R}^d;\mathbb{R}^d)$.

Proof. Let $\{\psi_k\} \subset C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$ be a countable dense subset of $L^2(\mathbb{R}^d;\mathbb{R}^d)$ and $k \in \mathbb{N}$. Let $w_k \in C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ be the Newtonian potential of ψ_k (cf. Section 9.4 in [28]), for which we have

$$\Delta w_k = \psi_k \tag{4.26}$$

and

$$|\nabla w_k(x)| \le C|x|^{1-d}, \qquad |\nabla^2 w_k(x)| \le C|x|^{-d} \tag{4.27}$$

for $|x|$ sufficiently large and $C = C(d,k)$ a constant.

Owing to a well-known formula for the Laplacian of a vector field, (4.26) implies

$$\psi_k = \Delta w_k = \nabla \operatorname{div} w_k + \operatorname{div}\operatorname{curl} w_k$$

(there is a slight abuse of notation here: the first “div” is the usual divergence of a vector field, whereas the second “div” is the row-wise defined divergence of a matrix. The d-dimensional curl is defined as the antisymmetric $d \times d$-matrix field with components $(\operatorname{curl} w)_{ij} = \partial_j w_i - \partial_i w_j$).

Defining $\phi_k := \operatorname{div}\operatorname{curl} w_k$ and $p_k := \operatorname{div} w_k$, a simple calculation yields $\operatorname{div}\phi_k = 0$, and (4.27) implies that ϕ_k and ∇p_k are in $C^\infty \cap L^2$. Hence

$$\psi_k = \phi_k + \nabla p_k$$

is the usual Helmholtz decomposition. For $R > 0$ let now η^R be a smooth function on $\mathbb{R}^d$ with $\eta^R \equiv 1$ in $B_R(0)$, $\operatorname{supp}\eta^R \subset B_{R+1}(0)$, and $0 \le \eta^R \le 1$ everywhere. We may also assume $|\nabla\eta^R| < 2$. Define now $\phi_k^R = \eta^R\phi_k$ and $p_k^R = \eta^R p_k$. Then we have the estimates

$$\left\|\phi_k - \phi_k^R\right\|_{L^2} = \left\|(1-\eta^R)\phi_k\right\|_{L^2} \to 0$$

as $R \to \infty$ and

$$\left\|\nabla p_k^R - \nabla p_k\right\|_{L^2} \le \left\|(1-\eta^R)\nabla p_k\right\|_{L^2} + \left\|p_k \nabla \eta^R\right\|_{L^2}.$$

The first term on the right hand side clearly converges to zero as $R \to \infty$ because $\nabla p_k \in L^2$. The second term may be estimated as follows, keeping in mind $|p_k| \le C|\nabla w_k|$ and (4.27):

$$\begin{aligned}
\left\|p_k \nabla \eta^R\right\|_{L^2}^2 &= \int_{B_{R+1}\setminus B_R} |\nabla \eta^R|^2 |p_k|^2 dx \\
&\le C \int_{B_{R+1}\setminus B_R} |x|^{2-2d} dx \\
&= C \int_R^{R+1} r^{1-d} dr \le C R^{1-d} \to 0
\end{aligned}$$

as $R \to \infty$. The assertion now follows by taking the union

$$\bigcup_{R\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \{\phi_k^R + \nabla p_k^R\}.$$

□

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