On the Γ-convergence of the energies and the convergence of almost minimizers in infinite magnetic cylinders

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Abstract

In this thesis we study static 180 degree domain walls in infinite thin magnetic wires with either a rectangular or a centrally symmetric Lipschitz cross section. We explore the magnetization energy minimization problem by finding an approximation for the magnetostatic energy. Two different pattern formations of the magnetization have been observed. In dependence of the thickness of the wire, different pattern formations of the magnetization vector are observed. We prove an existence of global minimizers(even for Lipschitz cross sections). We prove a Γ-convergence result for both types of thin wires. For rectangular cross sections we distinguish two different regimes and establish the minimal energy scaling in terms of the cross section edge's lengths. For a centrally symmetric cross section we establish as well the minimal energy scaling in terms of the diameter of the cross section and some geometric parameters relating to it. We prove as well a rate of convergence for the minimal energies for all cases. For thick wires with a rectangular cross section we prove an upper bound and give a reference for a lower bound on the minimal energy. For thin wires a Néel wall occurs and for thick wires a vortex wall is expected to occur.

Contents

Chapter 1

Introduction

The aim of this thesis is to study static 180 degree domain walls in infinite thin magnetic wires. We explore the magnetization energy minimization problem by finding an approximation to the magnetostatic energy. Two different pattern formations of the magnetization have been observed. In dependence of the the thickness of the wire different pattern formations of the magnetization vector are observed. We make a detailed study for thin wires, where a Néel wall occurs, and give lower and upper bounds on the minimal energy for thick wires, where a vortex wall is expected to occur.

1.1 Pattern formation and the reversal process

In the last years there has been significant progress in production and investigation of thin magnetic wires, e.g. [30,32,34]. Such arrays of nanowires are considered as future high density storage devices, e.g. [2]. It is known that the magnetization pattern switching time is closely related to the writing and reading speed of such a device, thus it is crucial to understand the magnetization switching process. The reversal of the magnetization typically starts at one end of the wire creating a domain wall, which moves along the wire. The domain wall separates the reversed and the not yet reversed parts of the wire (Fig. 1.1). Because it is difficult to do experiments with thin wires, there are few results on the speed of the moving wall. It has been observed experimentally and in numerical simulations, that there is a distinctive crossover between two different modes of magnetization switching at a critical diameter: in particular, for nickel the crossover occurs at the diameter of about 50nm. For thin wires the transverse mode is observed: the magnetization is constant on each cross section and it is rotating and moving

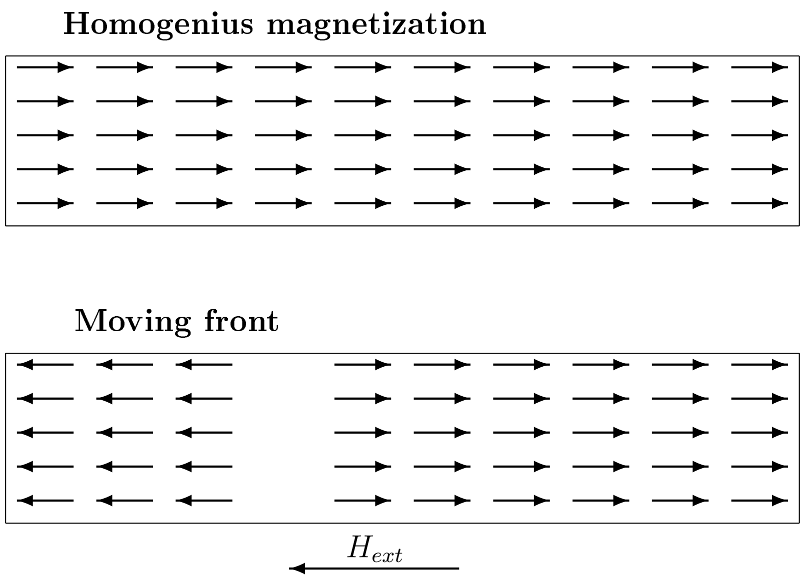


Figure 1.1

along the wire (see Fig 1.2). For thick wires the vortex mode is observed: the magnetization is almost tangential to the boundary and develops a vortex which propagates along the wire (see Fig 1.2). The vortex mode appeared to be much faster than the transverse mode.

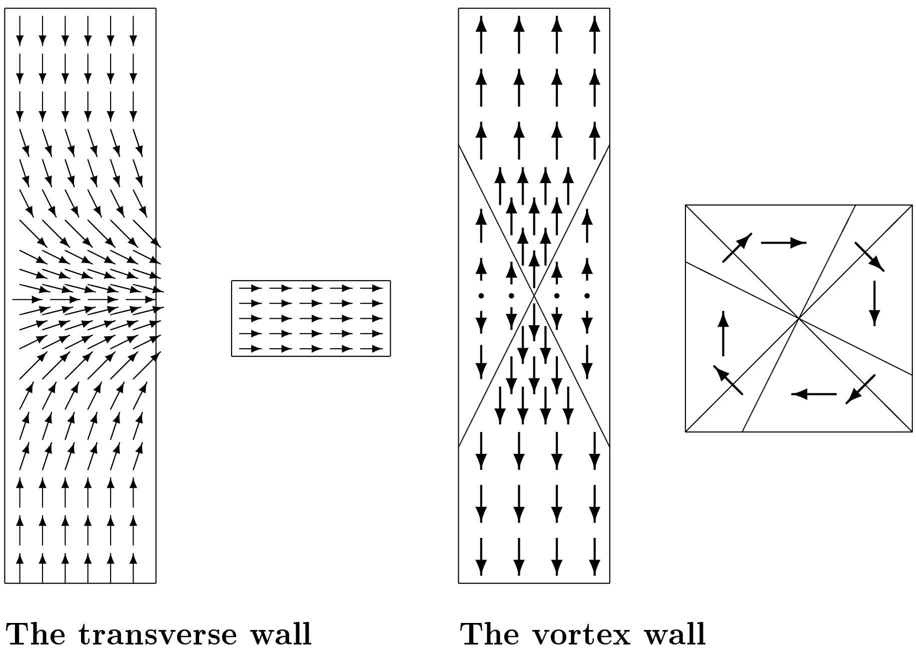


Figure 1.2 (Longitudinal section and cross section)

It is well known that the pattern formation of the magnetization can be understood from the behavior of the energy minimizing profiles and it has been suggested in [26,27] that the magnetization reversal process can be understood by studying the Landau-Lifshitz-Gilbert equation of the micromagnetics. A justification for a circular cross section has been done there by K.Kühn using the results on the static domain walls obtained in [24,25] and then studying the dynamics of the magnetization(Landau-Lifshits-Gilbert equation). In this work we study the static domain walls in a more general setting, namely when the cross section is either an arbitrary centrally symmetric Lipschitz domain or a rectangle with various aspect ratios.

1.2 Brief introduction to micromagnetics

Micromagnetics is a theory that assigns a nonlocal energy to each magnetization m from the domain Ω to $\mathbb{R}^3$, where the domain Ω represents a magnetized body in $\mathbb{R}^3$. The vector field m represents the magnetization of the body and has a unit length in Ω. It as extended as zero outside Ω. It is assumed that the body Ω is ferromagnetic. The energy functional of micromagnetics is given by the following expression:

$$E(m) = \epsilon^2 \int_\Omega |\nabla m|^2 + Q \int_\Omega \varphi(m) + \int_{\mathbb{R}^3} |\nabla u|^2 - 2 \int_\Omega H_{ext} \cdot m. \tag{1.1}$$

The four summands in (1.1) are called exchange energy, anisotropy energy, magnetostatic(or demagnetizing) energy and Zeeman(or external field) energy respectively. The numbers ϵ and Q are material parameters, the vector H_{ext} is an applied magnetic field, while ∇u is magnetic field generated by the magnetized body Ω. Here $u\colon \mathbb{R}^3 \to \mathbb{R}$ is a scalar function that is obtained from m by solving the Maxwell's equation of micromagnetics:

$$\mathrm{div}(\nabla u + m) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

which is equivalent to

$$\triangle u = \mathrm{div}(m) \quad \text{in} \quad \mathbb{R}^3$$

in the distributional sense. It is known in physics that the ground states of the magnetization correspond the minimizers of the micromagnetic energy functional. The theory of micromagnetics is used for the analysis and design of magnetic devices. It explains observations on different length scales. For a more detailed discussion we refer to [9,22].

1.3 Overview of the thesis

In chapter 2 we study static 180 degree domain walls in infinite cylinders with a rectangular cross section . We distinguish three different regimes. The first regime corresponds to the case when both the hight d and the width l of the cross section are sufficiently small and comparable to each other. The second regime corresponds to the case when both d and l are small but d is much smaller than l: so $d << l$. The third regime corresponds to the case when both d and l are big and comparable to each other. In the first two regimes the optimal scaling of the minimal energy can be realized by a Néel wall(transverse wall) for which the magnetization is constant on each cross section. We prove that as $d, l \to 0$ and if in addition $\frac{d}{l} \to c$, where evidently $c > 0$ for the first regime and $c = 0$ for the second regime, the rescaled energy minimizing problem $\min \frac{E(m)}{\mu}$ (where $\mu = d \cdot l$ for the first regime and $\mu = d^{\frac{3}{2}} l^{\frac{1}{2}} |\ln d - \ln l|^{\frac{1}{2}}$ for the second regime) $\Gamma-$converges to a one dimensional problem which attains its minimum and can be solved explicitly. Moreover, we find a rate of convergence for the minimal energies. In the third regime we prove an upper bound on the minimal energy scaling by constructing an example. We also make a reference for a lower bound.
In Chapter 3 we study static 180 degree domain walls in infinite cylinders with a centrally symmetric and Lipschitz cross section. Like in the rectangular cross section case we prove a $\Gamma-$convergence for the rescaled minimization problem $\frac{E(m)}{d^2}$ as d goes to zero, where d is the diameter of the cross section. The optimal scaling turns out to be d^2 and is realized by a Néel wall(transverse wall). We prove as well an existence of the energy minimizer. We also establish a rate of convergence for the minimal energies.

1.4 General notation

In this section we point out the notations and some conventions we are going to use throughout this work. We will use the following conventions: The letter $\xi = (\xi_1, \xi_2, \xi_3)$ denotes a point in $\mathbb{R}^3$. A map f with values in $\mathbb{R}^3$ will have the components f_x, f_y, f_z, i.e, $f = (f_x, f_y, f_z)$. For $l, d > 0$ numbers we denote the rectangle $[-l, l] \times [-d, d]$ by $R(l, d)$, the rectangle $\{x\} \times [-l, l] \times [-d, d]$ by $R_x(l, d)$ and $\Omega(l, d) = \mathbb{R} \times R[l, d] \subset \mathbb{R}^3$-an infinite cylinder with rectangular cross section (note that the cross section is the intersection of Ω with any hyperplane orthogonal to the x axis). We denote as well

$$E_{ex}(m) = \epsilon^2 \int_\Omega |\nabla m|^2$$

and

$$E_{mag}(m) = \int_{\mathbb{R}^3} |\nabla u|^2$$

Chapter 2

The static domain walls in cylinders with a rectangular cross section

2.1 Inrtroduction

In this chapter we study the static energy functional for the magnetizations $m\colon \Omega(l,d) \to \mathbb{S}^2$. We consider three different regimes. The first two of them relate to thin wires and the third one relates to thick wires. We use many of the methods used in [9] and [24]. In [9] many different regimes corresponding to magnetic films are studied.

2.2 The model problem

We consider the micromagnetic energy without an external field and anisotropy energy:

$$E(m) = \epsilon^2 \int_\Omega |\nabla m|^2 \,\mathrm{d}\xi + \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}\xi$$

Let $A(l,d)$ be the set of magnetizations with finite energies:

$$A(l,d) = \{m\colon \Omega(l,d) \to \mathbb{S}^2 \mid E(m) < \infty\}.$$

We are interested in the magnetisations with a 180 degree domain wall, so we will consider a subset $\tilde{A}(l,d)$ of $A(l,d)$ containing the magnetisations of $A(l,d)$ satisfying the conditions $\lim_{x\to\pm\infty} m(x,\cdot) = \pm\vec{e_x}$, where the limits are understood in the following sense: $m - \bar{e} \in H^1(\Omega)$, and

$$\bar{e} = \begin{cases} -\overrightarrow{e_x} & \text{if} \quad x < -1 \\ x \cdot \overrightarrow{e_x} & \text{if} \quad -1 \le x \le 1 \\ \overrightarrow{e_x} & \text{if} \quad 1 < x \end{cases}$$

We will sometimes leave out l and d in Ω, A, and $\tilde{A}$, provided it is certain which domain is being considered.

We study the minimization problem

$$\inf_{m \in \tilde{A}(l,d)} E(m) \tag{2.1}$$

First of all we eliminate the material constant ϵ from the energy functional expression and we also try to find out which kind of magnetizations are favorable for thin and thick films respectively. To that end we consider the magnetization $m_k(t) = m(kt)$ for $k > 0$. It is easy to see that

$$E(m_k) = kE_{ex}(m) + k^3 E_{mag}(m),$$

where the integration on the left hand side is done over the domain $\frac{1}{k} \cdot \Omega$. This shows that if k is big then the major contribution to the energy comes from the magnetostatic energy, therefore the energy of a thick wire favors magnetizations with a vortex wall. If k is small then the major contribution to the energy comes from the exchange energy, thus the energy of a thin wire favors magnetizations that are almost constant on each cross section. We rescale our spatial variable by a constant factor $k = \epsilon$ which will yield to a situation when the coefficients of

$$\int_\Omega |\nabla m|^2 \, \mathrm{d}\xi \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}\xi$$

are the same. We will hereafter assume that

$$E(m) = \int_\Omega |\nabla m|^2 \, \mathrm{d}\xi + \int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}\xi \tag{2.2}$$

where u is the weak solution of $\triangle u = \mathrm{div} m$. We will consider an auxiliary subset A_x of A which consists of all the magnetizations from A that are constant on each cross section:

$$A_x = \{m \in A \mid m \textit{ is constant on each cross section}\},$$

and we define as well the set

$$\tilde{A}_x = \{m \in \tilde{A} \mid m \textit{ is constant on each cross section}\}.$$

Let E_{min} and $E_{min,x}$ be the infimums of $E(m)$ respectively in $\tilde{A}$ and A_x.

2.3 The main results

We study the existence of a minimizer for minimization problem (2.1). We consider as well the pattern formation of the optimal wall profile, the minimal energy scaling and we find a rate of convergence. We prove the following results.

Theorem 2.3.1 (Existence of minimizers). *For every $0 < d \leq l$ there exist minimizers of E in $\tilde{A}$ and $\tilde{A}_x$.*

Theorem 2.3.2 (Energy scaling). *The minimal energy scales like μ, where*

$$\mu = d \cdot l \qquad \text{in the first regime,}$$
$$\mu = d^{\frac{3}{2}} \cdot l^{\frac{1}{2}} |\ln d - \ln l|^{\frac{1}{2}} \qquad \text{in the second regime.}$$

Theorem 2.3.3 (Upper and lower bounds). *Assume that $\delta \leq \frac{d}{l}$. Then there exist two positive numbers d_0 and C, both depending on δ such that if $d > d_0$ then*

$$Cd^2(\ln d)^{\frac{1}{2}} \leq E(m) \leq 150 d^{\frac{5}{2}}(\ln d)^{\frac{1}{2}}.$$

The magnetization that admits the scaling shown in the upper bound is tangential to the boundary and forms a vortex. We expect it to be the optimal scaling in the third regime.

Instead of energy minimizing problem (2.1) we consider the rescaled problem

$$\inf_{m \in \tilde{A}} \frac{E(m)}{\mu}. \tag{2.3}$$

Theorem 2.3.4 (Γ-convergence). *In the first two regimes the rescaled energy minimizing problem Γ-converges to a one dimensional problem as d goes to zero, provided*

$$\lim_{d \to 0} \frac{d}{l} = c$$

and

$$c > 0 \quad \text{in the first regime,} \quad c = 0 \quad \text{in the second regime.}$$

Moreover, the limit problem can be solved explicitly.

Since $\Gamma-$convergence implies the convergence of the minimal energies as well as the convergence of minimizers under good compactness properties we obtain that

$$\lim_{d \to 0} \frac{E_{min}}{\mu} = E^0_{min} \tag{2.4}$$

where E^0_{min} is the minimal value of the limit energy. For thin cylinders any energy minimizer is almost constant on each cross section and forms a Néel wall (the transverse wall). We find a rate of convergence for limit (2.4) in the second regime. For the first regime we prove a rate of convergence theorem in a more general setting in Chapter 3.

Theorem 2.3.5 (Rate of convergence). *For sufficiently small d the following bound holds:*

$$\left|\frac{E_{min}}{\mu} - E^0_{min}\right| \leq \frac{64}{\sqrt{|\ln c|}} + 36l. \tag{2.5}$$

2.4 The characterization theorem

Hereafter we will consider not only the magnetizations but also all the bounded and measurable vector fields $m\colon \Omega \to \mathbb{R}^3$ satisfying

$$m(x) = 0 \ \text{ in } \ \mathbb{R}^3 \setminus \Omega.$$

We denote by M_Ω the set of such vector fields and by M^x_Ω the set of all vector fields in M_Ω which are constant on each cross section. For any $m \in M_\Omega$ the divergence of m consists of two parts: the body charges v and the surface charges s, i.e., the distributional divergence from the normal component of the magnetisation on the surface.

$$v(\xi) = \begin{cases} -\mathrm{div} m & \text{in } \ \Omega \\ 0 & \text{in } \ \mathbb{R}^3 \backslash \Omega \end{cases}$$

$$s(\xi) = \begin{cases} m(\xi) \cdot \nu(\xi) & \text{on } \ \partial\Omega \\ 0 & \text{in } \ \mathbb{R}^3 \backslash \partial\Omega \end{cases}$$

where $\nu(\xi)$ is the outward normal to the boundary of Ω at point ξ. Recall that the map u is the weak solution of

$$\triangle u = \mathrm{div} m \qquad \text{in} \qquad \mathbb{R}^3 \tag{2.6}$$

if and only if

$$\nabla u \in {}^2(\mathbb{R}^3) \ \text{ and } \ \int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi = \int_{\mathbb{R}^3} m \cdot \nabla\varphi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3) \tag{2.7}$$

which is itself equivalent to

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi = \int_\Omega v \cdot \varphi + \int_{\partial\Omega} s \cdot \varphi \qquad \text{for all} \qquad \varphi \in C_0^\infty(\mathbb{R}^3). \tag{2.8}$$

This defines u up to a constant, but we deal with the gradient of u so that constant does not effect the energy functional. The next lemma gives in particular a bound on $\|s\|_{L^2(\partial\Omega)}$.

Lemma 2.4.1. *If the vector field $m \in M^x_{\Omega(l,d)}$ satisfies*

$$|m| \leq 1 \quad in \quad \Omega$$

and

$$E(m) < \infty$$

then there exists a positive number M depending only on l, d and $E(m)$ such that

$$\|m_y\|^2_{L^2(\mathbb{R})} + \|m_z\|^2_{L^2(\mathbb{R})} \leq M.$$

Proof. We have

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = \int_\Omega v \cdot \varphi + \int_{\partial\Omega} s \cdot \varphi \qquad \text{for all} \qquad \varphi \in C_0^\infty(\mathbb{R}^3). \tag{2.9}$$

By the density argument one can show that this equality stays valid also for such functions φ which have compact support and are weakly differentiable with gradient in $L^2(\mathbb{R}^3)$. We prove the lemma by taking suitable test functions φ in (2.9) and using the finiteness of the norms $\|\nabla u\|_{L^2(\mathbb{R}^3)}$ and $\|\nabla m\|_{L^2(\Omega)}$. The idea is to choose the test functions φ close to s. Note that

$$s(\xi) = \begin{cases} m_y(\xi) & \text{on } \Gamma_{left} \\ -m_y(\xi) & \text{on } \Gamma_{right} \\ m_z(\xi) & \text{on } \Gamma_{up} \\ -m_z(\xi) & \text{on } \Gamma_{down} \end{cases}$$

where

$$\Gamma_{right} = \mathbb{R} \times \{l\} \times [-d, d], \quad \Gamma_{left} = \mathbb{R} \times \{-l\} \times [-d, d],$$
$$\Gamma_{up} = \mathbb{R} \times [-l, l] \times \{d\}, \quad \Gamma_{low} = \mathbb{R} \times [-l, l] \times \{-d\}$$

and it is clear that $\partial\Omega = \Gamma_{right} \cup \Gamma_{left} \cup \Gamma_{up} \cup \Gamma_{low}$. For convenience we choose test functions having support close to each of the surfaces $\Gamma_{right}, \Gamma_{left}, \Gamma_{up}$ and Γ_{low}. For any $r > 0$ there exists a function $\psi_r \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that

$$\psi_r = 1 \ \text{ in } \ [-r, r] \times \Big[-\frac{l}{2}, \frac{l}{2}\Big] \times \{d\}, \quad 0 \leq \psi_r \leq 1 \quad \text{in} \quad \mathbb{R}^3$$

$$\text{supp}\psi_r \subset \Big[-r - \frac{d}{2}, r + \frac{d}{2}\Big] \times \Big[-\frac{l+d}{2}, \frac{l+d}{2}\Big] \times \Big[\frac{d}{2}, \frac{3d}{2}\Big] \ \text{ and } \ |\nabla \psi_r| \leq \frac{10}{d}.$$

Note that m is strongly differentiable a.e. in $\mathbb{R}$ since it depends only on x and is weakly differentiable. We choose $\varphi_r = m_z\psi_r$. It is clear that

$$|\partial_x\varphi_r| = |\partial_x m_z\psi_r + \partial_x\psi_r m_z| \leq |\partial_x m_z| + \frac{10}{d}|m_z| \quad \text{in} \quad \operatorname{supp}(\varphi),$$

$$|\partial_y\varphi_r| = |\partial_y\psi_r m_z| \leq \frac{10}{d}|m_z| \quad \text{in} \quad \operatorname{supp}(\varphi),$$

$$|\partial_z\varphi_r| = |\partial_y\psi_r m_z| \leq \frac{10}{d}|m_z| \quad \text{in} \quad \operatorname{supp}(\varphi),$$

thus

$$|\nabla\varphi_r|^2 \leq \frac{400}{d^2}|m_z|^2 + 2|\partial_x m_z|^2 \quad \text{in} \quad \operatorname{supp}(\varphi). \tag{2.10}$$

We denote $I_r = \int_{-r}^{r} |m_z(x)|^2 \,\mathrm{d}x$. We have on one hand

$$\int_{\partial\Omega} s\cdot\varphi_r \,\mathrm{d}\xi = \int_{\partial\Omega} m_z^2\cdot\psi_r \,\mathrm{d}\xi \geq l\cdot\int_{-r}^{r} |m_z(x)|^2 \,\mathrm{d}x \tag{2.11}$$

and on the other hand

$$\Big|\int_{\partial\Omega} s\cdot\varphi_r \,\mathrm{d}\xi\Big| \leq \int_{\mathbb{R}^3} |\nabla u|\cdot|\nabla\varphi_r| \,\mathrm{d}\xi + \int_{\Omega} |v|\cdot|\varphi_r| \,\mathrm{d}\xi$$

$$\leq \|\nabla u\|_{L^2(\mathbb{R}^3)}\cdot\|\nabla\varphi_r\|_{L^2(\mathbb{R}^3)} + \|v\|_{L^2(\Omega)}\cdot\|\varphi_r\|_{L^2(\Omega)} \tag{2.12}$$

We have as well

$$\|\varphi_r\|^2_{L^2(\Omega)} = \int_{\Omega} |\varphi_r|^2 \,\mathrm{d}\xi = \int_{\Omega\cap supp(\varphi_r)} m_z^2\cdot\psi_r^2 \,\mathrm{d}\xi$$

$$\leq d(l+d)\int_{-r-\frac{d}{2}}^{r+\frac{d}{2}} |m_z(x)|^2 \,\mathrm{d}x \leq d(l+d)(I_r+d),$$

$$\|\nabla\varphi_r\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla\varphi_r|^2 \,\mathrm{d}\xi \leq \int_{supp(\varphi_r)} \Big(\frac{400}{d^2}|m_z|^2 + 2|\partial_x m_z|^2\Big) \,\mathrm{d}\xi$$

$$\leq \frac{400}{d}(l+d)(I_r+d) + 2d(l+d)\int_{\mathbb{R}} |\partial_x m_z|^2 \,\mathrm{d}x \leq \frac{400}{d}(l+d)(I_r+d) + \int_{\Omega} |\nabla m|^2 \,\mathrm{d}\xi$$

$$\leq \frac{400}{d}(l+d)(I_r+d) + E(m),$$

$$\|\nabla u\|^2_{L^2(\mathbb{R}^3)} \leq E(m) \quad \text{and} \quad \|v\|^2_{L^2(\Omega)} = \int_{\Omega} |\partial_x m|^2 \,\mathrm{d}\xi \leq E(m).$$

Using now (2.12) and the inequalities for $\|\nabla u\|_{L^2(\mathbb{R}^3)}$, $\|\nabla\varphi_r\|_{L^2(\mathbb{R}^3)}$, $\|v\|_{L^2(\Omega)}$, $\|\varphi_r\|_{L^2(\Omega)}$ we obtain

$$\Big|\int_{\partial\Omega} s\cdot\varphi_r \,\mathrm{d}\xi\Big|^2 \leq 2E(m)\Big(\frac{400}{d}(l+d)(I_r+d)+E(m)+d(l+d)(I_r+d)\Big). \quad (2.13)$$

Inequalities (2.11) and (2.13) yield an inequality of the form

$$I_r^2 \leq c_1 I_r + c_2$$

where c_1 and c_2 are constants depending only on l, d and $E(m)$. This implies that $I_r \leq x_0$ where x_0 is the biggest root of the equation $x^2 - c_1 x - c_2 = 0$. In the same way one can show that $J_r \leq y_0$ where $J_r = \int_{-r}^{r} |m_y|^2 \,\mathrm{d}x$ and y_0 depends only on l, d and $E(m)$. This completes the proof since r was arbitrary.

□

We investigate the average function $\bar{m}$ which is the mean value of m over the rectangle $R_x(l,d)$ and thus depends only on the first variable x:

$$\bar{m}(x,y,z) = \int_{R_x(l,d)} m \,\mathrm{d}y\,\mathrm{d}z, \qquad (x,y,z)\in\Omega(l,d).$$

Like m we extend $\bar{m}$ as 0 outside Ω. This function will play a crucial role in the proofs of the foregoing theorems. Actually it is the key point to the extensions of several lemmas that hold for the magnetizations constant on each cross section to the general case. It is easy to see that if m is weakly differentiable in x then so is $\bar{m}$ and

$$\partial_x \bar{m}(x,y,z) = \frac{1}{|R(l,d)|}\int_{R(l,d)} \partial_x m(x,y_1,z_1)\,\mathrm{d}y_1\,\mathrm{d}z_1, \quad (x,y,z)\in\Omega(l,d).$$

We also prove some auxiliary lemmas which allow us to prove some properties of the energy functional provided we have proven them for the magnetizations constant on each cross section. The first lemma shows that if two magnetizations are closed to each other in $L^2(\Omega)$ then so are their magnetostatic energies. The second lemma allows us to estimate from above the energy of the average magnetization as well as the sum $\|\bar{m}_y\|_{L^2(\Omega)}+\|\bar{m}_z\|_{L^2(\Omega)}$ in terms of l, d and $E(m)$ and hence it yields the finiteness of the sum $\|\bar{m}_y\|_{L^2(\Omega)}+\|\bar{m}_z\|_{L^2(\Omega)}$. The third lemma describes some properties of a magnetization with 180 degree domain wall and with a finite energy. It shows that the average function $\bar{m}$ is almost ±1 at respectively $\pm\infty$ and also that its first component can not have a lot of oscillations in a certain sense.

Lemma 2.4.2. *For any vector fields $m_1, m_2 \in M_\Omega$ with finite energies the following statements hold:*

- $E_{mag}(m_1+m_2) \leq 2(E_{mag}(m_1)+E_{mag}(m_2))$

- $|E_{mag}(m_1)-E_{mag}(m_2)| \le E_{mag}(m_1-m_2)+2\sqrt{E_{mag}(m_1)E_{mag}(m_1-m_2)}$
- $|E_{mag}(m_1)-E_{mag}(m_2)| \le \|m_1-m_2\|^2_{L^2(\Omega)}+2\|m_1-m_2\|_{L^2(\Omega)}\sqrt{E_{mag}(m_1)}$ *if* $m_1-m_2 \in L^2(\Omega)$

Proof. Assume that u_1 and u_2 are the weak solutions of $\triangle u = \mathrm{div} m_1$ and $\triangle u = \mathrm{div} m_2$ respectively. It is clear that

$$E_{mag}(m_1+m_2) = \int_{\mathbb{R}^3} |\nabla(u_1+u_2)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^3} (|\nabla u_1|^2+|\nabla u_2|^2+2\nabla u_1\cdot\nabla u_2)\,\mathrm{d}\xi$$

$$\le 2\int_{\mathbb{R}^3}(|\nabla u_1|^2+|\nabla u_2|^2)\,\mathrm{d}\xi = 2(E_{mag}(m_1)+E_{mag}(m_2)),$$

$$|E_{mag}(m_1)-E_{mag}(m_2)| = \Big|\int_{\mathbb{R}^3}(|\nabla u_1|^2-|\nabla u_2|^2)\,\mathrm{d}\xi\Big|$$

$$= \Big|\int_{\mathbb{R}^3}(|\nabla(u_1-u_2)|^2+2\nabla u_1\cdot\nabla u_2-2|\nabla u_1|^2)\,\mathrm{d}\xi\Big|$$

$$\le \int_{\mathbb{R}^3}|\nabla(u_1-u_2)|^2\,\mathrm{d}\xi + 2\int_{\mathbb{R}^3}|\nabla u_1(\nabla u_2-\nabla u_1)|\,\mathrm{d}\xi$$

$$\le E_{mag}(m_1-m_2)+2\sqrt{\int_{\mathbb{R}^3}|\nabla u_1|^2\,\mathrm{d}\xi\cdot\int_{\mathbb{R}^3}|\nabla(u_1-u_2)|^2\,\mathrm{d}\xi}$$

$$= E_{mag}(m_1-m_2)+2\sqrt{E_{mag}(m_1)E_{mag}(m_1-m_2)}$$

the last inequality is a consequence of Schwartz inequality. The third statement is a consequence of the second one and $E_{mag}(m) \le \|m\|_{L^2(\Omega)}$.

□

Lemma 2.4.3. *For any $m \in M_\Omega$ with a finite energy the following statements hold:*

- $\int_{R_x(l,d)}(|m|^2-|\bar{m}|^2)\,\mathrm{d}y\,\mathrm{d}z = \int_{R_x(l,d)}|m-\bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z \le \acute{C}(d^2+l^2)\int_{R_x(l,d)}|\nabla_{yz}m|\,\mathrm{d}y\,\mathrm{d}z$ *for all $x \in \mathbb{R}$, where $\acute{C}$ is an absolute constant(the Poincaré constant)*
- $E_{ex}(\bar{m})+E_{ex}(m-\bar{m}) = E_{ex}(m)$
- *There exists a constant C_1 depending only on l and d such that*

$$E(\bar{m}) \le C_1 E(m) \tag{2.14}$$

- *There exists a constant C_2 depending only on l, d and $E(m)$ such that*

$$\|\bar{m}_y\|^2_{L^2(\Omega(l,d))} + \|\bar{m}_z\|^2_{L^2(\Omega(l,d))} \leq C_2 \tag{2.15}$$

Proof. We have for any $x \in \mathbb{R}$

$$\int_{R_x(l,d)} (m - \bar{m})\,\mathrm{d}y\,\mathrm{d}z = \int_{R_x(l,d)} m\,\mathrm{d}y\,\mathrm{d}z - |R_x(l,d)| \cdot \bar{m}(x) = 0$$

thus

$$\int_{R_x(l,d)} |m|^2\,\mathrm{d}y\,\mathrm{d}z = \int_{R_x(l,d)} |\bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z$$
$$+ \int_{R_x(l,d)} |m - \bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z + 2\bar{m}(x) \int_{R_x(l,d)} (m - \bar{m})\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_{R_x(l,d)} |\bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z + \int_{R_x(l,d)} |m - \bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z.$$

Taking into account now that the weak derivative of the average function is the average of the original function's weak derivative we get the second equality. We have according to Lemma 2.4.2

$$E_{mag}(\bar{m}) \leq 2E_{mag}(\bar{m} - m) + 2E_{mag}(m) \leq 2E_{mag}(m) + 2\|m - \bar{m}\|^2_{L^2(\Omega(l,d))} \tag{2.16}$$

and the Poincaré inequality gives us the following

$$\int_{R_x(l,d)} |m - \bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z \leq \acute{C}(l^2 + d^2) \int_{R_x(l,d)} |\nabla_{yz} m|^2\,\mathrm{d}y\,\mathrm{d}z \quad \text{for any} \quad x \in \mathbb{R}$$

Integrating the last inequality over $\mathbb{R}$ we obtain

$$\|m - \bar{m}\|^2_{L^2(\Omega(l,d))} = \int_{\Omega(l,d)} |m - \bar{m}|^2\,\mathrm{d}\xi \leq \acute{C}(l^2 + d^2) \int_{\Omega(l,d)} |\nabla_{yz} m|^2\,\mathrm{d}\xi \leq \acute{C} E_{ex}(m).$$

Applying (2.16) and the last inequality we get in conclusion

$$E(\bar{m}) = E_{ex}(\bar{m}) + E_{mag}(\bar{m}) = E_{ex}(m) - E_{ex}(m - \bar{m}) + E_{mag}(\bar{m})$$
$$\leq E_{ex}(m) + E_{mag}(\bar{m}) \leq E_{ex}(m) + 2E_{mag}(m) + 2\acute{C}(l^2 + d^2)E_{ex}(m)$$
$$\leq (2 + 2\acute{C}(l^2 + d^2))E(m).$$

The forth statement is a consequence of the third one and Lemma 2.4.1.

□

Corollary 2.4.4. *For any $m \in A$ and $x \in \mathbb{R}$*

$$\int_{R_x(l,d)} |\bar{m}|^2 \,\mathrm{d}y\,\mathrm{d}z \leq \int_{R_x(l,d)} |m|^2 \,\mathrm{d}y\,\mathrm{d}z$$

$$\leq \int_{R_x(l,d)} |\bar{m}|^2 \,\mathrm{d}y\,\mathrm{d}z + \acute{C}(l^2+d^2) \int_{R_x(l,d)} |\nabla_{yz} m|^2 \,\mathrm{d}y\,\mathrm{d}z \qquad (2.17)$$

Lemma 2.4.5.

- *Let $m \in A$ be a magnetization and α and β be real numbers such that $-1 < \alpha < \beta < 1$. Assume $\Re$ is family of disjoint intervals (a,b) satisfying the conditions*
  $\{\bar{m}_x(a), \bar{m}_x(b)\} = \{\alpha, \beta\}$, $|\bar{m}_x(x)| \leq max(|\alpha|, |\beta|)$ in (a,b). Then

  $$card(\Re) \leq M_2 \quad and \quad \sum_{(a,b)\in\Re} (b-a) \leq M_2 \qquad (2.18)$$

  where M is a constant depending on l, d, α, β and $E(m)$.

- *If $m \in \tilde{A}$ then for any $0 < \delta < 1$ there exists a positive number N_δ such that two of the following properties hold:*
  $-1 \leq \bar{m}_x \leq -1+\delta$ in $(-\infty, -N_\delta)$
  $-1 \leq \bar{m}_x \leq -1+\delta$ in $(N_\delta, +\infty)$
  $1-\delta \leq \bar{m}_x \leq 1$ in $(N_\delta, +\infty)$
  $1-\delta \leq \bar{m}_x \leq 1$ in $(-\infty, -N_\delta)$
  (note that only two of them can simultaneously hold.)

- *For any $m \in \tilde{A}$ the function $\bar{m}_x$ has a constant sign at $\pm\infty$.*

Proof. We first prove that the sum of the lengths of the intervals in $\Re$ is bounded. We have that $|\bar{m}_x(x)| \leq max(|\alpha|, |\beta|) = \rho$ with $0 < \rho < 1$. As we have mentioned $\bar{m}$ is weakly differentiable in x and taking into account that every weakly differentiable function of one variable is locally absolutely continuous in $\mathbb{R}$ we get that so is $\bar{m}$. Let $(a,b) \in \Re$. It is clear that

$$\int_{(a,b)\times R(l,d)} \bar{m}_x^2 \,\mathrm{d}\xi \leq 4ld\rho^2(b-a) \qquad (2.19)$$

Integrating (2.17) over (a,b) and taking into account (2.19) we get

$$4ld(b-a) = \int_{(a,b)\times R(l,d)} |m|^2 \,\mathrm{d}\xi$$

$$\leq \int_{(a,b)\times R(l,d)} |\bar{m}|^2 \,\mathrm{d}\xi + \acute{C}(l^2+d^2) \int_{(a,b)\times R(l,d)} |\nabla_{yz} m|^2 \,\mathrm{d}\xi$$

$$\leq 4ld\rho^2(b-a) + \int_{(a,b)\times R(l,d)} (\bar{m}_y^2 + \bar{m}_z^2)\,\mathrm{d}\xi + \acute{C}(l^2+d^2) \int_{(a,b)\times R(l,d)} |\nabla_{yz} m|^2\,\mathrm{d}\xi$$

We do this for all $(a,b) \in \Re$ and add the obtained inequalities. For convenience we put

$$\Sigma = \bigcup_{(a,b)\in\Re} (a,b)\times R(l,d).$$

Since $\Re$ is a family of disjoint intervals then $\Sigma \subset \Omega(l,d)$. In conclusion we get:

$$\begin{aligned} & 4ld \sum_{(a,b)\in\Re} (b-a) \\ & \leq 4ld\rho^2 \sum_{(a,b)\in\Re} (b-a) + \int_\Sigma (\bar{m}_y^2 + \bar{m}_z^2)\,\mathrm{d}\xi + \acute{C}(l^2+d^2) \int_\Sigma |\nabla_{yz} m|^2\,\mathrm{d}\xi \\ & \leq 4ld\rho^2 \sum_{(a,b)\in\Re} (b-a) + \int_{\Omega(l,d)} (\bar{m}_y^2 + \bar{m}_z^2)\,\mathrm{d}\xi + \acute{C}(l^2+d^2) \int_{\Omega(l,d)} |\nabla m|^2\,\mathrm{d}\xi \\ & \leq 4ld\rho^2 \sum_{(a,b)\in\Re} (b-a) + C_2 + \acute{C}(l^2+d^2)E(m) \end{aligned}$$

in the last step we used (2.15). Finally we get

$$\sum_{(a,b)\in\Re} (b-a) \leq \frac{C_2 + \acute{C}(l^2+d^2)E(m)}{4ld(1-\rho^2)}. \tag{2.20}$$

Now we prove that $\Re$ contains finitely many intervals namely we get an upper bound on the number of the entries of $\Re$. For any point $(y,z) \in R(l,d)$ and any interval $(a,b) \in \Re$ we have

$$\int_a^b |\partial_x m_x(x,y,z)|^2\,\mathrm{d}x \geq \frac{1}{b-a}\Big(\int_a^b |\partial_x m_x(x,y,z)|\,\mathrm{d}x\Big)^2 \tag{2.21}$$

Integrating (2.21) over $R(l,d)$ we get

$$\begin{aligned} \int_{(a,b)\times R(l,d)} |\partial_x m_x(x,y,z)|^2\,\mathrm{d}\xi & \geq \frac{1}{b-a} \int_{R(l,d)} \Big(\int_a^b |\partial_x m_x(x,y,z)|\,\mathrm{d}x\Big)^2\,\mathrm{d}y\,\mathrm{d}z \\ & \geq \frac{1}{b-a} \int_{R(l,d)} |m_x(a,y,z) - m_x(b,y,z)|^2\,\mathrm{d}y\,\mathrm{d}z \\ & \geq \frac{1}{4ld(b-a)} \bigg(\int_{R(l,d)} |m_x(a,y,z) - m_x(b,y,z)|\,\mathrm{d}y\,\mathrm{d}z\bigg)^2 \end{aligned}$$

$$\geq \frac{1}{4ld(b-a)}\bigg(\int_{R(l,d)}\big(m_x(a,y,z)-m_x(b,y,z)\big)\,\mathrm{d}y\,\mathrm{d}z\bigg)^2$$
$$=\frac{1}{4ld(b-a)}\Big(4ld\big(\bar{m}_x(a)-\bar{m}_x(b)\big)\Big)^2=\frac{4ld(\alpha-\beta)^2}{b-a},$$

thus

$$\int_{(a,b)\times A(l,d)}|\partial_x m_x(x,y,z)|^2\,\mathrm{d}\xi\geq\frac{4ld(\alpha-\beta)^2}{b-a}.$$

We add the obtained inequalities for all $(a,b)\in\mathfrak{R}$ to get

$$4ld(\alpha-\beta)^2\sum_{(a,b)\in\mathfrak{R}}\frac{1}{b-a}\leq\int_\Sigma|\partial_x m_x|^2\,\mathrm{d}\xi\leq\int_\Omega|\partial_x m_x|^2\,\mathrm{d}\xi\leq E(m). \tag{2.22}$$

Adding (2.20) and (2.22) we obtain

$$\sum_{(a,b)\in\mathfrak{R}}\Big(\frac{1}{b-a}+b-a\Big)\leq\frac{1}{4ld}\Big(\frac{E(m)}{(\alpha-\beta)^2}+\frac{C_2+\acute{C}(l^2+d^2)E(m)}{1-\rho^2}\Big):=M_2 \tag{2.23}$$

The fact that for any $(a,b)\in\mathfrak{R}$ the inequality $\frac{1}{b-a}+b-a\geq 2$ holds and (2.23) show that $M_2\geq 2N$ where N is the number of the entries of $\mathfrak{R}$ and M_2 depends only on l, d, α, β and $E(m)$,i.e., M_2 satisfies (2.18). The first statement is proven. Using now (2.15) and (2.17) we get

$$\int_\Omega(1-\bar{m}_x^2)\,\mathrm{d}\xi\leq\int_\Omega(\bar{m}_y^2+\bar{m}_z^2)\,\mathrm{d}\xi+\acute{C}(l^2+d^2)E(m)<\infty \tag{2.24}$$

and it is as well clear that

$$|\bar{m}_x(x)|=\frac{1}{4ld}\Big|\int_{R(l,d)}m_x(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\Big|\leq\frac{1}{4ld}\int_{A(l,d)}|m_x(x,y,z)|\,\mathrm{d}y\,\mathrm{d}z\leq 1$$

thus

$$0\leq 1-\bar{m}_x^2(x)\leq 1\quad\text{for all}\quad x\in\mathbb{R}.$$

We know that $\int_\mathbb{R}(1-\bar{m}_x^2)\,\mathrm{d}x<\infty$ which is equivalent to the finiteness of the two integrals: $\int_0^{+\infty}(1-\bar{m}_x^2)\,\mathrm{d}x$ and $\int_{-\infty}^0(1-\bar{m}_x^2)\,\mathrm{d}x$. The integrand is continuous and positive thus for any positive δ less than 1 and a natural number N there exists $x_0\in\mathbb{R}$ greater than N such that $|\bar{m}_x(x_0)|>1-\frac{\delta}{2}$. Therefore there exists an increasing sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ tending to $+\infty$ such that $|\bar{m}_x(x_n)|>1-\frac{\delta}{2}$. Hence for infinitely many indices n one of the following statements holds: $\bar{m}_x(x_n)>1-\frac{\delta}{2}$ or $\bar{m}_x(x_n)<-1+\frac{\delta}{2}$. Assume that for a subsequence (not relabeled) we have $\bar{m}_x(x_n)>1-\frac{\delta}{2}$.

We will prove that $\bar{m}_x(x) > 1-\delta$ for all $x > N_\delta$ for some N_δ. Assume in the contrary that for an increasing sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ tending to $+\infty$ $\bar{m}_x(\tilde{x}_n) \leq 1-\delta$. We can choose an infinite family of disjoint intervals (a_n, b_n) such that the value of $\bar{m}_x$ at one of the ends of (a_n, b_n) is less or equal than $1-\delta$ and at the other end is big than $1-\frac{\delta}{2}$ for all $n \in \mathbb{N}$. The construction of such a family of intervals goes in the following way: In the first step we take the smallest n such that $\tilde{x}_n > x_1$ and denote it by $\tilde{n}_1$ and take $a_1 = x_1$, $b_1 = \tilde{x}_{\tilde{n}_1}$. In the second step we take the smallest n such that $x_n > b_1$ and denote it by n_2 and then we take the smallest n such that $\tilde{x}_n > x_{n_2}$ and denote it by $\tilde{n}_2$ and take $a_2 = x_{n_2}$ and $b_2 = \tilde{x}_{\tilde{n}_2}$. We continue this process as long as possible. Since $(x_n)_{n\in\mathbb{N}}$ and $(\tilde{x}_n)_{n\in\mathbb{N}}$ tend to $+\infty$ this sequence of steps is infinite and thus we have constructed an infinite sequence of disjoint intervals (a_n, b_n) with the property that $\bar{m}_x(a_n) > 1-\frac{\delta}{2}$ and $\bar{m}_x(b_n) \leq 1-\delta$ for all $n \in \mathbb{N}$. Since $\bar{m}_x$ is continuous in $\mathbb{R}$ the new sequence of intervals $(\acute{a}_n, \acute{b}_n)$ where $\acute{a}_n = \sup\{x \in (a_n, b_n) \mid \bar{m}_x(x) \geq 1-\frac{\delta}{2}\}$ and $\acute{b}_n = \inf\{x \in (\acute{a}_n, b_n) \mid \bar{m}_x(x) \leq 1-\delta\}$ has the property $\bar{m}_x(\acute{a}_n) = 1-\frac{\delta}{2}$ and $\bar{m}_x(\acute{b}_n) = 1-\delta$ and they are disjoint because $(\acute{a}_n, \acute{b}_n) \subset (a_n, b_n)$. Moreover, the construction of $\acute{a}_n$ and $\acute{b}_n$ yields $\bar{m}_x(x) \leq 1-\frac{\delta}{2}$ for all $x \in (\acute{a}_n, \acute{b}_n)$. But this contradicts the first statement of the foregoing lemma which states that the number of such intervals must be finite. The same can be done for $-\infty$. The fourth statement is an obvious consequence of the third one taking for instance $\delta = \frac{1}{2}$.

$\square$

Remark 2.4.6. *In the proof of Lemma 2.4.5 we have actually shown that for an arbitrary magnetization m the finiteness of the three norms*

$$\|\nabla m\|_{L^2(\Omega)}, \quad \|\bar{m}_y\|_{L^2(\mathbb{R})}, \quad \|\bar{m}_z\|_{L^2(\mathbb{R})}$$

yields that $\bar{m}_x$ and $|\bar{m}_x|$ have a constant sign and tend to 1 respectively at both $\pm\infty$.

Corollary 2.4.7. *Assume that a magnetization $m \in A_x$ satisfies the conditions*

$$\lim_{x\to\pm\infty} m_x(x) = c_\pm$$

and

$$\|\nabla m\|_{L^2(\mathbb{R})}, \|m_y\|_{L^2(\mathbb{R})}, \|m_z\|_{L^2(\mathbb{R})} < \infty.$$

Denote

$$m^*(x) = \begin{cases} m_x(x) - c_- & \text{if} \quad x \in (-\infty, 0] \\ m_x(x) - c_+ & \text{if} \quad x \in (0, +\infty), \end{cases}$$

then $m^* \in L^2(\mathbb{R})$.

Proof. According to Remark 2.4.6 we have that $c_-, c_+ \in \{-1, 1\}$. We will show for the case $c_+ = 1$, the other cases are analogues. Utilizing once again Remark 2.4.6 we have that there exists a positive number N such that $m_x(x) > 0$ in $[N, +\infty)$. We have that

$$\int_0^{+\infty} (m^*(x))^2 \,\mathrm{d}x \leq 4N + \int_N^{+\infty} (1 - m_x^2(x)) \,\mathrm{d}x =$$

$$= 4N + \int_N^{+\infty} (m_y^2(x) + m_z^2(x)) \,\mathrm{d}x < \infty.$$

$\square$

In the next step we describe the magnetizations which are constant on each cross section and have finite energy.

Theorem 2.4.8 (Characterization)**.** *For any l and d if $m \in A(l, d)$ then one of the four functions $m \pm \overrightarrow{e_x}$, $m \pm \bar{e}$ belongs to $H^1(\Omega(l, d))$. (the function $\bar{e}$ is defined in Section 2.2).*

Proof. For any $m \in A$ we have

$$E(m) = \int_\Omega |\nabla m|^2 \,\mathrm{d}\xi + E_{mag} < \infty$$

thus $\nabla m \in L^2(\Omega)$. Note that the gradients of $\pm\overrightarrow{e_x}$ are zero and the gradients of $\pm\bar{e}$ are zero outside the bounded set $[-1, 1] \times R(l, d)$ and are $(\pm 1, 0, 0)$ in $(-1, 1) \times R(l, d)$ so they are all in $L^2(\Omega)$. Using triangle inequality we get that the gradients of all the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belong to $L^2(\Omega)$. It remains to prove that one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $L^2(\Omega)$. Denote

$$\Omega_- = (-\infty, 0] \times R(l, d) \quad \text{and} \quad \Omega_+ = [0, +\infty) \times R(l, d).$$

We have

$$\int_{\Omega_-} |m - \overrightarrow{e_x}|^2 \,\mathrm{d}\xi = \int_{\Omega_-} \left((m_x - 1)^2 + m_y^2 + m_z^2\right) \mathrm{d}\xi =$$

$$= 2\int_{\Omega_-} (1 - m_x) \,\mathrm{d}\xi = 8ld \int_{-\infty}^0 (1 - \bar{m}_x) \,\mathrm{d}x$$

and similarly

$$\int_{\Omega_-} |m + \overrightarrow{e_x}|^2 \,\mathrm{d}\xi = 8ld \int_{-\infty}^0 (1 + \bar{m}_x) \,\mathrm{d}x$$

It is now clear that $m \pm \overrightarrow{e_x} \in L^2(\Omega_-)$ if and only if $1 \pm \bar{m}_x \in L^1(-\infty, 0)$. Similarly we have that $m \pm \overrightarrow{e_x} \in L^2(\Omega_+)$ if and only if $1 \pm \bar{m}_x \in L^1(0, +\infty)$. According to Lemma 2.4.5 $\bar{m}_x$ has a constant sign at $\pm\infty$. Suppose that $\bar{m}_x(x) \geq 0$ for $x \geq N \geq 0$. According to (2.24) we have that

$$\int_0^{+\infty} (1 - \bar{m}_x^2)\,\mathrm{d}x < \infty$$

thus

$$\int_0^{+\infty} (1 - \bar{m}_x^2)\,\mathrm{d}x \geq \int_N^{+\infty} (1 - \bar{m}_x^2)\,\mathrm{d}x = \int_N^{+\infty} (1 - \bar{m}_x)(1 + \bar{m}_x)\,\mathrm{d}x \geq$$

$$\geq \int_N^{+\infty} (1 - \bar{m}_x)\,\mathrm{d}x$$

and thus

$$\int_0^{+\infty} (1 - \bar{m}_x)\,\mathrm{d}x \leq 2N + \int_N^{+\infty} (1 - \bar{m}_x)\,\mathrm{d}x < \infty.$$

Similarly we could prove that if we had $\bar{m}_x(x) < 0$ for $x \geq N > 0$ for some N then $1 + \bar{m}_x \in L^1(0, +\infty)$. Obviously the same can be done for Ω_-. Therefore we have obtained that exactly two of the four statements hold: $1 + \bar{m}_x \in L^1(\Omega_-)$, $1 + \bar{m}_x \in L^1(\Omega_+)$, $1 - \bar{m}_x \in L^1(\Omega_-)$, $1 - \bar{m}_x \in L^1(\Omega_+)$ which ends the proof.

$\square$

2.5 The magnetostatic energy

2.5.1 A representation of u and the magnetostatic energy

In this subsection we recall some theorems from [24] which give a representation of u and the magnetostatic anergy and also show that the inverse of the characterization theorem holds. Since we work in an infinite domain It is not clear under which conditions a weak solution of the equation

$$\triangle u = \mathrm{div} m$$

exists and has a finite L^2-norm. A very well known case is the case $m \in {}^2(\Omega)$. In this case the equation

$$\triangle u = \mathrm{div} m$$

has a weak solution u with $\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq \|m\|_{L^2(\Omega)}$.

Consider for all $c^-, c^+ \in \mathbb{R}$ the function $\chi_{c^-}^{c^+} : \mathbb{R} \to \mathbb{R}^3$ such that

$$\chi_{c^-}^{c^+} = (c^{sign(x)} \min(1, |x|), 0, 0)$$

and define the set

$$X(l, d) = \{m \colon \Omega(l, d) \to \mathbb{R}^3 \mid \exists\, c^-, c^+ \in \mathbb{R} \text{ such that } m - \chi_{c^-}^{c^+} \in H^1(\Omega(l, d))\}.$$

Recall that the Green function for $-\triangle$ in $\mathbb{R}^3$ is $\Gamma(\xi) = \frac{1}{4\pi|\xi|}$.

Lemma 2.5.1. *For $m \in X$ define the maps $u_v, u_s, u \colon \mathbb{R}^3 \to \mathbb{R}$ by*

$$u_v(\xi) = \int_\Omega \Gamma(\xi - \xi_1) v(\xi_1)\, \mathrm{d}\xi_1,$$

$$u_s(\xi) = \int_{\partial\Omega} \Gamma(\xi - \xi_1) s(\xi_1)\, \mathrm{d}\xi_1,$$

$$u(\xi) = u_v(\xi) + u_s(\xi).$$

Then the following statements hold:

- *The maps u_v and u_s satisfy the equalities*

$$\nabla u_v(\xi) = \sum_{i \in \{x,y,z\}} \int_\Omega \partial_i \Gamma(\xi - \xi_1) v(\xi_1) \overrightarrow{e_i}\, \mathrm{d}\xi \quad \textit{for all} \quad \xi \in \mathbb{R}^3, \tag{2.25}$$

$$\nabla u_s(\xi) = \sum_{i \in \{x,y,z\}} \int_{\partial\Omega} \partial_i \Gamma(\xi - \xi_1) s(\xi_1) \overrightarrow{e_i}\, \mathrm{d}\xi \quad \textit{for all} \quad \xi \in \mathbb{R}^3 \setminus \partial\Omega, \tag{2.26}$$

$$\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla \varphi = \int_\Omega v\varphi \quad \textit{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3), \tag{2.27}$$

$$\int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi = \int_{\partial\Omega} s\varphi \quad \textit{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3). \tag{2.28}$$

- *u is a weak solution of $\triangle u = divm$.*
- *∇u is in $L^2(\mathbb{R}^3)$.*

Proof. The validity of (2.25) and (2.26) is clear because the integrands are absolutely continuous for any $\xi \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^3 \setminus \partial\Omega$ respectively. For the proof of (2.27) and (2.28) we refer to [24]. The second statement is now clear if we take into account (2.27) and (2.28). For the proof of the third statement we again refer to [24].

$\square$

For any $m \in X$ we will hereafter consider the weak solution of $\triangle u = \text{div} m$ which is defined in Lemma 2.5.1. As a corollary we get a necessary and sufficient condition for a magnetization to have a finite energy.

Theorem 2.5.2 (Characterization)**.** *A magnetization $m \colon \Omega \to \mathbb{S}^2$ is in A if and only if one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$.*

Proof. The necessity is Theorem 2.4.8. To prove the sufficiency we note that if one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$ then $m \in X$ thus according to Lemma 2.5.1 m belongs to A.

□

Corollary 2.5.3. *A magnetization m belongs to A if and only if*

$$\nabla m, m_y, m_z \in L^2(\Omega).$$

Proof. Assume that $m \in A$. First of all note that

$$\|\nabla m\|^2_{L^2(\Omega)} \leq E(m) < \infty$$

Theorem 2.4.8 states that one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$. Assume for instance that

$$m - \overrightarrow{e_x} \in H^1(\Omega).$$

We have then that

$$\|m_y\|^2_{L^2(\Omega)} + \|m_z\|^2_{L^2(\Omega)} \leq \|m - \overrightarrow{e_x}\|^2_{H^1(\Omega)} < \infty.$$

Assume now that

$$\nabla m, m_y, m_z \in L^2(\Omega).$$

Applying the Poincaré inequality to the functions m_y and m_z we get

$$\|\bar{m}_y\|^2_{L^2(\Omega)} + \|\bar{m}_z\|^2_{L^2(\Omega)} \leq \|m_y\|^2_{L^2(\Omega)} + \|m_z\|^2_{L^2(\Omega)} + \|\bar{m}_y - m_y\|^2_{L^2(\Omega)} + \|\bar{m}_z - m_z\|^2_{L^2(\Omega)}$$

$$\leq \|m_y\|^2_{L^2(\Omega)} + \|m_z\|^2_{L^2(\Omega)} + \acute{C}(l^2 + d^2)\|\nabla_{yz} m\|^2_{L^2(\Omega)} < \infty.$$

According to Remark 2.4.6 we have that the function $\bar{m}_x$ must have a limit 1 or -1 at $\pm\infty$. Recall that in the proof of Theorem 2.4.8 we actually showed that once we know that $\bar{m}_x$ has a limit 1 or -1 at $\pm\infty$ and the norms $\|\bar{m}_y\|^2_{L^2(\Omega)}$ and $\|\bar{m}_y\|^2_{L^2(\Omega)}$ are finite then one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$. Therefore applying now Theorem 2.5.2 we establish $m \in A$.

□

We consider now the functional E_{mag} for the magnetisations which are constant on each cross section, i.e., for $m \in A_x$.

Lemma 2.5.4. *For any $m \in A_x$ the gradients ∇u_v and ∇u_s are orthogonal in $L^2(\mathbb{R}^3)$.*

Proof. Since v is independent of y and $s(x, y, z) = -s(x, -y, -z)$ then we have the following for u_v and u_s:

$u_s(x, y, z) = -u_s(x, -y, -z)$ and $u_v(x, y, z) = u_v(x, -y, -z)$,
$\partial_x u_s(x, y, z) = -\partial_x u_s(x, -y, -z)$, $\partial_y u_s(x, y, z) = \partial_y u_s(x, -y, -z)$
$\partial_z u_s(x, y, z) = \partial_z u_s(x, -y, -z)$, $\partial_x u_v(x, y, z) = \partial_x u_v(x, -y, -z)$
$\partial_y u_v(x, y, z) = -\partial_y u_v(x, -y, -z)$, $\partial_z u_v(x, y, z) = -\partial_z u_v(x, -y, -z)$

$$E_{vs}(m) = 2\int_{\mathbb{R}^3} \nabla u_v(x, y, z)\nabla u_s(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z =$$

$$\int_{\mathbb{R}^3} \nabla u_v(x, y, z)\nabla u_s(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathbb{R}^3} \nabla u_v(x, y, z)\nabla u_s(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Making the change of variables $y \to -y$, $z \to -z$ in the second summand and using the identities for the partial derivatives of u_v and u_s we get $E_{vs} = 0$.

$\square$

Thus for $m \in A_x$ the energy functional has the form

$$E(m) = 4ld\|\partial_x m\|^2_{L^2(\mathbb{R})} + E_v(m) + E_s(m).$$

2.5.2 The representation of E_s in Fourier space

In this section we find a representation of the magnetostatic energy in Fourier space. We do this because the expression $\int_{\mathbb{R}^3} |\nabla u|^2$ is hard to deal with but its representation in Fourier space will make it more transparent. First of all we would like to recall the Fourier transform in $\mathbb{R}^n$ and some of its properties. The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is denoted by $\hat{f}$ and equals to

$$\hat{f}(x) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} f(\xi)e^{-ix\cdot\xi}\,\mathrm{d}\xi \quad \text{for all} \quad x \in \mathbb{R}^n.$$

The set of all functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_x |x^\beta D^\alpha \varphi(x)| < \infty \text{ for all multi-indices } \alpha \text{ and } \beta$$

is denoted by $\mathcal{J}$ and called the "Schwartz class." Fourier transform has in particular the following properties:

$$1. \quad \widehat{\left(\frac{\partial f}{\partial \xi_j}\right)} = i\xi_j \hat{f} \quad \text{for all} \quad f \in \mathcal{J} \tag{2.29}$$

2.(Parseval's equality) $\displaystyle\int_{\mathbb{R}^n} |f|^2 \, \mathrm{d}\xi = \int_{\mathbb{R}^n} |\hat{f}|^2 \, \mathrm{d}\xi \quad \text{for all} \quad f \in \mathcal{J}$ (2.30)

$$3. \quad \int_{\mathbb{R}^n} |\nabla f|^2 \, \mathrm{d}\xi = \int_{\mathbb{R}^n} \frac{|\widehat{\triangle f}|^2}{|\xi|^2} \, \mathrm{d}\xi \quad \text{for all} \quad f \in \mathcal{J} \quad \text{and} \quad n \geq 3. \tag{2.31}$$

By the density argument the first equality is also valid for all $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $\frac{\partial f}{\partial \xi_j} \in L^2(\mathbb{R}^n)$. The third equality is valid if $\nabla f \in L^2(\mathbb{R}^n)$ and $\frac{\widehat{\triangle f}}{|\xi|} \in L^2(\mathbb{R}^n)$ even if $\triangle f$ is a distribution. For a detailed discussion of Fourier transform we refer to [21].
Let us get back to our problem. For a given surface $\Gamma \subset \mathbb{R}^3$ we denote the distribution H^2_Γ by δ_Γ. The next theorem gives the representation of E_s in Fourier space, which will play a crucial role in approximating the summand E_{mag} for magnetisations constant on each cross section.

Theorem 2.5.5. *If $m \in A_x$ then the following formula is valid:*

$$E_s(m) = \frac{4}{\pi^2} \int_{\mathbb{R}^3} \frac{\sin^2(ly)\sin^2(dz)}{x^2+y^2+z^2} \Big(\frac{|\hat{m}_y(x)|^2}{z^2} + \frac{|\hat{m}_z(x)|^2}{y^2} \Big) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

where $\hat{m}_y$ and $\hat{m}_z$ are the Fourier transforms of respectively m_y and m_z in the first coordinate.

Proof. Denote $\Gamma = \partial\Omega$. Note that (2.28) and is equivalent to $\triangle u_s = -s \cdot \delta_\Gamma$ in the distributional sense. Let us now compute the Fourier transform of $s \cdot \delta_\Gamma$. We have for any $k \in \mathbb{R}^3$

$$\widehat{s \cdot \delta_\Gamma}(k) = \frac{1}{2\pi\sqrt{2\pi}} \int_{\mathbb{R}^3} e^{-i\xi k} (s \cdot \delta_\Gamma)(\xi) \, \mathrm{d}\xi.$$

$$\int_{\mathbb{R}^3} e^{-i\xi k} (s \cdot \delta_\Gamma)(\xi) \, \mathrm{d}\xi = \int_{\mathbb{R}\times[-l,l]} m_z(\xi_1) e^{-i(k_1\xi_1 + k_2\xi_2)} (e^{-ik_3 d} - e^{ik_3 d}) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2$$
$$+ \int_{\mathbb{R}\times[-d,d]} m_y(\xi_1) e^{-i(k_1\xi_1 + k_3\xi_3)} (e^{-ik_2 l} - e^{ik_2 l}) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_3.$$

We have that for any $a \in \mathbb{R}$

$$\int_{-a}^{a} e^{-ixt} \, \mathrm{d}t = \frac{e^{ixa} - e^{-ixa}}{ix},$$

thus

$$\int_{\mathbb{R}\times[-l,l]} m_z(\xi_1) e^{-i(k_1\xi_1 + k_2\xi_2)} (e^{-ik_3 d} - e^{ik_3 d}) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2$$

$$= \frac{(e^{ik_2 l} - e^{-ik_2 l})(e^{-ik_3 d} - e^{ik_3 d})}{ik_2} \int_{\mathbb{R}} m_z(\xi_1) e^{-ik_1\xi_1} \,\mathrm{d}\xi_1$$

and

$$\int_{\mathbb{R}\times[-d,d]} m_y(\xi_1) e^{-i(k_1\xi_1 + k_3\xi_3)} (e^{-ik_2 l} - e^{ik_2 l}) \,\mathrm{d}\xi_1 \,\mathrm{d}\xi_3$$
$$= \frac{(e^{-ik_2 l} - e^{ik_2 l})(e^{ik_3 d} - e^{-ik_3 d})}{ik_3} \int_{\mathbb{R}} m_y(\xi_1) e^{-ik_1\xi_1} \,\mathrm{d}\xi_1,$$

hence

$$\widehat{s\cdot\delta_\Gamma}(k) = -\frac{1}{2\pi i}(e^{ik_2 l} - e^{-ik_2 l})(e^{ik_3 d} - e^{-ik_3 d})\Big(\frac{\hat{m}_z(k_1)}{k_2} + \frac{\hat{m}_y(k_1)}{k_3}\Big).$$

Let us now compute $\int_{\mathbb{R}^3} \frac{|\widehat{s\cdot\delta_\Gamma}(k)|^2}{|k|^2} \,\mathrm{d}k$. After some computation we obtain

$$\frac{|\widehat{s\cdot\delta_\Gamma}(k)|^2}{|k|^2} = \frac{4\sin^2(k_2 l)\sin^2(k_3 d)}{\pi^2|k|^2}\Big(\frac{|\hat{m}_z|^2}{k_2^2} + \frac{|\hat{m}_y|^2}{k_3^2} + \frac{1}{k_2 k_3}(\hat{m}_y\overline{\hat{m}_z} + \hat{m}_z\overline{\hat{m}_y})\Big).$$

It is easy to see that

$$\int_{\mathbb{R}^2} \frac{4\sin^2(k_2 l)\sin^2(k_3 d)}{\pi^2 k_2 k_3 |k|^2} \,\mathrm{d}k_2 \,\mathrm{d}k_3 = 0 \quad \text{for any} \quad k_1 \in \mathbb{R}$$

thus

$$\int_{\mathbb{R}^3} \frac{|\widehat{s\cdot\delta_\Gamma}(k)|^2}{|k|^2} \,\mathrm{d}k = \frac{4}{\pi^2}\int_{\mathbb{R}^3} \frac{\sin^2(k_2 l)\sin^2(k_3 d)}{|k|^2}\Big(\frac{|\hat{m}_z|^2}{k_2^2} + \frac{|\hat{m}_y|^2}{k_3^2}\Big) \,\mathrm{d}k \tag{2.32}$$

We will see later that the right hand side integral of (2.32) is convergent therefore taking into account the facts $\int_{\mathbb{R}^3} |\nabla u_s|^2 < \infty$, $\triangle u_s = -s\cdot\delta_\Gamma$ and (2.31) we obtain

$$\int_{\mathbb{R}^3} |\nabla u_s(k)|^2 \,\mathrm{d}k = \int_{\mathbb{R}^3} \frac{|\widehat{\triangle u_s}(k)|^2}{|k|^2} \,\mathrm{d}k = \int_{\mathbb{R}^3} \frac{|\widehat{s\cdot\delta_\Gamma}(k)|^2}{|k|^2} \,\mathrm{d}k =$$
$$= \frac{4}{\pi^2}\int_{\mathbb{R}^3} \frac{\sin^2(k_2 l)\sin^2(k_3 d)}{|k|^2}\Big(\frac{|\hat{m}_z|^2}{k_2^2} + \frac{|\hat{m}_y|^2}{k_3^2}\Big) \,\mathrm{d}k.$$

$\square$

2.5.3 Lower and upper bounds on E_s

To simplify the expressions for and E_s we consider the integral:

$$I(l,d,x)=\int_{\mathbb{R}^2}\frac{\sin^2(ly)\sin^2(dz)}{y^2(x^2+y^2+z^2)}\,\mathrm{d}y\,\mathrm{d}z,$$

It is clear that

$$E_s(m)=\frac{4}{\pi^2}\int_{\mathbb{R}}\Big(I(l,d,x)|\hat{m}_z(x)|^2+I(d,l,x)|\hat{m}_y(x)|^2\Big)\,\mathrm{d}x.$$

The next lemma describes some properties of I. We prove upper and lower bounds on I for certain values of x. Using this lemma we establish an approximation for the magnetostatic energy.

Lemma 2.5.6. *Assume d and l are positive numbers with $0<d\leq l$. The following inequalities hold:*

$$I(l,d,x),I(d,l,x)\leq\pi^2 ld \quad \text{for all} \quad x\in\mathbb{R} \tag{2.33}$$

$$I(l,d,x),I(d,l,x)\geq\frac{4\pi d^2}{27} \quad \text{if} \quad |x|\leq\frac{1}{3l} \tag{2.34}$$

$$I(l,d,x)\geq 2\pi\Big(1-\sqrt{c}\Big)\Big(\frac{\pi}{2}-3\sqrt{c}\Big)ld \quad \text{if} \quad |x|\leq\frac{1}{3\sqrt{dl}}. \tag{2.35}$$

$$I(d,l,x)\leq\pi(1+\pi)ld\sqrt{c} \quad \text{for all} \quad x\in\mathbb{R} \tag{2.36}$$

If $c_n\to c_0>0$ then for any $\epsilon>0$ there exists a natural number n_ϵ such that if $n>n_\epsilon$ then

$$\begin{aligned}\frac{8}{\pi}l_n d_n\Big[(a_{c_0}-\epsilon)\int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}}|\hat{m}_y(x)|^2\,\mathrm{d}x+(b_{c_0}-\epsilon)\int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}}|\hat{m}_z(x)|^2\,\mathrm{d}x\Big]&\leq E_s(m)\\ \leq\frac{8}{\pi}l_n d_n\Big[(a_{c_0}+\epsilon)\int_{\mathbb{R}}|\hat{m}_y(x)|^2\,\mathrm{d}x+(b_{c_0}+\epsilon)\int_{\mathbb{R}}|\hat{m}_z(x)|^2\,\mathrm{d}x\Big]&\end{aligned} \tag{2.37}$$

and

$$E_s(\bar{m}^n)\geq\frac{4}{\pi}(1-\epsilon)^2(1-3\epsilon)l_n d_n c_n|\ln c_n|\int_{-\frac{1}{3l_n}}^{\frac{1}{3l_n}}(|\widehat{\bar{m}_y^n}(x)|^2+|\ln c_n|\cdot|\widehat{\bar{m}_z^n}(x)|^2)\,\mathrm{d}x. \tag{2.38}$$

Proof. First of all we would like to mention that we will use the following well known facts:

$$\frac{1}{2} \le 1 - \frac{t}{2} \le \frac{1-e^{-t}}{t} \le 1 \ \ \forall t \in [0,1] \ \text{ and } \ \frac{1-e^{-t}}{t} \le 1 \ \ \forall t > 0 \tag{2.39}$$

$$\text{the function } \ f(t) = \frac{1-e^{-t}}{t} \ \text{ is decreasing in } \ (0,+\infty) \tag{2.40}$$

$$|\sin t| \ge \frac{2}{3}|t| \ \text{ if } \ |t| \le 1 \tag{2.41}$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2} \ \text{ and } \ \int_0^\infty \frac{\sin^2(pt)}{t^2+q^2}\,\mathrm{d}t = \frac{\pi}{4q}(1-e^{-2pq}) \ \text{ if } \ p,q > 0. \tag{2.42}$$

Note that the integrand of I is an even function in both y and z thus

$$I(l,d,x) = 4\int_0^\infty \int_0^\infty \frac{\sin^2(ly)\sin^2(dz)}{y^2(x^2+y^2+z^2)}\,\mathrm{d}y\,\mathrm{d}z.$$

After making the change of variables $y \to |x|y$, $z \to |x|z$(we assume that $x \neq 0$) and denoting $a = l|x|$, $b = d|x|$ we get

$$I(l,d,x) = \frac{4}{x^2}\int_0^\infty \int_0^\infty \frac{\sin^2(ay)\sin^2(bz)}{y^2(1+y^2+z^2)}\,\mathrm{d}y\,\mathrm{d}z.$$

Using now the second identity of (2.42) and also making a change of variables $y = \frac{t}{a}$ we obtain

$$I(l,d,x) = \frac{\pi}{x^2}\int_0^\infty \frac{\sin^2(ay)}{y^2}\cdot\frac{1-e^{-2b\sqrt{y^2+1}}}{\sqrt{y^2+1}}\,\mathrm{d}y =$$

$$= \frac{2\pi ab}{x^2}\int_0^\infty \frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-\frac{2b}{a}\sqrt{t^2+a^2}}}{\frac{2b}{a}\sqrt{t^2+a^2}}\,\mathrm{d}t.$$

Using the second inequality of (2.39) and the first identity of (2.42) we get

$$I(l,d,x) \le \frac{2\pi ab}{x^2}\int_0^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi^2 ab}{x^2} = \pi^2 ld.$$

Similarly we get $I(d,l,x) \le \pi^2 ld$.
Suppose now $0 \le t \le \frac{d}{3l}$ and $|x| \le \frac{1}{3l}$. We have that $d \le l$ so $t \le \frac{l}{3d}$ and $|x| \le \frac{1}{3d}$ as well. We have in this case

$$\frac{2b}{a}\sqrt{t^2+a^2} = \frac{2d}{l}\sqrt{t^2+l^2x^2} \le \frac{2d}{l}\sqrt{\frac{l^2}{9d^2}+\frac{l^2}{9d^2}} = \frac{2\sqrt{2}}{3} < 1$$

and similarly $\frac{2a}{b}\sqrt{t^2+b^2} < 1$. Thus utilizing the first part of (2.39) we obtain

$$\frac{1-e^{-\frac{2b}{a}\sqrt{t^2+a^2}}}{\frac{2b}{a}\sqrt{t^2+a^2}} \geq \frac{1}{2} \quad \text{and} \quad \frac{1-e^{-\frac{2a}{b}\sqrt{t^2+b^2}}}{\frac{2a}{b}\sqrt{t^2+b^2}} \geq \frac{1}{2}.$$

Finally we get

$$I(l,d,x) \geq \frac{\pi ab}{x^2}\int_0^{\frac{d}{3l}} \frac{\sin^2 t}{t^2}\,\mathrm{d}t \quad \text{and} \quad I(d,l,x) \geq \frac{\pi ab}{x^2}\int_0^{\frac{d}{3l}} \frac{\sin^2 t}{t^2}\,\mathrm{d}t.$$

Now we utilize (2.41) to get

$$I(l,d,x) \geq \pi ld \cdot \frac{4}{9} \cdot \frac{d}{3l} = \frac{4\pi d^2}{27}$$

The proof of the inequality

$$I(d,l,x) \geq \frac{4\pi d^2}{27}$$

is analogues. Suppose now δ is a positive number less than 1, $0 \leq t \leq \frac{\delta l}{3d}$ and $|x| \leq \frac{\delta}{3d}$.
We have that

$$\frac{2b}{a}\sqrt{t^2+a^2} = \frac{2d}{l}\sqrt{t^2+l^2x^2} \leq \frac{2d}{l}\sqrt{\frac{l^2\delta^2}{9d^2}+\frac{l^2\delta^2}{9d^2}} = \frac{2\sqrt{2}}{3}\delta < \delta < 1$$

hence

$$\frac{1-e^{-\frac{2b}{a}\sqrt{t^2+a^2}}}{\frac{2b}{a}\sqrt{t^2+a^2}} \geq 1-\frac{\delta}{2}.$$

For the function I we get

$$I(l,d,x) = 2\pi ld\int_0^{\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2b}{a}\sqrt{t^2+a^2}}}{\frac{2b}{a}\sqrt{t^2+a^2}}\,\mathrm{d}t \geq 2\pi\Big(1-\frac{\delta}{2}\Big)ld\int_0^{\frac{\delta l}{3d}} \frac{\sin^2 t}{t^2}\,\mathrm{d}t.$$

Note that if $p > 0$ then

$$\int_0^p \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \int_0^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t - \int_p^\infty \frac{\sin^2 t}{t^2}\,\mathrm{d}t \geq \frac{\pi}{2} - \int_p^\infty \frac{1}{t^2}\,\mathrm{d}t = \frac{\pi}{2} - \frac{1}{p},$$

thus we obtain

$$I(l,d,x) \geq 2\pi\Big(1-\frac{\delta}{2}\Big)\Big(\frac{\pi}{2}-\frac{3d}{\delta l}\Big)ld.$$

Taking now $\delta = \sqrt{c}$ we get

$$I(l,d,x) \geq 2\pi\Big(1-\sqrt{c}\Big)\Big(\frac{\pi}{2} - 3\sqrt{c}\Big)ld.$$

Fix again a positive number δ less than 1. For $t \geq \frac{d}{2l\delta}$ we have

$$\frac{2a}{b}\sqrt{t^2+b^2} \geq \frac{2at}{b} = \frac{2lt}{d} \geq \frac{1}{\delta} > 1, \quad \text{thus}$$

$$I(d,l,x) \leq 2\pi ld \int_0^{\frac{d}{2l\delta}} \frac{\sin^2 t}{t^2}\,\mathrm{d}t + 2\pi ld \int_{\frac{d}{2l\delta}}^{\infty} \frac{\sin^2 t}{t^2} \cdot \delta\,\mathrm{d}t$$

$$\leq 2\pi ld \cdot \frac{d}{2l\delta} + 2\pi ld \int_0^{\infty} \frac{\sin^2 t}{t^2} \cdot \delta\,\mathrm{d}t = \pi ld\Big(\frac{c}{\delta} + \pi\delta\Big).$$

Taking now $\delta = \sqrt{c}$ we obtain

$$I(d,l,x) \leq \pi(1+\pi)ld\sqrt{c}.$$

Assume now $\frac{d_n}{l_n} = c_n \to c_0 > 0$. For any $n \in \mathbb{N}$ we get lower and upper bounds on $I(l_n,d_n,x)$ for $x \in [-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}]$ and $x \in \mathbb{R}$ respectively. It is clear that

$$\frac{2b_n}{a_n}\sqrt{t^2+a_n^2} = \frac{2d_n}{l_n}\sqrt{t^2+l_n^2x^2} \leq 2c_n\sqrt{t^2+l_n} \ \text{ if } \ t>0, x \in \Big[-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}\Big]$$

and

$$\frac{2b_n}{a_n}\sqrt{t^2+a_n^2} = \frac{2d_n}{l_n}\sqrt{t^2+l_n^2x^2} \geq 2c_n t \ \text{ if } \ t>0, x \in \mathbb{R}$$

thus taking into account (2.40) we get

$$2\pi l_n d_n \int_0^{\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-2c_n\sqrt{t^2+l_n}}}{2c_n\sqrt{t^2+l_n}}\,\mathrm{d}t \leq I(l_n,d_n,x) \ \text{ for any } \ x \in \Big[-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}\Big] \tag{2.43}$$

and

$$I(l_n,d_n,x) \leq 2\pi l_n d_n \int_0^{\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-2c_n t}}{2c_n t}\,\mathrm{d}t \quad \text{for any} \quad x \in \mathbb{R}. \tag{2.44}$$

Note that for any $t > 0$ we have

$$2c_n\sqrt{t^2+l_n} \to 2c_0 t \ \text{ and } \ 2c_n t \to 2c_0 t \ \text{ as } \ n \to \infty.$$

We utilize (2.39) to get

$$\left|\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_n\sqrt{t^2+l_n}}}{2c_n\sqrt{t^2+l_n}}\right|\leq\frac{\sin^2 t}{t^2},\qquad \left|\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_nt}}{2c_nt}\right|\leq\frac{\sin^2 t}{t^2},$$

$$\left|\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_0t}}{2c_0t}\right|\leq\frac{\sin^2 t}{t^2}\quad\text{for any}\quad t>0$$

and the function $\frac{\sin^2 t}{t^2}$ is integrable on $(0,+\infty)$, therefore by the dominated convergence theorem we establish

$$\int_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_n\sqrt{t^2+l_n}}}{2c_n\sqrt{t^2+l_n}}\,\mathrm{d}t\to\int_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_0t}}{2c_0t}\,\mathrm{d}t=b_{c_0}$$

and

$$\int_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_nt}}{2c_nt}\,\mathrm{d}t\to\int_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_0t}}{2c_0t}\,\mathrm{d}t=b_{c_0}.$$

The same argument can be done for $I(d_n,l_n,x)$ with a lower bound for $x\in[-\frac{1}{\sqrt{d_n}},\frac{1}{\sqrt{d_n}}]$ and an upper bound for any $x\in\mathbb{R}$. This yields that for any $\epsilon>0$ there exists a natural number n_ϵ such that if $n>n_\epsilon$ then

$$\frac{8}{\pi}l_nd_n\Big[(a_{c_0}-\epsilon)\int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}}|\hat{m}_y(x)|^2\,\mathrm{d}x+(b_{c_0}-\epsilon)\int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}}|\hat{m}_z(x)|^2\,\mathrm{d}x\Big]\leq E_s(m)$$

$$\leq\frac{8}{\pi}l_nd_n\Big[(a_{c_0}+\epsilon)\int_{\mathbb{R}}|\hat{m}_y(x)|^2\,\mathrm{d}x+(b_{c_0}+\epsilon)\int_{\mathbb{R}}|\hat{m}_z(x)|^2\,\mathrm{d}x\Big].$$

This inequality plays a crucial role in the proof of the first Γ-convergence theorem. One of important properties of this inequality is the fact that the number n_ϵ depends only on ϵ and the sequences $(l_n)_{n\in\mathbb{N}}$,$(d_n)_{n\in\mathbb{N}}$, namely if we have a sequence of domain-magnetization pairs $(\Omega(l_n,d_n),m^n)$ with finite energy each and satisfying the properties $l_n,d_n\to 0$, and $c_n\to c>0$ then (2.37) is fulfilled for any m^n with n greater than the same number n_ϵ. In the next step we obtain accurate lower and upper bounds for E_s which will be used in the third Γ-convergence theorem which corresponds to the case $d,l,\frac{d}{l}\to 0$. To obtain accurate bounds on E_s we need accurate bounds on $I(d_n,l_n,x)$. It is clear that

$$\frac{2l_n}{d_n}\sqrt{t^2+d_n^2x^2}\geq\frac{2l_n}{d_n}t=\frac{2t}{c_n}$$

hence

$$I(d_n, l_n, x) \leq 2\pi l_n d_n \int_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c_n}}}{\frac{2t}{c_n}} \,\mathrm{d}t$$

$$= \underbrace{\pi l_n d_n c_n \int_0^{c_n} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c_n}}}{t} \,\mathrm{d}t}_{I_1} + \underbrace{\pi l_n d_n c_n \int_{c_n}^{1} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c_n}}}{t} \,\mathrm{d}t}_{I_2}$$

$$+ \underbrace{\pi l_n d_n c_n \int_1^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c_n}}}{t} \,\mathrm{d}t}_{I_3}.$$

$$I_1 = 2\pi l_n d_n \int_0^{c_n} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c_n}}}{\frac{2t}{c_n}} \,\mathrm{d}t \leq 2\pi l_n d_n \int_0^{c_n} \mathrm{d}t = 2\pi l_n d_n c_n,$$

$$I_2 \leq \pi l_n d_n c_n \int_{c_n}^{1} \frac{1}{t} \,\mathrm{d}t = -l_n d_n c_n \ln c_n \quad \text{and}$$

$$I_3 \leq \pi l_n d_n c_n \int_1^{+\infty} \frac{\sin^2 t}{t^2} \,\mathrm{d}t \leq \pi l_n d_n c_n \int_1^{+\infty} \frac{1}{t^2} \,\mathrm{d}t = \pi l_n d_n c_n.$$

Concluding we obtain

$$I(d_n, l_n, x) \leq \pi l_n d_n c_n (3 - \ln c_n). \tag{2.45}$$

Remark 2.5.7. *We have as well shown*

$$\limsup_{c \to 0} \frac{a_c}{c|\ln c|} \leq \frac{1}{2}. \tag{2.46}$$

To get a lower bound on $I(d_n, l_n, x)$ we note that the main contribution to the integral comes from the interval $[c_n, 1]$. We have replaced $\frac{\sin^2 t}{t^2}$ and $1 - e^{-\frac{2t}{c_n}}$ by 1 in $[c_n, 1]$ to get an upper bound, but since near the endpoints $\frac{\sin^2 t}{t^2}$ as well as $1 - e^{-\frac{2t}{c_n}}$ can be much smaller than 1 we can not do the same to get a lower bound. That is why we choose another interval with suitable endpoints, namely we replace $[c_n, 1]$ by $[c_n^{1-\epsilon}, c_n^{\epsilon}]$ where ϵ is a small positive number yet to be chosen. Assume ϵ is any positive number smaller than $\frac{1}{3}$. We estimate $I(d_n, l_n, x)$ for the values $x \in \left[-\frac{1}{l_n}, \frac{1}{l_n}\right]$. For any $t \in [c_n^{1-\epsilon}, c_n^{\epsilon}]$ we have

$$\frac{2l_n}{d_n}\sqrt{t^2 + x^2 d_n^2} \geq \frac{2t}{c_n} \geq 2c_n^{-\epsilon}$$

and

$$\sqrt{t^2 + x^2 d_n^2} \leq t + |x| d_n \leq t + \frac{d_n}{l_n} = t + c_n$$

hence

$$I(d_n, l_n, x) \geq \pi l_n d_n c_n \int_{c_n^{1-\epsilon}}^{c_n^{\epsilon}} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-2c_n^{-\epsilon}}}{t + c_n} \, \mathrm{d}t. \tag{2.47}$$

Since

$$\lim_{n\to\infty} c_n = 0 \quad \text{and} \quad \lim_{t\to 0} \frac{\sin^2 t}{t^2} = 1$$

there exists $n_\epsilon \in \mathbb{N}$ such that if $n > n_\epsilon$ then

$$c_n^{\epsilon} < 1, \quad 1 - e^{-2c_n^{-\epsilon}} > 1 - \epsilon, \quad |\ln c_n| > \frac{\ln 2}{\epsilon}$$

and

$$\frac{\sin^2 t}{t^2} > 1 - \epsilon \quad \text{for} \quad t \in [0, c_n^{\epsilon}].$$

Thus we obtain for any $n > n_\epsilon$

$$I(d_n, l_n, x) \geq \pi l_n d_n c_n (1-\epsilon)^2 \int_{c_n^{1-\epsilon}}^{c_n^{\epsilon}} \frac{1}{t + c_n} \, \mathrm{d}t$$

$$= \pi (1-\epsilon)^2 l_n d_n c_n \big(\ln(c_n + c_n^{\epsilon}) - \ln(c_n + c_n^{1-\epsilon}) \big).$$

It is clear that

$$\ln(c_n + c_n^{1-\epsilon}) = \ln c_n + \ln(1 + c_n^{-\epsilon}) \leq \ln c_n + \ln(2c_n^{-\epsilon}) = (1-\epsilon)\ln c_n + \ln 2$$

$$\leq (1 - 2\epsilon) \ln c_n$$

and

$$\ln(c_n + c_n^{\epsilon}) \geq \ln c_n^{\epsilon} = \epsilon \ln c_n.$$

Concluding we obtain

$$I(d_n, l_n, x) \geq \pi (1-\epsilon)^2 (1 - 3\epsilon) l_n d_n c_n |\ln c_n|. \tag{2.48}$$

Remark 2.5.8. *We have also got that*

$$\liminf_{c\to 0} \frac{a_c}{c|\ln c|} \geq \frac{1}{2} \tag{2.49}$$

Corollary 2.5.9. *The function a_c has the property*

$$\liminf_{c\to 0} \frac{a_c}{c|\ln c|} = \frac{1}{2} \tag{2.50}$$

According to (2.35) we have for big n

$$I(l_n, d_n, x) \geq \pi l_n d_n \quad \text{if} \quad x \in \Big[-\frac{1}{3l_n}, \frac{1}{3l_n}\Big].$$

Coupling the last inequality with (2.48) we obtain for sufficiently big n

$$E_s(\bar{m}^n) \geq \frac{4}{\pi}(1-\epsilon)^2(1-3\epsilon) l_n d_n c_n |\ln c_n| \int_{-\frac{1}{3l_n}}^{\frac{1}{3l_n}} \big(|\widehat{\bar{m}_y^n}(x)|^2 + \frac{1}{c_n|\ln c_n|} \cdot |\widehat{\bar{m}_z^n}(x)|^2\big)\,\mathrm{d}x$$

$$\geq \frac{4}{\pi}(1-\epsilon)^2(1-3\epsilon) l_n d_n c_n |\ln c_n| \int_{-\frac{1}{3l_n}}^{\frac{1}{3l_n}} \big(|\widehat{\bar{m}_y^n}(x)|^2 + |\ln c_n| \cdot |\widehat{\bar{m}_z^n}(x)|^2\big)\,\mathrm{d}x.$$

□

The next lemmas give an upper bound on E_v

Lemma 2.5.10. *For any numbers $0 < d \leq l$ and any point $(y_1, z_1) \in R(l, d)$ the following bound holds:*

$$I = \int_{R(l,d)} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{(y-y_1)^2 + (z-z_1)^2}} < 10d\Big(1 + \ln\frac{l}{d}\Big).$$

Proof. It is clear that

$$I \leq \int_{R(2l,2d)} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{y^2+z^2}} = \int_{R(2d,2d)} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{y^2+z^2}} + \int_{R(2l,2d)\setminus R(2d,2d)} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{y^2+z^2}}$$

$$\leq \frac{1}{4}\int_{D_{4\sqrt{2}d}(0)} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{y^2+z^2}} + 8d\int_{2d}^{2l} \frac{\mathrm{d}y}{y} = 2\sqrt{2}\pi d + 8d\ln\frac{l}{d} < 10d\Big(1 + \ln\frac{l}{d}\Big).$$

□

Lemma 2.5.11. *For any $0 < d \leq l$ and $m \in A_x(l, d)$ the following bound holds:*

$$E_v(m) \leq M_m\Big(l^2d^2 + ld^2\Big(1 + \ln\frac{l}{d}\Big)\Big), \tag{2.51}$$

where M_m is a constant depending on the magnetization m.

Proof. According to (2.27) we have that

$$\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla\phi = \int_\Omega v \cdot \phi \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^3).$$

By the density argument we can transfer this equality to u_v, because $\nabla u_v \in L^2(\mathbb{R}^3)$ and $u_v \in L^6(\mathbb{R}^3)$, thus utilizing Lemma 2.5.1 we obtain

$$E_v(m) = \int_{\mathbb{R}^3} |\nabla u_v|^2 = \int_\Omega v \cdot u_v = \int_\Omega \int_\Omega \Gamma(\xi - \xi_1) v(\xi) v(\xi_1) \, \mathrm{d}\xi \, \mathrm{d}\xi_1.$$

We have that $m \in A_x$ so $v(x,y,z) = \partial_x m_x(x)$ thus

$$E_v(m) = \frac{1}{4\pi} \int_\Omega \int_\Omega \frac{\partial_x m_x(x) \partial_x m_{x_1}(x_1)}{|\xi - \xi_1|} \, \mathrm{d}\xi \, \mathrm{d}\xi_1$$

where $\xi = (x,y,z)$ and $\xi_1 = (x_1, y_1, z_1)$. It is clear that

$$\int_{\mathbb{R}} \frac{\partial_x m_x(x)}{|\xi - \xi_1|} \, \mathrm{d}x = \int_{-\infty}^0 \frac{\mathrm{d}m^*(x)}{|\xi - \xi_1|} + \int_0^{+\infty} \frac{\mathrm{d}m^*(x)}{|\xi - \xi_1|}$$

$$= \frac{2}{\sqrt{x_1^2 + (y-y_1)^2 + (z-z_1)^2}} - \int_{\mathbb{R}} \frac{(x - x_1) m^*(x)}{|\xi - \xi_1|^3} \, \mathrm{d}x,$$

hence for the energy we have

$$E_v(m) \leq \underbrace{\frac{1}{2\pi} \int_{R(l,d)} \int_\Omega \frac{|\partial_x m_x(x_1)|}{\sqrt{x_1^2 + (y-y_1)^2 + (z-z_1)^2}} \, \mathrm{d}\xi_1 \, \mathrm{d}y \, \mathrm{d}z}_{I_1}$$

$$+ \underbrace{\int_\Omega \int_\Omega \frac{|\partial_x m_x(x_1) m^*(x)|}{|\xi - \xi_1|^2} \, \mathrm{d}\xi \, \mathrm{d}\xi_1}_{I_2}.$$

We have

$$\int_{\mathbb{R}} \frac{|\partial_x m_x(x_1)|}{\sqrt{x_1^2 + (y-y_1)^2 + (z-z_1)^2}} \, \mathrm{d}x_1$$

$$\leq \frac{1}{2} \int_{\mathbb{R}} \Big(|\partial_x m_x(x_1)|^2 + \frac{1}{x_1^2 + (y-y_1)^2 + (z-z_1)^2} \Big) \, \mathrm{d}x_1$$

$$= \frac{1}{2} \|\partial_x m_x\|^2_{L^2(\mathbb{R})} + \frac{\pi}{2\sqrt{(y-y_1)^2 + (z-z_1)^2}}.$$

Utilizing now Lemma 2.5.11 we get

$$I_1 \leq \frac{4}{\pi} |\partial_x m_x\|^2_{L^2(\mathbb{R})} l^2 d^2 + \frac{1}{4} \int_{R(l,d)} \int_{R(l,d)} \frac{1}{\sqrt{(y-y_1)^2 + (z-z_1)^2}} \, \mathrm{d}y_1 \, \mathrm{d}z_1 \, \mathrm{d}y \, \mathrm{d}z$$

$$\leq \frac{4}{\pi} |\partial_x m_x\|^2_{L^2(\mathbb{R})} l^2 d^2 + 10 l d^2 \Big(1 + \ln \frac{l}{d} \Big).$$

By making a change of variables $\xi_2 = \xi_1 - \xi$ and utilizing Lemma 2.5.11 we get

$$
\begin{aligned}
I_2 &= \int_\Omega \int_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]} \frac{|m^\star(x)|\cdot|\partial_x m_x(x_2+x)|}{|\xi_2|^2}\,\mathrm{d}\xi_2\,\mathrm{d}\xi \\
&\leq \frac{1}{2}\int_{R(l,d)}\int_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]}\int_{\mathbb{R}} \frac{|m^*(x)|^2+|\partial_x m_x(x_2+x)|^2}{|\xi_2|^2}\,\mathrm{d}x\,\mathrm{d}\xi_2\,\mathrm{d}y\,\mathrm{d}z \\
&= 2ld(\|m^*\|^2_{L^2(\mathbb{R})}+\|\partial_x m_x\|^2_{L^2(\mathbb{R})})\int_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]}\frac{\mathrm{d}\xi_2}{|\xi_2|^2} \\
&= 2\pi ld(\|m^*\|^2_{L^2(\mathbb{R})}+\|\partial_x m_x\|^2_{L^2(\mathbb{R})})\int_{R(l,d)}\frac{1}{\sqrt{(y_1-y)^2+(z_1-z)^2}}\,\mathrm{d}y_1\,\mathrm{d}z_1 \\
&\leq 20\pi ld^2\Big(1+\ln\frac{l}{d}\Big)(\|m^*\|^2_{L^2(\mathbb{R})}+\|\partial_x m_x\|^2_{L^2(\mathbb{R})}).
\end{aligned}
$$

The summary of the estimates on I_1 and I_2 and Corollary 2.4.7 completes the proof.

□

2.6 The existence of minimizers

In the next step we prove a lemma which will be used in both the existence and the Γ-convergence theorems. It states a compactness for a sequence of magnetizations with bounded energies.

Lemma 2.6.1. *Suppose we are given a sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ defined in the same domain Ω and with energies bounded by the same constant C. Then there exists a magnetization $m^0\colon \Omega\to\mathbb{S}^2$ such that for a subsequence of $(m^n)_{n\in\mathbb{N}}$ (not relabeled) the following statements hold*

- $\nabla m^n \rightharpoonup \nabla m^0$ *weakly in* $L^2(\Omega)$
- $m^n \to m^0$ *strongly in* $L^2_{loc}(\Omega)$
- $E(m^0) \leq \liminf E(m^n)$.

Proof. Let u_n be the weak solution of $\triangle u = \mathrm{div} m^n$. We have that

$$\int_\Omega |\nabla m^n|^2\,\mathrm{d}\xi \leq E(m^n) \leq C$$

thus $(\nabla m^n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence (not relabeled),i.e.,

$$\nabla m^n \rightharpoonup f \quad \text{weakly in} \quad L^2(\Omega)$$

for some $f \in L^2(\Omega)$. Similarly the new subsequence $(\nabla u_n)_{n\in\mathbb{N}}$ contains a weakly convergent subsequence (not relabeled),i.e.,

$$\nabla u_n \rightharpoonup g \quad \text{weakly in} \quad L^2(\mathbb{R}^3)$$

for some $g \in L^2(\mathbb{R}^3)$. Since $|m^n| = 1$ in Ω we have that

$$m^n \in W^{1,2}\big([-N,N] \times R(l,d)\big) \quad \text{for any} \quad N \in \mathbb{N}.$$

Taking into account the fact that the embedding

$$W^{1,2}\big([-N,N] \times R(l,d)\big) \hookrightarrow L^2\big([-N,N] \times R(l,d)\big)$$

is compact, one can extract a subsequence from the new subsequence $(m^n)_{n\in\mathbb{N}}$ (not relabeled) converging to some m^0 in $L^2\big([-N,N] \times R(l,d)\big)$. We do this giving N all the natural values and then apply diagonal argument to the extracted subsequences. Finally we obtain a subsequence of $(m^n)_{n\in\mathbb{N}}$ (not relabeled) with the following properties:

- $\nabla m^n \rightharpoonup f \quad \text{weakly in} \quad L^2(\Omega)$

- $\nabla u_n \rightharpoonup g \quad \text{weakly in} \quad L^2(\mathbb{R}^3)$

- $m^n \to m^0 \quad \text{strongly in} \quad L^2_{loc}(\Omega).$

Applying a standard argument we can deduce that m^0 is weakly differentiable and $\nabla m^0 = f$. We extend m^0 outside Ω as zero. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi \,\mathrm{d}\xi = \int_\Omega m^n \cdot \nabla\varphi \,\mathrm{d}\xi,$$

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla\varphi \,\mathrm{d}\xi \to \int_{\mathbb{R}^3} g \cdot \nabla\varphi \,\mathrm{d}\xi$$

and

$$\int_\Omega m^n \cdot \nabla\varphi \,\mathrm{d}\xi \to \int_\Omega m^0 \cdot \nabla\varphi \,\mathrm{d}\xi$$

as n goes to infinity hence we establish

$$\int_{\mathbb{R}^3} m^0 \cdot \nabla\varphi \,\mathrm{d}\xi = \int_{\mathbb{R}^3} g \cdot \nabla\varphi \,\mathrm{d}\xi.$$

Since $g \in L^2(\mathbb{R}^3)$ we have that the equation $\triangle u = \mathrm{div} g$ has a weak solution u_0 which is equivalent to

$$\int_{\mathbb{R}^3} g \cdot \nabla\varphi \, \mathrm{d}\xi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla\varphi \, \mathrm{d}\xi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3)$$

thus

$$\int_{\mathbb{R}^3} m^0 \cdot \nabla\varphi \, \mathrm{d}\xi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla\varphi \, \mathrm{d}\xi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3)$$

which means that u_0 is a weak solution of

$$\triangle u = \mathrm{div} m^0.$$

Since $g \in L^2(\mathbb{R}^3)$ we already know that

$$\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^2(\mathbb{R}^3)}$$

and we have as well

$$\nabla u_n \rightharpoonup g \quad \text{weakly in} \quad L^2(\mathbb{R}^3),$$

$$\nabla m^n \rightharpoonup \nabla m^0 \quad \text{weakly in} \quad {}^2(\Omega).$$

Taking into account the fact that any norm is lower semi-continuous under the weak convergence we obtain

$$\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^2(\mathbb{R}^3)} \leq \liminf_{n\to\infty} \|\nabla u_n\|_{L^2(\mathbb{R}^3)}$$

$$\|\nabla m^0\|_{L^2(\mathbb{R}^3)} \leq \liminf_{n\to\infty} \|\nabla m^n\|_{L^2(\mathbb{R}^3)}$$

which yields

$$E(m^0) \leq \liminf_{n\to\infty} E(m^n).$$

□

We proceed now to the existence theorem.

Theorem 2.6.2 (Existence). *For every $0 < d \leq l$ there exist minimizers of E is $\tilde{A}$ and $\tilde{A}_x$.*

Proof. We will first prove the existence of a minimizer in $\tilde{A}$. Let m^n be a minimizing sequence, i.e.,

$$\lim_{n\to\infty} E(m^n) = E_{min}.$$

Since $(E(m^n))_{n\in\mathbb{N}}$ is bounded, applying the preceding lemma we extract a subsequence from $(m^n)_{n\in\mathbb{N}}$ (denoted again by $(m^n)_{n\in\mathbb{N}}$) such that for a magnetization $m^0 \in W^{1,2}_{loc}(\Omega)$ we have:

- $\nabla m^n \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$

- $m^n \to m^0$ strongly in $L^2_{loc}(\Omega)$
- $E(m^0) \leq \liminf E(m^n)$

If we could show that $m^0 \in \tilde{A}$ then m^0 would be the desired minimizer because of the fact that

$$E(m^0) \leq \liminf E(m^n) = E_{min}$$

and E_{min} is the infimum of the energy functional in $\tilde{A}$ so $E(m^0) = E_{min}$. But m^0 does not have to belong to $\tilde{A}$ in general. For instance the boundary conditions could fail, we could have $\|m - \bar{e}\|_{H^1(\Omega)} = \infty$. At the end of the proof we will give an example of a sequence of minimizers for which the limit function m^0 does not satisfy the boundary conditions. To overcome this difficulty we construct a minimizing sequence so that its limit belongs to $\tilde{A}$. To that end we choose any minimizing sequence $(m^n)_{n\in\mathbb{N}}$ as above and suppose that it has a limit m^0 in the described sense. The key point is to show that the desired minimizing sequence can be constructed by translating every vector m^n by a factor x_n in the x coordinate direction. First of all note that if $m \in \tilde{A}$ then obviously $m_c(x,y,z) = m(x-c,y,z) \in \tilde{A}$ and $E(m_c) = E(m)$ (the minimization problem is invariant under translations in the first coordinate). Since $E(m^n) \to E_{min}$, the sequence $(E(m^n))_{n\in\mathbb{N}}$ is bounded by some number M for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ we consider three auxiliary sets A_n, B_n and C_n defined in the following way:

$$A_n = \{x \in \mathbb{R} \ \mid -1 \leq \bar{m}^n_x(x) \leq -\frac{1}{2}\}$$

$$B_n = \{x \in \mathbb{R} \ \mid -\frac{1}{2} < \bar{m}^n_x(x) < \frac{1}{2}\}$$

$$C_n = \{x \in \mathbb{R} \ \mid \ \frac{1}{2} \leq \bar{m}^n_x(x) \leq 1\}$$

Since $\bar{m}^n_x$ is continuous in $\mathbb{R}$ for all $n \in \mathbb{N}$, A_n and C_n are a finite or countable union of disjoint closed intervals and B_n is a finite or countable union of disjoint open intervals. According to Lemma 2.4.5 one of the intervals in A_n has the form $(-\infty, a_n]$ and one of the intervals in C_n has the form $[c_n, +\infty)$ (note that $\bar{m}^n_x$ is negative at $-\infty$ and positive at $+\infty$.) We distinguish two types of intervals in B_n. The interval $(a,b) \subset B_n$ is said to be of the first type if $|\bar{m}^n(a) - \bar{m}^n(b)| = 1$, and of the second type otherwise. According to Lemma 2.4.5 the sum of the lengths of all intervals, as well as the number of the first type intervals in B_n is bounded by a number depending only on M, l and d, i.e., a constant not depending on n. Suppose first that there are no second type intervals in B_n for all $n \in \mathbb{N}$. Let us paint all the point of

A_n, B_n and C_n with respectively black, yellow and red color for all $n \in \mathbb{N}$. We call a sequence $(n_k)_{k\in\mathbb{N}}$ "good" if for any $k \in \mathbb{N}$ there exist two intervals $[a_1^k, a_2^k] \subset A_{n_k}$ and $[c_1^k, c_2^k] \subset C_{n_k}$ such that

$$a_2^k - a_1^k \to +\infty, \quad c_2^k - c_1^k \to +\infty \quad \text{and} \quad 0 < c_1^k - a_2^k \le C$$

for a constant C not depending on k. The endpoints a_1^k and c_2^k can also take values $-\infty$ and $+\infty$ respectively. We prove that for any minimizing sequence $(m^n)_{n\in\mathbb{N}}$, with $m^n \in \tilde{A}$ there exists a "good" subsequence $(n_k)_{k\in\mathbb{N}}$. We fix a natural number n and take the two intervals $(-\infty, a_n]$ and $[c_n, +\infty)$. There are some black, yellow and white intervals between this two. Note that if the number of yellow intervals is less than s then the number of both black and red intervals are less than $s+1$ because there is obviously at least one yellow interval between any two black and any two red intervals. Therefore the number of all intervals is less than $3s+2$. Since n was arbitrary we get that the number of all the intervals in the n-th family of the constructed intervals is bounded by the same number S. Let us number both the red and the black intervals in any family of intervals. We prove the existence of a "good" subsequence by induction in S but we first reformulate the problem as follows: Suppose we are given a sequence of natural numbers S_n and a sequence of families of S_n disjoint intervals on the real line pained with black and red color for all $n \in \mathbb{N}$. Assume $S_n \le S$ and the sum of the lengths of $S_n - 1$ gaps between the intervals of the n-th family is bounded by the same number M for all $n \in \mathbb{N}$. Assume furthermore that for any $n \in \mathbb{N}$ the far left placed interval is black and the far right placed interval is red and their lengths tend to $+\infty$ as n goes to infinity. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and two intervals (a_1^k, a_2^k) and (c_1^k, c_2^k) in the n_k-th family such that

$$(a_1^k, a_2^k) \ \text{ is black, } \ (c_1^k, c_2^k) \ \text{ is red,}$$

$$a_2^k - a_1^k \to +\infty, \quad c_2^k - c_1^k \to +\infty \quad \text{and} \quad 0 < c_1^k - a_2^k \le M_2 \tag{2.52}$$

for a constant M_2 and all $k \in \mathbb{N}$. We prove this statement by induction in S. The case $S = 2$ is evident. Assume it is true for $S \le N$ and let us prove it for $S = N+1$. Since $S \ge 3$, in every family there are at least two intervals of the same color. Assume that for infinitely many indices n there are at least two black intervals in the n-th family. We consider now the subsequence of the families with such indices. We consider the far right placed black intervals for all such families. There are two possible cases:

Case 1. *For a subsequence their lengths tend to* $+\infty$

In this case we can omit all the intervals placed on their left side which leads to a situation with less intervals in every family (in such a subsequence) fulfilling the requirements of the statement, so by induction the existence of a "good" subsequence is proven.

Case 2. *Their lengths are bounded by the same number* M_3

In this case we can omit this intervals and this will lead us to a situation with less intervals in any family fulfilling the requirements of the statements so by the induction the existence of a "good" subsequence is proven

Let us get now back to our situation. If we omit all the yellow intervals from the real line for all $n \in \mathbb{N}$ then the families of the black and the red intervals fulfill the requirements of the statement proven above, thus the existence of a "good" sequence is proven. We take the two intervals $[a_1^k, a_2^k]$ and $[c_1^k, c_2^k]$ for all $k \in \mathbb{N}$ and denote the the "good" sequence of the magnetizations again by $(m^k)_{k\in\mathbb{N}}$ which will also be a minimizing sequence. We transfer the origin of the real line to the point a_2^k for any m^k and denote

$$m_{good}^k(x, y, z) = m^k(x + a_2^k, y, z).$$

As we already know $(m_{good}^k)_{k\in\mathbb{N}}$ is a minimizing sequence and furthermore if we put $a_3^k = a_2^k - a_1^k$, $c_3^k = c_1^k - a_2^k$ and $c_4^k = c_2^k - a_2^k$ then

$$m_{good}^k(x) \le -\frac{1}{2} \text{ for } x \in [-a_3^k, 0] \text{ and } m_{good}^k(x) \ge \frac{1}{2} \text{ for } x \in [c_3^k, c_4^k]$$

where

$$a_3^k \to +\infty, c_4^k - c_3^k \to +\infty \text{ and } \ 0 < c_3^k < M \text{ for all } \ k \in \mathbb{N}.$$

By Lemma 2.6.1 one can extract a subsequence from $(m_{good}^k)_{k\in\mathbb{N}}$ (not relabeled) such that for some $m^0 \in A$ the three statements hold:

- $\nabla m_{good}^k \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$
- $m_{good}^k \to m^0$ strongly in $L_{loc}^2(\Omega)$
- $E(m^0) \le \liminf E(m_{good}^k)$.

We will prove that $m^0 \in \tilde{A}$. Recall that for any magnetization m the inclusions $m \pm \overrightarrow{e_x} \in L^2(\Omega_+)$ are equivalent to $1 \pm \bar{m}_x \in L^1(0, +\infty)$ respectively and the inclusions $m \pm \overrightarrow{e_x} \in L^2(\Omega_-)$ are equivalent to $1 \pm \bar{m}_x \in L^1(-\infty, 0)$ respectively. Since $m^0 \in A$ according to the characterization theorem two of the four statements must hold: $1 \pm \bar{m}_x^0 \in L^1(0, +\infty)$ and $1 \pm \bar{m}_x^0 \in L^1(-\infty, 0)$. We have for any fixed $R > 0$

$$
\begin{aligned}
\int_{-R}^{R} |\bar{m}_x^0 - \bar{m}_{good,x}^k| \, \mathrm{d}x &= \frac{1}{4ld} \int_{-R}^{R} \Big| \int_{R(l,d)} (m_x^0 - m_{good,x}^k) \, \mathrm{d}y \, \mathrm{d}z \Big| \, \mathrm{d}x \\
&\leq \frac{1}{4ld} \int_{-R}^{R} \int_{R(l,d)} |m_x^0 - m_{good,x}^k| \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x \\
&\leq \frac{1}{4ld} \left(8ldR \cdot \int_{[-R,R]\times R(l,d)} |m_x^0 - m_{good,x}^k|^2 \, \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \sqrt{\frac{R}{2ld}} \cdot \|m_x^0 - m_{good,x}^k\|_{L^2([-R,R]\times R(l,d))} \to 0
\end{aligned}
$$

as $k \to \infty$ because of the strong convergence $m_{good}^k \to m^0$ in $L_{loc}^2(\Omega)$. This means that a subsequence of $(\bar{m}_{x,good}^k(x))_{k\in\mathbb{N}}$ converges pointwise to $\bar{m}_x^0(x)$ almost everywhere in $[-R, R]$. Giving R all natural values and applying diagonal argument we obtain that a subsequence of $(\bar{m}_{x,good}^k(x))_{k\in\mathbb{N}}$ converges pointwise to $\bar{m}_x^0(x)$ almost everywhere in $\mathbb{R}$, therefore $\bar{m}_x^0(x) \leq -\frac{1}{2}$ a.e. in $(-\infty, 0)$ and $\bar{m}_x^0(x) \geq \frac{1}{2}$ a.e. in $[M, +\infty)$ which itself yields $1 - \bar{m}_x^0$ and $1 + \bar{m}_x^0$ can not belong to $L^1(-\infty, 0)$ and $L^1(0, +\infty)$ respectively, therefore $1 + \bar{m}_x^0 \in L^1(-\infty, 0)$ and $1 - \bar{m}_x^0 \in L^1(0, +\infty)$ which implies $m^0 \in \tilde{A}$. The theorem is proven for the case when there is no second type yellow interval. Assume now that there are such intervals. Throwing away all the second type yellow intervals from the real line we can regard the rest of the real line as a real line without gaps simply by shifting all the intervals to the left hand side such that after that operation no overlap occurs and there is no gap left. To be more precise, we shift each of the left intervals to the left hand side by a factor equal to the sum of the lengths of the gaps between that interval and $-\infty$. During that operation we unify the black and red intervals with the consecutive intervals of the same color but we regard the possible consecutive first type yellow intervals as separate. We get a situation like above and therefore we can prove the existence of a "good" subsequence. It is easy to show that since that sum of the lengths of the second type yellow intervals in each family is bounded by the same constant then the in Lemma 2.6.1 described limit of the obtained "good" subsequence will belong to $\tilde{A}$ and hence will be an energy minimizer. □

2.7 The Γ-convergence in the first regime

In this section we consider sequences of domain-magnetization-energy triples $(\Omega(l_n, d_n), m^n, E(m^n))$ such that $d_n, l_n \to 0$ and $c_n = \frac{d_n}{l_n} \to c > 0$ as n goes to infinity. we put

$$\acute{E}(m) = \frac{E(m)}{l_n d_n}.$$

For any $n \in \mathbb{N}$ we consider the minimization problem

$$\inf_{m\tilde{A}(l_n,d_n)} \acute{E}(m)$$

instead of the original problem

$$\inf_{m\tilde{A}(l_n,d_n)} E(m),$$

where for the admissible sets we take the sets $\tilde{A}(l_n, d_n)$ and call the new problem "rescaled". We continue with the description of the full and the reduced variational problems. As we have mentioned the full variational problem will be the minimization of the rescaled energy. We will scale the magnetizations in the y and z directions to keep the domain fixed in order to pass to the Γ-limit. We define the rescaled magnetization

$$\acute{m}(x, y, z) = m(x, ly, dz).$$

It is clear that $\acute{m}\colon \Omega(1,1) \to \mathbb{S}^2$. The admissible set for the rescaled variational problem is

$$\tilde{A}_1 = \tilde{A}_1(1,1) = \{\acute{m} \mid m \in \tilde{A}\}.$$

It is apparent that if $\acute{m} \in \tilde{A}_1$ then $\acute{m} - \bar{e} \in H^1(\Omega(1,1))$. The rescaled energy functional will have the form:

$$\acute{E}(\acute{m}) = \acute{E}(m) = \int_{\Omega(1,1)} \Big(|\partial_x \acute{m}(\xi)|^2 + \frac{1}{l^2}|\partial_y \acute{m}(\xi)|^2 + \frac{1}{d^2}|\partial_z \acute{m}(\xi)|^2\Big)\, \mathrm{d}\xi + \frac{1}{ld} E_{mag}(m).$$

The limit variational problem energy functional is given by

$$E_0(m) = \int_{\mathbb{R}} |\partial_x m|^2 \, \mathrm{d}x + \frac{2a_c}{\pi} \int_{\mathbb{R}} |m_y|^2 \, \mathrm{d}x + \frac{2b_c}{\pi} \int_{\mathbb{R}} |m_z|^2 \, \mathrm{d}x,$$

where

$$a_c = \frac{c}{2} \int_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{-2t}{c}}}{t} \, \mathrm{d}t \quad \text{and} \quad b_c = a_{\frac{1}{c}}.$$

The admissible set is

$$\tilde{A}_0 = \{m\colon \mathbb{R} \to \mathbb{S}^2 \mid m - \bar{e} \in H^1(\mathbb{R}) \ \text{ and } \ E_0(m) < \infty\}.$$

Define additionally the following sets:

$$A_0 = \{m\colon \mathbb{R} \to \mathbb{S}^2 \mid E_0(m) < \infty\}$$

and

$$X_0 = \{m\colon \mathbb{R} \to \mathbb{S}^2 \mid \partial_x m \in L^2(\mathbb{R}) \ \text{ and } \ m_y, m_z \in L^2_{loc}(\mathbb{R})\}.$$

The reduced variational problem is to minimize the reduced energy functional E_0 over the admissible set $\tilde{A}_0$. Now we define the notion of convergence of the magnetizations we are going to use for the Γ-convergence of the energies.

Definition 2.7.1. *Let $m^0(x) \in X_0$. Consider a sequence of domain-magnetization pairs (Ω_n, m^n) where $m^n \in \tilde{A}_n$ and define $\acute{m}^n(x,y,z) = m^n(x, l_n y, d_n z)$. Then $\acute{m}^n$ is said to converge to m^0 when n goes to infinity if the following statements hold:*

- $\partial_x \acute{m}^n \rightharpoonup \partial_x m^0$ *weakly in* $L^2(\Omega(1,1))$
- $\nabla_{yz} \acute{m}^n \to 0$ *strongly in* $L^2(\Omega(1,1))$
- $\acute{m}^n \to m^0$ *strongly in* $L^2_{loc}(\Omega(1,1))$

We can now formulate the Γ-convergence result.

Theorem 2.7.2 (Γ-convergence 1)**.** *The reduced variational problem is the Γ-limit of the full variational problem with respect to the convergence defined above. This amounts to the following three statements:*

- ***Lower semicontinuouty*** *If a sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ with entries in $A(l_n, d_n)$ converges to some $m^0 \in X_0$ in the sense of Definition 2.7.1 then*

$$E_0(m^0) \le \liminf_{n\to\infty} \acute{E}_n(\acute{m}^n)$$

- ***Construction*** *For every $m^0 \in \tilde{A}_0$ and every sequence of pairs $(l_n, d_n)_{n\in\mathbb{N}}$ with $l_n, d_n \to 0$, $c_n \to c$ there exists a sequence $(m^n)_{n\in\mathbb{N}}$ with entries in $\tilde{A}(l_n, d_n)$ such that*

$$\acute{m}^n \to m^0 \quad \textit{in the cense of Definition 2.7.1}$$

$$E_0(m^0) = \lim_{n\to\infty} \acute{E}_n(\acute{m}^n)$$

- ***Compactness*** *Let $(l_n, d_n)_{n\in\mathbb{N}}$ be a sequence of pairs such that $l_n, d_n \to 0$ and $c_n \to c > 0$. Let $m^n \in \tilde{A}(l_n, d_n)$ and let $(\acute{E}_n(\acute{m}^n))_{n\in\mathbb{N}}$ be bounded. Then there exists a subsequence of $(m^n)_{n\in\mathbb{N}}$ (not relabeled) such that $\acute{m}^n$ converges to some $m^0 \in \tilde{A}_0$ in the cense of Definition 2.7.1.*

Proof. **Lower semicontinuouty** The proof consists of two steps. In the first step we will prove an equality which allows us to extend (2.37) to the general case, once we know that the rescaled energies are bounded by the same number C. Namely we prove the following: Suppose $\acute{E}_n \leq C$ for all $n \in \mathbb{N}$ then

$$\liminf_{n\to\infty} \frac{E_{mag}(m^n)}{l_n d_n} = \liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{l_n d_n}.$$

According to Lemma 2.4.2 and the Poincaré inequality we have

$$|E_{mag}(m^n) - E_{mag}(\bar{m}^n)| \leq \|m^n - \bar{m}^n\|^2_{L^2(\Omega(l_n,d_n))} + 2\|m^n - \bar{m}^n\|_{L^2(\Omega(l_n,d_n))}\sqrt{E_{mag}(m^n)}$$

$$\leq C\acute{C} l_n d_n (l_n^2 + d_n^2) + 2C l_n d_n \sqrt{\acute{C}(l_n^2 + d_n^2)}$$

thus putting $R_n^2 = l_n^2 + d_n^2$ we obtain

$$\left| \frac{E_{mag}(m^n)}{l_n d_n} - \frac{E_{mag}(\bar{m}^n)}{l_n d_n} \right| \leq C\sqrt{\acute{C}} R_n(\sqrt{\acute{C}} R_n + 2) \to 0 \ \text{ as } \ n \to +\infty. \quad (2.53)$$

In the second step we prove that

$$\liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{l_n d_n} \geq \frac{8}{\pi}\Big(a_c \int_{\mathbb{R}} |m_y^0|^2 \,\mathrm{d}x + b_c \int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x\Big).$$

We have that

$$E_{mag}(\bar{m}^n) \geq E_s(\bar{m}^n) \ \text{ thus } \ \liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{l_n d_n} \geq \liminf_{n\to\infty} \frac{E_s(\bar{m}^n)}{l_n d_n}.$$

We estimate $E_s(\bar{m}^n) - E_s^\star(m^n)$ for big n, where

$$E_s^\star(m^n) = \frac{8}{\pi} l_n d_n \Big(a_{c_0} \int_{\mathbb{R}} |\bar{m}_y^n|^2 \,\mathrm{d}x + b_{c_0} \int_{\mathbb{R}} |\bar{m}_z^n|^2 \,\mathrm{d}x\Big).$$

We fix a positive number ϵ. According to Lemma 2.5.6 there exists a natural number N_ϵ such that when $n > N_\epsilon$ then

$$E_s(\bar{m}^n) \geq \frac{8}{\pi} l_n d_n \Big((a_c - \epsilon) \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} |\widehat{\bar{m}_y^n}(x)|^2 \,\mathrm{d}x + (b_c - \epsilon) \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} |\widehat{\bar{m}_z^n}(x)|^2 \,\mathrm{d}x\Big),$$

thus

$$E_s(\bar{m}^n) - E_s^\star(m^n) \geq -\frac{8}{\pi} l_n d_n \Big(\epsilon \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} |\widehat{\bar{m}_y^n}(x)|^2 \, \mathrm{d}x +$$

$$+ \epsilon \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} |\widehat{\bar{m}_z^n}(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}]} (|\widehat{\bar{m}_y^n}(x)|^2 + |\widehat{\bar{m}_z^n}(x)|^2) \, \mathrm{d}x \Big) =$$

$$= -\frac{8}{\pi} l_n d_n (\epsilon \cdot S_1^n + S_2^n), \tag{2.54}$$

where

$$S_1^n = \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} (|\widehat{\bar{m}_y^n}(x)|^2 + |\widehat{\bar{m}_z^n}(x)|^2) \, \mathrm{d}x \leq \int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2) \, \mathrm{d}x$$

and

$$S_2^n = \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}]} (|\widehat{\bar{m}_y^n}(x)|^2 + |\widehat{\bar{m}_z^n}(x)|^2) \, \mathrm{d}x$$

$$\leq l_n \int_{\mathbb{R} \setminus [-\frac{1}{\sqrt{l_n}}, \frac{1}{\sqrt{l_n}}]} (|x \cdot \widehat{\bar{m}_y^n}(x)|^2 + |x \cdot \widehat{\bar{m}_z^n}(x)|^2) \, \mathrm{d}x$$

$$\leq l_n \int_{\mathbb{R}} (|x \cdot \widehat{\bar{m}_y^n}(x)|^2 + |x \cdot \widehat{\bar{m}_z^n}(x)|^2) \, \mathrm{d}x = l_n \int_{\mathbb{R}} (|\partial_x \bar{m}_y^n(x)|^2 + |\partial_x \bar{m}_z^n(x)|^2) \, \mathrm{d}x$$

$$= \frac{1}{d_n} \int_{\Omega(l_n, d_n)} (|\partial_x \bar{m}_y^n(x)|^2 + |\partial_x \bar{m}_z^n(x)|^2) \, \mathrm{d}\xi \tag{2.55}$$

We estimate now

$$\frac{1}{d_n} \int_{\Omega(l_n, d_n)} (|\partial_x \bar{m}_y^n(x)|^2 + |\partial_x \bar{m}_z^n(x)|^2) \, \mathrm{d}\xi.$$

Note that for any $m \in A$ and $x \in \mathbb{R}$

$$\partial_x \bar{m}(x) = \frac{1}{4 l_n d_n} \int_{R(l_n, d_n)} \partial_x m(x, y, z) \, \mathrm{d}y \, \mathrm{d}z$$

thus

$$|\partial_x \bar{m}(x)|^2 \leq \frac{1}{4 l_n d_n} \int_{R(l_n, d_n)} |\partial_x m(x, y, z)|^2 \, \mathrm{d}y \, \mathrm{d}z.$$

Integrating the last inequality over $\mathbb{R}$ we get

$$\int_{\Omega(l_n, d_n)} |\partial_x \bar{m}|^2 \, \mathrm{d}\xi \leq \int_{\Omega(l_n, d_n)} |\partial_x m|^2 \, \mathrm{d}\xi. \tag{2.56}$$

Utilizing (2.55) and (2.56) we get for S_2^n the following

$$S_2^n \le \frac{1}{d_n}\int_{\Omega(l_n,d_n)} |\partial_x m^n|^2 \,\mathrm{d}\xi \le \frac{1}{d_n}E_{ex}(m^n) \le Cl_n \to 0.$$

It remainins to show that the sequence

$$\Big(\int_{\mathbb{R}} (|\bar{m}_y^n|^2 + |\bar{m}_z^n|^2)\,\mathrm{d}x\Big)_{n\in\mathbb{N}}$$

is bounded. Recall again Lemma 2.5.6. If we take $\epsilon = \min(\frac{a_c}{2}, \frac{b_c}{2})$ then for $n > N_\epsilon$ we have

$$E_{mag}(\bar{m}^n) \ge E_s(\bar{m}^n) \ge \frac{8}{\pi}\epsilon l_n d_n \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} \big(|\widehat{\bar{m}_y^n}(x)|^2 + |\widehat{\bar{m}_z^n}(x)|^2 \,\mathrm{d}x.$$

Now using (2.53) we obtain

$$E_{mag}(m^n) \ge \frac{8}{\pi}\epsilon l_n d_n \int_{-\frac{1}{\sqrt{l_n}}}^{\frac{1}{\sqrt{l_n}}} \big(|\widehat{\bar{m}_y^n}(x)|^2 + |\widehat{\bar{m}_z^n}(x)|^2\big)\,\mathrm{d}x - C\sqrt{\acute{C}}R_n(\sqrt{\acute{C}}R_n + 2). \tag{2.57}$$

We also have

$$E_{ex}(m^n) \ge \int_{\Omega(l_n,d_n)} |\partial_x m^n|^2 \,\mathrm{d}\xi \ge \int_{\Omega(l_n,d_n)} |\partial_x \bar{m}^n|^2 \,\mathrm{d}\xi$$

$$\ge 4l_n d_n \int_{\mathbb{R}} \big(|\partial_x \bar{m}_y^n|^2 + |\partial_x \bar{m}_z^n|^2\big)\,\mathrm{d}x = 4l_n d_n \int_{\mathbb{R}} \big(|x\cdot\widehat{\bar{m}_y^n}|^2 + |x\cdot\widehat{\bar{m}_z^n}|^2\big)\,\mathrm{d}x$$

$$\ge 4l_n d_n \int_{\mathbb{R}\setminus\big[-\frac{1}{\sqrt{l_n}},\frac{1}{\sqrt{l_n}}\big]} \big(|x\cdot\widehat{\bar{m}_y^n}|^2+|x\cdot\widehat{\bar{m}_z^n}|^2\big)\,\mathrm{d}x \ge 4d_n \int_{\mathbb{R}\setminus\big[-\frac{1}{\sqrt{l_n}},\frac{1}{\sqrt{l_n}}\big]} \big(|\widehat{\bar{m}_y^n}|^2+|\widehat{\bar{m}_z^n}|^2\big)\,\mathrm{d}x \tag{2.58}$$

Finally utilizing (2.57) and (2.58) we obtain

$$\int_{\mathbb{R}} (|\bar{m}_y^n|^2 + |\bar{m}_z^n|^2)\,\mathrm{d}x = \int_{\mathbb{R}} (|\widehat{\bar{m}_y^n}|^2 + |\widehat{\bar{m}_z^n}|^2)\,\mathrm{d}x$$

$$\le \frac{E_{ex}(m^n)}{4d_n} + \frac{E_{mag}(m^n)\cdot\pi}{8\epsilon l_n d_n} + \frac{\pi C\sqrt{\acute{C}}R_n(\sqrt{\acute{C}}R_n+2)}{8\epsilon}$$

$$\le C\Big(\frac{l_n}{4} + \frac{\pi}{8\epsilon}\Big) + \frac{\pi C\sqrt{\acute{C}}R_n(\sqrt{\acute{C}}R_n+2)}{8\epsilon} \tag{2.59}$$

we used the uniformly boundedness of the rescaled energies. Inequality (2.59) shows that the sequence $(S_1^n)_{n\in\mathbb{N}}$ is bounded. Concluding we have that since in (2.54) ϵ was arbitrary then the following inequality holds:

$$\liminf_{n\to\infty}\frac{E_s(\bar{m}^n)}{l_n d_n}\geq \liminf_{n\to\infty}\frac{E_s^\star(\bar{m}^n)}{l_n d_n}.$$

We would like now to show that

$$\liminf_{n\to\infty}\frac{E_s^\star(\bar{m}^n)}{l_n d_n}\geq \frac{8}{\pi}\Big(a_c\int_{\mathbb{R}}|m_y^0|^2\,\mathrm{d}x+b_c\int_{\mathbb{R}}|m_z^0|^2\,\mathrm{d}x\Big).$$

We fix a natural number N. Since $(\acute{m}^n)_{n\in\mathbb{N}}$ tends to m^0 in $L^1_{loc}(\Omega(1,1))$ we have

$$\int_{[-N,N]\times R(1,1)}|\acute{m}_y^n(x,y,z)-m_y^0(x)|^2\,\mathrm{d}\xi\to 0$$

which is equivalent to

$$\frac{1}{4l_n d_n}\int_{[-N,N]\times R(l_n,d_n)}|m_y^n(x,y,z)-m_y^0(x)|^2\,\mathrm{d}\xi\to 0 \quad \text{so}$$

$$\|m_y^n-m_y^0\|_{L^2([-N,N]\times R(l_n,d_n))}=o(\sqrt{l_n d_n}) \text{ as } n \text{ tends to infinity.}$$

We have already seen as well

$$\|m_y^n-\bar{m}_y^n\|_{L^2([-N,N]\times R(l_n,d_n))}\leq\|m_y^n-\bar{m}_y^n\|_{L^2(\Omega(l_n,d_n)}=o(\sqrt{l_n d_n}) \quad \text{thus}$$

$$\|\bar{m}_y^n-m_y^0\|_{L^2([-N,N]\times R(l_n,d_n))}=o(\sqrt{l_n d_n}) \quad \text{and this is equivalent to}$$

$\|\bar{m}_y^n-m_y^0\|_{L^2[-N,N]}=o(1)$ which itself yields

$$\liminf_{n\to\infty}\int_{\mathbb{R}}|\bar{m}_y^n|^2\,\mathrm{d}x\geq\liminf_{n\to\infty}\int_{[-N,N]}|\bar{m}_y^n|^2\,\mathrm{d}x\geq\int_{[-N,N]}|m_y^0|^2\,\mathrm{d}x.$$

Since N was arbitrary we obtain

$$\liminf_{n\to\infty}\int_{\mathbb{R}}|\bar{m}_y^n|^2\,\mathrm{d}x\geq\int_{\mathbb{R}}|m_y^0|^2\,\mathrm{d}x.$$

Similarly we can get the same inequality for $\bar{m}_z^n$. We can estimate now

$$\liminf_{n\to\infty}\frac{E_s^\star(\bar{m}^n)}{l_n d_n}=\frac{8}{\pi}\liminf_{n\to\infty}\Big(a_{c_0}\int_{\mathbb{R}}|\bar{m}_y^n|^2\,\mathrm{d}x+b_{c_0}\int_{\mathbb{R}}|\bar{m}_z^n|^2\,\mathrm{d}x\Big)$$

$$\geq\frac{8}{\pi}\liminf_{n\to\infty}a_{c_0}\int_{\mathbb{R}}|\bar{m}_y^n|^2\,\mathrm{d}x+\frac{8}{\pi}\liminf_{n\to\infty}b_{c_0}\int_{\mathbb{R}}|\bar{m}_z^n|^2\,\mathrm{d}x$$

$$\geq \frac{8}{\pi}\Big(a_{c_0}\int_{\mathbb{R}}|m_y^0|^2\,\mathrm{d}x + b_{c_0}\int_{\mathbb{R}}|m_z^0|^2\,\mathrm{d}x\Big)$$

which completes the proof of the second step. In the third step we prove that

$$\liminf_{n\to\infty}\frac{E_{ex}(m^n)}{l_n d_n} \geq 4\int_R |\partial_x m^0|^2\,\mathrm{d}x.$$

The weak convergence $\partial_x \acute{m}^n \rightharpoonup \partial_x m^0$ in $L^2(\Omega(1,1))$ yields the lower semi-continuity of the norms, i.e.,

$$\liminf_{n\to\infty}\int_{\Omega(1,1)}|\partial_x \acute{m}^n|^2\,\mathrm{d}\xi \geq \int_{\Omega(1,1)}|\partial_x m^0|^2\,\mathrm{d}\xi$$

but the exchange energy can be represented as follows

$$E_{ex}(m^n) = l_n d_n\Big(\int_{\Omega(1,1)}|\partial_x \acute{m}^n|^2\,\mathrm{d}\xi + \frac{1}{l_n^2}\int_{\Omega(1,1)}|\partial_y \acute{m}^n|^2\,\mathrm{d}\xi + \frac{1}{d_n^2}\int_{\Omega(1,1)}|\partial_z \acute{m}^n|^2\,\mathrm{d}\xi\Big)$$

$$\geq l_n d_n\int_{\Omega(1,1)}|\partial_x \acute{m}^n|^2\,\mathrm{d}\xi$$

thus

$$\liminf_{n\to\infty}\frac{E_{ex}(m^n)}{l_n d_n} \geq \liminf_{n\to\infty}\int_{\Omega(1,1)}|\partial_x \acute{m}^n|^2\,\mathrm{d}\xi \geq \int_{\Omega(1,1)}|\partial_x m^0|^2\,\mathrm{d}\xi = 4\int_{\mathbb{R}}|\partial_x m^0|^2\,\mathrm{d}x.$$

Construction We simply prove that the constant sequence

$$m^n(\xi) = m^0(x) \ \text{ if } \ \xi\in\Omega(l_n,d_n) \ \text{ and } m^n(\xi)=0 \ \text{ if } \xi\in\mathbb{R}^3\setminus\Omega(l_n,d_n)$$

satisfies the required condition. First of all note that by Corollary 2.5.3 $m^n \in A(l_n,d_n)$ and since $m^n - \bar{e} \in H^1(\Omega(l_n,d_n))$ then $m^n \in \tilde{A}(l_n,d_n)$. According to the "lower semi-continuity" part of the foregoing theorem we have that

$$E_0(m^0) \leq \liminf_{n\to\infty}\acute{E}_n(m^n),$$

thus it remains to only prove the opposite inequality. It is clear furthermore that

$$E(m^n) = E_{ex}(m^n) + E_{mag}(m^n) = \int_{\Omega(l_n,d_n)}|\partial_x m^0|^2\,\mathrm{d}\xi + E_{mag}(m^n)$$

$$= 4l_n d_n\int_{\mathbb{R}}|\partial_x m^0|^2\,\mathrm{d}x + E_{mag}(m^n)$$

so it remains to prove that

$$\limsup_{n\to\infty}\frac{E_{mag}(m^n)}{l_nd_n}\leq\frac{8}{\pi}\Big(a_c\int_{\mathbb{R}}|m_y^0|^2\,\mathrm{d}x+b_c\int_{\mathbb{R}}|m_z^0|^2\,\mathrm{d}x\Big).$$

According to Lemma 2.5.6 for any $\epsilon>0$ there exists an $N_\epsilon\in\mathbb{N}$ such that when $n>N_\epsilon$ then

$$\begin{aligned}E_s(m^n)&\leq\frac{8}{\pi}l_nd_n\Big[(a_c+\epsilon)\int_{\mathbb{R}}|\hat{m}_y^n(x)|^2\,\mathrm{d}x+(b_c+\epsilon)\int_{\mathbb{R}}|\hat{m}_z^n(x)|^2\,\mathrm{d}x\Big]\\&=\frac{8}{\pi}l_nd_n\Big[(a_c+\epsilon)\int_{\mathbb{R}}|m_y^n(x)|^2\,\mathrm{d}x+(b_c+\epsilon)\int_{\mathbb{R}}|m_z^n(x)|^2\,\mathrm{d}x\Big]\\&=\frac{8}{\pi}l_nd_n\Big[(a_c+\epsilon)\int_{\mathbb{R}}|m_y^0(x)|^2\,\mathrm{d}x+(b_c+\epsilon)\int_{\mathbb{R}}|m_z^0(x)|^2\,\mathrm{d}x\Big].\end{aligned}$$

Since ϵ was arbitrary we obtain

$$\limsup_{n\to\infty}\frac{E_s(m^n)}{l_nd_n}\leq\frac{8}{\pi}\Big[a_c\int_{\mathbb{R}}|m_y^0(x)|^2\,\mathrm{d}x+b_c\int_{\mathbb{R}}|m_z^0(x)|^2\,\mathrm{d}x\Big].$$

We show as well that

$$\limsup_{n\to\infty}\frac{E_v(m^n)}{l_nd_n}=0.$$

To that end we invoke Lemma 2.5.11. It is now clear that

$$\limsup_{n\to\infty}\frac{E_v(m^n)}{l_nd_n}\leq M_{m_0}\limsup_{n\to\infty}d_n(l_n+1+\ln l_n-\ln d_n)=0$$

because $l_n\to 0$ and $d_n\to 0$. The proof of the construction part is complete. We proceed now to the compactness part.

Compactness. Assume $m^n\in A(l_n,d_n)$, $l_n\to 0$, $\frac{d_n}{l_n}\to c>0$. Without loss of generality one can assume that

$$\liminf_{n\to\infty}\frac{E(m^n)}{l_nd_n}=\lim_{n\to\infty}\frac{E(m^n)}{l_nd_n}=C<\infty. \tag{2.60}$$

We are going to use now the relatively compactness of $(m^n)_{n\in\mathbb{N}}$ coupled with the idea of constructing a "good" subsequence without changing the energies to ensure that the limit function m^0 would belong to $\tilde{A}_0$. We have that

$$\acute{E}(m^n)=\int_{\Omega(1,1)}|\partial_x\acute{m}^n|^2\,\mathrm{d}\xi+\frac{1}{l_n^2}\int_{\Omega(1,1)}|\partial_y\acute{m}^n|^2\,\mathrm{d}\xi+\frac{1}{d_n^2}\int_{\Omega(1,1)}|\partial_z\acute{m}^n|^2\,\mathrm{d}\xi+\frac{E_{mag}}{l_nd_n}$$

hence for sufficiently big n we have

$$\|\partial_x\acute{m}^n\|^2_{L^2(\Omega(1,1))}\leq C+1,$$

$$\|\partial_y \acute{m}^n\|^2_{L^2(\Omega(1,1))} \le (C+1) l_n^2 \to 0 \quad \text{and} \quad \|\partial_z \acute{m}^n\|^2_{L^2(\Omega(1,1))} \le (C+1) d_n^2 \to 0. \tag{2.61}$$

Like in Lemma 2.6.1 one can prove that the sequence $(\acute{m}^n)_{n\in\mathbb{N}}$ is relatively compact with respect to the convergence defined in Definition 2.7.1, thus it remains to construct a subsequence which has the limit function in $\tilde{A}_0$. If we remember the proof of the existence lemma we will see that the key point to the existence of a "good" subsequence is inequality (2.23). Moreover it does not matter if the domain Ω is fixed or not, the point is that (2.23) is valid with a constant M_2 not depending on n. Therefore in order to be able to prove the existence of a "good" subsequence we have to show that inequality (2.23) holds for any l_n, d_n, m^n, $E(m^n)$ with M_2 not depending on n. We invoke (2.59) to have

$$\int_{\mathbb{R}} (|m_y^n|^2 + |m_z^n|^2)\,\mathrm{d}x \le C_1\Big(\frac{l_n}{4} + \frac{\pi}{8\epsilon}\Big) + \frac{\pi C\sqrt{\acute{C}}R_n(\sqrt{\acute{C}}R_n + 2)}{8\epsilon} \le C_2, \quad n \in \mathbb{N}$$

where C_2 is a constant. With this new definition of the constant C_2 inequality (2.19) will have the form

$$\sum_{(a,b)\in\mathfrak{R}} (b-a) \le \frac{C_2 l_n d_n + \acute{C}R_n^2 E(m^n)}{4 l_n d_n (1-\rho^2)} \quad \text{for all} \quad n \in \mathbb{N}$$

and (2.23) will have the form

$$\sum_{(a,b)\in\mathfrak{R}} \Big(\frac{1}{b-a} + b - a\Big) \le \frac{1}{4 l_n d_n}\Big(\frac{E(m^n)}{(\alpha-\beta)^2} + \frac{C_2 l_n d_n + \acute{C}R_n^2 E(m^n)}{1-\rho^2}\Big) \text{ for all } n \in \mathbb{N} \tag{2.62}$$

Coupling now (2.60) and (2.62) we obtain for sufficiently big n

$$\sum_{(a,b)\in\mathfrak{R}} \Big(\frac{1}{b-a} + b - a\Big) \le \frac{1}{4}\Big(\frac{C+1}{(\alpha-\beta)^2} + \frac{C_2+1}{1-\rho^2}\Big) \tag{2.63}$$

which was supposed to be proven. Thus we can assume that the sequence $(\acute{m}^n)_{n\in\mathbb{N}}$ is "good". Using the relatively compactness of $(\acute{m}^n)_{n\in\mathbb{N}}$ and (2.61) we obtain that a subsequence (not relabeled) converges to some $m^0 \in X_0$ in the sense of Definition 2.7.1, thus we can as well apply the "lower semi-continuity" part of the foregoing theorem to discover

$$E_0(m^0) = 4\int_{\mathbb{R}} |\partial_x m^0|^2\,\mathrm{d}x + \frac{8}{\pi} a_c \int_{\mathbb{R}} |m_y^0|^2\,\mathrm{d}x + \frac{8}{\pi} b_c \int_{\mathbb{R}} |m_z^0|^2\,\mathrm{d}x$$

$$\leq \liminf \frac{E(m^n)}{l_n d_n} = C,$$

which this yields that $m^0 \in A_0$. Since $(\acute{m}^n)_{n\in\mathbb{N}}$ is "good" m^0 must belong to $\tilde{A}_0$.

□

2.8 The minimal energy scaling

2.8.1 The minima of the limit energy

In this section we recall how one can determine the minima of the energy functional

$$E_\alpha(m) = \int_{\mathbb{R}} |\partial_x m(x)|^2 \,\mathrm{d}x + \alpha \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) \,\mathrm{d}x$$

where $\alpha > 0$ and the admissible set is

$$\tilde{A}_0 = \{m \colon \mathbb{R} \to \mathbb{R}^3 \mid |m| = 1, m - \bar{e} \in H^1(\mathbb{R})\}.$$

It is well known that the minimal value of $E_\alpha(m)$ is positive and attained in $\tilde{A}_0$. Remark 2.4.6 states that if $m \in \tilde{A}_0$ and depends only on x then m_x should tend to -1 and $+1$ respectively at $-\infty$ and $+\infty$. Therefore we can parameterize m in the following way:

$$\begin{cases} m_x(x) = \sin\varphi(x) \\ m_y(x) = \cos\varphi(x)\cos\theta(x) \\ m_z(x) = \cos\varphi(x)\sin\theta(x) \end{cases} \tag{2.64}$$

where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in [0, 2\pi)$ and $\varphi(x) \to \pm\frac{\pi}{2}$ as $x \to \pm\infty$. It is clear that

$$\begin{aligned} E_\alpha(m) &= \int_{\mathbb{R}} \varphi'^2(x) + \theta'^2(x)\cos^2\varphi(x) \,\mathrm{d}x + \alpha \int_{\mathbb{R}} \cos^2\varphi(x) \,\mathrm{d}x \\ &\geq \int_{\mathbb{R}} \varphi'^2(x) \,\mathrm{d}x + \alpha \int_{\mathbb{R}} \cos^2\varphi(x) \,\mathrm{d}x \\ &\geq 2\sqrt{\alpha} \int_{\mathbb{R}} |\varphi'(x)||\cos\varphi(x)| \,\mathrm{d}x \\ &\geq 2\sqrt{\alpha} \int_{\mathbb{R}} \varphi'(x)\cos\varphi(x) \,\mathrm{d}x \end{aligned}$$

$$= 2\sqrt{\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \,\mathrm{d}t = 4\sqrt{\alpha}$$

and the equality holds if and only if the following conditions hold:

$$\varphi'^2(x) = \alpha\cos^2\varphi(x), \;\; \varphi'(x)\cos\varphi(x) \geq 0 \;\text{ and }\; \theta'(x)\cos^2\varphi(x) = 0 \;\text{ for all }\; x \in \mathbb{R}. \tag{2.65}$$

Note that the first two conditions in (2.65) yield

$$\varphi\prime(x) = \sqrt{\alpha}\cos\varphi \;\;\text{for all}\;\; x \in \mathbb{R}$$

which has the only solution

$$\varphi_{\alpha,\beta} = \arcsin\frac{e^{2\sqrt{\alpha}x}\cdot\beta - 1}{e^{2\sqrt{\alpha}x}\cdot\beta + 1}, \quad \text{where} \;\; \beta > 0.$$

Note furthermore that $\cos\varphi_{\alpha,\beta}$ does not vanish, thus the third condition in (2.65) implies $\theta \equiv \text{const}$. Is is clear that $\varphi_{\alpha,\beta} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\varphi_{\alpha,\beta}(x) \to \pm\frac{\pi}{2}$ as $x \to \pm\infty$ for any $\alpha, \beta > 0$. We denote $\varphi_\alpha = \varphi_{\alpha,1}$ and $m^\alpha = m(\varphi_\alpha)$. The minimal value of E_α in $\tilde{A}_0$ will be $4\sqrt{\alpha}$.

Remark 2.8.1. *Neither the minimal energy(the infima of the energy) nor the second summand of the energy depend on the constant θ.*

2.8.2 The minimal energy scaling

In this subsection we determine the minimal energy scaling when l and d are small enough. We consider a sequence of domain-magnetization-energy triples $(\Omega(l_n, d_n), m^n_{min}, E(m^n_{min}))_{n\in\mathbb{N}}$, where m^n_{min} is a minimizer of the energy functional in $\tilde{A}(l_n, d_n)$. We would like to find the scaling of $E_{min}(l_n, d_n) = E(m^n_{min})$ in terms of l_n and d_n. We will show that the minimal energy scales like $l_n \cdot d_n$. We have for the limit energy functional

$$E_0(m^0) = 4\int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x + \frac{8}{\pi}\Big(a_c\int_{\mathbb{R}} |m^0_y|^2 \,\mathrm{d}x + b_c\int_{\mathbb{R}} |m^0_z|^2 \,\mathrm{d}x\Big),$$

where $c = \lim_{n\to\infty}\frac{d_n}{l_n} \leq 1 \leq \frac{1}{c}$. We will show later that a_c is increasing on $(0, +\infty)$ thus $a_c \leq b_c$. As we saw in the preceding section the limit energy can be estimated from below in the following way:

$$E_0(m^0) \geq 4\Big(\int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x + \frac{2a_c}{\pi}\int_{\mathbb{R}} (|m^0_y| + |m^0_z|^2) \,\mathrm{d}x\Big) \geq \frac{16\sqrt{2a_c}}{\sqrt{\pi}}. \tag{2.66}$$

In is clear that we have equalities in (2.62) if and only if when

$$c = 1, \quad m^0 = m^\alpha \text{ with } \alpha = \frac{2a_1}{\pi} \text{ and } \theta \equiv \text{const}$$

or

$$c < 1, \quad m^0 = m^\alpha \quad \text{with} \quad \alpha = \frac{2a_{c_0}}{\pi} \quad \text{and} \quad \theta \equiv 0.$$

Hence we establish that the infimum of the limit energy E_0 is attained and equals $\frac{16\sqrt{2a_c}}{\sqrt{\pi}}$. We already showed in the "construction" part of Γ-convergence Theorem 2.7.2 that for the constant sequence

$$m^n(\xi) = m^\alpha(x) \text{ in } \Omega(l_n, d_n) \text{ and } m^n(\xi) = 0 \text{ in } \mathbb{R}^3 \setminus \Omega(l_n, d_n)$$

the sequence of the corresponding energies satisfies the condition

$$\limsup_{n\to\infty} \frac{E(m_n)}{l_n d_n} \le E_0(m^\alpha) = \frac{16\sqrt{2a_c}}{\sqrt{\pi}}$$

which implies the same bound for the minimal energies:

$$\limsup_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} \le \frac{16\sqrt{2a_c}}{\sqrt{\pi}}. \tag{2.67}$$

Assume now $(m^n)_{n\in\mathbb{N}}$ is any sequence of magnetizations with $m^n \in \tilde{A}(l_n, d_n)$. We will show that

$$\liminf_{n\to\infty} \frac{E(m^n)}{l_n d_n} \ge \frac{16\sqrt{2a_c}}{\sqrt{\pi}}.$$

Without loss of generality one can assume that

$$\liminf_{n\to\infty} \frac{E(m^n)}{l_n d_n} = \lim_{n\to\infty} \frac{E(m^n)}{l_n d_n} < \infty.$$

According Theorem 2.7.2 we have the a subsequence of $(\acute{m}^n)_{n\in\mathbb{N}}$ converges to some $m^0 \in \tilde{A}_0$, therefore using once again Theorem 2.7.2 we establish

$$\frac{16\sqrt{2a_c}}{\sqrt{\pi}} \le E_0(m^0) \le \limsup_{n\to\infty} \frac{E(m^n)}{l_n d_n} = \liminf_{n\to\infty} \frac{E(m^n)}{l_n d_n}$$

and this completes the proof. Summarizing the obtained inequalities we obtain

$$\frac{16\sqrt{2a_c}}{\sqrt{\pi}} \le \liminf_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} \le \limsup_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} \le \frac{16\sqrt{2a_c}}{\sqrt{\pi}}$$

hence

$$\lim_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} = \frac{16\sqrt{2a_c}}{\sqrt{\pi}}. \tag{2.68}$$

2.9 The Γ-convergence and the minimal energy scaling in the second regime

2.9.1 An estimate on the energy scaling

In this subsection we study the case $c = 0$. Like in the previous case we consider a sequence of domain-magnetization-energy triples $(\Omega(l_n, d_n), m^n, E(m^n))$ for which all of the parameters l_n, d_n and $c_n = \frac{d_n}{l_n}$ tend to zero as n goes to infinity. In the first step we show that the minimal energies decay faster than $l_n d_n$ as n goes to infinity. To that end we fix a magnetization m^0 such that

$$\partial_x m^0, m_y^0, m_z^0 \in L^2(\mathbb{R}).$$

We show that the constant sequence $m^n \equiv m^0$ satisfies the following condition

$$\limsup_{n\to\infty} \frac{E(m^n)}{l_n d_n} \leq 4\int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x + 4\int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x.$$

It is clear that

$$E_{ex}(m^n) = 4 l_n d_n \int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x$$

thus it remains to prove that

$$\limsup_{n\to\infty} \frac{E_{mag}(m^n)}{l_n d_n} \leq 4\int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x.$$

We will prove it by showing that

$$\limsup_{n\to\infty} \frac{E_v(m^n)}{l_n d_n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{E_s(m^n)}{l_n d_n} \leq 4\int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x.$$

According to Lemma 2.5.6 we have that

$$I(l_n, d_n, x) \leq \pi^2 l_n d_n \quad \text{and} \quad I(d_n, l_n, x) \leq \pi(1+\pi) l_n d_n \sqrt{c_n} \quad \text{for all} \quad x \in \mathbb{R}.$$

This implies the following bound

$$\begin{aligned} E_s(m^n) &\leq l_n d_n \Big(4\int_{\mathbb{R}} |\widehat{\bar{m}_z^0}|^2 \,\mathrm{d}x + \frac{4(1+\pi)}{\pi}\sqrt{c_n}\int_{\mathbb{R}} |\widehat{\bar{m}_y^0}|^2 \,\mathrm{d}x\Big) \\ &= l_n d_n \Big(4\int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x + \frac{4(1+\pi)}{\pi}\sqrt{c_n}\int_{\mathbb{R}} |m_y^0|^2 \,\mathrm{d}x\Big), \end{aligned}$$

hence

$$\limsup_{n\to\infty} \frac{E_s(m^n)}{l_n d_n} \leq 4\int_{\mathbb{R}} |m_z^0|^2 \,\mathrm{d}x.$$

We have furthermore by Lemma 2.5.11

$$\limsup_{n\to\infty} \frac{E_v(\bar{m}^n)}{l_n d_n} \le \limsup_{n\to\infty}(l_n d_n + d_n(1+\ln l_n - \ln d_n))M_{m^0} = 0.$$

Consider now a sequence of domain-(minimal energy) pairs $(\Omega(l_n, d_n), E_{min}(l_n, d_n))$. Let ϵ be any positive number. We choose the angle θ for m^ϵ such that $m_z^\epsilon \equiv 0$, i.e., $\theta \equiv \frac{\pi}{2}$. We have that

$$E_\epsilon(m^\epsilon) = \int_{\mathbb{R}} |\partial_x m^\epsilon|^2 \,\mathrm{d}x + \epsilon \int_{\mathbb{R}} (|m_y^\epsilon|^2 + |m_z^\epsilon|^2) \,\mathrm{d}x = 4\sqrt{\epsilon}.$$

As we have proven for the constant sequence $m^n \equiv m^\epsilon$ the following inequality holds:

$$\limsup_{n\to\infty} \frac{E(m^n)}{l_n d_n} \le 4\int_{\mathbb{R}} |\partial_x m^\epsilon|^2 \,\mathrm{d}x + 4\int_{\mathbb{R}} |m_z^\epsilon|^2 \,\mathrm{d}x$$

$$= 4\int_{\mathbb{R}} |\partial_x m^\epsilon|^2 \,\mathrm{d}x \le 4\Big(\int_{\mathbb{R}} |\partial_x m^\epsilon|^2 \,\mathrm{d}x + \epsilon \int_{\mathbb{R}} (|m_y^\epsilon|^2 + |m_z^\epsilon|^2) \,\mathrm{d}x\Big) = 16\sqrt{\epsilon},$$

thus

$$\limsup_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} \le \limsup_{n\to\infty} \frac{E(m_n)}{l_n d_n} \le 16\sqrt{\epsilon}.$$

Since ϵ was arbitrary we obtain

$$\lim_{n\to\infty} \frac{E_{min}(l_n, d_n)}{l_n d_n} = 0. \tag{2.69}$$

This equality motivates us to rescale the sequence of magnetizations not only in the directions y and z but also in the x direction. Adopting that strategy we first establish a Γ-convergence on the energies and then we determine the minimal energy scaling. In the next section we observe some properties of the function

$$a_c = \frac{c}{2}\int_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1 - e^{-\frac{2t}{c}}}{t} \,\mathrm{d}t.$$

2.9.2 An observation on the function a_c

We consider $c \to a_c$ as a map from $(0, +\infty)$ to $(0, +\infty)$.

Lemma 2.9.1. *The function a_c has the following properties:*

- *a_c increases in $(0, +\infty)$*
- $\lim_{c\to 0} \frac{a_c}{c|\ln c|} = \frac{1}{2}$

- $\lim_{c\to+\infty} a_c = \frac{\pi}{2}$.

Let c_1 and c_2 be two positive numbers with $c_1 > c_2$. Since the function $f(t) = \frac{1-e^{-t}}{t}$ decreases in $(0,+\infty)$ we have

$$a_{c_1} = \int_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c_1}}}{\frac{2t}{c_1}} \,\mathrm{d}t \geq \int_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c_2}}}{\frac{2t}{c_2}} \,\mathrm{d}t = a_{c_2},$$

which is the first property. The second property is Corollary 2.5.9. To prove the third property we utilize (2.39). Assume now $c \geq 4$. We have that

$$\frac{1-e^{-\frac{2t}{c}}}{\frac{2t}{c}} \geq 1 - \frac{t}{c} \quad \text{if} \quad t \in \Big[0, \frac{c}{2}\Big] \quad \text{thus}$$

$$\frac{1-e^{-\frac{2t}{c}}}{\frac{2t}{c}} \geq 1 - \frac{t}{c} \geq 1 - \frac{1}{\sqrt{c}} \quad \text{if} \quad t \in [0, \sqrt{c}] \text{ (note that } \sqrt{c} \leq \frac{c}{2}).$$

Therefore for a_c we have on one hand

$$\liminf_{n\to\infty} a_c \geq \liminf_{n\to\infty} \Big(1 - \frac{1}{\sqrt{c}}\Big) \int_0^{\sqrt{c}} \frac{\sin^2 t}{t^2} \,\mathrm{d}t = \int_0^{+\infty} \frac{\sin^2 t}{t^2} \,\mathrm{d}t = \frac{\pi}{2},$$

but on the other hand

$$a_c \leq \int_0^{+\infty} \frac{\sin^2 t}{t^2} \,\mathrm{d}t = \frac{\pi}{2} \quad \text{for any} \quad c > 0.$$

The last two inequalities complete the proof.

2.9.3 The Γ-convergence

First of all we show how one can guess the scaling of the minimal energies $E_{min}(l_n, d_n)$ where $l_n, d_n, c_n \to 0$. As we have seen for sufficiently big n one can formally write

$$E_s(m^n) \approx \frac{8}{\pi} l_n d_n a_{c_n} \int_{\mathbb{R}} |m_y^n(x)|^2 \,\mathrm{d}x + \frac{8}{\pi} l_n d_n b_{c_n} \int_{\mathbb{R}} |m_z^n(x)|^2 \,\mathrm{d}x$$

We know that a_{c_n} scales like $c_n \ln c_n$ and $b_{c_n} \to \frac{\pi}{2}$. Furthermore, for a fixed $m^n = m^n(x)$ the summand $E_v(m^n)$ decays not slower than $l_n d_n^2 \ln^2 \frac{l_n}{d_n}$. We blow up m^n by a factor λ_n in the x direction where $\lambda_n \to +\infty$ and denote the blown up function by $\acute{m}^n$. We have

$$E_{ex}(m^n) = \frac{l_n d_n}{\lambda_n} \int_{\Omega(1,1)} \Big(|\partial_x \acute{m}^n|^2 + \frac{\lambda_n^2}{l_n^2} |\partial_y \acute{m}^n|^2 + \frac{\lambda_n^2}{d_n^2} |\partial_z \acute{m}^n|^2 \Big) \,\mathrm{d}t,$$

thus

$$E_{ex}(m^n) \approx \frac{4l_n d_n}{\lambda_n}\int_{\mathbb{R}} |\partial_x m^n|^2 \,\mathrm{d}x$$

and

$$E_s(m^n) \approx \frac{4}{\pi} l_n d_n c_n |\ln c_n|\lambda_n \int_{\mathbb{R}} (|m_y^n(x)|^2 + \frac{\pi}{c_n|\ln c_n|}|m_z^n(x)|^2)\,\mathrm{d}x.$$

It is now clear that the coefficients $\frac{l_n d_n}{\lambda_n}$ and $l_n d_n c_n|\ln c_n|\lambda_n$ should be taken equal and they will both be the scaling of $E(m^n)$. This yields $\lambda_n = \frac{1}{\sqrt{c_n|\ln c_n|}}$ and we set $\mu_n = \frac{l_n d_n}{\lambda_n}$. We proceed to do justification on this reasoning.

Like in the previous cases we consider the full minimization problem

$$\inf_{m\in\tilde{A}(l,d)} \acute{E}(\acute{m}) \quad \text{where} \quad \acute{E}(\acute{m}) = \frac{\lambda}{ld}E(m)$$

and l and d are small enough. It is clear that the admissible set will be

$$\tilde{A}_1(l,d) = \{\acute{m} \mid m \in \tilde{A}(l,d)\}.$$

We define as well the reduced energy functional E_0 and the admissible set $\tilde{A}_0$ for the reduced variational problem. We set

$$E_0(m^0) = \begin{cases} 4\int_{\mathbb{R}} |\partial_x m^0|^2\,\mathrm{d}x + \frac{4}{\pi}\int_{\mathbb{R}} |m_y^0|^2\,\mathrm{d}x, & \text{if } m_z^0 \equiv 0 \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$\tilde{A}_0 = \{m^0 \colon \mathbb{R} \to \mathbb{S}^2 \mid m^0 - \bar{e} \in H^1(\mathbb{R})\}$$

We also define the subset $\tilde{A}_0^z$ of $\tilde{A}_0$ in the following way:

$$\tilde{A}_0^z = \{m^0 \in \tilde{A}_0 \mid m_z^0 \equiv 0\}.$$

We introduce as well the following sets

$$X_0 = \{m^0 \colon \mathbb{R} \to \mathbb{S}^2 \mid \partial_x m^0 \in L^2(\mathbb{R}) \text{ and } m_y^0, m_z^0 \in L^2_{loc}(\mathbb{R})\}.$$

$$A_0 = \{m^0 \colon \mathbb{R} \to \mathbb{S}^2 \mid E_0(m^0) < \infty\}.$$

It is evident that

$$\min_{m^0\in\tilde{A}_0} E_0(m^0) = \min_{m^0\in\tilde{A}_0^z} E_0(m^0).$$

This allows us to consider the minimization problem $\min_{m^0 \in \tilde{A}_0^z} E_0(m^0)$ instead of $\min_{m^0 \in \tilde{A}_0} E_0(m^0)$. The notion of convergence that we use in the Γ-convergence theorem is the same:

Definition 2.9.2. *Assume we are given a sequence of domain-magnetization pairs $(\Omega(l_n, d_n), m^n)_{n\in\mathbb{N}}$ and a magnetization $m^0 \in X_0$. We define $\acute{m}^n(x, y, z) = m^n(\lambda_n x, l_n y, d_n z)$ for any $(x, y, z) \in \Omega(1, 1)$. The sequence $(\acute{m}^n)_{n\in\mathbb{N}}$ is said to converge to m^0 if the following statements hold:*

- $\partial_x \acute{m}^n \rightharpoonup \partial_x m^0$ *weakly in* $L^2(\Omega(1, 1))$
- $\nabla_{yz} \acute{m}^n \to 0$ *strongly in* $L^2(\Omega(1, 1))$
- $\acute{m}^n \to m^0$ *strongly in* $L^2_{loc}(\Omega(1, 1))$

Like in the previous case a Γ-convergence holds:

Theorem 2.9.3 (Γ-convergence)**.** *The reduced variational problem is the Γ-limit of the full variational problem with respect to the convergence stated in Definition 2.9.2. This amounts to the following three statements:*

- ***Lower semicontinuity*** *If a sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ with entries in $A(l_n, d_n)$ converges to some $m^0 \in X_0$ in the sense of Definition 2.9.2 then*
$$E_0(m^0) \le \liminf_{n\to\infty} \acute{E}_n(\acute{m}^n)$$
- ***Construction*** *For every $m^0 \in \tilde{A}_0$ and every sequence of pairs $(l_n, d_n)_{n\in\mathbb{N}}$ with $l_n, d_n \to 0$, $c_n \to c$, there exists a sequence $(m^n)_{n\in\mathbb{N}}$ with entries in $\tilde{A}(l_n, d_n)$ such that*
$$\acute{m}^n \to m^0 \quad \textit{in the cense of Definition 2.9.2}$$
$$E_0(m^0) = \lim_{n\to\infty} \acute{E}_n(\acute{m}^n)$$
- ***Compactness*** *Let $(l_n, d_n)_{n\in\mathbb{N}}$ be a sequence of pairs such that $l_n, d_n \to 0$ and $c_n \to c > 0$. Let $m^n \in \tilde{A}(l_n, d_n)$ and let $(\acute{E}_n(m^n))_{n\in\mathbb{N}}$ be bounded. Then there exists a subsequence of $(m^n)_{n\in\mathbb{N}}$ (not relabeled) such that $\acute{m}^n$ converges to some $m^0 \in \tilde{A}_0^z$ in the cense of Definition 2.9.2.*

Proof. **Lower semicontinuity** If $\liminf_{n\to\infty} \frac{E(m^n)}{\mu_n} = +\infty$ then there is nothing to prove, otherwise one can assume that $E(m^n) \le M \cdot \mu_n$ for some constant M and all $n \in \mathbb{N}$. In this case everything is analogues to the previous case except the lower bound on E_s with the right coefficient. It is clear that

$$\liminf_{n\to\infty} \frac{E(m^n)}{\mu_n} \geq \liminf_{n\to\infty} \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty} \frac{E_{mag}(m^n)}{\mu_n}$$

$$= \liminf_{n\to\infty} \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{\mu_n} \geq \liminf_{n\to\infty} \frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty} \frac{E_s(\bar{m}^n)}{\mu_n}.$$

Assume $(q_n)_{n\in\mathbb{N}}$ is a sequence with entries between 0 and 1 yet to be defined. We have that

$$q_n \frac{E_{ex}(m^n)}{\mu_n} \geq \frac{q_n}{\mu_n} \int_{\Omega(l_n,d_n)} |\partial_x m^n(\xi)|^2 \,\mathrm{d}\xi \geq \frac{q_n}{\mu_n} \int_{\Omega(l_n,d_n)} |\partial_x \bar{m}^n(\xi)|^2 \,\mathrm{d}\xi$$

$$= 4\frac{q_n l_n d_n}{\mu_n} \int_{\mathbb{R}} |\partial_x \bar{m}^n(x)|^2 \,\mathrm{d}x = 4\frac{q_n l_n d_n}{\mu_n} \int_{\mathbb{R}} |\widehat{\partial_x \bar{m}^n}(x)|^2 \,\mathrm{d}x$$

$$= 4\frac{q_n l_n d_n}{\mu_n} \int_{\mathbb{R}} |x \cdot \widehat{\bar{m}^n}(x)|^2 \,\mathrm{d}x \geq \frac{4 q_n d_n}{9 l_n \mu_n} \int_{\mathbb{R}\setminus[-\frac{1}{3l_n},\frac{1}{3l_n}]} (|\widehat{\bar{m}^n_y}(x)|^2 + |\widehat{\bar{m}^n_z}(x)|^2) \,\mathrm{d}x$$

and according to (2.38) we have for big n as well

$$\frac{E_s(\bar{m}^n)}{\mu_n} \geq \frac{4}{\pi\mu_n}(1-\epsilon)^2(1-3\epsilon) l_n d_n c_n |\ln c_n|^2 \int_{-\frac{1}{3l_n}}^{\frac{1}{3l_n}} \Big(\frac{1}{|\ln c_n|} |\widehat{\bar{m}^n_y}(x)|^2 + \cdot |\widehat{\bar{m}^n_z}(x)|^2\Big) \,\mathrm{d}x.$$

Now the choice of q_n is evident, we should make the coefficients of the integrals equal:

$$\frac{4 l_n d_n c_n |\ln c_n|^2}{\pi\mu_n} = \frac{4 q_n d_n}{9 l_n \mu_n} \quad \text{thus} \quad q_n = \frac{9 l_n d_n |\ln c_n|^2}{\pi} \to 0.$$

We split E_{ex} into the sum of $(1-q_n)E_{ex}$ and $q_n E_{ex}$ to obtain

$$\liminf_{n\to\infty} \frac{E(m^n)}{\mu_n} \geq \liminf_{n\to\infty} (1-q_n) \int_{\Omega(1,1)} |\partial_x \acute{m}^n|^2 \,\mathrm{d}\xi$$

$$+ \liminf_{n\to\infty} \frac{4}{\pi\mu_n}(1-\epsilon)^2(1-3\epsilon) l_n d_n c_n |\ln c_n| \int_{\mathbb{R}} (|\bar{m}^n_y(x)|^2 + |\ln c_n| \cdot |\bar{m}^n_z(x)|^2) \,\mathrm{d}x$$

$$\geq 4 \int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x + \frac{4}{\pi}(1-\epsilon)^2(1-3\epsilon) \liminf_{n\to\infty} \frac{1}{\lambda_n} \int_{\mathbb{R}} (|\bar{m}^n_y(x)|^2 + |\ln c_n| \cdot |\bar{m}^n_z(x)|^2) \,\mathrm{d}x \tag{2.70}$$

According to Lemma 2.4.3 we have

$$\int_{\mathbb{R}} (|\bar{m}^n_y(x)|^2 + |\ln c_n| \cdot |\bar{m}^n_z(x)|^2) \,\mathrm{d}x = \frac{1}{4 l_n d_n} \int_{\Omega(l_n,d_n)} (|\bar{m}^n_y|^2 + |\ln c_n| \cdot |\bar{m}^n_z|^2) \,\mathrm{d}\xi$$

$$\geq \frac{1}{4l_n d_n}\Big(\int_{\Omega(l_n,d_n)}(|m_y^n|^2+|\ln c_n|\cdot|m_z^n|^2)\,\mathrm{d}\xi - M\acute{C}R_n^2\mu_n|\ln c_n|\Big) \tag{2.71}$$

Like in the proof of Theorem 2.7.2 we can prove that for any fixed $N \in \mathbb{N}$ the following inequalities hold:

$$\int_{[-N,N]\times R(1,1)} |m_y^0(\xi)|^2\,\mathrm{d}\xi \leq \liminf_{n\to\infty}\frac{1}{l_n d_n \lambda_n}\int_{\Omega(l_n,d_n)}|m_y^n(\xi)|^2\,\mathrm{d}\xi$$

and

$$\int_{[-N,N]\times R(1,1)} |m_z^0(\xi)|^2\,\mathrm{d}\xi \leq \liminf_{n\to\infty}\frac{1}{l_n d_n \lambda_n}\int_{\Omega(l_n,d_n)}|m_z^n(\xi)|^2\,\mathrm{d}\xi.$$

We fix a number $L > 0$. Utilizing (2.71) we get

$$\begin{aligned}
4\int_{-N}^{N}(|m_y^0(x)|^2+L|m_z^0(x)|^2)\,\mathrm{d}x &= \int_{[-N,N]\times R(1,1)}(|m_y^0(\xi)|^2+L|m_z^0(\xi)|^2)\,\mathrm{d}\xi \\
&\leq \liminf_{n\to\infty}\frac{1}{l_n d_n\lambda_n}\int_{\Omega(l_n,d_n)}(|m_y^n(\xi)|^2+L|m_z^n(\xi)|^2)\,\mathrm{d}\xi \\
&\leq \liminf_{n\to\infty}\frac{4}{\lambda_n}\int_{\mathbb{R}}(|\bar{m}_y^n(x)|^2+L|\bar{m}_z^n(x)|^2)\,\mathrm{d}x + \limsup_{n\to\infty}\frac{M\acute{C}R_n^2}{\lambda_n^2} \\
&= \liminf_{n\to\infty}\frac{4}{\lambda_n}\int_{\mathbb{R}}(|\bar{m}_y^n(x)|^2+L|\bar{m}_z^n(x)|^2)\,\mathrm{d}x
\end{aligned}$$

and since N was arbitrary we obtain

$$\int_{\mathbb{R}}(|m_y^0(x)|^2+L|m_z^0(x)|^2)\,\mathrm{d}x \leq \liminf_{n\to\infty}\frac{1}{\lambda_n}\int_{\mathbb{R}}(|\bar{m}_y^n(x)|^2+L|\bar{m}_z^n(x)|^2)\,\mathrm{d}x. \tag{2.72}$$

Utilizing now (2.70) and (2.72) and taking into account that for sufficiently big n we have $|\ln c_n| > L$ and that ϵ was arbitrary we establish

$$\liminf_{n\to\infty}\frac{E(m^n)}{\mu_n} \geq 4\int_{\mathbb{R}}|\partial_x m^0(x)|^2\,\mathrm{d}x + \frac{4}{\pi}\int_{\mathbb{R}}(|m_y^0(x)|^2+L|m_z^0(x)|^2)\,\mathrm{d}x. \tag{2.73}$$

Note that (2.73) holds for any $L > 0$, thus

$$\liminf_{n\to\infty}\frac{E(m^n)}{\mu_n} \geq E_0(m^0)$$

which was supposed to be proven.

Construction Like in Theorem 2.7.2 we prove that the sequence $m^n(x,y,z) = m^0(\frac{x}{\lambda_n})$ where $m^0_z \equiv 0$ satisfies the condition

$$\limsup_{n\to\infty} \frac{E(m^n)}{\mu_n} \leq E_0(m^0).$$

The only difference is the upper bound on $I(d_n, l_n, x)$. Without loss of generality one can assume that $E_0(m^0) < \infty$, otherwise there is nothing to prove. Therefore we have $m^0_z \equiv 0$. Referring to (2.45) we recall that for any $x \in \mathbb{R}$

$$I(d_n, l_n, x) \leq \pi l_n d_n c_n (3 - \ln c_n),$$

thus

$$E_s(\bar{m}^n) \leq \frac{4}{\pi} l_n d_n c_n (3 - \ln c_n) \int_{\mathbb{R}} |\bar{m}^n_y|^2 \,\mathrm{d}x = \frac{4}{\pi} l_n d_n c_n (3 - \ln c_n)\lambda_n \int_{\mathbb{R}} |m^0_y|^2 \,\mathrm{d}x$$

hence

$$\limsup_{n\to\infty} \frac{E_s(\bar{m}^n)}{\mu_n} \leq \frac{4(\ln c_n - 3)}{\pi \ln c_n} \int_{\mathbb{R}} |m^0_y|^2 \,\mathrm{d}x = \frac{4}{\pi} \int_{\mathbb{R}} |m^0_y|^2 \,\mathrm{d}x.$$

Compactness Assume now we are given a sequence of magnetization-domain-energy triples $(m^n, \Omega(l_n, d_n), E(m^n))_{n\in\mathbb{N}}$ such that $l_n, c_n \to 0$ and $E(m^n) \leq M\mu_n$ for all $n \in \mathbb{N}$. Like in Theorem 2.7.2 one can prove the existence of a "good" subsequence of magnetizations (not relabeled) and of a magnetization $m^0 \in \tilde{A}_0$ such that $(m^n)_{n\in\mathbb{N}}$ converges to m^0 in the sense of Definition 2.9.2. It remains to prove that in this case $\acute{m}^n_z \to 0$ strongly in $\Omega(1,1)$ and thus $m^0_z \equiv 0$. To that end we recall lemma 2.5.6, and the lower semi-continuity part of proof of Theorem 2.7.2. Namely we have

$$I(l_n, d_n, x) \geq 2\pi_n d_n \Big(1 - \sqrt{c_n}\Big)\Big(\frac{\pi}{2} - 3\sqrt{c_n}\Big) \quad \text{if} \quad x \in \Big[-\frac{1}{3\sqrt{l_n d_n}}, \frac{1}{3\sqrt{l_n d_n}}\Big]$$

hance for big n we have

$$E_s(\bar{m}^n) \geq \frac{1}{8} l_n d_n \int_{[-\frac{1}{3\sqrt{c_n}}, \frac{1}{3\sqrt{c_n}}]} |\widehat{\bar{m}^n_y}|^2 \,\mathrm{d}x$$

and

$$E_{ex}(m^n) \geq \frac{4 l_n d_n}{9 c_n} \int_{\mathbb{R}\setminus[-\frac{1}{3\sqrt{c_n}}, \frac{1}{3\sqrt{c_n}}]} |\widehat{\bar{m}^n_y}|^2 \,\mathrm{d}x$$

therefore for big n we have

$$\int_{\mathbb{R}} |\bar{m}_y^n|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\widehat{\bar{m}_y^n}|^2 \,\mathrm{d}x \leq M\mu_n\Big(\frac{9c_n}{4l_n d_n} + \frac{8}{l_n d_n}\Big) \leq \frac{9M\mu_n}{l_n d_n}$$

which is equivalent to

$$\int_{\Omega(l_n,d_n)} |\bar{m}_y^n|^2 \,\mathrm{d}x \leq 36M\mu_n.$$

We also have that

$$\Big|\int_{\Omega(l_n,d_n)} |m_y^n|^2 \,\mathrm{d}\xi - \int_{\Omega(l_n,d_n)} |\bar{m}_y^n|^2 \,\mathrm{d}\xi\Big| \leq C_n^2\mu_n$$

where C_n is the diameter of $R(l_n, d_n)$ times a constant, therefore

$$\int_{\Omega(l_n,d_n)} |m_y^n|^2 \,\mathrm{d}x \leq (C_n^2 + 36M)\mu_n.$$

Finally we get

$$\int_{\Omega(1,1)} |\acute{m}_y^n(\xi)|^2 \,\mathrm{d}\xi = \frac{1}{l_n d_n \lambda_n} \int_{\Omega(l_n,d_n)} |m_y^n(\xi)|^2 \,\mathrm{d}\xi \leq \frac{C_n^2 + 36M}{\lambda_n^2} \to 0$$

as n goes to infinity. The proof is complete.

□

Now the minimal energy scaling for the case $c = 0$ can be found. It is easy to see that like in the first regime the following equality holds:

$$\lim_{n\to\infty} \frac{E_{min}(l_n, d_n)}{\mu_n} = \frac{16}{\sqrt{\pi}}, \tag{2.74}$$

therefore we can also state the the minimal energies scale like μ_n.

2.10 The rate of convergence

In this section we find a rate of convergence for limit (2.74). To that end we need an accurate lower bound on $E_{mag}(m)$ for any $m \in \tilde{A}(l_n, d_n)$ and an accurate upper bound for a suitable m. We choose $m(x, y, z) = m^0(\frac{x}{\lambda_n})$, where $m_z \equiv 0$ and m^0 is a minimizer of the energy functional

$$E_0(m) = \int_{\mathbb{R}} |\partial_x m|^2 \,\mathrm{d}x + \frac{1}{\pi} \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) \,\mathrm{d}x.$$

Utilizing estimate (2.45) and we obtain for big n

$$E(m) \leq \frac{4l_nd_n}{\lambda_n}\int_{\mathbb{R}}|\partial_x m^0|^2\,\mathrm{d}x + \frac{4l_nd_nc_n(3-\ln c_n)}{\pi}\int_{\mathbb{R}}|m_y^0(x)|^2\,\mathrm{d}x + E_v(m).$$

According to Lemma 2.5.11 we get for big n

$$\begin{aligned}\frac{E(m)}{\mu_n} &\leq 4E_0(m) + \frac{12}{\pi|\ln c_n|}\int_{\mathbb{R}}|m_z^0(x)|^2\,\mathrm{d}x + 2M_{m^0}d_n\lambda_n(1-\ln c_n)\\ &\leq \frac{16}{\sqrt{\pi}} + \frac{10}{|\ln c_n|} + 2\sqrt{l_nd_n|\ln c_n|},\end{aligned}$$

thus the minimal energy satisfies the inequality

$$\frac{E_{min}(l_n,d_n)}{\mu_n} - \frac{16}{\sqrt{\pi}} \leq \frac{10}{|\ln c_n|} + 2\sqrt{l_nd_n|\ln c_n|}. \tag{2.75}$$

To get a lower bound we use (2.48) but we now play a bit with ϵ. Assume now ϵ is a positive number smaller than 1. We have

$$I(d_n,l_n,x) \geq \pi l_nd_nc_n\int_{c_n^{1-\epsilon}}^{c_n^{\epsilon}}\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-2c_n^{-\epsilon}}}{t+c_n}\,\mathrm{d}t \quad x\in\Big[-\frac{1}{l_n},\frac{1}{l_n}\Big].$$

Using the inequalities

$$\sin t \geq t - \frac{t^2}{6} \quad \text{and} \quad e^t > t \quad \text{for} \quad t\in[0,+\infty)$$

and the argument used when proving (2.48) we get

$$I(d_n,l_n,x) \geq \pi(1-3\epsilon)(1-\frac{c_n^{2\epsilon}}{6})^2(1-c_n^{2\epsilon})l_nd_nc_n|\ln c_n|. \tag{2.76}$$

We now choose the a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that we have $\epsilon_n\to 0$ and $c_n^{\epsilon_n}\to 0$ simultaneously. An example of such a sequence is $\epsilon_n = \frac{1}{\sqrt{|\ln c_n|}}$. It is easy to see that

$$c_n^{2\epsilon_n} < \frac{1}{2\sqrt{|\ln c_n|}} \quad \text{and} \quad (1-3\epsilon_n)(1-\frac{c_n^{2\epsilon_n}}{6})^2(1-c_n^{2\epsilon_n}) > 1 - \frac{4}{\sqrt{|\ln c_n|}}.$$

Now with this choice of ϵ_n (2.76) will have the form

$$I(d_n,l_n,x) \geq \pi l_nd_nc_n|\ln c_n|\Big(1-\frac{4}{\sqrt{|\ln c_n|}}\Big) \quad \text{for} \quad x\in\Big[-\frac{1}{l_n},\frac{1}{l_n}\Big]. \tag{2.77}$$

Assume now m is a minimizer of $E(l_n,d_n)$. We have that

$$I(l_n,d_n,x)\geq I(d_n,l_n,x)$$

thus

$$E_{mag}(\bar{m})\geq\frac{4}{\pi}l_nd_nc_n|\ln c_n|\Big(1-\frac{4}{\sqrt{|\ln c_n|}}\Big)\int_{-\frac{1}{l_n}}^{\frac{1}{l_n}}(|\widehat{\bar{m}_y}|^2+|\widehat{\bar{m}_z}|^2)\,\mathrm{d}x.$$

According to (2.75) we have for big n

$$\frac{E_{min}(l_n,d_n)}{\mu_n}\leq\frac{16}{\sqrt{\pi}}+1<11. \tag{2.78}$$

We have furthermore for big n that

$$\int_{\mathbb{R}\setminus[-\frac{1}{l_n},\frac{1}{l_n}]}(|\widehat{\bar{m}_y}|^2+|\widehat{\bar{m}_z}|^2)\,\mathrm{d}x\leq l_n^2\int_{\mathbb{R}}(|x\cdot\widehat{\bar{m}_y}|^2+|x\cdot\widehat{\bar{m}_z}|^2)\,\mathrm{d}x$$

$$=l_n^2\int_{\mathbb{R}}(|\partial_x\bar{m}_y|^2+|\partial_x\bar{m}_z|^2)\,\mathrm{d}x\leq\frac{l_n}{4d_n}\int_{\Omega(l_n,d_n)}(|\partial_xm_y|^2+|\partial_xm_z|^2)\,\mathrm{d}x$$

$$\leq\frac{l_nE_{ex}(m)}{4d_n}\leq\frac{11l_n\mu_n}{4d_n},$$

thus

$$\frac{4}{\pi}l_nd_nc_n|\ln c_n|\int_{\mathbb{R}\setminus[-\frac{1}{l_n},\frac{1}{l_n}]}(|\widehat{\bar{m}_y}|^2+|\widehat{\bar{m}_z}|^2)\,\mathrm{d}x\leq\frac{11}{\pi}l_n^2c_n|\ln c_n|\mu_n$$

and

$$E_{mag}(\bar{m})\geq\frac{4}{\pi}l_nd_nc_n|\ln c_n|\Big(1-\frac{4}{\sqrt{|\ln c_n|}}\Big)\int_{\mathbb{R}}(|\bar{m}_y|^2+|\bar{m}_z|^2)\,\mathrm{d}x-\frac{11}{\pi}l_n^2c_n|\ln c_n|\mu_n$$

We have by Lemma 2.4.3 that

$$4l_nd_n\int_{\mathbb{R}}(|\bar{m}_y|^2+|\bar{m}_z|^2)\,\mathrm{d}x\geq\int_{\Omega(l_n,d_n)}(|m_y|^2+|m_z|^2)\,\mathrm{d}x-\acute{C}R_n^2E_{ex}(m)$$

and we have for big n analogues to (2.53) that

$$E_{mag}(m)\geq E_{mag}(\bar{m})-33\sqrt{\acute{C}}R_n\mu_n,$$

thus combining the last three inequalities and remembering that $E_{ex}(m) \leq 11\mu_n$ we discover

$$\begin{aligned}
&E_{mag}(m) \\
&\geq \frac{1}{\pi} c_n |\ln c_n| \Big(1 - \frac{4}{\sqrt{|\ln c_n|}}\Big) \int_{\Omega(l_n,d_n)} (|m_y|^2 + |m_z|^2)\,\mathrm{d}x \\
&-\frac{11\acute{C}+1}{\pi} c_n |\ln c_n| R_n^2 \mu_n - 33\sqrt{\acute{C}} R_n \mu_n.
\end{aligned}$$

For the whole energy we obtain

$$E(m) \geq \mu_n \Big(1 - \frac{4}{\sqrt{|\ln c_n|}}\Big)\Big(\int_{\Omega(1,1)} (|\partial_x \acute{m}|^2\,\mathrm{d}\xi + \frac{1}{\pi}\int_{\Omega(1,1)} (|\acute{m}_y|^2 + |\acute{m}_z|^2)\,\mathrm{d}\xi\Big) -$$

$$-\frac{11\acute{C}+1}{\pi} c_n |\ln c_n| R_n^2 \mu_n - 33\sqrt{\acute{C}} R_n \mu_n.$$

Finally taking into account Lemma 3.7.5 and the fact that $c_n|\ln c_n|R_n^2$ decays faster than R_n we establish for big n

$$\frac{E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}} \geq -\frac{64}{\sqrt{|\ln c_n|}} - 34\sqrt{\acute{C}} R_n. \tag{2.79}$$

Combining now (2.75) and (2.79) and taking into account the fact that the right hand side of (2.79) decays faster than the right hand side of (2.75) we establish for big n

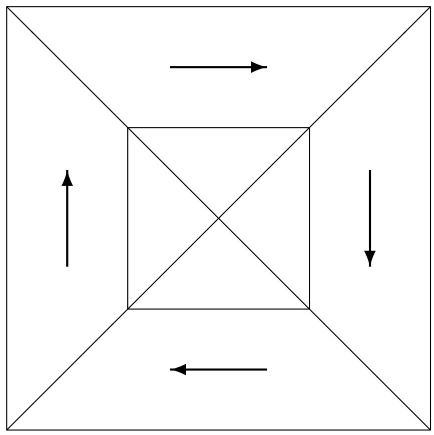
$$\Big|\frac{E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}}\Big| \leq \frac{64}{\sqrt{|\ln c_n|}} + 34\sqrt{\acute{C}} R_n. \tag{2.80}$$

2.11 Upper and lower bounds for thick wires

Throughout this section we assume that the parameters d and l are both big and comparable to each other. For convenience we will assume that $d = l$. We prove an upper bound on the minimal energy and refer to [24] for a lower bound. However it is not clear if the upper bound we get has the optimal scaling or not. We directly construct a magnetization m with the described energy. We start with some notation: Assume $L > 0$ and denote by Ω_L the domain $[-L, L] \times [-d, d] \times [-d, d]$. We take the rectangular parallelepiped Ω_L and cut off from it the two cones with the vertex at $(0, 0, 0)$ and the bases $-L \times [-d, d] \times [-d, d]$ and $L \times [-d, d] \times [-d, d]$ respectively and denote the obtained domain by R_L. The main diagonals of Ω_L divide R_L into four parts. Taking into account the orientation in the plane OYZ we denote that parts by R_L^{up}, R_L^{right}, R_L^{down} and R_L^{left} respectively. First we construct a magnetization $\tilde{m}$ which has infinite exchange energy but a magnetostatic energy easy to bound. We consider the following vector field:

$$\tilde{m} = \begin{cases} \left(\sin\frac{\pi dx}{2Lz}, \cos\frac{\pi dx}{2Lz}, 0\right) & \text{in} \quad R_L^{up} \\ \left(\sin\frac{\pi dx}{2Ly}, 0, -\cos\frac{\pi dx}{2Ly}\right) & \text{in} \quad R_L^{right} \\ \left(-\sin\frac{\pi dx}{2Lz}, -\cos\frac{\pi dx}{2Lz}, 0\right) & \text{in} \quad R_L^{down} \\ \left(-\sin\frac{\pi dx}{2Ly}, 0, \cos\frac{\pi dx}{2Ly}\right) & \text{in} \quad R_L^{left} \end{cases}$$

Note that the vector field $(0, \tilde{m}_y, \tilde{m}_z)$ is divergence free (see cross section Figure 2.1).



A cross section for $\tilde{m}$

Figure 2.1

Therefore

$$\mathsf{div}\tilde{m} = \frac{\partial \tilde{m}_x}{\partial x} \geq 0 \quad \text{in} \quad \Omega$$

and $s \equiv 0$, thus we have

$$E_{mag}(\tilde{m}) = \int_\Omega \int_\Omega \Gamma(\xi - \xi_1) \frac{\partial \tilde{m}_x(\xi)}{\partial x} \frac{\partial \tilde{m}_x(\xi_1)}{\partial x} \, \mathrm{d}\xi \, \mathrm{d}\xi_1.$$

The integrand is zero in the complement of R_L, so we first estimate it if the first integration is done over R_L^{up}. Note that in R_L^{up} we have

$$\frac{\partial \tilde{m}_x(\xi)}{\partial x} = \frac{\pi d}{2Lz} \cos\frac{\pi dx}{2Lz} \leq \frac{\pi d}{2Lz},$$

thus

$$\int_{R_L^{up}} \Gamma(\xi-\xi_1)\frac{\partial\tilde{m}_x(\xi)}{\partial x}\,\mathrm{d}\xi \leq \int_0^d \frac{\pi d}{2Lz}\,\mathrm{d}z \int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \Gamma(\xi-\xi_1)\,\mathrm{d}y\,\mathrm{d}x.$$

Recall Lemma 2.5.10. Apparently Lemma 2.5.10 is valid also when the point (y_1, z_1) does not belong to $R(l, d)$. Indeed, in that case we will replace (y_1, z_1) by the closest point of $R(l, d)$ to (y_1, z_1) which will not decrease the integral. Hence we have that

$$\int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \Gamma(\xi-\xi_1)\,\mathrm{d}y\,\mathrm{d}x \leq \frac{10z}{4\pi}\big(1+\ln\frac{L}{d}\big)$$

and

$$\int_{R_L^{up}} \Gamma(\xi-\xi_1)\frac{\partial\tilde{m}_x(\xi)}{\partial x}\,\mathrm{d}\xi \leq \frac{5d^2}{4L}\big(1+\ln\frac{L}{d}\big).$$

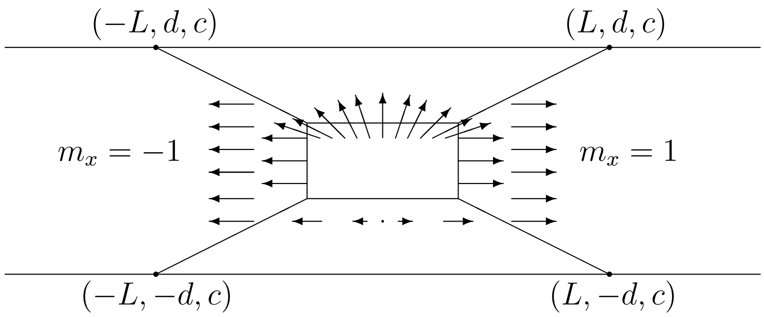
The integrals over the other parts of R_L have the same upper bound, thus we obtain

$$E_{mag}(\tilde{m}) \leq \frac{20d^4}{L}\big(1+\ln\frac{L}{d}\big). \tag{2.81}$$

The reason for $\tilde{m}$ having an infinite exchange energy is that it has singularities on the part of the boundary of R_L that belongs to Ω_L. We ignore for a moment this boundary charges and compute $E_{ex}(\tilde{m})$ taking into account only the volume charges. We have formally that

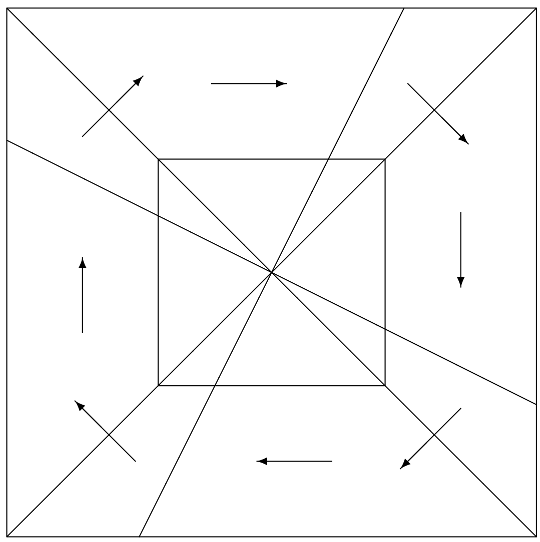
$$\begin{aligned} E_{ex}^{formal}(\tilde{m}) &= 4\int_0^d \frac{\pi^2 d^2}{4L^2z^2}\int_{-\frac{Lz}{d}}^{\frac{Lz}{d}}\int_z^z \Big(1+\frac{x^2}{z^2}\Big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z \leq \\ &\leq 4\int_0^d \frac{\pi^2 d^2}{4L^2z^2}\int_{-\frac{Lz}{d}}^{\frac{Lz}{d}}\int_z^z \Big(1+\frac{L^2}{d^2}\Big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z = \\ &= 4\pi^2\big(\frac{d^2}{L}+L\big). \end{aligned} \tag{2.82}$$

In the next step we build a magnetization m with finite exchange energy by slightly modifying $\tilde{m}$ near the singularity points. It works in the following way: We first take the planes $\{z = \frac{d}{d-1}y\}$ and $\{z = -\frac{d-1}{d}y\}$. To get a continuous m from $\tilde{m}$ we change $\tilde{m}$ in the following two regions: The first one is the intersection of Ω_L with the region between the planes $\{z = \frac{d}{d-1}y\}$ and $\{z = y\}$ and the second one is the intersection of Ω_L with the region between the planes $\{z = -\frac{(d-1)}{d}y\}$ and $\{z = -y\}$. For more transparency see Figures 2.2 and 2.3



A longitudinal section $\{z = c > 0\}$

Figure 2.2



A cross section for m.

Figure 2.3

We denote the upper part of the first region(where $z \geq 0$) by $\Omega_{L,1}^{up}$ and the lower part by $\Omega_{L,1}^{down}$. We make the same notation also for the second region. Finally we define the magnetization m in $\Omega_{L,1}^{up}$

$$m(x, y, z) = (\sin \frac{\pi dx}{2Lz}, \cos \frac{\pi dx}{2Lz} \sin \frac{\pi d(z-y)}{2z}, -\cos \frac{\pi dx}{2Lz} \cos \frac{\pi d(z-y)}{2z}).$$

The definition of m in the other three regions is analogues. Note that the vector field m has now only one singularity which is the origin. We estimate now the energy of m. Note first that by Lemma 2.4.2 we have

$$|E_{mag}(m) - E_{mag}(\tilde{m})| \leq \|m - \tilde{m}\|^2_{L^2(\Omega_L)} + 2\|m - \tilde{m}\|_{L^2(\Omega_L)} \sqrt{E_{mag}(\tilde{m})}$$
$$\leq 16dL + 16\sqrt{5}d^2\sqrt{d \ln L}. \tag{2.83}$$

Using the inequalities $|y| \leq z$ and $|x| \leq \frac{L}{d}z$ in $\Omega^{up}_{L,1}$ one can by direct calculation discover

$$|\partial_y m_y|^2 + |\partial_z m_y|^2 + |\partial_y m_z|^2 + |\partial_y m_z|^2 \leq \frac{\pi^2}{4}(2d^2+1) \cdot \frac{1}{z^2} \quad \text{in} \quad \Omega^{up}_{L,1}.$$

We calculate now

$$\int_{\Omega^{up}_{L,1}} \frac{1}{z^2}\,\mathrm{d}\xi = 2\int_0^L \int_{\frac{dx}{L}}^d \int_{\frac{d-1}{d}z}^z \frac{1}{z^2}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x = \frac{1}{d}\int_0^L (\ln L - \ln x)\,\mathrm{d}x = \frac{L}{d}.$$

We have furthermore

$$|E^{formal}_{ex}(\tilde{m}) - E_{ex}(m)| \leq \int_{\Omega^{up}_{L,1}} (|\partial_y m_y|^2 + |\partial_z m_y|^2 + |\partial_y m_z|^2 + |\partial_y m_z|^2)\,\mathrm{d}\xi \leq \pi^2 dL + \frac{\pi^2 L}{2d}. \tag{2.84}$$

Employing now (2.81)-(2.84) and choosing $L = d^{\frac{3}{2}}\sqrt{\ln d}$ we obtain for big d

$$E(m) \leq 150d^{\frac{5}{2}}\sqrt{\ln d}.$$

For a lower bound we refer to [24]. It is shown in [24] that there exists a number $R_0 > 0$ such that if $R \geq R_0$ then the minimal energy is bigger than a constant times $R^2\sqrt{\ln R}$, where the cross section of the domain Ω is a disc with radius R. It is easily seen that the proof in works also for a rectangular cross section, thus we obtain that there exist numbers $d_0, C > 0$ such that if $l, d > d_0$ then

$$Cd^2\sqrt{\ln d} \leq E(m) \leq 150d^{\frac{5}{2}}\sqrt{\ln d}.$$

Chapter 3

The static domain walls in cylinders with a centrally symmetric cross section

3.1 Introduction

In this chapter we study the static domain walls in a more general setting, namely we assume that the domain Ω has the form $\mathbb{R} \times \omega$, where ω is a centrally symmetric, bounded Lipschitz domain in $\mathbb{R}^2$. We consider sequences of homothetic cylinders $\mathbb{R} \times \omega_n$. Denote by d_n the diameter of ω_n and assume that the sequence $(d_n)_{n\in\mathbb{N}}$ converges to zero. We prove a Γ-convergence of the rescaled minimization problems

$$\inf_{m\in\tilde{A}_n} \frac{E(m)}{d_n^2}$$

and show that they converge to a one-dimensional problem which can be solved explicitly. Moreover, we prove a convergence result on the sequences of almost minimizers of the magnetization energy.

3.2 General Notation

We denote by d the length of the diameter of ω. We emphasize all the other notations that will differ from the ones in the previous chapter. We use the following notation:

- $A(\Omega)$ and $\tilde{A}(\Omega)$ instead of $A(l,d)$ and $\tilde{A}(l,d)$ respectively

- $A_x(\Omega)$ instead of $A_x(l,d)$,

- ω and ω_x instead of $R(l,d)$ and $R_x(l,d)$, respectively
- $\Omega_n = d_n \cdot \omega$, where ω has diameter 1,

We keep all the other notation of the previous chapter.

3.3 The main results

Like in the rectangular cross section case we establish an existence and a Γ-convergence result.

Theorem 3.3.1 (Existence)**.** *For any Lipschitz domain ω there exist minimizers of the energy functional in both $\tilde{A}$ and A_x.*

We fix a centrally symmetric Lipschitz domain $\omega \subset \mathbb{R}^2$ with a diameter 1. For any positive number d denote $\Omega_d = \mathbb{R} \times (d \cdot \omega)$. We consider the rescaled minimization problems

$$\inf_{m \in \tilde{A}(\Omega_d)} \frac{E(m)}{d^2}.$$

Theorem 3.3.2 (Γ-convergence)**.** *The rescaled minimization problems Γ-converge to a one dimensional problem as d goes to zero. The limit problem can be solved explicitly.*

As a consequence we obtain that the minimal energy scaling is d^2, moreover we establish

$$\lim_{d \to 0} \frac{E_{min}}{d^2} = E^0_{min}.$$

We prove as well a rate of convergence for the above limit.

Theorem 3.3.3 (Rate of convergence)**.** *The following rate of convergence holds:*

$$\Big|\frac{E_{min}}{d^2} - E^0_{min}\Big| \leq 120\pi^2 \sqrt{\frac{2c_\omega}{a_\omega}} (per(\omega))^2 d^{\frac{1}{6}}.$$

(The numbers a_ω and c_ω are defined in Chapter 3.7).

We establish furthermore a strong H^1 convergence for sequences of almost minimizers. (See the definition of a sequence of almost minimizers in Section

3.9) We consider a sequence $(m_n)_{n\in\tilde{A}_n}$ with $\omega_n = d_n \cdot \omega$ and assume that $d_n \to 0$.

Theorem 3.3.4. *For any sequence of almost minimizers $(m_n)_{n\in\tilde{A}_n}$ there exist a sequence of translations T_n in the x direction and a sequence of rotations R_n is the OYZ plane, such that for a magnetization $m^0 \in \tilde{A}_\omega$ strong H^1 convergence holds:*

$$\lim_{n\to\infty} \frac{1}{d_n} \|m^n(T_n(R_n)) - m^0\|_{H^1(\Omega_n)} = 0.$$

3.4 The characterization theorem

First of all note that $|\omega| = c_\omega \cdot d^2$ where c_ω is a constant depending only on the shape of ω. i.e., if another domain ω is homothetic to ω_1 then $c_{\omega_1} = c_\omega$. We claim that all the theorems and lemmas of the previous chapter hold also for this case, but formulated in another way if needed. We point out the theorems and lemmas that need to have another formulation and the changes that should be made in their proofs. We prove as well some new lemmas which will be used for the main $\Gamma-$convergence theorem.

Lemma 3.4.1. *If the vector field $m \in A^x_\Omega$ satisfies*

$$|m| \leq 1 \quad in \quad \Omega,$$

$$E(m) < \infty$$

then there exists a positive number M depending on ω and $E(m)$ such that

$$\|m_y\|^2_{L^2(\mathbb{R})} + \|m_z\|^2_{L^2(\mathbb{R})} \leq M.$$

Proof. The only idea that should be changed in the proof is choosing the suitable test functions φ_r. We choose a point (y_0, z_0) on $\partial\omega$ such that $\nu_y(y_0, z_0) \neq 0$ and $\nu_z(y_0, z_0) \neq 0$. If such a point does not exists then clearly there exist on $\partial\omega$ two points (y_1, z_1) and (y_2, z_2) such that $\nu_y(y_1, z_1) = 0$ and $\nu_z(y_2, z_2) = 0$. Consider the first case. Since $\partial\omega$ is Lipschitz one can choose an $\epsilon > 0$ such that for any $(y, z) \in B_\epsilon(y_0, z_0) \cap \partial\omega$ we have $\nu_y(y, z) > \frac{1}{2}\nu_y(y_0, z_0)$ and $\nu_z(y, z) > \frac{1}{2}\nu_z(y_0, z_0)$ and ν_y and ν_z keep their sign on $B_\epsilon(y_0, z_0) \cap \partial\omega$. The function ϕ can be chosen as follofs:

$$\phi_r = 1 \quad \text{in} \quad [-r, r] \times \left[y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2}\right] \times \left[z_0 - \frac{\epsilon}{2}, z_0 + \frac{\epsilon}{2}\right],$$

$$\text{supp}\phi \subset \left[r - \frac{\epsilon}{2}, r + \frac{\epsilon}{2}\right] \times [y_0 - \epsilon, y_0 + \epsilon] \times [z_0 - \epsilon, z_0 + \epsilon] \quad \text{and}$$

$$0 \leq \phi \leq 1, \quad |\nabla \phi_r| \leq \frac{10}{\epsilon}.$$

The choise of the function φ and the rest of the proof is the same as in the previous chapter, namely $\varphi_r = \phi_r \cdot s$. The same can be done for the two-point case.

□

Lemma 3.4.2. *For any vector fields $m_1, m_2 \in M_\Omega$ with finite energies the following statements hold:*

- $E_{mag}(m_1 + m_2) \leq 2(E_{mag}(m_1) + E_{mag}(m_2))$
- $|E_{mag}(m_1) - E_{mag}(m_2)| \leq E_{mag}(m_1 - m_2) + 2\sqrt{E_{mag}(m_1) E_{mag}(m_1 - m_2)}$
- $|E_{mag}(m_1) - E_{mag}(m_2)| \leq \|m_1 - m_2\|^2_{L^2(\Omega)} + 2\|m_1 - m_2\|_{L^2(\Omega)} \sqrt{E_{mag}(m_1)}$ *if* $m_1 - m_2 \in L^2(\Omega)$

Lemma 3.4.3. *For any $m \in M_\Omega$ with a finite energy the following statements hold:*

- $\int_{\omega_x} (|m|^2 - |\bar{m}|^2)\,\mathrm{d}y\,\mathrm{d}z = \int_{\omega_x} |m - \bar{m}|^2\,\mathrm{d}y\,\mathrm{d}z \leq \acute{C} d^2 \int_{\omega_x} |\nabla_{yz} m|\,\mathrm{d}y\,\mathrm{d}z$ *for all $x \in \mathbb{R}$, where $\acute{C}$ is an absolute constant (the Poincaree constant for bounden Lipschitz domains in $\mathbb{R}^2$).*
- $E_{ex}(\bar{m}) + E_{ex}(m - \bar{m}) = E_{ex}(m)$
- *There exists a constant C_1 depending only on ω such that*
$$E(\bar{m}) \leq C_1 E(m) \tag{3.1}$$
- *There exists a constant C_2 depending only on ω and $E(m)$ such that*
$$\|\bar{m}_y\|^2_{L^2(\Omega(l,d))} + \|\bar{m}_z\|^2_{L^2(\Omega(l,d))} \leq C_2 \tag{3.2}$$

Lemma 3.4.4.
- *Let $m \in A$ be a magnetization and α and β be real numbers such that $-1 < \alpha < \beta < 1$. Assume $\Re$ is a family of disjoint intervals (a, b) satisfying the conditions $\{\bar{m}_x(a), \bar{m}_x(b)\} = \{\alpha, \beta\}$ and $|\bar{m}_x(x)| \leq max(|\alpha|, |\beta|)$ in (a, b). Then*
$$card(\Re) \leq M_2 \quad and \quad \sum_{(a,b)\in\Re} (b - a) \leq M_2 \tag{3.3}$$
where M_2 is a constant depending on α, β, ω and $E(m)$.

- *If $m \in \tilde{A}$ then for any $0 < \delta < 1$ there exists a positive number N_δ such that two of the following properties hold:*
  $-1 \le \bar{m}_x \le -1+\delta \quad in \quad (-\infty, -N_\delta)$
  $-1 \le \bar{m}_x \le -1+\delta \quad in \quad (N_\delta, +\infty)$
  $1-\delta \le \bar{m}_x \le 1 \quad in \quad (N_\delta, +\infty)$
  $1-\delta \le \bar{m}_x \le 1 \quad in \quad (-\infty, -N_\delta)$
  (note that only two of them can simultaneously hold.)

- *For any $m \in \tilde{A}$ the function $\bar{m}_x$ has a constant sign at $\pm\infty$.*

Proof. In the proof the number $4ld$ must everywhere be replaced by $|\omega|$. □

Theorem 3.4.5. *If $m \in A(\Omega)$ then one of the four functions $m \pm \vec{e_x}$, $m \pm \bar{e}$ belongs to $H^1(\Omega)$.*

Proof. In the proof the number $4ld$ must everywhere be replaced by $|\omega|$. □

3.5 The magnetostatic energy

3.5.1 A representation of u and the magnetostatic energy

Recall first of all that Γ is the Green function for the Laplace operator in $\mathbb{R}^3$.

Lemma 3.5.1. *For $m \in X$ define the maps $u_v, u_s, u\colon \mathbb{R}^3 \to \mathbb{R}$ by*

$$u_v(\xi) = \int_\Omega \Gamma(\xi - \xi_1) v(\xi_1)\,\mathrm{d}\xi_1,$$

$$u_s(\xi) = \int_{\partial\Omega} \Gamma(\xi - \xi_1) s(\xi_1)\,\mathrm{d}\xi_1,$$

$$u(\xi) = u_v(\xi) + u_s(\xi).$$

Then the following statements hold:

- *The maps u_v and u_s satisfy the equalities*

$$\nabla u_v(\xi) = \sum_{i\in\{x,y,z\}} \int_\Omega \partial_i \Gamma(\xi - \xi_1) v(\xi_1) \vec{e_i}\,\mathrm{d}\xi \quad \textit{for all} \quad \xi \in \mathbb{R}^3, \tag{3.4}$$

$$\nabla u_s(\xi) = \sum_{i\in\{x,y,z\}} \int_{\partial\Omega} \partial_i \Gamma(\xi - \xi_1) s(\xi_1) \vec{e_i}\,\mathrm{d}\xi \quad \textit{for all} \quad \xi \in \mathbb{R}^3 \backslash \partial\Omega, \tag{3.5}$$

$$\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla\varphi = \int_\Omega v\varphi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3), \tag{3.6}$$

$$\int_{\mathbb{R}^3} \nabla u_s \cdot \nabla\varphi = \int_{\partial\Omega} s\varphi \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3). \tag{3.7}$$

- *u is a weak solution of $\triangle u = \mathrm{div} m$.*
- *∇u is in $L^2(\mathbb{R}^3)$.*

For any $m \in X$ we will hereafter consider the solution of $\triangle u = \mathrm{div} m$ which is defined in the previous lemma. As a corollary we get a necessary and sufficient condition for a magnetization to have a finite energy.

Theorem 3.5.2 (Characterization). *A magnetization $m\colon \Omega \to \mathbb{S}^2$ is in $A(\Omega)$ if and only if one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$.*

Proof. The necessity is Theorem 3.4.5. To prove the sufficiency we note that if one of the four functions $m \pm \overrightarrow{e_x}, m \pm \bar{e}$ belongs to $H^1(\Omega)$ then $m \in X$ thus according to Lemma 3.5.1 m belongs to A.

□

Corollary 3.5.3. *A magnetization m belongs to A if and only if $\nabla m, m_y, m_z \in L^2(\Omega)$.*

We consider now the functional E_{mag} for the magnetizations which are constant on each cross section, i.e., for $m \in A_x$.

Lemma 3.5.4. *For any $m \in A_x$ the gradients ∇u_v and ∇u_s are orthogonal in $L^2(\mathbb{R}^3)$.*

Thus for $m \in A_x$ the energy functional has the form

$$E(m) = c_\omega d^2 \|\partial_x m\|^2_{L^2(\mathbb{R})} + E_v(m) + E_s(m).$$

3.5.2 The representation of E_s in Fourier space

In this section we find a representation of E_s in Fourier space. Let the point $(0,0)$ be the center of symmetry of ω and let the parametrization

$$\begin{cases} y = y(t), & t \in [0,2] \\ z = z(t), & t \in [0,2] \end{cases}$$

of $\partial\omega$ be chosen so that $y(t+1) = -y(t)$, $z(t+1) = -z(t)$ and

$$\nu(t) = (\nu_y(t), \nu_z(t)) = \Big(\frac{z'(t)}{\sqrt{y'^2(t) + z'^2(t)}}, -\frac{y'(t)}{\sqrt{y'^2(t) + z'^2(t)}}\Big),$$

where $\nu(t)$ is the outward normal to $\partial\omega$ at $(y(t), z(t))$.

Theorem 3.5.5. *For any $m \in A_x$ the following formula is valid:*

$$E_s(m) = \frac{1}{4\pi^2}\int_{\mathbb{R}^3} \frac{1}{|k|^2}(|a|^2|\hat{m}_y(k_1)|^2 + |b|^2|\hat{m}_z(k_1)|^2 + \bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)} + \overline{\hat{m}_y(k_1)}\hat{m}_z(k_1))\,\mathrm{d}k,$$

where

$$a(k_2, k_3, \omega) = -2i\int_0^1 z'(t)\sin(k_2 y(t) + k_3 z(t))\,\mathrm{d}t$$

and

$$b(k_2, k_3, \omega) = 2i\int_0^1 y'(t)\sin(k_2 y(t) + k_3 z(t))\,\mathrm{d}t.$$

Proof. In order to calculate $\int_{\mathbb{R}^3}|\nabla u_s|^2$ we again use (2.30) and the distributional identity $\triangle u_s = -s\cdot\delta_{\partial\omega}$. Denote $\bar{x} = (x, y, z)$. We have for any $k \in \mathbb{R}^3$

$$\widehat{s\cdot\delta_{\partial\omega}}(k) = \frac{1}{2\pi\sqrt{2\pi}}\int_{\mathbb{R}^3} e^{-i\bar{x}k}(s\cdot\delta_{\partial\omega})(\bar{x})\,\mathrm{d}\bar{x} \tag{3.8}$$

It is clear that

$$\int_{\mathbb{R}^3} e^{-i\bar{x}k}(s\cdot\delta_{\partial\omega})(\bar{x})\,\mathrm{d}\bar{x} = \int_{\mathbb{R}}\int_{\partial\omega} e^{-i(k_2y+k_3z)}\nu(y,z)\,\mathrm{d}y\,\mathrm{d}z\cdot e^{-ik_1x}m(x)\,\mathrm{d}x$$

$$= \sqrt{2\pi}\hat{m}_y(k_1)\int_{\partial\omega} e^{-i(k_2y+k_3z)}\nu(y,z)\,\mathrm{d}y\,\mathrm{d}z + \sqrt{2\pi}\hat{m}_z(k_1)\int_{\partial\omega} e^{-i(k_2y+k_3z)}\nu(y,z)\,\mathrm{d}y\,\mathrm{d}z$$

$$= \sqrt{2\pi}\hat{m}_y(k_1)\int_0^2 z'(t)e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t - \sqrt{2\pi}\hat{m}_z(k_1)\int_0^2 y'(t)e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t.$$

For convenience we investigate the two parameters a and b as follows:

$$a(k_2, k_3, \omega) = \int_0^2 z'(t)e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t,$$

$$b(k_2, k_3, \omega) = -\int_0^2 y'(t)e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t.$$

Note that since the curve $\partial\omega$ is closed

$$k_3a - k_2b = \int_0^2 (k_3z'(t) + k_2y'(t))e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t = 0, \tag{3.9}$$

$$a(k_2,k_3,\omega) = \int_0^1 z'(t)e^{-i(k_2y(t)+k_3z(t))}\,\mathrm{d}t - \int_0^1 z'(t)e^{i(k_2y(t)+k_3z(t))}\,\mathrm{d}t$$

$$-2i\int_0^1 z'(t)\sin(k_2y(t)+k_3z(t))\,\mathrm{d}t.$$

Similarly we have

$$b(k_2,k_3,\omega) = 2i\int_0^1 y'(t)\sin(k_2y(t)+k_3z(t))\,\mathrm{d}t.$$

For the Fourier transform of $\triangle u_s$ we have

$$|\widehat{\triangle u_s}(k)|^2 = \frac{1}{4\pi^2}|a\hat{m}_y(k_1)+b\hat{m}_z(k_1)|^2$$

$$= \frac{1}{4\pi^2}(|a|^2|\hat{m}_y(k_1)|^2 + |b|^2|\hat{m}_z(k_1)|^2 + \bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)} + \overline{\hat{m}_y(k_1)}\hat{m}_z(k_1)).$$

Finally we obtain for E_s

$$E_s(m) = \int_{\mathbb{R}^3} |\nabla u_s(k)|^2\,\mathrm{d}k = \int_{\mathbb{R}^3} \frac{|\widehat{\triangle u_s}(k)|^2}{|k|^2}\,\mathrm{d}k$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^3}\frac{1}{|k|^2}(|a|^2|\hat{m}_y(k_1)|^2+|b|^2|\hat{m}_z(k_1)|^2+\bar{a}b(\hat{m}_y(k_1)\overline{\hat{m}_z(k_1)}+\overline{\hat{m}_y(k_1)}\hat{m}_z(k_1))\,\mathrm{d}k. \tag{3.10}$$

□

In the next step we recall some well-known facts and prove some auxialary lemmas which will be utilized to get lower and upper bounds on E_s. The following equalities are well known:

$$\int_0^{+\infty}\frac{\cos px}{x^2+q^2}\,\mathrm{d}x = \frac{\pi}{2q}e^{-pq}, \quad q>0, p>0 \tag{3.11}$$

$$\int_0^{+\infty}\frac{e^{-p_1x}\cos q_1x - e^{-p_2x}\cos q_2x}{x}\,\mathrm{d}x = \frac{1}{2}\ln\frac{p_1^2+q_1^2}{p_2^2+q_2^2}, \quad p_1,p_2>0, q_1,q_2\in\mathbb{R}, \tag{3.12}$$

Lemma 3.5.6. *For any $p,q,l>0$ the following inequality holds:*

$$\Big|\int_l^{+\infty}\frac{\sin qt}{t}e^{-pt}\,\mathrm{d}t\Big| \le \pi.$$

Proof. Making $t = \frac{x}{q}$ change of variables and denoting $r = \frac{p}{q}, L = ql$ we get

$$\int_l^{+\infty} \frac{\sin qt}{t} e^{-pt}\,\mathrm{d}t = \int_L^{+\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x$$

Denote $x_n = \pi n$ for $n = 0, 1, ...$ Assume $L \in [x_k, x_{k+1}]$ for some k. We have that

$$\int_L^{+\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x = \int_L^{x_{k+1}} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x + \int_{x_{k+1}}^{\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x.$$

Since the function

$$\phi(y) = \int_y^{x_{k+1}} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x$$

is either increasing or decreasing on $[x_k, x_{k+1}]$ then

$$\Big|\int_L^{+\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x\Big| \le \max\Big(\Big|\int_{x_k}^{\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x\Big|, \Big|\int_{x_{k+1}}^{\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x\Big|\Big),$$

thus it suffice to prove the lemma for $L = x_k$ for some k. We expand the integral in the following way:

$$\int_{x_k}^{+\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x = \sum_{i=k}^{\infty} \int_{x_i}^{x_{i+1}} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x = \sum_{i=k}^{\infty} \int_0^{\pi} \frac{(-1)^i \sin t}{t + \pi i} e^{-r(t+\pi i)}\,\mathrm{d}t$$

$$= \int_0^{\pi} \sin t \sum_{i=k}^{\infty} \frac{(-1)^i}{t + \pi i} e^{-r(t+\pi i)}\,\mathrm{d}t.$$

For a fixed t we have a sign-changing series with decreasing terms with their absolute value, therefore the absolute value of the sum of the series is not bigger than absolute value of its first term, e.i,

$$\Big|\int_{x_k}^{+\infty} \frac{\sin x}{x} e^{-rx}\,\mathrm{d}x\Big| \le \int_0^{\pi} \frac{\sin t}{t + \pi k} e^{-r(t+\pi k)}\,\mathrm{d}t \le \int_0^{\pi} \frac{\sin t}{t}\,\mathrm{d}t \le \pi.$$

□

Lemma 3.5.7. *For any $p \ge 0$ the function $I(p, y) = \int_0^{+\infty} \frac{\sin pt}{t^2+y^2}\,\mathrm{d}t$ is non-negative and decreasing in y in $(0, +\infty)$ and $I(p, y) \le \frac{7p^{\frac{1}{3}}}{y^{\frac{2}{3}}}$.*

Proof. The case $p = 0$ is evident. Suppose now $p > 0$. We make a change of variables $t = \frac{x}{p}$ to get

$$I(p, y) = p \int_0^{+\infty} \frac{\sin x\,\mathrm{d}x}{x^2 + p^2 y^2} = pI(1, py).$$

We consider now $I(1,y)$ for $y>0$. We have

$$I(1,y)=\int_0^{+\infty}\frac{\sin t}{t^2+y^2}\,\mathrm{d}t=\sum_{n=0}^{\infty}\int_0^{2\pi}\frac{\sin t}{(t+2\pi n)^2+y^2}\,\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\int_0^{\pi}\sin t\Big(\frac{1}{(t+2\pi n)^2+y^2}-\frac{1}{(t+\pi(2n+1))^2+y^2}\Big)\,\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\int_0^{\pi}\frac{(2\pi(t+2\pi n)+\pi^2)\sin t}{((t+2\pi n)^2+y^2)((t+\pi(2n+1))^2+y^2)}\,\mathrm{d}t$$

$$=\int_0^{\pi}\sin t\cdot\sum_{n=0}^{\infty}\frac{2\pi(t+2\pi n)+\pi^2}{((t+2\pi n)^2+y^2)((t+\pi(2n+1))^2+y^2)}\,\mathrm{d}t.$$

It is now evident that $I(1,y)$ in nonnegative and decreasing in y in $(0,+\infty)$ and therefore the same does $I(p,y)$. Note that for any $n\geq 1$ and $t\in[0,\pi]$ we have

$$\frac{2\pi(t+2\pi n)+\pi^2}{((t+2\pi n)^2+y^2)((t+\pi(2n+1))^2+y^2)}<\frac{\pi^2(4n+3)}{4\pi^2n^2(4\pi^2n^2+y^2)}$$

$$<\frac{2}{n(2\pi^2n^2+2\pi^2n^2+y^2)}\leq\frac{2}{3n(4\pi^4n^4y^2)^{\frac{1}{3}}}<\frac{1}{9n^2y^{\frac{2}{3}}},$$

hence

$$I(1,y)<\int_0^{\pi}\frac{\sin t(\pi^2+2\pi t)}{(t^2+y^2)((t+\pi)^2+y^2)}\,\mathrm{d}t+\sum_{n=1}^{\infty}\frac{1}{9n^2y^{\frac{2}{3}}}\int_0^{\pi}\sin t\,\mathrm{d}t$$

$$<\int_0^{\pi}\frac{3\pi^2 t}{3\pi^2\big(\frac{t^4y^2}{4}\big)^{\frac{1}{3}}}\,\mathrm{d}t+\frac{4}{9y^{\frac{2}{3}}}<\frac{7}{y^{\frac{2}{3}}}.$$

Finally we have

$$I(p,y)=pI(1,py)<\frac{7p}{(py)^{\frac{2}{3}}}=\frac{7p^{\frac{1}{3}}}{y^{\frac{2}{3}}}$$

$\square$

Lemma 3.5.8. *For any decreasing function $f\in C((0,+\infty),\mathbb{R}^+)$ and numbers $p>0, l\geq 0$ the following inequalities hold:*

$$\Big|\int_l^{+\infty}f(t)\cos pt\,\mathrm{d}t\Big|\leq\frac{4f(l)}{p},\quad\Big|\int_l^{+\infty}f(t)\sin pt\,\mathrm{d}t\Big|\leq\frac{4f(l)}{p}.$$

Proof. We determine the sequence $t_n = \frac{\pi n}{p}$, $n \in \mathbb{N}$. In every interval $[t_n, t_{n+1}]$ the function $\sin pt$ has a constant sign, therefore

$$\int_{t_n}^{t_{n+1}} f(t) \sin pt \, \mathrm{d}t = (-1)^n f(t_n') \int_{t_n}^{t_{n+1}} \sin pt \, \mathrm{d}t = 2 \cdot (-1)^n f(t_n')$$

for some point $t_n' \in [t_n, t_{n+1}]$. We have

$$\int_{t_n}^{+\infty} f(t) \sin pt \, \mathrm{d}t = \frac{2}{p} \sum_{k=n}^{\infty} (-1)^k f(t_k'),$$

thus

$$\Big| \int_{t_n}^{+\infty} f(t) \sin pt \, \mathrm{d}t \Big| \le \frac{2|f(t_n')|}{p} \le \frac{2|f(t_n)|}{p}.$$

Assume now that $l \in [t_m, t_{m+1}]$. It is clear that

$$\Big| \int_{l}^{+\infty} f(t) \sin pt \, \mathrm{d}t \Big| \le \Big| \int_{l}^{t_{m+1}} f(t) \sin pt \, \mathrm{d}t \Big| + \Big| \int_{t_{m+1}}^{+\infty} f(t) \sin pt \, \mathrm{d}t \Big|$$

$$\le f(l) \Big| \int_{t_m}^{t_{m+1}} \sin pt \, \mathrm{d}t \Big| + \frac{2f(t_{m+1})}{p} \le \frac{4f(l)}{p}.$$

The first integral can be estimated in the same way.

□

Lemma 3.5.9. *The following inequalities hold:*

$$\Big| \int_{l}^{+\infty} \int_{0}^{+\infty} \frac{\cos px \cos qy}{x^2 + y^2} \, \mathrm{d}x \, \mathrm{d}y \Big| \le \frac{2\pi}{ql} \quad \textit{for any} \quad p, q, l > 0,$$

$$\Big| \int_{l}^{+\infty} \int_{0}^{+\infty} \frac{\sin px \sin qy}{x^2 + y^2} \, \mathrm{d}x \, \mathrm{d}y \Big| \le \frac{28 p^{\frac{1}{3}}}{q l^{\frac{2}{3}}} \quad \textit{for any} \quad p, q, l > 0,$$

$$\Big| \int_{l}^{+\infty} \int_{0}^{+\infty} \frac{\cos px \sin qy}{x^2 + y^2} \, \mathrm{d}x \, \mathrm{d}y \Big| \le \frac{\pi^2}{2} \quad \textit{for any} \quad p, q, l > 0$$

$$\Big| \int_{l}^{+\infty} \int_{0}^{+\infty} \frac{\sin px \cos qy}{x^2 + y^2} \, \mathrm{d}x \, \mathrm{d}y \Big| \le \frac{28 p^{\frac{1}{3}}}{q l^{\frac{2}{3}}} \quad \textit{for any} \quad p, q, l > 0.$$

Proof. Using (3.11) and Lemma 3.5.8 we get

$$\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\cos px\cos qy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\Big| = \frac{\pi}{2}\Big|\int_l^{+\infty}\frac{e^{-py}\cos qy}{y}\,\mathrm{d}y\Big| \le \frac{\pi}{2}\cdot\frac{4e^{-pl}}{ql} < \frac{2\pi}{ql}.$$

To estimate the second and the forth integrals we use Lemma 3.5.7 and Lemma 3.5.8. We have

$$\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\sin px\sin qy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\Big| = \Big|\int_l^{+\infty}I(p,y)\sin qy\,\mathrm{d}y\Big| \le \frac{4I(p,l)}{q} \le \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}.$$

Similarly

$$\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\sin px\cos qy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\Big| \le \frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}.$$

To estimate the third integral we utilize (3.11) and Lemma 3.5.6, namely

$$\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\cos px\sin qy}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\Big| \le \frac{\pi}{2}\Big|\int_l^{+\infty}\frac{e^{-py}\sin qy}{y}\,\mathrm{d}y\Big| \le \frac{\pi^2}{2}.$$

$\square$

Theorem 3.5.10. *Determine*

$$I^1_\omega(k_1) = \int_0^{+\infty}\int_{\mathbb{R}}\frac{|a|^2}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3, \quad I^2_\omega(k_1) = \int_0^{+\infty}\int_{\mathbb{R}}\frac{|b|^2}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3,$$

$$I^3_\omega(k_1) = \int_0^{+\infty}\int_{\mathbb{R}}\frac{\bar{a}b}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3.$$

Then the following formulae are valid:

$$I^1_\omega(0) = \pi\int_{[0,1]^2}\ln\frac{(y(t)-y(t_1))^2+(z(t)-z(t_1))^2}{(y(t)+y(t_1))^2+(z(t)+z(t_1))^2}z'(t)z'(t_1)\,\mathrm{d}t\,\mathrm{d}t_1, \quad (3.13)$$

$$I^2_\omega(0) = \pi\int_{[0,1]^2}\ln\frac{(y(t)-y(t_1))^2+(z(t)-z(t_1))^2}{(y(t)+y(t_1))^2+(z(t)+z(t_1))^2}y'(t)y'(t_1)\,\mathrm{d}t\,\mathrm{d}t_1, \quad (3.14)$$

$$I^3_\omega(0) = \pi\int_{[0,1]^2}\ln\frac{(y(t)-y(t_1))^2+(z(t)-z(t_1))^2}{(y(t)+y(t_1))^2+(z(t)+z(t_1))^2}y'(t)z'(t_1)\,\mathrm{d}t\,\mathrm{d}t_1. \quad (3.15)$$

Proof. For convenience denote $y(t)$ and $y(t_1)$ by y and y_1 respectively and we make the same notation also for z. We have that

$$|a|^2 = 4\Big(\int_0^1 z'(t)\sin(k_2y(t)+k_3z(t))\,\mathrm{d}t\Big)^2$$

$$= 4\int_0^1\int_0^1 z'(t)z'(t_1)\sin(k_2y(t)+k_3z(t))\sin(k_2y(t_1)+k_3z(t_1))\,\mathrm{d}t\,\mathrm{d}t_1$$

$$= 2\int_0^1\int_0^1 z'z_1'(\cos(k_2(y-y_1)+k_3(z-z_1))-\cos(k_2(y+y_1)+k_3(z+z_1)))\,\mathrm{d}t\,\mathrm{d}t_1. \tag{3.16}$$

We have as well

$$I_\omega^1(0) = \int_0^{+\infty}\int_{\mathbb{R}} \frac{|a|^2}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3 = 2\int_0^1\int_0^1 z'z_1 I_1^\star\,\mathrm{d}t\,\mathrm{d}t_1,$$

where

$$I_1^\star = \int_0^{+\infty}\int_{\mathbb{R}} \frac{\cos(k_2(y-y_1)+k_3(z-z_1))-\cos(k_2(y+y_1)+k_3(z+z_1))}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3.$$

We make the following notation:

$$p=|y-y_1|, q=(z-z_1)\mathsf{sign}(y-y_1), r=|y+y_1|, s=(z+z_1)\mathsf{sign}(y+y_1).$$

Taking into account (3.11) and (3.12) we obtain

$$I_1^\star = \pi\int_0^{+\infty}\frac{1}{k_3}(e^{-pk_3}\cos qk_3 - e^{-rk_3}\cos sk_3)\,\mathrm{d}k_3 = \frac{\pi}{2}\ln\frac{p^2+q^2}{r^2+s^2}.$$

The same can be done also for $I_\omega^2(0)$ and $I_\omega^3(0)$.

□

The next theorem gives upper and lower bounds on I^1, I^2 and I^3.

Theorem 3.5.11. *Assume ω has a diameter d and $l>0$. Then for any $k_1\in[-l,l]$ the following bounds hold:*

$$|I_\omega^1(0)-I_\omega^1(k_1)| \le 8\pi(\pi+3)ld(\mathsf{per}(\partial\omega))^2, \tag{3.17}$$

$$|I_\omega^2(0)-I_\omega^2(k_1)| \le 8\pi(\pi+3)ld(\mathsf{per}(\partial\omega))^2, \tag{3.18}$$

$$|I_\omega^3(0)-I_\omega^3(k_1)| \le 60(ld+4(ld)^{\frac{4}{3}}+3(ld)^{\frac{1}{3}})(\mathsf{per}(\partial\omega))^2. \tag{3.19}$$

Proof. We estimate the difference $|I^1_\omega(k_1) - I^1_\omega(0)|$, the estimate for $I^2_\omega(k_1)$ is straightforward. The validity of the inequality $I^1_\omega(k_1) \leq I^1_\omega(0)$ for any $k_1 \in \mathbb{R}$ is evident. Note that if $k_1 \in [-l, l]$ then

$$I^1_\omega(k_1) \geq \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a|^2}{k_2^2 + (k_3 + l)^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3$$

thus taking account (3.16) we obtain

$$I^1_\omega(0) - I^1_\omega(k_1) \leq \int_0^{+\infty} \int_{\mathbb{R}} |a|^2 \Big(\frac{1}{k_2^2 + k_3^2} - \frac{1}{k_2^2 + (k_3 + l)^2} \Big) \, \mathrm{d}k_2 \, \mathrm{d}k_3$$

$$2 \int_0^1 \int_0^1 |z' z_1'| (|J_1^1| + |J_2^1| + |J_3^1|) \, \mathrm{d}t \, \mathrm{d}t_1,$$

where

$$J_1^1 = \int_0^l \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3) - \cos(rk_2 + sk_3)}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3,$$

$$J_2^1 = \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3) - \cos(pk_2 + q(k_3 - l))}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3,$$

$$J_3^1 = \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(rk_2 + sk_3) - \cos(rk_2 + s(k_3 - l))}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3.$$

We have that

$$|J_1^1| = \Big| 2 \int_0^l \int_0^{+\infty} \frac{\cos pk_2 \cos qk_3 - \cos rk_2 \cos sk_3}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3 \Big|$$

$$= \pi \Big| \int_0^l \frac{e^{-pk_3} \cos qk_3 - e^{-rk_3} \cos sk_3}{k_3} \, \mathrm{d}k_3 \Big|$$

$$\leq \pi \int_0^l \frac{e^{-pk_3} |\cos qk_3 - \cos sk_3|}{k_3} \, \mathrm{d}k_3 + \pi \int_0^l \frac{|\cos sk_3 (e^{-pk_3} - e^{-rk_3})|}{k_3} \, \mathrm{d}k_3 \leq$$

$$\leq 2\pi \int_0^l \frac{\big| \sin \frac{q+s}{2} k_3 \sin \frac{q-s}{2} k_3 \big|}{k_3} \, \mathrm{d}k_3 + \pi \int_0^l \frac{1}{k_3} \Big| \int_p^r \frac{\mathrm{d}}{\mathrm{d}t} (e^{-k_3 t}) \, \mathrm{d}t \Big| \, \mathrm{d}k_3 \leq$$

$$\leq \pi l |q - s| + \pi |p - r| \int_0^l \max(e^{-pk_3}, e^{-rk_3}) \, \mathrm{d}k_3 \leq 4\pi d l.$$

According to Lemma 3.5.9 we have

$$|J_2^1| \leq (1 - \cos ql) \Big| \int_l^{+\infty} \int_{\mathbb{R}} \frac{\cos(pk_2 + qk_3)}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3 \Big|$$

$$
\begin{aligned}
&+|\sin ql|\Big|\int_l^{+\infty}\int_{\mathbb{R}} \frac{\sin(pk_2+qk_3)}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3\Big|\\
&+4\sin^2\frac{ql}{2}\Big|\int_l^{+\infty}\int_0^{+\infty} \frac{\cos pk_2\cos qk_3}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3\Big|\\
&+2|\sin ql|\Big|\int_l^{+\infty}\int_0^{+\infty} \frac{\cos pk_2\sin qk_3}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3\Big|\\
&\le (ql)^2\cdot\frac{2\pi}{ql}+\frac{\pi^2}{2}2ql=\big(2\pi+\pi^2\big)ql\le(4\pi+2\pi^2)dl.
\end{aligned}
$$

Similarly $|J_3^1|\le(4\pi+2\pi^2)dl$. Concluding we obtain

$$
|J_1^1|+|J_2^1|+|J_3^1|\le 4\pi(\pi+3)dl,
$$

thus

$$
I_\omega^1(0)-I_\omega^1(k_1)\le 8\pi(\pi+3)dl\Big(\int_0^1|z'(t)|\,\mathrm{d}t\Big)^2\le 8\pi(\pi+3)dl(\mathsf{per}(\omega))^2.
$$

Analogously we have

$$
I_\omega^2(0)-I_\omega^2(k_1)\le 8\pi(\pi+3)dl(\mathsf{per}(\omega))^2.
$$

To estimate $|I_\omega^3(0)-I_\omega^3(k_1)|$ we recall that $b=\frac{k_3}{k_2}a$, thus

$$
I_\omega^3(k_1)=\int_0^{+\infty}\int_{\mathbb{R}}\frac{k_3|a|^2}{k_2|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3.
$$

Note that the integrand is positive if $k_2>0$ and negative if $k_2<0$, therefore

$$
\begin{aligned}
|I_\omega^3(0)-I_\omega^3(k_1)|\le&\int_0^{+\infty}\int_0^{+\infty}\frac{k_3|a|^2}{k_2}\Big(\frac{1}{k_2^2+k_3^2}-\frac{1}{|k^2|}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3\\
&+\int_0^{+\infty}\int_{-\infty}^{0}\frac{k_3|a|^2}{k_2}\Big(\frac{1}{|k^2|}-\frac{1}{k_2^2+k_3^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3\\
\le&\int_0^{+\infty}\int_0^{+\infty}\bar{a}b\Big(\frac{1}{k_2^2+k_3^2}-\frac{1}{k_2^2+(k_3+l)^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3\\
&+\int_0^{+\infty}\int_{-\infty}^{0}\bar{a}b\Big(\frac{1}{k_2^2+(k_3+l)^2}-\frac{1}{k_2^2+k_3^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3.
\end{aligned}
$$

We have

$$
\int_0^{+\infty}\int_0^{+\infty}\bar{a}b\Big(\frac{1}{k_2^2+k_3^2}-\frac{1}{k_2^2+(k_3+l)^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3
$$

$$\leq 2\int_0^1\int_0^1 |z'y_1'|(|J_1^3|+|J_2^3|+|J_3^3|)\,\mathrm{d}t\,\mathrm{d}t_1,$$

where

$$J_1^3=\int_0^l\int_0^{+\infty}\frac{\cos(pk_2+qk_3)-\cos(rk_2+sk_3)}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3,$$

$$J_2^3=\int_l^{+\infty}\int_0^{+\infty}\frac{\cos(pk_2+qk_3)-\cos(pk_2+q(k_3-l))}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3,$$

$$J_3^3=\int_l^{+\infty}\int_0^{+\infty}\frac{\cos(rk_2+sk_3)-\cos(rk_2+s(k_3-l))}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3.$$

Using lemma 3.5.6 and the estimate for J_1^1 we get

$$|J_1^3|\leq\frac{|J_1^1|}{2}+\Big|\int_0^l I(p,k_3)\sin qk_3\,\mathrm{d}k_3\Big|+\Big|\int_0^l I(r,k_3)\sin sk_3\,\mathrm{d}k_3\Big|$$

$$\leq 2\pi ld+7p^{\frac{1}{3}}\int_0^l\frac{qk_3}{k_3^{\frac{2}{3}}}\,\mathrm{d}k_3+7r^{\frac{1}{3}}\int_0^l\frac{sk_3}{k_3^{\frac{2}{3}}}\,\mathrm{d}k_3<2\pi ld+30(ld)^{\frac{4}{3}}.$$

According to Lemma 3.5.8 we have

$$|J_2^3|\leq 2\sin^2\frac{ql}{2}\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\cos(pk_2+qk_3)}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3\Big|$$

$$+|\sin ql|\Big|\int_l^{+\infty}\int_0^{+\infty}\frac{\sin(pk_2+qk_3)}{k_2^2+k_3^2}\,\mathrm{d}k_2\,\mathrm{d}k_3$$

$$\leq\frac{(ql)^2}{2}\Big(\frac{2\pi}{ql}+\frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}\Big)+ql\Big(\frac{28p^{\frac{1}{3}}}{ql^{\frac{2}{3}}}+\frac{\pi^2}{4}\Big)$$

$$<10(3ld+4(ld)^{\frac{1}{3}}+4(ld)^{\frac{4}{3}}.$$

Similarly we have

$$|J_3^3|<10(3ld+4(ld)^{\frac{1}{3}}+4(ld)^{\frac{4}{3}}.$$

Concluding we obtain

$$\int_0^{+\infty}\int_0^{+\infty}\bar{a}b\Big(\frac{1}{k_2^2+k_3^2}-\frac{1}{k_2^2+(k_3+l)^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3<20(7ld+8(ld)^{\frac{1}{3}}+11(ld)^{\frac{4}{3}})(\mathsf{per}(\partial\omega))^2.$$

The validity of the same estimate for

$$\int_0^{+\infty}\int_{-\infty}^{0} \bar{a}b\Big(\frac{1}{k_2^2+(k_3+l)^2}-\frac{1}{k_2^2+k_3^2}\Big)\,\mathrm{d}k_2\,\mathrm{d}k_3$$

is evident. For I_ω^3 we get

$$|I_\omega^3(0)-I_\omega^3(k_1)| \le 40(7ld+8(ld)^{\frac{1}{3}}+11(ld)^{\frac{4}{3}})(\mathsf{per}(\partial\omega))^2 \quad \text{for any} \quad k_1\in[-l,l].$$

□

Corollary 3.5.12. *Denote $u = d^{\frac{1}{6}}(per(\omega))^2$. Then for sufficiently small d and for any $k_1 \in [-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}]$ we have*

$$|I_\omega^1(0)-I_\omega^1(k_1)| \le u,$$

$$|I_\omega^2(0)-I_\omega^2(k_1)| \le u,$$

$$|I_\omega^3(0)-I_\omega^3(k_1)| \le 200u.$$

In the next step we find an approximation for the magnetostatic energy. For convenience we denote $A_\omega = I_\omega^1(0)$, $B_\omega = I_\omega^2(0)$, $C_\omega = I_\omega^3(0)$. According to Theorem 3.5.10 the parameters A_ω, B_ω and C_ω depend homogeneously on the diameter of ω with exponent 2, namely if $\omega = d\cdot\omega_0$ then $A_\omega = d^2A_{\omega_0}$, $B_\omega = d^2B_{\omega_0}$ and $C_\omega = d^2C_{\omega_0}$. For convenience we put $A_0 = A_{\omega_0}$, $B_0 = B_{\omega_0}$, $C_0 = C_{\omega_0}$.

Theorem 3.5.13. *Suppose $m\colon \mathbb{R}\to\mathbb{R}^3$ is measurable and $m_y, m_z \in L^2(\mathbb{R})$. Define*

$$E_s^\star(m) = \frac{1}{2\pi^2}\Big(A_\omega\int_{\mathbb{R}}|m_y(x)|^2\,\mathrm{d}x + B_\omega\int_{\mathbb{R}}|m_z(x)|^2\,\mathrm{d}x + C_\omega\int_{\mathbb{R}}(\hat{m}_y(x)\overline{\hat{m}_z(x)}+\overline{\hat{m}_y(x)}\hat{m}_z(x))\,\mathrm{d}x\Big).$$

For sufficiently small d the following inequality holds:

$$|E_s(m)-E_s^\star(m)| \le 12u\int_{\mathbb{R}}(|m_y(x)|^2+|m_z(x)|^2)\,\mathrm{d}x + \frac{(A_\omega+B_\omega)E_{ex}(m)}{\pi^2c_\omega d}.$$

Proof. We fix a positive l. We have that

$$E_{ex}(m) = c_\omega d^2\int_{\mathbb{R}}|\partial_x m(x)|^2\,\mathrm{d}x \ge c_\omega d^2\int_{\mathbb{R}}(|\partial_x m_y(x)|^2+|\partial_x m_z(x)|^2)\,\mathrm{d}x$$

$$= c_\omega d^2 \int_{\mathbb{R}} (|\widehat{\partial_x m_y}(x)|^2 + |\widehat{\partial_x m_z}(x)|^2)\,\mathrm{d}x = c_\omega d^2 \int_{\mathbb{R}} |x|^2(|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2)\,\mathrm{d}x$$

$$\geq c_\omega d^2 l^2 \int_{\mathbb{R}\setminus[-l,l]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2)\,\mathrm{d}x,$$

which implies for $l = \frac{1}{\sqrt{d}}$ the following

$$\int_{\mathbb{R}\setminus\left[-\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}}\right]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2)\,\mathrm{d}x \leq \frac{E_{ex}(m)}{c_\omega d}. \tag{3.20}$$

It is clear that for any $k_1 \in \mathbb{R}$

$$|I^3_\omega(k_1)| \leq \int_0^{+\infty}\int_{\mathbb{R}} \frac{|\bar{a}b|}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3 \leq \Big(\int_0^{+\infty}\int_{\mathbb{R}} \frac{|a|^2}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3 \cdot \int_0^{+\infty}\int_{\mathbb{R}} \frac{|b|^2}{|k|^2}\,\mathrm{d}k_2\,\mathrm{d}k_3\Big)^{\frac{1}{2}}$$

$$= (I^2_\omega(k_1)\cdot I^2_\omega(k_1))^2 \leq (A_\omega B_\omega)^{\frac{1}{2}} \leq \frac{A_\omega + B_\omega}{2}. \tag{3.21}$$

Utilizing Corollary 3.5.12 and inequalities (3.21), (3.22) we obtain

$$|E_s(m) - E^\star_s(m)| \leq \frac{1}{2\pi^2}\int_{\mathbb{R}} |A_\omega - I^1_\omega(x)||m_y(x)|^2\,\mathrm{d}x$$

$$+\frac{1}{2\pi^2}\int_{\mathbb{R}} |B_\omega - I^2_\omega(x)|m_z(x)|^2\,\mathrm{d}x + \frac{1}{2\pi^2}\int_{\mathbb{R}} |C_\omega - I^3_\omega(x)||\hat{m}_y(x)\overline{\hat{m}_z(x)} + \overline{\hat{m}_y(x)}\hat{m}_z(x)|\,\mathrm{d}x\Big)$$

$$\leq \frac{u}{2\pi^2}\int_{-\frac{1}{\sqrt{d}}}^{\frac{1}{\sqrt{d}}} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2)\,\mathrm{d}x + \frac{A_\omega}{2\pi^2}\int_{\mathbb{R}\setminus[-\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}}]} |\hat{m}_y(x)|^2\,\mathrm{d}x$$

$$+\frac{B_\omega}{2\pi^2}\int_{\mathbb{R}\setminus[-\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}}]} |\hat{m}_z(x)|^2\,\mathrm{d}x + \frac{200u}{2\pi^2}\int_{-\frac{1}{\sqrt{d}}}^{\frac{1}{\sqrt{d}}} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2)\,\mathrm{d}x$$

$$+\frac{A_\omega + B_\omega}{2\pi^2}\int_{\mathbb{R}\setminus[-\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}}]} (|\hat{m}_y(x)|^2 + |\hat{m}_z(x)|^2\,\mathrm{d}k_1$$

$$\leq 12u\int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2)\,\mathrm{d}x + \frac{(A_\omega + B_\omega)E_{ex}(m)}{\pi^2 c_\omega d}.$$

$\square$

3.6 The existence of minimizers

It is easy to check that Lemma 2.6.1 and Theorem 2.6.2 are valid also for the domains Ω with a bounded Lipschitz cross section ω. In fact in their proofs we did not use that the cross section is rectangular.

Lemma 3.6.1. *Suppose we are given a sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ defined in the same domain Ω and with energies bounded by the same constant C. Then there exists a magnetization $m^0\colon \Omega \to \mathbb{S}^2$ such that for a subsequence of $(m^n)_{n\in\mathbb{N}}$ (not relabeled) the following statements hold*

- $\nabla m^n \rightharpoonup \nabla m^0$ *weakly in* $L^2(\Omega)$
- $m^n \to m^0$ *strongly in* $L^2_{loc}(\Omega)$
- $E(m^0) \leq \liminf E(m^n)$.

Theorem 3.6.2 (Existence of minimixers)**.** *For any domain $\Omega = \mathbb{R}\times\omega$, where ω is bounded and Lipschitz, there exist minimizers of E in $\tilde{A}$ and $\tilde{A}_x$.*

3.7 The Γ-convergence

We start with the description of the full and the reduced variational problems. As we have mentioned the full variational problem is the minimization of the rescaled energy, which is $\frac{E(m)}{d^2}$ in this case. We will scale the magnetizations in the y and z directions to keep the domain fixed in order to pass to the Γ-limit. We define the rescaled magnetization $\acute{m}(x,y,z) = m(x,dy,dz)$. It is clear that $\acute{m}\colon \Omega_0 \to \mathbb{S}^2$ and that the admissible set for the full variational problem is

$$\tilde{A}_1 = \{\acute{m}\colon \Omega_0 \to \mathbb{S}^2 \mid \acute{m} - \bar{e} \in H^1(\Omega_0)\}.$$

It is clear that $m \in \tilde{A}$ if and only if $\acute{m} \in \tilde{A}_1$ and

$$\acute{E}(m) = \int_{\Omega_0} \Big(|\partial_x \acute{m}(\xi)|^2 + \frac{1}{d^2}|\partial_y \acute{m}(\xi)|^2 + \frac{1}{d^2}|\partial_z \acute{m}(\xi)|^2\Big)\,\mathrm{d}\xi + \frac{1}{d^2}E_{mag}(m).$$

The reduced variational problem energy functional is

$$E_0(m) = c_{\omega_0}\int_{\mathbb{R}} |\partial_x m|^2\,\mathrm{d}x + \frac{a_0}{2\pi^2}\int_{\mathbb{R}} (|m_y|^2 + |m_z|^2)\,\mathrm{d}x + \frac{C_0}{2\pi^2 t_0}\int_{\mathbb{R}} |t_0 m_y + m_z|^2\,\mathrm{d}x,$$

where the numbers a_0 and t_0 are defined as follows:

$$t_0 = \frac{A_0 - B_0 + \sqrt{(A_0 - B_0)^2 + 4C_0^2}}{2C_0}$$

and

$$a_0 = A_0 - C_0 t_0.$$

We will show later that $a_0, t_0 > 0$.

The admissible set is

$$\tilde{A}_0 = \{m\colon \mathbb{R} \to \mathbb{S}^2 \mid E_0(m) < \infty, \quad m - \bar{e} \in H^1(\Omega_0)\}.$$

Like in the previous chapter we define additionally the set X_0 as follows:

$$X_0 = \{m\colon \mathbb{R} \to \mathbb{R}^3 \mid \partial_x m \in L^2(\mathbb{R}) \ \text{ and } \ m_y, m_z \in L^2_{loc}(\mathbb{R})\}.$$

The reduced variational problem is to minimize the reduced energy functional E_0 over the admissible set $\tilde{A}_0$. Now we define the notion of convergence of the magnetizations we are going to use for the Γ-convergence of the energies.

Definition 3.7.1. *Let $m^0(x) \in X_0$. Consider a sequence of domain-magnetization pairs (Ω_n, m^n) where $m^n \in \tilde{A}_n$. Define $\acute{m}^n(x,y,z) = m^n(x, d_n y, d_n z)$. Then the sequence $(\acute{m}^n)_{n\in\mathbb{N}}$ is said to converge to m^0 as n goes to infinity if the following statements hold:*

- $\partial_x \acute{m}^n \rightharpoonup \partial_x m^0$ *weakly in* $L^2(\Omega_0)$
- $\nabla_{yz} \acute{m}^n \to 0$ *strongly in* $L^2(\Omega_0)$
- $\acute{m}^n \to m^0$ *strongly in* $L^2_{loc}(\Omega_0)$

Before passing to the main theorem we formulate an auxialary lemma which will allow us to switch from the one variable-dependent case to the general case.

Lemma 3.7.2. *For any Ω and $m \in A(\Omega)$ the following statements hold:*

- *There exists a constant C depending only on the geometry of ω such that $|E_{mag}(m) - E_{mag}(\bar{m})| \le d(Cd + 2\sqrt{C})E(m)$,*
- *If $E(m^n) \le M d_n^2$ for a constant M and $(\acute{m}^n)_{n\in\mathbb{N}}$ converges to m^0 in the cense of Definition 3.7.1 then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}} |\bar{m}_y^n(x)|^2 \,\mathrm{d}x \ge \int_{\mathbb{R}} |\bar{m}_y^0(x)|^2 \,\mathrm{d}x \quad \textit{and} \quad \liminf_{n\to\infty} \int_{\mathbb{R}} |\bar{m}_z^n(x)|^2 \,\mathrm{d}x \ge \int_{\mathbb{R}} |\bar{m}_z^0(x)|^2 \,\mathrm{d}x,$$

- *There exists a constant M_m depending only on m such that*
$$E_v(m) \leq M_m d^3(1+d).$$

Proof. By Poincaré inequality there exists a constant C depending only on the geometry of ω such that for any $x \in \mathbb{R}$

$$\int_{\omega_x} (m-\bar{m})^2 \,\mathrm{d}y\,\mathrm{d}z \leq Cd^2 \int_{\omega_x} |\nabla_{yz} m|^2 \,\mathrm{d}y\,\mathrm{d}z$$

thus

$$\int_{\Omega} (m-\bar{m})^2 \,\mathrm{d}\xi \leq Cd^2 \int_{\Omega} |\nabla_{yz} m|^2 \,\mathrm{d}\xi \leq Cd^2 E_{ex}(m).$$

Utilizing now Lemma 3.4.2 for m and $\bar{m}$ we obtain

$$\frac{|E_{mag}(m) - E_{mag}(\bar{m})|}{d^2} \leq dE(m)(Cd + 2\sqrt{C}).$$

The proof of the second statement can be found in the proof of the lower-semicontinuity part of the first $\Gamma-$convergence theorem in Chapter 2. Recalling the proof of Lemma 2.5.11 we note that the only difference between this case and the rectangular-cross section case is the estimate on

$$\int_{\omega_x} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{(y-y_1)^2 + (z-z_1)^2}}.$$

The domain ω can be put in a square with sides parallel to the y and z axis and lengths d, thus

$$\int_{\omega_x} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{(y-y_1)^2 + (z-z_1)^2}} \leq \int_{[0,d]^2} \frac{\mathrm{d}y\,\mathrm{d}z}{\sqrt{(y-y_1)^2 + (z-z_1)^2}} \leq 10d.$$

□

Theorem 3.7.3 (Γ-convergence)**.** *The reduced variational problem is the Γ-limit of the full variational problem with respect to the convergence defined above. This amounts to the following three statements:*

- ***Lower semicontinuity*** *If a sequence of rescaled magnetizations $(\acute{m}^n)_{n\in\mathbb{N}}$ with $m^n \in A(\Omega_n)$ converges to some $m^0 \in X_0$ in the sense of Definition 3.7.1 then*
$$E_0(m^0) \leq \liminf_{n\to\infty} \acute{E}_n(\acute{m}^n)$$

- ***Construction*** *For every $m^0 \in \tilde{A}_0$ and every infinitesimal sequence of positive numbers $(d_n)_{n\in\mathbb{N}}$, there exists a sequence $(m^n)_{n\in\mathbb{N}}$ with entries in $\tilde{A}(\Omega_n)$ such that*

$$\acute{m}^n \to m^0 \quad \textit{in the cense of Definition 3.7.1}$$

$$E_0(m^0) = \lim_{n\to\infty} \acute{E}_n(\acute{m}^n)$$

- ***Compactness*** *Let $(d_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence of positive numbers. Let $m^n \in \tilde{A}(\Omega_n)$ and let $(\acute{E}_n(m^n))_{n\in\mathbb{N}}$ be bounded. Then there exists a subsequence of $(m^n)_{n\in\mathbb{N}}$(not relabeled again) such that $(\acute{m}^n)_{n\in\mathbb{N}}$ converges to some $m^0 \in \tilde{A}_0$ in the cense of Definition 3.7.1.*

Proof. **Lower semicontinuouty**
The majority of the proof is the same as in the proof of Theorem 2.7.2. The idea is to represent the functional E_s^{star} as a sum of squares of L^2 norms with nonnegative coefficients, which is the key point to the establishment

$$\liminf_{n\to\infty} \frac{E_s^\star(m^n)}{d_n l_n} \geq a_c \int_{\mathbb{R}} |m_y^0|^2 \, \mathrm{d}x + b_c \int_{\mathbb{R}} |m_z^0|^2 \, \mathrm{d}x$$

as soon as we have the convergence $\acute{m}^n \to m^0$ in $L^2_{loc}(\Omega_0)$. To that end we need to first prove some inequalities on A_n, B_n, and C_n. First of all we claim that the numbers A_0 and B_0 are positive (recall that $A_n = d_n^2 A_0$ and $B_n = d_n^2 B_0$.) Indeed, suppose for instance that $A_0 = 0$ for some ω_0. Obviously the set $\tilde{A}_x(\Omega_0)$ is not empty. We fix a magnetization $m^0 \in \tilde{A}_x(\Omega_0)$. We have

$$A_0 = \int_0^{+\infty} \int_{\mathbb{R}} \frac{|a_0|^2}{k_2^2 + k_3^2} \, \mathrm{d}k_2 \, \mathrm{d}k_3 = 0$$

thus $a_0(k_2, k_3, \omega_0) = 0$ a.e. in $\mathbb{R}^2$. We have as well $b_0(k_2, k_3, \omega_0) = \frac{k_2}{k_3} a(k_2, k_3, \omega_0) = 0$ a.e. in $\mathbb{R}^2$. This means that

$$E_s(m^0) = 0 = \int_{\mathbb{R}^3} |\nabla u_s|^2 \, \mathrm{d}\xi \quad \text{thus} \quad \nabla u_s = 0 \ \text{ a.e. in } \ \mathbb{R}^3.$$

According to (2.27) we have

$$\int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi \, \mathrm{d}\xi = \int_{\partial\Omega_0} s\varphi \, \mathrm{d}\xi \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^3)$$

thus $s = 0$ a.e. on $\partial\Omega_0$ which means $m_y^0 = 0$ and $m_z^0 = 0$ a.e. in Ω_0. Taking into account that m_x^0 is a weakly differentiable function of one variable we get that m_x^0 must be continuous in $\mathbb{R}$, therefore it must be identically 1 or -1,

which contradicts the boundary conditions $m_x^0(-\infty) = -1$, $m_x^0(+\infty) = 1$. We distinguish now three different cases.
1) $C_0 = 0$.

If $\liminf_{n\to\infty} \acute{E}(\acute{m}^n) + \infty$ then there is nothing to prove. Assume now that $\liminf_{n\to\infty} \acute{E}(\acute{m}^n) < \infty$. Without loss of generality we can assume that

$$\liminf_{n\to\infty} \acute{E}(\acute{m}^n) = \lim_{n\to\infty} \acute{E}(\acute{m}^n),$$

thus

$$E(m^n) \le M d_n^2$$

for some constant M. According to Lemma 3.7.2 we have

$$E_{mag}(m) - E_{mag}(\bar{m}) = \delta_n \cdot d_n^2, \quad \text{where} \quad \lim_{n\to\infty} \delta_n = 0.$$

We have for sufficiently big n

$$\begin{aligned} M \ge \frac{E(m^n)}{d_n^2} \ge \frac{E_{mag}(m^n)}{d_n^2} &= \frac{E_{mag}(\bar{m}^n)}{d_n^2} + \delta_n \ge \frac{E_s(\bar{m}^n)}{d_n^2} + \delta_n \\ &\ge \frac{A_0}{2\pi^2}\int_{\mathbb{R}} |\bar{m}_y^n(x)|^2 \,\mathrm{d}x + \frac{B_0}{2\pi^2}\int_{\mathbb{R}} |\bar{m}_z^n(x)|^2 \,\mathrm{d}x \\ -12u\int_{\mathbb{R}} &(|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2)\,\mathrm{d}x - \frac{M(A_0+B_0)}{\pi^2 c_\omega} d_n - |\delta_n| \end{aligned}$$

$$\ge \Big(\frac{1}{2\pi^2}\min(A_0,B_0) - 12u\int_{\mathbb{R}}\Big)(|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2)\,\mathrm{d}x - \frac{M(A_0+B_0)}{\pi^2 c_\omega} d_n - |\delta_n| \tag{3.22}$$

thus

$$\int_{\mathbb{R}} (|\bar{m}_y^n(x)|^2 + |\bar{m}_z^n(x)|^2)\,\mathrm{d}x \le \frac{4\pi^2 M}{\min(A_0,B_0)}. \tag{3.23}$$

Utilizing now (3.22) and (3.23) we obtain

$$\liminf_{n\to\infty} \frac{E_{mag}(m^n)}{d_n^2} \ge \liminf_{n\to\infty} \Big(\frac{A_0}{2\pi^2}\int_{\mathbb{R}} |\bar{m}_y^n(x)|^2\,\mathrm{d}x + \frac{B_0}{2\pi^2}\int_{\mathbb{R}} |\bar{m}_z^n(x)|^2\,\mathrm{d}x\Big).$$

By using Lemma 3.7.2 the rest of the proof is analogues to the proof of Theorem 2.7.2.

2) $C_0 > 0$.
This case is a bit more tricky. First we prove that $C_0^2 < A_0 B_0$. We determine

$$C_0^- = \int_0^{+\infty}\int_{-\infty}^0 \frac{k_3|a_0|^2}{k_2(k_2^2+k_3^2)}\,\mathrm{d}k_2\,\mathrm{d}k_3 \text{ and } C_0^+ = \int_0^{+\infty}\int_0^{+\infty} \frac{k_3|a_0|^2}{k_2(k_2^2+k_3^2)}\,\mathrm{d}k_2\,\mathrm{d}k_3,$$

so that $C_0^- \leq 0$, $C_0^+ \geq 0$ and $C_0 = C_0^- + C_0^+$. We have by the Schwartz inequality $|C_0^-|^2 \leq A_0 B_0$, $|C_0^+|^2 \leq A_0 B_0$, moreover in both cases the equality is not possible because as we saw before neither the ratio $\frac{a_0}{b_0}$ is constant, nor any of a_0 and b_0 is identically 0 in the integration regions. Taking into account that $|C_0| \leq \max(|C_0^-|, |C_0^+|)$ we get $C_0^2 < A_0 B_0$. We have furthermore for any positive t_n,

$$\hat{m}_y^n \cdot \overline{\hat{m}_z^n} + \hat{m}_z^n \cdot \overline{\hat{m}_y^n} = \frac{1}{t_n}((t_n\hat{m}_y^n)\cdot \overline{\hat{m}_z^n} + \hat{m}_z^n \cdot (\overline{t_n\hat{m}_y^n}))$$

$$= \frac{1}{t_n}(|t_n\hat{m}_y^n + \hat{m}_z^n|^2 - t_n^2|\hat{m}_y^n|^2 - |\hat{m}_z^n|^2),$$

thus

$$E_s^\star(m^n) = \frac{A_n - t_nC_n}{2\pi^2}\int_{\mathbb{R}} |m_y^n(x)|^2\,\mathrm{d}x + \frac{B_n - \frac{C_n}{t_n}}{2\pi^2}\int_{\mathbb{R}} |m_z^n(x)|^2\,\mathrm{d}x$$
$$+\frac{C_n}{2\pi^2 t_n}\int_{\mathbb{R}} |t_n m_y^n(x) + m_z^n(x)|^2\,\mathrm{d}x.$$

We choose t_n such that

$$A_n - t_nC_n = B_n - \frac{C_n}{t_n} > 0 \quad \text{i.e.,} \quad t_n = \frac{A_n - B_n + \sqrt{(A_n-B_n)^2 + 4C_n^2}}{2C_n}$$

which is possible because $C_n^2 < A_n B_n$. Note that t_n does not depend on n, namely

$$t_n = \frac{A_0 - B_0 + \sqrt{(A_0-B_0)^2 + 4C_0^2}}{2C_0} = t_0. \tag{3.24}$$

We determine $a_n = A_n - t_nC_n = d_n^2(A_0 - t_0C_0)$. With this notation we have

$$E_s^\star(m) = \frac{a_n}{2\pi^2}\int_{\mathbb{R}}(|m_y(x)|^2 + |m_z(x)|^2)\,\mathrm{d}x + \frac{C_n}{2\pi^2 t_0}\int_{\mathbb{R}} |t_0 m_y(x) + m_z(x)|^2\,\mathrm{d}x.$$

Like in the case $C_0 = 0$ we can prove that

$$\liminf_{n\to\infty} \frac{E_{mag}(m^n)}{d_n^2}$$

$$\geq \liminf_{n\to\infty}\Big(\frac{a_0}{2\pi^2}\int_{\mathbb{R}}(|\bar{m}_y^n(x)|^2+|\bar{m}_z^n(x)|^2)\,\mathrm{d}x+\frac{C_0}{2\pi^2 t_0}\int_{\mathbb{R}}|t_0\bar{m}_y^n(x)+\bar{m}_z^n(x)|^2\,\mathrm{d}x\Big)$$

provided

$$\liminf_{n\to\infty}\frac{E(m^n)}{d_n^2}<\infty.$$

The rest is analogous to the proof of Theorem 2.7.2.
3) $C_0<0$.
Note that formula (3.24) defines a negative t_n, thus $\frac{C_n}{t_n}>0$. Note furthermore that $a_0>0$. The rest is analogous to the case $C_0>0$.

<u>**Construction**</u>

As a candidate for m^n we take as usual the constant sequence $m^n\equiv m^0$. The only difference from the "construction" part of the proof of Theorem 2.7.2 is the upper bounds on $E_s(m^n)$ and $E_v(m^n)$. In the lower-semi-continuity part we showed that for big n we have

$$\int_{\mathbb{R}}(|\bar{m}_y^n(x)|^2+|\bar{m}_z^n(x)|^2)\,\mathrm{d}x\leq\frac{4\pi^2 M}{a_0}$$

thus utilizing Theorem 3.5.13 we obtain

$$\limsup_{n\to\infty}\frac{E_s(m^n)}{d_n^2}\leq\limsup_{n\to\infty}\frac{E_s^\star(m^n)}{d_n^2}=E_0(m^0)-c_{\omega_0}\int_{\mathbb{R}}|\partial_x m^0(x)|^2\,\mathrm{d}x.$$

We have as well according to Lemma 3.7.2

$$0\leq\lim_{n\to\infty}\frac{E_v(m^n)}{d_n^2}\leq\lim_{n\to\infty}M_{m^0}\cdot d_n(1+d_n)=0.$$

The last two inequalities complete the proof.

<u>**Compactness**</u>

The proof of this part is completely similar to the one of Theorem 2.7.2.

Corollary 3.7.4. *If a sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ satisfies $E(m^n)\leq Cd_n^2$ for some constant C then*

$$E(m^n)\geq\int_{\Omega_n}|\nabla m^n|^2+\frac{a_0}{2\pi^2 c_{\omega_0}}\int_{\Omega_n}(|m_y^n|^2+|m_z^n|^2)+\alpha_n\cdot d_n^2,$$

where $\alpha_n\to 0$ as $n\to\infty$

□

3.7.1 The minimal energy scaling

For convenience we put $a_{\omega_0} = a_0$ and $b_{\omega_0} = \frac{C_0}{2\pi^2 t_0}$. We minimize the limit energy

$$E_0(m) = c_{\omega_0}\int_{\mathbb{R}} |\partial_x m(x)|^2\,\mathrm{d}x + \frac{a_{\omega_0}}{2\pi^2}\int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2)\,\mathrm{d}x + b_{\omega_0}\int_{\mathbb{R}} |t_0 m_y(x) - m_z(x)|^2\,\mathrm{d}x.$$

As we saw in subsection 2.8.1 in Chapter 2 the only minimizer of this functional is the vector

$$m_{\omega_0} = (\sin\varphi_{\omega_0}(x), \cos\varphi_{\omega_0}(x)\cos\theta_{\omega_0}, \cos\varphi_{\omega_0}(x)\sin\theta_{\omega_0}),$$

where

$$\varphi_{\omega_0}(x) = \arcsin\frac{e^{2\sqrt{\alpha}x} - 1}{e^{2\sqrt{\alpha}x} + 1}\quad ,\alpha = \frac{a_{\omega_0}}{2\pi^2 c_{\omega_0}}\ \text{and}\ \ \theta_{\omega_0} = \arctan t_0.$$

The minimum of the limit energy is then

$$E_0^{min} = \frac{2\sqrt{2c_{\omega_0}a_{\omega_0}}}{\pi}.$$

In conclusion we mention that like in Chapter 2 we can state that

$$\lim_{n\to\infty}\frac{E_n^{min}}{d_n^2} = \frac{2\sqrt{2c_{\omega_0}a_{\omega_0}}}{\pi}. \tag{3.25}$$

3.8 The rate of convergence

In this section we find a rate of convergence for limit (3.25). Theorem 3.5.13 will be useful to bound the energy functional from below and above. We first bound the minimal energy from above. Suppose we are given a centrally symmetric, bounded Lipschitz domain $\omega \in \mathbb{R}^2$. Consider any infinitesimal sequence of positive numbers $(d_n)_{n\in\mathbb{N}}$ and the sequence of domains $\Omega_n = \mathbb{R}\times(d_n\cdot\omega.)$ Consider furthermore the corresponding sequence of minimal energies $E^{min}(\Omega_n)$. We consider as usual the constant sequence of magnetizations $m^n \equiv m^\omega$ regarding m^n as a magnetization defined in Ω_n, where m^ω is a minimizer of the limit energy. It is clear that

$$m^n \in \tilde{A}_n \quad \text{and} \quad E^{min}(\Omega_n) \le E(m^n).$$

We estimate now $E(m^n)$ from above. We have that

$$E_{ex}(m^n) = c_\omega d_n^2 \int_{\mathbb{R}} |\partial_x m^\omega|^2\,\mathrm{d}x.$$

According to Lemma 3.7.2 we have

$$E_v(m^n) \leq M_{m^\omega} d_n^3 (1 + d_n).$$

m^{ω_0} is the minimizer of the limit energy E_0, thus

$$\frac{E_{ex}(m^n)}{d_n^2} = c_\omega \int_{\mathbb{R}} |\partial_x m^\omega|^2 \,\mathrm{d}x = \frac{\sqrt{2c_\omega a_\omega}}{\pi} \tag{3.26}$$

and

$$E_s^\star(m^n) = \frac{d_n^2 a_\omega}{2\pi^2} \int_{\mathbb{R}} (|m_y^\omega|^2 + |m_z^\omega|^2) \,\mathrm{d}x = \frac{d_n^2 E_0(m^\omega)}{2} = \frac{d_n^2 \sqrt{2c_\omega a_\omega}}{\pi},$$

hence

$$\int_{\mathbb{R}} (|m_y^\omega|^2 + |m_z^\omega|^2) \,\mathrm{d}x = 2\pi \sqrt{\frac{2c_\omega}{a_\omega}}. \tag{3.27}$$

Taking into account (3.27) and Theorem 3.5.13 we get

$$E_s(m^n) \leq E_s^\star(m^n) + 24\pi \sqrt{\frac{2c_\omega}{a_\omega}} \cdot u_n + \frac{d_n^3 (A_\omega + B_\omega)\sqrt{2a_\omega}}{\pi^3 \sqrt{c_\omega}}.$$

Recall that $u_n = d_n^{\frac{1}{6}} (\mathrm{per}(\omega_n))^2 = d_n^{\frac{13}{6}} (\mathrm{per}(\omega))^2$, therefore for big n we discover

$$\frac{E(m^n)}{d_n^2} - \frac{2\sqrt{2c_\omega a_\omega}}{\pi} \leq 25\pi \sqrt{\frac{2c_\omega}{a_\omega}} (\mathrm{per}(\omega))^2 d_n^{\frac{1}{6}}. \tag{3.28}$$

Suppose now $m \colon \mathbb{R} \to \mathbb{R}^3$ is bounded, measurable and $\partial_x m, m_y, m_z \in L^2(\mathbb{R})$. Assume furthermore that m regarded as a vector field from Ω_n to $\mathbb{R}^3$ has the energy $E_n(m)$ satisfying the condition

$$E_n(m) \leq M \cdot d_n^2 \quad \text{for any} \quad n \in \mathbb{N}$$

for a constant M. Then we have according to Theorem 3.5.13 that

$$E_s(m) \geq E_s^\star(m) - 12u_n \int_{\mathbb{R}} (|m_y|^2 + |m_z|^2) \,\mathrm{d}x - \frac{(A_{\omega_n} + B_{\omega_n}) E_{ex}(m)}{\pi^2 c_\omega d_n^2}. \tag{3.29}$$

We have as well that

$$E_s^\star(m) \geq \frac{d_n^2 a_\omega}{2\pi^2} \int_{\mathbb{R}} (|m_y|^2 + |m_z|^2) \,\mathrm{d}x,$$

thus we obtain for big n

$$\int_{\mathbb{R}}(|m_y|^2+|m_z|^2)\,\mathrm{d}x\leq\frac{3\pi^2M}{a_\omega}.$$

Coupling the last inequality with (3.29) we establish for big n

$$E_s(m)\geq E_s^\star(m)-\frac{40\pi^2M}{a_\omega}\cdot u_n. \tag{3.30}$$

Assume now $m^n\in\tilde{A}_n$ is a minimizer of the energy functional. We showed in the proof of Theorem 3.5.1 that in this case $\partial_x\bar{m}^n,\bar{m}_y^n,\bar{m}_z^n\in L^2(\mathbb{R})$. Utilizing (3.25) and (3.30) we discover for big n

$$E_s(m)\geq E_s^\star(m)-\frac{40\pi^2M}{a_\omega}\cdot u_n\quad\text{where}\quad M=\frac{3\sqrt{2a_\omega c_\omega}}{\pi}.$$

For the energy functional of $\bar{m}^n$ we obtain

$$E(\bar{m}^n)\geq E_{ex}(\bar{m}^n)+E_s^\star(\bar{m}^n)-119\pi^2\sqrt{\frac{2c_\omega}{a_\omega}}\cdot u_n$$

Recall that

$$\lim_{x\to\pm\infty}\bar{m}^n(x)=\pm1,$$

thus we get by Lemma 3.9.5

$$\frac{E(\bar{m}^n)}{d_n^2}-\frac{2\sqrt{2a_\omega c_\omega}}{\pi}\geq-119\pi^2\sqrt{\frac{2c_\omega}{a_\omega}}\cdot(\mathrm{per}(\omega))^2d_n^{\frac{1}{6}}. \tag{3.31}$$

Utilizing (2.49) we get for the energy of m^n

$$E(m^n)\geq E_{ex}(\bar{m}^n)+E_{mag}(m^n)\geq E(\bar{m}^n)-3M\sqrt{\acute{C}}d_n^3.$$

The last inequality and (3.31) imply for big n the following

$$\frac{E(m^n)}{d_n^2}-\frac{2\sqrt{2a_\omega c_\omega}}{\pi}\geq-120\pi^2\sqrt{\frac{2c_\omega}{a_\omega}}\cdot(\mathrm{per}(\omega))^2d_n^{\frac{1}{6}} \tag{3.32}$$

Finally we have by (3.28) and (3.32) that for big n the following bound holds

$$\Big|\frac{E(m^n)}{d_n^2}-\frac{2\sqrt{2a_\omega c_\omega}}{\pi}\Big|\leq120\pi^2\sqrt{\frac{2c_\omega}{a_\omega}}\cdot(\mathrm{per}(\omega))^2d_n^{\frac{1}{6}} \tag{3.33}$$

3.9 The convergence of almost minimizers

Throughout this section we will consider a sequence of domain-magnetization-energy triples $(\Omega_n, m^n, E(m^n))_{n\in\mathbb{N}}$ such that $\Omega_n = d_n \cdot \Omega_0$, $m^n \in \tilde{A}_n$, the sequence $(d_n)_{n\in\mathbb{N}}$ converges to zero and

$$\lim_{n\to\infty} \frac{E(m^n)}{d_n^2} = E_0^{min}. \tag{3.34}$$

We will call such a sequence an almost minimizing sequence.

Lemma 3.9.1. *If $(\acute{m}^n)_{n\in\mathbb{N}}$ converges to some $m^0 \in \tilde{A}_0$ in the sense of Definition 3.7.1 then*

- $$\lim_{n\to\infty} \frac{1}{d_n^2} \int_{\Omega_n} |\nabla m^n(\xi)|^2 \,\mathrm{d}\xi = \int_{\Omega_0} |\partial_x m^0(\xi)|^2 \,\mathrm{d}\xi$$
- $$\lim_{n\to\infty} \frac{1}{d_n^2} \int_{\Omega_n} |\bar{m}_y^n(\xi)|^2 \,\mathrm{d}\xi = \int_{\Omega_0} |m_y^0(\xi)|^2 \,\mathrm{d}\xi$$
- $$\lim_{n\to\infty} \frac{1}{d_n^2} \int_{\Omega_n} |\bar{m}_z^n(\xi)|^2 \,\mathrm{d}\xi = \int_{\Omega_0} |m_z^0(\xi)|^2 \,\mathrm{d}\xi.$$

Proof. We have already shown that the above limits with $\liminf$ are big or equal than the corresponding expected limits, thus it remains to only show the opposite inequalities with $\limsup$. Since

$$\lim_{n\to\infty} \frac{E(m^n)}{d_n^2} = E_0^{min}$$

we have

$$\lim_{n\to\infty} \frac{E_{mag}(m^n)}{d_n^2} = \lim_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2}.$$

Assume in contradiction that one of the three inequalities fails. Therefore we have for some $\delta > 0$

$$\limsup_{n\to\infty} \frac{E(m^n)}{d_n^2} \geq \max\Big(\limsup_{n\to\infty} \int_{\Omega_0} \|\partial_x \acute{m}^n(\xi)|^2 \,\mathrm{d}\xi + \liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2},$$

$$\liminf_{n\to\infty} \int_{\Omega_0} |\partial_x \acute{m}^n(\xi)|^2 \,\mathrm{d}\xi + \limsup_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{d_n^2} \Big)$$

$$\geq \int_{\Omega_0} |\partial_x m^0(\xi)|^2 \,\mathrm{d}\xi + \frac{a_{\omega_0}}{2\pi^2}\int_{\mathbb{R}} (|m_y^0(x)|^2 + |m_z^0(x)|^2)\,\mathrm{d}x + \delta \geq \frac{2\sqrt{2c_{\omega_0}a_{\omega_0}}}{\pi} + \delta$$

which contradicts (3.27).

□

Corollary 3.9.2. *Let* $(m^n)_{n\in\mathbb{N}}$ *and* m^0 *be as in Lemma 3.9.1. Then*

- $$\lim_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |m_y^n(\xi)|^2 \,\mathrm{d}\xi = \int_{\Omega_0} |m_y^0(\xi)|^2 \,\mathrm{d}\xi,$$

- $$\lim_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |m_z^n(\xi)|^2 \,\mathrm{d}\xi = \int_{\Omega_0} |m_z^0(\xi)|^2 \,\mathrm{d}\xi.$$

Proof. It follows from Lemmas 3.9.1 and 3.4.3

□

Lemma 3.9.3. *Let* $(m^n)_{n\in\mathbb{N}}$ *and* m^0 *be as in Lemma 3.9.1. Then*

- $$\lim_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |\nabla m^n(\xi) - \nabla m^0(\xi)|^2 \,\mathrm{d}\xi = 0$$

- $$\lim_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |m_y^n(\xi) - m_y^0(\xi)|^2 \,\mathrm{d}\xi = 0, \quad \lim_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |m_z^n(\xi) - m_z^0(\xi)|^2 \,\mathrm{d}\xi = 0.$$

Proof. We have that

$$\frac{1}{d_n^2}\int_{\Omega_n} |\nabla m^n(\xi) - \nabla m^0(\xi)|^2 \,\mathrm{d}\xi = \frac{1}{d_n^2}\int_{\Omega_n} |\nabla_{yz} m^n(\xi)|^2 \,\mathrm{d}\xi + \frac{1}{d_n^2}\int_{\Omega_n} |\partial_x (m^n(\xi) - m^0(\xi))|^2 \,\mathrm{d}\xi$$

$$= \frac{1}{d_n^2}\int_{\Omega_n} |\nabla_{yz} m^n(\xi)|^2 \,\mathrm{d}\xi + \Big(\frac{1}{d_n^2}\int_{\Omega_n} |\partial_x m^n(\xi)|^2 \,\mathrm{d}\xi - \int_{\Omega_0} |\partial_x m^0(\xi)|^2 \,\mathrm{d}\xi\Big)$$

$$+2\frac{1}{d_n^2}\int_{\Omega_0} \partial_x m_0(\xi)(\partial_x m^0(\xi) - \partial_x \acute{m}^n(\xi))\,\mathrm{d}\xi.$$

We have that each summand converges to zero and thus the same does the sum. In the next step we fix $l > 0$. We have

$$\limsup_{n\to\infty} \frac{1}{d_n^2}\int_{\Omega_n} |m_y^n(\xi) - m_y^0(\xi)|^2 \,\mathrm{d}\xi \leq \limsup_{n\to\infty} \int_{[-l,l]\times\omega_0} |\acute{m}_y^n(\xi) - m_y^0(\xi)|^2 \,\mathrm{d}\xi$$

$$+\limsup_{n\to\infty}\int_{\Omega_0\setminus[-l,l]\times\omega_0}|\acute{m}_y^n(\xi)-m_y^0(\xi)|^2\,\mathrm{d}\xi\leq 2\limsup_{n\to\infty}\int_{\Omega_0\setminus[-l,l]\times\omega_0}(|\acute{m}_y^n(\xi)|^2+|m_y^0(\xi)|^2)\,\mathrm{d}\xi$$

$$\leq 2\limsup_{n\to\infty}\Big(\frac{1}{d_n^2}\int_{\Omega_n}|m_y^n(\xi)|^2\,\mathrm{d}\xi+\int_{\Omega_0}|m_y^0(\xi)|^2\,\mathrm{d}\xi\Big)-2\liminf_{n\to\infty}\int_{[-l,l]\times\omega_0}(|\acute{m}_y^n(\xi)|^2+|m_y^0(\xi)|^2)\,\mathrm{d}\xi$$

$$=4|\omega_0|\int_{\mathbb{R}\setminus[-l,l]}|m_y^0(x)|^2\,\mathrm{d}x,$$

thus using the arbitrariness of l we get the validity of the second statement. The same can be done also for the third components of m^n and m^0.

□

Lemma 3.9.4. *Let $(m^n)_{n\in\mathbb{N}}$ and m^0 be as in Lemma 3.9.1. Assume in addition that for some $N\in\mathbb{N}$ and $l>0$ we have for all $n\geq N$*

$$\bar{m}^n(x)\leq 0,\ x\in(-\infty,-l]\quad\textit{and}\quad\bar{m}^n(x)\geq 0,\ x\in[l,+\infty).$$

Then

$$\lim_{n\to\infty}\frac{1}{d_n}\|m^n-m^0\|_{H^1(\Omega_n)}=0.$$

Proof. According to Lemma 3.9.2 it suffice to show that

$$\lim_{n\to\infty}\frac{1}{d_n^2}\int_{\Omega_n}|m_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi=0.$$

Since $m^0\in\tilde{A}_0$ there exists $l_1>0$ such that

$$m_x^0(x)\leq-\frac{1}{2}\quad x\in(-\infty,l_1]\quad\text{and}\quad m_x^0(x)\geq\frac{1}{2}\quad x\in[l_1,+\infty).$$

For any fixed $l_2>\max(l,l_1)$ we have that

$$\frac{1}{d_n^2}\int_{\Omega_n}|m_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi$$

$$=\int_{[-l_2,l_2]\times\omega_0}|\acute{m}_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi+\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}|m_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi.$$

The first summand converges to zero as n goes to infinity and we have furthermore that

$$\lim_{n\to\infty}\frac{1}{d_n^2}\int_{\Omega_n}|m_x^n(\xi)-\bar{m}_x^n(\xi)|^2\,\mathrm{d}\xi=0,$$

thus it suffice to show that

$$\lim_{n\to\infty}\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}|\bar{m}_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi=0.$$

For $n \geq N$ we have

$$\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}|\bar{m}_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi \leq \frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}||\bar{m}_x^n(\xi)|^2-|m_x^0(\xi)|^2|\,\mathrm{d}\xi$$

$$\leq \frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}||\bar{m}_x^n(\xi)|^2-|m_x^n(\xi)|^2|\,\mathrm{d}\xi+\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}||m_x^n(\xi)|^2-|m_x^0(\xi)|^2|\,\mathrm{d}\xi.$$

The first summand converges to zero, for the second summand we have

$$\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}||m_x^n(\xi)|^2-|m_x^0(\xi)|^2|\,\mathrm{d}\xi$$

$$\leq \frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}(|m_y^n(\xi)|^2+|m_z^n(\xi)|^2+|m_y^0(\xi)|^2+|m_z^0(\xi)|^2)\,\mathrm{d}\xi,$$

thus utilizing Lemma 3.9.1 and Corollary 3.9.2 we obtain

$$\limsup_{n\to\infty}\frac{1}{d_n^2}\int_{\Omega_n\setminus[-l_2,l_2]\times\omega_n}|\bar{m}_x^n(\xi)-m_x^0(\xi)|^2\,\mathrm{d}\xi \leq \limsup_{n\to\infty}\frac{1}{d_n^2}\int_{\Omega_n}(|m_y^n(\xi)|^2+|m_z^n(\xi)|^2)\,\mathrm{d}\xi$$

$$+\int_{\Omega_0}(|m_y^0(\xi)|^2+|m_z^0(\xi)|^2)\,\mathrm{d}\xi-\liminf_{n\to\infty}\frac{1}{d_n^2}\int_{[-l_2,l_2]\times\omega_n}(|m_y^n(\xi)|^2+|m_z^n(\xi)|^2)\,\mathrm{d}\xi$$

$$-\int_{[-l_2,l_2]\times\omega_0}(|m_y^0(\xi)|^2+|m_z^0(\xi)|^2)\,\mathrm{d}\xi = 2\int_{\mathbb{R}\setminus[-l_2,l_2]\times\omega_0}(|m_y^0(\xi)|^2+|m_z^0(\xi)|^2)\,\mathrm{d}\xi$$

which converges to zero as l_2 goes to infinity.

$\square$

Lemma 3.9.5. *Assume that $\omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain. Then for any interval $(a,b)\subset\mathbb{R}$, positive α and a vector field $f \in H^1\big((a,b)\times\omega,\mathbb{R}^3\big)$ the following inequality holds:*

$$\int_{(a,b)\times\omega}|\partial_x f(\xi)|^2\,\mathrm{d}\xi+\alpha^2\int_{(a,b)\times\omega}(|f_y(\xi)|^2+|f_z(\xi)|^2)\,\mathrm{d}\xi \geq 2\alpha|\omega||\bar{f}_x(a)-\bar{f}_x(b)|.$$

(The endpoints a and b can take values $-\infty$ and $+\infty$ respectively).

Proof. We fix a point $(y,z)\in\omega$ and consider the vector field f on the segment with endpoints (a,y,z) and (b,y,z). Being an H^1 vector field, it must be absolutely continuous on that segment as a function on one variable, thus denoting

$$m_x(x,y,z)=\sin\varphi(x),\;\; m_y(x,y,z)=\cos\varphi(x)\cos\theta(x),\;\; m_z(x,y,z)=\cos\varphi(x)\sin\theta(x)$$

we obtain that φ and θ are differentiable in $[a,b]$ a.e.. Thus we can calculate

$$\int_{(a,b)\times(y,z)} |\partial_x f(\xi)|^2 \,\mathrm{d}x + \alpha^2 \int_{(a,b)\times(y,z)} (|f_y(\xi)|^2 + |f_z(\xi)|^2)\,\mathrm{d}x$$

$$= \int_a^b (\varphi\prime^2(x) + \theta\prime^2(x)\cos^2\varphi(x))\,\mathrm{d}x + \alpha^2 \int_a^b \cos^2\varphi(x)\,\mathrm{d}x$$

$$\geq \int_a^b (\varphi\prime^2(x)\,\mathrm{d}x + \alpha^2 \int_a^b \cos^2\varphi(x)\,\mathrm{d}x \geq 2\alpha\Big|\int_a^b \varphi\prime(x)\cos\varphi(x)\,\mathrm{d}x\Big|$$

$$= 2\alpha|f_x(a,y,z) - f_x(b,y,z)|.$$

Integrating now the obtained inequality over ω we get

$$\int_{(a,b)\times\omega} |\partial_x f(\xi)|^2 \,\mathrm{d}\xi + \alpha^2 \int_{(a,b)\times\omega} (|f_y(\xi)|^2 + |f_z(\xi)|^2)\,\mathrm{d}\xi$$

$$\geq 2\alpha \int_\omega |f_x(a,y,z) - f_x(b,y,z)|\,\mathrm{d}y\,\mathrm{d}z$$

$$\geq 2\alpha\Big|\int_\omega (f_x(a,y,z) - f_x(b,y,z))\,\mathrm{d}y\,\mathrm{d}z\Big| = 2\alpha|\omega||\bar{f}_x(a) - \bar{f}_x(b)|.$$

□

Lemma 3.9.6. *Let the sequence of intervals* $\big([b_n^1, b_n^2]\big)_{n\in\mathbb{N}}$ *be such that*

$$\bar{m}_x^n(b_n^1) = -\frac{1}{2}, \quad \bar{m}_x^n(b_n^2) = \frac{1}{2} \quad \textit{and} \quad |\bar{m}_x^n(x)| \leq \frac{1}{2} \quad, x \in [b_n^1, b_n^2].$$

Then for sufficiently big n we have

$$\bar{m}_x^n(x) < -\frac{1}{3}, \quad x \in (-\infty, b_n^1] \quad \textit{and} \quad \bar{m}_x^n(x) > \frac{1}{3}, \quad x \in [b_n^2, +\infty).$$

Proof. Assume in contradiction that for some subsequence (not relabeled) there is a point $b_n^3 \in (-\infty, b_n^1)$ such that $\bar{m}_x^n(b_n^3) \geq -\frac{1}{3}$. Since $\bar{m}_x^n(-\infty) = -1$ and $\bar{m}_x^n$ is continuous we can without loss of generality assume that $\bar{m}_x^n(b_n^3) = -\frac{1}{3}$. Utilizing Lemma 3.9.5 for the intervals $(-\infty, b_n^3]$, $[b_n^3, b_n^1]$, $[b_n^1, +\infty)$ and Corollary 3.7.4 we get

$$E(m^n) \geq \int_{\Omega_n} |\nabla m^n|^2 + \frac{a_{\omega_0}}{2\pi^2 c_{\omega_0}} \int_{\Omega_n} (|m_y^n|^2 + |m_z^n|^2) + \alpha_n \cdot d_n^2 \geq$$

$$\geq \frac{2}{\pi}\sqrt{\frac{a_{\omega_0}}{2c_{\omega_0}}}|\omega_n|\Big(\big|-1+\frac{1}{3}\big| + \big|-\frac{1}{3}+\frac{1}{2}\big| + \big|-\frac{1}{2}-1\big|\Big) + \alpha_n \cdot d_n^2 =$$

$$= \frac{7\sqrt{2a_{\omega_0}c_{\omega_0}}}{3\pi}d_n^2 + \alpha_n \cdot d_n^2,$$

thus

$$\liminf_{n\to\infty} \frac{E(m^n)}{d_n^2} \geq \frac{7}{6}E^0_{min}$$

which is a contradiction.

□

Theorem 3.9.7. *Assume that the domain* ω_0 *is so that* $C_0^2 + (A_0 - B_0)^2 > 0$. *Then for any sequence of magnetizations* $(m^n)_{n\in\mathbb{N}}$ *satisfying (3.34) there exist a sequence* $(T_n)_{n\in\mathbb{N}}$ *of translations in the variable* x *and a sequence* $(R_n)_{n\in\mathbb{N}}$ *of rotations in the* OYZ *plane, each of which is either the identity or the rotation by* 180 *degree such that the sequence with the terms* $\tilde{m}^n(x,y,z) = m^n(T_n(R_n(x,y,z)))$ *converges to some* $m^0 \in \tilde{A}_0$ *in the sense of Definition 3.7.1.*

Proof. First of all note that the change of variables mentioned in the theorem translate the domain Ω to itself and preserve the energy. Let the intervals $[b_n^1, b_n^2]$ be as in Lemma 3.9.6. We prove the theorem by constructing such sequences. In the first step we prove that if a sequence of magnetizations converges to some $m^0 \in \tilde{A}_0$ in the sense of Definition 3.7.1 and satisfies the conditions $E(m^n) \leq Md_n^2$ and $\bar{m}_y^n(x_0) \geq 0$ for some $x_0 \in \mathbb{R}, M > 0$ and for big n then $m_y^0(x_0) \geq 0$. Assume in contradiction that $m_y^0(x_0) = \delta < 0$. We have for big n

$$\int_{[x_0-1,x_0+1]\times\omega_0} |\acute{m}_y^n(\xi) - m_y^0(\xi)|^2 \,\mathrm{d}\xi = \beta_n \to 0$$

and by the Poincaré inequality

$$\int_{[x_0-1,x_0+1]\times\omega_n} |m_y^n(\xi) - \bar{m}_y^n(\xi)|^2 \,\mathrm{d}\xi \leq Cd_n^2 \int_{\Omega_n} |\nabla_{yz} m^n(\xi)|^2 \,\mathrm{d}\xi \leq MCd_n^2$$

for some $C > 0$. Combining this two we get

$$\int_{x_0-1}^{x_0+1} |\bar{m}_y^n(x) - m_y^0(x)|^2 \,\mathrm{d}x \leq \frac{(\sqrt{MC} + \sqrt{\beta_n})^2}{|\omega_0|} \longrightarrow_{n\to\infty} 0. \tag{3.35}$$

On the other hand we have

$$\int_{\Omega_n} |\partial_x \bar{m}^n(\xi)|^2 \,\mathrm{d}\xi \leq \int_{\Omega_n} |\partial_x m^n(\xi)|^2 \,\mathrm{d}\xi \leq Md_n^2,$$

thus

$$\int_{\mathbb{R}} |\partial_x \bar{m}^n(x)|^2 \,\mathrm{d}x \leq \frac{M}{|\omega_0|} = M_1.$$

We have furthermore for any $x_1 < x_2$

$$|\bar{m}_y^n(x_1) - \bar{m}_y^n(x_2)| \leq \int_{x_1}^{x_2} |\partial_x \bar{m}_y^n(x)| \,\mathrm{d}x \leq \Big(\int_{x_1}^{x_2} \mathrm{d}x \int_{x_1}^{x_2} |\partial_x \bar{m}_y^n(x)|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}$$
$$\leq \sqrt{M_1(x_2 - x_1)}$$

which gives

$$\bar{m}_y^n(x) \geq \frac{\delta}{3} \quad \text{for all} \quad x \in \big[x_0 - \frac{\delta^2}{9M_1}, x_0 + \frac{\delta^2}{9M_1}\big].$$

Since m^0 is continuous there exists $\epsilon > 0$ such that

$$m_y^0(x) \leq \frac{2\delta}{3} \quad \text{for all} \quad x \in [x_0 - \epsilon, x_0 + \epsilon].$$

Combining the last inequality with the inequality for $\bar{m}_y^n$ we obtain

$$\int_{x_0-1}^{x_0+1} |\bar{m}_y^n(x) - m_y^0(x)|^2 \,\mathrm{d}x \geq 2\frac{\delta^2}{9} \min(\epsilon, \frac{\delta^2}{9M_1}, 1)$$

which contradicts Lemma 3.9.1. The same sing preserving property can be also proved for the first and the third component of $\bar{m}^n$ and also for the opposite sign. This means in particular that if $\bar{m}_x^n(x_0) = 0$ for big n then $m_x^0(x_0) = 0$. In the second step we construct the sequences $(T_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$. Let the intervals $[b_n^1, b_n^2]$ be as in Lemma 3.9.6 and $x_n \in [b_n^1, b_n^2]$ be such that $\bar{m}_x^n(x_n) = 0$. By continuity such intervals and points exist for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ we choose T_n to be the translation by x_n and the rotation R_n to be the identity if $\bar{m}_y^n(x_n) \geq 0$ and the rotation by 180 degree otherwise. We show now that the sequence $(\tilde{m}^n)_{n\in\mathbb{N}}$ converges to some $m^0 \in \tilde{A}_0$ in the sense of Definition 3.7.1. Utilizing the Γ-convergence theorem we get that the sequence $(\tilde{m}^n)_{n\in\mathbb{N}}$ is relatively compact thus what we have to actually show now is that every convergent subsequence (in the sense of Definition 3.7.1) of it has the same limit. Suppose $(\tilde{m}^{n_k})_{k\in\mathbb{N}}$ converges to some $m^0 \in A_0$. We first show that $m^0 \in \tilde{A}_0$. Lemma 3.4.4 states that there exists $M_2 > 0$ such that $b_{n_k}^2 - b_{n_k}^1 \leq M_2$ for any $k \in \mathbb{N}$, therefore utilizing Lemma 3.9.6 we obtain that $\bar{\tilde{m}}_x^{n_k}$ is negative in $(-\infty, -M_2]$ and is positive in $[M_2, +\infty)$ and hence using the fact that $\bar{\tilde{m}}^{n_k}$ converges to m_0 in $L^2_{loc}(\mathbb{R})$ we get that m_0 must be nonpositive in $(-\infty, -M_2]$ and is nonnegative in

$[M_2, +\infty)$ and therefore belongs to $\tilde{A}_0$. Now the above proved fact states that $m_x^0(0) = 0$ and $m_y^0(0) \geq 0$. Furthermore from the lower semi-continuity part of the Γ-convergence theorem we have that

$$E_0(m^0) \leq \liminf_{n\to\infty} \frac{E(\tilde{m}^{n_k})}{d_n^2} = \liminf_{n\to\infty} \frac{E(m^{n_k})}{d_n^2} = E_{min}^0$$

thus m^0 is a minimizer of E_0. We have seen in section 3.6 that any minimizer of E_0 must have the form

$$(\sin\varphi(x), \cos\varphi(x)\cos\theta(x), \cos\varphi(x)\sin\theta(x))$$

where

$$\varphi(x) = \arcsin\frac{e^{2\sqrt{\alpha}x+\beta} - 1}{e^{2\sqrt{\alpha}x+\beta} - 1}, \quad \alpha = \frac{a_{\omega_0}}{2\pi^2 c_{\omega_0}}, \quad \theta = \arctan t_0 \quad \text{and} \quad \beta \in \mathbb{R}$$

and we take $t_0 = 0$ if $C_0 = 0$. It is easy to see now that the properties $m_x(0) = 0$ and $m_y(0) \geq 0$ determine m^0 in the unique way, namely we get $\beta = 0$ and $m_y(0) = \frac{1}{\sqrt{1+t_0^2}}$.

$\square$

Theorem 3.9.8. *Assume that the domain ω_0 is so that $C_0^2 + (A_0 - B_0)^2 > 0$. Then for any sequence of magnetizations $(m^n)_{n\in\mathbb{N}}$ satisfying (3.34) there exist a sequence $(T_n)_{n\in\mathbb{N}}$ of translations in the variable x and a sequence $(R_n)_{n\in\mathbb{N}}$ of rotations in the OYZ plane, each of which is either the identity or the rotation by 180 degree such that for the sequence with the terms $\tilde{m}^n(x, y, z) = m^n(T_n(R_n(x, y, z)))$ and some $m^0 \in \tilde{A}_0$ we have*

$$\lim_{n\to\infty} \frac{1}{d_n} \|\tilde{m}^n - m^0\|_{H^1(\Omega_n)} = 0.$$

Proof. It is a consequence of Lemma 3.9.4, Lemma 3.9.6 and Theorem 3.9.7.

$\square$

Corollary 3.9.9. *Theorem 3.9.8 is valid for any sequence of minimizers $(m)_{n\in\mathbb{N}}$.*

In conclusion we mention that it is easy to see any rectangle that is not a square and any ellipse that is not a circle satisfies the condition

$$C_0^2 + (A_0 - B_0)^2 > 0.$$

It is also worth mentioning that one can prove a modified version of Theorem 3.9.8 in the case when ω_0 is a disc, namely due to the symmetry one can not state that the rotations R_n are either the identity or a rotation by 180 degree, but one can prove their existence.

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