On the relaxation of a variational principle for the motion of a vortex sheet in perfect fluid

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Inhaltsangabe

In der vorliegenden Arbeit soll ein Zweiphasenmodell aus der Strömungsmechanik besprochen werden. Seien s_+, s_- zwei Dichten mit Summe $s_+ + s_- = 1$, und $u_\pm$ zwei Geschwindigkeitsfelder. Wir interessieren uns für die partielle Differentialgleichung in Raum und Zeit,

$$\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0, \tag{1}$$

$$\partial_t u_\pm + Du_\pm \, u_\pm + \nabla p = 0. \tag{2}$$

Sie steht in einem gewissen, noch nicht genau spezifizierten Zusammenhang zur Eulergleichung für ein inkompressibles Fluid, und dessen Instabilität, siehe [10], [19], die Arbeiten von Brenier, und auch unsere vorliegende Diskussion.

Insbesondere kann sie begriffen werden als relaxiertes Modell für die zeitliche Entwicklung eines *vortex sheet*, d. i. eine Trennschicht, die durch konzentrierte Wirbelbildung entsteht. Wir möchten dies hier auf der Ebene eines Variationsproblems darstellen:
Die Euler–Gleichung, die dem Fluid zugrundeliegen soll, kann verstanden werden als Hamiltonsche Gleichung, die durch das Problem der kleinsten Wirkung beschrieben wird,

$$\iint |u|^2 dx dt \;\to\; \min .$$

Es soll nun mithilfe einer Materialpartition, die an das vortex sheet angepasst ist, das Geschwindigkeitsfeld zerlegt werden in $u = \chi_+ u_+ + \chi_- u_-$,
und folglich

$$\int |u|^2 dx \;=\; \int \chi_+ |u_+|^2 dx + \int \chi_- |u_-|^2 dx.$$

Hier sind $\chi_\pm$ zwei charakteristische Funktionen mit Summe $\chi_+ + \chi_- = 1$. Sie dienen dazu, die Lage des vortex sheet im Raum anzugeben. Wegen des Helmholtz'schen Erhaltungssatzes für die Vortizität müssen sie formal der Transportgleichung

$$\partial_t \chi_\pm + \nabla \cdot (\chi_\pm u) = 0$$

genügen. Auf diese Weise wird die Evolution als freies Randwertproblem interpretiert.

Es ist dann ein natürliches Vorgehen, zunächst zu den Variablen $\chi_\pm$ und $m_\pm = \chi_\pm u$ überzugehen, sodass die Transportgleichung linear wird,

$$\partial_t \chi_\pm + \nabla \cdot m_\pm = 0;$$

und den so beschriebenen Konfigurationsraum abzuschließen unter schwacher Konvergenz: Man geht über zum größeren Konfigurationsraum aller $s_\pm$, $m_\pm$, wobei nun $s_\pm \in [0, 1]$.

In diesem Raum wird folgendes Variationsproblem betrachtet,

$$\iint s_+ \left| \frac{m_+}{s_+} \right|^2 dxdt + \iint s_- \left| \frac{m_-}{s_-} \right|^2 dxdt \;\to\; \min . \tag{3}$$

Der Integrand $s_\pm \left| \frac{m_\pm}{s_\pm} \right|^2$ kann als konvexe Einhüllende der ursprünglichen Energie verstanden werden. Tatsächlich stimmt er überein mit $\chi_\pm |u_\pm|^2$, wann immer $s_\pm = \chi_\pm$ charakteristische Funktionen sind, und man hat die Unterhalbstetigkeit unter der schwachen Konvergenz, die zugrundegelegt wurde.

Ein solcher Relaxationsprozess ist dann vollständig befriedigend, wenn man rechtfertigen kann, dass für Folgen im Raum der $\chi_\pm$, $m_\pm$, der Grenzübergang von $\chi_\pm |u_\pm|^2$ zu $s_\pm \left| \frac{m_\pm}{s_\pm} \right|^2$ möglich ist. Wie unsere Diskussion anzudeuten versucht, ist eine solche Beziehung eine vernünftige Hypothese auf der Ebene des Variationsproblems, aber es ist nicht ganz offensichtlich, inwiefern sie eine fluidmechanische Interpretation besitzt.

Unabhängig davon hat das relaxierte Modell eine Berechtigung in sich, und soll im Weiteren untersucht werden. Wir möchten uns in dieser Arbeit mit ihm beschäftigen unter dem Gesichtspunkt seiner Geometrie als geodätischer Fluss. Tatsächlich lässt sich die Gleichung (1), (2) auffassen als Hamilton'sche Gleichung, die aus dem Variationsprinzip (3) hervorgeht.
Die zugrundeliegende Geometrie wird analog zu den Resultaten von Arnold für die Euler–Gleichung charakterisiert durch eine Riemannsche Mannigfaltigkeit; und der zugehörige Krümmungstensor wird bestimmt. Eine sorgfältige Abschätzung desselben produziert eine Instabilitätsanalyse, die die bekannte Instabilität dieses relaxierten Modells herausarbeitet. Unser Resultat reproduziert in gewisser Weise auch die Überlegungen von Kelvin und Helmholtz zur Instabilität des vortex sheet, ist aber eigenständig, und erlaubt den Vergleich von Eulergleichung und relaxiertem Problem.

Die Arbeit enthält ferner Bemerkungen zu speziellen 1–dimensionalen Lösungen der relaxierten Gleichung, sowie auch zu Entropien.

Contents

1 Introduction: Homogenized Equations

We discuss in this work a certain two–phase model for the motion of a perfect fluid. It stands in a loose relation to fluid–dynamical instability in a more general sense of the term. The resulting equations are in some sense canonical, and hence different ways to justify them seem possible. We would like to take the approach to introduce them as a possible homogenized but kinetic description of the motion that follows a vortex sheet configuration. This attempt is indeed instructive, but we will have to consider such a description to be problematic and to leave open questions. Therefore we will only try to give a sketch of such a derivation in simple terms.

Consider, let us say, a two–dimensional domain, $x \in \mathbb{R}/\mathbb{Z}$, $y \in \mathbb{R}$, and let $u_0(x, y) = (\pm 1, 0)$ be the discontinuous shear flow with x–component $+1$ for $y > 0$ and -1 for $y < 0$. This configuration is the simplest case of a *vortex sheet*: its tangential component has a jump across the line $\{y = 0\}$. It is a stationary solution of the Euler equation

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \tag{4}$$

$$\nabla \cdot u = 0. \tag{5}$$

Similar situations in dimension $d > 2$ can of course be considered. Let u_ε be a perturbation, $\varepsilon \to 0$, consisting of smooth divergence–free fields so that $\|u_\varepsilon - u_0\| \to 0$ in $L^2(dxdy)$. Here, one may assume periodicity on a small scale, probabilistic data may be introduced, or a small viscosity, but since we keep this discussion informal, we would like to leave these details open.

Let then $u_\varepsilon(t)$ be the solution of the Euler equation, which exists for all time in the present case of dimension 2, and may be assumed to exist otherwise. Since one has that $\int |u_\varepsilon|^2 dxdy$ is bounded uniformly, there is a sub–sequence with a weak L^2–limit $u_\infty(t)$. It is the consequence of the instability of the configuration u_0, that in general, this limit is not a strong limit, and that in general moreover, $u_\infty \neq u_0$. We may say that the Euler equation with datum u_0 is *ill–posed*, in the sense of an L^2–topology (even in a strong–weak sense). This ill–posedness is known since Kelvin and Helmholtz, although they considered merely the free boundary value problem which arises if the velocity is required to be always of the vortex sheet type. It is a different question, whether nevertheless, u_∞ will be a solution to the Euler equation (4), (5). This has actually shown to be true in dimension $d = 2$, [8], and is probably false if $d > 2$.

The question that leads to the model that we would like to investigate may be asked as follows: Is it possible to understand this limit process by introducing a *material partition*: i. e. a pair of functions $\chi^\varepsilon_\pm(t)$, valued in $\{0, 1\}$ and with sum 1 for all (t, x, y), so that

$$\partial_t \chi^\varepsilon_\pm + \nabla \cdot (\chi^\varepsilon_\pm u_\varepsilon) = 0 \tag{6}$$

holds, the transport along the solution of the Euler equation. The property (6) is consistent in view of (5). This is motivated as follows: As long as u_ε is in the class of solutions of the vortex sheet type, the vorticity will be concentrated on a line at every instant in time, and the evolution can be understood as a free boundary value problem. Then $\chi^\varepsilon_\pm$ should be chosen so that u_ε is curl–free in the support of both χ^ε_+ and χ^ε_- (see also our Remark 1). In this case, (6) is formally a consequence of the conservation of vorticity. A similar two–phase problem, which is however different in character, has been investigated in [17].

One then always has the weak limits of the measures

$$\chi^\varepsilon_\pm \rightharpoonup s_\pm, \tag{7}$$

$$\chi^\varepsilon_\pm \, u_\varepsilon \rightharpoonup m_\pm, \tag{8}$$

$$\chi^\varepsilon_\pm \, (u_\varepsilon \otimes u_\varepsilon) \rightharpoonup T_\pm. \tag{9}$$

Here, $s_\pm$ is a pair of functions, valued in $[0,1]$ and with sum 1. The question may be asked more precisely: Is it possible to choose the material partition in such a way that in this limit not too much information is lost, so that a consistent description results? We mean that

$$\begin{aligned} & m_\pm = s_\pm u_\pm \\ \Longrightarrow \quad & T_\pm = s_\pm \, (u_\pm \otimes u_\pm). \end{aligned} \tag{10}$$

This amounts to the information that by means of such a partition, one may understand the Reynolds tensor

$$- \operatorname{wlim} \, u_\varepsilon \otimes \operatorname{wlim} \, u_\varepsilon \, + \, \operatorname{wlim} \, (u_\varepsilon \otimes u_\varepsilon). \tag{11}$$

We emphasize that the condition (10) can be characterized simply in terms of the kinetic energy, without further knowledge of curl u_ε.

If such a description is possible, it leads to a closed equation which is a two–phase model of fluid motion. It may be described both in terms of a partial differential equation, and of a variational problem. We first give this equation:

$$\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0, \tag{12}$$

$$\partial_t (s_\pm u_\pm) + \nabla \cdot (s_\pm u_\pm \otimes u_\pm) + s_\pm \nabla p = 0. \tag{13}$$

Here, the pressure p is common to both phases, and is self–consistently defined in such a way as to assure $s_+ + s_- = 1$. Hence apparently, in such a limit process, compactness of the pressure gradients $\nabla p_\varepsilon = -\partial_t u_\varepsilon - \nabla \cdot (u_\varepsilon \otimes u_\varepsilon)$ would be needed in a strong topology (say L^1). In this case, (12), (13) are indeed the limit of (4) and (6), in view of (7) – (9) and (10).

This equation still has variational structure, and can in fact be understood as the optimality equation for the variational problem which follows,

Definition 1. *(Relaxed variational problem)*
We denote

$$I = \inf \left\{ \int \iint s_+ |u_+|^2 + s_- |u_-|^2 \, dxdy \, dt \right\} \tag{14}$$

where the infimum is taken over $s_\pm(t)$, $u_\pm(t)$ *so that*

$$\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0, \;\; s_+ + s_- = 1.$$

It may be regarded as the limit of the variational problem which gives rise to the Euler equation on the level of the u_ε,

$$\inf \left\{ \int \iint |u|^2 dxdy \, dt \right\}, \tag{15}$$

because indeed, according to (9), (10), where the limit of the second moments of u_ε was described, one has

$$\int \iint \chi_\pm^\varepsilon \, |u_\varepsilon|^2 \, dxdy \, dt \; \rightarrow \; \int \iint s_\pm \, |u_\pm|^2 \, dxdy \, dt.$$

The variational problem in Definition 1 has to be equipped with time boundary data. Here, different choices seem possible. In particular, one may choose as boundary data the pair $s_\pm$ with sum 1, to be prescribed at say $t = 0$ and $t = 1$.
As is shown in [4], minimizing I in this way, one can produce solutions to (12), (13). These are special in that the velocities $u_\pm$ are curl–free: they satisfy $u_\pm = \nabla \phi_\pm$, at least in the case of positive densities $s_\pm$. We mention here immediately that the solutions to (12), (13) with positive densities $s_\pm$ and curl–free $u_\pm = \nabla \phi_\pm$ enjoy stronger properties, [4]: they are always equal to the unique minimizer of I with respect to their endpoints, and also they are stable with respect to their boundary data.

If one has thus constructed a homogenized solution, the following reverse question makes sense, which we leave as a question. A good answer will clarify the amount of information which is contained in the homogenized quantities $s_\pm$, $u_\pm$.
The variational problem I gives rise to a distance in the space of pairs $s_\pm$. One may ask whether there is a *relaxation gap*: Can the infimum in (14) (which is attained) be realized by smooth fluid flow? More precisely, we try to formulate in the simplest possible way

Question 1. *Given a smooth trajectory* $s_\pm(t)$, $u_\pm(t)$, *which satisfies* $s_+ + s_- = 1$ *and*

$$\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0,$$

is there a sequence of smooth velocities, $u_\varepsilon(t)$*, with*

$$\nabla \cdot u_\varepsilon = 0,$$

and of pairs of $\{0,1\}$*–valued functions with sum* 1*,* $\chi^\varepsilon_\pm(t)$*, so that*

$$\partial_t \chi^\varepsilon_\pm + \nabla \cdot (\chi^\varepsilon_\pm \, u_\varepsilon) = 0, \tag{16}$$

and it holds true that

$$\chi^\varepsilon_\pm(t) \rightharpoonup s_\pm(t), \tag{17}$$

$$\limsup_{\varepsilon \to 0} \int \iint |u_\varepsilon|^2 \, dxdy \, dt \;\le\; \int \iint s_+|u_+|^2 + s_-|u_-|^2 \, dxdy \, dt. \tag{18}$$

That is, the smooth flow should join in the limit the same data $s_\pm(0)$ *and* $s_\pm(1)$*, and have not larger action.*

It should be reasonably added that

$$\chi^\varepsilon_\pm u^\varepsilon \rightharpoonup s_\pm u_\pm, \tag{19}$$

although it is not needed to formulate the question.

If one believes that, justifiably, the relaxed infimum I in (14) is already described by the infimum over smooth curves $s_\pm(t)$, $u_\pm(t)$, a positive answer to Question 1 indeed means that the relaxed distance problem encodes the behaviour of minimizing sequences for the original action (15) – in the class of smooth flows u, given the initial and final position of the material partition which is then subject to (16). These sequences are in general quite complicated, since the homogenized quantities $s_\pm$, $u_\pm$ are to be understood in the sense of a Young measure: the construction to answer Question 1 should consequently use fine laminates where the velocity oscillates between u_+ and u_-.

We remark finally, that the interpretation of the homogenized solutions in the sense of Question 1 can be extended to any curve $s_\pm(t)$, $u_\pm(t)$, in particular to any solution to (12), (13), rather than just to minimizers of I. Indeed, when in Question 1, we require in addition (19), it becomes in fact equivalent to the same question, but where (18) is replaced by (9), (10). Hence the focus is then both narrowed and enlarged, from the study of a relaxation gap to the mere realization of a Young measure. A good answer may eventually constitute a mechanical interpretation of such a Young measure in a simple case, a question that was raised in [19], see also [21].

1.1 Discussion: Variational formulation, consistency

We have hence seen where the merit of the homogenized solutions lies, be they applied to a perfect fluid, directly as a relaxed model for a vortex sheet

or possibly more generally to a description of its instability, or possibly even to other related situations, involving more explicitly two distinct phases.
Notice also that the study of the variational problem is closely linked to the study of the homogenized equations, in that it provides existence, and in some sense also uniqueness. More importantly still, so far for us, it is the only way to express that the interpretation as smooth flow according to Question 1 of a homogenized solution, would actually be related to the Euler equation (4), (5). This last point seems to be less problematic in a framework of the Euler equations as a differential inclusion, [21].
Going now back to our original motivation, the consideration of the stratified sheet u_0, we would like to use this subsection to resume that several points in the correspondence of homogenized solutions and smooth fluid flow remain problematic. Also, we comment on the nature of the variational solutions.

We would like to remark here that a limit of incompressible flows in the form (9), (10) is likely to be restricted to a certain regime. It should be noted for example that under the condition (10), no energy can be lost in a concentration process, but all information is contained in a kind of oscillatory pattern, see [9]. Also, we notice that it may be considered as the content of (10), that the Young–measure $\mu_{(x,y)}(du)$, associated to the sequence u_ε, is only concentrated on two velocities, $u_+(x,y)$ with probability $s_+(x,y)$, and $u_-(x,y)$ with probability $s_-(x,y)$. It can then be easily seen that in this case, the tensor in (11) must have rank 1. This is in contrast to the result of [8], which states that in the case of signed vorticity, and dimension $d = 2$, this Reynolds tensor must be isotropic.
It may be for this reason, that in [10], such a two–phase model was only judged to be a first step in this kind of study of fluid dynamic instability. To be more flexible in this respect, one may extend the model to allow the Young–measure to have more than two atoms, as was already suggested in [4].
Remarkably, the solutions to (12), (13) with positive densities and curl–free velocities are in fact minimal in the larger class of pairs of generalized flow, given their endpoints, [4].

Let us emphasize, that following this line of thought leads to a contrast to the original deliberation of homogenizing a free boundary value problem. As we will mention below, such a point of view is indeed justified on the level of the variational problem; one should then *a priori* only work with (7) – (9), without (10).
Furthermore, it may be worth pointing out in this context, that the special solution which we discuss in Problem 1 below actually uses velocities in the normal, the y–direction. One could then even try to make a more concise point, again in the context of the dynamic equation: that such a situation of a 'saturated instability' be on the one hand typical because of the very unstable nature of the vortex sheet configuration, and that on the other

hand, this situation be the only reasonable scenario where actually, the Young–measure would be two–atomic.

We turn now to the discussion of the class of variational solutions. It can be remarked that the choice of a time–boundary datum changes the nature of the problem, and the choice of a space boundary condition also changes the nature of the problem.

Let us first comment, that already in the unrelaxed formulation, the relation of the variational problem to the motion of an incompressible fluid is not quite clear, although as a matter of fact, the correct equations (4), (5), (6) are produced. Indeed, minimizing the action (15), given the material partition at the time boundary, plays a yet somewhat unspecified role, and has rather the character of a transport problem: Neither does it mean to prescribe the vorticity at the time boundary, which would lead to an over–determined problem, nor is it identical to the classical problem of Euler–Arnold.

We underline however that this point may supportably be ignored, and the variational problem may be isolated: One may investigate the question whether there is a relaxation gap in the passage from (15) to (14) in its own right, as it is expressed by Question 1. From this point of view, we may understand (14) as the natural relaxation of the free boundary value problem (15), (16), because the action functional in (14) is seen to be convex, and one has the property of lower semi–continuity as stated in Lemma 2.

The following can then be said about the relation to the incompressible fluid, *a posteriori*. The choice of a solution periodic in space, of the form

$$u = \chi_+ \nabla\phi_+ + \chi_- \nabla\phi_-, \text{ with } \nabla \cdot u = 0, \tag{20}$$

is restrictive in that it only allows for *modulated* vortex sheets rather than a configuration like the stratified sheet u_0. We mean that curl u is concentrated on a line but changes sign. It was mentioned in [8] that a configuration of this form is probably even more unstable than u_0, for which the curl is signed. It seems fine to consider solutions of the form (20), then maybe rather as an intrinsic instability of the fluid. The homogenized velocities which arise as the limit of solutions of the form (20) need not automatically be themselves curl–free.

Also, the choice of a material partition as time boundary datum restricts the solution – in particular, the stratified velocity field u_0 would not occur because the solution has no reason to move at all if the material partition does not change. Let us elaborate on this remark: The homogenized solution which minimizes I subject to given material partition, was seen to be formally a particular solution of (12), (13) of the form $u_\pm = \nabla\phi_\pm$. It is then still to be interpreted in the sense that it produces at least some sequence which belongs to the original variational problem (15), according to Question 1 – if ever the question has a positive answer, this recovering sequence should then be taken to be of the form (20), up to a small error in L^2, see our Remark 3. Hence although in Question 1, the material partition is apparently only introduced to verify the time boundary data, it is in this reconstruction, that the original idea reappears that it be the purpose of such a material partition to separate two regions of curl–free flow.

We suggest finally that it may be reasonable to discuss further the boundary conditions for the variational problem, to adapt it to more precise situations, such as the stratified sheet u_0. Two possible extensions of the variational problem I seem appropriate in this context: On the one hand, one may prescribe a boundary condition in space, such as $u_\pm = (\pm 1, 0)$ at $y = \pm\infty$. On the other hand, the time boundary condition may be more precisely specified to be a transport plan, allowing two final positions for each particle, rather than just the two densities $s_\pm$. Such a suggestion is inspired by our discussion in Section 2.
The existence of minimizers is in both cases unknown. In the first case, this is a question in its own right because such boundary condition in space may or may not change the nature of the variational problem. In the second case, we see two questions: Firstly, is the variational problem consistent in this case, that is will the infimum be attained in the class of two–flows? Secondly, in a limit process as in our motivation, (7) – (9), would one have enough information about the transport plan at some positive time?

1.2 Contribution of the present work

In the present work, we will not try to answer Question 1. We refer to [19] and [1], where similar questions were discussed. It seems that the answer is positive in dimension $d \geq 3$, and unclear if $d = 2$. Moreover, we concentrate on the equation (12), (13), rather than to contribute to the study of the variational problem. We are interested more precisely in the unstable nature of equation (12), (13): It can be shown that is not of hyperbolic character, and in fact leads still to an *ill–posed Cauchy–problem* – following Hadamard, we indicate by this term that the solution for data $s_\pm$, $u_\pm$ at a time $t = 0$ cannot be determined in a stable way. It is in this sense that the model even in itself may be considered as yet somewhat unsatisfactory.
It seems worth to try and understand the nature of this instability, also because the equations themselves are of a canonical form.

In Section 2, we give the formal relation of the variational problem with the equation (12), (13). In Section 3, it is shown that the system (12), (13) is of a character elliptic in space–time, at least in a special class of solutions in one space variable. This shows the ill–posed stability property in a special case, and means also that it is in fact favourable to consider the variational problem I in this situation.
We then proceed to describe in Section 4 an interpretation of the relaxed

distance problem as a geodesic problem on a formal Riemannian manifold, following the work of Arnold et al. for the Euler flow. The outcome is the identification of a Riemannian curvature tensor, whose sectional curvatures describe in a way that is more related to the original metric setting (distance in action) the instability of both the original problem (4), (5), and the relaxed problem (12), (13). We devote the major part of this work, Section 6 and 7, to an analysis of this object, which produces an expression of the instability in terms of a pair of variational fields along a given smooth pair of velocity fields. We give a rigorous estimate for this instability. It makes discernible the elliptic character, for two representative choices of the phase functions $s_\pm$.
The same tensor is capable of describing both the sharp and the relaxed situation. In a certain sense, we show also that it is already well–understood when computed for smooth $s_\pm$, as is expressed by the continuity property in Sections 6.5, 7.2.

As an additional remark, in Section 5, an entropy identity is noted as a formal identity. It shows that a mixing entropy is displacement convex along smooth solutions. A similar remark was made in [10]. We do not enter here seriously the discussion of this fact.

Finally, one can clarify some basic behaviour of the system through the study of special solutions, notably in one space dimension. Also here, we make only minor contributions.

2 Derivation of the geodesic equations from the principle of least action

We give now, to be explicit, a formal derivation of the geodesic equations from the variational problem. We do this in four versions: we distinguish a sharp interface, this is, a sharp material partition, as in Proposition 1, and a relaxed material partition, that means, two interpenetrating phases, as in Proposition 2. And moreover we distinguish a general pair of velocities for the two phases, as in point i), and the case of potential velocity fields, as in point ii).
Let in this section $x \in \mathbb{T}^d$ denote a general space variable. We obtain the following two propositions.

Proposition 1. *(Description of a vortex sheet)*
i) The system

$$\partial_t \chi_\pm + \nabla \cdot (\chi_\pm u_\pm) = 0, \tag{21}$$

$$\partial_t u_\pm + D u_\pm u_\pm + \nabla p = 0 \tag{22}$$

describes the stationary points of the variational problem

$$A = \int_0^T \int \chi_+ |u_+|^2 + \chi_- |u_-|^2 dx\, dt \longrightarrow \min,$$

where always, $\partial_t \chi_\pm + \nabla \cdot (\chi_\pm u_\pm) = 0$, and $\chi_\pm \in \{0, 1\}$ with $\chi_+ + \chi_- = 1$, and where the endpoints of the curve $t \mapsto (\Phi_+(t), \Phi_-(t))$ are prescribed. Here, the diffeomorphism $\Phi_\pm(t, x)$ integrates the velocity: $\partial_t \Phi_\pm = u_\pm \circ \Phi_\pm$.
ii) The system

$$\partial_t \chi_\pm + \nabla \cdot (\chi_\pm \nabla \phi_\pm) = 0, \tag{23}$$

$$\partial_t \phi_\pm + \frac{1}{2} |\nabla \phi_\pm|^2 + p = 0 \tag{24}$$

describes the stationary points of the same variational problem

$$A = \int_0^T \int \chi_+ |u_+|^2 + \chi_- |u_-|^2 dx dt \longrightarrow \min,$$

if only the endpoints of the curve $t \mapsto (\chi_+(t), \chi_-(t))$ are prescribed.

Remark 1. *i) This system describes still solutions to the Euler equation in weak form: If $u = \chi_+ u_+ + \chi_- u_-$, then since $\chi_\pm$ are characteristic functions, one has $u \otimes u = \chi_+ u_+ \otimes u_+ + \chi_- u_- \otimes u_-$, and it follows that in fact*

$$\nabla \cdot u = 0, \quad \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0.$$

ii) In its potential version (23), (24), this system describes the same solutions as the so–called Birkhoff–Rott equation, where the velocity is recovered

from a vorticity concentrated solely on an interface, through a principal value integral. We refer to the book of Marchioro and Pulvirenti [15], where details of this relation are given.
One may say that the Birkhoff–Rott solutions are rare, because they require a smooth (analytic) interface. Moreover, they would in general exist only for short time.
iii) We consider the property of stationarity formal, because we believe that the infimum generically is not attained in the class of $\chi_\pm$.
iv) The pressure is common to both phases, and satisfies an elliptic equation

$$\nabla \cdot \nabla \cdot (\chi_+ u_+ \otimes u_+ + \chi_- u_- \otimes u_-) + \Delta p = 0.$$

Proposition 2. *(Homogenized Vortex Sheet Equations)*
i) The system

$$\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0, \tag{25}$$
$$\partial_t u_\pm + D u_\pm u_\pm + \nabla p = 0 \tag{26}$$

describes the stationary points of the variational problem

$$A = \int_0^T \int s_+ |u_+|^2 + s_- |u_-|^2 dx\, dt \longrightarrow \min,$$

where always, $\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0$, and $s_\pm \in [0,1]$ with $s_+ + s_- = 1$, and where the endpoints of the curve $t \to (\Phi_+(t), \Phi_-(t))$ are prescribed. Again, the diffeomorphisms $\Phi_\pm(t)$ integrate the two velocities: $\partial_t \Phi_\pm = u_\pm \circ \Phi_\pm$.
ii) The system

$$\partial_t s_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0, \tag{27}$$
$$\partial_t \phi_\pm + \frac{1}{2} |\nabla \phi_\pm|^2 + p = 0 \tag{28}$$

describes the stationary points of the same variational problem

$$A = \int_0^T \int s_+ |u_+|^2 + s_- |u_-|^2 dx\, dt \longrightarrow \min,$$

if only the endpoints of the curve $t \mapsto (s_+(t), s_-(t))$ are prescribed.

Remark 2. *i) The sum $s_+ u_+ + s_- u_- = j$ describes the mean flux of particles. It is not anymore in general a solution to the Euler equations.*
ii) It is in the potential version (27), (28), that Brenier [4] obtains an existence result for the Homogenized Vortex Sheet equations (HVSE): An argument based on general convexity and duality produces a so–concipied (and slightly more precisely defined) variational solution, for any data $s_\pm \in L^\infty$ given at $t = 0$, $t = T$.

iii) We interpret these HVSE as geodesic equations in a Riemannian context in Section 4.
iv) The pressure is common to both phases and satisfies an elliptic equation

$$\nabla \cdot \nabla \cdot (s_+ u_+ \otimes u_+ + s_- u_- \otimes u_-) + \Delta p = 0. \tag{29}$$

Argument. To prove the two propositions, we rewrite the variational problem in Lagrangian form, so that it reads

$$\int_0^T \int \chi_+(0,x)|\partial_t \Phi_+(t,x)|^2 + \chi_-|\partial_t \Phi_-(t,x)|^2 dxdt,$$

and

$$\int_0^T \int s_+(0,x)|\partial_t \Phi_+(t,x)|^2 + s_-(0,x)|\partial_t \Phi_-(t,x)|^2 dxdt,$$

respectively.
We obtain a formula for the first variation along a variational field $(\xi_+(t), \xi_-(t))$: If $\Phi_\pm(t,\varepsilon)$ is a family of curves so that $\frac{\partial \Phi_\pm}{\partial \varepsilon} = \xi_\pm \circ \Phi_\pm$, then

$$\frac{\partial}{\partial \varepsilon} A = \int_0^T \int \chi_+ u_+ \cdot (\partial_t \xi_+ + D\xi_+ u_+) + \chi_- u_- \cdot (\partial_t \xi_- + D\xi_- u_-) dx\, dt,$$

and

$$\frac{\partial}{\partial \varepsilon} A = \int_0^T \int s_+ u_+ \cdot (\partial_t \xi_+ + D\xi_+ u_+) + s_- u_- \cdot (\partial_t \xi_- + D\xi_- u_-) dx\, dt,$$

respectively. Here, we transformed back to Eulerian variable, using $s_\pm(t) = \Phi_\pm(t)\# s_\pm(0)$.
We infer that if we require $\xi_\pm$ to vanish at the endpoints, then a stationary trajectory satisfies

$$\int_0^T \int \big\{\partial_t(s_+u_+) + \nabla\cdot(s_+u_+\otimes u_+)\big\}\cdot\xi_+ + \big\{\partial_t(s_-u_-) + \nabla\cdot(s_-u_-\otimes u_-)\big\}\cdot\xi_-\, dxdt = 0, \tag{30}$$

for all $\xi_\pm$ so that $\nabla \cdot (s_+\xi_+ + s_-\xi_-) = 0$. The same holds true if $s_\pm = \chi_\pm$, only the integral is to be read as a distributional pairing

$$\langle \partial_t(\chi_+u_+) + \nabla\cdot(\chi_+u_+\otimes u_+),\, \xi_+\rangle + \langle \partial_t(\chi_-u_-) + \nabla\cdot(\chi_-u_-\otimes u_-),\, \xi_-\rangle. \tag{31}$$

If the variation is not required to fix endpoints, we obtain in addition

$$\int s_+ u_+ \cdot \xi_+ + s_- u_- \cdot \xi_- dx = 0 \tag{32}$$

at $t = 0$, $t = T$. The same holds true for $\chi_\pm$.

It remains to deduce the systems in i), that means (25), (26), from (30), and (21), (22) from (31); and moreover to use (32) to interpret the systems in ii), (27), (28) and (23), (24). These latter are always a special case of the systems in i): if $\phi_\pm$ solve the system in ii), then $u_\pm = \nabla\phi_\pm$ solve the system in i). We assume here that $u_\pm$ are smooth and stationary in the sense of (30) and (31), respectively, and, as for the case of ii), additionally in the sense of (32). We achieve both aims by a consideration of the space of tangent vectors and its decompositions.

Let us first argue that (32), as valid for all $\xi_\pm$ so that both $\nabla\cdot(s_+\xi_+) = 0$, $\nabla\cdot(s_-\xi_-) = 0$, implies that $u_\pm$ are of the form $u_\pm = \nabla\phi_\pm$. This is indeed the consequence of a classical Helmholtz–decomposition. We may hence interpret the special cases ii): If $\xi_\pm$ are allowed to be non–zero at the time boundary, only subject to fixing $s_\pm$, we obtain $u_\pm = \nabla\phi_\pm$ as an optimality condition. This property in fact, for the minimizer, holds at every instant in time, because given a solution $s_\pm$, $u_\pm$, replacing $u_\pm$ by the projection,

$$-\partial_t s_\pm = \nabla\cdot(s_\pm u_\pm) = \nabla\cdot(s_\pm\nabla\phi_\pm),$$

produces the same curve $s_\pm(t)$, but with less action.

We are hence left with showing i), and we argue separately for the two propositions in this step. To prove the first proposition, two classes of variational fields are needed: If we use independently ξ_+ and ξ_- with $\nabla\cdot(\chi_\pm\xi_\pm) = 0$, we infer from

$$\langle\partial_t(\chi_\pm u_\pm) + \nabla\cdot(\chi_\pm u_\pm\otimes u_\pm), \xi_\pm\rangle = 0,$$

that the distribution on the left hand side equals a gradient, which must be smooth in $\mathrm{supp}\chi_\pm$, if $u_\pm$ are assumed to be smooth,

$$\partial_t(\chi_\pm u_\pm) + \nabla\cdot(\chi_\pm u_\pm\otimes u_\pm) + \chi_\pm\nabla p_\pm = 0.$$

This is guaranteed by the Helmholtz–decomposition in the domain $\mathrm{supp}\chi_\pm$. If secondly we use global smooth divergence–free test fields ξ, $\xi_\pm = \chi_\pm\xi$, we can infer, now by virtue of a Helmholtz–decomposition in $\mathbb{T}^d$, that the sum on the left hand side equals some distribution ∇p,

$$\partial_t(\chi_+u_+ + \chi_-u_-) + \nabla\cdot(\chi_+u_+\otimes u_+ + \chi_-u_-\otimes u_-) + \nabla p = 0. \tag{33}$$

But the comparison of the two results shows

$$\nabla p = \chi_+\nabla p_+ + \chi_-\nabla p_-,$$

whence in particular ∇p is integrable. This yields a continuity condition for p across the interface, and makes meaningful the assertion that p is common to both phases. We may also remark that the statement (33) means that $u = \chi_+u_+ + \chi_-u_-$ is a weak solution to the Euler equation.

We have shown that

$$\partial_t(\chi_\pm u_\pm) + \nabla \cdot (\chi_\pm u_\pm \otimes u_\pm) + \chi_\pm \nabla p = 0,$$

which translates to our claim (22), since u_+, u_- are assumed to be smooth in the support of $\chi_\pm$.

We now address the second proposition. We need again two classes of variational fields in this case. Let us assume here that both $s_\pm$ and $u_\pm$ are smooth functions, and $s_\pm > 0$. Then we use firstly independent fields $\xi_\pm$ with $\nabla \cdot (s_\pm \xi_\pm) = 0$, to infer via a Helmholtz–decomposition that there are gradients $\nabla p_\pm$ so that

$$\partial_t(s_\pm u_\pm) + \nabla \cdot (s_\pm u_\pm \otimes u_\pm) + s_\pm \nabla p_\pm = 0. \tag{34}$$

Secondly, and only for the relaxed model, we may construct variational fields so that $s_+\xi_+ + s_-\xi_- = 0$. Indeed since $s_\pm > 0$, we are free to arbitrarily choose $m = s_+\xi_+ = -s_-\xi_-$. Stationarity with respect to these fields implies

$$\frac{1}{s_+}\big(\partial_t(s_+u_+) + \nabla \cdot (s_+u_+ \otimes u_+)\big) = \frac{1}{s_-}\big(\partial_t(s_-u_-) + \nabla \cdot (s_-u_- \otimes u_-)\big). \tag{35}$$

We remark that in particular we have thus included the use of global divergence–free variational fields ξ, in the sense that $\xi_- = \xi_+ = \xi$, which produces the information that there is a gradient ∇p so that

$$\partial_t(s_+u_+ + s_-u_-) + \nabla \cdot (s_+u_+ \otimes u_+ + s_-u_- \otimes u_-) + \nabla p = 0. \tag{36}$$

It now follows, combining (34) with first (35) and then additionally with (36), that in fact

$$\nabla p_+ = \nabla p_- = \nabla p.$$

We conclude with the assertion, because by means of the formula

$$\nabla \cdot (su \otimes u) = \nabla \cdot (su)\, u + s\, Du\, u,$$

the equations obtained for the momenta,

$$\partial_t(s_\pm u_\pm) + \nabla \cdot (s_\pm u_\pm \otimes u_\pm) + s_\pm \nabla p = 0.$$

translate into equations for the velocities, (26). □

Let us finally make explicit the following simple observations.

Lemma 1. *(Kinetic energy of the mean flux)*
Let $j = s_+u_+ + s_-u_-$ be the flux. Then we have
i) $\int |j|^2 dx \le \int s_+|u_+|^2 + s_-|u_-|^2 dx$, with equality only if for every x, either $s_+(x)s_-(x) = 0$ or $u_+(x) = u_-(x)$.
ii) Likewise for the tensor product, $j \otimes j \le s_+u_+ \otimes u_+ + s_-u_- \otimes u_-$ in the sense of symmetric matrices, with equality only if either $s_+s_- = 0$ or $u_+ = u_-$.

Proof. We have $|j|^2 = |s_+u_+ + s_-u_-|^2 = s_+^2|u_+|^2 + 2s_+s_-u_+\cdot u_- + s_-^2|u_-|^2$. By the Cauchy-Schwarz inequality, $2u_+\cdot u_- \le |u_+|^2 + |u_-|^2$, with equality only if $u_+ = u_-$. Hence we have $|j|^2 \le s_+^2|u_+|^2 + s_+s_-|u_+|^2 + s_+s_-|u_-|^2 + s_-^2|u_-|^2$, with equality only if either $s_+s_- = 0$ or $u_+ = u_-$. But since $s_+ + s_- = 1$, we have $s_+^2|u_+|^2 + s_+s_-|u_+|^2 + s_+s_-|u_-|^2 + s_-^2|u_-|^2 = s_+(s_+ + s_-)|u_+|^2 + s_-(s_+ + s_-)|u_-|^2 = s_+|u_+|^2 + s_-|u_-|^2$. This shows i) after an integration, and the proof of ii) is similar, if one considers $j\cdot\xi$ for an arbitrary $\xi \in \mathbb{R}^d$. $\square$

Lemma 2. *(Lower semi–continuity)*
Let $\chi_\pm^\varepsilon$, u^ε be a sequence of velocity fields, together with a material partition. Assume there are $s_\pm$, $u_\pm$, so that

$$\chi_\pm^\varepsilon \rightharpoonup s_\pm, \quad \chi_\pm^\varepsilon u^\varepsilon \rightharpoonup s_\pm u_\pm \quad \textit{weakly.}$$

Then it holds

$$\int s_+|u_+|^2 + s_-|u_-|^2 dx \;\le\; \liminf_{\varepsilon\to 0} \int |u^\varepsilon|^2 dx.$$

Proof. We have that the relaxed action is convex, as stated in Proposition 5. The lower semi–continuity then follows with a straightforward argument. $\square$

Remark 3. *(Property of the Young–measure)*
Consider a sequence of velocity fields with material partition, so that the limit

$$\begin{aligned} \chi_\pm^\varepsilon &\rightharpoonup s_\pm, \\ \chi_\pm^\varepsilon u_\varepsilon &\rightharpoonup s_\pm u_\pm, \\ \chi_\pm^\varepsilon |u_\varepsilon|^2 &\rightharpoonup s_\pm |u_\pm|^2 \end{aligned}$$

exists. Then one has indeed, up to an error term in $L^2(dx)$, that

$$u_\varepsilon \;=\; \chi_+^\varepsilon u_+ + \chi_-^\varepsilon u_- + o(1).$$

Proof. It is straightforward, expanding the square, that by the assumptions,

$$\int |u_\varepsilon - \chi_+^\varepsilon u_+ - \chi_-^\varepsilon u_-|^2\,dx \;=\; \int \chi_+^\varepsilon |u_\varepsilon - u_+|^2 dx \;+\; \int \chi_-^\varepsilon |u_\varepsilon - u_-|^2 dx$$

must converge to zero. $\square$

3 Linear stability analysis, ill–posedness of the Cauchy–problem, elliptic character

It is the first main insight that the relaxed model (25), (26) is in itself subject to instability and in fact leads to an ill–posed Cauchy–problem.

It can be understood that we obtained the relaxed formulation as the convex envelope of the action minimization problem. The relaxation then is to be interpreted in the sense that now two individual phases of fluid are allowed to interpenetrate. As we have seen, the relaxed model allows for a larger class of tangent vectors, describing also the mixing of the phases. It can be said that it is this class of mixing variations that makes up a one–dimensional class of solutions. It can be easily seen that the relaxed model is unstable in this class, in the sense that it leads to an ill–posed Cauchy problem.
Nevertheless, solutions can be produced by means of the variational problem, and summarizing, it can be said that the model is of a character elliptic in space–time. The purpose of this section is to make this fact apparent in the simplest possible way.

3.1 Homogenized Vortex Sheet Equations in one spatial variable

We would like to introduce briefly a special solution of the HVSE which only depends on one spatial variable. More precisely we discuss the following problem which is of some interest of its own, as a model problem as well as, possibly, as a building block for more general constructions. It motivates the study of the equations of 1–dimensional HVSE.

Problem 1. *(Solution on two unit cubes)*
Consider two spatial variables $-1 \leq y \leq 1$ *and* $0 \leq x \leq 1$*, and a time variable* $-1 \leq t \leq 1$*. Let* $\chi(x, y) = \chi(y) = 1_{\{y \leq 0\}}$*. We ask for the solution of the variational problem I with* (s_+, s_-) *equal to* $(\chi, 1-\chi)$ *at* $t = -1$*, and equal to* $(1-\chi, \chi)$ *at* $t = 1$*.*

We will take for granted

Proposition 3. *(after Brenier)*
There is a unique minimizer $s_\pm(t, x, y)$*,* $u_\pm(t, x, y)$*, solution to* (27)*,* (28)*.*

We refer also to [5], where this solution is characterized as the solution of some degenerate elliptic problem.

We would like to collect the simplest properties of this solution. We emphasize that in particular, the solution is a mixture.

Lemma 3. *(Symmetries)*
i) $s_\pm = s_\pm(t, y)$ *and* $u_\pm = u_\pm(t, y)$ *do not depend on* x*. Moreover,* $u_\pm =$

$(0, u_y^\pm)$ has only the normal component.
ii) One has the symmetries

$$s_+(-t, y) = s_-(t, y) = s_+(t, -y).$$

In particular, $s_+(0, y) = \frac{1}{2} = s_+(t, 0)$.
iii) The flux $j = s_+u_+ + s_-u_-$ vanishes: $j = 0$.
iv) The pressure can be chosen as an even function, $p(-t, y) = p(t, y) = p(t, -y)$, equal to $p = u_+u_-$.

Proof. *ad i).* Introduce the momenta $m_\pm = s_\pm u_\pm$, and consider the averaged quantities $\bar{s}_\pm(t, y) = \int s_\pm(t, x, y)dx$, $\bar{m}_\pm(t, y) = \int m_\pm(t, x, y)dx$. Then still the constraint is satisfied: $\partial_t s_\pm + \nabla \cdot m_\pm = 0$ implies $\partial_t \bar{s}_\pm + \partial_y \bar{m} = 0$; and of course $\bar{s}_+ + \bar{s}_- = 1$. But the averaged trajectory has less action: by convexity,

$$\int \bar{s}_\pm |\frac{\bar{m}_\pm}{\bar{s}_\pm}|^2 dy \le \int s_\pm |\frac{m_\pm}{s_\pm}|^2 dxdy.$$

Since the averaging respects also the time boundary datum, it follows that already $s_\pm = \bar{s}_\pm$. It follows then that $\partial_t s_\pm = -\partial_y(s_\pm u_y^\pm)$ does not depend on $u_x^\pm$, whence one must have $u_x^\pm = 0$ so that the energy $\int s_\pm(|u_x^\pm|^2 + |u_y^\pm|^2)dy$ is least.
ad ii). This is true since $(s_-(-t, y), s_+(-t, y))$ and $(s_-(t, -y), s_+(t, -y))$ match the same boundary data as $(s_+(t, y), s_-(t, y))$. As together with the according velocities $(-u_-, -u_+)$, they produce the same action, they actually solve the same variational problem, hence by uniqueness coincide with (s_+, s_-).
ad iii). One has $s_+ + s_- = \chi + (1 - \chi)$ on the two cubes, in particular $\partial_t(s_+ + s_-) = 0$. Therefore,

$$0 = \partial_t(s_+ + s_-) = -\partial_y j$$

holds distributionally on $\mathbb{R}$, and implies $j = \mathit{const}$, hence $j = 0$.
ad iv). Since $j = 0$, one has that

$$-s_+u_+^2 - s_-u_-^2 = s_-u_-u_+ + s_+u_+u_- = u_+u_- \tag{37}$$

To see first the symmetry, observe that the symmetry of $s_\pm$ implies

$$-\partial_t s_+(-t, y) = \partial_t s_-(t, y) = \partial_t s_+(t, -y),$$

whence also

$$-\partial_y m_+(-t, y) = \partial_y m_-(t, y) = \partial_y m_+(t, -y).$$

Since again, this is an identity distributionally on $\mathbb{R}$, by the boundary condition at $\pm\infty$,

$$-m_+(-t, y) = m_-(t, y) = -m_+(t, -y).$$

Therefore,

$$-u_+(-t,y) = u_-(t,y) = -u_+(t,-y),$$

which implies

$$\partial_t u_+(-t,y) = \partial_t u_-(t,y) = -\partial_t u_+(t,-y),$$

and

$$-\partial_y u_+(-t,y) = \partial_y u_-(t,y) = \partial_y u_+(t,-y).$$

It follows that

$$\partial_y p(-t,y) = \partial_y p(t,y) = -\partial_y p(t,-y).$$

Now, by the optimality equation, namely (29), one has

$$\partial_{yy}(s_+ u_+^2 + s_- u_-^2 + p) = 0,$$

which according to (37) implies

$$\partial_y(p - u_+ u_-) = const.$$

But the constant function is odd in y and must in fact vanish, whence we obtain $p = u_+ u_-$, and

$$p(-t,y) = p(t,y) = p(t,-y),$$

as claimed. □

Remark 4. *i) This motivates the study of the PDE*

$$\partial_t s_\pm + \partial_y(s_\pm u_\pm) = 0, \tag{38}$$

$$\partial_t u_\pm + u_\pm \partial_y u_\pm + \partial_y(u_+ u_-) = 0. \tag{39}$$

It takes the form of a local conservation law and is the object of our discussion in the remaining paragraphs 3.2 and 3.3 of this section.
ii) One may investigate the behaviour at the singularity in $(t=-1, y=0)$. We will not do this in the present work, but it seems reasonable to conjecture a self–similar behaviour of the form

$$s(t,y) = \hat{s}\left(\frac{y}{(t+1)^{\frac{2}{3}}}\right), \quad u(t,y) = \frac{1}{(t+1)^{\frac{1}{3}}}\,\hat{u}\left(\frac{y}{(t+1)^{\frac{2}{3}}}\right).$$

Indeed, this scaling respects both the dimension of u as a velocity, and the fact that the kinetic energy $\int s_+ u_+^2 + s_- u_-^2 dy$ is constant in time. An example of such a solution has been given independently by Brenier and Duchon/Robert [10].

We make a humble contribution to the study of (38), (39), and give a special, homogeneous solution in Appendix 9.

3.2 Linear instability and ill–posedness

In this section, we report the result of a linear stability analysis of the partial differential equation (38), (39), subject to $s_+ + s_- = 1$, $s_+u_+ + s_-u_- = 0$. It takes the form of a local conservation law

$$\partial_t q + A(q)\,\partial_y q = 0, \tag{40}$$

where $q = (s, u) = (s_+, u_+)$ denotes the two effective variables in this constrained system. We show in Appendix 10,

Proposition 4. *The matrix A has genuinely complex eigenvalues, for every $1 > s > 0$, $u \neq 0$. Consequently, the conservation law (40) is ill–posed.*

Indeed, the linearized equation takes to leading order the form

$$\partial_t q + ik\lambda q = 0,$$

if λ is an eigenvalue of the matrix A and q an eigenvector. Moreover here, k denotes the Fourier–variable replacing y. If the eigenvalue $\lambda = \Re\lambda + i\Im\lambda$ is genuinely complex, then the system is said to be of an elliptic character. As a consequence, the perturbation is a travelling wave which is amplified exponentially as $\exp(k\ \Im\lambda\ t)$, hence with uncontrollably large rate.

This shows that the Cauchy–problem for (38), (39), hence also, in general, for (25), (26), is ill–posed. On the other hand, the variational problem with time boundary data has a solution, and it must even display an interior regularity, as is shown in [4].
In a sense, we reproduce this instablity analysis in Section 6, and generalize it to the nonlocal system (25), (26), on the level of a Riemannian curvature tensor.

3.3 Convexity of the action functional

The system (38), (39) is simpler because the mean flux $j = 0$. At least for this system, we show also in the appendix, that one gains an additional convexity of the kinetic energy density, with respect to the phase functions $s_\pm$. This fact is related to the elliptic character of the system in the following sense: If one writes $\partial_y h = s$, for a function $h(t, y)$ suitably chosen, one finds that $\partial_t h = -m = -su$. Hence the action functional takes the form $\int\int F(s, u, v)dydt = \int\int F(s, m)dydt = \int\int F(\partial_y h, \partial_t h)dydt$. Thus strict convexity of F corresponds to the elliptic property of the system. See also [5] for a similar observation.

We have

Proposition 5. *Consider for pairs* $s_\pm$, $m_\pm$ *so that* $s_+ + s_- = 1$, $m_+ + m_- = 0$, *the function*

$$F : (s_\pm, m_\pm) \mapsto s_+ |\frac{m_+}{s_+}|^2 + s_- |\frac{m_-}{s_-}|^2.$$

Then F *is convex, and strictly convex for* $s_\pm > 0$*:* $D^2 F > 0$*.*

A proof is shown in Appendix 10.

4 Riemannian interpretation

4.1 Preliminaries. Classical fluid mechanics

We would like to give, for convenience and in view in particular of the next paragraph 4.2, a short informal overview following Arnold et al., [2], [3], and summarize certain notions and notations to give a geometric interpretation of smooth fluid flow.

To this purpose, let D denote the set of diffeomorphisms of $\mathbb{T}^d$. Moreover, denote by P the space of densities on $\mathbb{T}^d$, and by μ the Lebesgue measure. Then by D_μ we denote the subset of all volume–preserving diffeomorphisms. Now D can be regarded as a Riemannian manifold, if equipped with a metric

$$\langle u, u\rangle = \int |u|^2 \rho \, d\mu, \tag{41}$$

for any $\rho \in P$ fixed, and for any vector field u, considered as a tangent vector in the identity to D. The metric is extended to TD as follows: if $\Phi \in D$, $u \circ \Phi$ the tangent vector, then the metric equals

$$\int |u \circ \Phi|^2 \rho \, d\mu \; = \int |u|^2 \, \Phi \# \rho \, d\mu, \tag{42}$$

where we use the notation $\Phi\#\rho = \det D\Phi^{-1} \rho \circ \Phi^{-1}$ for the push forward of the measure $\rho \, d\mu$. We stress that this structure is flat w.r.t. the underlying Euclidean space, $D \subset L^2(\mathbb{T}^d)$, but not right-invariant as w.r.t. the group structure of D.

Denote by D_ρ the set of all diffeomorphisms which preserve the measure $\rho \, d\mu$. It is regarded as a submanifold of D, with its metric

$$\int |u|^2 \rho \, d\mu,$$

which is right–invariant on D_μ since $\det D\Phi = 1$. Here, the field u is in the tangent space to the identity, $V_\rho = \{u, \nabla \cdot (\rho u) = 0\}$.

On the other hand we have a Riemannian submersion

$$\pi : D \longrightarrow P, \quad \Phi \mapsto \Phi\#\rho, \tag{43}$$

if P is equipped with the so–defined image metric, which induces the Wasserstein distance on the space of probability measures. The fibres of the submersion are sets of diffeomorphisms that preserve a given measure, and in particular,

$$\pi^{-1}(\rho) = D_\rho.$$

Notice that the tangent space in the identity, V, which contains all vector fields on $\mathbb{T}^d$, splits into the set V_ρ of vectors tangential to D_ρ, which we call vertical – they satisfy a constraint $\nabla \cdot (\rho\, u) = 0$, and the set of vectors normal to TD_ρ, which we call horizontal, and which can be expressed as a gradient $u = \nabla \eta$.

Then on a level of formal computation, we have the following notions. The embedding $D_\rho \hookrightarrow D$ gives rise to a *second fundamental* form $B(u, u) = \nabla p$, expressed by the formula $\nabla \cdot (\rho\, (Du\, u + \nabla p)) = 0$ for any field $u \in V_\rho$. It is the normal projection of the covariant derivative $Du\, u$. Notice that owing to $\nabla \cdot (\rho u) = 0$, we have $\rho\, Du\, u = \nabla \cdot (\rho\, u \otimes u)$, so that

$$\nabla \cdot (\nabla \cdot (\rho u \otimes u) + \rho \nabla p) = 0. \tag{44}$$

We give explicitly the equations of geodesics. In D, geodesics are characterized by the Burgers equation

$$\partial_t u + Du\, u = 0. \tag{45}$$

The normal bundle is an image (gradient of functions) and, in the solution space, it is invariant. This is the same as to say that we have an *equation of horizontal geodesics*, the Hamilton–Jacobi equation, or potential Burgers equation, on the level of η:

$$\partial_t \eta + \frac{1}{2} |\nabla \eta|^2 = 0. \tag{46}$$

Indeed one verifies easily that if η is a solution to (46), then $u = \nabla \eta$ solves (45). The horizontal equation (46) describes in fact also the geodesics in the image P of the submersion, in the sense that any path in D with velocity u projects to P by means of a transport equation

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$

and the equations of geodesics in P are given by

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \nabla \eta) &= 0, \\ \partial_t \eta + \frac{1}{2} |\nabla \eta|^2 &= 0. \end{aligned} \tag{47}$$

We refer here to the careful exposition in [16].

On the other hand, the *geodesics in* D_μ are given by the vertical projection of the covariant derivative,

$$\partial_t u + Du\, u + B(u, u) = 0.$$

Explicitly, this means the Euler equations

$$\begin{aligned} \partial_t u + Du\, u + \nabla p &= 0, \qquad (48) \\ \nabla \cdot (\rho u) &= 0, \end{aligned}$$

where the case $\rho = 1$ corresponds to the classical notion of incompressible fluid.

Finally, by means of the Gauss–formula, we are able to relate the exterior geometry of D_ρ in D to the *interior geometry of* D_ρ: Notice that D is flat as expressed by (45), so the sectional curvature of in plane spanned by normalized vectors $u,\ v \in V_\mu$ is given by

$$\begin{aligned} R(u,v) \ \equiv \ R(u,v,u,v) \ &= \ \langle B(u,u), B(v,v) \rangle \ - \ |B(u,v)|^2 \\ &= \int \nabla p_u \cdot \nabla p_v \, \rho \, d\mu - \int |\nabla q|^2 \, \rho \, d\mu. \end{aligned} \tag{49}$$

Here, p_u and p_v are defined as in (44), whereas q uses the polarization, explicitly

$$\nabla \cdot (\nabla \cdot (\rho u \otimes v) + \rho \nabla q) = 0,$$

or with a test function,

$$\int \rho \, \nabla q \cdot \nabla \eta \, d\mu \ = \ \int \rho \, u \otimes v : D^2 \eta \, d\mu, \tag{50}$$

see Lemma 4 below.

Notice on the other hand that according to Otto [16], translating a formula by O'Neill to the submersion onto P allows one to understand the interior geometry of P. Its curvature is non–negative, and the difference to the flat situation in D owes to the fact that the normal bundle is not integrable. Precisely, the vertical part of the commutator accounts for this phenomenon, and it is found that in any given point $\rho \in P$, the sectional curvature of a plane in TP spanned by normalized vector fields $\nabla\eta_1$,$\nabla\eta_2$ is given by

$$R(\eta_1, \eta_2) = \inf_\pi \Big\{ \int |\, [\nabla \eta_1, \nabla \eta_2] - \nabla \pi \,|^2 \, \rho \, d\mu \Big\}. \tag{51}$$

This expression is nonnegative and in general non–zero. So one can see that such a process of submersion can only produce more positive curvature in the base space than in the original space.

To conclude this introduction, let us include the following basic assertion. It can be interpreted as to say that TD_ρ is integrable in a way which is compatible with its structure as a subgroup, and in particular identifies the expression in (50) as the symmetrization of (44).

Lemma 4. *(Polarization identity)*
i) The expression $\nabla \cdot \nabla \cdot (\rho u \otimes v)$ *is symmetric in* u *and* v.
ii) Let $u,\ v$ *be vector fields which satisfy* $\nabla \cdot (\rho u) = 0$ *and* $\nabla \cdot (\rho v) = 0$. *Then the commutator again satisfies* $\nabla \cdot (\rho \, [u,v]) = 0$.

Proof. *Ad i).* This is clear because by symmetry of the second derivative $D^2\eta$, the expression

$$\int D^2\eta : u \otimes v \, \rho \, d\mu,$$

for any test function η, is symmetric in u and v. *Ad ii),* notice that

$$\nabla \cdot (\rho u \otimes v) = \nabla \cdot (\rho u) \, v + \rho \, Dv \, u. \tag{52}$$

So by the assumption on u and v,

$$\nabla \cdot (\rho u \otimes v - \rho v \otimes u) = \rho \, Dv \, u - \rho \, Du \, v = \rho \, [u, v].$$

Taking the divergence, we find that ii) follows from i). □

4.2 Homogenized Vortex Sheet Equations

We show now an interpretation of the Homogenized Vortex Sheet Equations as geodesic equations on a Riemannain manifold. In particular, we aim to identify the curvature tensor associated to this geometry. It is on the basis of this object, that we present our instability analysis in Section 6. In this section, we aim to make it valid in the sense of a formal computation with smooth functions and positive densities.

Let in this section $x \in \mathbb{T}^d$. We would like to convince the reader of the following two propositions.

Proposition 6. *The HVSE system,*

$$\begin{aligned} \partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) &= 0, \\ \partial_t u_\pm + D u_\pm u_\pm + \nabla p &= 0, \end{aligned} \tag{53}$$

can be understood as the equations of geodesics in a submanifold $M \subset D \times D$. Its geometry is characterized by a curvature tensor of the form

$$R(u_\pm^{(1)}, u_\pm^{(2)}) = -\int |\nabla q|^2 dx + \int \nabla p_1 \cdot \nabla p_2 \, dx. \tag{54}$$

This gives immediately the formula for the sectional curvature in the plane spanned by $u_\pm^{(1)}$ and $u_\pm^{(2)}$, if one assumes normalized vectors,

$$\int s_+ |u_+^{(i)}|^2 + s_- |u_-^{(i)}|^2 \, dx = 1, \quad i = 1, 2.$$

Moreover, the pressure terms are determined by the self–consistent reconstruction relation

$$\nabla \cdot \big(\nabla \cdot (s_+ u_+ \otimes u_+ + s_- u_- \otimes u_-) + \nabla p \big) = 0, \tag{55}$$

and its symmetrization.

Proposition 7. *The potential HVSE system,*

$$\partial_t s_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0,$$
$$\partial_t \phi_\pm + \frac{1}{2}|\nabla \phi_\pm|^2 + p = 0, \tag{56}$$

can be understood as the equations of geodesics in a submanifold $N \subset P \times P$. Its geometry is characterized by a curvature tensor of the form

$$R(\phi_\pm^{(1)}, \phi_\pm^{(2)}) = -\int |\nabla q|^2 dx + \int \nabla p_1 \cdot \nabla p_2 \, dx$$
$$+\inf_{\pi_+} \int s_+ |[\nabla \phi_+^{(1)}, \nabla \phi_+^{(2)}] - \nabla \pi_+|^2 \, dx + \inf_{\pi_-} \int s_- |[\nabla \phi_-^{(1)}, \nabla \phi_-^{(2)}] - \nabla \pi_-|^2 \, dx. \tag{57}$$

This gives immediately a formula for the sectional curvature in the plane spanned by $\phi_\pm^{(1)}$ and $\phi_\pm^{(2)}$, if one assumes normalized vectors,

$$\int s_+ |\nabla \phi_+^{(i)}|^2 + s_- |\nabla \phi_-^{(i)}|^2 \, dx = 1, \quad i = 1, 2.$$

Moreover, the pressure terms are determined by the self–consistent reconstruction relation

$$\nabla \cdot \big(\nabla \cdot (s_+ \nabla \phi_+ \otimes \nabla \phi_+ + s_- \nabla \phi_- \otimes \nabla \phi_-) + \nabla p\big) = 0, \tag{58}$$

and its symmetrization. Finally, $[u, v] = Du\, v - Dv\, u$ denotes the commutator of two vector fields.

Justification. Let us make more precise the first proposition. Let an initial datum $s_\pm^{(0)}$ with $s_+^{(0)} + s_-^{(0)} = 1$ be fixed, and consider $D \times D$, equipped with the metric, given for a tangent vector $u_\pm$ to a point $\Phi_\pm$ as

$$g_{\Phi_\pm}(u_\pm, u_\pm) = \int s_+ |u_+|^2 + s_- |u_-|^2 dx = \int s_+^{(0)} |\partial_t \Phi_+|^2 + s_-^{(0)} |\partial_t \Phi_-|^2 \, dx. \tag{59}$$

Here, we gave two expressions which are equivalent due to the change of variables formula, and mean more precisely that $s_\pm = \Phi_\pm \# s_\pm^{(0)}$, and $\partial_t \Phi_\pm = u_\pm \circ \Phi_\pm$. With this metric, $D \times D$ is a flat manifold, and its geodesics are described by the 'straight lines'

$$\partial_t u_\pm + D u_\pm u_\pm = 0. \tag{60}$$

Denote more precisely $D \times D = D_+ \times D_-$ to distinguish the phases, and let $M \subset D_+ \times D_-$ be given by

$$M = \{\Phi_\pm \mid \Phi_+ \# s_+^{(0)} + \Phi_- \# s_-^{(0)} = 1\}. \tag{61}$$

The relation means precisely that $s_+ + s_- = 1$ along the motion, so that by what we showed earlier, (53) are indeed the geodesic equations in the set M.

Notice that $TM \subset TD_+ \times TD_-$ is given as

$$T_{\Phi_\pm}M = \big\{u_\pm \in T_{\Phi_+}D_+ \times T_{\Phi_-}D_- = V \times V \;\big|\; \nabla\cdot(s_+u_+ + s_-u_-) = 0\big\}. \quad (62)$$

We now interpret (60) and (53) as the covariant derivative in $D_+ \times D_-$ and M respectively. Indeed this is justified simply because both were derived as geodesic equations. (It means more precisely that the connection must respect the metric and be torsion–free.)

This justifies to regard their difference ∇p as the projection onto M. More precisely, the vector $w_\pm \in TD_+ \times TD_-$ given by $w_+ = \nabla p = w_-$, is orthogonal to TM, since

$$\int (s_+u_+ + s_-u_-)\cdot\nabla p\, dx = 0,$$

and its value defined by (55) is indeed to be understood as the second fundamental form associated to the embedding $M \subset D_+ \times D_-$, in other words,

$$B(u_\pm, u_\pm) = \nabla p.$$

Finally, we find the remaining values of the quadratic form B by polarization. In particular, in (54), p_1 and p_2 are defined by

$$\nabla\cdot\big(\nabla\cdot(s_+u_+^{(i)}\otimes u_+^{(i)} + s_-u_-^{(i)}\otimes u_-^{(i)}) + \nabla p_i\big) = 0, \quad i = 1, 2,$$

and q by the symmetrization

$$\nabla\cdot\big(\nabla\cdot(s_+u_+^{(1)}\otimes u_+^{(2)} + s_-u_-^{(1)}\otimes u_-^{(2)}) + \nabla q\big) = 0,$$

where we refer to Lemma 4 to identify this symmetrization.

This allows us to justify (54) as a formula for the sectional curvature, by means of the Gauss–formula, analogous to (49). Indeed, the metric then reduces to

$$\int s_+\nabla p_1\cdot\nabla p_2 + s_-\nabla p_1\cdot\nabla p_2\, dx = \int \nabla p_1\cdot\nabla p_2\, dx,$$

and likewise for ∇q. □

We next turn to the second proposition. Here, let $P \times P$ be equipped with the product metric, given in a point $s_\pm$ as

$$g_{s_\pm}(\phi_\pm, \phi_\pm) = \int s_+|\nabla\phi_+|^2 + s_-|\nabla\phi_-|^2\, dx, \quad (63)$$

if we understand the potentials $\phi_\pm$ to realize the tangent vector $\partial s_\pm \in T(P \times P)$ according to $\partial s_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0$.
With this metric, which induces the Wasserstein distance, $P \times P$ is not flat, but indeed by an O'Neill formula as in Section 4.1, carries a nonnegative curvature

$$\begin{aligned} R(\phi_\pm^{(1)}, \phi_\pm^{(2)}) = \inf_{\pi_+} \int s_+ |[\nabla \phi_+^{(1)}, \nabla \phi_+^{(2)}] - \nabla \pi_+|^2 \, dx \\ + \inf_{\pi_-} \int s_- |[\nabla \phi_-^{(1)}, \nabla \phi_-^{(2)}] - \nabla \pi_-|^2 \, dx. \end{aligned} \tag{64}$$

Its geodesics nevertheless, similar with the above, are 'straight lines' as described by

$$\partial_t \phi_\pm + \frac{1}{2} |\nabla \phi_\pm|^2 = 0. \tag{65}$$

Let now $N \subset P \times P$ be given as

$$N = \{s_\pm | s_+ + s_- = 1\}. \tag{66}$$

Then by construction, (56) are the geodesic equations in N. Again, $TN \subset TP \times P$ is given as

$$T_{s_\pm} N = \{\phi_\pm \in T_{s_+} P \times T_{s_-} P \;\big|\; \nabla \cdot (s_+ \nabla \phi_+ + s_- \nabla \phi_-) = 0\}.$$

We argue analogously as for the first proposition: one may interpret (65) and (56) as the covariant derivative in N and $P \times P$, respectively. This justifies to regard their difference p as the projection onto N. More precisely, the vector $\psi_\pm \in TP \times P$ given by $\psi_+ = p = \psi_-$, is orthogonal to TN, and its value defined by (58) is to be understood as the second fundamental form associated to the embedding $N \subset P \times P$,

$$B(\phi_\pm, \phi_\pm) = p.$$

We may hence define the symmetrization by

$$\nabla \cdot \big(\nabla \cdot (s_+ \nabla \phi_+^{(1)} \otimes \nabla \phi_+^{(2)} + s_- \nabla \phi_-^{(1)} \otimes \nabla \phi_-^{(2)}) + \nabla q\big) = 0,$$

and justify by the Gauss–formula, that the curvature in N can be expressed in terms of the curvature in $P \times P$, (64), and the second fundamental form B, precisely as the sum in (57) . □

Let us make finally the following remark. The double copy of the operation of projecting from D onto P, $\pi : \Phi \mapsto \Phi \# s^{(0)}$, maps $D \times D$ onto $P \times P$, may be restricted to a projection $M \longrightarrow N$. One may say that it commutes with the inclusions $M \subset D \times D$, $N \subset P \times P$: It is equally fine to notice that (58) is simply the restriction of (55) to horizontal vectors $\nabla \phi_\pm$, and apply first the Gauss–formula for $M \subset D \times D$, and then the O'Neill–formula for $\pi : M \longrightarrow N$. Since this latter O'Neill formula is more complicated (in that it involves two projections), we propose the argument given above.

5 Digression: Entropy

We aim here to study the behaviour of functionals of the form

$$\int h(s_+) + h(s_-)\, dx \tag{67}$$

for some convex function h, along the geodesic flow given by the potential HVSE system. In the sense of Remark 6 below, the result sheds an additional light on the relaxation process.
In particular, we are interested in the functional

$$E = \int s_+ \ln s_+ + s_- \ln s_-\, dx, \tag{68}$$

the Boltzmann mixing entropy, which is bounded as $s_\pm \in [0,1]$. We are going to show that E is convex on the Riemannian manifold $N \subset P \times P$. It is only in 1 space dimension that we can prove a corresponding result for more general convex functions h.

We can think of two ways to see that E is convex with respect to the geodesic flow

$$\partial_t s_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0, \tag{69}$$

$$\partial_t \phi_\pm + \frac{1}{2}|\nabla \phi_\pm|^2 + p = 0. \tag{70}$$

Of course one may perform the direct calculation. This is possible, and a result is obtained in explicit local terms because the entropy functional is such that the term ∇p producing the projection onto $N \subset P \times P$ is orthogonal to the gradient of E.
We present here an alternative reasoning using the gradient flow of E. This has some advantage, in that it involves a parabolic evolution rather than a conservative one, and hence is at least consistent regarding the regularity of the solution. Again, the Boltzmann entropy is particular in that this gradient flow is linear in s. We obtain

Proposition 8. *(Displacement convexity)*
Along any smooth solution to the potential HVSE system (69), (70) *with positive densities $s_\pm$, one has*

$$\partial_{tt}\left(\int s_+ \ln s_+ + s_- \ln s_-\, dx\right) = \int s_+ |D^2\phi_+|^2 + s_-|D^2\phi_-|^2\, dx. \tag{71}$$

This is the analogous result as is known in the theory of optimal mass transportation. We mean that by the same reasoning, E is convex on $P \times P$, and we find here that it is equally convex on $N \subset P \times P$. The explanation for this fact may be formally expressed as $\nabla E(s) \in T_s N \quad \forall s \in N$.

Remark 5. *(Mixing entropy is finite)*
i) E is bounded according to $-E_0 \leq E \leq 0$. The extreme values are attained for $s_\pm = \frac{1}{2} \implies E = -E_0 = \int dx \ln \frac{1}{2}$, and for $s_\pm = \chi_\pm \implies E = 0$.
ii) Of course the kinetic energy $K = \int s_+|\nabla\phi_+|^2 + \int s_-|\nabla\phi_-|^2 dx$ is conserved: $\partial_t K = 0$.

5.1 Discussion: Displacement convexity

It would be a natural question whether it is possible to justify at least the inequality

$$\partial_{tt} E \geq \int s_+|D^2\phi_+|^2 + s_-|D^2\phi_-|^2 \, dx$$

for a more general, distributional solution of the potential HVSE, with only nonnegative densities, or in particular, for the minimizing geodesic. This question does not, to our knowledge, have an obvious answer.

Let us briefly mention three implications:

Remark 6. *Such an entropy inequality would express in a quantitative way the fact that the solution of the shortest distance problem will not remain in the class of characteristic functions (because then $E = 0$ and hence $D^2\phi_\pm = 0$), but make use of the possibility of passing mixed states. Although it may be considered a crude example, indeed the solution to Problem 1 above was shown to be a mixture.*

It is worth pointing out on the other hand a notable exception to this behaviour: The analytical solutions to the Birkhoff–Rott equation, which as mentioned is equivalent to the potential Vortex Sheet system (23), (24)*, satisfy $E = 0$ but $\nabla\phi_\pm \neq 0$, although they solve the geodesic equation. We could then give the interpretation that these non–generic analytical short–time solutions are actually not to be viewed as the shortest geodesics in the sense of a Riemannian interpretation of the system* (27), (28)*. In fact a conjecture seems plausible that they would not even for short time be minimizers of I.*

Moreover, such an inequality expresses a certain regularity of the solution. We refer to [4] for a rigorous result in this direction.

Finally, the existence of a bounded and convex entropy functional is related to the fact that solutions to the potential HVSE system on a compact domain can only exist for finite time. This assertion, too, is shown in [4].

5.2 A formal proof of the formula for the Hessian

We now give the formal proof of the formula (71) in Proposition 8. Consider the gradient flow of the functional $E = \int s_+ \ln s_+ + \int s_- \ln s_-$, that is the dynamics for $s(t) \in N$ given by

$$\langle \partial_t s, \xi \rangle = -\frac{\partial}{\partial \xi} E, \quad \forall \text{ tangent vectors } \xi.$$

Explicitly, the tangent vector $\xi = (\xi_+, \xi_-)$ is to be understood in the sense $\partial s_\pm + \nabla \cdot (s_\pm \xi_\pm) = 0$, so that the right hand side equals

$$-\int \ln s_+ \, \partial s_+ - \int \ln s_- \partial s_- = -\int s_+ \nabla \ln s_+ \cdot \xi_+ - \int s_- \nabla \ln s_- \cdot \xi_-.$$

The left hand side uses the metric, so if $\partial_t s_\pm + \nabla \cdot (s_\pm u_\pm) = 0$, the left hand side is equal to

$$\int s_+ u_+ \cdot \xi_+ + \int s_- u_- \cdot \xi_-.$$

The identity holds true for all $\xi_\pm$ with $\nabla \cdot (s_+ \xi_+ + s_- \xi_-) = 0$. By definition, also $\nabla \cdot (s_+ u_+ + s_- u_-) = 0$, and moreover it is the property of the Boltzmann entropy to be extensive: Its gradient is consistent with $s_+ + s_- = 1$, since automatically, $s_+ \nabla \ln s_+ + s_- \nabla \ln s_- = \nabla s_+ + \nabla s_- = 0$. Hence we deduce that $s_\pm u_\pm = -s_\pm \nabla \ln s_\pm = -\nabla s_\pm$, and the gradient flow turns out to be

$$\partial_t s_\pm - \nabla \cdot \nabla s_\pm = 0.$$

So it follows simply the heat equation, in particular is linear, and respects the product structure of $P \times P$.

To compute the Hessian of E, we compute the evolution of the metric tensor along this gradient flow, for a variation of a fixed solution. This yields indeed an expression for the Hessian of E, because if $\partial_t s = -\nabla E(s)$, then

$$\partial_t \dot{s} = -\mathrm{Hess} E(s) \, \dot{s} \quad \Longrightarrow \quad \partial_t \frac{1}{2} \langle \dot{s}, \dot{s} \rangle = -\langle \dot{s}, \, \mathrm{Hess} E(s) \, \dot{s} \rangle.$$

Since the heat flow is linear, also the variation $\dot{s}_\pm$ of $s_\pm$ (understood simply with respect to the underlying vector space, $\dot{s}_+ + \dot{s}_- = 0$), satisfies

$$\partial_t \dot{s}_\pm - \nabla \cdot \nabla \dot{s}_\pm = 0.$$

We thus observe that

$$\partial_t \frac{1}{2} \langle \dot{s}, \dot{s} \rangle \; = \; \partial_t \left(\frac{1}{2} \int s_+ |\nabla \phi_+|^2 + \frac{1}{2} \int s_- |\nabla \phi_-|^2 \right)$$

is to be computed along the heat flow, if we understand now $\dot{s}_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0$. By a classical calculation, the result is

$$-\int s_+ |D^2 \phi_+|^2 - \int s_- |D^2 \phi_-|^2. \tag{72}$$

This establishes the claim of the proposition, because the second derivative along a geodesic coincides by definition with the evaluation of the Hessian. Obviously, at this point, our argument is formal.

For convenience, we reproduce the classical [18] calculation which leads to (72). For this, we need an expression for $\partial_t \phi_\pm$ and find it as

$$\dot{s}_\pm + \nabla \cdot (s_\pm \nabla \phi_\pm) = 0 \quad \Longrightarrow \tag{73}$$

$$\partial_t \dot{s}_\pm + \nabla \cdot (\partial_t s_\pm \nabla \phi_\pm) + \nabla \cdot (s_\pm \nabla \partial_t \phi_\pm) = 0. \tag{74}$$

We proceed to compute for each phase, say $\phi = \phi_+$, $s = s_+$, $\dot{s} = \dot{s}_+$,

$$\partial_t \frac{1}{2} \int s |\nabla \phi|^2 = \text{two terms.}$$

Indeed, the derivative would apply on the one hand to s, to yield

$$\frac{1}{2} \int \Delta s |\nabla \phi|^2 = \int s \Delta \frac{1}{2} |\nabla \phi|^2. \tag{75}$$

On the other hand, it would apply to ϕ, to give via (74) the contribution

$$\int s \nabla \phi \cdot \nabla \partial_t \phi = \int \Delta \dot{s}\, \phi - \int \Delta s \, |\nabla \phi|^2 = \int s \, \nabla \phi \cdot \nabla \Delta \phi - \int s \, \Delta \, |\nabla \phi|^2. \tag{76}$$

The argument is concluded with help of the Bochner formula

$$\Delta \frac{1}{2} |\nabla \phi|^2 = \nabla \phi \cdot \nabla \Delta \phi + |D^2 \phi|^2,$$

summing (75) and (76) to arrive at (72). □

5.3 Entropies in dimension 1

The one–dimensional system (38), (39) is simpler for three reasons: obviously, the mean flux $j = 0$, moreover, in dimension 1, length and volume coincide, and also $u_\pm = \partial_y \phi_\pm$ are automatically curl–free. This allows us to obtain a more general result.

Let $h(s)$ be a convex function. Notice that in the case $h(s) = s \ln s + (1-s) \ln(1-s)$, one has $h''(s)\, s(1-s) = 1$. Consider the equation as written for $s = s_+$, $u = u_+$:

$$\partial_t s + \partial_y (su) = 0, \tag{77}$$

$$\partial_t u + u \, \partial_y u - \partial_y \left(\frac{s}{1-s} u^2 \right) = 0. \tag{78}$$

This is for example a special case of (153) in the appendix.

Denote here nevertheless as an abbreviation, $v = -\frac{s}{1-s} u$. Then the following identity holds true:

Proposition 9.

$$\frac{d^2}{dt^2} \int h(s)dy = \int h''(s)s(1-s) \, \left(s\,|\partial_y u|^2 + (1-s)|\partial_y v|^2\right) \, dy. \tag{79}$$

For completeness we include

Remark 7. *If h is of the form $h(s) = g(s) + g(1-s)$, then*
i) $g'' \geq 0 \quad \Longrightarrow \quad h'' \geq 0$,
ii) $g''(s)\,s \, \geq 1 \quad \Longrightarrow \quad h''(s)\,s(1-s) \, \geq 1$,
and the reverse implications are both false.

Proof of the Proposition. We have

$$\partial_t \int h(s)dy = \int h'(s)\partial_t s \, dy = \int \partial_y(h'(s))\, su \, dy.$$

We write this as

$$\int h''(s)\, s\, \partial_y s \, u \, dy = \int \partial_y(F(s))\, u \, dy,$$

if $F'(s) = h''(s)s$. From there we obtain

$$\partial_{tt} \int h(s)dy = \int \partial_y \partial_t(F(s))\, u\, dy + \int \partial_y(F(s))\, \partial_t u \, dy,$$

which by the equation equals

$$-\int \partial_y(F'(s)\, \partial_y(su))\, u\, dy - \int \partial_y(F(s))\, u\, \partial_y u \, dy + \int \partial_y(F(s))\, \partial_y \left(\frac{s}{1-s}u^2\right) \, dy.$$

The first term, integrating first by parts and then differentiating the bracket $\partial_y(su)$, is found to equal

$$\int F'(s)\partial_y s \, u \, \partial_y u \, dy + \int F'(s) s \, |\partial_y u|^2 \, dy.$$

The fact that the square $|\partial_y u|^2$ appears, is specific of dimension 1. Hence the first term produces a cancellation with the second term. Carrying out then the differentiation in the third term, and inserting back the definition of F, we are left with

$$\begin{aligned}
\partial_{tt} \int h(s)dy &= \int F'(s) s \, |\partial_y u|^2 \, dy + \int F'(s)\partial_y s \, \partial_y \left(\frac{s}{1-s}u^2\right) \, dy \\
&= \int h''(s) s^2 \, |\partial_y u|^2 \, dy + \int h''(s) s \, \partial_y s \, \partial_y \left(\frac{s}{1-s}u^2\right) \, dy.
\end{aligned}$$

From here, we split the first term into two parts and differentiate the bracket in the second term, using that $\partial_y \frac{s}{1-s} = \frac{1}{(1-s)^2}\partial_y s$. We hence obtain

$$\partial_{tt} \int h(s)\,dy = \int h''(s)s(1-s)\,s|\partial_y u|^2\,dy$$
$$+\int h''(s)s^3\,|\partial_y u|^2\,dy + 2\int h''(s)\frac{s^2}{1-s}\partial_y s\,u\,\partial_y u\,dy + \int h''(s)\frac{s}{(1-s)^2}|\partial_y s|^2\,u^2 dy. \tag{80}$$

On the other hand, the second term in the claim, replacing $\partial_y v = -\partial_y(\frac{su}{1-s})$, equals

$$\int h''(s)s(1-s)\,(1-s)|\partial_y v|^2 dy = \int h''(s)s(1-s)^2 \left|\partial_y\left(\frac{s}{1-s}u\right)\right|^2 dy$$
$$= \int h''(s)s(1-s)^2 \left(\frac{s^2}{(1-s)^2}|\partial_y u|^2 + \frac{2}{(1-s)^2}\,\partial_y s\,\frac{s}{1-s}\,u\,\partial_y u + \frac{1}{(1-s)^4}|\partial_y s|^2 u^2\right) dy,$$

hence matches precisely the three terms in (80). This concludes the proof of the identity (79). $\square$

6 A Lagrangian instability analysis

We show in this chapter how to give a linear stability analysis in a Lagrangian sense by means of the curvature tensor. For simplicity, we restrict ourselves to the case of two dimensions with coordinates x and y. We assume x to be a periodic variable. Similar results are valid in dimension $d \geq 2$.

We show how the vortex sheet is responsible for an instability. The result relates to the findings of Section 3.2 in the following sense: The geometry in which vortex sheet dynamics can be viewed as a geodesic flow, is still more hyperbolic than it is the case for classical fluid dynamics. This can be expressed by the fact that the Jacobi equation has elliptic character in space–time. We will not try to be rigorous about this last point, but still we give the correspondent Remark 8.

We focus here on negative curvature. Indeed we identify the same type of unstable behaviour in the case of two representative examples: we study the case of a single sharp interface, which corresponds to the original geometry of the Euler flow, and the case of a homogeneous mixture. Our results give the impression that the unbounded sectional curvatures are in fact always negative; at least this is the case for our two examples.

6.1 Curvature in the classical case

Here we address the Euler system as in Section 4.1 with $\rho = 1$, where the quantities p, q, R were defined. Let $u(x, y) = f(y)\,\frac{\partial}{\partial x}$ be a fixed shear flow with flow direction x, shear direction y. We are interested in the stability of this stationary flow with respect to variations in direction of a second divergence–free field $w(x, y)$. The second variation of the action then is related to the sectional curvature of the plane spanned by u and w. Since u is a stationary solution with zero pressure (indeed, $Du\,u = 0$), we have

$$R(u, w) = -\int |\nabla q|^2\, dx\, dy,$$

where q is given by the relation

$$\nabla \cdot (Du\, w + \nabla q) = 0. \tag{81}$$

Indeed, the polarization was identified in Lemma 4 of Section 4.1, which entails in particular $\nabla \cdot (Du\, w - Dw\, u) = 0$; and (50) together with (52) gives (81).

We observe in this section simply, that this implies that $|R(u, w)|$ can be estimated as

$$\int |\nabla q|^2 dxdy \,\leq\, \int |Du\, w|^2 dxdy \,\leq\, \sup |Du|^2 \int |w|^2 dxdy. \tag{82}$$

Now to obtain the sectional curvature, one has to divide this by $\int |u|^2 dxdy$ and $\int |w|^2 dxdy$. Hence if we fix u with $\int |u|^2 = 1$ and $\sup |Du| < \infty$, we find that the sectional curvature is negative and bounded (independent of w). It is not summable in the sense of a Ricci tensor as observed by Malliavin [7].

6.2 The case of a vortex sheet, in the classical setup

We are now interested in the case where f is close to the jump function $-1_{\{y<0\}} + 1_{\{y>0\}}$, that is u is close to the discontinuous shear flow which constitutes a vortex sheet (the line $\{y = 0\}$). We use the same formula as above, with reverse roles,

$$\nabla \cdot (Dw\, u + \nabla q) = 0. \tag{83}$$

The estimate (82) then becomes useless. In fact we will see, that this time, the curvature is unbounded and only controlled by $\int |Dw|^2 dxdy$, which in turn is obvious with a similar argument as above. In fact since $Dw\, u = f\partial_x w$, we even have:

$$\int |\nabla q|^2 dxdy \;\leq\; \int |Dw\, u|^2 dxdy \;\leq\; \sup |u|^2 \int |\partial_x w|^2 dxdy. \tag{84}$$

To argue that this is sharp, it is convenient to have w of the form $w = \nabla\phi$, where ϕ is the harmonic function in the half space with normal velocity equal to a function $g(x)$ on $\{y = 0\}$. Explicitly if for $k \in \mathbb{Z}$, $g(x) = \cos kx$, then

$$w(x, y) = e^{-|k||y|}(\mathrm{sign}y\, \mathrm{sign}k\, \sin kx, \cos kx) \tag{85}$$

is divergence–free, curl–free in $\{y \neq 0\}$, and has second component $\cos kx$ on $\{y = 0\}$. That is, w is of the form $w = \chi_+ \nabla\phi_+ + w_- \nabla\phi_-$. In this case,

$$R(u, w) = -\int |\nabla q|^2 dxdy$$

is to be devided by $\int |u|^2 dxdy$ and $\int |w|^2 dxdy$. We will ignore for the moment the fact that u is not square–integrable.

To proceed, we have to compute q from the elliptic equation (83). Because u is divergence–free, we find not surprisingly, that $\nabla\cdot(Du\, w) = f'(y)\partial_x w_y(x, y)$ depends only on Du and Dw. We assume now that $f'(y)$ is close to 2δ, a Dirac mass in $y = 0$, in the sense for example of a convolution, on a scale $[y] \sim \varepsilon$, and k is large but $|k|^{-1} \gg \varepsilon$. Observe then that $w_y(x, 0) = g(x)$, hence $\partial_x w_y(x, 0) = \partial_x g$, so that we have justified that (83) turns into

$$2\delta_{y=0}\, \partial_x g + \Delta q = 0. \tag{86}$$

In a terminology introduced in the Appendix 11, q is then the Hilbert transform of $g = w_y(\cdot, 0)$, and one has

$$\int |\nabla q|^2 dxdy = \int |\nabla w_y|^2 dxdy, \tag{87}$$

which in turn equals $\int |\partial_x w|^2 dxdy$. This means that in fact, (84) is sharp.

Explicitly, if $g(x) = \cos kx$, then $\partial_x g(x) = -k \sin kx$, and we have $q(x,y) = -\text{sign}\, k e^{-|k||y|} \sin kx$, whence

$$\nabla q = |k| e^{-|k||y|} (\cos kx, -\text{sign} k \,\text{sign} y\, \sin kx),$$

since indeed the jump of $\partial_y q$ at $y = 0$ is $-2k \sin kx = 2\partial_x g(x)$.
Thus $\int |\nabla q|^2 dxdy = |k|^2 \int |w|^2 dxdy$, so that we have found that

the sectional curvature in the plane spanned by u and w is of order $O(|k|^2)$.

Remark 8. *We now show how this finding is related to the Jacobi equation for the variation along geodsics. It reads*

$$-\frac{D^2}{dt^2} J - R(\dot\gamma, J)\dot\gamma = 0, \tag{88}$$

where $\gamma(t)$ is the geodesic, and J the Jacobi field. We are thus interested in the spectrum of the symmetric operator

$$J \mapsto -R(\dot\gamma, J)\dot\gamma. \tag{89}$$

It is the symmetric operator which generates the quadratic form

$$J \mapsto -R(\dot\gamma, J, \dot\gamma, J). \tag{90}$$

We will show below that it behaves like a certain (degenerate) Dirichlet integral.
Now this implies that the index–form

$$\int_0^T \left\| \frac{D}{dt} J \right\|^2 - R(\dot\gamma, J, \dot\gamma, J)\, dt$$

behaves like a Dirichlet–integral in space–time; and on the other hand the operator (89) *has unbounded spectrum - or indeed, behaves like negative second space derivatives. Thus, equation* (88) *will be (degenerate) elliptic in character.*

It is a natural question what this implies about the regularity of solutions. Indeed, we understand that we may interpret more or less the regularity result of [4] in this way: Brenier obtains interior regularity through an inner, or geometric, variation.

6.3 Instabilty of a 2 phase–model

As we have seen in the discussion of ellipticity in Section 3.2, the relaxed model must be expected to be already unstable (ill–posed) for intrinsic reasons. The result of Proposition 4 has indeed its counterpart on the level of the curvature tensor. For this discussion, we are interested in generic smooth positive densities $s_\pm$. For simplicity, we content ourselves here to the homogeneous case $s_+ = \frac{1}{2} = s_-$. Let moreover the pair (u_+, u_-) be fixed with $\nabla \cdot (\frac{1}{2}u_+ + \frac{1}{2}u_-) = 0$.

We will preferably use a natural class $w_\pm \in L^\infty \cap H^1$, and sometimes also even $\sup |Du_\pm|$. Although there is no reason in the evolution why in particular $u_\pm \in L^\infty$, it is in this class that one has estimates of the curvature symmetric in $u_\pm$ and $w_\pm$. For this section, let $x \in \mathbb{T}^d$ denote the (homogeneous isotropic) space variable.

We are then interested in the Rayleigh quotient

$$(w_+, w_-) \longmapsto \frac{\int |\nabla q|^2}{\int |w_+|^2 + |w_-|^2\, dx},$$

among $w_\pm$ with

$$\nabla \cdot \left(\frac{1}{2}w_+ + \frac{1}{2}w_-\right) = 0. \tag{91}$$

Here, q is defined by

$$\nabla \cdot \left(\nabla \cdot \left(\frac{1}{2}u_+ \otimes w_+ + \frac{1}{2}u_- \otimes w_-\right) + \nabla q\right) = 0. \tag{92}$$

It was argued in Section 4 that the so–defined q is indeed symmetric in $u_\pm$ and $w_\pm$.

We have

Proposition 10. *(Estimate for homogeneous mixture)*

$$i) \int |\nabla q|^2\, dx \le \frac{1}{2}\int |Dw_+u_+|^2 + |Dw_-u_-|^2 dx \tag{93}$$

$$+\frac{1}{2}\left(\int |\nabla \cdot u_+|^2 + |\nabla \cdot u_-|^2 dx\right)\left(\sup |w_+|^2 + |w_-|^2\right). \tag{94}$$

ii) If $(w_+, w_-) = (\nabla\phi_+, \nabla\phi_-)$ is potential flow, then in fact

$$\int |\nabla q|^2\, dx = \frac{1}{4}\int |Dw_+u_+ + Dw_-u_-|^2 dx - r, \tag{95}$$

with a remainder r which can be estimated as

$$|r| \le C\left(\int |Du_+|^2 + |Du_-|^2 dx\right)\left(\sup |w_+|^2 + |w_-|^2\right), \tag{96}$$

$$|r| \le C\left(\sup |Du_+|^2 + |Du_-|^2\right)\left(\int |w_+|^2 + |w_-|^2 dx\right), \tag{97}$$

with a numerical constant C.

Lemma 5. *The other contribution to the curvature tensor,*

$$\int \nabla p_u \cdot \nabla p_w \, dx,$$

is controlled by

$$\sup |u_\pm| \left(\int |Du_+|^2 + |Du_-|^2 \right)^{\frac{1}{2}} \sup |w_\pm| \left(\int |Dw_+|^2 + |Dw_-|^2 \right)^{\frac{1}{2}}.$$

If $u_\pm$ are smooth, there is even a constant $C = C(u_\pm)$, so that

$$\int \nabla p_u \cdot \nabla p_w \, dx \,\leq\, C \int |w_+|^2 + |w_-|^2 dx. \tag{98}$$

As an immediate consequence one can see

Corollary 1. *(Curvature for homogeneous mixture)*
Let $u_\pm$ be smooth, and let $R = -\int |\nabla q|^2 dx + \int \nabla p_u \cdot \nabla p_w dx$ be defined as by Proposition 6 of Section 4.2. If both $w_\pm = \nabla \phi_\pm$ and $u_\pm = \nabla \psi_\pm$ are potential velocities, then

$$R(u,w) \,=\, - \int \frac{1}{2} |Dw_+ u_+|^2 + \frac{1}{2} |Dw_- u_-|^2 dx \,+\, O\Big(\int |w_\pm|^2 dx \Big). \tag{99}$$

Remark 9. *We also have*

$$4 \int |\nabla q|^2 dx = \int \Big| \mathbb{P}[u_+(\nabla \cdot w_+)] + \mathbb{P}[u_-(\nabla \cdot w_-)] \Big|^2 dx \,+\, r, \tag{100}$$

with a remainder

$$r = O\left(\min\Big\{ \int |Du_\pm|^2 dx \sup |w_\pm|^2,\, \sup |Du_\pm|^2 \int |w_\pm|^2 dx \Big\} \right).$$

Here, $\mathbb{P} w_\pm = \nabla \Delta^{-1} \nabla \cdot w_\pm$ is the Helmholtz projection onto gradients.
In particular, in view of (91), this instability only exists, if $u_+ \neq u_-$. $\square$

Proof of Proposition 10. The first statement is obvious by continuity in L^2 of the Helmholtz–projection. Indeed we have according to (92) that

$$2\nabla q = \mathbb{P}\,(Dw_+ u_+ + Dw_- u_-) + \mathbb{P}\,(w_+(\nabla \cdot u_+) + w_-(\nabla \cdot u_-)).$$

Notice that reversing the roles of $u_\pm$ and $w_\pm$, we may also immediately deduce Remark 9.
To see the second point, we write

$$\mathbb{P}\big[D^2 \phi_\pm \, u_\pm\big] \,=\, D^2 \phi_\pm u_\pm + (-id + \mathbb{P})(D^2 \phi_\pm u_\pm).$$

It now remains to estimate the last projection against (96) or (97). But since $D^2\phi\, u + Du^T\nabla\phi = \nabla(\nabla\phi\cdot u)$ is a gradient,

$$(-id + \mathbb{P})(D^2\phi_\pm u_\pm) = (id - \mathbb{P})(Du_\pm^T\nabla\phi_\pm),$$

which finally is controlled by $\int |Du_\pm|^2|\nabla\phi_\pm|^2dx$, by continuity in L^2 of the projection $(id - \mathbb{P})$. □

Proof of Lemma 5. The first statement is an application of Proposition 10. The second statement follows from the definition of p_w, which entails

$$\int \nabla p_u \cdot \nabla p_w\, dx \;=\; \int D^2 p_u : (\frac{1}{2}w_+ \otimes w_+ + \frac{1}{2}w_- \otimes w_-)\, dx,$$

and the fact that p_u is smooth if $u_\pm$ are. □

Proof of Corollary 1. By Lemma 5, it suffices to consider the first term $\int |\nabla q|^2 dx$. By Proposition 10, ii), it can be replaced up to bounded contributions by $\frac{1}{4}\int |Dw_+u_+ + Dw_-u_-|^2dx$. But then if $u_\pm$, $w_\pm$ are potential, the divergence–constraint means actually $u_+ = -u_-$, $w_+ = -w_-$, so that

$$\frac{1}{4}\int |Dw_+u_+ + Dw_-u_-|^2dx = \frac{1}{2}\int |Dw_+u_+|^2dx + \frac{1}{2}\int |Dw_-u_-|^2dx.$$

This proves the claim. □

6.3.1 Construction of unstable directions

We have shown in Corollary 1, that the map $w \mapsto |R(u, w)|$ behaves like a certain Dirichlet integral. We would like to establish now that this lower bound indeed is larger than the L^2–norm of $w_\pm$, more precisely matches $\int |Dw_\pm|^2dx$ in terms of scaling. We obtain an asymptotic expression for the rescaled quantity.

Precisely, let $(u_+, u_-) = (\nabla\psi_+, \nabla\psi_-)$ be a fixed pair of smooth vector fields, and $(w_+, w_-) = (\nabla\phi_+, \nabla\phi_-)$ be fixed potential flows, which are supposed to be nontransversal to u in the sense

$$\int Dw_\pm^T(z)Dw_\pm(z)dz : \int u_\pm(x) \otimes u_\pm(x)dx \;\neq\; 0. \qquad (101)$$

We propose the following rescaled vector fields

$$(w_+^N(x), w_-^N(x)) = (w_+(Nx), w_-(Nx)), \quad N \in \mathbb{N},$$

which are again periodic in the torus. Then the L^2–norm of $w_\pm^N$ is independent of N. Hence by Corollary 1, it is enough to discuss the leading term: We then have to check that $\int |Dw_+^N u_+|^2 dx = O(N^2)$. This is true, because

$$\int |Dw_+^N u_+|^2dx = N^2\int |Dw_+(Nx)u_+(x)|^2dx \qquad (102)$$

$$= N^2\left(\int Dw_+^T(z)Dw_+(z)dz : \int u_+(x) \otimes u_+(x)dx \;+\; o(1)\right). \qquad (103)$$

Here we used, that for the torus, $\int dz = 1$. So for large N, the integral decomposes into short and long wavelength, and if we require that the product in (101) is nonzero, indeed the right hand side of (99) is of order N^2. Precisely, we used that a rescaled function $f(x) = Dw^T Dw(x)_{ij}$, periodic in $L^1(\mathbb{T}^d)$, converges to its average, $f(Nx) \to \int f(z)dz$ weakly.

Arguably, also if w_+ is not a gradient field but satisfies some weaker condition, a suitable choice of a test function on two scales may show an estimate of the form

$$\sup_{\eta} \frac{\int Dw^N u \cdot \nabla \eta dx}{\left(\int |\nabla \eta|^2 dx\right)^{\frac{1}{2}}} \geq O(N).$$

We thus found that the bound in Corollary 1 matches $\int |Dw_\pm|^2 dx$ in terms of scaling, and the asymptotic expression (103) is a less degenerate Dirichlet–integral than the one in Corollary 1. More precisely even, the generic situation is that the tensor $\int u(x) \otimes u(x)\, dx$ has full rank, in which case the asymptotic expression (103) is actually equivalent to the full Dirichlet integral $\int |Dw_\pm^N|^2 dx$.

6.4 The case of sharp interface

Here, we consider a model interface, which is the x-axis, so let $\chi_+ = 1_{\{y>0\}}$. We keep the discussion two–dimensional, but it can be made valid for any d dimensions through an integration over the remaining coordinates. For the velocities, we write $u = \chi_+ u_+ + \chi_- u_-$, and likewise for w. The derivate, symbolically

$$Dw = \chi_+ Dw_+ + \chi_- Dw_- + \left([w_x] \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x}\right) dH^1(x),$$

has a regular and a singular part. Here, $[w_x] = (w_+ - w_-)_x$ is the jump of the tangential velocity component across $\Gamma = \{y = 0\}$, and H^1 denotes the Hausdorff measure on Γ. Moreover $\frac{\partial}{\partial x}$ is the unit vector in x–direction. Let us denote finally by w_y the well defined trace of the normal velocity. We discuss this case of sharp interface in some detail, but the main result is Corollary 3, which complements Corollary 1 of the last section.

We first show that a natural space is $w \in L^2$ and moreover $w \in L^\infty$ and $\chi_+ Dw_+ + \chi_- Dw_- \in L^2$. This is expressed by the following proposition, which we try to keep symmetric in u and w.

Proposition 11. *(Estimate for sharp interface)*
Let q be defined by

$$\nabla \cdot (\nabla \cdot (u \otimes w) + \nabla q) = 0 \text{ distributionally in } S^1 \times \mathbb{R}.$$

Then for any smooth test function η, one has the formula

$$\int \nabla q \cdot \nabla \eta \, dxdy$$
$$= -\int (\chi_+ Dw_+ u_+ + \chi_- Dw_- u_-) \cdot \nabla \eta \, dxdy \; - \; \int [w_x] \, u_y \, \partial_x \eta \, dH^1(x). \tag{104}$$

Moreover, the following estimate holds true

$$\int |\nabla q|^2 dxdy \;\leq\; C \sup_{x,y} |u|^2 \left(\int \chi_+ |Dw_+|^2 + \chi_- |Dw_-|^2 dxdy \right) \tag{105}$$
$$+ \sup_{x \in \Gamma} |[w_x]|^2 \int |\nabla u_y|^2 dxdy, \tag{106}$$

where C is a numerical constant.

The proof makes use of the following classical observation.

Lemma 6. *(Trace estimate)*
Let $f = f(x,y)$, $g = g(x,y)$ be two functions in the cylinder $\{x \in S^1\} \times \{y > 0\}$ with values $f(x,0)$, $g(x,0)$ on $\{y = 0\}$. Then

$$\left| \int f \, \partial_x g \, dH^1(x) \right| \;\leq\; \left(\int |\nabla f|^2 dxdy \right)^{\frac{1}{2}} \left(\int |\nabla g|^2 dxdy \right)^{\frac{1}{2}}.$$

In particular, we say that $f \in H^{\frac{1}{2}}$, if the seminorm

$$\left(\int |\partial_x^{1/2} f|^2 \, dH^1(x) \right)^{\frac{1}{2}} := \sup_g \frac{\int f \partial_x g \, dH^1(x)}{(\int |\nabla g|^2 dxdy)^{\frac{1}{2}}}$$

is finite, and a pairing $\langle f, \partial_x g \rangle$ is defined in this completion.

We discuss this lemma in Appendix 11.

Proposition 11 implies

Proposition 12. *(Formula for sharp interface)*
The traces $\lim_{y \to 0\pm} q$ are defined in $H^{\frac{1}{2}}$, and coincide, $[q] = 0$. One has

$$\int |\nabla q|^2 dxdy \;=\; \int (\chi_+ Dw_+ u_+ + \chi_- Dw_- u_-) \cdot \nabla q \, dxdy \; + \langle [w_x] \, u_y, \, \partial_x q \rangle.$$

Discussing further the term $\int |\nabla q|^2 dxdy$, on the other hand, reversing the roles of u and w, similar as in (83), (84), we obtain the following estimate.

Proposition 13. *(Lower bound)*
We have for smooth η the formula

$$\int \nabla q \cdot \nabla \eta \, dxdy$$
$$= -\int (\chi_+ D u_+ w_+ + \chi_- D u_- w_-) \cdot \nabla \eta \, dxdy - \int [u_x] w_y \partial_x \eta \, dH^1(x). \quad (107)$$

Moreover, if $u_\pm$ are C^1, then an equality

$$\int |\nabla q|^2 dxdy = \int \left| \partial_x^{\frac{1}{2}} ([u_x] w_y) \right|^2 dH^1(x) - r, \quad (108)$$

holds true, where

$$|r| \leq C(u) \int |w|^2 dxdy.$$

We can see in particular that we have an instability only if $[u_x] \neq 0$. It is then obvious that the expression (108) is controlled up to terms of order $\int |w|^2 dxdy$ by the norm $\int |\nabla w_y|^2 dxdy$ of the normal velocity component:

Remark 10. *If $u_\pm$ are smooth, there is a constant $C = C(u)$, so that*

$$\int |\nabla q|^2 dxdy \;\leq\; C \int |\nabla w_y|^2 dxdy + C \int |w|^2 dxdy.$$

This upper bound is in accordance with (87). We show on the other hand in Appendix 11, that we have the following control,

Proposition 14. *If $u_\pm \in C^1$, and $g(x, y)$ denotes a smooth continuation of $[u_x]|_\Gamma$, then there is a constant $C = C(g) = C(u)$ so that for all potential $w_\pm = \nabla \phi_\pm$ one has*

$$C \int |w|^2 dxdy + 4 \int \left| \partial_x^{\frac{1}{2}} ([u_x] w_y) \right|^2 dH^1(x) \;\geq\; \int g^2 |\nabla w_y|^2 dxdy. \quad (109)$$

The constant depends essentially on the upper bound $\sup |\partial_x [u_x]|^2$.

Thus the instability due to the vortex sheet is nonlocal, and behaves indeed as a weighted $H^{\frac{1}{2}}$–norm of the normal component of the perturbation velocity, w_y, weighted with $[u_x]$, the intensity of vorticity on the sheet. It allows a lower bound in terms of a degenerate Dirichlet–integral, degenerate only in terms of zeroes of $[u_x]$.

As a further remark, and also for later reference, we give an additional formula for this situation of a sharp interface, with more regular velocity fields.

Proposition 15. *If $u_\pm \in C^1$, then one has for all smooth η the symmetric formula*

$$\int \nabla q \cdot \nabla \eta \, dxdy \;=\; \int (\chi_+ tr\; Du_+ Dw_+ + \chi_- tr\; Du_- Dw_-)\, \eta \, dxdy \quad (110)$$

$$+ \int ([w_x]\partial_x u_y + [u_x]\partial_x w_y)\, \eta \, dH^1(x). \quad (111)$$

In particular, q can be interpreted as the solution of a Neumann problem of the form

$$\Delta q = f \;\; in \; y \neq 0, \quad (112)$$

$$[\partial_y q] = g \;\; across \; y = 0, \quad (113)$$

if we set $f = \chi_+ tr\; Du_+ Dw_+ + \chi_- tr\; Du_- Dw_-$, and $g = [w_x]\partial_x u_y + [u_x]\partial_x w_y$.

By general theory of the Neumann–problem, this assures

Corollary 2. *If $u_\pm$ and $w_\pm$ are smooth, then q is smooth and bounded up to the interface with derivatives, and the jump of the derivative is a smooth function of x equal to*

$$[\partial_y q] = [w_x]\partial_x u_y + [u_x]\partial_x w_y. \quad (114)$$

Let us finally make explicit

Remark 11. *These results, in particular formula* (107)*, or Proposition 15, generalize the one obtained in Section 6.2, in particular based on* (86)*, which corresponds to the special case that the normal velocity $u_y = 0$, the jump $[u_x] = 1$, and $Du_\pm = 0$.*

Proof of Proposition 11. If we remember that for the moment, ∇q denotes only a measure, the defining relation for q may be written as

$$\int \nabla q \cdot \nabla \eta \, dxdy \;=\; \int (\chi_+ u_+ \otimes w_+ + \chi_- u_- \otimes w_-) : D^2 \eta \, dxdy. \quad (115)$$

We proceed to derive the expression (104) through an integration by parts. We find

$$\int \chi_\pm u_\pm \otimes w_\pm : D^2 \eta \, dxdy$$

$$= - \int \chi_\pm D w_\pm u_\pm \cdot \nabla \eta \, dxdy - \int (\pm u_\pm)_y (w_\pm \cdot \nabla \eta) dH^1(x). \quad (116)$$

Indeed, more precisely,

$$\int \chi_+ (u_x (w_x \partial_{xx}\eta + w_y \partial_{xy}\eta) + u_y (w_x \partial_{yx}\eta + w_y \partial_{yy}\eta))\, dxdy \quad (117)$$

$$= \int \chi_+ (u_x \partial_x (w_x \partial_x \eta + w_y \partial_y \eta) + u_y \partial_y (w_x \partial_x \eta + w_y \partial_y \eta)\, dxdy \quad (118)$$

$$- \int \chi_+ ((u_x \partial_x w_x + u_y \partial_y w_x)\partial_x \eta + ((u_x \partial_x w_y + u_y \partial_y w_y)\partial_y \eta)\, dxdy, \quad (119)$$

and the term (118) equals further

$$-\int \chi_+(\partial_x u_x + \partial_y u_y)(w_x\partial_x\eta + w_y\partial_y\eta)\,dxdy \tag{120}$$

$$-\int_{\Gamma^+} u_y(w_x\partial_x\eta + w_y\partial_y\eta)\,dH^1(x). \tag{121}$$

Then (120) vanishes because of $\nabla \cdot u = 0$ for $y > 0$. Thus we are left with (119) and (121). A similar reasoning is valid for χ_- and concludes the proof of (116).
As u_y is continuous across Γ, the first contribution to the boundary term (121) sums up to yield $-\int u_y[w_x]\partial_x\eta\,dx$. As also w_y is continuous across Γ, the second contribution to the boundary term vanishes after summation. This shows (104).

We now justify $\nabla q \in L^2$ and show the estimate. To achieve this, we show that the expression (104) defines a bounded functional on $\eta \in H^1$. The norm of the first term in the formula (104) is obviously estimated against the first term in the claim, (105). It remains to estimate the second term using the trace estimate. For this, we use Lemma 6 with the functions $f = [w_x]u_y$, $g = \eta$. We may conclude with the expression (105), (106), if we make a careful choice for the continuation $f(x,y)$: Let $h(x,y)$ the harmonic function with boundary values $[w_x]$, and $f(x,y) = h(x,y)u_y(x,y)$. Then

$$\int |\nabla f|^2 dxdy \;\le\; \sup_{x,y} |h|^2 \int |\nabla u_y|^2 dxdy \;+\; \sup |u_y|^2 \int |\nabla h|^2 dxdy. \tag{122}$$

For the first term, we have that by the maximum principle, $\sup_{x,y}|h| \le \sup_x |[w_x]|$, so that the first term is estimated against the second term (106) in the claim. Concerning the second term on the other hand, the Dirichlet integral satisfies

$$\int |\nabla h|^2 dxdy = \int |\partial_x^{\frac{1}{2}}[w_x]|^2 dx \le \int \chi_+|Dw_+|^2 + \chi_-|Dw_-|^2,$$

so that the second term is estimated against the first term (105) in the claim. $\square$

Proof of Proposition 12. Having shown the estimate (105), (106), we may invoke Lemma 6. This shows $[q] = 0$, and we may deduce the formula as the limit $\nabla\eta \to \nabla q$ of formula (104). $\square$

Proof of Proposition 13. We obviously have formula (107), which follows as above from the definition of q. We then appeal to

$$\left(\int |\nabla q|^2 dxdy\right)^{\frac{1}{2}} = \sup_\eta \frac{\int \nabla q \cdot \nabla\eta\,dxdy}{(\int |\nabla\eta|^2 dxdy)^{\frac{1}{2}}}.$$

Taking a look at the right hand side of (107) reveals that the first term is bounded against $\sup |Du_\pm|(\int |w|^2 dxdy)^{\frac{1}{2}}$. But the second term, by definition, gives rise to the $H^{\frac{1}{2}}$–norm. □

Proof of Remark 10. We start with the expression (107), and argue similarly as for the estimate (105), (106). The first term in (107) is bounded by $C(u) \int |w|^2 dxdy$, as a H^1–functional. For the second term, we invoke Lemma 6 with the functions $f = [u_x]w_y$, $g = \eta$, and with some continuation $f = hw_y$. It remains to estimate

$$\int |\nabla f|^2 dxdy \;\le\; \sup |h|^2 \int |\nabla w_y|^2 dxdy + \sup |\nabla h|^2 \int |w_y|^2 dxdy,$$

and to notice that $\sup |\nabla h|$ only depends on u, for a suitable choice of the continuation. □

Proof of Proposition 15. We start from (107). An integration by parts identifies the first term with

$$\int (\chi_+ \mathrm{tr} Du_+ Dw_+ + \chi_- \mathrm{tr} Du_- Dw_-)\eta\, dxdy + \int [w \cdot \nabla u_y]\eta\, dH^1(x). \quad (123)$$

Indeed more precisely,

$$-\int \chi_+((w_x\partial_x + w_y\partial_y)u_x\, \partial_x\eta) + (w_x\partial_x + w_y\partial_y)u_y\, \partial_y\eta)\, dxdy \quad (124)$$

$$= \int \chi_+\big(\partial_x((w_x\partial_x + w_y\partial_y)u_x) + \partial_y((w_x\partial_x + w_y\partial_y)u_y)\big)\, \eta\, dxdy \quad (125)$$

$$+ \int (w_x\partial_x + w_y\partial_y)u_y\, \eta\, dH^1(x). \quad (126)$$

This shows (123) in view of $\nabla \cdot u = 0$ in $\{y > 0\}$, if one differentiates the products in the first term (125). Moreover, due to the regularity assumption, the trace $(\partial_y u_y)_+$ is defined, and must equal $-\partial_x(u_+)_x$ because of the divergence condition.
A similar reasoning is valid for χ_-. Hence summing the two phases, the boundary term (126) gives rise to

$$\int [w_x]\, \partial_x u_y\, \eta - w_y\, \partial_x[u_x]\, \eta\, dx.$$

The sum with the second term in (107), after an integration by parts in x, yields the expression in (111). □

We may now proceed to discuss the tensor R.

Lemma 7. *We have that the contribution*

$$\int \nabla p_u \cdot \nabla p_w\, dxdy$$

is controlled by

$$\sup|u|\left(\int \chi_+|Du_+|^2+\chi_-|Du_-|^2dxdy\right)^{\frac{1}{2}}\sup|w|\left(\int \chi_+|Dw_+|^2+\chi_-|Dw_-|^2dxdy\right)^{\frac{1}{2}}.$$

If $u_\pm$ are smooth, then there is a constant $C=C(u)$ so that

$$\begin{aligned}&\left|\int \nabla p_u\cdot\nabla p_w\,dxdy\right|\\ &\qquad\le C\int|w|^2dxdy+C\left(\int g^2|\nabla w_y|^2dxdy\right)^{\frac{1}{2}}\left(\int|w|^2dxdy\right)^{\frac{1}{2}},\end{aligned}\qquad(127)$$

if g is a suitable C^1–continuation of $[u_x]$.

Proof. The first point is an application of Proposition 11. If $u_\pm$ are smooth, then by Corollary 2, D^2p_u is smooth up to the boundary, and the jump $[\partial_y p_u]=2[u_x]\partial_x u_y$ is also a smooth function (which vanishes if $[u_x]$ does). Consider the definition of p_w as in (115), and choose as test functions an approximation of p_u. We find that in the limit,

$$\begin{aligned}&\int\nabla p_u\cdot\nabla p_w\,dxdy\\ &=\int\chi_+D^2p_u:w_+\otimes w_++\chi_-D^2p_u:w_-\otimes w_-\,dxdy+\int[\partial_y p_u]\,w_y^2\,dH^1(x).\end{aligned}\qquad(128)$$

Indeed, this must be true by Corollary 2, if $w_\pm$ are smooth. In particular we are claiming then, as may also be seen as an application of Proposition 11 to the derivative $\partial_x q$, that $\partial_{xy}p_u\in L^2$, and only $\partial_{yy}p_u$ has a singular part. In a second step, we see, again by Corollary 2, that the function $D^2p_u|_{y\neq0}$ is actually bounded, and so is $[\partial_y p_u]$, so that by a similar trace estimate as above, it is clear that (128) holds also true if only $w\in L^2$, $w_y\in H^1$.

The first term in (128) is then estimated in an obvious way, and the relevant term is a boundary term, which has an explicit expression,

$$2\int[u_x]\,\partial_x u_y\,w_y^2\,dH^1(x).$$

We estimate it as follows: If g is a continuation of $[u_x]$, then $|g|$ is a continuation of $|[u_x]|$, $|\partial_y|g|| \leq |\partial_y g|$, and we have

$$\begin{aligned}
&\left|\int [u_x]\partial_x u_y\, w_y^2\, dH^1(x)\right| \\
&\qquad\qquad\qquad \leq \sup|\partial_x u_y| \int |[u_x]| w_y^2 dx \\
&\qquad\qquad\qquad = C(u) \int_{-\infty}^{0} \int \partial_y(|g| w_y^2)\, dxdy \\
&\qquad\qquad\qquad \leq C \iint |g||w_y||\partial_y w_y| dxdy + C(g) \iint w_y^2 dxdy.
\end{aligned}$$

We conclude by the Cauchy–Schwarz inequality. □

Our results may be summarized as follows:

Corollary 3. *The curvature tensor,*

$$R(u,w) = -\int |\nabla q|^2 dxdy + \int \nabla p_u \cdot \nabla p_w\, dxdy,$$

for smooth $u_\pm$ and potential $w_\pm = \nabla\phi_\pm$, satisfies

$$-R(u,w) \,+\, C(u)\int |w|^2 dxdy \;\sim\; \int |\partial_x^{\frac{1}{2}}([u_x]w_y)|^2\, dH^1(x) \,+\, \int |w|^2 dxdy. \tag{129}$$

Here, $\sim$ means equivalence of norms.

Proof. The statement follows from an application of Proposition 13 to the first term $\int |\nabla q|^2 dxdy$, and the combination of Proposition 14 and Lemma 7 for the second term $\int \nabla p_u \cdot \nabla p_w dxdy$. □

6.4.1 Construction of unstable directions

As in the previous discussion in Section 6.3, we obtained a lower bound, Corollary 3, and are now interested in showing that it matches (87) in terms of scaling. That is, we provide a choice of $u_\pm$, $w_\pm$, and derive an asymptotic expression.

It turns out that for the situation with an interface, the construction of unstable directions is slightly more delicate. To see why, consider a similar rescaling procedure as in the last section,

$$w^N(x,y) = N^{\frac{1}{2}} w(Nx, Ny), \tag{130}$$

for a given velocity w. This time, we normalized so that $\int w^2 dxdy = O(1)$. Note that because of $y \in \mathbb{R}$, an additional re–normalizing factor appears. In this case, the quantity $\int \chi_+|Dw_+|^2 + \chi_-|Dw_-|^2 dxdy$ is of order N^2, while $\sup|w|^2$ is still of order N.

We will now argue for smooth $u_\pm$, and for potential $w_\pm$, so that the conclusion of Corollary 3 is valid, hence avoiding this difficulty. Precisely fix a pair $w_\pm = \nabla\phi_\pm$ and define $w_\pm^N$ by (130). Let g denote a suitable smooth continuation of $[u_x]|_\Gamma$.

It remains then, in view of Proposition 14, to show that the semi–norm

$$\int g^2 |\nabla w_y^N|^2 dxdy \;\geq\; O(N^2).$$

Indeed, it may be written as

$$N^3 \int |g(x,y)|^2 |\nabla w_y(Nx,Ny)|^2 dxdy \tag{131}$$

$$= N^2 \left(\int_\Gamma |[u_x]|^2 dH^1(x) \int |\nabla w_y(\tilde{x},\tilde{y})|^2 d\tilde{x}d\tilde{y} \;+\; o(1) \right), \tag{132}$$

by a similar argument about decomposition of the integral as in the previous section. More precisely, here the measure $N|\nabla w_y(Nx,Ny)|^2 dxdy$ is bounded uniformly, and converges to a measure which is supported only on $\{y=0\}$ (in view of the rescaled y–variable), and has constant density (in view of the rescaled x–variable).

Arguably, also if $w_y^\pm$ are not harmonic but satisfy some weaker condition, a suitable choice of test function on two scales may show an estimate of the form

$$\sup_\eta \frac{\int [u_x] w_y^N \partial_x \eta \, dH^1(x)}{\left(\int |\nabla \eta|^2 dxdy\right)^{\frac{1}{2}}} \;\geq\; O(N).$$

We thus found that Corollary 3 matches (87) in terms of scaling. Moreover, the asymptotic expression (132) is actually the full $H^{\frac{1}{2}}$–norm of w_y, which equals some multiple of the full Dirichlet integral $\int \chi_+ |Dw_+|^2 + \chi_- |Dw_-|^2 dxdy$, as long as $[u_x]$ does not vanish altogether.

6.5 Continuity of the second fundamental form, hence the curvature, for sharp interface

After the discussion of the last section, we are in a position to verify the following continuity property. We mention that the limit in this section somehow resembles the one in Section 6.2 above, where the vorticity function concentrates on an interface ($f' \to 2\delta$, which led to (86)), but is different in nature.

We would like to consider the question of a sequence of relaxed material partitions recovering again a sharp interface. More precisely, let us consider

Requirement–Conjecture 1. *Let (χ_+, χ_-) be a given interface with smooth boundary Γ. Let (u_+, u_-) be a given velocity, of class $H^1 \cap L^\infty$, so that*

$$\nabla \cdot (\chi_+ u_+ + \chi_- u_-) = 0.$$

Then there exists a sequence of smooth functions $(s_+^\varepsilon, s_-^\varepsilon)$, $(u_+^\varepsilon, u_-^\varepsilon)$, *with*

$$s_+^\varepsilon + s_-^\varepsilon = 1, \quad \nabla \cdot (s_+^\varepsilon u_+^\varepsilon + s_-^\varepsilon u_-^\varepsilon) = 0 \qquad \forall \varepsilon \geq 0,$$

so that as $\varepsilon \to 0$,

1. $s_\pm^\varepsilon \to \chi_\pm$ *in* L^{1*}, *that means* $\int h s_\pm^\varepsilon dxdy \to \int h\chi_\pm dxdy \ \ \forall\ h \in L^1(dxdy)$,

2. $s_\pm^\varepsilon u_\pm^\varepsilon \to \chi_\pm u_\pm, \quad s_\pm^\varepsilon Du_\pm^\varepsilon \to \chi_\pm Du_\pm \qquad$ *in* L^2,

3. $u_\pm^\varepsilon \in L^\infty$ *uniformly, with* $\chi_\pm(u_\pm^\varepsilon - u_\pm) \to 0$ *in* L^{1*}, *that means* $\int \chi_\pm h \cdot (u_\pm^\varepsilon - u_\pm) dxdy \to 0 \ \ \forall\ h \in L^1(dxdy)$,

4. $\int s_\pm |u_\pm^\varepsilon|^2 dxdy \to \int \chi_\pm |u_\pm|^2 dxdy$, *and* $\int s_\pm^\varepsilon |Du_\pm^\varepsilon|^2 dxdy \to \int \chi_\pm |Du_\pm|^2 dxdy.$

We consider this statement partly as a proposition, and actually suggest it is true. In particular we are again interested in the case where the interface is the line $\{y = 0\}$, and $s_\pm = s_\pm(y)$ depend only on the normal coordinate. We show in the Appendix 12, that a natural canditate for such an approximation can be obtained through a convolution, and indeed prove the proposition for the special case $s_\pm = s_\pm(y)$.

In this limit, we are interested in the continuity of second fundamental form. Here, we content ourselves to consider the listed properties as an assumption.

Proposition 16. *For two sequences,* $s_\pm^\varepsilon = s^\varepsilon(y)$, $u_\pm^\varepsilon$, $w_\pm^\varepsilon$ *as in the Requirement, we have*
i) $s_+^\varepsilon u_+^\varepsilon \otimes u_+^\varepsilon + s_-^\varepsilon u_-^\varepsilon \otimes u_-^\varepsilon \to \chi_+ u_+ \otimes u_+ + \chi_- u_- \otimes u_-$, *in the sense of measures.*
ii) If q_ε, q *are defined by*

$$\nabla \cdot \nabla \cdot (s_+^\varepsilon u_+^\varepsilon \otimes w_+^\varepsilon + s_-^\varepsilon u_-^\varepsilon \otimes w_-^\varepsilon) + \Delta q_\varepsilon = 0,$$

$$\nabla \cdot \nabla \cdot (\chi_+ u_+ \otimes w_+ + \chi_- u_- \otimes w_-) + \Delta q = 0,$$

we have that

$$\int |\nabla q_\varepsilon|^2 dxdy \longrightarrow \int |\nabla q|^2 dxdy. \tag{133}$$

In particular, we claim that the expression $\int |\nabla q_\varepsilon|^2 dxdy$ is bounded uniformly, above and below, thus establishing a sort of compatibility between the estimates for homogeneous mixture of Section 6.3 and those for sharp interface in Section 6.4.

Proof of Proposition 16. The first point is an obvious consequence of the assumptions. It implies in particular that $q_\varepsilon \to q$ in the sense of measures.

We next establish (133), that is $\nabla q_\varepsilon \to \nabla q$ in L^2. Notice that we have shown already in Proposition 11, that the limit $\int |\nabla q|^2 dxdy$ must be finite. Let thus η be a smooth test function. Then on the level of the s_ε, one has after an integration by parts,

$$\int \nabla q_\varepsilon \cdot \nabla \eta \, dxdy = \int (s_+^\varepsilon D w_+^\varepsilon u_+^\varepsilon + s_-^\varepsilon D w_-^\varepsilon u_-^\varepsilon) \cdot \nabla \eta \, dxdy \tag{134}$$

$$+ \int (\nabla \cdot (s_+^\varepsilon u_+^\varepsilon))(w_+^\varepsilon)_x \partial_x \eta \; + (\nabla \cdot (s_-^\varepsilon u_-^\varepsilon))(w_-^\varepsilon)_x \partial_x \eta \, dxdy \tag{135}$$

$$+ \int (\nabla \cdot (s_+^\varepsilon u_+^\varepsilon))(w_+^\varepsilon)_y \partial_y \eta + (\nabla \cdot (s_-^\varepsilon u_-^\varepsilon))(w_-^\varepsilon)_y \partial_y \eta \, dxdy. \tag{136}$$

We claim that this expression for ∇q_ε converges to the corresponding one found for ∇q in (104), weakly, and indeed strongly in the sense of H^1–functionals, which is equivalent to the claim. The first term (134) obviously converges to the first term in (104), since $s_\pm^\varepsilon D w_\pm^\varepsilon \to \chi_\pm D w_\pm$ converges in L^2, and $u_\pm^\varepsilon \to u_\pm$ converge in L^{1*}, finally $\nabla \eta \in L^2$. More precisely, the difference of the first term to its limit is the sum of two terms of the form

$$\int (s_\pm^\varepsilon D w_\pm^\varepsilon - \chi_\pm D w) \, u_\pm^\varepsilon \cdot \nabla \eta \, dxdy + \int \chi_\pm D w_\pm \, (u_\pm^\varepsilon - u_\pm) \cdot \nabla \eta \, dxdy,$$

which converge to zero because of assumptions 2. and 3.

We claim further that the second term (135) converges to the second term in (104), and that finally, the third term (136) converges to zero, by virtue of the side constraint.

We address more precisely the second term (135), which after an integration by parts is found to equal

$$\begin{aligned} &\int (s_+ u_+ \cdot \nabla w_x^+ + s_- u_- \cdot \nabla w_x^-) \partial_x \eta \, dxdy + \int (s_+ u_+ w_x^+ + s_- u_- w_x^-) \cdot \nabla \partial_x \eta \, dxdy \\ = &\int (s_+ u_+ \cdot \nabla w_x^+ + s_- u_- \cdot \nabla w_x^-) \partial_x \eta \, dxdy - \int [s_+ \partial_x (u_+ w_x^+) + s_- \partial_x (u_- w_x^-)] \cdot \nabla \eta \, dxdy, \end{aligned} \tag{137}$$

where we used that $s_\pm = s_\pm(y)$. This shows that it is bounded in terms of the assumed quantities, as a functional w.r.t. $\eta \in H^1$. In particular, it must be weakly convergent.

As for the third term, differentiating the bracket yields an expression

$$\int [s_+ (\nabla \cdot u_+) w_y^+ + s_- (\nabla \cdot u_-) w_y^-] \partial_y \eta \, dxdy + \int [\partial_y s_+ u_y^+ w_y^+ + \partial_y s_- u_y^- w_y^-] \partial_y \eta \, dxdy.$$

Using then all three manifestations of the side constraint, we find that

$$\begin{aligned} &\partial_y s_+ \, u_y^+ \, w_y^+ + \partial_y s_- \, u_y^- \, w_y^- \\ = \; &(\partial_y s_+ u_y^+ + \partial_y s_- u_y^-) w_y^+ + (\partial_y s_- + \partial_y s_+) u_y^- w_y^+ + u_y^- (\partial_y s_+ w_y^+ + \partial_y s_- w_y^-) \\ &\quad = \; -[s_+ (\nabla \cdot u_+) + s_- (\nabla \cdot u_-)] w_y^+ - u_y^- [s_+ (\nabla \cdot w_+) + s_- (\nabla \cdot w_-)]. \end{aligned}$$

Hence also the third term is a bounded H^1–functional. It is even strongly convergent by assumption, with limit zero, as $Du_\pm$, $Dw_\pm$ must be trace–free in the limit. Here, for example,

$$\int s_-^\varepsilon\,(\nabla\cdot u_-^\varepsilon)\,(w_y)_+^\varepsilon\,\partial_y\eta\,dxdy \;=\; \int (s_-^\varepsilon\,(\nabla\cdot u_-^\varepsilon) - \chi_-(\nabla\cdot u_-))\,(w_y)_+^\varepsilon\,\partial_y\eta\,dxdy \;\to\; 0,$$

because $s_-^\varepsilon Du_-^\varepsilon$ converges in L^2, and w_+^ε is bounded in L^∞.

Finally we identify the limit of the second term (135), and see also that the convergence must be strong in the sense of H^1–functionals. This holds because for the limit, the same expansion holds true, precisely

$$\begin{aligned}&\int u_y\,[w_x]\,\partial_x\eta\,dH^1(x)\\ =&\int(\chi_+u_+\cdot\nabla w_x^+ + \chi_-u_-\cdot\nabla w_x^-)\,\partial_x\eta\,dxdy - \int(\chi_+\partial_x(u_+w_x^+) + \chi_-\partial_x(u_-w_x^-))\cdot\nabla\eta\,dxdy,\end{aligned}$$

so that (137), hence (135), for similar reasons as before, must converge by the given assumptions to the pairing $\langle u_y[w_x],\eta\rangle = \int u_y\,[w_x]\,\partial_x\eta\,dH^1(x)$, uniformly in $\eta\in H^1$. $\square$

7 Potential HVSE

For the potential HVSE model, only slight modifications of our discussion are necessary. Since we discuss in particular the term from the submersion, let for this section again $x \in \mathbb{T}^d$ denote a general spatial variable.

7.1 The expression of O'Neill/Otto

Here we address first the term (51) from the introductory Section 4.1,

$$N(\phi_1, \phi_2) = \inf_{\pi} \int \rho |[\nabla\phi_1, \nabla\phi_2] - \nabla\pi|^2 \, dx.$$

We expect that the following quotient is divergent:

$$\sup_{\phi_1, \phi_2} \frac{\inf_\pi \int \rho |[\nabla\phi_1, \nabla\phi_2] - \nabla\pi|^2 \, dx}{\int \rho |\nabla\phi_1|^2 dx \int \rho |\nabla\phi_2|^2 dx} = \infty.$$

We show here only

Proposition 17. *(Contribution from the submersion)*

$$\inf_{\pi} \int \rho |[\nabla\phi_1, \nabla\phi_2] - \nabla\pi|^2 \, dx \;\leq\; \sup |D^2\phi_1|^2 \int \rho |\nabla\phi_2|^2 \, dx.$$

In particular, if ϕ_1 is smooth, then $\nabla\phi_2 \mapsto N(\phi_1, \phi_2)$ is a bounded quadratic form, and yields bounded sectional curvature.

Proof. This statement follows from continuity of the projection, as applied to the asymmetric formula

$$N(\phi_1, \phi_2) = \inf_{\pi} \int \rho |D^2\phi_1 \nabla\phi_2 - \nabla\pi|^2 \, dx, \tag{138}$$

which in turn is due to $D^2\phi_1 \nabla\phi_2 + D^2\phi_2 \nabla\phi_1 = \nabla\left(\nabla\phi_1 \cdot \nabla\phi_2\right)$. $\square$

We obtain in the same way the following estimate, uniform in $s_\pm$,

Proposition 18. *(HVSE)*
For all pairs (s_+, s_-) we have with constant 1,

$$\begin{aligned} &\inf_{\pi_+} \int s_+ |[\nabla\phi_+^{(1)}, \nabla\phi_+^{(2)}] - \nabla\pi_+|^2 \, dx \,+\, \inf_{\pi_-} \int s_- |[\nabla\phi_-^{(1)}, \nabla\phi_-^{(2)}] - \nabla\pi_-|^2 \, dx \\ &\qquad \leq\; \sup |D^2\phi_+^{(1)}|^2 \int s_+ |\nabla\phi_+^{(2)}|^2 \, dx \,+\, \sup |D^2\phi_-^{(1)}|^2 \int s_- |\nabla\phi_-^{(2)}|^2 \, dx. \end{aligned}$$

7.2 Continuity for sharp interface

The same continuity property for the curvature tensor as in the last section is addressed. As we will show, it is no problem to see that the O'Neill–term is a contribution which is continuous with respect to the identified natural convergences. The continuity property analogous with Section 6.5 hence follows, if we may rely on the slightly more delicate set of properties,

Requirement–Conjecture 2. *Let (χ_+, χ_-) be a given interface with smooth boundary Γ. Let $(\nabla\phi_+, \nabla\phi_-)$ be a given velocity, of class H^1 and, say, $C^{0,\alpha}$ $\exists\, \alpha > 0$. Then there exists a sequence $(s_+^\varepsilon, s_-^\varepsilon)$, $(\phi_+^\varepsilon, \phi_-^\varepsilon)$, with*

$$\nabla \cdot (s_+^\varepsilon \nabla\phi_+^\varepsilon + s_-^\varepsilon \nabla\phi_-^\varepsilon) = 0 \quad \forall\varepsilon \geq 0,$$

so that as $\varepsilon \to 0$,

1. $s_\pm^\varepsilon \to \chi_\pm$ *in* L^{1*},
2. $s_\pm^\varepsilon \nabla\phi_\pm^\varepsilon \to \chi_\pm \nabla\phi_\pm, \quad s_\pm^\varepsilon D^2\phi_\pm^\varepsilon \to \chi_\pm D^2\phi_\pm$ *in* L^2,
3. $\nabla\phi_\pm^\varepsilon \in L^\infty$ *uniformly, with* $\chi_\pm(\nabla\phi_\pm^\varepsilon - \nabla\phi_\pm) \to 0$ *in* L^{1*},
4. $\int s_\pm |\nabla\phi_\pm^\varepsilon|^2 dx \to \int \chi_\pm |\nabla\phi_\pm|^2 dx, \quad \int s_\pm^\varepsilon |D^2\phi_\pm^\varepsilon|^2 dx \to \int \chi_\pm |D^2\phi_\pm|^2 dx.$

Again, we view this statement partly as a conjecture. We will not insist on its proof, but we propose a natural candidate sequence in Appendix 12, and at least prove parts of its statement.

To argue for the continuity of the curvature–tensor, we view the listed properties as an assumption. We then have

Proposition 19. *(Continuity of the O'Neill term)*
Under the assumptions of the Requirement, it holds that

$$\inf_{\pi_+} \int s_+^\varepsilon |[\nabla\phi_+^{(1)}, \nabla\phi_+^{(2)}]^\varepsilon - \nabla\pi_+|^2\, dx + \inf_{\pi_-} \int s_-^\varepsilon |[\nabla\phi_-^{(1)}, \nabla\phi_-^{(2)}]^\varepsilon - \nabla\pi_-|^2\, dx$$

converges to

$$\inf_{\pi_+} \int \chi_+ |[\nabla\phi_+^{(1)}, \nabla\phi_+^{(2)}] - \nabla\pi_+|^2\, dx + \inf_{\pi_-} \int \chi_- |[\nabla\phi_-^{(1)}, \nabla\phi_-^{(2)}] - \nabla\pi_-|^2\, dx$$

Together with Proposition 16, this asserts the continuity of the sectional curvatures, as defined by Proposition 7 in Section 4.2.

By means of the asymmetric formula (138), Proposition 19 is reduced to

Lemma 8. *For $u_\pm^\varepsilon$, $w_\pm^\varepsilon$ as in the Requirement 1 (as in Section 6.5),*

$$\lim_{\varepsilon\to 0}\inf_\pi \int s_\pm^\varepsilon |Dw_\pm^\varepsilon u_\pm^\varepsilon - \nabla\pi|^2 dx \;=\; \inf_\pi \int \chi_\pm |Dw_\pm\, u_\pm - \nabla\pi|^2 dx.$$

Proof. Let us first show that

$$\lim_{\varepsilon\to 0}\int s_\pm^\varepsilon |Dw_\pm^\varepsilon u_\pm^\varepsilon - \chi_\pm Dw_\pm\, u_\pm|^2 dx = 0. \tag{139}$$

This is clear because similar with the argument in the last section, by assumptions 2., 3., 4., and 1.,

$$\int s_\pm^\varepsilon |Dw_\pm^\varepsilon - \chi_\pm Dw_\pm|^2 |u_\pm^\varepsilon|^2 dx + \int s_\pm^\varepsilon \chi_\pm |Dw_\pm|^2\, |u_\pm^\varepsilon - u_\pm|^2 dx$$

converges to zero (expanding the square for the first term).
It is then obvious (commuting limit and infimum) that

$$\limsup_{\varepsilon\to 0}\inf_\pi \int s_\pm^\varepsilon |Dw_\pm^\varepsilon u_\pm^\varepsilon - \nabla\pi|^2 dx \;\le\; \inf_\pi \int \chi_\pm |Dw_\pm\, u_\pm - \nabla\pi|^2 dx.$$

This would already be sufficient for our discussion, which is concluded with this upper bound on the positive part of the curvature tensor.
To show the equality, it is necessary to understand the optimal π. This is also possible, using more explicitly the construction in the appendix which produces the recovery sequence. Precisely, let $\pi_\pm^\varepsilon$ and $\pi_\pm$ be the optimal functions in the infimum, that is, the solutions of

$$\nabla\cdot(s_\pm^\varepsilon Dw_\pm^\varepsilon u_\pm^\varepsilon - s_\pm^\varepsilon \nabla\pi_\pm^\varepsilon) = 0, \quad \nabla\cdot(\chi_\pm Dw_\pm u_\pm - \chi_\pm \nabla\pi_\pm) = 0.$$

Then, as is argued at the end of Appendix 12, the convergence in (139) entails

$$\int s_\pm^\varepsilon |\nabla\pi_\pm^\varepsilon - \chi_\pm \nabla\pi_\pm|^2 dx \;\to 0.$$

This shows the claim of the lemma, hence also Proposition 19. $\square$

7.3 Summary and discussion

We showed that in the two prototype cases of a homogeneous mixture and a sharp interface, the sectional curvature of M or N is bounded above and unbounded below. More precisely, if $u_\pm$ denote a smooth pair of velocities, the sectional curvature is bounded from above by the metric tensor as applied to the perturbation fields $w_\pm$.
The passage to infinite negative sectional curvature is then governed by the leading term $-\int |\nabla q|^2 dx$, which behaves as a certain H^1–type norm of the perturbation that was identified in the two special cases. We believe that it

follows with similar arguments that such a behaviour is common to all $s_\pm$ which are fixed positive smooth functions. We leave as a conjecture, that this be indeed the general behaviour, uniform in $s_\pm$, and we suggest that the approximation procedure presented is an indication that this may be true. Indeed we showed in Sections 6.5 and 7.2 that the expression for the sectional curvature behaves in a continuous way in the limit $s_\pm \to \chi_\pm$.

8 As an additional consideration: Asymptotic directions

In this section, we discuss the occurrence of *asymptotic directions*: For an embedding of a Riemannian manifold in some ambient space, this notion describes those tangent vectors to the sub–manifold, for which the geodesic in the sub–manifold with this start velocity does not differ up to second order in time from the geodesic in the ambient space with this start velocity. It means precisely, that in the direction of this tangent vector X, the second fundamental form vanishes, more precisely that the normal vector $B(X, X)$ is zero. In the case of co–dimension 1, and flat ambient space, the presence of such asymptotic directions is directly linked to the sign of the sectional curvatures: Presence of asymptotic directions implies presence of semi–negative curvature, and their absence implies positive curvature. In the case of larger codimension, in the case to be discussed even of infinite co–dimension, the first implication remains true, but the second does not.

We wish to discuss the embedding $M \subset D \times D$: Here, there are many asymptotic directions, since any shear flow $u = f(y)\frac{\partial}{\partial x}$ gives rise to a stationary solution with zero pressure. This indicates the presence of a lot of negative sectional curvatures, and such shear flows even span L^2. This is analogous to what was oberved for the classical embedding $D_\mu \subset D$, in [2] and also in [7].

In the case of the embedding $N \subset P \times P$ for the potential HVSE–system, however, the presence of such asymptotic directions is not so clear, at least not in a sufficiently regular class. We would like to formulate this as a problem

Conjecture 1. *Let (s_+, s_-) be a pair of smooth functions on the torus, valued in $(0, 1)$ and so that $s_+ + s_- = 1$. Let (ϕ_+, ϕ_-) be a pair of functions which satisfies*

$$\nabla \cdot (s_+ \nabla \phi_+ + s_- \nabla \phi_-) = 0, \tag{140}$$

and solves

$$\nabla \cdot (\nabla \cdot (s_+ \nabla \phi_+ \otimes \nabla \phi_+ + s_- \nabla \phi_- \otimes \nabla \phi_-)) = 0. \tag{141}$$

Then $(\nabla \phi_+, \nabla \phi_-) = (0, 0)$.

In the homogeneous case, $s_\pm = \frac{1}{2}$, the statement simplifies, and we can prove at least the following fact.

Lemma 9. *Let $\phi(x)$ be a smooth function on the torus, $x \in \mathbb{T}^d$, which solves*

$$\nabla \cdot (\nabla \cdot (\nabla \phi \otimes \nabla \phi)) = 0. \tag{142}$$

Then ϕ is constant.

Proof. For a smooth solution, we may carry out the differentiation, to find

$$2\nabla\phi \cdot \nabla\Delta\phi + (\Delta\phi)^2 + |D^2\phi|^2 = 0.$$

Consider a point where $\Delta\Phi$ has a local maximum or minimum. In such a point in fact,

$$(\Delta\phi)^2 + |D^2\phi|^2 = 0,$$

whence in particular $\Delta\phi = 0$. This implies that

$$\Delta\phi = 0 \quad \text{on the torus.}$$

We conclude with the assertion, now by the maximum principle applied to ϕ. □

More generally, we can remark that the constrained equation (141) in the conjecture can be reduced to a single equation,

$$\nabla \cdot \nabla \cdot (L\phi \otimes L\phi + \alpha\nabla\phi \otimes \nabla\phi) \;=\; 0. \tag{143}$$

This follows from the following lemma which makes more precise Lemma 1,

Lemma 10. *The pair $\nabla\phi_\pm$ is recovered from the two quantities*

$$u = s_+\nabla\phi_+ + s_-\nabla\phi_-, \qquad \phi = \phi_+ - \phi_-.$$

In particular, the tensor of second moments has the expression

$$s_+\nabla\phi_+ \otimes \nabla\phi_+ + s_-\nabla\phi_- \otimes \nabla\phi_- = u \otimes u + s_+s_-\nabla\phi \otimes \nabla\phi.$$

Finally in (143), we wrote $\alpha = s_+s_-$ and $u = L\phi$, on the basis of

Remark 12. *The mean velocity u only depends on the pair $\phi_\pm$ through their difference ϕ, since*

$$\nabla \cdot u = 0, \quad \nabla \times u = \nabla s_+ \times \nabla\phi.$$

Remark 13. *(Regularity class)*
Lemma 9 holds true, if $\phi \in C^3$. Notice on the other hand that the lemma is false, if one requires only $\nabla\phi \in L^2$: any solution to the Eikonal–equation in one dimension, $|\varphi'|^2 = 1$, will produce a solution to (142) of the form

$$\phi(x) = \varphi(k \cdot x),\ k \in \mathbb{Z}^d,$$

and these functions even span $L^2(dx)$. On the other hand they would each give rise to an infinite sectional curvature, and it remains open whether (142) has non–trivial solutions, if one restricts for example to $D^2\phi \in L^2$.

9 Appendix: A spatially homogeneous solution to the HVSE

We seek a particular solution of the one–dimensional system

$$\partial_t u + u\,\partial_y u + \partial_y(uv) = 0, \tag{144}$$
$$\partial_t v + v\,\partial_y v + \partial_y(uv) = 0, \tag{145}$$

where this time $y \in \mathbb{R}$. Whenever there is a solution in the elliptic case $uv < 0$, it is possible to reconstruct the densities $s_\pm$ from $s_+ + s_- = 1$ and $s_+u + s_-v = 0$, and by construction they will satisfy the transport equation. We make now the following *ansatz*; it is a generalization of the formula $u = \frac{y}{t}$, which describes a kind of fundamental solution in the hyperbolic case of a single mass transport. Let thus u, v be of the form

$$u(t,y) = a(t)y, \quad v(t,y) = b(t)y,$$

for functions a and b. The merit of this solution must remain somewhat open, since it has infinite energy. The system then reduces to two ordinary differential equations for a, b:

$$\partial_t a + a^2 + 2ab = 0, \tag{146}$$
$$\partial_t b + b^2 + 2ab = 0. \tag{147}$$

We are interested in solutions in the region $a > 0$, $b < 0$. (On reversing time they correspond directly to solutions with $a < 0$, $b > 0$.) The vector field $(\partial_t a, \partial_t b)$ is homogeneous, so if one understands one solution, all others can be obtained by scaling.
There are five straight lines in the (a,b)–plane which confine an orbit. $\{a = 0\}$, which will not be passed because there, $\partial_t a = 0$. $\{2a + b = 0\}$, which is a line on which $\partial_t b = 0$. $\{a + b = 0\}$, which is a line of symmetry: if (t,a,b) is a solution, so is $(-t,-b,-a)$, in particular, if $-a = b$ for $t = 0$, the trajectory has this symmetry. $\{2b + a = 0\}$, where $\partial_t a = 0$. $\{b = 0\}$, which cannot be passed since there, $\partial_t b = 0$.

Proposition 20. *The solution through* $(t = 0, a = 1, b = -1)$ *behaves as follows: It is defined on* $t \in (-\infty, +\infty)$*, and*
i) for $t \to -\infty$*, it approaches the origin* $(0,0)$*, more precisely,* $a(t) \sim \frac{1}{t^2}$*, and* $b \sim \frac{1}{t}$*,*
ii) it passes the three lines mentioned, so that afterwards, $\partial_t a$*,* $\partial_t b$ *have changed sign, in particular, in* $t = 0$*,* $(\partial_t a, \partial_t b) = (1,1)$*,*
iii) for $t \to +\infty$*, it approaches again the origin* $(0,0)$*, more precisely,* $a(t) \sim \frac{1}{t}$*, and* $b \sim -\frac{1}{t^2}$*.*

Proof. It is enough to show iii). For large times, by what was said before, one must have $b \to 0$ and $|b| \ll a$. Hence one determines the behaviour of a from

$$\partial_t a + a^2 = 0,$$

which yields $a \sim \frac{1}{t}$. Then b is approximately subject to

$$0 = \partial_t b + 2ab = \partial_t b + \frac{2}{t}\, b,$$

whence $-b \sim \frac{1}{t^2}$. Indeed one may check also that $\frac{a^2}{b}$ is approximately constant via

$$\partial_t \frac{a^2}{b} = -3b\frac{a^2}{b}.$$

If $\int(-b)dt < \infty$, this implies $-b \sim a^2$. □

These solutions are bounded, in contrast to the one–phase hyperbolic case, where $u = \frac{y}{t}$ has a singularity in $t = 0$.
On the level of s, these solutions are very simple, since due to the momentum balance $as_+ + bs_- = 0$, $s_\pm = s_\pm(t)$ must be homogeneous in space. Such a solution exchanges phases in the sense

$$s_+ \to 1, s_- \to 0, \quad \text{as } t \to -\infty, \qquad s_+ \to 0, s_- \to 1, \quad \text{as } t \to +\infty.$$

On the level of the particles, if for large positive times, $a(t) \sim \frac{1}{t}$, then particles follow straight lines:

$$\dot{y} \sim \frac{y}{t} \implies y(t) \sim u_0 t.$$

On the other hand, if for large negative times, $a(t) \sim \frac{1}{t^2}$, then particles move only a finite distance:

$$\dot{y} \sim \frac{y}{t^2} \implies \int \partial_t \log y \, dt < \infty.$$

A similar fact holds for the particles in the $(-)$–phase: for negative times they are straight lines, but for positive times, trajectories are bounded.

10 Appendix: Calculations for the linear stability analysis

10.1 Computation of a discriminant

Here, we prove Proposition 4. In order to be able to make an according remark we introduce in this section an additional parameter $\rho_i \in [0, 1]$. Hence we study

$$\int s|u|^2 d\mu + \rho_i \int (1-s)|v|^2 d\mu,$$

subject to the transport equations

$$\partial_t s + \nabla\cdot(su) = 0, \quad \partial_t(1-s) + \nabla\cdot((1-s)v) = 0.$$

Here, in favour of the formulation (154) below, we used the notation $(s, 1-s) = (s_+, s_-)$, $(u, v) = (u_+, u_-)$.
The geodesic equations are then changed according to

$$\partial_t u + Du\, u \;=\; \rho_i(\partial_t v + Dv\, v) \;=\; -\nabla p.$$

Hence ρ_i plays the role of a density of inertial mass in the $(-)$–phase with respect to the $(+)$–phase. As $\rho_i \to 0$, the particles in the $(-)$–phase are infinitely easy to move and we are left with an equation $\partial_t u + Du\, u = 0$ for the velocity transporting a single density function $s \le 1$. This case corresponds to the classical optimal mass transport.
In this section we study only the 1-dimensional system

$$\begin{aligned}
\partial_t s + \partial_y(su) &= 0, && (148)\\
\partial_t u + u\,\partial_y u + \partial_y p &= 0, && (149)\\
\partial_t(1-s) + \partial_y((1-s)v) &= 0, && (150)\\
\rho_i(\partial_t v + v\,\partial_y v) + \partial_y p &= 0, && (151)\\
\text{with} \quad su + (1-s)v &= 0. && (152)
\end{aligned}$$

In the case $\rho_i = 1$, there is a symmetry between the two phases and it is possible to substitute the pressure according to $p = -uv$. In the present case we prefer the equally fine approach to write the system for two effective variables s and u. Let us assume here that $s < 1$. This is reasonable since whenever $s(y) = 1$ we must have $u(y) = 0$. Thus we compute the pressure as follows: Note that $\rho_i = 0$ implies $\partial_y p = 0$, and otherwise

$$0 = -\partial_t(s_+u) - \partial_t(s_-v) = \partial_y(s_+u^2) + s_+\partial_y p + \partial_y(s_-v^2) + \frac{1}{\rho_i}s_-\partial_y p,$$

whence $\partial_y p = -\frac{1}{s+\frac{1}{\rho_i}(1-s)}\partial_y(su^2 + (1-s)v^2) = -\frac{\rho_i}{\rho_i s + 1 - s}\partial_y\left(\frac{s}{1-s}u^2\right)$.
Therefore, we find that one can replace the system (148) ff. by $\partial_t s + \partial_y(su) = 0$ and

$$0 \;=\; \partial_t u + u\,\partial_y u - \frac{\rho_i}{\rho_i s + 1 - s}\partial_y\left(\frac{s}{1-s}u^2\right). \tag{153}$$

We used here that $su^2 + (1-s)v^2 = su^2 + \frac{s^2}{1-s}u^2 = \frac{s}{1-s}u^2$. Notice moreover that $\partial_y \frac{s}{1-s} = \frac{1}{(1-s)^2}\partial_y s$, so that we can write the system as a conservation law:

$$\begin{pmatrix}\partial_t s\\ \partial_t u\end{pmatrix} + \begin{pmatrix}u & s\\ 0 & u\end{pmatrix}\begin{pmatrix}\partial_y s\\ \partial_y u\end{pmatrix} + \begin{pmatrix}0 & 0\\ -f(s,\rho_i)\,u^2 & -2g(s,\rho_i)\,u\end{pmatrix}\begin{pmatrix}\partial_y s\\ \partial_y u\end{pmatrix} = 0, \tag{154}$$

where we introduced the functions

$$f(s,\rho_i) = \frac{\rho_i}{\rho_i s + 1 - s}\frac{1}{(1-s)^2}, \quad g(s,\rho_i) = \frac{\rho_i}{\rho_i s + 1 - s}\frac{s}{1-s}.$$

We see that if $\rho_i = 0$, (153) is just the Burgers equation, which is hyperbolic: the first matrix in (154) has a double (degenerate) eigenvalue equal to the velocity u. For $\rho_i > 0$, we are interested in the spectrum of the sum of the two matrices. We have that the trace equals $2u - 2gu$, which turns into $2(u+v)$ in the symmetric case $\rho_i = 1$. Moreover the determinant is $(1-2g)\,u^2 + sfu^2$, so that we find a discriminant

$$\left(\frac{1}{2}\text{trace}\right)^2 - \text{determinant} \;=\; \left[(1-g)^2 - (1-2g) - sf\right]\, u^2.$$

We compute the expression

$$\begin{aligned}
& (1-g)^2 - (1-2g) - sf \\
= \quad & (1 - \frac{\rho_i s}{(\rho_i s + 1 - s)(1-s)})^2 - 1 + \frac{2\rho_i s}{(\rho_i s + 1 - s)(1-s)} - \frac{\rho_i s}{(\rho_i s + 1 - s)}\frac{1}{(1-s)^2} \\
= \quad & \frac{1}{(\rho_i s + 1 - s)^2\,(1-s)^2}[\rho_i^2 s^2 - \rho_i s(\rho_i s + 1 - s)] \\
= \quad & -\frac{\rho_i s(1-s)}{(\rho_i s + 1 - s)^2\,(1-s)^2}.
\end{aligned} \tag{155}$$

We hence see that the discriminant is non–positive and negative as soon as $\rho_i > 0$. Let us mention that in the case $\rho_i = 1$ it turns into $-\frac{s_+ s_-}{s_-^2}u^2 = uv$ (and is negative since u and v have opposite signs). We found that the system (154), hence (148) ff., has a double real eigenvalue u, for $\rho_i = 0$ (in which case it is hyperbolic), but a pair of genuinely complex eigenvalues, as soon as $\rho_i > 0$. In particular, in the case $\rho_i = 1$ it is elliptic. $\square$

It may be worth to include the

Remark 14. *(Hyperbolic scaling)*
It is a consequence of this elliptic property, that there cannot be a self–similar solution to (40) *of the form*

$$s(t,y) = s\left(\frac{y}{t}\right), \quad u(t,y) = u\left(\frac{y}{t}\right).$$

Indeed, in this case, denoting $\xi = \frac{y}{t}$, and $q = q(\xi)$, $q' = \frac{\partial q}{\partial \xi}$, (40) turns into

$$-\xi q' + A(q)q' = 0,$$

which cannot hold if A has no real eigenvectors.

10.2 Convexity of the kinetic energy density

To round up the discussion, we take in this section a look at the action functional and prove Proposition 5. As we have seen, its strict convexity is related to the elliptic character of the system (38), (39).
We will keep again the parameter $\rho_i \in [0,1]$. Hence we are interested in the function

$$F(s_+, s_-, m_+, m_-) = s_+|u|^2 + \rho_i s_-|v|^2 = \frac{1}{s_+}|m_+|^2 + \rho_i \frac{1}{s_-}|m_-|^2.$$

We will compute the Hessian and show that it is only positive semidefinite for $\rho_i = 0$, but positive definite as soon as $\rho_i > 1$. This is linked to the elliptic property: in the case of classical mass transport it gives only information on the velocity variable $\partial_t h = -m$, thus giving rise to a hyperbolic system, whereas the coupling of the phases ($\rho_i > 1$) produces a convexity also with respect to the space derivative $\partial_y h = s$, so that the resulting system becomes elliptic.
Explicitly, let us compute the Hessian of the function $F_0(s,m) = \frac{1}{s}|m|^2$. We have

$$\begin{pmatrix} \partial_{ss}F_0 & \partial_{sm}F_0 \\ \partial_{sm}F_0 & \partial_{mm}F_0 \end{pmatrix} = \frac{2}{s}\begin{pmatrix} \frac{|m|^2}{s^2} & -\frac{m}{s} \\ -\frac{m}{s} & 1 \end{pmatrix}.$$

Let $\gamma = \dot{s}$, $\alpha = \dot{m}$ be the linearized variables. The Hessian of F_0 is thus characterized by the quadratic form

$$\frac{2}{s}\left(\frac{|m|^2}{s^2}\gamma^2 - 2\frac{m}{s}\gamma\cdot\alpha + |\alpha|^2\right) \quad = \quad \frac{2}{s}\left|\frac{m}{s}\gamma - \alpha\right|^2.$$

We see that it is positive semidefinite. Now if accordingly $\gamma_\pm$, $\alpha_\pm$ denote linearized variables, we have that the Hessian of F gives rise to a quadratic form

$$\frac{2}{s_+}\left|\frac{m_+}{s_+}\gamma_+ - \alpha_+\right|^2 + \rho_i \frac{2}{s_-}\left|\frac{m_-}{s_-}\gamma_- - \alpha_-\right|^2. \tag{156}$$

Again it is positive semidefinite. But as soon as $\rho_i > 0$, it can only be zero if

$$\frac{m_+}{s_+}\gamma_+ - \alpha_+ = 0, \quad \frac{m_-}{s_-}\gamma_- - \alpha_- = 0. \tag{157}$$

Let us identify the kernel of the Hessian. Since the constraint $s_+ + s_- = 1$ implies $\gamma_+ + \gamma_- = 0$, we may write $\gamma_+ = \gamma = -\gamma_-$, and from γ, both α_+ and α_- are already determined. Now if $j = 0$, this gives the constraint $m_+ + m_- = 0$, as well as $\alpha_+ + \alpha_- = 0$. So in this case, if the form (156) is zero we infer in particular

$$0 = \frac{m_+}{s_+}\gamma_+ + \frac{m_-}{s_-}\gamma_- - \alpha_+ - \alpha_- = \frac{m_+}{s_+}\gamma - \frac{m_-}{s_-}\gamma.$$

It follows that either γ is zero (and hence $\alpha_\pm$ by (157)), or

$$\frac{m_+}{s_+} - \frac{m_-}{s_-} = 0,$$

which means $u = v$. But this cannot happen if $m_+ + m_- = j = 0$, unless $u = v = 0$. So in the case that $j = 0$, the kernel is trivial, and the action is strictly convex.

The proof of Proposition 5, also for the case $s_+ = 0$, is completed in Appendix 12, according to the note following (178).

11 Appendix: Control by means of a weighted $H^{\frac{1}{2}}$–norm

In this section we prove Proposition 14 of Section 6.4. We believe that there can be a simpler argument for this fact, but we are quite willing to introduce the Hilbert transform.
For this, let w be a smooth harmonic function in the cylinder $(x, y) \in S^1 \times \{y \geq 0\}$, that vanishes at infinity as expressed by $\int |w|^2 dxdy < \infty$. Then in particular, $w(x, 0)$ must be mean–free: $\int w \, dH^1(x) = 0$, and also one has $\int |\nabla w|^2 dxdy < \infty$. We introduce

Lemma 11. *Consider the unique smooth integrable solution F of the Neumann-problem*

$$\Delta F = 0 \ \textit{in} \ y > 0 \tag{158}$$

$$\partial_y F = \partial_x w \ \textit{for} \ y = 0. \tag{159}$$

Then $F(x, 0)$, and also $\partial_y F(x, 0)$, are again mean–free, and the Cauchy–Riemann equations

$$\partial_y F = \partial_x w \tag{160}$$

$$\partial_x F = -\partial_y w \tag{161}$$

hold in $S^1 \times \{y \geq 0\}$. In particular, we have the point–wise identity $|\nabla F|^2 = |\nabla w|^2$.
Moreover, in terms of a Fourier expansion,

$$\textit{if } w = e^{ikx} e^{-|k|y}, \ \textit{then } F = -i \, (\mathrm{sign} k) \, w. \tag{162}$$

This implies that also

$$\int |w|^2 dxdy = \int |F|^2 dxdy. \tag{163}$$

This assignment, $w \mapsto F$, is also called the Hilbert transform.

Proof. The first equation, (160), holds because both $\partial_y F$ and $\partial_x w$ are harmonic and integrable, with the same Dirichlet datum by definition. We deduce (161) from (160) because we know already that w and F are harmonic: One has

$$-\partial_{yy} w = \partial_{xx} w = \partial_{xy} F,$$

and

$$\partial_{xy} w = \partial_{yy} F = -\partial_{xx} F,$$

whence $\partial_x F + \partial_y w$ is constant. Since it is also mean–free on $\{y = 0\}$, we obtain (161).

The formula (162) is obvious. To see (163), we have only to notice that the Fourier modes in (162) are not only orthogonal as elements of $L^2(dH^1(x))$, but also in $L^2(dxdy)$. Since under the assumptions given, $\int w^2 dH^1(x)$ is finite, and e^{ikx} form a basis of $L^2(dH^1(x))$, $w(x,y)$ possesses an expansion in these modes. Hence (163) follows. $\square$

Remark 15. *i) The function F is the test function which realizes*

$$\sup_f \frac{\int \partial_x w f dH^1(x)}{\left(\int |\nabla f|^2 dxdy\right)^{\frac{1}{2}}}. \tag{164}$$

ii) If $w = \partial_y \phi$ for a harmonic potential ϕ, then in fact $F = \partial_x \phi$.

Let us sketch for completeness a **proof of Lemma 6**. Expand $f(x,0) = \sum_{k\in\mathbb{Z}} f_k\, e^{ikx}$ in a Fourier series. Then $f(x,y) = \sum_{k\in\mathbb{Z}} f_k\, e^{ikx} e^{-|k|y}$ is the harmonic continuation in the half plane. Denote moreover by $\tilde{f}(x,y)$ any other continuation. Then one sees
1) $\int |\nabla \tilde{f}|^2 dxdy \geq \int |\nabla f|^2 dxdy$,
as a consequence of $f = \tilde{f}$ at $y = 0$, and $\Delta f = 0$,
2) $\int |\nabla f|^2 dxdy = \sum_{k\in\mathbb{Z}} |k|\, |f_k|^2$,
evaluating the integral $\int e^{-2|k|y} dy$ and using a Plancherel identity,
3) $\int_{\{y=0\}} f\, \partial_x g\, dx = i \sum_{k\in\mathbb{Z}} k\, f_k\, g_k$, also by Plancherel's identity.
The claim then follows as an application of the Cauchy Schwarz – inequality. $\square$

Next let g be a smooth function, vanishing at infinity. We obtain the desired estimate (109) as follows.

Lemma 12. *There is a constant $C = C(g)$, depending only on the upper bound* $\sup |\nabla g|$, *so that one has i)*

$$\int \partial_x (gw)\, gF\, dH^1(x) + C \int w^2 dxdy \;\geq\; \frac{1}{2} \int g^2 |\nabla w|^2 dxdy,$$

and ii)

$$\int |\nabla (gF)|^2 dxdy \;\leq\; 2 \int g^2 |\nabla w|^2 dxdy + C \int w^2 dxdy.$$

Proof. The second point is obvious:

$$\int |\nabla(gF)|^2 dxdy \;\le\; 2\int g^2|\nabla F|^2 dxdy + 2\int |\nabla g|^2 F^2 dxdy \tag{165}$$

$$\le\; 2\int g^2|\nabla F|^2 dxdy + 2\sup|\nabla g|^2 \int F^2 dxdy \tag{166}$$

$$=\; 2\int g^2|\nabla w|^2 dxdy + 2\sup|\nabla g|^2 \int w^2 dxdy, \tag{167}$$

by Lemma 11. To see i), we compute

$$\int \partial_x(gw)\, gF\, dH^1(x) \;=\; \int \partial_x g\, w\, g\, F\, dx + \int g\, \partial_x w\, g\, F\, dx \tag{168}$$

$$= \int \frac{1}{2}\partial_x(g^2)\, w\, F\, dx + \int g^2 \partial_x w\, F\, dx \tag{169}$$

$$= -\frac{1}{2}\int g^2 \partial_x(wF)\, dx + \int g^2\, \partial_x w\, F\, dx \tag{170}$$

$$= \frac{1}{2}\int g^2\, \partial_x w\, F\, dx - \frac{1}{2}\int g^2\, w\, \partial_x F\, dx. \tag{171}$$

According to Lemma 11, we identify this last expression as

$$\frac{1}{2}\int g^2\, \partial_y F\, F\, dx + \frac{1}{2}\int g^2\, w\, \partial_y w\, dx.$$

In view of the harmonicity of w, F, an application of the Gauss formula reveals that it can be written and estimated as

$$\begin{aligned}
&\frac{1}{2}\int \nabla\cdot(g^2 F\,\nabla F) dxdy + \frac{1}{2}\int \nabla\cdot(g^2 w \nabla w) dxdy \\
&= \frac{1}{2}\int g^2|\nabla F|^2 dxdy + \int g\nabla g\cdot F\nabla F dxdy + \frac{1}{2}\int g^2|\nabla w|^2 dxdy + \int g\nabla g\cdot w\nabla w dxdy \\
&\ge \frac{1}{4}\int g^2|\nabla F|^2 dxdy - \sup|\nabla g|^2 \int F^2 dxdy + \frac{1}{4}\int g^2|\nabla w|^2 dxdy - \sup|\nabla g|^2 \int w^2 dxdy \\
&\qquad = \frac{1}{2}\int g^2|\nabla w|^2 dxdy \;-\; 2\sup|\nabla g|^2 \int w^2 dxdy.
\end{aligned}$$

This proves i). □

Corollary 4.

$$\int g^2|\nabla w|^2 dxdy \;\le\; 4\int |\partial_x^{\frac{1}{2}}(gw)|^2 dH^1(x) \;+\; C(g)\int w^2 dxdy.$$

Proof. Denoting generic constants C_1, C_2, we have using Lemma 12, i) and ii),

$$
\begin{aligned}
C_1\left(\int w^2 dxdy\right)^{\frac{1}{2}} + \left(\int |\partial_x^{\frac{1}{2}}(gw)|^2 dx\right)^{\frac{1}{2}} \\
\geq\ C_1\|w\|_2 + \frac{\int \partial_x(gw)\, gF\, dx}{\left(\int |\nabla(gF)|^2 dxdy\right)^{\frac{1}{2}}} \\
\geq\ C_1\|w\|_2 + \frac{\frac{1}{2}\int g^2|\nabla w|^2 dxdy - C_2(\|w\|_2)^2}{(2\int g^2|\nabla w|^2 dxdy)^{\frac{1}{2}} + C_2\|w\|_2} \\
\geq\ \frac{1}{4}\left(\int g^2|\nabla w|^2 dxdy\right)^{\frac{1}{2}}. \quad \square
\end{aligned}
$$

This proves Proposition 14, because we may take w, g to be w_y, and g, the continuation of $[u_x]$. It is then apparent that the constant $C = C(\sup|\nabla g|)$ in fact depends only on $\sup|\partial_x[u_x]| + \sup|[u_x]|$. $\square$

12 Appendix: Estimate of a convolution operation

Let $y \in \mathbb{R}$. Let $\eta(y)$ be a nonnegative, symmetric kernel, $\int \eta \, dy = 1$. Let $\chi(y) = 1_{y>0}$ be the indicator function of the right half–line, and let $f(y)$ be a function.
Let then $m = \chi f$, and consider for $\varepsilon > 0$ the operation

$$s_\varepsilon = \eta_\varepsilon * \chi, \tag{172}$$

$$m_\varepsilon = \eta_\varepsilon * m, \tag{173}$$

$$f_\varepsilon = \frac{1}{s_\varepsilon} m_\varepsilon, \tag{174}$$

which assigns to each function f a convolute f_ε. Here,

$$\eta_\varepsilon(y) = \frac{1}{\varepsilon} \eta\left(\frac{y}{\varepsilon}\right),$$

denotes as usual the rescaled convolution kernel, and it is worth noticing that also $s_\varepsilon(y) = s(\frac{y}{\varepsilon})$ for $s = s_1 = \eta * \chi$.
We denote this linear operator as $f_\varepsilon = L_\varepsilon f$. Note that the quotient in (174) is to be read as an ordinary quotient, if $\eta > 0$ on $\mathbb{R}$, and defines a Radon-Nykodim–derivative f_ε of m_ε with respect to s_ε otherwise.

Lemma 13. *(Estimate in energy)*
i) If f is bounded, we have that

$$\sup_{y \in \mathbb{R}} |f_\varepsilon(y)| \le \sup_{y \ge 0} |f(y)|$$

with constant 1, and indeed if $f \in C^0$ then $f_\varepsilon \to f$ point–wise in $y > 0$.
ii) If $\chi |f|^2$ is integrable, we have

$$\int s_\varepsilon |f_\varepsilon|^2 dy \le \int \chi |f|^2 dy \tag{175}$$

with constant 1, and indeed

$$\int s_\varepsilon |f_\varepsilon|^2 dy \to \int \chi |f|^2 dy \tag{176}$$

as $\varepsilon \to 0$. Moreover, the functions $s_\varepsilon |f_\varepsilon|^2$ are uniformly integrable, that is if $B \subset \mathbb{R}$, then

$$\int_B s_\varepsilon |f_\varepsilon|^2 dy \to 0,$$

uniformly in ε as $|B| \to 0$.

Remark 16. *We have more precisely, that $f \in L^\infty$ implies*

$$\int \chi \, h \, (f_\varepsilon - f) \, dy \to 0, \tag{177}$$

for any $h \in L^1(dy)$.

Proof. Ad i). Since η and χ, hence also s_ε, are nonnegative,

$$s_\varepsilon(y)\,|f_\varepsilon(y)| = |(s_\varepsilon\, f_\varepsilon)(y)| = \left|\int \eta_\varepsilon(z)\chi(y-z)f(y-z)\,dz\right| \le \int \eta_\varepsilon(z)\chi(y-z)\,|f(y-z)|\,dz,$$

which is less than

$$\int \eta_\varepsilon(z)\chi(y-z)dz\,\sup|f| = s_\varepsilon(y)\,\sup|f|.$$

To see the convergence, it suffices to combine

$$m_\varepsilon \to \chi f, \quad \text{point–wise in } y > 0$$

with $s_\varepsilon \to \chi = 1$, point–wise in $y > 0$.

Proof of the remark. To prove the remark, we need $s_\varepsilon \ge \frac{1}{2}$ in $y \ge 0$, which holds by symmetry of the kernel η, and $s_\varepsilon \to \chi = 1$ in $y > 0$. The remark follows from the formula

$$f_\varepsilon - f = \frac{1}{s_\varepsilon}\,(m_\varepsilon - m) + \left(\frac{1}{s_\varepsilon} - \frac{1}{\chi}\right)\, m,$$

which holds in $y > 0$. Indeed, the second term then passes to the limit because $m \in L^\infty$. For the first term, we appeal to the fact that $m_\varepsilon \to m$ in L^{1*}, which holds by duality:

$$\int \chi h\,(m_\varepsilon - m)\,dy \;=\; \int \big(\eta_\varepsilon * (\chi h) - \chi h\big)\, m\, dy,$$

and the convolution converges in L^1.

Ad ii). To see the estimate (175), we invoke Jensen's inequality. More precisely, consider the function

$$F(s,m) = \begin{cases} \frac{|m|^2}{s}, & \text{if } |m| > 0,\ s > 0, \\ \infty, & \text{if } |m| > 0,\ s = 0, \\ 0, & \quad\text{if } m = 0. \end{cases} \tag{178}$$

We have seen that this is a convex function for $s > 0$. Due to its homogeneity, it is actually a convex function also for $s \ge 0$: Along any straight line in the (m,s)–plane, one must have

$$\lim_{m\to 0,\ s\to 0} F(s,m) = 0.$$

With this knowledge we find a function $r(z,y)$, so that

$$\forall\ y, z, \quad F(\chi(z), m(z)) \;\ge\; F(s_\varepsilon(y), m_\varepsilon(y)) + r(z,y), \tag{179}$$

$$\forall\ y, \quad \int \eta_\varepsilon(z) r(y-z, y) dz = 0. \tag{180}$$

Precisely, r is the affine function

$$r(z,y) \;=\; \frac{2m_\varepsilon(y)}{s_\varepsilon(y)}(m(z)-m_\varepsilon(y)) - \frac{m_\varepsilon^2(y)}{s_\varepsilon^2(y)}(\chi(z)-s_\varepsilon(y)).$$

Integrating inequality (179) against the convolution kernel yields

$$\int \eta_\varepsilon(z)F(\chi(y-z),m(y-z))dz \;\geq\; F(s_\varepsilon(y),m_\varepsilon(y)) \quad \forall y\in\mathbb{R}.$$

Integrating finally over $y\in\mathbb{R}$, and using $\int \eta_\varepsilon(z)dz = 1$,

$$\int F(\chi(y),m(y))\,dy \geq \int F(s_\varepsilon(y),m_\varepsilon(y))\,dy.$$

By the definition of F, this means

$$\int \chi|f|^2dy \geq \int s_\varepsilon|f_\varepsilon|^2dy.$$

To show the uniform integrability, we integrate over $B\subset\mathbb{R}$ and find

$$\int_B s_\varepsilon|f_\varepsilon|^2dy \;\leq\; \int \eta_\varepsilon(z)\int_B F(\chi(y-z),m(y-z))dydz \;\leq\; \sup_z \int_{B-z} F(\chi(y),m(y))\,dy,$$

thus identifying a modulus of integrability in terms of the limit f, because the Lebesgue measure is translation invariant.
To see that this implies convergence in L^2, we show that on the other hand

$$\int \chi|f|^2dy \;\leq\; \liminf_{\varepsilon\to 0}\int s_\varepsilon|f_\varepsilon|^2dy \tag{181}$$

must hold true. This is a consequence of the convergence of $m_\varepsilon\to m$ in $L^2(dy)$, which holds weakly (and in fact strongly, as is well known and follows by a similar argument as the one presented here). Indeed, since $0\leq s_\varepsilon\leq 1$,

$$\int \chi|f|^2dy \;=\; \int |\chi f|^2dy \;\leq\; \liminf_{\epsilon\to 0}\int |s_\varepsilon f_\varepsilon|^2dy \;\leq\; \liminf_{\varepsilon\to 0}\int s_\varepsilon|f_\varepsilon|^2dy,$$

which completes the proof. Notice that we have shown in particular as a straightforward consequence that $\int_{y\leq 0} s_\varepsilon|f_\varepsilon|^2dy \to 0$, and (expanding the square) that

$$\int s_\varepsilon|f_\varepsilon-\chi f|^2dy \;\to\; 0. \tag{182}$$

$\square$

We now proceed to discuss the derivative. For this let $\int \chi|f'|^2dy < \infty$. It turns out that the choice of convolution kernel now plays a certain role. Any reasonable kernel η will work, either piecewise polynomial with compact support, or of algebraic or exponential decay. We have

Lemma 14. *(Estimate for the derivative)*
i) There is a constant $C = C(\eta)$, so that for all $\varepsilon \geq 0$,

$$\int s_\varepsilon |f_\varepsilon'|^2 dy \;\leq\; C \int \chi |f'|^2 dy. \tag{183}$$

ii) The value $f(0)$ is defined, and it holds

$$\frac{1}{\varepsilon^{1/2}} |f_\varepsilon(0) - f(0)| \;\to\; 0. \tag{184}$$

iii) One has uniform integrability of $s_\varepsilon |f_\varepsilon'|^2$, and

$$\lim_{\varepsilon \to 0} \int s_\varepsilon |f_\varepsilon'|^2 dy \;=\; \int \chi |f'|^2 dy. \tag{185}$$

iv) $s_\varepsilon\, f_\varepsilon' \to \chi\, f'$ in $L^2(dy)$.

Proof. The first point will be seen along the proof of iii). The fact that $f(0)$ is defined is due to the embedding into the Hoelder–space $C^{0,\frac{1}{2}}$. The second point also follows as a by–product. We proceed directly to the proof of iii), which relies on a formula for the derivative. One can say that the strategy is to compare $m_\varepsilon' = s_\varepsilon' f_\varepsilon + s_\varepsilon f_\varepsilon'$ with $m' = f(0)\delta_0 + \chi f'$. Precisely we claim that

$$f_\varepsilon'(y) \;=\; -\frac{\eta_\varepsilon(y)}{s_\varepsilon(y)} (f_\varepsilon(y) - f(0)) + (L_\varepsilon f')(y). \tag{186}$$

Here, we denoted as above,

$$L_\varepsilon f' = \frac{1}{s_\varepsilon}\, \eta_\varepsilon * (\chi f'). \tag{187}$$

Indeed, notice first that $s_\varepsilon' = \eta_\varepsilon$. Then by the definition of f_ε,

$$f_\varepsilon' = -\frac{s_\varepsilon'}{s_\varepsilon^2}\, \eta_\varepsilon * (\chi f) + \frac{1}{s_\varepsilon}\, \eta_\varepsilon * (\chi f)'.$$

The first term can be written as $-\frac{\eta_\varepsilon}{s_\varepsilon}\, f_\varepsilon$. For the second term, the derivative falls onto f, and yields precisely $L_\varepsilon f'$, or falls onto χ, and then yields $\frac{\eta_\varepsilon(y)}{s_\varepsilon(y)}\, f(0)$.
It was the result of Lemma 13 that the sequence $L_\varepsilon f'$ is uniformly integrable, and realizes already the limit

$$\int s_\varepsilon |L_\varepsilon f'|^2 dy \to \int \chi |f'|^2 dy. \tag{188}$$

Moreover obviously, $s_\varepsilon\, L_\varepsilon f' \to \chi f'$ in $L^2(dy)$.

Hence to prove iii), (and thus also iv)), it remains only to show that the first term in (186) converges to zero. To show this, we rely on another expression, precisely in terms of $L_\varepsilon f'$. Indeed we have

$$\frac{\eta_\varepsilon(y)}{s_\varepsilon(y)}\left(f_\varepsilon(y)-f(0)\right)=\frac{\eta_\varepsilon(y)}{s_\varepsilon(y)^2}\int_{-\infty}^{y}(\eta_\varepsilon*(\chi f'))(z)dz, \tag{189}$$

or equivalently,

$$f_\varepsilon(y)-f(0)=\int_{-\infty}^{y}\frac{s_\varepsilon(z)}{s_\varepsilon(y)}\left(L_\varepsilon f'\right)(z)dz. \tag{190}$$

In this second form, we deduce this formula directly from the differential equation (186). Indeed, (190) is Duhamel's formula and provides clearly a solution to the differential equation (186). Hence one verifies (190) if one checks that the integral is finite, and the limit at $y\to-\infty$ is consistent. But this is obvious from its last reformulation as

$$s_\varepsilon(y)f_\varepsilon(y)-s_\varepsilon(y)f(0)=\int_{-\infty}^{y}(\eta_\varepsilon*(\chi f')(z)dz. \tag{191}$$

Indeed, on the left hand side, both s_ε and $s_\varepsilon f_\varepsilon=m_\varepsilon$ must decay to zero. On the right hand side, the integrand is the convolute of a function that vanishes for $y<0$.
Let us more precisely give an estimate of this right hand side. It reads by virtue of the Fubini lemma

$$\int_{-\infty}^{y}\int_{\mathbb{R}}\eta_\varepsilon(w)(\chi f')(z-w)dw\,dz=\int_{z\leq y}dz\int_{w\leq z}dw\,\eta_\varepsilon(w)f'(z-w) \tag{192}$$

$$=\int_{w\leq y}dw\int_{w\leq z\leq y}dz\,f'(z-w)\,\eta_\varepsilon(w), \tag{193}$$

and is estimated according to Cauchy–Schwarz as

$$\int_{-\infty}^{y}|(\eta_\varepsilon*(\chi f')(z)|\,dz\;\leq\int_{-\infty}^{y}dw\,\eta_\varepsilon(w)\,|y-w|^{\frac{1}{2}}\left(\int_{w}^{y}|f'(z-w)|^2dz\right)^{\frac{1}{2}}. \tag{194}$$

Let us show that the evaluation in $y=0$ leads to the claim ii). Indeed, due to the symmetry of η, $s_\varepsilon(0)=\frac{1}{2}$, hence according to (191) and (194),

$$\frac{1}{2}\,|f_\varepsilon(0)-f(0)|\;\leq\int_{-\infty}^{0}dw\,\eta_\varepsilon(w)\,|w|^{\frac{1}{2}}\left(\int_{w}^{0}|f'(z-w)|^2dz\right)^{\frac{1}{2}}.$$

The right hand side in turn we rewrite according to the rescaling procedure $\varepsilon w=\hat{w}$ as

$$\varepsilon^{\frac{1}{2}}\int_{-\infty}^{0}d\hat{w}\,\eta(\hat{w})|\hat{w}|^{\frac{1}{2}}\left(\int_{\varepsilon\hat{w}}^{0}|f'(z-\varepsilon\hat{w})|^2dz\right)^{\frac{1}{2}}.$$

We see that if

$$|\hat{w}|^{\frac{1}{2}}\eta(\hat{w}) \in L^1(d\hat{w}), \tag{195}$$

and of course $\chi f' \in L^2(dz)$, then

$$\frac{1}{\varepsilon^{1/2}}\,|f_\varepsilon(0) - f(0)| \,\le\, 2\int_{-\infty}^{0} d\hat{w}\,\eta(\hat{w})|\hat{w}|^{\frac{1}{2}} \left(\int_{\varepsilon\hat{w}}^{0} |f'(z-\varepsilon\hat{w})|^2 dz\right)^{\frac{1}{2}}$$

is finite independent of ε, and even tends to zero as $\varepsilon \to 0$ by dominated convergence: For every fixed $\hat{w}$, the dz–integral converges to zero as $\varepsilon \to 0$. Under the same assumption on η, the right hand side of (194) is finite, and this proves the formula (190).

In view of (186), we finally need to show that the norm

$$\int_{\mathbb{R}} s_\varepsilon(y)\,\frac{\eta_\varepsilon^2(y)}{s_\varepsilon^2(y)}\,|f_\varepsilon(y) - f(0)|^2 dy \to 0, \tag{196}$$

as $\varepsilon \to 0$. Indeed this will show claim iii), and make obvious *a forteriori* claim iv). We rely again on the estimate (194) to see

$$|f_\varepsilon(y) - f(0)| \,\le\, \frac{1}{s_\varepsilon(y)}\int_{-\infty}^{y} dw\,\eta_\varepsilon(w)\,|y-w|^{\frac{1}{2}} \left(\int_{w}^{y} |f'(z-w)|^2 dz\right)^{\frac{1}{2}},$$

so that the norm in (196) is less than

$$\int_{\mathbb{R}} dy\,\frac{\eta_\varepsilon^2(y)}{s_\varepsilon^3(y)}\left|\int_{-\infty}^{y} dw\,\eta_\varepsilon(w)\,|y-w|^{\frac{1}{2}} \left(\int_{w}^{y} |f'(z-w)|^2 dz\right)^{\frac{1}{2}}\right|^2.$$

A similar rescaling procedure as above turns this into

$$\int_{\mathbb{R}} d\hat{y}\,\frac{\eta^2(\hat{y})}{s^3(\hat{y})}\left|\int_{-\infty}^{\hat{y}} d\hat{w}\,\eta(\hat{w})\,|\hat{y}-\hat{w}|^{\frac{1}{2}} \left(\int_{\varepsilon\hat{w}}^{\varepsilon\hat{y}} |f'(z-\varepsilon\hat{w})|^2 dz\right)^{\frac{1}{2}}\right|^2.$$

We see that if in addition

$$\frac{\eta^2(\hat{y})}{s^3(\hat{y})}\left(\int_{-\infty}^{\hat{y}} d\hat{w}\,\eta(\hat{w})\,|\hat{y}-\hat{w}|^{\frac{1}{2}}\right)^2 \,\in L^1(d\hat{y}), \tag{197}$$

then the norm (196) must be bounded by $C\int \chi|f'|^2 dz$. This proves claim i), in view of (186) and Lemma 13. More precisely, the norm (196) must converge to zero by dominated convergence: For fixed $\hat{y}$, $\hat{w}$, the dz–integral tends to zero, hence by dominated convergence, for fixed $\hat{y}$, the $d\hat{w}$–integral converges to zero, and for the same reason, the $d\hat{y}$–integral converges to zero. This proves claim iii), again in view of (186) and Lemma 13. $\square$

Let us show in addition that (197) can indeed be satisfied, by a function η of algebraic decay $|\hat{y}|^{-(\alpha+1)}$, $\alpha > 0$. Then the integral is finite for $\hat{y} > 0$, since there, $s \geq \frac{1}{2}$, if only $\eta^2(1+|\hat{y}|) \in L^1$, which is obviously true.
And for $\hat{y} \ll -1$ we have, dropping the hats, up to constants depending on α,

$$\frac{|y|^{-2(\alpha+1)}}{|y|^{-3\alpha}} \left(\int |w|^{-(\alpha+1)}(|y|^{1/2} + |w|^{1/2})dw \right)^2,$$

which behaves as $|y|^{-(\alpha+1)}$ and is hence in L^1. Indeed, the exponent is

$$-2\alpha - 2 + 3\alpha - 2(\alpha - 1 + \tfrac{1}{2} + 1) = -\alpha - 1.$$

It is not difficult to produce from these arguments the
Proof of Conjecture 1 (of Section 6.5, in the case $s = s(y)$). Given the pair $\chi_\pm(y)$, choose $\eta(y)$ as a mollifier in the normal direction, and let as above, $s_\pm^\varepsilon = \eta_\varepsilon * \chi_\pm$. This respects $s_+ + s_- = 1$. Given moreover the pair $u_\pm$ with $\int \chi_\pm |u_\pm|^2 dxdy < \infty$ and $\nabla \cdot (\chi_+ u_+ + \chi_- u_-) = 0$, consider the same operation, the convolution to be understood in the y–variable:

$$m_\pm = \chi_\pm u_\pm, \tag{198}$$
$$m_\pm^\varepsilon = \eta_\varepsilon * m_\pm, \tag{199}$$
$$u_\pm^\varepsilon = \frac{1}{s_\pm^\varepsilon} m_\pm^\varepsilon. \tag{200}$$

It respects

$$\nabla \cdot (s_+ u_+ + s_- u_-) = 0.$$

Then by the first lemma, applied to the two components of u, one has the L^∞–bound, and convergence in L^{1*}, moreover the L^2–convergence. For the derivative, notice that ∂_x commutes with the operator L_ε, hence also the L^2–convergence of the x–derivative follows from the first lemma. For the y–derivative, we invoke the second lemma, which also yields the L^2–convergence. $\square$

We address finally the **Conjecture 2** (in Section 7.2). Here, we suggest that the natural operation would be

$$m_\pm = \chi_\pm \nabla \phi_\pm, \tag{201}$$
$$m_\pm^\varepsilon = \eta_\varepsilon * m_\pm, \tag{202}$$
$$-\nabla \cdot (s_\pm^\varepsilon \nabla \phi_\pm^\varepsilon) + \nabla \cdot m_\pm^\varepsilon = 0. \tag{203}$$

It respects the side constraint. Let us show that one has convergence in L^2, which at least justifies the definition of the $\phi_\pm^\varepsilon$. We suggest that one can show in a similar way the L^2–convergence for the derivative, combining

the arguments of Lemma 14 with those following below. It is a different question whether there is an estimate for the elliptic equation (203) which assures that $\nabla \phi_{\pm}^{\varepsilon} \in L^{\infty}$, and we leave it aside.
To show the convergence in L^2, we remark that on the one hand,

$$\int s_\varepsilon |\nabla \phi_\varepsilon|^2 dx \leq \int s_\varepsilon \left|\frac{m_\varepsilon}{s_\varepsilon}\right|^2 dx \leq \int \chi |\nabla \phi|^2 dx,$$

where the first inequality is the property of the Dirichlet integral, and the second is the result of Lemma 13.
On the other hand,

$$\int s_\varepsilon \nabla \phi_\varepsilon \cdot \nabla \eta \, dx = \int m_\varepsilon \cdot \nabla \eta \, dx \to \int \chi \nabla \phi \cdot \nabla \eta \, dx$$

holds true for any test smooth function η, and implies

$$\int \chi |\nabla \phi|^2 dx \leq \liminf_{\varepsilon \to 0} \int s_\varepsilon |\nabla \phi_\varepsilon|^2 dx,$$

because indeed

$$\left(\int \chi |\nabla \phi|^2 dx\right)^{\frac{1}{2}} = \sup_\eta \frac{\int \chi \nabla \phi \cdot \nabla \eta \, dx}{(\int \chi |\nabla \eta|^2 dx)^{\frac{1}{2}}} = \sup_\eta \lim_{\varepsilon \to 0} \frac{\int s_\varepsilon \nabla \phi_\varepsilon \cdot \nabla \eta \, dx}{(\int s_\varepsilon |\nabla \eta|^2 dx)^{\frac{1}{2}}}$$

is not larger than

$$\liminf_{\varepsilon \to 0} \sup_\eta \frac{\int s_\varepsilon \nabla \phi_\varepsilon \cdot \nabla \eta \, dx}{(\int s_\varepsilon |\nabla \eta|^2 dx)^{\frac{1}{2}}} = \liminf_{\varepsilon \to 0} \left(\int s_\varepsilon |\nabla \phi_\varepsilon|^2 dx\right)^{\frac{1}{2}}.$$

As in (182), this implies

$$\int s_\varepsilon \, |\nabla \phi_\varepsilon - \chi \nabla \phi|^2 dx \to 0.$$

□

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