The Farrell-Jones conjecture for some general linear groups

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Contents

1 Introduction

1.1 The main result

In this thesis I will prove the Farrell-Jones conjecture for all groups that are linear over $F[t][S^{-1}]$ for a finite field F and a finite set of primes $S \subset F[t]$ (Theorem 8.21). This means all subgroups of $GL_n(F[t][S^{-1}])$. Furthermore I will prove a relative version of the Farrell-Jones conjecture for subgroups of $GL_n(\mathbb{Z}[S^{-1}])$ for a finite set of primes $S \subset \mathbb{Z}$.

The Farrell-Jones conjecture makes predictions about the algebraic K-theory of group rings. The Baum-Connes conjecture about the K-theory of the reduced group C^*-algebra is still open for $GL_n(\mathbb{Z})$.

The action of $\mathrm{GL}_n(\mathbb{Z})$ on its symmetric space has been used to show the Farrell-Jones in [8].

I will show the strongest version of this conjecture for those groups; the version with coefficients in any additive category with a group action and with finite wreath products. This version has strong inheritance properties, for example any group commensurable to a subgroup of one of the groups mentioned above will satisfy the Farrell-Jones conjecture. This includes in particular S-arithmetic groups over function fields.

These following conjectures about torsionfree groups are still open; the main result implies that they hold if the group is linear over $F[t][S^{-1}]$. Let G be a torsionfree group.

- Any finitely generated, projective $\mathbb{Z}[G]$-module is stably free.
- Any matrix $A \in GL_n(\mathbb{Z}[G])$ can stably be written as a product of elementary matrices. Stabilization means passing to a matrix of the form $\begin{pmatrix} A & 0 \\ 0 & I_m \end{pmatrix}$.

1.2 Applications of the algebraic K-theory of group rings

It often happens that some construction does not work a priori, but only if a certain obstruction vanishes. Strangely those obstructions tend to live in some group. Algebraic K-groups appear quite often in such situations. Let me briefly state some examples:

1.2.1 Walls finiteness obstruction and K_0

The zeroth K-group of a ring R is defined to be the Grothendieck group of the monoid of isomorphism classes of direct summands of some R^n where composition is given by the direct sum. The reduced zeroth K-group $\tilde{K}_0(R)$ is the quotient of $K_0(R)$ by the subgroup generated by the representatives of free modules.

How can we decide whether a given space is homotopy equivalent to a compact space? In the world of CW-complexes we can wonder whether a given CW-complex X is homotopy equivalent to a finite CW-complex. Let us restrict to *finitely dominated* CW-complexes, meaning that there is a finite CW-complex Y and maps $s\colon X \to Y$ and $d\colon Y \to X$ such that $d \circ s\colon X \to X \simeq id_X$.

Out of these data one can construct a finite chain complex of finitely generated, projective $\mathbb{Z}[\pi_1(X)]$-modules P_* chain equivalent to the cellular chain complex $C_*(\tilde{X})$. The *finiteness obstruction* of X is defined to be $o(X) := \sum_i (-1)^i [P_i] \in \tilde{K}_0(\mathbb{Z})$. The finiteness obstruction is independent of the choice of P.

If X itself was a finite CW-complex we could take $P_* = C(\tilde{X})$. But this is a free chain complex. So $o(X) = 0$ in $K_0(X)$. For more details and proofs see for example [15].

1.2.2 The s-cobordism theorem and K_1

The first K-group of a ring R is defined to be the Abelianization of $\mathrm{GL}(R)$, where $\mathrm{GL}(R)$ is defined to be the union of all $\mathrm{GL}_n(R)$ under the inclusions $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Hence each element of $K_1(R)$ can be represented by an invertible matrix with entries in R. In the case where R is a group ring $R = R'[G]$ the *Whitehead group* is defined as

$$\mathrm{Wh}_{R'}(G) := K_1(R[G]/\{[\varepsilon g] \mid \varepsilon \in R^*, g \in G\}).$$

Let $f\colon X \to Y$ be a homotopy equivalence of finite, connected CW-complexes. It induces a map of $\mathbb{Z}[\pi_1(X)]$ chain complexes from the cellular chain complex of the universal covering of X to the chain complex of the universal covering of Y.

This map is a $\mathbb{Z}[\pi_1(X)]$-chain homotopy equivalence and so its mapping cone (C_*, c_*) is contractible. If γ_* is a chain contraction of C_* it turns out that the map

$$(\gamma_* + c_*)_{\mathrm{odd}}\colon \bigoplus_{n\in\mathbb{Z}} C_{2n+1} \to \bigoplus_{n\in\mathbb{Z}} C_{2n}$$

is an isomorphism of finitely generated, free, based $\mathbb{Z}[\pi_1(X)]$-modules. The bases correspond to the cells. Its matrix represents an element $\tau(f)$ in the Whitehead group $\mathrm{Wh}_{\mathbb{Z}}(\pi_1(X))$ — the so called *Whitehead torsion* of f.

The simplest way to obtain from one CW-complex X another homotopy equivalent CW-complex is an elementary extension along a map $m\colon D^n \to X$. First attach a cell along the boundary $m|_{\partial D^n}$. Now we can define a map $S^n \to D^n \cup_{m|_{\partial D^n}} X$ using m on the upper hemisphere and id_{D^n} on the lower hemisphere to attach an $n+1$-cell. The resulting complex X' is homotopy equivalent to X. It is called an *elementary extension* of X. A homotopy inverse of the inclusion $X \hookrightarrow X'$ is called *elementary collapse*.

A composition of such elementary extensions and elementary collapses is called a *simple homotopy equivalence*. We can check that the Whitehead torsion is compatible with compositions and that the Whitehead torsion of an elementary extension respectively collapse is zero. Furthermore homotopic homotopy equivalences have the same

Whitehead torsion. So a homotopy equivalence with nonvanishing Whitehead torsion cannot be homotopic to a simple one.

The topological invariance of the Whitehead torsion shows that any homeomorphism of finite, connected CW-complexes has vanishing Whitehead torsion (see [13]). So a homotopy equivalence with nonvanishing Whitehead torsion cannot be homotopic to a homeomorphism. Furthermore it shows that the Whitehead torsion of a homotopy equivalence of two CW-complexes is independent of the cell structure.

Conversely we can use the s-cobordism theorem to produce a homeomorphism. A *s-cobordism* (W, M, M') is a cobordism W from one manifold M to another manifold M' with the property that both inclusions $M, M' \hookrightarrow W$ are simple homotopy equivalences.

Theorem 1.1 (s-cobordism theorem)**.** *A s-cobordism between closed, connected, compact manifolds of dimension* ≥ 5 *is* trivial*. This means that it is homeomorphic to a cylinder.*

Especially the two components of the boundary are homeomorphic.

1.2.3 The surgery obstruction, L-groups and the Borel conjecture

Another interesting question is whether manifolds are determined by their homotopy type. Given a closed topological manifold M we can define its *topological structure set* $\mathcal{S}^{\mathrm{Top}}(M)$ as the set of all homotopy equivalences $N \to M$ from another closed topological manifold N into M where two maps $(f_i \colon N_i \to M)_{i=1,2}$ are identified if there is a homeomorphism $h\colon N_1 \to N_2$ such that f_1 and $f_2 \circ h$ are homotopic.

Note that the structure set $\mathcal{S}^{\mathrm{Top}}(M)$ is pointed with the basepoint $[\mathrm{id}_M]$. So $\mathcal{S}^{\mathrm{Top}}(M)$ is trivial if and only if any homotopy equivalence $f\colon N \to M$ from another manifold to M is homotopic to $h \circ id_M$ for some homeomorphism $h\colon N \to M$. In this case M is called *topologically rigid*.

The structure set appears in the so called surgery long exact sequence.

Let M be a *aspherical* manifold of dimension greater than 5. Aspherical means that M is a model for $B\pi_1(M)$, or equivalently $\pi_i(M) = 0$ for $i \geq 2$. The Borel conjecture states that closed aspherical manifolds are topologically rigid.

Provided that the K- and L-theoretic Farrell-Jones conjecture holds we can understand some maps in the surgery exact sequence for M to deduce that M is *topologically rigid*. More details can be found in [21, Theorem 7.28].

1.3 Statement of the Farrell-Jones conjecture

Let us fix a ring R for this section. The goal is to compute $K_*(R[G])$ from $K_*(R)$ in some way. Let us first consider the infinite cyclic group $\mathbb{Z}$. The Bass-Heller-Swan decomposition [29, Theorem 3.2.22] says that

$$(f, g, h_+, h_-)\colon K_1(R) \oplus K_0(R) \oplus \mathrm{NK}_1(R) \oplus \mathrm{NK}_1(R) \cong K_1(R[\mathbb{Z}]),$$

where the so-called *NIL-terms* $\mathrm{NK}_1(R)$ are defined as the cokernel of the map $K_1(R) \hookrightarrow K_1(R[x])$ where x denotes an indeterminate. Note that this map is a split injection; the map sending x to zero induces a right inverse. This yields a map $K_1(R[x]) \to \mathrm{NK}_1(R)$. The maps in the Bass-Heller-Swan decomposition are given by

- $f = K_1(R \to R[t, t^{-1}], \quad t \mapsto 0)$,
- $h_\pm \colon \mathrm{NK}_1(R) \to K_1(R[x]) \to K_1(R[t, t^{-1}])$ where the last map sends x to t (respectively t^{-1}),
- $g \colon K_0(R) \to K_1(R[\mathbb{Z}]), \qquad [P] \mapsto [\mathrm{id}_P \otimes t : P \otimes_R R[\mathbb{Z}] \to P \otimes_R R[\mathbb{Z}]]$.

This suggests defining $K_{n-1}(R)$ as the cokernel of the map

$$\mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R) \oplus K_n(R) \to K_n(R[\mathbb{Z}])$$

and $\mathrm{NK}_n(R)$ as the cokernel of $K_n(R) \to K_n(R[t])$. This gives inductively a definition of negative K-theory and it turns out that the Bass-Heller-Swan decomposition also holds in negative degrees [29, Theorem 3.3.3.].

The upper formula looks a bit like the computation of the homology of the circle

$$H_*(B\mathbb{Z}) = H_*(S^1) = H_*(\mathrm{pt}) \oplus H_{*-1}(\mathrm{pt}).$$

We could hope to compute the algebraic K-groups of a group ring $R[G]$ by evaluating a certain homology theory, which depends on R, on the classifying space BG.

If the NIL-terms do not vanish this conjecture cannot be true. An infinite cyclic subgroup Z of G gives maps

$$\mathrm{NK}_n(R) \to K_n(R[Z]) \to K_n(R[G]).$$

We need a way to keep track of all infinite cyclic subgroups of G. Then we could ask whether all elements in $K_*(R[G])$ come from an infinite cyclic subgroup.

We would like to "spacify" the collection of infinite cyclic subgroups. Indeed we have to take all virtually cyclic subgroups into account. This leads to the notion of classifying spaces over families and to define equivariant homology theories that also take these subgroups into account.

This will be explained in the sequel. A *family of subgroups* of a group G is a nonempty collection of subgroups closed under conjugation and taking subgroups.

Definition 1.2. A *classifying space* for a family of subgroups $\mathcal{F}$ of a group G is a G-CW-complex that is a terminal object $E_{\mathcal{F}}(G)$ in the category whose objects are G-CW-complexes with isotropy in $\mathcal{F}$ and whose morphisms are G-homotopy classes of G-maps.

Such a classifying space always exists and it is up to G-homotopy equivalence characterized by the property

$$E_{\mathcal{F}}(G)^H \simeq \begin{cases} \mathrm{pt} & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{cases}.$$

The classifying space with respect to the family of trivial subgroups is just EG and the classifying space for the family of all subgroups is $\mathrm{pt} = G/G$. An inclusion of families $\mathcal{F} \subset \mathcal{F}'$ gives an up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{F}'}(G)$ by the defining property.

Definition 1.3. A *G-homology theory* is a functor $\mathcal{H}_*^G$ from the category of G-CW-pairs to $\mathbb{Z}$-graded abelian groups together with natural transformations $\partial_n \colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A,\emptyset)$ satisfying the following axioms:

(i) G-Homotopy invariance. Let X, Y be G-CW-complexes. Let $[0,1]$ be be equipped with the trivial G-action. For any G-map $H \colon X \times [0,1] \to Y$ we have $\mathcal{H}_*^G(H|_{\{0\}}) = \mathcal{H}_*^G(H|_{\{1\}})$.

(ii) Long exact sequence of a pair. The sequence

$$\ldots \to \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(p)} \mathcal{H}_n^G(X,A) \xrightarrow{\partial_n} \mathcal{H}_{n-1}^G(A) \to \ldots$$

is exact for a G-CW-pair (X,A) and the obvious inclusions $A \overset{i}{\hookrightarrow} X \xrightarrow{p} (X,A)$.

(iii) Excision. For a G-CW-pair (X,A) and a cellular G-map $f \colon A \to B$ the induced map $F \colon (X,A) \to (X \cup_F B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(X \cup_F B, B) \to \mathcal{H}_n^G(X,A).$$

(iv) Disjoint union axiom. For a family $\{X_i \mid i \in I\}$ of G-CW-complexes the inclusions $X_j \hookrightarrow \amalg_{i\in I} X_i$ induce an isomorphism

$$\bigoplus_{j\in I} \mathcal{H}_n^G(X_j) \to \mathcal{H}_n^G(\amalg_{i\in I} X_i).$$

The orbit category $Or(G)$ of a group G is the category whose objects are the G-sets G/H for some $H \leq G$ and whose maps are G-maps. Especially $Or(G)$ is a subcategory of the category of G-CW-complexes.

Any natural transformation of G-homology theories that induces isomorphisms for every object in the orbit category is a natural isomorphism. This can be proved completely analogous to the nonequivariant case. The restriction of the G-homology theory $\mathcal{H}_*^G$ to the orbit category should be thought of as the coefficients of $\mathcal{H}_*^G$.

Any G-CW-complex can be considered as a contravariant functor from $Or(G)$ to CW-complexes via $G/H \mapsto X^H = \mathrm{map}(G/H, X)$. Any covariant functor $F \colon Or(G) \to \mathsf{Spectra}$ gives rise to a G-homology theory via

$$X \mapsto \pi_*(X_+ \wedge_{Or(G)} F) =: H_*^G(X;F).$$

Up to this point we have only worked for one specific group. Usually constructions of those homology theories will work for any group G and the results will be related. Recall that given group homomorphism $\alpha : G \to H$ we can *induce* an H-space X up to the G-space $\mathrm{ind}_\alpha(X) := G \times_\alpha X := G \times X/(g, hx) = (g\alpha(h), x)$.

Definition 1.4 (Equivariant homology theory)**.** A *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$ and to any group homomorphism $\alpha : G \to H$ and any H-CW pair (X, A) a map

$$\mathrm{ind}_\alpha : \mathcal{H}_*^H(X, A) \to \mathcal{H}_*^G(G \times_\alpha (X, A)),$$

such that

- ind_α is a bijection if $\ker(\alpha)$ acts freely on X,
- ind_α is compatible with the boundary homomorphisms,
- $\mathrm{ind}_{\alpha\circ\beta} = \mathrm{ind}_\alpha \circ \mathrm{ind}_\beta$ for two composable group homomorphisms α, β,
- they are compatible with conjugation, i.e.

  $$\mathrm{ind}_{c(g)} : \mathcal{H}_*^H(X, A) \to \mathcal{H}_*^H(\mathrm{ind}_{c(g)}(X, A))$$

  agrees with $\mathcal{H}_*^H(f)$, where c_g denotes conjugation with g and $f : X \to \mathrm{ind}_{c(g)} X$ is given by $x \mapsto (1, g^{-1}x)$.

The assignment ind_* is called *induction structure*.

Let us examine when a family of functors $\{Or(G) \to \mathtt{Spectra}\}_{G \in \mathrm{Groups}}$ gives rise to an equivariant homology theory. We can associate to a G-set S its *transport groupoid* $C^g(S)$ whose objects are the elements of S and whose morphisms from s to s' is $\{g \in G \mid gs = s'\}$.

For any group G the transport groupoid defines a functor C^G from $Or(G)$ to the category $\mathtt{Groupoids}^{\mathtt{inj}}$ of small groupoids with injective functors. A functor $f : \mathcal{G}_0 \to \mathcal{G}_1$ between groupoids is said to be injective if the map $\mathrm{mor}_{\mathcal{G}_0}(x, y) \to \mathrm{mor}_{\mathcal{G}_1}(f(x), f(y))$ is injective for any $x, y \in \mathcal{G}_0$.

Given a functor $F : \mathtt{Groupoids}^{\mathtt{inj}} \to \mathtt{Spectra}$ sending equivalences to weak equivalences we obtain for any group G a functor $F \circ C^G : Or(G) \to \mathtt{Spectra}$. These functors fit together in a nice way to get an equivariant homology theory ([22, Proposition 6.8] and [30, Theorem 2.10 on page 26]).

Associated to a ring R functors $\mathbf{K}_R^{alg}, \mathbf{L}_R$ from $\mathtt{Groupoids}^{\mathtt{inj}}$ to $\mathtt{Spectra}$ have been constructed in [14, Section 2]. They give rise to the desired equivariant homology theories. Its coefficients are by construction $H_*^G(G/H, \mathbf{K}_R^{alg}) = K_*(R[H])$ respectively $H_*^G(G/H, \mathbf{L}_R) = L_*(R[H])$.

Furthermore there are generalizations that associates to any additive category $\mathcal{A}$ a functor $\mathbf{K}_\mathcal{A}^{alg}$ in [9, Definition 3.1] and to any additive category with involution [9, Section 5] a functor $\mathbf{L}_\mathcal{A}$.

Now we are ready to state the Farrell-Jones conjecture.

Conjecture 1.5 (K- and L-theoretic Farrell-Jones conjecture)**.** *Let G be a group and $\mathcal{A}$ be an additive category with a right G-action. The maps $E_{\mathcal{V}Cyc}(G) \to G/G$ induce isomorphisms*

$$H_*^G(E_{\mathcal{V}Cyc}G, \mathbf{K}_\mathcal{A}^{alg}) \to H_*^G(G/G, \mathbf{K}_\mathcal{A}^{alg}) = K_*(\mathcal{A} \rtimes G),$$

$$H_*^G(E_{\mathcal{V}Cyc}G, \mathbf{L}_\mathcal{A} \to H_*^G(G/G, \mathbf{L}_\mathcal{A}) = L_*(\mathcal{A} \rtimes G).$$

1.4 Reformulating the conjecture in terms of controlled algebra

Basically K-theory and L-theory have very similar properties (compare [4, Theorem 5.1]). Let us concentrate on the K-theoretic setting in this section.

An *additive category* is a category enriched over abelian groups with finite biproducts. An inclusion of a full, additive subcategory $\mathcal{A}$ of an additive category $\mathcal{U}$ is called a *Karoubi-filtration* if every object $U \in \mathcal{U}$ has a family of decompositions

$$\{\varphi_i : U \xrightarrow{\cong} A_i \oplus U_i \mid i \in I_U\}$$

(called a *filtration* of U) with $A_i \in \mathcal{A}$ such that

(i) For each object $U \in \mathcal{U}$ the relation

$$(E_i \oplus U_i) \leq (E_{i'} \leq U_{i'}) \Leftrightarrow E_i \subseteq E_i' \wedge U_{i'} \subseteq U_i$$

is a partial order on the family of decompositions of U (i.e. on I_U) where any two elements have a common upper bound. The notation $E_i \subseteq E_{i'}$ means that there is a factorization

$$\begin{array}{ccc} E_i & \longrightarrow & E_i \oplus U_i \\ \downarrow & & \downarrow \cong \\ E_{i'} & \longrightarrow & E_{i'} \oplus U_{i'} \end{array}$$

The right vertical isomorphism $E_i \oplus U_i \cong U \cong E_{i'} \oplus U_{i'}$ is given by the isomorphisms in the filtration.

(ii) Every map $A \to U$ from an object $A \in \mathcal{A}$ factors as $A \to E_i \hookrightarrow E_i \oplus U_i \cong U$ for some $i \in I_U$.

(iii) Every map $U \to A$ to an object $A \in \mathcal{A}$ factors as $U \cong E_i \oplus U_i \to E_i \to A$ for some $i \in I_U$.

(iv) For each $U, V \in \mathcal{U}$ the filtration on $U \oplus V$ is equivalent to the sum

$$\{U \oplus V \cong (E_i \oplus F_j) \oplus (U_i \oplus V_j) \mid (i, j) \in I_U \times I_V\}$$

of the filtrations $\{U = E_i \oplus U_i \mid i \in I_U\}$ and $\{V = F_j \oplus V_j \mid j \in I_V\}$.

Let me postpone examples of Karoubi filtrations to the next sections where they arise naturally.

A Karoubi filtration allows us to define a quotient category $\mathcal{U}/\mathcal{A}$. It has the same objects as $\mathcal{U}$ and two morphisms $f, g \in \mathrm{Mor}_{\mathcal{U}}(U, V)$ are identified if and only if their difference factors through an object in $\mathcal{A}$. The conditions appearing in the definition of a Karoubi filtration ensure that this is again an additive category.

Let $K_n(\mathcal{A})$ denote the nonconnective K-theory i.e. the n-th homotopy group of the nonconnective K-theory spectrum of $\mathcal{A}$ as defined in [27]. The assignment K_n : AddCat $\to$ Ab is a functor. We need only the following properties:

(i) [12, 1.0.2] For a Karoubi filtration $\mathcal{A} \xrightarrow{i} \mathcal{U} \xrightarrow{p} \mathcal{U}/\mathcal{A}$ there is a long exact sequence

$$\ldots \to K_n(\mathcal{A}) \xrightarrow{K_n(i)} K_n(\mathcal{U}) \xrightarrow{K_n(p)} K_n(\mathcal{U}/\mathcal{A}) \to K_{n-1}(\mathcal{A}) \to \ldots;$$

(ii) a weak equivalence of additive categories induces an isomorphism in K-theory;

(iii) an additive category $\mathcal{A}$ is called flasque if there is an endofunctor $F : \mathcal{A} \to \mathcal{A}$ and a natural isomorphism $F \oplus \mathrm{ID} \cong F$. We have $K_*(\mathcal{A}) = 0$ if $\mathcal{A}$ is flasque.

The first example of a flasque category is the category of all $\mathbb{Z}$-modules where F is given by $F(V) := \bigoplus_{\mathbb{N}} V$.

Our next goal is to interpret the source of the assembly map as the K-theory of some additive category. The basic tool to construct this category is controlled algebra:

Out of a space X and an additive category $\mathcal{A}$ we can construct a new additive category $C(X;\mathcal{A})$. Its objects are locally finite collections of objects $\{A_x\}_{x\in X}$ and a morphism from $\{A_x\}_{x\in X}$ to $\{B_y\}_{y\in X}$ is a collection $\{\varphi_{x,y} : A_y \to B_x\}_{(x,y)\in X\times X}$ of morphisms such that for each $x \in X$ the sets

$$\{y \in X \mid \varphi_{x,y} \neq 0\}, \{y \in X \mid \varphi_{y,x} \neq 0\}$$

are finite. The composition is given by matrix-multiplication, i.e.

$$(\varphi \circ \psi)_{x,z} := \sum_y \varphi_{x,y} \circ \psi_{y,z}.$$

Note that any of the finiteness conditions mentioned above ensures that this is a finite sum.

A left G-action on X and a (strict) right G-action on $\mathcal{A}$ gives rise to a (strict) right G-action on $C(X;\mathcal{A})$ via $(g^*A)_x := (g^*A)_{gx}$ and $(g^*\varphi)_{x,y} : g^*\varphi_{gx,gy}$. Strict means that h^*g^*A and $(gh)^*A$ are equal and not only isomorphic. Let $C^G(X;\mathcal{A})$ denote the fixed point category.

Now we can impose several restrictions on objects and morphisms of $C^G(X;\mathcal{A})$. An *object control condition* on X is a collection $\mathcal{F}$ of subsets of X that is a directed poset, e.g. for any two $F_1, F_2 \in \mathcal{F}$ there is an $F \in \mathcal{F}$ with $F_1 \cup F_2 \subset \mathcal{F}$.

A morphism control condition on X is a collection $\mathcal{E}$ of subsets of $X \times X$ satisfying

(i) For $E, E' \in \mathcal{E}$ there is an $E'' \in \mathcal{E}$ such that $E \circ E' \subset E''$, where
$E \circ E' := \{(x,z) \mid \exists\, y \in X : (x,y) \in E \wedge (y,z) \in E'\}$,

(ii) for $E, E' \in \mathcal{E}$ there is an $E'' \in \mathcal{E}$ such that $E \cup E' \subset E''$,

(iii) $\{(x,x) \mid x \in X\} \in \mathcal{E}$.

A morphism control condition on X is also known as a coarse structure on X. Define the *support* of an object $A \in \mathcal{A}$ to be the set $\{x \in X \mid A_x \neq 0\} \subset X$ and the support of an morphism φ to be the set $\{(x,y) \in X \times X \mid \varphi_{(x,y)} \neq 0\}$. We can define the category $C^G(X,\mathcal{E},\mathcal{F};\mathcal{A})$ as the subcategory of $C^G(X;\mathcal{A})$ consisting of those objects whose support is contained in some member of $\mathcal{F}$ and those morphisms whose support is contained in some element of $\mathcal{E}$.

Let us first have a look at an example.

Example 1.6. Let G be the trivial group and let $X := \mathbb{N}$. Let $\mathcal{F}$ denote the finite subsets of $\mathbb{N}$. Then $C(X,\mathcal{F};\mathcal{A}) \simeq \mathcal{A}$, the inclusion $C(X,\mathcal{F};\mathcal{A}) \hookrightarrow C(X;\mathcal{A})$ is a Karoubi filtration and $C(X;\mathcal{A})$ is flasque. We could pick as $F : C(X;\mathcal{A}) \to C(X;\mathcal{A})$, $\quad M \mapsto (n \mapsto \bigoplus_{n'<n} M_{n'})$. Then the natural isomorphism $\phi : F \oplus \mathrm{Id} \cong F$ is given by

$$\varphi_{n,n+1} = \mathrm{id}_{\bigoplus_{n'<n+1} M_{n'}} \text{ and } \varphi_{n,n'} = 0 \text{ for } n' \neq n+1.$$

Furthermore the inclusion of $C(X,\mathcal{F};\mathcal{A})$ in $C(X;\mathcal{A})$ is a Karoubi filtration. The decompositions are given by

$$M \cong M|_{\{1,\dots,n\}} \oplus M|_{\{n+1,\dots\}}$$

where for a subset $S \subset X$ the term $M|_S$ is defined to be

$$x \mapsto \begin{cases} M_x & x \in S \\ 0 & x \notin S \end{cases}.$$

Note that

$$\mathcal{A} \to C(X,\mathcal{F};\mathcal{A}), \qquad A \mapsto (n \mapsto \begin{cases} A & n = 0 \\ 0 & \text{else} \end{cases})$$

is a weak equivalence. Using the long exact sequence in K-theory we get isomorphisms

$$K_{n+1}(C(X;\mathcal{A})/C(X,\mathcal{F};\mathcal{A})) \cong K_n(\mathcal{A}).$$

Control conditions can be pulled back along a map. Some important control conditions are the cocompact object control condition $\mathcal{F}_{G-c}(X)$ on a G-space X consisting of the cocompact subsets of X, the metric morphism control condition $\mathcal{E}_d(X)$ on a metric space (X,d) consisting of those subsets S of $X \times X$ such that $d(S)$ is bounded and the so called equivariant continuous morphism control condition $\mathcal{E}_{G-cc}(X)$on $X \times [1,\infty)$ for a G-space X. It consists of those subsets J of $(X \times [1,\infty))^2$ with the following properties:

- For every x in X and every G_x-invariant open neighborhood of U of a point (x,∞) in $X \times [1,\infty]$ there exists a G_x-invariant open neighborhood $V \subseteq U$ of (x,∞) in $X \times [1,\infty]$ such that

$$((X \times [1,\infty] \setminus U) \times V) \cap J = \emptyset;$$

- The image of J under the map $(X \times [1,\infty))^2 \to [1,\infty)^2 \xrightarrow{(x,y)\mapsto|x-y|} \mathbb{R}$ is bounded.
- J is symmetric and invariant under the diagonal G-action.

This means that for a sequence of points $(x_n, t_n, x'_n, t'_n) \in J \in \mathcal{E}_{G-cc}(X)$ with $(x_n, t_n) \to (x, \infty)$ we also have $(x'_n, t'_n) \to (x, \infty)$.

Controlled algebra can be used to associate to a space X an additive category whose K-groups are $H^G_*(X, \mathbf{K}_{\mathcal{A}})$ in the following way:

$$\begin{aligned} \mathcal{T}^G(X;\mathcal{A}) &:= C^G(G\times X, \mathcal{F}_{G-c}(G\times X);\mathcal{A}),\\ O^G(X;\mathcal{A}) &:= C^G(G\times X\times[1,\infty),\\ &\qquad \mathrm{pr}^{-1}_{X\times[1,\infty)}(\mathcal{E}_{G-cc}(X)) \cap \mathrm{pr}_G^{-1}(\mathcal{E}_{d_G}(G), \mathcal{F}_{G-c}(G\times X);\mathcal{A}). \end{aligned}$$

Note that $\mathcal{T}^G(X;\mathcal{A})$ is weakly equivalent to the full subcategory of $O^G(X;\mathcal{A})$ consisting of those objects whose support is contained in $G\times X\times[1,n]$ for some n. Furthermore the inclusion of that subcategory is a Karoubi Filtration. Let $\mathcal{D}^G(Y)$ denote the quotient. The equivariant continuous control condition is constructed in such a way, that $K_n(\mathcal{D}^G(X;\mathcal{A}))$ is excisive. Indeed $X \to K_n(\mathcal{D}^G(X;\mathcal{A}))$ is an equivariant homology theory on G-CW-complexes [7, Theorem 3.7].

The assembly map can be identified with the boundary map in the long K-theory sequence [7, Proposition 3.8]. So the final goal is to show that the K-theory of the obstruction category $O^G(E_{\mathcal{F}}G;\mathcal{A})$ vanishes.

1.5 Analyzing the obstruction category

This section should make plausible how the construction of some systems of open sets will help in the proof of the Farrell-Jones conjecture. The precise proofs will be done in the later sections. This section is a short summary of the argument of [5] and [32].

The goal is to show that the K-theory of the obstruction category vanishes. The first step is to introduce yet another version, which allows more space for certain constructions. Define for a metric space (Y,d) with an isometric G-action the category $O^G(X,(Y,d);\mathcal{A})$ to be the category of G-invariant controlled objects over $G\times X\times Y\times[1,\infty)$ whose object control conditions are G-compact support and whose morphism control conditions are the metric control conditions on G, X and $[1,\infty)$, the G-equivariant continuous control condition on $X\times[1,\infty)$. The space Y will give room for certain constructions.

Define for a sequence of metric spaces $(Y_n, d_n)_{n\in\mathbb{N}}$

$$O^G(X,(Y_n,d_n)_{n\in\mathbb{N}};\mathcal{A})$$

to be the lluf subcategory of $\prod_{n=1}^\infty O^G(X,(Y_n,d_n);\mathcal{A})$ which contains only those morphisms $(\varphi(n))_{n\in\mathbb{N}}$ that are uniformly bounded in the following sense. There is a constant R and a finite subset $F\subset G$ such that $\varphi(n)_{(g,x,y,t),(g',x',y',t')} = 0$ whenever $gg'^{-1}\notin F$ or $d(y,y') > R$.

Clearly $\bigoplus_{n=1}^\infty O^G(X,(Y_n,d_n);\mathcal{A})$ is a subcategory of $O^G(X,(Y_n,d_n)_{n\in\mathbb{N}};\mathcal{A})$. The inclusion is a Karoubi filtration. Let $O^G(X,(Y_n,d_n)_{n\in\mathbb{N}};\mathcal{A})^{>\oplus}$ denote its quotient. Note that a sequence of maps $f_n : (Y_n, d_n) \to (Y'_n, d'_n)$ induces a functor

$$O^G(X,(Y_n,d_n)_{n\in\mathbb{N}};\mathcal{A}) \to O^G(X,(Y'_n,d'_n)_{n\in\mathbb{N}};\mathcal{A}),$$

when there is for each $\beta > 0$ an ε such that for each n and for each pair of points $y, z \in Y_n$ of distance at most β the distance between $f_n(y)$ and $f_n(z)$ is at most ε.

The key result is that

$$\mathcal{O}^G(X, (\Sigma_n, nd^1_n)_{n\in\mathbb{N}}; \mathcal{A})^{>\oplus} = 0,$$

if each Σ_n is an N-dimensional simplicial complex with a cell preserving G-action with a rescaled version of the L^1-metric (see [7, Theorem 7.2]). What remains to do is to

- Find spaces (Y_n, d_n) and a *transfer* map,

  $$\mathrm{trans}_* : K_*(\mathcal{O}^G(E_{\mathcal{F}}G; \mathcal{A})) \to K_*(\mathcal{O}^G(E_{\mathcal{F}}G, (Y_n, d_n)_{n\in\mathbb{N}}; \mathcal{A})^{>\oplus})$$

  such that the composite

  $$\mathrm{pr}_* \circ \mathrm{trans}_* : K_*(\mathcal{O}^G(E_{\mathcal{F}}G; \mathcal{A})^{>\oplus}) \to K_*(\mathcal{O}^G(E_{\mathcal{F}}G, (\mathrm{pt})_{n\in\mathbb{N}}; \mathcal{A})^{>\oplus})$$

  is injective. In the setting of a group acting on a CAT(0) space we can choose Y_n to be $G \times B_{R_n}(x_0)$, where the number R_n is chosen to be large enough and x_0 denotes some chosen basepoint. The G-action on the CAT(0) space and the projections to balls will induce a so called *strong homotopy action* on $B_{R_n}(x_0)$ which is used in the construction of the transfer. The metric on $G \times B_{R_n}(x_0)$ also uses the strong homotopy action.

- Find a system of maps $(Y_n, d_n) \to (\Sigma_n, nd^1_n)$ satisfying the metric conditions from above. This will give a factorization of the map

  $$\mathrm{pr} : \mathcal{O}^G(E_{\mathcal{F}}G, (Y_n, d_n)_{n\in\mathbb{N}}; \mathcal{A})^{>\oplus} \to \mathcal{O}^G(E_{\mathcal{F}}G, (\mathrm{pt})_{n\in\mathbb{N}}; \mathcal{A})^{>\oplus}$$

  through a category with vanishing K-theory. Hence it induces the zero map in K-theory. Since $\mathrm{trans}_* \circ \mathrm{pr}_*$ is injective, we see that the K-theory of $\mathcal{O}^G(E_{\mathcal{F}}G; \mathcal{A})$ vanishes. This shows that the Farrell-Jones assembly map with respect to the family $\mathcal{F}$ is an isomorphism.

The realization of the nerve will associate to any open cover $\mathcal{U}_n$ of Y_n a simplicial complex — its nerve. Its vertices are the elements of $\mathcal{U}_n$. A finite set of vertices spans a simplex if their intersection is not empty. The geometric realization of a simplicial complex can be defined as the set of all maps from its vertex set to $[0, \infty)$ whose support spans a simplex modulo rescaling. There is a map to the realization of the nerve sending a point $y \in Y$ to the map sending $U \in \mathcal{U}_n$ to the distance from y to the complement of U. If one open set consists of the entire space we would have to define the distance to the empty set to be some number.

So the goal is to construct for each Y_n a nice open cover $\mathcal{U}_n$ such that these maps satisfy the metric conditions from above.

The flow space of a metric space consists of certain “generalized geodesics” (see Section 2.1). Let me briefly give an overview where what has been done.

- Constructing long and thin covers of a cocompact part of the flow space of a CAT(0) space X — meaning that for a given real number $R > 0$ there is a cover $\mathcal{U}$ of $FS(X)$ and a real number $\beta > 0$ such that for each $x \in FS(X)$ there is an $U \in \mathcal{U}$ with $B_\beta(\Phi_{[-R,R]}(x)) \subset U$: [5, Chapter 4-5]. It relies on the technical paper [6].

- Such a nice cover can be pulled back to a nice cover of larger and larger balls $B_R(x_0)$. This shows that the group is (strongly) transfer reducible: [5, Chapter 6] and [32].

- The data from (strong) transfer reducibility gives nice maps into simplicial complexes. [4, Chapter 3] and [32].

- These maps can be used to show that the K-theory of the obstruction category vanishes and hence that the Farrell-Jones assembly map is an isomorphism (see Theorem 1.1 both in [4, Chapter 11] and [32, Theorem 1.1]).

1.6 What remains to be done

So finally we have to construct some covers of $FS(X)$. In the case where a group G acts properly, isometrically and cocompactly on a CAT(0) space X this has been done in [5]. These techniques can be used in the noncocompact case to get covers of some cocompact part.

The additional input needed if the group does not act cocompactly is a specific system of open sets. Although the $\mathrm{GL}_n(\mathbb{Z})$-case has already been dealt with in [8], it will be useful to rephrase the argument.

The groups $\mathrm{GL}_n(\mathbb{Z}), \mathrm{GL}_n(F[t]), \mathrm{GL}_n(\mathbb{Z}[S^{-1}]), \mathrm{GL}_n(F[t][S^{-1}])$ act properly, isometrically on a CAT(0) space where S denotes a finite set of primes.

The goal of Section 2 is to examine which conditions the system of open sets should satisfy. This is basically just a collection of all properties of the system used in [8] with the one difference: The fact that $\mathrm{GL}_n(\mathbb{Z})$ has a bound on the order of finite subgroups is used in [8]. It turns out that this condition is indeed superfluous; I have to track back where the argument is used precisely and give an alternative construction there. This construction is due to Adam Mole.

Section 3 deals with the interplay of volume and rank. Given an inner product on $\mathbb{R}^n$ and a submodule of $\mathbb{Z}^n$ we can consider its volume and its rank. These desired open sets can be constructed from those invariants. It contains an axiomatized version of Grayson's construction ([17]). I still have hope that there are other situations in which there are notions of the volume and the rank of certain subobjects of an object so that its automorphisms may be studied with this construction. I tried it for the free group, but did not succeed in defining a reasonable volume function.

Sections 4-6 deal with the definitions of volume in each of the cases $R = \mathbb{Z}$, $F[t]$, $\mathbb{Z}[S^{-1}]$, $F[t][S^{-1}]$. The last two cases can be dealt with simultaneously. Especially it is shown that they have all properties needed in Section 3.

Section 7 applies the construction to analyze the spaces on which $\mathrm{GL}_n(R)$ acts. Basically the CAT(0) metric on each of the spaces is defined and it is shown that that the functions c_W from Section 3 are Lipschitz. I verify all conditions for Proposition 2.4. So the group $\mathrm{GL}_n(R)$ satisfies the Farrell-Jones conjecture with respect to the family $\mathcal{VC}\mathrm{yc} \cup \mathcal{F}$, where $\mathcal{F}$ is the family of subgroups of $\mathrm{GL}_n(R)$ consisting of the normalizers of nontrivial direct summands of R^n. A direct summand is called nontrivial if it is neither 0 nor R^n.

Finally this family will be reduced in section 8 to some smaller family. In the case of $\mathbb{Z}[S^{-1}]$ this family will be $\mathcal{VS}\mathrm{ol}$ in all other cases the family will be reduced to $\mathcal{VC}\mathrm{yc}$. This shows the full Farrell-Jones conjecture in K- and L-theory for those groups.

1.7 Acknowlegdements

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2 Axiomatic setting

The goal of this chapter is to formulate and proof Proposition 2.4. It will apply to those general linear groups in consideration.

2.1 CAT(0)-spaces and their flow spaces

A metric space is called *geodesic* if any two points can be connected by a geodesic. This is a path whose length equals the distance of its endpoints. The metric space $\mathbb{R}^2$ has the following special property: Given any three numbers $a, b, c \in \mathbb{R}$ satisfying the triangular inequalities there is up to isometry a unique triangle $\Delta \subset \mathbb{R}^2$ with those side lengths. Such a triangle will be called a *comparison triangle* of a, b, c.

To three points x, y_1, y_2 in a geodesic metric space (X, d) we can choose unit speed geodesics $c_i : [0, d(x, y_i)] \to X$ from x to y_i. Let $\bar{c}_i : [0, d(x, y_i)] \to \mathbb{R}^2$ be the corresponding geodesics for some comparison triangle of $d(x, y_1), d(x, y_2), d(y_1, y_2)$.

Let us compare the distances in X with the distances in $\mathbb{R}^2$. We can wonder whether we have for some $t \in [0, d(x, y_1)], t' \in [0, d(x, y_2)]$

$$d(c_1(t), c_2(t')) \leq d_{\mathbb{R}^2}(\bar{c}_1(t), \bar{c}_2(t')).$$

If this inequality is satisfied for all choices of $x, y_1, y_2, c_1, c_2, t, t'$ then the space X is called a CAT(0) space, named after Élie Cartan, Aleksandr Danilovich Aleksandrov and Victor Andreevich Toponogov. We could also take rescaled versions of the two-dimensional hyperbolic space or the 2-sphere as comparison spaces. This leads to the notion of $CAT(\kappa)$-spaces for any $\kappa \in \mathbb{R}$. For positive κ the definition has to be modified a bit, since there are no comparison triangles if the side length are too large.

Let us study the isometries of CAT(0) spaces. An important tool is the *translation length* t_f of an isometry f of a metric space X. It is

$$t_f := \inf\{d(x, f(x)) \mid x \in X\}.$$

The isometry f is called *semisimple* if the translation length is attained somewhere. Note that if a group acts properly, isometrically and cocompactly on a metric space it automatically acts by semisimple isometries [11, II.6.10 (2)]. A group acting this way on a CAT(0) space is also called a CAT(0) group.

The semisimple isometries can be divided further into two classes: the *elliptic* isometries are those whose translation length is zero and the *hyperbolic* isometries are those with a positive translation length. An infinite unit-speed geodesic $c : \mathbb{R} \to X$ is an *axis* of f if f acts on the geodesic by translation with some number $r \in \mathbb{R}$. It is a nice exercise to show that this number r is the translation length of f if X is a geodesic metric space.

For CAT(0) spaces the converse also holds. Namely every hyperbolic isometry has an axis ([11, Theorem II.6.8]). So for example the group $\mathbb{Z}[\frac{1}{2}]$ can never appear as a subgroup of a CAT(0) group by the following argument: A cocompact action is always semisimple so every nontrivial element has an axis. The upper argument shows that $t_{\frac{1}{2^n}} = \frac{1}{2^n} t_1$. But a group acting properly and cocompactly must have a lower bound on the translation lengths. So $\mathbb{Z}[\frac{1}{2}]$ cannot appear as a subgroup of a CAT(0) group.

So we have seen that it can be beneficial to study the geodesics in a CAT(0) space X. Let us build a space out of them. For a metric space X consider the set of all *generalized geodesics*. These are those maps $\gamma : \mathbb{R} \to X$ such that there is an interval $[a, b]$ with $-\infty \leq a \leq b \leq \infty$ such that $\gamma|_{[a,b]}$ is a unitspeed geodesic path and $\gamma|_{\mathbb{R}\setminus[a,b]}$ is locally constant. The set of all generalized geodesics will be denoted by $FS(X)$ (as in [5, Section 1]). Define a metric on $FS(X)$ by

$$d_{FS}(\gamma, \gamma') \coloneqq \int_t \frac{d_X(\gamma(t), \gamma'(t))}{2e^{|t|}} dt.$$

Furthermore the space is equipped with an action of the topological group $\mathbb{R}$ via

$$\Phi_\tau(\gamma)(t) \coloneqq \gamma(t + \tau).$$

2.2 Long covers at infinity and periodic flow lines

Let G be a proper, finite dimensional CAT(0) space with a proper, isometric group action of a group G. For the proof we need to construct suitable covers of the flow space $FS(X)$. More precisely we have to find for every $\gamma > 0$ a cover of $FS(X)$ and a number $\varepsilon > 0$ such that we can find for any $c \in FS(X)$ an open set containing $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(c))$.

[5, Theorem 5.7] constructs such a cover. The basic idea of the construction appearing there is to require the existence of a nice collection of open sets dealing with everything except a cocompact part so that only a cocompact part needs to be covered. [5, Theorem 5.6] deals with the cocompact part.

If those nice systems of open sets exist, $FS(X)$ is said to admit *long covers at infinity and periodic flow lines* ([5, Definition 5.5]). Formally this is defined in the following way.

Definition 2.1 (Long $\mathcal{F}$-covers at infinity and periodic flow lines)**.** Let $FS_{\leq\gamma}(X)$ be the subspace of $FS(X)$ of those generalized geodesics c for which there exists for every $\epsilon > 0$ an element $\tau \in (0, \gamma + \epsilon]$ and $g \in G$ such that $g \cdot c = \Phi_\tau(c)$ holds. We will say that *FS admits long $\mathcal{F}$-covers at infinity and periodic flow lines* if the following holds:

There is $N > 0$ such that for every $\gamma > 0$ there is a collection $\mathcal{V}$ of open $\mathcal{F}$-subsets of FS and $\varepsilon > 0$ satisfying:

(i) $\mathcal{V}$ is G-invariant: $g \in G, V \in \mathcal{V} \implies gV \in \mathcal{V}$;

(ii) $\dim \mathcal{V} \leq N$;

(iii) there is a compact subset $K \subseteq FS$ such that
   - $FS_{\leq\gamma} \cap G \cdot K = \emptyset$;
   - for $z \in FS \setminus G \cdot K$ there is $V \in \mathcal{V}$ such that $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(z)) \subset V$.

For the groups $\mathrm{GL}_n(\mathbb{Z})$, $\mathrm{GL}_n(\mathbb{Z}[S^{-1}])$, $\mathrm{GL}_n(F[t])$ and $\mathrm{GL}_n(F[t][S^{-1}])$ these collections are constructed the same way. Basically I first construct a collection of open sets in X, pulls them back to $FS(X)$ along the evaluation map (compare 2.4). These sets will deal with the "at infinity"-part. For the "periodic flow line"-part the following theorem is needed. It is taken from [5, Theorem 4.2].

Theorem 2.2 (Cover of the periodic part with small G-period)**.** *Let G be a group which acts properly and isometrically on a* CAT(0) *space X. Then there is a natural number M such that for every compact subset $L \subseteq X$ and for every $\gamma > 0$ there exists a collection $\mathcal{U}$ of subsets of $FS(X)$ satisfying:*

(i) Each element $U \in \mathcal{U}$ is an open $\mathcal{V}Cyc$-subset of the G-space $FS(X)$;

(ii) $\mathcal{U}$ is G-invariant; i.e. for $g \in G$ and $U \in \mathcal{U}$ we have $g \cdot U \in \mathcal{U}$;

(iii) $G \backslash \mathcal{U}$ is finite;

(iv) We have $\dim \mathcal{U} \leq M$*;*

(v) There is $\varepsilon > 0$ with the following property: for $c \in FS_{\leq\gamma}$ such that $c(t) \in G \cdot L$ for some $t \in \mathbb{R}$ there is $U \in \mathcal{U}$ such that $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq U$.

We need the following lemma taken from [8, Lemma 3.4].

Lemma 2.3. *Consider $\delta, \tau > 0$ and $c \in FS(X)$. Then we get for $d \in B_\delta(\Phi_{[-\tau,\tau]}(c))$*

$$d_X(d(0), c(0)) < 4 + \delta + \tau.$$

Proof. Choose $s \in [-\tau, \tau]$ with $d_{FS(X)}(d, \Phi_s(c)) < \delta$. We compute using [5, Lemma 1.3 and Lemma 1.4].

$$\begin{aligned} d_X(d(0), c(0)) &\leq d_X(d(0), \Phi_s(c)(0)) + d_X(\Phi_s(c)(0), c(0)) \\ &\leq d_{FS(X)}(d, \Phi_s(c)) + 2 + d_{FS(X)}(\Phi_s(c), c) + 2 \\ &< \delta + 2 + |s| + 2 \\ &\leq 4 + \delta + \tau. \end{aligned}$$

□

The following proposition sums up all conditions that are used in [8] to prove that the group action of $\mathrm{GL}_n(\mathbb{Z})$ on the space of inner products admits long coverings at infinity and periodic flow lines.

We will see later that the general linear groups over $R[S^{-1}]$ where R is either $\mathbb{Z}$ or $F[t]$ for a finite field F and S is a finite set of primes in R also satisfy these conditions. The proof is just a slight modification of the proof given in [8]. It might be interesting to find other groups which also satisfy these assumptions.

Proposition 2.4. *Let*

- *G be a group,*
- *X be a G-space,*
- *N a natural number,*
- *$\mathcal{W}$ a collection of open subsets of X*

such that

(i) *X is a proper* CAT(0) *space,*

(ii) *the covering dimension of X is less or equal to N,*

(iii) *the group action of G on X is proper and isometric,*

(iv) *$G\mathcal{W} := \{gW \mid g \in G, W \in \mathcal{W}\} = \mathcal{W}$,*

(v) *the sets gW and W are either disjoint or equal for all $g \in G, W \in \mathcal{W}$,*

(vi) *the dimension of $\mathcal{W}$ is less or equal to N.*

(vii) *the G operation on*

$$X \setminus (\bigcup \mathcal{W}^{-\beta}) := \{x \in X \mid \nexists W \in \mathcal{W} : \overline{B}_\beta(x) \subset W\}$$

is cocompact for every $\beta \geq 0$.

Then $FS(X)$ admits long $\mathfrak{F}$-covers at infinity and periodic flow lines for the family $\mathfrak{F} := \mathcal{V}Cyc \cup \{H \leq G \mid \exists\, W \in \mathcal{W}\ \forall\, h \in H :\ hW = W\}$.

Proof. $FS(X)$ is a proper metric space by [5, Proposition 1.9]. Hence it is locally compact. Fix $\gamma \geq 1$. Let $\beta := 4 + \gamma + 1$. Pick a compact subset $L \subset X$ such that $G \cdot L = X \setminus \bigcup \mathcal{W}^{-\beta}$. For this compact subset L we obtain a natural number M, a real number $\varepsilon > 0$ and a set $\mathcal{U}$ of subsets of $FS(X)$ from Theorem 2.2. We can assume $\varepsilon \leq 1$. Let $\mathcal{V} := ev_0^{-1}(\mathcal{W}) := \{ev_0^{-1}(W) \mid W \in \mathcal{W}\}$. We have

(i) $\mathcal{V}$ is a G-set with $gV \cap V \in \{\emptyset, V\}$ for any $g \in G$ and any $V \in \mathcal{V}$,

(ii) every element $V \in \mathcal{V}$ is an open subset of $FS(X)$ since the evaluation map is continuous by [5, Lemma 1.4]),

(iii) the dimension of $\mathcal{V}$ is bounded by N,

(iv) the group action on $ev_0^{-1}(X \setminus \mathcal{W}^{-R}) = FS(X) \setminus ev_0^{-1}(\mathcal{W}^{-R})$ is cocompact (as the evaluation map is proper [5, Lemma 1.10]).

Consider the union $\mathcal{U} \cup \mathcal{V}$. Each element is an open $\mathcal{VC}\mathrm{yc} \cup \{H \leq G \mid \exists\, W \in \mathcal{W}\ \forall\, h \in H: \ hW = W\}$-subset. Define

$$S := \{c \in FS(X) \mid \exists\, Z \in \mathcal{U} \cup \mathcal{V} \text{ with } \overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq Z\}.$$

This set S contains $FS(X)_{\leq\gamma} \cup |ev_0^{-1}(\mathcal{W}^{-(5+\gamma)})|$ by the following argument. If $c \in |ev_0^{-1}(\mathcal{W}^{-R})|$ we get for any $c' \in \overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c))$ by Lemma 2.3 $d(c'(0), c(0)) \leq 4+\gamma+\varepsilon \leq 5+\gamma$ and hence $c'(0) \in W$. So $c' \in ev_0^{-1}(W)$. So we verified that $|\mathcal{W}^{-R}|$ is contained in S. If $c \in FS(X)_{\leq\gamma}$ and $c \notin |ev_0^{-1}(\mathcal{W}^{-(5+\gamma)})|$, then $c \in FS(X)_{\leq\gamma}$ and $c(0) \in G \cdot L$ and hence $c \in S$ by Theorem 2.2 (v).

Next we prove that S is open. Assume that this is not the case. Then there exists $c \in S$ and a sequence $(c_k)_{k\geq 1}$ of elements in $FS(X) - S$ such that $d_{FS(X)}(c, c_k) < 1/k$ holds for $k \geq 1$. Choose $Z \in \mathcal{U} \cup \mathcal{V}$ with $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c)) \subseteq Z$. Since $FS(X)$ is proper as metric space by [5, Proposition 1.9] and $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c))$ has bounded diameter, $\overline{B}_\epsilon(\Phi_{[-\gamma,\gamma]}(c))$ is compact. Hence we can find $\mu > 0$ with $B_{\epsilon+\mu}(\Phi_{[-\gamma,\gamma]}(c)) \subseteq Z$. We conclude from [5, Lemma 2.3] for all $s \in [-\gamma, \gamma]$

$$d_{FS(X)}(\Phi_s(c), \Phi_s(c_k)) \leq e^s \cdot d_{FS(X)}(c, c_k) < e^\tau \cdot 1/k.$$

Hence we get for $k \geq 1$

$$B_\epsilon(\Phi_{[-\gamma,\gamma]}(c_k)) \subseteq B_{\epsilon+e^\tau \cdot 1/k}(\Phi_{[-\gamma,\gamma]}(c)).$$

Since c_k does not belong to S, we conclude that $B_{\epsilon+e^\tau\cdot 1/k}(\Phi_{[-\gamma,\gamma]}(c))$ is not contained in Z. This implies $e^\tau \cdot 1/k \geq \mu$ for all $k \geq 1$, a contradiction. Hence $FS(X) - S$ is a closed G-subset of the cocompact set $FS(X) - |\mathcal{W}^{-R}|$. So it is also cocompact and there is a compact $K \subset FS(X)$ with $G \cdot K = FS(X) - S$. All in all the G-system of open sets $\mathcal{U} \cup \mathcal{V}$ of dimension $\leq M + N + 1$ has the following properties

- $FS_{\leq\gamma}(X) \cap G \cdot K = FS_{\leq\gamma}(X) \cap (FS(X) \setminus S) = \emptyset$ as $FS_{\leq\gamma}(X) \subset S$;
- for $z \in FS(X) \setminus G \cdot K = S$ there is a $V \in \mathcal{V}$ such that $B_\varepsilon(\Phi_{[-\gamma,\gamma]}(z))$.

Hence $FS(X)$ admits long $\mathfrak{F}$-covers at infinity and periodic flow lines. □

Let me now "quote" [5, Theorem 5.7].

Theorem 2.5. *Let G be a group that acts properly and isometrically on a locally compact metric space (FS, d_{FS}). Assume further that FS is equipped with a G-equivariant flow Φ such that*

- *$FS - FS^{\mathbb{R}}$ is locally connected,*
- *the covering dimension of $FS - FS^{\mathbb{R}}$ is finite,*
- *the flow is uniformly continuous in the following sense: for $\alpha > 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$d_{FS}(z, z') \leq \delta, |\tau| \leq \alpha \Rightarrow d_{FS}(\Phi_\tau(z), \Phi_\tau(z')) \leq \varepsilon,$$

- *FS admits long $\mathcal{F}$-covers at infinity and periodic flow lines for a family $\mathcal{F}$ containing the family of virtually cyclic subgroups.*

Then there is $\hat{N} \in \mathbb{N}$ such that for every $\alpha > 0$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of dimension at most $\hat{N}$ and an $\varepsilon > 0$ (depending on α) such that the following holds:

(i) For every $z \in FS$ there is $U \in \mathcal{U}$ such that $B_\varepsilon(\Phi_{[-\alpha,\alpha]}) \subset U$,

(ii) $\mathcal{U}/G$ is finite.

Proof. The only difference is that I left out the assumption on orders of finite subgroups. The idea is to modify the proof of Bartels-Lück so that it does not need the bound on the order of finite subgroups. First we have to track back where this condition is really needed. This happens in [6, Proposition 3.2-3.3]. The key step is when one wants to produce for a finite group F from a non-F-equivariant cover $\mathcal{U}$ an equivariant cover. Instead of taking $F \cdot \mathcal{U}$ whose dimension depends on the order of F one should rather push it down to a cover of the quotient, refine it there and pull it back again. Of course then one has to show that all the other properties are still satisfied. This is nontrivial. This workaround is due to Adam Mole and will be written up soon. □

Remark 2.6. The flow space associated to a finite-dimensional, proper CAT(0) space automatically satisfies the first three conditions of Theorem 2.5. This is explained in [5, Section 6.2]. Cocompactness is not needed in any of these statements.

2.3 Transfer reducibility

Let X be a finite-dimensional, proper CAT(0) space equipped with an proper and isometric action such that $FS(X)$ admits long $\mathcal{F}$ covers at infinity and periodic flow lines. Let $x_0 \in X$ be some base point.

The goal of this section is to motivate how the covers of the flow space can be used to obtain nice covers of $G \times B_R(x_0)$. This has basically all been done in [5] respectively [32]. Since x_0 might not be a fixed point of the G-action on X the ball $B_R(x_0)$ might not be G-invariant and hence we cannot simply restrict the G-action. Let $\mathrm{pr} : X \to B_R(x_0)$ denote the projection to the convex subset $B_R(x_0)$. We might try to define the action of a group element g as

$$\varphi_g : B_R(x_0) \to B_R(x_0) \qquad x \mapsto \mathrm{pr}(gx).$$

Of course, this does not give a group action since the associativity can fail. Nevertheless we can define for $g, h \in G$ a homotopy $H_{g,h} : B_R(x_0) \times [0,1] \to B_R(x_0)$ from $\varphi_g \circ \varphi_h$ to φ_{gh} in the following way. For a point $x \in X$ connect $g\varphi_h(x)$ and ghx by a constant speed geodesic and define $H_{g,h}(x, -)$ to be its postcomposition with the projection to $B_R(x_0)$.

The data $(\varphi_g)_{g\in G}, (H_{g,h})_{g,h\in G}$ is called a *homotopy action* on $B_R(x)$. If we only specify those maps for some (finite) set of group elements S containing the neutral element, i.e. $(\varphi_s)_{g\in S}, (H_{g,h})_{g,h\in S \text{ with } gh\in S}$ it is called a *homotopy S-action* (as in [5, Definition 0.1]).

Now the idea is that for two group elements g, h there is a large number R so that the paths $(H_{g,h}(x, -))_{x\in B_R(x)}$ are short ([5, Proposition 3.8]). This uses the CAT(0) inequality.

This observation is used to show that nice covers of $FS(X)$ yield nice covers of $G \times B_R(x_0)$.

In the context of a group action a group element g moves a point x to gx. The analog for a homotopy S-action is the following: An element $s \in S$ can move a point x to any point of the form $H_{s',s''}(x,t)$ for some $t \in [0,1]$, $s, s', s'' \in S$ with $s = s's''$. Denote by $F_s(\varphi, H)$ for $s \in S$ the set of all maps of the form $H_{a,b}(-,t)$ with $t \in [0,1]$ and $a, b \in S$.

Note that a group action induces a homotopy action via $\varphi_g(x) := gx$ and $H_{g,h}(x,t) := ghx$. So in the case where the homotopy action comes from an group action both notions of "movement" agree. Now let us finally describe what a "nice" cover of $G \times B_R(x_0)$ really is (as in [4, Definition 1.4]).

Definition 2.7. Let G be a group and Y be a space equipped with a homotopy S-action (φ, H).

(i) Define $S^1_{\varphi,H}(g,x)$ as the subset of $G \times X$ consisting of all points $(ga^{-1}b, y)$ such that there are $a, b \in S$ and maps $f \in F_a(\varphi, H)$, $\tilde{f} \in F_b(\varphi, H)$ and $f(x) = \tilde{f}(y)$.

(ii) Define inductively
$$S^n_{\varphi,H}(g,x) := S^1_{\varphi,H}(S^{n-1}_{\varphi,H}(g,x)),$$
where for a subset $A \subset G \times X$ the term $S^1_{\varphi,H}(A)$ stands for $\bigcup_{(h,y)\in A} S^1_{\varphi,H}(h,y)$.

(iii) A cover $\mathcal{U}$ of $G \times X$ is called *S-long* if there is for every $(g,x) \in G \times X$ an open set $U \in \mathcal{U}$ containing $S^{|S|}_{\varphi,H}(h,y)$.

If the homotopy action comes from an honest group action the set $S^1_{\varphi,H}(g,x)$ is nothing but $\{ga^{-1}b, b^{-1}ax \mid a, b \in S\}$. Finally the notion of transfer reducibility [5, Definition 0.4] is introduced which is used in [4, Theorem 1.1] to show the L-theoretic Farrell-Jones conjecture and the K-theoretic Farrell-Jones conjecture up to dimension one:

Definition 2.8. Let G be a group and $\mathcal{F}$ be a family of subgroups. We will say that G is transfer reducible over $\mathcal{F}$ if there is a number N with the following property: For every finite subset S of G there are

- a contractible compact controlled N-dominated metric space X;
- a homotopy S-action (φ, H) on X;
- a G-invariant cover $\mathcal{U}$ of $G \times X$ by open sets,

such that the following holds for the G-action on $G \times X$ given by $g \cdot (h,x) = (gh, x)$:

(i) $\dim \mathcal{U} \leq N$;

(ii) U is S-long with respect to (φ, H);

(iii) $\forall\, U \in \mathcal{U} :\ gU \cap U \neq \emptyset \Rightarrow gU = U$

(iv) The group $G_U = \{g \in G \mid gU = U\}$ is in $\mathcal{F}$ for each $U \in \mathcal{U}$.

A metric space is called *controlled N-dominated* if there are for each $\varepsilon > 0$ maps $X \to K \to X$ where K is a finite CW-complex of dimension at most N such that there is a homotopy H from the composition to id_X such that for every $x \in X$ the diameter of $\{H(x,t) \mid t \in [0,1]\}$ is at most ε.

There is also a stronger version of homotopy actions introduced by Wegner in [32] to show the full K-theoretic Farrell-Jones conjecture in all dimensions. It includes higher homotopies corresponding to factorizations of an element $s \in S$ into more than two elements.

Definition 2.9. A *strong* homotopy action on a topological space consists of a continuous map

$$\Psi : \bigcup_{n \in \mathbb{N}} (G \times [0,1])^n \times G \times X \to X$$

such that

(i) $\Psi(\ldots, g_l, 0, g_{l+1}, \ldots) = \Psi(\ldots, g_l, \Psi(g_{l+1}, \ldots))$,

(ii) $\Psi(\ldots, g_l, 1, g_{l+1}, \ldots) = \Psi(\ldots, g_l g_{l+1}, \ldots)$,

(iii) $\Psi(e, t_j, g_{j+1}, \ldots) = \Psi(g_{j+1}, \ldots)$,

(iv) $\Psi(\ldots, t_l, e, t_{l+1}, \ldots) = \Psi(\ldots, t_l \cdot t_{l+1}, \ldots)$,

(v) $\Psi(\ldots, t_1, e, x) = \Psi(\ldots, x)$,

(vi) $\Psi(e, x) = x$

Note that an honest G action gives a strong homotopy G-action via

$$\Psi(g_n, t_n, \ldots, t_1, g_0, x) := g_n \ldots g_0 x$$

and that a strong homotopy action Ψ gives a homotopy action via

$$\varphi_g(x) := \Psi(g, x), \qquad H_{g,h}(x,t) := \Psi(g, t, h, x).$$

In the strong setting a group element $g \in G$ can move a point x to all points of the form $\Psi(g_n, t_n, \ldots, t_1, g_0, x)$ with $g_n \ldots g_0 = g$. Consequently define for a finite subset of G and a natural number k the set of maps $F_g(\Psi, S, k)$ to be

$$\{\Psi(g_k, t_k, \ldots, g_0, ?) : X \to X \mid g_i \in S, t_i \in [0,1], g_k \cdot \ldots \cdot g_0 = g\}.$$

The sets $S^n_{\Psi,S,k}(g,x)$ are defined analogously. Consequently we can also define what an (S,n,k)-long cover of $G \times X$ for a strong homotopy G-space X is. A group is called *strongly transfer reducible* over a family $\mathcal{F}$ if one can find a natural number N and for any choice of (S,n,k) a controlled N-dominated metric space X that is equipped with a strong homotopy G-action and an (S,n,k)-long cover of $G \times X$ with the properties from above.

Wegner explains how a proper and isometric G-action on a CAT(0) space gives a strong homotopy G-action on a ball $B_R(x_0)$. He uses them to show that CAT(0) groups are strongly transfer reducible over $\mathcal{VC}$yc. Basically his proof works as follows. His strategy to construct the desired covers is to take the covers from [5, Theorem 5.7] of $FS(X)$ and pull them back along a certain continuous map. The estimations used to show that the resulting covers are long do not use the cocompactness of the group action. Indeed the cocompactness is needed only to verify the assumption "long $\mathcal{F}$-covers at infinity and periodic flow lines" of [5, Theorem 5.7] for $\mathcal{F} = \mathcal{VC}$yc (using [5, Section 6.3]). So finally we can replace [5, Theorem 5.7] in his argument the "new version of Theorem 5.7" (Theorem 2.5) we get

Theorem 2.10. *Suppose G acts properly and isometrically on a proper* CAT(0) *space such that $FS(X)$ admits long $\mathcal{F}$-covers at infinity and periodic flow lines for $\mathcal{VC}yc \subset \mathcal{F}$. Then G is strongly transfer reducible over $\mathcal{F}$. In particular it is also transfer reducible over $\mathcal{F}$.*

Finally the L-theory case has a small flaw. Namely transfer reducibility over $\mathcal{F}$ does not imply the L-theoretic Farrell-Jones conjecture with respect to $\mathcal{F}$ but only with respect to the family of index at most 2 overgroups of $\mathcal{F}$. This makes the induction step more complicated. So if we want to reduce the family in the L-theory setting we have to consider index 2 overgroups of groups from $\mathcal{F}$. Since there is no general inheritance property of the Farrell-Jones conjecture to finite index overgroups known, we have to introduce a slightly stronger version.

Definition 2.11. A group G is said to *satisfy the Farrell-Jones conjecture with finite wreath products* in K- and/or L-theory if $G \wr F$ satisfies the Farrell-Jones conjecture in K- and/or L-theory for any finite group F.

It would be nice if transfer reducibility was also stable under wreath products in some sense. The idea is the following. A G-action on X induces a $G \wr F$ action on X^F. And so we would like to produce from the cover of $G \times X$ a cover of $G \wr F \times X^F$. The construction will have the flaw that the condition $gU \cap U \neq \emptyset \Rightarrow gU = U$ is not preserved.

The notion almost (strongly) transfer reducible results from the notion of "(strongly) transfer reducible" by dropping that condition. ([8, Definition 5.3]). So we get:

Proposition 2.12. *Let $\mathcal{F}$ be a family of subgroups of the group G and let F be a finite group. Denote by $\mathcal{F}^{\wr}$ the family of subgroups H of $G \wr F$ that contain a subgroup of finite index that is isomorphic to a subgroup of $H_1 \times \cdots \times H_n$ for some n and $H_1, \ldots, H_n \in \mathcal{F}$.*

If G is almost (strongly) transfer reducible over a family $\mathcal{F}$, then $G \wr F$ is almost (strongly) transfer reducible over $\mathcal{F}^{\wr}$.

Proof. This has been proven in the proof of [8, Theorem 5.1]. □

The "almost versions" also do not give the K and L-theoretic Farrell-Jones conjecture with respect to the whole family $\mathcal{F}$ but only with respect to finite extensions of groups from the family. We have the following list of versions of "transfer reducible" and which versions of the Farrell-Jones conjecture they imply (as in [8, Proposition 5.4]).

Proposition 2.13. *Let $\mathcal{F}$ be a family of subgroups of a group G and let $\mathcal{F}'$ be the family of subgroups of G that contain a member of $\mathcal{F}$ as a finite index subgroup.*

(i) Let G be transfer reducible over $\mathcal{F}$. Then the K-theoretic Farrell-Jones assembly map

$$H_n^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right)$$

is an isomorphism in degree $n \leq 1$. The L-theoretic Farrell-Jones assembly map

$$H_n^G(E_{\mathcal{F}_2}G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}\left(\int_G \mathcal{A}\right)$$

is an isomorphism in all degrees, where

$$\mathcal{F}_2 := \{H \leq G \mid \exists\, H' \leq H : \ [H : H'] \leq 2 \wedge H' \in \mathcal{F}\}.$$

(ii) Let G be strongly transfer reducible over $\mathcal{F}$. Then the K-theoretic assembly map

$$H_n^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right)$$

is an isomorphism in all degrees.

(iii) Let G be almost transfer reducible over $\mathcal{F}$. Then the K-theoretic Farrell-Jones assembly map

$$H_n^G(E_{\mathcal{F}'}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right)$$

is an isomorphism in degree $n \leq 1$. The L-theoretic Farrell-Jones assembly map

$$H_n^G(E_{\mathcal{F}'}G; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \to H_n^G(\mathrm{pt}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}\left(\int_G \mathcal{A}\right)$$

is an isomorphism in all degrees.

(iv) Let G be almost strongly transfer reducible over $\mathcal{F}$. Then the K-theoretic assembly map

$$H_n^G(E_{\mathcal{F}'}G; \mathbf{K}_{\mathcal{A}}) \to H_n^G(\mathrm{pt}; \mathbf{K}_{\mathcal{A}}) = K_n\left(\int_G \mathcal{A}\right)$$

is an isomorphism in all degrees.

Proof. (i) This is [4, Theorem 1.1].

(ii) This is [32, Theorem 1.1].

(iii) Basically the only modification to part (i) is that the realizations of the nerves of the covers are replaced by their barycentric subdivisions (see [8, Proposition 5.4]). Of course, the underlying space stays the same, but the l^1-metric is different.

(iv) see [8, Proposition 5.4].

□

3 The canonical filtration

This section shows how the systems of open sets used in Proposition 2.4 are constructed. The ideas of this section can all be found in [17]. Let V be a free $\mathbb{Z}$-module and s an inner product on $\mathbb{R}\otimes_{\mathbb{Z}} V$. The size of submodules can be measured in two different ways – by its rank and its volume. The desired open sets in the space of homothety classes of inner products are constructed by comparing these two quantities. This section is formulated in a very general way, since the same constructions also apply for the rings $\mathbb{Z}[S^{-1}], F[t][S^{-1}]$.

An *order-theoretic lattice* $\mathfrak{L}$ is a poset such that any finite subset has a least upper bound and a greatest lower bound. For any two elements $W, W' \in \mathfrak{L}$ let $W + W'$ denote their least upper bound and let $W \cap W'$ denote their greatest lower bound. Let $\mathbb{0}$ denote the minimal element. It is the least upper bound of the empty set. Let $\mathbb{1}$ denote the maximal element which is the greatest lower bound of the empty set. An order-theoretic lattice $\mathfrak{L}$ can be viewed as a category whose objects are the elements of $\mathfrak{L}$. The morphism set from W to W' consists of an unique element if $W \leq W'$ and is empty otherwise. Least upper bounds and greatest lower bounds are product and coproducts.

Convention 3.1. *Let $\mathfrak{L}$ be an order-theoretic lattice. Suppose furthermore there are functions* $\mathrm{rk} : \mathfrak{L} \to \mathbb{N}$ *and* $\log \mathrm{vol} : \mathfrak{L} \to \mathbb{R}$ *such that*

(i) rk *is strictly monotone. This means that for all $W, W' \in \mathfrak{L}$:*

$$W < W' \Rightarrow \mathrm{rk}(W) < \mathrm{rk}(W').$$

(ii) rk *is additive. This means that for all $W, W' \in \mathfrak{L}$:*

$$\mathrm{rk}(W \cap W') + \mathrm{rk}(W + W') = \mathrm{rk}(W) + \mathrm{rk}(W').$$

(iii) $\log \mathrm{vol}(-) : \mathfrak{L} \to \mathbb{R}$ *is subadditive. This means that for all $W, W' \in \mathfrak{L}$:*

$$\log \mathrm{vol}(W \cap W') + \log \mathrm{vol}(W + W') \leq \log \mathrm{vol}(W) + \log \mathrm{vol}(W').$$

(iv) For each $C \in \mathbb{R}$ there are only finitely many $L \in \mathfrak{L}$ with $\log \mathrm{vol}(W) \leq C$.

(v) $\mathrm{rk}(\mathbb{0}) = 0, \log \mathrm{vol}(\mathbb{0}) = 0$.

Remark 3.2. (i) Note that the strict monotonicity holds for example for the lattice of direct summands of $\mathbb{Z}^n$ whereas it fails for the lattice of all submodules of $\mathbb{Z}^n$.

(ii) Note that in the lattice of direct summands of $\mathbb{Z}^n$ the least upper bound $V + W$ is *not* the sum of the modules but the direct summand spanned by the sum of the modules.

(iii) It follows that $\mathbb{0}$ and $\mathbb{1}$ are the only elements of rank zero resp. $\operatorname{rk}(\mathbb{1})$.

(iv) Later the volume will also depend on the choice of a inner product. Thus we will view log vol as a real function on the space of inner products.

Definition 3.3. We can plot every element $W \in \mathfrak{L}$ on the (x, y)-plane with x-coordinate equal to its rank and y-coordinate equal to $\log \operatorname{vol}(W)$. For any fixed rank between zero and $\operatorname{rk}(\mathbb{1})$ there is a lowest point among all points with that rank.

We can omit those elements which lie above or on a line connecting two other points of this set and call the remaining points the *canonical path*.

Of course, it might happen that there are several elements from $\mathfrak{L}$ with the same rank and volume. We will see that this will not be the case for the points in the canonical path.

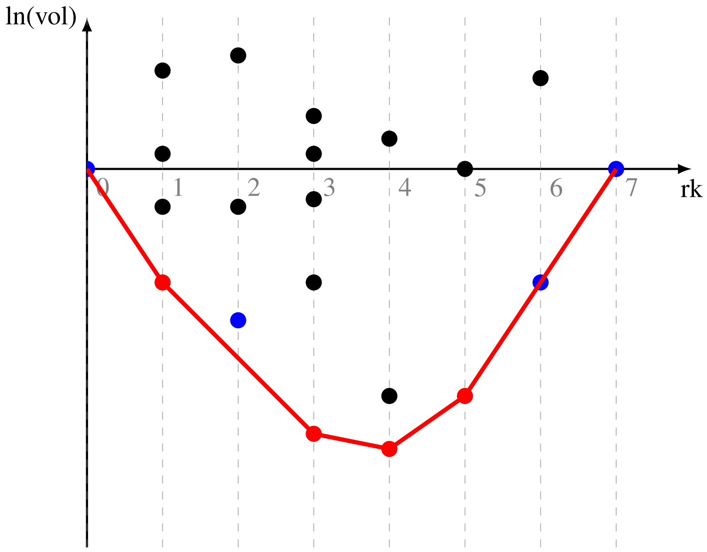

Figure 3.1: The canonical plot.

Definition 3.4. We can define for $W \in \mathfrak{L} \setminus \{\mathbb{0}, \mathbb{1}\}$ a number

$$c_W := \inf_{\binom{W_0 \lneq W}{W \lneq W_2}} \operatorname{slope}(W_2, W) - \operatorname{slope}(W, W_0),$$

where $\operatorname{slope}(W, W')$ is defined as $\frac{\log \operatorname{vol}(W) - \log \operatorname{vol}(W')}{\operatorname{rk}(W) - \operatorname{rk}(W')}$.

Note that the denominators of $\operatorname{slope}(W, W')$ are nonzero for $W \lneq W'$ by the strict monotonicity of the rank. If W represents a vertex in the canonical path we get $c_W > 0$. Otherwise W would lie above the edge from W_0 to W_2. The following lemma leads to the converse.

Lemma 3.5. *Given two incomparable elements $V, W \in \mathfrak{L}$. Then $c_W \leq 0$ or $c_V \leq 0$.*

Proof. Incomparable means that $V \not\lesssim W$ and $W \not\lesssim V$. Especially $V \cap W$ is smaller than V, so it can't be W since W is not smaller than V. So we have by the same argument

$$V \cap W \lneq V \lneq \mathrm{lub}(V, W) \text{ and } V \cap W \lneq W \lneq \mathrm{lub}(V, W).$$

Let us assume that $c_W > 0$. So we have to show that $c_V \leq 0$. We have

$$\begin{aligned}
0 &< c_W \\
&= \inf_{\substack{W_0 \lneq W \\ W \lneq W_2}} \mathrm{slope}(W_2, W) - \mathrm{slope}(W, W_0) \\
&\leq \mathrm{slope}(\mathrm{lub}(V, W), W) - \mathrm{slope}(W, V \cap W) \\
&= \frac{\log \mathrm{vol}(\mathrm{lub}(V, W)) - \log \mathrm{vol}(W)}{\mathrm{rk}(\mathrm{lub}(V, W)) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}(W) - \log \mathrm{vol}(V \cap W)}{\mathrm{rk}(W) - \mathrm{rk}(V \cap W)} \\
&\overset{3.1(ii),(iii)}{\leq} \frac{\log \mathrm{vol}(V) - \log \mathrm{vol}(V \cap W)}{\mathrm{rk}(V) - \mathrm{rk}(V \cap W)} - \frac{\log \mathrm{vol}(\mathrm{lub}(V, W)) - \log \mathrm{vol}(V)}{\mathrm{rk}(\mathrm{lub}(V, W)) - \mathrm{rk}(V)} \\
&= -(\mathrm{slope}(\mathrm{lub}(V, W), V) - \mathrm{slope}(V, V \cap W)) \\
&\leq -\inf_{\substack{V_0 \lneq V \\ V \lneq V_2}} \mathrm{slope}(V_2, V) - \mathrm{slope}(V, V_0) \\
&= -c_V.
\end{aligned}$$

So $c_V < 0$. □

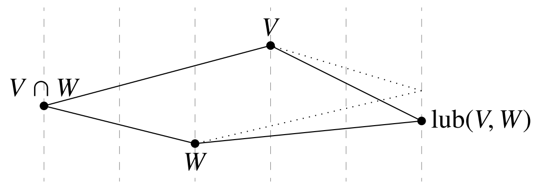

Figure 3.2: Subadditivity gives an upper bound for the logarithmic volume of $\mathrm{lub}(V, W)$ indicated by the dotted line. It completes a the parallelogram so subadditivity can also be called the "parallelogram rule".

Corollary 3.6. *We have that*

(i) every vertex in the canonical path is represented by a unique element $V \in \mathfrak{L}$ and

(ii) those elements form a chain.

(iii) Furthermore an element $V \in \mathfrak{L} \setminus \{\mathbb{0}, \mathbb{1}\}$ represents a vertex in the canonical path if and only if $c_V > 0$.

Proof. (i) By definition any element $V \in \mathfrak{L}$ has $c_V > 0$. The slope of the outgoing line must be larger than the slope of the incoming line. Given two elements $V, V' \in \mathfrak{L} \setminus \{\mathbb{0}, \mathbb{1}\}$ that represent the same vertex of the canonical path. They cannot be incomparable by the last lemma. So either $V \leq V'$ or $V' \leq V$. But they have the same rank. So the strict monotonicity of the rank gives $V = V'$.

(ii) Let $V_0, \ldots, V_m$ be the list of elements ordered by rank that represent the vertices of the canonical path whose rank is at least one and at most $\operatorname{rk} \mathbb{1} - 1$. By the last item the ranks of those elements are all distinct. So $\operatorname{rk}(V_i) < \operatorname{rk}(V_j)$ if $i < j$. As in the last item we know that either $V_i \leq V_j$ or $V_j \leq V_i$. The monotonicity of the rank gives that $V_i \leq V_j$ for $i < j$. So

$$\mathbb{0} \lneq V_0 \lneq \ldots \lneq V_m \lneq \mathbb{1}$$

is a chain.

(iii) Given $V \in \mathfrak{L}$ with $c_V > 0$. Assume V does not represent a vertex in the canonical filtration. So V lies above a line segment of the canonical path. Say that segment starts at W and ends at W'. By the last lemma we again know that V and W (resp. V and W') are not incomparable. As in the last item we get $W \leq V \leq W'$. Because V lies above the edge from W to W' we have

$$\begin{aligned} 0 &< \operatorname{slope}(W', V) - \operatorname{slope}(V, W) \\ &\leq \inf_{\binom{W_0 \lneq W}{W \lneq W_2}} \operatorname{slope}(W_2, W) - \operatorname{slope}(W, W_0) = c_V. \end{aligned}$$

□

Definition 3.7. The chain of elements $\mathbb{0} = V_0 \leq V_1 \leq \ldots V_m = \mathbb{1}$ that represent the vertices in the canonical path is called the *canonical filtration* of $(\mathfrak{L}, \operatorname{rk}, \log \operatorname{vol})$.

4 Volume: The integral case

Let V be a finitely generated, free $\mathbb{Z}$-module V and let m be its rank. Let s be an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$.

Definition 4.1. The volume of a submodule $W \subset V$ with respect to s can be defined as

$$\mathrm{vol}_W(s) := \det((s(b_i, b_j))_{1 \le i,j \le m})^{\frac{1}{2}},$$

where $b_1, \ldots, b_{\mathrm{rk}(W)}$ is a basis for W.

Alternatively we could equip $\Lambda^m \mathbb{R} \otimes_{\mathbb{Z}} V$ with the inner product $\Lambda^m s$ given on elementary exterior products by

$$\Lambda^m s(v_1 \wedge \ldots \wedge v_{\mathrm{rk}(W)}, w_1 \wedge \ldots \wedge w_{\mathrm{rk}(W)}) := \det((s(v_i, w_j))_{1 \le i,j \le \mathrm{rk}(V)}).$$

Let me omit the computation showing that $\Lambda^m s$ is indeed a scalar product. Using this scalar product the volume is just the length of the vector $b_1 \wedge \ldots \wedge b_m \in \Lambda^m \mathbb{R} \otimes_{\mathbb{Z}} V$. Choosing a different basis might change this vector by a multiplication with -1. Its length is independent of the choice of the basis. This shows that the volume is well defined.

Furthermore the volume is always positive. The volume of the trivial Abelian group is one as the determinant of the 0×0 -matrix is defined to be one.

Remark 4.2. We have for a submodule $W \subset V$ and an automorphism $\varphi \in \mathrm{aut}_{\mathbb{Z}}(V)$ and any inner product s on $\mathbb{R} \otimes_{\mathbb{Z}} V$

$$\mathrm{vol}_{\varphi W}(s \circ (\varphi^{-1} \otimes \varphi^{-1})) = \mathrm{vol}_W(s)$$

since $s \circ (\varphi^{-1} \otimes \varphi^{-1})(\varphi(b_i), \varphi(b_j)) = s(b_i, b_j)$.

Let us start with an easy but useful lemma.

Lemma 4.3. *Let* $\mathrm{pr} : \mathbb{R} \otimes_{\mathbb{Z}} V \to \mathbb{R} \otimes_{\mathbb{Z}} V$ *be an orthogonal projection on a linear subspace. Then the map*

$$\Lambda^m \mathrm{pr} : \Lambda^m \mathbb{R} \otimes_{\mathbb{Z}} V \to \Lambda^m \mathbb{R} \otimes_{\mathbb{Z}} V$$

is the orthogonal projection on $\Lambda^m \mathrm{pr}(\mathbb{R} \otimes_{\mathbb{Z}} V)$.

Proof. We can pick an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R} \otimes_{\mathbb{Z}} V$ such that $e_1, \ldots, e_k$ is a basis for $\mathrm{pr}(\mathbb{R} \otimes_{\mathbb{Z}} V)$ and $e_{k+1}, \ldots, e_n$ is a basis for $\ker(\mathrm{pr})$. Note that for a given tuple $1 \le i_1 < \ldots < i_m \le n$ the vector $e_{i_1} \wedge \ldots \wedge e_{i_m}$ lies in the kernel of $\Lambda^m \mathrm{pr}$ if and only if $i_m > k$. Otherwise it is fixed by $\Lambda^m \mathrm{pr}$. Thus we have found bases for the eigenspaces corresponding to the eigenvalues zero and one of the projection pr such that their union is a basis for V. Hence Λ^m pr is really a projection.

We still have to show that the kernel and the image are orthogonal. However it is not harder to show that this basis is an orthogonal basis. So consider $1 \leq i_1 < \ldots < i_m \leq n$, $1 \leq i'_1 < \ldots < i'_m \leq n$. We get

$$\Lambda^m s(e_{i_1} \wedge \ldots \wedge e_{i_m}, e_{i'_1} \wedge \ldots \wedge e_{i'_m}) = \det(s(e_{i_j}, e_{i'_{j'}})_{j,j'}).$$

Note that if there is an index j such that i_j does not occur in $i'_1, \ldots, i'_m$, then there is a vanishing column in the matrix on the right hand side and thus its determinant is zero. If this is not the case, then every i_j must occur somewhere and since the tuples are sorted we get $i_j = i'_j$. Hence the basis is indeed an orthogonal basis and thus pr is an orthogonal projection. □

Lemma 4.4. *Let $W \subset V$ be a projective submodule. We can identify $\mathbb{R} \otimes_{\mathbb{Z}} (V/W)$ with the orthogonal complement of $\mathbb{R} \otimes_{\mathbb{Z}} W \subset \mathbb{R} \otimes_{\mathbb{Z}} V$. So we can restrict any inner product s on $\mathbb{R} \otimes_{\mathbb{Z}} V$ to an inner product s' on $\mathbb{R} \otimes_{\mathbb{Z}} (V/W)$. Furthermore we have*

$$\mathrm{vol}_V(s) = \mathrm{vol}_W(s) \cdot \mathrm{vol}_{V/W}(s').$$

Proof. Let $\mathbb{R} \otimes_{\mathbb{Z}} V \to (\mathbb{R} \otimes_{\mathbb{Z}} W)^{\perp}$ denote the orthogonal projection. Choose a basis $v_1, \ldots, v_n$ of V such that $v_1, \ldots v_m$ is a basis of W. Then $\mathrm{pr}(v_i) - v_i$ for $i = m+1, \ldots, n$ can be written as a linear combination of the $v_1, \ldots, v_m$ and hence

$$v_1 \wedge \ldots \wedge v_n = v_1 \wedge \ldots \wedge v_m \wedge \mathrm{pr}(v_{m+1}) \wedge \ldots \wedge \mathrm{pr}(v_m).$$

Thus we can use the right hand side in the definition of the volume (Definition 4.1) to get a block matrix since $s(v_i, \mathrm{pr}(v_j)) = 0$ for $i \leq m < j$. Thus

$$\mathrm{vol}_V(s) = \det((s(v_i, v_j))_{1 \leq i,j \leq m})^{\frac{1}{2}} \cdot \det(s(\mathrm{pr}(v_i), \mathrm{pr}(v_j))_{m+1 \leq i,j, \leq n})^{\frac{1}{2}}.$$

The left factor is just $\mathrm{vol}_W(s)$ and the right factor is $\mathrm{vol}_{V/W}(s')$. □

Consider the lattice $\mathfrak{L}$ of all direct summands of V. For a direct summand $W \subseteq V$ let $\mathrm{rk}(W)$ denote its rank as an $\mathbb{Z}$-module and let $\ln \mathrm{vol}_V(s)$ denote the natural logarithm of the volume defined above. Let us fix an inner product s and verify that the functions

$$\mathrm{rk}_{\mathbb{Z}} : \mathfrak{L} \to \mathbb{Z}, \qquad \ln \mathrm{vol}_?(s) : \mathfrak{L} \to \mathbb{R}$$

satisfy all properties needed in Convention 3.1.

Lemma 4.5. *Let V be a free $\mathbb{Z}$-module of rank n and let W be a submodule of the same rank. Then* $\mathrm{vol}_W(s) = \mathrm{vol}_V(s) \cdot |V/W|$

Proof. By the elementary divisor theorem there is a basis $e_1, \ldots, e_n$ for V and natural numbers $r_1, \ldots, r_n$, such that $r_1 e_1, \ldots, r_n e_n$ is a basis for W. Inserting this in the definition of the volume (Definition 4.1) we get

$$\mathrm{vol}_W(s) = \|r_1 e_1 \wedge \ldots \wedge r_n e_n\|_{\Lambda^n s} = \left(\prod_{i=1}^{n} r_i\right) \|e_1 \wedge \ldots \wedge e_n\|_{\Lambda^n s} = \prod_{i=1}^{n} r_i \, \mathrm{vol}_V(s).$$

Clearly $V/W \cong \bigoplus_{i=1}^{n} \mathbb{Z}/r_i$ and hence we get the desired result. □

Proposition 4.6. *Let V be a finitely generated free $\mathbb{Z}$-module and s an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$. Consider the lattice of direct summands of V. Let* $\operatorname{lub}(W, W')$ *denote the least upper bound of two elements $W, W' \in \mathfrak{L}$. The logarithmic volume function $W \mapsto$* $\ln \operatorname{vol}_W(B)$ *and the rank $W \mapsto$* $\operatorname{rk}_{\mathbb{Z}}(W)$ *have the following properties.*

(i) rk *is strictly monotone, i.e.* $\operatorname{rk}(W) < \operatorname{rk}(W')$ *for all $W, W' \in \mathfrak{L}$ with $W < W'$.*

(ii) rk *is additive, i.e.* $\operatorname{rk}(W \cap W') + \operatorname{rk}(\operatorname{lub}(W, W')) = \operatorname{rk}(W) + \operatorname{rk}(W')$ *for all $W, W' \in \mathfrak{L}$.*

(iii) The function $\ln \operatorname{vol}(-) : \mathfrak{L} \to \mathbb{R}$ *is subadditive. This means*

$$\ln \operatorname{vol}_{W \cap W'}(s) + \ln \operatorname{vol}_{\operatorname{lub}(W,W')}(s) \leq \ln \operatorname{vol}_W(s) + \ln \operatorname{vol}_{W'}(s) \text{ for all } W, W' \in \mathfrak{L}.$$

(iv) For each $C \in \mathbb{R}$ there are only finitely many $L \in \mathfrak{L}$ with $\ln \operatorname{vol}_W(s) \leq C$.

(v) $\operatorname{rk}(\mathbb{0}) = 0, \ln \operatorname{vol}_{\mathbb{0}}(s) = 0$.

Proof. (i) We clearly have for $W \subset W'$ that $\operatorname{rk}(W) \leq \operatorname{rk}(W')$. Their quotient W'/W is a submodule of the module V/W'. The module V/W' is again free by the structure theorem for finitely generated modules over a principal ideal domain since it is a direct summand of the free module V. Hence W'/W is torsionfree. The structure theorem tells us that its rank is bigger than zero if W'/W is not trivial. Hence the additivity of the rank implies:

$$\operatorname{rk}_{\mathbb{Z}}(W) \leq \operatorname{rk}_{\mathbb{Z}}(W) + \operatorname{rk}_{\mathbb{Z}}(W'/W) = \operatorname{rk}_{\mathbb{Z}}(W')$$

and the inequality is strict if $W \neq W'$.

(ii) Note that this is not exactly the additivity of the rank applied to the exact sequence

$$0 \to W \cap W' \to W \oplus W' \to W + W' \to 0$$

since $\operatorname{lub}(W, W')$ is not the sum $W + W'$ but the direct summand spanned by the sum. It is just the preimage of the torsion subgroup of $V/(W + W')$. So we get another exact sequence

$$0 \to W + W' \to \operatorname{lub}(W, W') \to \operatorname{tors}(V/W + W') \to 0.$$

Using additivity and the fact that the rank of a torsion group vanishes we get

$$\operatorname{rk}(\operatorname{lub}(W, W')) = \operatorname{rk}(W + W') = \operatorname{rk}(W) + \operatorname{rk}(W') - \operatorname{rk}(W \cap W').$$

(iii) Using Lemma 4.4 we can restrict to the case where $W \cap W' = 0$. If this was not the case we would have to pass to quotients by $W \cap W'$.

Since $W + W'$ is a finite index submodule of $\operatorname{lub}(W + W')$ its volume is larger (by Lemma 4.5). Let us show the stronger statement

$$\ln \operatorname{vol}_{W+W'}(s) \leq \ln \operatorname{vol}_W(s) + \ln \operatorname{vol}_{W'}(s)$$

instead. So we can pick a basis $w_1, \dots, w_n$ of $W + W'$ such that $w_1, \dots, w_m$ is a basis of W and $w_{m+1}, \dots, w_n$ is a basis of W'. Let $\mathrm{pr} : \mathbb{R} \otimes_{\mathbb{Z}} V \to \mathbb{R} \otimes_{\mathbb{Z}} W^{\perp}$ be the orthogonal projection. Then

$$\begin{aligned}
\mathrm{vol}_{W+W'}(s) &= \|w_1 \wedge \ldots \wedge w_n\|_{\Lambda^n s} \\
&= \|w_1 \wedge \ldots \wedge w_m \wedge \mathrm{pr}(w_{m+1}) \wedge \ldots \wedge \mathrm{pr}(w_n)\|_{\Lambda^n s} \\
&= \|w_1 \wedge \ldots \wedge w_m\|_{\Lambda^m s} \cdot \| \mathrm{pr}(w_{m+1}) \wedge \ldots \wedge \mathrm{pr}(w_n)\|_{\Lambda^{n-m} s} \\
&= \|w_1 \wedge \ldots \wedge w_m\|_{\Lambda^m s} \cdot \|(\Lambda^{n-m} \mathrm{pr})(w_{m+1} \wedge \ldots \wedge w_n)\|_{\Lambda^{n-m} s}.
\end{aligned}$$

The first factor is just $\mathrm{vol}_W(s)$. By Lemma 4.3 $\Lambda^{n-m}\,\mathrm{pr}$ is also an orthogonal projection and hence it is length decreasing. Thus the second factor is bounded by

$$\|w_{m+1} \wedge \ldots \wedge w_n\|_{\Lambda^{n-m} s} = \mathrm{vol}_{W'}(s).$$

Since all volumes are positive we can pass to logarithms and obtain the desired result.

(iv) $\Lambda^m \mathbb{R} \otimes_{\mathbb{Z}} V$ equipped with the inner product $\Lambda^m s$ is a proper metric space and $\Lambda^m V$ is a discrete subset. So the intersection of this subset with any ball is compact and discrete and hence finite. So the set

$$S := \{v \in \Lambda^m V \mid \|v\|_{\Lambda^m s} \leq e^C\}/w \sim -w$$

is finite. We can assign to a direct summand $W \subset V$ of rank m the element $[w_1 \wedge \ldots \wedge w_m]$ in this set, where $w_1, \dots, w_m$ is a basis of W. This element is independent of this choice. On the other hand we can assign to an element $[w]$ from this set the submodule

$$r(w) = \ker(- \wedge w : V \to \Lambda^{m+1} V).$$

Given any direct summand $W \subset V$ we can pick a basis $w_1, \dots, w_n$ of V such that $w_1, \dots, w_m$ is a basis for W. Then $w_1, \dots, w_m$ lie in the kernel of the map

$$- \wedge w_1 \wedge \ldots \wedge w_m : V \to \Lambda^{m+1} V.$$

The vectors $w_{m+1}, \dots, w_n$ get mapped to a linear independent subset of $\Lambda^{m+1} V$. Hence the kernel is really W. So the set of all direct summands W of rank m with $\mathrm{vol}_W(s) \leq e^C$ can be identified with a subset of the finite set S. Hence we obtain the desired result.

(v) see Definition 4.1.

□

Hence we have shown that for a fixed inner product s the lattice $\mathfrak{L}$ of direct summands of V and the functions $\mathrm{rk}_{\mathbb{Z}}$ and $W \mapsto \ln \mathrm{vol}_W(s)$ satisfy all conditions needed in Convention 3.1. So we can use the numbers c_W from Definition 3.4. So we get a function that assigns to an inner product s the number $c_W(s)$. This function will be used to analyze the space of homothety classes of inner products in section 7.1.

5 Volume: The function field case

Let F be a finite field and consider the ring $Z := F[t]$ and its quotient field Q. Let us examine the function

$$v : Q \to \mathbb{Z} \cup \{\infty\}, \qquad \frac{p}{q} \mapsto \deg(q) - \deg(p).$$

We use the convention that the degree of the zero polynomial is $-\infty$.

Lemma 5.1. *v is a valuation, i.e. it satisfies*

(i) $v(x) = \infty$ *if and only if* $x = 0$*,*

(ii) $v(ab) = v(a) + v(b)$ *for any* $a, b \in Q$*,*

(iii) $v(a + b) \geq \min(v(a), v(b))$ *for any* $a, b \in Q$*.*

Consider the valuation ring of this valuation

$$R := \{\frac{p}{q} \in Q \mid \deg(p) \leq \deg(q)\} = \{x \in Q \mid v(x) \geq 0\}.$$

The following definition is the analogue of a "lattice" from [17, section 1] in the integral case; but I would like to avoid this term because it will also appear with different meanings.

Definition 5.2. A Z-volume space (V, S) is a finitely generated free Z-module V with the choice of an R-lattice S in $Q \otimes_Z V$. This means that S is a finitely generated R-submodule with $\mathrm{rk}_Z(V) = \mathrm{rk}_R(S)$.

Remark 5.3.

- We will usually think of V as a Z-submodule of $Q \otimes_Z V$ via $v \mapsto 1 \otimes v$.
- Note first that we get $\langle S \rangle_Q = Q \otimes_Z V$ for a volume space (V, S). The R-module S is torsionfree since it is an R-submodule of a Q-vector space. By the structure theorem for finitely generated R-modules we see that S is free. An R-basis of S is Z-linear independent and hence also Q-linear independent. So it is also a Q-basis for $Q \otimes_Z V$ since $\mathrm{rk}_R(S) = \mathrm{rk}_Z(V) = \dim_Q(Q \otimes_Z V)$.

Definition 5.4. We say that (W, S') is a *sub-volume space* of (V, S) (written $(W, S') \subset (V, S)$) if $W \subset V$ is a Z-submodule and $S' = S \cap Q \otimes_Z W =: \mathrm{res}_W(S)$ for the inclusion $i : W \hookrightarrow V$.

If V/W is projective the *quotient volume space* of $(W, S') \subset (V, S)$ is defined as

$$(V/W, S/(S \cap (Q \otimes_Z W))).$$

Let us denote it by $(V, S)/(W, S')$. Let $\mathrm{quot}_W(S) := S/(S \cap (Q \otimes_Z W))$.

Remark 5.5. (i) The R-lattice occuring in the definition of sub-volume space can be omitted. More precisely, any submodule W of V can be turned into a sub-volume space with the choice $\mathrm{res}_W(S)$.

As R is a principal ideal domain, submodules of finitely generated modules are again finitely generated i.e. R is Noetherian. We have to show that $\mathrm{rk}_R(\mathrm{res}_W(S)) = \mathrm{rk}_Z(W)$:

Let us first show that the R-module $Q \otimes_Z V/S$ is torsion; i.e. its rank is zero.

$$\begin{aligned}
\mathrm{rk}_R(Q \otimes_Z V/S) &= \mathrm{rk}_R(Q \otimes_Z V) - \mathrm{rk}_R(S) && \text{additivity of the rank}\\
&= \dim_Q(Q \otimes_R Q \otimes_Z V) - \mathrm{rk}_R(S) && \text{by definition of } \mathrm{rk}_R\\
&= \dim_Q(Q \otimes_Z V) - \mathrm{rk}_R(S) &&\\
&= \mathrm{rk}_Z(V) - \mathrm{rk}_R(S) && \text{structure theorem}\\
&= 0 && \text{as } (V, S) \text{ is a volume space.}
\end{aligned}$$

Consider

$$S \cap (Q \otimes_Z W) = \ker (Q \otimes_Z W \to Q \otimes_Z W/(S \cap Q \otimes_Z W) \hookrightarrow Q \otimes_Z V/S).$$

Hence $Q \otimes_Z W/(S \cap Q \otimes_Z W)$ is a submodule of $Q \otimes_Z V/S$. By Definition 5.2 we have $\mathrm{rk}_R(S) = \mathrm{rk}_Z(V) = \mathrm{rk}_Q(Q \otimes_Z V) = \mathrm{rk}_R(Q \otimes_Z V)$ and hence the additivity of the rank shows that $Q \otimes_Z V/S$ has rank zero. So the rank of $W/W \cap B$ is also zero. Additivity implies that $\mathrm{rk}_R(S \cap (Q \otimes_Z W)) = \mathrm{rk}_R(Q \otimes_Z W) = \mathrm{rk}_Z(W)$.

So $(W, \mathrm{res}_W(S))$ is indeed a volume space.

(ii) The R-module $S \cap (Q \otimes_Z W)$ is a direct summand in S. Equivalently (structure theorem for finitely generated modules over a principal ideal domain) we could show that the quotient $S/((Q \otimes_Z W) \cap S)$ is torsionfree. But this is clear since it is an R-submodule of the torsionfree module $(Q \otimes_Z V)/(Q \otimes_Z W)$ which is a Q-vector space.

(iii) Consequently there is an R-basis $b_1, \ldots, b_n$ of $S \subset Q \otimes_Z V$ with

$$\langle b_1, \ldots, b_{\mathrm{rk}(W)} \rangle_Q = Q \otimes_Z W.$$

(iv) An implicit claim in the definition of a quotient lattice has to be verified: Let

$$\mathrm{pr} : Q \otimes_Z V \to Q \otimes_Z V/Q \otimes_Z W$$

denote the projection. We have to check that $\mathrm{rk}_R(\mathrm{quot}(S)) = \mathrm{rk}_Z(V) - \mathrm{rk}_Z(W)$. By definition we have $\mathrm{quot}(S) = S/((Q \otimes_Z W) \cap S)$ and the additivity of the rank implies:

$$\begin{aligned}
\mathrm{rk}_R(S/((Q \otimes_Z W) \cap S)) &= \mathrm{rk}_R(S) - \mathrm{rk}_R((Q \otimes_Z W) \cap S)\\
&= \mathrm{rk}_Z(V) - \mathrm{rk}_R((Q \otimes_Z W) \cap S) && (V, S) \text{ is a volume space}\\
&= \mathrm{rk}_Z(V) - \mathrm{rk}_Z(W) && \text{using the previous item}\\
&= \mathrm{rk}_Z(V/W).
\end{aligned}$$

Note that $W/W \cap S$ has rank zero as it is a submodule of the rank zero module $Q \otimes_Z V/S$.

Lemma 5.6 (Analog of [17, Lemma 1.1]). *Let $L = (V, S)$ be a volume space. For $V_1 \subset V_2 \subset V$ we have* $\mathrm{res}_{V_1} \circ \mathrm{res}_{V_2}(S) = \mathrm{res}_{V_1}(S)$.

Proof. Since Q is a flat Z-module we have $Q \otimes_Z V_1 \subset Q \otimes_Z V_2 \subset Q \otimes_Z V$. Thus $Q \otimes_Z V_1 \cap (Q \otimes_Z V_2 \cap S) = Q \otimes_Z V_1 \cap S$. □

Remark 5.7. Not all of the isomorphism theorems for modules also work for volume spaces. When dealing with subquotients first passing to subgroups and then to quotients might result in a different volume space than first passing to quotients and then to subgroups. For example if V is the free $F[t]$-module on generators e_1, e_2 and let V_1 be the submodule spanned by $1e_1 + t^n e_2$ and let V_2 be the submodule spanned by e_2.

Let S be the R-module spanned by $1 \otimes e_1, 1 \otimes e_2$. In this example we have

$$\mathrm{quot}_{V_1 \cap V_2}(\mathrm{res}_{V_1}(S)) \neq \mathrm{res}_{V_1/(V_1 \cap V_2)} \mathrm{quot}_{V_2}(S).$$

The situation is better if we assume that V_2 is a submodule of V_1:

Lemma 5.8. *Let $L = (V, S)$ be a volume space and let $V_2 \subset V_1 \subset V$ be submodules L_1 such that V/V_2 is projective. Then the quotient V_1/V_2 is a sub-volume space of V/V_2, i.e.*

$$\mathrm{quot}_{V_2}(\mathrm{res}_{V_1}(S)) = \mathrm{res}_{V_1/V_2}(\mathrm{quot}_{V_2}(S)).$$

If additionally V/V_1 is also projective, then it is the quotient of V/V_2 by V_1/V_2, i.e.

$$\mathrm{quot}_{V_1/V_2}(\mathrm{quot}_{V_2}(S)) = \mathrm{quot}_{V_1}(S).$$

Proof. Clearly V_1/V_2 is a submodule of V/V_2. So let us now compare the lattices. Let

$$\mathrm{pr} : Q \otimes_Z V \to Q \otimes_Z V/(S \cap Q \otimes_Z V_2)$$

denote the projection. Note that in general $\mathrm{pr}(A \cap B) = \mathrm{pr}(A) \cap \mathrm{pr}(B)$ holds only if A and B are pr-saturated, i.e. $\mathrm{pr}(\mathrm{pr}^{-1})(A)) = A$. In the case where A is a submodule this simply means that $S \cap Q \otimes_Z V_2 \subset A$. This is where the condition $V_2 \subset V_1$ enters.

$$\begin{aligned}
\mathrm{quot}_{V_2}(\mathrm{res}_{V_1}(S)) &:= (S \cap Q \otimes_Z V_1)/(S \cap Q \otimes_Z V_2) \\
&= \mathrm{pr}(S \cap Q \otimes_Z V_1) \\
&= \mathrm{pr}(S) \cap pr(Q \otimes_Z V_1) \\
&= S/(S \cap Q \otimes_Z V_2) \cap (Q \otimes_Z V_1/V_2) \\
&=: \mathrm{res}_{V_1/V_2}(\mathrm{quot}_{V_2}(S)).
\end{aligned}$$

This proves the first claim. Note that

$$\begin{aligned}
\mathrm{quot}_{V_1/V_2}(\mathrm{quot}_{V_2}(S)) &= (S/(S \cap Q \otimes_Z V_2))/((S \cap Q \otimes_Z V_1)/(S \cap Q \otimes_Z V_2)) \\
&= S/(S \cap Q \otimes_Z V_1) \\
&= \mathrm{quot}_{V_1}(S).
\end{aligned}$$

gives the second claim. □

Remark 5.9. Let (V, S) be a volume space. Then $(\Lambda^m V, \Lambda^m S)$ is a volume space.

This motivates the following definition:

Definition 5.10 (logarithmic volume)**.** Let (V, S) be a volume space. Pick an Z-basis $v_1, \ldots, v_n$ for V and an R-basis $b_1, \ldots, b_n$ of S. The Q-vector space $Q \otimes_Z \Lambda^n V \cong \Lambda^n(Q \otimes_Z V)$ is one dimensional. Consider the element $q \in Q$ with $v_1 \wedge \ldots \wedge v_n = q(b_1 \wedge \ldots \wedge b_n)$ (It exists since $b_1 \wedge \ldots \wedge b_n \neq 0$). Define

$$\log \mathrm{vol}_V(S) := -\nu(q).$$

Clearly the volume is independent of the involved choices. Choosing different bases will change q by a multiplication with an element in $Z^* = F^*$ resp. $R^* = \{q \in Q \mid \nu(q) = 0\}$. This change does not affect the valuation.

Remark 5.11. We clearly have the following equivariance property. For any $\varphi \in \mathrm{aut}_Z(V)$ we have

$$\log \mathrm{vol}_{\varphi(W)}(\varphi(S)) = \log \mathrm{vol}_W(S).$$

Let us now find a way to compute the volume of a sub-volume space $W \subset (V, S)$ without constructing a basis for $S \cap Q \otimes_Z W$.

Proposition 5.12 (Formula for the logarithmic volume)**.** *Let (V, S) be an Z-volume space and let $(b_1, \ldots, b_n)$ be an R-basis of S. Let $W \subset Q \otimes V$ be a finitely generated Z-submodule. Choose a Z-basis $w_1, \ldots, w_m$ of W. The set*

$$\{b_{i_1} \wedge \ldots \wedge b_{i_m} \mid 1 \leq i_1 < \ldots < i_m \leq n\}$$

is a Q-basis for $\Lambda^m Q \otimes_Z V$. Write $w_1 \wedge \ldots \wedge w_m$ as a linear combination of this basis

$$w_1 \wedge \ldots \wedge w_m = \sum_{1 \leq i_1 < \ldots < i_m \leq n} \lambda_{i_1, \ldots, i_m} \cdot b_{i_1} \wedge \ldots \wedge b_{i_m}.$$

Then the logarithmic volume of W with respect to S is

$$\log \mathrm{vol}_W(\mathrm{res}_W(S)) = \sup_{1 \leq i_1 < \ldots < i_m \leq n} -\nu(\lambda_{i_1, \ldots, i_m}).$$

Proof. First we show that the right hand side does not depend on the choice of bases and afterwards we show that it agrees with Definition 5.10. Because $F[t]$ is a principal ideal domain W is again free. Furthermore R is also a principal ideal domain; so the same holds for the R-module S.

Picking a different basis for W changes $w_1 \wedge \ldots \wedge w_m$ by multiplication with an element in $Z^* = F^*$. The new coefficients can be obtained from the old ones by multiplication with that element. This change does not affect the valuation.

If we pick a different R-basis $b'_1, \ldots, b'_n$ of S we get a base change matrix $A = (a_{i,j})$ from $\{b_{i_1} \wedge \ldots \wedge b_{i_m} \mid 1 \leq i_1 < \ldots < i_m \leq n\}$ to $\{b'_{i_1} \wedge \ldots b'_{i_m} \mid 1 \leq i_1 < \ldots < i_m \leq n\}$ with entries in R. Hence $\nu(a_{i,j}) \geq 0$. Let us abbreviate $I := \{(i_1, \ldots, i_m) \mid i_1 < \ldots < i_m\}$

and $c_{(i_1,\dots,i_m)} := b_{i_1} \wedge \dots \wedge b_{i_m}$ and $c'_{(i_1,\dots,i_m)} := b'_{i_1} \wedge \dots \wedge b'_{i_m}$. Let λ_i be the coefficients of $w_1 \wedge \dots \wedge w_m$ with respect to the basis c'_i. We get

$$w_1 \wedge \dots \wedge w_m = \sum_{i \in I} \lambda_i c'_i = \sum_{i \in I} \lambda_i \sum_{j \in I} a_{i,j} c_j = \sum_{j \in I} (\sum_{i \in I} \lambda_i a_{i,j}) c_j.$$

Now we can compare the length of $w_1 \wedge \dots \wedge w_m$ with respect to $(c_i)_{i \in I}$ and with respect to $(c'_i)_{i \in I}$.

$$\begin{aligned} & \sup_{j \in I} -\nu(\sum_{i \in I} \lambda_i a_{i,j}) \\ \leq \; & \sup_{j \in I} \sup_{i \in I} -\nu(\lambda_i a_{i,j}) \\ \leq \; & \sup_{j \in I} \sup_{i \in I} -\nu(\lambda_i) - \nu(a_{i,j}) \\ \leq \; & \sup_{i \in I} -\nu(\lambda_i). \end{aligned}$$

Using the invertibility of the matrix $(a_{i,j})_{1 \leq i,j \leq n}$ we get symmetrically the converse. Hence we have shown the independence of the choices. Now let us choose a special basis for S: Since $Q \otimes_Z V$ is a Q-module the quotient $S/S \cap (Q \otimes_Z W)$ is torsionfree. So the structure theorem tells us that $S \cap (Q \otimes_Z W)$ is a direct summand of S. Hence a basis $b_1, \dots, b_s$ of $S \cap (Q \otimes_Z W) \subset S$ can be completed to a basis of $S \cap Q \otimes_Z W \subset S$. Let us write $w_1 \wedge \dots \wedge w_m$ as a linear combination of the completed basis.

The b_i have been chosen such that each w_i can be written as a linear combination of $b_1, \dots, b_s$. So $w_1 \cap \dots \cap w_s$ can be written as a linear combination of $b_1 \cap \dots \cap b_s$. Hence the only nonzero coefficient belongs to $b_1 \wedge \dots \wedge b_s$. Hence the volume is the negative of the valuation of that coefficient. But this is $\log \mathrm{vol}_W(S \cap Q \otimes_Z W)$ by definition. □

Remark 5.13. We will sometimes use the abbreviation $\log \mathrm{vol}_W(S)$ for $\log \mathrm{vol}_W(\mathrm{res}_W(S))$.

Remark 5.14. Let us define the logarithmic volume of the zero volume space to be zero.

Remark 5.15. There is an explicit formula for the logarithmic volume. Let S, W, $(b_i)_i$, $(w_i)_i$, m, n be as above and let $w_i = \sum_{j=1}^{n} \lambda_{i,j} b_j$ with $\lambda_{i,j} \in Q$. Inserting this in the definition of the logarithmic volume yields

$$\log \mathrm{vol}_W(\mathrm{res}_W(S)) = \sup_{1 \leq i_1 < \dots < i_m \leq n} -\nu(\sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma) \lambda_{\sigma(1),i_1} \cdot \dots \cdot \lambda_{\sigma(m),i_m})$$

This means the following. We consider the non-square matrix $(\lambda_{i,j})_{i,j}$. Consider all $m \times m$ minors, i.e. square matrices obtained from this matrix by deleting rows/columns, and the valuation of their determinants. The logarithmic volume of $\langle w_1, \dots, w_m \rangle$ is just the negative of their minimum.

Lemma 5.16. *Let (V, S) be a volume space and let $W' \subset W \subset V$ be a chain of Z-modules of the same rank m. Let A denote a matrix that represents the inclusion after choice of bases for W and W'. Then*

$$\log \mathrm{vol}_{W'}(\mathrm{res}_{W'}(S)) = \log \mathrm{vol}_W(\mathrm{res}_W(S)) + (-\nu(\det(A))).$$

Furthermore $-\nu(\det(A)) = \dim_F(W/W')$.

Proof. The matrix of the inclusion has diagonal form for a suitable choice of bases $w_1, \ldots, w_m$ of W and $w'_1, \ldots, w'_m$ of W' (invariant factor theorem). Let $d_1, \ldots, d_m$ be its diagonal entries. We obtain by definition of the determinant:

$$w'_1 \wedge \ldots \wedge w'_m = \det(i)(w_1 \wedge \ldots \wedge w_m)$$

and hence

$$\log \mathrm{vol}_{W'}(S) = -\nu(\det(i)) + \log \mathrm{vol}_W(S).$$

The equality $-\nu(\det(A)) = \dim_F(W/W')$ follows directly from the invariant factor theorem. □

Lemma 5.17 (Volume of a quotient)**.** *Let* (V, S) *be a volume space and let* $(W, S \cap (Q \otimes_Z W))$ *be a sub-volume space such that* V/W *is projective. Then*

$$\log \mathrm{vol}_V(S) = \log \mathrm{vol}_W(\mathrm{res}_W(W)) + \log \mathrm{vol}_{V/W}(\mathrm{quot}_W(S)).$$

Proof. Since $Q \otimes_Z W$ is a Q-module the quotient of S by $S \cap Q \otimes_Z W$ is torsionfree. So $S \cap Q \otimes_Z W$ is a direct summand of S. Hence an R-basis $b_1, \ldots, b_s$ of $S \cap (Q \otimes_Z W)$ can be extended to an R-basis $b_1, \ldots, b_n$ of S. We can also extend a Z-basis $w_1, \ldots, w_m$ of W to a Z-basis $w_1, \ldots, w_n$ for V.

Let $\mathrm{pr} : Q \otimes_Z V \to Q \otimes_Z (V/W)$ denote the projection. The tuple $(\mathrm{pr}(w_{s+1}), \ldots, \mathrm{pr}(w_n))$ is an Z-basis of V/W and the cosets of $\mathrm{pr}(b_{s+1}), \ldots, \mathrm{pr}(b_n)$ form an R-basis of $\mathrm{pr}(S) = S/S \cap Q \otimes_Z W$. Now write $w_i = \sum_j \lambda_{i,j} b_i$. Then the base change matrix has block form; all coefficients $\lambda_{i,j}$ with $i \leq s$ and $j > s$ vanish. Especially we get for $i > s$

$$\mathrm{pr}(w_i) = \sum_{j=s+1}^{n} \lambda_{i,j} pr(b_j).$$

We can use the formula for the volume from Remark 5.15. Since we are in the case of a submodule of maximal rank the formula simplifies to

$$\log \mathrm{vol}_V(S) = -\nu \det((\lambda_{i,j})_{1 \leq i,j \leq n}),$$

$$\log \mathrm{vol}_W(S \cap (Q \otimes_Z W)) = -\nu \det((\lambda_{i,j})_{1 \leq i,j \leq s}),$$

$$\log \mathrm{vol}_{V/W}(S/S \cap (Q \otimes_Z W)) = -\nu \det((\lambda_{i,j})_{s+1 \leq i,j \leq n}).$$

The formula for the determinant of block matrices implies the lemma. □

Lemma 5.18 (Parallelogram constraint/subadditivity)**.** *Let* (V, S) *be a volume space and let* W_1, W_2 *be finitely generated* Z*-submodules of* V*. Then*

$$\log \mathrm{vol}_{W_1 \cap W_2}(S) + \log \mathrm{vol}_{W_1 + W_2}(S) \leq \log \mathrm{vol}_{W_1}(S) + \log \mathrm{vol}_{W_2}(S).$$

Proof. Let us first consider the case of $W_1 \cap W_2 = 0$. Hence $W_1 + W_2$ is isomorphic to $W_1 \oplus W_2$. Choose Z-bases $w_1, \ldots, w_r$ of W_1 and $w_{r+1}, \ldots, w_{r+s}$ of W_2. We can extend an R-basis $b_1, \ldots, b_r$ of $S \cap Q \otimes_Z W_1$ to an R-basis $b_1, \ldots, b_{r+s}$ of $S \cap (Q \otimes_Z (W_1 + W_2))$ (compare Remark (ii)). Furthermore we can write $w_i =: \sum_{j=1}^n \lambda_{i,j} b_j$ with $\lambda_{i,j} \in Q$. The formula from Remark 5.15 gives

$$\log \mathrm{vol}_{W_1+W_2}(S) = -\nu\Big(\sum_{\sigma\in\Sigma_{r+s}} \mathrm{sign}(\sigma)\lambda_{\sigma(1),1} \cdot \ldots \cdot \lambda_{\sigma(r+s),r+s}\Big).$$

We get $\lambda_{i,j} = 0$ for $i \leq r$ and $j > r$. Hence we can restrict to those permutations that leave $\{1, \ldots, r\}$ invariant. So we can write the sum above as a product:

$$\begin{aligned}
& \log \mathrm{vol}_{W_1+W_2}(S) \\
= \;& -\nu\left(\left(\sum_{\sigma\in\Sigma_r} \mathrm{sign}(\sigma)\lambda_{\sigma(1),1} \cdot \ldots \cdot \lambda_{\sigma(r),r}\right) \cdot \left(\sum_{\sigma\in\Sigma_s} \mathrm{sign}(\sigma)\lambda_{r+\sigma(1),r+1} \cdot \ldots \cdot \lambda_{r+\sigma(s),r+s}\right)\right) \\
= \;& -\nu\left(\sum_{\sigma\in\Sigma_r} \mathrm{sign}(\sigma)\lambda_{\sigma(1),1} \cdot \ldots \cdot \lambda_{\sigma(r),r}\right) - \nu\left(\sum_{\sigma\in\Sigma_s} \mathrm{sign}(\sigma)\lambda_{r+\sigma(1),r+1} \cdot \ldots \cdot \lambda_{r+\sigma(s),r+s}\right) \\
= \;& \log \mathrm{vol}_{W_1}(S) - \nu\left(\sum_{\sigma\in\Sigma_s} \mathrm{sign}(\sigma)\lambda_{r+\sigma(1),r+1} \cdot \ldots \cdot \lambda_{r+\sigma(s),r+s}\right).
\end{aligned}$$

Let us compare the second summand with

$$\log \mathrm{vol}_{W_2}(S) := \sup_{1\leq i_1<\ldots<i_s\leq r+s} -\nu\Big(\sum_{\sigma\in\Sigma_{r+s}} \mathrm{sign}(\sigma)\lambda_{r+\sigma(1),i_1} \cdot \ldots \cdot \lambda_{r+\sigma(m),i_m}\Big).$$

By picking $i_j := r+j$ we see that the second summand occurs in the set whose supremum is considered. Using $W_1 \cap W_2 = 0$ and $\log \mathrm{vol}_0 = 0$ we get:

$$\log \mathrm{vol}_{W_1+W_2}(S) + \log \mathrm{vol}_{W_1\cap W_2}(S) \leq \log \mathrm{vol}_{W_1}(S) + \log \mathrm{vol}_{W_2}(S).$$

Let us now consider the slightly more general case where W_1 and W_2 are direct summands of $W_1 + W_2$. The structure theorem for finitely generated R-modules gives that $W_1 \cap W_2$ is also a direct summand of $W_1 + W_2$ since their quotient is torsionfree.

Now we can use Lemma 5.17:

$$\begin{aligned}
\log \mathrm{vol}_{W_1}(\mathrm{res}_{W_1}(S)) &= \log \mathrm{vol}_{W_1/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1}(S)) \\
&\quad + \log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S)) \\
\log \mathrm{vol}_{W_2}(\mathrm{res}_{W_2}(S)) &= \log \mathrm{vol}_{W_2/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_2}(S)) \\
&\quad + \log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S)) \\
\log \mathrm{vol}_{W_1+W_2}(\mathrm{res}_{W_1\cap W_2}(S)) &= \log \mathrm{vol}_{(W_1+W_2)/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1+W_2}(S)) \\
&\quad + \log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S).
\end{aligned}$$

We want to show that

$$\begin{aligned} & \log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S)) + \log \mathrm{vol}_{W_1+W_2}(\mathrm{res}_{W_1+W_2}(S)) \\ \leq\ & \log \mathrm{vol}_{W_1}(\mathrm{res}_{W_1}(S)) + \log \mathrm{vol}_{W_2}(\mathrm{res}_{W_2}(S)). \end{aligned}$$

We can insert the upper equations and subtract $2\log \mathrm{vol}_{W_1\cap W_2}(S)$ on both sides. So we have to show that

$$\begin{aligned} & \log \mathrm{vol}_{(W_1+W_2)/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1+W_2}(S)) \\ & \leq \log \mathrm{vol}_{W_1/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1}(S)) \\ & + \log \mathrm{vol}_{W_2/(W_1\cap W_2)}(\mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_2}(S)). \end{aligned}$$

Lemma 5.8 tells us that

$$\begin{aligned} \mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1}(S) &= \mathrm{res}_{W_1/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S) \\ \mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_2}(S) &= \mathrm{res}_{W_2/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S) \\ \mathrm{quot}_{W_1\cap W_2} \mathrm{res}_{W_1+W_2}(S) &= \mathrm{res}_{W_1+W_2/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S). \end{aligned}$$

So we have to show that

$$\begin{aligned} & \log \mathrm{vol}_{(W_1+W_2)/(W_1\cap W_2)}(\mathrm{res}_{(W_1+W_2)/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S)) \\ & \leq \log \mathrm{vol}_{W_1/(W_1\cap W_2)}(\mathrm{res}_{W_1/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S)) \\ & + \log \mathrm{vol}_{W_2/(W_1\cap W_2)}(\mathrm{res}_{W_2/(W_1\cap W_2)} \mathrm{quot}_{W_1\cap W_2}(S)). \end{aligned}$$

So we are in the situation considered before for the submodules $W_1/(W_1 \cap W_2)$ and $W_2/(W_1 \cap W_2)$ of the volume space $(V/(W_1 \cap W_2), \mathrm{quot}_{W_1\cap W_2}(S))$. The intersection of the two submodules is zero. So we can use the previous case to get the desired result.

Now consider finally the general case. We want to reduce it to the previous case. So let W_1', W_2', W_3' be the direct summands of $W_1 + W_2$ generated by W_1 resp. W_2 resp. $W_3 := W_1 \cap W_2$. They are defined to be the preimage of the torsion subgroup under the maps $W_1 + W_2 \to W_1 + W_2/W_i$.

The isomorphism $((W_1+W_2)/W_1)\oplus((W_1+W_2)/W_2) \cong (W_1+W_2)/(W_1\cap W_2)$ restricts to the torsion subgroups. Recall that $Z = F[t]$. So we get:

$$\begin{aligned} & \dim_F(\mathrm{tors}((W_1 + W_2)/W_1)) + \dim_F(\mathrm{tors}((W_1 + W_2)/W_2)) \\ & \qquad = \dim_F(\mathrm{tors}((W_1 + W_2)/(W_1 \cap W_2))). \end{aligned}$$

Note that by definition $\mathrm{tors}((W_1 + W_2)/W_i) \cong W_i'/W_i$. Furthermore an element in a direct sum is a torsion element if each entry is. Hence $W_3' = W_1' \cap W_2'$. Now the previous case tells us that

$$\begin{aligned} & \log \mathrm{vol}_{W_1'\cap W_2'}(\mathrm{res}_{W_1'\cap W_2'}(S)) + \log \mathrm{vol}_{W_1'+W_2'}(\mathrm{res}_{W_1'+W_2'}(S)) \\ & \qquad \leq \log \mathrm{vol}_{W_1'}(\mathrm{res}_{W_1'}(S)) + \log \mathrm{vol}_{W_2'}(\mathrm{res}_{W_2'} S). \end{aligned}$$

Lemma 5.16 yields

$$\begin{aligned}
\log \mathrm{vol}_{W_1}(\mathrm{res}_{W_1}(S)) &= \log \mathrm{vol}_{W_1'}(\mathrm{res}_{W_1'}(S) + \dim_F(W_1'/W_1) \\
\log \mathrm{vol}_{W_2}(\mathrm{res}_{W_2}(S)) &= \log \mathrm{vol}_{W_2'}(\mathrm{res}_{W_2'}(S) + \dim_F(W_2'/W_2) \\
\log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S)) &= \log \mathrm{vol}_{W_1'\cap W_2'}(\mathrm{res}_{W_1'\cap W_2'}(S)) \\
&\quad + \dim_F((W_1' \cap W_2')/(W_1 \cap W_2).
\end{aligned}$$

Inserting this we obtain with help of the formula for $\dim_F$ from above:

$$\begin{aligned}
&\log \mathrm{vol}_{W_1\cap W_2}(\mathrm{res}_{W_1\cap W_2}(S)) + \log \mathrm{vol}_{W_1+W_2}(\mathrm{res}_{W_1+W_2}(S)) \\
&\qquad\qquad \leq \log \mathrm{vol}_{W_1}(\mathrm{res}_{W_1}(S)) + \log \mathrm{vol}_{W_2}(\mathrm{res}_{W_2}(S)).
\end{aligned}$$

Here we used that by construction $W_1' + W_2' = W_1 + W_2$. This proves the proposition. □

Lemma 5.19 (special bases)**.** *Let (V, S) be a volume space and let $w_1, \ldots, w_n$ be a Z-basis of V. Then there is an R-basis of S such that the base change matrix $(\lambda_{i,j}) \in M_n(Q)$ has upper triangular form ($\lambda_{i,j} = 0$ for $j > i$). The base change matrix is defined by $w_i = \sum_j \lambda_{i,j} b_j$.*

Proof. Consider the filtration

$$S \cap Q \otimes_Z \langle w_1\rangle_Z \subset S \cap Q \otimes_Z \langle w_1, w_2\rangle_Z \subset \ldots \subset S \cap Q \otimes_Z \langle w_1, \ldots, w_n\rangle_Z = S.$$

Each module in this filtration is a direct summand of the next one — they are all finitely generated R-modules and the quotient is torsionfree. The i-th module of this filtration has rank i. Hence we can choose an R-basis $b_1 \ldots, b_n$ that respects this filtration. The R-module $Q \otimes_Z \langle w_1 \ldots, w_i\rangle_Z/(S \cap Q \otimes_Z \langle w_1 \ldots, w_i\rangle_Z)$ has rank zero as an R-module and hence it is torsion. So there is an $r \in R \setminus \{0\}$ with $r' w_i \in (S \cap Q \otimes_Z \langle w_1 \ldots, w_i\rangle_Z)$. So $r' w_i$ can be written as a linear combination of the $b_1, \ldots, b_i$ with coefficients in R. Dividing by R shows that w_i can be expressed as a linear combination of $b_1, \ldots, b_i$ with coefficients in Q. This shows that the base change matrix has triangular form. □

Corollary 5.20. *Given a volume space (V, S) and a real number C. Then there are only finitely many elements $v \in V \setminus \{0\}$ with $\log \mathrm{vol}_{\langle v\rangle_Z}(S) \leq C$.*

Proof. The statement is trivially true if $\mathrm{rk}(V) = 0$ which means that V is trivial. Let us assume by induction that the statement is true for all volume spaces of rank $n - 1$ and for all $C \in \mathbb{R}$. Let $\mathrm{rk}(V) = n$. Let b_i be chosen as above and let $v = \sum_i \mu_i v_i$ with $\mu_i \in Z$ be any such vector. We get

$$v = \sum_j \left(\sum_i \mu_i \lambda_{i,j} \right) b_j.$$

Since $\lambda_{i,j} = 0$ for $j > i$ we see that the coefficient of b_n is $\mu_n \cdot \lambda_{n,n}$. By Proposition 5.12 we know that

$$C \geq \log \mathrm{vol}_{\langle v\rangle_Z}(S) \geq -\nu(\mu_n \cdot \lambda_{n,n})$$

and hence $\deg(\mu_n) = -\nu(\mu_n) \leq C + \nu(\lambda_{n,n})$. There are only finitely many elements in $Z = F[t]$ whose degree is less or equal to the number $C + \nu(\lambda_{n,n})$.

Let μ_n be one of those and consider $v - \mu_n v_n \in \langle v_1, \ldots, v_{n-1}\rangle_Z$. By Proposition 5.12 we get:

$$\begin{aligned} \log \mathrm{vol}_{\langle v-\mu_n v_n\rangle_Z}(S) &\leq \max(\log \mathrm{vol}_{\langle v\rangle_Z}(S), \log \mathrm{vol}_{\langle \mu_n v_n\rangle_Z}(S)) \\ &\leq \max(C, C + \nu(\lambda_{n,n}) + \log \mathrm{vol}_{\langle v_n\rangle_Z}(S)) \\ &=: C'. \end{aligned}$$

By induction hypothetis there are only finitely many vectors in the rank $(n-1)$ volume space $(\langle v_1, \ldots, v_{n-1}\rangle_Z, S \cap Q \otimes_Z \langle v_1, \ldots, v_{n-1}\rangle_Z)$ of length $\leq C'$. So v is contained in the finite set

$$\{v' + \mu v_n \mid v' \in \langle v_1, \ldots, v_{n-1}\rangle_Z, \log \mathrm{vol}_{\langle v'\rangle}(S) \leq C', \mu \in Z, \deg(\mu) \leq C + \nu(\lambda_{n,n})\}.$$

□

Corollary 5.21. *Given a volume space (V, S) and a real number C. Then there are only finitely many submodules $W \subset V$ with* $\log \mathrm{vol}_W(\mathrm{res}_W(S)) \leq C$.

Proof. This follows directly from the previous result and the following claim:

For every $m \leq n := \mathrm{rk}(V)$ and every $v \in \Lambda^m V$ there are only finitely many submodules $W \subset V$ such that $v = w_1 \wedge \ldots \wedge w_m$ for a Z-basis $w_1, \ldots, w_m$ of W. If there is no such W the claim is true. If there is at least one, then $K := \ker(- \wedge v : V \to \lambda^{m+1} V)$ has at least rank m as $w_1, \ldots, w_m$ are linearly independent and lie in K. Complete $w_1, \ldots, w_m$ to a linear independent subset $w_1, \ldots, w_n$ of V. The images of $w_{m+1}, \ldots, w_n$ are linearly independent and hence the kernel can have rank at most $n - (n - m) = m$.

Hence every such W is contained in K as a submodule of the same rank. As Z is a PID the module K is finitely generated free. Furthermore the index W is also prescribed by Lemma 5.16 and hence the cardinality of $|K/W|$. But there are only finitely many isomorphism types of Z-modules of a given cardinality (use the structure theorem and the fact that $|Z/(x)| = |F|^{\deg(x)}$). And as K is finitely generated there are only finitely many maps from K to any finite Z-module.

But W occurs as the kernel of the map $K \to K/W$. Hence there are only finitely many such W. □

Now if we are also allowed to change the basis of V we get an even simpler normal form:

Proposition 5.22 (diagonal bases)**.** *Let (V, S) be a volume space. Then there is an R-basis $b_1, \ldots, b_n$ of S and an Z-basis $w_1, \ldots, w_n$ of V such that $w_i = t^{r_i} b_i$ for some $r_i \in \mathbb{Z}$ with $r_1 \leq r_2 \leq \ldots \leq r_n$.*

Proof. The proof is done by induction on $\mathrm{rk}(V)$ and there is nothing to show in the case of $\mathrm{rk}(V) = 0$.

Let $v \in V$ be a shortest nontrivial vector. This means that $\log \mathrm{vol}_{\langle v\rangle_Z}(S)$ is minimal. Then v cannot be written as $\lambda v'$ with $\lambda \in Z \setminus Z^*$. This means exactly that $V/\langle v\rangle$ is torsionfree and hence $\langle v\rangle$ is a direct summand of V. Let b_1 be a basis vector of the R-module $(Q \otimes_Z \langle v\rangle_Z) \cap S$. Hence w_1 is of the form λb_1 for some $0 \neq \lambda \in Q$ since $w_1 \in Q \otimes_Z V$ and b_1 generate the same Q-vector space. Without loss of generality we can assume that λ is of the form t^{r_1} — otherwise replace b_1 by $\lambda \cdot t^{\nu(\lambda)} b_1$. We get the following two split exact sequences:

$$0 \to \langle v\rangle_Z \to V \to V/\langle v\rangle_Z \to 0,$$

$$0 \to S \cap (Q \otimes_Z \langle v\rangle_Z) \to S \to S/S \cap (Q \otimes_Z \langle v\rangle_Z) \to 0.$$

By induction we already get such bases for the quotient volume space. Let $b_2, \dots, b_n$ be preimages of the basis of $S/(Q \otimes_Z \langle v\rangle) \cap S$ and let $w_2, \dots, w_n \in W$ be preimages of the basis of $W/\langle v\rangle_Z$ under the projection map. We get the following linear combinations

$$w_1 = t^{r_1} b_1,$$

$$w_i = s_i b_1 + t^{r_i} b_i.$$

for some $s_i \in Q$. Let us consider a fixed $i \in \{2, \dots, n\}$. Now we can write s_i as a mixed fraction $s_i = \sum_{j=r_1}^{m} a_j t^j + t^{r_1 - 1} \cdot r$ with $r \in R$. If we now improve the choice of the preimages by replacing w_i by $w_i - \sum_{j=r_1}^{m} a_j t^{j-r_1} w_1$ we may assume $s_i = t^{r_1-1} \cdot r$. Now we have to use the fact that v was chosen to be a shortest vector and the formula 5.12 for the computation of the volume to get

$$r_1 = \log \mathrm{vol}_{\langle w_1\rangle_Z}(S) \leq \log \mathrm{vol}_{\langle w_i\rangle_Z}(S) = \max(-\nu(s_i), r_i).$$

We have already achieved that $-\nu(s_i) = (r_1 - 1) - \nu(s_i) \leq (r_1 - 1)$. Hence $\frac{s_i}{t^{r_i}} \in r$ and $r_1 \leq r_i$. If we finally replace b_i by $b_i + \frac{s_i}{t^{r_i}} b_1$, we can assume that $s_i = 0$. Hence we have found a basis of the desired form. □

An isomorphism of volume spaces $(V, S) \to (V', S')$ is a Z-linear isomorphism $\varphi : V \to V'$ with $(1_Q \otimes \varphi)(S) = S'$. Let us now classify all isomorphism types of volume spaces with a fixed prescribed underlying module V. This is just understanding the cosets of the action of $\mathrm{aut}_Z(V)$ on the set of all R-lattices in $Q \otimes_Z V$.

Proposition 5.23. *The numbers $r_1, \dots, r_n$ from the last proposition uniquely describe the isomorphism type of a volume space. Furthermore the canonical filtration of (V, S) consists exactly of the modules $\{\langle\{w_i \mid r_i \leq C\}\rangle_Z \mid C \in \mathbb{Z}\}$. The integral volume of such a module $\langle\{w_i \mid r_i \leq C\}\rangle_Z$ is just $\sum_{i \in \{j \mid r_j \leq C\}} r_i$.*

Furthermore $c_{\langle w_1, \dots, w_m\rangle}(S) = r_{m+1} - r_m$.

Unlike in the integral case there is in every dimension a module on the canonical path; this can be seen as an implication of the ultrametric inequality.

Proof. An isomorphism of two volume spaces maps such bases again onto such bases. So we only have to show that two volume spaces with such bases and different r_i's are not isomorphic. First let us show that the modules of the form $\langle w_1, \dots, w_m\rangle_Z$ have minimal (logarithmic) volume among all submodules of rank m.

Given any submodule $W' \subset V$ of rank m. Let $w'_1, \ldots, w'_m$ be a Z-basis of W'. We get

$$w'_1 \wedge \ldots w'_m = \sum_{1 \le i_1 < \ldots < i_m \le n} \lambda_{i_1,\ldots,i_m} w_{i_1} \wedge \ldots w_{i_m}$$

with $\lambda_{i_1,\ldots,i_m} \in Z$. Inserting $w_j = t^{r_j} b_j$ gives

$$w'_1 \wedge \ldots w'_m = \sum_{1 \le i_1 < \ldots < i_m \le n} \lambda_{i_1,\ldots,i_m} t^{\sum_{j=1}^m r_{i_j}} b_{i_1} \wedge \ldots b_{i_m}.$$

Now we can use the formula from Proposition 5.12 to compute the volume:

$$\begin{aligned}
\log \mathrm{vol}_{W'}(S) &= \max_{1 \le i_1 < \ldots < i_m \le n} -\nu(\lambda_{i_1,\ldots,i_m} \cdot t^{\sum_{j=1}^m r_{i_j}}) \\
&\ge \max_{1 \le i_1 < \ldots < i_m \le n} -\nu(t^{\sum_{j=1}^m r_{i_j}}) \\
&= \max_{1 \le i_1 < \ldots < i_m \le n} \sum_{j=1}^m r_{i_j} \\
&= \sum_{j=1}^m r_j \\
&= \log \mathrm{vol}_{\langle w_1,\ldots,w_m \rangle}(S).
\end{aligned}$$

So the logarithmic volume of a rank m module is at least $\sum_{i=1}^m r_i$. This value is obtained by $\langle w_1, \ldots, w_m \rangle$. Hence the number r_m can be obtained from the canonical plot as the difference of the minimal logarithmic volume of a rank m submodule minus the minimal logarithmic volume of a rank $(m-1)$ submodule. Hence each number r_m just depends on the isomorphism type of (V, S). The slope of the line segment from the module $\langle w_1, \ldots, w_{m-1} \rangle$ to $\langle w_1, \ldots, w_m \rangle$ is exactly r_m. The slope does not decrease and it increases at rank m if and only if $r_{m+1} > r_m$. Thus the modules $\{\langle \{w_i \mid r_i \le C\} \rangle_Z \mid C \in \mathbb{Z}\}$ are really the canonical filtration of (V, S). The value of $c_{\langle w_1,\ldots w_m \rangle}$ is $r_{m+1} - r_m$ because r_{m+1} is the slope of the canonical path from m to $m+1$ and r_m is the slope from $m-1$ to m. □

Lemma 5.24 (Monotonicity in the second coordinate)**.** *Let $(V, S), (V, S')$ be two volume spaces. If $S \subset S'$, then*

$$\log \mathrm{vol}_W(\mathrm{res}_W S) \le \log \mathrm{vol}_W(\mathrm{res}_W(S')).$$

Proof. Pick R-bases $b_1, \ldots, b_n$ of S and $b'_1, \ldots, b'_n$ of S'. Then

$$b_1 \wedge \ldots \wedge b_n = \lambda \cdot b'_1 \wedge \ldots \wedge b'_n \text{ for some } \lambda \in R.$$

Inserting this in the formula to compute the volume from Proposition 5.12 we get

$$\log \mathrm{vol}_V(S') = -\nu(\lambda) + \log \mathrm{vol}_V(S).$$

As $\lambda \in R$ we get $\nu(\lambda) \ge 0$. Hence the claim follows. □

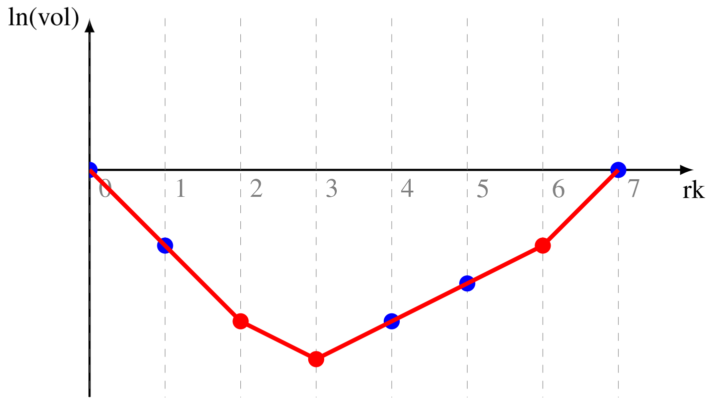

Figure 5.1: Sketch of the canonical plot of a volume space with $r_* = (-2,-2,-1,1,1,1,2)$. The slope of the canonical path from $i-1$ to i is r_i.

Corollary 5.25. *Let (V,S), (V,S') be volume spaces and let $W \subset V$ be any submodule. Let us assume $S \subset S' \subset tS$ (t is the variable in $F[t] = Z$). Then*

$$\log \mathrm{vol}_W(\mathrm{res}_W(S)) \le \log \mathrm{vol}_W(\mathrm{res}_W(S')) \le \mathrm{rk}_Z(W) + \log \mathrm{vol}_W(\mathrm{res}_W(S)).$$

Proof. Clearly $S \cap Q \otimes_Z W \subset S' \cap Q \otimes_Z W \subset tS \cap Q \otimes_Z W$. Hence we can apply the previous lemma to get

$$\log \mathrm{vol}_W(S) \le \log \mathrm{vol}_W(S') \le \log \mathrm{vol}_W(tS) = \mathrm{rk}_Z(W) + \log \mathrm{vol}_W(S).$$

□

So we have shown that the logarithmic volume function satisfies all conditions from Convention 3.1:

Proposition 5.26. *Let (V,S) be a volume space. Consider the lattice of direct summands of V. The logarithmic volume function $W \mapsto \log \mathrm{vol}_W(\mathrm{res}_W(S))$ and the rank $W \mapsto \mathrm{rk}_{F[t]}(W)$ have the following properties.*

(i) rk *is strictly monotone, i.e.* $\mathrm{rk}(W) < \mathrm{rk}(W')$ *for all* $W, W' \in \mathfrak{L}$ *with* $W < W'$.

(ii) rk *is additive, i.e.* $\mathrm{rk}(W \cap W') + \mathrm{rk}(\mathrm{lub}(W,W')) = \mathrm{rk}(W) + \mathrm{rk}(W')$ *for all* $W, W' \in \mathfrak{L}$.

(iii) The function $\log \mathrm{vol}(-) : \mathfrak{L} \to \mathbb{R}$ *is subadditive. This means that for all* $W, W' \in \mathfrak{L}$

$$\log \mathrm{vol}_{W\cap W'}(\mathrm{res}_{W\cap W'}(S)) + \log \mathrm{vol}_{\mathrm{lub}(W,W')}(\mathrm{res}_{\mathrm{lub}(W,W')}(S)) \le \log \mathrm{vol}_W(\mathrm{res}_W(S)) + \log \mathrm{vol}_{W'}(\mathrm{res}_{W'}(S)).$$

(iv) For each $C \in \mathbb{R}$ *there are only finitely many* $L \in \mathfrak{L}$ *with* $\log \mathrm{vol}_W(\mathrm{res}_W(S)) \le C$.

(v) $\mathrm{rk}(\mathbb{0}) = 0, \log \mathrm{vol}(\mathbb{0}) = 0$.

Proof. (i) Let $W \subset W'$. The map $Q \otimes_{F[t]} W \hookrightarrow Q \otimes_{F[t]} W'$ is injective because Q is flat as an $F[t]$ module. Hence

$$\mathrm{rk}_{F[t]}(W) := \dim_Q(Q \otimes_{F[t]} W) \leq \dim_Q(Q \otimes_{F[t]} W') =: \mathrm{rk}_{F[t]}(W').$$

(ii) Let two direct summands W, W' be given. The least upper bound is the preimage of the torsion submodule under the projection map $V \to V/(W+W')$. So we have a short exact sequence

$$1 \to W + W' \to \mathrm{lub}(W, W') \to \mathrm{tors}(V/(W+W')) \to 1.$$

Since a torsion module has rank 0 we get by the additivity of the rank

$$\mathrm{rk}_{F[t]}(\mathrm{lub}(W, W')) = \mathrm{rk}_{F[t]}(W + W').$$

We use again the flatness of Q to see that the sequence

$$0 \to Q \otimes (W \cap W') \to Q \otimes (W \oplus W') \to Q \otimes (W + W') \to 0$$

is exact. Hence

$$\begin{aligned}
& \mathrm{rk}_{F[t]}(W \cap W') + \mathrm{rk}_{F[t]}(W + W') \\
:= \; & \dim_Q(Q \otimes_{F[t]} (W \cap W')) + \dim_Q(Q \otimes_{F[t]} (W + W')) \\
= \; & \dim_Q(Q \otimes_{F[t]} (W)) + \dim_Q(Q \otimes_{F[t]} (W')) \\
=: \; & \mathrm{rk}_{F[t]}(W) + \mathrm{rk}_{F[t]}(W').
\end{aligned}$$

(iii) Proposition 5.18 shows the statement with $\mathrm{lub}(W, W')$ replaced by $W + W'$. Since $W + W' \subset \mathrm{lub}(W, W')$ is a submodule of finite index ≥ 1 we can use the formula from Lemma 5.16 to get the result.

(iv) This has been done in Corollary 5.21

(v) This is clear from the definitions (and Remark 5.14).

□

Remark 5.27. It follows directly from Definition 5.10 that

$$\log \mathrm{vol}_W(qS) = \mathrm{rk}(W) \cdot \nu(q) + \log \mathrm{vol}_W(S)$$

for $q \in Q$. The function c_W from Definition 3.4 is defined as an infimum over functions of the form

$$S \mapsto \frac{\log \mathrm{vol}_{W_2}(S) - \log \mathrm{vol}_W(S)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}_W(S) - \log \mathrm{vol}_{W_0}(S)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)}.$$

Using the upper formula we see that replacing S by qS does not affect c_W. Hence $c_W(S) = c_W(qS)$.

6 Volume: The localized case

Now let start to study groups of the form $GL_n(\mathbb{Z}[\frac{1}{2}])$. Again we want to assign to a scalar product its volume in a $GL_n(\mathbb{Z}[\frac{1}{2}])$ invariant way. But the volume of the parallelepiped spanned by a $\mathbb{Z}[\frac{1}{2}]$-basis of $\mathbb{Z}[\frac{1}{2}]^n$ depends on the choice of the basis. The solution is to add additional structure that tells us which bases are allowed. The set of possible choices for this additional information also carries a $GL_n(\mathbb{Z}[\frac{1}{2}])$-action, so it makes sense to pick the volume in a $GL_n(\mathbb{Z}[\frac{1}{2}])$-invariant way.

Convention 6.1. *Let*

- *Z denote either the integers $\mathbb{Z}$ or the polynomial ring $F[t]$ for a finite field F,*
- *an element $z \in Z$ be called* normalized *if it is positive in the case of $Z = \mathbb{Z}$ resp. if its leading coefficient is one in the case of $F[t]$.*
- *$\mathfrak{P}$ denote the set of all normalized primes in Z,*
- *$T \subset \mathfrak{P}$ denote a finite subset,*
- *Q denote the quotient field of Z,*
- *$Z[T^{-1}]$ be the ring $\{\frac{a}{b} \in Q \mid v_p(\frac{a}{b}) \geq 0$ for all $p \in \mathfrak{P} \setminus T\}$,*
- *Z_T be the ring $Z[(\mathfrak{P} \setminus T)^{-1}]$,*
- *n be a fixed integer,*
- *(z, T) denote the product of all normalized prime factors of $z \in Z$ that lie in T,*
- $\operatorname{ord}(m)$ *denote a generator of the ideal $Ker(Z \to M \quad r \mapsto rm)$ for an element m of a Z-module.*

Remark 6.2. (i) Every nonzero ring element $z \in Z$ is associated to a unique normalized element.

(ii) The rings $Z, Z_T, Z[T^{-1}]$ are all Euclidean rings and hence principal ideal domains. For $\mathbb{Z}$ a degree function is given by the absolute value and for $F[t]$ it is given by the degree of a polynomial. A degree function on $Z[S^{-1}]$ is for example given by

$$\frac{a}{b} \mapsto \deg((a, \mathfrak{P} \setminus S)) - \deg((b, \mathfrak{P} \setminus S))$$

where deg denotes a degree function on Z.

Definition 6.3. A *integral structure* with respect to T on a finitely generated free $Z[T^{-1}]$-module V of rank n is a finitely generated Z_T-submodule of $Q \otimes_{Z[T^{-1}]} V$ of rank n.

Remark 6.4. Let $V := Z[T^{-1}]^n$.

(i) Let B be an integral structure on V. Thus B is a submodule of the Q-vector space Q^n and so it is torsionfree. Hence the structure theorem for finitely generated modules over a principal ideal domain tells us that $B \cong Z_T^n$ as an Z_T-module.

(ii) $\mathrm{aut}_Q(Q \otimes V) \cong \mathrm{GL}_n(Q)$ acts transitively on the set of all integral structures on V; for any two integral structures B, B' we can pick Z_T-bases and a matrix $A \in \mathrm{GL}_n(Q)$ that maps one basis to the other. The stabilizer of the standard integral structure $Z[(\mathfrak{P} \setminus T)^{-1}]^n \subset Q^n$ is $\mathrm{GL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$. Hence every other stabilizer is conjugate to $\mathrm{GL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$ in $\mathrm{GL}_n(Q)$.

(iii) For any integral structure B we get that $Q^n/B \cong Q^n/Z[(\mathfrak{P} \setminus T)^{-1}]^n$ is T-torsion. It has rank zero as an abelian group by the additivity of the rank and there is no element $x \in Q^n$ with $x \notin B$ and $mx \in B$ for some prime factor $m \in \mathfrak{P} \setminus T$ as B is a Z_T-submodule of Q^n and m is a unit in $Z[T^{-1}]$.

We will later see that intersection with $Z[T^{-1}]^n \subset Q^n$ gives a map from the set of all integral structures to the set of all finitely generated free Z-submodules of rank n of $Z[T^{-1}]^n$ and that we can use the previous definition of volume. This map is $\{A \in \mathrm{GL}_n(Q) \mid A \cdot Z[T^{-1}]^n = Z[T^{-1}]^n\} = \mathrm{GL}_n(Z[T^{-1}])$-equivariant. So we need to investigate the assignment $W \mapsto W \cap B$ further.

6.1 Some posets

We have to figure out what happens if one changes the set of primes in consideration. This is done in this section.

Fix an integer $n \in \mathbb{N}$ for this section. For a Z-module M and a set of primes T let

$$\begin{aligned} T - \mathrm{tors}(M) &:= \ker(M \to Z[T^{-1}] \otimes_Z M \quad m \mapsto 1 \otimes m) \\ &= \{m \in M \mid \text{All prime factors of } \mathrm{ord}(m) \text{ lie in } T\}. \end{aligned}$$

Let us fix sets of primes T_1 and $T_2 \subset T_2'$ and a finitely generated $Z[T_1^{-1}]$-submodule M of Q^n.

Definition 6.5. Let $\mathfrak{L}^M_{Z[T_1^{-1}]}[T_2^{-1}]$ denote the poset of all $Z[T_1^{-1}]$ submodules V of M such that $T_2 - \mathrm{tors}(M/V) = 0$.

Remark 6.6. (i) $V \in \mathfrak{L}^M_{Z[T_1^{-1}]}[T_2^{-1}]$ is automatically finitely generated free. This follows from the structure theorem applied to the $Z[T_1^{-1}]$-module M/V.

(ii) By the structure theorem for finitely generated modules over a PID we know that any submodule of a finitely generated free module M is a direct summand if and only if the quotient is torsionfree. Hence in this case $\mathfrak{L}^{M}_{Z[T_1^{-1}]}[\mathfrak{P}^{-1}]$ is the subposet of direct summands of M.

(iii) Note that for a $Z[T_1^{-1}]$-module M' its $Z[T_1^{-1}]$-torsion submodule

$$\mathrm{tors}_{Z[T^{-1}]}(M') := \{x \in M' \mid cx = 0 \text{ for a } c \in Z[T^{-1}]\}$$

is the same as its Z-torsion part $\mathrm{tors}_Z(\mathrm{res}_Z(M'))$.

(iv) There is a retract $r : \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}] \to \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2'^{-1}]$ of posets left inverse to the inclusion $i : \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2'^{-1}] \to \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}]$. It is given by $V \mapsto \pi^{-1}((T_2' \setminus T_2) - \mathrm{tors}(M/V))$, where π is the canonical projection $M \to M/V$.

(v) The retract is rank preserving. Consider the short exact sequences

$$0 \to V \to V \to 0 \to 0,$$
$$0 \to V \to \pi^{-1}((T_2 \setminus T_2') - \mathrm{tors}(M/V)) \to (T_2 \setminus T_2') - \mathrm{tors}(M/V) \to 0.$$

Now the additivity of the rank and the fact that any torsion group has rank zero implies $\mathrm{rk}(V) = \mathrm{rk}(r(V))$.

(vi) We get for $T_2 \subset T_2'$ and $V \in \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}]$, $W \in \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2'^{-1}]$

$$V \subseteq i(W) \Rightarrow r(V) \subseteq r \circ i(W) = W.$$

and we always have $i \circ r(V) \supset V$.

(vii) The abelian group V has finite index in $r(V)$ for any $V \in \mathfrak{L}_Z$

$$\begin{aligned} [r(V) : V] &= [\pi^{-1}((T_2' \setminus T_2) - \mathrm{tors}(M/V) : \pi^{-1}(0)] \\ &= |(T_2' \setminus T_2) - \mathrm{tors}(M/V)|. \end{aligned}$$

M/V is a finitely generated $Z[T_1^{-1}]$ module. By the structure theorem its $(T_2' \setminus T_2)$-torsion part is isomorphic to a direct sum of finitely many copies of modules of the form $Z[T_1^{-1}]/p^k Z[T_1^{-1}]$ with $k \in \mathbb{N}$, $p \in T_2' \setminus T_2$. Note that all those summands are finite abelian groups. Hence $|(T_2' \setminus T_2) - \mathrm{tors}(M/V)| < \infty$.

Lemma 6.7. *$\mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}]$ is a lattice in the order theoretic sense. This means that any finite subset has a greatest lower bound and a least upper bound.*

Proof. Let S denote the chosen subset. The greatest lower bound is given by the intersection $\bigcap S = \bigcap_{A \in S} A$.

Note that $M/\bigcap S$ embeds into $\prod_{A \in S} M/A$ and hence it is also T_2-torsionfree.

In the case of $T_2 = \emptyset$ the least upper bound is given by the sum of all submodules in S. In general this sum need not lie in $\mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}]$. Let $i : \mathfrak{L}^{M}_{Z[T_1^{-1}]}[T_2^{-1}] \hookrightarrow \mathfrak{L}^{M}_{Z[T_1^{-1}]}[\emptyset^{-1}]$

denote the inclusion and let r denote the retract from Remark 6.6. Let us now find the least upper bound of S:

Let M' be an upper bound of all elements in S. Since i is order preserving we get $i(M) \leq i(M')$ for all $M \in S$. So since $\mathfrak{L}^{M}_{Z[T_1^{-1}]}[\emptyset^{-1}]$ is a lattice we get $\sum_{M \in S} i(M) \leq i(M')$. Since the retract is also order preserving we get

$$r(\sum_{M \in S} i(M)) \leq r(i(M')) = M.$$

Clearly $r(\sum_{M \in S} i(M))$ is an upper bound. So we have shown that $r(\sum_{M \in S} i(M))$ is really the least upper bound. □

Remark 6.8. Hence the inclusion $\mathfrak{L}_Z \hookrightarrow \mathfrak{L}_Z[S^{-1}]$ is just a morphism of posets and not a morphism of lattices as it doesn't preserve the least upper bounds.

The following lemma gives a criterion to decide whether a given $Z[T^{-1}]$-submodule of Q^n is finitely generated by looking at the denominators.

Lemma 6.9. *For $q \in Q$ let $f_T(q)$ denote the T-primary part of the denominator of q; i.e. $:= c$ where c is normalized, $q = \frac{a}{bc}$ and*

$$(a, bc) = 1 = (c, T) = (b, \mathfrak{P} \setminus T).$$

Then a $Z[T^{-1}]$-submodule M of Q^n is finitely generated if and only if the set

$$\{f_T(x_i) \mid (x_1, \ldots, x_n) \in M, i = 1, \ldots, n\} \subset Z$$

is finite. Equivalently we may ask for an element $N \in Z \setminus \{0\}$ such that all elements of this set divide N.

Proof. Let M be finitely generated and let $((x_{i,j})_{i=1,\ldots,n})_{j=1,\ldots,m}$ be a finite generating set. If $y = (y_1, \ldots, y_n)$ lies in the $Z[T^{-1}]$-span of the generating set we get

$$y_i = \sum_{j=1}^{m} \lambda_j x_{i,j}$$

So the denominator of y_i divides the product of the denominators of $x_{i,j}$. Thus the same holds for the T-primary parts. Hence the element $f_T(y_i)$ divides $\prod_{i=1,\ldots,n} \prod_{j=1,\ldots,m} f_T(x_{i,j})$ for every $(y_1, \ldots, y_n) \in M$. Hence the set is finite.

Conversely let this set be finite and let S denote the product of its elements. Hence M is a submodule of $\frac{1}{S} \cdot Z[T^{-1}]^n \subset Q^n$ and over a principal ideal domain submodules of finitely generated modules are finitely generated. □

Proposition 6.10. *Let T be a set of primes. Let B be an integral structure with respect to T. Let M be a finitely generated $Z[T^{-1}]$-submodule of Q^n. The map*

$$\mathfrak{L}^{M}_{Z[T^{-1}]} \to \mathfrak{L}^{M \cap B}_{Z}[T^{-1}] \qquad W \mapsto W \cap B$$

is an isomorphism of posets (and hence of lattices) that *(i) is rank preserving,*

(ii) is index preserving (considering the modules just as abelian groups),

(iii) restricts to an isomorphism $\mathfrak{L}^{M}_{Z[T^{-1}]}[T_2^{-1}] \to \mathfrak{L}^{M\cap B}_{Z}[(T \cup T_2)^{-1}]$ for another set of primes T_2. Especially for $T_2 = \mathfrak{P}$ this gives an isomorphism of the lattices of direct summands.

Proof. The first implicit claim is that $M \cap B$ is a finitely generated Z-module. By the last lemma there are $N, N' \in Z$ such that if $x = (\frac{a_1}{b_1 c_1}, \ldots, \frac{a_n}{b_n c_n})$ with $(a_i, b_i c_i) = 1 = (b_i, T) = (c_i, \mathfrak{P} \setminus T)$, then $b_i | N$ and $c_i | N'$. Hence $b_i c_i | NN'$ and the last lemma implies that M is a finitely generated Z-module.

The inverse is given by

$$\mathfrak{L}^{M\cap B}_{Z}[T^{-1}] \to \mathfrak{L}^{M}_{Z[T^{-1}]} \qquad V \mapsto \langle V \rangle_{Z[T^{-1}]}.$$

Let us check both compositions: Pick $W \in \mathfrak{L}^{M}_{Z[T^{-1}]}$ and pick an element $w \in W$. Because $W/(W \cap B) \subset Q^n/B$ we know, that $W/W \cap B$ is T-torsion by Remark 6.4 (iii). Hence there is an element $n \in Z \setminus \{0\}$ whose prime factors are in T such that $nw \in W \cap B$. But n is a unit in $Z[T^{-1}]$ and so $w = n^{-1} n w \in \langle W \cap B \rangle_{Z[T^{-1}]}$. Hence we get the chain of inclusions:

$$W \subset \langle W \cap B \rangle_{Z[T^{-1}]} \subset \langle W \rangle_{Z[T^{-1}]} = W.$$

So the first composition is the identity. Let us now pick a $V \in \mathfrak{L}^{M\cap B}_{Z}[T^{-1}]$ and note that

$$\langle V \rangle_{Z[T^{-1}]} = \{x \in Q^n \mid \exists\, n \in Z \setminus \{0\} : \text{ all prime factors of } n \text{ are in } T, nx \in V\},$$

$$\langle V \rangle_{Z[T^{-1}]} \cap B = \{x \in B \mid \exists\, n \in Z \setminus \{0\} : \text{ all prime factors of } n \text{ are in } T, nx \in V\}.$$

This is just V as M/V does not contain T-torsion since $V \in \mathfrak{L}_Z[T^{-1}]$.

Hence those maps are inverse bijections. They are obviously order preserving. As they are isomorphisms of posets they have to map a greatest lower bound to a greatest lower bound and a least upper bound to a least upper bound. Hence they are also isomorphisms of lattices.

(i) Note first that $\mathrm{rk}_{Z[T^{-1}]} = \mathrm{rk}_Z$. Pick $W \in \mathfrak{L}^{M}_{Z[T^{-1}]}$. We have the following short exact sequences of Z-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W \cap B & \longrightarrow & B & \longrightarrow & B/(W \cap B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & Q^n & \longrightarrow & Q^n/W & \longrightarrow & 0 \end{array}$$

The kernel of the map $B \to Q^n/W$ is $W \cap B$. So the last vertical map is injective. The quotient $(Q^n/W)/(B/W \cap B)$ is a quotient of the T-torsion module Q^n/B (see Remark (iii)) and hence its rank is zero. Applying the additivity of the rank to the last column shows that $B/(W \cap B)$ and Q^n/W have the same rank. The rank of both B and Q^n is n (see Definition 6.3). The additivity of the rank implies $\mathrm{rk}(W) = \mathrm{rk}(W \cap B)$.

(ii) Let $W, W' \in \mathfrak{L}^M_{Z[T^{-1}]}$ with $W \subset W'$. Note that $W' \cap B/W \cap B$ is a Z-submodule of W'/W. The inclusion is induced by $W' \cap B \hookrightarrow W'$. We want to show that the canonical map $\operatorname{tors}(W' \cap B/W \cap B) \to \operatorname{tors}(W'/W)$ is an isomorphism.

We only have to consider the surjectivity since the map is the restriction of the injective map $W' \cap B/W \cap B \to W'/W$. So let $[v] \in W'/W$ be any torsion element. No element of T divides the order of $[v]$ as it is an element of the $Z[T^{-1}]$-module W'/W. We already know that $W = \langle W \cap B\rangle_{Z[T^{-1}]}$ and hence $v' = tv$ for some $v \in W \cap B$ and an element $t \in Z \setminus \{0\}$ whose prime factors lie in T. W'/W is T-torsionfree and so is its subgroup $W' \cap B/W \cap B$.

Pick an element s with $st = \lambda \operatorname{ord}([v]) + 1$ for some $\lambda \in Z$. This exists as t and $\operatorname{ord}([v])$ are coprime. Hence

$$[v] = (1 + \operatorname{ord}([v]))[v] = st[v] = s[v'] \in W' \cap B/W \cap B.$$

So the canonical map $W' \cap B/W \cap B \to W'/W$ restricts to an isomorphism $\operatorname{tors}(W' \cap B/W \cap B) \to \operatorname{tors}(W'/W)$. If W'/W is a torsion group, so is its subgroup $W' \cap B/W \cap B$ and hence we get

$$W' \cap B/W \cap B \cong \operatorname{tors}(W' \cap B/W \cap B) \cong \operatorname{tors}(W'/W) \cong W'/W.$$

Especially $[W' \cap B : W \cap B] = [W' : W]$. If it is not a torsion group its rank is at least one. By the previous item the rank of $W' \cap B/W \cap B$ is also at least one and hence $[W' \cap B : W \cap B] = \infty = [W' : W]$.

(iii) The isomorphism $\operatorname{tors}(M/V) \cong \operatorname{tors}(M \cap B/V \cap B)$ from the last item shows that M/V is T_2-torsionfree if and only if $M \cap B/V \cap B$ is. This means exactly $V \in \mathfrak{L}^M_{Z[T^{-1}]}[T_2^{-1}]$ if and only if $V \cap B \in \mathfrak{L}^M_Z[(T \cup T_2)^{-1}]$.

□

Remark 6.11. Let V be a free $Z[S^{-1}]$-module of rank n and let B be an integral structure on V. Then $V \cap B$ is a free Z-module of rank n by the last lemma. Let us show that any Z-basis of $V \cap B$ is automatically a $Z[S^{-1}]$-basis of V. Clearly it is linear independent. Since $V/V \cap B$ is a Z-module of rank zero we can find for any $v \in V$ a number $\lambda \in Z$ such that $\lambda v \in V \cap B$. Write $\lambda = \lambda' \cdot \lambda''$ where λ' is a product of primes from S and λ'' is coprime to each element of S. Since B is a Z_S-module and λ'' is a unit in Z_S we can assume that $\lambda = \lambda'$. Hence we can write $\lambda' v$ as a Z-linear combination of the given basis of $V \cap B$. Since λ' is a unit in $Z[S^{-1}]$ we can multiply all coefficients λ'^{-1}. This shows that it is also a generating system.

Analogously it is also an Z_S-basis of B.

6.2 The localized case

Convention 6.12. *Let*

- $n \in \mathbb{N}$ *be a fixed nonnegative integer.*

- *Z be either $\mathbb{Z}$ (integral case) or $F[t]$ (function field case) for a finite field F,*
- *V be a finitely generated free $Z[T^{-1}]$-module of rank n,*
- *$\tilde{X}(V)$ denote the set of all inner products on $\mathbb{R} \otimes_{\mathbb{Z}} V$ in the integral case or the set of all $\{\frac{a}{b} \in Q \mid \deg(b) \geq \deg(a)\}$-lattices in $Q \otimes_Z V$ for a finitely generated, free $Z[T^{-1}]$-module in the function field case.*
- *$\mathfrak{L}$ denote the order-theoretic lattice of direct summands of the $Z[T^{-1}]$-module V.*

Now we are ready to define the volume function.

Definition 6.13. Let $\tilde{Y}_T(V)$ denote the set of all integral structures on V relative to T. Define the logarithmic volume function of V as

$$\log\mathrm{vol} : \mathfrak{L} \times \tilde{X}(V) \times \tilde{Y}_T(V) \to \mathbb{R}; \qquad (W, s, B) \mapsto \log\mathrm{vol}_{W\cap B}(s) =: \log\mathrm{vol}_W(B, s).$$

Remark 6.14. (i) An element in $\tilde{Y}_T(Z[T^{-1}]^n)$ is just a choice of an equivalence class of a system of n linear independent vectors in Q^n, where two such systems are equivalent if and only if their $Z[(\mathfrak{P} \setminus T)^{-1}]$-span agrees. Hence

$$\tilde{Y}_T(Z[T^{-1}]^n) \cong \mathrm{GL}_n(Q)/\mathrm{GL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$$

as left-$\mathrm{GL}_n(Q)$-sets.

(ii) Note that $W \cap B$ is just a finitely generated free Z-module. In the integral case s is an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$ and hence it can be restricted to $W \cap B \subset V \subset \mathbb{R} \otimes_{\mathbb{Z}} V$.

In the function field case s is a lattice in $Q \otimes_Z V$. The inclusion $V \cap B \to V$ induces an isomorphism $Q \otimes_Z (V \cap B) \to Q \otimes_Z V = Q \otimes_{Z[T^{-1}]} V$ since $Q \otimes_Z -$ is exact. So s can also be considered as a lattice in $V \cap B$. Hence $(V \cap B, s)$ is a volume space. So the definition of volume (Definition 5.10) for the function field case can be used here.

(iii) For any $\varphi \in \mathrm{aut}_{Z[S^{-1}]}(V)$, any submodule $W \subset V$ and any integral structure B we get:

$$\begin{aligned} \log\mathrm{vol}_{\varphi(W)}((\varphi \cdot s), \varphi \cdot B) &:= \log\mathrm{vol}_{\varphi(W\cap B)}(\varphi \cdot s) \\ &= \log\mathrm{vol}_{W\cap B}(s) \\ &=: \log\mathrm{vol}_W(s, B). \end{aligned}$$

The middle inequality has been explained in the number field case in Remark 4.2. In the function field case this has been done in Lemma 5.11.

Remark 6.15. (i) For two submodules $W \subset W'$ of V with the same rank we have in the case of $Z = \mathbb{Z}$:

$$\mathrm{vol}_W = [W' : W]\,\mathrm{vol}_{W'}.$$

(Note that the index $[W' : W]$ is finite if W, W' have the same rank). We get in the case of $Z = F[t]$:

$$\log\mathrm{vol}_W = \log_{|F|}([W' : W]) + \log\mathrm{vol}_{W'} = \dim'_F(W'/W) + \log\mathrm{vol}_{W'}.$$

(ii) For any constant C and a fixed element $s \in \tilde{X}(V)$ and an integral structure B there are only finitely many $W \in \mathfrak{L}$ with $\log \mathrm{vol}_W(s, B) \leq C$.

(iii) The minimal volume in each dimension is obtained by a direct summand.

Proof. (i) Note that intersection is index preserving by Proposition 6.10. In the integral case the claim follows from Lemma 4.5 and in the function field case it follows from Lemma 5.16.

(ii) Proposition 6.10 shows that the map

$$\mathfrak{L}^V_{Z[T^{-1}]} \to \mathfrak{L}^{V \cap B}_Z \subset \mathfrak{L}_Z; \qquad W \mapsto W \cap B$$

is an isomorphism of lattices and that it restricts to an isomorphism of the subposets of direct summands. The definition of the volume function (Definition 6.13) says that it is just the composition of the old volume function for Z and this isomorphism of lattices. Hence the statement follows directly from the statement for Z (see Proposition 4.6 for the integral case and Corollary 5.20 for the function field case).

(iii) Any element $V \in \mathfrak{L}^V_{Z[T^{-1}]}$ is a submodule of the direct summand spanned by V of the same rank. Hence we can use the first item of this remark to get the result.

□

6.3 Properties of the volume function for $Z[T^{-1}]$

Fix an integral structure B on V relative to a set of primes T and an element $s \in \tilde{X}(V)$. We want to show that the function

$$\log \mathrm{vol}_?(s, B) : \mathfrak{L} \to \mathbb{R}$$

satisfies all conditions from Convention 3.1 so that we can consider the canonical filtration.

Proposition 6.16 (Parallelogram constraint/subadditivity)**.** *For $W_1, W_2 \in \mathfrak{L}$ and any $s \in \tilde{X}(V), B \in \tilde{Y}_T(V)$ we have*

$$\mathrm{vol}_{W_1}(s, B) \cdot \mathrm{vol}_{W_2}(s, B) \geq \mathrm{vol}_{W_1 \cap W_2}(s, B)\, \mathrm{vol}_{\mathrm{lub}(W_1, W_2)}(s, B).$$

Proof. Using Definition 6.13 of the volume function, we really have to show that:

$$\log \mathrm{vol}_{W_1 \cap B}(s) + \log \mathrm{vol}_{W_2 \cap B}(s) \geq \log \mathrm{vol}_{W_1 \cap W_2 \cap B}(s) + \log \mathrm{vol}_{\mathrm{lub}(W_1, W_2) \cap B}(s).$$

This equation just involves the definition of the volume of a Z-module. Since $\_ \cap B$ is an isomorphism of the lattice of all direct summands of the $Z[T^{-1}]$-module V to the lattice of all direct summands of the Z-module $V \cap B$ by Proposition 6.10 we get $\mathrm{lub}(W_1, W_2) \cap B = \mathrm{lub}(W_1 \cap B, W_2 \cap B)$. We have already shown

$$\log \mathrm{vol}_{W_1 \cap B}(s) + \log \mathrm{vol}_{W_2 \cap B}(s) \geq \log \mathrm{vol}_{W_1 \cap W_2 \cap B}(s) + \log \mathrm{vol}_{\mathrm{lub}(W_1 \cap B, W_2 \cap B)}(s)$$

in Proposition 4.6 for the integral case and in Proposition 5.26(iii) for the function field case. □

So we can now show that this volume function and the $Z[T^{-1}]$-rank have all the properties required to construct the canonical filtration as in Convention 3.1.

Proposition 6.17. *Let $\mathfrak{L}$ denote the order-theoretic lattice of direct summands of V and for $W \in \mathfrak{L}$ let* $\mathrm{rk}(W)$ *denote the $Z[T^{-1}]$-rank of W. Let* $\log \mathrm{vol}_W(s, B)$ *denote the logarithmic volume as above. We have:*

(i) rk *is strictly monotone, i.e.* $\mathrm{rk}(W) < \mathrm{rk}(W')$ *for all* $W, W' \in \mathfrak{L}$ *with* $W \subsetneq W'$.

(ii) rk *is additive, i.e.* $\mathrm{rk}(W \cap W') + \mathrm{rk}(\mathrm{lub}(W, W')) = \mathrm{rk}(W) + \mathrm{rk}(W')$ *for all* $W, W' \in \mathfrak{L}$.

(iii) The function $\log \mathrm{vol}_{\_}(s, B) : \mathfrak{L} \to \mathbb{R}$ *is subadditive. This means that for all* $W, W' \in \mathfrak{L}$

$$\log \mathrm{vol}_{W \cap W'}(s, B) + \log \mathrm{vol}_{\mathrm{lub}(W,W')}(s, B) \leq \log \mathrm{vol}_W(s, B) + \log \mathrm{vol}_{W'}(s, B).$$

(iv) For each $C \in \mathbb{R}$ there are only finitely many $L \in \mathfrak{L}$ with $\log \mathrm{vol}_W(s, B) \leq C$.

(v) $rk(\mathbb{0}) = 0, \log \mathrm{vol}(\mathbb{0}) = 0$.

Proof. (i) By definition of the rank a submodule cannot have bigger rank than the entire module. If two direct summands W, W' of V have the same rank and $W \subseteq W'$, then W'/W is a submodule of the torsionfree module V/W and hence itself torsionfree. Since $Q \otimes_{Z[T^{-1}]} \_$ is exact we get a short exact sequence of Q-vector spaces

$$0 \to Q \otimes_{Z[T^{-1}]} W \to Q \otimes_{Z[T^{-1}]} W' \to Q \otimes_{Z[T^{-1}]} (W'/W) \to 0.$$

So the additivity of the dimension gives

$$\begin{aligned} \mathrm{rk}(W'/W) &:= \dim_Q(Q \otimes_{Z[T^{-1}]} W'/W) \\ &= \dim_Q(Q \otimes_{Z[T^{-1}]} W') - \dim_Q(Q \otimes_{Z[T^{-1}]} W) \\ &=: \mathrm{rk}_{Z[T^{-1}]}(W') - \mathrm{rk}_{Z[T^{-1}]}(W) \\ &= 0. \end{aligned}$$

Hence it has to be the trivial module by the structure theorem for finitely generated modules over a principal ideal domain. Hence $W = W'$. So rk is strictly monotone.

(ii) The rank of a finitely generated $Z[T^{-1}]$-module M is defined as $\dim_Q(Q \otimes_Z M)$. First note that we have a short exact sequence

$$0 \to W + W' \to \mathrm{lub}(W, W') \to \mathrm{lub}(W, W')/W + W \to 0.$$

Since Q is a flat Z-module we obtain a short exact sequence

$$0 \to Q \otimes_Z (W + W') \to Q \otimes_Z \mathrm{lub}(W, W') \to Q \otimes_Z (\mathrm{lub}(W, W')/W + W) \to 0.$$

Since $\mathrm{lub}(W, W')/W + W' = \mathrm{tors}(V/(W + W'))$ is torsion (see the construction of lub in Lemma 6.7) it vanishes after tensoring with Q. Hence the additivity of $\dim_Q$ gives $\mathrm{rk}_Z(W + W') = \mathrm{rk}_Z(\mathrm{lub}(W, W'))$.

Now consider the short exact sequence

$$0 \to W \cap W' \to W \oplus W' \to W + W' \to 0.$$

The flatness of Q and the additivity of $\dim_Q$ gives

$$\mathrm{rk}(W \cap W') + \mathrm{rk}(W + W') = \mathrm{rk}(W \oplus W') = \mathrm{rk}(W) + \mathrm{rk}(W').$$

This finishes the proof.

(iii) This has been proven in the last proposition with $\mathrm{lub}(W, W')$ replaced by $W + W'$.The module $W + W'$ is a submodule of $\mathrm{lub}(W, W')$ with finite quotient. We obtain from Remark 6.15 $\log \mathrm{vol}_{\mathrm{lub}(W,W')}(s, B) \leq \log \mathrm{vol}_{W+W'}(s, B)$. This completes the proof.

(iv) See Remark 6.15.

(v) The zero module is the minimal element in the lattice and its rank is zero and its logarithmic volume is defined to be zero.

□

Remark 6.18. So we can use section 3 to get for each $W \in \mathfrak{L}$ a number $c_W(s, B)$. We have

$$\begin{aligned}
& c_W(s,B) \\
:= & \inf_{\binom{W_0 \subsetneq W}{W \subsetneq W_2}} \frac{\log \mathrm{vol}_{W_2}(s,B) - \log \mathrm{vol}_W(s,B)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}_W(s,B) - \log \mathrm{vol}_{W_0}(s,B)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
:= & \inf_{\binom{W_0 \subsetneq W}{W \subsetneq W_2}} \frac{\log \mathrm{vol}_{W_2}(s,B) - \log \mathrm{vol}_W(s,B)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}_W(s,B) - \log \mathrm{vol}_{W_0}(s,B)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
:= & \inf_{\binom{W_0 \subsetneq W}{W \subsetneq W_2}} \frac{\log \mathrm{vol}_{W_2 \cap B}(s) - \log \mathrm{vol}_{W \cap B}(s)}{\mathrm{rk}(W_2 \cap B) - \mathrm{rk}(W \cap B)} - \frac{\log \mathrm{vol}_{W \cap B}(s) - \log \mathrm{vol}_{W_0 \cap B}(s)}{\mathrm{rk}(W \cap B) - \mathrm{rk}(W_0 \cap B)} \\
:= & \inf_{\binom{W_0 \subsetneq W}{W \subsetneq W_2}} \frac{\log \mathrm{vol}_{W_2 \cap B}(s) - \log \mathrm{vol}_{W \cap B}(s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}_{W \cap B}(s) - \log \mathrm{vol}_{W_0 \cap B}(s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
=: & \; c_{W \cap B}(s).
\end{aligned}$$

This used that the map $- \cap B$ from the lattice of direct summands of the $Z[S^{-1}]$-module V to the lattice of direct summands of $V \cap B$ is a rank-preserving isomorphism by Proposition 6.10.

Furthermore we have the following properties:

Lemma 6.19. *In the number field case ($Z = \mathbb{Z}$) we have* *(i)* $\mathrm{vol}_W(\lambda s, B) = \lambda^{\mathrm{rk}\, W} \mathrm{vol}_W(s, B)$ *for* $\lambda \in \mathbb{R}, \lambda > 0$*,*

(ii) $\mathrm{vol}_W(s, pB) = p^{\mathrm{rk}\, W} \mathrm{vol}_W(s, B)$ *for any* $p \in T$*.*

In the function field case ($Z = F[t]$) we have

(i) for $\lambda \in Z[T^{-1}] \setminus \{0\}$ *that*

$$\mathit{log}\, \mathrm{vol}_W(\lambda S, B) = \mathrm{rk}(W) \cdot \nu(\lambda) + \log \mathrm{vol}_W(S, B),$$

(ii) $\log \mathrm{vol}_W(S, pB) = -\mathrm{rk}(W)\nu(p) + \log \mathrm{vol}(S, B)$ *for any* $p \in T$*.*

Proof. We get in the number field case:

(i)
$$\mathrm{vol}_W(\lambda s, B) := \mathrm{vol}_{W\cap B}(\lambda s) = \lambda^{\mathrm{rk}\, W} \mathrm{vol}_{W\cap B}(s) =: \lambda^{\mathrm{rk}\, W} \mathrm{vol}_W(s, B).$$

The equality in the middle follows directly from the definition of the volume (see Definition 4.1).

(ii) As W is a $\mathbb{Z}[T^{-1}]$ module we get $pW = W$ and hence $W \cap pB = pW \cap pB = p(W \cap B)$ and consequently

$$\begin{aligned} \mathrm{vol}_W(s, pB) &= \mathrm{vol}_{p(W\cap B)}(s) \\ &= [W \cap B : p(W \cap B)]\, \mathrm{vol}_{W\cap B}(s) \\ &= p^{\mathrm{rk}(W\cap B)} \mathrm{vol}_{W\cap B}(s) \\ &= p^{\mathrm{rk}\, W} \mathrm{vol}_W(s, B). \end{aligned}$$

Let us now consider the function field case:

(i) We can use the same chain of equalities as in the number field case

$$\begin{aligned} \log \mathrm{vol}_W(\lambda S, B) &:= \log \mathrm{vol}_{W\cap B}(\lambda S) \\ &= \mathrm{rk}(W) \cdot \nu(\lambda) + \log \mathrm{vol}_{W\cap B}(S) \\ &=: \mathrm{rk}(W) \cdot \nu(\lambda) + \log \mathrm{vol}_W(S, B) \end{aligned}$$

and the middle equality is given by Lemma 5.16.

(ii) As W is a $Z[T^{-1}]$ module we get $tW = W$ and hence $W \cap pB = pW \cap pB = p(W \cap B)$ and consequently

$$\begin{aligned} \log \mathrm{vol}_W(S, pB) &= \log \mathrm{vol}_{p(W\cap B)}(S) \\ &\overset{5.16}{=} \dim_F((W \cap B)/p(W \cap B)) + \log \mathrm{vol}_{W\cap B}(S) \\ &= \mathrm{rk}(W) \deg(p) + \log \mathrm{vol}_{W\cap B}(S) \\ &= -\mathrm{rk}(W)\nu(p) + \log \mathrm{vol}_{W\cap B}(S). \end{aligned}$$

□

Corollary 6.20. *Given two integral structures B, B' such that $zB \subset B' \subset B$ for some $z \in Z$. Since B is a $Z[\mathfrak{P} \setminus T]$-module we get $pB = B$ for any $p \in \mathfrak{P} \setminus T$. Thus we can leave out all prime factors of z from $\mathfrak{P} \setminus T$. So let us assume that no element of $\mathfrak{P} \setminus T$ divides z. We have*

- *in the number field case*

$$\mathrm{rk}(W) \cdot \ln(z) + \mathrm{vol}_W(s, B) = \ln \mathrm{vol}_W(s, zB) \geq \ln \mathrm{vol}_W(s, B') \geq \ln \mathrm{vol}_W(s, B),$$

- *in the function field case*

$$-\mathrm{rk}(W) \cdot v(z) + \log \mathrm{vol}_W(s, B) = \log \mathrm{vol}_W(s, zB) \geq \log \mathrm{vol}_W(s, B') \geq \log \mathrm{vol}_W(s, B).$$

Corollary 6.21 (Scaling invariance of c_W)**.** *We get in the number field case:*

(i) $c_W(\lambda s, B) = c_W(s, B)$ *for any* $\lambda \in \mathbb{R}, \lambda > 0$

(ii) $c_W(s, pB) = c_W(s, B)$ *for any* $p \in T$.

and in the function field case

(i) $c_W(\lambda s, B) = c_W(s, B)$ *for any* $\lambda \in F[t] \setminus \{0\}$

(ii) $c_W(\lambda s, pB) = c_W(s, B)$ *for any* $p \in T$

Proof. This is just inserting the last lemma into the definition of c_W. The function c_W is defined as a supremum over a family of functions. So we just have to show that each function from this family is scaling invariant.

Let us for example show (i):

$$\begin{aligned}
& \frac{\ln \mathrm{vol}_{W_2 \cap B}(\lambda s) - \ln \mathrm{vol}_{W \cap B}(\lambda s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\ln \mathrm{vol}_{W \cap B}(\lambda s) - \ln \mathrm{vol}_{W_0 \cap B}(\lambda s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
= \; & \frac{\ln(\lambda)\,\mathrm{rk}(W_2 \cap B) + \ln \mathrm{vol}_{W_2 \cap B}(s) - \ln(\lambda)\,\mathrm{rk}(W \cap B) - \ln \mathrm{vol}_{W \cap B}(\lambda s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} \\
- \; & \frac{\ln(\lambda)\,\mathrm{rk}(W \cap B) + \ln \mathrm{vol}_{W \cap B}(s) - \ln(\lambda)\,\mathrm{rk}(W_0 \cap B) - \ln \mathrm{vol}_{W_0 \cap B}(\lambda s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
= \; & \ln(\lambda) \cdot \frac{\mathrm{rk}(W_2 \cap B) - \mathrm{rk}(W \cap B)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} + \frac{\ln \mathrm{vol}_{W_2 \cap B}(s) - \ln \mathrm{vol}_{W \cap B}(\lambda s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} \\
- \; & \ln(\lambda) \cdot \frac{\mathrm{rk}(W \cap B) - \mathrm{rk}(W_0 \cap B)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} - \frac{\ln \mathrm{vol}_{W \cap B}(s) - \ln \mathrm{vol}_{W_0 \cap B}(\lambda s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
= \; & \frac{\ln \mathrm{vol}_{W_2 \cap B}(s) - \ln \mathrm{vol}_{W \cap B}(\lambda s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\ln \mathrm{vol}_{W \cap B}(s) - \ln \mathrm{vol}_{W_0 \cap B}(\lambda s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)}.
\end{aligned}$$

The last equality holds since intersection with B is rank preserving (see Proposition 6.10). The other items can be proved analogously with the last lemma. □

Definition 6.22. Let $X(\mathbb{R}^n)$ denote the quotient of $\tilde{X}(V)$ under the group action

$$(\mathbb{R}^+, *) \times \tilde{X}(\mathbb{R}^n) \to \tilde{X}(\mathbb{R}^n) \qquad (\lambda, s) \mapsto \lambda s.$$

Let T be a set of primes. Let $Y_T(n)$ denote the quotient of $\tilde{Y}_T(n)$ under the group action of the group of positive units in $\mathbb{Z}[T^{-1}] \subset \mathbb{Q}$.

Remark 6.23. The scaling invariance from Corollary 6.21 shows that the function c_W descends to a function

$$c_W : X(\mathbb{R}^n) \times Y_T(n) \to \mathbb{R}.$$

The following lemma will be needed to study the action of $GL_n(Q)$ on a specific CAT(0)-space.

Lemma 6.24. *Let T be a set of primes.*

(i) Every matrix $A \in \mathrm{GL}_n(Q)$ can be written as a product of a matrix in $\mathrm{GL}_n(Z[T^{-1}])$ and a matrix in $\mathrm{GL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$.

(ii) Every matrix $A \in \mathrm{SL}_n(Q)$ can be written as a product of a matrix in $\mathrm{SL}_n(Z[T^{-1}])$ and a matrix in $\mathrm{SL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$.

(iii) Furthermore if a subgroup G is conjugate to $\mathrm{SL}_n(Z[(\mathfrak{P} \setminus T)^{-1}])$ in $\mathrm{GL}_n(Q)$ we can also decompose any matrix $A \in \mathrm{SL}_n(\mathbb{Q})$ as a product of a matrix in $\mathrm{SL}_n(Z[T^{-1}])$ and a matrix in G.

Proof. (i) This is obvious for diagonal matrices. If A is not diagonal let m be the least common multiple of the denominators of all entries of M. By the invariant factor theorem applied to the matrix $mA \in M_n(Z)$ we can find integral matrices $B, C, D \in M_n(Z)$ such that B, C are invertible matrices of determinant one and D is a diagonal matrix and $mA = BDC$. Hence $A = B \cdot (\frac{1}{m}D) \cdot C$. Then we apply this lemma to the diagonal matrix $\frac{1}{m}D$ to obtain the result.

(ii) The product of the two determinants of the two matrices obtained like in the last item is one. One of them lies in $Z[(\mathfrak{P} \setminus T)^{-1}]$ and the other one lies in $Z[T^{-1}]$. Hence they both have to be one.

(iii) Assume $G = B \cdot \mathrm{SL}_n(Z[(\mathfrak{P} \setminus T)^{-1}]) \cdot B^{-1}$. We can first decompose $B = B'B''$ like in the first item. Especially we get $B \cdot \mathrm{SL}_n(\mathbb{Z}[(\mathfrak{P} \setminus T)^{-1}]) \cdot B^{-1} = B' \cdot \mathrm{SL}_n(\mathbb{Z}[(\mathfrak{P} \setminus T)^{-1}]) \cdot B'^{-1}$. Hence without loss of generality we may assume $B \in \mathrm{GL}_n(\mathbb{Z}[T^{-1}])$. We decompose $B^{-1}AB$ as in the second item and conjugate each factor with B. This gives the desired decomposition.

□

Proposition 6.25. *For any finite set of primes S the group action of $\mathrm{saut}_{Z[S^{-1}]}(V)$ on $Y_S(V)$ is cofinite.*

Proof. A choice of a basis gives an isomorphism $\varphi V \cong Z[S^{-1}]^n$ and an isomorphism φ' from $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ to $\cong \mathrm{GL}_n(Q)$. We can associate to any $B \in \tilde{Y}_S(V)$ after choice of a Z_S-basis $b_1, \dots, b_n$ of B the matrix whose columns are $\varphi(b_i)$. Choosing a different basis results in right multiplication with a matrix in $\mathrm{GL}_n(Z_S)$. This gives an bijection from $\tilde{Y}_S(V)$ to the right cosets $\mathrm{GL}_n(Q)/\mathrm{GL}_n(Z_S)$. It is compatible with the left $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$-action where we use φ' to turn the left $\mathrm{GL}_n(Q)$-action on the cosets to an $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$-action.

The group action of $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ on $\tilde{Y}_S(n)$ is transitive; for $B, B' \in \tilde{Y}_S(n)$ we can choose Z_S-bases and define a Q-linear map mapping the first basis to the second basis.

Two elements $A \cdot \mathrm{GL}_n(Z_S), B \cdot \mathrm{GL}_n(Z_S)$ lie in the same $\mathrm{saut}_{Z[S^{-1}]}(V)$-orbit if and only if the p-adic valuation v_p of $\det(AB^{-1})$ is zero for all $p \in S$; i.e. $\det(AB^{-1})$ is a unit in Z_S.

To see this equivalence note that this condition does not depend on the choice of representatives of the right cosets and is clearly satisfied for $A = MB$ with $M \in \mathrm{SL}_n(Q)$. If it is satisfied, then we can achieve $\det(AB^{-1}) = 1$ for $p \notin S$ by replacing the representative A of its right $\mathrm{GL}_n(Z_S)$-coset by $A \cdot M'$ for a diagonal matrix M' with exactly one entry $\det(AB^{-1})^{-1} \in Z_S$ and the remaining diagonal entries ones. Hence we have infinitely many orbits.

Thus $\det(AB^{-1}) = 1$ which shows that the right $\mathrm{GL}_n(Z_s)$-cosets of A and B lie in the same $\mathrm{SL}_n(Q)$ orbit.

Let us now consider homothety classes. We can now replace the representative A by $p^k A$ for some $p \in S$. We obtain $\det(p^k A) = p^{nk} \det(A)$. So two elements $[A], [B] \in Y_S(n)$ lie in the same orbit of the $\mathrm{SL}_n(Q)$ action if and only if $v_p(\det(AB^{-1})) \equiv 0 \pmod{n}$ for all $p \in S$. There are exactly $n^{|S|}$ orbits.

Now restrict the group action to $\mathrm{saut}_{Z[S^{-1}]}(V) \cong \mathrm{SL}_n(Z[S^{-1}])$. For two elements $[A], [B] \in Y_S(n)$ lying in the same $\mathrm{SL}_n(Q)$-orbit, there are representatives A, B and a matrix $C \in \mathrm{SL}_n(Q)$ with $CA = B$. The stabilizer of $[A]$ is conjugate to $\mathrm{SL}_n(Z_S)$ in $\mathrm{GL}_n(Q)$ (compare Remark 6.4). Let us use the decomposition from Lemma 6.24. Hence we can write $C = C'C''$ with $C' \in \mathrm{SL}_n(Z[S^{-1}])$ and $C'' \in \mathrm{Stab}_{[A]}$. Finally $[B] = C'[A]$ with $C' \in \mathrm{SL}_n(Z[S^{-1}])$.

So any two points that lie in the same $\mathrm{SL}_n(Q)$ orbit also lie in the same $\mathrm{SL}_n(Z[S^{-1}])$-orbit. This completes the proof. □

7 Spaces with actions of general linear groups

In this section several spaces are introduced on which the general linear group of certain rings acts. Previously we have studied the volume functions for a specific scalar product. Let us examine in this section how fast the volume changes when we vary the scalar product. This will be used to construct a system of open sets satisfying all conditions from Proposition 2.4.

7.1 $\mathrm{GL}_n(\mathbb{Z})$ acts on the space of homothety classes of inner products

This section will analyze the metric on the space of homothety classes of scalar products (defined for example in [10, p. 314 ff.]). Furthermore certain properties of the volume functions will be established. Apart from the growth condition, which was analyzed in [8, Section 1], these have basically been shown in [17]. It still makes sense to restate them in precisely this form. Then the localized version for $\mathbb{Z}$ and for $F[t]$ can be treated simultaneously in Section 6.

Let V be finitely generated, free $\mathbb{Z}$-module of rank n and consider the space $\tilde{X}(V)$ of all inner products on $\mathbb{R} \otimes_{\mathbb{Z}} V$. We will think of an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$ either as a symmetric map $\mathbb{R} \otimes_{\mathbb{Z}} V \to (\mathbb{R} \otimes_{\mathbb{Z}} V)^*$ or as a bilinear form.

After a choice of a $\mathbb{Z}$-basis for $V \subset \mathbb{R} \otimes_{\mathbb{Z}} V$ we can write such an inner product as a matrix. This gives $\tilde{X}(V)$ the structure of a manifold. Rescaling gives a group action of $(\mathbb{R}^{>0}, \cdot)$ on $\tilde{X}(V)$ via

$$(\lambda, s) \mapsto \lambda s.$$

Let $X(V)$ be the quotient of $\tilde{X}(V)$ under this group action. An element of $X(V)$ is called a homothety class of inner products. The projection map has a section that sends a homothety class to the inner product whose representing matrix with respect to some basis of V has determinant one. The group $\mathrm{aut}_{\mathbb{Z}}(V) \cong \mathrm{GL}_n(\mathbb{Z})$ acts on the space of homothety classes of inner products.

$\tilde{X}(V)$ is a subset of the vector space $\mathrm{sym}(\mathbb{R} \otimes_{\mathbb{Z}} V)$ of symmetric linear maps $(\mathbb{R} \otimes_{\mathbb{Z}} V) \to (\mathbb{R} \otimes_{\mathbb{Z}} V)^*$. Symmetric means that for any $s \in \mathrm{sym}(V)$ the map

$$(\mathbb{R} \otimes_{\mathbb{Z}} V) \xrightarrow{\cong} (\mathbb{R} \otimes_{\mathbb{Z}} V)^{**} \xrightarrow{s^*} (\mathbb{R} \otimes_{\mathbb{Z}} V)^*$$

is again s. The isomorphism on the left is the inverse of the canonical evaluation isomorphism.

Indeed $\tilde{X}(V)$ is an open subset of $\mathrm{sym}(\mathbb{R} \otimes_{\mathbb{Z}} V)$. So we get a canonical trivialization of the tangent bundle

$$\tilde{X}(V) \times \mathrm{sym}(\mathbb{R} \otimes_{\mathbb{Z}} V) \xrightarrow{\cong} T_*\tilde{X}(V) \qquad (s, v) \mapsto [t \mapsto s + tv].$$

Let us now define a Riemannian metric on $\tilde{X}(V)$. So we have to define for each $s \in X(V)$ an inner product g_s on $\tilde{X}(V)$:

$$g_s(u, v) := \mathrm{tr}(s^{-1} \circ u \circ s^{-1} \circ v).$$

It is obviously bilinear and symmetric. Furthermore the endomorphism $s^{-1} \circ u$ is self adjoint with respect to the inner product s on $(\mathbb{R} \otimes_{\mathbb{Z}} V)$ since

$$s^{-1} \circ (s^{-1} \circ u)^* \circ s = s^{-1}u.$$

Hence there is an orthonormal basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$. Then the eigenvalues of $(s^{-1} \circ u)^2$ are $\lambda_1^2, \dots, \lambda_n^2$ and its trace is just the sum. Hence its trace is nonnegative and it vanishes only if $s^{-1}u$ is zero. In this case u is zero since s is invertible. So g_s is indeed a scalar product.

Remark 7.1. We get for any $\varphi \in \mathrm{aut}_{\mathbb{Z}}(V)$, $s \in \tilde{X}(V)$, $W \subset V$:

$$\mathrm{vol}_W(s \cdot \varphi) = \mathrm{vol}_{\varphi(W)}(s).$$

If we insert the definition of vol this follows directly. Consequently we get also the same equivariance property for c_W.

Lemma 7.2. *The right action of the group* $\mathrm{aut}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} V)$ *on* $\tilde{X}(V)$ *given by*

$$(f, s) \mapsto f^* \circ s \circ f$$

is an isometric action.

Proof. Pick $s \in \tilde{X}(V)$, $f \in \mathrm{aut}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} V)$, $u, v \in \mathrm{sym}(\mathbb{R} \otimes_{\mathbb{Z}} V)$. Let $r_f : \tilde{X}(V) \to \tilde{X}(V)$ be the map given by the action of f. We get:

$$\begin{aligned}
& g_{f^* \circ s \circ f}(Tr_f(u), Tr_f(v)) \\
= \quad & \mathrm{tr}((f^* \circ s \circ f)^{-1} \circ Tr_f(u) \circ (f^* \circ s \circ f)^{-1} \circ Tr_f(v)) \\
= \quad & \mathrm{tr}((f^* \circ s \circ f)^{-1} \circ f^* \circ u \circ f \circ (f^* \circ s \circ f)^{-1} \circ f^* \circ v \circ f)) \\
= \quad & \mathrm{tr}(f^{-1} \circ s^{-1} \circ u \circ s \circ v \circ f) \\
= \quad & \mathrm{tr}(s^{-1} \circ u \circ s \circ v) \\
= \quad & g_s(u, v).
\end{aligned}$$

Hence the group action is isometric. □

We will use the Riemannian metric on $X(V)$ coming from the section $X(V) \to \tilde{X}(V)$ mentioned above. The group action commutes with homotheties and hence descends to an group action on $X(V)$. After restricting the group action to $\mathrm{aut}_{\mathbb{Z}}(V)$ the section $X(V) \to \tilde{X}(V)$ is equivariant.

Lemma 7.3. *The group actions of* $\mathrm{aut}_{\mathbb{Z}}(V)$ *on* $\tilde{X}(V)$ *and* $X(V)$ *are proper.*

Proof. Again choosing a basis of V gives an isomorphism $\mathrm{aut}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} V) \cong \mathrm{GL}_n(\mathbb{R})$ and an diffeomorphism $\mathrm{GL}_n(\mathbb{R})/O(n) \cong \tilde{X}(V)$. Let $K \subset \tilde{X}(V)$ be any compact subset. Let $K' \subset \mathrm{GL}_n(\mathbb{R})$ be the preimage of K. The quotient map $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})/O(n)$ is proper. So K' is compact. We get with the upper identifications:

$$\{A \in \mathrm{GL}_n(\mathbb{R}) \mid AK \cap K \neq \emptyset\} = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid AK' \cap K' \neq \emptyset\} = K' \cdot K'^{-1}$$

and thus the group action of $\mathrm{aut}_{\mathbb{R}}(\mathbb{R} \otimes_{\mathbb{Z}} V)$ is proper. Hence the same is true for the restricted group action to $\mathrm{aut}_{\mathbb{Z}}(V)$. Since $X(V)$ is a closed, $\mathrm{aut}_{\mathbb{Z}}(V)$-invariant subspace of $\tilde{X}(V)$ the restricted group action on $X(V)$ is also proper. □

Now we can use the volume functions to construct a family of open sets with nice properties. Let us first determine how fast the volume grows.

Proposition 7.4. *Let W be a direct summand of V. Consider the function*

$$\mathrm{vol}_W^2 : \tilde{X}(V) \to \mathbb{R}.$$

With the upper identification of $T_s\tilde{X}(V)$ with $\mathrm{sym}(V)$ *its gradient at s is given by*

$$\mathrm{vol}_W^2(s) \cdot \mathrm{pr}^* \circ s \circ \mathrm{pr},$$

where $\mathrm{pr} : V \to V$ *denotes the orthogonal projection onto W with respect to the inner product s.*

Proof. We have to verify the defining property of the gradient for $\mathrm{vol}_W^2(s) \cdot \mathrm{pr}^* \circ s \circ \mathrm{pr}$. So let $s \in \tilde{X}(V)$ be any point and let $u \in T_s\tilde{X}(V)$ be any tangent vector. We have to show that the directional derivate of vol_W^2 along u can be computed as the inner product g_s of u and the desired term, i.e.

$$\lim_{t \to 0} \frac{\mathrm{vol}_W^2(s + tu) - \mathrm{vol}_W^2(s)}{t} = g_s(u, \mathrm{vol}_W^2(s) \cdot \mathrm{pr}^* \circ s \circ \mathrm{pr}).$$

Pick a basis $v_1, \dots, v_m$ of W. So let us first simplify the left hand side. For any $n \times n$-matrix A we have

$$\lim_{t \to 0} \frac{\det(I_n + tA) - \det(I_n)}{t} = \mathrm{tr}(A).$$

With the Leibniz rule $\det(I_n + tA)$ can be expressed as a polynomial in t whose coefficients depend on the entries on A. The zeroth coefficient is one and the first coefficient is $\mathrm{tr}(A)$. This implies the formula. So we get

$$\begin{aligned}
& \lim_{t \to 0} \frac{\mathrm{vol}_W^2(s + tu) - \mathrm{vol}_W^2(s)}{t} \\
:= \; & \lim_{t \to 0} \frac{\det((s + tu)(v_i, v_j))_{i,j \le m} - \det(s(v_i, v_j))_{i,j \le m}}{t} \\
= \; & \det(s(v_i, v_j))_{i,j \le m} \cdot \lim_{t \to 0} \frac{\det(I_m + t(s(v_i, v_j))_{i,j \le m}^{-1} \cdot (u(v_i, v_j))_{i,j \le m} - \det(I_m)}{t} \\
= \; & \mathrm{vol}_W^2(s) \cdot \mathrm{tr}\left(s(v_i, v_j))_{i,j \le m}^{-1} \cdot (u(v_i, v_j))_{i,j \le m}\right).
\end{aligned}$$

So we have found the factor $\mathrm{vol}_W^2(s)$ on the left hand side. So we still have to show that

$$\mathrm{tr}(s(v_i, v_j))_{i,j\leq m}^{-1} \cdot (u(v_i, v_j))_{i,j\leq m} = g_s(u, \mathrm{pr}^* \circ s \circ \mathrm{pr}).$$

Since pr is an orthogonal projection of $(\mathbb{R} \otimes_{\mathbb{Z}} V, s)$ we know that it equals its adjoint $s^{-1} \circ \mathrm{pr}^* \circ s$ and it equals pr^2. So let us now simplify the right hand side.

$$\begin{aligned} & g_s(u, \mathrm{vol}_W^2(s) \cdot \mathrm{pr}^* \circ s \circ \mathrm{pr}) \\ := \; & \mathrm{tr}(s^{-1} \circ u \circ s^{-1} \circ \mathrm{pr}^* \circ s \circ \mathrm{pr}) \\ = \; & \mathrm{tr}(s^{-1} \circ u \circ \mathrm{pr}). \end{aligned}$$

Let us now consider the decomposition $\mathbb{R} \otimes_{\mathbb{Z}} V \cong \mathbb{R} \otimes_{\mathbb{Z}} W \oplus (\mathbb{R} \otimes_{\mathbb{Z}} W)^{\perp}$. It gives a decomposition of the dual spaces $(\mathbb{R} \otimes_{\mathbb{Z}} V)^* \cong (\mathbb{R} \otimes_{\mathbb{Z}} W)^* \oplus ((\mathbb{R} \otimes_{\mathbb{Z}} W)^{\perp})^*$. And the map s has block form with respect to these decompositions since $s(w, w^{\perp}) = 0$ for $w \in \mathbb{R} \otimes_{\mathbb{Z}} W, W^{\perp} \in \mathbb{R} \otimes_{\mathbb{Z}} W)^{\perp}$. Write it as $s = s_W \oplus s_{W^{\perp}}$. Let us now extend the basis $v_1, \ldots, v_n$ of $\mathbb{R} \otimes_{\mathbb{Z}} W$ by a basis $v_{m+1}, \ldots, v_n$ of $(\mathbb{R} \otimes_{\mathbb{T}} W)^{\perp}$ to a basis of the whole vector space. Let $v_1^*, \ldots, v_n^*$ denote the dual basis. Let us now compute the upper trace with respect to this basis. The vectors $v_{m+1}, \ldots, v_n$ can be omitted since they lie in the kernel of pr. The matrix of the map s with respect to the bases $v_1, \ldots, v_n$ and its dual basis is given by $(s(v_i, v_j))_{i,j\leq n}$ and it has block form. So the matrix of s^{-1} can be computed by inverting the blocks separately. Hence if we write the vector $s^{-1} \circ u \circ \mathrm{pr}(v_i)$ for $i \leq m$ as a linear combination of the basis, then the coefficient of v_i is given by the (i, i)-th entry of the matrix $s(v_i, v_j))_{i,j\leq m}^{-1} \cdot (u(v_i, v_j))_{i,j\leq m}$. Thus we finally get

$$\mathrm{tr}(s(v_i, v_j))_{i,j\leq m}^{-1} \cdot (u(v_i, v_j))_{i,j\leq m} = g_s(u, \mathrm{pr}^* \circ s \circ \mathrm{pr})$$

which completes the proof. □

Corollary 7.5. *Using the chain rule we see that the gradient of* $\ln \mathrm{vol}_W$ *at* $s \in \tilde{X}$ *is given by* $\frac{1}{2}\mathrm{pr}^* \circ s \circ \mathrm{pr}$. *Its length is just*

$$\begin{aligned} g_s(\frac{1}{2}\mathrm{pr}^* \circ s \circ \mathrm{pr}, \frac{1}{2}\mathrm{pr}^* \circ s \circ \mathrm{pr})^{\frac{1}{2}} &= \frac{1}{2}\mathrm{tr}(s^{-1} \circ \mathrm{pr}^* \circ s \circ \mathrm{pr} \circ s^{-1} \circ \mathrm{pr}^* \circ s \circ \mathrm{pr})^{\frac{1}{2}} \\ &= \frac{1}{2}\mathrm{tr}(\mathrm{pr} \circ \mathrm{pr} \circ \mathrm{pr} \circ \mathrm{pr})^{\frac{1}{2}} \\ &= \frac{1}{2}\mathrm{tr}(\mathrm{pr})^{\frac{1}{2}} = \frac{1}{2}\sqrt{\mathrm{rk}(W)} \\ &\leq \frac{n}{2} \leq n \end{aligned}$$

Hence $\ln \mathrm{vol}_W$ *is n-Lipschitz. For a nontrivial direct summand W the function* c_W *is defined to be*

$$c_W(s) := \inf_{\binom{W_0 \leq W}{W \lneq W_2}} \frac{\ln \mathrm{vol}_{W_2}(s) - \ln \mathrm{vol}_W(s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\ln \mathrm{vol}_W(s) - \ln \mathrm{vol}_{W_0}(s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)}.$$

So the function c_W is $4n$-Lipschitz since it is defined as the infinum of a family of $4n$-Lipschitz functions. The functions $\{c_W \mid W \subset \mathbb{Z}^n$ direct summand$\}$ descend to homothety classes as the function c_W is invariant under scaling:

$$\begin{aligned}
& c(W)(\lambda \cdot s) \\
:= & \inf_{\binom{W_0 \lneq W}{W \lneq W_2}} \frac{\ln \mathrm{vol}_{W_2}(\lambda \cdot s) - \ln \mathrm{vol}_W(\lambda \cdot s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\ln \mathrm{vol}_W(\lambda \cdot s) - \ln \mathrm{vol}_{W_0}(\lambda \cdot s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
:= & \inf_{\binom{W_0 \lneq W}{W \lneq W_2}} \frac{\mathrm{rk}(W_2) \cdot \ln(\lambda) + \ln \mathrm{vol}_{W_2}(s) - \mathrm{rk}(W) \cdot \ln(\lambda) - \ln \mathrm{vol}_W(s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} \\
- & \frac{\mathrm{rk}(W) \cdot \ln(\lambda) + \ln \mathrm{vol}_W(s) - \mathrm{rk}(W_0) \cdot \ln(\lambda) - \ln \mathrm{vol}_{W_0}(s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} \\
:= & \inf_{\binom{W_0 \lneq W}{W \lneq W_2}} \frac{\ln \mathrm{vol}_{W_2}(s) - \ln \mathrm{vol}_W(s)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} + \ln(\lambda) - \frac{\ln \mathrm{vol}_W(s) - \ln \mathrm{vol}_{W_0}(s)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)} - \ln(\lambda) \\
= & c_W(s).
\end{aligned}$$

Using the section $X(V) \subset \tilde{X}(V)$ mentioned above we can also view the induced function from the quotient as a restriction. The gradient of the restriction at s is just the orthogonal projection of the gradient to the tangent space of the submanifold. Hence the length of the gradient can only be smaller. So the function $c_W : X(V) \to \mathbb{R}$ is also $4n$-Lipschitz.

So the functions give rise to the family $\{c_W^{-1}((0,\infty)) \mid W \subset \mathbb{Z}^n$ direct summand$\}$ of open sets. We need a preliminary lemma to show that they satisfy all conditions from Proposition 2.4.

Lemma 7.6. *Let X be a proper, inner metric space and let $U \subset X$ be an open subset and $\beta \in \mathbb{R}$ be any real number. Then*

$$U^{-\beta} := \{x \in U \mid \overline{B}_\beta(x) \subset U\}$$

is open.

Proof. We have to show that there is for $x \in U^{-\beta}$ an $\varepsilon' > 0$ such that $B_{\varepsilon'}(x) \subset U^{-\beta}$.

Since U is open there is for each $z \in \overline{B}_\beta(x)$ an $\varepsilon(z) \in \mathbb{R}$ with $B_{\varepsilon(z)}(z) \subset U$. The set $\overline{B}_\beta(x)$ is compact as the metric space is proper. So there is a uniform $\varepsilon > 0$ with $B_\varepsilon(z) \subset U$ for all $z \in \overline{B}_\beta(x)$.

Since the metric space is inner we get

$$B_{\beta+\varepsilon}(x) = \bigcup_{z \in B_\beta(x)} B_\varepsilon(z)$$

and hence it is contained in U. Hence by the triangular inequality $B_{\frac{\varepsilon}{2}}(x)$ is contained in $U^{-\beta}$ and hence it is open. □

Let us now show that for a given $C \in \mathbb{R}$ the group action of $\mathrm{aut}_{\mathbb{Z}}(V)$ on

$$\{s \in \tilde{X}(V) \mid \mathrm{vol}_V(s) = 1, c_W(x) \leq C \ \forall\ W \subset V\}$$

is cocompact. Pick a basis $v_1, \ldots, v_n$ of V. For a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ we obtain an inner product $f(A)$ on $\mathbb{R} \otimes_{\mathbb{Z}} V$ by pulling back the standard inner product on $\mathbb{R}^n$ using the map that sends v_i to the i-th column of A. This defines a continuous map $\mathrm{GL}_n(\mathbb{R}) \to \tilde{X}(V)$.

Let me give an outline of the proof first. We show that we can find a basis of short vectors of V — say of length $\leq R$. Thus we can obtain the inner product as $f(A)$ for a matrix whose first column a_1 is in $\overline{B_R(0)} \setminus \{0\}$, whose second column a_2 is in $B_R(0) \setminus \langle a_1 \rangle$, and so on. That matrix is invertible, since its columns are linearly independent. But the set of those matrices is not compact. To correct this we have to find an $\varepsilon > 0$ such that the first column is in $\overline{B_R(0)} \setminus B_\varepsilon(0)$, the second column is in $B_R(0) \setminus B_\varepsilon(\langle a_1 \rangle)$, and so on. Then the set of such matrices is compact.

Theorem 7.7 (Minkowski 1889)**.** *[25, 4.4 on page 27] Let S be a convex subset of $\mathbb{R}^n$ with volume $> 2^n$ that is symmetric with respect to the origin. Then S contains a nontrivial vector from $\mathbb{Z}^n$.*

Proof. Consider the map $S \hookrightarrow \mathbb{R}^n \to \mathbb{R}^n/(2\mathbb{Z}^n)$. It is locally an isometry onto its image. If it was injective, then the volume of its image equals the volume of the source. But S has volume $> 2^n$ and its image lies in $\mathbb{R}^n/2\mathbb{Z}^n$, which has volume 2^n. So S can't be injective and thus there are $p_1, p_2 \in S$ with $p_1 - p_2 \in 2\mathbb{Z}^n$. Using the reflection symmetry at the origin and the convexity we see that

$$S \ni \frac{1}{2}p_1 + \frac{1}{2}(-p_2) \in \mathbb{Z}^n.$$

Thus we have found the desired vector. □

Corollary 7.8. *Let V be a free $\mathbb{Z}$-module and let s be an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$. There is a nontrivial vector $v \in V$ of length $\leq 2(n+1)\sqrt[n]{\mathrm{vol}_V(s)}$.*

Proof. The standard ball in $\mathbb{R}^n$ of radius $2(n+1)\sqrt[n]{\mathrm{vol}_V(s)}$ has volume

$$\begin{aligned} &= \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \cdot \left(2(n+1)\sqrt[n]{\mathrm{vol}_V(s)}\right)^n \\ &\geq \pi^{\frac{n}{2}} \cdot 2^n \,\mathrm{vol}_V(s) \\ &> 2^n \,\mathrm{vol}_V(s). \end{aligned}$$

Here I used that $\Gamma(\frac{n}{2}+1) \leq (n+1)^n$ for $n \geq 1$. This is true for $n = 1, 2$ and can be shown inductively using the functional equation $\Gamma(z+1) = z\Gamma(z)$ of the gamma function.

A choice of a basis of V gives an $\mathbb{R}$-linear isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R} \otimes_{\mathbb{Z}} V$ mapping $\mathbb{Z}^n$ to $1 \otimes_{\mathbb{Z}} V$. If we equip $\mathbb{R}^n$ with the standard inner product this map scales the volume with the factor $\mathrm{vol}_V(s)$. Let $R := 2(n+1)\sqrt[n]{\mathrm{vol}_V(s)}$. So $\mu(B_R(0)) > 2^n \cdot \mathrm{vol}_V(s)$, where $\mu(-)$ denotes the Lebesgue measure on $(\mathbb{R} \otimes_{\mathbb{Z}} V, s)$. Consider the set $S := \varphi^{-1}(B_R(0))$. It is a convex set that is symmetric with respect to the origin as φ is linear. Its volume is at least 2^n. Thus we can apply Minkowski's theorem to conclude that there is a vector of $v' \in \mathbb{Z}^n$ contained in S. But this means that $v := \varphi(v') \in \varphi(\mathbb{Z}^n) = V$ is contained in $B_R(0)$, i.e. $\mathrm{vol}_{\langle v \rangle}(s) < R = 2(n+1)\sqrt[n]{\mathrm{vol}_V(s)}$. □

Lemma 7.9 (Existence of a short basis)**.** *Let V be a free $\mathbb{Z}$ module of rank n and s an inner product on $\mathbb{R} \otimes_{\mathbb{Z}} V$ with $\mathrm{vol}_V(s) = 1$. Suppose that there is a constant $C \in \mathbb{R}$, $C \geq 0$ such that $c_W(s) \leq C$ for any direct summand W of V. Then there is a basis $v_1, \ldots, v_n$ of V such that*

$$\mathrm{vol}_{\langle v_i \rangle}(s) \leq 2^i \cdot (n+1)e^{Cn^2}.$$

Proof. First let us find a lower bound on the volumes of all submodules $W \subset V$. Let a_i denote the slope in the canonical filtration for (V, s) from rank i to rank $i+1$. Then $\sum_{i=0}^{n-1} a_i = \ln \mathrm{vol}_V(s) = 0$ and $0 \geq a_{i+1} - a_i \leq C$.

Thus there is an index j with $a_i < 0$ for $i \leq j$ and $a_i \geq 0$ for $i \geq j$. Not all of the a_i's can be positive since their sum is zero. So $a_j \in [0, C]$ and since $|a_i - a_j| \leq C \cdot |i - j|$ we see that $a_i \in [-Cn, (C+1)n]$. The logarithm of the smallest volume of a rank k subgroup is bigger than the y-coordinate of the canonical path at k. This is just $\sum_{i=0}^{k} a_i \geq -Cn^2$. Thus we have found the desired bound.

An upper bound for the length of a shortest nontrivial vector is given by the previous corollary. Let us now construct inductively a basis $v_1, \ldots, v_n$ with the desired properties. Assume we have already constructed $v_1, \ldots, v_{m-1}$ such that their span is a direct summand of V. Let us choose a rank m-submodule V' that contains $v_1, \ldots, v_{m-1}$ of minimal volume. Since $V'/\langle v_1, \ldots, v_{m-1} \rangle$ is a rank one subspace of the finitely generated free module $V/\langle v_1, \ldots, v_{m-1} \rangle$ we can complete $v_1, \ldots, v_{m-1}$ to a basis of V' by some vector $v'_m \in V$. Using Lemma 4.4 we get

$$\mathrm{vol}_{V'}(s) = \mathrm{vol}_{\langle v_1, \ldots, v_{m-1} \rangle}(s) \cdot \mathrm{vol}_{[v'_m]}(s'),$$

where $[v'_m]$ denotes the class of v'_m in $V/\langle v_1, \ldots, v_{m-1} \rangle$. Hence we have to find a smallest vector in $(V/\langle v_1, \ldots, v_{m-1} \rangle, s')$. By the last corollary there exists a v'_m such that

$$\begin{aligned} \mathrm{vol}_{[v'_m]}(s') &\leq 2(n+1) \sqrt[n-m+1]{\mathrm{vol}_{V/\langle v_1, \ldots, v_{m-1} \rangle}(s')} \\ &\leq \frac{2(n+1)}{\sqrt[n-m+1]{\mathrm{vol}_{\langle v_1, \ldots, v_{m-1} \rangle}(s)}} \\ &\leq \frac{2(n+1)}{e^{\frac{-Cn^2}{n-m+1}}}. \end{aligned}$$

The last estimation used the lower bound e^{-Cn^2} on the volume of any submodule from above. By definition $\mathrm{vol}_{[v'_m]}(s')$ is the length of the orthogonal projection $\mathrm{pr}(v'_m)$ onto the orthogonal complement of $\langle v_1, \ldots, v_{m-1} \rangle$. Since $\mathrm{pr}(v'_m) - v'_m \in \ker(\mathrm{pr}) = \langle v_1, \ldots, v_{m-1} \rangle_{\mathbb{R}}$ we can write it in the form $\sum_{i=1}^{m-1} \mu_i v_i$. Let us consider the vector

$$v_m := v'_m + \sum_{i=1}^{m-1} \lfloor \mu_i \rfloor v_i.$$

We get

$$\begin{aligned}
|v_m| &\le |p(v'_m)| + \sum_{i=1}^{m-1} |v_i| \\
&\le \frac{2(n+1)}{e^{\frac{-Cn^2}{n-m+1}}} + \sum_{i=1}^{m-1} |v_i| \\
&\le 2(n+1)e^{Cn^2} + \sum_{i=1}^{m-1} |v_i| \\
&\le 2(n+1)e^{Cn^2}(1 + \sum_{i=0}^{m-2} 2^i) \\
&\le 2(n+1)e^{Cn^2} \cdot 2^{m-1} \\
&= 2^m(n+1)e^{Cn^2}.
\end{aligned}$$

Clearly $[v_m]$ spans a direct summand in $V/\langle v_1, \ldots, v_{m-1}\rangle$. The structure theorem tells us that otherwise v_m would be of the form λv with $\lambda \notin \{-1, 0, 1\}$. Hence $\langle v_1, \ldots, v_{m-1}, v\rangle$ would have smaller volume than V', which contradicts the choice of V'. □

Remark 7.10. The condition on c_W is necessary as the inner product

$$(v, w) \mapsto \langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \rangle$$

for $\lambda \gg 0$) shows. Any basis will have a vector of length $\ge \sqrt{\lambda}$.

Proposition 7.11. *Let V be a free $\mathbb{Z}$-module of rank n. Let $C \ge 0$ be given. Then the group action of* $\operatorname{aut}_{\mathbb{Z}}(V)$ *on the space*

$$\{s \in \tilde{X}(V) \mid \operatorname{vol}_V(s) = 1, c_W(s) \le C \textit{ for any nontrivial direct summand } W \subset V\}$$

is cocompact. Note that this space is $\operatorname{aut}_{\mathbb{Z}}(V)$*-equivariantly diffeomorphic to*

$$\{s \in X(V) \mid c_W(s) \le C \textit{ for any nontrivial direct summand } W \subset V\}.$$

Hence the $\operatorname{aut}_{\mathbb{Z}}(V)$*-action on this space is also cocompact.*

Proof. After choosing a basis $b_1, \ldots, b_n$ for V we can think of the space of isomorphisms $\mathbb{R} \otimes_{\mathbb{Z}} V \to \mathbb{R}^n$ as $\operatorname{GL}_n(\mathbb{R})$. We can assign to any such isomorphism the pullback of the standard inner product on $\mathbb{R}^n$. This defines a continuous map $\operatorname{GL}_n(\mathbb{R}) \to \tilde{X}(V)$. Let $\varepsilon := e^{-Cn^2}/(2^n(n+1)e^{Cn^2})^n$. Consider the compact set K of those matrices $A \in \operatorname{GL}_n(\mathbb{R})$ whose i-column a_i lies in the compact set $\overline{B}_{2^n(n+1)e^{Cn^2}} \setminus B_\varepsilon(\langle a_0, \ldots, a_{i-1}\rangle)$. By the previous lemma there is for any $s \in \tilde{X}(V)$ with $\operatorname{vol}_V(s) = 1$ and $c_W(s) \le C$ for any submodule $W \subset V$ a basis $v_1, \ldots, v_n$ of V such that each basis vector has length at most $2^n(n+1)e^{Cn^2}$. Furthermore we have seen in the proof of the previous lemma that any submodule $W \subset V$ has volume at most e^{-Cn^2}. Hence the projection of v_m on the orthogonal complement of $\langle v_1, \ldots, v_{m-1} \rangle$ has the length

$$\frac{\mathrm{vol}_{v_1,\ldots,v_m}(s)}{\mathrm{vol}_{v_1,\ldots,v_{m-1}}(s)} \leq \frac{e^{-Cn^2}}{(2^n(n+1)e^{Cn^2})^n} =: \varepsilon.$$

If we let $\varphi \in \mathrm{aut}_{\mathbb{Z}}(V)$ be the automorphism mapping b_i to v_i we see that $s \cdot \varphi \in K$. Thus the group operation is cocompact as desired. □

Proposition 7.12. *The space $X(V)$ satisfies all assumptions from Proposition 2.4. Let*

$$\mathcal{W} := \{\{x \in X(V) \mid c_W(x) > 0\} \mid W \subset V \textit{ is a nontrivial direct summand}\}.$$

This is a collection of open sets as the map $c_W : X(V) \to \mathbb{R}$ is continuous. We have

(i) *$X(V)$ is a proper* CAT(0) *space,*

(ii) *the covering dimension of $X(V)$ is less or equal to $\frac{(n+1)n}{2} - 1$,*

(iii) *the group action of $\mathrm{aut}_{\mathbb{Z}}(V) \cong \mathrm{GL}_n(\mathbb{Z})$ on X is proper and isometric,*

(iv) $\mathrm{aut}_{\mathbb{Z}}(V) \cdot \mathcal{W} := \{gW \mid g \in \mathrm{aut}_{\mathbb{Z}}(V), W \in \mathcal{W}\} = \mathcal{W}$,

(v) *gW and W are either disjoint or equal for all $g \in \mathrm{aut}_{\mathbb{Z}}(V), W \in \mathcal{W}$,*

(vi) *the dimension of $\mathcal{W}$ is less or equal to $n - 2$.*

(vii) *the $\mathrm{aut}_{\mathbb{Z}}(V)$ operation on*

$$X \setminus (\bigcup \mathcal{W}^{-\beta}) := \{x \in X \mid \nexists W \in \mathcal{W} : \overline{B}_\beta(x) \subset W\}$$

is cocompact for every $\beta \geq 0$.

Proof. (i) See for example [11, Chapter II Theorem 10.39].

(ii) After choosing a basis for V we can identify the space $X(V)$ with the set of positive definite, symmetric $n \times n$ matrices of determinant one. This is a Riemannian manifold of dimension $\frac{(n+1)n}{2} - 1$. Its covering dimension is at most $\frac{(n+1)n}{2} - 1$ by [23, Corollary 50.7].

(iii) This has been shown in Lemma 7.2 and Lemma 7.3.

(iv) Pick an element $g \in \mathrm{aut}_{\mathbb{Z}}(V)$ and an open set $U \in \mathcal{W}$. It has the form $U = \{x \in X(V) \mid c_W(x) > 0\}$ for a nontrivial direct summand $W \subset V$. We get $c_W(s \cdot g) = c_{gW}(s)$ (see Remark 7.1) and hence $gU = \{x \in X(V) \mid c_{gW}(x) > 0\} \in \mathcal{W}$.

(v) Assume $x \in gU \cap U$ for some $U \in \mathcal{W}, g \in \mathrm{aut}_{\mathbb{Z}}(V)$. Thus $c_W(x) > 0$ and $c_{gW}(x) > 0$. By Corollary 3.6 this means that W, gW are both contained in the canonical filtration. And since they have the same rank they have to be equal.

(vi) Suppose $x \in \bigcap_{i=1}^m U_i$ for some $U_i \in \mathcal{W}$. Then U_i can be written as $\{x \in X(V) \mid c_{W_i}(x) > 0\}$ for some nontrivial direct summands $(W_i)_{i=1...,m}$. Hence they all have to occur in the canonical filtration. The canonical filtration can have at most one module for each rank between one and $n-1$. Thus $m \le n-1$. So the dimension of $\mathcal{W}$ is at most $n-2$.

(vii) Let us show that $X(V) \setminus (\bigcup \mathcal{W}^{-\beta})$ is a closed subset of a cocompact set. We have already shown in (*iv*) that it is G-invariant. By Lemma 7.6 it is a closed subset of $X(V)$. By Corollary 7.5 we know that each function c_W is $4n$-Lipschitz. Hence

$$\begin{aligned} & X \setminus (\bigcup \mathcal{W}^{-\beta}) \\ \subset \quad & \{x \in X(V) \mid c_W(x) \le 4n\beta \text{ for each nontrivial direct summand } W \subset V\}. \end{aligned}$$

The group operation on the right hand side is cocompact by Proposition 7.11. Hence the group operation on the closed subset $X \setminus (\bigcup \mathcal{W}^{-\beta})$ is also cocompact. □

7.2 Preliminaries about affine buildings

Most of this subsection can be found in [16]. Basics about Euclidean simplicial complexes or more generally about M_k-polyhedral complexes can be found in [11, Chapter I.7].

Let us begin with some preliminaries about affine buildings. Let O be a discrete valuation ring with fractional field k. Let m be the unique nonzero prime ideal of O and let κ denote its residue field O/m. Let t be a generator of m. Let V be an n-dimensional vector space over k.

A homothety is a k-linear map of the form

$$V \to V \qquad v \mapsto \lambda v$$

for some $\lambda \in k \setminus \{0\}$. Two O-lattices $L_1, L_2 \subset V$ are homothetic if there is a homothety $f : V \to V$ with $f(L_1) = L_2$. Being homothetic is an equivalence relation and we write $[L_1]$ for the homothety class of L_1.

Now we can consider a simplicial complex whose vertex set is the set of all homothety classes of O-lattices in V and where a sequence $[L_1], \dots, [L_m]$ of equivalence classes spans a simplex if there are representatives such that

$$L_1 \subset L_2 \subset \ldots \subset L_m \subset t^{-1} L_n.$$

Lemma 7.13. *The set of neighbors of a vertex $[L]$ can be identified with the set of κ-subspaces of the n-dimensional κ-space $t^{-1}L/L$.*

Especially if κ is finite the complex $X(V)$ is locally finite. This condition is automatically satisfied for $k = \mathbb{Q}$ or $k = \mathrm{Quot}(F[t])$ for a finite field F.

Proof. By definition $m \cdot t^{-1}L = (t) \cdot t^{-1}L = L$ and hence $t^{-1}L/L$ has the structure of a κ-module. Any isomorphism $L \cong O^n$ induces $t^{-1}L/L \cong t^{-1}O^n/O^n \cong (O/tO)^n = \kappa^n$.

For two adjacent vertices $[L]$ and $[L']$ and a representative L of $[L]$ we can find a unique representative L' of $[L']$ such that $L \subset L' \subset t^{-1}L$. Assigning to it the κ-subspace $L'/L \subset t^{-1}L/L$ gives the desired bijection. □

Definition 7.14. We can furthermore label the vertices with elements in $\mathbb{Z}/n$. Let us first pick a base vertex $[L]$ with a representative L. Since $\bigcup_{n\in\mathbb{N}} t^{-n}L' = V$ we find an n such that $t^{-n}L'$ contains all generators of L. By changing the representative L' we thus may assume that $L \subset L'$.

Define the label of $[L']$ to be $l([L']) := \dim_\kappa(L'/L) \mod n$. We can check that this labeling does not depend on the choice of representatives. Furthermore it can also be expressed as the valuation of the determinant of a base change matrix from an O-basis of L to an O-basis of L'.

The difference between the labeling $([L''], [L']) \mapsto l([L']) - l([L''])$ is even independent of the choice of the base vertex. It can be expressed as $\dim_\kappa(L'/L'') \mod n$ where L', L'' are representatives of $[L'], [L'']$ with $L'' \subset L'$.

For an edge e with endpoints $[L], [L']$ let the label difference of e denote $\pm(l([L']) - l([L'']))$ in the set $(\mathbb{Z}/n)/x \sim -x$.

Lemma 7.15. *An edge e of label difference k is contained in*

$$\prod_{i=1}^{k} \frac{r^i - 1}{r - 1} \cdot \prod_{i=1}^{n-k} \frac{r^i - 1}{r - 1}$$

$n - 1$-dimensional simplices. The number r denotes the cardinality of κ. Especially the label differences of two edges with isomorphic links are equal.

Proof. Let L, L' be representatives of the endpoints of the e with $L \subset L' \subset t^{-1}L$. Then $\dim_\kappa(L/L') \in \{k, n-k\}$. So we can assume that this dimension is k. Otherwise replace L by L' and L' by $t^{-1}L$. Each such $(n-1)$-dimensional simplex then corresponds to a flag of the form

$$L \subset L_1 \subset \ldots \subset L_{k-1} \subset L' \subset L_{k+1} \subset \ldots L_{n-1} \subset t^{-1}L.$$

By dividing L out each such flag corresponds to a flag

$$0 \subset V_1 \subset \ldots \subset V_{k-1} \subset L'/L \subset V_{k+1} \subset \ldots V_{n-1} \subset (t^{-1}L)/L.$$

of the n-dimensional κ-vector space $(t^{-1}L)/L$ containing $V_k := L'/L$. Assume we already picked V_i and we want to pick V_{i+1} for $i + 1 < k$. So we have to pick a vector $v_{i+1} \in V_k$ that does not lie in V_i. There are $p^k - p^i$ choices for such a vector. And two vector yield the same vector space $V_{i+1} := \langle v_{i+1}, V_i \rangle$ if they differ multiplicatively by a unit in κ and additively by some element of V_i. So there are $\frac{r^k - r^i}{r^i(r-1)} = \frac{r^{k-i}-1}{r-1}$ such choices possible. The analogous argument holds for $i \geq k$ and yields

$$\prod_{i=0}^{k-1} \frac{r^{k-i} - 1}{r - 1} \cdot \prod_{i=0}^{n-k-1} \frac{r^{n-k-i} - 1}{r - 1}.$$

A final substitution yields the desired result. Now assume that $k \leq n/2$. Thus $k \leq n-k$. Let

$$f(k) := \prod_{i=0}^{k-1} \frac{r^{k-i}-1}{r-1} \cdot \prod_{i=0}^{n-k-1} \frac{r^{n-k-i}-1}{r-1}.$$

We have

$$\frac{f(k-1)}{f(k)} = \frac{r^{n-k+1}-1}{r^k - i} > 1.$$

Thus f is monotonically decreasing on $1, \ldots, \lfloor \frac{n}{2} \rfloor$. This is a complete system of representatives of $(\mathbb{Z}/n)/x \sim -x$. So the induced map $(\mathbb{Z}/n)/x \sim -x \to \mathbb{N}$ is injective. This proves the last claim. □

A Euclidean n-simplex is the convex hull of $n+1$ points in $\mathbb{R}^n$ in general position. An Euclidean simplicial complex is a simplicial complex where any simplex carries additionally the structure of an Euclidean simplex. This means that we can identify the vertices of the simplex with the vertices of the given Euclidean simplex. Furthermore the inclusions of the faces are required to be isometries. See [11, Chapter I, Definition 7.2] for the precise definition.

Let us recall the definition of a building as given in [11, Chapter I Definition 10A.1]. It is not the usual definition of an affine building; for example it already requires a metric.

Definition 7.16. A Euclidean building of dimension $n-1$ is a piecewise Euclidean simplicial complex X such that:

(i) X is the union of a collection $\mathcal{A}$ of subcomplexes E, called apartments, such that the intrinsic metric d_E on E makes (E, d_E) isometric to the Euclidean space E^n and induces the given Euclidean metric on each simplex. The $n-1$-simplices of E are called its chambers.

(ii) Any two simplices B and B' of X are contained in at least one apartment.

(iii) Given two apartments E and E' containing both the simplices B and B', there is a simplicial isometry from (E, d_E) onto $(E', d_{E'})$ which leaves both B and B' pointwise fixed.

The building X is called thick if the following extra condition is satisfied:

(iv) Thickness Condition: Any $(n-2)$-simplex is a face of at least three $n-1$-simplices.

Up to now the affine building is just a simplicial complex. We can furthermore equip the simplicial complex with the structure of an Euclidean simplicial complex. But first we need a preliminary lemma:

Lemma 7.17. *Every $x \in \mathbb{R}^n$ can be written uniquely as a convex combination*

$$x = \sum_{i=0}^{m} \mu_i p_i$$

with

- $p_i \in \mathbb{Z}^n$,
- $0 < \mu_i \leq 1$
- $\sum_{i=0}^{m} \mu_i = 1$,
- $p_0 < \ldots < p_m \leq p_0 + (1, \ldots, 1)$, *where* $a \leq b$ *if and only if* $a_i \leq b_i$ *for all* i. *Especially this implies* $m \leq n$.

Proof. This triangulation of $\mathbb{R}^n$ is obtained from the tesselation with cubes by a certain subdivision into simplices.

Let me just give a sketch of the proof. The statement is trivial for $n = 0$. So let $x \in \mathbb{R}^n$ be given. Without loss of generality we can assume that $\lfloor x_i \rfloor = 0$ for all i. By permuting the coordinates we can assume that

$$1 > x_1 \geq \ldots \geq x_n \geq 0.$$

Now let m be the number of different entries of x and let $(x'_1, \ldots, x'_m)$ be obtained from x by leaving out coordinates that occur twice. Let $\chi_{>x'_i}$ be the characteristic function

$$y \mapsto \begin{cases} 1 & y > x_i \\ 0 & \text{else} \end{cases}$$

and let $p_i := \chi_{>x'_i}$. Then x can be written as a convex combination of the p_i. Conversely if you know that x can be written as a convex combination of a totally ordered subset of $\{0, 1\}^n$ with nonzero coefficients you can read off that subset by comparing the coordinates of x. □

Remark 7.18. The convex combination for $x + \lambda(1, \ldots, 1)$ can be obtained from the convex combination for x in the following way. Let us assume without loss of generality that λ is positive; otherwise swap the roles. Make the coefficient of p_0 smaller and increase the coefficient of $p_0 + (1, \ldots, 1)$ correspondingly until the coefficient of p_0 becomes zero. Then p_1 is the smallest element from $\mathbb{Z}^n$ needed and we can continue this way: Now decrease the coefficient of p_1 and increase the coefficient of $p_1 + (1, \ldots, 1)$.

Now we are ready to define the metric on the affine building:

Proposition 7.19. *The affine building* X *has the following properties:*

(i) *For each basis* $b_1, \ldots, b_n$ *of* V *we can consider the full subcomplex* X' *spanned by all vertices of the form* $[t^{m_1} b_1, \ldots, t^{m_n} b_n]$ *for* $m \in \mathbb{Z}^n$. *This will be an apartment of the building.*

   We can map such a vertex to $\operatorname{pr}(m_1, \ldots, m_n) \in \mathbb{R}^n$, *where* $\operatorname{pr} : \mathbb{R}^n \to \langle (1, \ldots, 1) \rangle^\perp$ *denotes the orthogonal projection with respect to the standard inner product on* $\mathbb{R}^n$. *The linear extension* $f : X' \to \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ *of this map is a bijection. We can pull the metric on* $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ *back to each simplex to obtain an Euclidean simplicial complex.*

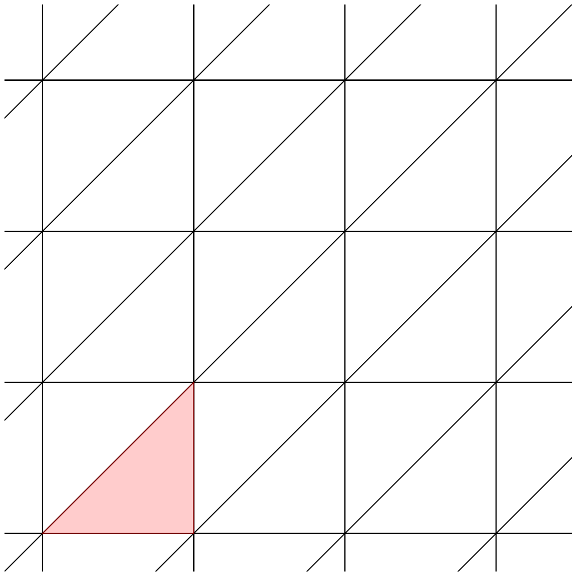

Figure 7.1: The tesselation of $\mathbb{R}^2$. The marked simplex corresponds to the chain $(0,0) < (1,0) < (1,1)$.

(ii) *The length of an edge in X' depends only on the label difference of its endpoints.*

(iii) *If a simplex is contained in two apartments we get the same metric on that simplex.*

(iv) *A simplicial automorphism $g : X \to X$ is an isometry.*

(v) $\mathrm{aut}_k(V)$ *acts isometrically on X.*

(vi) *Any two simplices are contained in at least one apartment.*

(vii) *Given two apartments E and E' containing both the simplices B and B', there is a simplicial isometry from (X, d_X) onto $(X', d_{X'})$ which leaves both B and B' pointwise fixed.*

(viii) *The affine building X is a* CAT(0) *space.*

Proof. (i) We have to show that each point $p \in \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ lies in the image of a unique open simplex. Let us first we can apply Lemma 7.17 to write it as a convex combination of certain points $p_1, \ldots, p_m$ of $\mathbb{Z}^n$: $\sum_{i=1}^m \mu_i \cdot p_i = p = \mathrm{pr}(p) = \sum_{i=1}^m \mu_i \cdot \mathrm{pr}(p_i)$. Hence p lies in the convex hull of the points $(f([t^{p_{i,1}} b_1, \ldots, t^{p_{i,n}} b_n]))_{i=1\ldots m}$. The conditions on p_i from Lemma 7.17 mean exactly that the vertices $[t^{p_{i,1}} b_1, \ldots, t^{p_{i,n}} b_n]$ span a simplex. Uniqueness follows from Remark 7.18. So f is really a continuous bijection. Since f is proper it is a homeomorphism.

The images vertices of each simplex are in general position since otherwise there would be a point that can be written as a convex combination of those vertices in two different ways which we have already ruled out. So one obtains the structure of an Euclidean simplicial complex.

The space $\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$ is a convex subset of $\mathbb{R}^n$. Hence the restriction of the standard metric to it is inner. The metric on the realization on an Euclidean simplicial complex is the unique inner metric whose restriction to each simplex agrees with the metrics given on it. So the realization of X' is really isometric to $\mathbb{R}^n$.

(ii) Let $e \in X'$ be any edge. Pick representatives

$$\langle t^{m_1} b_1, \ldots, t^{m_n} b_n\rangle \text{ and } \langle t^{m'_1} b_1, \ldots, t^{m'_n} b_n\rangle$$

of its endpoints p, p' with

$$\langle t^{m_1} b_1, \ldots, t^{m_n} b_n\rangle \subset \langle t^{m'_1} b_1, \ldots, t^{m'_n} b_n\rangle \subset t^{-1}\langle t^{m_1} b_1, \ldots, t^{m_n} b_n\rangle.$$

This means exactly that m'_i is either $m_i - 1$ or m_i. Note that

$$\dim_\kappa(\langle t^{m'_1} b_1, \ldots, t^{m'_n} b_n\rangle / \langle t^{m_1} b_1, \ldots, t^{m_n} b_n\rangle) = \sum_{i=1}^n m'_i - m_i.$$

Now we can consider the distance between $f(p)$ and $f(p')$. It is $\| \operatorname{pr}(m'_i - m_i)\|$. The length of a vector whose entries are either zero or one depends only on the number r of ones. Not all entries can be simultaneously zero (or one) since then the endpoints of e would be the same. This is impossible in a simplicial complex. But we now the residue $r \bmod n$ is just the label difference of the vertices. Since the desired number must be at least one and can be at most $n - 1$ this determines r. So length of an edge depends only on its label difference.

(iii) The metric on an Euclidean simplex is uniquely determined by the length of its edges. As shown before the length of an edge depends only on the label difference and not on the choice of some apartment.

(iv) Each simplicial automorphism of X preserves the label difference by Lemma 7.15. Thus it is an isometry.

(v) $\varphi \in \operatorname{aut}_k(V)$ preserves the label difference since for two lattices L, L' with $L \subset L' \subset t^{-1}L$ we have $\varphi(L) \subset \varphi(L') \subset t^{-1}\varphi(L)$ and $\varphi(L'/L) \cong L'/L$.

(vi) The proof can be found in [16, chapter 19, p. 289].

(vii) The proof goes as in [16, chapter 19, p. 290]. The automorphism constructed there is simplicial. Hence it is an isometry from one apartment to the other by the same argument as above.

(viii) [11, Chapter I Theorem 10A.4(ii)].

□

We need the following lemma to deal with the properness of the affine building.

Lemma 7.20. *A locally compact, complete, inner metric space is proper.*

Proof. Let (X, d) be a locally compact, complete, inner metric space. Assume X is not proper. Then there is an $x \in X$ and an $R \in \mathbb{R}$ such that $B_R(x)$ is not compact. Define a function

$$f : X \to \mathbb{R} \qquad x \mapsto \inf\{R \in [0, \infty) \mid B_R(x) \text{ is not compact}\}.$$

The set is nonempty for any $y \in X$, since $B_R(x)$ is a closed subset of $B_{R+d(x,y)}(y)$. So this function is well defined. The same argument shows that it is 1-Lipschitz and hence continuous. Since the space is locally compact there is for each $y \in X$ a number $\varepsilon \in R$ such that $B_\varepsilon(x)$ is compact. For $\delta < \varepsilon$ the set $B_\delta(x)$ is a closed subset of $B_\varepsilon(x)$ and hence compact. So $f(y) > 0$. So f is bigger than zero everywhere.

Now we want to construct a Cauchy-sequence of points x_i such that $\lim_{i\to\infty} f(x_i) = 0$. Then $\lim_{i\to\infty} x_i$ exists by completeness and the function value at this point has to be zero by continuity. This gives the desired contradiction. The existence of such a sequence follows from the following lemma. □

Lemma 7.21. *For any $x \in X$ there is a point y with $d(x, y) = f(x)/2$ and $f(y) \leq \frac{3}{4} f(x)$.*

Proof. Pick any $x \in X$ and let $R := f(x), \varepsilon := \frac{1}{12}R$. Then $B_{\frac{R}{2}}(x)$ is compact by definition of f and $B_{R+\varepsilon}(x)$ is not compact. So there is a sequence of points $z_i \in B_{R+\varepsilon}(x)$ for $i \in \mathbb{N}$ without an accumulation point. Since $B_{\frac{R}{2}}(x)$ is compact there can be only finitely many of the $z_i's$ in $B_{\frac{R}{2}}(x)$. By leaving them out we can assume that none of the $z_i's$ lie in $B_{\frac{R}{2}}(x)$. Because the metric space is inner we can choose a path from x to z_i of length at most $R + 2\varepsilon$. Let y_i be a point on this path that lies on $\partial B_{\frac{R}{2}}(x)$. All the $y_i's$ have an accumulation point y since they are contained in the compact set $B_{\frac{R}{2}}(x)$. We get

$$d(x, y_i) = \frac{R}{2}, \qquad d(z_i, y_i) \leq \frac{R}{2} + 2\varepsilon.$$

By again restricting to a subsequence we can assume that $d(y, y_i) < \varepsilon$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} y_i = y$. Then $(z_i)_{i\in\mathbb{N}}$ is a sequence in $B_{\frac{R}{2}+3\varepsilon}(y)$ that does not have an accumulation point; since $d(z_i, x) \leq R + \varepsilon$ any accumulation point would also lie in $R + \varepsilon$. Hence $B_{\frac{R}{2}+3\varepsilon}(y) = B_{\frac{3}{4}R}(y)$ is not compact and so $f(y) \leq \frac{3}{4}R$. □

Corollary 7.22. *The affine building X is a proper metric space if the local field κ is finite.*

Proof. We want to use Lemma 7.20. The metric on the affine building is defined to be the inner metric induced by the Euclidean structure on the simplices. If κ is finite, the simplicial complex is locally finite and hence locally compact. Furthermore the metric space is complete as mentioned above and shown in [11, Chapter I Theorem 7.13]. □

Furthermore we need another property of Euclidean simplicial complexes.

Proposition 7.23. *Let X be an Euclidean simplicial complex with finitely many isometry types of simplices. Fix any $C \in \mathbb{R}$. Then there is a $C' \in \mathbb{R}$ such that the linear extension $\overline{f}$ of any function $f : X^{(0)} \to \mathbb{R}$ with the property that $|f(x) - f(y)| \leq C$ for any two adjacent vertices $x, y \in X^{(0)}$ is C'-Lipschitz.*

Proof. Pick a simplex s and let n be its dimension. Each simplex can be isometrically embedded into Euclidean space. Divide the set of its vertices of s into two sets. Assign the value zero to all vertices of the first set and the value C to all vertices in the second set. Since the vertices of s are in general position there is a unique affine function $f : \mathbb{R}^n \to \mathbb{R}$ extending this map. We want to compute its gradient.

Put two parallel planes in $\mathbb{R}^n$ such that all vertices with value zero lie on the first plane and all vertices with value C lie on the other plane. The planes are levelsets of f and so the gradient of f is orthogonal to that plane. Let d denote the distance from one plane to the other. Then the gradient has length C/d. Note that $d > 0$ since the vertices are in general position. And so they do not lie on an $n-1$-dimensional plane.

So we got for any isometry class of a simplex and any division of the vertices into two sets a number C/d. Let C' be the maximum of all those numbers varying over all isometry types of simplices in X and over all subdivisions of the vertex set. Now we have to verify that $\overline{f}$ is C' -Lipschitz.

Any map from the vertices of a simplex to $[0, C]$ is a convex combination of maps to $\{0, C\}$. Consequently its linear extension to $\mathbb{R}^n$ is also a convex combination of the linear extensions above: So the length of the gradient is bounded by the lengths of the gradients of functions of the upper form. They are bounded by C'.

Let us now consider the general case. Assume there is a map $f : X^0 \to \mathbb{R}$ with $|f(x, y)| \leq C$. We have to show that its linear extension is C'-Lipschitz.

Given any two points x, y in the realization of X there is a geodesic $\gamma : [0, d(x, y)] \to |X|$ connecting them by [11, Chapter I Theorem 7.19]. Furthermore there are $t_0, \ldots, t_m \in [0, d(x, y)]$ such that

- $0 = t_0 \leq \ldots \leq t_m = d(x, y)$,
- $\gamma|_{[t_i, t_{i+1}]}$ is contained in a simplex.

Hence we get by the previous case where we considered only a single simplex

$$|f(\gamma(t_i)) - f(\gamma(t_{i+1}))| \leq C' \cdot d(\gamma(t_i), \gamma(t_{i+1})).$$

Now we use that γ is a geodesic to get

$$|f(\gamma(t_i)) - f(\gamma(t_{i+1}))| \leq C' \cdot d(\gamma(t_i), \gamma(t_{i+1})),$$

$$|f(x) - f(y)| \leq C' \cdot \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) = C' \cdot d(x, y).$$

□

7.3 $\mathrm{GL}_n(F[t])$ acts on a building

Let F be a finite field and let V be an n-dimensional free $F[t]$-module. The group $\mathrm{aut}_{F[t]}(V) \cong \mathrm{GL}_n(F[t])$ acts on the affine building $X(V)$ associated to the valuation v with

$$v(\frac{f}{g}) := \deg(g) - \deg(f) \qquad \text{for } \frac{f}{g} \in Q := \mathrm{Quot}(F[t]).$$

It is a simplicial complex whose vertex set consists of all homothety classes of R-lattices in Q^n where R denotes the valuation ring with respect to this valuation. A generator for the maximal ideal in R of v is given by $\frac{1}{t}$. Consequently we get from section 7.2 that a subset $[S_1], \ldots, [S_m]$ spans a simplex if and only if there are representatives with $S_1 \subset S_2 \subset \ldots \subset S_m \subset tS_1$.

The goal of this section is to show that this space satisfies all assumptions from Proposition 2.4.

Lemma 7.24. *The affine building $X(V)$ has the following properties*

(i) The group action of $\mathrm{aut}_Q(Q \otimes_{F[t]} V)$ *is simplicial.*

(ii) The group action of $\mathrm{aut}_{F[t]}(V)$ *on $X(V)$ is proper.*

Proof. (i) Let $\varphi \in \mathrm{aut}_Q(Q \otimes_{F[t]} V)$. The vertex set is equipped with a well defined $\mathrm{aut}_Q(Q \otimes_{F[t]} V)$ action via $(\varphi, [S]) \mapsto [\varphi(S)])$.

Let $x_1, \ldots, x_m$ be a set of vertices that span a simplex. So there are representatives $S_1, \ldots, S_m$ with $S_1 \subset S_2 \subset \ldots \subset S_m \subset tS_1$. Hence $\varphi(S_1) \subset \varphi(S_2) \subset \ldots \subset \varphi(S_m) \subset \varphi(tS_1) = t\varphi(S_1)$ and so $[\varphi(x_1)], \ldots, [\varphi(x_m)]$ also span a simplex.

(ii) The group action is the restricted action under

$$\mathrm{aut}_{F[t]}(V) \hookrightarrow \mathrm{aut}_Q(Q \otimes_{F[t]} V) \qquad \varphi \mapsto 1 \otimes \varphi.$$

Let us now consider the stabilizer of some $[S] \in Y(V)$. Let $\varphi \in \mathrm{aut}_{F[t]}(V)$ be any automorphism with $\varphi(S) = t^m S$ for some m. So $\det(\varphi) = t^m n$. But the determinant has to be invertible in $F[t]$, so $m = 0$. So the stabilizer of a vertex agrees with the stabilizer of any representative. Pick an $F[t]$-basis $v_1, \ldots, v_n$ of V and let $R := \max_i \log \mathrm{vol}_{\langle v_i \rangle_{F[t]}}(S)$. If φ stabilizes S it will preserve the volumes of those subspaces:

$$\log \mathrm{vol}_{\langle \varphi(v_i) \rangle_{F[t]}}(S) = \log \mathrm{vol}_{\langle \varphi(v_i) \rangle_{F[t]}}(\varphi(S)) = \log \mathrm{vol}_{\langle v_i \rangle_{F[t]}}(S).$$

But the set

$$M := \{v \in V \mid \log \mathrm{vol}_{\langle v \rangle_{F[t]}}(S) \leq R\}$$

is finite by Corollary 5.20. Since φ is uniquely determined by the images of $v_1, \ldots, v_n$ we inject $\mathrm{stab}(S)$ into M^n. Hence the stabilizer of S must be finite. A group action on a simplicial complex with finite vertex stabilizers is proper by [20, Theorem 1.23]. □

Remark 7.25. Nevertheless there is no bound on the order of the stabilizers. Consider the R-lattice $S := \langle(1,0),(0,t^m)\rangle \subset Q^2 = Q \otimes_{F[t]} F[t]^2$ for some $m \in \mathbb{Z}$. We see that $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ stabilizes S whenever $\deg(x) \leq m$. So the stabilizers get arbitrarily large if we choose m bigger and bigger. Especially this also shows that the group action of $\mathrm{aut}_{F[t]}(V)$ on $X(V)$ is not cocompact for $V \cong F[t]^2$.

We want to use the volume function from section 5 resp. the function c_W from section 3 the to construct certain open subsets. We can associate to any R-lattice $S \subset Q \otimes_{F[t]} V$ and any submodule $W \subset V$ a real number $c_W(S)$. By homothety invariance (Remark 5.27) this function descends to a function from the vertices of $X(V)$ to the real numbers. We can extend it linearly to get a function from the whole of $X(V)$ to $\mathbb{R}$ which is also called c_W.

Lemma 7.26. *For any two adjacent vertices $x, x' \in X(V)$ we have*

$$|c_W(x) - c_W(x')| \leq 4n.$$

Furthermore there is a number $C \in \mathbb{R}$ such that the function $c_W : X(V) \to \mathbb{R}$ is C-Lipschitz for all nontrivial direct summands W.

Proof. Let S, S' be representatives of the homothety classes of x, x' with $B \subset B' \subset tB$. Then we have by Corollary 5.25

$$\log \mathrm{vol}_W(B) \leq \log \mathrm{vol}_W(B') \leq \mathrm{rk}_R(W) + \log \mathrm{vol}_W(B)$$

for any direct summand $W \subset V$. Inserting this in the definition of c_W gives

$$|c_W(B) - c_W(B')| \leq 4n$$

for any nontrivial direct summand $W \subset V$. So Proposition 7.23 gives the desired result. □

Proposition 7.27. *The affine building $X(V)$ satisfies all assumptions from 2.4. Let*

$$\mathcal{W} := \{\{x \in X(V) \mid c_W(x) > 4n\} \mid W \subset V \text{ is a nontrivial direct summand}\}.$$

This is a collection of open sets as the map $c_W : X(V) \to \mathbb{R}$ is continuous. We have

(i) $X(V)$ *is a proper* CAT(0) *space,*

(ii) the covering dimension of $X(V)$ is less or equal to $n-2$,

(iii) the group action of $\mathrm{aut}_{F[t]}(V) \cong \mathrm{GL}_n(F[t])$ *on X is proper and isometric,*

(iv) $G\mathcal{W} := \{gW \mid g \in G, W \in \mathcal{W}\} = \mathcal{W}$.

(v) Let W, W' be two submodules of the same rank. Then the open sets $c_W^{-1}([4n,\infty))$ and $c_{W'}^{-1}([4n,\infty))$ do not intersect. Especially

$$gW \cap W \neq \emptyset \Rightarrow gW = W$$

for all $g \in \mathrm{aut}_{F[t]}(V)$, $W \in \mathcal{W}$.

(vi) The dimension of $\mathcal{W}$ is less or equal to $n-1$.

(vii) The $\mathrm{aut}_{F[t]}(V)$ *operation on*

$$X \setminus (\bigcup \mathcal{W}^{-\beta}) := \{x \in X \mid \nexists W \in \mathcal{W} : \overline{B}_\beta(x) \subset W\}$$

is cocompact for every $\beta \geq 0$.

Proof. (i) It is a CAT(0) space by Proposition 7.19 and properness has been shown in Corollary 7.22 since the residue field $R/t^{-1}R \cong F$ is finite.

(ii) $X(V)$ is a simplicial complex of dimension $n-1$. Hence its covering dimension is also $n-1$ by [24, Corollary 7.3].

(iii) It is proper and simplicial by Lemma 7.24. It furthermore preserves the label difference since for any two vertices $[L], [L']$ with representatives L, L' such that $L' \subset L$ and any $\varphi \in \mathrm{aut}_{F[t]}(V)$ we have

$$L/L' \cong \varphi(L)/\varphi(L').$$

(compare Definition 7.14). In Proposition 7.19 we have shown that any label difference preserving simplicial automorphism of the building is an isometry.

(iv) We have $\log \mathrm{vol}_{gW}(gS) = \log \mathrm{vol}_W(S)$ and consequently

$$\begin{aligned}
& G\mathcal{W} \\
= \;& \{\{gx \in X(V) \mid c_W(x) \geq 0\} \mid W \subset V \text{ is a nontrivial direct summand}, g \in G\} \\
= \;& \{\{x \in X(V) \mid c_W(g^{-1}x) \geq 0\} \mid W \subset V \text{ is a nontrivial direct summand}, g \in G\} \\
= \;& \{\{x \in X(V) \mid c_{gW}(x) \geq 0\} \mid W \subset V \text{ is a nontrivial direct summand}, g \in G\} \\
= \;& \{\{x \in X(V) \mid c_W(x) \geq 0\} \mid W \subset V \text{ is a nontrivial direct summand}, g \in G\} \\
= \;& \mathcal{W}.
\end{aligned}$$

Lemma 5.11 is used for the third equality.

(v) The idea is to use that for a vertex $[L]$ the condition $c_W([L]) > 0$ means that W occurs in the canonical filtration for (V, L). But at most one module of a given rank can occur in the canonical filtration. So $c_W([L])$ and $c_{W'}([L])$ cannot be both larger than zero.

So let $g \in \mathrm{aut}_{F[t]}(V), U = \{x \in X(V) \mid c_W(x) > 4n\} \in \mathcal{W}$ for a nontrivial direct summand W of V be give. Let $x \in X(V)$ be given with $c_W(x) > 4n$ and $c_{W'}(x) > 4n$. The point x is contained in a simplex s. The value of c_W at x is a convex combination of the values of c_W at the vertices of s. Let us recall the definition of c_W (Definition 3.4):

$$c_W(-) := \inf_{\binom{W_0 \subsetneq W}{W \subsetneq W_2}} \frac{\log \mathrm{vol}_{W_2}(-)) - \log \mathrm{vol}_W(-)}{\mathrm{rk}(W_2) - \mathrm{rk}(W)} - \frac{\log \mathrm{vol}_W(-) - \log \mathrm{vol}_{W_0}(-)}{\mathrm{rk}(W) - \mathrm{rk}(W_0)}.$$

Since Corollary 5.25 tells us that the value of $\log \mathrm{vol}_{W'}$ at two of those vertices can differ at most by $\mathrm{rk}(W') \leq n$, we see that the value of c_W at those vertices can differ at most by $4n$. Thus $c_W(v) > 4n - 4n = 0$ for each vertex v of s and for the same reason $c_{W'}(v) > 0$. Hence we can use the argument from above (see Corollary 3.6) to conclude that W and W' occur in the canonical filtration of (V, L) where L is any R-lattice representing the vertex $v = [L]$. So $gW = W$ since they both have the same rank. The second statement follows if we pick gW as W'.

(vi) We have already seen in the previous item that x cannot be contained in two sets from $\mathcal{W}$ corresponding to modules of the same rank. Thus any point can be an element of at most $|\{1, \ldots, n-1\}| = n-1$ sets and hence the covering dimension is at most $n-2$.

(vii) Let $\beta > 0$ be given. We use Lemma 7.26 to conclude that there is a constant $C > 0$ such that the function c_W is C-Lipschitz for every nontrivial direct summand W. Now let us pick a set $U = \{x \in X(V) \mid c_W(x) > 4n\} \in \mathcal{W}$. Note first that $U^{-\beta}$ is open by Lemma 7.6.

Consider the set $U' = \{x \in X(V) \mid c_W(x) > 4n + C \cdot \beta\}$. Since c_W is C-Lipschitz the closed ball of radius β around each $x \in U'$ is entirely contained in U and thus $U' \subset U^{-\beta}$. So let us consider the system

$$\mathcal{W}' := \{\{x \in X(V) \mid c_W(x) > 4n + C\beta\} \mid W \subset V \text{ is a nontrivial direct summand}\}$$

first. We have already shown that $\bigcup \mathcal{W}' \subset \bigcup \mathcal{W}^{-\beta}$ and hence $X \setminus (\bigcup \mathcal{W}^{-\beta})$ is a closed subset of $X \setminus (\bigcup \mathcal{W}')$. Thus it suffices to show that the group action on $X \setminus (\bigcup \mathcal{W}')$ is cocompact.

It suffices to show that there are finitely many $\mathrm{aut}_{F[t]}(V)$ orbits of vertices such that each $x \in X \setminus (\bigcup \mathcal{W}^{-\beta})$ lies in a simplex with at least one vertex from the given finite set of vertices. We could pick a finite set of representatives for those orbits and consider the union of the closed stars around each of the points. Each of them is a finite simplicial complex since $X(V)$ is locally finite by Lemma 7.13. So we have found a compact set whose translates cover $X \setminus (\bigcup \mathcal{W}')$. So let us find the finite set of orbits.

Let $x \in X \setminus (\bigcup \mathcal{W}')$ be given. Thus each vertex v of the simplex which contains x satisfy $c_W(v) \leq 8n + C\beta =: C'$ for each nontrivial direct summand $W \subset V$. Otherwise if one of them was bigger than they all would be bigger than $4n + C\beta$ by Lemma 7.26. But $c_W(x)$ is defined to be a convex combination of those values. So it would also be bigger than $4n + C\beta$ which contradicts the choice of x.

So let L be an R-lattice representing one of the vertices. By rescaling with a suitable power of t we can assume that $\log \mathrm{vol}_V(L) \in [0, n-1]$.

We can use Proposition 5.23. We make use of the numbers r_i occurring there. Recall that they have the property that

$$\sum_{i=1}^{n} r_i = \log \mathrm{vol}_V(L) \in [0, n-1],$$

$$(7.28) \qquad 0 \leq r_{m+1} - r_m = c_{\langle w_1,\dots,w_m\rangle}([L]) \leq C'.$$

This gives a bound on the size of the numbers r_i by the following consideration. At least one of the r_i is ≥ 0 since their sum is nonnegative. Also at least one of the r_i's is < 1 since their sum is smaller than $n-1$. Since the r_i's are monotonically increasing there is an index j such that $r_j \leq 0$ and $r_{j+1} \geq 0$. Using the bound on the growths (7.28) we get

$$|r_i - r_j| \leq C' \cdot |i-j| \qquad \text{and} \qquad r_j \in [-C', 0].$$

Hence each r_i lies in $[-C' - n \cdot C', n \cdot C']$. This means that there are only finitely many isomorphism types of such R-lattices possible that could occur as L by Proposition 5.23. And an isomorphism $(V, L) \cong (V, L')$ is just an element $g \in \operatorname{aut}_{F[t]}(V)$ with $\operatorname{id} \otimes g(L) = L'$. So we have found the desired finite s et of orbits. This completes the proof.

□

7.4 $\mathrm{GL}_n(Z[T^{-1}])$ acts on a product of CAT(0)-spaces

Convention 7.29. *Let*

- *Z denote either $\mathbb{Z}$ or $F[t]$ for a finite field F and let Q denote its quotient field,*
- *T be a finite set of primes in Z,*
- *V be a free $Z[T^{-1}]$ module of rank n,*
- *$X(V)$ be the space of homothety classes of inner products on V (as in section 7.1) in the case of $Z = \mathbb{Z}$ respectively the affine building for the valuation $v(\frac{f}{g}) = \deg(g) - \deg(f)$ on Q in the case of $Z = F[t]$,*
- *$Y_T(V)$ denote the product of the affine buildings of V for each p-adic valuation v_p on Q with $p \in T$ metrized as a product of CAT(0)-spaces,*
- *$\tilde{Y}_T(V)$ denote the set of all integral structures on V with respect to T, i.e. the set of all finitely generated Z_T-submodules of $Q \otimes_{Z[T]} V$ of rank n,*
- *D be the space $\mathbb{R}^T$ equipped with the $\operatorname{aut}_{Z[T^{-1}]}(V)$-action*

$$(f, (x_p)_{p \in T}) \mapsto (v_p(\det(f)) + x_p).$$

We will show in this section that the space $X(V) \times Y_T(V) \times D$ satisfies all requirements of Proposition 2.4. Let us first establish a connection between the vertices of $Y_S(V)$ and the set of integral structures on the $Z[T^{-1}]$-module V. Let $\operatorname{vert}(B)$ denote the vertex set of a simplicial complex B.

Proposition 7.30. *Let S be a (finite) set of primes. Then* *(i) the map*

$$\Psi : \prod_{s\in S} \tilde{Y}_{\{s\}}(Z_s \otimes_{Z_S} V) \to \tilde{Y}_S(V) \qquad (B_s)_{s\in S} \mapsto \bigcap_{s\in S} B_s$$

is an isomorphism of $\mathrm{aut}_Q(Q\otimes_{Z[S^{-1}]}V) \cong \mathrm{GL}_n(Q)$*-sets with inverse* $B \mapsto (\langle B\rangle_{Z_s})_{s\in S}$.

(ii) It induces an isomorphism of $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$*-sets*

$$\mathrm{vert}(\prod_{s\in S} Y_{\{s\}}(Z_s \otimes_{Z_S} V)) \to \mathrm{vert}(Y_S(V)) \qquad [B_s]_{s\in S} \mapsto [\bigcap_{s\in S} B_s]$$

with inverse $B \mapsto (\langle B\rangle_{Z_s})_{s\in S}$.

Proof. (i) An element B of $\tilde{Y}_S(V)$ is a Z_S-submodule of $Q \otimes_{Z[S^{-1}]} V$ of rank n. Since it is torsionfree it can be expressed as $\langle b_1, \ldots, b_n\rangle_{Z_S}$ for a system of linear independent vectors $b_1, \ldots, b_n \in Q^n$. Hence

$$\begin{aligned}
&\bigcap_{s\in S}\langle B\rangle_{Z_s} \\
= \;& \bigcap_{s\in S}\{\sum_{i=1}^{n} \lambda_i b_i \mid \lambda_i \in Z_s\} \\
= \;& \{\sum_{i=1}^{n} \lambda_i b_i \mid \lambda_i \in \bigcap_{s\in S} Z_s\} \qquad \text{as } b_1, \ldots, b_n \text{ are } Q\text{-linear independent} \\
= \;& \{\sum_{i=1}^{n} \lambda_i b_i \mid \lambda_i \in Z_S\} \\
= \;& B.
\end{aligned}$$

Hence one composition is the identity.

Let us now consider the other composition: Let

$$(B_s)_{s\in S} \in \prod_{s\in S} Y_{\{s\}}(Z_s \otimes_{Z_S} V)$$

be given. We want to show that $B_s = \langle \bigcap_{s'\in S} B'_s\rangle_{Z_s}$. The inclusion $\supset$ is obvious as the left hand side is a Z_s-module that contains $\bigcap_{s'\in S} B'_s$.

So let us consider the other inclusion. Note that $Q \otimes Z[S^{-1}]V/B_s$ is s-torsion by Remark (iii). Hence we have the implication

$$\forall\, m \in Z : \; mx \in B_s, \nu_s(m) = 0 \Rightarrow x \in B_s.$$

So

$$
\begin{aligned}
&\langle \bigcap_{s'\in S} B_{s'} \rangle_{Z_s} \\
= &\left\{ \sum_{i=1}^{r} \lambda_i b_i \;\middle|\; r \in \mathbb{N}, b_i \in \bigcap_{s'\in S} B_{s'}, \lambda_i \in Z_s \right\} \\
= &\left\{ \frac{1}{m} \sum_{i=1}^{r} \lambda_i b_i \;\middle|\; r \in \mathbb{N}, b_i \in \bigcap_{s'\in S} B_{s'}, \lambda_i \in Z_S, \right. \\
&\qquad \left. m \text{ is product of elements from } S \setminus \{s\} \right\} \\
= &\left\{ x \in Q^n \;\middle|\; mx \in \bigcap_{s'\in S} B'_s, \text{ and } m \text{ is product of elements from } S \setminus \{s\} \right\} \\
= &\ B_s.
\end{aligned}
$$

Hence also the other composition is the identity.

Clearly the map is $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ invariant.

(ii) The vertices of the simplicial complexes in consideration are homothety classes of integral structures. A homothety is an element in the center of $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V) \cong \{\lambda \cdot \mathrm{id} \mid \lambda \in Q^*\}$. We have to show that $B, B' \in \tilde{Y}_S(V)$ are homothetic if and only if $\langle B\rangle_{Z_s}$ and $\langle B'\rangle_{Z_s}$ are homothetic for each $s \in S$. If $\lambda B = B'$, then $\lambda\langle B\rangle_{Z_s} = \langle \lambda B\rangle_{Z_s} = \langle B'\rangle_{Z_s}$.

Conversely suppose there are $\lambda_s \in Q^*$ with $\lambda_s\langle B\rangle_{Z_s} = \langle B'\rangle_{Z_s}$. We can write λ_s in the form $\lambda_s = s^{\nu_s(\lambda)} \cdot \frac{\lambda_s}{s^{\nu_s(\lambda)}}$. $\frac{\lambda_s}{s^{\nu_s(\lambda)}}$ is a unit in Z_s since $\nu_s(\frac{\lambda_s}{s^{\nu_s(\lambda)}}) = 0$. So $\lambda_s\langle B\rangle_{Z_s} = s^{\nu_s(s)}\langle B\rangle_{Z_s}$. Thus we can assume without loss of generality that λ_s is of the form $\lambda_s = s^{n_s}$ with $n_s \in \mathbb{Z}$.

So we get

$$
\begin{aligned}
&\prod_{s\in S} s^{n_s} \cdot B \\
= &\prod_{s'\in S} s^{n_{s'}} \cdot \bigcap_{s\in S} \langle B\rangle_{Z_s} \\
= &\bigcap_{s\in S} \prod_{s'\in S} s'^{n_{s'}} \cdot \langle B\rangle_{Z'_s} \\
= &\bigcap_{s\in S} s^{n_s} \cdot \langle B\rangle_{Z'_s} \\
= &\bigcap_{s\in S} \langle B'\rangle_{Z'_s} \\
= &\ B'.
\end{aligned}
$$

Since homotheties commute with elements from $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ we get an induced action on the quotient.

□

Lemma 7.31. *Let V be a finitely generated free $Z[T^{-1}]$-module. The group action of*

$$\mathrm{saut}_{Z[T^{-1}]}(V) := \{\varphi \in \mathrm{saut}_{Z[T^{-1}]}(V) \mid \det(\varphi) = 1\}$$

on $\prod_{s\in S} |Y_s(V)|$ is cocompact.

Proof. $\prod_{s\in S} |Y_s(V)|$ has the structure of a locally finite simplicial complex by Lemma 7.13 equipped with a simplicial group action. Thus it suffices to show that the action on the vertex set is cofinite.

The previous lemma identifies this set with the set of all homothety classes of integral structures $Y_S(V)$ and the $\mathrm{saut}_{Z[T^{-1}]}(V)$-action on $Y_S(V)$ is cofinite by Proposition 6.25. □

We can consider for any nontrivial direct summand $W \subset V$ the function $c_W : X(V) \times Y_S(V) \to \mathbb{R}$ that is defined in the following way. If $y = (y_s)_{s\in S}$ is a tuple of vertices we can pick representatives and use the bijection from Proposition 7.30 to obtain an integral structure B. Corollary 6.21 shows that $c_W(x, B)$ is independent of the chosen representatives. So we can assign to a point (x, y) the value $c_W(x, B)$.

For general $y = (y_s)_{s\in S}$ we can write each y_i as a convex combination of the vertices $v_1^s, \dots, v_n^s$ of the open simplex in $Y_s(V)$ containing y_s, say $y_s = \sum_{i=1}^n \lambda_i^s v_i^s$. Then define $c_W(x, y)$ as the linear extension in y-direction. More precisely

$$c_W(x, y) := \sum_{i\in\{1,\dots,n\}} (\prod_{s\in S} \lambda_{i_s}^s) \cdot c_W(x, v_{i_s}^s).$$

Furthermore $Y_S(V)$ is a product of Euclidean simplicial complexes and thus it can be viewed as an Euclidean simplicial complex after a choice of simplex orientation that tells us how to subdivide the products of simplices.

Lemma 7.32. *(i) Given $y := (y_s)_{s\in S}, y' := (y'_s)_{s\in S} \in Y_S(V)$ such that each y_s, y'_s is a vertex of $Y_s(V)$. Suppose that for each s the vertices y_s and y'_s are either adjacent or equal. Then we have*

$$|c_W(x, y) - c_W(x, y')| \leq 4n \cdot \begin{cases} \ln(\prod_{s\in S} s) & Z = \mathbb{Z} \\ -\nu(\prod_{s\in S} s) & Z = F[t] \end{cases}.$$

(ii) There is a constant C (independent of W and x) such that $c_W(x, -)$ is C-Lipschitz.

Proof. (i) Pick for each $s \in S$ a Z_s-lattice B_s in $Q \otimes_{Z[S^{-1}]} V$ representing y_s. As y_s is adjacent or equal to y'_s we can find a representative B'_s of y_s such that $sB_s \subseteq B'_s \subseteq B_s$. Now we have to consider the intersections $B := \bigcap_{s\in S} B_s$ and $B' := \bigcap_{s\in S} B'_s$. Let $z := \prod_{s\in S} s$. We obtain since B_s is a Z_s-module

$$zB = \bigcap_{s\in S} zB_s = \bigcap_{s\in S} sB_s \subset B' \subset B.$$

By definition z is a product of elements from S. So we can use Corollary 6.20 to get

$$|\log \mathrm{vol}_{W'}(x, B) - \log \mathrm{vol}_{W'}(x, B)| \leq \mathrm{rk}(W') \cdot (-\nu(z))$$

in the function field case; respectively

$$|\log \mathrm{vol}_{W'}(x, B) - \log \mathrm{vol}_{W'}(x, B)| \leq \mathrm{rk}(W') \cdot \ln(z)$$

in the number field case. If we insert this into the definition of c_W we get

$$|c_W(x, B) - c_W(x, B)| \leq 4n \cdot \begin{cases} \ln(z) & Z = \mathbb{Z} \\ -\nu(z) & z = F[t] \end{cases}.$$

(ii) $Y_S(V)$ is by definition a product of Euclidean simplicial complexes with finitely many isometry types of simplices. After subdividing products of simplices into simplices it inherits the structure of Euclidean simplicial complex with finitely many isometry types of simplices. Note that vertices in the product can only be adjacent if they are adjacent or equal in each coordinate.

We have already computed a bound on the difference on two adjacent vertices in (i). Hence we can use Proposition 7.23 to conclude that there is a constant C depending only on n, S such that $c_W : X(V) \times Y_S(V) \to \mathbb{R}$ is C-Lipschitz.

□

Proposition 7.33. *The space $X(V) \times |Y_S(V)| \times D$ satisfies all assumptions from Proposition 2.4. Let $R \in \mathbb{R}$ be either $\ln(\prod_{s \in S} s)$ in the number field case or $\nu(\prod_{s \in S} s)$ in the function field case. Let $\mathcal{W} := \{\{(x, y, d) \in X(V) \times |Y_S(V)| \times D \mid c_W(x, y) > 4n(R + 1)\} \mid W \subset V$ is a nontrivial direct summand$\}$. This is a collection of open sets as the map $c_W : X(V) \to \mathbb{R}$ is continuous. We have*

(i) $X(V) \times |Y_S(V)| \times D$ is a proper $\mathrm{CAT}(0)$ space,

(ii) the covering dimension of $X(V) \times |Y_S(V)| \times D$ is less or equal to $\frac{n(n+1)}{2} - 1 + |S|n$,

(iii) the group action of $\mathrm{aut}_{Z[S^{-1}]}(V) \cong \mathrm{GL}_n(Z[S^{-1}])$ on $X(V) \times |Y_S(V)| \times D$ is proper and isometric,

(iv) $G\mathcal{W} := \{gW \mid g \in \mathrm{aut}_{Z[S^{-1}]}(V), W \in \mathcal{W}\} = \mathcal{W}$,

(v) $gW \cap W \neq \emptyset \Rightarrow gW = W$ for all $g \in \mathrm{aut}_{Z[S^{-1}]}(V), W \in \mathcal{W}$,

(vi) the dimension of $\mathcal{W}$ is less or equal to $n - 2$,

(vii) the $\mathrm{aut}_{Z[S^{-1}]}(V)$ operation on

$$X(V) \times |Y_S(V)| \times D \setminus (\bigcup \mathcal{W}^{-\beta}) = \{(x, y, d) \in X(V) \times |Y_S(V)| \times D | \nexists W \in \mathcal{W} : \overline{B}_\beta(x) \subset W\}$$

is cocompact for every $\beta \geq 0$.

Proof. (i) Each of the spaces $X(V)$ (Proposition 7.12) and $|Y_S(V)|$ (Proposition 7.19 for the CAT(0) condition and Corollary 7.22 for the properness) and D ($\cong \mathbb{R}^n$) is a proper CAT(0) space. Products of proper CAT(0) spaces are proper CAT(0) spaces (see for example [11, Chapter II Example 1.15(iii)]).

(ii) All the spaces $X(V)$, D and $Y_s(V)$ for $s \in S$ can be equipped with a CW-structure with countably many cells; in the number field case $X(V)$ is a smooth manifold and hence it can be triangulated. If a triangulation consisted of uncountably many cells we can take all barycenters of the cells to obtain a uncountable discrete subset. This contradicts the second countability. The same argument holds for D. The other space are defined to be the geometric realization of countable simplicial complexes and hence they also can be equipped with the structure of a CW-complex with countably many cells.

By [18, Theorem A.6] the product CW-structure on $X(V) \times |Y_S(V)| \times D$ really induces the product topology. $X(V)$ is a $n(n+1)/2-1$-dimensional manifold in the number field case or a $n-1$-dimensional simplicial complex in the function field case. Each $Y_s(N)$ is a simplicial complex of dimension $n-1$. So $Y_S(V) := \prod_{s \in S} Y_s(V)$ is a CW-complex of dimension $|S| \cdot (n-1)$ and D is a $|S|$-dimensional manifold. So the CW-dimension of $X(V) \times |Y_S(V)| \times D$ is at most $n(n+1)/2 - 1 + |S| \cdot n$. By [24, Corollary 7.3] its covering dimension equals its dimension as a CW-complex.

$|Y_S(V)|$ is a product of $|S|$ simplicial complexes of dimension $n-1$. Each of the factors has covering dimension $n-1$ by [24, Corollary 7.3]. Thus it has covering dimension $|S| \cdot (n-1)$ by . Since D is just $\mathbb{R}^{|S|}$ its covering dimension is $|S|$. In the function field case $X(V)$ is the realization of an $n-1$ dimensional simplicial complex and in the number field case it is an $n(n+1)/2-1$-dimensional manifold. Hence its covering dimension is at most $n(n+1)/2-1$.

So the covering dimension of the product is at most $n(n+1)/2 - 1 + |S| \cdot n$.

(iii) The group action on each factor is isometric by Lemma 7.2 and Proposition 7.19(v). So the action on the product is isometric. We have to show that it is a proper action (compare [11, Chapter I 8.2-8.3] for the subtilities of the definition of a proper action).

So let a point (x, y, d) be given. The group acts on D by translations via the group homomorphism

$$\operatorname{aut}_{Z[S^{-1}]}(V) \to \mathbb{Z}^{|S|} \quad \varphi \mapsto (\nu_s \det(\varphi))_{s \in S}.$$

Hence we see that for any $g \in \operatorname{aut}_{Z[S^{-1}]}(V) \setminus \operatorname{stab}(d)$ we have either $B_{\frac{1}{2}}(d) \cap gB_{\frac{1}{2}}(d) = \emptyset$. The group acts simplicially on $|Y_S(V)|$ and so the orbit of y is a discrete subset. Hence $B_{\frac{1}{2}}(y) \cap gB_{\frac{1}{2}}(y) = \emptyset$ for $g \in \operatorname{aut}_{Z[S^{-1}]}(V) \setminus \operatorname{stab}(y)$. Let us now consider the group $\operatorname{stab}(y) \cap \operatorname{stab}(d)$. Let B be a free Z_S-module representing a vertex of the open simplex containing y. *Claim:* The group $\operatorname{stab}([B]) \cap \operatorname{stab}(d)$ is $\operatorname{aut}_Z(V \cap B) \cong \mathrm{GL}_n(Z)$ in $\operatorname{aut}_Q(Q \otimes_{Z[S^{-1}]} V) \cong \mathrm{GL}_n(Q)$.

Let for $\varphi \in \mathrm{stab}(d) = \{\varphi \in \mathrm{aut}_{Z[S^{-1}]}(V) \mid \nu_p(\det(\varphi)) = 0 \text{ for all } p \in S\}$ be an group element with $[\varphi B] = [B]$. This means that $\varphi B = \lambda B$ for some $\lambda \in Z[S^{-1}]^*$. We have $0 = \nu_p(\det(\varphi)) = n \cdot \nu_p(\lambda)$ for any $p \in S$. But the only units in $Z[S^{-1}]$ with p-adic valuation zero for all $p \in S$ are ± 1 in the number field case and F^* in the function field case. In both cases they are also units in Z_S and so $\lambda B = B$.

Conversely the determinant of any element $\varphi \in \mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ with $\varphi(B) = B$ is a unit in Z_S which means that $\nu_p(\det(\varphi)) = 0$ for any $p \in S$. Thus

$$\begin{aligned} & \mathrm{stab}(y) \cap \mathrm{stab}([B]) = \mathrm{stab}(B) \\ = \; & \{\varphi \in \mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V) | \varphi(V) = V, \varphi(B) = B\}. \end{aligned}$$

The group $\mathrm{aut}_{Z[S^{-1}]}(V)$ consists of all linear maps in $\mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V)$ that map a (and hence any) $Z[S^{-1}]$-basis of V to another $Z[S^{-1}]$-basis of V. The analogous statement holds for $\mathrm{aut}_{Z_S}(B)$. By Remark 6.11 any Z-basis of $V \cap B$ is also a $Z[S^{-1}]$-basis of V and a $Z[S]$-basis of B. Thus

$$\mathrm{stab}(B) = \{\varphi \in \mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V) \mid \varphi(V \cap B) = V \cap B\} = \mathrm{aut}_Z(V \cap B).$$

So we just have to show that $\mathrm{aut}_Z(V \cap B)$ acts properly on $X(V)$. This has been done for the number field case in Lemma 7.3 and for the function field case in Lemma 7.24.

(iv) Let $g \in \mathrm{aut}_{Z[S^{-1}]}(V)$ and $W \in \mathfrak{L}$ be given. We have

$$\begin{aligned} & g \cdot \{(x, y, d) \in X(V) \mid c_W(x, y) > 4n(R + 1)\} \\ = \; & \{(x, y, d) \in X(V) \mid c_W(g^{-1}x, g^{-1}y) > 4n(R + 1)\} \\ = \; & \{(x, y, d) \in X(V) \mid c_{gW}(x, y) > 4n(R + 1)\}. \end{aligned}$$

The last equality uses Remark 6.14.

(v) Let us proof first that two nontrivial direct summands W, W' of V of rank m with

$$c_W(x, y) > 4n(R + 1) \text{ and } c_{W'}(x, y) > 4n(R + 1)$$

for some point $(x, y) \in X(V) \times Y_S(V)$ are equal. This will prove the statement since we have shown in the previous item that we get

$$gU = \{(x, y, d) \in X(V) \times |Y_S(V)| \times D \mid c_{gW}(x, y) > 4n(R + 1)\}.$$

for $U = \{(x, y, d) \in X(V) \times |Y_S(V)| \times D | c_W(x, y) > 4n(R + 1)\}$ and $g \in \mathrm{aut}_{Z[S^{-1}]}(V)$.

As mentioned above $Y_S(V)$ is an Euclidean simplicial complex. Let s denote the open simplex containing y. The value of c_W at (x, y) is defined to be a convex combination of the values of $c_W(x, -)$ at the vertices of s. By Lemma 7.32 we see that all their values can differ at most by $4nR$. Thus the value at any vertex v must be greater than $4n(R + 1) - 4nR = 4n$. So $c_W(x, v) > 4n, c_{gW}(x, v) > 4n$.

Fixing now the second coordinate we can consider the function $c_W(-,v) : X(V) \to \mathbb{R}$. Let B be an representative of the homothety class of integral structures y. We have by Remark 6.18 $c_W(-,y) = c_{W\cap B}(-)$. So $c_{W\cap B}(x) > 4n, c_{W'\cap B}(x) > 4n$. By Proposition 6.10 the two Z-submodules $W' \cap B, W \cap B$ of $V \cap B$ have the same rank. For the case of $Z = F[t]$ we can use Proposition 7.27 to conclude that $W' \cap B = W \cap B$. For the number field case we can use Proposition 7.12 instead. Using Proposition 6.10 this means that $W' = W$.

(vi) Let (x, y, d) be any point in $X(V) \times |Y_S(V)| \times D$. We have shown in the previous item that there can be at most one direct summand W for each rank between one and $n-1$ with $c_W(x,y) > 4n(R+1)$. So there can be at most $n-1$ open sets in $\mathcal{W}$ containing (x, y, d).

(vii) Of course, it suffices to show that $X(V)\times|Y_S(V)|\times D\setminus(\bigcup \mathcal{W}^{-\beta})$ is a closed subset of a cocompact set. It is a closed subset of the whole space by Lemma 7.6. For any nontrivial direct summand W the function c_W is C-Lipschitz for a constant C by Lemma 7.32. Hence

$$\begin{aligned}\{(x,y,d) \in X(V) \times |Y_S(V)| \times D \mid c_W(x,y) > 4n(R+1) + C\beta\}\\ \subset \{(x,y,d) \in X(V) \times |Y_S(V)| \times D \mid c_W(x,y) > 4n(R+1)\}^{-\beta}\end{aligned}$$

and consequently $X(V) \times |Y_S(V)| \times D \setminus (\bigcup \mathcal{W}^{-\beta})$ is a subset of

$$\{(x,y,d) \in X(V) \times |Y_S(V)| \times D \mid c_W(x,y) > 4n(R+1) + C\beta\}.$$

So we still have to show that the group action on the last set is cocompact. The group action of $\mathrm{aut}_{Z[S^{-}]}(V)$ on D is cocompact. A fundamental domain is given by $K_D := [0,1]^{|S|}$.

Consider the subgroup that stabilizes K_D pointwise. It is

$$H := \{\varphi \in \mathrm{aut}_{Z[S^{-1}]}(V) \mid \det(\varphi) \in Z^*\} \text{ with } Z^* = \begin{cases}\{\pm 1\} & Z = \mathbb{Z}\\ F^* & Z = F[t]\end{cases}.$$

It acts cocompactly on $Y_S(D)$ by Lemma 7.31. Thus there is a finite subcomplex $K_Y \subset Y_S(V)$ such that $H \cdot K_Y = Y_S(V)$. Let us consider the group that stabilizes every point in K_Y pointwise. It is the intersection of the stabilizers of all vertices of K_Y and thus it has finite index in the stabilizer of any vertex $v \in K_Y$ by Lemma 7.34. So consider

$$\{g \in \mathrm{saut}_{Z[S^{-1}]} \mid gy = y\}.$$

As shown in (iii) before, this is just $\mathrm{stab}(B)$ for any representative B of the homothety class y. Again we have shown before that

$$\{g \in \mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V) \mid gV = V, gB = B\} = \{g \in \mathrm{aut}_Q(Q \otimes_{Z[S^{-1}]} V \mid gV \cap B = V \cap B\}.$$

The group on the right hand side is just $\mathrm{aut}_Z(V \cap B)$.

So let us analyze the action of this group on $X(V)$. First note that we have by Proposition 6.10

$$
\begin{aligned}
& \{x \in X(V) \mid c_W(x, B) > 4n(R+1) \text{ for a nontrivial direct summand } W \subset V\} \\
= \quad & \{x \in X(V) \mid c_{W'}(x) > 4n(R+1) \text{ for a nontrivial direct summand } W' \subset V \cap B\}.
\end{aligned}
$$

The group action on the complement of this set is cocompact. For the number field case this is shown in Proposition 7.11. For the function field case this is shown in Proposition 7.27.

□

Lemma 7.34. *If a group G acts simplicially on a locally finite simplicial complex X the stabilizer groups of any two vertices are commensurable.*

Proof. Given any two vertices x, y let R denote the combinatorial distance between x and y. As the simplicial complex is locally finite the set of all vertices of combinatorial distance $\leq R$ to x is finite and it contains y. Now the stabilizer group G_x acts on this set. The isotropy group of y under this restricted action is $G_x \cap G_y$. So we get an injection. Hence the index of $G_x \cap G_y$ in G_x is finite. Analogously for y. Hence the subgroups G_x and G_y are commensurable. □

8 Reducing the family

Let Z be either $\mathbb{Z}$ or the polynomial ring over a finite field. Let S be a finite set of primes in Z and F be any finite group.

Definition 8.1. In the following the term "class of groups" will denote a class of groups with the following two properties:

- If a group is in the class, so are its subgroups.
- Any group isomorphic to a group in the class is also in the class.

A class of groups determines a family of subgroups; namely those which are in this class. Examples are the class of trivial groups, the class $\mathcal{F}$in of finite groups, the class $\mathcal{VC}$yc of virtually cyclic groups and the class $\mathcal{VS}$ol of virtually solvable groups. For a family $\mathcal{F}$ let $\mathcal{F}_2$ denote the family of those groups containing a group from $\mathcal{F}$ of index at most two.

Notation 8.2. Let us say that a triple $(\mathcal{H}^?_*, G, \mathcal{F})$ satisfies the isomorphism conjecture (in certain degrees), if the map

$$\mathcal{H}^G_*(E_{\mathcal{F}}G) \to \mathcal{H}^G_*(\mathrm{pt})$$

is an isomorphism (in those degrees). Let us say that a group G satisfies the K-theoretic Farrell-Jones conjecture if the isomorphism conjecture holds for $(H^?_*(-;\mathbf{K}_{\mathcal{A}}), G, \mathcal{VC}\mathrm{yc})$ for any additive category $\mathcal{A}$. A group G satisfies the L-theoretic Farrell-Jones conjecture, if the isomorphism conjecture holds for $(H^?_*(-;\mathbf{L}^{\langle -\infty\rangle}_{\mathcal{A}}), G, \mathcal{VC}\mathrm{yc})$ for any additive category $\mathcal{A}$ with involution (compare [9, Section 3 and Section 5]). A group G satisfies the Farrell-Jones conjecture if it satisfies both the K- and L-theoretic Farrell-Jones conjecture.

Let us say that a group G satisfies the Farrell-Jones conjecture relative to a family $\mathcal{F}$ if we replace $\mathcal{VC}$yc by $\mathcal{F}$. A group G is said to satisfy the Farrell-Jones conjecture with finite wreath products, if the group $G \wr F$ satisfies the Farrell-Jones conjecture for any finite group F.

Theorem 8.3. *Let F be a finite group and let $\mathcal{F}$ denote the family*

$$\mathcal{VC}yc \cup \{\mathrm{stab}(W) \mid W \text{ is a nontrivial direct summand of } Z[S^{-1}]^n\}.$$

(i) The group $\mathrm{GL}_n(Z[S^{-1}]) \wr F$ satisfies the K-theoretic Farrell-Jones conjecture with respect to the family $\mathcal{F}$ in all degrees.

(ii) The group $\mathrm{GL}_n(Z[S^{-1}]) \wr F$ satisfies the L-theoretic Farrell-Jones conjecture with respect to the family $\mathcal{F}_2$ in all degrees,

(iii) The group $\mathrm{GL}_n(Z[S^{-1}]) \wr F$ *satisfies the K- and L-theoretic Farrell-Jones conjecture in all degrees with respect to the family* $\mathcal{F}^{\wr}$*, which consists of those subgroups that have a finite index subgroup which is abstractly isomorphic to a finite product of groups from* $\mathcal{F}$*.*

Proof. We have found a space satisfying the conditions from Proposition 2.4 (see Proposition 7.12 for the case of $\mathbb{Z}$, Proposition 7.27 for the case of $F[t]$ and Proposition 7.33 for the localized versions). So $\mathrm{GL}_n(Z[S^{-1}])$ admits long $\mathcal{F}$-covers at infinity and periodic flow lines. Hence it is strongly transfer reducible over $\mathcal{F}$ by Theorem 2.10. So it satisfies the K-theoretic Farrell-Jones conjecture in all degrees with respect to the family $\mathcal{F}$ and the L-theoretic Farrell-Jones conjecture by Theorem 2.13.

The group $G \wr F$ is strongly transfer reducible over the family $\mathcal{F}^{\wr}$ by Proposition 2.12 and hence it satisfies the K- and L-theoretic Farrell-Jones conjecture by Theorem 2.13. Note that by definition $(\mathcal{F}^{\wr})_2 = \mathcal{F}^{\wr}$. □

The goal of this section is to reduce the family further as far as possible. We need the following two key properties:

Theorem 8.4 (Transitivity principle)**.** *Let* $\mathcal{H}^?_*$ *be an equivariant homology theory and let* G *be a group and let* $\mathcal{F} \subset \mathcal{F}'$ *be two families of subgroups. Suppose that each* $H \in \mathcal{F}'$ *satisfies the isomorphism conjecture for the family* $\mathcal{F}|_H$*.*

Then G *satisfies the isomorphism conjecture for the family* $\mathcal{F}$ *if and only if it satisfies the isomorphism conjecture for the family* $\mathcal{F}'$*.*

Proof. This proof can also be found in [22, Theorem 2.9]. Note first that $E_{\mathcal{F}}G \times E_{\mathcal{F}'}G$ with the diagonal G-action is another model for $E_{\mathcal{F}}G$. Consider the map

$$E_{\mathcal{F}}G \xrightarrow{\simeq_G} E_{\mathcal{F}'}G \times E_{\mathcal{F}}G \xrightarrow{\mathrm{pr}} E_{\mathcal{F}'}G \to \mathrm{pt}$$

We have to show that applying $\mathcal{H}^G_*$ to this composition yields an isomorphism. The first map induces an isomorphism by G-homotopy invariance. The last map induces the assembly map for $(\mathcal{H}^?_*, G, \mathcal{F}')$ which is an isomorphism by assumption. Let us examine the map in the middle. Applying the natural transformation

$$\mathrm{pr}_* : \mathcal{H}^G_*(E_{\mathcal{F}'}G \times -) \to \mathcal{H}^G_*(-)$$

of G-homology theories to the G-CW-complex $E_{\mathcal{F}}G$ yields that map. We want to show that pr_* is a natural isomorphism. So it suffices to consider the coefficients; i.e. the restriction to homogeneous objects G/H for some $H \le G$. We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}^G_*(G/H \times E_{\mathcal{F}}G) & \xleftarrow{\quad} & \mathcal{H}^G_*(G \times_H \mathrm{res}^H_G(E_{\mathcal{F}}G)) & \xrightarrow{\mathrm{ind}} & \mathcal{H}^H_*(\mathrm{res}^H_G(E_{\mathcal{F}}G)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}^G_*(G/H \times \mathrm{pt}) & \xleftarrow{\quad} & \mathcal{H}^G_*(G \times_H \mathrm{pt}) & \xrightarrow{\mathrm{ind}} & \mathcal{H}^H_*(pt). \end{array}$$

The left horizontal maps are induced by the G-homeomorphisms

$$G \times_H X \mapsto G/H \times X \qquad (g, x) \mapsto (gH, gx)$$

(for $X = E_{\mathcal{F}}G, pt$). So the left square commutes as it is induced by a commutative square of G-CW-complexes. The other horizontal maps are the natural induction isomorphisms (see Definition 1.4).

Note that $\mathrm{res}_G^H(E_{\mathcal{F}}G)$ is a model for $E_{\mathcal{F}|_H}H$ and hence the right vertical map is the assembly map for $(\mathcal{H}_*^?, H, \mathcal{F}|_H)$. It is an isomorphism by assumption. So all vertical maps are isomorphisms, especially the left one which is exactly

$\mathrm{pr}_*(E_{\mathcal{F}}G) = \mathcal{H}_*^G(pr : E_{\mathcal{F}'}G \times E_{\mathcal{F}}G \to E_{\mathcal{F}'}G)$.

□

Proposition 8.5. *Let $f : G \to H$ be a group homomorphism. If H satisfies the isomorphism conjecture (with finite wreath products) for a family $\mathcal{F}$, then G satisfies the isomorphism conjecture (with finite wreath products) for the family $f^*\mathcal{F}$.*

Proof. This is [9, Corollary 4.3]. □

Remark 8.6. If G satisfies the isomorphism conjecture with respect to a family $\mathcal{F}$, then each subgroup $H \leq G$ satisfies the isomorphism conjecture with respect to the family $\mathcal{F}|_H$.

So if $\mathcal{F}$ is a subfamily of $\mathcal{F}'$ and if a group G satisfies the isomorphism conjecture with respect to $\mathcal{F}$, then it also satisfies the isomorphism conjecture with respect to $\mathcal{F}'$.

Theorem 8.7. *The Farrell-Jones conjecture with finite wreath products holds for any CAT(0)-group.*

Proof. If group G acts properly, isometrically and cocompactly on a CAT(0) space X we can let $G \wr F$ act on X^F via

$$((g \in \mathrm{map}(F, G), f), h \in \mathrm{map}(F, X)) \mapsto g(-) \cdot h(f^{-1}-).$$

The group action is again proper isometric and cocompact. So $G \wr F$ is also a CAT(0) group. So it suffices to consider the version without wreath products.

This is then [4, Theorem B] for the L-theoretic setting and the K-theoretic setting up to dimension 1 and [32, Theorem 1.1 and Theorem 3.4] for the higher dimensional K-theoretic setting. □

Proposition 8.8. *Let $(G_i)_{i\in\mathbb{N}}$ be a directed system of groups indexed over the natural numbers. Suppose that the Farrell-Jones conjecture (with finite wreath products) holds for every G_i. Then it also holds for* $\mathrm{colim}_{i\in\mathbb{N}} G_i$.

Proof. First note that

$$(\mathrm{colim}_{i\in\mathbb{N}} G_i) \wr F \cong \mathrm{colim}_{i\in\mathbb{N}}(G_i \wr F)$$

for a finite group F and so it suffices to consider the version without wreath products.

This is basically [1, Theorem 0.7] with the minor problem that this reference does not deal with the version with coefficients in any additive category but only in the category of free R-modules for some ring R.

First it is shown in [1, Theorem 3.5] that Isomorphism conjectures are compatible with colimits if the given equivariant homology theory is strongly continuous in the sense of [1, Definition 2.3]. It is shown in [1, Lemma 5.2] that $H^?(-;\mathbf{K}_R^{alg})$ and $H^?(-;\mathbf{L}_R^{\langle-\infty\rangle})$ are strongly continuous for any ring R.

The crucial point is to verify that the canonical maps

$$\operatorname{colim}_i K_n(R \rtimes G_i) \to K_n(R \rtimes \operatorname{colim}_i G_i).$$

and

$$\operatorname{colim}_i L_n^{\langle-\infty\rangle}(R \rtimes G_i) \to L_n^{\langle-\infty\rangle}(R \rtimes \operatorname{colim}_i G_i).$$

are isomorphisms. The ring $R \rtimes G_i$ denotes the twisted group ring where the G_i action is the restriction of the $\operatorname{colim}_j G_j$-action along the canonical map $G_i \to \operatorname{colim}_j G_j$. More briefly let us say that the functor $K_n(R \rtimes -)$ is continuous. It is a functor from the category of groups over $\operatorname{aut}_{\mathtt{Rings}}(R)$ to the category of abelian groups.

The same statements also fold, if we allow coefficients in any additive category; the functor $\mathcal{A} \rtimes -$ is continuous by Lemma 10.16 and the functor K_n is continuous for all n by Proposition 10.23.

The proof for the L-theory part from [1, Lemma 5.2] works also in the setting of additive categories. □

Lemma 8.9. *The Farrell-Jones conjecture (with finite wreath products) holds for any virtually abelian group.*

Proof. The Farrell-Jones conjecture holds for $\mathbb{Z}^n$ since it is a CAT(0)-group by Theorem 8.7. Any finitely generated abelian group has a finitely generated, free abelian subgroup of finite index. So the Farrell-Jones conjecture with finite wreath products holds for finitely generated abelian groups by Remark 8.12. Proposition 8.8 shows the Farrell-Jones conjecture with finite wreath products for abelian groups. Using Remark 8.12 again, this shows the Farrell-Jones conjecture for virtually abelian groups. □

Lemma 8.10. *Let F,G be two groups. If F satisfies the isomorphism conjecture with respect to a family $\mathcal{F}$ and G satisfies the isomorphism conjecture with respect to a family $\mathcal{G}$, then $F \times G$ satisfies the isomorphism conjecture with respect to the family*

$$\mathcal{F} \times \mathcal{G} := \{H \mid H \leq F' \times G' F \in \mathcal{F}, G \in \mathcal{G}\}.$$

Proof. Consider the group homomorphism $p_G : F \times G \to G \quad (f,g) \mapsto g$. By Proposition 8.5 it suffices to show that for any subgroup $H \leq G$ with $H \in \mathcal{G}$ the group $p_G^{-1}(H) = F \times H$ satisfies the isomorphism conjecture relative to the family $\mathcal{F} \times \mathcal{G}$. Applying the same argument to the projection $p_H : F \times H \to H$ it suffices to consider $H' \times H$ with $H' \in \mathcal{F}, H \in \mathcal{G}$. This group trivially satisfies the isomorphism conjecture relative to $\mathcal{G} \times \mathcal{F}$ since it is an element of the family $\mathcal{G} \times \mathcal{F}$. □

Corollary 8.11. *Let $\mathcal{F}$ be a class of groups. Suppose that a product of two groups from $\mathcal{F}$ satisfies the isomorphism conjecture relative to $\mathcal{F}$. Then the class of groups satisfying the isomorphism conjecture (with finite wreath products) relative to $\mathcal{F}$ is closed under finite products.*

Especially this shows that the class of groups satisfying the Farrell-Jones conjecture (with finite wreath products) relative to the family $\mathcal{V}\mathcal{S}ol$ is closed under finite products, since the class $\mathcal{V}\mathcal{S}ol$ is. The class of groups satisfying the Farrell-Jones conjecture (with finite wreath products) is also closed under finite products.

Proof. Let two groups G, G' be given. Suppose both of them satisfy the isomorphism conjecture relative to the class $\mathcal{F}$. By the last lemma their product satisfies the isomorphism conjecture relative to the family $\mathcal{F} \times \mathcal{F}$. By assumption any group in $\mathcal{F} \times \mathcal{F}$ satisfies the isomorphism conjecture relative to $\mathcal{F}$. So we can reduce the family from $\mathcal{F} \times \mathcal{F}$ to $\mathcal{F}$ by the transitivity principle.

The version for the wreath products follows from the observation $(G \times G') \wr F \subset (G \wr F) \times (G' \wr F)$.

The final claim follows from the fact that a finite product of virtually cyclic groups is virtually abelian hence it is a CAT(0) group. So it satisfies the Farrell-Jones conjecture by Theorem 8.7. □

Remark 8.12. Let $\mathcal{F}$ be a class of groups. Then the class of groups satisfying the isomorphism conjecture with finite wreath products relative to $\mathcal{F}$ is closed under finite index overgroups.

Proof. Suppose G satisfies the isomorphism conjecture with respect to finite wreath products and H is a finite index overgroup of G. Let F be an arbitrary finite group. By Lemma 10.2 we have an embedding $H \hookrightarrow (G \wr F')$ for some finite group F'. This induces

$$H \wr F \hookrightarrow (G \wr F) \wr F' \hookrightarrow G \wr (F \wr F').$$

The last map is defined in Lemma 10.3. □

We can also combine several of those inheritance properties to get:

Lemma 8.13. *Let $f : G \to H$ be a group homomorphism.*

(i) If H satisfies the Farrell-Jones conjecture and every preimage $f^{-1}(V)$ of a virtually cyclic subgroup V satisfies the Farrell-Jones conjecture, so does G.

(ii) If H, $\ker(f) = f^{-1}(1)$ satisfy the Farrell-Jones conjecture with finite wreath products and every preimage $f^{-1}(Z)$ of an infinite cyclic subgroup Z satisfies the Farrell-Jones conjecture with finite wreath products, so does G.

Proof. (i) We know by Proposition 8.5 that G satisfies the Farrell-Jones conjecture relative to the family $f^*\mathcal{VC}yc$. Since every group in $f^*\mathcal{F}$ is a subgroup of a group of the form $f^{-1}(V)$ for some virtually cyclic subgroup V we can apply the transitivity principle (Proposition 8.4). So G satisfies the Farrell-Jones conjecture.

(ii) By the same argument we have to show that every preimage $f^{-1}(V)$ of a virtually cyclic subgroup $V \subset H$ satisfies the Farrell-Jones conjecture. If V was finite $f^{-1}(V)$ contains $\ker(f)$ as a finite index subgroup. The group $f^{-1}(V)$ satisfies the Farrell-Jones conjecture with finite wreath products by Remark 8.12.

Otherwise V contains an infinite cyclic subgroup Z of finite index. So the index of $f^{-1}(Z)$ in $f^{-1}(V)$ is also finite. $f^{-1}(V)$ satisfies the Farrell-Jones conjecture with finite wreath products by Remark 8.12.

□

Let us now reduce the family occuring in Theorem 8.3 to the class of all virtually solvable groups:

Theorem 8.14. *Let V be a finitely generated free $Z[S^{-1}]$-module of rank n and let F be a finite group. The group $\mathrm{aut}_{Z[S^{-1}]}(V) \wr F$ which is isomorphic to $\mathrm{GL}_n(Z[S^{-1}])$ satisfies the K- and L-theoretic Farrell-Jones conjecture with respect to the class $\mathcal{VS}ol$.*

Proof. We will show this theorem via induction on n. If $\mathrm{rk}(V) = 1$ we get that $\mathrm{aut}_{Z[S^{-1}]}(V) \wr F \cong \mathrm{GL}_1(Z[S^{-1}]) \wr F$ is virtually abelian. Hence the group itself is virtually solvable. So a point is a model for $E_{\mathcal{VS}\mathrm{ol}}\, \mathrm{GL}_1(Z[S^{-1}]) \wr F$ and the isomorphism conjecture is trivially true. Let us now consider the case of general n:

Let $\mathcal{F}$ denote the family

$$\mathcal{VC}\mathrm{yc} \cup \{H \mid H \leq \mathrm{stab}(W), W \text{ is a nontrivial direct summand of } Z[S^{-1}]\}.$$

We already know that $\mathrm{GL}_n(Z[S^{-1}]) \wr F$ satisfies the isomorphism conjecture with respect to the family $\mathcal{F}^{\wr}$ by Theorem 8.3. Using the transitivity principle we have to show that any group in this family satisfies the Farrell-Jones conjecture with finite wreath products relative to the family $\mathcal{VS}$ol.

Since the isomorphism conjecture with finite wreath products passes to finite index overgroups by Remark 8.12, it suffices to consider a product of groups from $\mathcal{F}$. By Corollary 8.11 we may further restrict to the case of a group $G \in \mathcal{F}$.

We have to show that G satisfies the Farrell-Jones conjecture with finite wreath products for any $G \in \mathcal{F}$. If G is virtually cyclic $G \wr F$ is virtually abelian and hence virtually solvable. So the statement is trivial in this case.

Otherwise G is a subgroup of $\mathrm{stab}(W)$ for some nontrivial direct summand $W \subset Z[S^{-1}]^n$. So we can assume by Remark 8.6 that $G = \mathrm{stab}(W)$. Let F be any finite group and let $W^{\perp}$ denote any complement of $W \subset V$. We get an isomorphism $W \oplus W^{\perp} \cong V$ sending (w, w') to $w + w'$. All elements of $\mathrm{stab}(W)$ have block form with respect to this decomposition. Hence we get a short exact sequence:

$$1 \to \mathrm{hom}_{Z[S^{-1}]}(W^{\perp}, W) \to \mathrm{stab}(W) \xrightarrow{p} \mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp}) \to 1.$$

The map $\mathrm{stab}(W) \to \mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp})$ is given by

$$f \mapsto (f|_W, \mathrm{pr}_{W^{\perp}} \circ f \circ inc_{W^{\perp}}).$$

The isomorphism from $\mathrm{hom}_{Z[S^{-1}]}(W^{\perp}, W)$ to $\ker(p)$ is given by

$$f \mapsto ((w, w') \mapsto (w + f(w'), w').$$

Applying $- \wr F$ to the epimorphism in the upper short exact sequence we get

$$1 \to \hom_{Z[S^{-1}]}(W^{\perp}, W)^F \to \mathrm{stab}(W) \wr F \xrightarrow{p} (\mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp})) \wr F \to 1.$$

Both factors of $\mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp})$ satisfy the isomorphism conjecture with respect to the family $\mathcal{VS}$ol and hence also $\mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp})$ satisfies the isomorphism conjecture with respect to the family $\mathcal{VS}$ol. We want to apply Proposition 8.5. So we have to check that the preimage of any virtually solvable subgroup H of $\mathrm{aut}_{Z[S^{-1}]}(W) \times \mathrm{aut}_{Z[S^{-1}]}(W^{\perp})$ satisfies the isomorphism conjecture with respect to the family $\mathcal{VS}$ol.

We get a short exact sequence

$$1 \to \hom_{Z[S^{-1}]}(W^{\perp}, W)^F \to p^{-1}(H) \xrightarrow{p} H \to 1.$$

We can identify $\hom_{Z[S^{-1}]}(W^{\perp}, W)$ with the additive group of $\mathrm{rk}(W) \times \mathrm{rk}(W^{\perp})$-matrices over $Z[S^{-1}]$ since $W, W^{\perp}$ are finitely generated free $Z[S^{-1}]$-modules. Especially it is an abelian group.

The group $p^{-1}(H')$ is a solvable subgroup of $p^{-1}(H)$ of finite index where $H' \leq H$ denotes a solvable subgroup of finite index. Hence $p^{-1}(H)$ is also virtually solvable. So it trivially satisfies the isomorphism conjecture for the family of all virtually solvable subgroups. This completes the proof. □

Reducing the family further

The Farrell-Jones isomorphism conjecture has not been proved for all virtually solvable groups. If we consider the case of $\mathrm{GL}_n(\mathbb{Z})$ first we can use a theorem of Mal'cev that shows that every solvable subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is polycyclic.

Alternatively the proof of Theorem 8.14 can be carried out with the family $\mathcal{VS}$ol replaced by the family of virtually polycyclic groups. For two free $\mathbb{Z}$-modules $W, W^{\perp}$ the group $(\hom(W, W^{\perp}), +)$ is a finitely generated, free abelian group.

Hence any extension $1 \to (\hom(W, W^{\perp}), +) \to G \to P \to 1$ for a virtually polycyclic group is again virtually polycyclic.

The Farrell-Jones conjecture has been proved for virtually polycyclic groups in [3, Theorem 0.1] and so we can reduce the family to the family of virtually cyclic subgroups by the transitivity principle 8.4. So we obtain the following theorem as in [8, Main theorem]:

Theorem 8.15. *The group* $\mathrm{GL}_n(\mathbb{Z})$ *satisfies the Farrell-Jones conjecture in K- and L-theory with finite wreath products.*

The Farrell-Jones conjecture is unknown for the solvable group $\mathbb{Z}[\frac{1}{p}] \rtimes_{\cdot p} \mathbb{Z}$ and it occurs as the subgroup of $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{p}])$ generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

So we cannot reduce the family further for the ring $\mathbb{Z}[S^{-1}]$ for nonempty S.

Let us now consider the function field case. For two finitely generated free $F[t][S^{-1}]$-modules $W, W^\perp$ we have that $\hom(W, W^\perp)$ is an $F[t][S^{-1}]$-module. So it is a vector space over the prime field $K \cong \mathbb{F}_{\mathrm{char}(F)}$ of F and thus it is isomorphic to $\bigoplus \mathbb{Z}/\mathrm{char}(F)$ as an abelian group. Let us now consider group extensions of the form

$$1 \to \bigoplus_{\mathbb{N}} K \to G \to \mathbb{Z} \to 1.$$

A group G that fits into such a short exact sequence of groups will be called a $\bigoplus_{\mathbb{N}} K$ by $\mathbb{Z}$-group. Let us now consider some special cases. The definition of the restricted wreath product $A \wr' F$ of two groups A, F can be found in Section 10.1.

Lemma 8.16. *Let A be any finite abelian group.*

(i) The lamplighter group of A

$$A \wr' \mathbb{Z} := \langle \mathrm{map}(\mathbb{Z}, A), t \mid f = 0 \textit{ almost everywhere }, tft^{-1} = f(\_ + 1)\rangle$$

is a colimit of CAT(0) *groups with noninjective stucture maps. So it satisfies the Farrell-Jones conjecture with finite wreath products.*

(ii) A group that has a subgroup of finite index isomorphic to $A \wr' \mathbb{Z}$ satisfies the Farrell-Jones conjecture.

Proof. (i) Let G_i be the HNN-extension of A^{i+1} relative to α, where α is the isomorphism

$$\alpha : A^i \times \{0\} \to \{0\} \times A^i \qquad (a, 0) \mapsto (0, a).$$

Consider the diagram

$$\begin{array}{ccccc} \{0\} \times A^i & \longrightarrow & A^{i+1} & \longleftarrow & A^i \times \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} \times A^{i+1} & \longrightarrow & A^{i+2} & \longleftarrow & A^{i+1} \times \{0\} \end{array}$$

where the vertical maps are all given by adding zero in the last coordinate. This makes the diagram commute and thus we get a map of HHN-extensions $s_i : G_i \to G_{i+1}$. So we still have to identify $\mathrm{colim}\, G_i$ with $A \wr' \mathbb{Z}$. Consider the maps

$$f_n : G_n = \langle A^{n+1}, t \mid t(a_0, \ldots, a_{n-1}, 0)t^{-1} = (0, a_0, \ldots, a_{n-1})\rangle \to A \wr' \mathbb{Z} = \mathrm{map}(\mathbb{Z}, A) \rtimes \mathbb{Z}$$

given by

$$(a_0, \ldots, a_n) \mapsto \left(i \mapsto \begin{cases} a_i & 0 \le i \le n \\ 0 & \text{else} \end{cases} \right),$$

$$t \mapsto t.$$

For the sake of brevity let us denote by a_z We can easily verify $f_{i+1} \circ s_i = f_i$ on generators. This defines a group homomorphism $f : \operatorname{colim}_{i \in \mathbb{N}} A_i \to A \wr' \mathbb{Z}$. We have to show that it is bijective. The generating set $A \cup \{t\}$ of $A \wr' \mathbb{Z}$ is already contained in the image of f_0. So f is surjective.

To show the injectivity let us first consider elements of a special form. We have

$$f_n(t^m(a_0, \ldots, a_n)t^s) = ((z \mapsto \begin{cases} a_{z+m} & z \in [-m, n-m] \\ 0 & \text{else} \end{cases}, t^{m+s}).$$

So if two elements $t^m(a_0, \ldots, a_n)t^s$,$t^{m'}(a'_0, \ldots, a'_n)t^{s'}$ of this form have the same image under f we have

$$m + s = m' + s' \text{ and } a_{z+m} = a'_{z+m'} \text{ for all } z \in \mathbb{Z}$$

where a_{z+m} is defined to be zero whenever $z + m \notin [0, n]$; respectively $a'_{z+m'} := 0$ for $z + m' \notin [0, n]$. So the relation $t(a_0, \ldots, a_{n-1}, 0)t^{-1} = (0, a_0, \ldots, a_{n-1})$ already implies that both group elements are equal.

Now we have to consider general preimages. It suffices to show that any group element can be written in that form after stabilizing. Any element in G_n can be written as a product of the form

$$t^{r_1} b_1 t^{r_2} b_2 \ldots t^{r_m} b_m t^{r_{m+1}}$$

with $b_i \in A^{n+1}$ for some m. Let us say that such an element is of complexity $\leq m$. Let us now examine how stabilizing can be used to reduce the complexity from two to one:

$$\begin{aligned} & t^{r_1} b_1 t^{r_2} b_2 t^{r_3} \qquad \text{stabilize } |r_s|\text{-times} \\ \mapsto\ & t^{r_1}(b_{1,0}, \ldots, b_{1,n}, 0, \ldots, 0)t^{r_2}(b_{2,0}, \ldots, b_{2,n}, 0, \ldots, 0)t^{r_3} \\ =\ & \begin{cases} t^{r_1}(b_{1,0}, \ldots, b_{1,n}, 0, \ldots, 0)(0, \ldots, 0, b_{2,0}, \ldots, b_{2,n})t^{r_2+r_3} & r_2 \geq 0 \\ t^{r_1+r_2}(0, \ldots, 0, b_{1,0}, \ldots, b_{1,n})(b_{2,0}, \ldots, b_{2,n}, 0, \ldots, 0)t^{r_3} & r_2 < 0 \end{cases}. \end{aligned}$$

By iterating this procedure one can show that any group element gets mapped to a group element of complexity ≤ 1 after a finite amount of stabilizing steps. This finishes the proof of the injectivity of f.

G_i acts properly and cocompactly on a tree with vertex stabilizers conjugate to A^{i+1}, edge stabilizers conjugate to A^i and compact quotient. So the action is proper, isometric and cocompact. Hence G_i it is a CAT(0) group. So $A \wr' \mathbb{Z}$ is a colimit of CAT(0) group and it satisfies the Farrell-Jones conjecture with finite wreath products by Theorem 8.7 and Proposition 8.8.

(ii) Since $A \wr' \mathbb{Z}$ is finitely generated, so is any finite index overgroup G. Hence we can apply Corollary 10.2 to get an embedding $G \hookrightarrow (A \wr' \mathbb{Z}) \wr F$ for some finite group F. Hence the last item implies the claim since the Farrell-Jones conjecture passes to subgroups by Remark 8.6.

□

So we have showed that one specific $\bigoplus_{\mathbb{N}} K$ by $\mathbb{Z}$ group satisfies the Farrell-Jones conjecture. Let us now find a way how to build any $\bigoplus_{\mathbb{N}} K$ by $\mathbb{Z}$ out of virtually cyclic groups and the group $K \wr \mathbb{Z}$. Since $\mathbb{Z}$ is free any $\bigoplus_{\mathbb{N}} K$ by $\mathbb{Z}$-group arises as a semidirect product $\bigoplus_{\mathbb{N}} K \rtimes_{\varphi} \mathbb{Z}$ for some automorphism $\varphi \in \operatorname{aut}(\bigoplus_{\mathbb{N}} K)$. Picking an automorphism means choosing a $K[\mathbb{Z}]$-module structure on $\bigoplus_{\mathbb{N}} K$.

Lemma 8.17. *Let V be a $K[\mathbb{Z}]$-module whose underlying module is isomorphic to $\bigoplus_{\mathbb{N}} K$. Then V is the colimit of a family of $K[\mathbb{Z}]$-modules $(V_i)_{i\in\mathbb{N}}$ with $V_0 = 0$ such that V_{i+1}/V_i is a finite dimensional K-vector space or V_{i+1}/V_i is isomorphic to $K[\mathbb{Z}]$.*

Proof. The proof is straight forward. Pick a basis $(b_i)_{i\in\mathbb{N}}$ of $\bigoplus_{\mathbb{N}} K$. Let $V_0 := 0$. Assume we already constructed V_i. Let v_0 be the first basis vector of $(b_i)_{i\in\mathbb{N}}$ that is not contained in V_i. If there is none we have $V_i = \bigoplus_{\mathbb{N}} K$ and we can set $V_j := V_i$ for $j > i$.

This choice of v_0 guarantees that $V = \bigcup_{i\in\mathbb{N}} V_i$. Now let

$$V_{i+1} := V_i + K[\mathbb{Z}] \cdot v_0.$$

We have $K[\mathbb{Z}]$ surjects onto V_{i+1}/V_i via $x \mapsto x \cdot [v_0]$ and its kernel is an ideal of $K[\mathbb{Z}]$. Either it is trivial in which case $V_{i+1}/V_i \cong K[\mathbb{Z}]$ or it is nontrivial. Since $K[\mathbb{Z}]$ is a principal ideal domain we can pick a generator g. So we get

$$V_{i+1}/V_i \cong K[\mathbb{Z}]/(g)$$

and the latter is isomorphic to $K^{\deg(g)}$ as a K-vector space. □

Corollary 8.18. *Each $\bigoplus_{\mathbb{N}} K$ by $\mathbb{Z}$-group $V \rtimes_{\varphi} \mathbb{Z}$ can be written as a colimit of subgroups $V_i \rtimes_{\varphi} \mathbb{Z}$ with $V_0 = 0$ such that we have for any i:*

$$1 \to V_i \to V_{i+1} \rtimes \mathbb{Z} \to V_{i+1}/V_i \rtimes \mathbb{Z} \to 1.$$

The right term is either virtually cyclic if V_{i+1}/V_i is finite abelian or it is isomorphic to the lamplighter group $K \wr' \mathbb{Z}$.

Proposition 8.19. *Any virtually $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$ group G satisfies the Farrell-Jones conjecture with finite wreath products.*

Proof. Let us first consider the case when G is finitely generated.

First note that we embed G into a wreath product of a $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$-group H with a finite group F by Corollary 10.2. The last Corollary 8.18 tells us that we can write H as a colimit of groups of the form $V_i \rtimes_{\varphi} \mathbb{Z}$ with certain properties. So we can write $H \wr F$ as a colimit of groups $(V_i \rtimes_{\varphi} \mathbb{Z}) \wr F$. Since the Farrell-Jones conjecture with finite wreath products inherits to colimits by Proposition 8.8 it suffices to show it for those groups. Let us now show the following statement by induction. *The group $(V_i \rtimes \mathbb{Z}) \wr F$ satisfies the Farrell-Jones conjecture for each finite group F.*

V_0 is trivial. So $(V_0 \rtimes_{\varphi} \mathbb{Z}) \wr F$ is virtually abelian and hence a CAT(0) group.

For the induction step use the short exact sequence from the previous corollary. We can apply the functor $- \wr F$ to obtain the short exact sequence

$$1 \to \operatorname{map}(F, V_i) \to (V_{i+1} \rtimes \mathbb{Z}) \wr F \xrightarrow{p} (V_{i+1}/V_i \rtimes \mathbb{Z}) \wr F \to 1.$$

The right term satisfies the Farrell-Jones conjecture, since it is either virtually abelian if V_{i+1}/V_i is finite or isomorphic to the group $(K \wr' \mathbb{Z}) \wr F$. Hence it satisfies the Farrell-Jones conjecture by Lemma 8.9 respectively Lemma 8.16). So we only have to show that $p^{-1}(V)$ satisfies the Farrell-Jones conjecture with finite wreath products.

We want to apply Proposition 8.5. Let W be a virtually cyclic subgroup of $V_{i+1}/V_i \rtimes \mathbb{Z} \wr F$.

- If W is finite $p^{-1}(W)$ is virtually abelian and so it satisfies the Farrell-Jones conjecture with finite wreath products.
- Consider now the case where W is infinite cyclic and contained in the subgroup

  $$\mathrm{map}(F, V_{i+1}/V_i \rtimes \mathbb{Z}) \subset V_{i+1}/V_i \rtimes \mathbb{Z} \wr F.$$

  We can pick a preimage x of a generator of W to get a splitting of

  $$1 \to \mathrm{map}(F, V_i) \to p^{-1}(W) \xrightarrow{p} W \to 1.$$

  Now let us define an embedding

  $$p^{-1}(V) \hookrightarrow \mathrm{map}(F, V_i \rtimes_\varphi \mathbb{Z})$$

  sending a preimage x of a generator of W to $f \mapsto (0, pr_{\mathbb{Z}}(x(f)))$. On $\mathrm{map}(F, V_i)$ it is given by the inclusion into $\mathrm{map}(F, V_i \rtimes_\varphi \mathbb{Z})$. Here we used that W is contained in $\mathrm{map}(F, V_{i+1}/V_i \rtimes \mathbb{Z})$ so that we can evaluate x at some group element $f \in F$.

  Since $\mathrm{map}(F, V_i \rtimes_\varphi \mathbb{Z})$ is a subgroup of $(V_i \rtimes_\varphi \mathbb{Z}) \wr F$ it satisfies the Farrell-Jones conjecture by induction hypothesis.
- If W is an infinite virtually cyclic group we can find a finite index, infinite cyclic subgroup $W' \subset W$ contained in $\mathrm{map}(F, V_{i+1}/V_i \rtimes \mathbb{Z})$ as before. So $p^{-1}(W')$ is also a finite index subgroup of $p^{-1}(W)$. Hence we can find an embedding $p^{-1}(W) \to p^{-1}(W') \wr F'$ for some finite group F'. So we get the chain of embeddings:

  $$p^{-1}(W) \hookrightarrow p^{-1}(W') \wr F' \hookrightarrow ((V_i \rtimes_\varphi \mathbb{Z}) \wr F) \wr F' \hookrightarrow (V_i \rtimes_\varphi \mathbb{Z}) \wr (F \wr F').$$

  The last group in this chain of embeddings satisfies the Farrell-Jones conjecture by induction hypothesis. This completes the proof of the finitely generated case.

Now suppose G is not finitely generated. Then it can be written as a colimit of its finitely generated subgroups. But a subgroup S of the virtually $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$-group G is either virtually $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$ or virtually abelian by the following argument. Let G' be a $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$-subgroup of G of finite index.

- If $S \cap \bigoplus_{\mathbb{N}} k^n$ is finite then S is virtually finite by $\mathbb{Z}$ and hence virtually abelian. The term $\bigoplus_{\mathbb{N}} k^n$ denotes a choice of a $\bigoplus_{\mathbb{N}} k^n$ subgroup of G' with infinite cyclic quotient.
- If $S \cap G'$ is contained in the abelian group $S \cap \bigoplus_{\mathbb{N}} k^n$, S would be virtually abelian since

  $$[S : S \cap G'] = [S \cap G : S \cap G'] \leq [G : G'].$$

- Otherwise the subgroup $\bigoplus_{\mathbb{N}} k^n \cap S$ of $S \cap G'$ is an k-vector space of countable infinite dimension and the quotient $S/(\bigoplus_{\mathbb{N}} k^n \cap S)$ is an nontrivial infinite cyclic subgroup of $\mathbb{Z}$. Hence $S \cap G'$ is also a $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$-group. Since it has finite index in S it is finitely generated and S is a virtually $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$-group.

Thus S satisfies the Farrell-Jones conjecture with finite wreath products by Lemma 8.9 if it was virtually abelian and by the previous case if it was not virtually Abelian.

Hence G can be written as a colimit of groups satisfying the Farrell-Jones conjecture with finite wreath products. So G satisfies the Farrell-Jones conjecture with finite wreath products by Proposition 8.8. □

Theorem 8.20. *Let F be a finite field and S be a finite set of primes in the polynomial ring $F[t]$. Let V be a finitely generated, free $F[t][S^{-1}]$-module. Then the group $\mathrm{aut}_{F[t][S^{-1}]}(V) \cong \mathrm{GL}_n(F[t][S^{-1}])$ satisfies the Farrell-Jones conjecture.*

Proof. The proof is completely analogous to the proof of Theorem 8.14. Let us proceed by induction on $\mathrm{rk}(V)$. We already know that it satisfies the isomorphism conjecture with respect to the family

$$\mathcal{VC}\mathrm{yc} \cup \{H \leq \mathrm{stab}(W) \mid W \subset V \text{ is a nontrivial direct summand}\}.$$

By the transitivity principle we only have to show that the group $\mathrm{stab}(W)$ satisfies the Farrell-Jones conjecture for any nontrivial direct summand $W \subset V$. We again use the short exact sequence from the proof of Theorem 8.14:

$$1 \to \mathrm{hom}_{F[t][S^{-1}]}(W^\perp, W) \to \mathrm{stab}(W) \xrightarrow{p} \mathrm{aut}_{F[t][S^{-1}]}(W) \times \mathrm{aut}_{F[t][S^{-1}]}(W^\perp) \to 1.$$

$\mathrm{hom}_{F[t][S^{-1}]}(W^\perp, W)$ is a free $F[t][S^{-1}]$-module of rank $\mathrm{rk}(W)\,\mathrm{rk}(W^\perp)$. If we restrict the module structure to F we get that $\mathrm{hom}_{F[t][S^{-1}]}(W^\perp, W) \cong \bigoplus_{\mathbb{Z}} F$.

Since W is nontrivial we know that the rank of W and $W^\perp$ can be at most $\mathrm{rk}(V) - 1$. So the groups $\mathrm{aut}_{F[t][S^{-1}]}(W)$ and $\mathrm{aut}_{F[t][S^{-1}]}(W^\perp)$ satisfy the Farrell-Jones conjecture by induction hypothesis.

So does their product by Corollary 8.11. Now let $V \subset \mathrm{aut}_{F[t][S^{-1}]}(W) \times \mathrm{aut}_{F[t][S^{-1}]}(W^\perp)$ be any virtually cylic subgroup. If V is finite the group $p^{-1}(V)$ is virtually abelian and hence satisfies the Farrell-Jones conjecture by Lemma 8.9.

If V is infinite we can find an infinite cyclic subgroup V' of finite index. So $p^{-1}(V')$ has finite index in $p^{-1}(V)$ and it fits into the exact sequence

$$1 \to \mathrm{hom}_{F[t][S^{-1}]}(W^\perp, W) \to p^{-1}(V') \xrightarrow{p} V' \to 1.$$

So $p^{-1}(V')$ is a $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$ group and hence $p^{-1}(V)$ is a virtually $\bigoplus_{\mathbb{N}} k^n$ by $\mathbb{Z}$ group and hence it satisfies the Farrell-Jones conjecture by Proposition 8.19. So we can apply Lemma 8.5 to conclude that the group $\mathrm{stab}(W)$ satisfies the Farrell-Jones conjecture. This completes the proof. □

Let us summarize what has been shown in this section:

Theorem 8.21. *Let*

$$\mathcal{F} := \begin{cases} \mathcal{VS}ol & Z = \mathbb{Z}, S \neq \emptyset \\ \mathcal{VC}yc & else \end{cases}.$$

Then $\mathrm{GL}_n(Z[S^{-1}]) \wr F$ *satisfies the isomorphism conjecture in K- and L-theory relative to the family* $\mathcal{F}$ *for any finite group* F.

Proof. The function field case has been done in the previous theorem; the case of $\mathrm{GL}_n(\mathbb{Z})$ has been considered in Theorem 8.15 and the case of $\mathbb{Z}[S^{-1}]$ in Theorem 8.14. □

9 Extensions

9.1 Ring extensions

Again let Z be either $\mathbb{Z}$ or $F[t]$ for a finite field F and let S be a finite set of primes in Z. Assume that we have a ring R and an injective ring homomorphism $i : Z[S^{-1}] \hookrightarrow \mathrm{Cent}(R) \subset R$. This gives R the structure of a left-$Z[S^{-1}]$-module via

$$(x, r) \mapsto i(x) \cdot r.$$

In this situation multiplication with an element $r \in R$ is a $Z[S^{-1}]$-linear map. This gives a ring homomorphism $f : R \to \mathrm{End}_{Z[S^{-1}]}(R)$. Such rings R are also called an associative $Z[S^{-1}]$-algebras.

If we further assume that R is a finitely generated free $Z[S^{-1}]$-module of rank n we have that $\mathrm{End}_{Z[S^{-1}]}(R) \cong M_n(Z[S^{-1}])$.

The ring homomorphism f is injective since

$$f(r) = 0 \Rightarrow 0 = f(r)(1) := r \cdot 1 = r.$$

Hence we get in this situation an induced, injective ring homomorphism

$$M_m(R) \to M_m(M_n(Z[S^{-1}])) \cong M_{mn}(Z[S^{-1}]).$$

If we restrict to the group of units, we obtain an injective group homomorphism

$$\mathrm{GL}_m(R) \to \mathrm{GL}_{mn}(Z[S^{-1}]).$$

So $\mathrm{GL}_m(R)$ also satisfies the Farrell-Jones conjecture with finite wreath products as in Theorem 8.21.

The following lemma shows that the ring of S-integers in a finite field extension of Q has these properties. It uses the idea from [19].

Note that any ring homomorphism $\mathbb{Z} \to R$ automatically factors through the center of R. This is not true in general for polynomial rings over finite fields.

Lemma 9.1. *Let K be a finite extension of the quotient field Q of Z and let R be the ring of $Z[S^{-1}]$-integers.*

(i) Given any $x \in K$. Then there is a $\lambda \in Z$ such that λx is a Z-integer.

(ii) For any $Z[S^{-1}]$-integer x we have that $\mathrm{tr}_Q(\cdot x : K \to K)$ is also an $Z[S^{-1}]$-integer.

(iii) Given any Q-basis $\alpha_1, \dots, \alpha_n$ of K. Then

$$f : (K, +) \to Q^n \qquad \lambda \mapsto (\mathrm{tr}(\cdot\lambda\alpha_1), \dots, \mathrm{tr}(\cdot\lambda\alpha_n))$$

is an isomorphism of Q-vector spaces.

(iv) The ring of $Z[S^{-1}]$- integers in K is finitely generated as a $Z[S^{-1}]$-module.

Proof. (i) Since K/Q is a finite extension, it is algebraic. So we can find a polynomial $y^m + \sum_{i=0}^{m-1} \frac{a_i}{b_i} y^i$ with root x. Let $\lambda \in Z$ be the product of all denominators $b_0, \dots, b_{m-1}$. Then λx is a root of

$$\frac{1}{\lambda^m} y^m + \sum_{i=0}^{m-1} \frac{a_i}{b_i \lambda^i} y^i.$$

Multiplying the polynomial by the constant λ gives

$$y^m + \sum_{i=0}^{m-1} \frac{a_i \lambda^{m-i}}{b_i} y^i$$

Note that all coefficients are integral by the choice of λ and that the leading coefficient is a unit in Z. So λx is a Z-integer.

(ii) Let

$$p = t^n + \sum_{i=0}^{n-1} a_i t^i \in Z[S^{-1}][y]$$

be an polynomial with root y. Without loss of generality we may assume that it is irreducible. The ring $Z[S^{-1}]$ is a unique factorization domain. So Gauss' Lemma tells us that p is also irreducible over $Q = \mathrm{Quot}(Z[S^{-1}])$. So it is the minimal polynomial of x. We have $Q \subset Q(x) \subset K$. Pick a $Q(x)$- basis $b_1, \dots, b_m$ of K. Then

$$\{x^i b_j \mid 0 \le i < \deg(p), 1 \le j \le m\}$$

is a Q-Basis of K. The matrix of $\cdot x$ with respect to this basis has just m nonempty entries on the diagonal. They are all $-a_{n-1}$. Thus $\mathrm{tr}_Q(\cdot x : K \to K) = -m a_{n-1} \in Z[S^{-1}]$.

(iii) The upper map is a Q-linear map between two Q-vector spaces of the same dimension. So it suffices to show that it is injective.

Assume that there is a nonzero element γ in the Kernel of f. Write $\gamma^{-1} = \sum_{i=0}^{n} a_i \alpha_i$. Then we get

$$n = \mathrm{tr}_Q(\cdot 1) = \mathrm{tr}_Q(\cdot \gamma\gamma^{-1}) = \sum_{i=0}^{n} a_i \, \mathrm{tr}_Q(\cdot \gamma \alpha_i) = 0$$

The last equality holds since $f(\gamma) = 0$. Contradiction. So the map is injective and hence an isomorphism.

(iv) By (i) we can pick a Q-Basis of K consisting of S-integers. Let us restrict the map f from (iii) to the ring of $Z[S^{-1}]$-integers. By (ii) its image lies in $Z[S^{-1}]^n$. Over a principal ideal domain submodules of finitely generated modules are finitely generated. So the image is a finitely generated $Z[S^{-1}]$-module. The map f is injective by (iii). So the ring of $Z[S^{-1}]$-integers in K is a finitely generated as a $Z[S^{-1}]$-module.

□

9.2 Short exact sequences

The goal of this section is to show the Farrell-Jones conjecture for certain extension of groups. Let again Z be either $\mathbb{Z}$ or the polynomial ring $F[t]$ over a finite field F. Let S be a finite set of primes in Z.

Let us start with a brief observation:

Lemma 9.2. *Let G be a group and φ an automorphism of G. Then*

(i) the isomorphism type of $G \rtimes_{\varphi} \mathbb{Z}$ depends only on the class $[\varphi] \in \mathrm{Out}(G)$;

(ii) if $[\varphi] \in \mathrm{Out}(G)$ is a torsion element, then $G \rtimes_{\varphi} \mathbb{Z}$ contains a subgroup isomorphic to $G \times \mathbb{Z}$ of finite index.

Proof. (i) Let c_g denote conjugation with $g \in G$. Then the isomorphism

$$G \rtimes_{\varphi} \mathbb{Z} \to G \rtimes_{\varphi \circ c_g} \mathbb{Z}$$

is given by id_G and $(e, 1) \mapsto (g, 1)$, where e denotes the neutral element of G.

(ii) Let n be the order of φ. Using the previous item we get

$$G \rtimes_{\varphi} n\mathbb{Z} \cong G \rtimes_{\varphi^n} \mathbb{Z} \cong G \times \mathbb{Z}.$$

So we have found the desired subgroup of finite index.

□

Let us now consider extensions of $\mathbb{Z}$ first.

Lemma 9.3. *Let $n \geq 3$. If $Z = \mathbb{Z}$ assume that $S = \emptyset$.*

(i) Any extension of $\mathbb{Z}$ by $\mathrm{SL}_n(Z[S^{-1}])$ satisfies the Farrell-Jones conjecture with finite wreath products.

(ii) Any extension of $\mathbb{Z}$ by $\mathrm{GL}_n(Z[S^{-1}])$ satisfies the Farrell-Jones conjecture with finite wreath products.

Proof. (i) Let $G = \mathrm{SL}_n(Z[S^{-1}]) \rtimes_{\varphi} \mathbb{Z}$ be any such extension. We know by Proposition 10.14 that any automorphism has finite order in the outer automorphism group of $\mathrm{SL}_n(Z[S^{-1}])$. So by Lemma 9.2 we know that G has a finite index subgroup isomorphic to $\mathrm{SL}_n(Z[S^{-1}]) \times \mathbb{Z}$. $\mathrm{SL}_n(Z[S^{-1}])$ satisfies the Farrell-Jones conjecture with finite wreath products by Theorem 8.21 and $\mathbb{Z}$ does since it is a CAT(0) group by Theorem 8.7.

(ii) Abelianization induces a group homomorphism

$$f : \mathrm{GL}_n(Z[S^{-1}]) \rtimes_{\varphi} \mathbb{Z} \to \mathrm{GL}_n(Z[S^{-1}])_{ab} \rtimes_{\varphi_{ab}} \mathbb{Z}$$

$\mathrm{GL}_n(Z[S^{-1}])_{ab} \rtimes_{\varphi_{ab}} \mathbb{Z}$ is virtually polycyclic and hence it satisfies the Farrell-Jones conjecture with finite wreath products by [3][Theorem 0.1]. Note that the class of virtually polycyclic groups is closed under taking wreath products with finite groups. By Lemma 8.13(ii) it suffices to show that the kernel of f, which is isomorphic to $SL_n(Z[S^{-1}])$, and every preimage of an infinite cyclic subgroup satisfy the Farrell-Jones conjecture with finite wreath products. Theorem 8.21 shows this for $\ker(f)$ and every preimage of an infinite cyclic subgroup does so by Lemma 9.3(i).

□

Extensions of $\mathbb{Z}$ are the basic building blocks for the next proposition.

Proposition 9.4. *Suppose that a group G satisfies the Farrell-Jones conjecture with finite wreath products. Then*

(i) Any extension of G by $\mathrm{SL}_n(Z[S^{-1}])$ *satisfies the Farrell-Jones conjecture with finite wreath products.*

(ii) Any extension of G by $\mathrm{GL}_n(Z[S^{-1}])$ *satisfies the Farrell-Jones conjecture with finite wreath products.*

Proof. Given such an extension G'. Let $f : G' \to G$ be the projection map. This is just an application of Proposition 8.13(ii). We have to verify its assumptions. First $p^{-1}(1)$ which is either $\mathrm{SL}_n(Z[S^{-1}])$ or $\mathrm{GL}_n(Z[S^{-1}])$ satisfies the Farrell-Jones conjecture with finite wreath products by Theorem 8.21. Second we have to verify that each preimage of an infinite cyclic group satisfies the Farrell-Jones conjecture with finite wreath products. This has been done in Lemma 9.3. □

Remark 9.5. It turns out that the outer automorphism group of GL_n is not always torsion. Some explicit examples are given in Proposition 10.8(ii).

10 Appendix

10.1 Wreath product and group extensions

It might be very hard to classify all extensions of a group H by a group G. However, they all embed in a big group $G \wr H := G^H \rtimes H$ — the so called *wreath product* of G and H. The left-action of H on G^H is given by $(h, f) \mapsto f(h^{-1}-)$. There is also a so called *restricted wreath product* $G \wr' H$ defined the same way with G^H replaced by

$$\{f \in \mathrm{map}(G, H) \mid |f^{-1}(G \setminus \{0\})| < \infty\}.$$

Clearly both definitions agree if H is finite. As mentioned above we have

Lemma 10.1 (Universal embedding theorem)**.** *Given a short exact sequence of groups* $1 \to G \xrightarrow{i} H \xrightarrow{p} K \to 1$*. Then H embeds into $G \wr K$.*

Proof. Let us choose representatives of the cosets $H/i(G)$ such that the chosen representative of $(hi(G))^{-1} = h^{-1}i(G)$ is the inverse of the representative of $hi(G)$. Let $q : H \to H$ be the set-theoretic map that assigns to $h \in H$ the representative of $hi(G)$.

Define a map

$$j : H \to G \wr K \qquad h \mapsto (\sigma_h, p(h)),$$

where $\sigma_h \in G^K$ is defined as

$$\sigma_h(yi(G)) := i^{-1}(q(y^{-1}) \cdot h \cdot q(h^{-1}y)).$$

The element $q(y^{-1}) \cdot h \cdot q(h^{-1}y)$ lies in the image of the injective map i since it lies in the kernel of p. Especially $\sigma_h(1 \cdot i(G)) = h \cdot q(h^{-1})$. Let us show that the map j is injective. Given two group element $h, h' \in H$ with $j(h) = j(h')$. So $p(h) = p(h')$. Inserting the neutral element into $\sigma_h = \sigma_{h'}$ we get $h \cdot q(h^{-1}) = h' \cdot q(h'^{-1})$. But $p(h) = p(h')$ means that h and h' lie in the same coset. So do h^{-1} and h'^{-1}. So the chosen representatives $q(h^{-1})$ and $q(h'^{-1})$ agree and hence we get $h = h'$.

It remains to show that this j is a group homomorphism. This is just an explicit calculation and can be done. □

Corollary 10.2. *Suppose $G \leq H$ is a subgroup of finite index. Then H embeds into $G \wr F$ for some finite group F.*

Proof. Consider the conjugates of G in H. Note that hGh^{-1} depends only on the right coset hG of h. So there are only finitely many conjugates of G in H. Hence their intersection N is a finite index normal subgroup of H and so we get by Lemma 10.1 an embedding

$$H \to N \wr H/N \hookrightarrow G \wr F,$$

where $F := H/N$. The last map is induced by the inclusion $N \subset G$. □

Lemma 10.3. *Let G, H, K be groups. There is an embedding*

$$(G \wr H) \wr K \hookrightarrow G \wr (H \wr K).$$

Proof. This is basically just bookkeeping. As a set the left hand side is $(G^H \times H)^K \times F = G^{H\times K} \times H^K \times K$ and the right hand side is $G^{H^K \times K} \times H^K \times K$. The map is given by

$$i : (g, h, k) \mapsto ((h', k') \mapsto g(h'(k'), k'), h, k).$$

A lengthy but straightforward computation shows that this map is an injective group homomorphism. □

10.2 The Outer automorphism groups of $\mathrm{GL}_n(Z[S^{-1}])$

Let Z denote either $\mathbb{Z}$ or the polynomial ring $F[t]$ over a finite field F. Let S be a finite set of primes in Z, let $n \geq 3$ and let $V := Z[S^{-1}]^n$.

Let us apply Proposition 9.4 to short exact sequences of the form

$$1 \to GL_n(Z[S^{-1}]) \to G \to G'' \to 1.$$

Thatfor we need that the outer automorphism group of $GL_n(Z[S^{-1}])$ is finite. In this section we will examine when this is the case. O'Meara gave an explicit description of all automorphisms of $GL_n(Z[S^{-1}])$ in [26, Theorem A-D]. He shows that every automorphism has some specific form. So we can just check whether any automorphism of that form has finite order in the outer automorphism group.

Recall that for a ring automorphism $\sigma : R \to R$ and two R-modules a homomorphism $M \to \sigma^* N$ is called a σ-semilinear homomorphism from M to N. The restriction of the R-module structure along σ is denoted by σ^*. In other words a σ-semilinear homomorphism f from M to N is a homomorphism $M \to N$ of abelian groups such that

$$f(rm) = \sigma(r) f(m) \text{ for } r \in R, m \in M.$$

If f_1, f_2 are composable σ_1- resp. σ_2-semilinear homomorphisms, then $f_1 \circ f_2$ is $(\sigma_1 \circ \sigma_2)$-semilinear.

Define the dual of a σ-semilinear map $f : V \to W$ to be

$$f^* : W^* \to V^* \qquad w^* \mapsto \sigma^{-1} \circ w^* \circ f.$$

This definition is actually a bit tricky. A semilinear map could a priori be σ-semilinear for several ring automorphisms σ. The zero map for example is semilinear for any ring automorphism σ. But we have

Lemma 10.4. *Suppose R does not contain zero divisors and let $f : M \to R$ be semilinear. If there are two different ring automorphisms σ_1, σ_2 such that f is σ_i-semilinear for $i = 1, 2$, then $f = 0$.*

Proof. Pick an $r \in R$ with $\sigma_1(r) \neq \sigma_2(r)$. Then we have for any $m \in M$

$$\sigma_1(r) f(m) = f(rm) = \sigma_2(r) f(m).$$

Hence $(\sigma_1(r) - \sigma_2(r)) f(m) = 0$. Since R does not have zero divisors we get $f(m) = 0$ for each $m \in M$. So $f = 0$. □

The dual f^* of a σ-semilinear morphism is σ^{-1}-semilinear. A semilinear homomorphism is a σ-semilinear homomorphism for some σ. Furthermore the usual composition rules $(f \circ g)^* = g^* \circ f^*$ also hold in the semilinear world. Let us now introduce some specific kinds of self-homomorphisms of $GL_n(Z[S^{-1}])$.

Notation 10.5. We have the following self-homomorphisms of $G = GL_n(Z[S^{-1}])$ respectively $G = \mathrm{SL}_n(Z[S^{-1}])$.

(i) For a group homomorphism $\chi : G \to \mathrm{Cent}(G)$ we have a group homomorphism

$$P_\chi : G \to G \qquad A \mapsto \chi(A)A.$$

(ii) For a semilinear group homomorphism $g : V \to V$ we get an automorphism

$$\Phi_g : G \to G \qquad A \mapsto gAg^{-1}.$$

(iii) For a semilinear automorphism $g : V \to V^*$ we get an automorphism

$$\Psi_g : G \to G \qquad A \mapsto g^{-1}(A^{-1})^{tr} g.$$

Note that P_χ is in general neither injective nor surjective. Furthermore if Φ_g is id_R-semilinear, then Φ_g is an inner automorphism of G. Let us first establish some general properties of $GL_n(Z[S^{-1}])$.

Lemma 10.6. *(i) The group of units of* $Z[S^{-1}]$ *is isomorphic to* $Z^* \times \mathbb{Z}^S$ *via*

$$Z^* \times \mathbb{Z}^S \to Z[S^{-1}]^* \qquad (z, (a_s)_{s \in S}) \mapsto z \cdot \prod_{s \in S} s^{a_s}.$$

(ii) The center of $\mathrm{GL}_n(Z[S^{-1}])$ *consists of all elements of the form*

$$M(\lambda) : Z[S^{-1}]^n \to Z[S^{-1}]^n, \qquad v \mapsto \lambda v \text{ for some } \lambda \in Z[S^{-1}]^*.$$

It is isomorphic to $Z[S^{-1}]^*$ *via*

$$M : Z[S^{-1}]^* \to \mathrm{Cent}(GL_n(Z[S^{-1}]), \lambda \mapsto M(\lambda)$$

.

(iii) The determinant induces an isomorphism $\mathrm{GL}_n(Z[S^{-1}])_{Ab} \to Z[S^{-1}]^*$. *Equivalently the map* $\det : \mathrm{GL}_n(Z[S^{-1}]) \to Z[S^{-1}]^*$ *has the universal property of the Abelianization.*

Proof. (i) The inverse is given by $z \mapsto (z \cdot \prod_{s\in S} s^{-\nu_s(z)}, (\nu_s(z))_{s\in S})$ where ν_s denotes the s-adic valuation.

(ii) If a matrix commutes with all elementary matrices, then all off-diagonal entries have to be zero and all entries on the diagonal have to be equal. So the center has the desired form.

(iii) Let $E_{i,j}(z)$ denote the elementary matrix whose (i, j)-th entry is z, whose other off-diagonal entries are zero and whose other diagonal entries are one. Clearly the map is surjective as $E_{1,1}(z)$ is a preimage of $z \in Z[S^{-1}]^*$. For the injectivity we have to show any matrix with determinant one can be expressed as a product of commutators. Note that the following computation uses $n \geq 3$:

- Any off-diagonal elementary matrix $E_{i,j}(r)$ lies in the commutator subgroup:
$$[E_{i,k}(1), E_{k,j}(r)] = E_{i,j}(r) \text{ for } k \notin \{i, j\}.$$
- We have
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
And analogously any such signed swap matrices lies in the commutator subgroup.
- Whitehead Lemma: For any $z \in Z[S^{-1}]^*$ we have
$$\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1-z & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1-z^{-1} & 1 \end{pmatrix}.$$

$Z[S^{-1}]$ is an Euclidean ring (see Remark 6.2). The (extended) Euclidean Algorithm can be used to write any invertible matrix $A \in \mathrm{GL}_n(Z[S^{-1}])$ of determinant one as a product of matrices of the upper forms.

□

Let us identify $Z[S^{-1}]$ with the Abelianization of $\mathrm{GL}_n(Z[S^{-1}])$ and det with the Abelianization map. Let us now consider those self homomorphisms of the form P_χ. Especially we want to figure out when they are automorphisms.

Lemma 10.7. *(i) Given a homomorphism*
$$\chi : \mathrm{GL}_n(Z[S^{-1}]) \to \mathrm{Cent}(\mathrm{GL}_n(Z[S^{-1}])).$$
By Lemma 10.6(iii) and the identification in Lemma 10.6(ii) we may assume that it factors as
$$\chi = M \circ \overline{\chi} \circ \det$$
for some endomorphism χ' of $Z[S^{-1}]$. Then
$$(P_\chi)_{Ab} = n \circ \overline{\chi} + \mathrm{id} : Z[S^{-1}]^* \to Z[S^{-1}]^*.$$
We used the isomorphism from Lemma 10.6(ii) here. Especially if P_χ is invertible, so is $n \cdot \overline{\chi} + \mathrm{id}$,

(ii) For two such homomorphisms χ_1, χ_2 *we have*

$$P_{\chi_1} \circ P_{\chi_2} = P_{(n \cdot \overline{\chi_1} \circ \overline{\chi_2} + \overline{\chi_1} + \overline{\chi_2}) \circ \det}.$$

(iii) Suppose we pick χ *such that* P_χ *is an automorphism. Then* P_χ *is an inner automorphism if and only if* $\chi = 0$.

Proof. (i)

$$(P_\chi)_{Ab}(\det(A)) := \det(P_\chi(A)) = \det(M(\overline{\chi}(\det(A))) \circ A) = \overline{\chi}^n(\det(A)) \cdot \det(A).$$

It might be a bit confusing that the last term is written multiplicatively. Additively it is just $(D(n) \circ \overline{\chi} + \mathrm{id})(\det(A))$.

(ii) This is basically a straightforward computation, which unfortunately involves all possible structure on $\hom(Z[S^{-1}]^*, Z[S^{-1}]^*)$, namely the composition, addition and multiplication by scalars in $Z[S^{-1}]$.

$$\begin{aligned}
& P_{\chi_1} \circ P_{\chi_2}(f) \\
= \;& P_{\overline{\chi}_1 \circ \det}(M(\overline{\chi_2}(\det(f))) \circ f) \\
= \;& M(\overline{\chi}_1 \circ \det(M(\overline{\chi_2}(\det(f))) \circ f)) \circ M(\overline{\chi_2}(\det(f))) \circ f \\
= \;& M(\overline{\chi}_1(\overline{\chi_2}(\det(f))^n \cdot \det(f))) \circ M(\overline{\chi_2}(\det(f))) \circ f \\
= \;& M(\overline{\chi}_1 \circ \overline{\chi_2}(\det(f))^n \cdot \overline{\chi}_1(\det(f))) \circ M(\overline{\chi_2}(\det(f))) \circ f \\
= \;& M(\overline{\chi}_1 \circ \overline{\chi_2}(\det(f))^n \cdot \overline{\chi}_1(\det(f)) \cdot \overline{\chi_2}(\det(f))) \circ f \\
= \;& P_{(n \cdot \overline{\chi_1} \circ \overline{\chi_2} + \overline{\chi_1} + \overline{\chi_2}) \circ \det}(f).
\end{aligned}$$

Again in the step I used additive notation.

(iii) Clearly $P_0 = \mathrm{id}$ is inner. So let χ be chosen such that P_χ is an automorphism and $\chi \neq 0$. Then $\overline{\chi} \neq 0$ and we can find a $z \in Z[S^{-1}]^*$ such that $\overline{\chi}(z)$ is not the neutral element. The following endomorphism has trace one and determinant z:

$$f : V \to V \qquad v_i \mapsto \begin{cases} v_1 & i = 1 \\ (-1)^n z v_n & i = 2 \\ v_{i-1} & i \geq 3 \end{cases}.$$

Thus $\mathrm{tr}(P_\chi(f)) = \mathrm{tr}(M(\overline{\chi}(z)) \circ f) = \overline{\chi}(z) \cdot \mathrm{tr}(f) = \overline{\chi}(z)$. But $\overline{\chi}(z)$ is by construction not the neutral element of $Z[S^{-1}]^*$. An inner automorphism preserves traces. So P_χ cannot be inner since $\mathrm{tr}(f) = 1$.

□

Proposition 10.8. *(i) Let* $S \leq 1$. *Every automorphism of* $\mathrm{GL}_n(Z[S^{-1}])$ *of the form* P_χ *for some* $\chi : \mathrm{GL}_n(Z[S^{-1}]) \to \mathrm{cent}(\mathrm{GL}_n(Z[S^{-1}]))$ *has finite order in the outer automorphism group.*

(ii) Let $S \geq 2$. There is a homomorphism $\chi : \mathrm{GL}_n(Z[S^{-1}]) \to \mathrm{cent}(\mathrm{GL}_n(Z[S^{-1}]))$ such that P_χ is an automorphism of infinite order in the outer automorphism group.

Proof. (i) Using the isomorphism $Z[S^{-1}]^* \cong Z^* \times \mathbb{Z}^S$ we can write $\overline{\chi}$ as a block matrix

$$\begin{pmatrix} \overline{\chi}_{Z^*} & g \\ 0 & \overline{\chi}_{\mathbb{Z}^S} \end{pmatrix}$$

The zero occurs since there is no nontrivial homomorphism from the finite group Z^* to $\mathbb{Z}^S$. Since P_χ was assumed to be an automorphism, we have by Lemma 10.7(i) that $n\overline{\chi}+\mathrm{id}$ is invertible. Since it has block form we get that $n\overline{\chi}_{Z^*}+\mathrm{id}$ and $n\overline{\chi}_{\mathbb{Z}^S}+\mathrm{id}$ are invertible.

First let us show that $\overline{\chi}_{\mathbb{Z}^S}$ is zero. If $|S| = 0$ there is nothing to show. Otherwise note that there is no nontrivial endomorphism m of $\mathbb{Z}$ such that $mn+1$ is invertible since we assumed $n \geq 3$.

Second since the group Z^* is finite we may assume by passing to a suitable power of P_χ that $n\overline{\chi}_{Z^*} + \mathrm{id} = \mathrm{id}$, i.e. $n\overline{\chi} = 0$. But now the formula from Lemma 10.7(ii) simplifies to $P_\chi \circ P_{\chi'} = P_{\chi+\chi'}$. This gives inductively $(P_\chi)^m = P_{m\chi}$. Hence by passing to the $|Z^*|$-th power we can also assume that $\overline{\chi}_{Z^*} = 0$. Luckily $|Z^*|g : \mathbb{Z}^S \to Z^*$ also vanishes since the target is an abelian group of order $|Z^*|$. So after passing to a suitable power we get $\chi = 0$ and hence P_χ has finite order.

(ii) Let $\overline{\chi}$ be the endomorphism $0_{Z^*} \times \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} \times 0_{\mathbb{Z}^{|S|-2}}$ of $Z^* \times \mathbb{Z}^S \cong Z[S^{-1}]^*$. Then

$$P_{\overline{\chi}\circ\det}$$

is an automorphism of infinite order in $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}]))$.

An inverse is induced by $\overline{\chi}_2 := 0_{Z^*} \times \begin{pmatrix} 0 & -1 \\ -1 & n \end{pmatrix} \times 0_{\mathbb{Z}^{|S|-2}}$. We have

$$\begin{aligned} & n\overline{\chi} \circ \overline{\chi}_2 + \overline{\chi} + \overline{\chi}_2 \\ = \; & 0_{Z^*} \times (n\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ -1 & n \end{pmatrix} + \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & n \end{pmatrix}) \times 0_{\mathbb{Z}^{|S|-2}} \\ = \; & 0_{Z^*} \times (n\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}) \times 0_{\mathbb{Z}^{|S|-2}} \\ = \; & 0 \end{aligned}$$

And hence

$$P_{\overline{\chi}\circ\det} \circ P_{\overline{\chi}_2\circ\det} = P_0 = \mathrm{id}\,.$$

Analogously the other composition is also the identity. So $P_{\overline{\chi}\circ\det}$ is indeed an automorphism. An inner automorphism induces the identity on the Abelianization. But

$$(P_{\overline{\chi}\circ\det})_{Ab} = n \cdot \overline{\chi} + \mathrm{id} = \mathrm{id}_{Z^*} \times \begin{pmatrix} n^2+1 & n \\ n & 1 \end{pmatrix} \times \mathrm{id}_{\mathbb{Z}^{|S|-2}}$$

has infinite order since it has an eigenvalue of absolute value $\neq 1$. So no power of $P_{\overline{\chi}\circ\det}$ can be inner. Hence we have found an element in $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}]))$ of infinite order.

□

Note that the examples from Proposition 10.8(ii) also exist for $n = 0, 1, 2$. Let us now consider automorphisms of the form Φ_g:

Lemma 10.9. *Consider either* $\mathrm{GL}_n(Z[S^{-1}])$ *or* $\mathrm{SL}_n(Z[S^{-1}])$. *Let* $g : V \to V, h : V \to V^*$ *be* σ*-semilinear isomorphisms and let* χ *be given such that* P_χ *is an automorphism. Let* $f \in \mathrm{aut}(V)$ *be given.*

(i) $\mathrm{tr}(\Phi_g(f)) = \sigma(\mathrm{tr}(f))$,

(ii) g is linear (i.e. id*-semilinear) if and only if* Φ_g *is inner,*

(iii) $\Phi_g^m = \Phi_{g^m}$ *and* g^m *is* σ^m*-semilinear,*

(iv) Suppose there is a ring automorphism σ *of* $Z[S^{-1}]$ *of infinite order. Pick a basis* $v_1, \ldots, v_n$ *of* V *and let* $g' : V \to V$ *be the* σ*-semilinear automorphism of* V *fixing* $v_1, \ldots, v_n$*, i.e.*

$$\sum \lambda_i v_i \mapsto \sum \sigma(\lambda_i) v_i.$$

Then $\Phi_{g'}$ *has infinite order in* $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}])$.

Proof. (i) Pick a basis $v_1, \ldots, v_n$ of V and let $A = (a_{i,j})_{1\leq i,j\leq n}$ be the matrix of f with respect to this basis, i.e.

$$f(v_i) = \sum_{1\leq j\leq n} a_{i,j} v_j.$$

The matrix of $\Phi_g(f) = gfg^{-1}$ with respect to the basis $g(v_1), \ldots, g(v_n)$ is $\sigma(A)$ since

$$gfg^{-1}(g(v_i)) = g(\sum_{1\leq j\leq n} a_{i,j} v_j) = \sum_{1\leq j\leq n} \sigma(a_{i,j}) g(v_j).$$

So $\mathrm{tr}(\Phi_g(f)) = \sigma(\mathrm{tr}(f))$.

(ii) If $\sigma = \mathrm{id}$, then $g \in \mathrm{GL}_n(Z[S^{-1}])$ and hence Φ_g is inner.

Let us now consider the case $\sigma \neq \mathrm{id}$, i.e. there is a $z \in Z[S^{-1}]$ not fixed by σ. Consider an endomorphism f of V represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \times I_{n-2}.$$

This matrix has determinant one. We have

$$\mathrm{tr}(f) = x + (n-2) \neq \sigma(x) + (n-2) = \mathrm{tr}(\Phi_g(f)).$$

So Φ_g cannot be an inner automorphism since it changes the trace of a matrix.

(iii) $\Phi_g^m(f) = g^m \circ f \circ g^{-m} = \Phi_{g^m}(f)$ and $g^m(\lambda v) = \sigma^m(\lambda)g^m(v)$.

(iv) Let σ be an infinite order ring automorphism and let $n \geq 1$ be arbitrary. Then g^n is σ^n-semilinear and hence not inner by (ii).

□

Lemma 10.10. *Let k be a field. The group* $\mathrm{Gal}(k(t)/k)$ *is isomorphic to* $PGl_2(k)$ *where* $k(t)$ *denotes the field of rational function in the indeterminate t.*

Proof. We have a group injective homomorphism

$$\mathrm{PGL}_2(k) \to \mathrm{Gal}(k(t)/k) \qquad \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \mapsto (t \mapsto \frac{at+b}{ct+d}).$$

The crucial part is to show that each element φ of $\mathrm{Gal}(k(t) : k)$ is of that form. Let $t' \in k(t) \setminus k$ be any element. Write it in the form $\frac{p}{q}$ with $p, q \in k[t]$ coprime. Since $t' \notin k$ not both of p and q can be constant.

Then t is a root of $x \mapsto q(x)t' - p(x) \in k(t')[x]$. By Gauss' Lemma it is irreducible in $k(t')[x]$, iff it is irreducible in $k[t'][x]$. But since its t'-degree is one, any factorization of it would have one factor of t'-degree zero. Hence this factor would lie in $k[x]$. Especially it would divide both $p(x)$ and $q(x)$. Since they are coprime that factor has to a unit. Hence we have found the minimal polynomial of t over $k(t')$. I used here that t' cannot be algebraic over k, since otherwise t would also be. Thus $k[t']$ is really a polynomial ring and the t'-degree is well defined.

But now $[k(t) : k(t')] = \deg_x(q(x) - t'p(x)) = \max(\deg_x(p(x), \deg_x(q(x))$. Any automorphism $\varphi \in \mathrm{Gal}(k(t)/k)$ would map t to some element t' with $k(t) = k(t')$. Thus t' must be of the form $\frac{at+b}{ct+d}$ where $ad - bc \neq 0$ since otherwise we could cancel down the fraction to an element in k. This contradicts $k(t') = k(t)$. □

So let us now briefly consider ring automorphisms of $Z[S^{-1}]$.

Lemma 10.11. *(i) Every ring automorphism of* $\mathbb{Z}[S^{-1}]$ *is trivial,*

(ii) Every ring automorphism of $F[t][S^{-1}]$ *has finite order.*

Proof. (i) Let σ be a ring automorphism. I want to show that it fixes any element $\frac{a}{b}$. This is the unique element x satisfying $bx = a$. Any ring automorphism fixes the prime ring and hence $a = \sigma(a) = \sigma(bx) = b\sigma(x)$. So $\sigma(x) = x$.

(ii) First note that F is the algebraic closure of the prime ring in $F[t][S^{-1}]$. So any ring automorphism φ of $F[t][S^{-1}]$ will map the finite field F to itself. So by passing to a suitable power we may assume that $\varphi|_F = \mathrm{id}_F$.

Any ring automorphism φ will extend to an automorphism of the quotient field $F(t)$. The group of automorphisms of $F(t)$ that fix F are isomorphic to the finite group $\mathrm{PGL}_2(F)$ by Lemma 10.10. So by passing to a suitable power we may assume that $\varphi = \mathrm{id}$.

□

If we argued a bit more carefully we get a short exact sequence

$$1 \to \mathrm{PGL}_2(F) \to \mathrm{aut}(F[t][S^{-1}]) \to \mathrm{Gal}(F, \mathbb{F}_{\mathrm{char}(F)}) \to 1.$$

Let us now come to a positive result. Let me first state [26][Theorem A-D] without the TL_n case. Since O'Meara deals with the more general setting of any integral domain he had to consider more automorphisms, namely he allows for g also semilinear automorphisms of $Q \otimes V$ with the property $gV = \mathfrak{a}V$ for some fractional ideal $\mathfrak{a}$. But since we only consider principal ideal domains such a fractional ideal is of the form aV for some $a \in Q$ and we can consider $\frac{1}{a}g$. It is a semilinear automorphism of V and $\Phi_g = \Phi_{\frac{1}{a}g}$.

Theorem 10.12 ([26][Theorem A-D)**.** *] Let R be an integral domain and let $n \geq 3$. Let G be either* $\mathrm{GL}_n(R)$ *or* $\mathrm{SL}_n(R)$*. Then any automorphism of G can be expressed as $P_\chi \circ \Phi_g$ for some semilinear automorphism g of V or as $P_\chi \circ \Psi_h$ for some semilinear isomorphism $h : V \to V^*$.*

It might be a nice exercise to find an automorphism of $\mathrm{GL}_2(\mathbb{Z})$ that can not be expressed in this form. Let us now use this theorem to show that $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}])$ is a torsion group in all the remaining cases.

Proposition 10.13. *Let $n \geq 3$. Suppose that $|S| \leq 1$. Then* $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}]))$ *is torsion.*

Proof. Given any automorphism φ of $\mathrm{GL}_n(Z[S^{-1}])$. We have to find a power of it that is inner. Let us first assume that φ is of the form $P_\chi \circ \Psi_h$ for some σ-semilinear isomorphism $h : V \to V^*$. Let $ev : V \to V^{**}$ be the natural evaluation isomorphism. Thus $f^{**} = ev \circ f \circ ev^{-1}$ for any $f \in \mathrm{GL}_n(Z[S^{-1}])$. Then

$$\begin{aligned} & (P_\chi \circ \Psi_h)^2(f) \\ = \quad & P_\chi \circ \Psi_h(\chi(h^{-1}(f^*)^{-1}h)h^{-1}(f^*)^{-1}h) \\ = \quad & \chi(\ldots) \cdot \sigma^{-1} \circ \chi(h^{-1}(f^*)^{-1}h) \cdot h^{-1}((h^{-1}(f^*)^{-1}h)^*)^{-1}h \\ = \quad & \chi(\ldots) \cdot \sigma^{-1} \circ \chi(h^{-1}(f^*)^{-1}h) \cdot h^{-1}h^* ev f ev^{-1} h^{-*} h \\ = \quad & P_? \circ \Phi_{h^{-1}h^* ev}(f) \end{aligned}$$

So we see that φ^2 is of the other form. So it suffices to consider that case.

Let $\varphi = P_\chi \circ \Phi_g$ for some σ-semilinear automorphism $V \to V$. The same computation shows inductively that φ^n is of the form $P_{\chi'} \circ \Phi_{g^n}$. By our assumptions and Lemma 10.11 we know that the group of ring automorphisms of $Z[S^{-1}]$ is finite.

So there is an m such that $\sigma^m = \mathrm{id}$ and hence that g^m is linear. Hence Φ_{g^m} is an inner automorphism and $[\varphi^m] = P_{\chi'}$ in $\mathrm{Out}(\mathrm{GL}_n(Z[S^{-1}]))$. So it finally suffices to consider the following case.

Let $\varphi = P_\chi$ for some group homomorphism $\chi : \mathrm{GL}_n(Z[S^{-1}]) \to Z[S^{-1}]^*$. Then φ has finite order by Proposition 10.8. □

Proposition 10.14. $\mathrm{Out}(\mathrm{SL}_n(Z[S^{-1}]))$ *is torsion for $n \geq 3$.*

Proof. First recall that the Abelianization of $\mathrm{SL}_n(Z[S^{-1}])$ is trivial, so there are no nontrivial homomorphisms from $\mathrm{SL}_n(Z[S^{-1}])$ to its center.

Consequently Theorem 10.12 gives that any automorphism of $\mathrm{SL}_n(Z[S^{-1}])$ is of the form Ψ_h or Φ_g. The same computation as in Proposition 10.13 shows that the square of an automorphism of the form Ψ_h is of the other form. So it suffices to consider that case.

So consider Φ_g for a σ-semilinear automorphism $V \to V$. We know that the group of ring automorphisms of $Z[S^{-1}]$ is torsion, so there is an m such that $\sigma^m = \mathrm{id}$. Hence Φ_{g^m} is linear and hence inner by 10.9. □

10.3 Additive Categories and directed continuity

Let AddCat denote the category of small additive categories with additive functors as morphisms. All additive categories mentioned here will be small and the term "additive category" will mean small, additive category. Recall that a partially ordered set I is *directed*, if any two elements have an upper bound. A directed system of objects in a category $\mathcal{A}$ indexed over some directed set I consists of

- An object $X_i \in \mathcal{A}$ for each $i \in I$ and
- a morphism $s_{i,j} : X_i \to X_j$ for each pair $(i, j) \in I^2$ with $i \leq j$ such that for any triple (i, j, k) with $i \leq j \leq k$ we have $s_{j,k} \circ s_{i,j} = s_{i,k}$.

Those morphisms will be called structure maps. All our posets will be directed in this chapter and I will always denote a directed set.

Let us call a functor $F : \mathcal{C} \to \mathcal{D}$ between two categories with directed colimits *directed continuous* if the natural map

$$\mathrm{colim}_{i\in I} F(X_i) \to F(\mathrm{colim}_{i\in I} X_i)$$

is an isomorphism for any directed system of objects $(X_i)_{i\in I}$.

Lemma 10.15. *The category* AddCat *has directed colimits.*

Proof. Given a directed system of additive categories $\mathcal{A}_i$ with structure maps $S_{i,j} : \mathcal{A}_i \to \mathcal{A}_j$. Define a new category $\mathcal{C}$ whose objects are $\mathrm{colim}_{i\in I} \mathrm{Obj}(\mathcal{A}_i)$ and where the morphisms from an object represented by some $A_i \in \mathcal{A}_i$ to an object represented by some object A_j in $\mathcal{A}_j$ is given by $\mathrm{colim}_k \hom_{\mathcal{A}_k}(S_{i,k}(A_i), S_{j,k}(A_j))$. Denote the canonical functors $\mathcal{A}_i \to \mathcal{C}$ by $S_{i,\infty}$. Both colimits are taken in the category of sets. The morphism sets inherit the structure of an Abelian group and composition is still bilinear. A zero object is given by $S_{1,\infty}(0)$ where 0 is a zero object in $\mathcal{A}_1$.

The category $\mathcal{C}$ has finite biproducts. For a finite set of objects $A^1, \ldots, A^n$ we can pick representatives $A^1_k, \ldots, A^n_k$ in some $\mathcal{A}_k$, take their biproduct and map it to $\mathcal{C}$ via $S_{k,\infty}$. Taking the representatives $S_{m,k}(A^1_k), \ldots, S_{m,k}(A^n_k)$ in $\mathcal{A}_m$ instead, we would still get the same object.

We have to verify that these are really coproducts. Given a collection of morphisms $f^i : A^i \to B$ into some object B. We can pick representatives $f^i_k : A^i_k \to B_k$ in some $\mathcal{A}_k$. Then there is a unique morphism in $\mathcal{C}$ which makes the coproduct diagram commute.

Applying $S_{k,\infty}$ will then result in a commuting diagram in C. Thus we have found one morphism which fits in the coproduct diagram.

Suppose we have two morphisms f, g in C that both make the diagram commute. A diagram commutes, if certain compositions are equal. Since the morphisms in C are defined to be the directed colimits of the morphisms in $\mathcal{A}_*$, we can find a large number K and representatives of all morphisms such that the diagram in $\mathcal{A}_K$ obtained by replacing each morphism by its representative is already a commuting diagram in $\mathcal{A}_K$. But since the biproducts in $\mathcal{A}_K$ are coproducts there is a unique morphism that makes this diagram commute. Thus the two representatives of our two morphisms in C are equal. Hence $f = g$ and we have shown the coproduct property.

The universal property of a product can be verified completely analogously.

This completes the proof that C is an additive category. □

Given an additive category $\mathcal{A} \in \mathtt{AddCat}$ and a group G we can construct a new additive category $\mathcal{A} \rtimes G$ that generalizes the group ring construction ([9, Definition 2.1 and example 2.6]). This means that if $\mathcal{A}$ is the category of free R-modules than $\mathcal{A} \rtimes G$ is isomorphic to the category of free $R[G]$-modules. Furthermore twists are built into this construction to deal with twisted group rings. This means that the natural source of the functor $\mathcal{A} \rtimes -$ is the category of groups over $\mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$.

Let us recall the definition of $\mathcal{A} \rtimes G$ for an additive category and a group G over $\mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$ from [9, Definition 2.1 and example 2.6]. It is called there $\mathcal{A} *_G \mathrm{pt}$.

First note that the homomorphism $G \to \mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$ gives a right-G-action on the additive category $\mathcal{A}$. The objects of $\mathcal{A} \rtimes G$ are just the objects of $\mathcal{A}$ and a morphism from A to B is given by a collection of morphisms

$$(\varphi_g : A \to g^* B)_{g \in G}$$

where almost all φ_g are zero. The composition of two composable morphisms is given by

$$(\varphi'_g)_{g\in G} \circ (\varphi_g)_{g\in G} = (\sum_{g=kh} h^*(\varphi'_k) \circ \varphi_h)_{g\in G}.$$

Note this composition really uses the addition on the morphism sets of $\mathcal{A}$. A group homomorphism $f : G \to H$ over $\mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$) gives rise to a functor

$$\mathcal{A} \rtimes G \to \mathcal{A} \rtimes H$$

in the following way. It is defined to be the identity on objects and it sends a morphism $\varphi : A \to B$ to

$$\left(\sum_{g \in f^{-1}(h)} \varphi_g : A \to h^* B \right)_{h \in H} .$$

Since f is a homomorphism over $\mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$ the actions of G and H on $\mathcal{A}$ are related via $g^* B = f(g)^* B$. Thus this functor is really well defined. A computation shows that it is compatible with composition.

Lemma 10.16. *Given a directed system of groups $(G_i)_{i\in I}$ and a additive category $\mathcal{A}$. Then the functor*

$$\mathcal{A} \rtimes - : \mathtt{Groups}\ \textit{over}\ \mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A}) \to \mathtt{AddCat}$$

is directed continuous.

Proof. Given a directed system of groups $(f_{i,j} : G_i \to G_j)$ over $\mathrm{aut}_{\mathtt{AddCat}}(\mathcal{A})$. Let $f_{i,\infty} : G_i \to \mathrm{colim}_{i\in I} G_i$ denote the structure maps of the colimit. Then the natural functor

$$\mathrm{colim}_{i\in I} \mathcal{A} \rtimes G_i \to \mathcal{A} \rtimes \mathrm{colim}_{i\in I} G_i$$

is the identity on objects by definition of $\mathcal{A} \rtimes -$ and by construction of the directed colimit of additive categories in Lemma 10.15. We have to show that it induces a bijection on morphism sets. It sends a morphism represented by some $(\varphi = (\varphi_g)_{g\in G_i} : A \to B) \in \mathcal{A} \rtimes G_i$, to

$$(\sum_{g\in s_n^{-1}(h)} \varphi_g : A \to h^*B)_{h\in \mathrm{colim}_{i\in I} G_i}.$$

Let $\Psi : A \to B$ be any morphism in $\mathcal{A} \rtimes \mathrm{colim}_{i\in I} G_i$. By definition the set

$$S := \{g \in \mathrm{colim}_{i\in I} G_i \mid \Psi_g \neq 0\}$$

is finite. So we can find an $n \in I$ and for each g in this set a representative $r(g) \in G_n$. So $r : S \to G_n$ is just a map of sets. Now the morphism

$$\left(\begin{cases} \Psi_{s_m(h)} & h \in R(S) \\ 0 & \text{else} \end{cases}\right)_{h\in G_m} \in \hom_{\mathcal{A}\rtimes G_m}(A, B)$$

is a preimage of φ. This shows the surjectivity.

Assume that $\varphi \in \hom_{\mathcal{A}\rtimes G_n}(A, B)$ represents some element in the kernel of $\mathrm{colim}_{i\in I} \mathcal{A} \rtimes G_i \to \mathcal{A} \rtimes \mathrm{colim}_{i\in I} G_i$. Let $S \subset G_n$ be the set of all $g \in G_n$ with $\varphi_g \neq 0$. It is a finite set. By construction of a directed colimit of sets we can find for any two elements $g, g' \in S$ whose images in $\mathrm{colim}_{i\in I} G_i$ are equal some index $m_{g,g'}$ such that already their images under $f_{n,m_{g,g'}}$ are equal. Let m be the maximum of those $m_{g,g'}$. Thus we have that

$$f_{m,\infty}|_{f_{n,m}(S)} : f_{n,m}(S) \to f_{n,\infty}(S)$$

is a bijection.

Since φ represents some element in the Kernel

$$0 = \sum_{g\in f_{n,\infty}^{-1}(h)} \varphi_g \text{ for all } h \in \mathrm{colim}_{i\in I} G_i.$$

Of course, we could add the additional condition $g \in S$ in the sum since $\varphi_g = 0$ for $g \notin S$.

Thus either some element $x \in G_n$ is not contained in $f_{n,m}(S)$ in which case

$$\sum_{g\in S, g\in (f_{n,m})^{-1}(x)} \varphi_g = 0$$

since it is an empty sum or it is contained in $f_{n,m}(S)$ in which case

$$\sum_{g\in S, g\in (f_{n,m})^{-1}(x)} \varphi_g = \sum_{g\in S, g\in f_{n,\infty}^{-1}(f_{m,\infty}(x))} \varphi_g = 0$$

by assumption. Thus $(f_{n,m})_*(\varphi) = 0$ and hence φ represents the zero map in the colimit on the left hand side. This shows the injectivity. This completes the proof that $\mathcal{A} \rtimes -$ is compatible with directed colimits. □

The definition of a Karoubi filtration can be found in [12, Definition 3.2]. Note that strictly one cannot say that a inclusion of a subcategory is a Karoubi filtration. There is more data required, thus we should say, that a certain family of direct sum decompositions turns an inclusion of a subcategory into a Karoubi filtration. However in most cases it is quite obvious which decompositions we should take.

Definition 10.17. Let $\mathcal{A}$ denote an additive category. Let $C^b(\mathcal{A})$ denote the category of *controlled $\mathcal{A}$-objects over* $\mathbb{N}$. This means that an object is a sequence $(A_n)_{n\in\mathbb{N}}$ of objects in $\mathcal{A}$ and a morphism $\varphi : (A_m)_{m\in\mathbb{N}} \to (B_m)_{m\in\mathbb{N}}$ is given by a locally finite collection of morphisms $\varphi_{m,n} : A_m \to B_n$. Locally finite means that there are only finitely many nonzero morphisms starting in each object A_m and that there are only finitely many nonzero morphisms ending in each object B_n.

An object $(A_n)_{n\in\mathbb{N}}$ in $C(\mathcal{A})$ is called *bounded* iff only finitely many A_i are nonzero. Let $C^b(\mathcal{A})$ denote the full subcategory of bounded objects.

Remark 10.18. The category $C(\mathcal{A})$ is flasque. An Eilenberg swindle is given by

$$T : C(\mathcal{A}) \to C(\mathcal{A}) \qquad T(M)_n := \bigoplus_{m<n} M_m$$

on objects and on morphisms by

$$T(f : M \to N) : T(M) \to T(N) \qquad T(f)_{m,n} = \bigoplus_{0<k\le \min(m,n)} f_{m-k,n-k}.$$

The natural isomorphism $T \oplus \mathrm{id} \cong T$ is defined on a object $(A_i)_{i\in I}$ by the collection of morphisms $\varphi_{m,n}$ with

$$\varphi_{n,n+1} : \bigoplus_{m\le n} A_m \xrightarrow{\mathrm{id}} \bigoplus_{m<n+1} A_m$$

and $\varphi_{m,n} = 0$ for $n \neq m + 1$.

For an object $A \in C(\mathcal{A})$ let $A|_{\le n}$ denote the object defined by

$$(A|_{\le n}) = \begin{cases} A_m & m \le n \\ 0 & m > n \end{cases}.$$

Define $A_{\ge n}$ the analogous way. Note that $-_{\le n}$ is not a functor. We then have canoncial decompositions $A \cong A|_{\le n} \oplus A_{\ge n+1}$. These turn the inclusion $C^b(\mathcal{A}) \hookrightarrow C(\mathcal{A})$ into a Karoubi filtration. Denote its quotient by $C^{/b}(\mathcal{A})$.

Remark 10.19. The functor C^b is directed continuous, and the functor C is in general not directed continuous.

An object in $C(\mathrm{colim}_{i\in I}\mathcal{A}_i)$ is represented by a sequence of objects $A_i \in \mathcal{A}_{j(i)}$ whereas a object in $\mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ is represented by a sequence of objects $A_i \in \mathcal{A}_k$ for some k. It might happen that one cannot achieve a uniform bound on the sequence $(j(i))_{i\in\mathbb{N}}$ by changing the representative. Thus there is no hope that C is directed continuous.

However C^b behaves better. An object in $C^b(\mathrm{colim}_{i\in I}\mathcal{A}_i)$ is represented by a sequence of objects $A_i \in \mathcal{A}_{j(i)}$ where only finitely many objects are nonzero. An object in $\mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ is represented by a sequence of objects $A_i \in \mathcal{A}_k$ for some k. In this case we can take k to be the maximum of the set

$$\{j(i) \mid i \in \mathbb{N}, A_i \neq 0\}.$$

Similarly one can verify that two representatives of objects in $C^b(\mathrm{colim}_{i\in I}\mathcal{A}_i)$ represent the same object if and only if they do in $\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i)$. Thus both categories have the same object sets. The same argument then can be used to show that the morphisms in both categories are also the same. Thus $\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i)$ and $C^b(\mathrm{colim}_{i\in I}\mathcal{A}_i)$ are the same categories. The natural functor between them is the identity functor.

Lemma 10.20. *Given a directed system of additive categories $\mathcal{A}_i$. Let $j_{m,n} : \mathcal{A}_m \to \mathcal{A}_n$ denote the structure maps for $m \leq n$ and let $j_{m,\infty} : \mathcal{A}_m \to \mathrm{colim}_{i\in I}\mathcal{A}_i$ denote the canonical map to the colimit. Then $\mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ is flasque, the functor $\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i) \hookrightarrow \mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ is also a Karoubi filtration and the natural map*

$$\mathrm{colim}_{i\in I} C^{/b}(\mathcal{A}_i) \to \mathrm{colim}_{i\in I} C(\mathcal{A}_i)/\,\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i)$$

is an isomorphism.

Proof. Note that the Eilenberg-swindles $(T_i : C(\mathcal{A}_i) \to C(\mathcal{A}_i))$ as defined in remark 10.18 are compatible with the structure maps in the colimit. Thus they induce a functor

$$T : \mathrm{colim}_{i\in I} C(\mathcal{A}_i) \to \mathrm{colim}_{i\in I} C(\mathcal{A}_i).$$

The natural isomorphisms $T_i \oplus \mathrm{id} \cong T_i$ are also compatible with the structure maps and hence define a natural isomorphism $T \oplus \mathrm{id} \cong T$. Thus $\mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ is flasque.

The inclusion of $C^b(\mathcal{A}_i)$ into $C(\mathcal{A}_i)$ is a Karoubi filtration for each i. The induced functor $\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i) \to \mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ will also be a Karoubi filtration by the following argument. First we have to show that this functor is an inclusion of a full subcategory.

The map $\mathrm{colim}_{i\in I} S'_i \to \mathrm{colim}_{i\in I} S_i$ is injective for a directed system S_i of sets and a directed system S'_i of subsets. The explicit models for colimits given in Lemma 10.15 say that the object and morphism sets of a colimit of additive categories is just the colimit in the category of sets. Thus given a directed system of additive categories $\mathcal{D}_i$ and a sub-directed system of subcategories $\mathcal{D}'_i$ the functor $\mathrm{colim}_{i\in I}\mathcal{D}'_i \to \mathrm{colim}_{i\in I}\mathcal{D}_i$ is an inclusion of a subcategory. Let $f : M \to N$ be a morphism in $\mathrm{colim}_{i\in I} C(\mathcal{A}_i)$ such that the objects M and N lie in $\mathrm{colim}_{i\in I} C^b(\mathcal{A}_i)$. We can find a representative f_k :

$M' \to N'$ in $C(\mathcal{A}_k)$ such that M', N' lie in $C^b(\mathcal{A}_k)$. Since $C^b(\mathcal{A}_k)$ is a full subcategory of $C(\mathcal{A}_k)$ this morphism already lies in $C^b(\mathcal{A}_k)$ and thus f also lies in $\operatorname{colim}_{i \in I} C^b(\mathcal{A}_i)$. So we have shown that $\operatorname{colim}_{i \in I} C^b(\mathcal{A}_i)$ is a full subcategory of $\operatorname{colim}_{i \in I} C(\mathcal{A}_i)$.

Second we have to obtain a family of direct sum decompositions for each object $M \in \operatorname{colim}_{i \in I} C(\mathcal{A}_i)$. These are all decompositions of the form

$$M \cong j_{k,\infty}(M'|_{\leq n}) \oplus j_{k,\infty}(M'|_{\geq n+1}),$$

where M' is a representative of M in some $C(\mathcal{A}_k)$ and $M' \cong X' \oplus Y'$ with $X' \in C^b(\mathcal{A}_k), Y \in C(\mathcal{A}_k)$ is one of the given direct sum decompositions in $\mathcal{A}_k$ and where $j_{k,\infty} : C(\mathcal{A}_k) \to \operatorname{colim}_{i \in I} C(\mathcal{A}_i)$ is the structure map of the colimit. Note that it is compatible with direct sums since it is an additive functor. A choice (M', k, n) gives the same direct decomposition of M as the choice $(j_{k,k'}(M'), k', n)$. This will be called stabilization. Now we have to verify that the axioms of a Karoubi filtration hold (compare [12, Definition 3.2]).

(i) Recall that the set of those decompositions is ordered in the following way. Call $M \cong j_{k,\infty}(M'|_{\leq n}) \oplus j_{k,\infty}(M'|_{\geq n+1})$ smaller than $M \cong j_{\overline{k},\infty}(\overline{M}'|_{\leq \overline{n}}) \oplus j_{\overline{k},\infty}(\overline{M}'|_{\geq \overline{n}+1})$ if the composition

$$f : j_{k,\infty}(M'|_{\leq n}) \oplus j_{k,\infty}(M'|_{\geq n+1}) \cong M \cong j_{\overline{k},\infty}(M'|_{\leq \overline{n}}) \oplus j_{\overline{k},\infty}(M'|_{\geq \overline{n}+1})$$

has the property that

$$f(j_{\overline{k},\infty}(M'|_{\leq n})) \subset j_{\overline{k},\infty}(\overline{M}'|_{\leq \overline{n}}) \text{ and } j_{\overline{k},\infty}(M'|_{\geq n+1}) \subset f^{-1}(j_{\overline{k},\infty}(\overline{M}'|_{\geq \overline{n}+1})).$$

M' and $\overline{M}'$ denote representatives of M in $C(\mathcal{A}_k)$ resp. $C(\mathcal{A}_{k'})$ for some k, k'. We have to show that any two elements have a common upper bound.

Since M' and $\overline{M}'$ represent the same object in the colimit we can assume without loss of generality that they are equal by stabilizing. By stabilizing we can assume that $k = \overline{k}$. If $n \neq \overline{n}$ then the second decomposition is larger than the first one. Otherwise the first one is larger. Thus we have found the desired upper bound.

(ii) Given a map $A \to U$ with $A \in \operatorname{colim}_{i \in I} C^b(\mathcal{A}_i)$ and $U \in \operatorname{colim}_{i \in I} C(\mathcal{A}_i)$. Then there is a representative $f_k : A' \to U'$ in $C(\mathcal{A}_k)$ such that A' is an object of $C^b(\mathcal{A}_k)$. Since $C^b(\mathcal{A}_k) \subset C(\mathcal{A}_k)$ is a Karoubi filtration we can find a direct sum decomposition $U' \cong E_\alpha \oplus U_\alpha$ with $E_\alpha \in C^b(\mathcal{A}_k)$ such that f_k factors as $E' \to E_\alpha \hookrightarrow E_\alpha \oplus U_\alpha \cong U'$. Applying $j_{k,\infty}$ we get a factorization of

$$j_{k,\infty}(f_k) : j_k(E') \to j_k(E_\alpha) \hookrightarrow j_{k,\infty}(E_\alpha) \oplus j_k(U_\alpha) \cong j_k(U').$$

This can be simplified to

$$f : E \to j_{k,\infty}(E_\alpha) \hookrightarrow j_{k,\infty}(E_\alpha) \oplus j_{k,\infty}(U_\alpha) \cong U$$

and so we have found the desired direct sum decomposition.

(iii) This is exactly the same statement except that the arrows go in the other direction. Therefore the same proof works except that the arrows go in the other direction.

(iv) Given two objects M and N in $\operatorname{colim}_{i\in I} C(\mathcal{A}_i)$. A choice of a direct sum decomposition of these two objects also gives a direct sum decomposition of $M \oplus N$. We have to show that the poset of decompositions that arises this way is equivalent to the given poset of decompositions on $M \oplus N$. "Equivalent" means that both subposets of the poset of all decompositions are cofinal in each other.

Using again stabilization we get that new family of decompositions is given by

$$M \oplus N \cong (j_{k,\infty}(M'|_{\leq m}) \oplus j_{k,\infty}(N'|_{\leq n}) \oplus (j_{k,\infty}(M'|_{\geq m+1}) \oplus j_{k,\infty}(N'|_{\geq n+1}),$$

where $M', N' \in C(\mathcal{A}_k)$ are representatives of M, N for some k and m and n are some integers. Let us say that this decomposition is given by the choice (k, M', N', m, n). Then the old decompositions of $M \oplus N$ are given by all those choices where m and n are equal. Especially all the old choices arise this way. Thus we just have to check that they are cofinal in this bigger poset. The direct sum decomposition given by (k, M', N', m, n) is smaller than the direct sum decomposition given by $(k, M', N', \max(m, n), \max(m, n))$ and thus we have the cofinality.

So the inclusion of $\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i) \hookrightarrow \operatorname{colim}_{i\in I} C(\mathcal{A}_i)$ is a Karoubi filtration. We still have to show that the natural functor

$$J : \operatorname{colim}_{i\in I} C^{/b}(\mathcal{A}_i) \to \operatorname{colim}_{i\in I} C(\mathcal{A}_i)/\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i)$$

is an isomorphism. The object set of both categories is $\operatorname{colim}_{i\in I} \operatorname{Obj}(C(\mathcal{A}_i))$ and the functor induces the identity map on objects. A morphism is represented on both sides by a map $f \in \hom_{\mathcal{A}_k}(A, B)$ for some objects $A, B \in C(\mathcal{A}_k)$ for large k. The functor J sends the class represented by f on the left to the class represented by f on the right. Especially this shows the surjectivity on hom-sets. We have to show the injectivity.

On the left side two representatives f, f' are identified if there is some bigger number $\overline{k}$ such that $j_{k,\overline{k}}(f - f')$ factors over an object in $C^b(\mathcal{A}_{\overline{k}})$. On the right hand side they are identified if there is a $[g_{k'}] \in \hom_{\operatorname{colim}_{i\in I} C(\mathcal{A}_i)}(j_{k,\infty}(A), j_{k,\infty}(B))$ represented by some $g_{k'} : j_{k,k'}(A) \to j_{k,k'}(B)$ that factors over some object in $\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i)$ such that $j_{k',\infty}(g_{k'}) = j_{k,\infty}(f - f')$. But by the construction of the colimit this holds if and only if there is a $k'' \in \mathbb{N}$ such that $j_{k',k''}(g_{k'}) = j_{k,k''}(f - f')$. If we set $\overline{k} := k''$, then $j_{k,\overline{k}}(f - f')$ factors over some object in $\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i)$. This shows the injectivity on hom-sets. This completes the proof. □

Remark 10.21. First I thought that one could think of $\operatorname{colim}_{i\in I} C(\mathcal{A}_i)$ as the full subcategory of $C(\operatorname{colim}_{i\in I} \mathcal{A}_i)$ whose objects are the union of all the objects in the images of $C(\mathcal{A}_k) \to C(\operatorname{colim}_{i\in I} \mathcal{A}_i)$. Indeed the functor $\operatorname{colim}_{i\in I} C(\mathcal{A}_i) \to C(\operatorname{colim}_{i\in I} \mathcal{A}_i)$ is injective on object sets and the image consists of those objects mentioned above, but in general it is not injective on morphism sets. Thus we really need the whole argument from above to construct Karoubi filtrations.

Let us now focus on the directed continuity of those functors. We have the following observation:

Remark 10.22. The composition of two composable, directed continuous functors

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} C$$

is directed continuous. For a directed system of objects in $\mathcal{A}$ the natural map

$$\operatorname{colim}_{i\in I} FG(A_i) \to FG(\operatorname{colim}_{i\in I} A_i)$$

factors as

$$\operatorname{colim}_{i\in I} FG(A_i) \to F(\operatorname{colim}_{i\in I} G(A_i)) \to F(G(\operatorname{colim}_{i\in I} A_i)).$$

The left map is an isomorphism since F is directed continuous and the right map is an isomorphism since G is directed continuous. However it would also suffice to assume that applying F to the map $\operatorname{colim}_{i\in I} G(A_i) \to G(\operatorname{colim}_{i\in I} A_i)$ gives an isomorphism. Informally speaking the functor F should not see that G was not directed continuous. With this weaker assumption we can still conclude that $F \circ G$ is directed continuous even if G is not directed continuous. This idea will play a key role in the next proposition.

Proposition 10.23. *Non-connective K-Theory of additive categories is directed continuous. This means that for any directed system of additive categories $\mathcal{A}_i$ the canonical map*

$$\operatorname{colim}_{i\in I} K_n(\mathcal{A}_i) \to K_n(\operatorname{colim}_{i\in I} \mathcal{A}_i)$$

is an isomorphism.

Proof. The key idea of this proof can be found in [31, Lemma 6.3]. The proof will rely on the following facts, which can be found for example in [2, 2.1]:

(i) A Karoubi filtration induces a long exact sequence in non-connective K- theory.

(ii) A flasque additive category has vanishing K-theory.

(iii) A weak equivalence induces an isomorphism in K-theory.

(iv) The functor K_n is directed continuous for $n \geq 1$ ([28, (9) on page 20]).

The goal is to show by induction on n that K_{-n} is directed continuous.

We have a natural isomorphism

$$K_{-(n+1)} \cong K_{-n} \circ C^{/b} \text{ in } \operatorname{Fun}(\mathsf{AddCat}, \mathsf{Ab} - \mathsf{Groups}).$$

We can use Remark 10.22. So it suffices to show that the natural map $\operatorname{colim}_{i\in I} C^{/b}(\mathcal{A}_i) \to C^{/b}(\operatorname{colim}_{i\in I} \mathcal{A}_i)$ induces an isomorphism in K-theory. Consider the following commutative diagram of additive categories

$$\begin{array}{ccccc}
\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i) & \longrightarrow & \operatorname{colim}_{i\in I} C(\mathcal{A}_i) & \longrightarrow & \operatorname{colim}_{i\in I} C(\mathcal{A}_i)/\operatorname{colim}_{i\in I} C^b(\mathcal{A}_i)) \\
\downarrow & & \downarrow & & \downarrow \\
C^b(\operatorname{colim}_{i\in I} \mathcal{A}_i) & \longrightarrow & C(\operatorname{colim}_{i\in I} \mathcal{A}_i) & \longrightarrow & C^{/b}(\operatorname{colim}_{i\in I} \mathcal{A}_i)
\end{array}$$

We get natural maps between the long exact sequences associated to both Karoubi filtrations.

The additive categories $\operatorname{colim}_{i \in I} C(\mathcal{A}_i)$ and $C(\operatorname{colim}_{i \in I} \mathcal{A}_i)$ are flasque. We identified the quotient of $C(\operatorname{colim}_{i \in I} \mathcal{A}_i)$ by $C^b(\operatorname{colim}_{i \in I} \mathcal{A}_i)$ with $\operatorname{colim}_{i \in I} C^{/b}(\mathcal{A}_i)$ in Lemma 10.20. Thus we get:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_n(\operatorname{colim}_{i \in I} C(\mathcal{A}_i)/\operatorname{colim}_{i \in I} C^b(\mathcal{A}_i)) & \xrightarrow{\cong} & K_{n-1}(\operatorname{colim}_{i \in I} C^b(\mathcal{A}_i)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & K_n(C^{/b}(\operatorname{colim}_{i \in I} \mathcal{A}_i)) & \xrightarrow{\cong} & K_{n-1}(C^b(\operatorname{colim}_{i \in I} \mathcal{A}_i)) & \longrightarrow & 0. \end{array}$$

The right map is an isomorphism since C^b is directed continuous. Thus also the left map is an isomorphism. This completes the proof. □

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Zusammenfassung

In dieser Arbeit zeige ich die Farrell-Jones Vermutung für K- und L-Theorie in der allgemeinsten Form für die allgemeine lineare Gruppe $\mathrm{GL}_n(R)$ für einige Ringe R.

Der Beweis orientiert sich an dem Beweis der Farrell-Jones Vermutung für $\mathrm{GL}_n(\mathbb{Z})$ aus [8]. Dort wurde die Gruppenoperation von $\mathrm{GL}_n(\mathbb{Z})$ auf dem Raum der Skalarprodukte auf $\mathbb{R}^n$ benutzt. Ähnliche Räume findet man auch für die allgemeine lineare Gruppe von Polynomringen über endlichen Körpern. Dies sind die sogennanten affinen Gebäude. Desweiteren kann man in $\mathbb{Z}$ bzw. $F[t]$ für einen endlichen Körper F auch eine endliche Menge von Primidealen invertieren. Auch für diese Ringe findet man solche Räume.

Bartels-Lück haben in [4, Theorem 1.1] nachgewiesen, dass die Existenz bestimmter Überdeckungen die Farrell-Jones Vermutung impliziert. Für CAT(0) Gruppen wurden dann diese Überdeckungen in [5, Main Theorem] konstruiert. Nun sind die hier betrachteten Gruppen keine CAT(0) Gruppen. Sie operieren zwar proper und isometrisch auf einem CAT(0) Raum, jedoch ist diese Operation nicht kokompakt. Man kann mit [5, Main Theorem] einen kokompakten Teil überdecken und muss noch Überdeckungen vom Komplement definieren. Dazu werden Volumenfunktionen benutzt. Es wird nachgewiesen, dass diese Volumenfunktionen alle notwendigen Eigenschaften besitzen.

Leider funktioniert der Beweis nicht in allen Fällen. Für $\mathbb{Z}[S^{-1}]$ funktioniert er nur relativ zur Familie der virtuell auflösbaren Untergruppen. Dies liefert eine schwächere Version der Farrell-Jones Vermutung. Desweiteren kann man auch viele Ringerweiterungen bzw. Gruppenerweiterungen behandeln.

Im Wesentlichen liefert die Farrell-Jones Vermutung eine Möglichkeit die K-Theorie von Gruppenringen aus der K-Theorie von den Gruppenringen bestimmter Untergruppen zu berechnen. Viele interessante Hindernisse liegen in der K-Theorie von Gruppenringen - wie zum Beispiel Walls Endlichkeitshindernis oder die Whiteheadtorsion. Die Farrell-Jones Vermutung für eine Gruppe G impliziert diverse andere Vermutungen wie zum Beispiel die Borel-Vermutung für Mannigfaltigkeiten mit Fundamentalgruppe G, die Novikov-Vermutung für die Gruppe G und die Serre-Vermutung für G.