On rigidity of the ring spectra $P_m\mathbb{S}_{(p)}$ and ko

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We study the rigidity question for modules over certain ring spectra. A stable model category is rigid if its homotopy category determines the Quillen equivalence type of the model category. Amongst others, we prove that the model category Mod-S is rigid if S is the m^{th} Postnikov section of a p-localized sphere spectrum for a prime p and for a sufficiently large integer m. Moreover, we prove that the underlying spectrum of the ring spectrum ko is determined by the ring $\pi_*(ko)$ and certain Toda brackets.

Contents

Introduction

One main goal of algebraic topology is to understand and classify spaces up to homotopy equivalence. Depending on the context there are different definitions of homotopy, as for example homotopy between two maps of topological spaces or between two maps of simplicial sets. Moreover, the notion of homotopy does not only occur in topology but also in algebra, for example as a chain homotopy between two maps of chain complexes.

In order to generalize and axiomatize these different definitions of homotopy, Quillen developed model categories [Qu]. An important part of the structure of a model category is a certain class of morphisms, the weak equivalences. Using the model structure one can define a notion of homotopy between two maps in a model category such that all weak equivalences between so-called bifibrant objects are homotopy equivalences. There are model structures on the category of topological spaces, the category of simplicial sets and the category of unbounded chain complexes of modules over a ring R, whose definitions of homotopy coincide with the classical ones on certain full subcategories. For example, the category of topological spaces can be endowed with a model structure whose weak equivalences are the morphisms that induce isomorphisms on all homotopy groups. The resulting new definition of homotopy coincides with the classical definition on all maps between cell complexes.

The structure of a model category $\mathcal{M}$ ensures that it is possible to localize $\mathcal{M}$ with respect to its class of weak equivalences. Thereby, one obtains a new category $\mathrm{Ho}(\mathcal{M})$ which is called the homotopy category of $\mathcal{M}$.

In this process one usually looses 'higher homotopy information' about the model category such as its algebraic K-theory and the homotopy types of its mapping spaces. An example for the former is given by Schlichting [Sl]. He considers the category of finitely generated R-modules for the rings $(\mathbb{Z}/p)[x]/x^2$ and $\mathbb{Z}/p^2$ where p is an odd prime. These categories can be endowed with stable model structures such that their triangulated homotopy categories are triangulated equivalent. Moreover, the model categories have different algebraic K-theories and hence are not Quillen equivalent. An example for two model categories with equivalent homotopy categories and different homotopy types of mapping spaces can be constructed by using the n^{th} Morava K-theory ring spectrum $K(n)$ for a fixed prime p and a positive integer n (see [Sc01]): The category of $K(n)$-modules and the category of differential graded $\pi_*(K(n))$-modules admit stable model structures such that their homotopy categories are triangulated equivalent. However, all mapping spaces of the model category of differential graded $\pi_*(K(n))$-modules have the homotopy types of wedges of Eilenberg-MacLane spectra. This is not the case for the model category of $K(n)$-modules. In particular, these two model categories can not be Quillen equivalent.

In general, an equivalence between the homotopy categories of two model categories does not imply that the model categories themselves are Quillen equivalent. As we have seen above, this is not even the case if we additionally require the model categories to be stable and their homotopy categories to be triangulated equivalent. However, Schwede shows in [Sc07] that the stable model category of spectra $Sp^{\mathbb{N}}$ in the sense of Bousfield and Friedlander [BF] is rigid. That is, every stable model category $\mathcal{N}$ whose homotopy category is triangulated equivalent to the homotopy category $\mathrm{Ho}(Sp^{\mathbb{N}})$ is Quillen equivalent to $Sp^{\mathbb{N}}$. Another example of a rigid model category is the category of $K_{(2)}$-local spectra, $L_1 Sp^{\mathbb{N}}$ [Roi07]. In these proofs, one uses that the homotopy categories $\mathrm{Ho}(Sp^{\mathbb{N}})$ and $\mathrm{Ho}(L_1 Sp^{\mathbb{N}})$ are triangulated categories and have one compact generator $\mathbb{S}$ and $L_1\mathbb{S}$, respectively. In particular, the proofs are based on the rings of graded self maps of these generators and on Toda bracket relations between graded endomorphisms of $\mathbb{S}$ and $L_1\mathbb{S}$, respectively.

In this thesis, we consider model categories Mod-S of right S-modules for certain symmetric ring spectra S. As in the two examples above, the model category Mod-S has one compact generator, the right S-module S itself. In particular, every stable model category whose homotopy category is triangulated equivalent to the homotopy category Ho(Mod-S) also has a compact generator. By

a result of Weiner, a stable model category with one compact generator is Quillen equivalent to the model category of modules over a certain ring spectrum [We]. Thus, the question if the model category Mod-S is rigid reminds us of the problem of Morita theory. There, for two rings R and S the categories of right R-modules and right S-modules are equivalent if and only if there exists a small projective generator T in the category of right R-modules whose endomorphism ring is isomorphic to S. Schwede and Shipley generalize this to homotopy theory [ScSh03]: They prove that for two ring spectra S and R the model categories Mod-S and Mod-R are Quillen equivalent if there exists a compact, cofibrant and fibrant generator T of Mod-R such that its endomorphism ring spectrum $\mathrm{End}_{\text{Mod-}R}(T)$ is stably equivalent to the ring spectrum S. It turns out that the model category Mod-S is rigid if the ring spectrum S is determined by its ring of homotopy groups $\pi_*(S)$ and some Toda bracket relations.
In the following, we call a ring spectrum S rigid if the model category Mod-S is rigid. Examples of rigid ring spectra are the sphere ring spectrum $\mathbb{S}$ and the p-localized sphere spectrum $\mathbb{S}_{(p)}$ for all primes p [Sc07]. Moreover, the Eilenberg-MacLane spectra HR are also rigid for all rings R [ScSh03]. Since the p-localized sphere spectrum and its zeroth Postnikov section are rigid, we want to know if the other Postnikov sections $P_m\mathbb{S}_{(p)}$ of $\mathbb{S}_{(p)}$ are also rigid. That is, we want to know if these ring spectra are determined by their homotopy group rings and some Toda bracket relations. With the following two theorems, we are able to prove that this is the case if the integer m is large enough:

Theorem 2.2.3. *The ring spectra $P_m(\mathbb{S}_{(2)})$ are rigid for all integers $m \geq 0$.*

Theorem 2.3.11. *Let p be an odd prime. Then the ring spectra $P_m(\mathbb{S}_{(p)})$ are rigid for all integers $m \geq p^2(2p-2)-1$.*

Moreover, we consider the 2-localized real connective K-theory ring spectrum $ko_{(2)}$. Its ring of homotopy groups $\pi_*(ko_{(2)})$ and its Toda brackets seem to be sufficiently rich to expect that the ring spectrum $ko_{(2)}$ is rigid. We are able to prove that the ring spectrum $ko_{(2)}$ is determined as a *spectrum* by its homotopy group ring and its triple Toda brackets:

Theorem 3.3.7. *Let R be a ring spectrum whose ring of homotopy groups $\pi_*(R)$ is isomorphic to the ring $\pi_*(ko_{(2)})$ by an isomorphism which preserves Toda brackets (see Def. 1.2.14). Then there exists a stable equivalence of spectra $F\colon R \longrightarrow ko_{(2)}$.*

A result of Patchkoria suggests that for an odd prime p the p-localized real connective K-theory ring spectrum $ko_{(p)}$ should not be rigid. In [Pa], he proves that the homotopy category of right $ko_{(p)}$-modules is equivalent to the derived category of the homotopy ring $\pi_*(ko_{(p)})$. However, the question whether this equivalence is triangulated remains open.

Organization In the next section, we fix some notation and conventions and recall some necessary prerequisites for the proof of the three theorems above. In particular, we give the exact definition of rigidity of model categories and ring spectra and we recall constructions for Postnikov sections of spectra and ring spectra.
In the second section we prove that all Postnikov sections of the 2-localized sphere spectrum are rigid (Thm. 2.2.3) and that for an odd prime p the m^{th} Postnikov section of the p-localized sphere spectrum is rigid if the integer m is at least $p^2(2p-2)-1$ (Thm. 2.3.11). In the first part of the last section, we approximate the ring spectrum $ko_{(2)}$ by attaching ring spectrum cells to the sphere spectrum. Using this method, we are able to prove that the 4^{th} and 9^{th} Postnikov section of $ko_{(2)}$ are rigid. Afterwards, we prove Theorem 3.3.7 by calculating the cohomology of R with $\mathbb{Z}/2$-coefficients for every ring spectrum R whose ring of homotopy groups is isomorphic to $\pi_*(ko_{(2)})$ by an isomorphism which preserves triple Toda brackets.

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1 Prerequisites

First, we will fix some notation and conventions for this thesis. Afterwards, we define rigidity of model categories and ring spectra. Moreover, we recall a a characterization of rigid ring spectra. In the end of this section, we recall constructions for Postnikov sections of connective spectra and ring spectra.

1.1 Notation and conventions

In this thesis, we use the stable model structure on the category of symmetric spectra Sp^Σ which was introduced by Hovey, Shipley and Smith in [HSS, §3.4]. For an integer n, we define the n-dimensional sphere spectrum $\mathbb{S}^n$ by

$$\mathbb{S}^n = \begin{cases} \Sigma^\infty S^n & \text{if } n \geq 1, \\ \mathbb{S} & \text{if } n = 0, \text{ and} \\ F_{-n} S^0 & \text{if } n \leq -1, \end{cases} \tag{1.1}$$

where $\Sigma^\infty K$ denotes the symmetric suspension spectrum of a pointed simplicial set K and the free functor $F_m \colon sSet_* \longrightarrow Sp^\Sigma$, $m \geq 0$, is the left adjoint of the evaluation functor $Ev_m \colon Sp^\Sigma \longrightarrow sSet_*$ at level m [HSS, Def. 2.2.5]. For an integer n, we define the n^{th} stable homotopy group $\pi_n(X)$ of a symmetric spectrum X as the group $[\mathbb{S}^n, X]^{\mathrm{Ho}(Sp^\Sigma)}$ of morphisms from $\mathbb{S}^n$ to X in the category $\mathrm{Ho}(Sp^\Sigma)$.

More generally, let X and Y be two objects in a model category M. Throughout this thesis, $\mathsf{M}(X,Y)$ denotes the set of morphism from X to Y in M and $[X,Y]^{\mathrm{Ho}(\mathsf{M})}$ denotes the group of morphisms in the homotopy category $\mathrm{Ho}(\mathsf{M})$. Suppose that the model category M is stable and enriched over the model category of symmetric spectra Sp^Σ. We define the graded homotopy group of morphisms from X to Y by $[X,Y]_n^{\mathrm{Ho}(\mathsf{M})} = \left[\mathbb{S}^n \wedge^L X, Y\right]^{\mathrm{Ho}(\mathsf{M})}$.

In the following, we fix a ring structure on the graded group $[X,X]_*^{\mathrm{Ho}(\mathsf{M})}$ for every object X in the model category M. One can choose isomorphisms $\alpha_{m,n} \colon \mathbb{S}^{m+n} \longrightarrow \mathbb{S}^m \wedge^L \mathbb{S}^n$, $m,n \in \mathbb{Z}$, in the stable homotopy category $\mathrm{Ho}(Sp^\Sigma)$ such that the following properties hold for all integers l, m and n:

1. The morphisms $\alpha_{m,0}$ and $\alpha_{0,n}$ are inverse to the right and left unit morphisms, respectively.
2. The following diagram commutes in the homotopy category $\mathrm{Ho}(Sp^\Sigma)$.

$$\begin{tikzcd} \mathbb{S}^{l+m+n} \arrow[rr, "\alpha_{l+m,n}"] \arrow[d, "\alpha_{l,m+n}"] & & \mathbb{S}^{l+m} \wedge^L \mathbb{S}^n \arrow[d, "\alpha_{l,m} \wedge^L \mathbb{S}^n"] \\ \mathbb{S}^l \wedge^L \mathbb{S}^{m+n} \arrow[r, "\mathbb{S}^l \wedge^L \alpha_{m,n}"] & \mathbb{S}^l \wedge^L (\mathbb{S}^m \wedge^L \mathbb{S}^n) \arrow[r, phantom, "\cong"] & (\mathbb{S}^l \wedge^L \mathbb{S}^m) \wedge^L \mathbb{S}^n \end{tikzcd}$$

3. The diagram

$$\begin{tikzcd} \mathbb{S}^{m+n} \arrow[r, "(-1)^{mn}"] \arrow[d, "\alpha_{m,n}"] & \mathbb{S}^{n+m} \arrow[d, "\alpha_{n,m}"] \\ \mathbb{S}^m \wedge^L \mathbb{S}^n \arrow[r, phantom, "\cong"] & \mathbb{S}^n \wedge^L \mathbb{S}^m \end{tikzcd}$$

   commutes in $\mathrm{Ho}(Sp^\Sigma)$.

For objects X, Y and Z in the stable model category M, composition induces a morphism

$$\begin{aligned} [Y,Z]_m^{\mathrm{Ho}(\mathsf{M})} \times [X,Y]_n^{\mathrm{Ho}(\mathsf{M})} &\longrightarrow [X,Z]_{n+m}^{\mathrm{Ho}(\mathsf{M})} \\ (f,g) &\longmapsto f \circ (\mathbb{S}^m \wedge^L g) \circ (\alpha_{m,n} \wedge^L X). \end{aligned}$$

This defines a ring structure on the graded group $[X,X]_*^{\mathrm{Ho}(\mathsf{M})}$ for every object X in the model category M. Moreover, the graded group $\pi_*(X)$ is a right $\pi_*(\mathbb{S})$-module for every symmetric spectrum X.
Let R be a symmetric ring spectrum. The ring structure of the graded group $[R,R]_*^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)}$ induces a ring structure on $\pi_*(R)$ due to the Quillen adjunction $-\wedge_{\mathbb{S}} R\colon \mathrm{Mod}\text{-}\mathbb{S} \rightleftarrows \mathrm{Mod}\text{-}R : U$ which is induced by the unit $\iota\colon \mathbb{S} \longrightarrow R$ of the ring spectrum R (see Example 1.2.5(3)):

$$\pi_*(R) = [S,R]_*^{\mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S})} \cong [R,R]_*^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)}\,.$$

1.2 Model categories and rigidity

In this subsection, we define rigidity of model categories and ring spectra (Def. 1.2.12). Before, we give some examples of model categories and Quillen adjunctions (§1.2.1) and recall the definitions of a compact generator of a triangulated category and of Toda brackets (§1.2.2).

1.2.1 Model categories and Quillen adjunctions

Model categories were defined by Quillen in order to axiomatize homotopy theory [Qu]. We mostly refer to Hovey's book [Ho] about model categories as a reference. A *model category* $\mathcal{M}$ is a bicomplete category with three classes of maps, namely *weak equivalences* ($\xrightarrow{\simeq}$), *cofibrations* ($\rightarrowtail$) and *fibrations* ($\twoheadrightarrow$), which satisfy certain axioms (see [Ho, Def. 1.1.3]). This structure ensures that we can localize $\mathcal{M}$ with respect to the class of weak equivalences and obtain a new category, which is called the homotopy category of $\mathcal{M}$ and is denoted by $\mathrm{Ho}(\mathcal{M})$ (see [Ho, Def. 1.2.1]).

Notation 1.2.1. Let $\mathcal{M}$ be a model category and let X and Y be two objects in $\mathcal{M}$. Throughout this thesis, $\mathcal{M}(X,Y)$ denotes the set of morphism from X to Y in $\mathcal{M}$ and $[X,Y]^{\mathrm{Ho}(\mathcal{M})}$ denotes the group of morphisms in the homotopy category $\mathrm{Ho}(\mathcal{M})$.

Definition 1.2.2. An adjunction

$$F\colon \mathcal{M} \rightleftarrows \mathcal{N} : U$$

between two model categories $\mathcal{M}$ and $\mathcal{N}$ is a *Quillen adjunction* if the left adjoint functor F preserves cofibrations and acyclic cofibrations (e.g. [Ho, Def. 1.3.1]). In particular, it induces an adjunction

$$\mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N})$$

on the homotopy categories. A Quillen adjunction is a *Quillen equivalence* if it induces an equivalence of the homotopy categories. Two model categories are called *Quillen equivalent* if there exists a zig-zag of Quillen equivalences between them.

Example 1.2.3. The category of (pointed) topological spaces can be endowed with a model structure, where a morphism $f\colon X \longrightarrow Y$ is a weak equivalence if the maps $\pi_n(f,x)\colon \pi_n(X,x) \longrightarrow \pi_n(Y,f(x))$ are isomorphisms for all integers $n \geq 0$ and all points $x \in X$, and the fibrations are the Serre fibrations (e.g. [Ho, §2.4]). Moreover, the category of (pointed) simplicial sets admits a model structure, where the weak equivalences are all morphisms whose geometric realizations are weak equivalences in $\mathrm{Top}_{(*)}$ and the fibrations are Kan fibrations (e.g. [Ho, §3]). These model categories are Quillen equivalent by the adjunction

$$|\text{-}|\colon \mathrm{sSet}_{(*)} \rightleftarrows \mathrm{Top}_{(*)} : \mathcal{S},$$

where $|\text{-}|$ is the geometric realization functor and $\mathcal{S}$ is the singular functor.

A model category is called *pointed* if the map from the initial object to the terminal object is an isomorphism. Recall that one can define a suspension functor Σ together with a right adjoint loop functor Ω on the homotopy category of a pointed model category ([Qu, Thm. I.2.2] or see [Ho, Def. 6.1.1]).

Definition 1.2.4. A *stable* model category $\mathcal{M}$ is a pointed model category such that the suspension functor and the loop functor are inverse equivalences on the homotopy category $\mathrm{Ho}(\mathcal{M})$.

Example 1.2.5.

1. An important example of a stable model category is the category of chain complexes $Ch(R)$ with the weak equivalences being the quasi-isomorphisms. Here, the suspension functor is given by the shift functor. The associated homotopy category is the derived category $\mathcal{D}(R)$.

2. The category of symmetric spectra over simplicial sets Sp^{Σ} admits several stable model structures, where the weak equivalences are the stable equivalences. In this thesis, we work with a stable model structure defined by Hovey, Shipley and Smith [HSS, §3.4]. This model category is proper, cofibrantly generated, monoidal and satisfies the monoid axiom. Furthermore, the sphere spectrum is a compact generator of Sp^{Σ} (Def. 1.2.9) since the functors $\pi_*(-) = [\mathbb{S}, -]_*$ preserve arbitrary coproducts and detect trivial objects. We call an object in the category of symmetric spectra Sp^{Σ} *symmetric spectrum* or *spectrum*.

3. Thus, for any monoid S in Sp^{Σ} the category of right S-modules Mod-S admits a stable model structure, where a map is a fibration or weak equivalence if and only if it is a fibration or weak equivalence in the underlying category Sp^{Σ} ([HSS, Cor. 5.5.2] or [ScSh00, Thm. 4.1]). Moreover, a morphism $f\colon S \longrightarrow R$ of ring spectra induces a Quillen adjunction
$$- \wedge_S R\colon \text{Mod-}S \rightleftarrows \text{Mod-}R : U$$
which is a Quillen equivalence if f is a stable equivalence, by [HSS, Thm. 5.5.9] or [ScSh00, Thm. 4.3]. Due to the Quillen adjunction induced by the unit of the ring spectrum S, the right S-module S is a compact generator of the model category Mod-S.

4. If R is a commutative monoid in Sp^{Σ}, then there is a model structure on the category of R-algebras, where a map is a weak equivalence or fibration if and only if it is one in the underlying category Sp^{Σ}. Moreover, any cofibration of R-algebras whose source is cofibrant as an R-module is also a cofibration of R-modules [ScSh00, Thm. 4.1]. In particular, the category of $\mathbb{S}$-algebras admits a model structure like above since the monoid $\mathbb{S}$ is commutative.

Notation 1.2.6. In this thesis, a (commutative) monoid in the category of symmetric spectra is also called a (commutative) ring spectrum. Note that the category of $\mathbb{S}$-algebras and the category of monoids in Sp^{Σ} are isomorphic since the category of $\mathbb{S}$-modules is isomorphic to Sp^{Σ}.

Definition 1.2.7. Two (ring) spectra are called *stably equivalent* if there exists a zig-zag of stable equivalences of (ring) spectra between them.

1.2.2 Triangulated categories

The homotopy category $\mathrm{Ho}(\mathcal{M})$ of a stable model category can canonically be equipped with the structure of a triangulated category (see [Ho, Prop. 7.1.6]).

Remark 1.2.8. Let $\mathcal{M}$ be a stable model category which is enriched over the model category of symmetric spectra Sp^{Σ}. For a cofibrant object X in $\mathcal{M}$ the suspension of X is isomorphic to the object $\mathbb{S}^1 \wedge X$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$. We choose an isomorphism $\mathbb{S}^1 \wedge^L X \xrightarrow{\cong} \Sigma X$ for every object X in $\mathcal{M}$.

Definition 1.2.9. Let $\mathcal{T}$ be a triangulated category which has infinite coproducts. An object P in the category $\mathcal{T}$ is called *compact* if the canonical map of abelian groups

$$\bigoplus_{i\in I}[P, X_i] \longrightarrow \left[P, \bigoplus_{i\in I} X_i\right]$$

is an isomorphism for every family $\{X_i\}_{i\in I}$ of objects in $\mathcal{T}$.
A *localizing subcategory* of the triangulated category $\mathcal{T}$ is a full triangulated subcategory of $\mathcal{T}$ which is closed under coproducts. An object P of $\mathcal{T}$ is a *generator* if every localizing subcategory of $\mathcal{T}$ which contains P is all of $\mathcal{T}$.
Recall that a stable model category $\mathcal{M}$ is cocomplete and its homotopy category thus has infinite coproducts. An object P of a stable model category $\mathcal{M}$ is called *compact* or a *generator* if its image in the triangulated category $\mathrm{Ho}(\mathcal{M})$ is.

Definition 1.2.10.

1. Let $x\colon X \longrightarrow Y$, $y\colon Y \longrightarrow Z$ and $z\colon Z \longrightarrow W$ be composable morphisms in a triangulated category such that $yx = 0 = zy$. Then the *Toda bracket* $\langle z, y, x\rangle$ is defined to be the subset of $[\Sigma X, W]$ containing all morphisms $\Sigma X \longrightarrow W$ which can be constructed in the following way

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, dashed] & Y \arrow[r] \arrow[d, equal] & C_x \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, dashed] \arrow[dd, dashed, bend left=60, "\in\langle z,y,x\rangle"] \\
\Sigma^{-1}C_y \arrow[r] \arrow[d, dashed] & Y \arrow[r, "y"] \arrow[d, dashed] & Z \arrow[r] \arrow[d, equal] & C_y \arrow[d, dashed] \\
\Sigma^{-1}W \arrow[r] & \Sigma^{-1}C_z \arrow[r] & Z \arrow[r, "z"] & W,
\end{tikzcd}$$

   where the horizontal lines are exact triangles. This set $\langle z, y, x\rangle \subset [\Sigma X, W]$ is a coset of the group $[\Sigma Y, W] \circ (\Sigma x) + z \circ [\Sigma X, Z]$, which is called the *indeterminacy* of the Toda bracket.

2. Let S be a ring spectrum and M a right S-module. The unit ι of S induces an adjunction

$$- \wedge^L_{\mathbb{S}} S\colon\ \mathrm{Ho}(\text{Mod-}\mathbb{S}) \rightleftarrows \mathrm{Ho}(\text{Mod-}S) : U$$

   as we have seen in Example 1.2.5(3). Using this adjunction, we now define Toda brackets in $\pi_*(M)$: Let $\tilde{x}, \tilde{y} \in \pi_*(S)$ and $\tilde{z} \in \pi_*(M)$ be elements of homotopy groups of S and M and denote their adjoint morphisms by x, y and z, respectively. Moreover, suppose that the Toda bracket

$$\langle z, \Sigma^{|z|}y, \Sigma^{|y|+|z|}x\rangle \subset [S, M]_*^{\mathrm{Ho}(\text{Mod-}S)}$$

   is defined, where '$|x|$' denotes the degree of a morphism x. The Toda bracket

$$\langle \tilde{z}, \tilde{y}, \tilde{x}\rangle \subset \pi_*(M)$$

   is defined as the image of this set under the isomorphism $\pi_*(M) \cong [S, M]^{\mathrm{Ho}(\text{Mod-}S)}$, which is induced by the adjunction above. The indeterminacy of this Toda bracket is defined similarly.

Later, we will need the so-called juggling formulas:

Theorem 1.2.11. *Let $X \xrightarrow{x} Y \xrightarrow{y} Z \xrightarrow{z} W \xrightarrow{w} V$ be composable morphisms in a triangulated category $\mathcal{T}$. Then the following inclusions hold if the involved Toda brackets are defined:*

$$\begin{array}{rcl} \langle w,z,y\rangle \circ (\Sigma x) & \subset & \langle w,z,y\circ x\rangle \\ w\circ \langle z,y,x\rangle & \subset & \langle wz,y,x\rangle \\ -w\circ \langle z,y,x\rangle & = & \langle w,z,y\rangle \circ (\Sigma x) \\ \langle w,z,y\circ x\rangle & \subset & \langle w,z\circ y,x\rangle \\ \langle w\circ z,y,x\rangle & \subset & \langle w,z\circ y,x\rangle . \end{array}$$

The author does not know a reference for these well-known facts. Some references for similar statements are: [Ko, Prop. 5.7.4], [Ra, Thm. A1.4.6] and [To62, equations (3.5) and (3.6)].
Proof. These statements hold due to the axioms of triangulated categories. As an example, we prove the third statement of this theorem.
We consider the following diagram, where the horizontal triangles are exact.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, dashed] & Y \arrow[r, "i_x"] \arrow[d, equal] & C_x \arrow[r, "p_x"] \arrow[d, dashed, "a"] & \Sigma X \arrow[d, dashed, "b"] \arrow[dd, dashed, bend left=60, "\in\langle z,y,x\rangle"] \\
\Sigma^{-1}C_y \arrow[r, "\Sigma^{-1}p_y"] \arrow[d, dashed] & Y \arrow[r, "y"] \arrow[d, dashed, "c"] & Z \arrow[r, "i_y"] \arrow[d, equal] & C_y \arrow[d, dashed, "d"] \\
\Sigma^{-1}W \arrow[r, "\Sigma^{-1}i_z"] & \Sigma^{-1}C_z \arrow[r, "\Sigma^{-1}p_z"] & Z \arrow[r, "z"] & W \arrow[d, equal] \\
 & & & W \arrow[r, "w"] & V
\end{tikzcd} \tag{1.2}$$

In particular, the composite $w\circ(d\circ b)$ is an element of the set $w\circ\langle z,y,x\rangle$. We define the element $w\circ d\circ(-b) = f\circ(\Sigma c)\circ(\Sigma x)$ of the set $\langle w,z,y\rangle\circ(\Sigma x)$ by rotating the two lower exact triangles in the diagram (1.2) above and by using the morphisms a, b, c and d, which are defined in diagram (1.2):

$$\begin{tikzcd}
 & & \Sigma X \arrow[r, "\Sigma x"] \arrow[d, dashed, "-b"] & \Sigma Y \arrow[d, equal] \\
Y \arrow[r, "y"] \arrow[d, dashed, "c"] & Z \arrow[r, "i_y"] \arrow[d, equal] & C_y \arrow[r, "-p_y"] \arrow[d, dashed, "d"] & \Sigma^{-1}Y \arrow[d, dashed, "\Sigma c"] \arrow[dd, dashed, bend left=60, "\in\langle w,z,y\rangle"] \\
\Sigma^{-1}C_z \arrow[r, "\Sigma^{-1}p_z"] \arrow[d, dashed] & Z \arrow[r, "z"] \arrow[d, dashed, "e"] & W \arrow[r, "-i_z"] \arrow[d, equal] & C_z \arrow[d, dashed, "f"] \\
\Sigma^{-1}V \arrow[r, "\Sigma^{-1}p_w"] & \Sigma^{-1}C_w \arrow[r, "i_w"] & W \arrow[r, "w"] & V.
\end{tikzcd} \tag{1.3}$$

The signs in this diagram arise from the rotation of the two lower exact triangles in the diagram (1.2). It follows that every element of the set $-w\circ\langle z,y,x\rangle$ lies in $\langle w,z,y\rangle\circ(\Sigma x)$. Similarly, one shows the inclusion $\langle w,z,y\rangle\circ(\Sigma x)\subset -w\circ\langle z,y,x\rangle$. ☐

1.2.3 Rigidity of model categories and ring spectra

The stable model category of symmetric spectra (see Example 1.2.5(2)) is special in the sense that one does not 'loose information' by passage to its homotopy category (see [Sc01] and [Sc07]). In general, it is not possible to recover the Quillen equivalence type of the model category $\mathcal{M}$ from its homotopy category $\mathrm{Ho}(\mathcal{M})$ — even if one also demands that the model category $\mathcal{M}$ is stable and hence its homotopy category triangulated. Our aim is to find some stable model categories where this is the case.

Definition 1.2.12. We call a stable model category $\mathcal{M}$ *rigid* if every stable model category $\mathcal{N}$, whose homotopy category $\mathrm{Ho}(\mathcal{N})$ is triangulated equivalent to $\mathrm{Ho}(\mathcal{M})$, is already Quillen equivalent to $\mathcal{M}$. A ring spectrum R is said to be *rigid* if the model category of right R-modules is rigid.

Remark 1.2.13. Results of Schwede, Shipley and Weiner provide a characterization for rigid ring spectra which is independent of model categories. Before we give the precise statement (Thm. 1.2.16), we need the following definition.

Definition 1.2.14. Let S and T be two ring spectra. A morphism of graded rings

$$\psi\colon \pi_*(S) \longrightarrow \pi_*(T)$$

is said to *preserve Toda brackets* if for all elements $a, b, c \in \pi_*(S)$ with $ab = 0 = bc$ the set $\psi(\langle a, b, c\rangle)$ is contained in the set $\langle \psi(a), \psi(b), \psi(c)\rangle$ in $\pi_*(T)$.

Remark 1.2.15. If the morphism ψ is an isomorphism then the set $\psi(\langle a, b, c\rangle)$ equals $\langle \psi(a), \psi(b), \psi(c)\rangle$. Those kind of isomorphisms are important in this thesis. They occur, for example, in the setting of Theorem 1.2.16. Actually, the isomorphism in the second part of this theorem is even compatible with higher Toda brackets since it is induced by a triangulated equivalence between triangulated categories. However, we do not include this in our definition since we will not use higher Toda brackets.

The following theorem is a direct conclusion of some results of Schwede, Shipley ([ScSh03, Thm. 4.1.2] and [ScSh00, Thm. 4.3]) and Weiner [We, Thm. 4.6.3].

Theorem 1.2.16. *Let S be a ring spectrum, $\mathcal{N}$ a stable model category and*

$$\Phi\colon \mathrm{Ho}(\mathrm{Mod}\text{-}S) \longrightarrow \mathrm{Ho}(\mathcal{N}).$$

an equivalence of triangulated categories. Let P be a cofibrant-fibrant object in $\mathcal{N}$ whose image in the homotopy category $\mathrm{Ho}(\mathcal{N})$ is isomorphic to the image of S under the triangulated equivalence Φ.

(1) *Then the object P is a compact generator of the model category $\mathcal{N}$. Moreover, there exists a cofibrant and fibrant ring spectrum R and a chain of Quillen equivalences between $\mathcal{N}$ and the model category of right R-modules*
$$\mathcal{N} \simeq_Q \mathrm{Mod}\text{-}R.$$

(2) *The resulting triangulated equivalence $\mathrm{Ho}(\mathrm{Mod}\text{-}S) \simeq_\triangle \mathrm{Ho}(\mathrm{Mod}\text{-}R)$ maps the object S in $\mathrm{Ho}(\mathrm{Mod}\text{-}S)$ to an object which is isomorphic to R in $\mathrm{Mod}\text{-}R$. In particular, it induces an abstract isomorphism*
$$\pi_*(S) \cong [S, S]_*^{\mathrm{Ho}(\mathrm{Mod}\text{-}S)} \cong [R, R]_*^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)} \cong \pi_*(R),$$
which is multiplicative and preserves Toda brackets (see Def. 1.2.14).

(3) *The model categories $\mathcal{N}$ and $\mathrm{Mod}\text{-}S$ are Quillen equivalent if there exists a zig-zag of stable equivalences of ring spectra between R and S.*

Proof. This is a special case of Theorem 4.6.3 in [We] and Theorem 4.3 in [ScSh00].
First observe that the cofibrant-fibrant object P is a compact generator of the stable model category $\mathcal{N}$ since S as a right S-module is a compact generator of the model category Mod-S. In [We], Weiner constructs a symmetric ring spectrum R and a zig-zag of Quillen equivalences between the stable model category $\mathcal{N}$ and the model category Mod-R of R-modules [We, Thm. 4.6.2]. Moreover, we can assume without loss of generality that the ring spectrum R is cofibrant-fibrant since it is stably equivalent to a cofibrant-fibrant ring spectrum R^{cf} and this zig-zag of stable equivalences induces a zig-zag of Quillen equivalences Mod-$R \simeq_Q$ Mod-R^{cf} [HSS, Thm. 5.5.9]. This chain of Quillen equivalences $\mathcal{N} \simeq$ Mod-R induces a triangulated equivalence on homotopy categories which maps the object P in $\mathrm{Ho}(\mathcal{N})$ to an object isomorphic to R in the homotopy category Mod-R (see [We, proof of Thm. 4.4.2]). Thus, the first and second part of this theorem hold.
By [HSS, Thm. 5.5.9] or [ScSh00, Thm. 4.3], a zig-zag of stable equivalences of ring spectra induces a zig-zag of Quillen equivalences between the stable model categories of right modules over these ring spectra. This proves (3). □

Remark 1.2.17. In [ScSh03], Schwede and Shipley prove that every cofibrantly generated, simplicial, proper, stable model category $\mathcal{N}$ which has a compact generator P is Quillen equivalent to the category of right R-modules for a certain ring spectrum R [ScSh03, Thm. 3.1.1]. Later, Weiner was able to omit the additional technical assumptions on the model category $\mathcal{N}$ [We, Thm. 4.6.2].

The following corollary is a direct consequence of Theorem 1.2.16.

Corollary 1.2.18. *A ring spectrum S is rigid if it is stably equivalent to every cofibrant-fibrant ring spectrum R whose ring of homotopy groups is isomorphic to the ring $\pi_*(S)$ by an abstract isomorphism that preserves Toda brackets.*

Remark/Notation 1.2.19. Due to [HSS, Thm. 5.5.9], we can assume without loss of generality that all these ring spectra are cofibrant and fibrant. We will sometimes do that without emphasizing it.

Example 1.2.20. In the following, we give some examples of rigid and non-rigid ring spectra.

1. In [Sc01] and [Sc07], Schwede proves that the model category of spectra in the sense of Bousfield and Friedlander [BF, §2] is rigid and hence so is the model category of symmetric spectra [HSS, Prop. 4.3.1]. In particular, the sphere spectrum $\mathbb{S}$ is rigid. Moreover, Schwede proves that for any prime p the p-localized sphere spectrum $\mathbb{S}_{(p)}$ is rigid.

2. For any ring R, the Eilenberg-MacLane ring spectrum HR is rigid as it is uniquely determined by its homotopy ring $\pi_*(HR)$ [ScSh03, Theorem 5.1.1.].

3. In contrast, the Morava K-theory ring spectra are not rigid (see [Sc01, §2.1]). To see this, let $K(n)$ be the n^{th} Morava K-theory ring spectrum for a fixed prime p and a natural number $n > 0$. The homotopy category of $K(n)$-modules is triangulated equivalent to the derived category of the graded field $\pi_*(K(n)) = \mathbb{F}_p\left[v_n,\ v_n^{-1}\right]$, where $|v_n| = 2p^n - 2$. This derived category is the homotopy category of a model structure on the category of differential graded $\mathbb{F}_p\left[v_n,\ v_n^{-1}\right]$-modules. However, this model category can not be Quillen equivalent to the model category of $K(n)$-modules since the homotopy types of the mapping spaces do not coincide ([DwK, Prop. 5.4], see [Roi08, §5]). Indeed, for dg-modules all mapping spaces are products of Eilenberg-MacLane spaces, which is not the case for $K(n)$-modules.

1.3 Postnikov sections of spectra and ring spectra

In sections 2 and 3, we will need Postnikov sections of connective spectra and ring spectra. For example, we need Postnikov sections of spectra in the construction of a spectral sequence (see Thm. 3.2.8) and Postnikov sections of ring spectra in section 2, where we consider Postnikov sections of p-localized sphere ring spectra $\mathbb{S}_{(p)}$. Therefore, we recall constructions of Postnikov sections and some properties, that will be useful in subsection 3.2. We start with the following definitions:

Definition 1.3.1. Let n be an integer.

1. A morphism of spectra $f\colon X \longrightarrow Y$ is called a $\pi_{<n}$*-isomorphism* if the map $\pi_m(f)$ is an isomorphism for every integer $m < n$ and surjective for $m = n$.
2. A spectrum X is called n*-connected* if the groups $\pi_m(X)$ are trivial for $m \leq n$. A spectrum X is called *connective* if it is (-1)-connected. Further, a spectrum X is n*-coconnected* if $\pi_k(X) = 0$ for all $k \geq n$.

Clearly, a cone of a $\pi_{<n}$-isomorphism is n-connected.

Definition 1.3.2. We denote the free associative ring spectrum on a symmetric spectrum X

$$T(X) = \mathbb{S} \vee X \vee X^{\wedge 2} \vee \cdots = \bigvee_{n \geq 0} X^{\wedge n}$$

by $T(X)$. For a pointed simplicial set K, we abbreviate the ring spectrum $T(\Sigma^\infty K)$ on the symmetric suspension spectrum $\Sigma^\infty K$ of K by $T(K)$.

Recall that a cofibration $K \rightarrowtail L$ of pointed simplicial sets induces cofibrations $\Sigma^\infty K \rightarrowtail \Sigma^\infty L$ and $T(K) \rightarrowtail T(L)$ in the category of spectra and ring spectra, respectively (see [HSS, Prop. 3.4.2] and [ScSh00]). Now we recall constructions of Postnikov sections.

Construction 1.3.3. Let n be a non-negative integer.

1. The n^{th} *Postnikov section*
$$\bar{p}_n\colon X \longrightarrow \bar{P}_n X$$
of a cofibrant spectrum X is constructed by killing all higher homotopy groups of X. One way to do this is by applying the small object argument (e.g. [Ho, §2.1.2]) to the morphism $X \longrightarrow *$ with respect to the set of maps
$$I_n = \{\, \Sigma^\infty \partial\Delta^k_+ \longrightarrow \Sigma^\infty \Delta^k_+ \mid k \geq n+2\}$$
together with the generating trivial cofibrations for Sp^Σ [HSS, Def. 3.4.9]. We choose this construction since it ensures the spectrum $\bar{P}_n X$ to be cofibrant and fibrant. Moreover, there are natural maps
$$\bar{\varrho}^m_n\colon \bar{P}_m X \longrightarrow \bar{P}_n X$$
such that the equalities $\bar{\varrho}^m_n \circ \bar{p}_m = \bar{p}_n$ and $\bar{\varrho}^n_k \circ \bar{\varrho}^m_n = \bar{\varrho}^m_k$ hold, for all integers $m \geq n \geq k \geq 0$. To see this, observe that the morphism $\bar{p}_m\colon X \longrightarrow \bar{P}_m X$ is constructed by applying the small object argument to the subset $I_m = \{\, \Sigma^\infty \partial\Delta^k_+ \longrightarrow \Sigma^\infty \Delta^k_+ \mid k \geq m+2\}$ of I_n and that the spectra $\Sigma^\infty \partial\Delta^k_+$ are $\aleph_0$-small.

2. Now we recall the construction of *Postnikov sections of ring spectra.* In [DuSh, §2.1], Dugger and Shipley construct an n^{th} Postnikov section for a connective cofibrant $\mathbb{S}$-algebra S

$$p_n\colon S \longrightarrow P_n S.$$

They obtain this morphism by applying the small object argument to the morphism $S \longrightarrow *$ with respect to the set of maps

$$\Big\{\, T(\partial\Delta^k_+) \longrightarrow T(\Delta^k_+) \mid k \geq n+2 \Big\}$$

together with the generating trivial cofibrations for $\mathbb{S}$-algebras (see [ScSh00, Thm. 4.1]). By construction, the ring spectrum $P_n S$ is cofibrant, fibrant, $(n+1)$-coconnected and the morphism p_n is a $\pi_{<n+1}$-isomorphism [DuSh, 2.1]. Moreover, there are natural maps

$$\varrho^m_n\colon P_m S \longrightarrow P_n S$$

such that the equalities $\varrho^m_n \circ p_m = p_n$ and $\varrho^n_k \circ \varrho^m_n = \varrho^m_k$ hold, for all integers $m \geq n \geq k \geq 0$.

3. The n^{th} Postnikov section of an arbitrary connective ring spectrum X can be defined by $P_n(X^c)$, where $X^c \xrightarrow{\simeq} X$ is a fixed functorial cofibrant replacement for X in the category of ring spectra. Note that in this case, one only has a zig-zag of morphisms

$$X \xleftarrow{\simeq} X^c \xrightarrow{p_n} P_n(X^c).$$

Similarly, one defines the n^{th} Postnikov section of an arbitrary connective spectrum X to be the spectrum $\bar{P}_n(X^c)$.

Clearly, the n^{th} Postnikov section $\bar{p}_n$ (or p_n) of a connective cofibrant (ring) spectrum X is a $\pi_{<n+1}$-isomorphism with an $(n+1)$-coconnected (ring) spectrum as target. Since the construction of Postnikov sections is functorial, the morphisms $\bar{p}_n$ and p_n are 'universal' in the following sense:

Lemma 1.3.4. *Let $f\colon X \longrightarrow Y$ be a $\pi_{<n+1}$-isomorphism between connective cofibrant ring spectra with an $(n+1)$-coconnected target Y. Then the morphism $p_n f\colon P_n X \longrightarrow P_n Y$ is a stable equivalence such that the diagram*

$$\begin{tikzcd} & X \arrow[dl, "p_n"'] \arrow[dr, "f"] & \\ P_n X \arrow[r, "\simeq", "P_n f"'] & P_n Y & Y \arrow[l, "\simeq"', "p_n"] \end{tikzcd}$$

commutes. In particular, there exists a zig-zag of stable equivalences $P_n X \simeq Y$. An analogous statement holds for symmetric spectra.
In particular, this applies to the $\pi_{<n}$-isomorphisms $\bar{\varrho}^m_n$ and ϱ^m_n of Construction 1.3.3.

Corollary 1.3.5. *For a connective cofibrant ring spectrum X, the Postnikov sections $\bar{P}_n(X)$ and $P_n(X)$ are stably equivalent as symmetric spectra:*

$$\begin{tikzcd} & X \arrow[dl, "\bar{p}_n"'] \arrow[dr, "p_n"] & \\ \bar{P}_n X \arrow[rr, dash, "\simeq"] & & P_n X. \end{tikzcd}$$

Proof. This follows from Lemma 1.3.4 since the underlying spectrum of a cofibrant ring spectrum is cofibrant. □

Since the functors $\bar{P}_m\colon Sp^\Sigma \longrightarrow Sp^\Sigma$, $m \geq 0$, take stable equivalences between cofibrant spectra to stable equivalences, we can define derived functors $\bar{P}^L_m\colon \mathrm{Ho}(Sp^\Sigma) \longrightarrow \mathrm{Ho}(Sp^\Sigma)$, $m \geq 0$: In order to do this, we choose a cofibrant replacement functor $Q\colon Sp^\Sigma \longrightarrow Sp^\Sigma$ and a natural transformation $q\colon Q \longrightarrow \mathrm{Id}$ such that the morphism $q_X\colon QX \longrightarrow X$ is an acyclic fibration for every spectrum X. Precomposing the functor $\bar{P}_m\colon Sp^\Sigma \longrightarrow Sp^\Sigma$ with Q defines a derived functor $\bar{P}^L_m\colon \mathrm{Ho}(Sp^\Sigma) \longrightarrow \mathrm{Ho}(Sp^\Sigma)$ for every non-negative integer m (see [Ho, Def. 1.3.6]). Moreover, there is a natural transformation $\bar{p}_m\colon \mathrm{Id}_{\mathrm{Ho}(Sp^\Sigma)} \longrightarrow \bar{P}^L_m$ where the morphism $(\bar{p}_m)_X$ is the composite $(\bar{p}_m)_{QX} \circ q_X^{-1}\colon X \xleftarrow{\simeq} QX \longrightarrow \bar{P}_m(QX)$ for every spectrum X. The natural transformations $\bar{\varrho}^m_n\colon \bar{P}_m \longrightarrow \bar{P}_n$, $m \geq n \geq 0$, induce derived natural transformations $\bar{\varrho}^m_n\colon \bar{P}^L_m \longrightarrow \bar{P}^L_n$ such that the equalities $\bar{\varrho}^m_n \circ \bar{p}_m = \bar{p}_n$ and $\bar{\varrho}^n_k \circ \bar{\varrho}^m_n = \bar{\varrho}^m_k$ hold, for all integers $m \geq n \geq k \geq 0$.

Notation 1.3.6. To facilitate notation, we denote the derived functor $\bar{P}^L_m$ by $\bar{P}_m$ as well.

Lemma 1.3.7. *Let m be a non-negative integer and let $f\colon X \longrightarrow Y$ be a morphism in the homotopy category $\mathrm{Ho}(Sp^\Sigma)$ which is a $\pi_{<m+1}$-isomorphism with a $(m+1)$-coconnected target Y. Then the morphism $\bar{P}_m f$ is an isomorphism and the diagram*

$$\begin{tikzcd} & X \arrow[dl, "\bar{p}_m"'] \arrow[dr, "f"] & \\ \bar{P}_m X \arrow[r, "\cong", "\bar{P}_m f"'] & \bar{P}_m Y & Y \arrow[l, "\cong"', "\bar{p}_m"] \end{tikzcd}$$

commutes in $\mathrm{Ho}(Sp^\Sigma)$.

Corollary 1.3.8. *Let m be a non-negative integer. Let $f\colon X \longrightarrow Y$ be a morphism in the homotopy category $\mathrm{Ho}(Sp^\Sigma)$ such that $\pi_n(f)$ is an isomorphism for every integer $n > m$ and X is m-connected.*

(i) Then there exists an exact triangle in $\mathrm{Ho}(Sp^\Sigma)$, where the first morphism is f and the second morphism is $\bar{p}_m\colon Y \longrightarrow \bar{P}_m Y$:

$$X \xrightarrow{f} Y \xrightarrow{\bar{p}_m} \bar{P}_m Y \longrightarrow \Sigma X.$$

(ii) Let $M \geq m$ be an integer and Y be the M^{th} Postnikov section $\bar{P}_M Z$ of a spectrum Z. Then there exist exact triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & \bar{P}_M Z \arrow[r, "\bar{p}_m"] \arrow[d, equal] & \bar{P}_m \bar{P}_M Z \arrow[r] \arrow[d, "\cong"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f"] & \bar{P}_M Z \arrow[r, "\bar{\varrho}^M_m"] & \bar{P}_m Z \arrow[r] & \Sigma X \end{tikzcd}$$

that are isomorphic in $\mathrm{Ho}(Sp^\Sigma)$, where the isomorphism $\bar{P}_m \bar{P}_M Z \cong \bar{P}_m Z$ is the one of Lemma 1.3.7 applied to the morphism $\bar{\varrho}^M_m$.

Proof. (i): We choose an exact triangle $X \xrightarrow{f} Y \xrightarrow{i} C \xrightarrow{q} \Sigma X$. It follows that the triangle

$$X \xrightarrow{f} Y \xrightarrow{\bar{p}_m} \bar{P}_m Y \xrightarrow{q \circ \bar{p}_m^{-1} \circ \bar{P}_m i} \Sigma X$$

is exact, by applying Lemma 1.3.7 to the morphism $i\colon Y \longrightarrow C$.
(ii): There is an exact triangle $X \xrightarrow{f} \bar{P}_M Z \xrightarrow{\bar{p}_m} \bar{P}_m \bar{P}_M Z \xrightarrow{q} \Sigma X$ in $\mathrm{Ho}(Sp^\Sigma)$, by (i). As above, we replace the morphism $\bar{p}_m\colon \bar{P}_M Z \longrightarrow \bar{P}_m \bar{P}_M Z$ by the morphism $\bar{\varrho}^M_m\colon \bar{P}_M Z \longrightarrow \bar{P}_m Z$. □

2 On rigidity of some Postnikov sections $P_m\mathbb{S}_{(p)}$

Schwede proves in [Sc01] and [Sc07], that the sphere spectrum $\mathbb{S}_{(p)}$ is rigid. We adapt his proof to show that all Postnikov sections $P_m(\mathbb{S}_{(2)})$ are rigid (Thm. 2.2.3) and that for an odd prime p the Postnikov sections $P_m(\mathbb{S}_{(p)})$ are rigid for all $m \geq p^2(2p-2)-1$ (Thm. 2.3.11).
The proofs of these statements are divided into a sequence of lemmas and theorems. Recall that the ring spectrum $P_m\mathbb{S}_{(p)}$ is rigid if it is stably equivalent to certain ring spectra R (Thm. 1.2.16). These ring spectra R have the property that their rings of homotopy groups are isomorphic to the ring $\pi_*(P_m\mathbb{S}_{(p)})$ by isomorphisms which preserve Toda brackets. Let us consider the units

$$\iota\colon \mathbb{S} \longrightarrow R$$

of these ring spectra R. In Section 2.1, we prove that if all those units ι induce isomorphisms $\pi_k(\iota)\otimes\mathbb{Z}_{(p)}$ for all integers $k \leq m$ then the ring spectrum $P_m\mathbb{S}_{(p)}$ is rigid (Lemma 2.1.1).
Now let ι be such a unit. We prove in Theorem 2.1.5 that the morphisms $\pi_k(\iota)\otimes\mathbb{Z}_{(p)}$, $k \leq m$, are isomorphisms if the maps $\pi_k(\iota)\otimes\mathbb{Z}_{(p)}$ are bijective for certain integers k. In Sections 2.2 and 2.3, we consider these special cases and prove that the spectra $P_m\mathbb{S}_{(2)}$, $m \geq 0$, and $P_m\mathbb{S}_{(p)}$, $m \geq p^2(2p-2)-1$, are rigid (Theorems 2.2.3 and 2.3.11).

Let us fix the following notations:

Notation 2.0.9.

(i) Let p be a prime. We choose a p-localization $\mathbb{S}_{(p)}$ of the Sphere ring spectrum $\mathbb{S}$ that is a cofibrant and fibrant ring spectrum. (More precisely, let $S\mathbb{Z}_{(p)}$ be a Moore spectrum for $\mathbb{Z}_{(p)}$, that is the groups $H_k(S\mathbb{Z}_{(p)},\mathbb{Z})$ are trivial for all integers $k \neq 0$ and the group $H_0(S\mathbb{Z}_{(p)},\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{(p)}$. Using Bousfield localization $L_{S\mathbb{Z}_{(p)}}$ of $\mathbb{S}$-algebras at the $\mathbb{S}$-module $S\mathbb{Z}_{(p)}$ (see [EKMM, §VIII.2]), we construct a p-localized sphere ring spectrum $L_{S\mathbb{Z}_{(p)}}(\mathbb{S})$. We define the ring spectrum $\mathbb{S}_{(p)}$ to be a cofibrant-fibrant replacement of $L_{S\mathbb{Z}_{(p)}}(\mathbb{S})$. Note that every p-local ring spectrum S whose unit $\iota\colon \mathbb{S} \longrightarrow S$ induces an isomorphism $\pi_*(\iota)\otimes\mathbb{Z}_{(p)}$ is stably equivalent to $\mathbb{S}_{(p)}$.)
Let m be a non-negative integer. The ring spectrum $P_m = P_m(\mathbb{S}_{(p)})$ denotes the m-th Postnikov section of $\mathbb{S}_{(p)}$ (Constr. 1.3.3).

(ii) Let m be a non-negative integer and p be a prime number. Recall that the ring spectrum $P_m\mathbb{S}_{(p)}$ is rigid if every ring spectrum R, which has the property that the homotopy categories Ho(Mod-R) and Ho(Mod-$P_m\mathbb{S}_{(p)}$) are triangulated equivalent, is stably equivalent to the ring spectrum $P_m\mathbb{S}_{(p)}$ (Thm. 1.2.16). Throughout this section, the ring spectrum R denotes a cofibrant and fibrant ring spectrum such that there exists a triangulated equivalence $\Phi\colon$ Ho(Mod-$P_m\mathbb{S}_{(p)}$) $\longrightarrow$ Ho(Mod-R) (see Theorem 1.2.16). In particular, there exists a ring isomorphism

$$\psi_R\colon \pi_*(P_m\mathbb{S}_{(p)}) \longrightarrow \pi_*(R)$$

which preserves Toda brackets. Without loss of generality we can assume that the unit $i\colon \mathbb{S} \longrightarrow R$ of the p-local ring spectrum R factors through $\mathbb{S}_{(p)}$

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[rr, bend left, "i"] & \mathbb{S}_{(p)} \arrow[r, "\iota"'] & R \end{tikzcd}$$

by replacing the ring spectrum R with a cofibrant-fibrant replacement of the ring spectrum $L_{S\mathbb{Z}_{(p)}}(R) \simeq R$.

(iii) Recall that a morphism of ring spectra $f\colon S \longrightarrow R$ induces a Quillen adjunction

$$- \wedge_S R\colon \text{Mod-}S \rightleftarrows \text{Mod-}R : U.$$

In this section, the adjoint of a morphism $y\colon M \wedge_S^L R \longrightarrow N$ in Ho(Mod-R) is denoted by

$$\tilde{y}\colon M \longrightarrow U(N) \in \text{Ho}(\text{Mod-}S).$$

We often use this Quillen adjunction without stating it explicitly.

(iv) We abbreviate '$[-,-]^{\text{Ho}(\text{Mod-}\mathbb{S}_{(p)})}$' by '$[-,-]$' when it is clear from the context. Furthermore, we define $\pi_*(-) = \left[\mathbb{S}_{(p)}, -\right]_*^{\text{Ho}(\text{Mod-}\mathbb{S}_{(p)})}$. This definition makes sense since the unit of $\mathbb{S}_{(p)}$ induces a Quillen adjunction $- \wedge_{\mathbb{S}} \mathbb{S}_{(p)}\colon$ Mod-$\mathbb{S} \rightleftarrows$ Mod-$\mathbb{S}_{(p)}$ $:U$.

2.1 General statements about rigidity of $P_m\mathbb{S}_{(p)}$ for all primes p

Let R be a p-local cofibrant and fibrant ring spectrum as in Notation 2.0.9(ii). In particular, the homotopy categories Ho(Mod-R) and Ho(Mod-$P_m\mathbb{S}_{(p)}$) are triangulated equivalent and there exists a morphism of ring spectra $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$. Consider the diagram of ring spectra

$$\begin{tikzcd} \mathbb{S}_{(p)} \arrow[r, "\iota"] \arrow[d, "p_m"] & R \arrow[d, "p_m"] \\ P_m = P_m(\mathbb{S}_{(p)}) \arrow[r, "\tilde{\iota}:=P_m(\iota)"] & \tilde{R} = P_m(R), \end{tikzcd} \tag{2.4}$$

where $\tilde{R}$ and P_m are the m^{th} Postnikov sections of $\mathbb{S}$-algebras of R and $\mathbb{S}_{(p)}$, respectively (see Constr. 1.3.3). Our candidate for a zig-zag of stable equivalences between the ring spectra P_m and R is

$$P_m \xrightarrow{\tilde{\iota}} \tilde{R} \xleftarrow{p_m} R.$$

Lemma 2.1.1. *The ring spectra $P_m\mathbb{S}_{(p)}$ and R are stably equivalent if the morphism $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$ is a $\pi_{<m+1}$-isomorphism.*

Proof. Suppose that ι is a $\pi_{<m+1}$-isomorphism.
Thus, the morphism $\tilde{\iota}$ is a $\pi_{<m+1}$-isomorphism like the other three morphisms in diagram (2.4). Since the spectra $P_m = P_m\mathbb{S}_{(p)}$, R and $\tilde{R} = P_mR$ are $(m+1)$-coconnected, the $\pi_{<m+1}$-isomorphisms $\tilde{\iota}\colon P_m \longrightarrow \tilde{R}$ and $p_m\colon R \longrightarrow \tilde{R}$ are π_*-isomorphisms and hence stable equivalences. □

Therefore, it remains to prove that the morphism $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$ is a $\pi_{<m+1}$-isomorphism. In order to do this, let us recall the following well-known fact, which is used by Cohen [Co, Thm. 4.2] who cites a conversation with Adams. A proof of this fact can be found in [Sc01, Lemma 4.1].

Lemma 2.1.2. *Let p be a prime and $f\colon \mathbb{S}^k_{(p)} \longrightarrow \mathbb{S}_{(p)}$ be a map with $k \geq 1$ and mod-p Adams filtration at least two. Then the map f factors through some finite p-local spectrum whose mod-p cohomology is concentrated in dimensions 1 through $k-1$. More precisely, this spectrum is the $(k-1)^{th}$-skeleton of the fiber of the Hurewicz map $\mathbb{S}_{(p)} \longrightarrow H\mathbb{Z}_{(p)}$.*

For the prime $p = 2$, Adams proves that all elements in $\pi_k(\mathbb{S}_{(2)})$ with $k > 0$, except for unit multiples of the Hopf maps $\eta \in \pi_1(\mathbb{S}_{(2)})$, $\nu \in \pi_3(\mathbb{S}_{(2)})$ and $\sigma \in \pi_7(\mathbb{S}_{(2)})$, have Adams filtration at least two [Ad60, Thm. 1.1.1]. For prime $p > 2$, the only elements in $\pi_k(\mathbb{S}_{(p)})$, $k > 0$, with Adams filtration smaller than two are in the first non-trivial p-torsion homotopy group $\pi_{2p-3}(\mathbb{S}_{(p)}) \cong \mathbb{Z}/p\{\alpha_1\}$ (see [Li, Thm. 1.2.1]). In Corollary 2.1.4, we adapt this statement to our situation.

Definition 2.1.3. Define a function K on prime numbers by

$$K(p) = \begin{cases} 7 & \text{if } p = 2, \\ 2p-3 & \text{if } p > 2. \end{cases}$$

Corollary 2.1.4. *Let p be a prime and let k be an integer which is larger than $K(p)$.*

(i) All maps $f\colon \mathbb{S}^k_{(p)} \longrightarrow P_m\mathbb{S}_{(p)}$ factor through some finite p-local spectrum like in Lemma 2.1.2.

(ii) All maps in $\left[\Sigma^k P_m\mathbb{S}_{(p)}, P_m\mathbb{S}_{(p)}\right]^{\mathrm{Ho}(\mathrm{Mod}\text{-}P_m\mathbb{S}_{(p)})}$ and $\left[\Sigma^k R, R\right]^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)}$ factor through a finite $P_m\mathbb{S}_{(p)}$-cell and R-cell complex, respectively, with cells in dimensions $1, \dots, k-1$.

Proof. (i): Since the morphism $\mathbb{S}_{(p)} \longrightarrow P_m\mathbb{S}_{(p)}$ is a $\pi_{<m+1}$-isomorphism, the map f is either trivial or factors through the p-localized sphere spectrum and the claim follows from Lemma 2.1.2.
(ii): The second claim holds for all elements in $\left[\Sigma^k P_m, P_m\right]^{\mathrm{Ho}(\mathrm{Mod}\text{-}P_m)}$ due to (i) and the adjunction

$$- \wedge^L P_m \colon \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)}) \rightleftarrows \mathrm{Ho}(\mathrm{Mod}\text{-}P_m) : U$$

between triangulated categories. Moreover, recall that the triangulated equivalence

$$\mathrm{Ho}(\mathrm{Mod}\text{-}P_m) \simeq_\triangle \mathrm{Ho}(\mathrm{Mod}\text{-}R)$$

maps the object $P_m \in \mathrm{Ho}(\mathrm{Mod}\text{-}P_m)$ to an object isomorphic to $R \in \mathrm{Ho}(\mathrm{Mod}\text{-}R)$ (Thm. 1.2.16). Thus, the claim holds for morphisms in $\left[\Sigma^k R, R\right]^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)}$ as well. □

Theorem 2.1.5. *Let m be a non-negative integer. Then the morphism $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$ is a $\pi_{<m+1}$-isomorphism if the maps $\pi_l(\iota)$ are bijective for $l \leq \min\{K(p), m\}$.*

Remark 2.1.6. For an odd prime p, the only non-trivial homotopy groups $\pi_*(\mathbb{S}_{(p)})$ with dimensions at most $K(p) = 2p-3$ are $\pi_0(\mathbb{S}_{(p)}) \cong \mathbb{Z}_{(p)}$ and $\pi_{2p-3}(\mathbb{S}_{(p)}) \cong \mathbb{Z}/p\{\alpha_1\}$. Thus, in this case we only have to show bijectivity of the two maps $\pi_0(\iota)$ and $\pi_{2p-3}(\iota)$.

The theorem above follows inductively from

Lemma 2.1.7. *Let k and m be integers such that $K(p) < k \leq m$. If the maps $\pi_l(\iota)\colon \pi_l(\mathbb{S}_{(p)}) \longrightarrow \pi_l(R)$ are bijective for all $l < k$, then so is $\pi_k(\iota)$.*

Proof. It suffices to prove that the map $\pi_k(\iota)$ is surjective since the groups $\pi_k(\mathbb{S}_{(p)}) \cong \pi_k(P_m) \cong \pi_k(R)$ are finite. Let $\tilde{y}$ be a morphism in $\pi_k(R) = \left[\Sigma^k\mathbb{S}_{(p)}, R\right]$. We want to prove that this morphism factors through $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$. This proof is divided into three steps.
First, recall that the morphism ι induces an adjunction

$$- \wedge^L R \colon \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)}) \rightleftarrows \mathrm{Ho}(\mathrm{Mod}\text{-}R) : U$$

and that the adjoint $y \in \left[\Sigma^k R, R\right]^{\mathrm{Ho}(\mathrm{Mod}\text{-}R)}$ of the morphism $\tilde{y}$ factors through a finite R-cell complex Y with cells in dimension $1, \dots, k-1$ (Cor. 2.1.4):

$$\begin{tikzcd} \Sigma^k R \arrow[r, "a"] \arrow[rr, bend left, "y"] & Y \arrow[r, "b"] & R. \end{tikzcd}$$

Second, we inductively construct a finite $\mathbb{S}_{(p)}$-cell complex X with cells in dimensions $1, \dots, k-1$ and a $\pi_{<k}$-isomorphism $x\colon X \longrightarrow U(Y)$ in the homotopy category $\mathrm{Ho}(\text{Mod-}\mathbb{S}_{(p)})$. In particular, the map $\pi_k(x)$ is surjective and hence there exists an element $a' \in \pi_k(X)$ such that the diagram

$$\begin{tikzcd} \Sigma^k\mathbb{S}_{(p)} \arrow[r, "\tilde{a}"] \arrow[rr, bend left, "\tilde{y}"] \arrow[dr, dashed, "a'"'] & U(Y) \arrow[r, "U(b)"] & U(R) \\ & X \arrow[u, "x"'] & \end{tikzcd} \tag{2.5}$$

commutes. In order to do this, we fix a finite R-cell complex structure of Y with cells in dimensions 1 through $k-1$, that is we choose distinguished triangles in $\mathrm{Ho}(\text{Mod-}R)$:

$$\bigvee_{I_j} \Sigma^{j-1}R \xrightarrow{\quad s_j \quad} \mathrm{Sk}_{j-1} Y \xrightarrow{\quad i_j \quad} \mathrm{Sk}_j Y \xrightarrow{\quad p_j \quad} \bigvee_{I_j} \Sigma^j R$$

for every natural number $1 \le j \le k-1$. Here, I_j are finite (possibly empty) sets, the zeroth skeleton of Y equals $\mathrm{Sk}_0 Y = *$ and its $(k-1)^{th}$ skeleton is Y itself. Accordingly, we define the zeroth skeleton of X to be $\mathrm{Sk}_0 X = *$.
Let l be an integer with $1 \le l < k$. Suppose we have already constructed the $(l-1)^{th}$ Skeleton $\mathrm{Sk}_{l-1}(X)$ of X together with a $\pi_{<k}$-isomorphism $x_{l-1}\colon \mathrm{Sk}_{l-1} X \longrightarrow U(\mathrm{Sk}_{l-1} Y)$ in $\mathrm{Ho}(\text{Mod-}\mathbb{S}_{(p)})$. Then there exists a morphism s'_l such that the diagram

$$\begin{tikzcd} U(\bigvee_{I_l} \Sigma^{l-1}R) \arrow[r, "U(s_l)"] & U(\mathrm{Sk}_{l-1} Y) \arrow[r, "U(i_l)"] & U(\mathrm{Sk}_l Y) \arrow[r, "U(p_l)"] & U(\bigvee_{I_l} \Sigma^l R) \\ \bigvee_{I_l} \Sigma^{l-1}\mathbb{S}_{(p)} \arrow[u, "\bigvee \Sigma^{l-1}\iota"] \arrow[r, dashed, "s'_l"] & \mathrm{Sk}_{l-1} X \arrow[u, "x_{l-1}"'] & & \end{tikzcd}$$

commutes. We define the l^{th} skeleton $\mathrm{Sk}_l X$ by completing the morphism s'_l to a distinguished triangle in the homotopy category $\mathrm{Ho}(\text{Mod-}\mathbb{S}_{(p)})$:

$$\begin{tikzcd} U(\bigvee_{I_l} \Sigma^{l-1}R) \arrow[r, "U(s_l)"] & U(\mathrm{Sk}_{l-1} Y) \arrow[r, "U(i_l)"] & U(\mathrm{Sk}_l Y) \arrow[r, "U(p_l)"] & U(\bigvee_{I_l} \Sigma^l R) \\ \bigvee_{I_l} \Sigma^{l-1}\mathbb{S}_{(p)} \arrow[u, "\bigvee \Sigma^{l-1}\iota"] \arrow[r, "s'_l"] & \mathrm{Sk}_{l-1} X \arrow[u, "x_{l-1}"'] \arrow[r, "i'_l"] & \mathrm{Sk}_l X \arrow[u, dashed, "x_l"'] \arrow[r, "p'_l"] & \bigvee_{I_l} \Sigma^l \mathbb{S}_{(p)}. \arrow[u, "\bigvee \Sigma^l \iota"'] \end{tikzcd}$$

Observe that the upper triangle in the diagram above is exact since the forgetful functor

$$U\colon \text{Mod-}R \longrightarrow \text{Mod-}\mathbb{S}_{(p)}$$

induces an exact functor. Thus, there exists a morphism x_l such that this diagram commutes. By the Five Lemma, the morphism x_l is a $\pi_{<k}$-isomorphism since the morphisms $\bigvee \Sigma^{l-1}\iota$ and x_{l-1} are $\pi_{<l-1+k-1}$- and $\pi_{<k}$-isomorphisms, respectively.
It follows that there exists a spectrum X and a $\pi_{<k}$-isomorphism

$$X := \mathrm{Sk}_{k-1} X \xrightarrow{x := x_{k-1}} U(Y).$$

In particular, the morphism $\tilde{a}\colon \Sigma^k\mathbb{S}_{(p)} \longrightarrow U(R)$ factors through the morphism x as in diagram (2.5).

Third, we prove that the morphism

$$[X, \iota] : \big[X, \mathbb{S}_{(p)}\big] \longrightarrow [X, U(R)]$$

is surjective. Then there exists a morphism b' such that the diagram

$$\begin{tikzcd}
\Sigma^k\mathbb{S}_{(p)} \arrow[r, "\tilde{a}"] \arrow[dr, "a'"'] \arrow[rr, bend left, "\tilde{y}"] & U(Y) \arrow[r, "U(b)"] & U(R) \\
& X \arrow[u, "x"] \arrow[r, dashed, "b'"] & \mathbb{S}_{(p)} \arrow[u, "\iota"']
\end{tikzcd} \tag{2.6}$$

commutes and hence the element $b' \circ a' \in \pi_k(\mathbb{S}_{(p)})$ is in the preimage of $\tilde{y} \in \pi_k(R)$.

We prove the surjectivity of the map $[X, \iota]$ by induction over the skeleta of X. The morphism $[\mathrm{Sk}_0\, X, \iota] = [*, \iota]$ is bijective.
Let l be an integer with $0 < l < k$ and assume that the morphism $[\mathrm{Sk}_{l-1}\, X, \iota]_0$ is surjective. The exact triangle

$$\bigvee_I \Sigma^{l-1}\mathbb{S}_{(p)} \longrightarrow \mathrm{Sk}_{l-1}\, X \longrightarrow \mathrm{Sk}_l\, X \longrightarrow \bigvee_I \Sigma^l\mathbb{S}_{(p)}$$

in $\mathrm{Ho}(\text{Mod-}\mathbb{S}_{(p)})$ induces the following morphism of exact sequences:

$$\begin{tikzcd}
\big[\bigvee_I \Sigma^{l-1}\mathbb{S}_{(p)}, \mathbb{S}_{(p)}\big] \arrow[d, "\cong"] & \big[\mathrm{Sk}_{l-1}\, X, \mathbb{S}_{(p)}\big] \arrow[l] \arrow[d] & \big[\mathrm{Sk}_l\, X, \mathbb{S}_{(p)}\big] \arrow[l] \arrow[d] & \big[\bigvee_I \Sigma^l\mathbb{S}_{(p)}, \mathbb{S}_{(p)}\big] \arrow[l] \arrow[d, "\cong"] & \big[\Sigma\, \mathrm{Sk}_{l-1}\, X, \mathbb{S}_{(p)}\big] \arrow[l] \arrow[d] \\
\big[\bigvee_I \Sigma^{l-1}\mathbb{S}_{(p)}, R\big] & [\mathrm{Sk}_{l-1}\, X, R] \arrow[l] & [\mathrm{Sk}_l\, X, R] \arrow[l] & \big[\bigvee_I \Sigma^l\mathbb{S}_{(p)}, R\big] \arrow[l] & [\Sigma\, \mathrm{Sk}_{l-1}\, X, R] \arrow[l]\,.
\end{tikzcd}$$

In this diagram, the first and the fourth vertical morphism are bijective since the map $\pi_l(\iota)$ is bijective for every integer $l < k$. As the second morphism is surjective, the third morphism is also surjective by the Five Lemma.
It follows that the map $[X, \iota] = [\mathrm{Sk}_{k-1}\, X, \iota]$ is surjective and hence the element $\tilde{y} \in \pi_l(R)$ lies in the image of the map $\pi_k(\iota)$ (see diagram (2.6)). This finishes the proof of Lemma 2.1.7 and hence of Theorem 2.1.5. $\square$

2.2 The ring spectra $P_m(\mathbb{S}_{(2)})$ are rigid for all $m \geq 0$

In the last subsection, we prove that the ring spectrum $P_m\mathbb{S}_{(2)}$ is rigid if certain maps $\pi_l(\iota)$, $0 \leq l \leq \min\{7, m\}$, are bijections (Theorem 1.2.16, Lemma 2.1.1 and Theorem 2.1.5). Now, we show that this is indeed the case (Lemma 2.2.1) and thereby complete the proof of the rigidity of $P_m\mathbb{S}_{(2)}$, $m \geq 0$ (Thm. 2.2.3).

Lemma 2.2.1. *Let m be a non-negative integer and let R be a p-local cofibrant-fibrant ring spectrum as in Notation 2.0.9(ii). In particular, there exists an equivalence* $\Phi\colon \mathrm{Ho}(\text{Mod-}P_m\mathbb{S}_{(p)}) \longrightarrow \mathrm{Ho}(\text{Mod-}R)$

of triangulated categories and a ring isomorphism $\psi_R\colon \pi_*(P_m\mathbb{S}_{(p)}) \longrightarrow \pi_*(R)$ *that preserves Toda brackets. In addition, the unit* $i\colon \mathbb{S} \longrightarrow R$ *of* R *factorizes through* $\mathbb{S}_{(p)}$*:*

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[rr, bend left, "i"] & \mathbb{S}_{(p)} \arrow[r, "\iota"'] & R \end{tikzcd}$$

(see Notation 2.0.9). Then the morphism ι *is a* $\pi_{<m+1}$*-isomorphism.*

Proof. In order to avoid confusion, let us first fix the following notation: Let k be an integer which is at most m and a an element of the group $\pi_k(P_m\mathbb{S}_{(2)}) \cong \pi_k(\mathbb{S}_{(2)})$. Then we denote the element $\psi_R(a) \in \pi_k(R)$ with the same symbol 'a'.

To prove this lemma, it suffices to show that the morphism $\iota\colon \mathbb{S}_{(2)} \longrightarrow R$ induces isomorphisms $\pi_k(\iota)$ for all integers $k \leq \min\{m, K(2)\} = \min\{m, 7\}$ (Thm. 2.1.5). We do this consecutively for every integer $0 \leq k \leq 7$ with $k \leq m$:

$k = 0$: Since the unit $i\colon \mathbb{S} \longrightarrow R$ of the ring spectrum R factors over the morphism ι (see Notation 2.0.9(ii)), the map $\pi_0(\iota)$ sends the element $1 \in \pi_0(\mathbb{S}_{(2)})$ to $1 \in \pi_0(R) \cong \mathbb{Z}_{(2)}$ and is hence bijective.

$k = 1$: We prove the bijectivity of $\pi_1(\iota)$ in the following lemma using properties of the 2-local mod-2 Moore spectrum $M := \mathbb{S}_{(2)}/2$.

Lemma 2.2.2.

(i) The morphisms $2\,\mathrm{Id}_M \in [M, M]$ *and* $2\,\mathrm{Id}_{M\wedge R} \in [M \wedge R, M \wedge R]^{\mathrm{Ho(Mod}\text{-}R)}$ *factors through the morphisms* $\eta \in \left[\Sigma\mathbb{S}_{(2)}, \mathbb{S}_{(2)}\right]$ *and* $\eta \in [\Sigma R, R]^{\mathrm{Ho(Mod}\text{-}R)}$*, respectively.*

(ii) In particular, the map $\pi_1(\iota)\colon \pi_1(\mathbb{S}_{(2)}) \longrightarrow \pi_1(R)$ *is an isomorphism.*

$k = 2$: Since the group $\pi_2(\mathbb{S}_{(2)})$ is generated by the element η^2, the map $\pi_2(\iota)$ is bijective if $2 \leq m$.

$k = 3$: The map $\iota_*\colon \pi_3(\mathbb{S}_{(2)}) \cong \mathbb{Z}/8\mathbb{Z}\{\nu\} \longrightarrow \pi_3(R) \cong \mathbb{Z}/8\mathbb{Z}\{\nu\}$ sends the element $4\nu = \eta^3$ to $4\nu = \eta^3$ and hence it equals multiplication with a unit u in $\mathbb{Z}/8$.

$k = 4, 5$: Here is nothing to prove since the groups $\pi_4(\mathbb{S}_{(2)}) \cong \pi_4(R)$ and $\pi_5(\mathbb{S}_{(2)}) \cong \pi_5(R)$ are trivial.

$k = 6$: Since the group $\pi_6(\mathbb{S}_{(2)})$ is generated by ν^2, the map $\pi_6(\iota)$ is bijective if $6 \leq m$.

$k = 7$: Consider the Toda bracket $8\sigma = \langle \nu, 8, \nu\rangle \in \pi_7(\mathbb{S}_{(2)})$ [To62, §V]. It has trivial indeterminacy since the group $\pi_4(\mathbb{S}_{(2)})$ is trivial. As ι is a morphism of ring spectra it preserves Toda brackets and it follows that $\iota_*(8\sigma) = \iota_*(\langle \nu, 8, \nu\rangle) = \langle u\nu, 8, u\nu\rangle = \langle \nu, 8, \nu\rangle = 8\sigma$. Thus, the map $\pi_7(\iota)$ equals multiplication with a unit in $\mathbb{Z}/16$ and is hence an isomorphism.

□

Proof of Lemma 2.2.2.
(i): This is a well-known fact, which can be seen as follows: Due to the exact triangle

$$\mathbb{S}_{(2)} \xrightarrow{2} \mathbb{S}_{(2)} \xrightarrow{i} M \xrightarrow{q} \Sigma\mathbb{S}_{(2)}$$

in $\mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(2)})$, the group $[M, M]$ has exactly 4 elements and it suffices to show that the Identity map Id_M has 4-torsion in order to prove that the composition

$$M \xrightarrow{q} \Sigma\mathbb{S}_{(2)} \xrightarrow{\eta} \mathbb{S}_{(2)} \xrightarrow{i} M$$

equals $2\,\mathrm{Id}_M$.
As the group $[M,M]$ has exactly 4 elements, the morphism Id_M has either 2- or 4-torsion. Moreover, the morphism $2\,\mathrm{Id}_M$ can not be trivial since the cohomology of its cone $\mathrm{C}(2\,\mathrm{Id}_M) \simeq M \wedge M$ with $\mathbb{Z}/2$-coefficients does not split as a module over the Steenrod algebra. Thus, the morphism Id_M has 4-torsion and hence the first part of claim (i) follows.

In order to prove the claim for the morphism $2\,\mathrm{Id}_{M\wedge R}$, we consider the exact triangle

$$R \cong \mathbb{S}_{(2)} \wedge R \xrightarrow{2} R \xrightarrow{i\wedge R} M \wedge R \xrightarrow{q\wedge R} \Sigma R$$

in the homotopy category Ho(Mod-R). It follows that the image of $M \wedge P_m$ under the triangulated equivalence $\mathrm{Ho}(\text{Mod-}P_m) \simeq_\triangle \mathrm{Ho}(\text{Mod-}R)$ is isomorphic to $M \wedge R$ in Ho(Mod-R). Due to the $\pi_{<m+1}$-isomorphism $M \longrightarrow M \wedge P_m$, the group $[M, M \wedge P_m]$ is cyclic of order four and hence so is the group

$$[M, M\wedge R] \cong [M\wedge R, M \wedge R]^{\mathrm{Ho}(\text{Mod-}R)} \cong [M \wedge P_m, M \wedge P_m]^{\mathrm{Ho}(\text{Mod-}P_m)} \cong [M, M \wedge P_m]\,.$$

Like above, it follows that the element $2\,\mathrm{Id}_{M\wedge R} \in [M \wedge R, M \wedge R]^{\mathrm{Ho}(\text{Mod-}R)}$ factors as

$$\begin{tikzcd} M \wedge R \arrow[r, "q\wedge R"] \arrow[rrr, bend left, "2"] & \Sigma R \arrow[r, "\eta"] & R \arrow[r, "i\wedge R"] & M \wedge R \end{tikzcd}$$

in Ho(Mod-R).
(ii): The map $\pi_1(\iota)$ sends the element $\eta \in \pi_1(\mathbb{S}_{(2)})$ non-trivially to $\pi_1(R)$ since the morphisms $2\,\mathrm{Id}_M \in [M,M]$ and $(2\,\mathrm{Id}_M) \wedge \iota \in [M, M \wedge R] \cong [M \wedge R, M \wedge R]^{\mathrm{Ho}(\text{Mod-}R)}$ factor through the morphisms $\eta \in \pi_1(\mathbb{S}_{(2)})$ and $\eta \in \pi_1(R)$, by (i). Thus, the map $\pi_1(\iota)$ is bijective. ☐

Theorem 2.2.3. *The ring spectra $P_m(\mathbb{S}_{(2)})$ are rigid for all $m \geq 0$.*

Proof. Fix a non-negative integer m. In order to prove that $P_m\mathbb{S}_{(2)}$ is rigid, it suffices to prove that the morphism $\iota\colon \mathbb{S}_{(2)} \longrightarrow R$ induces an isomorphism $\pi_k(\iota)$ for every integer $k \leq \min\{m, K(2)\} = \min\{m, 7\}$ and for every ring spectrum R as in Notation 2.0.9(ii) (Theorem 1.2.16, Lemma 2.1.1 and Theorem 2.1.5). This is the case due to Lemma 2.2.1. ☐

2.3 The ring spectra $P_m(\mathbb{S}_{(p)})$ are rigid for odd primes p and $m \geq p^2q-1$

In subsection 2.1, we prove that the ring spectrum $P_m\mathbb{S}_{(p)}$ is rigid if certain morphisms $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$ (see Notation 2.0.9(ii) and Lemma 2.1.1) induce isomorphisms $\pi_k(\iota)$ of stable homotopy groups for $k \leq \min\{m, 2p-3\}$. Now we prove that this is the case if the integer m is at least p^2q-1, where q denotes the integer $2p-2$ (Theorem 2.3.11).
Since the stable homotopy groups of the spectrum $\mathbb{S}_{(p)}$ are trivial in dimensions 1 through $2p-4$, it remains to prove that the maps $\pi_0(\iota)$ and $\pi_{2p-3}(\iota)$ are bijective. Clearly, the former is since the unit $i\colon \mathbb{S} \longrightarrow R$ of the ring spectrum R factors over the morphism ι (see proof of Thm. 2.3.11). In order to prove the bijectivity of the map

$$\pi_{2p-3}(\iota)\colon \pi_{2p-3}(\mathbb{S}_{(p)}) \cong \mathbb{Z}/p\{\alpha_1\} \longrightarrow \pi_{2p-3}(R) \cong \mathbb{Z}/p,$$

we only need to show that the element α_1 is non-trivially mapped into the group $\pi_{2p-3}(R)$. The proof of this fact requires some work and is similar to Schwede's proof of the rigidity of the sphere ring spectrum [Sc07, Thm. 3.1 and Prop. 4.1]. One motivation for this proof is the following: By an unpublished result of Schwede (and probably by others), the $(2p-1)$-fold Toda bracket

$$\langle p, \beta_1, p, \beta_1, \dots, \beta_1, p\rangle \subset \pi_{2p-3+(p-1)(pq-2)}(\mathbb{S}_{(p)}), \tag{2.7}$$

is defined and contains the element $\alpha_1\beta_1^{p-1}$, where β_1 is a non-trivial element in $\pi_{pq-2}(\mathbb{S}_{(p)})$ (see the proof of Prop. 2.3.6). Now let us assume that the element α_1 is trivially mapped into the group $\pi_*(R)$ by the map $\pi_*(\iota)$. Then the Toda bracket (2.7) and the triangulated equivalence $\mathrm{Ho}(\text{Mod-}P_m\mathbb{S}_{(p)}) \cong_\triangle \mathrm{Ho}(\text{Mod-}R)$ indicate the existence of a $2p$-cell complex whose attaching maps correspond to the morphisms $p \cdot \mathrm{Id}_{\mathbb{S}_{(p)}}$ and β_1. This motivates the construction of a cell complex with these properties which, hopefully, can not exist and hence gives a contradiction to our assumption. In his proof of the rigidity of $\mathbb{S}_{(p)}$, Schwede constructs a cell complex with the necessary properties [Sc07, Thm. 3.1] using coherent M-modules [Sc07, Def. 2.1]. We recall their definition and some related statements in the following subsection. Afterwards, we prove that the map $\pi_{2p-3}(\iota)$ is bijective (Prop. 2.3.6) and hence finish the proof of the rigidity of certain Postnikov sections of the p-localized sphere spectrum (Thm. 2.3.11).

2.3.1 Coherent M-modules

For the proof of Theorem 2.3.11, we will need the following definitions and statements [Sc07]:

Definition 2.3.1. [Sc07, Def 1.1 and 1.2]

1. Let G be a group. We denote by EG the nerve of the transport category with object set G and exactly one morphism between any ordered pair of objects. So EG is a contractible simplicial set with a free G-action. We are mainly interested in the case $G = \Sigma_n$, the symmetric group on n letters.

2. The n^{th} *extended power* of a pointed simplicial set X is defined as the homotopy orbit construction
$$\mathsf{D}_n X = X^{\wedge n} \wedge_{\Sigma_n} E\Sigma_n^+,$$
where the symmetric group Σ_n permutes the smash factors, and the '+' denotes a disjoint basepoint. We often identify the first extended power $\mathsf{D}_1 X$ with X and use the convention $\mathsf{D}_0 X = S^0$.

3. The injection $\Sigma_i \times \Sigma_j \longrightarrow \Sigma_{i+j}$ induces a $\Sigma_i \times \Sigma_j$-equivariant map of simplicial sets
$$E\Sigma_i \times E\Sigma_j \longrightarrow E\Sigma_{i+j}$$
and thus a map of extended powers
$$\mu_{i,j} \colon \mathsf{D}_i X \wedge \mathsf{D}_j X \cong X^{\wedge(i+j)} \wedge_{\Sigma_i\times\Sigma_j} (E\Sigma_i \times E\Sigma_j)^+ \longrightarrow X^{\wedge(i+j)} \wedge_{\Sigma_{i+j}} (E\Sigma_{i+j})^+ = \mathsf{D}_{i+j}X.$$
These maps are associative in the sense that the diagram
$$\begin{tikzcd}
\mathsf{D}_i X \wedge \mathsf{D}_j X \wedge \mathsf{D}_k X \arrow[r, "\mathrm{Id}\wedge\mu_{j,l}"] \arrow[d, "\mu_{i,j}\wedge\mathrm{Id}"] & \mathsf{D}_i X \wedge \mathsf{D}_{j+l} X \arrow[d, "\mu_{i,j+l}"] \\
\mathsf{D}_{i+j} X \wedge \mathsf{D}_l X \arrow[r, "\mu_{i+j,l}"] & \mathsf{D}_{i+j+l} X
\end{tikzcd}$$

commutes for all $i, j, k \geq 0$. The maps $\mu_{i,j}$ are also unital, so that after identifying $\mathsf{D}_i X \wedge \mathsf{D}_0 X$ and $\mathsf{D}_0 X \wedge \mathsf{D}_i X$ with $\mathsf{D}_i X$ the maps $\mu_{i,0}$ and $\mu_{0,i}$ become the identity.

Throughout this section, M is a fixed simplicial set which is finite, pointed and of the homotopy type of a mod-p-Moore space with bottom cell in dimension two $i\colon S^2 \longrightarrow \mathsf{M}$, where S^2 is $\Delta[2]/\partial\Delta[2]$ (see [Sc07, p. 840]).

Definition 2.3.2. [Sc07, Def 2.1]
Let $\mathcal{C}$ be a pointed simplicial model category, p a prime and k an integer $1 \leq k \leq p$. A *k-coherent M-module* X consists of a sequence $X_{(1)}, X_{(2)}, \ldots, X_{(k)}$ of cofibrant objects in $\mathcal{C}$, together with morphisms in $\mathcal{C}$

$$\mu_{i,j}\colon \mathsf{D}_i\mathsf{M} \wedge X_{(j)} \longrightarrow X_{(i+j)}$$

for $1 \leq i, j$ and $i + j \leq k$, subject to the following two conditions

(C1) (Unitality) The composite

$$S^2 \wedge X_{(j-1)} \xrightarrow{i\wedge \mathrm{Id}} \mathsf{M} \wedge X_{(j-1)} \xrightarrow{\mu_{1,j-1}} X_{(j)}$$

is a weak equivalence for each $2 \leq j \leq l$, where we identify M with $\mathsf{D}_1\mathsf{M}$.

(C2) (Associativity) The square

$$\begin{tikzcd}
\mathsf{D}_i\mathsf{M} \wedge \mathsf{D}_j\mathsf{M} \wedge X_{(l)} \arrow[r, "\mathrm{Id}\wedge\mu_{j,l}"] \arrow[d, "\mu_{i,j}\wedge\mathrm{Id}"] & \mathsf{D}_i\mathsf{M} \wedge X_{(j+l)} \arrow[d, "\mu_{i,j+l}"] \\
\mathsf{D}_{i+j}\mathsf{M} \wedge X_{(l)} \arrow[r, "\mu_{i+j,l}"] & X_{(i+j+l)}
\end{tikzcd}$$

commutes for all $1 \leq i, j, l$ and $i + j + l \leq k$.

The *underlying object* of a k-coherent M-module X is the object $X_{(1)}$ of $\mathcal{C}$. We say that an object Y of $\mathcal{C}$ *admits a k-coherent M-action* if there exists a k-coherent M-module whose underlying $\mathcal{C}$-object is weakly equivalent to Y.
A *morphism* $f\colon X \longrightarrow Y$ of k-coherent M-modules consists of $\mathcal{C}$-morphisms $f_{(j)}\colon X_{(j)} \longrightarrow Y_{(j)}$ for $j = 1, \ldots, k$, such that the diagrams

$$\begin{tikzcd}
\mathsf{D}_i\mathsf{M} \wedge X_{(j)} \arrow[r, "\mathrm{Id}\wedge f_{(j)}"] \arrow[d, "\mu_{i,j}"] & \mathsf{D}_i\mathsf{M} \wedge Y_{(j)} \arrow[d, "\mu_{i,j}"] \\
X_{(i+j)} \arrow[r, "f_{(i+j)}"] & Y_{(i+j)}
\end{tikzcd}$$

commute for $1 \leq i, j$ and $i + j \leq k$.

Example 2.3.3. (Schwede, [Sc07, Example 2.4])
The mod-p Moore space acts on itself in a $(p-1)$-coherent fashion, which is also refered to as the *tautological* $(p-1)$-coherent M-module. We define a $(p-1)$-coherent M-module $\underline{\mathsf{M}}$ by setting

$$\underline{\mathsf{M}}_{(j)} = \mathsf{D}_j\mathsf{M}$$

for $1 \leq j \leq p-1$. In particular, the underlying object $\underline{\mathsf{M}}_{(1)}$ is just the Moore space M. The action maps

$$\mu_{i,j}\colon \mathsf{D}_i\mathsf{M} \wedge \underline{\mathsf{M}}_{(j)} = \mathsf{D}_i\mathsf{M} \wedge \mathsf{D}_j\mathsf{M} \longrightarrow \mathsf{D}_{i+j}\mathsf{M} = \underline{\mathsf{M}}_{(i+j)}$$

are the maps between extended powers in Definition 2.3.1(3). The unitality condition holds by [Sc07, Lemma 1.4]. Now let Y be a cofibrant object in a pointed simplicial model category $\mathcal{C}$. Then we can define a tautological $(p-1)$-coherent M-action on $\mathsf{M}\wedge Y$ by smashing the tautological module $\underline{\mathsf{M}}$ with Y. More precisely, we define a $(p-1)$-coherent M-module $\underline{\mathsf{M}\wedge Y}$ in $\mathcal{C}$ by

$$\underline{\mathsf{M}\wedge Y}_{(j)} = \underline{\mathsf{M}}_{(j)} \wedge Y = \mathsf{D}_j\mathsf{M}\wedge Y$$

for $1 \leq j \leq p-1$, and similarly for the structure maps. The associativity and unitality conditions are inherited from $\underline{\mathsf{M}}$.

We fix fibrant simplicial models S^{2p} and S^3 of the $2p$- and 3-sphere. We denote by $\alpha_1\colon S^{2p} \longrightarrow S^3$ a generator of the p-primary part of the homotopy group $\pi_{2p}(S^3)$ (see [Ra, Cor. 1.2.4]).

Proposition 2.3.4. *(Schwede, [Sc07, Thm. 2.5])*
Let Y be a cofibrant object of a simplicial stable model category $\mathcal{C}$ and let p be a prime. If the map

$$\alpha_1 \wedge Y\colon S^{2p}\wedge Y \longrightarrow S^3 \wedge Y$$

is trivial in the homotopy category of $\mathcal{C}$, then the tautological $(p-1)$-coherent M-action on $\mathsf{M}\wedge Y$ defined in Example 2.3.3 can be extended to a p-coherent M-action.

Proposition 2.3.5. *(cf. Schwede, [Sc07, Prop. 3.2])*
Let $\mathcal{C}$ be a simplicial stable model category and

$$\Phi\colon \mathrm{Ho}(\text{Mod-}P_m) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

an exact functor of triangulated categories that is fully faithful. Let $a\colon \Sigma^n P_m \longrightarrow E$ be a morphism in the stable homotopy category of right P_m-modules. Suppose that $\Sigma^l(\Phi E)$ admits a k-coherent M-action with $k\geq 2$ in $\mathrm{Ho}(\mathcal{C})$. Then there exists a morphism $\bar{a}\colon \mathsf{M}\wedge\Sigma^{n-2}P_m \longrightarrow E$ such that the following diagram commutes in $\mathrm{Ho}(\text{Mod-}P_m)$

$$\begin{tikzcd} \Sigma^n P_m \arrow[r, "a"] \arrow[d, "i\wedge\Sigma^{n-2}P_m"'] & E \\ \mathsf{M}\wedge\Sigma^{n-2}P_m \arrow[ur, dashed, "\exists\bar{a}"'] & \end{tikzcd}$$

and such that the object $\Sigma^{2+l}\Phi(C(\bar{a}))$ admits a $(k-1)$-coherent M-action, where $\mathrm{C}(\bar{a})$ is any mapping cone of $\bar{a}$ in $\mathrm{Ho}(\text{Mod-}P_m)$.

Proof. Replacing the stable homotopy category of spectra $\mathrm{Ho}(\text{Mod-}\mathbb{S})$ by $\mathrm{Ho}(\text{Mod-}P_m(\mathbb{S}_{(p)}))$ in Schwede's proof of Prop. 3.2. in [Sc07], provides the proof of this Proposition. □

2.3.2 The morphism ι is a $\pi_{<2p-2}$-isomorphism for $m \geq p^2(2p-2)-1$

In this subsection, we prove Theorem 2.3.11, which follows from

Proposition 2.3.6. *Let p be an odd prime and define q as the integer $2p-2$. Let m be an integer which is at least p^2q-1 and R a ring spectrum as in Notation 2.0.9(ii). In particular, there exists an exact functor*

$$\Phi\colon \mathrm{Ho}(\text{Mod-}P_m) \longrightarrow \mathrm{Ho}(\text{Mod-}R)$$

of triangulated categories that is fully faithful and that maps $P_m = P_m\mathbb{S}_{(p)}$ to an object isomorphic to R in the homotopy category of right R-modules. Then the map

$$\alpha_1 \wedge R\colon S^{2p} \wedge R \longrightarrow S^3 \wedge R$$

is non-trivial in $\mathrm{Ho}(\mathrm{Mod}\text{-}R)$.

The proof of this Proposition is analogous to the proof of Schwede for his Theorem ([Sc07, Thm. 3.1]). In our case, we need some knowledge about cohomology operations on the cohomology theory

$$\hat{H}^*(-) := [-, H\mathbb{F}_p]_{-*}^{\mathrm{Ho}(\mathrm{Mod}\text{-}P_m)}$$

where the right P_m-module structure on $H\mathbb{F}_p$ is defined by $H\mathbb{F}_p \wedge P_m \longrightarrow H\mathbb{F}_p \wedge H\mathbb{Z}_{(p)} \longrightarrow H\mathbb{F}_p$. Note that the graded group $\hat{H}^*(P_m)$ is concentrated in degree zero and is isomorphic to $H^*(\mathbb{S}_{(p)}) \cong \mathbb{Z}/p$.

Lemma 2.3.7. *The morphism of algebras $\hat{\mathcal{A}}^* := \hat{H}^*(H\mathbb{F}_p) \longrightarrow H^*(H\mathbb{F}_p) = \mathcal{A}^*$ which is induced by the forgetful functor $U\colon$ $\mathrm{Ho}(\mathrm{Mod}\text{-}P_m) \longrightarrow \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)})$ is bijective in degrees $* \leq m+1$.*

Remark/Notation 2.3.8. The cohomology operations in $\hat{\mathcal{A}}^*$ corresponding to $P^n \in \mathcal{A}^{nq}$ are denoted by $\hat{P}^n$. Since — among other things — we will need the relations $i\hat{P}^{ip} + \hat{P}^{ip-1}\hat{P}^1 = \hat{P}^p\hat{P}^{(i-1)p}$ for all $i = 1, \ldots, p$, we have to require that $m \geq p^2q - 1$.

Corollary 2.3.9. *For $m \geq p^2q-1$, the elements $\hat{P}^1, \hat{P}^2, \ldots, \hat{P}^{p^2}$ in $\hat{H}^*(H\mathbb{F}_p)$ corresponding to $P^1, P^2, \ldots, P^{p^2} \in H^*(H\mathbb{F}_p)$ are non-trivial and the relations $i\hat{P}^{ip} + \hat{P}^{ip-1}\hat{P}^1 = \hat{P}^p\hat{P}^{(i-1)p}$ hold for all $i = 1, \ldots, p$.*

Proof of Lemma 2.3.7. We prove that the forgetful functor

$$U\colon\ \mathrm{Ho}(\mathrm{Mod}\text{-}P_m) \longrightarrow \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)})$$

induces isomorphisms $U\colon \hat{H}^n(H\mathbb{F}_p) \longrightarrow H^n(H\mathbb{F}_p)$ for $n \leq m+1$, by using the adjunction

$$- \wedge P_m\colon\ \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)}) \rightleftarrows \mathrm{Ho}(\mathrm{Mod}\text{-}P_m)\ :U.$$

We denote the counit of this adjunction with ε. The morphism

$$\varepsilon_{H\mathbb{F}_p}\colon H\mathbb{F}_p \wedge P_m \longrightarrow H\mathbb{F}_p$$

is a $\pi_{<m+2}$-isomorphism for the following reasons: Observe that the group $\pi_n(H\mathbb{F}_p \wedge P_m)$ is trivial for every $0 < n \leq m+1$ since the morphism $H\mathbb{F}_p \wedge p_m\colon H\mathbb{F}_p \wedge \mathbb{S}_{(p)} \longrightarrow H\mathbb{F}_p \wedge P_m$ is a $\pi_{<m+1}$-isomorphism. Moreover, the map $\pi_0(\varepsilon_{H\mathbb{F}_p})\colon \pi_0(H\mathbb{F}_p \wedge P_m) \longrightarrow \pi_0(H\mathbb{F}_p)$ is bijective. It follows that the morphism $\varepsilon_{H\mathbb{F}_p}$ is a $\pi_{<m+2}$-isomorphism and that its Cone $\mathrm{C}(\varepsilon_{H\mathbb{F}_p})$ is $(m+2)$-connected.
Now we compare the groups $\hat{H}^n(H\mathbb{F}_p)$ and $H^n(H\mathbb{F}_p)$ using the commutative diagram

$$\begin{tikzcd}
\hat{H}^{n+1}(\mathrm{C}(\varepsilon_{H\mathbb{F}_p})) & \hat{H}^n(H\mathbb{F}_p \wedge P_m) \arrow[l] & \hat{H}^n(H\mathbb{F}_p) \arrow[l, "\hat{H}^n(\varepsilon_{H\mathbb{F}_p})"'] \arrow[d, "U"] & \hat{H}^n(\mathrm{C}(\varepsilon_{H\mathbb{F}_p})) \arrow[l] \\
 & & H^n(H\mathbb{F}_p), \arrow[ul, "\cong"', "\varphi"] &
\end{tikzcd}$$

where the right vertical map is induced by the forgetful functor U and the left diagonal map φ is part of the adjunction above. The group $\hat{H}^k(\mathrm{C}(\varepsilon_{H\mathbb{F}_p}))$ is trivial for every integer $k \leq m+2$ since the cone $\mathrm{C}(\varepsilon_{H\mathbb{F}_p})$ is $(m+2)$-connected, the ring spectrum P_m is connective and the spectrum $H\mathbb{F}_p$ is 1-coconnected [EKMM, IV.1.4.(i)]. Thus, the morphism

$$U\colon \hat{H}^n(H\mathbb{F}_p) \longrightarrow H^n(H\mathbb{F}_p)$$

is bijective in degrees $n \leq m+1$ since the morphism $\hat{H}^n(\varepsilon_{H\mathbb{F}_p})$ is. □

Proof for Proposition 2.3.6. We assume, contrary to our claim, that the morphism

$$\alpha_1 \wedge R\colon S^{2p} \wedge R \longrightarrow S^3 \wedge R$$

is trivial in the homotopy category of right R-modules. We will reach a contradiction by constructing right P_m-modules E_i for $i = 0, 1, \ldots, p-1$ with the following properties:

(a) The P_m-module E_i has exactly one stable P_m-cell in dimensions $jpq, jpq+1$ for all $j = 0, 1, \ldots, i$ — and no others.

(b) The map $\hat{P}^{ip}\colon \hat{H}^0(E_i) \longrightarrow \hat{H}^{ipq}(E_i)$ is non-trivial.

(c) The object $\Sigma^{2i+2}\Phi(E_i)$ admits a $(p-i)$-coherent M-action.

(d) There exists a morphism $a_i\colon \mathbb{S}^{(i+1)pq-1}_{(p)} \wedge P_m \longrightarrow E_i$ in $\mathrm{Ho}(\text{Mod-}P_m)$ such that the map

$$\hat{P}^p\colon \hat{H}^{ipq}(Ca_i) \longrightarrow \hat{H}^{(i+1)pq}(Ca_i)$$

is non-trivial.

We construct these P_m-modules $E_0, \ldots, E_{p-1}$ by induction.

For $\boldsymbol{i = 0}$: The zeroth right P_m-module E_0 is defined to be the P_m-cell complex $\mathsf{M} \wedge \mathbb{S}^{-2}_{(p)} \wedge P_m$, which satisfies properties (a) and (b). Moreover, the object $\Sigma^2\Phi(E_0) = \Sigma^2\Phi(\mathsf{M} \wedge \mathbb{S}^{-2}_{(p)} \wedge P_m)$ admits a p-coherent M-action since it is isomorphic to $\mathsf{M} \wedge \Phi(P_m)$ in the homotopy category $\mathrm{Ho}(\text{Mod-}R)$ and since the morphism $\alpha_1 \wedge R$ is trivial (Prop. 2.3.4).

We choose a morphism $\tilde{\beta}_1\colon \mathbb{S}^{pq-1}_{(p)} \longrightarrow \mathsf{M} \wedge \mathbb{S}^{-2}_{(p)}$ which is detected by the operation P^p. One possible morphism is the one constructed by Toda, whose composite with the pinch map $\mathsf{M} \wedge \mathbb{S}^{-2}_{(p)} \longrightarrow \mathbb{S}^1_{(p)}$ is a unit multiple of the generator β_1 of the homotopy group $\pi_{pq-2}(\mathbb{S}_{(p)})$ [To71, section 5, p. 60]. It follows that the map

$$a_0\colon \mathbb{S}^{pq-1}_{(p)} \wedge P_m \xrightarrow{\tilde{\beta}_1 \wedge P_m} \mathsf{M} \wedge \mathbb{S}^{-2}_{(p)} \wedge P_m$$

satisfies property (d).

For $\boldsymbol{0 < i < p}$: Suppose we have constructed the right P_m-modules E_j for $1 \leq j \leq i-1$. By Proposition 2.3.5, an extension $\bar{a}_{i-1}$ exists, such that the following diagram commutes in $\mathrm{Ho}(\text{Mod-}P_m)$

$$\begin{tikzcd} \mathbb{S}^{ipq-1}_{(p)} \wedge P_m \arrow[r, "a_{i-1}"] \arrow[d] & E_{i-1} \\ \mathsf{M} \wedge \mathbb{S}^{ipq-3}_{(p)} \wedge P_m \arrow[ur, "\exists \bar{a}_{i-1}"'] & \end{tikzcd} \tag{2.8}$$

and such that $\Sigma^{2i+2}\Phi(C(\bar{a}_{i-1}))$ admits a $(p-i)$-coherent M-action. We define $E_i = C(\bar{a}_{i-1})$ and hence get the following exact triangle in $\mathrm{Ho}(\text{Mod-}P_m)$:

$$\mathsf{M} \wedge \mathbb{S}^{ipq-3}_{(p)} \wedge P_m \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow \mathsf{M} \wedge \mathbb{S}^{ipq-2}_{(p)} \wedge P_m. \tag{2.9}$$

Thus properties (c) and (a) hold for E_i. To prove (b), consider the relation

$$\hat{P}^p\hat{P}^{(i-1)p} = i\hat{P}^{ip} + \hat{P}^{ip-1}\hat{P}^1.$$

For degree reasons, the operation $\hat{P}^1$ acts trivially on $\hat{H}^*(E_i)$ and hence the operation $i\hat{P}^{ip}$ is non-trivial if and only if $\hat{P}^p\hat{P}^{(i-1)p}$ is non-trivial. In the following, we prove that the operation $\hat{P}^p\hat{P}^{(i-1)p}$ is indeed non-trivial. Consider the commutative diagram

$$\begin{tikzcd}
 & \hat{H}^{(i-1)pq}(Ca_{i-1}) \arrow[r, "\hat{P}^p \neq 0"'] & \hat{H}^{ipq}(Ca_{i-1}) \\
\hat{H}^0(E_i) \arrow[r, "\text{④}", "\hat{P}^{(i-1)p}\neq 0"'] \arrow[d, "\text{③}"', "\cong"] & \hat{H}^{(i-1)pq}(E_i) \arrow[r, "\text{②}", "\hat{P}^p\neq 0"'] \arrow[u, "\text{①}"', "\cong"] \arrow[d, "\text{③}"', "\cong"] & \hat{H}^{ipq}(E_i) \cong \mathbb{F}_p \arrow[u] \\
\mathbb{F}_p \cong \hat{H}^0(E_{i-1}) \arrow[r, "\hat{P}^{(i-1)p}\neq 0"'] & \hat{H}^{(i-1)pq}(E_{i-1}) \cong \mathbb{F}_p. &
\end{tikzcd}$$

The morphisms in this diagram are bijective or non-trivial for the following reasons:

①: By the octahedral axiom and the commutative diagram (2.8), there is an exact triangle

$$\mathbb{S}^{ipq}_{(p)} \wedge P_m \longrightarrow C(a_{i-1}) \longrightarrow C(\bar{a}_{i-1}) = E_i \longrightarrow \mathbb{S}^{ipq+1}_{(p)} \wedge P_m. \tag{2.10}$$

Thus, the morphism $C(a_{i-1}) \longrightarrow E_i$ induces an isomorphism $\hat{H}^{(i-1)pq}(E_i) \xrightarrow{\cong} \hat{H}^{(i-1)pq}(Ca_{i-1})$ since the groups $\hat{H}^k(\mathbb{S}_{(p)} \wedge P_m) \cong H^k(\mathbb{S}_{(p)})$ are trivial for $k \neq 0$.

②: Therefore, the operation $\hat{P}^p\colon \hat{H}^{(i-1)pq}(E_i) \longrightarrow \hat{H}^{ipq}(E_i)$ is non-trivial since the operation $\hat{P}^p\colon \hat{H}^{(i-1)pq}(Ca_{i-1}) \longrightarrow \hat{H}^{ipq}(Ca_{i-1})$ is.

③: Due to the exact triangle (2.9), the morphisms $\hat{H}^n(E_i) \longrightarrow \hat{H}^n(E_{i-1})$ are bijective for all integers $n \leq ipq-2$ since the groups $\hat{H}^n(\mathsf{M} \wedge \mathbb{S}^{ipq-3}_{(p)} \wedge P_m) \cong H^n(\mathsf{M} \wedge \mathbb{S}^{ipq-3}_{(p)})$ are trivial for $n \neq ipq-1, ipq$.

④: Thus, the operation

$$\hat{P}^{(i-1)p}\colon \hat{H}^0(E_i) \longrightarrow \hat{H}^{(i-1)pq}(E_i)$$

is non-trivial since the operation $\hat{P}^{(i-1)p}\colon \hat{H}^0(E_{i-1}) \longrightarrow \hat{H}^{(i-1)pq}(E_{i-1})$ is.

Therefore, the composite

$$\hat{P}^p\hat{P}^{(i-1)p} = i\hat{P}^{ip}\colon \hat{H}^0(E_i) \longrightarrow \hat{H}^{ipq}(E_i)$$

and hence $\hat{P}^{ip}$ are non-trivial. Thus property (b) follows.
To prove (d), we consider the morphism

$$\mathbb{S}^{(i+1)pq-1}_{(p)} \wedge P_m \xrightarrow{S^1 \wedge \tilde{\beta}_1 \wedge \mathbb{S}^{ipq-1}_{(p)} \wedge P_m} S^1 \wedge \mathsf{M} \wedge \mathbb{S}^{ipq-3}_{(p)} \wedge P_m \cong E_i/E_{i-1}.$$

The obstruction to lifting this morphism to a morphism $a_i \colon \mathbb{S}^{(i+1)pq-1}_{(p)} \wedge P_m \longrightarrow E_i$ in the homotopy category $\mathrm{Ho}(\text{Mod-}P_m)$

$$\begin{tikzcd}
 & & \mathbb{S}^{(i+1)pq-1}_{(p)} \wedge P_m \arrow[dl, dashed, "\exists a_i"'] \arrow[d, "S^1\wedge\tilde{\beta}_1\wedge\mathbb{S}^{ipq-1}_{(p)}\wedge P_m"] \arrow[dr, bend left, "\simeq 0"] & \\
E_{i-1} \arrow[r] & E_i \arrow[r] & E_i/E_{i-1} \arrow[r] & \Sigma E_{i-1}
\end{tikzcd} \tag{2.11}$$

lies in the group

$$\left[\mathbb{S}^{(i+1)pq-1}_{(p)} \wedge P_m, \Sigma E_{i-1}\right]^{\mathrm{Ho}(\text{Mod-}P_m)} \cong \pi_{(i+1)pq-2}(E_{i-1}). \tag{2.12}$$

This group is trivial. To see this, recall that the stable homotopy groups $\pi_n(\mathbb{S}_{(p)})$ are trivial for $n = jpq-3, jpq-2$, $j = 2, \ldots, p$ [Ra, Thm. 4.4.20]. Since the same is true for the stable homotopy groups of P_m and since the spectrum E_{i-1} has P_m-cells in dimensions jpq and $jpq+1$ for $0 \leq j \leq i-1$, it follows that the group $\pi_{(i+1)pq-2}(E_{i-1})$ is trivial.

Therefore, it remains to show that every lift a_i is detected by the operation $\hat{P}^p$. By the octahedral axiom and the commutative diagram (2.11), the triangle

$$E_{i-1} \longrightarrow Ca_i \longrightarrow S^1 \wedge C\tilde{\beta}_1 \wedge \mathbb{S}^{ipq-1}_{(p)} \wedge P_m \longrightarrow \Sigma E_{i-1} \tag{2.13}$$

is exact in the homotopy category of right P_m-modules. Thus, the operation

$$\hat{P}^p \colon \hat{H}^{ipq}(Ca_i) \longrightarrow \hat{H}^{(i+1)pq}(Ca_i)$$

is non-trivial since the cone $\mathcal{C}\tilde{\beta}_1$ is detected by P^p and since the morphisms

$$\hat{H}^n(S^1 \wedge C\tilde{\beta}_1 \wedge \mathbb{S}^{ipq-1}_{(p)} \wedge P_m) \longrightarrow \hat{H}^n(Ca_i)$$

are bijective for all $n > (i-1)pq+1$.
Therefore, the right P_m-modules $E_0, E_1, \ldots, E_{p-1}$ exist.

Remark 2.3.10. In this argument, we needed that $m+1 \geq (p-1)pq$. In order to get a contradiction, we will need that $m+1 \geq p^2q$.

In the following, we consider cohomology operations on Ca_{p-1} in order to reach a contradiction. Recall that the relation of operations

$$\hat{P}^p\hat{P}^{(p-1)p} = p\hat{P}^{p^2} + \hat{P}^{p^2-1}\hat{P}^1$$

holds. Moreover, the two morphisms

$$p\hat{P}^{p^2}, \hat{P}^{p^2-1}\hat{P}^1 \colon \hat{H}^0(Ca_{p-1}) \longrightarrow \hat{H}^{p^2q}(Ca_{p-1})$$

are trivial for the following reasons: The map $p\hat{P}^{p^2}$ is trivial since its source and target are $\mathbb{F}_p$-vector spaces: $\hat{H}^0(Ca_{p-1}) \cong \hat{H}^0(E_{p-1}) \cong \mathbb{F}_p$ and $\hat{H}^{p^2q}(Ca_{p-1}) \cong \hat{H}^{p^2q}(S^1 \wedge C(\tilde{\beta}_1) \wedge \mathbb{S}^{(p-1)pq-1}_{(p)} \wedge P_m) \cong \hat{H}^{pq}(C(\tilde{\beta}_1) \wedge P_m) \cong H^{pq}(C(\tilde{\beta}_1)) \cong \mathbb{F}_p$ (see (2.13)). The second map $\hat{P}^{p^2-1}\hat{P}^1$ factors through the group $\hat{H}^q(Ca_{p-1}) \cong \hat{H}^q(E_{p-2}) = 0$ (see properties (a) and (d)) and is hence trivial. However, the sum

$$p\hat{P}^{p^2} + \hat{P}^{p^2-1}\hat{P}^1 = \hat{P}^p\hat{P}^{(p-1)p} \colon \hat{H}^0(Ca_{p-1}) \longrightarrow \hat{H}^{p^2q}(Ca_{p-1})$$

of these trivial maps is non-trivial since the properties (b) and (d) hold for E_{p-1}:

$$\begin{tikzcd}
\hat{H}^0(\mathbb{S}^{p^2q}_{(p)} \wedge P_m) = 0 \arrow[d] & & \hat{H}^{(p-1)pq}(\mathbb{S}^{p^2q}_{(p)} \wedge P_m) = 0 \arrow[d] & \\
\hat{H}^0(Ca_{p-1}) \arrow[rr, "\hat{P}^{(p-1)p}"] \arrow[d, "\cong"] & & \hat{H}^{(p-1)pq}(Ca_{p-1}) \arrow[r, "\hat{P}^p \neq 0"] \arrow[d, "\cong"] & \hat{H}^{p^2q}(Ca_{p-1}) \\
\mathbb{F}_p \cong \hat{H}^0(E_{p-1}) \arrow[rr, "\hat{P}^{(p-1)p} \neq 0"] \arrow[d] & & \hat{H}^{(p-1)pq}(E_{p-1}) \cong \mathbb{F}_p \arrow[d] & \\
\hat{H}^0(\mathbb{S}^{p^2q-1}_{(p)} \wedge P_m) = 0 & & \hat{H}^{(p-1)pq}(\mathbb{S}^{p^2q-1}_{(p)} \wedge P_m) = 0. &
\end{tikzcd}$$

This is a contradiction and hence the morphism $\alpha_1 \wedge R$ has to be non-trivial in Ho(Mod-R). □

Finally, we are able to prove that the Postnikov sections $P_m(\mathbb{S}_{(p)})$ are rigid for $m \geq p^2q-1$:

Theorem 2.3.11. *Let p be an odd prime and define $q = 2p-2$. Then the ring spectrum $P_m(\mathbb{S}_{(p)})$ is rigid for every integer $m \geq p^2q-1$.*

Remark 2.3.12. As mentioned above, the ring spectrum $P_0\mathbb{S}_{(p)} = H\mathbb{Z}_{(p)}$ is also rigid (see example 1.2.20(2)). The author does not know whether the ring spectra $P_m\mathbb{S}_{(p)}$, $2p-3 \leq m < p^2q-1$, are rigid.

Proof. By Theorem 2.1.5 and Remark 2.1.6, it suffices to prove that every morphism $\iota\colon \mathbb{S}_{(p)} \longrightarrow R$ as in Notation 2.0.9(ii) induces isomorphisms $\pi_0(\iota)$ and $\pi_{2p-3}(\iota)$. Let ι be such a morphism. The map $\pi_0(\iota)$ is an isomorphism since the unit $i\colon \mathbb{S} \longrightarrow R$ of the ring spectrum R whose zeroth homotopy group is $\mathbb{Z}_{(p)}$ factors over the morphism ι. Now we prove that the map

$$\pi_{2p-3}(\iota)\colon \pi_{2p-3}(\mathbb{S}_{(p)}) \cong \mathbb{Z}/p\{\alpha_1 \wedge \mathbb{S}_{(p)}\} \longrightarrow \pi_{2p-3}(R) \cong \mathbb{Z}/p$$

is non-trivial and hence bijective. It sends the element $\alpha_1 \wedge \mathbb{S}_{(p)}$, which is represented by the morphism

$$\alpha_1 \wedge \mathbb{S}_{(p)}\colon S^{2p} \wedge \mathbb{S}_{(p)} \longrightarrow S^3 \wedge \mathbb{S}_{(p)}$$

in the category of right $\mathbb{S}_{(p)}$-modules, to the element $(S^3 \wedge \iota) \circ (\alpha_1 \wedge \mathbb{S}_{(p)}) = \alpha_1 \wedge \mathbb{S}_{(p)} \in \pi_{2p-3}(R)$. This element is the adjoint of the non-trivial morphism $\alpha_1 \wedge R \in \left[\Sigma^{2p-3}R, R\right]^{\mathrm{Ho(Mod}\text{-}R)}$ (Prop. 2.3.6) under the adjunction

$$- \wedge^L R\colon \mathrm{Ho}(\mathrm{Mod}\text{-}\mathbb{S}_{(p)}) \rightleftarrows \mathrm{Ho}(\mathrm{Mod}\text{-}R) : U^R.$$

Thus, it is non-trivial and the claim follows. □

3 Towards rigidity of the real connective K-theory ring spectrum

In this section we consider the 2-local real connective K-theory ring spectrum $ko_{(2)}$. Recall that its graded ring of homotopy groups is given by $\pi_*(ko_{(2)}) = \mathbb{Z}_{(2)}[\eta, \omega, \beta]/(2\eta, \eta^3, \eta\omega, \omega^2 - 4\beta)$, where the generators η, ω and β have degree 1, 4 and 8, respectively. We would like to prove that the ring spectrum $ko_{(2)}$ is rigid. One possible approach is to show that $ko_{(2)}$ is stably equivalent to every ring spectrum R whose ring of homotopy groups $\pi_*(R)$ is isomorphic to $\pi_*(ko_{(2)})$ so that this isomorphism preserves triple Toda brackets (Cor. 1.2.18).
In this section, we prove that the ring spectra R and $ko_{(2)}$ are stably equivalent as spectra (Thm. 3.3.7). First, we show in subsection 3.1 that the 4^{th} Postnikov sections $P_4 ko_{(2)}$ and $P_4 R$ are stably equivalent (Cor. 3.1.7). In particular, their cohomologies $H^*(P_4 ko_{(2)}, \mathbb{Z}/2) \cong H^*(ko) \oplus H^*(ko)[9]$ and $H^*(P_4 R, \mathbb{Z}/2)$ are isomorphic as $\mathcal{A}^*$-modules.
Using this and the periodicity of R and $ko_{(2)}$, we deduce in subsection 3.2 that the $\mathcal{A}^*$-modules $H^*(ko_{(2)}, \mathbb{Z}/2)$ and $H^*(R, \mathbb{Z}/2)$ are abstractly isomorphic (Thm. 3.2.8). Thus, in the last subsection 3.3 we obtain a morphism of spectra

$$f : R_2^\wedge \longrightarrow ko_2^\wedge$$

(Cor. 3.3.2) due to the Adams spectral sequence and a result of Milgram [Mi, equation 5.20]. This morphism induces an isomorphism on cohomology with $\mathbb{Z}/2$-coefficients and is hence a stable equivalence. In the last subsection 3.3, we modify this stable equivalence such that it lifts to a stable equivalence between the 2-localized spectra R and $ko_{(2)}$. Unfortunately, this stable equivalence is in general not an equivalence of ring spectra.

Notation 3.0.13. In this section, R always denotes a ring spectrum as in Theorem 1.2.16. In particular, there is an isomorphism of rings

$$\psi\colon \pi_*(ko_{(2)}) \xrightarrow{\cong} \pi_*(R)$$

which preserves Toda brackets (see Def. 1.2.14). To facilitate notation, we denote the elements in $\pi_*(ko_{(2)}) \cong \left[ko_{(2)}, ko_{(2)}\right]_*^{\mathrm{Ho}(ko_{(2)}\text{-mod})}$ and their corresponding elements in $\pi_*(R)$ with the same symbols.

Moreover, we assume without loss of generality that the ring spectrum R is cofibrant and fibrant by taking a cofibrant and fibrant replacement in the model category of $\mathbb{S}$-algebras (see [ScSh00, Thm. 4.1.(3)]). We use a cofibrant and fibrant model of the ring spectrum $ko_{(2)}$.

Notation 3.0.14. All the (ring) spectra in this chapter are 2-local. To simplify notation, we often omit the corresponding index and denote the ring spectra $ko_{(2)}$ and $\mathbb{S}_{(2)}$ by 'ko' and '$\mathbb{S}$', respectively. Moreover, the 2-localization of the integers $\mathbb{Z}_{(2)}$ is denoted by '$\mathbb{Z}$' as well.

3.1 Cell-approximation of ko, P_4ko and P_8ko

In this subsection, we prove that the ring spectra $P_4 ko$ and $P_4 R$ are stably equivalent (Cor. 3.1.7). We need this statement in subsection 3.2, in order to calculate the cohomology groups of $\bar{P}_4 R$ (Cor. 3.2.6) and R (Thm. 3.2.8). Moreover, we prove in this subsection that the ring spectrum $P_4 ko$ is rigid (Thm. 3.1.12).
In order to prove Corollary 3.1.7, we need to construct a zig-zag of stable equivalences between the ring spectra $P_4 ko$ and $P_4 R$. This will be done inductively by gluing ring spectrum cells to the sphere spectrum $\mathbb{S}$ in such a way that the colimit C of the resulting ring spectra C_i is stably equivalent to

the ring spectra P_4ko and P_4R (see subsection 3.1.2):

$$\begin{tikzcd}
 & & & P_4R & & \\
\mathbb{S} \arrow[r] \arrow[urrr, "\iota"] \arrow[drrr, "\iota"'] & C_1 \arrow[r] \arrow[urr, dotted] \arrow[drr, dotted] & C_2 \arrow[r] \arrow[ur, dotted] \arrow[dr, dotted] & \cdots \arrow[r] & C = \operatorname{colim} C_i \arrow[ul, dotted, "\simeq"] \arrow[dl, dotted, "\simeq"'] \\
 & & & P_4ko. & &
\end{tikzcd}$$

In this diagram, the morphisms ι are the unit morphisms of the ring spectra P_4R and P_4ko, respectively. Recall that $\pi_*(ko)$ is isomorphic to $\mathbb{Z}[\eta, \omega, \beta]/(2\eta, \eta^3, \omega\eta, \omega^2 - 4\beta)$, where η, ω and β have degree 1, 4 and 8, respectively, and hence $\pi_*(P_4ko) \cong \mathbb{Z}[\eta, \omega]/(2\eta, \eta^3, \omega\eta, \omega^2)$. Thus, the first step in the process is to kill a generator ν of the group $\pi_3(\mathbb{S}) \cong \mathbb{Z}/8\mathbb{Z}\nu$ since $\pi_*(\iota)$ maps ν to zero. Gluing the ring spectrum $T(\mathbb{D}^4)$ to $\mathbb{S}$ via the map $\nu\colon \mathbb{S}^3 \longrightarrow \mathbb{S}$ gives a ring spectrum C_1 together with $\pi_{<5}$-isomorphisms $P_4R \leftarrow C_1 \longrightarrow P_4ko$ (see subsection 3.1.2).
We explain what is meant by 'gluing a ring cell to a ring spectrum' in the next subsection 3.1.1. Afterwards, we prove Corollary 3.1.7 and Theorem 3.1.12. In the rest of this subsection, we approximate the ring spectrum ko using this method. More precisely, we prove that there exist $\pi_{<10}$-isomorphisms of ring spectra

$$ko \longleftarrow C \longrightarrow R$$

(Thm. 3.1.15). In particular, the ring spectra P_8ko and P_8R are stably equivalent (Thm. 3.1.16). We need this statement in order to calculate the first differential of a spectral sequence, which converges to $H^*(R, \mathbb{Z}/2)$ (Lemma 3.2.11). It follows that the cohomology of R with $\mathbb{Z}/2$-coefficients is isomorphic to $H^*(ko, \mathbb{Z}/2)$ (Thm. 3.2.8).

3.1.1 Gluing ring spectra cells to a ring spectrum

In this subsection we consider pushouts of ring spectra

$$T(\mathbb{D}^{m+1}) \xleftarrow{T(i)} T(\mathbb{S}^m) \longrightarrow Z,$$

where Z is a ring spectrum and i denotes the cofibration $i\colon \mathbb{S}^m \longrightarrow \mathbb{D}^{m+1} = \Delta[1] \wedge \mathbb{S}^m$. The ring spectra $T(\mathbb{D}^{m+1})$ and $T(\mathbb{S}^m)$ denote the free associative ring spectra on the spectra $\mathbb{D}^{m+1}$ and $\mathbb{S}^m$ (Def. 1.3.2). In order to calculate some homotopy groups of the resulting ring spectrum we use a description of Schwede and Shipley for those kinds of pushouts [ScSh00, 6.2]. We recall the parts which are relevant for our work.

Let Z be a ring spectrum and $K \longrightarrow L$ a cofibration in the category of symmetric spectra. The pushout of the diagram

$$TL \longleftarrow TK \longrightarrow Z$$

in the category of ring spectra is given by the colimit $C = \operatorname{colim}_{n \geq 0} C_n$ of a sequence

$$C_0 = Z \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n \longrightarrow \cdots$$

in the underlying category Sp^{Σ}. The spectrum C_0 is defined to be the ring spectrum Z. The spectra C_n are inductively defined as pushouts of the diagrams

$$C_{n-1} \longleftarrow Q_n \longrightarrow (Z \wedge L)^{\wedge n} \wedge Z. \tag{3.14}$$

In order to define the spectra Q_n, we need to define punctured cubes in the category of spectra:

Definition 3.1.1. Let $n > 0$ be a natural number.

(i) An *n-dimensional cube* in the category of spectra is a functor

$$W_n\colon \mathcal{P}(\{1,2,\dots,n\}) \longrightarrow \text{Spectra},$$

from the poset category $\mathcal{P}(\{1,2,\dots,n\})$ of subsets of $\{1,2,\dots,n\}$ and inclusions to the category of spectra. The vertex of the cube at the subset $A \subseteq \{1,2,\dots,n\}$ is defined to be

$$W_n(A) = Z \wedge X_1 \wedge Z \wedge X_2 \wedge \cdots \wedge X_n \wedge Z,$$

with

$$X_i = \begin{cases} K & \text{if } i \notin A, \\ L & \text{if } i \in A. \end{cases}$$

The map $K \longrightarrow L$ induces a morphism $W_n(A) \longrightarrow W_n(B)$ for an inclusion of subsets $A \subseteq B$.

(ii) The *punctured cube* $\hat{W}_n$ denotes the cube W_n without the terminal vertex $W_n(\{1,2,\dots,n\})$.

(iii) The spectrum Q_n is defined as a colimit of this punctured cube $\hat{W}_n$. The morphism

$$Q_n \longrightarrow (Z \wedge L)^{\wedge n} \wedge Z$$

is induced by the maps

$$\hat{W}_n(A) \longrightarrow W_n(\{1,2,\dots,n\}) = (Z \wedge L)^{\wedge n} \wedge Z, \;\; A \subsetneqq \{1,2,\dots,n\}\,.$$

We refer the reader to [ScSh00, p. 508] for the definition of the map $Q_n \longrightarrow C_{n-1}$ in (3.14).

Lemma 3.1.2. *The morphism $Q_n \longrightarrow (Z \wedge L)^{\wedge n} \wedge Z$ of Def. 3.1.1(iii) is a cofibration and the quotient $((Z \wedge L)^{\wedge n} \wedge Z)/Q_n$ is isomorphic to $(Z \wedge L/K)^{\wedge n} \wedge Z$ for every natural number $n > 0$.*

Proof. The claim is clear for $n = 1$ since the spectrum Q_1 equals $Z \wedge K \wedge Z$. For n bigger than 1, the spectrum Q_n is isomorphic to the pushout of

$$((Z \wedge L)^{\wedge(n-1)} \wedge Z) \wedge K \wedge Z \longleftarrow Q_{n-1} \wedge K \wedge Z \longrightarrow Q_{n-1} \wedge L \wedge Z.$$

Thus, the lemma follows by induction on n since the following sublemma holds. □

Sublemma 3.1.3. *Let $\tilde{Y} \longrightarrow Y$ and $\tilde{Z} \longrightarrow Z$ be two cofibrations of symmetric spectra and denote the pushout $(Y \wedge \tilde{Z}) \cup_{\tilde{Y} \wedge \tilde{Z}} (\tilde{Y} \wedge Z)$ by Q. Then the induced morphism $Q \longrightarrow Y \wedge Z$ is a cofibration and the quotient $(Y \wedge Z)/Q$ is isomorphic to $(Y/\tilde{Y}) \wedge (Z/\tilde{Z})$.*

In the following subsections, we want to kill some elements in the homotopy groups of connective ring spectra. Let Z be a connective ring spectrum and $a\colon \mathbb{S}^m \longrightarrow Z$ a morphism of spectra. The forgetful functor from the category of ring spectra to the category of spectra has a left adjoint, the functor T (see Def. 1.3.2). The adjoint morphism $\tilde{a}\colon T(\mathbb{S}^m) \longrightarrow Z$ of a is denoted by $\tilde{a}$. We consider the following pushout in the category of ring spectra

$$\begin{tikzcd} T(\mathbb{S}^m) \arrow[r, "\tilde{a}"] \arrow[d] & Z \arrow[d] \\ T(\mathbb{D}^{m+1}) \arrow[r] & C(a). \end{tikzcd}$$

Recall that the underlying spectrum of the ring spectrum $C(a)$ equals the colimit $\operatorname{colim}_{n\geq 0} C_n$, where the spectrum C_1 is defined as the pushout of the diagram

$$\mathbb{S}\wedge\mathbb{D}^{m+1}\wedge\mathbb{S} \longleftarrow \mathbb{S}\wedge\mathbb{S}^{m}\wedge\mathbb{S} \xrightarrow{\mu\circ(1\wedge a\wedge 1)} Z\ .$$

Since the cofiber of the inclusion $\mathbb{S}^m \longrightarrow \mathbb{D}^{m+1}$ is $\mathbb{S}^{m+1}$, the triangle

$$\Sigma^{-1}(Z\wedge\mathbb{S}^{m+1})^{\wedge n}\wedge Z \longrightarrow C_{n-1} \longrightarrow C_n \longrightarrow (Z\wedge\mathbb{S}^{m+1})^{\wedge n}\wedge Z$$

is exact for $m\geq 0$, $n\geq 1$ (Lemma 3.1.2). Thus, the morphisms $C_{n-1}\longrightarrow C_n$ are $\pi_{<n(m+1)-1}$-isomorphisms since the spectra $(Z\wedge\mathbb{S}^{m+1})^{\wedge n}\wedge Z$ are $(n(m+1)-1)$-connected. Therefore, the colimit $\operatorname{colim}_n \pi_m(C_n)\cong\pi_m(C(a))$ stabilizes and the morphism $i\colon C_1\longrightarrow C(a)$ is a $\pi_{<2m+1}$-isomorphism.

Every morphism of ring spectra $f\colon Z\longrightarrow T$ which maps the element $[a]\in\pi_m(Z)$ to zero in $\pi_m(T)$ factors through the additive cone $\mathbb{D}^{m+1}\cup_a Z$ of a. Similarly, the morphism f factors through the pushout $C(a)=T(\mathbb{D}^{m+1})\cup_{T(\mathbb{S}^m)} Z$ in the category of ring spectra:

Construction 3.1.4. Let $f\colon Z\longrightarrow T$ be a morphism between cofibrant-fibrant ring spectra which maps the element $[a]\in\pi_m(Z)$ to zero in $\pi_m(T)$. We fix a morphism of spectra $a\colon\mathbb{S}^m\longrightarrow Z$ representing $[a]$ and denote the pushout $T(\mathbb{D}^{m+1})\cup_{T(\mathbb{S}^m)} Z$ in the category of ring spectra by $\mathrm{C}(a)$. Then choosing a null-homotopy $H\colon\mathbb{D}^{m+1}\longrightarrow T$ of the composite $f\circ a\colon\mathbb{S}^m\longrightarrow T$ defines a morphism of ring spectra G which extends f:

$$\begin{tikzcd}
T(\mathbb{S}^m) \arrow[r, "\tilde{a}"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "f"] & \\
T(\mathbb{D}^{m+1}) \arrow[r] \arrow[drr, bend right, "\tilde{H}"'] & \mathrm{C}(a) \arrow[dr, dotted, "\exists!\, G"] & \\
 & & T.
\end{tikzcd}$$

In the following, we need to determine some parts of the ring structure and the right $\pi_*(\mathbb{S})$-module structure of the ring $\pi_*(Z)$. Among others, we use Toda brackets for this purpose:

Lemma 3.1.5. *Let $x\colon X\longrightarrow Y$, $y\colon Y\longrightarrow Z$ and $z\colon Z\longrightarrow W$ be morphisms in a triangulated category such that $yx=0=zy$. Let*

$$Z\xrightarrow{z} W\xrightarrow{i}\mathrm{C}\xrightarrow{q}\Sigma Z$$

be an exact triangle. Consider the preimage of the element $y\in[Y,Z]$ under the morphism $[Y,q]$:

$$q_*^{-1}(y)=\left\{a\in\left[Y,\Sigma^{-1}\mathrm{C}\right]|q\circ a=y\right\},$$

which is non-empty since the element y lies in the kernel of the morphism $[Y,z]$. Then the set $\Sigma^{-1}\langle z,y,x\rangle\subset\left[X,\Sigma^{-1}W\right]$ is sent to the set $q_*^{-1}(y)\circ x\subset\left[X,\Sigma^{-1}\mathrm{C}\right]$ by the morphism i_*.

Proof. The proof follows directly from the definition of Toda brackets. □

3.1.2 Approximation of P_4ko: Killing the element $\nu \in \pi_3(\mathbb{S})$

Our aim is to approximate the ring spectra ko and P_4ko by gluing ring spectra cells to the sphere spectrum $\mathbb{S}$. In this subsection, we kill the element $\nu \in \pi_3(\mathbb{S})$ by attaching the ring spectrum $T(D^4)$ to the sphere spectrum. That is, we consider the ring spectrum $\mathrm{C}(\nu)$, which is defined by the following pushout diagram in the category of ring spectra (see section 3.1.1):

$$\begin{tikzcd}
T(\mathbb{S}^3) \arrow[r, "\tilde{\nu}"] \arrow[d, "T(j)"'] & \mathbb{S} \arrow[d] \\
T(\mathbb{D}^4) \arrow[r] & \mathrm{C}(\nu).
\end{tikzcd}$$

In the first part of this subsection, we construct morphisms of ring spectra $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow ko$ and $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow R$ which are $\pi_{<5}$-isomorphisms (Thm. 3.1.6(iii)) and calculate some homotopy groups of $\mathrm{C}(\nu)$. In the second part, the rigidity of P_4ko is deduced (Thm. 3.1.12).

The homotopy groups $\pi_n(\mathrm{C}(\nu))$, $0 \leq n \leq 9$: By the description of the ring spectrum $\mathrm{C}(\nu)$ as a colimit $\mathrm{colim}_n\, C_n$ (see subsection 3.1.1), it is sufficient to calculate the homotopy groups $\pi_n(C_1)$ and $\pi_n(C_2)$, $0 \leq n \leq 9$, in order to calculate the groups $\pi_n(\mathrm{C}(\nu))$ for $0 \leq n \leq 9$. Recall that the spectrum C_0 is the sphere spectrum $\mathbb{S}$. The spectra C_1 and C_2 are given by the following two pushout diagrams in the category of spectra:

$$\begin{tikzcd}
\mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S} \arrow[r, "\nu"] \arrow[d] & \mathbb{S} \arrow[d, "i"] \\
\mathbb{S} \wedge \mathbb{D}^4 \wedge \mathbb{S} \arrow[r, "H"] & C_1
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Q_2 \arrow[r, "\bar{l}"] \arrow[d] & C_1 \arrow[d] \\
(\mathbb{S} \wedge \mathbb{D}^4)^{\wedge 2} \wedge \mathbb{S} \arrow[r] & C_2.
\end{tikzcd}$$

Here, Q_2 and $\bar{l}$ are defined by the pushout diagram:

$$\begin{tikzcd}
\mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S} \arrow[r, "\mathrm{Id} \wedge j \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id}_{\mathbb{S}} \wedge j \wedge \mathrm{Id}_{(\mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S})}"] \arrow[rrdd, bend left, "i \circ (\nu \wedge \nu)"] & \mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S} \wedge \mathbb{D}^4 \wedge \mathbb{S} \arrow[d] \arrow[rdd, bend left, "\nu \wedge H"] & \\
\mathbb{S} \wedge \mathbb{D}^4 \wedge \mathbb{S} \wedge \mathbb{S}^3 \wedge \mathbb{S} \arrow[r] \arrow[rrd, bend right, "H \wedge \nu"'] & Q_2 \arrow[rd, dotted, "\exists!\, \bar{l}"] & \\
 & & C_1.
\end{tikzcd}$$

Thus, the spectrum Q_2 is weakly equivalent to $\mathbb{S}^7$. We fix such a weak equivalence and define l to be the composite $\mathbb{S}^7 \simeq Q_2 \xrightarrow{\bar{l}} C_1$ in the stable homotopy category. Up to a sign, this composite is independent of the choice of the weak equivalence $\mathbb{S}^7 \simeq Q_2$. Using the two resulting exact triangles

$$\mathbb{S}^3 \xrightarrow{\nu} \mathbb{S} \xrightarrow{i} C_1 \xrightarrow{p} \mathbb{S}^4$$

and

$$\mathbb{S}^7 \xrightarrow{l} C_1 \xrightarrow{j} C_2 \xrightarrow{q} \mathbb{S}^8, \tag{3.15}$$

we can calculate the homotopy groups $\pi_n(\mathrm{C}(\nu)) \cong \pi_n(C_2)$ for $0 \leq n \leq 9$.

Theorem 3.1.6. *(i) For $0 \leq n \leq 9$, the homotopy groups $\pi_n(C_1)$ and $\pi_n(C_2) \cong \pi_n(\mathrm{C}(\nu))$ are given in the following table*

n	0	1	2	3	4	5	6	7	8	9
$\pi_n(\mathbb{S})$	$\mathbb{Z}$	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	$\mathbb{Z}/8\{\nu\}$	0	0	$\mathbb{Z}/2\{\nu^2\}$	$\mathbb{Z}/16\{\sigma\}$	$\mathbb{Z}/2\{\eta\sigma\} \oplus \mathbb{Z}/2\{\epsilon\}$	$\mathbb{Z}/2\{\eta^2\sigma\} \oplus \mathbb{Z}/2\{\nu^3\} \oplus \mathbb{Z}/2\{\mu_1\}$
										$\eta\epsilon + \eta^2\sigma = \nu^3$
							$\big\downarrow 0$	$\big\downarrow$	$\big\downarrow \cong$	$\big\downarrow$
$\pi_n(C_1)$	$\mathbb{Z}$	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	0	$\mathbb{Z}\{\overline{8}\}$	$\mathbb{Z}/2\{\bar{\eta}\}$	$\mathbb{Z}/2\{\bar{\eta}\eta\}$	$\mathbb{Z}/16\{\sigma\} \oplus \mathbb{Z}/4\{L-\sigma\}$	$\mathbb{Z}/2\{\sigma\eta\} \oplus \mathbb{Z}/2\{\epsilon\}$	$\mathbb{Z}/2\{\epsilon\eta\} \oplus \mathbb{Z}/2\{\mu_1\}$
						$\overline{8}\cdot\eta = 0$		$\bar{\eta}\eta^2 = 2(L-\sigma)$	$L\eta = \sigma\eta + \epsilon$	$L\eta^2 = 0$
								$\big\downarrow$	$\sigma\eta, \epsilon \mapsto \epsilon$	$\big\downarrow \cong$
$\pi_n(C_2) \cong \pi_n(\mathrm{C}(\nu))$	$\mathbb{Z}$	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	0	$\mathbb{Z}\{\overline{8}\}$	$\mathbb{Z}/2\{\bar{\eta}\}$	$\mathbb{Z}/2\{\bar{\eta}\eta\}$	$\mathbb{Z}/4\{\sigma\}$	$\mathbb{Z}/2\{\epsilon\} \oplus \mathbb{Z}\{\overline{16}\}$	$\mathbb{Z}/2\{\mu_1\} \oplus \mathbb{Z}/2\{\epsilon\eta\}$
								$\bar{\eta}\eta^2 = 2\sigma$	$\sigma\eta = \epsilon$, $\overline{8}^2 = 4\cdot\overline{16}\ (\epsilon)$	$\overline{16}\cdot\eta = \mu_1$

where L denotes a multiple of $l\colon \mathbb{S}^7 \longrightarrow C_1$ with an odd natural number.

(ii) The following relations hold in $\pi_*(C_1)$*:*

$$\begin{aligned}
\overline{8}\eta &= 0 && (3.16)\\
\bar{\eta}\eta^2 &= 2(L-\sigma) && (3.17)\\
\langle\, i\colon \mathbb{S} \longrightarrow C_1\,, \nu, 8\rangle &= \overline{8} + 8\mathbb{Z}\overline{8} \quad \subseteq \quad \pi_4(C_1) && (3.18)\\
\overline{8}\nu &= 8\sigma && (3.19)\\
4L &= 4\sigma && (3.20)\\
L\eta &= \sigma\eta + \epsilon && (3.21)\\
L\eta^2 &= 0.
\end{aligned}$$

(iii) Recall that R denotes a ring spectrum whose ring of homotopy groups $\pi_*(R)$ is isomorphic to $\pi_*(ko_{(2)})$ so that this isomorphism preserves Toda brackets (Notation 3.0.13). There exist morphisms of ring spectra $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow ko$ and $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow R$ with the following properties:

a) The maps $\pi_*(\iota_\nu)$ send the element $\overline{8} \in \pi_4(C_1)$ to unit multiples of the elements $\omega \in \pi_4(ko)$ and $\omega \in \pi_4(R)$.

b) The restrictions of the maps ι_ν to the sphere spectrum $\mathbb{S}$ equal the unit map ι of ko and R, respectively.

In particular, the morphisms ι_ν are $\pi_{<5}$-isomorphisms.

(iv) The following relations hold in $\pi_*(C_2)$ and hence in $\pi_*(\mathrm{C}(\nu))$:

$$\begin{aligned}
\overline{16}\cdot\eta &= \mu_1 && (3.22)\\
\overline{8}^2 &= 4\cdot\overline{16} \ (\mathit{mod}\ \epsilon). && (3.23)
\end{aligned}$$

Moreover, the morphisms $\pi_*(\iota_\nu)\colon \pi_*(\mathrm{C}(\nu)) \longrightarrow \pi_*(ko)$ and $\pi_*(\iota_\nu)\colon \pi_*(\mathrm{C}(\nu)) \longrightarrow \pi_*(R)$ map the element $\overline{16} \in \pi_8(\mathrm{C}(\nu))$ to unit multiples of $\beta \in \pi_8(ko)$ and $\beta \in \pi_8(R)$, respectively.

The following corollary of Theorem 3.1.6(iii) will be used in the proof of Corollary 3.2.6.

Corollary 3.1.7. *The ring spectra P_4ko and P_4R are stably equivalent as ring spectra.*

Proof of Theorem 3.1.6.
(i) and (ii): The homotopy groups of C_1 can be calculated using the exact triangle

$$\mathbb{S}^3 \xrightarrow{\nu} \mathbb{S} \xrightarrow{i} C_1 \xrightarrow{p} \mathbb{S}^4.$$

We omit most of these calculations. However, we define the new elements in the list above and prove the relations of part (ii). First, we define the new elements in $\pi_*(C_1)$ in the table above:
The element $\overline{8}$ in $\pi_4(C(\nu)) \cong \pi_4(C_1)$ arises when killing ν. It is defined to be a lift of the morphism $8\cdot \mathrm{Id}_{\mathbb{S}}$ by p:

$$\begin{tikzcd}
 & & \mathbb{S}^4 \arrow[d, "\overline{8}"] \arrow[dr, "8"] & \\
\mathbb{S}^3 \arrow[r, "\nu"] & \mathbb{S} \arrow[r, "i"] & C_1 \arrow[r, "p"] & \mathbb{S}^4.
\end{tikzcd}$$

Since the homotopy group $\pi_4(\mathbb{S})$ is trivial, this gives a well-defined element in the homotopy group $\pi_4(C_1)$. The element $\bar{\eta} \in \pi_5(C(\nu))$ is uniquely defined by:

$$\begin{tikzcd}
 & & \mathbb{S}^5 \arrow[d, "\bar{\eta}"] \arrow[dr, "\eta"] & \\
\mathbb{S}^3 \arrow[r, "\nu"] & \mathbb{S} \arrow[r, "i"] & C_1 \arrow[r, "p"] & \mathbb{S}^4.
\end{tikzcd}$$

The element $\bar{\nu} \in \pi_7(C(\nu))$ is a lift of $2\nu \in \pi_7(\mathbb{S}^4)$ by the morphism p:

$$\begin{tikzcd}
 & & \mathbb{S}^7 \arrow[d, "\bar{\nu}"] \arrow[dr, "2\nu"] & \\
\mathbb{S}^3 \arrow[r, "\nu"] & \mathbb{S} \arrow[r, "i"] & C_1 \arrow[r, "p"] & \mathbb{S}^4.
\end{tikzcd}$$

This does not determine the element $\bar{\nu}$ uniquely since the group $\pi_7(\mathbb{S})$ is isomorphic to $\mathbb{Z}/16\mathbb{Z}$. However, we choose the element $\bar{\nu}$ such that the element $\bar{\eta}\eta^2$ equals $2\bar{\nu}$. This is possible for the following reasons:
Let $\bar{\nu} \in \pi_7(C_1)$ be a lift of 2ν. The element $\bar{\eta}\eta^2$ is (up to homotopy) a lift of $4\nu \in \pi_7(\mathbb{S}^4)$ by the morphism p: $\pi_*(p)(\bar{\eta}\eta^2) = \eta^3 = 4\nu$. Thus the difference $\bar{\eta}\eta^2 - 2\bar{\nu}$ is sent to zero by the map $\pi_*(p)$ and hence equals $n\sigma \in \pi_7(C_1)$ for some natural number n:

$$\bar{\eta}\eta^2 - 2\bar{\nu} = n\sigma.$$

The element $n\sigma \cdot \eta$ is trivial in $\pi_8(C_1)$:

$$(n\sigma)\eta = (\bar{\eta}\eta^2 - 2\bar{\nu})\eta = \bar{\eta} \cdot 4\nu - \bar{\nu} \cdot 2\eta = \bar{\eta} \cdot 0 - \bar{\nu} \cdot 0 = 0.$$

On the other hand, the element $\sigma\eta$ is non-trivial in $\pi_8(C_1)$ since the morphism $\pi_8(i)\colon \pi_8(\mathbb{S}) \longrightarrow \pi_8(C_1)$ is injective. Thus n has to be even and hence the element $(\bar{\nu} + \frac{n}{2} \cdot \sigma)$ is a lift of $2\nu \in \pi_7(\mathbb{S}^4)$ such that $2(\bar{\nu} + \frac{n}{2} \cdot \sigma)$ equals $\bar{\eta}\eta^2$ in $\pi_7(C_1)$. It follows that we can choose the element $\bar{\nu}$ such that $2\bar{\nu}$ equals $\bar{\eta}\eta^2$. Later, we define the element L such that $\bar{\nu}$ is a linear combination of the elements L and σ (equation (3.27)) and deduce relation (3.17).

The calculations for the homotopy groups $\pi_n(C_1)$ in the list above, except for the group $\pi_7(C_1)$, are very easy. The latter is isomorphic to the group $\mathbb{Z}/16\{\sigma\} \oplus \mathbb{Z}/4\{\bar{\nu}\}$ due to the short exact sequence

$$0 \longrightarrow \pi_7(\mathbb{S}) = \mathbb{Z}/16\{\sigma\} \xrightarrow{\pi_*(i)} \pi_7(C_1) \xrightarrow{\pi_*(p)} {}_{\nu}\{\pi_7(\mathbb{S}^4)\} = \mathbb{Z}/4\{2\nu\} \longrightarrow 0$$

and since the non-trivial element $2\bar{\nu} = \bar{\eta}\eta^2$ is 2-torsion. It follows that the group $\pi_7(C_1)$ is isomorphic to $\mathbb{Z}/16\{\sigma\} \oplus \mathbb{Z}/4\{L - \sigma\}$ under the identification $L = \sigma + (a^{-1}b)\bar{\nu}$, where $a^{-1}b$ is an invertible element in the ring $\mathbb{Z}/4$ (equation (3.27)).

Now we prove some relations between the elements of $\pi_*(C_1)$:
The relation $\overline{8} \cdot \eta = 0$ in $\pi_5(C_1)$ holds since the bijective morphism $\pi_5(p)\colon \pi_5(C_1) \longrightarrow \pi_5(\mathbb{S}^4)$ maps the element $\overline{8} \cdot \eta$ to $8\eta = 0$ in $\pi_5(\mathbb{S}^4)$. The third relation of (ii) holds since the Toda bracket $\langle i, \nu, 8\rangle$ has indeterminacy $8 \cdot \mathbb{Z}\overline{8}$ and contains the element $\overline{8}$ by the definition of Toda brackets (Def. 1.2.10):

$$\begin{tikzcd}
\mathbb{S}^3 \arrow[r, "8"] \arrow[d, dotted, "\overline{8}"] & \mathbb{S}^3 \arrow[r] \arrow[d, equal] & C(8) \arrow[r] \arrow[d, dotted] & \mathbb{S}^4 \arrow[d, dotted] \arrow[dd, dotted, bend left, "\in\langle i,\nu,8\rangle"] \\
\Sigma^{-1}C_1 \arrow[r, "p"] \arrow[d, dotted, "\mathrm{Id}"] & \mathbb{S}^3 \arrow[r, "\nu"] \arrow[d, dotted, "-\mathrm{Id}"] & \mathbb{S} \arrow[r, "i"] \arrow[d, equal] & C_1 \arrow[d, dotted, "\mathrm{Id}"] \\
\Sigma^{-1}C_1 \arrow[r, "-p"] & \mathbb{S}^3 \arrow[r, "\nu"] & \mathbb{S} \arrow[r, "i"] & C_1.
\end{tikzcd}$$

By Lemma 3.1.5, the element $\overline{8}\nu$ equals 8σ since the Toda bracket $\langle \nu, 8, \nu\rangle \in \pi_7(\mathbb{S})$ is 8σ with trivial indeterminacy [To62, §V]. The remaining relations of (ii) follow from

Sublemma 3.1.8. *Recall that l is an element in $\pi_7(C_1)$ (see (3.15)). The following relations hold in $\pi_*(C_1)$:*

$$
\begin{aligned}
l\eta &= \sigma\eta + \epsilon && (3.24)\\
l &= a\sigma + b\bar{\nu},\ \text{where } a \in (\mathbb{Z}/16)^{\times},\ b \in (\mathbb{Z}/4)^{\times}. && (3.25)\\
\bar{\nu}\eta &= \epsilon && (3.26)
\end{aligned}
$$

Proof. The idea of this proof is to show that some elements in $\pi_k(C_1)$, $7 \leq k \leq 10$, have to be zero in $\operatorname{colim}_n \pi_k(C_n) = \pi_k(\mathrm{C}(\nu))$ due to the ring spectrum structure of $\mathrm{C}(\nu)$. Since the morphism $C_2 \longrightarrow \mathrm{C}(\nu)$ is a $\pi_{<11}$-isomorphism, these elements have to be trivial in $\pi_k(C_2)$ and hence lie in the image of $\pi_k(l)$.

First, we prove the relation $l\eta = \sigma\eta + \epsilon$. The element $\epsilon + \sigma\eta = \langle \nu, \eta, \nu\rangle \in \pi_8(\mathbb{S})$ [To62, §VI] is sent to zero in $\pi_*(\mathrm{C}(\nu))$ by the morphism $\pi_*(\iota_\nu)\colon \pi_*(\mathbb{S}) \longrightarrow \pi_*(\mathrm{C}(\nu))$. Since the morphism $C_2 \longrightarrow \mathrm{C}(\nu)$ is a $\pi_{<11}$-isomorphism, the element $\epsilon+\sigma\eta$ has to lie in the image of the map $\pi_*(l)$ and hence has to equal $l\eta$.

Second, we prove relation (3.25). Let $a \in \mathbb{Z}/16$ and $b \in \mathbb{Z}/4$ be two elements such that $l = a\sigma + b\bar{\nu}$. We want to prove that a and b are invertible in $\mathbb{Z}/16$ and $\mathbb{Z}/4$, respectively. By equation (3.24), the relation

$$\sigma\eta + \epsilon = l\eta = a\sigma\eta + b\bar{\nu}\eta$$

holds in $\pi_8(C_1)$ and hence the element $b\bar{\nu}\eta$ equals the non-trivial element $(1-a)\cdot\sigma\eta + \epsilon$. It follows that the element b is invertible in $\mathbb{Z}/4$. The element $a \in \mathbb{Z}/16$ is invertible for the following reasons: The Toda bracket $\langle \nu \wedge C_1, i\colon \mathbb{S} \longrightarrow C_1, \nu\rangle \subset \pi_7(C_1)$ has indeterminacy 8σ since the element $(\nu\wedge C_1)\circ\overline{8}$ differs only in sign from $\overline{8}\nu$ and hence equals 8σ by relation (3.19). This Toda bracket is sent to the set

$$\{0\} = \langle \nu \wedge \mathrm{C}(\nu),\ \iota\colon \mathbb{S} \longrightarrow \mathrm{C}(\nu)\ ,\nu\rangle \subset \pi_7(\mathrm{C}(\nu))$$

by the map $\pi_*(C_1) \longrightarrow \pi_*(\mathrm{C}(\nu))$. Thus the set $\langle \nu \wedge C_1, i, \nu\rangle$ has to lie in the image of $\pi_*(l)$ and is hence of the form $\langle \nu \wedge C_1, i, \nu\rangle = ml + 8(\mathbb{Z}/16\,\sigma) \subset \pi_7(C_1)$ for some 2-local integer m. By the juggling formula (Lemma 1.2.11) and relation (3.18) in Theorem 3.1.6(ii), it follows that the element $8ml$ equals $8\sigma \in \pi_*(C_1)$:

$$\{8m\cdot l\} = \langle \nu \wedge C_1, i, \nu\rangle 8 = (\nu \wedge C_1)\circ\langle i, \nu, 8\rangle = \{(\nu\wedge C_1)\circ(\overline{8} + 8\mathbb{Z}\overline{8})\} = \{8\sigma\}.$$

Due to the equation $8\sigma = 8ml = 8ma\cdot\sigma$, the elements a and m have to be invertible and hence relation (3.25) follows. In particular, the equation

$$\sigma\eta + \epsilon = l\eta = (a\sigma + b\bar{\nu})\eta = \sigma\eta + \bar{\nu}\eta$$

and hence $\epsilon = \bar{\nu}\eta$ holds. This finishes the proof of Sublemma 3.1.8. □

Let us now continue with the proof of the relations in Theorem 3.1.6(ii). We define the element $L \in \pi_7(C_1)$ as

$$L = a^{-1}\cdot l = \sigma + (a^{-1}b)\bar{\nu}. \tag{3.27}$$

Under this identification, the relations

$$
\begin{aligned}
4L &= 4\sigma + 4\bar{\nu} = 4\sigma\\
L\eta &= l\eta = \sigma\eta + \epsilon\\
L\eta^2 &= \sigma\eta^2 + \epsilon\eta = \nu^3 = 0\\
\bar{\eta}\eta^2 &= 2\bar{\nu} = 2(L-\sigma)
\end{aligned}
$$

hold.
(iii): By Construction 3.1.4, there exists a morphism of ring spectra ι_ν such that the diagram

$$\begin{tikzcd} & ko \\ \mathbb{S} \arrow[ur, "\iota"] \arrow[r, "\iota"] & C(\nu) \arrow[u, "\iota_\nu"] \end{tikzcd}$$

commutes in the category of ring spectra and hence condition (b) of Theorem 3.1.6(iii) holds.
Now we prove that the map $\pi_*(\iota_\nu)$ sends the element $\overline{8} \in \pi_4(\mathrm{C}(\nu))$ to a unit multiple of $\omega \in \pi_4(ko)$. Consider the Toda bracket $\langle i\colon \mathbb{S} \longrightarrow C_1, \nu, 8\rangle = \overline{8} + 8\mathbb{Z}\overline{8} \subseteq \pi_4(C_1)$ (equation (3.18)). Thus, the Toda bracket $\langle \iota\colon \mathbb{S} \longrightarrow \mathrm{C}(\nu), \nu, 8\rangle \subseteq \pi_4(C(\nu))$ equals $\overline{8} + 8\mathbb{Z}\overline{8}$ as well. The map $\pi_*(\iota_\nu)$ sends this Toda bracket into the set $\langle \iota\colon \mathbb{S} \longrightarrow ko, \nu, 8\rangle \subseteq \pi_4(ko)$ and hence into $\omega + 2\mathbb{Z}\omega$ (Lemma 3.1.10(i)). Thus, the element $\overline{8} \in \pi_*(C(\nu))$ is mapped to a unit multiple of $\omega \in \pi_*(ko)$ by $\pi_*(\iota_\nu)$.
Similarly, one can prove claim (iii) for the ring spectrum R by using the inclusion

$$\langle \iota\colon \mathbb{S} \longrightarrow R, \nu, 8\rangle \subset \omega + 2\mathbb{Z}\omega \subseteq \pi_4(R) \qquad \text{(Lemma 3.1.10(ii))}.$$

(iv): Now we calculate the homotopy groups $\pi_n(C_2)$ for $n \leq 9$ (see page 37) and prove the relations in $\pi_*(C_2)$ (Theorem 3.1.6(iv)).
The element $\overline{16} \in \pi_8(C_2)$ is defined as a lift of the morphism $16 \cdot \mathrm{Id}_{\mathbb{S}^8}$ by the morphism q:

$$\begin{tikzcd} & & \mathbb{S}^8 \arrow[d, dotted, "\overline{16}"] \arrow[dr, "16"] & \\ \mathbb{S}^7 \arrow[r, "l"] & C_1 \arrow[r, "j"] & C_2 \arrow[r, "q"] & \mathbb{S}^8. \end{tikzcd}$$

This definition is not unique up to homotopy since $\pi_8(C_1) \cong (\mathbb{Z}/2)^2$. In the following lemma, we redefine the element $\overline{16}$ as a lift of a unit multiple of $16 \cdot \mathrm{Id}_{\mathbb{S}^8}$ in order to obtain useful relations.

Lemma 3.1.9. *One can choose the element $\overline{16} \in \pi_8(C_2)$ such that the following relations hold*

$$\overline{16}\eta = \mu_1 \tag{3.28}$$
$$\overline{8}^2 = 4 \cdot \overline{16}\,(mod\ \epsilon) \tag{3.29}$$
$$\pi_*(\iota_\nu)(\overline{16}) = \beta \in \pi_8(ko). \tag{3.30}$$

Proof. First, we prove relation (3.28). By Lemma 3.1.5, the Toda bracket $\langle l, 16, \eta\rangle \subset \pi_9(C_1)$ is sent to the set $\pi_*(q)^{-1}(16) \cdot \eta \subset \pi_9(C_2)$ by the morphism $\pi_*(j)\colon \pi_*(C_1) \longrightarrow \pi_*(C_2)$:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_9(C_1) \cong \mathbb{Z}/2\{\mu_1\} \oplus \mathbb{Z}/2\{\sigma\eta^2\} \arrow[r] & \pi_9(C_2) \cong \mathbb{Z}/2\{\mu_1\} \oplus \mathbb{Z}/2\{\epsilon\eta\} \arrow[r, "\pi_*(q)=0"] & \pi_8(\mathbb{S}^7) \arrow[r] & \cdots \\ & \langle l, 16, \eta\rangle \arrow[r, mapsto] & \pi_*(q)^{-1}(16) \cdot \eta. & & \end{tikzcd}$$

The juggling formula (Lemma 1.2.11) gives:

$$\langle l, 16, \eta\rangle = \langle 8l, 2, \eta\rangle = \langle 8\sigma, 2, \eta\rangle = \langle \sigma, 16, \eta\rangle = \mu_1 + \mathbb{Z}/2\epsilon\eta \subset \pi_9(C_1), \tag{3.31}$$

where all the Toda brackets have indeterminacy $\epsilon\eta$ since the element $\sigma\eta^2$ equals $\epsilon\eta$ in $\pi_*(C_1)$. The last equation in (3.31) holds since the Toda bracket

$$\langle \sigma, 16, \eta\rangle = \mu_1 + \mathbb{Z}/2\,\eta^2\sigma + \mathbb{Z}/2\,\eta\epsilon \subset \pi_9(\mathbb{S})$$

(see [Ko, §5.7]) is sent to $\langle \sigma, 16, \eta\rangle \subset \pi_9(C_1)$ by the morphism $\pi_*(\mathbb{S}) \longrightarrow \pi_*(C_1)$. Thus the element $\overline{16}\cdot\eta$ lies in the set $\pi_*(q)^{-1}(16)\cdot\eta = \mu_1 + \mathbb{Z}/2\,\epsilon\eta$. The relation $\overline{16}\cdot\eta = \mu_1$ follows by redefining the element $\overline{16}$ as $\overline{16}+\epsilon$ if necessary.

Using this relation (3.28), we deduce that the element $\overline{8}^2$ equals the element $4\cdot\overline{16}$ (modulo ϵ): Recall that the ring morphism $\pi_*(\iota_\nu)\colon \pi_*(C(\nu)) \longrightarrow \pi_*(ko)$ maps the element $\overline{8}\in\pi_*(\mathrm{C}(\nu))$ to $u\cdot\omega\in\pi_*(ko)$, where u denotes an invertible element in $\mathbb{Z}$ (Theorem 3.1.6(iii)). Therefore, the element $\overline{8}^2\in\pi_*(\mathrm{C}(\nu))$ is sent to $u^2\omega^2 = u^2 4\beta$ in $\pi_*(ko)$. Define n and m to be some (2-local) integers such that the relations

$$\begin{aligned} \pi_*(\iota_\nu)(\overline{16}) &= u^2 n\cdot\beta \\ \overline{8}^2 &= m\cdot\overline{16}\,(\mathrm{mod}\ \epsilon) \end{aligned}$$

hold. Summarizing gives

$$u^2 4\beta = (u\cdot\omega)^2 = \pi_*(\iota_\nu)(\overline{8}^2) = \pi_*(\iota_\nu)(m\overline{16}) = u^2 nm\beta$$

and hence $nm=4$. Moreover, the morphism $\pi_*(\iota_\nu)\colon \pi_*(\mathrm{C}(\nu)) \longrightarrow \pi_*(ko)$ maps the element μ_1 to $\beta\eta\in\pi_*(ko)$ since this is the case for the morphism $\pi_*(\iota)\colon\pi_*(\mathbb{S})\longrightarrow\pi_*(ko)$. Thus, the relation

$$\beta\eta = \pi_*(\iota_\nu)(\mu_1) = \pi_*(\iota_\nu)(\overline{16}\eta) = u^2 n\beta\eta$$

holds and n must not be a multiple of 2. Therefore, the number n is invertible in $\mathbb{Z}_{(2)}$ and m equals $4\cdot n^{-1}$. By redefining the element $\overline{16}$ as $n^{-1}\cdot\overline{16}$, the relation $\overline{8}^2 = m\cdot n\cdot\overline{16} = 4\cdot\overline{16}$ follows. Note that $\overline{16}$ is a lift of the morphism $n^{-1}\cdot 16\cdot\mathrm{Id}_{\mathbb{S}^8}$ by the morphism q. □

In particular, the $\pi_{<5}$-isomorphisms of ring spectra $\iota_\nu\colon \mathrm{C}(\nu)\longrightarrow ko$ and $\iota_\nu\colon \mathrm{C}(\nu)\longrightarrow R$, which were defined in part (iii), map the elements $\overline{8}$ and $\overline{16}\in\pi_*(\mathrm{C}(\nu))$ to unit multiples of the elements ω and β. This finishes the proof of Theorem 3.1.6. □

Lemma 3.1.10.

(i) The Toda bracket $\langle\iota\colon\mathbb{S}\longrightarrow ko,\nu,8\rangle\subseteq\pi_4(ko)$ *is contained in the set* $\omega+2\mathbb{Z}\omega$.

(ii) Similarly, the Toda bracket $\langle\iota\colon\mathbb{S}\longrightarrow R,\nu,8\rangle\subseteq\pi_4(R)$ *is contained in the set* $\omega+2\mathbb{Z}\omega$.

Proof. (i): Recall that the Toda bracket $\langle\eta^2,\eta,2\rangle\subset\pi_4(ko)$ equals the set $\omega+2\mathbb{Z}\omega$. This can be seen as follows: By [Ko, Prop. 5.7.5], the Toda bracket $\langle\eta^2,\eta,2\rangle\subset\pi_*(ko_2^\wedge)$ equals $\omega+2\mathbb{Z}_2^\wedge\omega$ since the Massey product $\langle h_1^2,h_1,h_0\rangle$ contains the non-trivial element in $\mathrm{Ext}^{3,7}_{\mathcal{A}^*}(H^*(ko),\mathbb{F}_2)\cong\mathbb{F}_2$ of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(H^*(ko),\mathbb{F}_2)\cong\mathbb{F}_2 \Longrightarrow [\mathbb{S},ko_2^\wedge]_{t-s}\,.$$

The juggling formula (Lemma 1.2.11) gives:

$$\langle\iota,\nu,8\rangle\subseteq\langle\iota,4\nu,2\rangle = \langle\iota,\eta^3,2\rangle = \langle\eta^2,\eta,2\rangle = \omega+2\mathbb{Z}\omega\subset\pi_*(ko).$$

(ii): The Toda bracket $\langle\eta^2,\eta,2\rangle\subset\pi_*(R)$ equals $\omega+2\mathbb{Z}\omega\subset\pi_*(R)$ since the homotopy groups $\pi_*(R)$ and $\pi_*(ko)$ are isomorphic by an isomorphism which preserves triple Toda brackets (see Def. 1.2.14). By the juggling formula (Lemma 1.2.11), the Toda bracket $\langle\iota\colon\mathbb{S}\longrightarrow R,\nu,8\rangle\subseteq\pi_4(R)$ is contained in the set $\langle\eta^2,\eta,2\rangle\subset\pi_*(R)$. Here, we use that the unit $\iota\colon\mathbb{S}\longrightarrow R$ maps the element $\eta\in\pi_1(\mathbb{S})$ to $\eta\in\pi_1(R)$ since the unit

$$p_2\circ\iota\colon\mathbb{S}\longrightarrow R\longrightarrow P_2R$$

of the ring spectrum P_2R does (Lemma 2.2.2). □

Remark 3.1.11. One can even show that the Toda brackets in Lemma 3.1.10 equal the set $\omega+8\mathbb{Z}\{\omega\}$ in $\pi_*(ko)$ and $\pi_*(R)$, respectively. In particular, it is possible to choose morphisms of ring spectra $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow ko$ and $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow R$ that map the element $\overline{8} \in \pi_*(\mathrm{C}(\nu))$ to the elements $\omega \in \pi_*(ko)$ and $\omega \in \pi_*(R)$, respectively (see proof of Theorem 3.1.6(iii)). However, we omit the rather long and technical proof since we do not need this statement.

The ring spectrum P_4ko is rigid: Now we show that the ring spectrum P_4ko is rigid by modifying the proof of Theorem 3.1.6. In this proof, we have constructed $\pi_{<5}$-isomorphisms of ring spectra $ko \leftarrow \mathrm{C}(\nu) \longrightarrow R$ (see page 41). Similarly, one can prove the following

Theorem 3.1.12. *The ring spectrum P_4ko is rigid.*

Proof. By Corollary 1.2.18, it suffices to show that the ring spectrum P_4ko is stably equivalent to every ring spectrum $\tilde{R}$ whose ring of homotopy groups $\pi_*(\tilde{R})$ is abstractly isomorphic to $\pi_*(P_4ko)$ by an isomorphism which preserves Toda brackets.
Let $\tilde{R}$ be such a ring spectrum. One can replace the ring spectra ko and R in the proof of Theorem 3.1.6(iii) by the ring spectra P_4ko and $\tilde{R}$ since only information about the homotopy groups $\pi_n(ko)$ and $\pi_n(R)$ for $n \leq 4$ were used. Thus, there exist morphisms of ring spectra $P_4ko \leftarrow \mathrm{C}(\nu) \longrightarrow \tilde{R}$, which are $\pi_{<5}$-isomorphisms. These morphisms induce a zig-zag of stable equivalences of ring spectra

$$P_4ko \xleftarrow{\simeq} P_4\,\mathrm{C}(\nu) \xrightarrow{\simeq} P_4\tilde{R} \xleftarrow{\simeq} \tilde{R}.$$

$\square$

3.1.3 Approximation of P_8ko and P_9ko

In this subsection, we prove that the ring spectra P_8ko and P_8R are stably equivalent (Theorem 3.1.16). To this end, we kill the elements in $\pi_*(\mathrm{C}(\nu))$ which are obstructions to the morphism of ring spectra $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow ko$ being a $\pi_{<9}$-isomorphism. For calculating some homotopy groups of the resulting ring spectra, we will need

Lemma 3.1.13.

(i) Let k be a non-negative integer and $f\colon A \longrightarrow B$ a $\pi_{<k}$-isomorphism between connective spectra. Then the morphism $f \wedge f$ is also a $\pi_{<k}$-isomorphism.

(ii) Some homotopy groups of the smash product $C_1^\nu \wedge C_1^\nu$ are:

n	0	1	2	3	4
$\pi_n(C_1^\nu \wedge C_1^\nu)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}^2$

Proof. (i): The morphisms $A \wedge f$ and $f \wedge B$ are $\pi_{<k}$-isomorphisms since the cone $\mathcal{C}(f)$ is k-connected.
(ii): The triangle $C_1^\nu \wedge \mathbb{S}^3 \xrightarrow{C_1^\nu \wedge \nu} C_1 \wedge \mathbb{S} \longrightarrow C_1^\nu \wedge C_1^\nu \longrightarrow C_1^\nu \wedge \mathbb{S}^4$ is exact. $\square$

First, we kill the element $\bar{\eta} \in \pi_5(\mathrm{C}(\nu))$.

Lemma 3.1.14. *There exists a ring spectrum* $\mathrm{C}(\overline{\eta})$ *together with two morphisms of ring spectra* $\iota_{\overline{\eta}}\colon \mathrm{C}(\overline{\eta}) \longrightarrow ko$ *and* $\iota_{\overline{\eta}}\colon \mathrm{C}(\overline{\eta}) \longrightarrow R$ *which are* $\pi_{<6}$*-isomorphisms such that the diagram*

$$\begin{tikzcd} & \mathbb{S} \arrow[dl, "\iota"'] \arrow[d] \arrow[dr, "\iota"] & \\ R & \mathrm{C}(\overline{\eta}) \arrow[l, "\iota_{\overline{\eta}}"] \arrow[r, "\iota_{\overline{\eta}}"'] & ko \end{tikzcd}$$

commutes. In particular, these morphisms map the elements $\overline{8}$, $\overline{16} \in \pi_*(\mathrm{C}(\overline{\eta}))$ *to unit multiples of the elements* ω *and* β.

Proof. The ring spectrum $\mathrm{C}(\overline{\eta})$ is defined as the pushout of the diagram

$$T(\mathbb{D}^6) \longleftarrow T(\mathbb{S}^5) \xrightarrow{\widetilde{\overline{\eta}}} \mathrm{C}(\nu)$$

in the category of ring spectra. Since the composite $\mathbb{S}^5 \xrightarrow{\overline{\eta}} \mathrm{C}(\nu) \xrightarrow{\iota_\nu} ko$ is null-homotopic, there exists a morphism of ring spectra

$$\iota_{\overline{\eta}}\colon \mathrm{C}(\overline{\eta}) \longrightarrow ko$$

extending the morphism $\iota_\nu\colon \mathrm{C}(\nu) \longrightarrow ko$ of Lemma 3.1.6(iii) by Construction 3.1.4. The morphism of ring spectra $\iota_{\overline{\eta}}\colon \mathrm{C}(\overline{\eta}) \longrightarrow R$ is defined similarly.

Recall that the morphism $C_1^{\overline{\eta}} \longrightarrow \mathrm{C}(\overline{\eta}) = \operatorname{colim} C_n^{\overline{\eta}}$ is a $\pi_{<11}$-isomorphism and that we can hence calculate some homotopy groups of the ring spectrum $\mathrm{C}(\overline{\eta})$ using the exact triangle

$$\mathrm{C}(\nu) \wedge \mathbb{S}^5 \wedge \mathrm{C}(\nu) \xrightarrow{\mu\circ(\mathrm{Id}\wedge\overline{\eta}\wedge\mathrm{Id})} \mathrm{C}(\nu) \longrightarrow C_1^{\overline{\eta}} \xrightarrow{q} \Sigma\, \mathrm{C}(\nu) \wedge \mathbb{S}^5 \wedge \mathrm{C}(\nu)$$

(see subsection 3.1.1). The homotopy groups $\pi_n(\mathrm{C}(\nu) \wedge \mathrm{C}(\nu))$ for $n \leq 4$, which we need for these calculations, are known due to the $\pi_{<7}$-isomorphism $C_1^\nu \longrightarrow \mathrm{C}(\nu)$ and Lemma 3.1.13.

n	0	1	2	3	4	5	6	7	8	9
$\pi_n(\mathrm{C}(\nu))$	$\mathbb{Z}$	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	0	$\mathbb{Z}\{\overline{8}\}$	$\mathbb{Z}/2\{\overline{\eta}\}$	$\mathbb{Z}/2\{\overline{\eta}\eta\}$	$\mathbb{Z}/4\{\sigma\}$	$\mathbb{Z}/2\{\epsilon\} \oplus \mathbb{Z}\{\overline{16}\}$	$\mathbb{Z}/2\{\mu_1\}\oplus \mathbb{Z}/2\{\epsilon\eta\}$
								$\overline{\eta}\eta^2 = 2\sigma$	$\sigma\eta = \epsilon$, $\overline{8}^2 = 4\cdot\overline{16}\ (\epsilon)$	$\overline{16}\cdot\eta = \mu_1$
							$\downarrow 0$	$\downarrow$	$\downarrow \cong$	$\downarrow$
$\pi_n(C_1^{\overline{\eta}}) \cong \pi_n(\mathrm{C}(\overline{\eta}))$	$\mathbb{Z}$	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta^2\}$	0	$\mathbb{Z}\{\overline{8}\}$	0	$\mathbb{Z}\{\overline{2}\}$	$\mathbb{Z}/2\{\sigma\}$	$\mathbb{Z}/2\{\epsilon\} \oplus \mathbb{Z}\{\overline{16}\}$	$\mathbb{Z}/2\{\mu_1\}$
								$\overline{2}\eta = \sigma$	$\sigma\eta = \epsilon$, $\overline{8}^2 = 4\cdot\overline{16}\ (\epsilon)$	$\overline{16}\cdot\eta = \mu_1$

The element $\bar{2} \in \pi_6(C_1^{\bar{\eta}})$ is (up to homotopy) uniquely defined as a lift of the morphism $\iota \wedge 2 \wedge \iota$ by q since the morphism $\pi_6(\mu \circ (\mathrm{Id} \wedge \bar{\eta} \wedge \mathrm{Id}))$ is surjective:

$$\begin{tikzcd}
 & & & \mathbb{S}^6 \arrow[d, dotted, "\bar{2}"] \arrow[dr, "\iota\wedge 2\wedge\iota"] & \\
\mathrm{C}(\nu) \wedge \mathbb{S}^5 \wedge \mathrm{C}(\nu) \arrow[r, "\mu\circ(\mathrm{Id}\wedge\bar{\eta}\wedge\mathrm{Id})"] & \mathrm{C}(\nu) \arrow[r] & & C_1^{\bar{\eta}} \arrow[r, "q"] & \Sigma\, \mathrm{C}(\nu) \wedge \mathbb{S}^5 \wedge \mathrm{C}(\nu).
\end{tikzcd}$$

The relation $\bar{2}\eta = \sigma$ holds since the Toda bracket

$$\langle \mu \circ (\mathrm{Id} \wedge \bar{\eta} \wedge \mathrm{Id}), \iota \wedge 2 \wedge \iota, \eta \rangle \subset \pi_7(\mathrm{C}(\nu))$$

is sent to the set $\pi_*(q)^{-1}(\iota \wedge 2 \wedge \iota) \circ \eta = \{\bar{2}\eta\}$:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(\mathrm{C}(\nu)) \arrow[r] & \pi_*(C_1^{\bar{\eta}}) \arrow[r, "\pi_*(q)"] & \pi_*(\Sigma\, \mathrm{C}(\nu) \wedge \mathbb{S}^5 \wedge \mathrm{C}(\nu)) \arrow[r] & \cdots \\
 & \langle \mu \circ (\mathrm{Id} \wedge \bar{\eta} \wedge \mathrm{Id}), \iota \wedge 2 \wedge \iota, \eta \rangle \arrow[r, mapsto] & \{\bar{2}\eta\},\ \bar{2} \arrow[r, mapsto] & \iota \wedge 2 \wedge \iota. &
\end{tikzcd}$$

We prove that this Toda bracket equals the set $\{\sigma, 3\sigma\}$ and hence the relation $\bar{2}\eta = \sigma$ holds in $\pi_7(C_1^{\bar{\eta}}) \cong \pi_7(\mathrm{C}(\bar{\eta}))$: By the juggling formula (Lemma 1.2.11), this Toda bracket equals

$$\langle \mu \circ (\mathrm{Id} \wedge \bar{\eta} \wedge \mathrm{Id}), \iota \wedge 2 \wedge \iota, \eta \rangle = \langle \bar{\eta}, 2, \eta \rangle \subset \pi_7(\mathrm{C}(\nu))$$

since the morphism $\mu \circ (\mathrm{Id} \wedge \bar{\eta} \wedge \mathrm{Id}) \circ (\iota \wedge \mathrm{Id} \wedge \iota)$ is $\bar{\eta}$. The Toda bracket $\langle \bar{\eta}, 2, \eta \rangle$ has the indeterminacy $\bar{\eta}\eta^2$. Recall that the Toda bracket $\langle 2, \eta, 2 \rangle \in \pi_*(\mathbb{S})$ has trivial indeterminacy and equals η^2 (see [Ko, §5.7]). By the juggling formula

$$\langle \bar{\eta}, 2, \eta \rangle 2 = \bar{\eta} \langle 2, \eta, 2 \rangle = \bar{\eta} \cdot \{\eta^2\} = \{2\sigma\} \subset \pi_7(C_1^{\bar{\eta}}),$$

the Toda bracket $\langle \bar{\eta}, 2, \eta \rangle \subset \pi_7(\mathrm{C}(\nu)) \cong \mathbb{Z}/4\sigma$ is $\{\sigma, 3\sigma\}$ and hence the relation $\bar{2}\eta = \sigma$ holds.
Due to this relation $\bar{2}\eta = \sigma$, the element $\epsilon\eta = \sigma\eta^2 = \bar{2}\eta^3 = 0$ has to be trivial in $\pi_*(C_1^{\bar{\eta}})$. The image of the element $\mu_1 \in \pi_9(\mathrm{C}(\nu))$ in $\pi_*(C_1^{\bar{\eta}})$ is non-trivial since $\mu_1 \in \pi_9(\mathrm{C}(\nu))$ is mapped by $\pi_*(\iota_\nu)$ to the non-trivial element $\beta\eta \in \pi_9(ko)$ and since the diagram

$$\begin{tikzcd}
\mathrm{C}(\nu) \arrow[d] \arrow[dr, "\iota_\nu"] & \\
\mathrm{C}(\bar{\eta}) \arrow[r, "\iota_{\bar{\eta}}"'] & ko
\end{tikzcd}$$

commutes.
The ring morphisms $\pi_*(\iota_{\bar{\eta}})\colon \pi_*(\mathrm{C}(\bar{\eta})) \longrightarrow \pi_*(ko)$ and $\pi_*(\iota_{\bar{\eta}})\colon \pi_*(\mathrm{C}(\bar{\eta})) \longrightarrow \pi_*(R)$ map the elements 1, η, $\overline{8}$ and $\overline{16}$ to unit multiples of the elements 1, η, ω and β since the ring morphisms $\pi_*(\iota_\nu)$ do. Therefore, the morphisms of ring spectra $\iota_{\bar{\eta}}$ are $\pi_{<6}$-isomorphisms. □

Similarly, one can prove the following

Lemma 3.1.15. *There exists a ring spectrum $\mathrm{C}(\bar{2})$ together with $\pi_{<10}$-isomorphisms of ring spectra $\iota_{\bar{2}}\colon \mathrm{C}(\bar{2}) \longrightarrow ko$ and $\iota_{\bar{2}}\colon \mathrm{C}(\bar{2}) \longrightarrow R$ such that the diagram*

$$\begin{tikzcd}
 & \mathbb{S} \arrow[dl, "\iota"'] \arrow[d] \arrow[dr, "\iota"] & \\
R & \mathrm{C}(\bar{2}) \arrow[l, "\iota_{\bar{2}}"] \arrow[r, "\iota_{\bar{2}}"'] & ko
\end{tikzcd} \tag{3.32}$$

commutes.

Proof. The ring spectrum $\mathrm{C}(\bar{2})$ is defined as the pushout of the diagram

$$T(\mathbb{D}^7) \longleftarrow T(\mathbb{S}^6) \xrightarrow{\tilde{\bar{2}}} \mathrm{C}(\bar{\eta})$$

in the category of ring spectra. Since the composite $\mathbb{S}^6 \xrightarrow{\bar{2}} \mathrm{C}(\bar{\eta}) \xrightarrow{\iota_{\bar{\eta}}} ko$ is null-homotopic, there exists a morphism of ring spectra

$$\iota_{\bar{2}} \colon \mathrm{C}(\bar{2}) \longrightarrow ko$$

extending the morphism $\iota_{\bar{\eta}} \colon \mathrm{C}(\bar{\eta}) \longrightarrow ko$ of Lemma 3.1.14 by Construction 3.1.4. The morphism of ring spectra $\iota_{\bar{2}} \colon \mathrm{C}(\bar{2}) \longrightarrow R$ is defined similarly. In particular, it follows that diagram (3.32) commutes.

We can calculate some homotopy groups of the ring spectrum $\mathrm{C}(\bar{2})$ due to the exact triangle

$$\mathrm{C}(\bar{\eta}) \wedge \mathbb{S}^6 \wedge \mathrm{C}(\bar{\eta}) \xrightarrow{\mu \circ (\mathrm{Id} \wedge \bar{2} \wedge \mathrm{Id})} \mathrm{C}(\bar{\eta}) \longrightarrow C_1^{\bar{2}} \xrightarrow{q} \Sigma\, \mathrm{C}(\bar{\eta}) \wedge \mathbb{S}^6 \wedge \mathrm{C}(\bar{\eta}),$$

the $\pi_{<13}$-isomorphism $C_1^{\bar{2}} \longrightarrow \mathrm{C}(\bar{2})$ (see subsection 3.1.1) and the $\pi_{<5}$-isomorphism $C_1^{\nu} \longrightarrow \mathrm{C}(\nu)$ (see Lemma 3.1.13):

n	0	1	2	3	4	5	6	7	8	9
$\pi_n(\mathrm{C}(\bar{\eta}))$	$\mathbb{Z}$	$\mathbb{Z}/2\,\{\eta\}$	$\mathbb{Z}/2\,\{\eta^2\}$	0	$\mathbb{Z}\{\bar{8}\}$	0	$\mathbb{Z}\{\bar{2}\}$	$\mathbb{Z}/2\,\{\sigma\}$	$\mathbb{Z}/2\,\{\epsilon\} \oplus \mathbb{Z}\{\overline{16}\}$	$\mathbb{Z}/2\,\{\mu_1\}$
								$\bar{2}\eta = \sigma$	$\sigma\eta = \epsilon$, $\bar{8}^2 = 4 \cdot \overline{16}\ (\epsilon)$	$\overline{16} \cdot \eta = \mu_1$
									$\downarrow$	$\downarrow \cong$
$\pi_n(C_1^{\bar{2}}) \cong \pi_n(\mathrm{C}(\bar{2}))$	$\mathbb{Z}$	$\mathbb{Z}/2\,\{\eta\}$	$\mathbb{Z}/2\,\{\eta^2\}$	0	$\mathbb{Z}\{\bar{8}\}$	0	0	0	$\mathbb{Z}\{\overline{16}\}$	$\mathbb{Z}/2\,\{\mu_1\}$
									$\bar{8}^2 = 4 \cdot \overline{16}$	

Due to Lemma 3.1.14 and the commutative diagram (3.32), the morphisms $\iota_{\bar{2}}$ induce isomorphisms $\pi_n(\iota_{\bar{2}})$ for all integers smaller than 9. Moreover, the map

$$\pi_{10}(\iota_{\bar{2}}) \colon \pi_{10}(\mathrm{C}(\bar{2})) \longrightarrow \pi_{10}(ko)$$

is surjective since the diagram (3.32) commutes and since the map $\pi_{10}(\iota) \colon \pi_{10}(\mathbb{S}) \longrightarrow \pi_{10}(ko)$ is bijective. Thus, the element $\mu_1 \cdot \eta \in \pi_{10}(\mathrm{C}(\bar{2}))$ is non-trivial and hence the morphism

$$\pi_{10}(\iota_{\bar{2}}) \colon \pi_{10}(\mathrm{C}(\bar{2})) \longrightarrow \pi_{10}(R)$$

is also surjective. $\square$

The previous lemma has the following two implications, where the first is obtained by taking the 8^{th} Postnikov section $\mathrm{C} := P_8\,\mathrm{C}(\bar{2})$ of the ring spectrum $\mathrm{C}(\bar{2})$.

Theorem 3.1.16. *The ring spectra P_8ko and P_8R are stably equivalent: $P_8ko \xleftarrow{\simeq} P_8\,\mathrm{C} \xrightarrow{\simeq} P_8R$.*

Theorem 3.1.17. *Let $\hat{R}$ be a ring spectrum whose ring of homotopy groups is abstractly isomorphic to the ring $\pi_*(P_9ko)$ by an abstract isomorphism which preserves Toda brackets. Then there exists a zig-zag of stable equivalences of ring spectra $\hat{R} \simeq P_9ko$. In particular, the ring spectrum P_9ko is rigid.*

Proof. Replacing the ring spectra ko and R in the proof of Lemma 3.1.15 by the ring spectra P_9ko and $\hat{R}$, respectively, proves that there exists a zig-zag of ring spectra

$$P_9ko \xleftarrow{\iota_{\bar{2}}} \mathrm{C}(\bar{2}) \xrightarrow{\iota_{\bar{2}}} \hat{R},$$

which induces isomorphisms $\pi_n(\iota_{\bar{2}})$ for all integers $n \leq 9$. Taking the 9^{th} Postnikov section $P_9\,\mathrm{C}(\bar{2})$ of the ring spectrum $\mathrm{C}(\bar{2})$ proves the theorem. □

3.2 Cohomology of R with $\mathbb{Z}/2$-coefficients

In this section, we calculate the cohomology groups with $\mathbb{Z}/2$-coefficients of the Postnikov sections $\bar{P}_{8n-4}ko \simeq \bar{P}_{8n-1}ko$ for $n > 0$ (Thm. 3.2.4) and hence of $\bar{P}_4R \simeq \bar{P}_4ko$ (Cor. 3.2.6). Using these groups, we prove that the cohomology groups $H^*(ko, \mathbb{Z}/2)$ and $H^*(R, \mathbb{Z}/2)$ of the ring spectra ko and R are isomorphic as $\mathcal{A}^*$-modules (Thm. 3.2.8).

Notation 3.2.1. In the following, the cohomology and homology with $\mathbb{Z}/2$-coefficients will be denoted by $H^*(-)$ and $H_*(-)$, respectively. To simplify notation, we denote the $(8n-4)^{th}$ Postnikov sections $P_{8n-4}ko$ and $P_{8n-4}R$ of the ring spectra ko and R (see Construction 1.3.3) by $\mathbb{P}_nko$ and $\mathbb{P}_nR$, respectively. Similarly, we denote the Postnikov section $\bar{P}_{8n-4}X$ of a spectrum X by $\bar{\mathbb{P}}_nX$. We also denote the morphisms p_{8n-4} and $\bar{p}_{8n-4}$ by $\mathbb{p}_n$ and $\bar{\mathbb{p}}_n$, respectively. Moreover, we denote the natural morphisms $\bar{\varrho}^{8n+4}_{8n-4}$ and ϱ^{8n+4}_{8n-4} (see Construction 1.3.3) by $\bar{\mathbb{r}}_n$ and $\mathbb{r}_n$, respectively.

Recall that the cohomology of ko with coefficients in $\mathbb{Z}/2$ is $H^*(ko) \cong H^*(ko_{(2)}) \cong \mathcal{A}^*/\mathcal{A}^*(\mathrm{Sq}^1, \mathrm{Sq}^2)$, where $\mathcal{A}^*$ denotes the Steenrod algebra [Sto, Thm. A]. Using this, we can calculate the cohomology groups of the Postnikov sections $\mathbb{P}_nko$ for every integer $n \geq 1$: We consider an exact triangle

$$\Sigma^{8n}ko \xrightarrow{\beta^n} ko \xrightarrow{\bar{\mathbb{p}}_n} \bar{\mathbb{P}}_nko \xrightarrow{k_n} \Sigma^{8n+1}ko$$

in the stable homotopy category of spectra (Corollary 1.3.8). Note that the $\mathcal{A}^*$-module morphism $H^*(\beta^n)$ is trivial for degree reasons and since $H^*(ko)$ is a cyclic $\mathcal{A}^*$-module. Thus, there exists a short exact sequence of $\mathcal{A}^*$-modules

$$\begin{array}{ccccccccc} 0 & \longleftarrow & H^*(ko) & \xleftarrow{H^*(\bar{\mathbb{p}}_n)} & H^*(\bar{\mathbb{P}}_nko) & \xleftarrow{H^*(k_n)} & H^*(\Sigma^{8n+1}ko) & \longleftarrow & 0 \\ & & 1 & \longleftarrow\!\shortmid & 1,\ \lambda_n & \longleftarrow\!\shortmid & 1, & & \end{array}$$

where the elements $1 \in H^0(\bar{\mathbb{P}}_nko)$ and $\lambda_n \in H^{8n+1}(\bar{\mathbb{P}}_nko)$ are defined as follows:

Definition 3.2.2. For every integer $n \geq 1$, we define the element $\lambda_n \in H^{8n+1}(\bar{\mathbb{P}}_nko)$ to be the image of the element $1 \in H^{8n+1}(\Sigma^{8n+1}ko)$ by the map $H^*(k_n)$. Moreover, the element $1 \in H^0(\bar{\mathbb{P}}_nko)$ is defined to be the preimage of the element $1 \in H^0(ko)$ by the bijective map $H^0(\bar{\mathbb{p}}_n)$.

Notation 3.2.3. We mostly consider the case $n = 1$ and abbreviate the element λ_1 by λ.

Theorem 3.2.4.

(i) For every natural number $n \geq 1$, there is an isomorphism of $\mathcal{A}^*$-modules

$$H^*(\bar{\mathbb{P}}_n(ko)) \cong H^*(ko)\{1\} \oplus H^*(ko)\,[8n+1]\,\{\lambda_n\}.$$

The morphism $H^*(\bar{\mathbb{p}}_n)\colon H^*(\bar{\mathbb{P}}_n ko) \longrightarrow H^*(ko)$ of $\mathcal{A}^*$-modules sends the elements 1 and λ_n to 1 and zero, respectively.

(ii) The morphism

$$H^*(\bar{\mathbb{r}}_n)\colon H^*(\bar{\mathbb{P}}_n ko) \longrightarrow H^*(\bar{\mathbb{P}}_{n+1} ko)$$

of $\mathcal{A}^*$-modules maps the element 1 to 1 and the element λ_n to zero.

Proof. (i) Recall that the exact triangle

$$\Sigma^{8n} ko \xrightarrow{\beta^n} ko \xrightarrow{\bar{\mathbb{p}}_n} \bar{\mathbb{P}}_n ko \xrightarrow{k_n} \Sigma^{8n+1} ko$$

induces a short exact sequence of $\mathcal{A}^*$-modules

$$\begin{tikzcd} 0 & H^*(ko) \arrow[l] \arrow[r, dashed, shift right, "s"'] & H^*(\bar{\mathbb{P}}_n ko) \arrow[l, shift right, "H^*(\bar{\mathbb{p}}_n)"'] & H^*(\Sigma^{8n+1} ko) \arrow[l, "H^*(k_n)"'] & 0 \arrow[l] \end{tikzcd}$$

since the $\mathcal{A}^*$-module morphism $H^*(\beta^n)$ is trivial. This short exact sequence of $\mathcal{A}^*$-modules splits uniquely since the map $H^0(\bar{\mathbb{p}}_n)$ is bijective, $H^*(ko)$ is a cyclic $\mathcal{A}^*$-module, and the Steenrod operations Sq^1 and Sq^2 act trivially on the generator of the group $H^0(\bar{\mathbb{P}}_n ko) \cong \mathbb{Z}/2$. In particular, the section s of the morphism $H^*(\bar{\mathbb{p}}_n)$ is an $\mathcal{A}^*$-module morphism which sends the element $1 \in H^*(ko)$ to $1 \in H^*(\bar{\mathbb{P}}_n ko)$. Therefore, the $\mathcal{A}^*$-module $H^*(\bar{\mathbb{P}}_n ko)$ is isomorphic to $H^*(ko)\{1\} \oplus H^*(ko)[8n+1]\{\lambda_n\}$. Moreover, the inclusion

$$H^*(k_n)\colon H^*(\Sigma^{8n+1} ko) \longrightarrow H^*(\bar{\mathbb{P}}_n ko)$$

maps $1 \in H^*(\Sigma^{8n+1} ko)$ to λ_n and the projection $H^*(\bar{\mathbb{p}}_n)$ maps the element $1 \in H^*(\bar{\mathbb{p}}_n)$ to $1 \in H^*(ko)$.
(ii) In order to determine the morphism $H^*(\bar{\mathbb{r}}_n)\colon H^*(\bar{\mathbb{P}}_n ko) \longrightarrow H^*(\bar{\mathbb{P}}_{n+1} ko)$ of $\mathcal{A}^*$-modules, we consider the commutative diagram

$$\begin{tikzcd} H^*(ko)\{1\} \oplus H^*(ko)\,[8n+1]\,\{\lambda_n\} \arrow[d, dotted, "\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)"] \arrow[r, phantom, "\cong"] & H^*(\bar{\mathbb{P}}_n ko) \arrow[r, "H^*(\bar{\mathbb{p}}_n)=(1\,0)"] \arrow[d, "H^*(\bar{\mathbb{r}}_n)"] & H^*(ko) \arrow[d, equal] \\ H^*(ko)\{1\} \oplus H^*(ko)\,[8n+9]\,\{\lambda_{n+1}\} \arrow[r, phantom, "\cong"] & H^*(\bar{\mathbb{P}}_{n+1} ko) \arrow[r, "H^*(\bar{\mathbb{p}}_{n+1})=(1\,0)"] & H^*(ko). \end{tikzcd}$$

Recall that the morphisms $H^*(\bar{\mathbb{p}}_n)$ and $H^*(\bar{\mathbb{p}}_{n+1})$ are projections on the direct summand $H^*(ko)$ (see part (i)). Moreover, the image of the morphism $H^*(\bar{\mathbb{r}}_n)$ in the direct summand $H^*(ko)\,[8n+9]$ of $H^*(\bar{\mathbb{P}}_{n+1} ko)$ is trivial for degree reasons. Thus, the morphism $H^*(\bar{\mathbb{r}}_n)$ maps the element $1 \in H^*(\bar{\mathbb{P}}_n ko)$ to $1 \in H^*(\bar{\mathbb{P}}_{n+1} ko)$ and the element λ_n to zero. □

Corollary 3.2.5. *The cohomology of $\mathbb{P}_n ko$ is isomorphic to $H^*(ko) \oplus H^*(ko)\,[8n+1]$.*

Proof. The spectra $\mathbb{P}_n ko$ and $\bar{\mathbb{P}}_n ko$ are stably equivalent (Cor. 1.3.5). □

Corollary 3.2.6. *The cohomology of $\mathbb{P}_1 R$ is isomorphic to $H^*(\mathbb{P}_1 ko) \cong H^*(ko)\{1\} \oplus H^*(ko)\,[9]\,\{\lambda\}$ as an $\mathcal{A}^*$-module.*

Proof. The zig-zag of stable equivalences of ring spectra $\mathbb{P}_1 ko \simeq \mathbb{P}_1 R$ of Corollary 3.1.7 induces an isomorphism of $\mathbb{Z}/2$-coalgebras and $\mathcal{A}^*$-modules on cohomology: $H^*(\mathbb{P}_1 ko) \cong H^*(\mathbb{P}_1 R)$. □

Remark/Notation 3.2.7. The zig-zag of stable equivalences $\bar{\mathbb{P}}_1 R \simeq \mathbb{P}_1 R \simeq \mathbb{P}_1 ko \simeq \bar{\mathbb{P}}_1 ko$ induces an $\mathcal{A}^*$-module isomorphism $H^*(\bar{\mathbb{P}}_1 R) \cong H^*(\mathbb{P}_1 R) \cong H^*(\mathbb{P}_1 ko) \cong H^{(}\bar{\mathbb{P}}_1 ko)$. This isomorphism is uniquely determined since the $\mathcal{A}^*$-module $H^*(\bar{\mathbb{P}}_1 ko)$ is generated by the elements $1 \in H^0(\bar{\mathbb{P}}_1 ko) \cong \mathbb{Z}/2$ and $\lambda \in H^9(\bar{\mathbb{P}}_1 ko) \cong \mathbb{Z}/2$. We write this isomorphism as identity $H^*(\mathbb{P}_1 R) = H^*(\mathbb{P}_1 ko)$. Moreover, we denote the elements in $H^*(\bar{\mathbb{P}}_1 ko)$ and $H^*(\bar{\mathbb{P}}_1 R)$ with the same symbols.

Now we can prove that the cohomologies of ko and R are isomorphic as $\mathcal{A}^*$-modules by using a spectral sequence which has as input the cohomology of $\bar{\mathbb{P}}_1 R \simeq \bar{\mathbb{P}}_1 ko$ and converges to $H^*(R)$.

Theorem 3.2.8. *(i) There is a spectral sequence of $\mathcal{A}^*$-modules with E_1-term given by*

$$E_1^{*,*} = \bigoplus_{n\geq 0} H^*(\bar{\mathbb{P}}_1 ko)\,[8n] \cong H^*(\bar{\mathbb{P}}_1 ko) \otimes \mathbb{Z}/2[x]\,,\ |x| = 8$$

converging strongly to $H^*(R)$.
(ii) The cohomology of R is isomorphic to $H^*(ko)$ as an $\mathcal{A}^*$-module. Under this isomorphism, the map

$$H^*(ko) \oplus H^*(ko)\,[9] \cong H^*(\bar{\mathbb{P}}_1 R) \xrightarrow{H^*(\bar{\mathbb{p}}_1)} H^*(R) \cong H^*(ko)$$

equals the projection on the direct summand $H^*(ko)$ of $H^*(\bar{\mathbb{P}}_1 R)$.

Notation 3.2.9. The isomorphism $H^*(ko) \longrightarrow H^*(R)$ of Theorem 3.2.8(ii) is denoted by φ. It is uniquely determined since $H^*(ko)$ is a cyclic $\mathcal{A}^*$-module and hence there exists only one non-trivial morphism from $H^*(ko)$ to $H^*(R)$.

Proof. (i): We construct this spectral sequence using the tower of Postnikov sections

$$\cdots \longrightarrow \bar{\mathbb{P}}_{n+1} R \xrightarrow{\bar{r}_n} \bar{\mathbb{P}}_n R \longrightarrow \cdots \longrightarrow \bar{\mathbb{P}}_2 R \xrightarrow{\bar{r}_1} \bar{\mathbb{P}}_1 R \longrightarrow *\,.$$

For every integer $n > 0$, there exists an exact triangle

$$\Sigma^{8n} \bar{\mathbb{P}}_1 R \xrightarrow{q_n} \bar{\mathbb{P}}_{n+1} R \xrightarrow{\bar{r}_n} \bar{\mathbb{P}}_n R \xrightarrow{k_n} \Sigma^{8n+1} \bar{\mathbb{P}}_1 R \tag{3.33}$$

for the following reason. Let $\bar{\mathbb{P}}_{n+1}(\beta^n)$ denote the morphism induced by taking the Postnikov section $\bar{\mathbb{P}}_{n+1}(-)$ of the map $\beta^n \colon \Sigma^{8n} R \longrightarrow R$. By Corollary 1.3.8(ii), there exists an exact triangle

$$\bar{\mathbb{P}}_{n+1}(\Sigma^{8n} R) \xrightarrow{\bar{\mathbb{P}}_{n+1}(\beta^n)} \bar{\mathbb{P}}_{n+1} R \xrightarrow{\bar{r}_n} \bar{\mathbb{P}}_n R \longrightarrow \Sigma \bar{\mathbb{P}}_{n+1}(\Sigma^{8n} R)$$

for every integer $n > 0$. Since the spectra $\bar{\mathbb{P}}_{n+1}(\Sigma^{8n} R)$ and $\Sigma^{8n} \bar{\mathbb{P}}_1 R$ are isomorphic in the homotopy category $\mathrm{Ho}(Sp^\Sigma)$ (Lemma 1.3.7), there exists an exact triangle (3.33) for every integer $n > 0$. Clearly, the triangle

$$\bar{\mathbb{P}}_1 R \xrightarrow{q_0 = \mathrm{Id}} \bar{\mathbb{P}}_1 R \longrightarrow * \xrightarrow{k_0} \Sigma^1 \bar{\mathbb{P}}_1 R$$

is exact, as well. These exact triangles induce an exact couple

$$\begin{tikzcd}
\bigoplus_{n\geq 0} H^*(\bar{\mathbb{P}}_{n+1}R) \arrow[rr, "{(0,1)}"] & & \bigoplus_{n\geq 0} H^*(\bar{\mathbb{P}}_{n+1}R) \arrow[dl, "{(0,0)}"] \\
& \bigoplus_{n\geq 0} H^*(\Sigma^{8n}\bar{\mathbb{P}}_1 R), \arrow[ul, "{(1,-1)}"] &
\end{tikzcd}$$

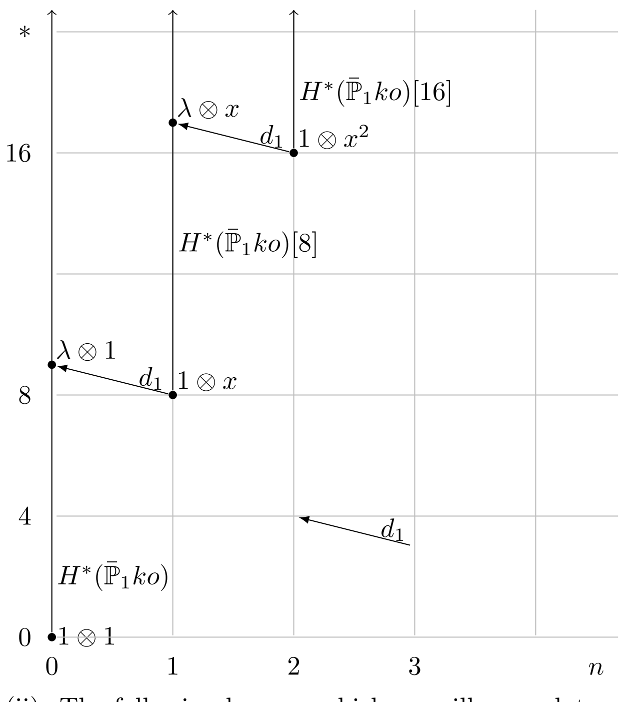
where $(*, n)$ denotes the bidegree of the morphisms. The E_1-term is isomorphic to

$$\bigoplus_{n\geq 0} E_1^{*,n} = \bigoplus_{n\geq 0} H^*(\Sigma^{8n}\bar{\mathbb{P}}_1 R) \cong \bigoplus_{n\geq 0} H^*(\bar{\mathbb{P}}_1 ko)\,[8n] \cong H^*(\bar{\mathbb{P}}_1 ko) \otimes \mathbb{Z}/2[x] \tag{3.34}$$

as an $\mathcal{A}^*$-module, where the variable x has degree 8.

Notation 3.2.10. The only purpose for the variable x is to distinguish the different shifted copies of $H^*(\bar{\mathbb{P}}_1 ko)$ in the E_1-term of the spectral sequence. For example, the generators of the $\mathcal{A}^*$-module $H^*(\bar{\mathbb{P}}_1 ko)\,[8m] \subset \bigoplus_{n\geq 0} H^*(\bar{\mathbb{P}}_1 ko)\,[8n]$ are denoted by $1 \otimes x^m$ and $\lambda \otimes x^m$, where m is a non-negative integer.

The differentials $d_r : E_r^{*,n} \longrightarrow E_r^{*+1,n-r}$ have bidegree $(1, -r)$. Therefore, the spectral sequence converges strongly to the colimit $\operatorname{colim}_n H^*(\bar{\mathbb{P}}_n R)$ since all except finitely many differentials leaving any point $(*, n)$ vanish (see [Bo, Theorem 6.1]). As all morphisms $H^*(\bar{\mathbb{P}}_n R) \longrightarrow H^*(R)$ are isomorphisms for $* < 8n$, this colimit $\operatorname{colim}_n H^*(\bar{\mathbb{P}}_n R)$ is isomorphic to $H^*(R)$.



(ii): The following lemma, which we will prove later, gives a more specific description of the differentials $d_1 \colon E_1^{*,n} \longrightarrow E_1^{*+1,n-1}$. Recall from part (i) that the entries on the E_1-page of the spectral sequence are given by

$$\bigoplus_{n\geq 0} E_1^{*,n} \cong H^*(\bar{\mathbb{P}}_1 ko) \otimes \mathbb{Z}/2[x]$$

and that the cohomology of $\bar{\mathbb{P}}_1 ko$ equals $H^*(\bar{\mathbb{P}}_1 ko) \cong H^*(ko)\{1\} \oplus H^*(ko)\,[9]\,\{\lambda\}$ (Thm. 3.2.4).

Lemma 3.2.11. *The differentials*

$$d_1 : E_1^{*,n} \longrightarrow E_1^{*+1,n-1}$$

map the elements $(y \cdot 1) \otimes x^n$ *to* $(y \cdot \lambda) \otimes x^{n-1}$ *and the elements* $(y \cdot \lambda) \otimes x^n$ *to zero for all integers* $n \geq 1$ *and all elements* $y \in \mathcal{A}^*$*. The elements* $(y \cdot 1) \otimes x^0$ *and* $(y \cdot \lambda) \otimes x^0$*,* $y \in \mathcal{A}^*$*, are sent to zero.*

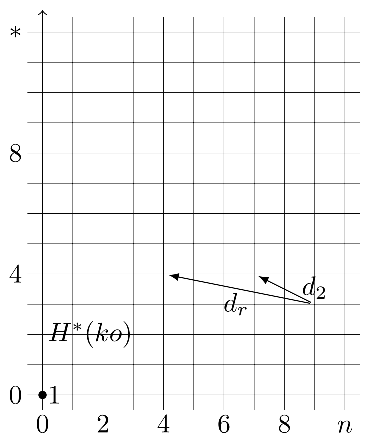
Consequently, the chain complex

$$E_1^{*,0} \xleftarrow{d_1} E_1^{*,1} \xleftarrow{d_1} \cdots \xleftarrow{d_1} E_1^{*,n} \longleftarrow \cdots$$

is exact and the entries on the E_2-page of the spectral sequence are given by

$$E_2^{*,n} \cong \begin{cases} H^*(ko) & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the isomorphism $E_2^{*,0} \cong H^*(ko)$ is an isomorphism of $\mathcal{A}^*$-modules.



For degree reasons all differentials $d_r \colon E_r^{*,n} \longrightarrow E_r^{*+1,n-r}$ with $r \geq 2$ are trivial and hence the spectral sequence collapses. It follows that $H^*(R)$ is isomorphic to $H^*(ko)$ as an $\mathcal{A}^*$-module since the spectral sequence converges strongly to $H^*(R)$: The filtration of $H^*(R)$

$$0 = F^{*,0} \subseteq F^{*,1} \subseteq \cdots \subseteq F^{*,s} \subseteq F^{*,s+1} \subseteq H^*(R)$$

which is defined by

$$F^{*,s} := \operatorname{Im}\Big(H^*(\bar{\mathrm{p}}_s) \colon H^*(\bar{\mathbb{P}}_s R) \longrightarrow H^*(R) \Big)$$

equals

$$0 = F^{*,0} \subseteq F^{*,1} = F^{*,2} = \cdots = F^{*,s} = \cdots = H^*(R),$$

since the quotients $F^{*,s+1}/F^{*,s}$ are isomorphic to

$$F^{*,s+1}/F^{*,s} \cong E_\infty^{*,s} \cong \begin{cases} H^*(ko) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, there are no extension problems and the cohomology of R equals

$$H^*(R) = F^{*,1} = \operatorname{Im}\Big(H^*(\bar{\mathbb{p}}_1)\colon H^*(\bar{\mathbb{P}}_1 R) \longrightarrow H^*(R)\Big) \cong E_\infty^{*,0} \cong H^*(ko).$$

Moreover, the morphism

$$H^*(ko) \oplus H^*(ko)\,[9] \cong H^*(\bar{\mathbb{P}}_1 R) \xrightarrow{H^*(\bar{\mathbb{p}}_1)} H^*(R) \cong H^*(ko)$$

is surjective and hence equals the projection on $H^*(ko)$ for degree reasons since the group $H^9(ko)$ is trivial. This finishes the proof of Theorem 3.2.8. □

Corollary 3.2.12. *The isomorphism $\varphi\colon H^*(ko) \cong H^*(R)$ of Theorem 3.2.8(ii) is an isomorphism of $\mathbb{Z}/2$-coalgebras.*

Proof. The diagram

$$\begin{tikzcd} H^*(\mathbb{P}_1 R) \arrow[r, equal] \arrow[d, two heads, "H^*(\mathbb{p}_1)"'] & H^*(\mathbb{P}_1 ko) \arrow[d, two heads, "H^*(\mathbb{p}_1)"'] \\ H^*(R) \arrow[r, phantom, "\cong"] & H^*(ko) \end{tikzcd}$$

is commutative since all maps are morphisms of $\mathcal{A}^*$-modules, and the two vertical morphisms $H^*(\mathbb{p}_1)$ map the elements 1 and λ to the elements 1 and 0, respectively. Thus, the isomorphism $H^*(R) \cong H^*(ko)$ is a morphism of $\mathbb{Z}/2$-coalgebras since all other maps in the diagram are morphisms of $\mathbb{Z}/2$-coalgebras and the two vertical morphisms are surjective. □

We finish this subsection with the remaining proof of Lemma 3.2.11:
Proof. The differentials $d_1\colon E_1^{*-1,n} \longrightarrow E_1^{*,n-1}$ are defined by the composite

$$\begin{tikzcd} H^{*-1}(\Sigma^{8n}\bar{\mathbb{P}}_1 R) \arrow[r, "H^*(k_n)"] \arrow[rr, bend left, dotted, "d_1"] & H^*(\bar{\mathbb{P}}_n R) \arrow[r, "H^*(q_{n-1})"] & H^*(\Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R). \end{tikzcd}$$

As the differentials d_1 are morphisms of $\mathcal{A}^*$-modules, it suffices to prove that they send

(i) the elements $1 \otimes x^n \in E_1^{*-1,n}$ to $\lambda \otimes x^{n-1} \in E_1^{*,n-1}$ and

(ii) the elements $\lambda \otimes x^n \in E_1^{*-1,n}$ to zero

for all $n \geq 1$. Clearly, part (i) implies (ii) since $(E_1^{*,n}, d_1)$ is a chain complex. Thus, it remains to prove (i). We prove it first for the case $n = 1$ and afterwards for the general case $n > 1$.

$\boldsymbol{n = 1}$**:** Our proof is motivated by the fact that the claim holds for $n = 1$ if the ring spectrum R is ko. One way to see this is the following. Observe that the element $\lambda \otimes 1 \in E_1^{*,0} \cong H^*(\bar{\mathbb{P}}_1 ko)$ has to be killed by some differential since the spectral sequence converges to $H^*(ko)$. Since the differentials d_r can not hit the element $\lambda \otimes 1$ for $r \geq 2$, it has to equal $d_1(1 \otimes x)$.
In order to prove the claim for a general ring spectrum R, we use that the ring spectra $P_8 ko$ and $P_8 R$ are stably equivalent (Thm. 3.1.16). Note that the differential $d_1\colon E_1^{*,1} \longrightarrow E_1^{*+1,0}$ equals $H^*(\Sigma^{-1} k_1)$ since the morphism q_0 is the identity $\mathrm{Id}_{\bar{\mathbb{P}}_1 R}$. The morphism k_1 is defined by the exact triangle

$$\Sigma^8 \bar{\mathbb{P}}_1 R = \Sigma^8 \bar{P}_4 R \longrightarrow \bar{P}_{12} R \xrightarrow{\bar{\varrho}_4} \bar{P}_4 R \xrightarrow{k_1} \Sigma^9 \bar{P}_4 R.$$

In order to use that the spectrum $\bar{P}_8 R$ is stably equivalent to $\bar{P}_8 ko$, we compare this triangle with the exact triangle

$$\Sigma^8 H\mathbb{Z} \longrightarrow \bar{P}_8 R \xrightarrow{\bar{\varrho}_4} \bar{P}_4 R \xrightarrow{k} \Sigma^9 H\mathbb{Z} :$$

Recall that the diagram

$$\begin{tikzcd}
\bar{P}_{12} R \arrow[d, "\bar{\varrho}_8"] \arrow[dr, "\bar{\varrho}_4"] & \\
\bar{P}_8 R \arrow[r, "\bar{\varrho}_4"] & \bar{P}_4 R
\end{tikzcd}$$

commutes (see Construction 1.3.3(1)). Thus, the following morphism of exact triangles

$$\begin{tikzcd}
\Sigma^8 \bar{P}_4 R \arrow[r] \arrow[d, dashed, "a_R"] & \bar{P}_{12} R \arrow[r, "\bar{\varrho}_4"] \arrow[d, "\bar{\varrho}_8"] & \bar{P}_4 R \arrow[r, "k_1"] \arrow[d, "\mathrm{Id}"] & \Sigma^9 \bar{P}_4 R \arrow[d, dashed, "\Sigma a_R"] \\
\Sigma^8 H\mathbb{Z} \arrow[r] & \bar{P}_8 R \arrow[r, "\bar{\varrho}_4"] & \bar{P}_4 R \arrow[r, "k"] & \Sigma^9 H\mathbb{Z}
\end{tikzcd}$$

exists. In this diagram, we can 'replace' the morphism a_R by the morphism $\Sigma^8 \bar{\varrho}_0$: By the Five Lemma, the morphism $a_R \colon \Sigma^8 \bar{P}_4 R \longrightarrow \Sigma^8 H\mathbb{Z}$ is a $\pi_{<8}$-isomorphism. Thus, there exists a zig-zag of isomorphisms $\Sigma^8 H\mathbb{Z} \simeq \Sigma^8 H\mathbb{Z}$, such that the diagram

$$\begin{tikzcd}
 & \Sigma^8 \bar{P}_4 R \arrow[dl, "\Sigma^8 \bar{\varrho}_0"'] \arrow[d, "\bar{p}_8"] \arrow[dr, "a_R"] & \\
\Sigma^8 H\mathbb{Z} \arrow[r, phantom, "\cong"] & \bar{P}_8(\Sigma^8 \bar{P}_4 R) \arrow[r, phantom, "\cong"] & \Sigma^8 H\mathbb{Z}
\end{tikzcd}$$

commutes (Lemma 1.3.7). Clearly, this isomorphism $\Sigma^8 H\mathbb{Z} \cong \Sigma^8 H\mathbb{Z}$ induces the identity on cohomology $1_{H^*(\Sigma^8 H\mathbb{Z})}$ since it induces an isomorphism between cyclic $\mathcal{A}^*$-modules. Thus, the morphisms $\Sigma^8 \bar{\varrho}_0$ and a_R induce the same morphism $H^*(\Sigma^8 \bar{\varrho}_0) = H^*(a_R)$ on cohomology. Summarizing, there exists a morphism of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(\Sigma^9 \bar{P}_4 R) \arrow[r, "H^*(k_1)"] & H^*(\bar{P}_4 R) \arrow[r] & H^*(\bar{P}_{12} R) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^*(\Sigma^9 H\mathbb{Z}) \arrow[r, "H^*(k)"] \arrow[u, "H^*(\Sigma^9 \bar{\varrho}_0)"] & H^*(\bar{P}_4 R) \arrow[r] \arrow[u, equal] & H^*(\bar{P}_8 R) \arrow[r] \arrow[u, "H^*(\bar{\varrho}_8)"'] & \cdots
\end{tikzcd}$$

Replacing the ring spectrum R by ko gives an analogous statement for ko. The zig-zag of stable equivalences $\bar{P}_8 ko \simeq P_8 ko \simeq P_8 R \simeq \bar{P}_8 R$ (Thm. 3.1.16) induces a commutative diagram

$$\begin{tikzcd}
 & \bar{P}_8 ko \arrow[r, phantom, "\simeq"] \arrow[dl, "\bar{\varrho}_4"'] \arrow[d, "\bar{p}_4"] & \bar{P}_8 R \arrow[d, "\bar{p}_4"'] \arrow[dr, "\bar{\varrho}_4"] & \\
\bar{P}_4 ko \arrow[r, phantom, "\simeq"] & \bar{P}_4(\bar{P}_8 ko) \arrow[r, phantom, "\simeq"] & \bar{P}_4(\bar{P}_8 R) \arrow[r, phantom, "\simeq"] & \bar{P}_4 R
\end{tikzcd}$$

by Lemma 1.3.4. Thus, the exact triangles

$$\Sigma^8 H\mathbb{Z} \longrightarrow \bar{P}_8 R \xrightarrow{\bar{\varrho}_4} \bar{P}_4 R \xrightarrow{k} \Sigma^9 H\mathbb{Z}$$

and

$$\Sigma^8 H\mathbb{Z} \longrightarrow \bar{P}_8 ko \xrightarrow{\bar{\varrho}_4} \bar{P}_4 ko \xrightarrow{k} \Sigma^9 H\mathbb{Z}$$

are equivalent. Summarizing gives the following commutative diagram of long exact sequences.

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(\Sigma^9 \bar{P}_4 R) \arrow[rr, "H^*(k_1)"'] & & H^*(\bar{P}_4 R) \arrow[r] & H^*(\bar{P}_{12} R) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^*(\Sigma^9 H\mathbb{Z}) \arrow[u, "H^*(\Sigma^9 \bar{\varrho}_0)"] \arrow[rr, "H^*(k)"'] \arrow[d, equal] & & H^*(\bar{P}_4 R) \arrow[u, equal] \arrow[r] \arrow[d, equal] & H^*(\bar{P}_8 R) \arrow[u, "H^*(\bar{\varrho}_8)"'] \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & H^*(\Sigma^9 H\mathbb{Z}) \arrow[rr, "H^*(k)"'] \arrow[ddd, "H^*(\Sigma^9 \bar{\varrho}_0)"'] & & H^*(\bar{P}_4 ko) \arrow[r] \arrow[ddd, equal] & H^*(\bar{P}_8 ko) \arrow[r] \arrow[ddd, "H^*(\bar{\varrho}_8)"'] & \cdots \\
& & 1 \arrow[d, mapsto] & & & \\
& & 1 \arrow[r, mapsto] & \lambda \arrow[r, mapsto] & 0 & \\
\cdots \arrow[r] & H^*(\Sigma^9 \bar{P}_4 ko) \arrow[rr, "H^*(k_1)"'] & & H^*(\bar{P}_4 ko) \arrow[r, "H^*(\bar{\varrho}_4)"'] & H^*(\bar{P}_{12} ko) \arrow[r] & \cdots
\end{tikzcd}$$

Finally, a diagram chase proves that the morphism $H^*(k_1)\colon H^*(\Sigma^9 \bar{P}_4 R) \longrightarrow H^*(\bar{P}_4 R)$ maps the element $1 \otimes x$ to $\lambda \otimes 1$: The morphism

$$(\mathcal{A}^*/\mathcal{A}^* \operatorname{Sq}^1)\,[9] \cong H^*(\Sigma^9 H\mathbb{Z}) \xrightarrow{H^*(\Sigma^9 \bar{\varrho}_0)} H^*(\Sigma^9 \bar{P}_4 ko) \cong H^*(ko)\,[9]\,\{1\} \oplus H^*(ko)\,[18]\,\{\lambda\}$$

maps the element $1 \in H^*(\Sigma^9 H\mathbb{Z})$ to $1 \in H^*(\Sigma^9 \bar{P}_4 ko)$ since the $\pi_{<10}$-isomorphism $\Sigma^9 \bar{\varrho}_0$ induces an isomorphism on the 9^{th} cohomology group. Recall that the morphism $H^*(\bar{\varrho}_4)\colon H^*(\bar{P}_4 ko) \longrightarrow H^*(\bar{P}_{12} ko)$ maps λ to zero (Theorem 3.2.4). Thus, the element $\lambda \in H^*(\bar{P}_4 ko)$ has to be hit by the morphism $H^*(k_1)$, which therefore has to send $1 \in H^*(\Sigma^9 \bar{P}_4 ko)$ to $\lambda \in H^*(\bar{P}_4 ko)$. It follows that the element $\lambda \otimes 1 \in H^*(\bar{P}_4 R)$ lies in the image of the morphism $H^*(k_1)\colon H^*(\Sigma^9 \bar{P}_4 R) \longrightarrow H^*(\bar{P}_4 R)$ and hence has to be hit by $1 \otimes x \in H^*(\Sigma^9 \bar{P}_4 R)$. Therefore, the differential $d_1 = H^*(q_0) \circ H^*(k_1)$ maps $(y \cdot 1) \otimes x \in E_1^{*-1,1} \cong H^{*-1}(\Sigma^8 \bar{P}_4 R)$ to the element $(y \cdot \lambda) \otimes 1 \in E_1^{*,0} \cong H^*(\bar{P}_4 R)$. This proves the claim for $n = 1$.

$n > 1$: In order to prove that all differentials $d_1 = H^*(q_{n-1}) \circ H^*(k_n)$, $n > 1$, send the element $1 \otimes x^n \in E_1^{*-1,n} \cong H^{*-1}(\Sigma^{8n} \bar{P}_4 R)$ to $\lambda \otimes x^{n-1} \in E_1^{*,n-1} \cong H^*(\Sigma^{8(n-1)} \bar{P}_4 R)$, it suffices to show that the diagram

$$\begin{tikzcd}
H^*(\Sigma^{8n-8}\Sigma^9 \bar{\mathbb{P}}_1 R) \arrow[r, "H^*(\Sigma^{8n-8} k_1)"] \arrow[d, equal] \arrow[rr, bend left, dotted, "d_1"] & H^*(\Sigma^{8n-8} \bar{\mathbb{P}}_1 R) \arrow[r, "H^*(\mathrm{Id})"] & H^*(\Sigma^{8n-8} \bar{\mathbb{P}}_1 R) \arrow[d, equal] \\
H^*(\Sigma^{8n+1} \bar{\mathbb{P}}_1 R) \arrow[r, "H^*(k_n)"] \arrow[rr, bend right, dotted, "d_1"'] & H^*(\bar{\mathbb{P}}_n R) \arrow[u, "H^*(q_{n-1})"'] \arrow[r, "H^*(q_{n-1})"] & H^*(\Sigma^{8(n-1)} \bar{\mathbb{P}}_1 R)
\end{tikzcd} \tag{3.35}$$

commutes. For the right square, commutativity is directly clear. For the left square, recall that the diagram

$$\begin{tikzcd}
\bar{\mathbb{P}}_{n+1}R \arrow[d, "\bar{\mathbb{r}}_n"] \arrow[dr, "\bar{\mathbb{r}}_{n-1}"] & \\
\bar{\mathbb{P}}_n R \arrow[r, "\bar{\mathbb{r}}_{n-1}"] & \bar{\mathbb{P}}_{n-1}R
\end{tikzcd}$$

commutes (see subsection 1.3) and hence the morphism of exact triangles

$$\begin{tikzcd}
\bar{\mathbb{P}}_{n+1}R \arrow[r, "\bar{\mathbb{r}}_n"] \arrow[d, equal] & \bar{\mathbb{P}}_n R \arrow[r, "k_n"] \arrow[d, "\bar{\mathbb{r}}_{n-1}"] & \Sigma^{8n+1}\bar{\mathbb{P}}_1 R \arrow[r, "\Sigma q_n"] \arrow[d, dashed, "\exists \hat{q}"] & \Sigma\bar{\mathbb{P}}_{n+1}R \arrow[d, equal] \\
\bar{\mathbb{P}}_{n+1}R \arrow[r, "\bar{\mathbb{r}}_{n-1}"] & \bar{\mathbb{P}}_{n-1}R \arrow[r, "k"] & \Sigma^{8n-7}\bar{\mathbb{P}}_2 R \arrow[r, "\Sigma q"] & \Sigma\bar{\mathbb{P}}_{n+1}R
\end{tikzcd}$$

exists, where $\hat{q}$ is a $\pi_{<8(n-1)}$-isomorphism by the Five Lemma. Due to the octahedral axiom of triangulated categories, the diagram

$$\begin{tikzcd}
 & \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[r, equal] \arrow[d, "q_{n-1}"] & \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[d, "\hat{k}"] & \\
\bar{\mathbb{P}}_{n+1}R \arrow[r, "\bar{\mathbb{r}}_n"] \arrow[d, equal] & \bar{\mathbb{P}}_n R \arrow[r, "k_n"] \arrow[d, "\bar{\mathbb{r}}_{n-1}"] & \Sigma^{8n+1}\bar{\mathbb{P}}_1 R \arrow[r, "\Sigma q_n"] \arrow[d, dashed, "\exists \hat{q}"] & \Sigma\bar{\mathbb{P}}_{n+1}R \arrow[d, equal] \\
\bar{\mathbb{P}}_{n+1}R \arrow[r, "\bar{\mathbb{r}}_{n-1}"] & \bar{\mathbb{P}}_{n-1}R \arrow[r, "k"] \arrow[d, "k_{n-1}"] & \Sigma^{8n-7}\bar{\mathbb{P}}_2 R \arrow[r, "\Sigma q"] \arrow[d, dashed, "\exists\, \Sigma\hat{p}"] & \Sigma\bar{\mathbb{P}}_{n+1}R \arrow[d, "\Sigma\bar{\mathbb{r}}_n"] \\
 & \Sigma^{8(n-1)+1}\bar{\mathbb{P}}_1 R \arrow[r, equal] \arrow[d, "\Sigma q_{n-1}"] & \Sigma^{8n-7}\bar{\mathbb{P}}_1 R \arrow[r, "\Sigma q_{n-1}"] \arrow[d, "\Sigma\hat{k}"] & \Sigma\bar{\mathbb{P}}_n \\
 & \Sigma\bar{\mathbb{P}}_n R \arrow[r, "\Sigma k_n"] & \Sigma^{8n+2}\bar{\mathbb{P}}_1 R &
\end{tikzcd}$$

commutes and the four triangles in this diagram are exact. In particular, the composite $k_n \circ q_{n-1}$ equals the morphism $\hat{k}$, which can be replaced by the morphism $\Sigma^{8n-8}k_1$ for the following reason: The triangle $(\hat{p}, \hat{k}, \hat{q})$ is isomorphic to the triangle

$$\Sigma^{8(n-1)}\bar{\mathbb{P}}_2 R \xrightarrow{\Sigma^{8(n-1)}\bar{\mathbb{r}}_1} \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \xrightarrow{\Sigma^{8(n-1)}k_1} \Sigma^{8n+1}\bar{\mathbb{P}}_1 R \xrightarrow{-\Sigma^{8n-7}q_1} \Sigma^{8n-7}\bar{\mathbb{P}}_2 R$$

since there exists a zig-zag of isomorphisms $\Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \cong \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R$ such that the diagram

$$\begin{tikzcd}
 & \Sigma^{8(n-1)}\bar{\mathbb{P}}_2 R \arrow[dl, "\Sigma^{8(n-1)}\bar{\mathbb{r}}_1"'] \arrow[d, "\bar{\mathbb{P}}_n"] \arrow[dr, "\hat{p}"] & \\
\Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[r, phantom, "\cong"] & \bar{\mathbb{P}}_n(\Sigma^{8(n-1)}\bar{\mathbb{P}}_2 R) \arrow[r, phantom, "\cong"] & \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R
\end{tikzcd}$$

commutes (Lemma 1.3.7). Therefore, also the diagram

$$\begin{tikzcd}
\Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[r, equal] \arrow[d, "q_{n-1}"] & \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[r, phantom, "\cong"] \arrow[d, "\hat{k}"] & \Sigma^{8(n-1)}\bar{\mathbb{P}}_1 R \arrow[d, "\Sigma^{8(n-1)}k_1"] \\
\bar{\mathbb{P}}_n R \arrow[r, "k_n"] & \Sigma^{8n+1}\bar{\mathbb{P}}_1 R \arrow[r, phantom, "\cong"] & \Sigma^{8n+1}\bar{\mathbb{P}}_1 R
\end{tikzcd}$$

commutes. The two isomorphisms in this diagram induce the identity morphism $\mathrm{Id}_{H^*(\bar{\mathbb{P}}_1 R)}$ on cohomology since they induce isomorphisms of $\mathcal{A}^*$-modules mapping the elements 1 and $\lambda \in H^*(\bar{\mathbb{P}}_1 R)$ to themselves. Thus, the diagram (3.35) commutes. It follows that the differentials d_1 map the elements $1 \otimes x^n \in E_1^{*,n} \cong H^*(\Sigma^{8n}\bar{\mathbb{P}}_1 R)$ to the elements $\lambda \otimes x^{n-1} \in E_1^{*+1,n-1} \cong H^*(\Sigma^{8n-9}\bar{\mathbb{P}}_1 R)$ for all $n \geq 1$. □

3.3 The ring spectra $ko_2^\wedge$ and $R_2^\wedge$ are stably equivalent as spectra

In the last subsection, we have proved that the cohomologies $H^*(R)$ and $H^*(ko)$ of the ring spectra R and ko are isomorphic as $\mathcal{A}^*$-modules (Thm. 3.2.8). Using this and the Adams spectral sequence, we now construct a stable equivalence $R_2^\wedge \longrightarrow ko_2^\wedge$ of spectra using the

Theorem 3.3.1. *Let X be a connective, 2-completed spectrum whose cohomology is isomorphic to the cohomology of $ko_2^\wedge$ as an $\mathcal{A}^*$-module. Then there exists a stable equivalence of spectra $f\colon X \longrightarrow ko_2^\wedge$.*

Proof. Let $\varphi\colon H^*(ko_2^\wedge) \longrightarrow H^*(X)$ be an isomorphism between the $\mathcal{A}^*$-modules $H^*(ko_2^\wedge)$ and $H^*(X)$. This isomorphism represents the non-trivial element in the group

$$\mathrm{Ext}_{\mathcal{A}^*}^{0,0}(H^*(ko_2^\wedge), H^*(X)) \cong \mathbb{Z}/2\,.$$

This element is a permanent cycle in the Adams spectral sequence $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(ko_2^\wedge), H^*(X))$ since the groups $\mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(ko_2^\wedge), H^*(X)) \cong \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(ko), H^*(ko))$ are trivial for $t - s = -1$ [Mi, 5.20]. Thus, there exists a morphism $f\colon X \longrightarrow ko_2^\wedge$ of spectra which induces an isomorphism on cohomology. This can bee seen as follows: Let us consider an Adams tower as in Adams' proof of [Ad74, Thm. 15.1] for the spectrum $ko_2^\wedge$

$$\begin{tikzcd}
ko_2^\wedge = Y_0 \arrow[dr, "i"] & & Y_1 \arrow[ll] \arrow[dr, "i"] & & Y_2 \arrow[ll] & \cdots \arrow[l] & Y_n \arrow[l] \arrow[dr, "i"] & & Y_{n+1} \arrow[ll] & \cdots \arrow[l] \\
& W_0 \arrow[ur, "q", "-1"'] & & W_1 \arrow[ur, "q", "-1"'] & & \cdots & & W_n, \arrow[ur, "q", "-1"'] & &
\end{tikzcd}$$

where each triangle is exact and the morphisms q have degree -1: $W_n \longrightarrow \Sigma Y_{n+1}$. Let $g\colon X \longrightarrow W_0$ be a morphism such that the map $H^*(g)$ represents the non-trivial element in $E_2^{0,0}$. Since this element is a permanent cycle, the composite $q \circ g$ is in the image of the projection $\lim_i [X, \Sigma Y_i] \longrightarrow [X, \Sigma Y_1]$ and hence trivial (see [Ad74, proof of Thm. 15.1]). Thus, the morphism g factors through the spectrum $Y_0 = ko_2^\wedge$ and hence a morphism f exists:

$$\begin{tikzcd}
& & X \arrow[dl, dashed, "\exists f"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
Y_1 \arrow[r] & ko_2^\wedge \arrow[r, "i"] & W_0 \arrow[r] & \Sigma Y_1.
\end{tikzcd}$$

Since the morphisms g and i induce surjective maps on cohomology, the morphism f induces a bijective map $H^*(f)$ and is hence a stable equivalence. □

By Theorem 3.2.8, the cohomology of R is isomorphic to $H^*(ko) \cong H^*(ko_2^\wedge)$ and hence the 2-completed spectra $R_2^\wedge$ and $ko_2^\wedge$ are stably equivalent:

Corollary 3.3.2. *There exists a stable equivalence of spectra $f\colon R_2^\wedge \longrightarrow ko_2^\wedge$.*

Unfortunately, the morphism f is not a morphism of $\mathbb{S}$-algebras in general. Moreover, it does not lift to a morphism between the 2-local spectra R and ko in general. However, we will prove that the stable equivalence f can be changed in a way that it lifts to a morphism between 2-local spectra (Thm. 3.3.6). The idea is to compose f with an endomorphism $\Phi \in [ko_2^\wedge, ko_2^\wedge]$ such that the map

$$\pi_*(\Phi \circ f) \colon \pi_*(R_2^\wedge) \longrightarrow \pi_*(ko_2^\wedge)$$

sends the subgroup $\pi_*(R) \subset \pi_*(R_2^\wedge)$ into the subgroup $\pi_*(ko)$ of $\pi_*(ko_2^\wedge)$. The endomorphism Φ will be constructed using the Adams operation ψ^3.
In the following, we recall some properties of Adams operations and the polynomials $\phi_n = \theta_n(\psi^3)$ (Def. 3.3.3). Afterwards, we prove the existence of an operation $\Phi \in [ko_2^\wedge, ko_2^\wedge]$ such that the morphism $\Phi \circ f$ lifts to a morphism between 2-local spectra (Thm. 3.3.6).

In 1962, Adams has constructed an unstable operation ψ^k of the (real) K-theory of spaces for every integer k [Ad62, §5]. Later, the Adams operation ψ^k was constructed as a stable operation under the condition that k is a unit in the coefficient ring one is working over [AHS, §4]. In particular, there exist stable operations ψ^k on $KO_{(2)}$ for odd integers k. These Adams operations ψ^k act on $\pi_{4i}(KO_{(2)})$ by multiplication with k^{2i} and as the identity on the torsion groups of $\pi_*(KO_{(2)})$. Similarly, one can construct Adams operations ψ^k on the 2-localized and 2-completed real connective K-theories ko and $ko_2^\wedge$ for an odd integer k (see for example [Mi]).

In the following, we consider the Adams operation ψ^3 on ko and $ko_2^\wedge$. As mentioned above, this operation acts on the homotopy groups $\pi_{4i}(ko_2^\wedge)$ and $\pi_{4i}(ko)$ by multiplication with $3^{2i} = 9^i$. Using this property, one can construct operations ϕ_n which induce trivial morphisms $\pi_i(\phi_n)$ for i smaller than $4n$:

Definition 3.3.3. For each non-negative integer n, we define the polynomial θ_n by

$$\theta_n(x) = \prod_{i=0}^{n-1} (x - 9^i)$$

and the corresponding the operation $\phi_n \in [ko_2^\wedge, ko_2^\wedge]^0$ by $\phi_n = \theta_n(\psi^3)$.

For example, ϕ_0 is the identity and ϕ_1 equals $\psi^3 - 1$. Thus, the maps $\pi_0(\phi_1)$, $\pi_4(\phi_1)$ and $\pi_{4i}(\phi_1)$ correspond to a multiplication with 0, $9-1=8$ and 9^i-1, respectively, since $\pi_{4i}(\psi^3)$ is multiplication with 9^i. In general, the operation ϕ_n acts on $\pi_{4i}(ko_2^\wedge)$ by multiplication with $\theta_n(9^i)$. Clearly, this term is zero for $i < n$. Moreover, the maps $\pi_*(\phi_n)$, $n > 0$, are zero on the torsion groups of $\pi_*(ko_2^\wedge)$ since ψ^3 induces the identity on these torsion groups. Thus, the maps $\pi_i(\phi_n)$ are trivial for i smaller than $4n$.

In order to understand the maps $\pi_{4i}(\phi_n)$, $i \geq n$, Milgram has calculated the exponent of 2 in the prime factorization of $\theta_n(9^i)$:

$$\theta_n(9^i) = \begin{cases} 0 & \text{if } i < n, \\ 2^{4n-\alpha(i)+\alpha(i-n)} \cdot (odd) & \text{if } i \geq n, \end{cases} \tag{3.36}$$

where $\alpha(n)$ denotes the number of ones in the binary representation of n and '(odd)' denotes some odd number [Mi, Lemma 3.5]. Recall the two inequalities

$$\begin{aligned} \alpha(l) + \alpha(n) &\geq \alpha(l+n) \\ n &\geq \alpha(n), \end{aligned}$$

which hold for all non-negative integers l and n. Thus, $\theta_n(9^i)$ is divisible by $2^{4n-\alpha(n)}$. Therefore, composing the map $\pi_*(f)$ with an endomorphism $\pi_*(\mathrm{Id} + c_n\phi_n)$, $c_n \in \mathbb{Z}_2^\wedge$, fixes the values in degree $* < 4n$ and changes them by a multiple of $2^{4n-\alpha(n)}$ in degree $* \geq 4n$. This will allow us to define 2-adic integers c_n such that the map $\pi_*((\sum_{n\geq 0} c_n\phi_n) \circ f)\colon \pi_*(R_2^\wedge) \longrightarrow \pi_*(ko_2^\wedge)$ restricts to a morphism $\pi_*(R) \longrightarrow \pi_*(ko)$ (see Thm. 3.3.6).

Remark 3.3.4. In the proof of Theorem 3.3.6, we will need infinite sums of the form $\sum_{n\geq 0} c_n\phi_n$, where c_n are 2-adic integers. Let us first observe that the sum $\pi_i(\sum_{n\geq 0} c_n\phi_n) = \sum_{n\geq 0} c_n\pi_i(\phi_n)$ is finite in every degree $i \geq 0$ since the maps $\pi_i(\phi_n)$ are trivial for i smaller than $4n$. One way to define these sums $\sum_{n\geq 0} c_n\phi_n$ in the stable homotopy category is the following: The operations ϕ_{2m} and ϕ_{2m+1} factor through $\Sigma^{8m} ko_2^\wedge$ ([Mi, Thm. B]):

$$\begin{tikzcd} ko_2^\wedge \arrow[r] \arrow[rr, bend left, "\phi_n"] & \Sigma^{8\lfloor \frac{n}{2} \rfloor} ko_2^\wedge \arrow[r, "\beta^{\lfloor \frac{n}{2} \rfloor}"'] & ko_2^\wedge. \end{tikzcd}$$

Moreover, the canonical morphism $\bigvee_{n\geq 0} \Sigma^{8\lfloor \frac{n}{2} \rfloor} ko_2^\wedge \longrightarrow \prod_{n\geq 0} \Sigma^{8\lfloor \frac{n}{2} \rfloor} ko_2^\wedge$ is a stable equivalence since it is a π_*-isomorphism. Thus, the composite

$$ko_2^\wedge \xrightarrow{\triangle} \prod_{n\geq 0} ko_2^\wedge \longrightarrow \prod_{n\geq 0} \Sigma^{8\lfloor \frac{n}{2} \rfloor} ko_2^\wedge \xleftarrow{\cong} \bigvee_{n\geq 0} \Sigma^{8\lfloor \frac{n}{2} \rfloor} ko_2^\wedge \xrightarrow{c_n\beta^{\lfloor \frac{n}{2} \rfloor}} ko_2^\wedge$$

defines a morphism in the stable homotopy category. This morphism induces the map $\sum_{n\geq 0} c_n\pi_*(\phi_n)$ on homotopy groups since for every degree i the sum $\sum_{n\geq 0} c_n\pi_i(\phi_n)$ is finite and the claim holds for finite sums $\sum c_n\phi_n$ by the universal properties of products and coproducts.

Before we state Theorem 3.3.6, we introduce the following

Notation 3.3.5. For convenience, we denote the canonical generators of the $\mathbb{Z}_2^\wedge$-modules $\pi_{4i}(ko_2^\wedge)$ and $\pi_{4i}(R_2^\wedge)$ by g_i, for $i \geq 0$. That is, the elements g_{2i} and g_{2i+1} equal the elements β^i and $\omega\beta^i$, respectively. Thus, the map $\pi_*(f)\colon \pi_*(R_2^\wedge) \longrightarrow \pi_*(ko_2^\wedge)$ defines a sequence of 2-adic integers $(a_i)_{i\geq 0}$ by

$$\pi_*(f)(g_i) = a_i \cdot g_i \in \pi_*(ko_2^\wedge).$$

These 2-adic integers a_i are units in the ring $\mathbb{Z}_2^\wedge$ since f is a stable equivalence and the maps $\pi_{4i}(f)$ are $\mathbb{Z}_2^\wedge$-module maps for all integers $i \geq 0$.

Theorem 3.3.6.

(i) There exists a stable equivalence $\Phi = \sum_{n\geq 0} c_n\phi_n \in [ko_2^\wedge, ko_2^\wedge]$ with $c_n \in \mathbb{Z}_2^\wedge$ such that the map $(\Phi \circ f)_*$ sends

(a) $1 \in \pi_0(R_2^\wedge)$ to $1 \in \pi_0(ko_2^\wedge)$ and

(b) $g_i \in \pi_{4i}(R_2^\wedge)$ to $u_i \cdot g_i \in \pi_{4i}(ko_2^\wedge)$, where u_i is an odd integer and hence a unit in $\mathbb{Z}_{(2)} \subset \mathbb{Z}_2^\wedge$, for every integer $i \geq 0$.

In particular, the subgroup $\pi_*(R) \subset \pi_*(R_2^\wedge)$ is sent into the subgroup $\pi_*(ko) \subset \pi_*(ko_2^\wedge)$ by the morphism $(\Phi \circ f)_*$.

(ii) The stable equivalence $\Phi \circ f$ lifts to a stable equivalence $F\colon R \longrightarrow ko$ such that the diagram

$$\begin{tikzcd} R \arrow[r, dashed, "F"] \arrow[d, "i"] & ko \arrow[d, "i"] \\ R_2^\wedge \arrow[r, "\Phi\circ f"] & ko_2^\wedge \end{tikzcd}$$

commutes up to homotopy, where the vertical maps i are 2-completion maps of R and ko.

(iii) The diagram

$$\begin{tikzcd} & \mathbb{S} \arrow[dl, "\iota"'] \arrow[dr, "\iota"] & \\ R \arrow[rr, "F"'] & & ko \end{tikzcd}$$

commutes up to homotopy.

Before we prove this theorem, let us note that it implies that the ring spectrum $ko_{(2)}$ is determined as a spectrum by its ring of homotopy groups and some Toda brackets:

Theorem 3.3.7. *Let R be a ring spectrum whose ring of homotopy groups $\pi_*(R)$ is isomorphic to the ring $\pi_*(ko_{(2)})$ by an isomorphism which preserves Toda brackets. Then there exists a stable equivalence of spectra $F\colon R \longrightarrow ko_{(2)}$.*

Proof of Theorem 3.3.6.
(i): Let us first observe that a morphism $\Phi = \sum_{n\geq 0} c_n\phi_n \in [ko_2^\wedge, ko_2^\wedge]$ is a stable equivalence if the 2-adic integer c_0 is invertible in $\mathbb{Z}_2^\wedge$. Suppose that c_0 is such a 2-adic integer, that is c_0 equals 1 modulo $2\mathbb{Z}_2^\wedge$. Then the operation Φ acts on $\pi_{4i}(ko_2^\wedge)$ by multiplication with the 2-adic integer

$$c_0 + \sum_{n>0} c_n\theta_n(9^i) \equiv c_0 \equiv 1 \,(\mathrm{mod}\ 2\mathbb{Z}_2^\wedge) \qquad \text{(see (3.36))}$$

which is invertible in $\mathbb{Z}_2^\wedge$. Moreover, Φ acts as the identity on the torsion groups of $\pi_*(ko_2^\wedge)$ since it acts on them by multiplication with c_0. Thus, the morphism Φ is a π_*-isomorphism and hence a stable equivalence.
Now we define the 2-adic integer c_0 such that condition (a) holds and Φ is a stable equivalence: Recall that the map $\pi_*(f)$ defines a sequence of invertible 2-adic integers $(a_i)_{i\geq 0}$ by (see Notation 3.3.5):

$$\pi_*(f)(g_i) = a_i \cdot g_i \in \pi_*(ko_2^\wedge).$$

Define c_0 to be the 2-adic integer a_0^{-1}. Since c_0 is a unit in $\mathbb{Z}_2^\wedge$, all operations $a_0^{-1} + \sum_{n>0} c_n\phi_n$, $c_n \in \mathbb{Z}_2^\wedge$, are stable equivalences. Moreover, all these operations satisfy condition (a) since the morphisms $\pi_0(\sum_{n>0} c_n\phi_n)$ are trivial for all sequences of 2-adic integers $(c_n)_{n>0}$ and hence the maps $\pi_*\left((a_0^{-1} + \sum_{n>0} c_n\phi_n) \circ f\right)$ send the element $1 \in \pi_0(R_2^\wedge)$ to $a_0^{-1}\cdot a_0 = 1 \in \pi_0(ko_2^\wedge)$.
It remains to define the 2-adic integers c_n for $n>0$ such that condition (b) is satisfied. We do this by induction. Let $N \geq 1$ be a natural number and suppose that the integers c_n, $0<n<N$, are defined such that (b) holds. Let a'_N be the 2-adic integer defined by the equation

$$\left(\left(\sum_{0\leq n<N} c_n\phi_n\right)\circ f\right)_* (g_N) = a'_N \cdot g_N \in \pi_{4n}(ko_2^\wedge).$$

Since the morphisms f and $\sum_{0\leq n<N} c_n\phi_n$ are stable equivalences, the 2-adic integer a'_N is invertible and hence equals 1 modulo $\mathbb{Z}_2^\wedge$. The value of the element $\left(\left(\sum_{0\leq n\leq N} c_n\phi_n\right)\circ f\right)_* (g_N)$ equals

$$a'_N g_N + (c_N \cdot \phi_N \circ f)_*(g_N) = \left(a'_N + c_N\cdot\theta_N(9^N)\cdot a_N\right)\cdot g_N.$$

We want to choose the 2-adic integer c_N such that $a'_N + c_N\cdot\theta_N(9^N)\cdot a_N \in \mathbb{Z}_2^\wedge$ is a natural number. Recall that the integer $\theta_N(9^N)$ is the product of $2^{4N-\alpha(N)}$ and an odd number u (see equation (3.36)). Thus the number $a'_N + c_N\cdot\theta_N(9^N)\cdot a_N$ equals a'_N modulo $2^{4N-\alpha(N)}$. We define u_N to be such a natural number, that is

$$u_N \equiv a'_N \ \mathrm{mod}(2^{4N-\alpha(N)}\mathbb{Z}_2^\wedge).$$

In particular, the natural number u_N is odd since the 2-adic integer a'_N equals 1 modulo $2\mathbb{Z}_2^\wedge$. We will define c_N to be a 2-adic integer such that the equation

$$u_N - a'_N \overset{!}{=} c_N \cdot \theta_N(9^N) \cdot a_N = c_N \cdot 2^{4N-\alpha(N)} \cdot u \cdot a_N$$

holds. Since $u_N - a'_N$ is divisible by $2^{4N-\alpha(N)}$, we get:

$$(u_N - a'_N)/2^{4N-\alpha(N)} \overset{!}{=} c_N \cdot u \cdot a_N.$$

Recall that u and a_N are invertible in $\mathbb{Z}_2^\wedge$. Thus, by defining the 2-adic integer c_N to be

$$(u_N - a'_N)/(a_N \cdot u \cdot 2^{4N-\alpha(N)}) = (u_N - a'_N)/\big(a_N \cdot \theta_N(9^N)\big),$$

the element $g_N \in \pi_{4N}(R_2^\wedge)$ is sent to the element

$$\left(\left(\sum_{0 \le n \le N} c_n \phi_n\right) \circ f\right)_* (g_N) = u_N \cdot g_N.$$

Note that the elements $g_i \in \pi_{4i}(R_2^\wedge)$ are still sent to $u_i g_i \in \pi_{4i}(ko_2^\wedge)$ for all integers $0 \le i < N$ since the value of $(\phi_N \circ f)_*(g_i)$ is trivial.

(ii): Let us consider an exact triangle

$$ko \xrightarrow{i} ko_2^\wedge \xrightarrow{q} ko_2^\wedge/ko \longrightarrow \Sigma ko.$$

It suffices to prove that the composition

$$R \xrightarrow{i} R_2^\wedge \xrightarrow{\Phi\circ f} ko_2^\wedge \xrightarrow{q} ko_2^\wedge/ko$$

is null-homotopic, in order to prove that the morphism $\Phi \circ f$ lifts to a morphism between the 2-local spectra R and ko in the stable homotopy category of spectra. Since the spectra R and ko are cofibrant and fibrant, this lift is represented by a morphism F in the category of spectra such that the diagram

$$\begin{tikzcd}
 & & ko \arrow[d] \\
R \arrow[urr, dashed, bend left, "\exists F?"] \arrow[r, "i"] \arrow[drr, bend right, "\simeq 0?"'] & R_2^\wedge \arrow[r, "\Phi\circ f"] & ko_2^\wedge \arrow[d, "q"] \\
 & & ko_2^\wedge/ko \arrow[d] \\
 & & \Sigma ko
\end{tikzcd} \tag{3.37}$$

commutes up to homotopy.
Clearly, the map $\pi_*\big(q \circ (\Phi f) \circ i\big)$ is trivial since $\pi_*(\Phi f)$ maps the subgroup $\pi_*(R) \subset \pi_*(R_2^\wedge)$ to $\pi_*(ko)$. Moreover, the spectrum $ko_2^\wedge/ko$ is rational since all its homotopy groups are rational, that is, they are uniquely divisible:

$$\pi_i(ko_2^\wedge/ko) = \begin{cases} \mathbb{Z}_2^\wedge/\mathbb{Z}_{(2)} & \text{if } i \ge 0 \text{ and } i \equiv 0 \,(\mathrm{mod}\ 4) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map

$$\pi_*(-)\colon\ [R, ko_2^\wedge/ko] \longrightarrow \mathrm{Hom}_{gr.Ab}\big(\pi_*(R), \pi_*(ko_2^\wedge/ko)\big)$$

is bijective and hence the composition $q \circ (\Phi f) \circ i$ is null-homotopic since its image under the map $\pi_*(-)$ is trivial. It follows that the morphism Φf lifts to a morphism

$$F\colon R \longrightarrow ko$$

in the category of spectra such that the diagram above (3.37) commutes up to homotopy. Thus, this morphism F is a π_*-isomorphism due to property (b) of $\Phi \circ f$ (see (i)).

(iii): The units in the rings $\pi_*(R)$ and $\pi_*(ko)$ are represented by the morphisms $\iota\colon \mathbb{S} \longrightarrow R$ and $\iota\colon \mathbb{S} \longrightarrow ko$, respectively. By (i) and (ii), the morphism $\pi_*(F)$ maps the element $1 = [\iota] \in \pi_0(R)$ to $1 = [\iota] \in \pi_0(ko)$ and hence (iii) holds. □

Remark 3.3.8. We would like the morphisms $\Phi \circ f\colon R_2^\wedge \longrightarrow ko_2^\wedge$ and $F\colon R \longrightarrow ko_{(2)}$ of Theorem 3.3.6 to be morphisms of ring spectra. However, the morphism $\pi_*(\Phi \circ f)$ and $\pi_*(F)$ need not even be ring morphisms. Like above, one can try to change the morphisms $\Phi \circ f$ and F in such a way that they induce ring morphisms $\pi_*(\Phi \circ f)$ and $\pi_*(F)$. Unfortunately, we were not able to do this since the maps $\pi_{4n}(\phi_n)$ equal multiplication with $\theta_n(9^n) = 2^{4n-\alpha(n)} \cdot (odd)$ for all $n \geq 0$ and hence the possibilities to modify the maps $\pi_*(\Phi \circ f)$ and $\pi_*(F)$ by composing them with a map of the form $\pi_*(\sum d_n\phi_n)$, $d_n \in \mathbb{Z}_2^\wedge$, are limited.
Using the example of the stable equivalence $h := \Phi \circ f$, we sketch our approach and explain where it encounters the difficulties mentioned above. Recall that the 2-adic integer a_i is defined by the equation $\pi_*(h)(g_i) = a_i \cdot g_i \in \pi_*(ko_2^\wedge)$ for every non-negative integer i and that the 2-adic integer a_0 equals one (Thm. 3.3.6). Without loss of generality, we can assume that the element a_1 equals one since we can redefine the elements $g_i \in \pi_{4i}(R_2^\wedge)$ by $a_1^{-i} \cdot g_i$ as a_1 is invertible in $\mathbb{Z}_2^\wedge$. Thus, it suffices to change the morphism h in such a way that the 2-adic integers a_i are one for all integers $i \geq 0$. Like above, we try to do this by composing h with a suitable operation $\Psi = \mathrm{Id} + \sum_{n\geq 2} c_n\phi_n\colon ko_2^\wedge \longrightarrow ko_2^\wedge$. We choose the coefficients c_n inductively:
Let m be an integer which is at least 1. Suppose that there are coefficients $c_2, c_3, \ldots, c_{m-1}$ such that the associated 2-adic integers a_i of the morphism

$$\tilde{h} := \left(\mathrm{Id} + \sum_{m>n\geq 2} c_n\phi_n\right) \circ h$$

equal one for all integers i with $0 \leq i < m$. We want to choose a 2-adic integer c_m such that the 2-adic integer a_m of the morphism $\tilde{h} + c_m\phi_m \circ h$ equals one, as well. This is possible if and only if the 2-adic integer a_m of $\tilde{h}$ equals one modulo $(2^{4m-\alpha(m)}) \cdot \mathbb{Z}_2^\wedge$ since the operation ϕ_m induces multiplication with $\theta_m(9^m) = 2^{4m-\alpha(m)} \cdot (odd)$ on $\pi_{4m}(ko_2^\wedge)$.
Unfortunately, we are only able to prove that the 2-adic integer a_m is one modulo $2^{4m-\alpha(m)}\mathbb{Z}_2^\wedge$ for integers $m \geq 2$ which are not powers of two. Here is a sketch of this proof: Let us consider the morphism

$$ko_2^\wedge \wedge ko_2^\wedge \xrightarrow{\tilde{h}^{-1}\wedge\tilde{h}^{-1}} R_2^\wedge \wedge R_2^\wedge \xrightarrow{\mu} R_2^\wedge \xrightarrow{\tilde{h}} ko_2^\wedge$$

which is an element in the group $[ko_2^\wedge \wedge ko_2^\wedge, ko_2^\wedge]$. Note that this $\mathbb{Z}_2^\wedge$-module $[ko_2^\wedge \wedge ko_2^\wedge, ko_2^\wedge]$ contains the elements $\mu \circ (\phi_l \wedge \phi_k)\colon ko_2^\wedge \wedge ko_2^\wedge \longrightarrow ko_2^\wedge$, $l, k \geq 0$ which are linearly independent. Using a similar

method as in [Mi], one can calculate the second page $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}^*}^{s,t}(H^*(ko_2^\wedge), H^*(ko_2^\wedge) \otimes H^*(ko_2^\wedge))$ of the Adams spectral sequence which converges to the graded group $[ko_2^\wedge \wedge ko_2^\wedge, ko_2^\wedge]_*$. It follows that the elements $\mu \circ (\phi_l \wedge \phi_k)$, $l, k \geq 0$, generate the $\mathbb{Z}_2^\wedge$-module $[ko_2^\wedge \wedge ko_2^\wedge, ko_2^\wedge]$. In particular, there exist 2-adic integers $c_{k,l}$, $k, l \geq 0$, such that the morphism $\tilde{h} \circ \mu \circ (\tilde{h}^{-1} \wedge \tilde{h}^{-1})$ equals $\sum_{k,l \leq 0} c_{k,l} \cdot \mu \circ (\phi_k \wedge \phi_l)$ in $[ko_2^\wedge \wedge ko_2^\wedge, ko_2^\wedge]$.
By comparing the images of the elements $g_i \wedge g_j \in \pi_*(ko_2^\wedge \wedge ko_2^\wedge)$, $i, j \geq 0$, under these two morphisms we obtain equations

$$a_i^{-1} \cdot a_j^{-1} \cdot a_{i+j} = \sum_{i \geq l, j \geq k} c_{k,l} \cdot \theta_k(9^i) \cdot \theta_l(9^j)$$

for all pairs of non-negative integers (i, j). In particular, the 2-adic integer $c_{0,0}$ equals one and the 2-adic integers $c_{k,l}$ are trivial for all integers $k, l > 0$ with $k + l \leq m - 1$ since the 2-adic integers $a_0, a_1, \ldots, a_{m-1}$ equal one. It follows that the associated 2-adic integer a_m of the morphism $\tilde{h}$ equals one modulo $(2^{4m-\alpha(i)-\alpha(m-i)}) \cdot \mathbb{Z}_2^\wedge$ for every integer i with $0 < i < m$:

$$\begin{aligned} a_i^{-1} \cdot a_{m-i}^{-1} \cdot a_m = a_m &= \sum_{i \geq l, m-i \geq k} c_{k,l} \cdot \theta_k(9^i) \cdot \theta_l(9^{m-i}) \\ &= c_{0,0} + c_{i,m-i} \cdot \theta_i(9^i) \cdot \theta_{m-i}(9^{m-i}) \\ &= 1 + c_{i,m-i} \cdot 2^{4m-\alpha(i)-\alpha(m-i)} \cdot (odd). \end{aligned}$$

Thus, the 2-adic integer a_m equals 1 modulo $(2^{4m-\alpha(m)}) \cdot \mathbb{Z}_2^\wedge$ if there exists an integer i with $0 < i < m$ such that $\alpha(m) = \alpha(i) + \alpha(m - i)$. This is the case if and only if m is not a power of 2.

Let us suppose that it is possible to choose a stable equivalence $\Psi\colon ko_2^\wedge \longrightarrow ko_2^\wedge$ such that the map $\pi_*(\Psi \circ h)$ sends the elements $g_i \in \pi_{4i}(R_2^\wedge)$ to $g_i \in \pi_{4i}(ko_2^\wedge)$ for all integers $i \geq 0$. In particular, the map $\pi_*(\Psi \circ h)$ is a morphism of rings. Moreover, the composite $(\Psi \circ h) \circ \mu \circ \big((\Psi \circ h)^{-1} \wedge (\Psi \circ h)^{-1}\big)$ is homotopic to the morphism $\mu \circ (\phi_0 \wedge \phi_0) \simeq \mu\colon ko_2^\wedge \wedge ko_2^\wedge \longrightarrow ko_2^\wedge$ and hence the diagram

$$\begin{tikzcd} R_2^\wedge \wedge R_2^\wedge \arrow[r, "\mu"] \arrow[d, "(\Psi\circ h)\wedge(\Psi\circ h)"'] & R_2^\wedge \arrow[d, "\Psi\circ h"'] \\ ko_2^\wedge \wedge ko_2^\wedge \arrow[r, "\mu"] & ko_2^\wedge \end{tikzcd}$$

commutes up to homotopy.
In this case, one can hope to prove that the ring spectra $R_2^\wedge$ and $ko_2^\wedge$ are stably equivalent as ring spectra by using the obstruction theory of Robinson ([Rob89] and [Rob04]) or Angeltveit ([An08] and [An11]). One possible approach could be to first prove the uniqueness of the A_∞-structure of the 2-localized periodic K-theory ring spectrum $KO_2^\wedge$ and then deduce that $R_2^\wedge$ and $ko_2^\wedge$ have the same A_∞-structures by a similar method as in [BaRi]. However, this is only possible if one can invert the element $\beta \in \pi_8(R_2^\wedge)$ in the ring spectrum $R_2^\wedge$.

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Zusammenfassung

Ein Hauptziel der algebraischen Topologie ist die Klassifizierung von topologischen Räumen. Dabei spielt der Begriff der Homotopieäquivalenz von topologischen Räumen eine wesentliche Rolle. Andererseits wird das Konzept der Homotopie zum Beispiel auch in der Kategorie der simplizialen Mengen und in der Kategorie der Kettenkomplexe verwendet. Eine Möglichkeit diese verschiedenen Definitionen von Homotopie zu vereinheitlichen und zu axiomatisieren ist Quillen's Begriff der Modellkategorie [Qu]. Ein wichtiger Bestandteil einer Modellkategorie $\mathcal{M}$ ist eine Klasse von speziellen Morphismen, den sogenannten schwachen Äquivalenzen. Die Modellkategorie $\mathcal{M}$ kann an dieser Klasse von Morphismen lokalisiert werden und die daraus resultierende Kategorie $\mathrm{Ho}(\mathcal{M})$ wird Homotopiekategorie von $\mathcal{M}$ genannt.

In dieser Dissertation beschäftigen wir uns mit der Frage ob bestimmte Modellkategorien starr sind. Eine stabile Modellkategorie $\mathcal{M}$ heißt *starr*, falls sie zu jeder stabilen Modellkategorie $\mathcal{N}$, deren Homotopiekategorie trianguliert-äquivalent zu $\mathrm{Ho}(\mathcal{M})$ ist, Quillen-äquivalent ist. Ein Ringspektrum R heißt *starr* falls die stabile Modellkategorie Mod-R der Moduln über diesem Ringspektrum starr ist. Dies ist unter anderem der Fall, falls das Ringspektrum R von seinem Homotopiegruppenring $\pi_*(R)$ und Todaklammer-Relationen bis auf stabile Äquivalenz eindeutig bestimmt wird [ScSh03].

Beispiele für starre Ringspektren sind das Sphärenspektrum $\mathbb{S}$ und die p-lokalisierten Sphärenspektren $\mathbb{S}_{(p)}$ für alle Primzahlen p [Sc07]. Des Weiteren sind die Eilenberg-MacLane Ringspektren HS für alle Ringe S und insbesondere die nullten Postnikovschnitte $P_0\mathbb{S}_{(p)} \simeq H\mathbb{Z}_{(p)}$ starr.

Im ersten Teil dieser Arbeit (Abschnitt 2) untersuchen wir die Starrheit der anderen Postnikovschnitte $P_m\mathbb{S}_{(p)}$, $m > 0$, der Ringspektren $\mathbb{S}_{(p)}$. Dazu verwenden wir, dass das Ringspektrum $P_m\mathbb{S}_{(p)}$ starr ist, falls es zu bestimmten Ringspektren R stabil äquivalent ist (Thm. 1.2.16). Letztere zeichnen sich durch Ringisomorphismen $\Psi_R \colon \pi_*(R) \longrightarrow \pi_*(P_m\mathbb{S}_{(p)})$ aus, die Todaklammern erhalten. Durch diese Isomorphismen beweisen wir, dass das Ringspektrum $P_m\mathbb{S}_{(p)}$ starr ist, falls die Einheiten $i \colon \mathbb{S}_{(p)} \longrightarrow R$ der Ringspektren R bijektive Abbildungen $\pi_k(i) \otimes \mathbb{Z}_{(p)}$, $k \leq m$, induzieren (Lemma 2.1.1). Danach zeigen wir, dass diese Bedingungen erfüllt sind, falls die Abbildungen $\pi_k(i) \otimes \mathbb{Z}_{(p)}$ für bestimmte Zahlen k bijektiv sind (Thm. 2.1.5). Wir betrachten diese speziellen Fälle und beweisen, dass die Ringspektren $P_m\mathbb{S}_{(2)}$, $m \geq 0$, und $P_m\mathbb{S}_{(p)}$, $m \geq p^2(2p-2)-1$, starr sind.

Im zweiten Teil dieser Arbeit (Abschnitt 3) untersuchen wir die Starrheit des 2-lokalisierten reellen konnektiven K-Theorie-Ringspektrums $ko_{(2)}$. Dessen Homotopiegruppenring $\pi_*(ko_{(2)})$ hat zusammen mit den Todaklammern eine ausreichend reiche Struktur, um annehmen zu können, dass $ko_{(2)}$ starr ist. Wir beweisen, dass das unterliegende Spektrum von $ko_{(2)}$ durch den Ring $\pi_*(ko_{(2)})$ und einigen Todaklammer-Relationen bis auf stabile Äquivalenz eindeutig bestimmt ist (Thm. 3.3.7).

Diese Aussage folgt aus mehreren Theoremen: Wir betrachten jedes Ringspektrum R, für das es einen Ringisomorphismus $\pi_*(R) \longrightarrow \pi_*(ko_{(2)})$ gibt, der Todaklammern erhält. Zuerst werden die beiden Ringspektren R und $ko_{(2)}$ durch Ankleben von Ringspektren-Zellen an $\mathbb{S}_{(2)}$ approximiert. Mit dieser Methode können wir beweisen, dass die Ringspektren $P_4ko_{(2)}$ und $P_9ko_{(2)}$ starr sind (Cor. 3.1.7 und Thm. 3.1.17) und dass der Postnikovschnitt P_8R stabil äquivalent zu $P_8ko_{(2)}$ ist (Thm. 3.1.16).

Daraus folgt insbesondere, dass die Kohomologie von P_4R mit $\mathbb{Z}/2$-Koeffizienten isomorph zum $\mathcal{A}^*$-Modul $H^*(ko_{(2)}, \mathbb{Z}/2) \cong H^*(ko_{(2)}) \oplus H^*(ko_{(2)})[9]$ ist (Cor. 3.2.6). Im nächsten Schritt zeigen wir mit Hilfe der Periodizität von $ko_{(2)}$ und R, dass diese beiden Ringspektren isomorphe Kohomologiegruppen haben (Thm. 3.2.8). Durch die Adams Spektralsequenz $\mathrm{Ext}^{s,t}_{\mathcal{A}^*}(H^*(ko_2^\wedge), H^*(R_2^\wedge)) \Rightarrow [R_2^\wedge, ko_2^\wedge]_{t-s}$ erhalten wir deshalb eine stabile Äquivalenz von Spektren $f : R_2^\wedge \longrightarrow ko_2^\wedge$ (Thm. 3.3.7). Zum Schluss verändern wir diesen Morphismus f so dass er zu einer stabilen Äquivalenz $R \longrightarrow ko_{(2)}$ zwischen den 2-lokalen Spektren liftet (Thm. 3.3.6). Damit haben wir das Theorem 3.3.7 bewiesen.