D-Brane Physics: From Weak to Strong Coupling

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Abstract

In this thesis we discuss two aspects of branes relevant to high-energy phenomenology. First, we consider a single D6-brane wrapping a special Lagrangian cycle and the background space compactified in a Calabi-Yau orientifold the conditions needed to obtain a four-dimensional $\mathcal{N} = 1$ supersymmetric theory. We calculate the bosonic part of the effective action by performing a Kaluza-Klein reduction of the brane seven-dimensional action, and obtain the $\mathcal{N} = 1$ characteristic data. To discuss the moduli, we first fix the moduli from deformations of the background Calabi-Yau and study the D-brane deformation moduli space. We next allow for Calabi-Yau deformations, and show that the moduli space for complex structure deformations is corrected by the fields living on the D6-brane. We also calculate the scalar potential from D- and F-terms generated from brane and background configurations that would break the supersymmetry condition. We then, via Mirror Symmetry, relate the spectrum obtained in our work to the spectrum in Type IIB effective theory with D3- D5- and D7-branes, and we propose a Kähler potential for the moduli space of brane deformations in Type IIB theories. In the second part of the thesis we discuss effects of brane intersections when the string coupling can become strong, and we work in the framework of F-theory. After reviewing the basics of F-theory constructions and a particular $SU(5)$ model already discussed in the literature, we construct a model which contains a point of E_8 singularity, and curves of E_6 singularity. By explictly resolving the space, we show that the resolution requires the introduction of higher dimensional fibers, and argue how we can circunvent this problem for the E_6 curve, leading to the expected resolution that generate an E_6 group, while at the E_8 point we cannot make the resolution lead to an expected E_8 structure.

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“Oh Susy Q, Oh Susy Q,
Susy Q, baby I love you,
Susy Q”
Creedence Clearwater Revival

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Chapter 1

Introduction

The quest for understanding Nature's mysteries have always played an important role in humankind. All around the world, since the most remote times, societies have tried to create explanations for why the world is as it is, either for practical technological purposes that could in principle favor societies to survive and flourish, either to satisfy an innate curiosity, and understand the broader picture of the Universe and our role as humans and part of the Cosmos. Much have changed since the ancient Greek philosophy, the Chinese schools of logic, the Persian medicine academies, the Aztec metaphysics and the Roman theology, but the desire for understanding the Universe still nowadays perseveres just as, or even more, strongly.

What often come with this ceaselessly search for answers are shifts in paradigms. Much more than just curiosities restricted to academic circles, these scientific shifts have cultural an social impact. Ancient cosmology saw, understandingly, a very limited picture of what is out there. Humankind discussed for millenia whether the Sun or the Earth would be the center of all things. The view of an Earth or the Sun as the (physical and philosophical) center of the whole Universe might now seem even ridiculous to most people. Our cosmology has advanced at such a fast pace, that took us from the first evidence for other galaxies outside the Milky Way by Hubble (the man) to the fascinating and breathtaking pictures by Hubble (the telescope) that showed us a countless number of galaxies of which our humble Milky Way seems to play no distinguishable role. We can now discuss the evolution and fate of the Universe as a whole, with new concepts that would probably be unimaginable some centuries ago. Down to Earth, the study of the basic constituents of matter also gave astonishingly new perspectives. In a little more than one century we changed dramatically our understanding of particles and forces, and often had to abandon our intuition when dealing with phenomena in the subatomic world. All these investigations are backed up by measurements with always increasing precision and based over a powerful mathematical background.

Together with the quest for understanding often comes a quest for simplicity and unification of phenomena in fewer fundamental ingredients. An idea present in different ancient cultures is that the universe could be described by a handful of basic "elements". with one of the best known examples being the four platonic elements. Moving fast forward in time, after the stablishment of the scientific method and the investigations of nature by what we now know as physical sciences, the probably greatest episodes in physics were the unification of earthly and heavenly mechanics in Newton's *Principia*, the unification of electricity and magnetism under Maxwell's theory, the unification of space, time and gravity via General Relativity and more recently the unification of particle physics under the elegant Standard Model of particles. Important to say is that unification comes often as much more than an obsession for elegance, but rather as a necessity when the existing theories become inconsistent by themselves or with phenomena. Today probably one of the most fascinating explorations in physics is String Theory, that tries to unify the two main pillars of fundamental physics, Quantum Theory and General Relativity. This proposal to describe everything as subsets of a single theory can seem too utopic and audacious, but there are in fact practical reasons for why such a pursuit is not only desired, but needed. Additionally, if String Theory is indeed correct, it will be a huge change to the way we understand particles, interactions, spacetime and even our Universe itself!

At the moment, this is a panorama on the status of fundamental physics: Quantum Field Theory (QFT) is the framework to describe the phenomena in small length scales, in which particles are point excitations of fields in spacetime. The Standard Model of particle physics is a QFT that has been extremely successful to explain with great detail what goes on in particle accelerators. It postulates as building blocks for matter three families of chiral fermions with two quarks and two leptons in each, each family being a heavier copy of the other. The vast majority of all the matter we see belongs to the first family, but the more massive particles from the other families can be produced in high energy collisions, but decay fast into the lighter "cousins". The fundamental forces of nature (outside gravity), namely strong nuclear, weak nuclear and electromagnetic forces are described by the bosonic spin one particles, respectively the gluons, W and Z bosons and the photon. As gravity is too weak when compared to the other forces, in the energy scales relevant to "earthly" particle physics the gravitational interaction can be safely neglected. The forces in the Standard Model arrange themselves elegantly as a gauge theory $SU(3) \times SU(2) \times U(1)$ under which the fermions are singlets or transform in the fundamental representation. The *electroweak* $SU(2) \times U(1)$ is spontaneously broken into the electromagnetic $U(1)$ and the massive (short range) weak force via a Higgs mechanism, under which the scalar Higgs field, charged under $SU(2)$, acquires a vacuum expectation value, giving a mass to the gauge bosons of $SU(2)$, breaking the symmetry. A surprising fact was that the Standard Model predicted correctly the existence of the top and bottom quark, the W and Z bosons and the gluon with amazing precision! It also predicts the existence of a remaining massive boson, the Higgs boson, and it is still the only piece of the Standard Model yet to be found. Recently however the ATLAS and the CMS collaboration at the LHC just claimed the discovery of a boson with a mass compatible with the predicted for the Higgs, ~ 125 GeV [1,2]. If its properties match with the predicted Higgs boson, this will show once more the strength of theoretical physics.

Leaving the micro-domain, we have General Relativity, a classical (that is, non-quantum) theory to describe phenomena when the masses and distances are large, i.e. when the gravitational interaction becomes relevant, and it has passed through a large number of experimental tests that makes it extremely difficult to challenge. The standard model of cosmology, called the Lambda-CDM model, assumes General Relativity as a framework and describes the evolution and fate of the Universe and what is in it. It has also been extremely successful in describing the formation of structures, the accelerated expansion of the universe, barionic formation, the cosmic microwave background and many other observed features. The model postulates a small cosmological constant in the general relativity equations, that could be described by a vacuum energy, now called *dark energy*. It also includes cold dark matter, some hypothetical non-relativistic[1] matter that does not interact electromagnetically and is therefore invisible directly, to account for the discrepancy between the seen matter in the universe and the necessary to match the observational data.

Quantum Theory and General Relativity are unfortunately not compatible. When one tries to describe General Relativity as a quantum field theory described by a spin two graviton (the quantized spacetime metric) problems emerge. The standard quantization of the theory, unlike what happens to the other forces, leads to a non-renormalizable theory, and one has to try other non-orthodox approaches. A quantum theory for gravity would be crucial when we want to describe situations of a high mass in small volumes, as it is expected to happen for example in the center of black-holes or at the Big-Bang singularity, or when we want to describe quantum particles in strong gravitational fields, as for example to try to describe the Hawking radiation of black holes. Also,the Standard Model has no particle which matches the criteria for cold dark matter, and the vacuum energy density calculated from the standard model is way much higher than the dark energy density. One could then expect that the Standard Model holds as long as we stay in a domain below the Planck energy scale $\sim 10^{19}$ GeV, where the gravity coupling is much smaller than the other couplings of the theory. However, even way below the Planck scale there are problems appearing in the Standard Model indicating that it might have to be modified already around or above the TeV energy.

[1] Proposals with hot and warm dark matter (where particles move at relativistic speeds) also exist, but are not as popular as cold dark matter

One of these problems is the hierarchy problem, that comes when one calculates the quantum corrections from loop amplitudes to the mass of the Higgs boson. If we introduce a cuttoff scale Λ to indicate the limit up to which we expect the Standard model to hold, already at one-loop the correction to the Higgs mass squared goes as Λ^2, while all the corrections to other massive particles go as $\ln(\Lambda^2/m^2)$, where m is the mass of the particle involved in the loop correction (for the Higgs boson, the main contribution comes from the mass of the tau lepton). This implies that to a Λ close to the Planck scale the logarithmic term contributes to a factor comparable to the mass of the particle or one maybe two orders of magnitude higher, while the Higgs mass correction runs to absurd values, and we need a strong fine-tuning to bring it back to the ~ 100 GeV scale.[2]

1.1 A novel symmetry and more unification

One way to address the Higgs mass problem is to introduce a new symmetry, *supersymmetry*, transforming fermions into bosons (and vice-versa), such that they would always appear in pairs of same mass. As a fermion loop caries opposite signal to a bosonic loop, the corrections to the Higgs could be cancelled. Of course, the bosons and fermions we see in nature do not have same mass *superpartners*, so supersymmetry must be broken, and approximately restored at at a scale ~ 1 TeV to solve the hierarchy problem. The same loop cancellations between supersymmetric partners can bring the vacuum energy contributing to the cosmological constant to a dramatic lower value. And since adding supersymmetry to the Standard Model adds a bunch of unseen massive particles, we have good candidates for dark matter. Experiments around the world are looking for signals from supersymmetric particles, and one hopes that the high energies and luminosities of the LHC give some hint on the existence of supersymmetric particles.

There is still another feature of the Standard Model that might indicate new physics below the Planck Scale, besides supersymmetry. When one calculates the quantum corrections to the gauge couplings, they run to large energies in such a way that they almost meet in Standard Model at an energy scale, and actually converge in the minimal supersymmetric version of the Standard Model (within the precision boundaries). Together with the fact that the fermions in the Standard Model arrange themselves nicely in families, one can formulate a Grand Unified Theory (GUT) in which the various particles inside a single family are actually different states of a single particle multiplet, and the gauge bosons are the remaining massless bosons after the breaking of a larger symmetry. The first proposal for such a unification of all the fundamental (except gravity) interactions was done by Georgi and Glashow [4], still inside non-supersymmetric theories, the $SU(5)$ Minimal GUT. Without supersymmetry however a precise calculation shows that the gauge couplings only almost unify. In the supersymmetric version of $SU(5)$ [5,6] the couplings do unify, but the new particles and couplings from the supersymmetric sector introduced the (unwanted) possibility of a proton decay. Although this idea might seem uncomfortable, there is still room for it, since the prediction for the decay time of the proton is within the actual experimental bounds. This problem can be completely avoided however if one introduces discrete symmetries to create selection rules for the allowed interactions.

Thus, Supersymmetry provides a solution to the hierarchy problem of the Higgs, has candidates for dark matter and favors a unification of all the quantum interactions in a single force. Besides, the introduction of a totally new symmetry, different from any other up now, is a fascinating possibility to explore. There is however still a problem. Supersymmetry might provide answers to many open questions, but it says nothing about the unification with General Relativity. When one tries to merge supersymmetry with general relativity one gets *supergravity*, that is however non-renormalizable. So

[2]Additionally, the running of the Higgs self-interaction at high energies ($\sim 10^{14}$ GeV or higher) might make the coupling negative, thus turning the value of ~ 125 GeV a metastable state of the Higgs, from which it might decay to a much larger value [3]. Other problems of the standard model not necessarily related to the Higgs include for example the strong CP problem or the neutrino masses and oscillations.

some other approach is still needed to describe the interplay between both frameworks. For an extensive review on supersymmetry, supergravity and their implications to particle physics we refer the reader to [7].

1.2 String Theory joins the game

String Theory [8,9] is an alternative proposal that leads to a quantized theory for gravity, which replaces the point particle by a tiny (up to now beyond experimental detectability) one-dimensional object, that can be either open or closed. It contains in its spectrum the graviton, the quantized excitation of the background spacetime metric, and it reproduces the Einstein's gravity equations not from equations of motion, but only from requiring the theory to be anomaly-free. Anomaly cancellations imply the background spacetime to be ten dimensional. Supersymmetry is also a requirement of String theory to get rid of unwanted tachyons in the spectrum. There is just one type of interaction, the joining and splitting of strings. The bad divergences of usual quantum theories arising from the point-like nature of the interactions are now absent present, since the interactions are now smeared out in a smooth worldsheet wiped by the interacting strings.

There are actually five consistent ways of quantizing the superstring that leads to a tachyon-free spectrum, or five Superstring theories, Type IIA, Type IIB, Type I $SO(32)$, Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$. The n-point amplitudes for string scatterings can be calculated and identified to the n-point amplitudes of effective supersymmetric quantum field theories as we use the string length as a small expansion parameter. In the limit of the String length going to zero, the effective theories are the Type IIA, Type IIB and Type I Supergravity theories in ten dimensions. So the non-renormalizability of supergravities theories are not a problem anymore, since they are just low energy effective actions. In Sting Theory the gauge groups are described by extra degrees of freedom at the end points of the open strings, or from the extra bosonic degrees of freedom appearing in the heterotic closed string. Type IIA and IIB however have the apparent problem of containing in the spectrum only closed strings and with possibility to include gauge degrees of freedom. So for a long time the attempts to describe the Standard Model as a limit of string theory were performed with Type I and heterotic strings. In particular the large $E_8 \times E_8$ is favorable for embedding a wide variety of extensions of the Standard Model, in particular Grand Unified Theories.

To arise at four dimensions, one has to perform a suitable compactification of the space, a generalization of the Kaluza-Klein reduction. If additionally we want a minimal amount of supersymmetry in the effective four-dimensional description, we have to compactify on a manifold with $SU(3)$ holonomy, a Calabi-Yau manifold. The effective theory therefore can also be expanded in a parameter of the order of the volume that will encode the Kaluza-Klein tower of states. In heterotic compactifications, in order to obtain a correct four dimensional coupling constant of gravity the string length and the length of the compactification manifold must be the order he Planck scale. Also, the length of the strings should be the order of the Planck length[3]. Phenomenology from heterotic strings are often referred to as *top-bottom* approach, in the sense that we construct the theory in a consistent internal space and extract the physical information of it. Often in compactified theories the size of the internal manifold is related to the coupling constants and the topological data of the manifold to the spectrum of the theory, i.e. for example the number of families, charges and spin of the fields. A concise review on the heterotic construction of the (supersymmetric) Standard Model can be found for example in [11] and references therein.

Heterotic constructions were unfortunately not perfect. As one of the issues, one can cite the moduli stabilization problem. In string compactifications usually (as was the case for heterotic compactifi-

[3]this can be avoided in *large-volume compactification* scenarios where the ten-dimensional Planck length can be made much smaller than the four-dimensional one via the relation $M_4^{\rm pl} \sim M_{10}^{\rm pl}{\rm Vol}$ while the Standard Model is located in a small subregion of the compactification space [10].

cations[4]) there are many moduli, massless scalar fields often related to particular deformations of the internal geometry that are not restricted by a potential, meaning that these fields can acquire any vacuum expectation value (vev) in a continuous. In other words, the geometry can be deformed with no impediment in a continuous way. This continuous space of possible deformations of the geometry define a manifold, the moduli space, that although extremely interesting from the theoretical and mathematical point of view, is phenomenologically undesired since the values for the physical couplings are related to the geometry, and thus a Universe with unfixed couplings would be incompatible with reality.

1.2.1 A Brane New World

The status of string theory changed dramatically in the mid 90's with the discovery of Dp-Branes in Type II theories, p-dimensional objects on which open strings end [15]. This has broadened the possibilities to construct interesting phenomenological actions in String Theory, since one could form a stack of N D-branes on top of each other, and the open strings would be in the adjoint representation of an $SU(N)$ group (or $Sp(2N)$ or $SO(2N)$ in orientifold compactifications), and their effective action described by a Super-Yang-Mills theory. Branes could intersect at some angle, and at the intersection of a stack of N D-branes with a stack of M D-Branes, open strings would be in the bi-fundamental representation $(\mathbf{N}, \mathbf{M})$, thus providing an elegant way of introducing barionic representations geometrically. Additionally, Type II theory with Branes contained a good variety of field-strength fluxes, topological solutions to the field strength that have become the main mechanism in the string theory literature to stabilize the moduli, give a chiral mass spectrum and induce breakings of the gauge groups.

In the last two decades there have been a huge number of papers written on the realization of four dimensional quantum field theories from Type II string theories with D-branes, reproducing Standard Model-like scenarios or extensions of it (as mentioned, the literature is extensive, so we just cite some early representatives [16–18] and the reviews [19–21]). Brane model constructions are often referred to as bottom-up, since one starts by specifying the local geometry (the configurations and intersections of branes at subregions of the compactification space), in opposition to the global constructions of heterotic strings.

Additionally, String Theory with D-branes have also become an interesting framework to study general properties of supersymmetric theories, such as phase transitions and non-perturbative corrections. D-brane constructions also provided a framework to compute the black-hole entropy [22], now one of the strongest theoretical evidences in favor of string theory. Type II strings with D-branes also supported a novel interpretation of gravity via holography (starting from the Maldacena conjecture [23] and the vast subsequent literature on AdS/CFT and related topics). Alternatively to these and many other physical applications, string theory has also contributed to pure mathematics, as for example in the computation of rational curves of the quintic manifold using string theory arguments [24].

Brane-world scenarios also brought a fascinating contribution to cosmology. Since the Standard Model is localized in the internal space at some crossing of branes (or branes at a singularities, as in [25]), the Universe as we see it could be just a tiny fraction of the whole. Matter representations away from the Standard Model region could account for dark sectors of the spectrum, connected to our world only via loop corrections, and the dynamical nature of branes could be behind cosmological episodes like the inflation epoch or even the Big Bang, also offering predictions to precision measurements of astronomical data (for a review, [26]).

Although much is known from the spectrum and the basic theory which branes give rise to, detailed calculations have shown non-trivial interactions between fields, that could in principle contribute to effects on our real world physics. As one example, one can cite the kinetic mixings between different $U(1)$ gauge bosons that often appear in brane compactifications [27, 28]. Such mixing could in principle be

[4]More recently many solutions to the moduli stabilization problem in heterotic strings have been proposed. As recent examples, we can cite the case for orbifold compactifications in [12] and for more general compactifications [13, 14].

detected in experiments, as for example the "light through walls" setups [29], where the electromagnetic photon could become a non-Standard Model boson (a "hidden photon", or for massive invisible $U(1)$s, Z-prime bosons) that could go through a barrier, mix back into a photon and be detected at the other side of the barrier. The non-detection of such phenomena impose limits on the coupling of the mixing, and therefore to the properties of the internal geometry. Also, many phenomenological examples treat the branes as fixed in the internal geometry, but this is not general, since the branes can be moved and deformed. These deformations can in some cases contribute to new loop couplings between the fields on the branes, as well as to generate potentials that break supersymmetry. Detailed calculations can also give explicit values for the cosmological constant, and in models with supersymmetry breaking with gravity or gauge mediation (in which there are two brane sectors, the Standard Model and a hidden sector that breaks Supersymmetry spontaneously) a more precise calculation is also important to understand quantitatively and not only qualitatively the induced masses after the breaking of supersymmetry. One of the focus of this thesis is to work explicitly the action for a single D-brane, and perform a careful analysis of the four dimensional field theory on it.

1.2.2 Getting stronger

The many different quantized string constructions Type IIA, Type IIB, Type I and the heterotics might seem incompatible with the idea of a single unifying theory. However an exciting consequence of String Theory is the existence of many dualities relating the various formulations [30]. It has also been postulated the existence of an eleven-dimensional theory, called M-theory, that has M2- and M5-Branes as the fundamental objects and reduce to eleven-dimensional supergravity in the low-energy limit [31]. From what it is known of M-theory, there are strong indications that it reproduces all the string theories in particular compactification limits or after a sequence of dualities. The most straightforward relation is between Type IIA and M-theory, in which it has been shown that eleven dimensional supergravity compactified on a circle reduces to Type IIA supergravity as the radius of the circle becomes tiny, and the various D-branes would be then compactification of the M-branes or special purely geometrical non-trivial solutions. M-theory would be then the strong-coupling "lift" of Type IIA, with the IIA string coupling given by the radius of the compactified eleventh dimension. Until this moment however only the supergravity limit of M-theory and the effective descriptions of single M2 and M5 branes are known [32]. Actions for the worldvolume of multiple M2- and M5-branes are still being proposed (for example in [33]).

The D-branes in Type IIB also obey a special type of duality that relates strong and weak string coupling. This duality acts on fields as an $SL(2,\mathbb{Z})$ symmetry, that is also the symmetry group lattice of a torus. The axio-dilaton, a combination of the string coupling with the Ramond-Ramond axion, would transform under this duality in the same way as the complex structure of a torus. This symmetry also implies the introduction of new non-perturbative objects, (p,q)-branes, around which the theory suffers a monodromy, described by the $SL(2,\mathbb{Z})$ group. All those facts induced string theorists to search for another theory with an underlying torus structure that could reduce to Type IIB in some limit. This hypothetical theory received the name F-theory [34]. Additionally, F-theory compactified on an elliptically fibered $K3$ manifold was inferred to be dual to Heterotic string compactified on a torus, strengthening the web of string dualities known. Finally, F-theory can also be seen as a T-dual picture of M-theory, in the same way as Type IIB is T-dual to Type IIA string theory. This comes from the fact that a special compactification limit of M-theory with a T-dualization leads to Type IIB theory with the desired underlying torus structure. And like M-theory, F-theory has no fundamental description. Even worse, there is no known low-energy action, as there is supergravity for M-theory. One could then ask oneself if it is really a physical theory, or just a nice mathematical way of seeing things. Independently on the answer, one can work with it. And for the moment, that is enough.

It is important to point out the geometrization that occurs as one moves from perturbative to strong coupling. The description of a D6 brane in M-theory is via a purely geometrical object, a generalization of the Kaluza-Klein monopole that is a non-trivial solution to the supergravity equations. When we T-dualize to F-theory, the lift of the D6-brane becomes an elliptically fibered divisor, with a singular fiber. But if we remain in perturbation theory, the D6-Branes are T-dual to D7-branes. That is, in F-theory the strongly coupled equivalents of the D7-branes become entirely part of the geometry.

F-theory has many other interesting features. String-theoretically, being a strong coupling regime of Type IIB, one can learn more about the non-perturbative effects of strings, branes and fluxes. The interplay among Type IIB string $\leftrightarrow$ M-Theory $\leftrightarrow$ Heterotic string under F-theory allow us to alternate among the available tools on each framework, with which we can obtain results beyond the perturbation theory. On the Heterotic side, one can use F/M-theory to solve the moduli problem described earlier, as we can incorporate fluxes. Also, one can explore the geometrical realizations of elements from heterotic theory, and use heterotic strings to understand better M- and F-theory.

Phenomenologically, the non-perturbative nature of F-theory allows for the realization of exceptional groups over the seven-branes, extending the previous possibilities $SU(N)$, $SO(2N)$ and $Sp(2N)$. This is crucial specially when constructing realizations of GUT models, since in previous perturbative brane constructions some of the couplings could not be realized. As a particular famous example, In $SU(5)$ GUTs localized on a brane, the coupling between the matter representations $\mathbf{10}\ \mathbf{10}\ \mathbf{5}$ was known to be obtained only via instantons. This however is the coupling necessary to couple the top-quark to the Higgs in $SU(5)$ models [35], so it is quite undesirable that the most massive fermion would receive its mass from highly suppressed corrections[5]. In GUTs constructed from F-theory, or simply called F-theory GUTs, such couplings are generated naturally, as in the intersection of branes the theory can become strongly coupled. F-theory as a non-perturbative lift of Type IIB was known since the end of the '90s, but the tools for constructing such GUT models with exceptional groups are very recent [36–38], and triggered a renewed interest in F-theory.

We could resume the standard constructions of F-theory GUTs in the following items:

- Start with an elliptically fibered Calabi-Yau fourfold that contains a divisor on the base over which the fiber becomes singular. The Calabi-Yau condition is the requirement for supersymmetry in four dimensions.
- One can analyse the gauge group on top of this divisor either by the Heterotic duality as an unbroken subgroup of E_8, either by using explicit algebraic or toric geometry to resolve the singularity and wrap M2 branes on the resolved fiber reproducing vector multiplets transforming in the adjoint of the corresponding group.
- One can turn on M-theory fluxes on the brane to break the GUT gauge group down to smaller groups (like the Standard Model groups).
- At curves on the base the singularity worsens, and the associated gauge group enhances, giving rise to matter curve representations. In Type IIB picture, these curves correspond to the intersection between two stack of seven-branes, reproducing bi-fundamental representations.
- From the M-theory perspective, one can localize the G_4 flux along the resolved curve, and make the matter representation chiral.
- The geometric localization of matter on curves allow a natural way to generate or suppress certain matter couplings in particular models, as they would simply correspond to intersecting or non-intersecting curves, respectively.
- The number of matter curves and their intersections can be encoded in the specifications of the fibration defining the topological data of the Calabi-Yau.

[5] The couplings might have the right hierarchical structure in flipped $SU(5)$ models [35].

It is important to point out the mixed local/global features of F-theory GUTs. First, the local realization of gauge groups as a singular elliptic fiber works only for non-Abelian groups. $U(1)$s in F-theory GUTs are generally massive (via coupling with the axion), and thus are only unbroken when they satisfy some global criteria [39–41]. Also, the fluxes (used for chirality or GUT breaking) must also obey some global restrictions to preserve the Calabi-Yau condition (and supersymmetry), and the local description of matter curves is not enough to give the full picture. The good thing about F-theory constructions is that one can explore the F-theory/Heterotic duality to obtain global informations. For a very incomplete list of examles of F-theory GUT constructions we refer to [42–47] and references there cited.

The possibility to describe in a globally consistent way matter curves reproducing interesting spectrum and Yukawa interactions has led many to try to reproduce more than just the spectrum and the gauge groups, but also open questions in the Standard Model like the neutrino mixings and the hierarchy for the lepton masses. The flavor hierarchy was explored in [48, 49], while the neutrino masses in [50, 51]. In order to solve both problems at the same time one has to consider the matter representations intersecting at a point that would enhance to an E_8 singularity (that is, a singular fiber whose associated gauge group after the resolution is an E_8 group). Kaluza-Klein excited states can couple to the matter representations, and are integrated out when we move below the GUT scale. Allowing extra $U(1)$ symmetries in the setup at high energies, the charges of the fields under these $U(1)$s can generate quark hierarchies in Froggatt-Nielsen models [52] (or reviews in supersymmetric models [53, 54]),

The idea of Froggatt-Nielsen works as follows: Close to the Planck scale there is an extra $U(1)$ under which a scalar *flavon* field S is charged. There are additionally massive fields G_i that have the same Standard Model quantum numbers as the quarks. These fields appear naturally in Strings/F-theory compactifications as the Kaluza-Klein excitations of the massless quarks (before the Higgs acquires a vev). By assigning correct charges to the Standard Model fields under the extra $U(1)$, the Lagrangian terms for the Yukawa couplings must be $U(1)$ invariant, and some couplings of massless quarks to the Higgs are not generated at the classical level. When the flavon acquires a vev $\langle S\rangle$ the $U(1)$ is broken, and in the low energy theory we can integrate out the fields G_i. This generates an effective coupling between the quarks and the Higgs with a coupling constant $(\langle S\rangle/M_{G_i})^n$, where n is some integer related to the $U(1)$ charge of the quarks. The interesting thing about this mechanism is that to generate large hierarchy between quarks we just need a very small hierarchy between $\langle S\rangle$ and M_{G_i}, that can be both close to the Planck scale. For example, with $(\langle S\rangle/M_{G_i}) \sim 0.2 = \epsilon$ and choosing the right charges under $U(1)$ one can generate charge hierarchies of the form $m_u : m_c : m_t \sim \epsilon^8 : \epsilon^4 : 1$ and $m_d : m_s : m_b \sim \epsilon^5 : \epsilon^2 : 1$, consistent with the experimental data.

Recently a problem with the construction of GUTs in F-theory was found out. It was worked out explicitly that the relation between the gauge group from the broken E_8 in the heterotic side and the M-theory interpretation via M2 branes in the adjoint of the group works fine at the GUT brane (divisor on the base), but might fail at the matter curves and Yukawa points [55]. In view of this fact, and with the phenomenological interest for flavor behind, we can try to construct explicitly some geometry to reproduce an E_8 point. Having an explicitly resolved geometry we can than proceed to calculate the allowed interactions at the Yukawa point, and that could in principle lead to non-obvious interactions between the matter representations. Besides, there are not many explicit examples for less generic fibers with stronger singularities, and such constructions might help understanding even better the non-perturbative effects of F-theory.

1.3 Outline

This thesis has two main parts. Both deal with aspects of brane physics. The first part focus on perturbative Type IIA theory, where we study the four dimensional physics of a D-brane in a particular setup, while the second part we introduce the lift to the hypotetical non-perturbative descriptions of Type II string theories, namely M- and F-theory, and we also discuss how to realize matter representatiosn and Yukawa couplings in F-theory models, and we discuss some tipically non-perturbative effects, not present in perturbative Type II picture with D-branes.

In the second chapter we will review the basics on supersymmetric theories. We will give a stronger focus on $\mathcal{N} = 1$ theories, as they are the most phenomenologically interesting. Since the remaining of the thesis will be focused on local descriptions of Brane actions and F-theory fibrations, we are not concerned with couplings to gravity. For that reason, we skip a discussion on supergravity.

In chapter 3 we present a calculation for the effective four dimensional theory of a single D6-brane in a Calabi-Yau orientifold, that fills the entire spacetime in a way that it preserves supersymmetry. We analyse the fields living on the brane from a four dimensional perspective, as well as the fields corresponding to deformations of the internal space and of the brane. Some of these deformations might break supersymmetry, and we find the conditions that leave it unbroken. The supersymmetric deformations in the absence of fluxes allow the brane to be deformed continuously, with no impediment. These continuous deformations form a moduli space, that has some specific geometrical properties that we describe. We also present corrections to the gauge coupling functions coming from the deformations and fluxes, and show how kinetic mixings between different $U(1)$s are generated. The results could be used in explicit phenomenological scenarios constructed from brane intersections, and the $U(1)$ mixings we calculated have been used in a nice description of the possible hidden photons appearing in different string constructions [56].

In chapter 4 we review M and F-theory, and present some tools needed for engineering Grand Unified Theories. Finally, in chapter 5 we first review the resolution for an $SU(5)$ model with an enhancement to an E_6 Yukawa point [55] and review the analysis of the matter curves and Yukawa couplings [57]. Then, using the spectral cover formalism imported from heterotic string, we construct a model that has a point with a singularity of the E_8 type, that we resolve explictly. The resolution cannot be performed in the same way as in [55], and we have to introduce new structure on the base. We analyse the possible implications of it, but up to this moment there are some unanswered questions, that still require attention. We plan to address the problems in an upcoming work.

Chapter 2

Supersymmetric Theories

Supersymmetry (SUSY) has become one of the main lines of thought in contemporary high-energy particle physics, and also in quantum field theories arising from or motivated by (super)string theory. In this chapter we review the background needed on supersymmetric theories that will be most relevant to the subsequent discussion along this thesis. For a longer discussion on supersymmetric theories we refer the readers to the main references used to write this chapter, [58–61].

First we construct the SUSY algebra for $\mathcal{N}$ independent supersymmetries. Then we focus on the particular case of $\mathcal{N} = 1$, i.e. *minimal supersymmetry*, and just the fact that we want a theory which is invariant under SUSY transformations imposes some conditions on the moduli space (the space of expectation values for the scalar fields in the theory, that classically correspond to minima of the potential). The most general four dimensional supersymmetric action without the inclusion of gravity can be written in a most simple form, described in section 2.5. This is the form pursued later in chapter 3, when we compactify type IIA supergravity and the D6-brane action (ten- and seven-dimensional, respectively) into a four-dimensional theory with an unbroken $\mathcal{N} = 1$ SUSY.

2.1 The SUSY Algebra

Supersymmetry is, up to this date, the only known unitary extension of the Poincaré spacetime symmetry (outside internal symmetries), and is generated by spinorial operators Q^A_α, that together with the momentum P^μ, the generator of spacetime translations, obey

$$\begin{aligned} \left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \\ \left[P_\mu, Q^A_\alpha\right] = \left[P_\mu, \bar{Q}_{\dot{\alpha}A}\right] &= 0 \\ \left\{Q^A_\alpha, Q^B_\beta\right\} &= \epsilon_{\alpha\beta} X^{AB}. \end{aligned} \tag{2.1.1}$$

Here we use the Weyl spinor convention for the indices ($\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$), μ are spacetime indices and A, B denote the number of independent supersymmetries $A = 1 \ldots \mathcal{N}$ we can start with. The supersymmetry algebra (2.1.1) contains a *central charge* described by the antisymmetric X^{AB} that appears only in $\mathcal{N} = 2$ or higher.

From the fact that the SUSY generators Q_α commute with P_μ, we can construct representations of the SUSY algebra (multiplets) from representations of the Casimir P^2, and therefore each component inside a supersymmetric representation will have the same mass. Taking the trace over the representation, one can also show that the number of bosonic and fermionic states of a supersymmetric multiplet is the same.

Massless representations

We first look at a massless supersymmetric representation. We can perform a spatial rotation (that commutes with the SUSY algebra) to put the state in the frame where $P_\mu = (-E, 0, 0, E)$, so

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta^A_B. \tag{2.1.2}$$

We introduce the fermionic creation and annihilation operators

$$a^A = \frac{1}{2\sqrt{E}} Q_1^A, \quad (a^A)^\dagger = \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}}^A. \tag{2.1.3}$$

that obey the Clifford algebra

$$\left\{a^A, (a^B)^\dagger\right\} = \delta_B^A. \tag{2.1.4}$$

The irreducible representation is constructed as usually is done with a Clifford algebra, by starting with some ground state $|\Omega\rangle$ that is annihilated by every annihilation operator a^A and act with the creation operators as many times as possible. In this case there is just one fermionic operator for each $A = 1, \ldots, \mathcal{N}$, so the SUSY representation consists simply in

$$|\Omega\rangle, \quad (a^{A_1})^\dagger|\Omega\rangle, \quad (a^{A_2})^\dagger(a^{B_2})^\dagger|\Omega\rangle, \ldots, (a^1)^\dagger(a^2)^\dagger \ldots (a^N)^\dagger|\Omega\rangle. \tag{2.1.5}$$

The $(a^A)^\dagger$ operator transforms as a spinor $(0, 1/2)$, and therefore when acting on $|\Omega\rangle$ it increases the helicity by $1/2$. Thus, if the state $|\Omega\rangle$ has helicity λ, the state

$$\frac{1}{\sqrt{n!}}(a^{A_1})^\dagger(a^{A_2})^\dagger \ldots (a^{A_n})^\dagger|\Omega\rangle \tag{2.1.6}$$

will carry a helicity $\lambda + \frac{1}{2}n$. The highest spin state in (2.1.5) will therefore carry $\lambda + \frac{1}{2}N$ helicity. The full representation will have a number of components given by the power set of the N creation operators, 2^N. The multiplet is in general not CPT invariant, since CPT takes λ to $-\lambda$. So, to have a CPT invariant representation we must add by hand the opposite helicity states to the spectrum. The only situations where the multiplet (2.1.5) is already CPT invariant is when the helicity of $|\Omega\rangle$ is $\lambda = -N/4$.

Massive Representations and BPS states

For a massive state, we can work in a frame where $P^\mu = (M, 0, 0, 0)$. Then (2.1.1) becomes

$$\left\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\right\} = 2M\delta_{\alpha\dot{\beta}}\delta_B^A. \tag{2.1.7}$$

We introduce fermionic creation and annihilation operators $(a_\alpha^A)^\dagger = (2M)^{-1/2}\bar{Q}_{\dot{\alpha}A}$ and $a_\alpha^A = (2M)^{-1/2}Q_\alpha^A$, that obey the usual Clifford algebra

$$\left\{a_\alpha^A, (a_\beta^B)^\dagger\right\} = \delta_\alpha^\beta\delta_B^A, \quad \left\{a_\alpha^A, a_\beta^B\right\} = \left\{(a_\alpha^A)^\dagger, (a_\beta^B)^\dagger\right\} = 0. \tag{2.1.8}$$

We construct the representation in the usual manner. There are $2\mathcal{N}$ creation operators, the 2 coming from the two spinorial components for a Weyl spinor. For a non-degenerate vacuum, the SUSY multiplet is $2^{2\mathcal{N}}$ dimensional. If we instead start with a "vacuum" that is a spin j multiplet, the dimension of the representation will be then $2^{2\mathcal{N}}(2j + 1)$.

For the case of a single supersymmetry, $\mathcal{N} = 1$, the states obtained from any massive ground state are

$$|\Omega\rangle, \quad a_1^\dagger|\Omega\rangle, \quad a_2^\dagger|\Omega\rangle, \quad \frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle. \tag{2.1.9}$$

Now consider a state with total spin j and component along the direction 3 of the spin $m = -j, (-j + 1), \ldots, (j - 1), j$. The spinorial operator $(a_\alpha)^\dagger$ has the following commutation property with the spin operator $S_3 = \frac{1}{2}\sigma^3$

$$\left[S_3, \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} a_1^\dagger \\ -a_2^\dagger \end{pmatrix}. \tag{2.1.10}$$

Thus, for a state with a particular spin component j_3,

$$S_3 a_1^\dagger|j_3\rangle = \left(j_3 + \frac{1}{2}\right) a_1^\dagger|j_3\rangle, \quad S_3 a_2^\dagger|j_3\rangle = \left(j_3 - \frac{1}{2}\right) a_1^\dagger|j_3\rangle. \tag{2.1.11}$$

In total, the irreducible representation (2.1.9) from a ground state $|j_3\rangle$ will consist then of states with spin $(j, j+1/2, j-1/2, j)$. For the particular case when $|\Omega\rangle$ has 0-spin, the representation under the supersymmetry algebra corresponds to two spin zero fields and the two components of a Weyl spinor (with spin $1/2$ and $-1/2$). The spin zero field $\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle$ corresponds to a pseudoscalar, since the parity operator interchanges $a_1^\dagger$ and $a_2^\dagger$, but from the anticommutation relation

$$\frac{1}{\sqrt{2}}a_1^\dagger a_2^\dagger|\Omega\rangle = -\frac{1}{\sqrt{2}}a_2^\dagger a_1^\dagger|\Omega\rangle. \tag{2.1.12}$$

For $\mathcal{N} > 1$, however, there might be even smaller massive representations than the $2^{2\mathcal{N}}(2j+1)$ dimensional ones, since now we have an extra parameter in our algebra, the central charges X^{AB},

$$\left\{Q_\alpha^A, Q_\beta^B\right\} = \epsilon_{\alpha\beta}X^{AB}. \tag{2.1.13}$$

For simplicity, we consider the case $\mathcal{N} = 2$. As the matrix X^{AB} has to be antisymmetric, we can write $X^{AB} = X^{12} \equiv Z$. We then define the operators

$$a_\alpha = \frac{1}{2}\left[Q_\alpha^1 + \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger\right], \quad b_\alpha = \frac{1}{2}\left[Q_\alpha^1 - \epsilon_{\alpha\beta}(Q_\beta^2)^\dagger\right] \tag{2.1.14}$$

and using (2.1.1) we can write the anticommutation relations

$$\left\{a_\alpha, a_\beta^\dagger\right\} = \delta_{\alpha\beta}(2M+Z)\,, \quad \left\{b_\alpha, b_\beta^\dagger\right\} = \delta_{\alpha\beta}(2M-Z). \tag{2.1.15}$$

So, for a normalized state with mass M and central charge Z $|M,Z\rangle$, $\langle M,Z|M,Z\rangle = 1$, and using the fact that the states constructed via $b_\alpha^\dagger|M,Z\rangle$ or $b_\alpha|M,Z\rangle$ also have norm ≥ 0, we have,

$$\langle M,Z|\left\{b_\alpha, b_\beta^\dagger\right\}|M,Z\rangle = \delta_{\alpha\beta}(2M-Z) \geq 0, \tag{2.1.16}$$

thus,

$$Z \leq 2M. \tag{2.1.17}$$

This is called the Bogomol'ny-Prasad-Sommerfield (BPS) bound. In the particular case when the bound is saturated, $Z = 2M$, the operators b_α and $b_\alpha^\dagger$ project the state to a zero norm state. So, they do not play a role when constructing the full SUSY multiplet for a *BPS-saturated state* (or simply, a *BPS state*). We can alternatively say that the state is invariant under half the supersymmetries. The supersymmetric representation is then constructed by acting only with $a_\alpha^\dagger$ on a ground state, and therefore has dimension $4(2j+1)$ for $\mathcal{N} = 2$. An analogous result holds in general for $\mathcal{N} > 1$, but there is one supercharge Z_n for each pair of supercharges, $n = 1, \ldots, N/2$, and the dimension of BPS states is $2^{\mathcal{N}}(2j+1)$ instead of $2^{2\mathcal{N}}(2j+1)$ for the non-BPS states. The multiplet constructed from a BPS state is often called a *short multiplet*, in opposition to the *long multiplet* constructed from $a_\alpha^\dagger$s and $b_\alpha^\dagger$s. For higher supersymmetric theories, $\mathcal{N} > 2$, the number of central charges is bigger and we can have more possibilities for the BPS bounds. For example, in $\mathcal{N} = 4$ there are two independent supercharges, Z_1 and Z_2, and a state with $M = Z_1 < Z_2$ is transformed only by 3/4s of the total number of independent supercharges, and is called a quarter-BPS state.

In chapter 3 we will start with ten-dimensional supergravity with two supercharges ($\mathcal{N} = 2$) and add to it a six-(spatial)-dimensional object, a D6-brane. In general, a single object localized in space breaks spacial translation, and also supersymmetry. There are however configurations for the brane in which it is invariant under the action of half the supercharges, breaking the other half. The brane, at these configurations, is a BPS object. The effective theory on such brane is thus $\mathcal{N} = 1$ supersymmetric. In chapter 3 this will be precisely the situation we will deal with.

Superspace

To work with supersymmetric field theory, it is more convenient to redefine the SUSY generators by introducing grassmanian variables θ^A_α and $\bar{\theta}^A_{\dot{\alpha}}$ (using the standard notation from the literature, as in [58]) such that (2.1.7) is replaced by the Lie algebra

$$\left[\theta^A Q, \bar{\theta}^B \bar{Q}\right] = 2\theta\sigma^\mu\bar{\theta}P_\mu\delta^{AB}.$$

A summation over the spinorial indices is understood. The grassmanian variables θ^A_α together with the usual spacetime coordinates x^μ define the *superspace*, the stardard framework of supersymmetric field theories.

The supersymmetry transformation maps fields of a given spin into fields with different spins. It is possible however to define *superfields* as a combination of fields covariant under the SUSY transformations, such that the supersymmetry transformation of a superfield is still a superfield of the same kind.

A representation for the SUSY generators in the *rigid* (that is, with global supersymmetry) $\mathcal{N} = 1$ superspace is

$$Q_\alpha = \partial_\alpha - i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu, \tag{2.1.18}$$

that act on superfields. We can also construct a covariant derivative that anticommutes with the SUSY generators

$$D_\mu = \partial_\mu, \quad D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu. \tag{2.1.19}$$

A superfield is in general not an irreducible representation of the supersymmetry algebra. From the fact that the covariant derivative commute with the supersymmetry generators, given a general superfield we can construct a smaller representation by imposing some constrain using the covariant derivatives. This explicit construction of the irreducible representations of superfields may not be directly related to the multiplets constructed from the SUSY algebra. The reason is that the superfields in general do not obey the mass-shell condition. Only after we impose the equations of motion we can relate the irreducible superfield to the $2^{2\mathcal{N}}$ dimensional multiplets from the one-particle representations discussed earlier.

We will in the following briefly describe the relevant superfields we shall be concerned with throughout this thesis.

2.2 $\mathcal{N} = 1$ Superfields

The first superfield we are interested in is the chiral multiplet Φ, defined by the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$. To construct explicitly such a field, it is convenient to introduce a translated bosonic coordinate

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}, \tag{2.2.1}$$

that obeys $\bar{D}_{\dot{\alpha}}y^\mu = 0$. Everywhere summations as $\theta\sigma^\mu\bar{\theta} = \theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}$ are understood. As $\bar{D}_{\dot{\alpha}}$ also satisfies $\bar{D}_{\dot{\alpha}}\theta^\beta = 0$, every combination of y and θ is chiral. So, the most general chiral superfield we can construct is (with factors added to match the literature)

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{2.2.2}$$

where ϕ and F are complex scalar fields and ψ_α is a left-handed Weyl spinor. Expanding y,

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + +i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x). \tag{2.2.3}$$

Analogously one can define an antichiral field from $D_\alpha \Phi^\dagger = 0$, and repeat the same steps, using $\bar{y} = x^\mu - i\theta\sigma^\mu\bar{\theta}$ instead of y.

The most general renormalizable Lagrangian containing only $i = 1...N$ chiral superfields is

$$\mathcal{L} = \int d^4\theta \Phi^i \Phi^{i\dagger} + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\Phi^\dagger), \tag{2.2.4}$$

where $W(\Phi) = \lambda_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k$ is the *superpotential*, and we have used the shorthand notation $d^4\theta = d\theta_\alpha d\theta^\alpha d\bar{\theta}_{\dot{\beta}} d\bar{\theta}^{\dot{\beta}}$ and $d^2\theta = d\theta_\alpha d\theta^\alpha$. in components,

$$\mathcal{L} = -i\psi\sigma^\mu\partial_\mu\psi - \partial_\mu\phi^*\partial^\mu\phi + F^*F - m_{ij}\frac{1}{2}\psi^i\psi^j + g_{ijk}\psi^i\psi^j\phi^k + \left(F\frac{\partial W(\phi)}{\partial\phi} + \text{h.c.} \right). \tag{2.2.5}$$

The field F is not dynamical, and can be removed from the action by the equations of motion $F^* = \partial W/\partial\phi$. When this is done, the Lagrangian becomes

$$\mathcal{L} = -i\psi\sigma^\mu\partial_\mu\psi - \partial_\mu\phi^*\partial^\mu\phi - m_{ij}\frac{1}{2}\psi^i\psi^j + g_{ijk}\psi^i\psi^j\phi^k - V(F, F^*), \tag{2.2.6}$$

where $V = F^*F = |\partial W/\partial\phi|^2$ is known as the *F-term potential*. As was stated in advance, the removal of F via the equations of motion leaves us with the complex scalar ϕ plus the Weyl fermion ψ, thus $2+2$ degrees of freedom, as expected for the lowest spin multiplet mentioned in the previous section.

We next want to construct supersymmetric gauge theories, and therefore we need a superfield that contains real vector fields. We start by defining a general superfield that obeys a reality condition

$$V = V^\dagger.$$

Since for our discussion the full form of the general expression for V is not relevant it is enough to say that, as inferred by the name, this multiplet contains a real vector field v_μ,

$$V(x, \theta, \bar{\theta}) \supset -\theta\sigma^\mu\bar{\theta}v_\mu(x) + ... \tag{2.2.7}$$

and therefore receives the name of *vector superfield*. Notice that we can also define a hermitian combination from any chiral field Φ,

$$\Phi + \Phi^\dagger,$$

that contains a term (2.2.3)

$$\Phi + \Phi^\dagger \supset i\theta\sigma^\mu\bar{\theta}\partial_\mu(\phi + \phi^*). \tag{2.2.8}$$

Comparing to (2.2.7), it is then natural to construct a supersymmetric version of a gauge transformation with a chiral field Λ,

$$V \to V + i(\Lambda + \Lambda^\dagger), \tag{2.2.9}$$

that reproduces the usual vector gauge transformation

$$A_\mu \to A_\mu + \partial_\mu(\phi + \phi^*). \tag{2.2.10}$$

By choosing a convenient Λ one can put the vector superfield in the *Wess-Zumino gauge*,

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x). \tag{2.2.11}$$

$\lambda(x)$ is the *gaugino* spinor and $D(x)$ is the D-term. This form, although much simpler, has no manifest supersymmetry. It is also worth to point out that under the transformation (2.2.9) the components $\lambda(x)$ and $D(x)$ are gauge invariant.

The supersymmetric version of the field strength for the vector superfield is defined as

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V, \tag{2.2.12}$$

and it is straightforward to show that it is a chiral superfield ($\bar{D}_{\dot{\beta}} W_\alpha = 0$) and gauge invariant. Similarly, we can also define an antichiral $\bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V$. Explicitly, in terms of the translated coordinate y, W_α reads, in the Wess-Zumino gauge,

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\sigma^\nu\theta)_\alpha(\partial_\mu A_\nu(y) - \partial_\nu A_\mu(y)) + \theta\theta\sigma^\mu\partial_\mu\bar{\lambda}(y), \tag{2.2.13}$$

that contains only the gauge invariant terms λ, D and the "usual" vector field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The *super-Yang-Mills* Lagrangian for Abelian gauge fields in the superspace reads

$$\mathcal{L}_{SYM} = \frac{1}{4}\left(\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta}\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}\right), \tag{2.2.14}$$

that in components translates to, up to total derivatives,

$$\mathcal{L}_{SYM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2. \tag{2.2.15}$$

At this stage, the field D is a non-dynamical free field, but when we couple this gauge theory to matter, it will play a role in generating a (*D-term*) scalar potential.

The generalization to non-Abelian theories works by introducing a trace over the adjoint representation in the action above, and by generalizing the gauge transformation

$$e^{V'} = e^{-i\Lambda^\dagger}e^V e^{i\Lambda}, \quad W'_\alpha = e^{-i\Lambda}W_\alpha e^{i\Lambda}, \tag{2.2.16}$$

where $\Lambda = T^a\Lambda_a$ and $V = V_a T^a$ are now matrices in the adjoint representation with the matrix generators T^a.

We next want to include chiral superfields charged under a gauge group. In non-supersymmetric theories, we would just add to the Yang-Mills action an action for the charged matter, with the kinetic terms modified, to include covariant derivatives under the new gauge bundle. Here the strategy is the same, and with the kinetic term replaced by

$$\Phi^\dagger\Phi \to \Phi^\dagger e^V\Phi,$$

where now the superfield V should be expanded in the representation under which Φ transforms. Notice that the kinetic term $\Phi^\dagger\Phi \to \Phi^\dagger e^V\Phi$ induces a coupling of the D-term with the scalar fields $\phi^* D\phi$, and similarly as the *F-term*, after eliminating D from the action one has a contribution of the form (after introduction of the gauge coupling g)

$$V_D = \frac{1}{2g^2}D^2 = \frac{g^2}{8}\left([\phi, \phi^*]\right)^2. \tag{2.2.17}$$

The scalar potential arises therefore from either self-interactions of the scalar components of the chiral fields giving rise to F-terms or via the interaction of the same scalar components with the gauge fields of the theory, generating D-terms. F- and D-term potentials are key elements in supersymmetric theories, since the mechanisms for breaking supersymmetry (and therefore establishing contact with our real universe) rely on building potentials that induce vacuum expectation values to the scalar components and spontaneously break supersymmetry. Alternatively, knowing the scalar potential of some theory and requiring the theory to remain supersymmetric may lead to restrictions on the allowed scalar fields of the theory, as in chapter 3

2.3 The moduli space for $\mathcal{N} = 1$ SUSY Theories

Often the scalar potential can be minimized not by single vacuum expectation values for the scalar fields, but by a continuous set. Since the scalar fields are also differentiable, this set of minima define a manifold called the *moduli space*. One interesting and very general result on the geometry of moduli spaces of supersymmetric theories is that, if we require $\mathcal{N} = 1$ supersymmetry to be preserved, the moduli space of every supersymmetric theory, renormalizable or not, has to be a Kähler space [59]. This argument is briefly reviewed here.

We start with the most general form for a supersymmetric Lagrangian, not necessarily renormalizable, with only chiral superfields Φ

$$\mathcal{L} = \int d^4\theta K(\Phi, \bar{\Phi}) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}), \tag{2.3.1}$$

where $K(\Phi, \bar{\Phi})$ and $W(\Phi)$ are the Kähler potential and the Superpotential, respectively. One can see that this Lagrangian is invariant under a (anti-)holomorphic transformation of the Kähler potential,

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + F(\Phi) + \bar{F}(\bar{\Phi}), \tag{2.3.2}$$

called a *Kähler transformation*. When expanded in components, only the Kähler potential contributes to a kinetic term to the scalar components,

$$\mathcal{L} \supset \frac{\partial^2 K}{\partial\phi^i \partial\bar{\phi}^{\bar{\jmath}}} d\phi^i d\bar{\phi}^{\bar{\jmath}}. \tag{2.3.3}$$

This term is the Lagrangian for a non-linear sigma model and can be seen as the line element of a manifold

$$ds^2 = g_{i\bar{\jmath}} d\phi^i d\bar{\phi}^{\bar{\jmath}}, \tag{2.3.4}$$

with the metric of the manifold given by

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial\phi^i \partial\bar{\phi}^{\bar{\jmath}}}. \tag{2.3.5}$$

If we require the kinetic term to come with the correct sign, we expect the metric to be positive definite, and to be nonsingular. Also, from the reality of the Lagrangian (2.3.1) the Kähler potential is hermitian, and so is the metric. A positive definite, nonsingular metric, that can be written as (2.3.5) defines a *Kähler metric*. A manifold which admits a Kähler metric is a *Kähler manifold*. Also notice that from (2.3.2) the Kähler potential does not define the metric completely, but only up to a (anti-)holomorphic translation.

It is instructive to see this statement in terms of the *Kähler form*, defined as,

$$J = i g_{i\bar{\jmath}} d\phi^i \wedge d\bar{\phi}^{\bar{\jmath}}. \tag{2.3.6}$$

If the metric is written as (2.3.5), this implies that the Kähler form is closed, since

$$\begin{aligned} dJ &= -i\frac{\partial g_{i\bar{\jmath}}}{\partial\phi^k} d\bar{\phi}^{\bar{\jmath}} \wedge d\phi^i \wedge d\phi^k + i\frac{\partial g_{i\bar{\jmath}}}{\partial\bar{\phi}^{\bar{k}}} d\phi^i \wedge d\bar{\phi}^{\bar{\jmath}} \wedge d\bar{\phi}^{\bar{k}} \\ &= -i\frac{\partial^3 K}{\partial\phi^k \partial\phi^i \partial\bar{\phi}^{\bar{\jmath}}} d\bar{\phi}^{\bar{\jmath}} \wedge d\phi^i \wedge d\phi^k + i\frac{\partial^3 K}{\partial\bar{\phi}^{\bar{k}} \partial\phi^i \partial\bar{\phi}^{\bar{\jmath}}} d\phi^i \wedge d\bar{\phi}^{\bar{\jmath}} \wedge d\bar{\phi}^{\bar{k}} \\ &= 0, \end{aligned} \tag{2.3.7}$$

where in the last step we used the symmetry of the derivatives and the antisymmetry of the wedge product. We then present an alternative definition of a Kähler manifold:

Definition: *A manifold is called Kähler the (1,1)-form J associated to the metric (i.e., the Kähler form) is closed.*

When the moduli manifold is compact, J cannot be exact, since the volume volume form of a compact n-complex dimensional Kähler manifold is proportional to J^n, and $J = d\alpha$ globally would imply a vanishing volume integral. Thus, in any compact Kähler manifold X, J is an element of the cohomology $H^{1,1}(X)$

2.4 Super Yang-Mills Action

As was discussed in section 2.2, the supersymmetric version of a field-strength for a vector field is given by the chiral superfield W_α, and the supersymmetric gauge transformation parameterized by the chiral superfield Λ is

$$W_\alpha \to e^{-i\Lambda} W_\alpha e^{i\Lambda}. \tag{2.4.1}$$

The most obvious way to construct a gauge invariant supersymmetric Lagrangian that reproduces the kinetic term for a vector field is via $\int d^2\theta \, \mathrm{tr}\, W_\alpha W^\alpha$. However, such a term in the Lagrangian is not real,

$$\int d^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha = \mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{1}{2}D^2\right], \tag{2.4.2}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

One way to construct a hermitian version of the vector superfield Lagrangian is to simply add the term $\bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}$, as was done in section 2.2. This definition however removes the topological theta term $iF_{\mu\nu}\tilde{F}^{\mu\nu}$. To construct a real version of the field strength Lagrangian that includes the gauge coupling g as well as a correct topological theta term, we introduce the complexified gauge coupling

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2}, \tag{2.4.3}$$

where θ_{YM} is the topological theta term, and define the Lagrangian (with the right coefficients adjusted),

$$\frac{1}{8\pi}\mathrm{Im}\left[\int d^2\theta\tau \, \mathrm{tr}\, W^\alpha W_\alpha\right] = \frac{1}{g^2}\mathrm{tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\nabla_\mu\bar{\lambda} + \frac{1}{2}D^2\right] - \frac{\theta_{YM}}{32\pi^2}\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{2.4.4}$$

By including charged chiral fields, one can write a more general kinetic term that replaces the constant coupling function τ with an holomorphic function of the chiral fields $f(\Phi)$

$$\mathcal{L} \supset \frac{1}{8\pi}\mathrm{Im}\left[\int d^2\theta f(\Phi) \, \mathrm{tr}\, W^\alpha W_\alpha\right]. \tag{2.4.5}$$

$f(\Phi)$ is called the *gauge kinetic coupling function.* In four dimensional effective actions arising from compactifying ten dimensional string theory theories, the coupling function for vector fields often depends on the fields that characterize the geometry of the compactification (internal) space.

In the particular configuration explored in chapter 3 the gauge fields are all Abelian, but there is still the possibility for mixings between different $U(1)$ fields,

$$\mathcal{L} \supset \frac{1}{8\pi}\mathrm{Im}\left[\int d^2\theta f_{AB}(\Phi) \, \mathrm{tr}\, W^{\alpha A}W^B_\alpha\right], \tag{2.4.6}$$

where A, B label the vector field. A non-diagonal f_{AB} matrix leads to kinetic mixings of different $U(1)$ vector bosons, that in turn may have interesting phenomenological consequences (for example, see [62, 63], where they explore the possibility of "Hidden Photons", massive vector bosons coupling to the Standard Model photon via kinetic mixing).

2.5 $\mathcal{N} = 1$ action and renormalization

We conclude this chapter by writing the bosonic part of a most general $\mathcal{N} = 1$ supersymmetric action in four dimensions in differential forms notation,

$$S = \int \Big(g_{i\bar{\jmath}}(\phi, \bar{\phi}) D_\mu \phi^i \wedge * \overline{D^\mu \phi^j} + \mathrm{Re}(f_{ab}) F^a \wedge * F^b + \mathrm{Im}(f_{ab}) F^a \wedge F^b - V_F - V_D \Big) \,. \qquad (2.5.1)$$

The quantities that define the theory, i.e. the *characteristic data*, are the spectrum (specified by the chiral and the vector multiplets), the superpotential that defines

$$V_F = g_{i\bar{\jmath}}(\phi, \bar{\phi}) D_i W \overline{D^{\bar{\jmath}} W}, \qquad (2.5.2)$$

the D-term potential V_D (that carries information on the charges of the scalars via the gauge groups, and how this can change the vacuum), the Kähler potential that locally specifies the metric

$$g_{i\bar{\jmath}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^{\bar{\jmath}}}, \qquad (2.5.3)$$

and the gauge coupling function f_{ab}, that usually is holomorphic on the chiral fields.

In general, the parameters of quantum field theories run with the energy scale, and the tracking of the running of all the couplings can become an annoyance as we change the scale we are working with. Supersymmetric theories however have powerful renormalization theorems that make things much easier and controlled.

It is possible to show via explicit loop calculations using supergraphs [64] that the superpotential is not renormalized in perturbation theory. There is also an alternative beautiful argument [65] that, motivated by string theory in which the coupling "constants" are actually dynamical fields, they consider the couplings of $\mathcal{N} = 1$ theories as background chiral fields with the same dimensions as the constants they are replacing. Then, they analise global symmetries that were explictly broken when the couplings were set to a fixed value, and that are restored when the couplings vev vanish. These broken global symmetries induce selection rules that tells what are the allowed corrections for the superpotential coming from loop contributions, leading to the conclusion that perturbatively the superpotential is given by its tree-level form. Corrections are however allowed non-perturbatively via instantons.

The (complexified) gauge coupling function (2.4.3) of super Yang-Mills is also constrained by $\mathcal{N} = 1$ supersymmetry. It can be shown [66] that it has perturbative couplings only at one-loop (calculated from the β-function), but non-perturbative corrections are present via instantons.

One of the main reasons behind the existence of such (non-)renormalization theorems for the superpotential and the gauge coupling come from the fact that these quantities are holomorphic/chiral, that strongly restricts the type of corrections allowed.

The Kähler potential, on the other hand, is not holomorphic, and there are no general $\mathcal{N} = 1$ results to infer restrictions on its loop corrections. However, in Calabi-Yau compactifications of effective actions of supersymmetry, the Kähler potential describing the complex structure deformations for type IIB theories gets no quantum corrections, that is, its classical description is exact! On the other hand, the Kähler potential that describes Kähler deformations (variations of the Kähler form) gets no perturbative corrections, only via (stringy) instantons. Mirror Symmetry maps both moduli spaces into one another, and therefore gives the powerful result [67] that the classical information of the one tells about the non-perturbative corrections of the other!

In the next chapter we study one particular scenario in the framework of (low energy) string theory, and one of the pursuits is to write the bosonic action in the standard form (2.5.1), and identify the $\mathcal{N} = 1$ characteristic data.

Chapter 3

Brane Effective Actions

In this chapter we start reviewing the effective action for ten-dimensional Type IIA supergravity when compactified to four dimensions. We choose the compactification space to be a six-dimensional Calabi-Yau manifold, an $SU(3)$ holonomy space in which a covariantly constant spinor can be globally defined, and each one of the two supercharges of $d = 10$ Type IIA theory has a component unbroken by the compactification. The Calabi-Yau compactification of Type IIA thus preserves 2 supercharges, so we obtain an $\mathcal{N} = 2$ supersymmetric theory in four dimensions.

To further reduce the supersymmetries of the four dimensional action to the more phenomenologically interesting $\mathcal{N} = 1$ we introduce on the Calabi-Yau threefold a discrete involution together with a reversing of the string orientation that projects out half of the degrees of freedom, and leaves just a symmetric combination of the two supercharges unbroken. The fixed point of the involution defines an Orientifold plane, and in the Type IIA case a O6-plane.

We next introduce a single D6-brane that fills completely the four dimensional spacetime and wraps a three-cycle in the Calabi-Yau space. To have unbroken supersymmetry in four dimensions, the three-cycle wrapped by the brane must be of a special kind, called *special Lagrangian*. The action of the D6-Brane is divided in two sectors: 1) the sector corresponding to fields living on the brane that come from open strings attached to the brane and generate super Yang-Mills fields and moduli from the brane deformations, 2) the Chern-Simons sector, that arises from the interaction between open and closed strings, respectively brane and bulk fields, and gives information on topological data and ("electric" and "magnetic" Ramond-Ramond charges of the D-brane).

The reduction to four dimensions leads to an $\mathcal{N} = 1$ supersymmetric theory of the form presented in section 2.5, and we find the characteristic data, together with the conditions on the brane deformations for unbroken supersymmetry. We also calculate corrections to the gauge coupling functions coming from mixings between the different gauge fields from different origins (open or closed strings), as well as corrections to the moduli space when we allow deformations of the Calabi-Yau manifold together with brane deformations.

Finally, we map via Mirror Symmetry [68] the moduli space for the D6-branes to the moduli space of D3-, D5- and D7-branes in the mirror type IIB geometry. We use the description of Mirror Symmetry described in [69], in which the Calabi-Yau is treated as a three-torus fibration over a three-(real)dimensional base, and Mirror Symmetry is simply T-duality along the three dimensions of the torus.

3.1 Calabi Yau compactification

To define our setup we start from ten dimensional type IIA supergravity, one of the two possible maximally supersymmetric gravity theories in ten dimensions, that arises from the low-energy effective description of (closed) Type IIA string theory (the second is Type IIB supergravity), and its bosonic field content consists of the NS-NS dilaton ϕ, the metric $g_{\mu\nu}$, the two-form $B_2^{(10)}$ and the R-R one- and three-forms C_1 and C_3. The bosonic part of the action reads [9]

$$
\begin{aligned}
S_{IIA}^{(10)} \;=\; & \int \frac{1}{2}R^{(10)} * \mathbf{1} - \frac{1}{4}d\phi^{(10)} \wedge *d\phi^{(10)} - \frac{1}{4}e^{-\phi^{(10)}} H_3^{(10)} \wedge *H_3^{(10)} \\
& -\frac{1}{2}e^{\frac{3}{2}\phi^{(10)}} G_2^{(10)} \wedge *G_2^{(10)} - \frac{1}{2}e^{\frac{1}{2}\phi^{(10)}} G_4^{(10)} \wedge *G_4^{(10)} - \frac{1}{2}B_2^{(10)} \wedge G_4^{(10)} \wedge G_4^{(10)},
\end{aligned} \tag{3.1.1}
$$

with the field strengths

$$H_3^{(10)} = dB_2^{(10)}, \quad G_2^{(10)} = dC_1^{(10)}, \quad F_4^{(10)} = dC_3^{(10)} - C_1^{(10)} \wedge H_3^{(10)}. \tag{3.1.2}$$

We next compactify the theory from ten to four dimensions, by requiring our ten dimensional space to be of the form $\mathcal{M}^{3,1} \times Y_3$, where $\mathcal{M}^{3,1}$ stands for a Minkowski four-dimensional spacetime and Y_3 is a Calabi-Yau threefold, that we will often refer as the *internal* space. This compactification leads to an $\mathcal{N} = 2$ action in four dimensions.

The metric splits thus in

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{i\bar{\jmath}} dy^i dy^{\bar{\jmath}}, \tag{3.1.3}$$

with $\eta_{\mu\nu}$ and $g_{i\bar{\jmath}}$ respectively a Minkowski and a Calabi-Yau metric.

The field content above splits in an internal and a 4d-spacetime part. As the internal space is compact, the field might become an infinite tower of Kaluza-Klein states. Usually in realistic compactification models we assume the compactification length scale to be small enough so that the Kaluza-Klein states are still beyond the energy scale of accelerators (otherwise we would have seen them already). A common compactification scale is around the GUT scale 10^{16}GeV[1] (although interesting phenomenology can also be obtained from Large Volume scenarios [71, 72]). The Kaluza-Klein states then are mostly irrelevant to the low-energy physics and we can consider only the massless modes. Additionally, the number of massless modes can be directly related to the number of independent harmonic forms on the internal space, and therefore to the cohomology of Y.

Zero mode deformations of the metric have three contributions, either coming from only spacetime deformations, only deformations of the Calabi-Yau space, or mixed deformations of the form $\delta g_{\mu i}$ or $\delta g_{\mu\bar{\imath}}$. The massless deformation modes could be decomposed in a basis of harmonic one-forms of the Calabi-Yau. However, a consequence of $SU(3)$ holonomy is the non-existence of harmonic one-forms in a Calabi-Yau manifold.

The Calabi-Yau deformations in turn can be decomposed in complex structure and Kähler deformations, arising respectively from deformations of the globally defined holomorphic 3-form Ω of the Calabi-Yau and from deformations of the Kähler form $J = i g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}$. In four dimensions, these deformations correspond to scalar fields, that locally define a moduli space of the splitted form

$$\mathcal{M}^{cs} \times \mathcal{M}^{ks}$$

Both factors are *special Kähler manifolds*. This type of manifold appear in general $\mathcal{N} = 2$ supergravity actions in four dimensions, and are defined as an n dimensional Hodge-Kähler manifold[2] in which a Kähler potential can be defined from a holomorphic section $\Omega \in \Gamma(\mathcal{SV} \otimes \mathcal{L}, M)$ (where $\mathcal{SV}$ is a simpletic vector bundle of rank $2n + 2$ and $\mathcal{L}$ a line bundle with $c_1(\mathcal{L}) = [J]$) as

$$K = -\ln\langle\Omega|\bar{\Omega}\rangle, \quad \text{with} \quad \langle\Omega|\partial_i\Omega\rangle = 0, \tag{3.1.4}$$

where $\langle\Omega|\bar{\Omega}\rangle$ is a symplectic inner product defined from the standard symplectic basis of $\mathcal{SV}$. Expanding Ω in the symplectic basis such that

$$\Omega = \begin{pmatrix} Z^{\hat{I}} \\ \mathcal{F}_{\hat{J}} \end{pmatrix}_{\hat{I},\hat{J}=0,1,\ldots,n}, \qquad K = -\ln i\left(Z^{\hat{I}}\bar{\mathcal{F}}_{\hat{I}} - \bar{Z}^{\hat{I}}\mathcal{F}_{\hat{I}}\right). \tag{3.1.5}$$

For the Calabi-Yau compactification, it turns out that the moduli spaces for the complex structure and Kähler deformations are precisely of such form [73, 74], as we will briefly describe (for a more

[1] A nice review on the energy scales involved in string phenomenology can be found in [70]

[2] that is, a Kähler manifold M with a line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = [J]$, where $[J]$ is the cohomology class of the Kähler form J.

detailed discussion, see [73,75–77]). The deformations of complex structure all lie inside the $H^{(1,2)}(Y)$ cohomology, and can be expanded in a basis of $(1,2)$ harmonic forms χ_K, with $K = 1, \dots, h^{(1,2)}$. The moduli space is an $h^{(1,2)}$-dimensional manifold with coordinates q,

$$ds^2 = G_{K\bar{L}} dq^K dq^{\bar{L}}, \quad \text{with} \quad G_{K\bar{L}} = \frac{\int_Y \chi_K \wedge \bar{\chi}_L}{\int_Y \Omega \wedge \bar{\Omega}}, \tag{3.1.6}$$

where Ω is the globally defined holomorphic (3,0)-form of the Calabi-Yau. It can also be shown that the metric $G_{K\bar{L}}$ can be written in terms of the Kähler potential

$$G_{K\bar{L}} = \frac{\partial^2 K^{cs}}{\partial q^K \partial \bar{q}^{\bar{L}}}, \quad K^{cs} = -\ln\left(i \int_Y \Omega \wedge \bar{\Omega}\right). \tag{3.1.7}$$

The complex structure deformations can also be described in terms of the deformation of $\Omega(q)$ via

$$\partial_{q^K} \Omega(q) = \chi_K + \Omega\, \partial_{q^K} K^{cs}. \tag{3.1.8}$$

The deformed holomorphic 3-form can be expanded in a complete basis of $H^3(Y)$, of dimension

$$\dim H^3(Y) = h^{(3,0)} + h^{(1,2)} + h^{(2,1)} + h^{(0,3)} = 2 + 2h^{(1,2)}, \tag{3.1.9}$$

using that in a Calabi-Yau $h^{(m,n)} = h^{(n,m)}$ and $h^{(3,0)} = h^{(0,3)} = 1$. One can introduce a symplectic basis of $H^3(Y)$, $(\alpha_{\hat{K}}, \beta^{\hat{L}})$, that satisfies[3]

$$\int_Y \alpha_{\hat{K}} \wedge *\beta^{\hat{L}} = -\int_Y \beta^{\hat{L}} \wedge *\alpha_{\hat{K}} = \delta^{\hat{L}}_{\hat{K}}, \qquad \hat{K}, \hat{L} = 0, \dots, h^{(1,2)}, \tag{3.1.10}$$

and expand the new holomorphic 3-form in this basis, by introducing the dual homology basis $(A^{\hat{K}}, B_{\hat{L}})$ of \$$H_3(Y)$ and defining the periods

$$Z^{\hat{K}} = \int_{A^{\hat{K}}} \Omega = \int_Y \Omega \wedge \beta^{\hat{K}}, \qquad \mathcal{F}_{\hat{L}} = \int_{B_{\hat{L}}} \Omega = \int_Y \Omega \wedge \alpha_{\hat{L}}, \tag{3.1.11}$$

we can write the deformed Ω as

$$\Omega(q) = Z^{\hat{K}}(q) \alpha_{\hat{K}} - \mathcal{F}_{\hat{K}}(q) \beta^{\hat{K}}. \tag{3.1.12}$$

The $\mathcal{F}_{\hat{K}}$ are not independent functions, but rather depend on $Z^{\hat{K}}$ and can be expressed as the derivative of a holomorphic prepotential $\mathcal{F}(Z)$,

$$\mathcal{F}_{\hat{K}} = \partial_{Z^{\hat{K}}} \mathcal{F}. \tag{3.1.13}$$

The expansion in the symplectic basis allows us to define a symplectic product and together with (3.1.8) and (3.1.7) reproduce (3.1.4). Notice also that $\Omega(q)$ is defined up to a rescaling $\Omega \to \Omega e^{-h(q)}$, since that translates into a Kähler transformation of K^{cs} leaving the metric $G_{K\bar{L}}$ unchanged. The rescaling defines a complex line bundle on the Calabi-Yau. The rescaling allow us to fix one of the $Z^{\hat{K}}$s, and in a particular patch[4] where $Z^0 \neq 0$ choose $Z^0 = 1$, and define $q^I = Z^I/Z^0$, making explicit the dependence on the moduli space coordinates q^I. With this redefinition, the Kähler potential can be put in the form

$$K^{cs} = -\ln i|Z^0| \left[2(f - \bar{f}) - (\partial_{q^K} f + \partial_{\bar{q}^K} \bar{f})(q^K - \bar{q}^K)\right], \quad \text{with} \quad \mathcal{F} = (Z^0)^2 f. \tag{3.1.14}$$

[3]the hat indicates that the index runs from 0 instead of 1.

[4]If Z^0 vanishes on this patch, just redefine another Z^I that is non-vanishing as the new Z^0.

The moduli space of Kähler structure deformations on the other hand is described by real fields v^A, from the expansion of the Kähler form $J = v^A w_A$. It does however get complexified by combining with the moduli b^A from the expansion of the two-form $B = b^A w_A$ via $J_c = B + iJ$, and defining $t^A = b^A + iv^A$. The complexified Kähler moduli t^i also can be written as the second derivative of a Kähler potential,

$$G_{A\bar{B}} = \frac{\partial^2 K^{ks}}{\partial t^A \partial \bar{t}^{\bar{B}}}, \quad K^{ks} = -\ln\left(\frac{i}{6}\mathcal{K}_{ABC}(t-\bar{t})^A(t-\bar{t})^B(t-\bar{t})^C\right), \tag{3.1.15}$$

where $\mathcal{K}_{ABC}$ is the triple intersection $\int \omega_A \wedge \omega_B \wedge \omega_C$, with the ω_A are basis elements for $H^{(1,1)}(Y)$. The Kähler potential can be re-expressed in terms of a holomorphic prepotential $f(t)$ as in (3.1.14),

$$K^{ks} = -\ln i\left[2(f-\bar{f}) - (\partial_{t^A} f + \partial_{\bar{t}^A}\bar{f})(t^A - \bar{t}^A)\right], \quad \text{where} \quad f = -\frac{1}{6}\mathcal{K}_{ABC}t^A t^B t^C. \tag{3.1.16}$$

It is important to point out that since the Kähler potential is described in terms of a chiral prepotential, it is also protected from perturbative loop corrections, receiving corrections only non-perturbatively.

We will not explore the complete $\mathcal{N} = 2$ effective action, since we are interested in the $\mathcal{N} = 1$ reduced case by the action of an Orientifold. For the reader who wants to read more on the $\mathcal{N} = 2$ action of type IIA, we suggest [78, 79] or the review in [80].

3.2 Type IIA Orientifold Compactification

We now introduce an orientifold projection on the Calabi-Yau, $\mathcal{O}$,

$$\mathcal{O} = (-1)^{F_L}\Omega_p \sigma^* \tag{3.2.1}$$

where Ω_p is the world-sheet parity reversal, F_L is the space-time fermion number in the left-moving sector, and σ is an anti-holomorphic and isometric involution of the compact Calabi-Yau manifold Y. If we require the projection $Y/\mathcal{O}$ to preserve $\mathcal{N} = 1$ supersymmetry, this implies [81, 82]

$$\sigma^* J = -J\ , \qquad \sigma^*\Omega = e^{2i\theta}\bar{\Omega}\ , \tag{3.2.2}$$

where θ is some real phase.

The four-dimensional scalars, vectors, two- and three-forms will arise in the expansions of the ten-dimensional fields into harmonic forms of Y which have to transform in a specified way under the orientifold parity to yield modes which remain in the orientifolded $\mathcal{N} = 1$ spectrum. More specifically, the ten-dimensional metric and the dilaton are invariant under the action of σ while the NS-NS B-field transforms as $\sigma^* B_2 = -B_2$. The R-R fields C_1, C_3, C_5, C_7 remain in the orientifold spectrum if they obey $\sigma^* C_p = (-1)^{(p+1)/2} C_p$. The R-R fields are however not all independent, as they obey an electric-magnetic duality,

$$G_{p+1} = (-1)^{(p+1)/2} *_{10} G_{9-p}\ , \tag{3.2.3}$$

where

$$G_2 = dC_1\ , \qquad G_{p+1} = dC_p - H_3 \wedge C_{p-2}\ , \qquad H_3 = dB_2\ . \tag{3.2.4}$$

with $G_{p+1}, p = 1...9$. One can use a democratic formulation of Type II supergravity [83] instead of the usual form (3.1.1) described[5] in section 3.1. The bosonic kinetic terms of the ten-dimensional action are then given by

$$S^{(10)}_{\text{dem}} = -\int \tfrac{1}{2}R *_{10} 1 + \tfrac{1}{4}H_3 \wedge *_{10}H_3 + \sum_{p=1}^{9}\tfrac{1}{8}G_{p+1} \wedge *_{10}G_{p+1}\ . \tag{3.2.5}$$

[5] We are also omitting the superscript (10) to avoid a dirty notation.

As the self duality condition of G_5, the duality conditions (3.2.3) do not arise from the equations of motion and have to be imposed by hand. When coupling the bulk supergravity to a D-brane it turns out to be useful to also introduce another basis $\mathcal{A}_q$ of q-forms with a redefined duality relation

$$\mathcal{A} = \sum_q \mathcal{A}_q = e^{-B_2} \wedge \sum_p C_p \,, \qquad d\mathcal{A}_q = (-1)^{(q+1)/2} (*_B \, d\mathcal{A})_q \,, \tag{3.2.6}$$

where the 'B-twisted' Hodge star is given by $*_B = e^{-B_2} *_{10} e^{B_2}$.

To perform the Kaluza-Klein expansion of the closed string fields we first decompose the de Rham cohomologies as even (denoted $H^n_+(Y)$)and odd (denoted $H^n_-(Y)$) cohomologies under the involution σ,

$$H^n(Y) = H^n_+(Y) \oplus H^n_-(Y). \tag{3.2.7}$$

From (3.2.2), we see that J is odd under the involution, and therefore can be expanded in a basis $\{\omega_a\}$ of $H^{(1,1)}_-(Y)$. The same happens for surviving components of the B_2 field, so the complexified holomorphic two-form J_c decomposes in a basis of $H^{(1,1)}_-(Y)$,

$$J_{\mathrm{c}} = B_2 + iJ = (b^a + iv^a)\,\omega_a = t^a \omega_a \,, \tag{3.2.8}$$

where $a = 1, \ldots, h^{(1,1)}_-$ labels a basis ω_a of $H^2_-(Y)$. We thus find the same complex structure as in the underlying $N = 2$ theory described in section 3.1 with the dimension of the Kähler moduli space truncated from $h^{(1,1)}$ to $h^{(1,1)}_-$.

To describe the three-form Ω, we first notice that $H^3(Y)$ splits as $H^3_+(Y) \oplus H^3_-(Y)$. Each component has dimension $\dim H^3_\pm(Y) = h^{(1,2)} + 1$. It is possible to write a symplectic basis $(\alpha_k, \beta^\lambda)$ that span H^3_+ and $(\alpha_\lambda, \beta^k)$ that span H^3_-. The intersections of the basis elements are

$$\int_Y \alpha_k \wedge \beta^l = \delta^l_k, \qquad \int_Y \alpha_\kappa \wedge \beta^\lambda = \delta^\lambda_\kappa, \tag{3.2.9}$$

and zero for all the others. We again split

$$\Omega = Z^k \alpha_k + Z^\lambda \alpha_\lambda - \mathcal{F}_\lambda \beta^\lambda - \mathcal{F}_k \beta^k \tag{3.2.10}$$

We introduce a complex 'compensator' $C \propto e^{-\phi + i\theta}$, as given in (3.2.16), that absorbs the phase of the orientifold action on Ω and contains the dilaton. Under the orientifold projection, we can easily see that

$$\mathrm{Im}(CZ^k) = \mathrm{Re}(C\mathcal{F}_k) = 0, \quad \mathrm{Re}(CZ^\lambda) = \mathrm{Im}(C\mathcal{F}_\lambda) = 0. \tag{3.2.11}$$

Thus,

$$C\Omega = \mathrm{Re}(CZ^k)\alpha_k - i\,\mathrm{Im}(CZ^\lambda)\alpha_\lambda - \mathrm{Re}(C\mathcal{F}_\lambda)\beta^\lambda - i\,\mathrm{Im}(C\mathcal{F}_k)\beta^k. \tag{3.2.12}$$

We then define a complexified three-form Ω_c that contains the degrees of freedom arising from the complex structure moduli, the dilaton as well as the scalars from the R-R forms. We combine these as

$$\Omega_c = 2\,\mathrm{Re}(C\Omega) + iC_3^{\mathrm{sc}} \;=\; N'^k \,\alpha_k - T'_\lambda \,\beta^\lambda \,, \tag{3.2.13}$$

where $k = 1, \ldots, n_-, \lambda = 1, \ldots, n_+$ label a basis $(\alpha_k, \beta^\lambda)$ of $H^3_+(Y, \mathbb{R})$. Here C_3^{sc} is the part R-R three-form which is also a three-form on the Calabi-Yau manifold Y and hence descents to scalars in four dimensions. We considered only the real part of $C\Omega$, since the imaginary part is redundant, and is connected to the real part via a Legendre transform (see the Appendix in [82]).

From the orientifold projection condition for the Ramond-Ramond fields $\sigma^* C_p = (-1)^{(p+1)/2} C_p$, we can also expand C_3^{sc} in the basis of $H^3_+(Y, \mathbb{R})$, $C_3^{\mathrm{sc}} = \xi^k \alpha_k + \tilde{\xi}_\lambda \beta^\lambda$. We thus find the explicit expressions

$$N'^k = 2\,\mathrm{Re}(CX^k) + i\,\xi^k \,, \qquad T'_\lambda = 2\,\mathrm{Re}(C\mathcal{F}_\lambda) + i\,\tilde{\xi}_\lambda \,. \tag{3.2.14}$$

Note that the split of the $h^{(2,1)}+1$ basis elements of $H^3_+(Y,\mathbb{R})$ into n_- elements α_k and n_+ elements β^λ does depend on the point in the complex structure moduli space on which one evaluates $C\Omega$. In fact, at the large complex structure point the precise split will determine whether this type IIA set-up is dual to an orientifold with O3/O7 planes or O5/O9 planes as we will discuss in detail in section 3.6. It is important to point out, that the complex coordinates (N'^k, T'_λ) are the correct complex scalars in the $\mathcal{N}=1$ chiral multiplets in the absence of D6-branes, but will receive corrections upon introducing dynamical D6-branes.

It is also interesting to note that the correct chiral coordinates now depend explicitly on $\mathcal{F}_\lambda$, while in the $\mathcal{N}=2$ case the coordinates were only Z^I, and the metric was encoded in a prepotential. This new fact is a consequence of the breakdown to $\mathcal{N}=1$, and the moduli space is not anymore special Kähler, although some structure of it can still be seen.

Before discussing the open string spectrum let us comment further on the complex function C appearing in (3.2.13). Since the orientifold projection is an anti-holomorphic involution the complex structure deformations will be real. In fact, C has a phase factor $e^{-i\theta}$ and is defined to compensate rescalings of Ω such that $C\Omega$ has a fixed normalization

$$e^{2\phi} C\Omega \wedge \overline{C\Omega} = \tfrac{1}{6} J \wedge J \wedge J \, . \tag{3.2.15}$$

It is convenient to introduce the four-dimensional dilaton D by setting $e^{-2D} = e^{-2\phi}\mathcal{V}$, where $\mathcal{V} = \frac{1}{6}\int_Y J \wedge J \wedge J$ is the string-frame volume of the Calabi-Yau space. The compensator field is then given by

$$C = e^{-D-i\theta} e^{K^{\rm cs}/2} = e^{-\phi-i\theta} \mathcal{V}^{1/2} e^{K^{\rm cs}/2}, \tag{3.2.16}$$

where $K^{\rm cs} = -\ln\big[-i\int \Omega\wedge\bar{\Omega}\big]$.

Let us note that the R-R three-form in general also leads to $U(1)$ vectors in four space-time dimensions via the expansion

$$C_3^{\rm vec} = A^\alpha \wedge \omega_\alpha \tag{3.2.17}$$

where ω_α is a basis of $H^2_+(Y,\mathbb{R})$. Their holomorphic gauge coupling function $f_{\alpha\beta}$ has also been determined in ref. [82]. Denoting by $\mathcal{K}_{\alpha\beta a} = \int_Y \omega_\alpha \wedge \omega_\beta \wedge \omega_a$, the intersection form of two elements of $H^2_-(Y,\mathbb{R})$ with one element of $H^2_+(Y,\mathbb{R})$ one finds that $f_{\alpha\beta} = i\mathcal{K}_{\alpha\beta a} t^a$.

3.3 The inclusion of a D6-brane

We have discussed up to now the degrees of freedom for the field theory on the bulk. In the string theory framework, the bulk theory consists of fields coming from closed strings. We next include a D6-brane in our setup. In subsection 3.3.1 we describe the conditions for unbroken supersymmetry and the spectrum coming from the massless modes of open strings attached to the brane. These modes decompose into longitudinal or normal modes to the brane, with the former corresponding to $U(1)$ vector fields living on the brane and the latter to modes that describe the geometrical deformations of the brane.

The deformations can be along the flat directions of the potential, preserving supersymmetry, or the brane can be deformed into configurations that are not anymore supersymmetric. The latter case contribute to a non-vanishing scalar potential, that we describe in subsection 3.3.2.

We conclude this section in 3.3.4 when we write the D6-brane action compactified to four dimensions. The result will not be yet in the standard $\mathcal{N}=1$ form (2.5.1). The identification of the $\mathcal{N}=1$ characteristic data will be discussed in the next sections, 3.4 and 3.5.

3.3.1 Open string sector: supersymmetric D6-branes

We want to include D6-branes in the background configuration such that they preserve the same supersymmetry as the O6-planes which arise as the fix-point set of the involution σ. In fact, since σ is an anti-holomorphic involution the O6-planes wrap special Lagrangian cycles satisfying

$$J|_{\text{O6-plane}} = 0 \, , \qquad \text{Im}(C\Omega)|_{\text{O6-plane}} = 0 \, . \tag{3.3.1}$$

Let us consider a single D6-brane wrapped on a three-cycle L in Y. We will consider the simple case where L is mapped under the orientifold map to a three-cycle $L' = \sigma(L)$ which is in a different cohomology class and does not intersect L.[6] For this situation the pair of the D6-brane and its image D6-brane is merely an auxiliary description of a single smooth D6-brane wrapping a cycle in the orientifold $Y/\mathcal{O}$. Note that the number of D6-branes is restricted by tadpole cancellation. In cohomology one has to satisfy[7]

$$\sum_{\text{D6}} [L + L'] = 4[L_{\text{O6}}] \, , \tag{3.3.2}$$

where the sum is over all D6-branes present in the compactification and L_{O6} is the fix-point set of the involution indicating the location of the O6-plane.

Supersymmetry implies that the D6-brane has to wrap a calibrated (i.e. minimal volume) cycle. These calibration conditions have been determined in [85], and they imply that the D6-brane must wrap a special Lagrangian submanifold $L_0 \subset Y$,

$$J|_{L_0} = 0 \, , \qquad \text{Im}(C\Omega)|_{L_0} = 0 \, , \qquad 2\,\text{Re}(C\Omega)|_{L_0} = e^{-\phi}\text{vol}_{L_0} \tag{3.3.3}$$

where $\text{vol}_{L_0} = \sqrt{\iota^* g_6}\, d^3\xi$ is the induced volume form on L. Note that the first condition in (3.3.3) implies that L_0 is Lagrangian, while the second condition makes it special Lagrangian. We fixed the coefficient, in particular the phase of $C\Omega$, such that the same supersymmetry is preserved as for the orientifold planes (3.3.1). The third equation is simply the calibration condition. Finally, we note that it was also shown in [86] that in a supersymmetric background one has

$$F_{\text{D6}} - B_2|_{L_0} = 0 \, , \tag{3.3.4}$$

where F_{D6} is the field strength of the $U(1)$ gauge field A living on the D6-brane. In the following we will always denote the background special Lagrangian cycle wrapped by a supersymmetric D6-brane by L_0.

For a *fixed* background complex and Kähler structure we can discuss supersymmetric deformations of the D6-branes. In fact, the deformations of L_0 preserving the special Lagrangian conditions (3.3.3) were studied by McLean [87]. When we deform a compact special Lagrangian cycle L_0 to L_η passing through a continuous family of cycles (not necessarily special Lagrangian), we can associate the deformation to a vector field η normal to L_0 (as figure 3.1 in page 35). The deformation is however performed only through special lagrangian cycles if and only if the one-form defined as $\theta_\eta = \eta \lrcorner J$ is harmonic. In other words, non-harmonic θ_η will correspond to deformations breaking the special Lagrangian conditions, and therefore breaking supersymmetry. This restriction to harmonic forms reduces the infinite dimensional space of maps from L_0 to L_η to a deformation space of dimension $b^1(L_0) = \dim H^1(L_0, \mathbb{R})$. Furthermore, there are no obstructions to extending an infinitesimal deformation to a finite deformation. The tangent space to such deformations can be identified through the cohomology class of the harmonic form with $H^1(L_0, \mathbb{R})$. We can thus write a basis of harmonic one-forms θ_i on L_0 as

$$\theta_i = s_i \lrcorner J|_{L_0} \, , \qquad *\theta_i = -2e^{\phi} s_i \lrcorner \text{Im}(C\Omega)|_{L_0} \, , \qquad i = 1, \dots, b^1(L_0) \, , \tag{3.3.5}$$

where s_i is a basis of the real special Lagrangian normal deformations. Let us recall the derivation of the expression for $*\theta_i$ [88]. We do this more generally, by determining the Hodge-dual of a one form

[6] This is a non-generic situation for a three-cycle in a six-dimensional manifold. Generically D6-branes on three-cycles will intersect in points. At these intersections matter fields can be localized and have to be included in the reduction.

[7] This condition will be modified in the presence of NS-NS background flux H_3 and the Romans mass parameter m^0 with an additional term proportional to $m^0 H_3$ (see, e.g. , ref. [84]).

$\alpha = (X \lrcorner J)|_{L_0}$ for some $X \in TY|_{L_0}$. Note that the vector dual to α by raising the index with the induced metric is IX where I is the complex structure on Y. Hence one checks

$$* (X \lrcorner J)|_{L_0} = (IX) \lrcorner \, \mathrm{vol}_{L_0} \ . \tag{3.3.6}$$

However, on L_0 the volume form is identical to $2e^{\phi}\mathrm{Re}(C\Omega)$ by (3.3.3). This implies

$$* (X \lrcorner J)|_{L_0} = 2e^{\phi}(IX \lrcorner \mathrm{Re}(C\Omega))|_{L_0} = -2e^{\phi}(X \lrcorner \mathrm{Im}(C\Omega))|_{L_0} \tag{3.3.7}$$

where the minus sign arises from evaluating I on the $(3,0)$-form Ω, $(IX) \lrcorner \Omega = iX \lrcorner \Omega$.

We have just introduced the general supersymmetric deformation encoded by $b^1(L_0)$ scalars η^i arising in the expansion $\theta_\eta = \eta^i \theta_i$ of the harmonic form θ_η. The $\eta^i(x)$ will be real scalar fields in the four-dimensional effective theory depending on the four space-time coordinates x. Let us next discuss the degrees of freedom due to $U(1)$ Wilson lines arising from non-trivial one-cycles on the D6-brane world-volume. Later on we will show that these real scalars will complexify the η^i, that is, the supersymmetric chiral coordinates will be a combination of brane deformations and Wilson line scalars. The latter arise in the expansion of the $U(1)$ gauge boson A_{D6} on the D6-brane as

$$A_{\mathrm{D6}} = A + a^i \, \tilde{\alpha}_i \ , \tag{3.3.8}$$

where A is a $U(1)$ gauge field and the $a^i(x)$ are $b^1(L_0)$ real scalars in four dimensions. The forms $\tilde{\alpha}_i$ provide a basis of $H^1(L_0)$. Note that in general the $U(1)$ field strength $F_{\mathrm{D6}} = dA_{\mathrm{D6}}$ can additionally admit a background flux $\langle F_{\mathrm{D6}} \rangle = f_{\mathrm{D6}}$ in $H^2(L_0, \mathbb{Z})$, which can be trivial or non-trivial in $H^2(Y, \mathbb{R})$. Since we will focus on the kinetic terms in the following we will set $f_{\mathrm{D6}} = 0$ for most of the discussion. Note that F_{D6} naturally combines with the NS–NS B-field into the combination $F_{\mathrm{D6}} - \iota^* B_2$.

To summarize, one finds as massless variations around a supersymmetric vacuum $h_-^{(1,1)} + h^{(2,1)} + 1$ chiral multiplets from the bulk and $b^1(L_0)$ chiral multiplets (η^i, a^i) from the D6-brane. The precise organization of these fields into $\mathcal{N} = 1$ complex coordinates is postponed to section 3.4.

3.3.2 General deformations of D6-branes

So far we have discussed the supersymmetric background D6-brane and its also supersymmetric deformations. However, in general L_0 admits an infinite set of deformations which will render the deformed D6-brane non-supersymmetric. These deformations will be included in the following and shown to be obstructed by a scalar potential. In order to do that, one recalls that the string-frame world-volume action for the D6-brane takes the form [8, 9, 89]

$$S_{\mathrm{D6}}^{\mathrm{SF}} = -\int_{\mathcal{W}_7} d^7\xi e^{-\phi} \sqrt{-\det\left(\iota^*\left(g_{10} + B_2\right) - F_{\mathrm{D6}}\right)} + \int_{\mathcal{W}_7} \sum_{q \text{ odd}} \iota^*(C_q) \wedge e^{F_{\mathrm{D6}} - \iota^*(B_2)} \ . \tag{3.3.9}$$

The first term of (3.3.9), the Dirac-Born-Infeld (DBI) action, can be understood very roughly as the stringy generalization of the action of an object wiping a seven dimensional worldvolume. A free-moving point particle, for example, has a Lagrangian $\mathcal{L} = \sqrt{-\iota^* g}$, where the integral is along a world line. ι is the pull back of the ambient metric to the world line (or for the case of the brane, the world-volume). The movement of the brane wiping the worldvolume is described by open strings with momentum normal to the brane, thus obeying Dirichlet boundary conditions along its momentum. Open strings are also responsible for introducing a $U(1)$ gauge field along the brane, coming from strings with Neumann boundary conditions, i.e., moving freely along the brane tangent directions.

A moving D0-brane is a relativistic point particle with three-dimensional velocity $\vec{v}$ and has a kinetic Lagrangian given in Minkowski space by

$$L_{\mathrm{pp}} = mc^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}}, \tag{3.3.10}$$

In the T-dual picture, when Dirichlet and Neumann boundary conditions are exchanged, the D0-brane becomes a D1-brane with a $U(1)$ field along the T-dualized direction. Just as the D0-brane had a velocity bounded by $v < 1$, the D1-brane also has a bounded electromagnetic field, described by the Born-Infeld action[8],

$$L_{\rm B} = b\sqrt{1 - (\vec{E}^2)/b^2}, \tag{3.3.11}$$

where b is the bound $(|\vec{E}|) < b$. The argument can be repeated for a D-brane of any dimension, and in more general spacetime backgrounds, leading to the Born-Infeld action

$$S_{\rm BI} = \int \sqrt{-\det(g_{mn} + F_{mn})}, \tag{3.3.12}$$

This Lagrangian however is not completely gauge invariant. In the string worldsheet action, There is a gauge transformation given by

$$\delta B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu\,, \quad \delta A_\mu = -\zeta_\mu, \tag{3.3.13}$$

under which $F_{\mu\nu}$ is not gauge invariant, but the combination $B_2 + F$ is. Therefore, the correct gauge invariant action for a Dp-brane is

$$S_{\rm DBI} = -\int_{\mathcal{W}_{p+1}} d^{p+1}\xi e^{-\phi}\sqrt{-\det\left(\iota^*\left(g_{10} + B_2\right) - F\right)}. \tag{3.3.14}$$

The second term in (3.3.9) is the Chern Simons action, that gives information on the Ramond-Ramond charge of the D-brane. $U(1)$ fields on the brane also couple to RR fields, and as before, gauge invariance imply the appearance of B_2.

In this subsection we calculate the scalar potential that arises when we study general deformations of special Lagrangian cycles, as was performed in [87]. The scalar potential corresponds thus to the obstruction for the deformed cycle to be special Lagrangian, and therefore supersymmetric. This scalar potential will appear in the reduction of the first term in (3.3.9) (the spacetime geometry term). In section 3.5 we show how this scalar potential descends from a superpotential, as required in supersymmetric theories.

Exponential map and normal coordinate expansion

A general fluctuation of L_0 to a nearby three-cycle L_η is described by real sections η of the normal bundle NL_0. Clearly, the space of such sections is infinite dimensional as is the space of all L_η. We can understand the deformation of L_0 to a neighboring L_η as a diffeomorphism mapping each point p on L_0 with a normal vector $\eta(p)$ to a point in L_η through a geodesics, given by the exponential map $\exp_\eta(p)$. We can construct then a vector field η for the deformations living in the normal vector bundle of L_0. We should also know how to describe the pullback of a bulk field onto L_η. In particular, we will be very interested in the pull back of the forms J and $\mathrm{Im}(C\Omega)$ that give us information on whether the cycle it is pulled back to is special Lagrangian or not, as in (3.3.3).

We then define the pullback of the exponential map $\exp_\eta$,

$$E_\eta(\gamma) = \exp^*_\eta(\gamma|_{L_\eta})\;, \tag{3.3.15}$$

where $\eta \in NL_0$, and $\gamma \in \Omega^p(Y)$ are p-forms on Y. E_η pulls back γ from L_η to a p-form $E_\eta(\gamma) \in \Omega^p(L_0)$ on L_0.

[8] Initially this proposal was made to try to solve the divergent electromagnetic field problem in classical electrodynamics [90, 91]. The electric field of a point charge would not be infinite, and in the limit $b \to \infty$ the Lagrangian gives the classical Maxwell theory.

It was shown in [87] that the pullback $E_\eta(J)$ and $E_\eta\big(C\Omega\big)$ are exact forms on L_0[9], that is, it is possible to find a 1-form $\hat\mu_1$ and a 2-form $\hat\mu_2$ such that

$$E_\eta(J) = d\hat\mu_1 \ , \qquad E_\eta\big(\mathrm{Im}(C\Omega)\big) = d\hat\mu_2 \ . \tag{3.3.16}$$

In order to study special Lagrangian deformations as in section 3.3.1 one thus has to consider the space of deformations $\eta_{\rm sp}$ such that $E_{\eta_{\rm sp}}(J) = 0$ and $E_{\eta_{\rm sp}}(\mathrm{Im}(C\Omega)) = 0$ [87], that is, μ_1 and μ_2 have to be closed forms for the mapping between special lagrangian cycles.

We can find easily explicit expressions for the deformations of J and $\mathrm{Im}(C\Omega)$ in the particular case of small (first order) deformations. We introduce a real parameter t to the deformation such that, for small deformations, $E'_\eta(\gamma) := \partial_t E_{t\eta}(\gamma)|_{t=0}$. A straightforward computation shows that for any closed for γ on Y one has

$$d\gamma = 0: \qquad E'_\eta(\gamma) = \mathcal{L}_\eta(\gamma)|_{L_0} = d(\eta\lrcorner\gamma)|_{L_0} \ . \tag{3.3.17}$$

Here we have used the standard formula for the Lie derivative on a form $\mathcal{L}_\eta\gamma = d(\eta\lrcorner\gamma) + \eta\lrcorner d\gamma$. Note that (3.3.17) immediately implies that

$$E'_\eta(J) = d\theta_\eta \ , \qquad E'_\eta(\mathrm{Im}(C\Omega)) = -2e^\phi d * \theta_\eta \ . \tag{3.3.18}$$

where $\theta_\eta = \eta\lrcorner J|_{L_0}$ and we have again used the fact that $*\theta_\eta = 2e^\phi \eta\lrcorner \mathrm{Im}C\Omega|_{L_0}$ as in (3.3.5). One can proceed with the expansion of the exponential map and determine the full normal coordinate expansion. In particular, for a p-form one finds the small t expansion

$$\begin{aligned} E_{t\eta}(C_p) \;=\; & \tfrac{1}{p!}\Big[C_{i_1\ldots i_p} + t\cdot\Big(\eta^n\partial_n C_{i_1\ldots i_p} - p\nabla_{i_1}\eta^n C_{ni_2\ldots i_p}\Big) \\ & + \tfrac{1}{2}t^2\cdot\Big(\eta^n\partial_n(\eta^m\partial_m C_{i_1\ldots i_p})p\nabla_{i_1}\eta^n\eta^m\partial_m C_{ni_2\ldots i_p} - \tfrac{p(p-1)}{2}\nabla_{i_1}\eta^n\nabla_{i_2}\eta^m C_{nmi_3\ldots i_p} \\ & + \tfrac{p-2}{2}R^j_{ni_1m}\eta^n\eta^m C_{ji_2\ldots i_p}\Big) + \mathcal{O}(t^3)\ \Big]\ d\xi^{i_1}\wedge\ldots\wedge d\xi^{i_p}. \end{aligned} \tag{3.3.19}$$

Such normal coordinate expansions have been used for D-branes of different dimensions, for example, in refs. [27, 92, 93].

The scalar potential for Lagrangian deformations

We can use the procedure above to describe how the volume form changes as we move from L_0 to L_η. At any L, we define the volume functional

$$V(L) = \int_L d^3\xi\, e^{-\phi}\sqrt{\det(\iota^* g)} = \int_L e^{-\phi}\,\mathrm{vol}_L \ . \tag{3.3.20}$$

The pullback ι is the same one described in the D-brane action, (3.3.9), that pulls the metric on the Calabi-Yau manifold onto the cycle L. Equation (3.3.20) is the geometrical part of the DBI action after compactification, that will lead to a scalar potential in four dimensions. This scalar potential will act as en obstruction to deformations of the brane that break the special Lagrangian condition, and therefore supersymmetry.

The idea will be to introduce again the real parameter t and calculate the deformation of $V(L_\eta)$. If the deformation is small, we can expand around L_0,

$$V(L_\eta) = V(L_0) + t\frac{dV}{dt}(L_0) + \frac{1}{2}t^2\frac{d^2V}{dt^2}(L_0) + \ldots . \tag{3.3.21}$$

[9]This can be deduced from the fact that J and $\mathrm{Im}(C\Omega)$ are closed and since L_0 and L_η are in the same cohomology class, one has in cohomology that $[E_\eta(\gamma)] = [\gamma|_{L_0}]$.

Also, the special Lagrangian manifold L_0 is by definition a calibrated manifold, that is, among all the manifolds in the same cohomology class, the calibrated ones are the ones with the minimal volume. That is, when deforming L_0, since it is a minimum of volume, the first derivative dV/dt evaluated on L_0 vanishes. So the first contribution to δV comes from the second derivative in (3.3.21), that we will calculate in the following.

The first case we will consider is when L_η is still Lagrangian, but not necessarily special. This implies that $\theta_\eta = \eta\lrcorner J$ is still closed, but the volume form now also has a contribution from the imaginary part of Ω. Actually, as the brane can also have a phase $\theta_{\rm D6}(\xi)$ different from the phase of the orientifold plane and dependent on the coordinates on the brane, we also define a compensator for the brane $C_{\rm D6}$ with which we can write

$$e^{-\phi}\mathrm{vol}_{L_\eta} = 2C_{\rm D6}\Omega|_{L_\eta} \ , \quad \text{with} \quad C_{\rm D6}(\xi) = |C|e^{-i\theta_{\rm D6}(\xi)} \ . \tag{3.3.22}$$

We can then calculate the first derivative of the volume,

$$\frac{d}{dt}(e^{-\phi+i\theta_{\rm D6}}\,\mathrm{vol}_L) = (\mathcal{L}_\eta|C|\Omega)|_L = e^{-\phi+i\theta_{\rm D6}}(id\theta_{\rm D6}\wedge\eta\lrcorner\,\mathrm{vol}_L + d(\eta\lrcorner\,\mathrm{vol}_L)\ , \tag{3.3.23}$$

where again $\mathcal{L}_\eta$ is the Lie derivative, $\mathcal{L}_\eta(\gamma) = d\eta\lrcorner\gamma + \eta\lrcorner d\gamma + \dots$ for any form γ, and we used the fact that vol_L is a closed form. As we will at the end evaluate this expressions on L_0 where we had $*\theta_\eta = \eta\lrcorner\,\mathrm{Im}(C\Omega)$, we can call already

$$\eta\lrcorner\,\mathrm{vol}_L = id*\theta_\eta, \tag{3.3.24}$$

that agrees with the usual result since at L_0 the normal vector field η is normal to $\mathrm{vol}_{L_0} \sim 2\,\mathrm{Re}\,C\Omega$, so $\eta\lrcorner\,\mathrm{Re}\,C\Omega$ vanishes there. Additionally, we can write the last term on (3.3.23) as $*d(\eta\lrcorner\,\mathrm{vol}_L)\,\mathrm{vol}_L$, where we used that, on the L cycle, the hodge dual $*$ imples $*1 = \mathrm{vol}_L$. We then calculate the derivative in the left-hand side of (3.3.23), and match imaginary and real terms,

$$\frac{d}{dt}\theta_{\rm D6} = -d^*\theta_\eta\ , \qquad \frac{d}{dt}\,\mathrm{vol}_L = -d\theta_{\rm D6}\wedge *\theta_\eta\ , \tag{3.3.25}$$

Note that a particularly interesting case is when $\theta_\eta = d\theta_{\rm D6}$, since in this case the second equation ensures that the volume of L is decreasing along this direction. In fact, this normal vector precisely parameterizes the directions to L in which its volume is most efficiently decreasing. This vector is known as mean curvature vector. Such Lagrangian mean curvature flows have been discussed intensively in the mathematical literature (see, e.g., refs. [94, 95], and references therein).

We now proceed to calculate the second derivative from (3.3.25) and evaluate at $t = 0$, where $L(t = 0) = L_0$,

$$\frac{d}{dt}\,\mathrm{vol}_L\,|_{t=0} = 0\ , \qquad \frac{d^2}{dt^2}\,\mathrm{vol}_L\,|_{t=0} = (dd^*\theta_\eta)\wedge *\theta_\eta\ . \tag{3.3.26}$$

In this computation it is crucial to use the fact that at $t = 0$ one has $\theta_{D6}(0) = \theta_{\rm O6}$ is constant on L_0. This immediately implies the vanishing of the first derivative of vol_L using (3.3.25). To evaluate the second derivative both equations (3.3.25) have to be applied successively. Finally, we can use (3.3.26) to evaluate the lowest order scalar potential for a Lagrangian brane on $L(t)$ as

$$\frac{d^2}{dt^2}V(L_{t\eta})|_{t=0} = e^{-\phi}\int_{L_0} d*\theta_\eta\wedge *d*\theta_\eta = 4e^{\phi}\int_{L_0} d(\eta\lrcorner\mathrm{Im}C\Omega)\wedge *d(\eta\lrcorner\mathrm{Im}C\Omega)\ , \tag{3.3.27}$$

where V is the volume functional (3.3.20). As we will show later on, this term provides a scalar potential which corresponds to a D-term in the four-dimensional $\mathcal{N} = 1$ effective theory for the D6-brane.

The scalar potential for general deformations

Before turning to the details of the Kaluza-Klein reduction let us recall that one can extend the analysis to deformations η for which $L(t)$ is no longer Lagrangian. In this case $d\eta \lrcorner J$ does not necessarily vanish and (3.3.22) is not generally possible. However, one can still evaluate the second derivative of the volume of $L(t)$ at the point $t = 0$ as [87]

$$\frac{d^2}{dt^2} V(L_{t\eta})|_{t=0} = e^{-\phi} \int_{L_0} d(\eta \lrcorner J) \wedge * d(\eta \lrcorner J) + 4e^{\phi} \int_{L_0} d(\eta \lrcorner \mathrm{Im} C\Omega) \wedge * d(\eta \lrcorner \mathrm{Im} C\Omega) \,. \qquad (3.3.28)$$

The new term depending on $d(\eta \lrcorner J)$ is the obstruction for $L(t)$ to be Lagrangian. In the four-dimensional $\mathcal{N} = 1$ effective theory for the D6-brane this term can be obtained as one of the F-term contributions from a superpotential which we determine in section 3.5.

The scalar potential including the B-field

So far we have discussed the scalar potential without the inclusion of the NS-NS B-field of Type IIA string theory and the brane field strength F_{D6}. To compute the leading order potential including F_{D6} we note that only the part $\tilde{F}$ of F_{D6} contributes to the potential which has indices on the internal three-cycle wrapped by the brane. We perform a Taylor expansion of the Dirac-Born-Infeld action using

$$\sqrt{\det(\mathfrak{A} + \mathfrak{B})} = \sqrt{\det(\mathfrak{A})} \Big[\mathbf{1} + \tfrac{1}{2} \mathrm{Tr}(\mathfrak{A}^{-1}\mathfrak{B}) + \tfrac{1}{8} \Big(\big[\mathrm{Tr}(\mathfrak{A}^{-1}\mathfrak{B})\big]^2 - 2\mathrm{Tr}\big([\mathfrak{A}^{-1}\mathfrak{B}]^2\big) \Big) + \ldots \Big] \qquad (3.3.29)$$

for small fluctuations $\mathfrak{B}$ and invertible $\mathfrak{A}$. The matrix $\mathfrak{B}$ we want to identify with the normal coordinate expansion of $B_2 - \tilde{F}$ in (3.3.9), while $\mathfrak{A}$ is the background metric of the Calabi-Yau space restricted to L_0. Recall that the normal coordinate expansion $E_{t\eta}(B_2)$ was given in (3.3.19). One notes that the first term in the expansion (3.3.29) is canceled by tadpole cancellation of the D6-branes with the O6-planes in the background. Moreover, the second and third term in (3.3.29) do not contribute to the potential since $\mathfrak{A}$ is symmetric while $\mathfrak{B}$ is anti-symmetric. Evaluating the remaining term $\mathrm{Tr}\big([\mathfrak{A}^{-1}\mathfrak{B}]^2\big)$ and adding the result (3.3.28) one finds

$$V^{\mathrm{SF}}_{\mathrm{DBI}} = e^{-\phi} \int_{L_0} \big[d * \theta_\eta \wedge * d * \theta_\eta + d\theta_\eta \wedge * d\theta_\eta + (\tilde{F} - B_2 - d\theta^B_\eta) \wedge * (\tilde{F} - B_2 - d\theta^B_\eta) \big] \,, \qquad (3.3.30)$$

which is still expressed in the ten-dimensional string frame. Here we have introduced the abbreviation

$$\theta^B_\eta = \eta \lrcorner B_2|_{L_0} \,, \qquad (3.3.31)$$

which is the B-field analog of $\theta_\eta = \eta \lrcorner J|_{L_0}$. This concludes the computation of the scalar potential from the Dirac-Born-Infeld action. In a next step we want to introduce a Kaluza-Klein basis and determine the complete leading order effective action including the kinetic terms.

3.3.3 A Kaluza-Klein basis

In performing a Kaluza-Klein reduction of the D6-brane action to four spacetime dimensions we would like to include all massive modes corresponding to arbitrary deformations of L_0 to L_η. This means that we include sections s_I of NL_0 which yield one-forms in the contraction with J

$$\theta_I = s_I \lrcorner J|_{L_0} \quad \in \quad \Omega^1(L_0) \,. \qquad (3.3.32)$$

For a compact L_0 it is possible to label these one-forms by indices $I = 1, \ldots, \infty$ by considering the Kaluza-Klein eigenmodes of the Laplacian Δ_{L_0}. In this case the zero modes $\Delta_{L_0}\theta_i = 0$ are precisely the harmonic forms θ_i introduced in (3.3.5). However, the basis adopted to Δ_{L_0} is not always useful, since it explicitly depends on the metric inherited form the ambient Calabi-Yau manifold. In the following we will therefore work with a general countable basis of $\Omega^1(L_0)$, and later use the induced metric to interpret the final expressions after performing the reduction. In general we will always demand that the one-forms θ_I are finite in the L^2-metric

$$\mathcal{G}(\tilde{\alpha}, \tilde{\beta}) = \int_{L_0} \tilde{\alpha} \wedge * \tilde{\beta} \ , \tag{3.3.33}$$

where $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(L_0)$.

Let us now turn to the discussion of the $U(1)$ gauge field on the D6-brane. It admits the general expansion

$$A_{\text{D6}} = A^J \, h_J + a^I \, \hat{\alpha}_I \ , \tag{3.3.34}$$

where $h_J \in C^\infty(L_0)$ is a basis of functions on L_0 and $\hat{\alpha}_J \in \Omega^1(L_0)$ is a basis of one-forms on L_0. Here again a countable basis can be chosen due to the compactness of L_0. Note that the field-strength of A_{D6} is given by

$$F_{\text{D6}} = F^J \, h_J - A^J \wedge dh_J + da^I \wedge \hat{\alpha}_I + \tilde{F} \ , \qquad \tilde{F} = a^I \, d\hat{\alpha}_I + f_{\text{D6}} \ , \tag{3.3.35}$$

where $f_{\text{D6}} \in H^2(L_0, \mathbb{Z})$ is a background flux of F_{D6} on L_0. The terms dh_J and $d\hat{\alpha}_I$ arise due to the fact that the functions h_J need not to be constant on L_0 and the one-forms $\hat{\alpha}_I$ need not to be closed.

We thus find that an infinite tower of scalars a^I which are coefficients of *exact* forms are actually gauged by the gauge fields A^J for which $dh_J \neq 0$. Moreover, scalars a^I arising in the expansion in *non-closed* forms appear without four-dimensional derivative in the expansion (3.3.35). To see this, we introduce a special basis adopted to the metric induced on L_0. More precisely, via the Hodge decomposition each one-form $\hat{\alpha}_I$ can be uniquely decomposed into a harmonic form, an exact form $d\hat{h}_I$ and an co-exact form $d^*\hat{\gamma}_I$ on L_0 as

$$\hat{\alpha}_I = \mu_I^i \tilde{\alpha}_i + d\hat{h}_I + d^*\hat{\gamma}_I \ , \tag{3.3.36}$$

where $\tilde{\alpha}_i$ are the $b^1(L_0)$ harmonic forms introduced in (3.3.8). We thus pick a basis of the space of exact forms $\Omega^1_{\text{ex}}(L_0)$ denoted by dh_I and a basis $d^*\gamma_I$ of the space $\Omega^1_{\text{co-ex}}(L_0)$ which are exact with respect to d^*. By appropriate redefinition we can introduce scalars $\hat{a}^I$ parameterizing the expansion in dh_I. Denoting the coefficients of the non-closed forms $d^*\gamma_I$ by $\tilde{a}^I$, and the coefficients of the harmonic forms by a^j the expansion (3.3.35) reads

$$\begin{aligned} F_{\text{D6}} &= F^I \, h_I + da^j \wedge \tilde{\alpha}_j + \mathcal{D}\hat{a}^I \wedge dh_I + d\tilde{a}^I \wedge d^*\gamma_I + \tilde{F} \ , \\ \mathcal{D}\hat{a}^I &= d\hat{a}^I - A^I \ , \qquad \tilde{F} = \tilde{a}^I dd^*\gamma_I + f_{\text{D6}} \ . \end{aligned} \tag{3.3.37}$$

From this we conclude that precisely the scalars $\hat{a}^I$ are gauged by A^I. Since the four-dimensional effective theory is an $\mathcal{N} = 1$ supersymmetric theory one infers that there will be D-terms induced due to these gaugings $\mathcal{D}\hat{a}^I$, while F-terms are induced due to $\tilde{F}$. We will determine the D-term in section 3.5, and check that it matches the moment map analysis of ref. [96].

3.3.4 The four-dimensional effective action

We can now determine the kinetic terms for the chiral multiplets of the D6-brane coupled to the bulk supergravity. Since the bulk action has been Kaluza-Klein reduced on the orientifold background in ref. [82] we will focus on the reduction of the D6-brane action (3.3.9). The contributions entirely due to bulk fields are later included in the determination of the $\mathcal{N} = 1$ characteristic data.

Dirac-Born-Infeld action

Let us start by considering the Kaluza-Klein reduction of the first term in (3.3.9), i.e. the Dirac-Born-Infeld action. We expand the determinant in (3.3.9) to quadratic order in the fluctuations around the supersymmetric background. These are precisely the fluctuations of the embedding ι of L parameterized by the fields η^i of (3.3.5) and the Wilson line scalars a^i introduced in (3.3.8). The normal coordinate expansions of the ten-dimensional metric on the D6-brane world-volume is given to leading order by

$$\iota^* g_{10} = \left(e^{2D}\eta_{\mu\nu} + g(\partial_\mu \eta, \partial_\nu \eta)\right) dx^\mu \cdot dx^\nu + (\iota^* g + \delta(\iota^* g))_{mn} d\xi^m \cdot d\xi^n \ , \tag{3.3.38}$$

where g_{mn} is the induced metric on L, and $\delta(\iota^* g)_{mn}$ is the metric variation induced by the variation of the background Kähler and complex structure. Note that the four-dimensional metric $\eta_{\mu\nu}$ is rescaled to the four-dimensional Einstein frame.[10] One first performs the Taylor expansion of the determinant while using (3.3.38). Inserting the result together with F_{D6} given in (3.3.37) into the first part of (3.3.9) we obtain the four-dimensional action

$$\begin{aligned} S^{(4)}_{\rm DBI} \ &= \ -\int \tfrac{1}{2}\mathrm{Re} f_{\mathrm{r}\,IJ}\, F^I \wedge *F^J + e^{2D}\mathcal{G}_{ij}\, da^i \wedge * da^j + e^{2D}\tilde{\mathcal{G}}_{IJ}\, d\tilde{a}^I \wedge * d\tilde{a}^J \\ &\qquad + e^{2D}\mathcal{G}_{IJ}\, \mathcal{D}\hat{a}^I \wedge * \mathcal{D}\hat{a}^J + e^{2D}\widehat{\mathcal{G}}_{IJ}\, d\eta^I \wedge * d\eta^J + V_{\rm DBI} * 1 \ , \end{aligned} \tag{3.3.39}$$

in the four-dimensional Einstein frame. The covariant derivative $\mathcal{D}\hat{a}^I$ was introduced in (3.3.37) and indicates the gauging of the infinite tower of scalars $\hat{a}^I$. The potential term $V_{\rm DBI}$ depends on the deformations $\delta(\iota^* g)_{mn}$ of the calibration conditions (3.3.3) induced by the variation of the induced metric on L_η which we computed in (3.3.28). Moreover, one obtains an additional term depending on the modes violating the background condition $F_{\rm D6} - B_2|_{L_0} = 0$ as in (3.3.30). Explicitly we find

$$V_{\rm DBI} = \frac{e^{3\phi}}{\mathcal{V}^2}\int_{L_0} d^*\theta_\eta \wedge * d^*\theta_\eta + \frac{e^{3\phi}}{\mathcal{V}^2}\int_{L_0}\left(d\theta_\eta \wedge * d\theta_\eta + (\tilde{F} - B_2 - d\theta^B_\eta)\wedge *(\tilde{F} - B_2 - d\theta^B_\eta)\right) , \tag{3.3.40}$$

where $\tilde{F}$ is defined in (3.3.37). In the following we will discuss the metric functions appearing in the kinetic terms of (3.3.39).

The first term in (3.3.39) is the kinetic term for the $U(1)$ gauge bosons A^I. The gauge coupling function is thus given to leading order by

$$\mathrm{Re}\, f_{\mathrm{r}\,IJ} = \int_{L_0} 2\,\mathrm{Re}(C\Omega) h_I h_J, \tag{3.3.41}$$

where the volume of L_0 has been replaced using (3.3.3). Note that $\mathrm{Re} f_{\mathrm{r}\,IJ}$ admits a simple geometrical interpretation as L^2-metric on the space of functions on L_0. More generally, without introducing a specific basis and restricting to a special Lagrangian one writes for two functions $h, \tilde{h}$ on L

$$\mathrm{Re} f_{\mathrm{r}}(h, \tilde{h})|_L = e^{-\phi}\int_L h \wedge * \tilde{h} \ , \tag{3.3.42}$$

which readily reduces to (3.3.53) on $L = L_0$ using $*1 = \mathrm{vol}_L$ and (3.3.3).

The second, third and fourth term in (3.3.39) are the kinetic terms for the Wilson line moduli $a^i, \tilde{a}^I, \hat{a}^I$, where the later appear with the covariant derivative $\mathcal{D}\hat{a}^I = d\hat{a}^I + A^I$ as introduced in (3.3.37). The appearing metrics take the form

$$\mathcal{G}_{ij} = \tfrac{1}{2} e^{-\phi}\mathcal{G}(\tilde{\alpha}_i, \tilde{\alpha}_j) \ , \qquad \tilde{\mathcal{G}}_{IJ} = \tfrac{1}{2} e^{-\phi}\mathcal{G}(d^*\gamma_I, d^*\gamma_J) \ , \qquad \mathcal{G}_{IJ} = \tfrac{1}{2} e^{-\phi}\mathcal{G}(dh_I, dh_J) \ , \tag{3.3.43}$$

[10] Recall that the four-dimensional metric in the Einstein frame η is related to the string frame metric $\eta^{\rm SF}$ via $\eta = e^{-2D}\,\eta^{\rm SF}$.

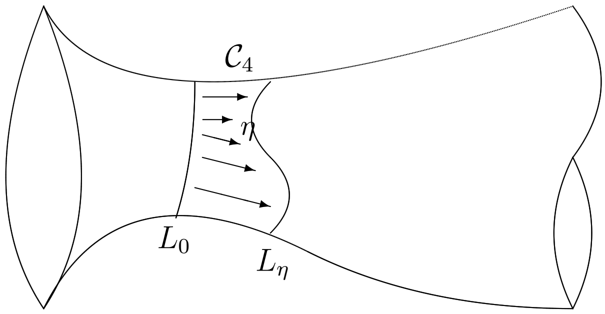


Figure 3.1: A diagrammatic representation of the original cycle L_0 wrapped by the brane, and its deformation until L_η, the $\mathcal{C}_4$ chain bounded by L_0 and L_η, and the vector field η normal to L_0.

where $\mathcal{G}$ is the L^2-metric defined in (3.3.33), and $\tilde{\alpha}_i$, dh^I and $d^*\gamma_I$ are the one-form basis introduced in (3.3.37). The fifth term in (3.3.39) contains the field space metric for the deformations η^I and is of the form

$$\widehat{\mathcal{G}}_{IJ} = \int_{L_0} g(s_I, s_J)\mathrm{Re}(C\Omega) = \tfrac{1}{2}e^{-\phi}\mathcal{G}(\theta_I, \theta_J) \ . \tag{3.3.44}$$

where θ_I are the one-forms on L_0 introduced in (3.3.32). Let us comment on the derivation of the second identity in (3.3.44). Here we first have to use the fact that $g(s_i, s_j) = J(s_i, Is_j) = (Is_j)\lrcorner\theta_i$, where J is the Kähler form and I is the complex structure on Y. Next we deduce from $J \wedge \mathrm{Re}(C\Omega) = 0$ that we can move the Is_j to obtain $\theta_i \wedge (Is_j)\lrcorner\mathrm{Re}C\Omega$. However, since $C\Omega$ is a $(3,0)$-form one deduces using

$$2(Is_j)\lrcorner\mathrm{Re}C\Omega = -2s_j\lrcorner\mathrm{Im}(C\Omega) = e^{-\phi} * \theta_j \ , \tag{3.3.45}$$

and the identity (3.3.5) the second equality in (3.3.44).

This completes our reduction of the Dirac-Born-Infeld action. Let us stress that the reduction so far only included the leading order terms. In order to fully extract the $\mathcal{N} = 1$ characteristic data, however, we will need to match also higher order terms. It turns out that an efficient strategy to proceed is to include these by using supersymmetry and a careful study of the the Chern-Simons action. We will turn to the Kaluza-Klein reduction of this part of the D-brane action in the following.

Chern-Simons action

Let us now turn to the dimensional reduction of the Chern-Simons part of the D6-brane action. In the reduction one can again perform a normal coordinate expansion of the form-fields appearing in the action. However, we will take here a somewhat different route and parameterize the normal variations by introducing a four-chain $\mathcal{C}_4$ which contains the three-cycle L_η in its boundary

$$\partial\mathcal{C}_4 = L_\eta - L_0 \ , \tag{3.3.46}$$

as in figure 3.1 where L_0 is the reference three-cycle, the supersymmetric background cycle.

We consider the Chern-Simons action containing the R-R forms C_3, C_5 and C_7 given by

$$S_{\mathrm{CS}} = \int_{\mathcal{W}_7^{(0)}} e^{F-B_2} \wedge (C_3 + C_5 + C_7) + S_{\mathrm{CS}}^{\mathcal{C}_4}. \tag{3.3.47}$$

Here $\mathcal{W}_7^{(0)} = \mathcal{M}^{3,1} \times L_0$,

$$S_{\mathrm{CS}}^{\mathcal{C}_4} = \int_{\mathcal{W}_8} d\big[e^{F-B_2} \wedge (C_3 + C_5 + C_7)\big] \ , \tag{3.3.48}$$

and $\mathcal{W}_8 = \mathcal{M}^{3,1} \times \mathcal{C}_4$ such that $\mathcal{W}_7 \subset \partial\mathcal{W}_8$. This is in a similar spirit as the constructions in [97]. To perform the Kaluza-Klein reduction of (3.3.48) we consider the expansion of $\mathcal{A}$, the wedge product between the R-R forms and the B-field introduced in (3.2.6), as

$$\begin{aligned}\sum_{p=3,5,7,9} e^{-B_2} \wedge C_p &= (\xi^k \alpha_k - \tilde{\xi}_\lambda \beta^\lambda) + (A^\alpha \wedge \omega_\alpha + A_\alpha \wedge \tilde{\omega}^\alpha) \\ &\quad + (C_2^\lambda \wedge \alpha_\lambda - \tilde{C}_k^2 \wedge \beta^k) + (C_3^0 + C_3^a \wedge \omega_a + C_a^3 \wedge \tilde{\omega}^a + C_0^3 \wedge \mathrm{vol}_Y) \, .\end{aligned} \tag{3.3.49}$$

In (3.3.49), $(\alpha_\lambda, \beta^k)$ is a basis of $H^3_-(Y,\mathbb{R})$, ω_a, ω_α are basis of $H^2_-(Y,\mathbb{R})$, $H^2_+(Y,\mathbb{R})$, and $\tilde{\omega}^a, \tilde{\omega}^\alpha$ are a basis of $H^4_+(Y,\mathbb{R})$, $H^4_-(Y,\mathbb{R})$. Here we introduced the four-dimensional two-forms $(C_2^\lambda, \tilde{C}_k^2)$ which are dual to the scalars $(\xi^k, \tilde{\xi}_\lambda)$, already introduced in (3.2.14). The vectors A^α have been already introduced in (3.2.17), and A_α are their four-dimensional duals. Moreover, the Kaluza-Klein expansion (3.3.49) also contains the four-dimensional three-forms $(C_3^0, C_3^a, C_a^3, C_0^3)$ which are non-dynamical, but will crucially contribute to the scalar potential as in ref. [93].

Note also that the fields defined in (3.3.49) are not the expansions from the R-R forms alone, but in general combine with the NS-NS two-form B_2. Denoting by a hat ˆ the fields which arise in the expansion of the R-R forms alone, one finds, for example, that

$$B_2\text{-corrected:} \quad \begin{cases} \text{vectors:} & A^\alpha = \hat{A}^\alpha \, , \qquad A_\alpha = \hat{A}_\alpha - \hat{A}^\beta b^a \mathcal{K}_{\beta a \alpha} \, , \\ \text{3-forms:} & C_3^0 = \hat{C}_3^0 \, , \qquad C_3^a = \hat{C}_3^a + \hat{C}_3^0 b^a \, , \qquad \text{etc.} \end{cases} \tag{3.3.50}$$

where $\hat{A}^\alpha$, $\hat{C}_3^0$ and $\hat{A}_\alpha$, $\hat{C}_3^a$ denote the space-time vector bosons and three-forms coming from the expansion of C_3 and C_5, respectively. In contrast, the scalars and two-forms in (3.3.49) have no mixing with the B-field such that

$$\text{no } B_2\text{-correction:} \qquad \text{scalars:} \;\; (\xi^k, \tilde{\xi}_\lambda) \qquad \text{2-forms:} \;\; (C_2^\lambda, \tilde{C}_k^2) \, . \tag{3.3.51}$$

As discussed in more detail in section 3.6 the situation is precisely reversed under mirror symmetry. In fact, using the results on the side without B_2 corrections mirror symmetry can be used to compute the corrected couplings.

The Chern Simons action is dimensionally reduced by inserting (3.3.49) into (3.3.48). Focusing on the couplings of A^α and $(C_2^\lambda, \tilde{C}_k^2)$ in favor over their duals, one finds [11]

$$\begin{aligned} S_{\text{CS}}^{(4)} &= \int \tfrac{1}{2} \mathrm{Im} f_{\mathrm{r}\,IJ}\, F^I \wedge F^J - (\delta_{I\lambda} dC_2^\lambda - \delta_I^k d\tilde{C}_k^2) \wedge A^I \\ &\quad - (\mathcal{I}_{I\lambda}\, dC_2^\lambda - \mathcal{I}_I^k\, d\tilde{C}_k^2) \wedge da^I + \mathcal{L}_{\text{mix}} + \mathcal{L}_3 \, . \end{aligned} \tag{3.3.52}$$

The first term is the theta term of the gauge theory on the D6-brane, with $\mathrm{Im}\, f_{\mathrm{r}\,IJ} = \int_{L_0} C_3^{\text{sc}} h_I h_J$ that combines with (3.3.41) to form

$$\mathrm{Re}\, f_{\mathrm{r}\,IJ} = \int_{L_0} (2\,\mathrm{Re}(C\Omega) + iC_3^{\text{sc}})\, h_I h_J = \int_{L_0} \Omega_c h_i h_J, \tag{3.3.53}$$

where we have used the definition for Ω_c in (3.2.13). $\mathcal{L}_{\text{mix}}$ in (3.3.52) corresponds to the mixing of the brane and bulk gauge bosons

$$\mathcal{L}_{\text{mix}} = \left(a^J \Delta_{(I)J\alpha} + \Gamma_{(I)\alpha}\right) dA^\alpha \wedge F^I + \tilde{\mathcal{J}}^\alpha_{(I)} dA_\alpha \wedge F^I \, , \tag{3.3.54}$$

and $\mathcal{L}_3$ is the term which depends on the three-form field strengths as

$$\mathcal{L}_3 = dC_3^0 \left(\tfrac{1}{2} a^I a^J \Delta_{IJ} + a^J \tilde{\Gamma}_J\right) + dC_3^a \left(a^J \Delta_{Ja} + \Gamma_a\right) + dC_a^3\, \tilde{\mathcal{J}}^a \, . \tag{3.3.55}$$

[11] One could also include the couplings to A_α and $(\xi^k, \tilde{\xi}_\lambda)$. In this case one has to analyze also the bulk action keeping all forms and their duals as in ref. [83].

In order to display the remaining couplings appearing in this action we first define the integral $\mathcal{I}(\tilde{\alpha},\alpha)$ between a one-form $\tilde{\alpha}$ on L_η and a three-form α on Y, as well as the integral $\mathcal{J}(\tilde{\beta},\omega)$ between a two-form $\tilde{\beta}$ on L_η and a two-form ω on Y. To do that we again extend the forms defined on L_0 to the chain $\mathcal{C}_4$ such that they are constant along the normal directions of L_η in Y. We define

$$\mathcal{I}(\tilde{\alpha},\alpha) = \int_{\mathcal{C}_4} \tilde{\alpha}\wedge\alpha \ , \qquad \mathcal{J}(\tilde{\beta},\omega) = \int_{\mathcal{C}_4} \tilde{\beta}\wedge\omega \ . \tag{3.3.56}$$

Furthermore, we will also need a pairing δ between a function h on L_0 and three-form α on Y, as well as a pairing Δ between a one-form γ on L_0 and a two-form on Y. Hence, we set

$$\delta(h,\alpha) = \int_{L_0} h\,\alpha + \mathcal{I}(dh,\alpha), \qquad \Delta(\gamma,\beta) = \int_{L_0} \gamma\wedge\beta + \mathcal{J}(d\gamma,\beta) \ . \tag{3.3.57}$$

Note that these latter definitions include terms supported on L_0 which are non-vanishing even in the limit of vanishing normal displacement η. This redefinition is necessary since $\mathcal{I}$ and $\mathcal{J}$ vanish for a vanishing normal displacement. In fact, we can expand (3.3.56) to first order in η for small normal displacement in $\partial\mathcal{C}_4 = L_\eta - L_0$ and obtain

$$\mathcal{I}(\tilde{\alpha},\alpha) = \int_{L_0} \tilde{\alpha}\wedge\eta\lrcorner\alpha \ + \dots \ , \qquad \mathcal{J}(\tilde{\beta},\omega) = \int_{L_0} \tilde{\beta}\wedge\eta\lrcorner\omega \ + \dots \ , \tag{3.3.58}$$

which has a leading term linear in η.

Having introduced the pairings we can display the couplings in (3.3.52), (3.3.54) and (3.3.55). Let us start with the couplings in (3.3.52) obtained as

$$\mathcal{I}_{I\lambda} = \mathcal{I}(\hat{\alpha}_I,\alpha_\lambda) \ , \qquad \mathcal{I}_I^k = \mathcal{I}(\hat{\alpha}_I,\beta^k) \ , \qquad \delta_{I\lambda} = \delta(h_I,\alpha_\lambda) \ , \qquad \delta_I^k = \delta(h_I,\beta^k) \ . \tag{3.3.59}$$

Furthermore, in the mixed term $\mathcal{L}_{\text{mix}}$, given in (3.3.54), for the gauge bosons one finds

$$\Delta_{(I)J\alpha} = \Delta(h_I\hat{\alpha}_J,\omega_\alpha) \ , \qquad \Gamma_{(I)\alpha} = \mathcal{J}(h_I f_{\text{D6}},\omega_\alpha) \ , \qquad \tilde{\mathcal{J}}^\alpha_{(I)} = \int_{\mathcal{C}_4} h_I\tilde{\omega}^\alpha \ . \tag{3.3.60}$$

Finally, we introduce the coefficients in (3.3.55) as

$$\Delta_{Ja} = \Delta(\hat{\alpha}_J,\omega_a) \ , \qquad \Gamma_a = \mathcal{J}(f_{\text{D6}},\omega_a) \ , \tag{3.3.61}$$

for couplings between the ambient space two-forms ω_a and forms $\hat{\alpha}_J$ and f_{D6} on the D6-brane. The remaining couplings are

$$\Delta_{IJ} = \int_{L_0} \hat{\alpha}_I\wedge d\hat{\alpha}_J \ , \qquad \tilde{\Gamma}_J = \int_{L_0} \hat{\alpha}_J\wedge f_{\text{D6}} \ , \qquad \tilde{\mathcal{J}}^a = \int_{\mathcal{C}_4} \tilde{\omega}^a \ . \tag{3.3.62}$$

It is not hard to interpret the different terms appearing in the action (3.3.52). The second term proportional to $(\delta_{I\lambda}dC_2^\lambda - \delta_I^k d\tilde{C}_k^2)\wedge A^I$ is a Green-Schwarz term which indicates that the scalar fields $(\xi^k,\tilde{\xi}_\lambda)$ dual to the two-forms $(C_k^2,\tilde{C}_2^\lambda)$ are gauged by the D6-brane vector fields A^I. In fact, upon elimination of $(C_k^2,\tilde{C}_2^\lambda)$ one finds the covariant derivative

$$D\xi^k = d\xi^k + \delta_I^k A^I \ , \qquad D\tilde{\xi}_\lambda = d\tilde{\xi}_\lambda + \delta_{I\lambda}A^I \ , \tag{3.3.63}$$

We will show in section 3.5 that these gaugings induce the corresponding D-terms in V_{DBI} as expected from a supersymmetric theory.

The third term in (3.3.52) proportional to da^I will be of importance for the derivation of the Kähler potential and complex coordinates on the $\mathcal{N}=1$ field space. Upon elimination of $(C_k^2,\tilde{C}_2^\lambda)$ it induces a mixing of the kinetic terms of $a^I = (a^i, \hat{a}^I)$ and $(\xi^k, \tilde{\xi}_\lambda)$. More precisely, one finds the modified four-dimensional kinetic terms

$$\mathcal{L}^{\text{kin}}_{C_3} = G_{kl} \nabla\xi^k \wedge *\nabla\xi^l + G^{\lambda\kappa} \nabla\tilde{\xi}_\lambda \wedge *\nabla\tilde{\xi}_\kappa + 2G_k^{\ \lambda} \nabla\xi^k \wedge *\nabla\tilde{\xi}_\lambda \tag{3.3.64}$$

where the modified derivatives ∇ are defined by

$$\nabla\xi^k \equiv D\xi^k + \mathcal{I}^k_I da^I \qquad \text{and} \qquad \nabla\tilde{\xi}_\lambda \equiv D\tilde{\xi}_\lambda + \mathcal{I}_{I\lambda} da^I , \tag{3.3.65}$$

with the metric G given as in the closed string case,

$$G_{kl} = \tfrac{1}{2}\, e^{2D} \int_Y \alpha_k \wedge * \alpha_l \, , \qquad G^{\lambda\kappa} = \tfrac{1}{2}\, e^{2D} \int_Y \beta^\lambda \wedge * \beta^\kappa \, , \qquad G_k^{\ \lambda} = -\tfrac{1}{2}\, e^{2D} \int_Y \alpha_l \wedge * \beta^\lambda \, . \tag{3.3.66}$$

Note that the form of the metric G for $\nabla\xi^k$ and $\nabla\tilde{\xi}_\lambda$ closely resembles the form of the metric $\mathcal{G}_{ij}$ for the scalars a^i as seen from (3.3.39) and (3.3.43). We will exploit this observation in the detailed study of the moduli space geometry later on. This similarity only occurs in the $\mathcal{N} = 1$ orientifold for which the field space metric is Kähler. In the underlying $\mathcal{N} = 2$ set-ups the moduli space containing the R-R scalars is a quaternionic manifold.

The $\mathcal{L}_{\text{mix}}$ is a kinetic mixing term between the $U(1)$ from the brane with the vector field from the C_3 expansion. This term will be important in the derivation of the gauge coupling function in section 3.4.

The term $\mathcal{L}_3$ given by (3.3.55) contains the four-dimensional three-forms which arise in the expansion of C_3, C_5, C_7. Very similar to the analysis in ref. [93] they will be crucial to complete the scalar potential contributions in V_{DBI} to supersymmetric F-terms which can be obtained from a superpotential. To find the scalar potential from the three-form potential one has to eliminate the forms dC^0_3, dC^3_a and dC^a_3 from the complete four-dimensional effective action. In particular, in addition to $\mathcal{L}_3$ one also has to include the reduction of the ten-dimensional kinetic term in (3.2.5). The resulting action for the three-forms will be given in terms of the matrix $\mathcal{N}_{\hat{A}\hat{B}}$ defined as

$$\mathcal{N}_{\hat{A}\hat{B}} = \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2}\mathcal{K}_{Bab} b^a b^b \\ \frac{1}{2}\mathcal{K}_{Aab} b^a b^b & -\mathcal{K}_{ABa} b^a \end{pmatrix} - i\mathcal{V} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{Ba} b^a \\ -4G_{Aa} b^a & 4G_{AB} \end{pmatrix} , \tag{3.3.67}$$

where $\hat{A} = \{0, a, \alpha\}$, and one has to use $\mathcal{K}_{ab\alpha} = \mathcal{K}_{\alpha\beta\gamma} = 0$. Using these definitions we find after rescaling to the Einstein frame that

$$S_{\text{3-form}} = \int \tfrac{1}{4} e^{-4D} (\text{Im}\mathcal{N})^{-1\,\hat{a}\hat{b}} (dC^3_{\hat{a}} - \mathcal{N}_{\hat{a}\hat{c}}\, dC^{\hat{c}}_3) \wedge *(dC^3_{\hat{b}} - \bar{\mathcal{N}}_{\hat{b}\hat{d}}\, dC^{\hat{d}}_3) + \mathcal{L}_3 \, , \tag{3.3.68}$$

where $C^{\hat{a}}_3 = (C^0_3, C^a_3)$ and $C^3_{\hat{a}} = (C^3_0, C^3_a)$, and $\mathcal{L}_3$ is the D-brane coupling defined in (3.3.55). As in ref. [93] we next dualize dC^0_3, dC^a_3 and dC^3_a, dC^3_0 into flux scalars e_0, e_a, m^a, m^0. In ref. [98] the interpretation of these scalars as quantized fluxes has been provided. They also arise as background values of the field strengths $F_2 = m^a \omega_a$, $F_4 = e_a \tilde{\omega}^a$ and $F_6 = e_0 \,\text{vol}_Y$ as their expansions into harmonic forms on Y. In addition there is Romans mass parameter $F_0 = G_0 = m^0$. After dualization of the three-forms on finds the scalar potential

$$V_{\text{flux+CS}} = \tfrac{1}{4} e^{-4D} (\text{Im}\mathcal{N})^{-1\,\hat{a}\hat{b}} (\tilde{e}_{\hat{a}} - \mathcal{N}_{\hat{a}\hat{c}}\, \tilde{m}^{\hat{c}}) \wedge *(\tilde{e}_{\hat{b}} - \bar{\mathcal{N}}_{\hat{b}\hat{d}}\, \tilde{m}^{\hat{d}}) \, , \tag{3.3.69}$$

where

$$\begin{aligned} \tilde{e}_0 &= e_0 + \tfrac{1}{2} \int_{\mathcal{C}_4} \tilde{F} \wedge \tilde{F} + \tfrac{1}{2} \int_{L_0} \tilde{F} \wedge a^I \hat{\alpha}_I , \\ \tilde{e}_a &= e_a + \int_{\mathcal{C}_4} \tilde{F} \wedge \omega_a + \int_{L_0} a^I \hat{\alpha}_I \wedge \omega_a , \\ \tilde{m}^a &= m^a + \int_{\mathcal{C}_4} \tilde{\omega}^a \, , \qquad \tilde{m}^0 = m^0 \, . \end{aligned} \tag{3.3.70}$$

The additional terms in the definitions (3.3.70) arise precisely because of the term $\mathcal{L}_3$ form the D6-brane. Luckily, apart from these shifts, the closed string moduli dependence of the potential (3.3.69) agrees with the analog expression found in ref. [82], and we will thus be able to integrate it into a superpotential without much effort.

Restriction of the brane action to harmonic modes

To conclude our reduction of the D6-brane action let us also give the result which is obtained by restricting to harmonic forms. This corresponds to a truncation of the Kaluza-Klein tower of the brane fields to include only the lightest states. The resulting action will be useful in the next section when analyzing the moduli space. The Kaluza-Klein Ansatz for the D6-brane field strength, eqn. (3.3.37), simplifies to

$$F_{\rm D6} = F + da^i \wedge \tilde{\alpha}_i + f_{\rm D6}\ . \tag{3.3.71}$$

This implies that the DBI action reduces to

$$S^{(4)}_{\rm DBI} = -\int \tfrac{1}{2}{\rm Re} f_{\rm r}\, F \wedge *F + e^{2D}\mathcal{G}_{ij}\, da^i \wedge *da^j + e^{2D}\widehat{\mathcal{G}}_{ij}\, d\eta^i \wedge *d\eta^j\ , \tag{3.3.72}$$

with the metric $\mathcal{G}_{ij}$ being the same as in (3.3.43), and $\widehat{\mathcal{G}}_{ij}$ the restriction of (3.3.44) to supersymmetric deformations (i.e., harmonic one-forms θ_i). The gauge coupling function (3.3.53) simplifies to

$${\rm Re}\, f_{\rm r} = \int_{L_0} 2\,{\rm Re}(C\Omega)\ , \tag{3.3.73}$$

as we restrict h_I to the only harmonic function, the constant function which we normalized to 1. We did not include the scalar potential $V_{\rm DBI}$ since it vanishes when restricting to the harmonic subset of forms, as we will show in section 3.5.

The truncation of the Chern-Simons action to the harmonic modes is

$$\begin{aligned} S^{(4)}_{\rm CS} \;=\; & \int \tfrac{1}{2}{\rm Im} f_{\rm r}\; F \wedge F - (\delta_\lambda dC_2^\lambda - \delta^k d\tilde{C}^2_k) \wedge A - (\mathcal{I}_{i\lambda}\, dC_2^\lambda - \mathcal{I}^k_i\, d\tilde{C}^2_k) \wedge da^i \\ & + \big(a^j \Delta_{j\alpha} + \Gamma_\alpha\big) dA^\alpha \wedge F + \tilde{\mathcal{J}}^\alpha dA_\alpha \wedge F + dC_3^a \big(a^j \Delta_{ja} + \Gamma_a\big) + dC^3_a\, \tilde{\mathcal{J}}^a + dC_3^0\, \big(a^j \tilde{\Gamma}_j\big)\ , \end{aligned} \tag{3.3.74}$$

with couplings

$$\begin{aligned} \delta_\lambda \;&=\; \int_{L_0} \alpha_\lambda, \qquad \delta^k = \int_{L_0} \beta^k, \qquad \mathcal{I}_{i\lambda} = \int_{\mathcal{C}_4} \tilde{\alpha}_i \wedge \alpha_\lambda, \qquad \mathcal{I}^k_i = \int_{\mathcal{C}_4} \tilde{\alpha}_i \wedge \beta^k\ , \\ \Delta_{iA} \;&=\; \int_{L_0} \tilde{\alpha}_i \wedge \omega_A, \qquad \Gamma_A = \int_{\mathcal{C}_4} f_{\rm D6} \wedge \omega_A, \quad A = \{a, \alpha\}, \qquad \tilde{\Gamma}_i = \int_{L_0} \tilde{\alpha}_i \wedge f_{\rm D6}\ , \end{aligned} \tag{3.3.75}$$

and $\tilde{\mathcal{J}}^A = \int_{\mathcal{C}_4} \tilde{\omega}^A$ as defined in (3.3.62). One realizes that the couplings $(\delta_\lambda, \delta^k)$ and $\Delta_{iA}, \tilde{\Gamma}_i$ are constants, while the couplings $(\mathcal{I}_{i\lambda}, \mathcal{I}^k_i)$ and Γ_A depend on the brane deformations through the chain $\mathcal{C}_4$.

Let us take a closer look at the three-form couplings coming from $\mathcal{L}_3$ (3.3.55), after the reduction given simply by $dC_3^a\big(a^j \Delta_{ja} + \Gamma_a\big) + dC^3_a\, \tilde{\mathcal{J}}^a + dC_3^0\, \big(a^j \tilde{\Gamma}_j$ in (3.3.74). We can expand the $\mathcal{C}_4$ chain around the L_0 cycle to see the explicit dependence on the brane deformations. Just as in (3.3.58), we obtain, up to first order in the open fields,

$$\mathcal{L}_3 = dC_3^a \int_{L_0} \big(a^j \tilde{\alpha}_j \wedge \omega_a + \eta^j s_j \lrcorner \omega_a \wedge f_{\rm D6}\big) + dC^3_a \int_{L_0} \eta^j s_j \lrcorner \tilde{\omega}^a + dC_3^0 \int_{L_0} a^j \tilde{\alpha}_j \wedge f_{\rm D6}. \tag{3.3.76}$$

Note that this implies that $\mathcal{L}_3$ is non-vanishing also in the case we restrict to harmonic forms only. However, note that (3.3.76) describes a coupling between the open and closed sector. In fact, the scalar potential (3.3.69) arising from (3.3.76) is obtained as an F-term potential when varying the superpotential with respect to the closed string fields t^a, as we will describe in section 3.5.

3.4 The open-closed moduli space and the Hitchin functionals

In this section we discuss the geometry of the moduli space of the bulk sector and brane sector in more detail. In the first part, section 3.4.1, we assume that the open moduli are frozen and discuss the geometry of the moduli space $\mathcal{M}^Q$ of the dilaton and the real complex structure deformations following [82]. In section 3.4.2 we discuss the moduli space of special Lagrangian deformations η^i following the work of Hitchin [88, 99]. This description will be slightly extended by including the NS-NS B-field. The open moduli space has finite dimension $b^1(L_0)$ and can be encoded by the variation of harmonic one- or two-forms on L_0.

In the complete set-up, with varying open and closed modes, the definition of being special Lagrangian crucially depends on both the Kähler as well as the complex structure moduli of Y. In fact, the normal vectors s_i used in order to define the one-forms $\theta_i = s_i \lrcorner J$ need to be chosen such that θ_i is harmonic. This notion changes when varying the complex and Kähler structure of Y. Nevertheless, if such a change does not alter the topology of Y and L_0, one expects to find a new embedding map ι' which makes L_η supersymmetric in Y and posses also $b^1(L_0)$ special Lagrangian deformations. This suggests to view the full moduli space as fibration of the open string moduli space $\mathcal{M}_o^{\mathbb{C}}$ over the closed string moduli space $\mathcal{M}_{\mathbb{C}}^K \times \mathcal{M}_{\mathbb{C}}^Q$, where $\mathcal{M}_{\mathbb{C}}^K$ is the space spanned by the complexified Kähler deformations. In section 3.4.3 we will explore the local geometry of this full moduli space in more detail. Note that we are still dealing with only a finite set of deformations. In the absence of background fluxes these remain massless due to the vanishing of the scalar potential.

In section 3.4.4 we also analyze the gauge coupling function and the kinetic mixing for the brane and bulk $U(1)$ gauge fields. In particular, we comment on its holomorphicity properties.

3.4.1 The orientifold moduli space

We start discussing the moduli space coming from the four dimensional dilaton D and the complex structure deformations, described by the scalars q^K. We will review how the complex structure deformations combine with the scalar part of the three form C_3, and justify our inclusion of Ω_c as the complexified complex structure that gives rise to the "correct" chiral fields of the theory. Here we will follow [82].

The moduli space for the dilaton and complex structure deformations is described by

$$\tfrac{1}{2}G = dD \cdot dD + K^{\text{cs}}_{KL}\, dq^K \cdot dq^L \, , \tag{3.4.1}$$

where K^{cs}_{KL} is the metric restricted to the deformations preserving the orientifold constraint (3.2.2). We will argue now how this moduli space can be encoded in the complexified Ω_c, (3.2.13),

$$\Omega_c = 2\,\text{Re}(C\Omega) + iC_3^{\text{sc}} \;=\; N'^k\, \alpha_k - T'_\lambda\, \beta^\lambda \, , \tag{3.2.13}$$

where N'^k and T'_λ were defined in (3.2.14)

$$N'^k = 2\,\text{Re}(CX^k) + i\,\xi^k \, , \qquad T'_\lambda = 2\,\text{Re}(C\mathcal{F}_\lambda) + i\,\tilde{\xi}_\lambda \, . \tag{3.2.14}$$

Here we have introduced the convenient notation for the periods of Ω_c,

$$U^k = 2\,\text{Re}(CX^k), \quad \text{and} \quad U_\lambda = 2\,\text{Re}(C\mathcal{F}_\lambda). \tag{3.4.2}$$

Importing the Kähler function from the underlying $\mathcal{N} = 2$ theory, the metric in (3.4.1) can be obtained as the second derivative with respect to U of the function [82]

$$K^Q(V) = -2\ln\left[i\int_Y C\Omega \wedge \overline{C\Omega}\right] = -2\log\left[e^{-2D}\right] . \tag{3.4.3}$$

Calculating the metric explictly,

$$G = \frac{\partial^2 K^Q}{\partial U^K \partial U^L}\, dU^K \cdot dU^L = G_{kl}\, dU^k \cdot dU^l + G^{\lambda\kappa}\, dU_\lambda \cdot dU_\kappa + 2G_k^{\ \lambda}\, dU^k \cdot dU_\lambda \, . \tag{3.4.4}$$

where the metrics are the same as in (3.3.66)

$$G_{kl} = \tfrac{1}{2}\, e^{2D} \int_Y \alpha_k \wedge * \alpha_l \, , \qquad G^{\lambda\kappa} = \tfrac{1}{2}\, e^{2D} \int_Y \beta^\lambda \wedge * \beta^\kappa \, , \qquad G_k^{\ \lambda} = -\tfrac{1}{2}\, e^{2D} \int_Y \alpha_l \wedge * \beta^\lambda \, . \tag{3.3.66}$$

Alternatively to the direct calculation from a truncation of the $\mathcal{N} = 2$ moduli space, one can calculate the above metrics as in [100] via the techniques introduced by Hitchin [101] The metrics in (3.3.66) were the metrics obtained from the kinetic terms of real scalars $\{\xi^k, \xi_\lambda\}$ coming from compactifying C_3,

$$C_3 = \xi^k \alpha_k + \xi_\lambda \beta^\lambda + \dots \, , \tag{3.4.5}$$

thus we can combine the real fields $\{\xi^k, \xi_\lambda\}$ with $\{U^k, U_\lambda\}$ in the chiral fields $\{N'^k, T'_\lambda\}$, (3.2.14). We can also see that the imaginary part of $C\Omega$ appears as the first derivative of K^Q,

$$\frac{1}{2}\frac{\partial K^Q}{\partial U^k} = 2\, e^{2D} \mathrm{Im}(C\mathcal{F}_k) \equiv V_k \, , \qquad \frac{1}{2}\frac{\partial K^Q}{\partial U_\lambda} = -2\, e^{2D} \mathrm{Im}(CX^\lambda) \equiv V^\lambda \, . \tag{3.4.6}$$

That is, $\{V_k, V^\lambda\}$ appear as a dual coordinates to $\{U^k, U_\lambda\}$ and can be alternated via a Legendre transform in the action.

Note that originally $\mathcal{M}^Q_{\mathbb{C}}$ was found as the $\mathcal{N} = 1$ field-space obtained by truncating the underling quaternionic geometry spanned by the $\mathcal{N} = 2$ hypermultiplets. Each hypermultiplet has been truncated to a single $\mathcal{N} = 1$ chiral multiplet such that $\mathcal{M}^Q$ has half the real dimension of the quaternionic space. But $\mathcal{M}^Q$ can also be viewed as a Lagrangian submanifold of an auxiliary vector space [28], similarly to what will be done in the next section 3.4.2 for the moduli space of brane deformations.

Let us conclude the discussion of the moduli space $\mathcal{M}^Q \times \mathcal{M}^K$ by presenting yet another way to motivate its geometrical structures. In an orientifold compactification it is well-known that the orientifold planes, located on the fix-points of the involution σ, are not dynamical and hence do not posses moduli at weak string coupling. Hence, all deformations in $\mathcal{M}^Q \times \mathcal{M}^K$ need to preserve the embedding of the fix-planes and thus the conditions (3.3.1). Also the real complex structure and Kähler structure deformations chosen such that $\mathrm{Im}(C\Omega)$ and J remain elements of $H^3_-(Y, \mathbb{R})$ and $H^2_-(Y, \mathbb{R})$ ensure that these forms vanishes on the fix-point locus of σ. In the discussion of the D6-brane moduli space we will turn the story around and consider the variations of the D-brane embedding maps ι which preserve the conditions (3.3.3) for fixed closed string fields.

3.4.2 The moduli space of D6-branes on special Lagrangian submanifolds

In the following we will discuss the moduli space of a supersymmetric D6-brane wrapped on a special Lagrangian cycle on a Calabi-Yau manifold Y with fixed complex and Kähler structure following [88, 99]. At the end of this subsection we propose a simple modification to include the B-field.

The geometry of the moduli space of special Lagrangian submanifolds

To begin with, recall that the space of deformations taking a three-dimensional submanifold L_0 into another L_η is infinite dimensional if no restrictions are imposed. We have seen however that there is a preferable subset of deformations that lie in the flat directions of the scalar potential, while all the others contribute to the scalar potential thus breaking supersymmetry. He have also seen that the fields corresponding to special Lagrangian (thus supersymmetric) deformations correspond to real massless scalar fields in the effective action, with the number of fields given by $b^1(L_0)$, while the remaining (infinite) fields that correspond to non-special Lagrangian deformations are massive in the compactified theory.

In this section we explore further the finite dimensional space corresponding to deformations that preserve the special Lagrangian conditions, described by the harmonic one-forms $\eta^i\theta_i = \eta^i s_i \lrcorner J$ on L_0. Here s_i is an element of the basis of normal vectors parameterizing a deformation through special Lagrangian submanifolds, and J is the *fixed* background Kähler form which vanishes on L_0. The Hodge dual to θ_i on L_0 can be obtained as contraction of Im$(C\Omega)$ with s_i as given in (3.3.45), $e^{-\phi} * \theta_i = -2s_i \lrcorner \mathrm{Im}(C\Omega)$. The variations of the θ_i and $*\theta_i$ are analyzed by expanding these forms in a basis $\tilde{\alpha}_i$ of $H^1(L_0)$ and $\tilde{\beta}^i$ of $H^2(L_0)$ respectively,

$$\theta_i = \lambda_i^j \, \tilde{\alpha}_j \, , \qquad \tfrac{1}{2} e^{-\phi} * \theta_i = \mu_{ji} \, \tilde{\beta}^j \, , \tag{3.4.7}$$

where $\lambda_i^j(\eta)$ and $\mu_{ij}(\eta)$ define the periods of θ_i and $e^{-\phi} * \theta_i$. Explicitly they are given by

$$\lambda_i^j = \int_{L_0} s_i \lrcorner J \wedge \tilde{\beta}^j \, , \qquad \mu_{ij} = -\int_{L_0} s_j \lrcorner \mathrm{Im}(C\Omega) \wedge \tilde{\alpha}_i \, . \tag{3.4.8}$$

Note that we have introduced an additional factor of the dilaton, which is constant for a fixed background, but will later allow us to make contact to the metrics found in section 3.3. Since J and Im$(C\Omega)$ are closed, one shows that there exist functions (u^i, v_i) such that [99]

$$\frac{\partial u^i}{\partial \eta^j} = \lambda_j^i \, , \qquad \frac{\partial v_i}{\partial \eta^j} = \mu_{ij} \, . \tag{3.4.9}$$

In fact, (u^i, v_i) are the analogs of (U^K, V_K) for the orientifold moduli space (3.4.6).

Let us point out that the harmonic one-forms θ_i^η can be constructed on each L_η obtained by a supersymmetric deformation of L_0 [99]. Generalizing (3.4.7) we can pull back θ_i^η from L_η to L_0 using the exponential map E introduced in section 3.3.2. Following the strategy of section 3.3.4 we can then use the chain $\mathcal{C}_4$ to write

$$\lambda_i^j = \partial_{\eta^i} \int_{\mathcal{C}_4} J \wedge \tilde{\beta}^j \, , \qquad \mu_{ji} = -\partial_{\eta^i} \int_{\mathcal{C}_4} \mathrm{Im} C\Omega \wedge \tilde{\alpha}_j \, . \tag{3.4.10}$$

which at linear order reproduces (3.4.7) on L_0. Inserting (3.4.10) into (3.4.9) this provides us with a chain integral expression for the coordinates (u^i, v_i).

To obtain the differential geometrical structure on $\mathcal{M}_o$ we introduce an embedding F_o,

$$\begin{array}{llll} F_o: & \mathcal{M}_o & \hookrightarrow & V \times V^*, \quad V = H^1(L, \mathbb{R}), \quad V^* = H^2(L, \mathbb{R}) \, , \\ & \eta^i & \mapsto & (\lambda_i^k \alpha_k, \mu_{ji} \tilde{\beta}^j) \, , \end{array} \tag{3.4.11}$$

where $V*$ is a dual vector space of V. We can construct a symplectic structure $\mathfrak{w}$ on the product $V \times V^*$ that acts on $a, b \in V$ and $a', b' \in V^*$ by [99]

$$\mathfrak{w}((a, a'), (b, b')) = a'(b) - b'(a), \quad a'(b) = \int_L a' \wedge b. \tag{3.4.12}$$

When pulled back to $\mathcal{M}_o$, the symplectic form $\mathfrak{w}$ is zero, $F_o^*(\mathfrak{w}) = 0$, the equivalent statement of $J|_{L_0} = 0$, thus $\mathcal{M}_o$ is special Lagrangian inside. We can show that, following [99]. We can act the pullback on two objects η^i and η^j in $\mathcal{M}_o$,

$$F_o^*(\mathfrak{w})(\eta^i, \eta^j) = \mathfrak{w}(F_o(\eta^i), F_o(\eta^j)). \tag{3.4.13}$$

Using the definition (3.4.11) and (3.4.12),

$$F_o^*(\mathfrak{w})(\eta^i, \eta^j) = \lambda_i^k \mu_{kj} - \lambda_j^k \mu_{ki}. \tag{3.4.14}$$

But from the fact that $J \wedge \mathrm{Im}(C\Omega) = 0$, we can contract with two vectors s_i and s_j and using that $J|_{L_0} = \mathrm{Im}(C\Omega)|_{L_0} = 0$ and the definitions $\theta_i = s_i \lrcorner J$, $e^{-\phi} * \theta_i = -2 s_i \lrcorner \mathrm{Im}(C\Omega)$ together with the identities (3.4.7) we show that the right-hand side of (3.4.14) is zero. Thus, $F_o^*(\mathfrak{w} = 0$.

We can also define a natural metric tensor in $V \times V^*$ as

$$\mathfrak{g}((a,b'),(a,b')) = b'(a), \tag{3.4.15}$$

that we can similarly pullback to $\mathcal{M}_o$ and obtain a natural metric for the moduli space,

$$(F_o^*\mathfrak{g})_{ij} d\eta^i d\eta^j = \widehat{\mathcal{G}}_{ij}\, d\eta^i \cdot d\eta^j = \mathcal{G}_{ij}\, du^i \cdot du^j\ , \tag{3.4.16}$$

where $\mathcal{G}_{ij}$ and $\widehat{\mathcal{G}}_{ij}$ are given in (3.3.43) and (3.3.44),

$$\mathcal{G}_{ij} = \tfrac{1}{2} e^{-\phi} \mathcal{G}(\tilde{\alpha}_i, \tilde{\alpha}_j)\ , \qquad \widehat{\mathcal{G}}_{IJ} = \int_{L_0} g(s_I, s_J) \mathrm{Re}(C\Omega) = \tfrac{1}{2} e^{-\phi} \mathcal{G}(\theta_I, \theta_J)\ . \tag{3.4.17}$$

It is straightforward to evaluate the metrics in terms of the periods λ_i^j and μ_{ij} using (3.4.7) and (3.4.9) as

$$\widehat{\mathcal{G}}_{ij} = \mu_{ki}\, \lambda_j^k\ , \qquad \mathcal{G}_{ij} = \mu_{ik}\, (\lambda^{-1})_j^k\ . \tag{3.4.18}$$

From the fact that $\mathcal{M}_o$ is a Lagrangian submanifold one finds that it can be locally represented by a single function K_o with $v_i = \partial K_o / \partial u^i$. This is the direct analog of (3.4.6).

As in the case of the orientifold moduli space, we next have to define a complexification of $\mathcal{M}_o$ to obtain the space $\mathcal{M}_o^{\mathbb{C}}$. Let us first consider the case of vanishing B-field. Since the metric $\mathcal{G}_{ij}$ in the coordinates u^i agrees with the metric for the Wilson line moduli a^i, found in (3.3.39), one defines complex coordinates ζ^i on $\mathcal{M}_o^{\mathbb{C}}$ as

$$\text{no B-field:} \qquad \zeta^i = u^i + i a^i\ , \tag{3.4.19}$$

and identifies $K_o(\zeta + \bar{\zeta})$ as a Kähler potential such that

$$\mathcal{G}_{ij} = \frac{\partial^2 K_o}{\partial u^i \partial u^j} = 4 \frac{\partial^2 K_o}{\partial \zeta^i \partial \bar{\zeta}^j}\ . \tag{3.4.20}$$

The metric $\mathcal{G}_{ij}$ on $\mathcal{M}_o^{\mathbb{C}}$ satisfies an important additional property. In fact, it turns out that $\mathcal{M}_o^{\mathbb{C}}$ is actually a non-compact Calabi-Yau manifold with non-vanishing holomorphic $b^1(L_0)$-form $\widehat{\Omega} = d\zeta^1 \wedge \ldots \wedge d\zeta^{b^1}$ with constant length with respect to the Kähler form on $\mathcal{M}_o^{\mathbb{C}}$ [99].

Open coordinates with B-field

So far we have analyzed in this subsection the open moduli space for vanishing B_2 and f_{D6}. We want to generalize this in the following. To include the B-field we note from (3.4.10) and (3.4.9) that u^i can be written by using the four-chain in (3.3.46) as

$$u^i = \int_{\mathcal{C}_4} J \wedge \tilde{\beta}^i = \int_{L_0} \eta \lrcorner J \wedge \tilde{\beta}^i + \ldots, \tag{3.4.21}$$

where we have also given the η expansion for small fluctuations around L_0. One can now replace J in (3.4.21) by $-iJ_c = J - iB_2$ as used for the closed coordinates in (3.2.8). This leads us to modify (3.4.19) as

$$\zeta^i = u_c^i + i a^i\ , \qquad u_c^i = -i \int_{\mathcal{C}_4} J_c \wedge \tilde{\beta}^i\ . \tag{3.4.22}$$

Note that u_c^i is the complexification of u^i with a B-field correction which can be absorbed by a shift of a^i. This implies that (3.4.20) remains to be valid.

In the definition (3.4.22) we have used the chain $\mathcal{C}_4$ with boundaries L_0 and L_η. It is desirable to introduce a similar extension which allows to include the gauge field. To do that we introduce an extension $\mathcal{F}_{\rm D6} = d\mathcal{A}_{\rm D6}$ of the gauge connection A_{D6} to the chain $\mathcal{C}_4$ such that

$$\mathcal{A}_{\rm D6}|_{L_0} = A^0_{\rm D6} \ , \qquad \mathcal{A}_{\rm D6}|_{L_\eta} = A^0_{\rm D6} - a^I \hat{\alpha}_I \ , \tag{3.4.23}$$

where $\hat{\alpha}_I$ and $A^0_{\rm D6}$ have been transported trivially from L_0 to L_η along the geodesic given by η. Here $A^0_{\rm D6}$ is a background gauge bundle on L_0 which for fixed B_2 allows to satisfy the supersymmetry conditions on L_0. In other words, for a constant B_2 along the chain, $\mathcal{F}_{\rm D6}$ might satisfy the supersymmetry conditions on L_0 but violate the supersymmetry conditions on L_η due to non-trivial Wilson line scalars a^I. Importantly this prescription can also be used for $\eta \to 0$. In this case, one does not deform L_0 but changes the gauge connection by modifying the Wilson line scalar a^I on a fixed brane. The imaginary part of the $\mathcal{N}=1$ coordinates arising from the gauge connection $A_{\rm D6}$ can now be also written as a chain integral $\int_{\mathcal{C}_4} \mathcal{F}_{\rm D6} \wedge \tilde{\beta}^i$. Thus, we find that the ζ^i are given by the elegant expression

$$\zeta^i = -i \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{\rm D6}) \wedge \tilde{\beta}^i \ . \tag{3.4.24}$$

At leading order in the η-expansion the complex coordinates ζ^i are encoded by a one-form $\mathcal{A}_c$ on L_0 with expansion

$$\mathcal{A}_c = -i\eta \lrcorner J_c + iA_{\rm D6} = \zeta^i \tilde{\alpha}_i \ , \tag{3.4.25}$$

into a basis $\tilde{\alpha}_i$ of $H^1(L_0, \mathbb{Z})$. Let us close by noting that (3.4.24) naturally includes a possible D6-brane flux. It would be interesting to evaluate all expressions found below including this flux. However, we will keep $f_{\rm D6} = 0$ in most of the computations.

3.4.3 The open-closed Kähler potential and $\mathcal{N}=1$ coordinates

In the following we determine the $\mathcal{N}=1$ data for the kinetic terms of the four-dimensional effective action by specifying the $\mathcal{N}=1$ complex coordinates, the Kähler potential and the gauge coupling function for the U(1) gauge theory on the D6-brane. We will do this by only including a finite set of deformations specified in the last two subsections. Note that these deformations will be obstructed by a scalar potential, since one always needs to impose the supersymmetry conditions (3.3.3) for the deformed D6-brane which depend on both the open as well as closed moduli. One thus expects that only a space of complex dimension smaller than $\frac{1}{2}b^3(Y) + h^{1,1}_-(Y) + b^1(L_0)$ can be studied as a true open-closed moduli space which is classically un-obstructed by a scalar potential in the absence of background fluxes. This can be also understood by noting that Type IIA compactifications with D6-branes will admit an M-theory embedding as a compactification on a G_2-manifold [98, 102, 103]. The finite number of massless deformations of this manifold will incorporate the subset of the closed and open deformations of section 3.4.1 and 3.4.2 which are flat directions of the supersymmetry conditions (3.3.3).

Let us start by noting that the D6-brane degrees of freedom are still encoded by the complex coordinates ζ^i which have been introduced in (3.4.19) and (3.4.22). From the closed string sector we find the complexified Kähler structure deformations t^a introduced in (3.2.8). As we will check later on, the definition of the remaining closed string complex coordinates is corrected by a functional depending on the open coordinates ζ^i. More precisely, they arrange very elegantly as

$$N^k = U^k - 2\,\partial_{V_k}(e^{2D}K_o) + i\xi^k \ , \qquad T_\lambda = U_\lambda - 2\,\partial_{V^\lambda}(e^{2D}K_o) + i\tilde{\xi}_\lambda, \tag{3.4.26}$$

where the real scalars $(\xi^k, \tilde{\xi}_\lambda)$ arise in the expansion (3.2.13), and we recall that $U^k = 2\text{Re}(CX^k)$, $U_\lambda = 2\text{Re}(C\mathcal{F}_\lambda)$ as well as $V_k = 2e^{2D}\text{Im}(C\mathcal{F}_k)$, $V^\lambda = -2e^{2D}\text{Im}(CX^\lambda)$ are periods of $C\Omega$. In summary, we can simply write

$$\zeta^i = u^i_c + ia^i \ , \qquad M^K = U^K - 2\,\partial_{V_K}(e^{2D}K_o) + i\xi^K \ , \tag{3.4.27}$$

where $\xi^K = (\xi^k, \tilde{\xi}_\lambda)$ and the abbreviations $U^K = (U^k, U_\lambda)$ and $V_K = (V_k, V^\lambda)$ are as in (3.4.6). The real function K_o is now dependent on both u^i as well as U^K (or rather V_K). To see this, note that $e^\phi * \theta_i = 2s_i \lrcorner \text{Im}(C\Omega)$ as introduced in (3.3.5), clearly depends on $\text{Im}(C\Omega)$. Performing the η-expansion of K_o around $\eta = 0$ one finds

$$\begin{aligned} -2\,\partial_{V_k}(e^{2D}K_o) &= -\partial_{V_k}(e^{2D}\mathcal{G}_{ij})|_{\eta=0}u^i u^j + \dots\ , \\ &= -\frac{1}{2}\int_{L_0}\tilde{\alpha}_i \wedge s_l \lrcorner \beta^k \Big(\int_{L_0}\tilde{\beta}^j \wedge s_l \lrcorner J\Big)^{-1} u^i u^j + \dots\ , \end{aligned} \quad (3.4.28)$$

as we derive in detail in appendix A.1. Together with a similar expression for $\partial_{V^\lambda}(e^{2D}K_o)$, replacing $\beta^k \to \alpha_\lambda$, one can use (3.4.28) to derive the leading order effective action. In order to do that, we also need to specify the Kähler potential, to which we will turn next. Realize that as a trivial check of (3.4.27) one recovers the bulk $\mathcal{N} = 1$ coordinates (N'^k, T'_k) given in (3.2.14) if $K_o = 0$.

To encode the leading order D6-brane effective action found in (3.3.39) and (3.3.52), we finally need to specify the Kähler potential. It is given by

$$K = K^{\text{ks}} + K^{\text{Q}} = -\ln\Big[\tfrac{4}{3}\int_Y J \wedge J \wedge J\Big] - 2\ln\Big[i\int_Y C\Omega \wedge \overline{C\Omega}\Big]\ , \qquad e^K = \frac{1}{8}e^{4D}\mathcal{V}^{-1}. \quad (3.4.29)$$

Note that K has to be evaluated in terms of the $\mathcal{N} = 1$ coordinates (3.4.26) and thus only depends on $\zeta^i + \bar{\zeta}^i$, $M^K + \bar{M}^K$ and $t^a - \bar{t}^a$. This can be done explicitly for the first term K^{ks} since

$$K^{\text{ks}}(t, \bar{t}) = -\ln\big[\tfrac{i}{6}\mathcal{K}_{abc}(t - \bar{t})^a(t - \bar{t})^b(t - \bar{t})^c\big]\ , \quad (3.4.30)$$

where $\mathcal{K}_{abc} = \int_Y \omega_a \wedge \omega_b \wedge \omega_c$ are the triple intersection numbers. It corresponds to the volume of the Calabi-Yau manifold Y and will be corrected by perturbative and non-perturbative string worldsheet contributions. For the second term K^{Q} it is in general hard to find an explicit expression in terms of the $\mathcal{N} = 1$ coordinates. However, we are nevertheless able to check that the general kinetic terms determined by the derivatives of K^{Q} match the leading order terms found by dimensional reduction.

Let us summarize the derivatives of the Kähler potential K^Q. We note that the derivatives with respect to the closed string moduli N^k, T_λ take the same form as in (3.4.6), $\partial_{N^k}K = V_k$, $\partial_{T_\lambda}K = V^\lambda$. However, (V_k, V^λ) now depend implicitly on the open string coordinates ζ^i through the evaluation of the closed string expressions in terms of the $\mathcal{N} = 1$ coordinates (3.4.26), i.e. one has to view $V_K(u^i, U^K)$. The derivatives with respect to ζ^i will be postponed to section 3.5. In summary one finds that

$$K_i = e^{2D}v_i\ , \qquad K_k = 2\,e^{2D}\text{Im}(C\mathcal{F}_k)\ , \qquad K^\lambda = -2\,e^{2D}\text{Im}(CX^\lambda)\ . \quad (3.4.31)$$

where $K_i = \partial K/\partial\zeta^i$, $K_k = \partial K/\partial N^k$ and $K_\lambda = \partial K/\partial T_\lambda$. Also the Kähler metric can be evaluated explicitly. One finds for the derivatives with respect to $(N^k, T_\lambda, \zeta^i)$ that

$$\begin{aligned} K_{k\bar{l}} &= G_{kl}\ , & K_{\lambda\bar{\kappa}} &= G^{\lambda\kappa}\ , & K_{k\bar{\lambda}} &= G_k^\lambda\ , \\ K_{i\bar{j}} &= e^{2D}\mathcal{G}_{ij} + \mathcal{I}_i^K G_{KL}\mathcal{I}_j^L\ , & K_{i\bar{k}} &= \mathcal{I}_i^L G_{Lk}\ , & K_{i\bar{\lambda}} &= \mathcal{I}_i^L G_L^\lambda\ , \end{aligned} \quad (3.4.32)$$

where $G_{KL} = (G_{kl}, G^{\lambda\kappa}, G_k^\lambda)$ was given in (3.3.66), and $\mathcal{I}_i^K = (\mathcal{I}_i^k, \mathcal{I}_{i\lambda})$ are the derivatives

$$\mathcal{I}_i^k = \frac{\partial^2 K_o}{\partial V_k \partial\zeta^i}\ , \qquad \mathcal{I}_{i\lambda} = \frac{\partial^2 K_o}{\partial V^\lambda \partial\zeta^i}\ . \quad (3.4.33)$$

In appendix A.1 we will check these expressions by an explicit computation, and match these data with the leading order effective action obtained in section 3.3.

Let us comment on the special form of the Kähler metric (3.4.32). It can be directly inferred by making use of the invariance of the kinetic terms under the shift symmetries

$$N^k \to N^k + i\Lambda^k\ , \qquad T_\lambda \to T_\lambda + i\Lambda_\lambda\ , \quad (3.4.34)$$

for arbitrary constants $(\Lambda^k, \Lambda_\lambda)$. If such shift symmetries exist in the full four-dimensional effective action one can replace the chiral multiplets N^k and T_λ) by linear multiplets (V_k, C_k^2) and (V^λ, C_2^λ), as described in more details in appendix A.2. Here $V_K = (V_k, V^\lambda)$ are the scalars dual to $(\text{Re}N^k, \text{Re}T_\lambda)$ given in (3.4.31) and (C_k^2, C_2^λ) are two-forms dual to the scalars from C_3. The chiral multiplets and linear multiplets are connected by a Legendre transform, and the new real function encoding the kinetic terms of the multiplets is given by

$$\begin{aligned}\tilde{K}(V, \zeta + \bar{\zeta}) &= K(V) - V_k(N^k + \bar{N}^k) - V^\lambda(T_\lambda + \bar{T}_\lambda) \\ &= K(V) + 4\frac{\partial(e^{2D}K_o)}{\partial V_K}V_K - 4\,,\end{aligned} \tag{3.4.35}$$

where we have inserted (3.4.27) and used that $V_k U^k + V^\lambda U_\lambda = e^{2D}\int C\Omega \wedge \overline{C\Omega} = 1$ to obtain the constant term -4. The key point to notice is that in this dual picture all quantities are functions of V_K, ζ^i. In particular, this implies that now $K(V) = -2\ln(e^{-2D}) = -2\ln(i\int C\Omega \wedge \overline{C\Omega})$ is independent of ζ^i, and all equalities found for the moduli space of special Lagrangian cycles of section 3.4.2 can be directly applied. Since the linear multiplet picture is just an equivalent dual description one can equally express the kinetic terms in the chiral multiplet picture in terms of the derivatives of $\tilde{K}$. Let us denote by $\tilde{K}^{KL} = \partial_{V_K}\partial_{V_L}\tilde{K}$, and by $\tilde{K}_{KL}$ its inverse. Similarly, we denote by $\tilde{K}^K_{\zeta^i}$ and $\tilde{K}_{\zeta^i\zeta^j}$ the remaining second derivatives with respect to ζ^i and V_K. The expression for the kinetic terms then has the form

$$\begin{aligned}\mathcal{L}^{\text{kin}} &= -(\tilde{K}_{\zeta^i\bar{\zeta}^j} + \tilde{K}^K_{\zeta^i}\tilde{K}_{KL}\tilde{K}^L_{\bar{\zeta}^j})\, d\zeta^i \wedge *d\bar{\zeta}^j + \tilde{K}_{KL}\left(d\text{Re}M^K \wedge *d\text{Re}M^L + d\xi^K \wedge *d\xi^J\right) \\ &\quad -2\,\tilde{K}_{KL}\tilde{K}^L_{\zeta^j}\left(d\text{Re}M^K \wedge *du^j + d\xi^K \wedge *da^j\right)\end{aligned} \tag{3.4.36}$$

This is precisely the form of the Kähler metric (3.4.32) and it remains to check that indeed $\tilde{K}_{KL} = G_{KL}$, $\tilde{K}_{\zeta^i\bar{\zeta}^j} = e^{2D}\mathcal{G}_{ij}$ and $\tilde{K}^K_{\zeta^i} = \mathcal{I}^K_i$. For the leading order actions found in section 3.3 this is done in appendix A.1. Note that the form of the metric (3.4.36) is also inherited if only a potential term breaks the shift-symmetries (3.4.34).

Let us make a brief comment on the appearance of the term $d\text{Re}M^I \wedge *du^j$. This term corresponds to a kinetic mixing between complex structure and brane deformations, and would be expected to appear in higher order expansions of the Dirac-Born-Infeld action. In this section however it was obtained by simply analyzing the $\mathcal{N} = 1$ characteristic data and the moduli space.

3.4.4 Gauge coupling functions and kinetic mixing for finite deformations

Having discussed the kinetic terms for the scalars in the $\mathcal{N} = 1$ effective theory we will now turn to an analysis of the kinetic terms for the $U(1)$ vectors fields. We have shown in section 3.3 in the case one focuses on harmonic modes in the reduction that the spectrum contains a D6-brane $U(1)$ vector A as well as $h_+^{(1,1)}$ bulk $U(1)$ vectors A^α. The leading gauge coupling function for the brane $U(1)$ was derived in section 3.3.4 and given by

$$f_{\text{r}} = \int_{L_0} \left(2\,\text{Re}(C\Omega) + iC_3\right) = \delta_k N'^k - \delta^\lambda T'_\lambda\,, \tag{3.4.37}$$

where $\delta_k = \int_{L_0}\alpha_k$ and $\delta^\lambda = \int_{L_0}\beta^\lambda$. However, as we have discussed in section 3.4.3, the inclusion of the open moduli forces us to introduce the modified complex coordinates N^k, T_λ given in (3.4.26). In order to obtain a holomorphic gauge coupling function it is expected that (3.4.37) is modified to

$$f = \delta_k N^k - \delta^\lambda T_\lambda\,. \tag{3.4.38}$$

The modifications in (3.4.38) did not appear in our leading order dimensional reduction, but are expected to arise a higher order in the brane deformations. As we will see shortly open moduli corrections to f_{r} are also obtained after a careful treatment of the two dual bulk gauge fields A^α, A_α introduced in (3.3.50). Recall that the gauge coupling function for the bulk R-R $U(1)$ vectors A^α is simply given by [82]

$$f_{\alpha\beta} = i \int_Y \omega_\alpha \wedge \omega_\beta \wedge \omega_a\, t^a = i\mathcal{K}_{\alpha\beta a} t^a = -i\bar{\mathcal{N}}_{\alpha\beta}\ . \tag{3.4.39}$$

where $\mathcal{N}_{\alpha\beta}$ is the complex matrix already introduced in (3.3.67). Clearly, $f_{\alpha\beta}$ is holomorphic in the complex fields t^a. Since the t^a are not corrected by the open moduli one expects the result (3.4.39) to remain valid also in the leading order reduction with a D6-brane. We will show in the following that this is indeed the case. More interestingly, we find that there are further corrections depending on the open moduli and D6-brane fluxes which induce a kinetic mixing of the brane and bulk $U(1)$ gauge fields.

Let us now turn to a more careful analysis of the gauge coupling functions including the brane moduli. In order to do that we summarize the action for all vector fields including the dual A_λ introduced in (3.3.50). The mixing terms proportional to $dA^\alpha \wedge F$ and $dA_\alpha \wedge F$ have appeared in the reduction of the Chern-Simons action in (3.3.74). The brane couplings have to be taken into account when eliminating A_α in favor of A^α by using vector-vector duality in four dimensions as enforced by (3.2.3). A detailed calculation can be found in appendix A.3 which uses a procedure similar to the one of ref. [27]. Here we just present the results. The action obtained after a careful elimination of A_λ is

$$\begin{aligned} S^{(4)}_{\text{vec}} = - & \int \tfrac{1}{2}\text{Re} f_\alpha dA^\alpha \wedge *F + \tfrac{1}{2}\text{Im} f_\alpha dA^\alpha \wedge F + \tfrac{1}{2}\text{Im}\mathcal{N}_{\alpha\beta} dA^\alpha \wedge * dA^\beta \\ & + \tfrac{1}{2}\text{Re}\mathcal{N}_{\alpha\beta} dA^\alpha \wedge dA^\beta + \tfrac{1}{2}\text{Re} f_{\text{cor}}\, F \wedge *F + \tfrac{1}{2}\text{Im} f_{\text{cor}} F \wedge F \end{aligned}$$

where the gauge coupling function f_α encoding the kinetic mixing between bulk and brane $U(1)$'s is given by

$$f_\alpha = -4(i\bar{\mathcal{N}}_{\alpha\beta}\tilde{\mathcal{J}}^\beta + i a^j \Delta_{j\alpha} + i\Gamma_\alpha)\ , \tag{3.4.40}$$

and the corrected gauge coupling function f_{cor} for the brane $U(1)$ is

$$f_{\text{cor}} = f_{\text{r}} + 4(i\bar{\mathcal{N}}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha + i a^j \Delta_{j\beta} + i\Gamma_\beta)\tilde{\mathcal{J}}^\beta\ . \tag{3.4.41}$$

The coefficient functions are given by $\tilde{\mathcal{J}}^\alpha = \int_{\mathcal{C}_4} \tilde{\omega}^\alpha$, $\Delta_{j\alpha} = \int_{L_0} \tilde{\alpha}_j \wedge \omega_\alpha$ and $\Gamma_\alpha = \int_{\mathcal{C}_4} \omega_\alpha \wedge f_{\text{D6}}$ as introduced in section 3.3. Recall that $\Delta_{j\alpha}$ is independent of the moduli, while $\tilde{\mathcal{J}}^\beta, \Gamma_\alpha$ depend on the brane deformations through the chain $\mathcal{C}_4$.

To study the holomorphicity properties of the gauge couplings we discuss f_α and f_{cor} in turn. One notes that the first term in (3.4.40) can be rewritten as

$$i\bar{\mathcal{N}}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha = \int_{\mathcal{C}_4} i\bar{\mathcal{N}}_{\alpha\beta}\tilde{\omega}^\alpha = \int_{\mathcal{C}_4} (J - iB) \wedge \omega_\beta = u^j_c\, \Delta_{j\beta}\ , \tag{3.4.42}$$

where we have used (3.4.22) to obtain the factor u^j_c. Using this expression it is straightforward to rewrite the gauge coupling f_α in the absence of brane fluxes as

$$f_\alpha = -4\zeta^j \Delta_{j\beta}\ , \tag{3.4.43}$$

which is clearly holomorphic on the open moduli $\zeta^i = u^i_c + ia^i$. It would be interesting to extend these arguments to include the D6-brane flux f_{D6}.

Let us now turn to the analysis of the corrected gauge coupling function f_{cor} of the brane $U(1)$. Using (3.4.41) and (3.4.40) one sees that it can be written as

$$f_{\text{cor}} = f_{\text{r}} - f_\alpha \tilde{\mathcal{J}}^\alpha\ , \tag{3.4.44}$$

the additional term is at least of second order in the open moduli. One notes that the real part of $f_{\rm cor}$ is given by

$$\text{Re}f_{\rm cor} = \text{Re}f_{\rm r} + 4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha\tilde{\mathcal{J}}^\beta = \text{Re}f_{\rm r} + \text{Re}f_\alpha \text{Re}f^{\alpha\beta}\text{Re}f_\beta \ , \tag{3.4.45}$$

which can be inferred from (3.4.40) and (3.4.44). This result generalizes to the space of infinite deformations by replacing $f_{\rm r}$ with $f_{{\rm r}\,IJ}$, and f_α with $f_{\alpha I}$. The expressions for these are straightforward generalizations of (3.4.40)-(3.4.45) with the abbreviations introduced in section 3.3.4. Hence, the real part of the gauge coupling function takes the form

$$\text{Re}\,\mathbf{f} = \begin{pmatrix} \text{Re}f_{{\rm r}\,IJ} + \text{Re}f_{\gamma I}\text{Re}f^{\gamma\delta}\text{Re}f_{\delta J} & \text{Re}f_{I\alpha} \\ \text{Re}f_{J\beta} & \text{Re}f_{\alpha\beta} \end{pmatrix}, \tag{3.4.46}$$

and can be easily inverted. This result will be important in section 3.5, when we compute the scalar potential coming from D-terms since it involved the inverse $(\text{Re}\,\mathbf{f})^{-1}$.

Let us close this section by making some general remarks about the holomorphicity of the gauge coupling function $f_{\rm cor}$ in (3.4.44). In order to do that, one has express it in terms of the $\mathcal{N} = 1$ coordinates N^k, T_λ, t^a and ζ^i. However, recall from (3.4.26) that also the N^k and T_λ receive corrections by the open deformations. In fact, we η-expand

$$\text{Re}(N^k - N'^k)\delta_k - \text{Re}(T_\lambda - T'_\lambda)\delta^\lambda = u^i\Big(-\tfrac{1}{2}\int_{L_0}\tilde{\alpha}_i \wedge \eta\lrcorner\beta^k \int_{L_0}\alpha_k + \tfrac{1}{2}\int_{L_0}\tilde{\alpha}_i\wedge\eta\lrcorner\alpha_\lambda\int_{L_0}\beta^\lambda\Big) + \dots \ , \tag{3.4.47}$$

where we have used (3.4.21) and (3.4.28). To compare this result, we also η-expand (3.4.44) to find

$$\text{Re}f_{\rm cor} - \text{Re}f_{\rm r} = 4u^i\int_{L_0}\tilde{\alpha}_i\wedge\omega_\alpha\int_{L_0}\eta\lrcorner\tilde{\omega}^\alpha + \dots \ . \tag{3.4.48}$$

This indicates that the result for $f_{\rm cor}$ cannot be complete. In particular, it is conceivable that a contribution from the two-forms ω_a is missing which arises at higher order in the Kaluza-Klein reduction. This is similar to what was found in [27,93] for D7- and D5-branes on the type IIB side. It would be interesting to complete this computation to higher order and determine the fully corrected gauge coupling function. For example, one loop corrections for the gauge-coupling function were calculated for orbifold models in [104].

3.5 General deformations and the D- and F-term potential

In the previous section we considered D6-branes with a finite number of deformations arising from the expansion into harmonic forms on the brane world-volume. Using harmonic modes one infers that the scalar potential (3.3.40) vanishes. A non-vanishing potential precisely arises for deformations which violate the supersymmetry conditions that the three-cycle is special Lagrangian. In this section we include such deformations into the discussion and analyze the $\mathcal{N} = 1$ encoding the geometry on the infinite field space. We discuss the Kähler potential and show that the scalar potential (3.3.40) indeed arises from a D-term, induced by a gauging, and a holomorphic superpotential. In order to do that we will keep the background geometry fixed and only consider the variations of the brane degrees of freedom.

3.5.1 A local Kähler metric for general deformations of L_0

In the general reduction performed in section 3.3.2 we already included a whole tower of normal deformations of L_0 as well as the whole tower of Kaluza-Klein modes in $F_{\rm D6}$ parameterizing variations around a background connection A_0. Together, these modes parameterize a neighborhood around

(L_0, A_0) in an infinite dimensional field-space $\mathcal{V}_o$. We will focus on the neighborhood around a supersymmetric L_0 and mainly be concerned with the local geometrical structure of $\mathcal{V}_o$. In order to do that we study the tangent space to $\mathcal{V}_o$ at the special Lagrangian L_0 with connection A_0. This tangent space is identified with

$$T_{(L_0,A_0)}\mathcal{V}_o \;\cong\; TY|_{L_0} \cong NL_0 \oplus TL_0 \; . \tag{3.5.1}$$

In this we can identify the s_I introduced in (3.3.32) as basis of sections of NL_0 and the $\tilde{s}^m_I = g^{mn}|_{L_0}(\hat{\alpha}_I)_n$ as sections of TL_0. Note that in defining the tangent vector $\tilde{s}_I$ we have simply raised the tangent index m of the one-form $\hat{\alpha}_I$ introduced in (3.3.34) by the inverse of the induced metric $g_{mn}|_{L_0}$. This also means that we can identify

$$T_{(L_0,A_0)}\mathcal{V}_o \;\cong\; \Omega^1(L_0) \oplus \Omega^1(L_0)\; , \tag{3.5.2}$$

which is naturally parameterized by the basis vectors θ_I and $\hat{\alpha}_I$ introduced in (3.3.32) and (3.3.34).

Using the first identification in (3.5.1) the tangent space $T_{(L_0,A_0)}\mathcal{V}_o$ admits a natural symplectic form

$$\varphi(X,Y) = \frac{1}{2} e^{-\phi} \int_{L_0} J(X,Y)|_{L_0}\, \mathrm{vol}_{L_0} \; . \tag{3.5.3}$$

for $X, Y \in TY|_{L_0}$. It was shown in [88] that the two-form φ on $\mathcal{V}_o$ is actually closed. The tangent space (3.5.1) also admits a natural complex structure I, which is the induced complex structure from the Calabi-Yau manifold Y. At L_0 the complex structure I identifies TL_0 with NL_0 such that complex tangent vectors in $T_{(L_0,A_0)}\mathcal{V}_o$ are given by

$$\partial_{z^I} = \tfrac{1}{2}(s_I - iIs_I)\; , \qquad \partial_{\bar{z}^{\bar{I}}} = \tfrac{1}{2}(s_I + iIs_I)\; . \tag{3.5.4}$$

Since this complex structure is formally integrable, the manifold $\mathcal{V}_o$ is Kähler, with Kähler form

$$\varphi(\partial_{z^I}, \partial_{\bar{z}^J}) = \frac{i}{2} e^{-\phi} \int_{L_0} g(s_I, s_J)\, \mathrm{vol}_{L_0} = i\widehat{\mathcal{G}}_{IJ}\; , \qquad \varphi(\partial_{z^I}, \partial_{z^J}) = \varphi(\partial_{\bar{z}^I}, \partial_{\bar{z}^J}) = 0\; . \tag{3.5.5}$$

Here we have used that $J(Is_I, s_J) = -g(s_I, s_J)$ and the fact that L_0 is Lagrangian such that $J(s_I, s_J) = -J(Is_I, Is_J) = 0$ for normal vectors s_I to L_0. This implies that $\widehat{\mathcal{G}}_{IJ}$ is a Kähler metric, which is locally the second derivative of a Kähler potential $K_o = K_o(z^I, \bar{z}^I)$. Explicitly this means that

$$\widehat{\mathcal{G}}_{IJ} = \partial_{z^I}\partial_{\bar{z}^J} K_o = \tfrac{1}{2} e^{-\phi} \int_{L_0} \theta_I \wedge *\theta_J\; , \tag{3.5.6}$$

with the forms θ_I as introduced in (3.3.32). Note that the real part of the complex coordinates z^I are the normal vectors η^I. This should be contrasted to the complex coordinates ζ^i which were the complexifications of the u^i as discussed in section 3.4.2.

It is interesting to note that there is a natural generalization of the finite-dimensional analysis of section 3.4.2 to the infinite dimensional deformation space. The key will be the use of the four-chain $\mathcal{C}_4$ which interpolates between L_0 and L_η. Clearly, the natural generalization of the complex coordinates ζ^i in (3.4.24) is

$$\zeta^I = -i \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{\mathrm{D6}}) \wedge \hat{\beta}^I\; , \tag{3.5.7}$$

where β^I is the infinite basis of two-forms on L_0 which has been trivially extended to the chain $\mathcal{C}_4$. We have also included the field strength $\mathcal{F}_{\mathrm{D6}}$ on $\mathcal{C}_4$ which is obtained from the gauge connection $\mathcal{A}_{\mathrm{D6}}$ introduced in (3.4.23). A natural proposal for the Kähler potential K_o is given by

$$K_o(\zeta + \bar{\zeta}) = -\frac{1}{2} \int_{\mathcal{C}_4} J \wedge \hat{\beta}^I \int_{\mathcal{C}_4} \mathrm{Im}(C\Omega) \wedge \hat{\alpha}_I\; . \tag{3.5.8}$$

This can be checked by performing an η-expansion around the supersymmetric cycle L_0. This yields the leading term

$$\begin{aligned} K_o(\zeta+\bar\zeta) &= -\tfrac{1}{2}\int_{L_0} s_L \lrcorner J\wedge\hat\beta^I \int_{L_0} s_K\lrcorner \mathrm{Im}(C\Omega)\wedge\hat\alpha_I\,\eta^L\eta^K+\ldots \\ &= \tfrac{1}{4}e^{-\phi}\int_{L_0}\theta_L\wedge\hat\beta^I\int_{L_0}*\theta_K\wedge\hat\alpha_I\,\eta^L\eta^K+\ldots \\ &= \tfrac{1}{2}\,\widehat{\mathcal{G}}_{LK}\,\eta^L\eta^K+\ldots \\ &= \tfrac{1}{8}\mathcal{G}_{LK}(\zeta+\bar\zeta)^L\,(\zeta+\bar\zeta)^K+\ldots \end{aligned} \tag{3.5.9}$$

where here we mean by $\widehat{\mathcal{G}}_{LK},\mathcal{G}_{LK}$ the leading order metrics independent of η. Here we have used (3.3.7) on L_0 to rewrite the contraction $s_K\lrcorner\mathrm{Im}(C\Omega)$ into the Hodge-star on L_0. Using (3.5.9) one sees that (3.5.6) is satisfied. Let us stress that in general the evaluation of K_o as a function of $\zeta^I+\bar\zeta^I$ is non-trivial due to the appearance of the chain $\mathcal{C}_4$ in both integrals of (3.5.8). It would be very interesting the compute K_o explicitly for specific orientifold examples, generalizing the superpotential computations of [105–115].

3.5.2 The superpotential and D-terms

Having discussed the Kähler potential determining the kinetic terms, we will now examine the scalar potential in more detail. More precisely, we will work in a fixed background geometry by fixing Kähler and complex structure deformations and focus on the leading scalar potential V_{DBI} given in (3.3.40). We will show that V_{DBI} splits into an F-term and a D-term piece as

$$V_{\mathrm{DBI}} = V_F + V_D\ , \tag{3.5.10}$$

with

$$V_D = \frac{e^{3\phi}}{\mathcal{V}^2}\int_{L_0} d^*\theta_\eta\wedge *d^*\theta_\eta \tag{3.5.11}$$

and

$$V_F = \frac{e^{3\phi}}{\mathcal{V}^2}\int_{L_0} d\theta_\eta\wedge *d\theta_\eta + (\tilde F - B_2 - d\theta^B_\eta)\wedge *(\tilde F - B_2 - d\theta^B_\eta)\ . \tag{3.5.12}$$

We will show momentarily that $V_F = e^K\mathcal{G}^{IJ}\partial_{\zeta^I}W\overline{\partial_{\zeta^J}W}$ can be obtained from a superpotential W and the metric determined from K_o using only the open string degrees of freedom.

To specify W we aim to define a functional which picks out deformations η such that L_η is a Lagrangian submanifold $J|_{L_\eta}=0$. In section 3.3.4 we defined a chain $\mathcal{C}_4$ with boundaries L_η and L_0. Recall also that we extended the gauge field A_{D6} from L_0 to $\mathcal{C}_4$ as in (3.4.23), such that the extension $\mathcal{F}_{\mathrm{D6}} = d\mathcal{A}_{\mathrm{D6}}$ satisfies

$$\mathcal{F}_{\mathrm{D6}}|_{L_0} = f_{\mathrm{D6}}\ , \qquad \mathcal{F}_{\mathrm{D6}}|_{L_\eta} = f_{\mathrm{D6}} + a^I d\hat\alpha_I\ . \tag{3.5.13}$$

In the following we will again set again the D-brane flux f_{D6} to zero. One next identifies the superpotential functional

$$W = \int_{\mathcal{C}_4}(J_c - \mathcal{F}_{\mathrm{D6}})\wedge(J_c-\mathcal{F}_{\mathrm{D6}}) \tag{3.5.14}$$

depending on the open string data as well as the complexified Kähler form (3.2.8). This is an extension of the functional introduced in ref. [96], since we have included the B-field through the complex two-form J_c. Note that a superpotential of this form has been already discussed in [116, 117].

Let us briefly study the holomorphicity properties of W. Clearly, W is holomorphic with respect to variations of the complexified Kähler form J_c parameterized by the scalars t^a in (3.2.8). However, note that one first has to express W as a function of the open fields $\zeta^I = u_c^I + ia^I$ introduced in (3.5.7). To check that W it is a holomorphic section in the ζ^I we show that $\partial_{\bar\zeta^I} W = (\partial_{u_c^I} + i\partial_{a^I})W = 0$. The derivative with respect to Wilson lines is

$$\partial_{a^I} W = 2\int_{L_\eta} (J_c - \mathcal{F}_{\text{D6}}) \wedge \hat\alpha_I = 2\int_{L_0} (J_c - \mathcal{F}_{\text{D6}}) \wedge \hat\alpha_I + 2\int_{L^0} d(\eta \lrcorner J_c - a^J \hat\alpha_J) \wedge \hat\alpha_I + \ldots \quad (3.5.15)$$

To evaluate the derivative with respect to u_c^I we expand the chain integral around the special Lagrangian cycle L_0 in terms of the deformations

$$\begin{aligned} W &= 2\int_{L_0} \eta\lrcorner J_c \wedge (J_c - \tilde F) + \int_{L_0} \eta\lrcorner J_c \wedge \mathcal{L}_\eta J_c + \ldots \\ &= 2\int_{L_0} (\eta\lrcorner J_c) \wedge (J_c - \mathcal{F}_{\text{D6}}) + \int_{L_0} \eta\lrcorner J_c \wedge d(\eta\lrcorner J_c + 2\, a^I \hat\alpha_I) + \ldots \end{aligned} \quad (3.5.16)$$

Recalling $\eta\lrcorner J_c = \theta^B_\eta + i\theta_\eta = iu_c^I \hat\alpha_I + \ldots$ one sees that by comparing (3.5.15) with $\partial_{u_c^I} W$ obtained from (3.5.16) that the superpotential is holomorphic in $\zeta^I = u_c^I + ia^I$.

It is now straightforward to determine the F-term potential using the expression (3.5.16). The real part of the derivative of (3.5.16) is given by

$$\text{Re}\, \partial_{\zeta^I} W = 2\int_{L_0} d\theta_\eta \wedge \hat\alpha_I \ . \quad (3.5.17)$$

Note that $d\theta_\eta$ is a 2-form in L_0 and therefore can be expanded in the infinite basis $*\hat\alpha_I$ as $d\theta_\eta = c^I * \hat\alpha_I$ The coefficients c^I can be obtained by taking on both sides the wedge product with α_J and integrate on L_0. Inverting this relation for c^I and taking the Hodge star one finds

$$* \, d\theta_\eta = \tfrac{1}{2} e^{-\phi} \hat\alpha_I\, \mathcal{G}^{IJ} \int_{L_0} \hat\alpha_J \wedge d\theta_\eta \ . \quad (3.5.18)$$

We proceed analogously with the imaginary part $\text{Im}\, \partial_{\zeta^I} W$ obtained from (3.5.16) and expand the two-form $(B - \tilde F + d\theta^B_\eta)$ in the $*\hat\alpha_I$ basis. The F-term potential is thus given by

$$\begin{aligned} V_F &= e^K \mathcal{G}^{IJ} \partial_{\zeta^I} W \overline{\partial_{\zeta^J} W} \\ &= \frac{e^{2D}}{2\mathcal{V}} \int_{L_0} d\theta_\eta \wedge \hat\alpha_I\, \mathcal{G}^{IJ} \int_{L_0} \hat\alpha_J \wedge d\theta_\eta \\ &\quad + \frac{e^{2D}}{2\mathcal{V}} \int_{L_0} (B - \tilde F + d\theta^B_\eta) \wedge \hat\alpha_I\, \mathcal{G}^{IJ} \int_{L_0} \hat\alpha_J \wedge (B - \tilde F + d\theta^B_\eta) \\ &= \frac{1}{\mathcal{V}^2} e^{3\phi} \int_{L_0} d\theta_\eta \wedge * d\theta_\eta + (B - \tilde F + d\theta^B_\eta) \wedge *(B - \tilde F + d\theta^B_\eta) \end{aligned} \quad (3.5.19)$$

which agrees with the result (3.5.12) obtained from dimensional reduction, and reduces to the result of McLean [87] in the limit of vanishing B field. As expected, the condition for vanishing of the potential and therefore to preserve supersymmetry is the closedness of θ_η and θ^B_η, as well as the condition $(B - \tilde F)|_{L_0} = 0$.

Finally, we also compute the D-term potential in (3.5.11) induced by the gaugings of the scalars $\hat a^I$ in (3.3.37) and $(\xi^k, \tilde\xi_\lambda)$ in (3.3.63). More precisely, these scalars are charged under the gauge transformations $A^I \to A^I + d\Lambda^I$ of the $U(1)$ vectors A^I as

$$\hat a^I \to \hat a^I - \Lambda^I \ , \qquad (\xi^k, \tilde\xi_\lambda) \to (\xi^k - \delta^k_I \Lambda^I, \tilde\xi_\lambda - \delta_{\lambda I}\Lambda^I) \quad (3.5.20)$$

The potential arising from D-terms can be calculated by

$$V_D = \tfrac{1}{2}\text{Re}f^{AB} D_A D_B\ , \qquad \partial_A D_I = K_{A\bar{B}} X_I^B\ , \tag{3.5.21}$$

where X_I^B are the Killing symmetries appearing in the covariant derivative $D\xi^k = d\xi^k + X_I^k A^I$. Explicitly they take the form

$$X_I^k = \int_{L_0} h_I \beta^k + \int_{\mathcal{C}_4} dh_I \wedge \beta^k\ , \qquad X_{I\lambda} = \int_{L_0} h_I \alpha_\lambda + \int_{\mathcal{C}_4} dh_I \wedge \alpha_\lambda\ . \tag{3.5.22}$$

The leading inverse gauge coupling function is simply

$$(\text{Re}f_{\text{r}}^{-1})^{IJ} = \left(\int_{L_0} h_I h_J 2\text{Re}(C\Omega)\right)^{-1}. \tag{3.5.23}$$

Integrating (3.5.21) we obtain the D-terms

$$D_I = -2e^{2D}\left(\int_{L_0} h_I \text{Im}C\Omega + \int_{\mathcal{C}_4} dh_I \wedge \text{Im}\, C\Omega\right)\ . \tag{3.5.24}$$

We can expand the chain along an infinite set of brane deformations and obtain

$$D_I = -2e^{2D}\int_{L_0} h_I \text{Im}C\Omega - 2e^{2D}\int_{L_0} h_I d(\eta \lrcorner \text{Im}C\Omega) + \dots\ , \tag{3.5.25}$$

where we have used that the functions h_I are translated constantly along the chain. Now we repeat a similar calculation as for the F-term, by expanding the three forms into $*h_I$ and noticing that on the L_0 cycle $\int h_J * h_I = e^\phi \int h_J h_I 2\text{Re}(C\Omega) = e^\phi \text{Re}f_{\text{r}IJ}$. The potential is then,

$$V_D = \frac{e^{3\phi}}{\mathcal{V}^2}\int_{L_0} 4\ \text{Im}C\Omega \wedge *\text{Im}C\Omega + 4\ \text{Im}C\Omega \wedge *d*\theta + d*\theta \wedge *d*\theta\ . \tag{3.5.26}$$

From the condition $\text{Im}C\Omega|_{L_0} = 0$ only the last term survives, yielding the remaining term obtained from dimensional reduction. The vanishing of the D-term potential, which is necessary in a supersymmetric vacuum, happens when the two-form $*\theta_\eta$ is closed.

3.6 Mirror Symmetry with D-branes

In this final section we relate the Type IIA $\mathcal{N} = 1$ characteristic data found in the previous sections with the data for Type IIB orientifolds with space-time filling D3-, D5- and D7-branes. In order to do that, we first review some basics of Type IIB orientifolds following [82]. To define the orientifold set-up starting with Type IIB string theory compactified on a Calabi-Yau manifold $\tilde{Y}$, one acts with a discrete involutive symmetry $\mathcal{O}$ containing worldsheet parity Ω_p. In Type IIB one still is left with two options of constructing such an involution. These correspond to the situations with O3/O7 or O5/O9 orientifold planes:

$$\begin{array}{lclcl} \mathcal{O}_1 &=& \Omega_p \sigma_B (-)^{F_L}\ , & \sigma_B^*\Omega = -\Omega\ , & \text{O3/O7}\ , \\ \mathcal{O}_2 &=& \Omega_p \sigma_B\ , & \sigma_B^*\Omega = \Omega\ , & \text{O5/O9}\ . \end{array} \tag{3.6.1}$$

Here σ_B is a holomorphic (instead of antiholomorphic, as in the Type IIA case) involutive symmetry $\sigma_B^2 = 1$ of the Calabi-Yau target space, and F_L is the space-time fermion number in the left-moving sector. The subspace of fields which are invariant under the orientifold projection has to satisfy

$$\begin{array}{lcl} \sigma_B^*\phi &=& \phi\ , \\ \sigma_B^* g &=& g\ , \\ \sigma_B^* B_2 &=& -B_2\ , \end{array} \qquad \begin{array}{lcl} & \underline{O3/O7} & \\ \sigma_B^* C_0 &=& C_0\ , \\ \sigma_B^* C_2 &=& -C_2\ , \\ \sigma_B^* C_4 &=& C_4\ , \end{array} \qquad \begin{array}{lcl} & \underline{O5/O9} & \\ \sigma_B^* C_0 &=& -C_0\ , \\ \sigma_B^* C_2 &=& C_2\ , \\ \sigma_B^* C_4 &=& -C_4\ , \end{array} \tag{3.6.2}$$

where the first column is identical for both involutions σ_B in (3.6.1). The involution allows us to separate the cohomologies into even and odd eigenspaces $H^{(p,q)} = H_+^{(p,q)} \oplus H_-^{(p,q)}$.

Let us focus on the closed string sector for the moment. Locally the truncated moduli space of Type IIB orientifolds can then be written as a direct product

$$\mathcal{M}_B^{\mathrm{K}} \times \mathcal{M}_B^{\mathrm{Q}} \, . \tag{3.6.3}$$

Here $\mathcal{M}_B^{\mathrm{Q}}$ is a Kähler manifold and spanned by the dilaton, the Kähler structure deformations, the NS-NS B-field and the R-R scalars. $\mathcal{M}_B^{\mathrm{K}}$ is a special Kähler manifold spanned by the complex structure deformations of $\tilde{Y}$ respecting the constraints (3.6.1). In contrast, recall that in Type IIA $\mathcal{M}_A^{\mathrm{Q}}$ is spanned by the dilaton, the complex structure deformations and the R-R scalars, while $\mathcal{M}_A^{\mathrm{K}}$ is spanned by the Kähler deformations and the NS-NS B-field. The Type IIB effective theory also contains $h_+^{(2,1)}(h_-^{(2,1)})$ vector multiplets for orientifolds with $O3/O7(O5/O9)$ planes, whereas in Type IIA one as $h_+^{(1,1)}$ vector multiplets. The number of multiplets from the closed string sector is shown in Table 3.6.1.

multiplets	IIA_Y $O6$	$\mathrm{IIB}_{\tilde{Y}}$ $O3/O7$	$\mathrm{IIB}_{\tilde{Y}}$ $O5/O9$
vector multiplets	$h_+^{(1,1)}$	$h_+^{(2,1)}$	$h_-^{(2,1)}$
chiral multiplets in $\mathcal{M}^{\mathrm{K}}$	$h_-^{(1,1)}$	$h_-^{(2,1)}$	$h_+^{(2,1)}$
chiral multiplets in $\mathcal{M}^{\mathrm{Q}}$	$h^{(2,1)}+1$	$h^{(1,1)}+1$	$h^{(1,1)}+1$

Table 3.6.1: Number of $N=1$ multiplets of orientifold compactifications.

Applying mirror symmetry to this $\mathcal{N}=1$ set-up one expects that the $\mathcal{M}_B^{\mathrm{Q}}$ space of type IIB should be identified with the $\mathcal{M}_A^{\mathrm{Q}}$ moduli space of the mirror IIA, and similarly $\mathcal{M}_B^{\mathrm{K}}$ with $\mathcal{M}_A^{\mathrm{K}}$. Requiring $\tilde{Y}$ to be the mirror manifold of Y, the mirror map between the moduli spaces implies that for the different orientifold setups

$$\begin{aligned} O3/O7 \quad &: \qquad h_-^{(1,1)}(Y) = h_-^{(2,1)}(\tilde{Y})\,, \qquad h_+^{(1,1)}(Y) = h_+^{(2,1)}(\tilde{Y})\,, \\ O5/O9 \quad &: \qquad h_-^{(1,1)}(Y) = h_+^{(2,1)}(\tilde{Y})\,, \qquad h_+^{(1,1)}(Y) = h_-^{(2,1)}(\tilde{Y})\,, \end{aligned} \tag{3.6.4}$$

as well as $h^{(2,1)}(Y) = h^{(1,1)}(\tilde{Y})$ for both set-ups. The mirror mapping for closed moduli is discussed in more detail in [82], and will be briefly recalled below.

In the following we want to extend the mirror identification to include the leading corrections due to the space-time filling D-branes. As we have seen, at leading order the moduli space $\mathcal{M}_A^{\mathrm{K}}$ remains unchanged after the inclusion of open string moduli. This is also true for $\mathcal{M}_B^{\mathrm{K}}$ on the Type IIB side. In section 3.4 we have shown that the open string moduli space of the D6-branes is fibered over the closed string moduli space $\mathcal{M}_A^{\mathrm{Q}}$. The mirror equivalent of this statement has been established in [27,92,93] for $\mathcal{M}_B^{\mathrm{Q}}$ and the moduli space of D3-, D5- or D7-branes. In the reminder of this section we will therefore focus on the discussion of the $\mathcal{M}^{\mathrm{Q}}$ and establish the mirror map including the open degrees of freedom.

3.6.1 Mirror of O3/O7 orientifolds

The moduli space $\mathcal{M}^{\mathrm{Q}}$ is obtained from the four-dimensional scalar parts of the fields J, B_2, C_2, C_4. To make this more precise, we expand

$$\begin{aligned} B_2 &= b^k\,\omega_k\,, \qquad C_2 \;=\; c^k\,\omega_k\,, \quad k=1,\ldots,h_-^{(1,1)}(\tilde{Y})\,, \\ J &= v^\lambda\,\omega_\lambda\,, \qquad C_4 = \rho_\lambda\,\tilde{\omega}^\lambda\,, \quad \lambda=1,\ldots,h_+^{(1,1)}(\tilde{Y})\,. \end{aligned} \tag{3.6.5}$$

The complex coordinates and the Kähler potential which encode the local geometry of $\mathcal{M}_B^{\mathrm{Q}}$ are [?]

$$\begin{aligned} \tau &= C_0 + ie^{-\phi_B}\,, \qquad G^k = c^k - \tau b^k\,, \\ T_\lambda'^B &= e^{-\phi_B}\tfrac{1}{2}\mathcal{K}_{\lambda\rho\sigma}v^\rho v^\sigma + i\rho_\lambda - i\tfrac{1}{2}\mathcal{K}_{\lambda kl}b^k G^l\,, \end{aligned} \tag{3.6.6}$$

and

$$K(\tau, G^k, T_\lambda'^B) = -2\ln\left[e^{-2\phi_B}\int_{\tilde{Y}} J\wedge J\wedge J\right] = \ln(e^{4D_B})\,. \tag{3.6.7}$$

Here D_B is the redefined four-dimensional dilaton. The Kähler potential has to be evaluated as a function of the moduli $\tau, G^k, T_\lambda'^B$ by solving (3.6.6) for v^a, ϕ_B and inserting the result into (3.6.7). The coefficients $\mathcal{K}_{\lambda bc}$ are the intersection numbers of the basis ω_λ of $H_+^{1,1}(\tilde{Y})$ and ω_a of $H_-^{1,1}(\tilde{Y})$, $\mathcal{K}_{\lambda bc} = \int \omega_\lambda\wedge\omega_b\wedge\omega_c$. Note that the above scalar fields can be also obtained from the expansion

$$-\,\mathrm{Re}\,\Phi^{\mathrm{ev}} + i\sum_n e^{-B}\wedge C_{2n} = i\tau + iG^k\omega_k + T_\lambda'^B\tilde{\omega}^\lambda\,, \tag{3.6.8}$$

which has to be evaluated by matching the parts of different form degrees on both sides. Here we have introduced the even form

$$\Phi^{\mathrm{ev}} = e^{-\phi_B}e^{-B_2+iJ} \tag{3.6.9}$$

following the notation of [100].

Let us now recall the mirror map to the Type IIA coordinates without inclusion of the open string degrees of freedom. The $\mathcal{N}=1$ coordinates (N'^k, T'_λ) have been introduced in (3.2.14). Note that on a Calabi-Yau manifold we can use the rescaling invariance of Ω to fix one of the X^I to be constant. At large complex structure there is a special real symplectic basis of $H^3(Y)$ which is distinguished by the logarithmic behavior of the solutions in the complex structure moduli of Y. In particular, this fixes a pair (α_0, β^0), by demanding that X^0, the fundamental period, has no logarithmic singularity. One can use the rescaling of Ω to set the α_0 period to a constant. Note that in the orientifold background $H^3(Y)$ splits into H^3_- and H^3_+. The component chosen to eliminate the rescaling property of Ω can be either in the positive or negative eigenspace of the orientifold projection. We will see momentarily these choices will correspond to different orientifold set-ups on the Type IIB side.

For the O3/O7 case we fix the component $X^0\alpha_0$ in $H^3_+(Y)$. We define then the special coordinates q and the scaling parameter g_A as

$$q^k = \frac{\mathrm{Re}CX^k}{\mathrm{Re}CX^0}\,, \qquad q^\lambda = \frac{\mathrm{Im}CX^\lambda}{\mathrm{Re}CX^0}\,, \qquad g_A^{-1} = 2\mathrm{Re}CX^0\,. \tag{3.6.10}$$

Recall that in the underlying $\mathcal{N}=2$ theory, the periods of Ω are determined by a holomorphic prepotential $\mathcal{F}(X)$. Due to the homogeneity property of $\mathcal{F}$ we can define a rescaled function f as

$$\mathcal{F}(2CX) = i(2\mathrm{Re}CX^0)^2 f(q^k, q^\lambda) \tag{3.6.11}$$

such that $C\Omega$ can be written as

$$2C\Omega = g_A^{-1}\left[1\alpha_0 + q^k\alpha_k + iq^\lambda\alpha_\lambda - f_\lambda\beta^\lambda - i(2f - q^k f_k - q^\lambda f_\lambda)\beta^0 - if_k\beta^k\right]\,, \tag{3.6.12}$$

$H^3(Y)$	$H^{\text{even}}(\tilde{Y})$
$\alpha_0 \in H^3_+(Y)$	1
$\alpha_k \in H^3_+(Y)$	$\omega_k \in H^2_-(\tilde{Y})$
$\alpha_\lambda \in H^3_-(Y)$	$\omega_\lambda \in H^2_+(\tilde{Y})$
$\beta^k \in H^3_-(Y)$	$\tilde{\omega}^k \in H^4_-(\tilde{Y})$
$\beta^\lambda \in H^3_+(Y)$	$\tilde{\omega}^\lambda \in H^4_+(\tilde{Y})$
$\beta^0 \in H^3_-(Y)$	$\mathcal{V}^{-1}\text{vol}_{\tilde{Y}}$

Table 3.6.2: The mirror mapping from the basis of $H^3(Y)$ to the basis of even cohomologies of the mirror Calabi-Yau $\tilde{Y}$ in $O3/O7$ orientifold setups.

where (f_λ, f_k) are the derivatives of f with respect to (q^k, q^λ). The coordinates (N'^k, T'_λ) become in terms of these special coordinates

$$N'^0 = g_A^{-1} + i\xi^0 \qquad N'^k = g_A^{-1} q^k + i\xi^k \qquad T'^A_\lambda = g_A^{-1} f_\lambda + i\tilde{\xi}_\lambda \,. \tag{3.6.13}$$

In order to provide complete match with the Type IIB side we need an explicit expression for f_λ at the large complex structure limit of the Calabi-Yau manifold Y. The results will then be identified with the large volume results of Type IIB. In this limit the $\mathcal{N} = 2$ pre-potential is given by

$$\mathcal{F}(X) = \frac{1}{6}\kappa_{IJK}\frac{X^I X^J X^K}{X^0} \,. \tag{3.6.14}$$

Therefore, inserting the orientifold constraints and switching to special coordinates we find

$$f(q) = -\tfrac{1}{6}\kappa_{\lambda\mu\rho} q^\lambda q^\mu q^\rho + \tfrac{1}{2}\kappa_{\lambda k l} q^\lambda q^k q^l \,, \tag{3.6.15}$$

such that one can readily evaluate the T'^A_λ using (3.6.13). Now it is straightforward to relate the Type IIA coordinates with the ones from the Type IIB side

$$(-i\tau, -iG^k) \leftrightarrow (N'^0, N'^k) \qquad \text{and} \qquad -T'^B_\lambda \leftrightarrow T'^A_\lambda \,, \tag{3.6.16}$$

with the matching of the cohomologies for the pair of mirror Calabi-Yau manifolds given in Table 3.6.2. In terms of the string moduli, the above relations translate into

$$\begin{aligned} g_A^{-1} &= e^{-\phi_B} \,, & q^k &= -b^k \,, & q^\lambda &= v^\lambda \,, \\ \xi_0 &= -C_0 \,, & \xi^k &= -c^k + C_0 b^k \,, & \tilde{\xi}_\lambda &= -\rho_\lambda + \frac{1}{2}\mathcal{K}_{\lambda k l} c^k b^l - \frac{1}{2} C_0 \mathcal{K}_{\lambda k l} b^k b^l \,. \end{aligned} \tag{3.6.17}$$

Inclusion of D3 brane moduli

In the discussion of mirror symmetry with D-branes we first consider the setup with spacetime filling D3 branes. The $\mathcal{N} = 1$ characteristic data were analyzed in [92]. The brane is a point in the internal space $\tilde{Y}$, such that the brane deformations η are described by six scalar fields ϕ^I corresponding to the possible movements in $\tilde{Y}$. These fields naturally combine into complex fields ϕ^i, $\phi^{\bar{\jmath}}$ with $i, \bar{\jmath} = 1, 2, 3$ if one uses the inherited complex structure of the Calabi-Yau manifold. Clearly, there are no Wilson line moduli for D3-branes since there is no internal one-cycle on the brane. It turns out that, up to second order in the fields, only the coordinates T'^B_λ are corrected by the open moduli [92]

$$\text{Re}\, T^B_\lambda = \text{Re}\, T'^B_\lambda + i(\omega_\lambda)_{i\bar{\jmath}}(\phi_0)\, \phi^i \phi^{\bar{\jmath}} \,, \tag{3.6.18}$$

where the two-form $(\omega_\lambda)_{i\bar{\jmath}}$ has to be evaluated at the point ϕ_0 around which the D3-brane fluctuates. More generally, it was argued in ref. [118] that the D3-brane correction to T_α can be expressed through the Kähler potential $K_{\tilde{Y}}$ for the Calabi-Yau metric as

$$\mathrm{Re}\, T^B_\lambda = \mathrm{Re}\, T'^B_\lambda - \partial_{v^\lambda} K_{\tilde{Y}}(\phi_0 + \phi) \ , \tag{3.6.19}$$

where v^λ are the Kähler moduli introduced in (3.6.5). To obtain (3.6.18) one expands $K_{\tilde{Y}}$ around the point ϕ_0 as

$$K_{\tilde{Y}}(\phi_0+\phi) = K^0_{\tilde{Y}} + 2\mathrm{Re}\big[(K_{\tilde{Y}})^0_i \phi^i\big] + \mathrm{Re}\big[(K_{\tilde{Y}})^0_{ij}\phi^i\phi^j\big] + (K_{\tilde{Y}})^0_{i\bar{\jmath}}\phi^i\bar{\phi}^j + \dots \ , \tag{3.6.20}$$

where $K^0_{\tilde{Y}}$, and $(K_{\tilde{Y}})^0_i, (K_{\tilde{Y}})^0_{ij}, (K_{\tilde{Y}})^0_{i\bar{\jmath}}$ are the Kähler potential and its ϕ^i-derivatives evaluated at ϕ_0. Since the coefficients are constant, the first three terms in (3.6.20) can be absorbed by a holomorphic redefinition into a new T^B_λ. Clearly, this does not change the complex structure on the $\mathcal{N}=1$ moduli space. Using $(K_{\tilde{Y}})^0_{i\bar{\jmath}} = -iJ^0_{i\bar{\jmath}} = -iv^\lambda(\omega_\lambda)_{i\bar{\jmath}}(\phi_0)$ one then recovers (3.6.18).

Let us now turn to the discussion of mirror symmetry. We aim to match the corrected coordinates T^B_λ as well as the un-corrected G^k and τ with the Type IIA side. This implies that we must have up to quadratic order in the brane moduli that

$$\begin{aligned} -2\partial_{V^\lambda}(e^{2D_A}K_o) &= \partial_{v^\lambda}K_{\tilde{Y}}(\phi_0+\phi) \cong -i(\omega_\lambda)_{i\bar{\jmath}}\phi^i\bar{\phi}^{\bar{\jmath}} \\ \partial_{V_0}(e^{2D_A}K_o) &= \partial_{V_k}(e^{2D_A}K_o) = 0 \ , \end{aligned} \tag{3.6.21}$$

where the $\cong$ indicates that one has to apply the transformation which identifies (3.6.19) and (3.6.18). Using the fact that $V^\lambda = -e^{2D_B}e^{-\phi_B}v^\lambda$, as inferred from (3.6.12), the identification (3.6.21) implies

$$K_o(\phi,\bar{\phi}) = \tfrac{1}{2}e^{-\phi_B}K_{\tilde{Y}} \ . \tag{3.6.22}$$

The number of open moduli must coincide, so the number of brane deformations on the Type IIB must equal the number of brane and Wilson line moduli on the Type IIA side. Since this number is given by the number of non-trivial one-cycles in L_0, we must have $b^1(L_0)=3$. However, recall that the open moduli space in Type IIA has shift symmetries, $\mathrm{Im}\zeta^i \to \mathrm{Im}\zeta^i + c^i$, for constants c^i. These are not manifested in the Type IIB side for a general $K_{\tilde{Y}}$, since the Calabi-Yau metric has no continuous symmetries. As we recall below, this can be attributed to the fact that instanton contributions break these symmetries and are not included in this leading order identification.

Before commenting on the corrections to the mirror construction let us make contact to the chain integral form of the Kähler potential as given in (3.5.8). For a D3-brane we simply have to introduce a one-chain $\mathcal{C}_1$ which starts at ϕ_0 and ends at the point in $\tilde{Y}$ to which the D3-brane has moved. We also introduce a basis of complex normal vectors s_i to the point ϕ_0 and dual $(1,0)$-forms $s^j_{(1)}$ such that

$$s_i \lrcorner s^j_{(1)} = \delta^j_i \ . \tag{3.6.23}$$

Note that the index i,j are counting here the number of such normal vectors. In case we only include the massless modes, one has $i,j=1,\dots,3$. The complex structure of s_i and $s^i_{(1)}$ is induced by the complex structure of $\tilde{Y}$, and hence depends on the complex structure moduli. In fact one can use the no-where vanishing $(3,0)$-form Ω on $\tilde{Y}$ and introduce a bi-vector s^j such that $s^j_{(1)} = \bar{s}^j \lrcorner \Omega$. To propose a form for K_o one trivially extended $s_i, \bar{s}^i$ to the chain $\mathcal{C}_1$ and writes

$$K_o = \tfrac{i}{4}e^{-\phi_B}\int_{\mathcal{C}_1} s_i \lrcorner J \int_{\mathcal{C}_1} \bar{s}^i \lrcorner \Omega + c.c. \ . \tag{3.6.24}$$

This form of K_o is very suggestive and yields upon expanding the chain integral the desired leading order expression (3.6.18). Moreover, we will see in the following that a generalization of this K_o also arises for D7-brane, and one can generally write in O3/O7 orientifolds for the deformations of a $D(p+3)$-brane

$$K_o^{\text{def}} = \tfrac{i}{4} \int_{\mathcal{C}_{p+1}} s_I \lrcorner \text{Im}\Phi^{\text{ev}} \int_{\mathcal{C}_{p+1}} \bar{s}^I \cdot \Omega + c.c. \, . \tag{3.6.25}$$

where Φ^{ev} has been introduced in (3.6.9), and $\mathcal{C}_{p+1}$ is a $(p+1)$-chain which ends on the internal parts of the D-branes and its reference cycle. Moreover, s_I is an appropriate basis of complex normal vectors and s^J are their duals as we discuss below.

Before giving a more careful treatment of the other D-brane configurations let us first comment on a more intuitive understanding of mirror symmetry which we will apply below. It was argued by Strominger, Yau and Zaslow [69] that the Calabi-Yau manifold $\tilde{Y}$ can be viewed as a three-torus fibration with singular fibers. This manifold can be endowed with a semi-flat metric. In a local patch avoiding possible singular points the metric of the Calabi-Yau manifold can be written as

$$ds^2 = g_{ab}(\tilde{u}) d\tilde{u}^a d\tilde{u}^b + 2 g_{ia}(\tilde{u}) d\tilde{a}^i d\tilde{u}^a + g_{ij}(\tilde{u}) d\tilde{a}^i d\tilde{a}^j \ , \qquad i, a = 1, 2, 3 \ , \tag{3.6.26}$$

where $\tilde{a}^i$ are the coordinates on the T^3 fiber and $\tilde{u}^a$ of the base. Since the coefficient functions in (3.6.26) are independent of $\tilde{a}^i$ the shift symmetry is now manifest. In fact, introducing complex coordinates as in the Type IIA setting a Kähler metric in (3.6.26) can be obtained from a Kähler potential $K_{\tilde{Y}}(\tilde{u})$ which is independent of a^i. The argument for the existence of such a T^3-fibration with a metric of the form (3.6.26) away from singularities proceeds precisely via mirror symmetry of a pointlike D-brane on $\tilde{Y}$ which is mapped to a D-brane which wraps a three-torus [69]. Having found a T^3-fibration in the Type IIB set-up one can equally use T-duality along all T^3-directions to analyze the setting. Since T-duality exchanges Neumann and Dirichlet boundary conditions, it exchanges the dimensionality of the brane for each wrapped cycle that is T-dualized. Starting with a D3-brane on such a fibered Calabi-Yau manifold, T-duality on the fiber will turn the brane into a D6-brane wrapping the T^3-fiber. The D6-brane then has $b^1(L_0) = 3$ deformation moduli in the direction of the base, and there are also $b^1(L_0) = 3$ Wilson line moduli will be along the torus.

In the following it will be more important that we can use the SYZ-picture also for D7- and D5-branes present in a Type IIB reduction. Clearly, both types of branes will map to D6-branes under mirror symmetry. Away from the singular fibers one can obtain a clearer picture of the wrappings of the D6-branes as indicated in Table 3.6.3.

	D6	D3	D6	D7	D6	D5
T^3	×			×		×
	×			×	×	
	×		×		×	
Base						
			×	×		
			×	×	×	×

Table 3.6.3: It is summarized how mirror symmetry acts on different brane configurations. The table shows the six dimensions of the Calabi-Yau manifold, split into base and fiber. × indicate the directions wrapped by each brane. Mirror symmetry acts as T-duality on all directions of the T^3-fiber. It exchanges Dirichlet and Neumann boundary conditions, while it does not act on the base. Different wrappings of a D6-brane correspond to different branes in the Type IIB side.

Inclusion of D7 brane moduli

Let us now discuss mirror symmetry for the D7-brane case. The effective action for a pair of moving D7-branes was computed in [27]. In this setup, the brane wraps a four-cycle $S^{(1)}$ while its orientifold image wraps a non-intersecting $S^{(2)}$. One can view the whole configuration as a single D7-brane wrapping a divisor $S_+ = S^{(1)} + S^{(2)}$. Brane deformations and Wilson line moduli can be expanded in terms of

$$\begin{aligned} \chi &= \chi^A s_A + \bar{\chi}^{\bar{A}} \bar{s}_{\bar{A}}, \qquad A = 1, \dots, h_-^{(2,0)}(S_+), \\ a &= a^I \gamma_I + \bar{a}^{\bar{I}} \bar{\gamma}_{\bar{I}} \qquad I = 1, \dots, h_-^{(0,1)}(S_+), \end{aligned} \tag{3.6.27}$$

where s_A and γ_I are complex normal vectors to $S^{(1)}$ and $(0,1)$-forms on $S^{(1)}$, respectively. The complex type of s_A and γ_I is induced by the complex structure of $\tilde{Y}$. Moreover, one can use the holomorphic $(3,0)$-form Ω on $\tilde{Y}$ to map the s_A to $(2,0)$-forms $\mathcal{S}_A = s_A \lrcorner \Omega$ on $S^{(1)}$. The restriction to the odd cohomology comes from the fact that as the normal bundle of S_+ is even and the holomorphic 3-form Ω is odd under the orientifold action, the contraction $\mathcal{S}_A = s_A \lrcorner \Omega$ is odd. Also, the tangent bundle on S_+ is anti-invariant under the orientifold action, and therefore the Wilson moduli must be in $H^1_-(S_+)$. The four-dimensional fields are thus the $h_-^{(2,0)} + h_-^{(0,1)}$ complex scalars χ^A and a^I, and their complex conjugates.

Including the open string degrees of freedom, the chiral coordinates $(\tau, G_a, T'^B_\lambda)$ are shifted to [27]

$$\begin{aligned} S &= \tau + \mathcal{L}_{A\bar{B}} \chi^A \bar{\chi}^{\bar{B}}, \qquad G^k = c^k - \tau b^k, \\ T^B_\lambda &= \tfrac{1}{2} e^{-\phi_B} \mathcal{K}_{\lambda\rho\sigma} v^\rho v^\sigma + i\rho_\lambda - i\tfrac{1}{2} \mathcal{K}_{\lambda kl} b^k G^l + i\mathcal{C}_{\lambda I\bar{J}} a^I \bar{a}^{\bar{J}}. \end{aligned} \tag{3.6.28}$$

The coupling functions $\mathcal{L}_{A\bar{B}}$ and $\mathcal{C}_{\lambda I\bar{J}}$ for the basis of brane deformations and Wilson line moduli on the four-cycle are given by

$$\mathcal{L}_{A\bar{B}} = \frac{\int_{S_+} \mathcal{S}_A \wedge \bar{\mathcal{S}}_B}{\int_{\tilde{Y}} \Omega \wedge \bar{\Omega}}, \qquad \mathcal{C}_{\lambda I\bar{J}} = \int_{S_+} \omega_\lambda \wedge \gamma_I \wedge \bar{\gamma}_{\bar{J}}. \tag{3.6.29}$$

Since the closed moduli are the same, we proceed in the same way as we did for the closed and the D3-brane cases, identifying the coordinates as (3.6.16). Analogously to the D3-brane case, we expand up to second order in the open moduli and match both theories by

$$\partial_{V^\lambda}(e^{2D_A} G_{ij}) u^i u^j \cong i\mathcal{C}_{\lambda I\bar{J}} a^I \bar{a}^{\bar{J}}, \qquad \partial_{V_0}(e^{2D_A} G_{ij}) u^i u^j \cong i\mathcal{L}_{A\bar{B}} \chi^A \bar{\chi}^{\bar{B}}, \qquad \partial_{V_k}(e^{2D_A} G_{ij}) u^i u^j \cong 0, \tag{3.6.30}$$

where we have indicated that as in the D3-brane case one will need to make the shift symmetry manifest before finding complete match. Crucially one has to split the Type IIA coordinates into two sets ζ^I and ζ^A and identify

$$\zeta^I \cong a^I, \qquad \zeta^A \cong \chi^A. \tag{3.6.31}$$

One notes that Wilson line moduli and brane deformations do not mix on the Type IIB side which seems to be in contrast to the general form on the Type IIA side. We will argue later how this splitting can be understood from the SYZ-picture of mirror symmetry.

As already suggested in (3.6.25) one expects that the open corrections to the $\mathcal{N} = 1$ coordinates can again be given in terms of chain integrals. Let us first give the expression for K_o which encodes upon differentiation with respect to V^λ, V_0, V_k the corrections in T_λ, N^0, N^k. Explicitly, we propose

$$K_o = \tfrac{i}{4} \int_{\mathcal{C}_5} s_A \lrcorner \mathrm{Im}\, \Phi^{\mathrm{ev}} \int_{\mathcal{C}_5} \bar{s}^A \wedge \Omega + \tfrac{i}{4} \int_{\mathcal{C}_5} \mathcal{F}_{\mathrm{D7}} \wedge \gamma_I \wedge \mathrm{Im}\Phi^{\mathrm{ev}} \int_{\mathcal{C}_5} \mathcal{F}_{\mathrm{D7}} \wedge \bar{\gamma}^I + c.c., \tag{3.6.32}$$

where Φ^{ev} is given in (3.6.9). Here we have used a five-chain $\mathcal{C}_5$ ending on the D7-brane and a reference four-cycles S_+^0. Note that similar to the D6-brane case we have to introduce a dual basis s_A and s^A. To do that we use the fact that no-where vanishing $(3,0)$-form Ω provides an identification

$$\Omega: \quad NS_+ \to TS_+^* \wedge TS_+^*, \tag{3.6.33}$$

of normal vectors with two-forms of S_+. Hence, in the Type IIB setting we adopt this basis to the complex structure by demanding that s_A is a complex normal vector in $H^0_+(NS_+)$ and s^A is a $(2,0)$-form in $H^{(2,0)}_-(S_+)$ on S^0_+. Similarly, γ_I is a $(0,1)$-form as introduced above and γ^J is a $(1,2)$-form in $H^{(1,2)}_-(S^0_+)$. These forms are defined to be dual and hence satisfy

$$\int_{S^0_+} \bar{s}^A \wedge (s_B \lrcorner \Omega) = \delta^A_B \, , \qquad \int_{S^0_+} \gamma_I \wedge \bar{\gamma}^J = \delta^J_I \, . \tag{3.6.34}$$

As in the D6-brane case we have to extend these forms to the chain. It is interesting to note that the expression (3.6.32) indeed reproduces the leading order corrections after differentiating with respect to V^λ, V_0, V_k.

3.6.2 Mirror symmetry for O5-orientifolds and D5-branes

Let us now discuss the second Type IIB set-up which is obtained by an involution with O5-planes as fix-point set. The bulk $\mathcal{N}=1$ coordinates of the moduli space $\mathcal{M}^{\mathrm{Q}}$ are given as functions of the zero-modes in the expansion

$$\begin{aligned} J &= v^k\,\omega_k\,, \qquad C_2 = \tilde{C}_2 + c^k\,\omega_k\,, \quad k = 1,\ldots,h^{(1,1)}_+(\tilde{Y})\,, \\ B_2 &= b^\lambda\,\omega_\lambda\,, \qquad C_4 = \rho_\lambda\,\tilde{\omega}^\lambda\,, \qquad \lambda = 1,\ldots,h^{(1,1)}_-(\tilde{Y})\,. \end{aligned} \tag{3.6.35}$$

Note the difference that we have used forms of different σ-parity in the expansion for the R-R-fields, C_2 and C_4 as required for the second orientifold projection in (3.6.2). While C_0 has been projected out C_2 now contains a four-dimensional two-form $\tilde{C}_2(x)$ which together with the dilaton ϕ_B form the bosonic content of a linear multiplet. However, $\tilde{C}_2$ can be dualized to a scalar field h and form with ϕ_B a chiral multiplet. The $\mathcal{N}=1$ coordinates which span $\mathcal{M}^{\mathrm{Q}}$ are thus the $h^{(1,1)}+1$ complex fields

$$\begin{aligned} t'^k &= e^{-\phi_B}v^k - ic^k\,, \qquad P_\lambda = \mathcal{K}_{\lambda\rho k}b^\rho t^k + i\rho_\lambda\,, \\ S &= e^{-\phi_B}\mathcal{V} + ih - \tfrac{i}{2}\rho_\lambda b^\lambda - \tfrac{1}{2}P_\lambda b^\lambda, \end{aligned} \tag{3.6.36}$$

Formally the Kähler potential is the same as in the O3/O7-case given in (3.6.7). However, it now has to be evaluated as a function of the coordinates t'^k, P_λ and S by using there explicit form (3.6.36). Similar to (3.6.8) we can write

$$-\,\mathrm{Im}\,\Phi^{\mathrm{ev}} + i\sum_n e^{B_2}\wedge C_{2n} = -t'^k\omega_k + P_\lambda\tilde{\omega}^\lambda + S\mathrm{vol}_{\tilde{Y}}. \tag{3.6.37}$$

Let us turn to the discussion of the mirror Type IIA side to this construction. As explained above the second set-up with O5-planes is obtained by choosing the three-form α_0 for the fundamental period X^0 to lie in the negative eigenspace $H^3_-(\tilde{Y})$. Again we will perform a rescaling of Ω setting the coefficient of α_0 to be constant. The special coordinates are then given by

$$g_A^{-1} = 2\mathrm{Im}CX^0\,, \qquad q^k = \frac{\mathrm{Re}CX^k}{\mathrm{Im}CX^0}\,, \qquad q^\lambda = \frac{\mathrm{Im}CX^\lambda}{\mathrm{Im}CX^0}\,. \tag{3.6.38}$$

Now the rescaled prepotential f is given by $\mathcal{F}(2CX) = -i(2\mathrm{Im}CX^0)^2 f(q^k, q^\lambda)$. This allows us to rewrite $C\Omega$ in the rescaled coordinates as

$$2C\Omega = g_A^{-1}\left[q^k\alpha_k + i\alpha_0 + iq^\lambda\alpha_\lambda + f_\lambda\beta^\lambda - (-2f + q^k f_k + q^\lambda f_\lambda)\beta^0 + if_k\beta^k\right]\,. \tag{3.6.39}$$

Moreover, we can use the special coordinates to write $(N'^k, T'^A_\lambda, T'^A_0)$ as

$$N'^k = g_A^{-1}q^k + i\xi^k \qquad T'^A_0 = g_A^{-1}(-2f + q^\lambda f_\lambda + q^k f_k) + i\tilde{\xi}_0 \qquad T'^A_\lambda = -g_A^{-1}f_\lambda + i\tilde{\xi}_\lambda\,. \tag{3.6.40}$$

With f in the large complex structure limit

$$f(q) = \tfrac{1}{6}\kappa_{klm}q^k q^l q^m - \tfrac{1}{2}\kappa_{l\mu\rho}q^l q^\mu q^\rho \ . \tag{3.6.41}$$

this allows us to write

$$T_0'^A = g_A^{-1}\left(\tfrac{1}{6}\kappa_{klm}q^k q^l q^m - \tfrac{1}{2}\kappa_{\mu\lambda k}q^\mu q^\lambda q^k\right) + i\tilde{\xi}_0 \ , \qquad T_\lambda'^A = g_A^{-1}\kappa_{\lambda\mu k}q^\mu q^k + i\tilde{\xi}_\lambda \ . \tag{3.6.42}$$

The mirror mapping is then realized by

$$t'^k \leftrightarrow N'^k \qquad \text{and} \qquad (S, P_\lambda) \leftrightarrow (T_0'^A, T_\lambda'^A) \ . \tag{3.6.43}$$

In terms of the Kaluza-Klein modes this amounts to the identification of the closed moduli

$$\begin{array}{lll} g_A^{-1} = e^{-\phi_B} \ , & q^k = v^k \ , & q^\lambda = b^\lambda \ , \\ \tilde{\xi}_0 = h - \rho_\lambda b^\lambda \ + \tfrac{1}{2}\mathcal{K}_{l\lambda\kappa}c^l b^\lambda b^\kappa \ , & \xi^k = -c^k \ , & \tilde{\xi}_\lambda = \rho_\lambda \ - \mathcal{K}_{\lambda\kappa l}c^l b^\kappa \ . \end{array} \tag{3.6.44}$$

The identification of the basis elements on the Type IIA and Type IIB side is given in Table 3.6.4.

$H^3(Y)$	$H^{\text{even}}(\tilde{Y})$
$\alpha_0 \in H^3_-(Y)$	1
$\alpha_k \in H^3_+(Y)$	$\omega_k \in H^2_+(\tilde{Y})$
$\alpha_\lambda \in H^3_-(Y)$	$\omega_\lambda \in H^2_-(\tilde{Y})$
$\beta^k \in H^3_-(Y)$	$\tilde{\omega}^k \in H^4_+(\tilde{Y})$
$\beta^\lambda \in H^3_+(Y)$	$\tilde{\omega}^\lambda \in H^4_-(\tilde{Y})$
$\beta^0 \in H^3_+(Y)$	$\mathcal{V}^{-1} vol_{\tilde{Y}}$

Table 3.6.4: The mirror mapping from the basis of $H^3(Y)$ to the basis of even cohomologies of the mirror Calabi-Yau $\tilde{Y}$ in $O5/O9$ orientifold setups.

Inclusion of D5 brane moduli

We now consider a pair of D5-branes on curves $\Sigma^{(1)}$ and $\Sigma^{(2)}$ which are interchanged under the orientifold involution. We call the positive union of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ by $\Sigma_+ = \Sigma^{(1)} + \Sigma^{(2)}$ Again we view this as a single D5-brane on the quotient space. The open moduli for a single D5-brane [93], corresponding to complex brane deformations χ^A and Wilson line moduli a^I, correct the $\mathcal{N} = 1$ coordinates according to

$$\begin{aligned} t^k &= t'^k + \mathcal{L}^k_{A\bar{B}}\chi^A\bar{\chi}^{\bar{B}}, \\ P_\lambda &= \mathcal{K}_{\lambda\rho k}b^\rho t'^k + i\rho_\lambda \ , \\ S &= e^{-\phi_B}\mathcal{V} + ih - \tfrac{i}{2}\rho_\lambda b^\lambda - \tfrac{1}{2}P_\lambda b^\lambda + \mathcal{C}_{I\bar{J}}a^I\bar{a}^{\bar{J}} \ . \end{aligned} \tag{3.6.45}$$

The deformations χ^A are given by sections of the holomorphic normal bundle $N\Sigma_+$ that are invariant under the orientifold projection, and therefore the index $A = 1, \ldots, \dim H^0_+(\Sigma_+, N\Sigma_+)$. The Wilson line moduli a^I are in the tangent bundle of Σ_+, just like in the D7 brane case, and the index runs as $I = 1, \ldots, h^{(0,1)}_-(\Sigma_+)$. Here we have introduced the couplings

$$\mathcal{L}^k_{A\bar{B}} = -i\int_{\Sigma_+} s_A \lrcorner \bar{s}_B \lrcorner \tilde{\omega}^k \ , \qquad \mathcal{C}_{I\bar{J}} = i\int_{\Sigma_+} \gamma_I \wedge \bar{\gamma}_{\bar{J}} \tag{3.6.46}$$

The Kähler potential now has to be evaluated as a function of t^k, P_λ, S as well as the open coordinates χ^A and a^I.

In order to discuss mirror symmetry to the D6-brane set-up we again compare the form of the $\mathcal{N}=1$ coordinates. Expanding to second order in the open corrections we find

$$-\partial_{V_k}(e^{2D_A}G_{ij})u^iu^j \cong \mathcal{L}^k_{A\bar{B}}\chi^A\bar{\chi}^{\bar{B}}, \quad -\partial_{V_0}(e^{2D_A}G_{ij})u^iu^j \cong \mathcal{C}_{I\bar{J}}a^I\bar{a}^{\bar{J}}\ , \quad -\partial_{V^\lambda}(e^{2D_A}G_{ij})u^iu^j \cong 0\,. \tag{3.6.47}$$

More interestingly, we can also directly compare the open Kähler potential K_o. To do that, we give a chain integral expression for the D5-brane case. We introduce a the three-chain $\mathcal{C}_3$ ending on a reference cycle Σ^0_+ and the two-cycle to which the brane has moved. The open Kähler potential then takes the form

$$K_o = -\tfrac{i}{4}\int_{\mathcal{C}_3} s_A \lrcorner \mathrm{Re}\,\Phi^{\mathrm{ev}} \int_{\mathcal{C}_3} \bar{s}^A\cdot\Omega - \tfrac{i}{4}\int_{\mathcal{C}_3}\mathcal{F}_{\mathrm{D5}}\wedge\gamma_I\wedge\mathrm{Re}\,\Phi^{\mathrm{ev}}\int_{\mathcal{C}_3}\mathcal{F}_{\mathrm{D5}}\wedge\bar{\gamma}^I + c.c.\ , \tag{3.6.48}$$

where Φ^{ev} is given in (3.6.9). Note that this expression has a similar structure as (3.6.32). However, due to the lower dimensionality of the chain the four-form part of $\mathrm{Re}\,\Phi^{\mathrm{ev}}$ is picked up in the first term of (3.6.48), while the zero-form part of $\mathrm{Re}\,\Phi^{\mathrm{ev}}$ contributes in the second term of (3.6.48). In the case of a D5-brane the $(3,0)$-form Ω on $\tilde{Y}$ provides a map

$$\Omega: \quad N\Sigma_+\otimes N\Sigma_+ \ \to\ T\Sigma^*_+\ , \tag{3.6.49}$$

taking two normal vectors to a one-form on Σ_+. This allows us to introduce a basis s^A of $H^0(T\Sigma^0_+\otimes\overline{N\Sigma^0_+})$ which is dual to the normal vectors s_A. Hence, the $\cdot$ in (3.6.48) indicates that the vector part of s^A is inserted, while the form part of s^A is wedged with Ω. We also introduce complex one-forms γ^J on Σ^0_+ which are dual to the $(0,1)$-forms γ_I used in the expansion determining the complex Wilson line scalars a^I. Explicitly, the s^A, γ^I have to satisfy on the reference Σ^0_+ that

$$\int_{\Sigma^0_+} s_A\lrcorner\bar{s}^B\cdot\Omega = \delta^B_A\ , \qquad \int_{\Sigma^0_+}\gamma_I\wedge\bar{\gamma}^J = \delta^J_I\ , \tag{3.6.50}$$

As in the D6-brane case the basis forms and vectors have to be extended trivially to the chain $\mathcal{C}_3$ to evaluate the open Kähler potential (3.6.48). One can now check that the expansion (3.6.48) leads upon differentiation with respect to V_k, V^0, V^λ the leading order corrections in (3.6.45).

3.6.3 General remarks on the structure of the couplings

In this subsection we address the question if there is a simple way to understand the mappings of (3.6.47), (3.6.21) and (3.6.30) using the SYZ-picture of mirror symmetry. For example for D5-branes the $(\partial_{V_k}(e^{2D_A}G_{ij}), \partial_{V_0}(e^{2D_A}G_{ij}))$ correct the coordinates t^k and S by brane deformations and Wilson line moduli as demanded by the mirror identification (3.6.47). In contrast, the coordinates P_λ do not receive any contributions from open moduli and hence $\partial_{V^\lambda}(e^{2D_A}G_{ij})$ has to vanish in the D6-brane set-up mirror dual to a D5-brane. To analyze this question in the SYZ-picture, first let us look at the gauge coupling functions. In the limit of vanishing open string moduli they are given by the analogous to the D6-brane gauge coupling function $f_{\mathrm{D6}} = N^k\int_L\alpha_k - T_\lambda\int_L\beta^\lambda$,

$$f_{\mathrm{D3}} = \tau\ , \qquad f_{\mathrm{D5}} = t^\Sigma\int_{\Sigma_+}\omega_\Sigma\ , \qquad f_{\mathrm{D7}} = T_S\int_{S_+}\tilde{\omega}^S\ , \tag{3.6.51}$$

where $\Sigma_+(S_+)$ is the curve(divisor) wrapped by the D5(D7)-brane, and they are obtained from a basis of homology by

$$\begin{aligned}[]
[\Sigma_+] &= n^k\,[\Sigma_k]\ , \qquad \Sigma_k\in H^+_2(Y) \quad \text{and} \\
[S_+] &= n_\lambda[S^\lambda]\ , \qquad S^\lambda\in H^+_4(Y)\ .
\end{aligned} \tag{3.6.52}$$

Therefore the forms appearing in (3.6.51) are, in terms of the cohomology basis, $\omega_\Sigma = n^k\omega_k$ and $\tilde{\omega}^S = n_\lambda\tilde{\omega}^\lambda$.

From the four internal dimensions the D7-brane wraps, locally two of them are along the T^3-fiber and the other two on the base, as seen from table 3.6.3. The mirror D6-brane, on the other hand, wraps one dimension on T^3-fiber and two dimensions on the base. It is also inferred from the gauge coupling function of the D7-brane (3.6.51) that $\tilde{\omega}^\lambda$ sits on the brane, therefore having two "legs" on the 3-Torus and two on the base. We define thus the notation $\tilde{\omega}^\lambda : (bbtt)$, where b and t correspond to base and torus components. Table 3.6.2 shows that $\tilde{\omega}^\lambda$ on the Type IIB side is mapped on the Type IIA side to β^λ. Therefore, from table 3.6.3, since β^λ must sit on the mirror D6-brane, it should satisfy $\beta^\lambda : (bbt)$. β^λ must be dual to α_λ on the Calabi-Yau manifold Y, thus $\alpha_\lambda : (btt)$. A similar analysis can be done for the D5 and D3-Branes, from where we obtain $\alpha_k : (btt)$, $\beta^k : (bbt)$, $\beta^0 : (bbb)$ and $\alpha_0 : (ttt)$.

One can now analyze the open moduli corrections to the $\mathcal{N} = 1$ chiral coordinates from the metric derivatives $\partial_{V_0}\widehat{\mathcal{G}}_{ij}$, $\partial_{V_k}\widehat{\mathcal{G}}_{ij}$ and $\partial_{V^\lambda}\widehat{\mathcal{G}}_{ij}$. As a simple example we consider the D3-brane case. We can rewrite the corrections in terms of the normal deformations η^i

$$\mathrm{Re}(N'^0 - N^0) = \partial_{V_0}(e^{2D_A}\widehat{\mathcal{G}}_{ij})\eta^i\eta^j = \frac{1}{2}\int_{L_0}\hat{\alpha}_k \wedge \eta\lrcorner\beta^0 \int_{L_0}\hat{\beta}^k \wedge \eta\lrcorner J\,. \tag{3.6.53}$$

Since the brane wraps the three-torus, both integrands in (3.6.53) must be of the form (ttt). The normal directions of this D6-brane are all on the base, so $\eta\lrcorner\beta^0 : (bb)$, making the first integral vanish. Therefore there is no correction to $N'^0 = i\tau$ coming from $\partial_{V_0}\widehat{\mathcal{G}}_{ij}$, as was already seen in (3.6.21). By repeating the analysis to $\partial_{V^\lambda}\widehat{\mathcal{G}}_{ij}$ and $\partial_{V_k}\widehat{\mathcal{G}}_{ij}$ one shows that only the latter can be non-vanishing, and analysing in the same fashion the corrections for the D5 and the D7 cases we obtain the same relations as (3.6.47) and (3.6.30).

One can realize then that brane deformations with normal direction η along one cycle of the 3-torus on the Type IIA side are mapped to Wilson line moduli along the T-dual cycle on the Type IIB side, while brane deformations along the base are mapped to brane deformations on the Type IIB side, also along the base. In the opposite direction, brane deformations on the Type IIB side along the 3-torus are mapped to Wilson line moduli along the dual cycle on the Type IIA side.

We conclude this section by summarizing the results here obtained. It was a known result [87] how some deformations of a calibrated special lagrangian manifold can break the special lagrangian conditions. This can be encoded in a scalar potential, that give obstructions to these conditions. The same potential can be obtained from explicit calculation of the Dirac-Born-Infeld of a D6-brane wrapping a special lagrangian manifold.

The reduction of the action describes a four-dimensional $\mathcal{N} = 1$ supersymmetric theory, from which we identified the characteristic data. The deformations of the brane are described by real fields, but they naturally complexify with the Wilson line moduli to form a chiral multiplet. When we introduce closed string modes, these chiral fields correct the moduli of complex structure deformations.

At the end, we mapped the field content using Mirror Symmetry in view of the SYZ conjecture. We also proposed a Kähler potential to describe the brane moduli space in type IIB theory, inspired by the Kähler potential we obtained for the D6-brane moduli.

Chapter 4

M and F Theory: non-perturbative descriptions

In the previous chapter we have seen how one arises at a standard $\mathcal{N} = 1$ four-dimensional action from spacetime filling D-branes, when we compactify part of the ten-dimensional space in a three complex-dimensional Calabi-Yau orientifold. Models with intersecting branes in fact correspond to a great part of the strings phenomenology literature. However, one can point out that in the previous chapter we did not treat the whole story since we fixed a small background value for the dilaton (the string coupling) such that we could trust the perturbation string theory description. But in general this cannot be expected to work, since the dilaton is a dynamical field and not just a free parameter of the theory. It is important then to explore the domain of strong coupling, not only as a theoretical curiosity, but as a theoretical necessity.

In the last two decades much effort has been put in constructing strong coupling descriptions for string theory. We will dedicate this chapter to review the basics on two such constructions, M- and F-theory.

4.1 Type IIA as a limit of M-theory

In the mid 90's [31,119] it was proposed that there should be a theory that contains Type IIA string theory as a particular limit, and this hypothetical theory received the name of *M-Theory*. Although M-Theory does not yet have a fundamental perturbative description, it is postulated that M-theory has an effective description described by eleven dimensional supergravity, the maximally supersymmetric gravity theory known. When one compactifies eleven dimensional supergravity on a circle of radius R_{11}, one arises at Type IIA supergravity with the vacuum expectation value for the dilaton being the compactification radius R_{11} in Planck length units. So, sending the string coupling to infinity corresponds to the circle decompactification.

Eleven dimensional supergravity contains solitonic solutions that couple electrically and magnetically to the three-form potential A_3. These solutions are extremal black-branes of two and five dimensions. The extremal black-hole condition can be translated to a BPS condition, and leads to the conclusion that such objects are stable even away from the low energy limit. These objects are the M2 and M5 branes of M-Theory, up to now believed to be the only fundamental objects of this theory.

When compactified to ten dimensions the M2- and M5-branes become respectively to the D2 and NS5 Branes of Type IIA, and if one of the dimensions of the M2- and M5-Branes wrap the compactification circle they lead respectively to the fundamental superstring[1] and the D4 Brane . The D0 brane however does not come from a compactified M-brane, but rather from the first Kaluza-Klein (KK) excitation of the graviton multiplet. Its mass is given by $1/R$, where R is the radius of the compactification circle.

The massless condition for the 11-dimensional graviton reads

$$-p_i p^i = M^2 = 0, \qquad i = 0, ..., 9, 11. \tag{4.1.1}$$

[1] the relation between the supermembrane in supergravity and the superstring appeared in a much earlier work by Duff in 1987 [120]

Note that to agree with the usual notation of the literature, we skip the index 10. When compactified along the 11th direction on a circle of radius R, the quantization of the momentum along the compact dimension implies

$$-p_\mu p^\mu - p_{11}p^{11} = -p_\mu p^\mu - \left(\frac{N}{R}\right)^2 = 0, \qquad \mu = 0, ..., 9. \tag{4.1.2}$$

The first KK excited state has mass $1/R$. From the dimensional reduction of the eleven-dimensional action, one can identify the radius compactification circle in terms of the 11d and 10d gravitational coupling constant, and in turn relate it to the string coupling. It turns out that the radius R (in string length units) is precisely the inverse of the string coupling constant g_s. Therefore the mass of the first excited KK mode agrees with the expected mass of a D0 brane, $M_{D0} = g_s$.

The D6-brane is the "magnetic" dual of the D0-brane, so one might expect that it also has a geometrical origin. Indeed, the D6-brane comes from a eleven-dimensional Kaluza-Klein monopole, a topological defect solution of eleven-dimensional supergravity that is the magnetic dual of the Kaluza-Klein excitation.

This solution is just the eleven-dimensional version of the five-dimensional Kaluza-Klein monopole [121] that we briefly review here. Five-dimensional Einstein's equations has as a possible solution

$$ds_5^2 = -dt^2 + ds_{\text{TN}}^2, \tag{4.1.3}$$

where the Taub NUT metric [122]

$$ds_{\text{TN}}^2 = V(r)(dr^2 + r^2 d\Omega_2^2) + \frac{1}{V(r)}(dy + R\sin^2(\theta/2)d\phi)^2, \tag{4.1.4}$$

where dy is a periodic direction and

$$V(r) = 1 + R/2r. \tag{4.1.5}$$

From the Kaluza-Klein Ansatz, we can identify the Kaluza-Klein one-form $A = R\sin^2(\theta/2)d\phi$, and from it calculate the magnetic field around the origin,

$$B = -\nabla V = \nabla \times A. \tag{4.1.6}$$

One can also calculate the electric field, $E = \partial_t A = 0$, so the Taub-NUT solution is purely magnetically charged, a magnetic monopole.

One can extend this solution by adding six transverse flat directions, reproducing then an extended six-dimensional object inside the eleven-dimensional theory. Thus, Type IIA supergravity is consistently described as eleven dimensional supergravity compactified on a circle when the radius of compactification (and therefore the Type IIA string coupling $g_{s\,IIA}$ is very small. The string coupling is given by the background value of the dilaton, $g_s = e^\phi$. If we allow the dilaton value to depend on the spacetime coordinates, the lift to M-theory corresponds to replace the circle compactification of radius R by a circle fibration over ten-dimensions, where $R(x)$ now vary over the ten dimensional space.

Type IIB string theory can be obtained from Type IIA string theory via T-duality, so it is also a particular limit of M-theory. Type I, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic superstring theories can also be described as limits of the same underlying eleven dimensional theory [123, 124]. M-theory then is the hypothetical eleven-dimensional theory that has as fundamental objects M2- and M5-branes and under particular limits it reduces to all the known supersymmetric string theory constructions.

4.2 F-Theory from M-theory

Low Energy Type IIB also has an hypotetical strong coupling limit construction, named F-theory [34]. In this section we will briefly review its formulation and the necessary background for the remaining of this thesis. We will follow mainly [125–127].

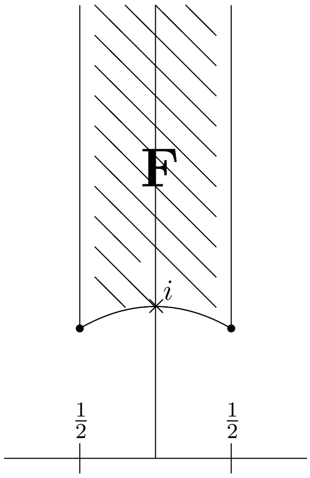


Figure 4.1: The fundamental domain of τ.

The motivations for F-theory may be traced to the $SL(2,\mathbb{Z})$ symmetry of Type IIB supergravity. In Type IIB string theory, the RR fields are all even forms C_{2n} (in contrast with the odd forms of Type IIA). Also, the axionic C_0 form joins with the dilaton $e^{-\phi}$ and we define the chiral axio-dilaton field

$$\tau = C_0 + ie^{-\phi}. \tag{4.2.1}$$

The effective action is

$$\begin{aligned} S^{(10)}_{IIB} &= \int \frac{1}{2}R * \mathbf{1} - \frac{1}{4}d\phi \wedge *d\phi - \frac{1}{4}H_3 \wedge *H_3 - \frac{1}{4}e^{2\phi}F_1 \wedge *F_1 \\ &\quad -\frac{1}{4}e^{\phi}F_3 \wedge *F_3 - \frac{1}{8}F_5 \wedge *F_5 - \frac{1}{4}C_4 \wedge H_3 \wedge F_3, \\ H_3 &= dB_2, \quad F_1 = dC_0, \quad F_{q+1} = dC_q - C_{q-2} \wedge H_3, \end{aligned} \tag{4.2.2}$$

This action can be rewritten in the more convenient form

$$\begin{aligned} S^{(10)}_{IIB} &= \int \frac{1}{2}R * \mathbf{1} - \frac{1}{2(\operatorname{Im}\tau)^2}d\tau \wedge *d\bar{\tau} - \frac{1}{2\operatorname{Im}\tau}G_3 \wedge *\overline{G}_3 - \frac{1}{8}\tilde{F}_5 \wedge *\tilde{F}_5 - \frac{1}{4}C_4 \wedge H_3 \wedge F_3, \\ G_3 &= dC_2 - \tau H_3, \qquad \tilde{F} = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge dC_2, \end{aligned} \tag{4.2.3}$$

that can be easily shown to be invariant under the $SL(2,\mathbb{R})$ transformation

$$\begin{aligned} \tau' &= \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1 \\ \begin{pmatrix} H' \\ F' \end{pmatrix} &= \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix}, \end{aligned} \tag{4.2.4}$$

The $SL(2,\mathbb{R})$ is a symmetry of the classical theory. Once we include D(-1)-instantons (that come with factors $\sim e^{2\pi i\tau}$) the $SL(2,\mathbb{R})$ symmetry is reduced to an $SL(2,\mathbb{Z})$.

The complexified gauge coupling function τ has thus a *fundamental domain*, a region F defined in the complex plane such that any point outside F can be mapped back to F via $SL(2,\mathbb{Z})$ transformations. The fundamental domain is shown in figure 4.1.

One particular choice for this transformation, namely $(a,b,c,d) = (0,-1,1,0)$, corresponds to the S duality

$$e^{\phi} \to e^{-\phi}, \quad B_2 \to C_2, \quad C_2 \to -B_2, \tag{4.2.5}$$

the exchange of a D1-brane (or a D-string) with a fundamental string (F-string). Both strings have the same vibrational spectrum, and same quantum numbers, but they differ on tension ($T_F = e^{\phi} T_D$) and the F-string is magnetically charged under B_2 while the D-string under C_2. In general, we can expect the existence of a (p, q)-string, with charge (p, q) under (B_2, C_2), where under this notation a $(0, 1)$- and an $(1, 0)$-string corresponds to a F- and a D-string, respectively.

Something interesting also happens with the D7-brane, still in the perturbative regime. The D7-brane is magnetically charged under C_0, and noticing that the space transverse to the D7-brane is two dimensional, we can integrate the brane charge in a complex plane z, with the D7-brane at the origin. Normalizing the charge to 1,

$$1 = \oint dC_1, \tag{4.2.6}$$

(analogous to the charge of a magnetic monopole, $\oint F$), that we can solve the residue and write a simple solution close to $z \to 0$,

$$\tau = \tau_0 + \frac{1}{2\pi i} \ln z + \ldots, \tag{4.2.7}$$

plus regular terms. Circling once around the origin gives the monodromy $\tau \to \tau + 1$, and therefore corresponds to the $SL(2, \mathbb{Z})$ transformation with $(a, b, c, d) = (1, 1, 0, 1)$. We can study the behavior of τ far from the brane, calculated from the backreaction of the brane on the spacetime metric, described by the warp factor $B(z, \bar{z})$, in

$$ds^2 = -dt^2 + \sum_i dx_i^2 + e^{B(z,\bar{z})} dz d\bar{z}, \tag{4.2.8}$$

where x_i are the 7 longitudinal directions to the brane. One can solve $B(z, \bar{z})$ in terms of τ, and conclude that [128] the space is asymptotically flat but has a deficit angle of $\pi/6$.

It is interesting to point out what is so special with the seven dimensional brane, compared to lower dimensional branes that do not introduce monodromies or asymptotic backreaction effects. This comes from the fact that when the transversal space to the brane is of dimension $d + 2$ ($d > 0$), the solution of (4.2.6) is of the form $\Phi \sim r^{-d}$, that vanishes at infinity and presents no monodromy.

In general, the $\tau \to \tau + N$ transformation corresponds to the existence of a stack of N D7-branes at the perturbative level. If we move to strong coupling, the perturbative description does not hold anymore, and we must include (p, q)-branes, objects of more complicated monodromies (corresponding to more general terms of the $SL(2, \mathbb{Z})$ action) on which the (p, q) strings can end.

Observing the $SL(2, \mathbb{Z})$ transformations, one could naively guess that Type IIB arises from a twelve dimensional theory compactified on a torus of complex structure τ, with some field strength G_4 wrapping each cycle A or B of the torus, corresponding to F_3 and H_3. The $SL(2, \mathbb{Z})$ transformations would then be simply the modular transformations of a torus. This guess turns out to be problematic, but the idea of a torus compactification survived in the now called *F-theory*.

There is up to now no fundamental description of F-theory, but there are different ways of extracting results in the strong coupling limit of Type IIB theory. One approach is to construct F-theory as a T-dual description of M-theory, as we will review in the following. Another explored approach is via Heterotic/F-theory duality, in which $E_8 \times E_8$ heterotic string theory compactified on a Calabi Yau threefold can be related to F-theory compactified on a fourfold. We will however not discuss Heterotic-F-theory duality here, but the reader may look for example in the original work by Vafa [34].

To formulate F-theory, we start from eleven dimensional M-theory compactified on a torus T^2. More precisely, we consider an elliptic fibration over a nine dimensional base. Locally, the prescription works as follows [129]:

1. As we shrink one of the cycles (say the "A-cycle"), M-theory on the torus becomes Type IIA compactified to nine dimensions on the remaining "B-cycle". The coupling is $g_{IIA} = R_A$, in string units;

2. We shrink the B-cycle to zero, and T-dualize it. The T-dual B'-cycle will have infinite radius, thus getting ten dimensional Type IIB, with coupling $g_{IIB} = g_{IIA}/R_B = R_A/R_B = \operatorname{Im}\tau$, as we wanted.

It is important to notice that the F-theory limit $R_A, R_B \to 0$ is taken in a way that τ is finite, and thus we can move into regions outside perturbation theory.

As a more phenomenologically interesting case, we consider M-theory compactified on an elliptically fibered Calabi-Yau fourfold that corresponds thus to an effective three (1+2) dimensional theory. After the dualization to F-theory, the B-cycle becomes a spacetime dimension, leaving us with four spacetime dimensions, (figure 4.2). It is not obvious that this asymmetric prescription leads to four dimensional Lorentz invariance, but the limit can be taken in a particular way that is Lorentz invariant [125].

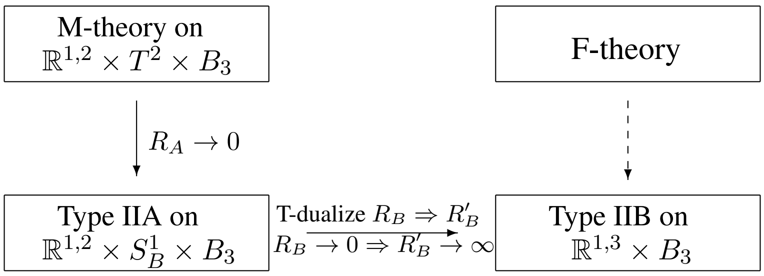


Figure 4.2: The steps from M-theory to Type IIB with an underlying torus structure. The limits are taken in such a way that $\tau = R_A/R_B$ is finite.

The branes in Type IIB, following this prescription, should also come from objects in M-theory. The M2 brane wrapped on the torus becomes an F- or a D-string, and when not wrapping the torus becomes a D3 brane. The M5 brane, depending on how it is wrapped, can correspond to D5- or NS5-branes. The 7-branes, however, do not come from M-branes. Looking at the intermediate Type IIA theory, the D7-branes of type IIB should come from D6-branes. The latter, as was seen in the previous section, are purely geometrical solutions of the eleven dimensional theory[2]. We should then expect that the 7-branes also have a geometrical origin, as we will describe.

We start with the algebraic construction of an *elliptic curve* (i.e., the torus), given by the Weierstrass function

$$y^2 = x^3 + fxz^4 + gz^6, \tag{4.2.9}$$

where $(y, x, z) \in \mathbb{C}^3$. More precisely, the elliptic curve is defined inside a weighted projective space $\mathbb{P}^2$, with homogeneous coordinates (y, x, z) obeying the rescaling conditions $(y, x, z) \simeq (\alpha^3 y, \alpha^2 x, \alpha z)$. In the patch where $z \neq 0$ we can use the rescaling condition to set $z = 1$,

$$y^2 = x^3 + fx + g = (x - e_1)(x - e_2)(x - e_3). \tag{4.2.10}$$

The equation above describes a double cover of the x plane, and by writing

$$y = \sqrt{(x - e_1)(x - e_2)(x - e_3)}, \tag{4.2.11}$$

one can easily see that it has branch points at $x = e_1, e_2, e_3$. There is also a monodromy around a circle of infinite radius, $x = |x|e^{i\theta}$ taking θ from 0 to 2π. We can alternatively describe this infinite circle by identifying the infinity in the complex plane to a point, so the infinite circle is a small circle around the infinity "point". But $\mathbb{C} \cup \infty \simeq \mathbb{P}^1$, and we can then represent the domain of $y(x)$ as a (double cover of

[2]The connection between the geometrical Taub-NUT spaces and D7-branes was explored for example in [130] to calculate non-perturbative contributions coming from $U(1)$ fluxes to the gauge-coupling function on a D7-brane.

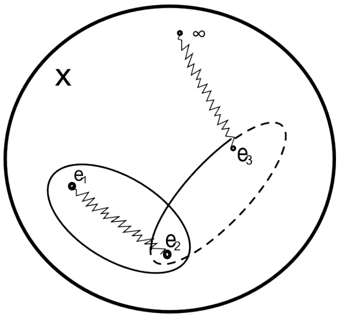


Figure 4.3: The x plane with the added point at infinity ($\sim \mathbb{P}^1$), as one cover of the $y(x)$ function. The branch cuts for (4.2.11) connects the branch points. It is also drawn the two non-homological 1-cycles, where the dashed line indicates the cycle passing through the second cover of $y(x)$.

a) sphere. We have then to define the branch cuts, and we can do it as represented in figure 4.3. The resulting manifold has then two independent 1-cycles, that we can immediately identify with the two independent cycles of the torus.

The complex structure of the torus can be calculated from

$$\tau = \frac{\oint_A \Omega}{\oint_B \Omega}, \tag{4.2.12}$$

where Ω is the globally defined meromorphic 1-form, and A and B are two independent one-cycles in figure 4.3. Clearly, τ is invariant up to $SL(2,\mathbb{Z})$ transformations when we consider any two combinations $A' = aA + bB$, $B' = cA + dB$ for the cycles defining (4.2.12).

The meromorphic form Ω can be obtained from the general equation

$$\Omega = \frac{1}{2\pi i} \oint_{P=0} \frac{w \cdot V}{P}, \tag{4.2.13}$$

where P is the defining equation for the elliptic curve, $P = y^2 - x^3 - fxz^4 - gz^6$, w is a meromorphic form defined on the ambient $\mathbb{C}^3$ (with w/P gauge invariant), $w = dx \wedge dy \wedge dz$, and V the vector fields generating the gaugings $(x, y, z) \simeq (\alpha^2 x, \alpha^3 y, \alpha z)$,

$$V = \sum_{x_i = x,y,z} Q_i x_i \frac{\partial}{\partial x_i} = 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \tag{4.2.14}$$

Inserting V, w and P into (4.2.13),

$$\Omega = \frac{1}{2\pi i} \oint_{P=0} \frac{2xdy \wedge dz + 3ydx \wedge dz + zdx \wedge dy}{-y^2 + x^3 + fxz^4 + gz^6}. \tag{4.2.15}$$

Since this holds globally, we can move to the particular patch $z = 1$, and easily solve

$$\Omega = \frac{1}{2\pi i} \oint_{P=0} \frac{dx \wedge dy}{P} = \frac{dx}{\partial P/\partial y} = \frac{dx}{2y}. \tag{4.2.16}$$

It is a known result from the mathematics literature that we can write a relation among the complex structure τ for the elliptic curve and the parameters f and g [128],

$$j(\tau) = \frac{4(24f)^3}{\Delta}, \quad \Delta = 27g^2 + 4f^3, \tag{4.2.17}$$

with $j(\tau) = e^{-2\pi i\tau} + 744 + e^{2\pi i\tau} + \ldots$, that is invariant under $SL(2,\mathbb{Z})$ transformation. $j(\tau)$ is a function that maps the fundamental domain of τ into $\mathbb{C}$. The Δ function is the discriminant of the curve, and the elliptic curve becomes singular when $\Delta = 0$. In terms of equation (4.2.11), this happens when two e_is coincide, and one of the cycles shrinks to zero size.

We now "promote" the elliptic curve to an *elliptic fiber*, by introducing a base B over which the parameters of the elliptic curve, f and g vary. We consider a simple case where the base has just one complex dimension represented locally by the coordinate u. In $K3$ fibrations, for example, the base is a $\mathbb{CP}^1$, a two-sphere, and $f(u)$ and $g(u)$ are polynomial functions of degree 12 and 8, respectively. Thus, Δ will be a degree 24 polynomial, and will have generically 24 order one zeros. We choose one of these particular zeros, namely $u = u_i$, such that near this point u_i,

$$j(\tau) \sim \frac{1}{u - u_i} \Rightarrow \tau \sim \frac{1}{2\pi i} \ln(u - u_i), \tag{4.2.18}$$

again reproducing the D7-brane monodromy. In fact, this solution is defined only up to an $SL(2,\mathbb{Z})$ transformation. When we consider more branes localized at different points u_i, we cannot in general fix τ to be of the form (4.2.18) for each brane. Not only this, but the monodromy present at one particular point u_i can change as we circle around another point u_j. So, in general, identifying locally a D7-brane (or a (p,q)-brane, with specific (p,q) values) is not something that holds globally.

There are however particular cases in which we can restore the perturbative type IIB description globally. In the beautiful work by Sen [131, 132] the conditions for a constant τ with large imaginary part (corresponding to small string coupling) are imposed, and a precise interpretation of Type IIB theory from the F-theory perspective is worked out. So, although the D-brane + O-plane picture can indeed be achieved, it is not a general feature, but rather a very particular subdomain of possible theories.

We also introduce a more general form for the Weierstrass function (with $z = 1$),

$$-y^2 + x^3 + a_6 + a_4 x + a_3 y + a_2 x^2 + a_1 xy = 0. \tag{4.2.19}$$

This form can be reduced back to (4.2.9) by completing the squares. To relate the new coefficients to f and g we first define

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \tag{4.2.20}$$

$$c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2 b_4 - 216b_6. \tag{4.2.21}$$

One can then show that

$$f = -\frac{1}{48}c_4, \quad g = -\frac{1}{864}c_6. \tag{4.2.22}$$

and from $\Delta = 27g^2 + 4f^3$ one computes directly

$$\Delta = \frac{1}{4}b_2^2(b_2 b_6 - b_4^2) - 8_4^3 - 27b_6^2 + 9b_2 b_4 b_6. \tag{4.2.23}$$

This form will be used later to introduce a new section that will correspond to a particular divisor in the base, via the Tate algorithm.

If we want our compactification to lead an effective four dimensional $\mathcal{N} = 1$ supersymmetric theory, we must compactify on a Calabi-Yau fourfold. This imposes some conditions on the elliptic fiber, that we review here. In a general F-theory compactification, there will be some divisors S_i on the base over which the fiber degenerates. In Type IIB theory picture, this corresponds to a D7-brane wrapping S_i. The first Chern class of such a fibration $\pi : Y_4 \to B_3$ is given by [126]

$$c_1(Y_4) \simeq \pi^*(c_1(B_3)) - \sum_i \frac{\delta_i}{12}[S_i], \tag{4.2.24}$$

where $\delta_i \sim \Delta|_{S_i}$, $\Delta|_{S_i}$ is the discriminant of the elliptic curve close to the divisor S_i, and $[S_i]$ is the Poincarè-dual two-form to the divisor S_i. We also used the common notation $c_1(X) = c_1(TX)$, where TX is the tangent bundle of any manifold X. If we want the fourfold to be Calabi-Yau, we must require $c_1(Y_4) = 0$ and therefore

$$\sum_i \delta_i [S_i] = 12 c_1(B_3), \tag{4.2.25}$$

that can be seen as a consequence of the tadpole cancellation of charges for D7-branes and O7-planes, (analogous to the condition for D6-branes in Type IIA orientifold compactifications, as in our setup (3.3.2)),

$$\sum_i N_i [S_i] = 4[O_7]. \tag{4.2.26}$$

An interesting consequence of F-theory compactifications is that the base itself is not Calabi-Yau, but only the full fourfold. As we move to the strong coupling regime of Type IIB theory, the D7-branes induce strong backreactions in the threefold geometry, thus breaking the Calabi-Yau condition.

The Calabi-Yau condition (4.2.25) implies that the variables of the Weierstrass form must be sections of specific bundles. Recall the first Chern class of the Canonical Line Bundle, $c_1(K_{B_3}) = -c_1(B_3)$. Also, the left-hand side of (4.2.25) is in homology equal to the full $[\Delta]$. Together with the relation $\Delta = 27g^2 + 4f^3$, this implies that f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$, respectively. Since the fiber is given by the Weierstrass equation,

$$y^2 = x^3 + xfz^4 + gz^6, \tag{4.2.27}$$

x, y must transform as $K_{B_3}^{-2}$ and $K_{B_3}^{-3}$, respectively. By construction, z is the section isomorphic to the base B_3. So, if we want our elliptic fourfold to be Calabi-Yau, the sections must transform as required above. In terms of the modified Weierstrass function (4.2.19), a_i, b_i and c_i each are sections of $K_{B_3}^{-1}$.

In perturbative type IIB, N coinciding D7-branes give SU(N), SO(2N) or Sp(2N) groups, depending on how the divisor wrapped by the brane behaves under the orientifold involution. In the F-theory perspective, putting branes on top of each other leads to higher vanishing order of Δ in (4.2.17). When this happens, not only the elliptic fiber degenerates, but it becomes singular, i.e., $P = dP = 0$, where P is the defining function of the elliptic fiber, (4.2.9). One can then resolve the singularity, and the resolution leads to a set of two-cycles that can be identified with the root system of some gauge group, as we describe in more detail in the next chapter. One of the most phenomenologically interesting consequences is that in F-theory there is the possibility of constructing gauge groups not realizable in perturbation theory. We will describe gauge groups via F-theory compactifications in the next chapter.

Chapter 5

Singularities in F-theory

In F-theory compatifications to four dimensions the Calabi-Yau threefold of the perturbative Type IIB is replaced by an elliptically fibered Calabi-Yau fourfold, and the regions on the base where the fiber degenerates correspond to a 7-brane. The total compactification space might be seen as a fourfold inside an ambient five dimensional manifold with a $\mathbb{P}^2$ fiber in which the elliptic curve is constructed. The gauge theory living on the brane is now encoded in the kind of singularity appearing in the elliptic fiber over the base regions where the fiber degenerates.

The way to identify the corresponding gauge group of a singularity in the elliptic fiber is by resolving it. The resolution in general replaces the singular point by a series of $\mathbb{P}^1$s on the fiber, that are two-cycles. From the M-theory perspective [133, 134], each 2-cycle can be wrapped by M2-branes of different orientations. These M2-branes are charged under the three-form C_3, that decomposes as

$$C_3 = A_1^k \wedge \omega_k \,, \tag{5.0.1}$$

where ω_k is a two-form on the k-th $\mathbb{P}^1$, and each A_1^k is a one-form corresponding to a vector gauge boson in the base of the fibration, and the M2-branes correspond to W bosons. When we shrink the $\mathbb{P}^1$ to zero size, the branes and the $U(1)$ boson become massless, and they enhance to an unbroken $SU(2)$. Thus, each $\mathbb{P}^1$ in the resolved singularity correspond to a broken $SU(2)$, and the blow-down limit arrange each $SU(2)$ as the root elements of a higher order group.

The classification of the singularities for compactifications on an elliptically fibered K3 (in which the base is the projective[1] space $\mathbb{P}^1$) was done by Kodaira [135, 136]. This classification identifies what sort of singularities appear at some point (divisor) on the base, depending on the vanishing order of Δ, f and g. The singularities appearing in this construction are ADE singularities, referring to the ADE groups associated to them. The classification is summarized in table 5.0.1.

With this prescription we can construct effective gauge theories in four dimensions with the groups described in table 5.0.1. But to build a realistic model one also has to include matter in the spectrum, as well as interactions among the matter representations. The matter representations are constructed in F-theory from intersections between singularities, that in the perturbative description correspond to 7-branes colliding, and a matter representation appearing at the intersection curve. Couplings among different matter curves happen as the matter curves intersect in points on the base, often called Yukawa points. There is also an equivalent brane picture, in which three or more branes intersect at a point.

The matter representations must be chiral, as is our real world. One standard way to construct chiral representations, also imported from model building with intersecting D-branes, is to add fluxes along the matter curve. In M-/F-theory, the only type of flux in the theory comes from the field strength G_4. This at first might seem as a contradiction, since the matter curve is two-real-dimensional, while G_4 is a four-form. However, at the matter curves the singularity enhancement can be interpreted as a collapsed two-cycle on the fiber that exists only on top of the matter curve. We can then localize the flux along the matter curve by decomposing it as

$$G_4 = F_2 \wedge \omega_0, \tag{5.0.2}$$

where F_2 is a two-form flux and ω_0 is the two-form dual to the collapsed two-cycle. Fluxes in general also play an important role on the breaking of GUT gauge theories to (extensions of) the Standard

[1] since we are dealing only with complex projective spaces, we write simply $\mathbb{P}^k$ instead of $\mathbb{CP}^k$. But it should be kept in mind that the homogeneous coordinates are always complex.

Type	ord(f)	ord(g)	ord(Δ)	$j(\tau)$	Group	Monodromy
I_0	≥ 0	≥ 0	0	$\mathbb{R}$	—	$\mathbf{1}_{2\times 2}$
I_1	0	0	1	∞	$U(1)$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
I_n	0	0	$n > 1$	∞	A_{n-1}	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
II	≥ 1	1	2	0	—	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
III	1	≥ 2	3	1	A_1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
IV	≥ 2	2	4	0	A_2	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_n^*	2	≥ 3	$n+6$	∞	D_{n+4}	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$
	≥ 2	3				
IV^*	≥ 3	4	8	0	E_6	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$
III^*	3	≥ 5	9	1	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
II^*	≥ 4	5	10	0	E_8	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Table 5.0.1: Kodaira Classification of singular fibers of an K3 elliptic fibration. Table extracted from [55, 137].

Model, and as in the example cited in chapter 3, fluxes can lead to corrections on the effective action characteristic data and to contributions to the scalar potential, with obvious and important consequences for phenomenology. A better understanding of the role of fluxes in F-theory model building has been an important field of research in the late years, and as some examples of the still growing literature we can cite [38, 40, 130, 138–144] and references therein.

5.1 The basics on Blow-ups

Since the construction of gauge groups and representations depend on the resolved manifold as seen in the M-theory picture, it is thus convenient to review some basics on blow-ups. We first look at simple double point singularities inside $\mathbb{C}^3$ that will allow us to set the ground for singularities arising in F-theory elliptic fibrations. A similar discusion focused on orbifold singularities appears in [145].

5.1.1 The Blow-up of a point

In this short section, we review the blow-up of the origin point in $\mathbb{C}^n$ [146]. We first introduce a $\mathbb{P}^{n-1}$ with homogeneous coordinates $[\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_n]$, and relating the original $\mathbb{C}^n$ coordinates z_i with the $\mathbb{P}^{n-1}$ coordinates via

$$z_i = \lambda \tilde{z}_i, \tag{5.1.1}$$

with λ complex. The relation (5.1.1) defines a subvariety inside $\mathbb{C}^n \times \mathbb{P}^{n-1}$, and we can define a projection π from the blow-up space to $\mathbb{C}^n$ a $\pi : (z_i, \tilde{z}_i) \to z_i$.

The origin is then replaced by the condition $\lambda = 0$, which implies that the coordinates x', y' and z' can take any value obeying the homogeneity condition. Thus, $\pi^{-1}(0) = \mathbb{P}^{n-1}$. Outside the origin, $\lambda \neq 0$ and (5.1.1) specifies a point in the $\mathbb{P}^{n-1}$. Therefore outside the origin the blow-up space is isomorphic to $\mathbb{C}^n$.

5.1.2 The A_1 singularity

Now we want to describe the resolution of a submanifold inside $\mathbb{C}^3$. A simple example is the A_1 singularity given by the singular point in the surface

$$X_{A_1} = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{C}^3. \tag{5.1.2}$$

It has a singular (double) point located at the origin $p_0 : x = y = z = 0$ (at this point, $X_{A_1} = 0$ and $dX_{A_1} = 0$). The resolution procedure consists in blowing up a $\mathbb{P}^2$ at the singular point of the embedding space,

$$x = \lambda\tilde{x}, \quad y = \lambda\tilde{y}, \quad z = \lambda\tilde{z}, \quad [\tilde{x}, \tilde{y}, \tilde{z}]. \tag{5.1.3}$$

Replacing the coordinates in (5.1.2) with (5.1.3), we get

$$\lambda^2(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2) = 0\,. \tag{5.1.4}$$

Notice the singularity $y = x = z = 0$ is now "encoded" in the condition $\lambda = 0$. We artificially remove the singularity by defining $\tilde{X}_{A_1}$, the blow-up space of X_{A_1}, as the *proper transform* of (5.1.2), that is,

$$\tilde{X}_{A_1} = \{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = 0 \quad | \quad [\tilde{x} : \tilde{y} : \tilde{z}]\}\,. \tag{5.1.5}$$

The new manifold $\tilde{X}_{A_1}$ is not singular ($X_{A_1} = 0$ and $dX_{A_1} = 0$ cannot be simultaneously satisfied).

We will now show that the singularity of the original curve X was replaced in the blown-up space by a $\mathbb{P}^1$ defined by $\{\lambda = 0\} \cap \tilde{X}_{A_1}$. The equation $\lambda = 0$ defines the *exceptional divisor* E in $\mathbb{C}^3$. To show that $E|_{\tilde{X}_{A_1}} = \mathbb{P}^1$, it is convenient to perform a change of coordinates such that the defining equation can be written in the form

$$\tilde{X}_{A_1} = \{-\tilde{u}\tilde{v} + \tilde{z}^2 = 0 \quad | \quad [\tilde{u} : \tilde{v} : \tilde{z}]\}\;. \tag{5.1.6}$$

Introducing the patches $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{Z}$, defined respectively by $\tilde{u} \neq 0$, $\tilde{v} \neq 0$ and $\tilde{z} \neq 0$, we can use the rescaling condition to fix $\tilde{u} = 1$ in the patch $\mathcal{U}$, and the same to $\mathcal{V}$ and $\mathcal{Z}$. In the patch $\mathcal{U}$

$$\tilde{v} = \tilde{z}^2, \tag{5.1.7}$$

which fixes $\tilde{z}$ in terms of $\tilde{v}$, so locally $E|_{\mathcal{U}} = \mathbb{C}$. Similarly, in the patch $\mathcal{V}$ equation $\tilde{u} = \tilde{z}^2$ implies $E|_{\mathcal{V}} = \mathbb{C}$. At the intersection $\mathcal{U} \cap \mathcal{V}$ the defining equation (5.1.6) implies that $\tilde{z} \neq 0$. The intersection region $\mathcal{U} \cap \mathcal{V}$ is therefore visible in the patch $\mathcal{Z}$,

$$\tilde{v} = \frac{1}{\tilde{u}}, \tag{5.1.8}$$

that is precisely the transition function for the two local patches $\mathbb{C}$ that define a $\mathbb{P}^1$.

5.1.3 A_2 Singularity

Another simple example is the singular surface defined by

$$X_{A_2} = \{x^2 + y^2 + z^3 = 0\} \subset \mathbb{C}^3. \tag{5.1.9}$$

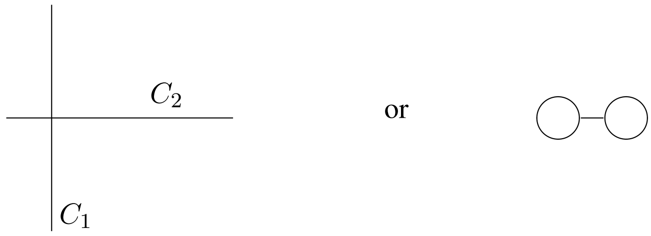
After blowing up a $\mathbb{P}^2$ at the origin as before,

$$\tilde{X}_{A_2} = \{\tilde{x}^2 + \tilde{y}^2 + \lambda\tilde{z}^3 = 0\} \subset \mathbb{C}^3 \times \mathbb{P}^2. \tag{5.1.10}$$

The origin was again replaced by the divisor $\lambda = 0$ in the ambient space, that now restricted to the surface $\tilde{X}_{A_2}$ corresponds to two independent curves,

$$C_1 \quad : \quad (\tilde{x} + i\tilde{y}) \quad , \qquad\qquad C_2 \quad : \quad (\tilde{x} - i\tilde{y}) \quad , \tag{5.1.11}$$

that intersect at the point $\lambda = x = y = 0$. Thus, the blow-up of X_{A_2} replaces the singular point by two intersecting $\mathbb{P}^1$s,



A_n singularity

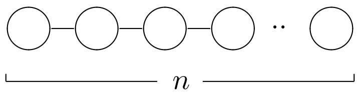
The general A_n singularity is given by the singular locus of the surface

$$X_{A_n} = \{x^2 + y^2 + z^{n+1} = 0\} \in \mathbb{C}^3. \tag{5.1.12}$$

After the first blow up,

$$x = \lambda_1 x', \quad y = \lambda_1 y', \quad z = \lambda_1 z', \quad [x' : y' : z']. \tag{5.1.13}$$

the surface might still be singular at $x' = y' = \lambda_1 = 0$. We need to blow-up again with new λ_i's until we get rid of all singularities, and the exceptional divisors will be given by restrictions of each $\lambda_i = 0$. One can easily show that this leads to a chain of n $\mathbb{P}^1$s, as the Dynkin diagram for an $SU(n+1)$ group,



5.1.4 The E_8 singularity

Analogously, we proceed to resolve the E_8 singularity, important in F-theory model building. In this example we come across a small resolution. The E_8 singularity is described by the point $\{x, y, z\} = \{0, 0, 0\}$ of the surface

$$X_{E_8} = \{y^2 + x^3 + z^5 = 0\} \in \mathbb{C}^3. \tag{5.1.14}$$

We perform the first blow up as

$$y = a_0 y_1, \qquad x = a_0 x_1, \qquad z_0 = a_0 z_1, \tag{5.1.15}$$

with the new homogeneous coordinates

$$[y_1 : x_1 : z_1]\,. \tag{5.1.16}$$

The subscript will indicate the number of times the coordinate was blown up. From now on, we will introduce the notation $a_0 : [x, y, z]$, corresponding to the blow up described above. After we take the proper transform, the defining equation for the blown-up surface is

$$y_1^2 + a_0 x_1^3 + a_0^3 z_1^5 = 0. \tag{5.1.17}$$

This is singular in $y_1 = x_1 = a_0 = 0$. We perform then a sequence of blow ups. Here we just show one particular choice of many possible resolutions, in the following order:

$$a_0 : [y, x, z], \quad b_0 : [y_1, x_1, a_0], \quad c_0 : [y_2, a_1, b_0], \quad d_0 : [y_3, c_0, a_2], \quad e_0 : [y_4, c_1, b_1]. \tag{5.1.18}$$

As said, the notation means for example that the third blow up is defined as $y_2 = c_0 y_3$, $a_1 = c_0 a_2$ and $b_0 = c_0 b_1$, with a projective relation $[y_3 : a_2 : b_1]$. The proper equation of X_{E_8} after the blow ups mentioned above is

$$y_5^2 + e_0 c_2 b_2 a_3 (b_2 x_2^3 + d_0^3 c_2 a_3^2 z_1^5) = 0 \tag{5.1.19}$$

There is still a singularity left, when $y = e_0 = (b_2 x_2^3 + d_0^3 c_2 a_3^2 z_1^5) = 0$. However, we do not need to perform a $\mathbb{P}^2$ blow up as before, since the equation is factorized. It is a binomial variety of the form $y^2 + e_0 U = 0$, that is a "conifold-like" equation. And just like the conifold, we can perform a *small resolution*, a $\mathbb{P}^1$ blow up instead of the $\mathbb{P}^2$ blow up.

A conifold $uv + w^2 = 0$ can be resolved either by blowing up $\{u, w\}$ or $\{v, w\}$. The two resolutions are connected via a *conifold transition*, when we start with a resolved conifold, say, in $\{u, w\}$, we blow down by shrinking the volume of the $\mathbb{P}^1$ to zero, thus restoring the singularity, and then blow up the coordinates $\{v, w\}$. Another way to de-singularize the conifold is to introduce a non-zero parameter in the equation as $uv + w^2 = \epsilon$, *deforming* the conifold. The deformation and the resolution are connected by a flop transition, sending the parameter ϵ to zero and then blowing up a $\mathbb{P}^1$, or vice-versa.

In terms of the ambient space $\mathbb{C}^3$ with coordinates u, v, w, a small resolution defined as $u = \lambda u_1$, $v = \lambda v_1$ with the projective relation $[u_1 : v_1]$ corresponds to replace every point on the complex plane $u = v = 0$ by a $\mathbb{P}^1$, instead of the replacement of the origin point by a $\mathbb{P}^2$, described in section 5.1.1.

Back to (5.1.19), we perform the small resolution $f : [y_5, e]$, obtaining the smooth space defined by

$$\tilde{X}_{E_8} \quad : \quad f\tilde{y}^2 + ecba(b\tilde{x}^3 + d^3 ca^2 \tilde{z}^5) = 0\,. \tag{5.1.20}$$

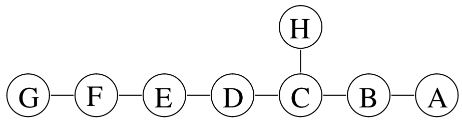
To clean the notation, we simply omitted all the subscripts and denoted the final coordinates $\tilde{y}$, $\tilde{x}$ and $\tilde{z}$ to differentiate from the original coordinates y, x and z.

The projective relations in terms of the final blown up coordinates are

$$[fedcb\tilde{y} : fedcb\tilde{x} : \tilde{z}]\,, \quad [fedc\tilde{y} : \tilde{x} : fedca]\,, \tag{5.1.21}$$

$$[fd\tilde{y} : da, feb]\,, \quad [fe\tilde{y} : fec : a]\,, \quad [f\tilde{y} : c : b]\,, \quad [\tilde{y} : e]\,.$$

The exceptional divisors are obtained as before by restricting the new coordinates a,...,f to zero. The restriction $\{\lambda_i = 0\} \cap \tilde{X}_{E_8}$ (where $\lambda_i = a \ldots f$) gives the curves from table 5.1.1, that intersect as the E_8 Dynkin diagram below.



The E_8 singularity from an elliptic fibration

In F-theory compactifications we can get an E_8 gauge group on a divisor $\{w\}$ of the base defined by the equation $w = 0$ if our curve for the elliptic fiber is written as

$$y^2 + x^3 + \beta_0 w^5 = 0, \tag{5.1.22}$$

with $\beta_0 \neq 0$. This is the same equation as above, and we can thus resolve it in the same way (we introduce back the λ_is to denote the coordinates introduced at each blow up. We will mantain this notation from now on),

$$\tilde{Y}_{E_8} \quad : \quad \lambda_6 \tilde{y}^2 + \lambda_5 \lambda_3 \lambda_2 \lambda_1 (\lambda_2 \tilde{x}^3 + \beta_0 \lambda_4^3 \lambda_3 \lambda_1^2 \tilde{w}^5) = 0\,, \tag{5.1.23}$$

together with the projective relations

$$[\lambda_6\lambda_5\lambda_4\lambda_3\lambda_2\tilde{y} : \lambda_6\lambda_5\lambda_4\lambda_3\lambda_2\tilde{x} : \tilde{w}]\,, \quad [\lambda_6\lambda_5\lambda_4\lambda_3\tilde{y} : \tilde{x} : \lambda_6\lambda_5\lambda_4\lambda_3\lambda_1]\,,$$
$$[\lambda_6\lambda_4\tilde{y} : \lambda_4\lambda_1, \lambda_6\lambda_5\lambda_2]\,, \quad [\lambda_6\lambda_5\tilde{y} : \lambda_6\lambda_5\lambda_3 : \lambda_1]\,, \quad [\lambda_6\tilde{y} : \lambda_3 : \lambda_2]\,, \quad [\tilde{y} : \lambda_5]\,.$$

The singularity located at the point $y = x = 0$ of the elliptic fiber on top of the $\{w\}$ divisor (that is, the divisor defined by the holomorphic equation $w = 0$) is replaced by a set of smooth intersecting

Curve		Multiplicity	Diagram
$A:$	$\lambda_1 = y = 0$	2	
$B:$	$\lambda_2 = \lambda_6 = 0$	3	
$C:$	$\lambda_2 = \tilde{y} = 0$	3	
$D:$	$\lambda_3 = \lambda_6 = 0$	4	
$E:$	$\lambda_3 = \tilde{y} = 0$	5	
$F:$	$\lambda_4 = \lambda_6\tilde{y}^2 + \lambda_5\lambda_3\lambda_2^2\lambda_1\tilde{x}^3 = 0$	2	
$G:$	$\lambda_5 = \lambda_6 = 0$	4	
$H:$	$\lambda_6 = \lambda_2\tilde{x}^3 + \lambda_4^3\lambda_3\lambda_1^2\tilde{z}^5 = 0$	6	
$X:$	$z = \lambda_2 y^2 + \lambda_1\lambda_4^2\lambda_5 x^3 = 0$	1	

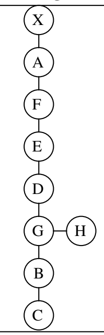


Table 5.1.1: Curves and intersections for an E8 singularity

exceptional divisors over which the fiber is described by a $\mathbb{P}^1$. The complete description of the resolved fiber on top of the original $\{w\}$ divisor can be seen from noticing that the restriction $w = 0$ becomes after the resolution

$$\tilde{w}\lambda_1\lambda_2\lambda_3\lambda_4^2\lambda_5^4\lambda_6^6 = 0. \tag{5.1.24}$$

The power appearing on each holomorphic variable λ_i must be taken into account, and they contribute to the multiplicity of each exceptional divisor. Formally, after blowing up an exceptional divisor E inside Y, an already existing divisor D inside Y becomes

$$\pi^{-1}D = \tilde{D} + nE \tag{5.1.25}$$

where π is the projection as defined in section 5.1.1, $\tilde{D}$ is the proper transform of D, and n of an exceptional divisor E is the multiplicity. In the case at hand, the divisor $\{w\}$ inside the fivefold X_5 becomes

$$\pi^{-1}\{w\} = \{\tilde{w}\} + \{\lambda_1\} + \{\lambda_2\} + \{\lambda_3\} + 2\{\lambda_4\} + 4\{\lambda_5\} + 6\{\lambda_6\}. \tag{5.1.26}$$

Note that we still have to intersect these divisors inside X_5 with the fiber (5.1.23). When we do that, it is straightforward to see that we obtain the curves and multiplicities of table 5.1.1, that intersect according to the depicted diagram. The intersections and multiplicities obtained agree exactly with the affine Dynkin diagram for the E_8 group, as in [147].

The fiber over the curve X in **??**, obtained by $\tilde{w} = 0$, corresponds to the affine node in the E_8 affine diagram, as described in picture 5.1. It cannot be interpreted as a broken $U(1)$, in the sense that an M2 brane wrapping X does not shrink to zero volume in the unbroken limit, when we blow down the resolution.

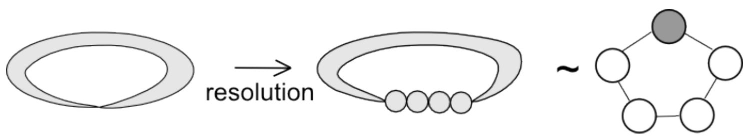


Figure 5.1: The resolution replaces the singularity by a series of $\mathbb{P}^1$s, and their intersections can be described in a Dynkin diagram. Here we show the A_4 (or SU(5)) case. The affine node is simply the proper transform of the singular elliptic curve, that is topologically a $\mathbb{P}^1$.

5.2 An SU(5) GUT Model

In this section we review an $SU(5)$ model with enhancement loci, where the enhancements can be resolved via small resolutions as done in [55] and further explored in [57]. We start by constructing the elliptic curve [2], introduced in previous chapter, by the Weierstrass equation defined as a submanifold inside a projective space $\mathbb{P}^2$,

$$y^2 - x^3 - fxz^4 - gz^6 = 0. \tag{5.2.1}$$

The complex structure of the elliptic curve τ can be obtained from f and g Now we fiber this elliptic curve over a base B_3. We construct the fibration in such a way that we can define a smooth section isomorphic to the base. This section will be given by $z = 0$, that restricts the elliptic curve to the divisor (a point) $(x, y) = (0, 1)$, where we used the rescaling condition to fix y. In F-Theory the base will be identified with the physical internal space of Type IIB String Theory, while the complex structure of the elliptic fiber is identified with the axion-dilaton in type IIB theory.

To geometrically engineer an SU(5) model, we specify a divisor on the base such that the fibration develops an SU(5) singularity. In type IIB, the divisor will correspond to the divisor wrapped by the stack of D7 Branes. Additionally, there might be regions on this divisor where this singularity is enhanced. In the Type IIB picture, this corresponds to other branes intersecting the SU(5) Brane.

We explicitly construct such fiber by specifying the f and g, and one way to do that is by the *Tate's Algorithm*. We start with a more general expression for the Weierstrass equation,

$$-zy^2 + x^3 + a_0z^3 + a_2xz^2 + a_3yz^2 + a_4zx^2 + a_5zxy = 0, \tag{5.2.2}$$

that can be obtained from (5.2.1) by completing squares. The paremeters a_i are related to the complex structure of the curve, and in general depend on the coordinates of the base. Since we want an SU(5) singularity on a divisor $w = 0$ of the base, we introduce the section w as

$$a_0 = \beta_0 w^5\,, \quad a_2 = \beta_2 w^3\,, \quad a_3 = \beta_3 w^2\,, \quad a_4 = \beta_4 w\,, \quad a_5 = \beta_5\,. \tag{5.2.3}$$

Thus, the elliptic fiber reads

$$-y^2 + x^3 + \beta_0 z^3 w^5 + \beta_2 xz^2 w^3 + \beta_3 yz^2 w^2 + \beta_4 x^2 zw + \beta_5 zxy = 0. \tag{5.2.4}$$

According to the Tate's algorithm, this model gives an elliptic fiber with an $SU(5)$ singularity over the divisor defined by $w = 0$. The restriction of this divisor to the base we will call S_{GUT}, that in the Type IIB limit corresponds to the divisor wrapped by the brane that holds the GUT group.

We can see that by recalling the definitions in chapter 4, the discriminant Δ and the functions f and g become

$$\begin{aligned}
\Delta =& w^5\Delta',\\
f =& \frac{1}{48}(-\beta_5^4 - 8\beta_4\beta_5^2 w - 16\beta_4^2 w^2 + 24\beta_3\beta_5 w^2 + 48\beta_2 w^3),\\
g =& \frac{1}{864}(\beta_5^6 + 12\beta_4\beta_5^4 w + 48\beta_4^2\beta_5^2 w^2 - 36\beta_3\beta_5^3 w^2 +\\
& + 64\beta_4^3 w^3 - 144\beta_3\beta_4\beta_5 w^3 - 72\beta_2\beta_5^2 w^3 + 216\beta_3^2 w^4 - 288\beta_2\beta_4 w^4 + 864\beta_0 w^5),
\end{aligned} \tag{5.2.5}$$

with

$$\Delta' = \beta_5 P + w\beta_5(8\beta_4 P + \beta_5 R) + w^2(16\beta_3^2\beta_4^3 + \beta_5 Q) + w^3 S + w^4 T + w^5 U, \tag{5.2.6}$$

[2] Often in the literature (as in the review [126] and in the previous chapter) the elliptic curve is constructed in a weighted projective space $\mathbb{WP}^2_{2,3,1}$, with rescaling relations $(x, y, z) \equiv (\lambda^2 x, \lambda^3 y, \lambda z)$, and $\lambda \in \mathbb{C}^*$.

where P, R, Q, S, T and U are polynomials related to β_i and w as

$$\begin{aligned}
&P = \beta_3^2\beta_4 - \beta_2\beta_3\beta_5 + \beta_0\beta_5^2, \quad R = 4\beta_0\beta_4\beta_5 - \beta_3^3 - \beta_2^2\beta_5, \\
&Q = -2(18\beta_3^3\beta_4 + 8\beta_2\beta_3\beta_4^2 - 15\beta_2\beta_3^2\beta_5 + 4\beta_2^2\beta_4\beta_5 - 24\beta_0\beta_4^2\beta_5 + 18\beta_0\beta_3\beta_5^2), \\
&S = 27\beta_3^4 - 72\beta_2\beta_3^2\beta_4 - 16\beta_2^2\beta_4^2 + 64\beta_0\beta_4^3 + 96\beta_2^2\beta_3\beta_5 - 144\beta_0\beta_3\beta_4\beta_5 - 72\beta_0\beta_2\beta_5^2, \\
&T = 8\left(8\beta_2^3 + 27\beta_0\beta_3^2 - 36\beta_0\beta_2\beta_4\right), \quad U = 432\beta_0^2.
\end{aligned} \tag{5.2.7}$$

So on top of the GUT divisor $w = 0$, for general values of β_is, and as long as Δ' remains non-vanishing, the vanishing orders of Δ, f and g as we approach $w \to 0$ are

$$\text{ord}(\Delta) = 5, \qquad \text{ord}(f) = 0, \qquad \text{ord}(g) = 0. \tag{5.2.8}$$

According to the Kodaira classification described in table 5.0.1, this corresponds to a singular curve of type I_5, thus an A_4 (or $SU(5)$) singularity.

Notice also that if we just fix all the parameters β_i to zero except β_0, we reproduce the E_8 singularity described in section 5.1.4. In fact, such an $SU(5)$ model described by (5.2.4) can be seen as a higgsing of an E_8 down to $SU(5)$,

$$E_8 \to SU(5) \times SU(5)_\perp, \tag{5.2.9}$$

and the vevs for the Higgs are related to the sections β_i. In the Type IIB picture, this can be interpreted locally as a stack of five D7-branes separated by some distance (encoded by the vev of a Higgs field), to other branes. As we move them close together the singularity enhances. However, from the pure perturbative description we cannot reproduce an E_8-brane. If however we allow ourselves to deepen in the strong coupling regime, we need to include (p,q)-7-branes, and open strings that attach to 3 or more branes. These bizarre open strings configurations can reproduce an E_8 algebra [148, 149].

In the following, we proceed to resolve the curve explicitly. First notice that

$$-zy^2 + x^3 + \beta_0 z^3 w^5 + \beta_2 x z^2 w^3 + \beta_3 y z^2 w^2 + \beta_4 x^2 z w + \beta_5 z x y = 0, \tag{5.2.10}$$

is singular when $w = x = y = 0$. Since we are interested in what happens at the singularity, we can from now on use the rescaling condition of the ambient $\mathbb{P}^2$ $[x : y : z]$ to set $z = 1$,

$$-y^2 + x^3 + \beta_0 w^5 + \beta_2 x w^3 + \beta_3 y w^2 + \beta_4 x^2 w + \beta_5 x y = 0. \tag{5.2.11}$$

The β_is depend on the coordinates on the base of the elliptic fibration B_3. They define holomorphic sections of the bundle $\mathcal{O}([6-i]K_{B_3} - [5-m]S_{\text{GUT}})$, where K_{B_3} is the canonical bundle on the base B_3.

Even after we resolve completely the singularity located in $x = y = w = 0$, there will be singularities remaining in particular subloci on the GUT divisor, specified by particular values of the β_i's. Esole and Yau [55] worked out such resolutions, that will be reviewed in the following.

We perform the first blow up $\lambda_1 : [y, x, w]$, that is, introducing a coordinate λ_1 as

$$y = \lambda_1 \hat{y}\,, \qquad x = \lambda_1 \hat{x}\,, \qquad w = \lambda_1 \hat{w}\,. \tag{5.2.12}$$

together with the projective relations $[\hat{y} : \hat{x} : \hat{w}]$. The defining equation (5.2.11) becomes

$$\lambda_1^2(-\hat{y}^2 + \lambda_1\hat{x}^3 + \beta_0\lambda_1^3\hat{w}^5 + \beta_2\lambda_1^2\hat{x}\hat{w}^3 + \beta_3\lambda_1\hat{y}\hat{w}^2 + \beta_4\lambda_1\hat{x}^2\hat{w} + \beta_5\hat{x}\hat{y}) = 0. \tag{5.2.13}$$

The expression in brackets is the proper transform that defines $\hat{Y}_4$ (we reserve the tilde to the fully resolved space), and as in (5.1.25)

$$Y_4 \to \hat{Y}_4 + 2E_1, \tag{5.2.14}$$

where $E_1 = \{\hat{\lambda}_1\}$.

$\hat{Y}_4$ is still singular in $\hat{y} = \hat{x} = \hat{\lambda}_1$, and we blow up (we will reuse primed coordinates to denote all the intermediate blow ups, to avoid adding too many symbols and because we are mainly interested in the original space and the final resolved one),

$$-\hat{y}^2 + \lambda_2\hat{\lambda}_1\hat{x}^3 + \beta_0\lambda_2\hat{\lambda}_1^3\hat{w}^5 + \beta_2\lambda_2\hat{\lambda}_1^2\hat{x}\hat{w}^3 + \beta_3\hat{\lambda}_1\hat{y}\hat{w}^2 + \beta_4\lambda_2\hat{\lambda}_1\hat{x}^2\hat{w} + \beta_5\hat{x}\hat{y} = 0, \qquad (5.2.15)$$

or, rearranging,

$$-\hat{y}\left(\hat{y} + \beta_3\hat{\lambda}_1\hat{w}^2 + \beta_5\hat{x}\right) + \lambda_2\hat{\lambda}_1\left(\hat{x}^3 + \beta_0\hat{\lambda}_1^2\hat{w}^5 + \beta_2\hat{\lambda}_1\hat{x}\hat{w}^3 + \beta_4\hat{x}^2\hat{w}\right) = 0. \qquad (5.2.16)$$

The space is smooth in general points of the GUT divisor, but it acquires further singularities when we approach particular values for the β_is, corresponding to subregions on S_{GUT}. To resolve this additional singularities we follow the work of Esole and Yau [55]. First, notice that we can rewrite the defining equation as

$$\hat{y}s + \lambda_2\hat{\lambda}_1 t = 0, \qquad (5.2.17)$$

with

$$s = \hat{y} + \beta_3\hat{\lambda}_1\hat{w}^2 + \beta_5\hat{x}\,, \qquad t = \hat{x}^3 + \beta_0\hat{\lambda}_1^2\hat{w}^5 + \beta_2\hat{\lambda}_1\hat{x}\hat{w}^3 + \beta_4\hat{x}^2\hat{w}. \qquad (5.2.18)$$

The equation (5.2.17) is a binomial equation, and similarly to the E_8 resolution discussed in section 5.1.4, we resolve such a space through small resolutions. However, since s and t are not independent sections, but involve the other coordinates, (5.2.18) must always be observed.

The introduction of s and t can also be interpreted as replacing the elliptic fiber by a higher dimensional auxiliary binomial variety (5.2.17), and the elliptic fiber is defined as the complete intersection of the binomial variety with (5.2.18).

Notice that the ways of resolving the singularities are not unique. Namely, one can perform one of the six following pair of blow ups,

$$\begin{array}{c|c|c|c|c|c} [\hat{y},\hat{\lambda}_1] & [\hat{y},\hat{\lambda}_1] & [\hat{y},\lambda_2] & [\hat{y},\lambda_2] & [\hat{y},t] & [\hat{y},t] \\ [s,\lambda_2] & [s,t] & [s,\hat{\lambda}_1] & [s,t] & [s,\hat{\lambda}_1] & [s,\lambda_2] \end{array}\,.$$

In the following we will work in detail only the first resolution in the list above. The other resolutions are connected to the one we are going to perform via conifold-like transitions [55]. We blow up as $\delta_1 : [\hat{y}, \hat{\lambda}_1]$ and $\delta_2 : [s, \tilde{\lambda}_2]$, thus obtaining (again, repeating the hatted notation)

$$\begin{cases} \tilde{y}\tilde{s} + \tilde{\lambda}_2\tilde{\lambda}_1(\delta_2\lambda_2\tilde{x}^3 + \beta_0\delta_1^2\tilde{\lambda}_1^2\tilde{w}^5 + \beta_2\delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^3 + \beta_4\tilde{x}^2\tilde{w}) = 0 \\ \delta_2\tilde{s} - \delta_1\tilde{y} + \beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2 + \beta_5\tilde{x} = 0 \end{cases}, \qquad (5.2.19)$$

with the projective relations

$$[\delta_1\delta_2\lambda_2\tilde{y} : \delta_2\lambda_2\tilde{x} : \tilde{w}], \quad [\delta_1\tilde{y} : \tilde{x} : \delta_1\tilde{\lambda}_1], \quad [\tilde{y} : \tilde{\lambda}_1], \quad [\tilde{s} : \tilde{\lambda}_2]. \qquad (5.2.20)$$

As long as the β_i's are non-factorizable holomorphic sections, (5.2.19) is smooth. A factorizable β_i might lead to even stronger singularities, and we will explore one particular model in section 5.3.

To see the resolved structure of the elliptic fiber, we first notice that

$$w = 0 \to \delta_1\delta_2\lambda_1\lambda_2\tilde{w} = 0. \qquad (5.2.21)$$

We now look for the the intersections of the divisor $\{w = 0\}$ inside the fivefold with the elliptically fibered fourfold, defined by (5.2.19), together with the conditions (5.2.20). Similarly to what was done in section (5.1.4),we obtain the curves described in table 5.2.1 with $\mathbb{P}^1$ fibers intersecting as an (affine) SU(5).

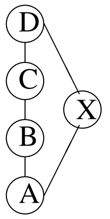
	Curve	Mult.	Diagram
$A:$	$\lambda_1 = s = \delta_1\tilde{y} - \beta_5\tilde{x} = 0$	1	
$B:$	$\delta_2 = \delta_1\tilde{y} + \beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2 + \beta_5\tilde{x} =$ $= \tilde{y}\tilde{s} + \tilde{\lambda}_2\tilde{\lambda}_1(\beta_0\delta_1^2\tilde{\lambda}_1^2\tilde{w}^5 + \beta_2\delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^3 + \beta_4\tilde{x}^2\tilde{w}) = 0$	1	
$C:$	$\lambda_2 = y = \delta_2\tilde{s} - \beta_3\delta_1\lambda_1\tilde{w}^2 + \beta_5\tilde{x} = 0$	1	
$D:$	$\delta_1 = \delta_2\tilde{s} + \beta_5\tilde{x} = \tilde{y}\tilde{s} + \tilde{\lambda}_2\tilde{\lambda}_1(\delta_2\lambda_2\tilde{x}^3 + \beta_4\tilde{x}^2\tilde{w}) = 0$	1	
$X:$	$\tilde{w} = \tilde{y}\tilde{s} + \tilde{\lambda}_2^2\tilde{\lambda}_1\tilde{x}^3 = \delta_2\tilde{s} - \delta_1\tilde{y} + \beta_5\tilde{x} = 0$	1	

Table 5.2.1: Curves on the resolved SU(5) divisor.

We can see explictly that each curve is a $\mathbb{P}^1$. Take for example the curve A. $\lambda_1 = \tilde{s} = 0$ allow us to use the projective relations $[\tilde{y} : \tilde{\lambda}_1]$ and $[\tilde{s} : \tilde{\lambda}_2]$ and fix $\tilde{y} = \lambda_2 = 1$. We still have the condition $\delta_1 = \beta_5\tilde{x}$ on the curve, that makes the condition $[\delta_1\tilde{y} : \tilde{x} : \delta_1\tilde{\lambda}_1]$ be trivially satisfied, and we can just fix the value of $\tilde{x}$. We are thus left with the first condition, that now becomes simply $[\delta_2 : \tilde{w}]$. The curve A is therefore defined by two patches $\mathbb{C}$ connected by $[\delta_2 : \tilde{w}]$, that is, a $\mathbb{P}^1$.

From the fact that the self-intersection of a $\mathbb{P}^1$ is -2, we can also identify each $\mathbb{P}^1$ with the (negative) roots of $SU(5)$ and write the Cartan matrix for the (extended) $SU(5)$,

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ X \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & X \end{array} \\ \left(\begin{array}{cccc:c} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ \hdashline 1 & 0 & 0 & 1 & -2 \end{array}\right) \end{array}$$

5.2.1 Matter Curves

In general, the model described up to now has an elliptic curve with a singularity of type A_4, that reconstructs the root system for an SU(5) group. But as was pointed out, there are special regions along the GUT divisor where the singularities enhances. From the expression for f, g and the discriminant Δ, (5.2.5), we can extract information on the location of the enhanced singularities. For convenience, we repeat (5.2.5) here,

$$\begin{aligned} \Delta =& w^5(\beta_5(\beta_3^2\beta_4 - \beta_2\beta_3\beta_5 + \beta_0\beta_5^2) + \mathcal{O}(w)), \\ f =& \frac{1}{48}(-\beta_5^4 - 8\beta_4\beta_5^2 w - 16\beta_4^2 w^2 + 24\beta_3\beta_5 w^2 + 48\beta_2 w^3), \\ g =& \frac{1}{864}(\beta_5^6 + 12\beta_4\beta_5^4 w + 48\beta_4^2\beta_5^2 w^2 - 36\beta_3\beta_5^3 w^2 + \\ & + 64\beta_4^3 w^3 - 144\beta_3\beta_4\beta_5 w^3 - 72\beta_2\beta_5^2 w^3 + 216\beta_3^2 w^4 - 288\beta_2\beta_4 w^4 + 864\beta_0 w^5), \end{aligned} \tag{5.2.22}$$

We see immediately that we get an enhancement when we reach the loci $\beta_5 = 0$ or $P = \beta_3^2\beta_4 - \beta_2\beta_3\beta_5 + \beta_0\beta_5^2 = 0$. We will explore each case in the following.

10 Matter

The first singularity enhancement we analyse is when we reach the region on the base where $\beta_5 = 0$. At this locus, the vanishing orders go as ord$(\Delta) = 6$, ord$(f) = 2$ and ord$(g) = 4$. From the Kodaira table 5.0.1 we see that naively we would expect an enhancement to an $SO(10)$ singularity.

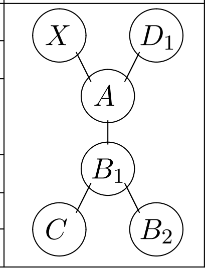
Curve		Mult.	Diagram
$A:$	$\lambda_1 = s = \delta_1 = 0$	2	
$B_1:$	$\delta_1 = \delta_2 = \tilde{y}\tilde{s} + \beta_4\tilde{\lambda}_2\tilde{\lambda}_1\tilde{x}^2\tilde{w} = 0$	2	
$B_2:$	$\delta_2 = \tilde{y} + \beta_3\tilde{\lambda}_1\tilde{w}^2 =$ $\beta_3\tilde{s} + \tilde{\lambda}_2(\beta_0\delta_1^2\tilde{\lambda}_1^2\tilde{w}^4 + \beta_2\delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^2 + \beta_4\tilde{x}^2) = 0$	1	
$C:$	$\tilde{\lambda}_2 = \tilde{y} = \delta_2\tilde{s} - \beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2 = 0$	1	
$D_1:$	$\delta_1 = \tilde{s} = \delta_2\lambda_2\tilde{x}^3 + \beta_4\tilde{x}^2\tilde{w} = 0$	1	
$X:$	$\tilde{w} = \tilde{y}\tilde{s} + \tilde{\lambda}_2^2\tilde{\lambda}_1\tilde{x}^3 = \delta_2\tilde{s} - \delta_1\tilde{y} = 0$	1	

Table 5.2.2: Curves over the β_5 matter curve

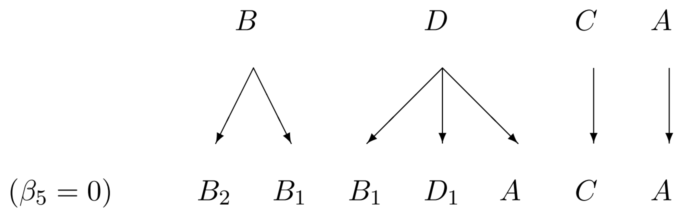
From the resolution, we can extract explicitly what happen to the curves at the locus $\beta_5 = 0$. Some curves remain the same, for example A that is now described by the simpler expression $A|_{\beta_5=0} : \tilde{\lambda}_1 = \tilde{s} = \delta_1 = 0$. Some curves however split into smaller components. Take for instance $\delta_1 = 0$, that previously corresponded to the curve D. It now has three possible solutions,

$$\begin{aligned} D_1 &: \delta_1 = \tilde{s} = \delta_2\lambda_2\tilde{x}^3 + \beta_4\tilde{x}^2\tilde{w} = 0 \\ B_1 &: \delta_1 = \delta_2 = \tilde{y}\tilde{s} + \beta_4\tilde{\lambda}_2\tilde{\lambda}_1\tilde{x}^2\tilde{w} = 0 \\ D_3(=A) &: \delta_1 = \tilde{s} = \tilde{\lambda}_1 = 0 \end{aligned} \tag{5.2.23}$$

When we reach a point on the base where $\beta_5 = 0$ the fibered $\mathbb{P}^1$ that we denoted by D splits into three curves, and one of them "merges" with the already existing curve A, changing its multiplicity. Another curve that suffers a splitting is B, into $B_1 = D_2$ and B_2. We sum up all the curves with their multiplicities and intersections at $\beta_5 = 0$ in table 5.2.2.

The curves intersect as the Dynkin diagram for the $SO(10)$, as expected. from the vanishing order of the coefficients.

It is also interesting to see how the **10** representation arises from the resolution. As was shown above, when we reach the β_5 locus some of the curves split as



By calculating the intersection of each new curve with the $\mathbb{P}^1$s that correspond to the roots of the $SU(5)$ we get the Cartan charge associated with each $\mathbb{P}^1$, that will correspond to the Cartan charge of a $M2$-brane wrapped on the corresponding $\mathbb{P}^1$. The intersection number of each splitted curve with the original roots will be either $+1$ or -1. Additionally, since there was no new curve appearing, but only splittings, the sum of charges of the "daughter curves" has to be equal to the "mother curve". For example

$$\begin{array}{ccccccccc} D & \to & A & + & D_1 & + & B_1 \\ (0,0,1,-2) & = & (-2,1,0,0) & + & (\pm1,0,0,\pm1) & + & (\pm1,\pm1,\pm1,\pm1)\,, \end{array}$$

that we can immediately solve and obtain

$$D_1(1,0,0,-1), \qquad B_1(1,-1,1,-1). \tag{5.2.24}$$

These charges correspond to weights of the $\mathbf{10}$ representation. We can analyse in the same way the splitting of the D curve, and we get the charges indicated below. The curves form a basis for the complete $\mathbf{10}$ representation.

Curve	Charge	Weight
$A_{\beta_5=0}$	$(-2,1,0,0)$	$-\alpha_1$
$B_{1\,\beta_5=0}$	$(1,-1,1,-1)$	$-(\mu_{10}-\alpha_1-\alpha_2-\alpha_3)$
$B_{2\,\beta_5=0}$	$(0,-1,0,1)$	$\mu_{10}-\alpha_1-2\alpha_2-\alpha_3$
$C_{\beta_5=0}$	$(0,1,-2,1)$	$-\alpha_3$
$D_{1\,\beta_5=0}$	$(1,0,0,-1)$	$\mu_{10}-\alpha_2-\alpha_3-\alpha_4$

$\bar{\mathbf{5}}$ Matter

When $P=\beta_3^2\beta_4-\beta_2\beta_3\beta_5+\beta_0\beta_5^2$ vanishes, the vanishing order Δ increases to $\mathrm{ord}(\Delta)=6$. This implies that, on the particular region in the base where the β_i's satisfy the particular constraint given by $P=0$, the $SU(5)$ singularity enhances to an $SU(6)$ (cf. table 5.0.1).

We can see explictly what happens to the curves on top of $P=0$. Notice that we can rewrite $\beta_3^2\beta_4-\beta_2\beta_3\beta_5+\beta_0\beta_5^2=0$ as the solution to the set of equations

$$\begin{cases}\beta_0\zeta^2+\beta_2\zeta+\beta_4=0\\ \beta_3\zeta+\beta_5=0,\end{cases} \tag{5.2.25}$$

with $\beta_3\neq 0$. Notice also, that in the resolved fourfold (5.2.19),

$$\begin{cases}\tilde{y}\tilde{s}+\tilde{\lambda}_2\tilde{\lambda}_1\tilde{x}^2\left[\delta_2\lambda_2\tilde{x}+\tilde{w}\left(\beta_0(\frac{\delta_1\tilde{\lambda}_1\tilde{w}^2}{\tilde{x}})^2+\beta_2(\frac{\delta_1\tilde{\lambda}_1\tilde{w}^2}{\tilde{x}})+\beta_4\right)\right]=0\\ \delta_2\tilde{s}-\delta_1\tilde{y}+\tilde{x}\left[\beta_3(\frac{\delta_1\tilde{\lambda}_1\tilde{w}^2}{\tilde{x}})+\beta_5\right]=0\end{cases}. \tag{5.2.26}$$

If we restrict $\delta_2=\tilde{y}=0$, we can immediately identify $\zeta=\delta_1\tilde{\lambda}_1\tilde{w}^2/\tilde{x}$. Keeping this in mind, we recall the expression for the curve B,

$$B:\quad \delta_2=\delta_1\tilde{y}+\left(\beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2+\beta_5\tilde{x}\right)=\tilde{y}\tilde{s}+\tilde{\lambda}_2\tilde{\lambda}_1\tilde{w}\left(\beta_0\delta_1^2\tilde{\lambda}_1^2\tilde{w}^4+\beta_2\delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^2+\beta_4\tilde{x}^2\right)=0.$$

We can solve $\beta_3(\delta_1\tilde{\lambda}_1\tilde{w}^2)=-(\delta_1\tilde{y}+\beta_5\tilde{x})$ in the second member of the equation, multiply the last one by β_3^2 and replace the solved $\beta_3(\delta_1\tilde{\lambda}_1\tilde{w}^2)$. We use the fact that $P=0$, and a $\tilde{y}$ factorizes, thus splitting the curve B into two curves

$$\begin{aligned}B_1:&\quad \delta_2=\beta_3^2 s+\tilde{\lambda}_2\tilde{\lambda}_1\tilde{w}\delta_1\left[\beta_0\delta_1\tilde{y}+(-2\beta_0\beta_5+\beta_2\beta_3)\tilde{x}\right]=\delta_1\tilde{y}+\beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2+\beta_5\tilde{x}=(P=)0,\\ B_2:&\quad \delta_2=\tilde{y}=\beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2+\beta_5\tilde{x}=(P=)0.\end{aligned} \tag{5.2.27}$$

The curves now structure themselves as the diagram of an $SU(6)$, again in agreement with the Kodaira classification, table 5.0.1. We can again calculate the intersections of B_1 and B_2 with the $SU(5)$ roots, table 5.2.3, and see that the $\mathbb{P}^1$s reproduce the $\mathbf{5}$ representation.

It is important to point out that the curves obtained explictly via the resolution agree with the Kodaira classification for the vanishing orders of f, g and Δ, even though the Kodaira classification was obtained for codimension one singularities.

5.2.2 Yukawa couplings (codimension 3)

Having successfully reproduced the matter curves, and seeing that they agree with the expected from a naive comparison with the Kodaira classification, we now proceed to show the behavior of the curves at the Yukawa couplings, or triple intersections on the base.

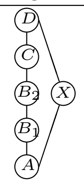
Curve	Charge	Weight	Diagram
$A_{P=0}$	$(-2,1,0,0)$	$-\alpha_1$	
$B_{1\,P=0}$	$(1,-1,0,0)$	$-(\mu_5-\alpha_1)$	
$B_{2\,P=0}$	$(0,-1,1,0)$	$\mu_5-\alpha_1-\alpha_2$	
$C_{P=0}$	$(0,1,-2,1)$	$-\alpha_3$	
$D_{P=0}$	$(0,0,1,-2)$	$-\alpha_4$	

Table 5.2.3: Curves and their Cartan charges over the $P=0$ matter curve

$\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ coupling

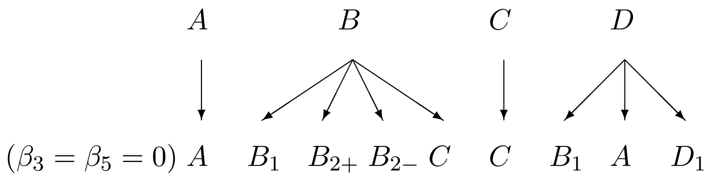
We first look at the point defined by the intersection $\tilde{w}=\beta_3=\beta_5=0$. Before writing the curves, we point out that when we restrict the curve B, one of its factorizations is

$$B_2 : \delta_2 = \tilde{y} = \beta_0\delta_1^2\tilde{\lambda}_1^2\tilde{w}^4 + \beta_2\delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^2 + \beta_4\tilde{x}^2 = 0. \tag{5.2.28}$$

The last equality is a second order polynomial in $\zeta = \delta_1\tilde{\lambda}_1\tilde{x}\tilde{w}^2/\tilde{x}$. We can solve it and we obtain two solutions,

$$B_{2\pm} \quad : \quad \delta_2 = \tilde{y} = \delta_1\tilde{\lambda}_1\tilde{w}^2 - \frac{\tilde{x}\beta_2 \mp \tilde{x}\sqrt{\beta_2^2-4\beta_4\beta_0}}{2\beta_0} = 0. \tag{5.2.29}$$

The curves and their structure are given in table 5.2.4 and, following the splitting



we can again compute the Cartan charges, as was done along the matter curves. The charges are shown in table 5.2.

As a more interesting phenomenological consequence, we can analyse the Yukawa couplings allowed at this locus. For that, we want to see how the matter representations couple at the enhancement point. So, instead of moving directly from $\tilde{w}=0$ to $\tilde{w}=\beta_3=\beta_5=0$, we follow the curves as they move to the Yukawa point.

We first look at the curves from the $\mathbf{10}$ representation, shown in table 5.2.2. Notice that

$$B_{2\,(\beta_5=0)} \xrightarrow{\beta_3=0} B_{2+\,(\beta_5=\beta_3=0)} + B_{2-\,(\beta_5=\beta_3=0)}. \tag{5.2.30}$$

But $B_{2\,(\beta_5=0)}$ corresponds to an element in the $\mathbf{10}$ representation, while $B_{2\pm\,(\beta_5=\beta_3=0)}$ are in $\mathbf{5}$. We can therefore write the "invariant" term at the $\beta_3=\beta_5=0$ point

$$(\mu_{10}-\alpha_1-2\alpha_2-\alpha_3)-2(\mu_5-\alpha_1-\alpha_2),$$

or simply $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$, the known coupling from the perturbative brane description.

It is more tricky to see the $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ coupling arising from the $\mathbf{5}$ matter curve. Notice that when we computed the daughter curves of B in $P=0$ (5.2.27), we had to use $\beta_3\neq 0$, which is not allowed now. But we can simply reconstruct the weight of $B_{1\,P=0}$ and $B_{2\,P=0}$ from the daughter curves of B in table 5.2. We thus get for the $\bar{\mathbf{5}}$ from $B_{1\,P=0}$, the decomposition $-(\mu_5-\alpha_1)\rightarrow(\mu_5-\alpha_1-\alpha_2)-(\alpha_3)-(\mu_{10}-\alpha_1-\alpha_2-\alpha_3)$, or simply $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$.

This might have looked redundant, since this result was known just from group theory arguments. Indeed this is true, but it is nevertheless instructive to explicitly see the couplings arising from a geometrical perspective.

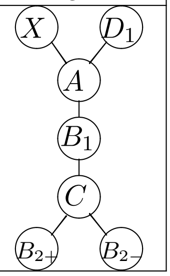
Curve			Mult.	Diagram
$A_{(\beta_3=\beta_5=0)}$	:	$\lambda_1 = s = \delta_1 = 0$	2	
$B_{1\,(\beta_5=\beta_3=0)}$	:	$\delta_2 = \delta_1 = \tilde{y}\tilde{s} + \beta_4\tilde{\lambda}_2\tilde{\lambda}_1\tilde{x}^2\tilde{w} = 0$	2	
$B_{2+\,(\beta_5=\beta_3=0)}$	:	$\delta_2 = \tilde{y} = 2\beta_0\delta_1\tilde{\lambda}_1\tilde{w}^2 - \tilde{x}\beta_2 + \tilde{x}\sqrt{\beta_2^2 - 4\beta_4\beta_0} = 0$	1	
$B_{2-\,(\beta_5=\beta_3=0)}$	:	$\delta_2 = \tilde{y} = 2\beta_0\delta_1\tilde{\lambda}_1\tilde{w}^2 - \tilde{x}\beta_2 - \tilde{x}\sqrt{\beta_2^2 - 4\beta_4\beta_0} = 0$	1	
$C_{(\beta_3=\beta_5=0)}$	:	$\lambda_2 = y = \delta_2 = 0$	2	
$D_{1\,(\beta_5=\beta_3=0)}$	:	$\delta_1 = \tilde{s} = \delta_2\lambda_2\tilde{x} + \beta_4\tilde{w} = 0$	1	
$X_{(\beta_3=\beta_5=0)}$	:	$\tilde{w} = \tilde{y}\tilde{s} + \tilde{\lambda}_2^2\lambda_1\tilde{x}^3 = \delta_2\tilde{s} - \delta_1\tilde{y} = 0$	1	

Table 5.2.4: Curves over the $\tilde{w} = \beta_5 = \beta_3 = 0$ point

Curve	Charge	Weight
$A_{(\beta_5=\beta_3=0)}$	$(-2, 1, 0, 0)$	$-\alpha_1$
$B_{1\,(\beta_5=\beta_3=0)}$	$(1, -1, 1, -1)$	$-(\mu_{10} - \alpha_1 - \alpha_2 - \alpha_3)$
$B_{2+\,(\beta_5=\beta_3=0)}$	$(0, -1, 1, 0)$	$\mu_5 - \alpha_1 - \alpha_2$
$B_{2-\,(\beta_5=\beta_3=0)}$	$(0, -1, 1, 0)$	$\mu_5 - \alpha_1 - \alpha_2$
$C_{(\beta_5=\beta_3=0)}$	$(0, 1, -2, 1)$	$-\alpha_3$
$D_{1\,(\beta_5=\beta_3=0)}$	$(1, 0, 0, -1)$	$\mu_{10} - \alpha_2 - \alpha_3 - \alpha_4$

Figure 5.2: Curves and Cartan charges over the $\beta_3 = \beta_5 = 0$ point.

A $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ coupling

As was mentioned earlier, one of the main phenomenological interests on F-theory is the possibility to construct exceptional groups that are not present in perturbative Type II theory. In particular, the $SU(5)$ elliptic fiber model was claimed to contain an E_6 enhancement locus, at $\beta_4 = \beta_5 = 0$. We will now see the explicit resolved structure at this point.

Since most of the discusion is similar to what was done for the $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ point, we will simply present the results. The curves split not as an affine E_6, but something similar (table 5.2.5), although not present in the Dynkin classification. The cartan charges are presented in table 5.2.6.

It is also straightforward to work the splitting of each "weight" $\mathbb{P}^1$ for the $\mathbf{5}$ and the $\mathbf{10}$ representation, and one sees that the only splittings allowed are the ones that form an intersection of the form $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$. That is, the explicit reproduction of an E_6 Dynkin root system is not needed, as long as the intersections of curves at the point reproduce the desired couplings.

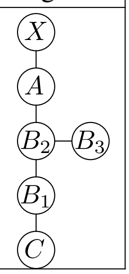
Curve			Mult.	Diagram
$A_{(\beta_3=\beta_5=0)}$	:	$\lambda_1 = s = \delta_1 = 0$	2	
$B_{1\,(\beta_5=\beta_4=0)}$	:	$\delta_2 = \delta_1 = \tilde{y} = 0$	2	
$B_{2\,(\beta_5=\beta_4=0)}$	:	$\delta_2 = \delta_1 = \tilde{s} = 0$	3	
$B_{3\,(\beta_5=\beta_4=0)}$	:	$\delta_2 = \tilde{y} - \beta_3\tilde{\lambda}_1\tilde{w}^2 =$ $= \beta_3\tilde{s} - \lambda_2\lambda_1\tilde{w}\delta_1(\beta_0\delta_1\lambda_1 w^2 + \beta_2 x) = 0$	1	
$C_{(\beta_3=\beta_5=0)}$	:	$\lambda_2 = y = \delta_2\tilde{s} - \beta_3\delta_1\tilde{\lambda}_1\tilde{w}^2 = 0$	1	
$X_{(\beta_3=\beta_5=0)}$	:	$\tilde{w} = \tilde{y}\tilde{s} + \tilde{\lambda}_2^2\lambda_1\tilde{x}^3 = \delta_2\tilde{s} - \delta_1\tilde{y} = 0$	1	

Table 5.2.5: Curves over the "E_6-like" point

Curves			
GUT divisor	"E_6" point	Charges	Repr.
A →	$A_{\beta_5=\beta_4=0}$	$(-2,1,0,0)$	$-\alpha_1$
B	$B_{1\,\beta_5=\beta_4=0}$	$(0,-1,1,0)$	$\mu_5-\alpha_1-\alpha_2$
	$B_{2\,\beta_5=\beta_4=0}$	$(1.0,0,-1)$	$\mu_{10}-\alpha_2-\alpha_3-\alpha_4$
	$B_{3\,\beta_5=\beta_4=0}$	$(0,-1,0,1)$	$\mu_{10}-\alpha_1-2\alpha_2-\alpha_3$
C →	$C_{\beta_5=\beta_4=0}$	$(0,1,-2,1)$	$-\alpha_3$
D	$B_{1\,\beta_5=\beta_4=0}$	$(0,-1,1,0)$	$\mu_5-\alpha_1-\alpha_2$
	$2\times B_{2\,\beta_5=\beta_4=0}$	$2\times(1.0,0,-1)$	$\mu_{10}-\alpha_2-\alpha_3-\alpha_4$
	$A_{\beta_5=\beta_4=0}$	$(-2,1,0,0)$	$-\alpha_1$

Table 5.2.6: Curve splittings and their charges and associated representation at the "E_6-like" point.

5.3 An $SU(5)$ model with an E_8 Yukawa Point

In particular, at the point on the base defined by $w=\beta_4=\beta_5=0$, when these coordinates are treated as non-factorizable holomorphic variables, the explicit resolution does not give the diagram for a E_6 group that would be expected from the counting of the vanishing order as in table 5.0.1. Further analysis [57] showed that although the resolution does not reproduce the exact diagram that would be naively expected from the Tate's algorithm, it still reproduces the **10 10 5** coupling, necessary to give mass to the top quark in an $SU(5)$ GUTs, the main reason for why one considers E_6 enhancements in the first place.

However, Esole-Yau resolution [55] does not contemplate further singularities, that can arise in particular regions on the moduli space of the base where the parameters β_i factorize. Such possibility of factorization was studied for example in [141, 150].

We also propose a similar factorization to reproduce a codimension 3 (a point on the base) enhancement to an E_8 singularity. Our construction of such splitting follows from the established connections between F-theory and heterotic string theory, where the coefficients β_i are related to the higgsings used to break E_8 down to $SU(5)$, as

$$E_8 \longrightarrow SU(5)\times SU(5)_\perp. \tag{5.3.1}$$

We will briefly review how are the coefficients in the elliptic fiber related to the Higgs vevs in the next section, following [151, 152].

5.3.1 The β_i Coefficients from the Spectral Cover

The rough idea of the spectral cover construction is to incorporate in a geometrical description the higgsing of a gauge group. Our starting point is the E_8 group, that we break down to an SU(5),

$$E_8 \to SU(5)\times SU(5)_\perp \to SU(5)\times U(1)^4. \tag{5.3.2}$$

There is a Higgs field responsible for the breaking, which can be locally described as a section of the canonical bundle over the SU(5) divisor S with values on the adjoint of E_8,

$$K_S \otimes Adj(E_8). \tag{5.3.3}$$

In standard geometrical engineering, the gauge groups are identified with singularities as the standard ADE classification, obtained as a blow-down of the resolved geometry. They are then broken to smaller subgroups by giving non-vanishing volume to some $\mathbb{P}^1$'s. This corresponds to giving a vev to the Cartans of $SU(5)_\perp$. Thus, being a Cartan root, the Higgs field we are interested in obeys $[\Phi,\Phi^\dagger]=0$. These solutions are also relevant since they usually leave $\mathcal{N}=1$ supersymmetry unbroken.

We next want to describe the Higgs field in terms of its eigenvalues and eigenvectors, that is, its spectral data. We introduce a section s of the canonical bundle over S, K_S, and we can write the eigenvalue equation

$$\det(sI - \Phi) = 0. \tag{5.3.4}$$

Since we restrict to the Higgs field that leaves the SU(5) but breaks $SU(5)_\perp$, we can expand (5.3.4) in the 5 eigenvalues t_i for the fundamental representation of $SU(5)_\perp$ as

$$\prod_i (s - t_i) = 0. \tag{5.3.5}$$

Expanding, we find

$$\beta_0 s^5 + \beta_2 s^3 + \beta_3 s^2 + \beta_4 s + \beta_5 = 0. \tag{5.3.6}$$

The β_1 is not present since $\beta_1 = t_1 + ... + t_5 = 0$, from the tracelessness condition of the roots in $SU(5)$. Although it is not yet clear, the β_is in (5.3.6) are the same as the elliptic fiber equation in the Tate form, when in the vicinity of the GUT divisor. To see that, we first define the "Tate divisor" [57], as the equation

$$\mathcal{C}_{\text{Tate}} \quad : \quad \beta_0 w^5 + \beta_2 x w^3 + \beta_3 y w^2 + \beta_4 x^2 w + \beta_5 x y = 0. \tag{5.3.7}$$

The equation for the elliptic fiber (5.2.11),

$$-y^2 + x^3 + \big(\beta_0 w^5 + \beta_2 x w^3 + \beta_3 y w^2 + \beta_4 x^2 w + \beta_5 x y\big) = 0, \tag{5.3.8}$$

when restricted to the Tate divisor implies $\frac{y^2}{x^3} = 1$. Also we can define the holomorphic section $u = y/x$ on the Tate divisor. This allows us to write the Tate divisor as

$$\mathcal{C}_{\text{Tate}} \quad : \quad \beta_0 w^5 + \beta_2 w^3 u^2 + \beta_3 w^2 u^3 + \beta_4 w u^4 + \beta_5 u^5 = 0. \tag{5.3.9}$$

We then restrict to the vicinity of $w \to 0$ and close to the singularity on the elliptic fiber $y \to 0$, $x \to 0$. This in turn implies $u \to 0$. To arise at the spectral curve (5.3.6), we consider the section $s = w/u$, and we arrive at

$$\beta_0 s^5 + \beta_2 s^3 + \beta_3 s^2 + \beta_4 s + \beta_5 = 0, \tag{5.3.10}$$

that is precisely our spectral cover construction (5.3.6). The coefficients β_i are given in terms of the eigenvalues t_i as

$$\begin{aligned} \beta_1 &= -\beta_0 \sum_i t_i = 0, \qquad \beta_2 = \beta_0 \sum_{i \neq j} t_i t_j, \\ \beta_3 &= -\beta_0 \sum_{i\neq j\neq k} t_i t_j t_k, \qquad \beta_4 = \beta_0 \sum_{i\neq j\neq k\neq l} t_i t_j t_k t_l, \qquad \beta_5 = -\beta_0 t_1 t_2 t_3 t_4 t_5. \end{aligned}$$

Recall also that the first enhancements that we encounter in the $SU(5)$ model happens when Δ' in $\Delta = w^5 \Delta'$ has a first order zero. This corresponds to the vanishing of

$$\begin{aligned} \beta_5 &= -\beta_0 t_1 t_2 t_3 t_4 t_5\,, \quad \text{or} \\ P_5 &= -\beta_0^3 \prod_{i\neq j} (-t_i - t_j) = \beta_3^2 \beta_4 - \beta_2 \beta_3 \beta_5 + \beta_0 \beta_5^2. \end{aligned} \tag{5.3.11}$$

The expansion of the f, g and Δ in terms of the variables on the singularity w and $\{t_i\}$ is given by

$$\Delta = -432 \beta_0^2 w^{10} + \dots \quad f = \beta_0 \sum_{i\neq j} t_i t_j w^3 + \dots, \quad g = \beta_0 w^5 + \dots, \tag{5.3.12}$$

where the dots indicate higher order terms in w. As we want an enhancement to an E_8, it follows from the Kodaira classification 5.0.1 that we should have

$$\text{ord}(f) \geq 4\,, \qquad \text{ord}(g) = 5\,, \qquad \text{ord}(\Delta) = 10\,. \tag{5.3.13}$$

In order to achieve $\text{ord}(f) \geq 4$ we require every term $t_i t_j$ to vanish, that in turn imposes that at least 4 of the t_is are zero. But then the tracelessness condition implies that all should be zero. We could have advanced this, since the t_is are related to higgsings of the underlying E_8 group, so setting all t_is to zero would un-break the E_8.

Since we want to describe this enhancement as a codimension 3 locus, we introduce sections p and q, that will be normal sections to curves on the SU(5) divisor. We could interpret this as the normal sections of divisors on the base B_3, but as we will argue later, having started from the Tate model does not allow this interpretation. Looking at β_2 relation with β_0 this means that $\sum_{i\neq j} t_i t_j$ must be a section of the $K_{B_3}^{-2} \otimes \mathcal{L}_{SU(5)}^2$.

We impose now that the t_i's can be written as

$$t_i = t_i^p p + t_i^q q. \tag{5.3.14}$$

The t_i^p and t_i^q could also be sections of some bundle on the base, but for simplicity of the model we consider them to be constant integer numbers. This implies that p and q are sections of $K_{B_3}^{-1} \otimes \mathcal{L}_{SU(5)}$, and could be homologically equivalent to each other. The tracelessness condition $\beta_1 = 0$ implies

$$p(t_1^p + t_2^p + t_3^p + t_4^p + t_5^p) + q(t_1^q + t_2^q + t_3^q + t_4^q + t_5^q) = 0.$$

We also want no trivial solution to (5.3.11), thus $t_i \neq 0$ and $t_i + t_j \neq 0$. One choice that satisfies all the above requirements is

$$t_1 = p,\;\; t_2 = q,\;\; t_3 = p+q,\;\; t_4 = -2q,\;\; t_5 = -2p.$$

This particular selection is symmetric under the exchange $p \leftrightarrow q$. The matter curves represented by $p = w = 0$ and $q = w = 0$ could be exchanged. Together with the fact that they are in the same homology class, they represent the same curve, as in figure 5.3. The role of exchange symmetries for the curves in phenomenological F-Theory models was explored in [51].

The β_is after the replacements (5.3.15) become

$$\begin{aligned} \beta_2 &= -\beta_0(3p^2 + pq + 3q^2), & \beta_3 &= \beta_0(p+q)(2p^2 - 3pq + 2q^2), \\ \beta_4 &= 2\beta_0 pq(p^2 + 4pq + q^2), & \beta_5 &= -4\beta_0 p^2 q^2 (p+q). \end{aligned}$$

It is also convenient, for reference, to write the polynomials P and R,

$$\begin{aligned} P =& 2\beta_0^3 p(p-2q)(p-q)^2(2p-q)q(p+q)^2(2p+q)(p+2q), \\ R =& -\beta_0^3(p-q)^2(p+q)(8p^6 - 4p^5q - 38p^4q^2 - 43p^3q^3 - 38p^2q^4 - 4pq^5 + 8q^6). \end{aligned} \tag{5.3.15}$$

Replacing the values for the β_i's, equation (5.3.15), and setting $\beta_0 = 1$,

$$\begin{aligned} y^2 - 4p^2q^2(p+q)xyw + \left[2(p^3+q^3) - pq(p+q)\right] yw^2 + x^3 + w^5 + & \\ + \left[-3(p^2+q^2) - pq\right] xw^3 + 2pq(p^2+q^2+4pq)x^2 w \;\; &= \;\; 0 \end{aligned} \tag{5.3.16}$$

We next calculate explictly f, g and Δ with respect to w, p and q. There will be codimension two and three loci on the base where the vanishing order of the functions will increase. If we extrapolate the result of the Kodaira classification (table 5.0.1) to codimension higher than one, we can extract the information on the "expected" gauge group over each enhancement locus. For $p \neq 0$ and $q \neq 0$, as expected for the SU(5) singularity (for the full expression, see Appendix A.4),

$$\begin{aligned} \Delta &= w^5 \left(-512p^9(p-2q)(p-q)^2(2p-q)q^9(p+q)^6(2p+q)(p+2q)\right) + \mathcal{O}(w^6), \\ f &= -\frac{16}{3}p^8q^8(p+q)^4 + \mathcal{O}(w), \\ g &= \frac{128}{27}p^{12}q^{12}(p+q)^6 + \mathcal{O}(w). \end{aligned} \tag{5.3.17}$$

So $\mathrm{ord}(\Delta) = 5$ and $\mathrm{ord}(f) = \mathrm{ord}(g) = 0$. On the codimension 2 locus $p = 0$ together with $w = 0$, we get

$$\begin{aligned} \Delta &= -432q^{12}w^8 + \ldots, \\ f &= -3q^2w^3 + \ldots, \\ g &= q^6w^4 + \ldots, \end{aligned} \tag{5.3.18}$$

where the ... are terms of higher vanishing order. Therefore $\mathrm{ord}(\Delta) = 8$, $\mathrm{ord}(f) = 3$ and $\mathrm{ord}(g) = 4$, the vanishing degrees for a E_6 singularity. There are also other codimension 2 enhancements appearing in $p \pm q = 0$, $p \pm 2q = 0$ and $2p \pm q = 0$, that we sumarise in table 5.3.1. We will call the $w = p = 0$ locus the E_6 *matter curve*, even if the explicit resolution lead to something different from an E_6. Similar notation will apply to the other codimension two loci.

At the point $p = q = w = 0$ in which we expect by construction to get a E_8 singularity,

$$\begin{aligned} \Delta &= -432w^{10} + \ldots, \\ f &= -3p^2w^3 - pqw^3 - 3q^2w^3 + \ldots, \\ g &= w^5 + \ldots, \end{aligned} \tag{5.3.19}$$

Thus $\mathrm{ord}(\Delta) = 10$, $\mathrm{ord}(f) = 5$ and $\mathrm{ord}(g) = 5$, the expected for a E_8 singularity. Similarly, we call this codimension three locus the E_8 Yukawa point.

Curve (in $w = 0$)	Codim	ord($\Delta/f/g$)	Sing. type
$p = 0$	2	8/3/4	E_6
$p + q = 0$		8/2/3	$SO(12)$
$p - q = 0$		7/0/0	$SU(7)$
$p \pm 2q = 0$		6/0/0	$SU(6)$
$p = q = 0$	3	10/5/5	E_8

Table 5.3.1: Codimension 2 and 3 enhancements for the particular model (5.3.15). Exchanging $p \leftrightarrow q$ gives the same results

5.3.2 Resolution

The expression for the elliptic curve becomes, after we replace the values of the β_i's for our chosen factorization (5.3.15),

$$\begin{aligned} & y(-y - 4p^2q^2(p+q)x + \left[2(p^3+q^3) - pq(p+q)\right]\lambda_1 w^2) + \\ & + \lambda_2\lambda_1(\lambda_2 x^3 + \lambda_1^2 w^5 + \left[-3(p^2+q^2) - pq\right]\lambda_1 w^2 x + 2pq(p^2+q^2+4pq)x^2 w) = 0\,. \end{aligned} \tag{5.3.20}$$

We again have a binomial variety of the form encountered before,

$$ys + \lambda_1\lambda_2(\ldots) = 0, \tag{5.3.21}$$

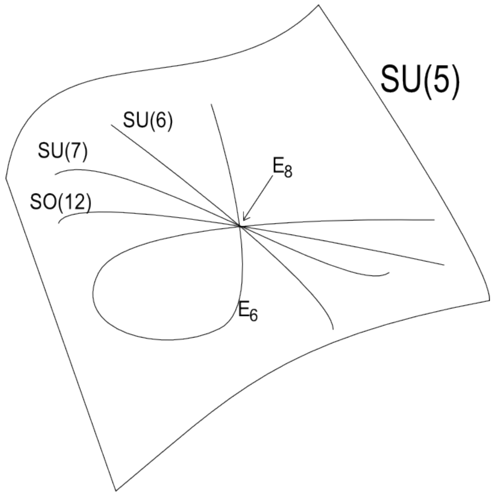


Figure 5.3: A diagram of the model considered. The curves $p = 0$ and $q = 0$ are in the same homology on the base, and correspond to the same matter curve, and they meet at a point of E_8 enhancement.

with $s = -y - 4p^2q^2(p+q)x + \left[2(p^3+q^3) - pq(p+q)\right]\lambda_1 w^2$. We perform the small resolutions $\delta_1 : [y, \lambda_1]$ and $\delta_2 : [s, \lambda_2]$,

$$\begin{cases} 0 &= ys + \lambda_2\lambda_1(\delta_2\lambda_2 x^3 + \delta_1^2\lambda_1^2 w^5 + \left[-3(p^2+q^2) - pq\right]\delta_1\lambda_1 w^2 x + 2pq(p^2+q^2+4pq)x^2 w) \\ 0 &= \delta_2 s + \delta_1 y + 4p^2q^2(p+q)x - \left[2(p^3+q^3) - pq(p+q)\right]\delta_1\lambda_1 w^2\,, \end{cases} \tag{5.3.22}$$

with the projective relations

$$[\delta_2\delta_1\lambda_2 y : \delta_2\lambda_2 x : w], \quad [\delta_1 y : x : \delta_1\lambda_1], \quad [y : \lambda_1], \quad [s : \lambda_2]. \tag{5.3.23}$$

At this point, in Esole-Yau resolution [55] we would have the fully resolved space. The enhancements would not worsen the singularities, but only split the already existing curves. Here, however, there are further singularities to be resolved. The reason for this is that in our model the β_i's split in a product of the sections p and q, thus enhancing the vanishing order of the previously smooth terms.

We now proceed to resolve the additional singularities. First, we rearrange the equation as

$$\begin{cases} 0 &= ys + \lambda_2\lambda_1(\delta_2\lambda_2 x^3 + \delta_1^2\lambda_1^2 w^5 + \left[-3(p^2+q^2) - pq\right]\delta_1\lambda_1 w^2 x + 2pq(p^2+q^2+4pq)x^2 w) \\ 0 &= \delta_2 s + \delta_1\left(y - \left[2(p^3+q^3) - pq(p+q)\right]\lambda_1 w^2\right) + 4p^2q^2(p+q)x\,. \end{cases}$$

We note that all the singularities of the second equation arise when $\delta_2 = \delta_1 = s = (...) = p = 0$. or $\delta_2 = \delta_1 = s = (...) = q = 0$. Similarly as was done to resolve the $SU(5)$, we introduce an auxiliary equation $t = y - \left[2(p^3+q^3) - pq(p+q)\right]\lambda_1 w^2$, and we now work with the system of three equations,

$$\begin{cases} 0 = & ys + \lambda_2\lambda_1(\delta_2\lambda_2 x^3 + \delta_1^2\lambda_1^2 w^5 + \left[-3(p^2+q^2) - pq\right]\delta_1\lambda_1 w^2 x + 2pq(p^2+q^2+4pq)x^2 w) \\ 0 = & \delta_2 s + \delta_1 t + 4p^2q^2(p+q)x \\ 0 = & -t + y - \left[2(p^3+q^3) - pq(p+q)\right]\lambda_1 w^2\,. \end{cases} \tag{5.3.24}$$

It is now straightforward to resolve this system of equations. First we resolve the singularity at $q = s = t = \delta_1 = \delta_2 = 0$ in the second equation, by performing the blow up $\chi_1 : [s, t, q]$, that is

$$s \to \chi_1 s,\ t \to \chi_1 t,\ q \to \chi_1 q, \quad \text{with the projective relation} \quad [s : t : q]. \tag{5.3.25}$$

Notice that this was a $\mathbb{P}^2$ blow up, not a small resolution as was done to resolve the curves and Yukawa enhancements in the previous section. As a consequence, we are in fact introducing a new (three-dimensional) divisor on the fourfold, but localized along codimension 2 on the base (the matter curve). That is, the new fiber will have to have dimension higher than one. Up to now, the effect of the resolutions was only to modify the one-dimensional fiber of the fourfold, replacing the singular points by one dimensional curves. In the resolution we will perform now, we then also modify the base of the fibration, introducing new submanifolds along the enhancement loci.

The difference here to the previous case where we only needed small resolutions lies on the fact that, in the brane picture, the collisions that lead to $SU(6)$ and $SO(10)$ matter curves come from collision of the $SU(5)$-brane with an $U(1)$ seven-brane or an O7-plane. Both correspond to a non-singular degeneration of the fiber, and therefore a resolution is not needed. In this case, however, the collision inducing an E_6 enhancement can be understood as

$$E_6 \to SU(5) \times SU(3), \tag{5.3.26}$$

and thus the colliding brane would carry with it a singularity from the F-theory perspective. Our local construction however does not allow us to see the colliding brane outside the $SU(5)$ locus $\{w = 0\}$.

One should also keep in mind that there is a large number of possibilities for the blow-ups, that might lead to different final resolved manifolds. Here we perform one of many choices, that leads to a resolved space in few steps. There is however no possibility to resolve this singular space only via small resolutions. To fully resolve the space, we then choose to perform other three blow ups, in the following order,

$$\pi_1 : [s, t, p],\ \pi_2 : [s, t, \pi_1],\ \chi_2 : [s, t, \chi_1], \tag{5.3.27}$$

thus introducing four new divisors to the ambient fivefold given by $\{\pi_1 = 0\}$, $\{\pi_2 = 0\}$, $\{\chi_1 = 0\}$ and $\{\chi_2 = 0\}$. The defining equations for the elliptic fiber consists now on the triple intersection

$$\begin{cases} 0 = & \pi_1\pi_2^2\chi_1\chi_2^2 sy + \lambda_1\lambda_2(-2p\pi_1\pi_2 q\chi_1\chi_2(p^2\pi_1^2\pi_2^2 + 4p\pi_1\pi_2 q\chi_1\chi_2 + q^2\chi_1^2\chi_2^2) - \delta_1^2\lambda_1^2 w^5 + \\ & +\delta_1\lambda_1(3p^2\pi_1^2\pi_2^2 + p\pi_1\pi_2 q\chi_1\chi_2 + 3q^2\chi_1^2\chi_2^2)w^2 x - \delta_2\lambda_2 x^3) \\ 0 = & 4p^2\pi_1 q^2\chi_1(p\pi_1\pi_2 + q\chi_1\chi_2) + \delta_2 s - \delta_1 t \\ 0 = & -\pi_1\pi_2^2\chi_1\chi_2^2 t - \lambda_1(p\pi_1\pi_2 + q\chi_1\chi_2)(2p^2\pi_1^2\pi_2^2 - 3p\pi_1\pi_2 q\chi_1\chi_2 + 2q^2\chi_1^2\chi_2^2)w^2 + y\,, \end{cases} \tag{5.3.28}$$

together with the list of projective relations

$$\begin{gathered} [\delta_1\delta_2\lambda_2 y : \delta_2\lambda_2 x : w] \neq [0:0:0], \quad [\delta_1 y : x : \delta_1\lambda_1] \neq [0:0:0], \quad [y : \lambda_1] \neq [0:0], \\ [\pi_1\pi_2^2\chi_1\chi_2^2 s : \lambda_2] \neq [0:0], \quad [\pi_1\pi_2^2\chi_2 s : \pi_1\pi_2^2\chi_2 t : q] \neq [0:0:0], \\ [\pi_2\chi_2 s : \pi_2\chi_2 t : p] \neq [0:0:0], \quad [\chi_2 s : \chi_2 t : \pi_1] \neq [0:0:0], \quad [s : t : \chi_1] \neq [0:0:0]. \end{gathered} \tag{5.3.29}$$

Since the last equation in (5.3.28) has a simple dependence on y, we can use it to eliminate y in the other equations, and return to a system of two equations,

$$\begin{cases} 0 = & \pi_1\pi_2^2\chi_1\chi_2^2 s\left[\pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1(p\pi_1\pi_2 + q\chi_1\chi_2)(2p^2\pi_1^2\pi_2^2 - 3p\pi_1\pi_2 q\chi_1\chi_2 + 2q^2\chi_1^2\chi_2^2)w^2\right] + \\ & \lambda_1\lambda_2\left[-2p\pi_1\pi_2 q\chi_1\chi_2(p^2\pi_1^2\pi_2^2 + 4p\pi_1\pi_2 q\chi_1\chi_2 + q^2\chi_1^2\chi_2^2) - \delta_1^2\lambda_1^2 w^5 + \right. \\ & \left.\delta_1\lambda_1(3p^2\pi_1^2\pi_2^2 + p\pi_1\pi_2 q\chi_1\chi_2 + 3q^2\chi_1^2\chi_2^2)w^2 x - \delta_2\lambda_2 x^3\right] \\ 0 = & 4p^2\pi_1 q^2\chi_1(p\pi_1\pi_2 + q\chi_1\chi_2) + \delta_2 s - \delta_1 t, \end{cases} \tag{5.3.30}$$

while the projective relations should be inverted to eliminate the dependence on y. The sections corresponding to the original GUT divisor and the two matter enhancements now become

$$\delta_2\delta_1\lambda_2\lambda_1 w = 0, \qquad \pi_2\pi_1 p = 0, \qquad \chi_2\chi_1 q = 0. \tag{5.3.31}$$

5.3.3 Codimension 1 - The GUT Divisor

At codimension one, when we restrict ourselves to $\pi_2\pi_1 p \neq 0$, $\chi_2\chi_1 q \neq 0$, we can simply blow down the four $\mathbb{P}^2$s since they sit on $p_0 = 0$ or $q_0 = 0$. Blowing down, we simply return to our space after the two small resolutions, (5.3.24). Even without blowing down, we have the same number of curves as before, with the same multiplicities and same intersecting properties. This is rather obvious, since the additional structure coming from the extra blow-ups appear only at particular values of p and q.

5.3.4 Codimension 2 - The "E_6" Matter Curve

Now we look at the interesting locus of codimension 2, where in the original blown-down space corresponded to $w = p = 0$. As mentioned above, this locus naively corresponds to an E_6 enhancement, as in table 5.3.1. In the blown-up space, for each curve in codimension one, table 5.2.1, we can take as restrictions to the E_6 matter enhancement either $p = 0$, $\pi_1 = 0$ or $\pi_2 = 0$ obeying the projective relations (5.3.29). At this locus some solutions are not simple $\mathbb{P}^1$s inside the resolved elliptic curve, as before. Take as an example the curve B. Along the $p_0 = \pi_1\pi_2 p = 0$ locus, β_4 and β_5 vanish, so the defining equation of the curve simplifies to

$$\delta_2 = \pi_1\pi_2^2\chi_1\chi_2^2 s(\pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1\beta_3 w^2) + \lambda_1\lambda_2(-\delta_1^2\lambda_1^2 w^5 + \delta_1\lambda_1\beta_2 w^2 x) = \delta_1 t = 0. \tag{5.3.32}$$

However, as mentioned, the restriction to the E_6 curve can be taken to be $p = 0$, $\pi_1 = 0$ or $\pi_2 = 0$. When $\pi_1 = 0$ the curve B reduces to

$$(\pi_1 = 0) \quad \delta_2 = \delta_1\lambda_1^2\lambda_2(-\delta_1\lambda_1 w^5 + \beta_2 w^2 x) = \delta_1 t = 0. \tag{5.3.33}$$

That has as one possible solution

$$E_* : \quad \delta_2 = \pi_1 = \delta_1 = 0. \tag{5.3.34}$$

The rescaling conditions that have to be obeyed for E_* are

$$[0:0:w],\ [0:x:0],\ [\lambda_1 q\chi_1\chi_2 w:\lambda_1],\ [0:\lambda_2],\ [0:0:q],\ [\pi_2\chi_2 s:\pi_2\chi_2 t:p],\ [\chi_2 s:\chi_2 t:0],\ [s:t:\chi_1].$$

Using the fact that $\chi_1\chi_2 q \neq 0$ and the rescaling conditions above, we can fix $\chi_1 = q = \lambda_2 = x = w = \lambda_1 = 1$, and we are still left with the unfixed coordinates π_2, s, t and p, together with the conditions

$$[\pi_2 s:\pi_2 t:p],\ [s:t]. \tag{5.3.35}$$

So, the solution E_* is actually a $\mathbb{P}^2$ blown up at the point $s = t = 0$ by a $\mathbb{P}^1$.

There are however minimal solutions that correspond to a $\mathbb{P}^1$ hypersurface inside this $\mathbb{P}^2$. As one concrete example, take the intersection of B with $p = 0$,

$$(p =)\delta_2 = \pi_1\pi_2^2\chi_1\chi_2^2 s(\pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1(q\chi_1\chi_2)^3 w^2) + \delta_1\lambda_1^2\lambda_2 w^2(-\delta_1\lambda_1 w^3 + 3(q\chi_1\chi_2)^2 x) = \delta_1 t = 0, \tag{5.3.36}$$

that has as one possible minimal solution

$$E_{*p} : \quad (p =)\delta_2 = \delta_1 = \pi_1 = 0. \tag{5.3.37}$$

We can see from the defining equations of E_* (5.3.34) and (5.3.35) that E_{*p} is a $\mathbb{P}^1$ hypersurface inside E_*.

Summarizing all the possible splittings, the codimension one curves decompose as they reach the matter curve $p_0 = 0$ as shown in figure 5.4.

The only curve that is not a $\mathbb{P}^1$ is E_* described above. We denoted $E_{*\hat{k}}$ minimal solutions that were hypersurfaces inside E_*, with $\hat{k} = s, t, p, \pi_2$.

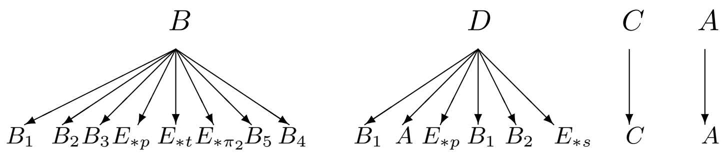


Figure 5.4: The splittings as we move to the E_6 curve.

The equations for the curves are given in table 5.3.4. The multiplicity in $E_{*(k)}$: is counted as the largest multiplicity among all the representatives $\mathbb{P}^1$ curves of the homology. The diagram has ten independent nodes, and do not correspond to any group in the Dynkin classification. One should keep in mind that the $\mathbb{P}^1$s are not all localized on the fiber of the resolved Calabi-Yau fourfold. The matter curve was also blown up, and what we are considering the "new" fiber is a non-trivial mixing of the old fiber with the new one-dimensional structure introduced on the base along the matter curve.

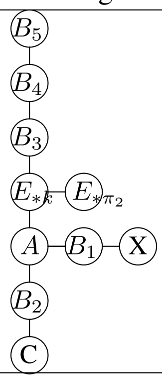
Curve			Mult.	Diagram
A	:	$\lambda_1 = p = s = \delta_1 = 0$	2	
$E_{*(k)}$	:	$\delta_2 = \pi_1 = \delta_1 = (k =)0, \quad k = s, t, p$	2*	
$E_{*\pi_2}$	:	$\delta_2 = \pi_1 = \delta_1 = \pi_2 = 0$	2	
B_1	:	$\delta_2 = p = s = \delta_1 = 0$	3	
B_2	:	$\delta_2 = p = \pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1\beta_3 w^2 = \delta_1 = 0$	2	
B_3	:	$\delta_2 = p = 2\pi_1\pi_2^2\chi_2^5 s + \lambda_2\delta_1\lambda_1(-\delta_1\lambda_1 w^3 + 3\chi_2^2 x) = t = 0$	1	
B_4	:	$\delta_2 = \pi_1 = -\delta_1\lambda_1 w^3 + (q\chi_1\chi_2)^2 x = t = 0$	1	
B_5	:	$\delta_2 = \pi_2 = -\delta_1\lambda_1 w^3 + (q\chi_1\chi_2)^2 x = pp^2\pi_1 q^3\chi_1^2\chi_2 - \delta_1 t = 0$	1	
C	:	$\lambda_2 = p = \pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1\beta_3 w^2 = \delta_2 s - \delta_1 t = 0$	1	
X	:	$w = p = \pi_1^2\pi_2^4\chi_1^2\chi_2^4 st + \lambda_1\lambda_2^2\delta_2 x^3 = \delta_2 s - \delta_1 t = 0$	1	

Table 5.3.2: Curves in codimension 2 $w_0 = p_0 = 0$. * the multiplicity is counted from the $\mathbb{P}^1$ with the highest multiplicity. This diagram should NOT be interpreted as the "Dynkin-like" diagram on the fiber.

Notice that while $k = s, t, p$ correspond to the $\mathbb{P}^1$ hypersurfaces of $\mathbb{P}^2$ and therefore are in the same homology class, $E_{*\pi_2}$ is not homologically equivalent. It intersects with the $\mathbb{P}^2$ at the point $s = t = 0$, and therefore with E_{*s} and E_{*t}, but not with E_{*p}. The intersections among the curves E_{*k} and A are represented in figure 5.5.

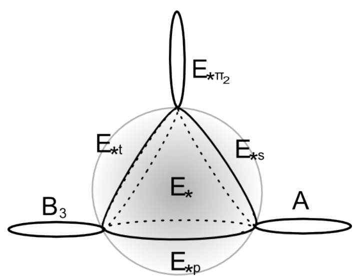


Figure 5.5: The intersections of A, E_{*s}, E_{*p}, $E_{*\pi_2}$, E_{*t} and B_3. For obvious dimensional difficulties, we represented the 2n-cycles $\mathbb{P}^n$ by n-cycles S^n.

In the M-theory perspective, the four-cycle $\mathbb{P}^2$ can be wrapped by M5-branes that correspond to a string in the remaining dimensions with tension given by the volume of the four-cycle [153]. When shrunken back to zero volume, the wrapped M5-branes become tensionless strings. In the effective theory, this corresponds to a tensor multiplet becoming massless, leading to a breaking of the low-energy effective theory and thus a phase transition. In the Type IIB picture, the blow-up introduce a one dimensional $\mathbb{P}^1$ on the fiber and also a $\mathbb{P}^1$ on the base along the matter curve (one non-trivially fibered over the other). This blown-up $\mathbb{P}^1$ can be wrapped by a D3-brane, that again in the blown-down limit give rise to a massless string. Additionally, we might have to worry about string worldsheet instantons wrapping the vanishing $\mathbb{P}^1$s. The $\mathbb{P}^1$ along the curve might also break the Calabi-Yau condition. A similar blow-up along a curve in Type IIB picture was studied in [115]. A more detailed exploration of the role of tensionless strings on the theory (or the phase transition) arising in this particular setup would be interesting, however we do not deal with it in this thesis.

Alternative interpretation - Matter Curve

Here we mention a somewhat *ad hoc* argument to obtain a structure of $\mathbb{P}^1$s in what we will call the "F-theory fiber", the fiber composed simply from the proper transform of the elliptic fiber X and a particular subset of the $\mathbb{P}^1$s. As we mentioned, the blow-ups that took $p \to \pi_2\pi_1 p$ introduced two-dimensional spaces located along the matter curve that could be interpreted as a mixed resolution of the fiber via an one-dimensional space and a resolution of the matter curve on the base. We assume that the $\mathbb{P}^1$s forming the "F-theory fiber" are the ones obtained only by intersection with $p = 0$, then the curves $B_{\text{out}} = \{E_{*\pi_2}, E_{*s}, E_{*t}, B_4, B_5\}$ of table 5.3.4 are not anymore solutions. One sees that the remaining curves intersect precisely as an affine E_6 Dynkin diagram, with the correct multiplicities.

Curve			Mult.	Diagram
A	:	$\lambda_1 = p = s = \delta_1 = 0$	2	
E_{*p}	:	$\delta_2 = p = \delta_1 = \pi_1 = 0$	2	
B_1	:	$\delta_2 = p = s = \delta_1 = 0$	3	
B_2	:	$\delta_2 = p = \pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1\beta_3 w^2 = \delta_1 = 0$	2	
B_3	:	$\delta_2 = p = 2\pi_1\pi_2^2\chi_2^5 s + \lambda_2\delta_1\lambda_1(-\delta_1\lambda_1 w^3 + 3\chi_2^2 x) = t = 0$	1	
C	:	$\lambda_2 = p = \pi_1\pi_2^2\chi_1\chi_2^2 t + \lambda_1\beta_3 w^2 = \delta_2 s - \delta_1 t = 0$	1	
X	:	$w = p = \pi_1^2\pi_2^4\chi_1^2\chi_2^4 st + \lambda_1\lambda_2^2\delta_2 x^3 = \delta_2 s - \delta_1 t = 0$	1	

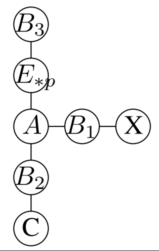


Table 5.3.3: Curves in codimension 2 $w_0 = p = 0$. The Diagram is precisely the Dynkin diagram of an affine E_6 group.

The removed curves, B_{out} could be related to the broken $SU(3)$ described in (5.3.26). We however do not address this possibility here, but leave it to a future work.

5.3.5 Codimension 3 - The Yukawa Point

We next restrict the elliptic fiber to the Yukawa point, that before the blow ups corresponded to the locus $w = p = q = 0$. Similarly, the restriction to the codimension three locus has as solutions some curves that are not $\mathbb{P}^1$s, but again, some of the internal $\mathbb{P}^1$ hypersurfaces appear as solutions. The curves at the Yukawa point with their multiplicities are described in table 5.3.5. It is also interesting to see how the $\mathbb{P}^2$s intersect to form the respective intersections for the $\mathbb{P}^1$ hypersurfaces. The resolution diagram at the Yukawa point is presented in figure 5.6.

The interpretation of this resolution at the Yukawa point is even more complicated. The blowups again introduce divisors on the fourfold, but now located on a point on the base. This implies a two-dimensional structure appearing on the base. However, no blow-up was performed directly at the Yukawa point, but only on the matter curves. And as we argued, the blow-ups introduced locally a one-dimensional $\mathbb{P}^1$ on the base. Thus, the collision of the two E_6 matter curves should then lead to the Hirzebruch surface $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, or even a more general Hirzebruch surfaces F_n.

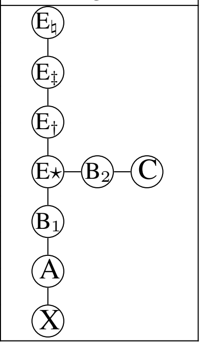
Curve			Mult.	Diagram
A	:	$\lambda_1 = p = q = s = \delta_1 = 0$	2	
$E\star_{(\hat{k})}$	:	$\delta_2 = p = \chi_1 = \delta_2 = (k=)0, \quad \hat{k} = s, t, q$	4	
$E_{\dagger(k)}$	:	$\delta_2 = \pi_1 = \chi_1 = \delta_2 = (k=)0, \quad k = s, t$	2	
$E_{\ddagger(k)}$	:	$\delta_2 = \pi_2 = \chi_1 = \delta_2 = (k=)0, \quad k = s, t$	2	
$E_{\natural(k)}$	:	$\delta_2 = \pi_2 = \chi_2 = \delta_2 = (k=)0, \quad k = s, t$	2	
B_1	:	$\delta_1 = p = q = \delta_2 = s = 0$	3	
B_2	:	$\delta_1 = p = q = \delta_2 = t = 0$	4	
C	:	$\delta_2 = p = q = \lambda_2 = t = 0$	2	
X	:	$w = p = q = \pi_1^2 \pi_2^4 \chi_1^2 \chi_2^4 st + \lambda_1 \lambda_2^2 \delta_2 x^3 = \delta_2 s - \delta_1 t = 0$	1	

Table 5.3.4: Curves in the expected E_8 Yukawa point.

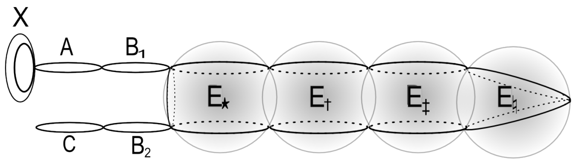


Figure 5.6: The intersection diagram with the higher dimensional surfaces.

Again, we can restrict ourselves to the curves obtained when we collide the curves given only by the restriction $p = 0$ and $q = 0$. The curves located at this intersection are $A, C, B_1, B_2, E\star_q$ and X, that arrange as shown in the diagram of table 5.3.5. We see that some of the $\mathbb{P}^2$s are completely removed from the set, so the hypothesis that on the E_8 Yukawa point we would have an F_n surface on the base seems to not be valid.

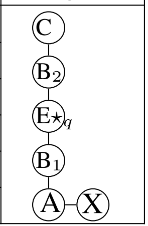
Curve			Mult.	Diagram
C	:	$\delta_2 = p = q = \lambda_2 = t = 0$	2	
B_2	:	$\delta_1 = p = q = \delta_2 = t = 0$	4	
$E\star_q$	:	$\delta_2 = p = q = \delta_2 = \chi_1 = 0$	4	
B_1	:	$\delta_1 = p = q = \delta_2 = s = 0$	3	
A	:	$\lambda_1 = p = q = s = \delta_1 = 0$	2	
X	:	$w = p = q = \pi_1^2 \pi_2^4 \chi_1^2 \chi_2^4 st + \lambda_1 \lambda_2^2 \delta_2 x^3 = \delta_2 s - \delta_1 t = 0$	1	

Table 5.3.5: Curves restricted to $p = q = 0$ over the Yukawa point.

We conclude this section and chapter with a summary on the results obtained so far. F-theory model building rely on the assumption that the matter curves and Yukawa couplings are reproduced at loci on the base where the gauge group is enhanced, importing what has been known from perturbative type IIB theory with intersecting branes and the spectral cover formalism of heterotic strings. However, the non-perturbative character of branes in F-theory might lead this naive perturbative picture inconsistent. The gauge groups in F-theory are constructed from ADE singularities appearing on the elliptic fiber at particular submanifolds on the base. Having an explicit algebraic description of the singularity, one can identify a gauge group associated to it. This gauge group is constructed in the M-dual picture by wrapping M2 branes on curves in the resolved fiber that arrange themselves as the roots of a gauge group. The identification of the resolved structure of curves and the corresponding gauge group gives an ADE classification for the singularities. The ADE classification however is only valid in codimension one on the base of the fibration. It might work for particular cases in higher codimension but in general can fail (as was shown in [55] for the matter curves, and reviewed here in section 5.2.1).

Even if the resolution at subloci does not lead to a series of curves intersecting in a way that reproduces the expected enhanced gauge group, once we know the explicit fiber resolution it is nevertheless possible to analyse the physical quantities we are interested in. Namely, we can identify to what representation a matter curve corresponds to, and identify how the representations couple at the Yukawa points. This was done for the SU(5) case going to an expected E_6 Yukawa point [57]. Not treated here in this thesis is the work [154] in which the matter representation and Yukawa coupling from a model starting with a divisor carrying an E_6 singularity is analyzed, and it reproduces the matter representation $\mathbf{27}$ and the coupling $\mathbf{272727}$.

Here we tried to construct an explicit model which would have a codimension 3 locus with an E_8 singularity. We imported results from F-theory/Heterotic duality, specially to the identification via the spectral cover of the coefficients in the elliptic fiber to with the vevs for the Higgs field responsible for breaking the E_8 gauge group in $E_8 \times E_8$ heterotic models. To fully resolve the space, we had to perform blow-ups that introduced two(-complex)-dimensional subspaces along the codimension two enhancement loci. We proposed that these two-dimensional spaces correspond partly to a resolution of the matter curve itself (and therefore a modification of the base) and partly to a resolution of the singular one-domplex-dimensional elliptic fiber. We have shown and ad hoc selection of curves that reproduce a E_6 diagram in codimension two, as naively expected from the Kodaira classification, but the same selection lead to a resolution in codimension three with a very small number of curves, while the non-selected curves on the Yukawa point at least agreed with a E_8 diagram in the number of nodes. As was pointed out however, we should not expect to obtain an agreement with the Dynkin diagrams, but we do have to check if the resolution leads to desired matter representations and couplings. We have not performed this computation yet, but this will be done in a next work.

Chapter 6

Conclusions and Outlook

In this thesis we presented two constructions worth of exploration in the framework of effective String Theory and its strongly coupled relatives M- and F-theory. In the first part we calculated the Kaluza-Klein reduction of the action for a spacetime filling D6-brane on a Calabi-Yau orientifold, and derived the $\mathcal{N} = 1$ characteristic data for the theory living on the brane reduced to four dimensions. We first discussed the geometrical space associated to the scalar fields for brane deformations ζ^I, when the background Calabi-Yau internal space is fixed. We were able to construct a local description for the Kähler potential of this space written in the elegant form

$$K_o(\zeta + \bar{\zeta}) = -\frac{1}{2}\int_{\mathcal{C}_4} J \wedge \hat{\beta}^I \int_{\mathcal{C}_4} \mathrm{Im}(C\Omega) \wedge \hat{\alpha}_I, \tag{6.0.1}$$

where $\{\beta^I, \alpha_I\}$ form an infinite basis of two- and one-forms on L_0. A subset of the deformation fields, namely $b^1(L_0)$ of them, generate deformations that preserve the supersymmetry conditions and are described by massless modes, and the base $\{\beta^I, \alpha_I\}$ reduces to the symplectic basis discussed in section 3.4.2. The remaining deformation fields induce a positive scalar potential. We saw how the scalar potential could be generated from a generalized version of the superpotential in (3.5.14),

$$W = \int_{\mathcal{C}_4} (J_c - \mathcal{F}_{\mathrm{D6}}) \wedge (J_c - \mathcal{F}_{\mathrm{D6}}) \tag{6.0.2}$$

where we included of B-fields. The scalar potential would also receive contributions from the D-terms, giving at the end

$$V = \frac{1}{\mathcal{V}^2} e^{3\phi} \int_{L_0} d\theta_\eta \wedge *d\theta_\eta + (B - \tilde{F} + d\theta^B_\eta) \wedge *(B - \tilde{F} + d\theta^B_\eta) + d * \theta \wedge *d * \theta. \tag{6.0.3}$$

In section 3.4 we showed how to combine complex structure deformations and brane deformations, and specified a Kähler potential for this open-closed moduli space. The deformations ζ corrected the complex structure deformation moduli $\{N'^k, T'_\lambda\}$, as

$$N^k = N'^k - 2\,\partial_{V_k}(e^{2D} K_o)\,, \qquad T_\lambda = T'_\lambda - 2\,\partial_{V^\lambda}(e^{2D} K_o). \tag{6.0.4}$$

The gauge coupling function for the $U(1)$ field living on a static brane,

$$f_{\mathrm{r}} = \int_{L_0} (2\,\mathrm{Re}(C\Omega) + iC_3) = \delta_k N'^k - \delta^\lambda T'_\lambda \tag{6.0.5}$$

is corrected to

$$f = \delta_k N^k - \delta^\lambda T_\lambda \tag{6.0.6}$$

when we allow brane deformations. Here δ_k and $delta^\lambda$ are the integral over L_0 of $\{\alpha_k, \beta^\lambda\}$, the base elements of $H^3(Y)$. We also calculated kinetic mixings between the massless $U(1)$s coming open and closed strings, A and A^α respectively, and the mixing coupling was found to be given in the simple holomorphic form

$$f_\alpha = -\zeta^i \Delta_{i\beta}, \tag{6.0.7}$$

where $\Delta_{i\beta}$ is a geometrical factor integrated over the cycle of the background brane. This coupling induced a correction to the gauge coupling function of the $U(1)$ living on the brane $f_{\rm r}$ as

$$f_{\rm cor} = f_{\rm r} - f_\alpha \tilde{\mathcal{J}}^\alpha . \tag{6.0.8}$$

The corrections are not holomorphic, since the factor $\mathcal{J}^\alpha$ defined in (3.3.60) is integrated over the chain $\mathcal{C}_4$, and therefore can be expanded in the real deformations. Some extra term to make it holomorphic should arise in higher order expansions of the DBI action. Alternatively, one could modify the chain in the definition of $\mathcal{J}^\alpha$ to include also a deformation of the gauge bundle as discussed at the end of section 3.4.2.

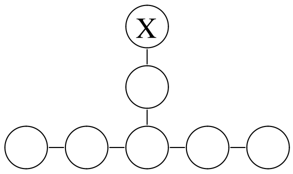
Finally, using Mirror Symmetry described by the SYZ conjecture [69], we showed how could we locally relate the obtained effective theory to the known results in the literature for the effective theory of D3-, D5- and D7-branes in Type IIB orientifolds, for special limits of the compactification manifold, we discussed the mappings of the fields in type IIA to type IIB and we proposed a mirror version for the Kähler potential for the open moduli space of branes in Type IIB theory (3.6.25),

$$K_o^{\rm def} = \tfrac{i}{4} \int_{\mathcal{C}_{p+1}} s_I \lrcorner {\rm Im}\Phi^{\rm ev} \int_{\mathcal{C}_{p+1}} \bar{s}^I \cdot \Omega + c.c. \,. \tag{6.0.9}$$

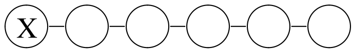
As a possible further direction to this work, one could study the lift of the dynamical brane described here to M-theory/eleven-dimensional supergravity compactified on a G_2 manifold (the compactification leading to an $\mathcal{N} = 1$ four dimensional theory). The results should be related to what was obtained here after we identify some one-cycle which could be used as the M-theory cycle, that reduces to Type IIA as we shrink the cycle. A similar discussion for the six dimensional effective action of an elliptically fibered Calabi-Yau in F-theory (the lift of a D7-brane) was performed in [155]. Additionally, in our calculations we often ignored more complicated contributions coming from brane flux, and we did not include bulk fluxes. Their introduction could induce corrections to many results in our setup. Also, a more phenomenologically interesting setup could be considered by intersecting two D6-branes, generating a matter representation at the intersection. Such configurations were explored to the level of conformal theories in particular compactifications, or had just their spectrum analyzed. A detailed effective action description, so far as we know, is still lacking.

In the second part of the thesis we moved beyond perturbative Type II theory, and introduced M- and F-theory. In particular, we studied the GUT model of an $SU(5)$ brane that enhances at curves and points. We saw explictly how the Kodaira classification cannot hold at higher codimensional locus, but saw that nevertheless the $\mathbf{10}\;\mathbf{10}\;\mathbf{5}$ and the $\bar{\mathbf{5}}\;\bar{\mathbf{5}}\;\mathbf{10}$ couplings are still generated in genetic models. Then, motivated by the interest in explaining flavor hierarchy from F-theory GUT models, we proceeded to construct an explicit model which had a codimension three enhancement to an E_8 singularity. We made an *ad hoc* choice of the coefficients β_i in the Tate model, which accounts for fixing a very particular point in the moduli space. Such choice introduced new sections p and q such that $p = q = w = 0$ the discriminant Δ of the elliptic curve would vanish as the predicted for an E_8 singularity, where $w = 0$ was the $SU(5)$ GUT divisor.

We saw that besides the obvious curves $w = p = 0$ and $w = q = p$ that reproduced the vanishing order of Δ for an E_6 singularity, our model had additional curves corresponding to $SU(6)$, $SU(7)$ and $SO(12)$ singularities. The singularities over these additional curves are resolved by the same small resolutions of [55]. However, the E_6 curve introduces stronger singularities. The resolution of the singularity changes the dimension of the fiber, making it two dimensional along the matter curve. We then proposed that this higher dimensional fiber could be splitted in an "F-theory fiber", corresponding to $\mathbb{P}^1$s, and a deformation of the base via some complementary $\mathbb{P}^1$. By this proposal, the "F-theory fiber" reproduces exactly the expected intersections at the E_6 matter curve, given by the affine E_6 Dynkin diagram of table 5.3.4 shown below, where X is the curve that becomes the singular elliptic fiber in the blow down limit.



A similar problem also appears at the Yukawa point, but there with a three-complex-dimensional fiber over it. We again proposed a decomposition between "F-theory fiber" and the blow-up of the base, where now the point on the base would be replaced by a divisor on the base. The "F-theory" fiber is described in table 5.3.5, and the diagram repeated below.



that does not correspond to any Dynkin diagram.

As an urgent next step, we should understand better what are the implications for these higher dimensional fibers. As was pointed out from other authors [153, 156], such resolutions do happen in F-theory at exceptional singularities and might correspond in the effective theory to tensionless strings modes connected to the gauge theory. Additionally, one would have to check if such resolutions do not break the Calabi-Yau condition. Since we are explictly deforming the base, this changes the Kähler cone and could induce a breaking of Kählerity. Only then it is sensible to analyse the phenomenological properties of such constructions. The rich intersection pattern, if realizable, could generate interesting physics.

Appendix A

A.1 Derivation of the Kähler metric

Let us now discuss the derivation of the Kähler metric and compare the result with the effective action for the D6-brane found by dimensional reduction, (3.3.72) and (3.3.74). Firstly, we note that the metrics for ReM^K and the pure ξ^K terms match the result found from the reduction of the closed string action, since $\tilde{K}^{KL} = (G_{kl}, G^{\lambda\kappa}, G^{\lambda}_{k})$, as described in [82]. We need then to check the terms involving open string moduli ζ^i. From the reduction of the action the metrics $\mathcal{G}_{ij}$ and $\widehat{\mathcal{G}}_{ij}$ are

$$\widehat{\mathcal{G}}_{ij} = \mu_{ki}\,\lambda^k_j \;, \qquad \mathcal{G}_{ij} = \mu_{ik}\,(\lambda^{-1})^k_j \;, \tag{A.1.1}$$

where, recalling equations (3.4.7) and (3.3.5),

$$e^{-\phi}\theta_i = \lambda^j_i\tilde{\alpha}_j \;\;, \qquad \theta_i = s_i\lrcorner J|_{L_0} \;,$$
$$\tfrac{1}{2}e^{-\phi} * \theta_i = \mu_{ji}\,\tilde{\beta}^j \;\;, \qquad *\theta_i = -2e^{\phi}s_i\lrcorner \mathrm{Im}(C\Omega)|_{L_0} \;.$$

The coefficients μ_{ij} and λ^j_i are calculated to be

$$e^{2D}\mu_{ij} = \frac{1}{2}\int_L \tilde{\alpha}_i \wedge s_j\lrcorner(V^\kappa\alpha_\kappa + V_k\beta^k) \;, \qquad \lambda^j_i = \int_L \tilde{\beta}^j \wedge s_i\lrcorner J \;, \tag{A.1.2}$$

also making use of the relations $\int \tilde{\alpha}_i \wedge \tilde{\beta}^j = \delta^j_i$. To leading order, the V derivatives of μ_{ij} are

$$\frac{\partial}{\partial V^\lambda}(e^{2D}\mu_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_i \wedge s_j\lrcorner\alpha_\lambda \;, \qquad \frac{\partial}{\partial V_k}(e^{2D}\mu_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_i \wedge s_j\lrcorner\beta^k \;, \tag{A.1.3}$$

On the other hand, λ^j_i is independent of (V^λ, V_k), at least for leading order complex structure deformations. This implies using (3.4.35), (3.4.22) and (A.1.1) that

$$\tilde{K}_{\zeta^i\bar{\zeta}^j} = \frac{\partial(e^{2D}\mathcal{G}_{ij})}{\partial V_K}V^K = e^{2D}\mathcal{G}_{ij} \;, \tag{A.1.4}$$

which is in accord with the result (3.3.72) found from dimensional reduction. The derivatives of the metric with respect to (V^λ, V_k) are given explicitly by (for first order deformations)

$$\partial_{V^\lambda}(e^{2D}\mathcal{G}_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_i\wedge s_l\lrcorner\alpha_\lambda\Big(\int_L \tilde{\beta}^j\wedge s_l\lrcorner J\Big)^{-1} \;, \quad \partial_{V_k}(e^{2D}\mathcal{G}_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_i\wedge s_l\lrcorner\beta^k\Big(\int_L \tilde{\beta}^j\wedge s_l\lrcorner J\Big)^{-1} \,. \tag{A.1.5}$$

The derivatives of the metric $\widehat{\mathcal{G}}_{ij}$ are, in turn,

$$\partial_{V^\lambda}(e^{2D}\widehat{\mathcal{G}}_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_l \wedge s_i\lrcorner\alpha_\lambda \int_L \tilde{\beta}^l \wedge s_j\lrcorner J \;, \qquad \partial_{V_k}(e^{2D}\widehat{\mathcal{G}}_{ij}) = \frac{1}{2}\int_L \tilde{\alpha}_l \wedge s_i\lrcorner\beta^k \int_L \tilde{\beta}^l \wedge s_j\lrcorner J \;. \tag{A.1.6}$$

To also check the mixing terms of the Wilson lines a^i with the scalars ζ^K we expand

$$\frac{\partial K_o}{\partial\zeta^i} = \tfrac{1}{2}\mu_{ij}|_{\text{fix}}\,\eta^j + \dots \;, \tag{A.1.7}$$

to lowest order in the η^i. This yields the lowest order expression for $\tilde{K}^K_{\zeta^i}$ evaluated to be

$$\tilde{K}^k_{\zeta^i} = \hat{\mathcal{I}}^k_i \ , \qquad \tilde{K}^\lambda_{\zeta^i} = \hat{\mathcal{I}}_{i\lambda} \ , \tag{A.1.8}$$

where were used equations (3.3.58) and (3.3.59)

$$\hat{\mathcal{I}}^k_i = \int_L \tilde{\alpha}_i \wedge \eta \lrcorner \beta^k + \dots \ , \qquad \hat{\mathcal{I}}_{i\lambda} = \int_L \tilde{\alpha}_i \wedge \eta \lrcorner \alpha_\lambda + \dots \ . \tag{A.1.9}$$

A.2 Supergravity with several linear multiplets

In this appendix we want to show, in a step by step way, how does the dualization from linear to chiral multiplets work, following [82]. We want to relate the effective action in terms of linear multiplets (V_K, C^2_K), obtained by generalizing a result in [157],

$$\begin{aligned} \mathcal{L} \ &= \ -\tilde{K}_{\zeta^i\bar{\zeta}^j}\, d\zeta^i \wedge *d\bar{\zeta}^j + \tfrac{1}{4}\tilde{K}_{V_K V_L}\, dV_K \wedge *dV_L \\ &\quad +\tilde{K}_{V_K V_L}\, dC^2_K \wedge *dC^2_L - i\, dC^2_K \wedge \big(\tilde{K}_{V_K\zeta^i}\, d\zeta^i - \tilde{K}_{V_K\bar{\zeta}^i}\, d\bar{\zeta}^i\big) \ , \end{aligned} \tag{A.2.1}$$

with the one with chiral multiplets, (3.4.36),

$$\begin{aligned} \mathcal{L}^{\text{kin}} \ &= \ -(\tilde{K}_{\zeta^i\bar{\zeta}^j} + \tilde{K}^K_{\zeta^i}\tilde{K}_{KL}\tilde{K}^L_{\bar{\zeta}^i})\, d\zeta^i \wedge *d\bar{\zeta}^j \\ &\quad +\tilde{K}_{KL}\left(d\text{Re}M^I \wedge *\text{Re}M^J + d\xi^K \wedge *d\xi^J\right) - 2\,\tilde{K}_{KL}\tilde{K}^L_{\zeta^i}\left(d\text{Re}M^I \wedge *du^j + d\xi^I \wedge *da^j\right) \ . \end{aligned}$$

In (A.2.1) $\tilde{K}(V,\zeta,\bar{\zeta})$ is a function of the scalars V_K and the chiral multiplets ζ^i. The function $\tilde{K}$ encodes the dynamics of the fields, and we would like to relate it to the Kähler potential from (3.4.36). The standard procedure is to eliminate the fields C^2_K in favor of its duals ξ^K by introducing an appropriate term to the action

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \ , \qquad \delta\mathcal{L} \ = \ -2\xi^K\, dC^3_K \ = \ -2C^3_K \wedge d\xi^K \ , \tag{A.2.2}$$

where $\xi^K(x)$ is a Lagrange multiplier. By solving the equations of motion for ξ^K one finds $dC^3_K = 0$ such that locally $C^3_K = dC^2_K$, giving $\delta\mathcal{L} = 0$ as expected. One can use the equations of motion of C^3_K,

$$*\, C^3_K = \tilde{K}^{V_K V_L}\Big(d\xi^L + \tfrac{i}{2}\big(\tilde{K}_{V_L\zeta^i}\, d\zeta^i - \tilde{K}_{V_L\bar{\zeta}^i}\, d\bar{\zeta}^i\big)\Big) \tag{A.2.3}$$

to eliminate it from (A.2.1),

$$\begin{aligned} \mathcal{L} \ &= \ -\tilde{K}_{\zeta^i\bar{\zeta}^j}\, d\zeta^i \wedge *d\bar{\zeta}^j + \tfrac{1}{4}\tilde{K}_{V_K V_L}\, dV_K \wedge *dV_L \\ &\quad +\tilde{K}^{V_K V_L}\Big(d\tilde{\xi}^L - \text{Im}\big(\tilde{K}_{V_L\zeta^j}\, d\zeta^j\big)\Big) \wedge *\Big(d\tilde{\xi}^L - \text{Im}\big(\tilde{K}_{V_L\zeta^i}\, d\zeta^i\big)\Big) \ . \end{aligned} \tag{A.2.4}$$

For our particular case, we can further simplify this equation. Comparing (3.3.52) with the Chern-Simons action (3.3.49), one can notice that the field C^2 couples, to first order, with the imaginary part of ζ^i, namely a^i. We can assume that $\tilde{K}$ is a function only of V_L and the real part of ζ^i, $\text{Re}\zeta^i = u^i$. We will see shortly that this assumption agrees with our results (indications that $\tilde{K}$ depends only on $\text{Re}\zeta^i$ can be inferred from section 3.4, as in equation (3.4.20)). The effective Lagrangian (A.2.4) thus simplifies to

$$\begin{aligned} \mathcal{L} \ &= \ -\tfrac{1}{4}\tilde{K}_{u^i u^j}\, d\zeta^i \wedge *d\bar{\zeta}^j + \tfrac{1}{4}\tilde{K}_{V_K V_L}\, dV_K \wedge *dV_L \\ &\quad +\tilde{K}^{V_K V_L}\Big(d\tilde{\xi}^K - \tfrac{1}{2}\tilde{K}_{V_K u^i}\, d\,\text{Im}\zeta^i\Big) \wedge *\Big(d\tilde{\xi}^L - \tfrac{1}{2}\tilde{K}_{V_L u^j}\, d\,\text{Im}\zeta^j\Big) \ . \end{aligned} \tag{A.2.5}$$

We would like to relate this $N = 1$ Lagrangian to the standard Lagrangian of chiral multiplets $\Phi = (M^I, \zeta^i)$

$$\begin{aligned}
\mathcal{L} &= -K_{\Phi\bar{\Phi}}\, d\Phi \wedge *d\bar{\Phi} \qquad \text{(A.2.6)}\\
&= -K_{\zeta^i\bar{\zeta}^j}\, d\zeta^i \wedge *d\bar{\zeta}^j - K_{M^I\bar{M}^J}\left(d\mathrm{Re}M^I \wedge *\mathrm{Re}M^J + d\xi^K \wedge *d\xi^J\right)\\
&\quad -2K_{M^I\bar{\zeta}^j}\left(d\mathrm{Re}M^I \wedge *du^j + d\xi^I \wedge *da^j\right).
\end{aligned}$$

and relate the Kähler metrics $K_{\Phi\bar{\Phi}}$ with derivatives of the function $\tilde{K}$, as in equation (3.4.36). This is obtained by performing a Legendre transform with respect to the fields M^K,

$$K(M,\zeta) = \tilde{K}(V, \zeta + \bar{\zeta}) + (M^K + \bar{M}^K)V_K \qquad \text{(A.2.7)}$$

where $V_K(\zeta, M)$ is written as a function of the complex fields ζ^i and implicitly of new field M^K, defined as

$$M^K = -\tfrac{1}{2}\tilde{K}_{V_K} + i\xi^K. \qquad \text{(A.2.8)}$$

One can see $(M^K + \bar{M}^K)$ as the conjugate coordinate to V_K. To see that equations (A.2.6) and (A.2.5) are indeed related by this Legendre transformation, one has to calculate the derivatives of K in terms of the derivatives of $\tilde{K}$. One starts by differentiating (A.2.8),

$$\begin{aligned}
\frac{\partial V_K}{\partial M^L} &= -\tilde{K}^{V_K V_L}, \qquad \text{(A.2.9)}\\
\frac{\partial V_K}{\partial \zeta^j} &= \frac{1}{2}\frac{\partial V_K}{\partial M^L}\frac{\partial M^L}{\partial u^j} = \tfrac{1}{2}\tilde{K}^{V_K V_L}\tilde{K}_{V_L u^j}.
\end{aligned}$$

Using these expressions one easily calculates the first derivatives of the Kähler potential (A.2.7) as

$$K_{M^K} = V_K, \qquad K_{\zeta^i} = \tfrac{1}{2}\tilde{K}_{u^i}. \qquad \text{(A.2.10)}$$

Applying the equations (A.2.9) once more when differentiating (A.2.10) one finds the Kähler metrics

$$\begin{aligned}
K_{M^K\bar{M}^L} &= -\tilde{K}^{V_K V_L}, \quad K_{M^K\bar{\zeta}^i} = \tfrac{1}{2}\tilde{K}^{V_K V_L}\tilde{K}_{V_L u^i},\\
K_{\zeta^i\bar{\zeta}^j} &= \tfrac{1}{4}\tilde{K}_{u^i u^j} + \tfrac{1}{4}\tilde{K}_{u^i V_K}\tilde{K}^{V_K V_L}\tilde{K}_{V_L u^j}, \qquad \text{(A.2.11)}
\end{aligned}$$

with inverses

$$\begin{aligned}
K^{M^K\bar{M}^L} &= -\tilde{K}_{V_K V_L} + \tilde{K}_{u^i V_K}\tilde{K}^{u^i u^j}\tilde{K}_{V_L u^j},\\
K^{M^K\bar{\zeta}^j} &= 2\tilde{K}^{u^i u^j}\tilde{K}_{u^i V_K}, \quad K^{\zeta^i\bar{\zeta}^j} = 4\tilde{K}^{u^i u^j}. \qquad \text{(A.2.12)}
\end{aligned}$$

Finally, one checks that $K(T, N)$ is indeed the Kähler potential for the Lagrangian (A.2.5). This is done by inserting in the definition of T_κ and the Kähler metrics obtained above into (A.2.6), yielding back (A.2.5).

A.3 Mixing of brane and bulk $U(1)$ vectors

In this Appendix we analyze the 4D effective action for all the massless spacetime vector fields that appear after dimensional reduction. They are the A^α and A_α components coming from the combination of RR and B_2 bulk fields (3.3.49), and A, the massless vector component of the U(1) field A_{D6} on the brane, (3.3.34). The duality relation between C_3 and C_5 implies a electric-magnetic duality between A^α

and A_α. To avoid the overcounting of degrees of freedom, we consider both fields, but each weighted by a factor of one half, as in [27]. This procedure gives the action

$$\begin{aligned} S^{(4)}_{\rm vec} &= -\int \tfrac{1}{2}{\rm Re}f_{\rm r}\, F\wedge *F + \tfrac{1}{2}{\rm Im}f_{\rm r}\, F\wedge F \\ &\quad + \tfrac{1}{4}({\rm Im}\mathcal{N}_{\alpha\beta} + {\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\delta}{\rm Re}\mathcal{N}_{\delta\beta}) dA^\alpha \wedge * dA^\beta \\ &\quad + \tfrac{1}{4}{\rm Im}\mathcal{N}^{\alpha\beta} dA_\alpha \wedge * dA_\beta - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta} dA_\beta \wedge * dA^\alpha - \Delta_\alpha dA^\alpha \wedge F - \tilde{\mathcal{J}}^\alpha dA_\alpha \wedge F\ , \end{aligned} \tag{A.3.1}$$

where $F = dA$, $\Delta_\alpha = (a^j \Delta_{j\alpha} + \Gamma_\alpha)$ and

$${\rm Im}\mathcal{N}_{\alpha\beta} = -\int_Y \omega_\alpha \wedge * \omega_\beta \qquad {\rm Im}\mathcal{N}^{\alpha\beta} = ({\rm Im}\mathcal{N}_{\alpha\beta})^{-1} = -\int_Y \tilde{\omega}^\alpha \wedge * \tilde{\omega}^\beta \qquad {\rm Re}\mathcal{N}_{\alpha\beta} = -b^a \mathcal{K}_{a\alpha\beta}\ . \tag{A.3.2}$$

Recalling the duality relation (3.2.6) for the $\mathcal{A}$ fields

$$e^B d\mathcal{A}\big|_6 = - *_{10} \left(e^B d\mathcal{A}\right)\big|_4\ , \tag{A.3.3}$$

we obtain, for A^α and A_α,

$$d(A_\alpha \tilde{\omega}^\alpha) + dA^\beta b^a \omega_a \wedge \omega_\beta = - * dA^\gamma *_6 \omega_\gamma\ . \tag{A.3.4}$$

We take the wedge product of the above expression with ω_α and integrate to obtain the duality relation

$$dA_\alpha = {\rm Im}\mathcal{N}_{\alpha\beta} * dA^\beta + {\rm Re}\mathcal{N}_{\alpha\beta} dA^\beta\ . \tag{A.3.5}$$

From the variation of action (A.3.1), we obtain the equations of motion for A_α and A^α,

$$\begin{gathered} \tfrac{1}{2}({\rm Im}\mathcal{N}_{\alpha\beta} + {\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\delta}{\rm Re}\mathcal{N}_{\delta\beta})\, d * dA^\beta - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta}\, d * dA_\beta - \Delta_\alpha dF = 0\ , \\ \tfrac{1}{2}{\rm Im}\mathcal{N}^{\alpha\beta}\, d * dA_\alpha - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta}\, d * dA^\alpha - \tilde{\mathcal{J}}^\beta dF = 0\ . \end{gathered} \tag{A.3.6}$$

However, if one takes the exterior derivative of equation (A.3.5) and compare with (A.3.6), one notes that the equations are not compatible. That is, the equations of motion and the duality constraints cannot be simultaneously satisfied. In order to make the duality relation consistent, one should modify the field strengths as

$$dA^\alpha \to G^\alpha = dA^\alpha - 2\tilde{\mathcal{J}}^\alpha F\ , \qquad dA_\alpha \to G_\alpha = dA_\alpha + 2\Delta_\alpha F\ , \tag{A.3.7}$$

as well as the duality relation (A.3.5) by the same redefinition. This modified action becomes then

$$\begin{aligned} S^{(4)}_{\rm vec} &\to -\int \tfrac{1}{4}({\rm Im}\mathcal{N}_{\alpha\beta} + {\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\delta}{\rm Re}\mathcal{N}_{\delta\beta}) G^\alpha \wedge * G^\beta - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta} G_\beta \wedge * G^\alpha \\ &\quad + \tfrac{1}{4}{\rm Im}\mathcal{N}^{\alpha\beta} G_\alpha \wedge * G_\beta + \tfrac{1}{2}{\rm Re}f_{\rm r}\, F\wedge *F + \tfrac{1}{2}{\rm Im}f_{\rm r}\, F\wedge F - \Delta_\alpha G^\alpha \wedge F - \tilde{\mathcal{J}}^\alpha G_\alpha \wedge F\ . \end{aligned} \tag{A.3.8}$$

The equations coming from this action are

$$\begin{gathered} dG^\alpha = -2\tilde{\mathcal{J}}^\alpha dF\ , \qquad dG_\alpha = 2\Delta_\alpha dF\ , \qquad G_\alpha = {\rm Im}\mathcal{N}_{\alpha\beta} * G^\beta + {\rm Re}\mathcal{N}_{\alpha\beta}\, G^\beta\ , \\ \tfrac{1}{2}({\rm Im}\mathcal{N}_{\alpha\beta} + {\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\delta}{\rm Re}\mathcal{N}_{\delta\beta})\, d * G^\beta - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta}\, d * G_\beta - \Delta_\alpha dF = 0\ , \\ \tfrac{1}{2}{\rm Im}\mathcal{N}^{\alpha\beta}\, d * G_\alpha - \tfrac{1}{2}{\rm Re}\mathcal{N}_{\alpha\gamma}{\rm Im}\mathcal{N}^{\gamma\beta}\, d * G^\alpha - \tilde{\mathcal{J}}^\beta dF = 0\ . \end{gathered} \tag{A.3.9}$$

The first two equations follow directly from (A.3.7), the third is the imposed duality condition, and the two remaining are the equations of motion for A^α and A_α. One can check that they are now consistent, by starting with the equation of motion for one of the fields and obtaining the equation for the dual field after imposing the duality conditions.

As was mentioned, the duality condition implies that the degrees of freedom for the fields are not independent. To eliminate the dependence of A_α in favor of its dual, we now treat the field strength G_α as an independent field, and add to the action the term

$$\delta S = -\tfrac{1}{2} dA^\alpha \wedge (G_\alpha - 2\Delta_\alpha F) + \lambda(dG_\alpha - 2\Delta_\alpha dF) \,, \tag{A.3.10}$$

where λ is an auxiliary field acting as a Lagrange multiplier. The equations for this modified action are the same as (A.3.9), but now they all come from variations on the fields A^α, G_α and λ. Having the equations for G_α, we now substitute them back into the action, and obtain

$$\begin{aligned} S^{(4)}_{\text{vec}} &= -\int \tfrac{1}{2}\text{Re} f_{\text{r}}\; F \wedge *F + \tfrac{1}{2}\text{Im} f_{\text{r}}\; F \wedge F \\ &\quad + \tfrac{1}{2} dA^\alpha \wedge (\text{Im}\mathcal{N}_{\alpha\beta} * G^\beta + \text{Re}\mathcal{N}_{\alpha\beta} G^\beta - 2\Delta_\alpha F) \\ &\quad - \Delta_\alpha (dA^\alpha - 2\tilde{\mathcal{J}}^\alpha F) \wedge F - (\text{Im}\mathcal{N}_{\alpha\beta} * G^\beta + \text{Re}\mathcal{N}_{\alpha\beta} G^\beta)\tilde{\mathcal{J}}^\alpha \wedge F \\ &= -\int \tfrac{1}{2}(\text{Re} f_{\text{r}} + 4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha\tilde{\mathcal{J}}^\beta)\, F \wedge *F + \tfrac{1}{2}(\text{Im} f_{\text{r}} + 4\Delta_\alpha\tilde{\mathcal{J}}^\alpha + 4\text{Re}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta\tilde{\mathcal{J}}^\alpha) F \wedge F \\ &\quad - 2\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta dA^\alpha \wedge *F - 2(\Delta_\alpha + \tilde{\mathcal{J}}^\beta \text{Re}\mathcal{N}_{\alpha\beta}) dA^\alpha \wedge F \\ &\quad + \tfrac{1}{2}\text{Im}\mathcal{N}_{\alpha\beta} dA^\alpha \wedge * dA^\beta + \tfrac{1}{2}\text{Re}\mathcal{N}_{\alpha\beta} dA^\alpha \wedge dA^\beta \,, \end{aligned} \tag{A.3.11}$$

from where we can extract a corrected gauge coupling function f_{cor} for the brane U(1) gauge fields,

$$\text{Re} f_{\text{cor}} = \text{Re} f_{\text{r}} + 4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\alpha\tilde{\mathcal{J}}^\beta \,, \quad \text{Im} f_{\text{cor}} = \text{Im} f_{\text{r}} + 4\Delta_\alpha\tilde{\mathcal{J}}^\alpha + 4\text{Re}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta\tilde{\mathcal{J}}^\alpha \,, \tag{A.3.12}$$

a gauge coupling function f_α for the mixing between brane and bulk gauge bosons,

$$\text{Re} f_\alpha = -4\text{Im}\mathcal{N}_{\alpha\beta}\tilde{\mathcal{J}}^\beta \,, \quad \text{Im} f_\alpha = -4(\Delta_\alpha + \tilde{\mathcal{J}}^\beta \text{Re}\mathcal{N}_{\alpha\beta}) \,, \tag{A.3.13}$$

and the gauge coupling function for the vector field A^α from the bulk (3.4.39),

$$f_{\alpha\beta} = -i\bar{\mathcal{N}}_{\alpha\beta} \,. \tag{A.3.14}$$

A.4 Δ, f and g in E_8 Enhancement Model

$$
\begin{aligned}
\Delta \;=\;& -2048p^{21}q^{9}w^{5} - 8192p^{20}q^{10}w^{5} + 512p^{19}q^{11}w^{5} + 43008p^{18}q^{12}w^{5} + 53248p^{17}q^{13}w^{5} - \\
& -34816p^{16}q^{14}w^{5} - 103424p^{15}q^{15}w^{5} - 34816p^{14}q^{16}w^{5} + 53248p^{13}q^{17}w^{5} + 43008p^{12}q^{18}w^{5} + \\
& +512p^{11}q^{19}w^{5} - 8192p^{10}q^{20}w^{5} - 2048p^{9}q^{21}w^{5} - 2560p^{18}q^{6}w^{6} - 13056p^{17}q^{7}w^{6} - \\
& -4224p^{16}q^{8}w^{6} + 73920p^{15}q^{9}w^{6} + 109440p^{14}q^{10}w^{6} - 60864p^{13}q^{11}w^{6} - 205312p^{12}q^{12}w^{6} - \\
& -60864p^{1}1q^{13}w^{6} + 109440p^{10}q^{14}w^{6} + 73920p^{9}q^{15}w^{6} - 4224p^{8}q^{16}w^{6} - 13056p^{7}q^{17}w^{6} - \\
& -2560p^{6}q^{18}w^{6} - 2816p^{15}q^{3}w^{7} - 6400p^{14}q^{4}w^{7} + 11456p^{13}q^{5}w^{7} + 41344p^{12}q^{6}w^{7} + \\
& +26976p^{11}q^{7}w^{7} - 36096p^{10}q^{8}w^{7} - 73536p^{9}q^{9}w^{7} - 36096p^{8}q^{10}w^{7} + 26976p^{7}q^{11}w^{7} + \\
& +41344p^{6}q^{12}w^{7} + 11456p^{5}q^{13}w^{7} - 6400p^{4}q^{14}w^{7} - 2816p^{3}q^{15}w^{7} - 432p^{12}w^{8} - \\
& -864p^{11}qw^{8} + 1944p^{10}q^{2}w^{8} + 10456p^{9}q^{3}w^{8} + 6805p^{8}q^{4}w^{8} - 11908p^{7}q^{5}w^{8} - \\
& -18962p^{6}q^{6}w^{8} - 11908p^{5}q^{7}w^{8} + 6805p^{4}q^{8}w^{8} + 10456p^{3}q^{9}w^{8} + 1944p^{2}q^{1}0w^{8} - \\
& -864pq^{1}1w^{8} - 432q^{1}2w^{8} + 864p^{6}w^{9} + 864p^{5}qw^{9} - 1080p^{4}q^{2}w^{9} - 4400p^{3}q^{3}w^{9} - \\
& -1080p^{2}q^{4}w^{9} + 864pq^{5}w^{9} + 864q^{6}w^{9} - 432w^{10}\,, \\
f \;=\;& \frac{1}{3}(16p^{12}q^{8} + 64p^{11}q^{9} + 96p^{10}q^{10} + 64p^{9}q^{11} + 16p^{8}q^{12} + 16p^{9}q^{5}w + 96p^{8}q^{6}w + \\
& +160p^{7}q^{7}w + 96p^{6}q^{8}w + 16p^{5}q^{9}w + 16p^{6}q^{2}w^{2} + 38p^{5}q^{3}w^{2} + \\
& +60p^{4}q^{4}w^{2} + 38p^{3}q^{5}w^{2} + 16p^{2}q^{6}w^{2} + 9p^{2}w^{3} + 3pqw^{3} + 9q^{2}w^{3})\,, \\
g \;=\;& \frac{1}{108}(512p^{18}q^{12} + 3072p^{17}q^{13} + 7680p^{16}q^{14} + \\
& +10240p^{15}q^{15} + 7680p^{14}q^{16} + 3072p^{13}q^{17} + 512p^{12}q^{18} + \\
& +768p^{15}q^{9}w + 6144p^{14}q^{10}w + 17664p^{13}q^{11}w + 24576p^{12}q^{12}w + \\
& +17664p^{11}q^{13}w + 6144p^{10}q^{14}w + 768p^{9}q^{15}w + 960p^{12}q^{6}w^{2} + \\
& +5280p^{11}q^{7}w^{2} + 14016p^{10}q^{8}w^{2} + 19392p^{9}q^{9}w^{2} + 14016p^{8}q^{10}w^{2} + \\
& +5280p^{7}q^{11}w^{2} + 960p^{6}q^{12}w^{2} + 352p^{9}q^{3}w^{3} + 2496p^{8}q^{4}w^{3} + \\
& +4848p^{7}q^{5}w^{3} + 5920p^{6}q^{6}w^{3} + 4848p^{5}q^{7}w^{3} + 2496p^{4}q^{8}w^{3} + \\
& +352p^{3}q^{9}w^{3} + 108p^{6}w^{4} + 108p^{5}qw^{4} + 855p^{4}q^{2}w^{4} + \\
& +990p^{3}q^{3}w^{4} + 855p^{2}q^{4}w^{4} + 108pq^{5}w^{4} + 108q^{6}w^{4} + 108w^{5})
\end{aligned}
$$

Bibliography

[1] **CMS Collaboration** Collaboration, S. Chatrchyan *et al.*, "Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC," *Phys.Lett.* **B716** (2012) 30–61, `arXiv:1207.7235 [hep-ex]`.

[2] **ATLAS Collaboration** Collaboration, G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Phys.Lett.* **B716** (2012) 1–29, `arXiv:1207.7214 [hep-ex]`.

[3] J. Ellis, J. Espinosa, G. Giudice, A. Hoecker, and A. Riotto, "The Probable Fate of the Standard Model," *Phys.Lett.* **B679** (2009) 369–375, `arXiv:0906.0954 [hep-ph]`.

[4] H. Georgi and S. Glashow, "Unity of All Elementary Particle Forces," *Phys.Rev.Lett.* **32** (1974) 438–441.

[5] N. Sakai and T. Yanagida, "Proton Decay in a Class of Supersymmetric Grand Unified Models," *Nucl.Phys.* **B197** (1982) 533.

[6] S. Weinberg, "Supersymmetry at Ordinary Energies. 1. Masses and Conservation Laws," *Phys.Rev.* **D26** (1982) 287.

[7] H. P. Nilles, "Supersymmetry, Supergravity and Particle Physics," *Phys.Rept.* **110** (1984) 1–162.

[8] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge University Press, 1998.

[9] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge University Press, 1998.

[10] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, "The Hierarchy problem and new dimensions at a millimeter," *Phys.Lett.* **B429** (1998) 263–272, `arXiv:hep-ph/9803315 [hep-ph]`.

[11] H. P. Nilles, S. Ramos-Sanchez, M. Ratz, and P. K. Vaudrevange, "From strings to the MSSM," *Eur.Phys.J.* **C59** (2009) 249–267, `arXiv:0806.3905 [hep-th]`.

[12] S. L. Parameswaran, S. Ramos-Sanchez, and I. Zavala, "On Moduli Stabilisation and de Sitter Vacua in MSSM Heterotic Orbifolds," *JHEP* **1101** (2011) 071, `arXiv:1009.3931 [hep-th]`.

[13] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "Stabilizing All Geometric Moduli in Heterotic Calabi-Yau Vacua," *Phys.Rev.* **D83** (2011) 106011, `arXiv:1102.0011 [hep-th]`.

[14] L. B. Anderson, J. Gray, A. Lukas, and B. Ovrut, "The Atiyah Class and Complex Structure Stabilization in Heterotic Calabi-Yau Compactifications," *JHEP* **1110** (2011) 032, `arXiv:1107.5076 [hep-th]`.

[15] J. Polchinski, "Tasi lectures on D-branes," `arXiv:hep-th/9611050 [hep-th]`.

[16] L. E. Ibanez, F. Marchesano, and R. Rabadan, "Getting just the standard model at intersecting branes," *JHEP* **0111** (2001) 002, `arXiv:hep-th/0105155 [hep-th]`.

[17] M. Cvetic, G. Shiu, and A. M. Uranga, "Three family supersymmetric standard - like models from intersecting brane worlds," *Phys.Rev.Lett.* **87** (2001) 201801, arXiv:hep-th/0107143 [hep-th].

[18] R. Blumenhagen, B. Kors, D. Lust, and T. Ott, "The standard model from stable intersecting brane world orbifolds," *Nucl.Phys.* **B616** (2001) 3–33, arXiv:hep-th/0107138 [hep-th].

[19] R. Blumenhagen, M. Cvetic, P. Langacker, and G. Shiu, "Toward Realistic Intersecting D-Brane Models," *Annual Review of Nuclear and Particle Science* **55** no. 1, (Jan., 2005) 85, arXiv:0502005 [hep-th]. http://arxiv.org/abs/hep-th/0502005.

[20] R. Blumenhagen, B. Kors, D. Lüst, and S. Stieberger, "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes," *Physics Reports* **445** no. 1-6, (Oct., 2006) 331, arXiv:0610327 [hep-th]. http://arxiv.org/abs/hep-th/0610327.

[21] D. Lüst, "Intersecting Brane Worlds – A Path to the Standard Model?," *Classical and Quantum Gravity* **21** no. 10, (May, 2004) 31, arXiv:0401156 [hep-th]. http://arxiv.org/abs/hep-th/0401156.

[22] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys.Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029 [hep-th].

[23] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[24] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nucl.Phys.* **B359** (1991) 21–74.

[25] M. J. Dolan, S. Krippendorf, and F. Quevedo, "Towards a Systematic Construction of Realistic D-brane Models on a del Pezzo Singularity," *JHEP* **1110** (2011) 024, arXiv:1106.6039 [hep-th].

[26] L. McAllister and E. Silverstein, "String Cosmology: A Review," *Gen.Rel.Grav.* **40** (2008) 565–605, arXiv:0710.2951 [hep-th].

[27] H. Jockers and J. Louis, "The effective action of D7-branes in N=1 Calabi-Yau orientifolds," *Energy* no. September 2004, (Sept., 2004) 47, arXiv:0409098 [hep-th]. http://arxiv.org/abs/hep-th/0409098.

[28] T. W. Grimm and D. Vieira Lopes, "The N=1 effective actions of D-branes in Type IIA and IIB orientifolds," arXiv:1104.2328. http://arxiv.org/abs/1104.2328.

[29] J. Redondo and A. Ringwald, "Light shining through walls," *Contemp.Phys.* **52** (2011) 211–236, arXiv:1011.3741 [hep-ph].

[30] R. Dijkgraaf, "Les Houches lectures on fields, strings and duality," arXiv:hep-th/9703136 [hep-th].

[31] E. Witten, "String theory dynamics in various dimensions," *Nucl.Phys.* **B443** (1995) 85–126, arXiv:hep-th/9503124 [hep-th].

[32] N. B. Copland, "Introductory Lectures on Multiple Membranes," arXiv:1012.0459 [hep-th].

[33] J. Bagger and N. Lambert, "Modeling Multiple M2's," *Phys.Rev.* **D75** (2007) 045020, arXiv:hep-th/0611108 [hep-th].

[34] C. Vafa, "Evidence for F theory," *Nucl.Phys.* **B469** (1996) 403–418, arXiv:hep-th/9602022 [hep-th].

[35] R. Blumenhagen, M. Cvetic, D. Lust, R. Richter, and T. Weigand, "Non-perturbative Yukawa Couplings from String Instantons," *Phys.Rev.Lett.* **100** (2008) 061602, arXiv:0707.1871 [hep-th].

[36] C. Beasley, J. J. Heckman, and C. Vafa, "GUTs and Exceptional Branes in F-theory - I," *JHEP* **0901** (2009) 058, arXiv:0802.3391 [hep-th].

[37] C. Beasley, J. J. Heckman, and C. Vafa, "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions," *JHEP* **0901** (2009) 059, arXiv:0806.0102 [hep-th].

[38] R. Donagi and M. Wijnholt, "Model Building with F-Theory," arXiv:0802.2969 [hep-th].

[39] T. W. Grimm and T. Weigand, "On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs,". http://arxiv.org/abs/1006.0226.

[40] T. W. Grimm, M. Kerstan, E. Palti, and T. Weigand, "Massive Abelian Gauge Symmetries and Fluxes in F-theory," *JHEP* **1112** (2011) 004, arXiv:1107.3842 [hep-th].

[41] M. J. Dolan, J. Marsano, N. Saulina, and S. Schafer-Nameki, "F-theory GUTs with U(1) Symmetries: Generalities and Survey," *Phys.Rev.* **D84** (2011) 066008, arXiv:1102.0290 [hep-th].

[42] M. J. Dolan, J. Marsano, and S. Schafer-Nameki, "Unification and Phenomenology of F-Theory GUTs with $U(1)_PQ$," *JHEP* **1112** (2011) 032, arXiv:1109.4958 [hep-ph].

[43] J. C. Callaghan, S. F. King, G. K. Leontaris, and G. G. Ross, "Towards a Realistic F-theory GUT," *JHEP* **1204** (2012) 094, arXiv:1109.1399 [hep-ph].

[44] R. Blumenhagen, T. W. Grimm, B. Jurke, and T. Weigand, "Global F-theory GUTs," *Nucl.Phys.* **B829** (2010) 325–369, arXiv:0908.1784 [hep-th].

[45] J. Marsano, N. Saulina, and S. Schafer-Nameki, "F-theory Compactifications for Supersymmetric GUTs," *JHEP* **0908** (2009) 030, arXiv:0904.3932 [hep-th].

[46] P. G. Camara, E. Dudas, and E. Palti, "Massive wavefunctions, proton decay and FCNCs in local F-theory GUTs," *JHEP* **1112** (2011) 112, arXiv:1110.2206 [hep-th].

[47] J. Knapp, M. Kreuzer, C. Mayrhofer, and N.-O. Walliser, "Toric Construction of Global F-Theory GUTs," *JHEP* **1103** (2011) 138, arXiv:1101.4908 [hep-th].

[48] J. J. Heckman and C. Vafa, "Flavor Hierarchy From F-theory," *Nucl.Phys.* **B837** (2010) 137–151, arXiv:0811.2417 [hep-th].

[49] L. Randall and D. Simmons-Duffin, "Quark and Lepton Flavor Physics from F-Theory," arXiv:0904.1584 [hep-ph].

[50] V. Bouchard, J. J. Heckman, J. Seo, and C. Vafa, "F-theory and Neutrinos: Kaluza-Klein Dilution of Flavor Hierarchy," *JHEP* **1001** (2010) 061, arXiv:0904.1419 [hep-ph].

[51] J. J. Heckman, A. Tavanfar, and C. Vafa, "The Point of E(8) in F-theory GUTs," *JHEP* **1008** (2010) 040, arXiv:0906.0581 [hep-th].

[52] C. Froggatt and H. B. Nielsen, "Hierarchy of Quark Masses, Cabibbo Angles and CP Violation," *Nucl.Phys.* **B147** (1979) 277.

[53] H. K. Dreiner and M. Thormeier, "Supersymmetric Froggatt-Nielsen models with baryon and lepton number violation," *Phys.Rev.* **D69** (2004) 053002, arXiv:hep-ph/0305270 [hep-ph].

[54] K. Babu, "TASI Lectures on Flavor Physics," arXiv:0910.2948 [hep-ph].

[55] M. Esole and S.-T. Yau, "Small resolutions of SU(5)-models in F-theory," arXiv:1107.0733 [hep-th].

[56] P. G. Camara, L. E. Ibanez, and F. Marchesano, "RR photons," *JHEP* **1109** (2011) 110, arXiv:1106.0060 [hep-th].

[57] J. Marsano and S. Schafer-Nameki, "Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau's," *JHEP* **1111** (2011) 098, arXiv:1108.1794 [hep-th].

[58] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton Univ Pr, 1992.

[59] B. Zumino, "Supersymmetry and kähler manifolds," *Physics Letters B* **87** no. 3, (1979) 203–206.

[60] J. D. Lykken, "Introduction to supersymmetry," arXiv:hep-th/9612114 [hep-th].

[61] J. Terning, *Modern supersymmetry: Dynamics and duality*. Oxford Science Publications, 2006.

[62] S. Abel, M. Goodsell, J. Jaeckel, V. Khoze, and A. Ringwald, "Kinetic Mixing of the Photon with Hidden U(1)s in String Phenomenology," *JHEP* **0807** (2008) 124, arXiv:0803.1449 [hep-ph].

[63] M. Goodsell, J. Jaeckel, J. Redondo, and A. Ringwald, "Naturally Light Hidden Photons in LARGE Volume String Compactifications," *JHEP* **0911** (2009) 027, arXiv:0909.0515 [hep-ph].

[64] M. T. Grisaru, W. Siegel, and M. Rocek, "Improved Methods for Supergraphs," *Nucl.Phys.* **B159** (1979) 429.

[65] N. Seiberg, "Naturalness versus supersymmetric nonrenormalization theorems," *Phys.Lett.* **B318** (1993) 469–475, arXiv:hep-ph/9309335 [hep-ph].

[66] N. Seiberg, "Supersymmetry and Nonperturbative beta Functions," *Phys.Lett.* **B206** (1988) 75.

[67] P. Candelas, X. C. De la Ossa, P. S. Green, and L. Parkes, "An Exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds," *Phys.Lett.* **B258** (1991) 118–126.

[68] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, *et al.*, "Mirror symmetry,".

[69] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror Symmetry is T-Duality," *Nuclear Physics B* **479** (1996) 20. http://arxiv.org/abs/hep-th/9606040.

[70] B. S. Acharya, G. Kane, and P. Kumar, "Compactified String Theories – Generic Predictions for Particle Physics," *Int.J.Mod.Phys.* **A27** (2012) 1230012, arXiv:1204.2795 [hep-ph].

[71] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications," *JHEP* **0503** (2005) 007, arXiv:hep-th/0502058 [hep-th].

[72] J. P. Conlon, F. Quevedo, and K. Suruliz, "Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking," *JHEP* **0508** (2005) 007, arXiv:hep-th/0505076 [hep-th].

[73] A. Strominger, "Special Geometry," *Commun.Math.Phys.* **133** (1990) 163–180.

[74] P. Candelas and X. de la Ossa, "Moduli Space of Calabi-Yau Manifolds," *Nucl.Phys.* **B355** (1991) 455–481.

[75] L. Castellani, R. D'Auria, and S. Ferrara, "Special Kahler Geometry: An Intrinsic Formulation from N=2 Space-Time Supersymmetry," *Phys.Lett.* **B241** (1990) 57.

[76] T. Mohaupt, "Black hole entropy, special geometry and strings," *Fortsch.Phys.* **49** (2001) 3–161, arXiv:hep-th/0007195 [hep-th].

[77] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, "What is special Kahler geometry?," *Nucl.Phys.* **B503** (1997) 565–613, arXiv:hep-th/9703082 [hep-th].

[78] S. Ferrara and S. Sabharwal, "Dimensional Reduction of Type II Superstrings," *Class.Quant.Grav.* **6** (1989) L77.

[79] S. Ferrara and S. Sabharwal, "Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces," *Nucl.Phys.* **B332** (1990) 317.

[80] T. W. Grimm, "The Effective action of type II Calabi-Yau orientifolds," *Fortsch.Phys.* **53** (2005) 1179–1271, arXiv:hep-th/0507153 [hep-th].

[81] I. Brunner and K. Hori, "Orientifolds and mirror symmetry," *JHEP* **0411** (2004) 005, arXiv:hep-th/0303135 [hep-th].

[82] T. W. Grimm and J. Louis, "The effective action of type iia calabi-yau orientifolds," *Nuclear Physics B* **718** no. 1-2, (2005) 153 – 202.

[83] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, "New Formulations of D=10 Supersymmetry and D8-O8 Domain Walls,". http://arxiv.org/abs/hep-th/0103233.

[84] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, "Type IIA Moduli Stabilization," *JHEP* **7** no. 07, (2005) 48. http://arxiv.org/abs/hep-th/0505160.

[85] K. Becker, M. Becker, and A. Strominger, "Fivebranes, membranes and non-perturbative string theory," *Nuclear Physics B* **456** no. 1-2, (Dec., 1995) 130–152, arXiv:9507158 [hep-th]. http://arxiv.org/abs/hep-th/9507158http://www.sciencedirect.com/science/article/pii/0550321395004871.

[86] M. Marino, R. Minasian, G. Moore, and A. Strominger, "Nonlinear Instantons from Supersymmetric p-Branes," *JHEP* **2000** no. 01, (1999) 35. http://arxiv.org/abs/hep-th/9911206.

[87] R. C. McLean, "Deformations of calibrated submanifolds," *Communications In Analysis And Geometry* **6** no. 4, (1998) 705–747.

[88] N. Hitchin, "Lectures on Special Lagrangian Submanifolds," *Winter school on mirror symmetry vector bundles and Lagrangian submanifolds proceedings of the Winter School on Mirror Symmetry January 1999 Harvard University Cambridge Massachusetts* **23** no. January, (1999) 151. `http://arxiv.org/abs/math/9907034`.

[89] R. Leigh, "Dirac-Born-Infeld Action from Dirichlet Sigma Model," *Mod.Phys.Lett.* **A4** (1989) 2767.

[90] M. Born, "Modified field equations with a finite radius of the electron," *Nature* **132** (1933) 282.

[91] M. Born and L. Infeld, "Foundations of the new field theory," *Proceedings of the Royal Society of London. Series A* **144** no. 852, (1934) 425–451.

[92] M. Graña, T. W. Grimm, H. Jockers, and J. Louis, "Soft Supersymmetry Breaking in Calabi-Yau Orientifolds with D-branes and Fluxes," *Nuclear Physics B* **690** no. 1-2, (Dec., 2003) 41, `arXiv:0312232 [hep-th]`. `http://arxiv.org/abs/hep-th/0312232`.

[93] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "The D5-brane effective action and superpotential in N=1 compactifications," *Nuclear Physics B* **816** no. 1-2, (Nov., 2008) 52, `arXiv:0811.2996`. `http://arxiv.org/abs/0811.2996`.

[94] R. Thomas and S.-T. Yau, "Special Lagrangians, stable bundles and mean curvature flow," *Commun.Anal.Geom.* **10** (2002) 1075–1113, `arXiv:math/0104197 [math.DG]`.

[95] M. Wang, "Some recent developments in Lagrangian mean curvature flows," *Surveys in differential geometry* **12** (2008) 333–347. `http://www.math.columbia.edu/~mtwang/survey_final.pdf`.

[96] R. P. Thomas, "Moment maps, monodromy and mirror manifolds,". `http://arxiv.org/abs/math/0104196`.

[97] E. Witten, "Five-Brane Effective Action In M-Theory," *Journal of Geometry and Physics* **22** no. 2, (1996) 103–133. `http://arxiv.org/abs/hep-th/9610234`.

[98] C. Beasley and E. Witten, "A Note on Fluxes and Superpotentials in M-theory Compactifications on Manifolds of G_2 Holonomy," *ReCALL* **2002** no. March 2002, (2002) 15. `http://arxiv.org/abs/hep-th/0203061`.

[99] N. Hitchin, "The moduli space of special Lagrangian submanifolds," `arXiv:9711002 [dg-ga]`. `http://arxiv.org/abs/dg-ga/9711002`.

[100] I. Benmachiche and T. W. Grimm, "Generalized N=1 Orientifold Compactifications and the Hitchin functionals,". `http://arxiv.org/abs/hep-th/0602241`.

[101] N. Hitchin, "The geometry of three-forms in six and seven dimensions," *ArXiv Mathematics eprints* **math.DG** (2000) 38. `http://arxiv.org/abs/math/0010054`.

[102] J. A. Harvey and G. Moore, "Superpotentials and Membrane Instantons,". `http://arxiv.org/abs/hep-th/9907026`.

[103] S. Gukov, "Solitons, Superpotentials and Calibrations," *Nuclear Physics B* **574** no. November 1999, (1999) 26. `http://arxiv.org/abs/hep-th/9911011`.

[104] F. Gmeiner and G. Honecker, "Complete gauge threshold corrections for intersecting fractional D6-branes: The and Standard Models," *Nuclear Physics B* **829** no. 1-2, (Apr., 2010) 225–297. `http://arxiv.org/abs/0910.0843`.

[105] J. Walcher, "Calculations for mirror symmetry with D-branes," *JHEP* **2009** no. 09, (Sept., 2009) 129–129. `http://arxiv.org/abs/0904.4905`.

[106] J. Knapp and E. Scheidegger, "Matrix Factorizations, Massey Products and F-Terms for Two-Parameter Calabi-Yau Hypersurfaces,". `http://arxiv.org/abs/0812.2429`.

[107] M. Baumgartl and S. Wood, "Moduli webs and superpotentials for five-branes," *JHEP* **2009** no. 06, (June, 2009) 052–052. `http://arxiv.org/abs/0812.3397`.

[108] H. Jockers and M. Soroush, "Relative periods and open-string integer invariants for a compact CalabiYau hypersurface," *Nuclear Physics B* **821** no. 3, (Nov., 2009) 535–552. `http://arxiv.org/abs/0904.4674`.

[109] H. Jockers and M. Soroush, "Effective Superpotentials for Compact D5-Brane Calabi-Yau Geometries," *Communications in Mathematical Physics* **290** no. 1, (Jan., 2009) 249–290. `http://arxiv.org/abs/0808.0761`.

[110] M. Alim, M. Hecht, P. Mayr, and A. Mertens, "Mirror symmetry for toric branes on compact hypersurfaces," *JHEP* **2009** no. 09, (Sept., 2009) 126–126. `http://arxiv.org/abs/0901.2937`.

[111] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens, and M. Soroush, "Hints for off-shell mirror symmetry in type II/F-theory compactifications," *Nuclear Physics B* **841** no. 3, (Dec., 2010) 303–338. `http://arxiv.org/abs/0909.1842`.

[112] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens, and M. Soroush, "Type II/F-theory superpotentials with several deformations and N = 1 mirror symmetry," *JHEP* **2011** no. 6, (June, 2011) 62. `http://arxiv.org/abs/1010.0977`.

[113] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "Five-brane superpotentials and heterotic/F-theory duality," *Nuclear Physics B* **838** no. 3, (Oct., 2010) 458–491. `http://arxiv.org/abs/0912.3250`.

[114] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "Computing Brane and Flux Superpotentials in F-theory Compactifications,". `http://arxiv.org/abs/0909.2025`.

[115] T. W. Grimm, A. Klemm, and D. Klevers, "Five-Brane Superpotentials, Blow-Up Geometries and SU(3) Structure Manifolds," *JHEP* **1105** (2011) 113, `arXiv:1011.6375 [hep-th]`.

[116] L. Martucci, "D-branes on general Script N = 1 backgrounds: superpotentials and D-terms," *JHEP* **2006** no. 06, (June, 2006) 033–033. `http://arxiv.org/abs/hep-th/0602129`.

[117] P. Koerber and L. Martucci, "From ten to four and back again: how to generalize the geometry," *Physics* (2007) , `arXiv:0707.1038v1`. `http://uk.arxiv.org/abs/0707.1038v1`.

[118] O. DeWolfe and S. Giddings, "Scales and hierarchies in warped compactifications and brane worlds," *Physical Review D* **67** no. 6, (Mar., 2003) 33. `http://arxiv.org/abs/hep-th/0208123`.

[119] P. Townsend, "The eleven-dimensional supermembrane revisited," *Phys.Lett.* **B350** (1995) 184–187, `arXiv:hep-th/9501068 [hep-th]`.

[120] M. Duff, P. S. Howe, T. Inami, and K. Stelle, "Superstrings in D=10 from Supermembranes in D=11," *Phys.Lett.* **B191** (1987) 70.

[121] R. Sorkin, "Kaluza-Klein Monopole," *Phys.Rev.Lett.* **51** (1983) 87–90.

[122] Misner, C. W. and Taub, A. H., "A Singularity-free Empty Universe," *Soviet Journal of Experimental and Theoretical Physics* **28** (Jan., 1969) 122.

[123] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl.Phys.* **B460** (1996) 506–524, `arXiv:hep-th/9510209 [hep-th]`.

[124] P. Horava and E. Witten, "Eleven-dimensional supergravity on a manifold with boundary," *Nucl.Phys.* **B475** (1996) 94–114, `arXiv:hep-th/9603142 [hep-th]`.

[125] F. Denef, "Les Houches Lectures on Constructing String Vacua," `arXiv:0803.1194 [hep-th]`.

[126] T. Weigand, "Lectures on F-theory compactifications and model building," *Class.Quant.Grav.* **27** (2010) 214004, `arXiv:1009.3497 [hep-th]`.

[127] R. Blumenhagen, "Basics of F-theory from the Type IIB Perspective," *Fortsch.Phys.* **58** (2010) 820–826, `arXiv:1002.2836 [hep-th]`.

[128] B. R. Greene, A. D. Shapere, C. Vafa, and S.-T. Yau, "Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds," *Nucl.Phys.* **B337** (1990) 1.

[129] E. Witten, "Nonperturbative superpotentials in string theory," *Nucl.Phys.* **B474** (1996) 343–360, `arXiv:hep-th/9604030 [hep-th]`.

[130] T. W. Grimm, D. Klevers, and M. Poretschkin, "Fluxes and Warping for Gauge Couplings in F-theory," `arXiv:1202.0285 [hep-th]`.

[131] A. Sen, "F theory and orientifolds," *Nucl.Phys.* **B475** (1996) 562–578, `arXiv:hep-th/9605150 [hep-th]`.

[132] A. Sen, "Orientifold limit of F theory vacua," *Phys.Rev.* **D55** (1997) 7345–7349, `arXiv:hep-th/9702165 [hep-th]`.

[133] K. A. Intriligator, D. R. Morrison, and N. Seiberg, "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces," *Nucl.Phys.* **B497** (1997) 56–100, `arXiv:hep-th/9702198 [hep-th]`.

[134] E. Witten, "Phase transitions in M theory and F theory," *Nucl.Phys.* **B471** (1996) 195–216, `arXiv:hep-th/9603150 [hep-th]`.

[135] K. Kodaira, "On compact analytic surfaces, iii," *The Annals of Mathematics* **78** no. 1, (1963) 1–40.

[136] K. Kodaira, "On compact analytic surfaces: Ii," *The Annals of Mathematics* **77** no. 3, (1963) 563–626.

[137] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, *et al.*, "Geometric singularities and enhanced gauge symmetries," *Nucl.Phys.* **B481** (1996) 215–252, `arXiv:hep-th/9605200 [hep-th]`.

[138] E. Witten, "On flux quantization in M theory and the effective action," *J.Geom.Phys.* **22** (1997) 1–13, `arXiv:hep-th/9609122 [hep-th]`.

[139] D. Lust, P. Mayr, S. Reffert, and S. Stieberger, "F-theory flux, destabilization of orientifolds and soft terms on D7-branes," *Nucl.Phys.* **B732** (2006) 243–290, arXiv:hep-th/0501139 [hep-th].

[140] A. Collinucci and R. Savelli, "On Flux Quantization in F-Theory," *JHEP* **1202** (2012) 015, arXiv:1011.6388 [hep-th].

[141] A. P. Braun, A. Collinucci, and R. Valandro, "G-flux in F-theory and algebraic cycles," *Nucl.Phys.* **B856** (2012) 129–179, arXiv:1107.5337 [hep-th].

[142] S. Krause, C. Mayrhofer, and T. Weigand, "Gauge Fluxes in F-theory and Type IIB Orientifolds," *JHEP* **1208** (2012) 119, arXiv:1202.3138 [hep-th].

[143] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, "Computing Brane and Flux Superpotentials in F-theory Compactifications," *JHEP* **1004** (2010) 015, arXiv:0909.2025 [hep-th].

[144] T. W. Grimm and H. Hayashi, "F-theory fluxes, Chirality and Chern-Simons theories," *JHEP* **1203** (2012) 027, arXiv:1111.1232 [hep-th].

[145] P. S. Aspinwall, "K3 surfaces and string duality," arXiv:hep-th/9611137 [hep-th].

[146] P. Griffiths and J. Harris, *Principles of algebraic geometry*, vol. 197. Wiley Online Library, 1978.

[147] R. Miranda, *The basic theory of elliptic surfaces*. ETS, 1989.

[148] M. R. Gaberdiel and B. Zwiebach, "Exceptional groups from open strings," *Nucl.Phys.* **B518** (1998) 151–172, arXiv:hep-th/9709013 [hep-th].

[149] O. DeWolfe and B. Zwiebach, "String junctions for arbitrary Lie algebra representations," *Nucl.Phys.* **B541** (1999) 509–565, arXiv:hep-th/9804210 [hep-th].

[150] S. Krause, C. Mayrhofer, and T. Weigand, "G_4 flux, chiral matter and singularity resolution in F-theory compactifications," *Nucl.Phys.* **B858** (2012) 1–47, arXiv:1109.3454 [hep-th].

[151] R. Donagi and M. Wijnholt, "Higgs Bundles and UV Completion in F-Theory," arXiv:0904.1218 [hep-th].

[152] E. Dudas and E. Palti, "On hypercharge flux and exotics in F-theory GUTs," *JHEP* **1009** (2010) 013, arXiv:1007.1297 [hep-ph].

[153] M. Bershadsky and A. Johansen, "Colliding singularities in F theory and phase transitions," *Nucl.Phys.* **B489** (1997) 122–138, arXiv:hep-th/9610111 [hep-th].

[154] M. Kuntzler and S. Schafer-Nameki, "G-flux and Spectral Divisors," arXiv:1205.5688 [hep-th].

[155] F. Bonetti and T. W. Grimm, "Six-dimensional (1,0) effective action of F-theory via M-theory on Calabi-Yau threefolds," *JHEP* **1205** (2012) 019, arXiv:1112.1082 [hep-th].

[156] P. Candelas, D.-E. Diaconescu, B. Florea, D. R. Morrison, and G. Rajesh, "Codimension three bundle singularities in F theory," *JHEP* **0206** (2002) 014, arXiv:hep-th/0009228 [hep-th].

[157] P. Binetruy, G. Girardi, and R. Grimm, "Supergravity couplings: A Geometric formulation," *Phys.Rept.* **343** (2001) 255–462, arXiv:hep-th/0005225 [hep-th].