

Rigidity in equivariant stable homotopy theory

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For any finite group G , we show that the 2-local G -equivariant stable homotopy category, indexed on a complete G -universe, has a unique equivariant model in the sense of Quillen model categories. This means that the suspension functor, homotopy cofiber sequences and the stable Burnside category determine all “higher order structure” of the 2-local G -equivariant stable homotopy category, such as the equivariant homotopy types of function G -spaces. The theorem can be seen as an equivariant version of Schwede’s rigidity theorem at the prime 2.

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1 Introduction

The homotopy theory of topological spaces has been developed since the beginning of the last century and is a significant tool in various fields of mathematics. There are notions in the classical homotopy theory that are very important in applications, like, for example, weak homotopy equivalences, fibrations or CW-complexes. There are also key facts that relate these concepts and make them very useful, like the Whitehead theorem or the long exact homotopy sequences. Since the 1950's people started to observe that there are also other categories beside the category of topological spaces with similar homotopical notions and where it is possible to develop homotopy theory. Examples of such categories include the category of simplicial sets and the category of chain complexes of modules over a ring. This led to an axiomatization of homotopy theory.

In [Qui67], Quillen introduced *model categories* which are an abstract framework for doing homotopy theory. (See Subsection 2.1 for a brief introduction to model categories.) The key notions and facts from homotopy theory of topological spaces are axiomatically encoded in the notion of a model category. Basic examples of a model category are the category of topological spaces and the category of simplicial sets with weak homotopy equivalences and the category of chain complexes with quasi-isomorphisms. The language of model categories also provides a unified and more conceptual treatment of derived functors that generalizes the classical theory of derived functors which is very useful in algebraic topology and algebraic geometry [Qui67].

The framework of model categories makes precise what it means to consider objects up to a given notion of weak equivalence. More precisely, for any model category \mathcal{C} , one defines its homotopy category $\mathrm{Ho}(\mathcal{C})$ which is the localization of \mathcal{C} at the class of weak equivalences. The objects which were weakly equivalent in \mathcal{C} now become isomorphic in $\mathrm{Ho}(\mathcal{C})$. For example, if \mathcal{C} is the model category of topological spaces or simplicial sets, then $\mathrm{Ho}(\mathcal{C})$ is the classical homotopy category of CW-complexes. If \mathcal{C} is the category of chain complexes of modules over a ring R , then $\mathrm{Ho}(\mathcal{C})$ is the derived category of R .

Generally, when passing from a model category \mathcal{C} to its homotopy category $\mathrm{Ho}(\mathcal{C})$, one loses “higher homotopical information” such as homotopy types of mapping spaces in \mathcal{C} or the algebraic K -theory of \mathcal{C} . In particular, an existence of a triangulated equivalence of homotopy categories does not necessarily imply that two given models are Quillen equivalent to each other. Here is an easy example of such a loss of information. Let $\mathbf{Mod}\text{-}K(n)$ denote the model category of right modules over the n -th Morava K -theory $K(n)$ and let $\mathbf{dgMod}\text{-}\pi_*K(n)$ denote the model category of differential graded modules over the graded homotopy ring $\pi_*K(n)$. Then the homotopy categories $\mathrm{Ho}(\mathbf{Mod}\text{-}K(n))$ and $\mathrm{Ho}(\mathbf{dgMod}\text{-}\pi_*K(n))$ are triangulated equivalent, whereas the model categories $\mathbf{Mod}\text{-}K(n)$ and $\mathbf{dgMod}\text{-}\pi_*K(n)$ are not Quillen equivalent. The reason is that the homotopy types of function spaces in $\mathbf{dgMod}\text{-}\pi_*K(n)$ are products of Eilenberg-MacLane spaces which is not the case for $\mathbf{Mod}\text{-}K(n)$ (see e.g. [Pat12, A.1.10]).

Another important example which we would like to recall is due to Schlichting. It

is easy to see that for any prime p , the homotopy categories $\mathrm{Ho}(\mathbf{Mod}\text{-}\mathbb{Z}/p^2)$ and $\mathrm{Ho}(\mathbf{Mod}\text{-}\mathbb{F}_p[t]/(t^2))$ are triangulated equivalent. In [Sch02] Schlichting shows that the algebraic K -theories of the subcategories of compact objects of $\mathbf{Mod}\text{-}\mathbb{Z}/p^2$ and $\mathbf{Mod}\text{-}\mathbb{F}_p[t]/(t^2)$ are different for $p \geq 5$. It then follows from [DS04, Corollary 3.10] that the model categories $\mathbf{Mod}\text{-}\mathbb{Z}/p^2$ and $\mathbf{Mod}\text{-}\mathbb{F}_p[t]/(t^2)$ are not Quillen equivalent. Note that there is also a reinterpretation of this example in terms of differential graded algebras [DS09].

There are cases when one can recover the “higher homotopical information” from the triangulated structure of the homotopy category. An important example for such a recovery is provided by Schwede’s rigidity theorem [Sch01, Sch07] about the uniqueness of models for the stable homotopy category. Before stating this theorem precisely we review some historical background.

One of the most difficult problems of algebraic topology is to calculate the stable homotopy groups of spheres. There has been an extensive research in this direction establishing some remarkable results. A very important object used to do these kind of computations is the classical *stable homotopy category* SHC . This category was first defined in [Kan63] by Kan. Boardmann in his thesis [Boa64] constructed the (derived) smash product on SHC whose monoids represent multiplicative cohomology theories. In [BF78], Bousfield and Friedlander introduced a stable model category Sp of spectra with $\mathrm{Ho}(\mathrm{Sp})$ triangulated equivalent to SHC . The category Sp enjoys several nice point-set level properties. However, it does not possess a symmetric monoidal product that descends to Boardmann’s smash product on SHC . This initiated the search for new models for SHC that possess symmetric monoidal products. In the 1990’s several such models appeared: S -modules [EKMM97], symmetric spectra [HSS00], simplicial (continuous) functors [Lyd98] and orthogonal spectra [MMSS01]. All these models turned out to be Quillen equivalent to Sp (and hence, to each other) and this naturally motivated the following

Question. How many models does SHC admit up to Quillen equivalence?

In [Sch07], Schwede answered this question. He proved that the stable homotopy category is rigid, i.e., if \mathcal{C} is a stable model category with $\mathrm{Ho}(\mathcal{C})$ triangulated equivalent to SHC , then the model categories \mathcal{C} and Sp are Quillen equivalent. In other words, up to Quillen equivalence, there is a unique stable model category whose homotopy category is triangulated equivalent to the stable homotopy category. This theorem implies that all “higher order structure” of the stable homotopy theory, like, for example, homotopy types of function spaces, is determined by the suspension functor and the class of homotopy cofiber sequences.

Initiated by Schwede’s result, in recent years much research has been done on establishing essential uniqueness of models for certain homotopy categories. In [Roi07], Roitzheim shows that the $K_{(2)}$ -local stable homotopy category has a unique model. For other theorems of this type see [BR12] and [Hut12].

The present work establishes a new uniqueness result. It proves an *equivariant version* of Schwede’s rigidity theorem at the prime 2. Before formulating our main result, we would like to say a few words on equivariant stable homotopy theory.

The G -equivariant stable homotopy category (indexed on a complete G -universe), for any compact Lie group G , was introduced in the book [LMSM86]. Roughly speaking, the objects of this category are G -spectra indexed on finite dimensional G -representations. In this thesis we will work with the stable model category Sp_G^O of G -equivariant orthogonal spectra indexed on a complete G -universe [MM02]. The homotopy category of Sp_G^O is the G -equivariant stable homotopy category. The advantage of this model is that it possesses a symmetric monoidal product compatible with the model structure. As in the non-equivariant case, the G -equivariant stable homotopy category has some other monoidal models, like, for example, the category of orthogonal G -spectra equipped with the \mathbb{S} -model structure (flat model structure) [Sto11, Theorem 2.3.27], the model category of S_G -modules [MM02, IV.2] and the model category of G -equivariant continuous functors [Blu06]. For a finite group G , the model categories of G -equivariant topological symmetric spectra in the sense of [Man04] and [Hau13] are also monoidal models for the G -equivariant stable homotopy category. Note that all these model categories are known to be G - \mathbf{Top}_* -Quillen equivalent to each other (see [MM02, IV.1.1], [Blu06, 1.3], [Sto11, 2.3.31], [Man04] and [Hau13]).

Now we return to the actual content of this thesis. Suppose G is a finite group and H a subgroup of G . For any $g \in G$, let gH denote the conjugate subgroup gHg^{-1} . Then the map

$$g: \Sigma_+^\infty G/{}^gH \longrightarrow \Sigma_+^\infty G/H$$

in the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G^O)$, given by $[x] \mapsto [xg]$ on the point-set level, is called the *conjugation* map associated to g and H . Further, if K is another subgroup of G such that $K \leq H$, then we have the *restriction* map

$$\mathrm{res}_K^H: \Sigma_+^\infty G/K \longrightarrow \Sigma_+^\infty G/H$$

which is just the obvious projection on the point-set level. Moreover, there is also a map backwards, called the *transfer* map

$$\mathrm{tr}_K^H: \Sigma_+^\infty G/H \longrightarrow \Sigma_+^\infty G/K,$$

given by the Pontryagin-Thom construction (see e.g. [LMSM86, IV.3] or [tD87, II.8]). These morphisms generate the *stable Burnside (orbit) category* which is the full preadditive subcategory of $\mathrm{Ho}(\mathrm{Sp}_G^O)$ with objects the stable orbits $\Sigma_+^\infty G/H$, $H \leq G$ [LMSM86, V.9] (see also [Lew98]).

Let G be a finite group. We say that a model category \mathcal{C} is a G -equivariant stable model category if it is enriched, tensored and cotensored over the category G - \mathbf{Top}_* of pointed G -spaces in a compatible way (i.e., the pushout-product axiom holds) and if the adjunction

$$S^V \wedge -: \mathcal{C} \rightleftarrows \mathcal{C} : \Omega^V(-)$$

is a Quillen equivalence for any finite dimensional orthogonal G -representation V .

All the models for the G -equivariant stable homotopy category mentioned above are G -equivariant stable model categories. Different kinds of equivariant spectra indexed

on incomplete universes provide examples of G - \mathbf{Top}_* -model categories which are not G -equivariant stable model categories but are stable as underlying model categories.

Here is the main result of this thesis:

Theorem 1.1.1. *Let G be a finite group, \mathcal{C} a cofibrantly generated, proper, G -equivariant stable model category, and let $\mathrm{Sp}_{G,(2)}^O$ denote the 2-localization of Sp_G^O . Suppose that*

$$\Psi: \mathrm{Ho}(\mathrm{Sp}_{G,(2)}^O) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C})$$

is an equivalence of triangulated categories such that

$$\Psi(\Sigma_+^\infty G/H) \cong G/H_+ \wedge^{\mathbf{L}} \Psi(\mathbb{S}),$$

for any $H \leq G$. Suppose further that the latter isomorphisms are natural with respect to the restrictions, conjugations and transfers. Then there is a zigzag of G - \mathbf{Top}_ -Quillen equivalences between \mathcal{C} and $\mathrm{Sp}_{G,(2)}^O$.*

In fact, we strongly believe that the following integral version of Theorem 1.1.1 should be true:

Conjecture 1.1.2. Let G be a finite group and let \mathcal{C} be a cofibrantly generated, proper, G -equivariant stable model category. Suppose that

$$\Psi: \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C})$$

is an equivalence of triangulated categories such that

$$\Psi(\Sigma_+^\infty G/H) \cong G/H_+ \wedge^{\mathbf{L}} \Psi(\mathbb{S}),$$

for any $H \leq G$. Suppose further that the latter isomorphisms are natural with respect to the restrictions, conjugations and transfers. Then there is a zigzag of G - \mathbf{Top}_* -Quillen equivalences between \mathcal{C} and Sp_G^O .

Note that if G is trivial, then the statement of Conjecture 1.1.2 is true. This is Schwede's rigidity theorem [Sch07]. (Or, more precisely, a special case of it, as the model category in Schwede's theorem need not be cofibrantly generated, topological or proper.) The solution of Conjecture 1.1.2 would in particular imply that all "higher order structure" of the G -equivariant stable homotopy theory such as, for example, equivariant homotopy types of function G -spaces, is determined by the suspension functor, the class of homotopy cofiber sequences and the basic π_0 -information of $\mathrm{Ho}(\mathrm{Sp}_G^O)$, i.e., the stable Burnside (orbit) category.

The proof of Theorem 1.1.1 is divided into two main parts: The first is categorical and the second is computational. The categorical part of the proof is mainly discussed in Section 3 and essentially reduces the proof of Conjecture 1.1.2 to showing that a certain exact endofunctor

$$F: \mathrm{Ho}(\mathrm{Sp}_G^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

is an equivalence of categories. The computational part shows that 2-locally the endofunctor is indeed an equivalence of categories. The proof starts by generalizing Schwede's arguments from [Sch01] to free (naive) G -spectra. From this point on, classical techniques of equivariant stable homotopy theory enter the proof. These include the Wirthmüller isomorphism, geometric fixed points, isotropy separation and the tom Dieck splitting. The central idea is to do induction on the order of subgroups and use the case of free G -spectra as the induction basis.

The only part of the proof of Theorem 1.1.1 which uses that we are working 2-locally is the part about free G -spectra in Section 4. The essential fact one needs here is that the self map $2 \cdot \text{id}: M(2) \rightarrow M(2)$ of the mod 2 Moore spectrum is not zero in the stable homotopy category. For p an odd prime, the map $p \cdot \text{id}: M(p) \rightarrow M(p)$ is equal to zero and this makes a big difference between the 2-primary and odd primary cases. Observe that the nontriviality of $2 \cdot \text{id}: M(2) \rightarrow M(2)$ amounts to the fact that $M(2)$ does not possess an A_2 -structure with respect to the canonical unit map $\mathbb{S} \rightarrow M(2)$. In fact, for any prime p , the mod p Moore spectrum $M(p)$ has an A_{p-1} -structure but does not admit an A_p -structure. The obstruction for the latter is the element $\alpha_1 \in \pi_{2p-3}\mathbb{S}_{(p)}$. This is used by Schwede to obtain the integral rigidity result for the stable homotopy category in [Sch07]. It seems to be rather nontrivial to generalize Schwede's obstruction theory arguments about coherent actions of Moore spaces [Sch07] to the equivariant case.

This thesis is organized as follows. Section 2 contains some basic facts about model categories and G -equivariant orthogonal spectra. We also review the level and stable model structures on the category of orthogonal G -spectra. In Section 3 we discuss the categorical part of the proof. Here we introduce the category of orthogonal G -spectra $\text{Sp}_G^O(\mathcal{C})$ internal to an equivariant model category \mathcal{C} and show that if \mathcal{C} is stable in an equivariant sense and additionally satisfies certain technical conditions, then \mathcal{C} and $\text{Sp}_G^O(\mathcal{C})$ are Quillen equivalent. This allows us to reduce the proof of Theorem 1.1.1 to showing that a certain exact endofunctor F of $\text{Ho}(\text{Sp}_{G,(2)}^O)$ is an equivalence of categories. In Section 4 we show that F becomes an equivalence when restricted to the full subcategory of free G -spectra.

In Section 5 we prove that it is sufficient to check that the induced map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G$$

is an isomorphism for any subgroup H of G . This is then verified inductively in Section 7. The results of Section 4 are used for the induction basis. The induction step uses geometric fixed points and a certain short exact sequence which we review in Section 6.

2 Preliminaries

2.1 Model categories

A *model category* is a bicomplete category \mathcal{C} equipped with three classes of morphisms called weak equivalences, fibrations and cofibrations, satisfying certain axioms. We will not list these axioms here. The point of this structure is that it allows one to “do homotopy theory” in \mathcal{C} . Good references for model categories include [DS95], [Hov99] and [Qui67].

The fundamental example of a model category is the category of topological spaces ([Qui67], [Hov99, 2.4.19]). Further important examples are the category of simplicial sets ([Qui67], [GJ99, I.11.3]) and the category of chain complexes of modules over a ring [Hov99, 2.3.11].

For any model category \mathcal{C} , one has the associated *homotopy category* $\mathrm{Ho}(\mathcal{C})$ which is defined as the localization of \mathcal{C} with respect to the class of weak equivalences (see e.g., [Hov99, 1.2] or [DS95]). The model structure guarantees that we do not face set theoretic problems when passing to localization, i.e., $\mathrm{Ho}(\mathcal{C})$ has Hom-sets.

A *Quillen adjunction* between two model categories \mathcal{C} and \mathcal{D} is a pair of adjoint functors

$$F: \mathcal{C} \rightleftarrows \mathcal{D} : E,$$

where the left adjoint F preserves cofibrations and acyclic cofibrations (or, equivalently, E preserves fibrations and acyclic fibrations). We refer to F as a left Quillen functor and to E as a right Quillen functor. Quillen’s total derived functor theorem (see e.g., [Qui67] or [GJ99, II.8.7]) says that any such pair of adjoint functors induces an adjunction

$$\mathbf{L}F: \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbf{R}E.$$

The functor $\mathbf{L}F$ is called the left derived functor of F and $\mathbf{R}E$ the right derived functor of E . If $\mathbf{L}F$ is an equivalence of categories (or, equivalently, $\mathbf{R}E$ is an equivalence), then the Quillen adjunction is called a *Quillen equivalence*.

Next, recall ([Qui67], [Hov99, 6.1.1]) that the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a pointed model category \mathcal{C} supports a *suspension* functor

$$\Sigma: \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

with a right adjoint *loop* functor

$$\Omega: \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

If the functors Σ and Ω are inverse equivalences, then the pointed model category \mathcal{C} is called a *stable model category*. For any stable model category \mathcal{C} , the homotopy category $\mathrm{Ho}(\mathcal{C})$ is naturally triangulated [Hov99, 7.1]. The suspension functor is the shift and the distinguished triangles come from the cofiber sequences. (We do not recall triangulated categories here and refer to [GM03, Chapter IV] or [Wei94, 10.2] for the necessary background.)

Examples of stable model categories are the model category of chain complexes and also various model categories of spectra (S -modules [EKMM97], orthogonal spectra [MMSS01], symmetric spectra [HSS00], sequential spectra [BF78]).

For any stable model category \mathcal{C} and objects $X, Y \in \mathcal{C}$, we will denote the abelian group of morphisms from X to Y in $\text{Ho}(\mathcal{C})$ by $[X, Y]^{\text{Ho}(\mathcal{C})}$.

Next, let us quickly review cofibrantly generated model categories. Here we mainly follow [Hov99, Section 2.1]. Let I be a set of morphisms in an arbitrary cocomplete category. A *relative I -cell complex* is a morphism that is a (possibly transfinite) composition of coproducts of pushouts of maps in I . A map is called *I -injective* if it has the right lifting property with respect to I . An *I -cofibration* is map that has the left lifting property with respect to I -injective maps. The class of I -cell complexes will be denoted by $I\text{-cell}$. Next, $I\text{-inj}$ will stand for the class of I -injective maps and $I\text{-cof}$ for the class of I -cofibrations. It is easy to see that $I\text{-cell} \subset I\text{-cof}$. Finally, let us recall the notion of smallness. An object K of a cocomplete category is small with respect to a given class \mathcal{D} of morphisms if the representable functor associated to K commutes with colimits of large enough transfinite sequences of morphisms from \mathcal{D} . See [Hov99, Definition 2.13] for more details.

Definition 2.1.1 ([Hov99, Definition 2.1.17]). Let \mathcal{C} be a model category. We say that \mathcal{C} is *cofibrantly generated*, if there are sets I and J of maps in \mathcal{C} such that the following hold:

- (i) The domains of I and J are small relative to $I\text{-cell}$ and $J\text{-cell}$, respectively.
- (ii) The class of fibrations is $J\text{-inj}$.
- (iii) The class of acyclic fibrations is $I\text{-inj}$.

Here is a general result that will be used in this thesis:

Proposition 2.1.2 (see e.g. [Hov99, Theorem 2.1.19]). *Let \mathcal{C} be a category with small limits and colimits. Suppose \mathcal{W} is a subcategory of \mathcal{C} and I and J are sets of morphisms of \mathcal{C} . Assume that the following conditions are satisfied:*

- (i) *The subcategory \mathcal{W} satisfies the two out of three property and is closed under retracts.*
- (ii) *The domains of I and J are small relative to $I\text{-cell}$ and $J\text{-cell}$, respectively.*
- (iii) *$J\text{-cell} \subset \mathcal{W} \cap I\text{-cof}$.*
- (iv) *$I\text{-inj} = \mathcal{W} \cap J\text{-inj}$.*

Then \mathcal{C} is a cofibrantly generated model category with \mathcal{W} the class of weak equivalences, $J\text{-inj}$ the class fibrations and $I\text{-cof}$ the class of cofibrations.

Note that the set I is usually referred to as a set of generating cofibrations and J as a set of generating acyclic cofibrations.

Further, we recall the definitions of monoidal model categories and enriched model categories.

Definition 2.1.3 (see e.g. [Hov99, Definition 4.2.6]). A *monoidal model category* is a closed symmetric monoidal category \mathcal{V} together with a model structure such that the following conditions hold:

(i) (The pushout-product axiom) Let $i: K \rightarrow L$ and $j: A \rightarrow B$ be cofibrations in the model category \mathcal{V} . Then the induced map

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \rightarrow L \wedge B$$

is a cofibration in \mathcal{V} . Furthermore, if either i or j is an acyclic cofibration, then so is $i \square j$.

(ii) Let $q: QI \rightarrow I$ be a cofibrant replacement for the unit I . Then the maps

$$q \wedge 1: QI \wedge X \rightarrow I \wedge X \quad \text{and} \quad 1 \wedge q: X \wedge QI \rightarrow X \wedge I$$

are weak equivalences for any cofibrant X .

Definition 2.1.4 (see e.g. [Hov99, Definition 4.2.18]). Let \mathcal{V} be a monoidal model category. A \mathcal{V} -*model category* is a model category \mathcal{C} with the following data and properties:

(i) The category \mathcal{C} is enriched, tensored and cotensored over \mathcal{V} (see [Kel05, Section 1.2 and Section 3.7]). This means that we have *tensors* $K \wedge X$ and *cotensors* X^K and mapping objects $\text{Hom}(X, Y) \in \mathcal{V}$ for $K \in \mathcal{V}$ and $X, Y \in \mathcal{C}$ and all these functors are related by \mathcal{V} -enriched adjunctions

$$\text{Hom}(K \wedge X, Y) \cong \text{Hom}(X, Y^K) \cong \text{Hom}(K, \text{Hom}(X, Y)).$$

(ii) (The pushout-product axiom) Let $i: K \rightarrow L$ be a cofibration in the model category \mathcal{V} and $j: A \rightarrow B$ a cofibration in the model category \mathcal{C} . Then the induced map

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \rightarrow L \wedge B$$

is a cofibration in \mathcal{C} . Furthermore, if either i or j is an acyclic cofibration, then so is $i \square j$.

(iii) If $q: QI \rightarrow I$ is a cofibrant replacement for the unit I in \mathcal{V} , then the induced map $q \wedge 1: QI \wedge X \rightarrow I \wedge X$ is a weak equivalence in \mathcal{C} for any cofibrant X .

Finally, let us recall the definition of a proper model category.

Definition 2.1.5. A model category is called *left proper* if weak equivalences are preserved by pushouts along cofibrations. Dually, a model category is called *right proper* if weak equivalences are preserved by pullbacks along fibrations. A model category which is left proper and right proper is said to be *proper*.

2.2 G-equivariant spaces

Convention 2.2.1. In this thesis G will always denote a finite group.

Convention 2.2.2. By a topological space we will always mean a compactly generated weak Hausdorff space.

The category $G\text{-Top}_*$ of pointed topological G -spaces admits a proper and cofibrantly generated model structure such that $f: X \rightarrow Y$ is a weak equivalence (resp. fibration) if the induced map on H -fixed points

$$f^H: X^H \rightarrow Y^H$$

is a weak homotopy equivalence (resp. Serre fibration) for any subgroup $H \leq G$ (see e.g. [MM02, III.1]). The set

$$(G/H \times S^{n-1})_+ \rightarrow (G/H \times D^n)_+, \quad n \geq 0, H \leq G$$

of G -maps generates cofibrations in this model structure. The acyclic cofibrations are generated by the maps

$$\text{incl}_0: (G/H \times D^n)_+ \rightarrow (G/H \times D^n \times I)_+, \quad n \geq 0, H \leq G.$$

The model category $G\text{-Top}_*$ is a closed symmetric monoidal model category [MM02, III.1]. The monoidal product on $G\text{-Top}_*$ is given by the smash product $X \wedge Y$, with the diagonal G -action, for any $X, Y \in G\text{-Top}_*$, and the mapping object is the nonequivariant pointed mapping space $\text{Map}(X, Y)$ with the conjugation G -action.

2.3 G-equivariant orthogonal spectra

We start by reminding the reader about the definition of an orthogonal spectrum [MMSS01]:

Definition 2.3.1. An *orthogonal spectrum* X consists of the following data:

- a sequence of pointed spaces X_n , for $n \geq 0$;
- a base-point preserving continuous action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$;
- continuous based maps $\sigma_n: X_n \wedge S^1 \rightarrow X_{n+1}$.

This data is subject to the following condition: For all $n, m \geq 0$, the iterated structure map $X_n \wedge S^m \rightarrow X_{n+m}$ is $O(n) \times O(m)$ -equivariant.

Next, let us recall the definition of G -equivariant orthogonal spectra (here we mainly follow [Sch13]. See also [MM02] which is the original source for G -equivariant orthogonal spectra):

Definition 2.3.2. An *orthogonal G -spectrum* (*G -equivariant orthogonal spectrum*) is an orthogonal spectrum X equipped with a categorical G -action, i.e., with a group homomorphism $G \rightarrow \text{Aut}(X)$.

The category of orthogonal G -spectra is denoted by Sp_G^O . Any orthogonal G -spectrum X can be evaluated on an arbitrary finite dimensional orthogonal G -representation V . The G -space $X(V)$ is defined by

$$X(V) = \mathbf{L}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n,$$

where the number n is the dimension of V , the vector space \mathbb{R}^n is equipped with the standard scalar product and $\mathbf{L}(\mathbb{R}^n, V)$ is the space of (not necessarily equivariant) linear isometries from \mathbb{R}^n to V . The G -action on $X(V)$ is given diagonally:

$$g \cdot [\varphi, x] = [g\varphi, gx], \quad g \in G, \quad \varphi \in \mathbf{L}(\mathbb{R}^n, V), \quad x \in X_n.$$

For the trivial G -representation \mathbb{R}^n , the pointed G -space $X(\mathbb{R}^n)$ is canonically isomorphic to the pointed G -space X_n . Next, let S^V denote the representation sphere of V , i.e., the one-point compactification of V . Using the iterated structure maps of X , for any finite dimensional orthogonal G -representations V and W , one can define G -equivariant *generalized structure maps*

$$\sigma_{V,W}: X(V) \wedge S^W \longrightarrow X(V \oplus W).$$

These are then used to define G -equivariant homotopy groups

$$\pi_k^G X = \mathrm{colim}_n [S^{k+n\rho_G}, X(n\rho_G)]^G, \quad k \in \mathbb{Z},$$

where ρ_G denotes the regular representation of G . Furthermore, for any subgroup $H \leq G$, one defines $\pi_k^H X$, $k \in \mathbb{Z}$, to be the k -th H -equivariant homotopy group of X considered as an H -spectrum.

Definition 2.3.3. A map $f: X \longrightarrow Y$ of G -equivariant orthogonal spectra is called a *stable equivalence* if the induced map

$$\pi_k^H(f): \pi_k^H X \longrightarrow \pi_k^H Y$$

is an isomorphism for any integer k and any subgroup $H \leq G$.

2.4 Comparison of different definitions

Before continuing the recollection, let us explain the relation of Definition 2.3.2 with the original definition of G -equivariant orthogonal spectra due to Mandell and May. For this we first recall G -universes:

Definition 2.4.1 (see e.g. [MM02, Definition II.1.1]). Let \mathcal{U} be a countable dimensional real inner product space with an invariant G -action. Then \mathcal{U} is said to be a G -universe if it satisfies the following conditions

- (i) The trivial representation \mathbb{R} embeds into \mathcal{U} ;
- (ii) If an orthogonal G -representation V (equivariantly) embeds into \mathcal{U} , then the countable sum of copies of V also embeds into \mathcal{U} .

A G -universe is called *complete* if all irreducible G -representations embed into \mathcal{U} and is called *trivial* if only trivial representations embed into \mathcal{U} .

The sum $\infty\rho_G$ of countable copies of the regular representation ρ_G is an example of a complete G -universe. The euclidean space \mathbb{R}^∞ with the trivial G -action is an example of a trivial universe.

In [MM02, II.2], Mandell and May define G -equivariant orthogonal spectra indexed on a universe \mathcal{U} . Such a G -spectrum is a collection of G -spaces indexed on those representations that embed into \mathcal{U} together with certain equivariant structure maps. It follows from [MM02, II.4.3] that the category of G -equivariant orthogonal spectra indexed on a universe \mathcal{U} is equivalent to a certain equivariant diagram category. We will now compare these diagram categories with the category Sp_G^O . For this we have to recall the definition of the indexing G - \mathbf{Top}_* -category $O_{G,\mathcal{U}}$. Note that this category will be also important in Section 3 for defining orthogonal G -spectra in equivariant model categories. The objects of $O_{G,\mathcal{U}}$ are finite dimensional orthogonal G -representations that embed into the universe \mathcal{U} . For any such orthogonal G -representations V and W , the pointed morphism G -space $O_{G,\mathcal{U}}(V, W)$ is defined to be the Thom complex of the G -equivariant vector bundle

$$\xi(V, W) \longrightarrow \mathbf{L}(V, W),$$

where $\mathbf{L}(V, W)$ is the space of linear isometric embeddings from V to W and

$$\xi(V, W) = \{(f, x) \in \mathbf{L}(V, W) \times W \mid x \perp f(V)\}.$$

For more details about this category see [MM02, II.4]. (Note that in [MM02], the category $O_{G,\mathcal{U}}$ is denoted by \mathcal{J}_G .)

Remark 2.4.2. If \mathcal{U} is a complete universe, then we will denote the category $O_{G,\mathcal{U}}$ just by O_G . Further, since the Thom spaces $O_{G,\mathcal{U}}(V, W)$ do not really depend on \mathcal{U} , the subscript \mathcal{U} will be omitted in the sequel and we will denote these spaces by $O_G(V, W)$.

Theorem II.4.3 of [MM02] tells us that the category of $O_{G,\mathcal{U}}$ -spaces (which is the category of G - \mathbf{Top}_* -enriched functors from $O_{G,\mathcal{U}}$ to G - \mathbf{Top}_*) is equivalent to the category of G -equivariant orthogonal spectra indexed on a universe \mathcal{U} . Next, consider any trivial G -universe, for example \mathbb{R}^∞ . Then for an arbitrary G -universe \mathcal{U} , there is an obvious fully faithful inclusion

$$O_{G,\mathbb{R}^\infty} \hookrightarrow O_{G,\mathcal{U}}.$$

This embedding is in fact a G - \mathbf{Top}_* -enriched embedding and hence induces a G - \mathbf{Top}_* -enriched adjunction between the categories of $O_{G,\mathcal{U}}$ -spaces and O_{G,\mathbb{R}^∞} -spaces. The right adjoint is the precomposition with the inclusion and the left adjoint is given by a G - \mathbf{Top}_* -enriched left Kan extension [Kel05, Section 4.1]. In fact, [MM02, Theorem V.1.5] (see also [HHR09, Proposition A.18]) implies that this G - \mathbf{Top}_* -enriched adjunction is a G - \mathbf{Top}_* -enriched equivalence of categories. On the other hand, one can immediately see that the category of O_{G,\mathbb{R}^∞} -spaces is equivalent to the category Sp_G^O (Definition 2.3.2). Hence, for any G -universe \mathcal{U} , the category of orthogonal G -spectra

indexed on \mathcal{U} , the category of $O_{G,\mathcal{U}}$ -spaces and the category Sp_G^O are equivalent. This shows that universes are not really relevant for the point-set level definition of an orthogonal G -spectrum. However, they become really important when one considers the homotopy theory of orthogonal G -spectra (see Subsection 2.6).

Next, the category Sp_G^O is a closed symmetric monoidal category. The symmetric monoidal structure on Sp_G^O is given by the smash product of underlying orthogonal spectra [MMSS01] with the diagonal G -action. Similarly, for any universe \mathcal{U} , the category of G -equivariant orthogonal spectra indexed on a universe \mathcal{U} as well as the category of $O_{G,\mathcal{U}}$ -spaces are closed symmetric monoidal categories (see Subsection 3.2 for the detailed construction of the smash product). It follows from [MM02, Theorem II.4.3, Theorem V.1.5] (see also [HHR09, Proposition A.18]) that all the equivalences discussed above are in fact equivalences of closed symmetric monoidal categories.

From this point on we will freely use all the results of [MM02] for the category Sp_G^O having the above equivalences in mind.

2.5 The level model structures on Sp_G^O

In this subsection we closely follow [MM02, III.2].

Let \mathcal{U} be a G -universe. For any finite dimensional orthogonal G -representation V , the evaluation functor $\mathrm{Ev}_V: \mathrm{Sp}_G^O \rightarrow G\text{-}\mathbf{Top}_*$, given by $X \mapsto X(V)$, has a left adjoint $G\text{-}\mathbf{Top}_*$ -functor

$$F_V: \mathrm{Sp}_G^O \rightarrow G\text{-}\mathbf{Top}_*$$

which is defined by (see [MM02, II.4])

$$F_V A(W) = O_G(V, W) \wedge A.$$

We fix (once and for all) a small skeleton $\mathrm{sk} O_{G,\mathcal{U}}$ of the category $O_{G,\mathcal{U}}$. Let $I_{\mathrm{lv}}^{G,\mathcal{U}}$ denote the set of morphisms

$$\{F_V(G/H \times S^{n-1})_+ \rightarrow F_V((G/H \times D^n)_+) \mid V \in \mathrm{sk} O_{G,\mathcal{U}}, n \geq 0, H \leq G\}$$

and $J_{\mathrm{lv}}^{G,\mathcal{U}}$ denote the set of morphisms

$$\{F_V((G/H \times D^n)_+) \rightarrow F_V((G/H \times D^n \times I)_+) \mid V \in \mathrm{sk} O_{G,\mathcal{U}}, n \geq 0, H \leq G\}.$$

In other words, the sets $I_{\mathrm{lv}}^{G,\mathcal{U}}$ and $J_{\mathrm{lv}}^{G,\mathcal{U}}$ are obtained by applying the functors F_V , $V \in \mathrm{sk} O_{G,\mathcal{U}}$, to the generating cofibrations and generating acyclic cofibrations of $G\text{-}\mathbf{Top}_*$, respectively. Further, we recall

Definition 2.5.1. Let $f: X \rightarrow Y$ be a morphism in Sp_G^O . The map f is called a \mathcal{U} -level equivalence if $f(V): X(V) \rightarrow Y(V)$ is a weak equivalence in $G\text{-}\mathbf{Top}_*$ for any $V \in \mathrm{sk} O_{G,\mathcal{U}}$. It is called a \mathcal{U} -level fibration if $f(V): X(V) \rightarrow Y(V)$ is a fibration in $G\text{-}\mathbf{Top}_*$ for any $V \in \mathrm{sk} O_{G,\mathcal{U}}$. A map in Sp_G^O is called a \mathcal{U} -cofibration if it has the left lifting property with respect to all maps that are \mathcal{U} -level fibrations and \mathcal{U} -level equivalences (i.e., \mathcal{U} -level acyclic fibrations).

Proposition 2.5.2 ([MM02, III.2.4]). *Let \mathcal{U} be a G -universe. The category Sp_G^O together with \mathcal{U} -level equivalences, \mathcal{U} -level fibrations and \mathcal{U} -cofibrations forms a cofibrantly generated, proper model category. The set $I_{\mathrm{lv}}^{G, \mathcal{U}}$ serves as a set of generating cofibrations and the set $J_{\mathrm{lv}}^{G, \mathcal{U}}$ serves as a set of generating acyclic cofibrations.*

2.6 The stable model structures on Sp_G^O

The reference for this subsection is [MM02, III.4].

Recall that for any G -equivariant orthogonal spectrum X we have the generalized structure maps

$$\sigma_{V,W}: X(V) \wedge S^W \longrightarrow X(V \oplus W).$$

Let $\tilde{\sigma}_{V,W}: X(V) \longrightarrow \Omega^W X(V \oplus W)$ denote the adjoint of $\sigma_{V,W}$.

Definition 2.6.1. Suppose \mathcal{U} is a G -universe. An orthogonal G -spectrum X is called a G - \mathcal{U} - Ω -spectrum if the maps $\tilde{\sigma}_{V,W}$ are weak equivalences in G - \mathbf{Top}_* for any V and W in $\mathrm{sk} O_{G, \mathcal{U}}$.

Further, for any G -universe \mathcal{U} and any orthogonal G -spectrum X , Mandell and May define H -equivariant homotopy groups $\pi_k^{H, \mathcal{U}}(X)$, $k \in \mathbb{Z}$, $H \leq G$ [MM02, Definition III.3.2]. We do not give the details here. A map $f: X \longrightarrow Y$ of orthogonal G -spectra is called a \mathcal{U} -stable equivalence, if $\pi_k^{H, \mathcal{U}}(f)$ is an isomorphism for any $k \in \mathbb{Z}$, $H \leq G$. Note that if the universe \mathcal{U} is complete, then [MM02, Definition III.3.2] recovers the definition of H -equivariant homotopy groups we gave in Subsection 2.3 and a \mathcal{U} -stable equivalence is the same as a stable equivalence (see Definition 2.3.3).

Before formulating the theorem about the \mathcal{U} -stable model structure on Sp_G^O , let us introduce certain sets of morphisms in Sp_G^O that will serve as generating sets for cofibrations and acyclic cofibrations for this model structures. Let $V, W \in \mathrm{sk} O_{G, \mathcal{U}}$ and

$$\lambda_{V,W}: F_{V \oplus W} S^W \longrightarrow F_V S^0$$

denote the map of G -equivariant orthogonal spectra that is adjoint to the map

$$S^W \longrightarrow \mathrm{Ev}_{V \oplus W}(F_V S^0) = O_G(V, V \oplus W)$$

that sends $z \in W$ to $(V \xrightarrow{(1,0)} V \oplus W, z)$ (see [MM02, III.4.3, III.4.5]). Using the mapping cylinder construction, the map $\lambda_{V,W}$ factors as a composite

$$F_{V \oplus W} S^W \xrightarrow{\kappa_{V,W}} M\lambda_{V \oplus W} \xrightarrow{r_{V \oplus W}} F_V S^0,$$

where $r_{V \oplus W}$ is a G -equivariant homotopy equivalence and $\kappa_{V,W}$ a cofibration and a stable equivalence [MM02, III.4.5-4.6]. Now consider any generating cofibration

$$i: (G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+.$$

Let $i \square \kappa_{V,W}$ denote the pushout-product induced from the commutative square:

$$\begin{array}{ccc} (G/H \times S^{n-1})_+ \wedge F_{V \oplus W} S^W & \longrightarrow & (G/H \times S^{n-1})_+ \wedge M\lambda_{V \oplus W} \\ \downarrow & & \downarrow \\ (G/H \times D^n)_+ \wedge F_{V \oplus W} S^W & \longrightarrow & (G/H \times D^n)_+ \wedge M\lambda_{V \oplus W}. \end{array}$$

Define

$$K^{G, \mathcal{U}} = \{i \square \kappa_{V,W} \mid H \leq G, \ n \geq 0, \ V, W \in \text{sk } O_{G, \mathcal{U}}\}.$$

Let $J_{\text{st}}^{G, \mathcal{U}}$ stand for the union $J_{\text{lv}}^{G, \mathcal{U}} \cup K^{G, \mathcal{U}}$. For convenience, we will also introduce the notation $I_{\text{st}}^{G, \mathcal{U}} = I_{\text{lv}}^{G, \mathcal{U}}$.

Finally, before formulating the main theorem of this subsection we need the following definition:

Definition 2.6.2. A map $f: X \longrightarrow Y$ of orthogonal G -spectra is called a \mathcal{U} -stable fibration, if it has the right lifting property with respect to the maps that are \mathcal{U} -cofibrations and \mathcal{U} -stable equivalences.

Theorem 2.6.3 ([MM02, III.4.2]). *The category Sp_G^O together with \mathcal{U} -cofibrations, \mathcal{U} -stable equivalences and \mathcal{U} -stable fibrations forms a proper, cofibrantly generated, stable model category. The set $I_{\text{st}}^{G, \mathcal{U}}$ generates cofibrations and the set $J_{\text{st}}^{G, \mathcal{U}}$ generates acyclic cofibrations. Furthermore, the fibrant objects are precisely the G - \mathcal{U} - Ω -spectra.*

The category Sp_G^O together with the latter model structure is referred to as the *model category of orthogonal G -spectra indexed on the universe \mathcal{U}* .

Remark 2.6.4. Since in this thesis we will mostly consider complete universes, let us introduce some notational conventions which will simplify the exposition. From this point on the notation Sp_G^O will stand for the model category of orthogonal G -spectra indexed on the complete universe $\infty\rho_G$ (ρ_G is the regular representation of G) and this model structure will be referred to as the stable model structure on Sp_G^O . Next, we will mostly omit the symbol \mathcal{U} from all subscripts and superscripts if \mathcal{U} is complete. In particular, the sets $I_{\text{lv}}^{G, \infty\rho_G}$, $J_{\text{lv}}^{G, \infty\rho_G}$, $K^{G, \infty\rho_G}$, $I_{\text{st}}^{G, \infty\rho_G}$, $J_{\text{st}}^{G, \infty\rho_G}$ will be denoted by I_{lv}^G , J_{lv}^G , K^G , I_{st}^G , J_{st}^G , respectively. Similarly, G - $\infty\rho_G$ - Ω -spectra will be referred to as G - Ω -spectra. A cofibration in Sp_G^O will mean an $\infty\rho_G$ -cofibration, a stable fibration in Sp_G^O stands for an $\infty\rho_G$ -fibration and as we already observed an $\infty\rho_G$ -stable equivalence is exactly a stable equivalence in the sense of Definition 2.3.3.

Finally, we recall that the stable model category Sp_G^O together with the smash product forms a closed symmetric monoidal model category [MM02, III.7]. In particular, the following holds:

Proposition 2.6.5. *Suppose that $i: K \longrightarrow L$ and $j: A \longrightarrow B$ are cofibrations in Sp_G^O . Then the pushout-product*

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \longrightarrow L \wedge B$$

is a cofibration in Sp_G^O . The map $i \square j$ is also a stable equivalence if in addition i or j is a stable equivalence.

2.7 The equivariant stable homotopy category

In this subsection we list some well known properties of the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G^O)$. Note that the category $\mathrm{Ho}(\mathrm{Sp}_G^O)$ is equivalent to the Lewis-May G -equivariant stable homotopy category of genuine G -spectra (see [MM02, IV.1]) introduced in [LMSM86].

As noted in the previous subsection, the model category Sp_G^O is stable and hence the homotopy category $\mathrm{Ho}(\mathrm{Sp}_G^O)$ is naturally triangulated. Further, since the maps

$$\lambda_V = \lambda_{0,V}: F_V S^V \longrightarrow F_0 S^0$$

are stable equivalences [MM02, Lemma III.4.5], it follows that the functor

$$S^V \wedge -: \mathrm{Ho}(\mathrm{Sp}_G^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

is an equivalence of categories for any finite dimensional orthogonal G -representation V .

Next, before continuing, let us introduce the following notational convention. For any G -equivariant orthogonal spectra X and Y , the abelian group $[X, Y]^{\mathrm{Ho}(\mathrm{Sp}_G^O)}$ of morphisms from X to Y in $\mathrm{Ho}(\mathrm{Sp}_G^O)$ will be denoted by $[X, Y]^G$.

An adjunction argument immediately implies that for any subgroup $H \leq G$ and an orthogonal G -spectrum X , there is a natural isomorphism

$$[\Sigma_+^\infty G/H, X]_*^G \cong \pi_*^H X.$$

As a consequence, we see that the set

$$\{\Sigma_+^\infty G/H \mid H \leq G\}$$

is a set of compact generators for the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G^O)$. Note that since G is finite, for $* > 0$ and any subgroups $H, H' \leq G$, the abelian group

$$[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H']_*^G$$

is finite (see e.g. [GM95, Proposition A.3]).

Finally, we recall the stable Burnside category. For any $g \in G$, let ${}^g H$ denote the conjugate subgroup gHg^{-1} . Then the map

$$g: \Sigma_+^\infty G/{}^g H \longrightarrow \Sigma_+^\infty G/H$$

in $\mathrm{Ho}(\mathrm{Sp}_G^O)$, given by $[x] \mapsto [xg]$ on the point-set level, is called the conjugation map associated to g and H . Further, if K is another subgroup of G such that $K \leq H$, then we have the restriction map

$$\mathrm{res}_K^H: \Sigma_+^\infty G/K \longrightarrow \Sigma_+^\infty G/H$$

which is just the obvious projection on the point-set level. Moreover, there is also a map backwards, called the transfer map

$$\mathrm{tr}_K^H: \Sigma_+^\infty G/H \longrightarrow \Sigma_+^\infty G/K,$$

given by the Pontryagin-Thom construction (see e.g. [LMSM86, IV.3] or [tD87, II.8]). These morphisms generate the stable Burnside (orbit) category which is a full preadditive subcategory of $\mathrm{Ho}(\mathrm{Sp}_G^O)$ with objects the stable orbits $\Sigma_+^\infty G/H$, $H \leq G$ [LMSM86, V.9] (see also [Lew98]).

The stable Burnside category plays an important role in equivariant stable homotopy theory as well as in representation theory. Indeed, the contravariant functors from this category to abelian groups are exactly *Mackey functors*. Note that the stable Burnside category shows up in the formulation and proof of Theorem 1.1.1.

3 Categorical Input

3.1 Outline

Recall that G is a finite group. We start with

Definition 3.1.1. A G - \mathbf{Top}_* -model category \mathcal{C} (see Definition 2.1.4 and Subsection 2.2) is said to be a G -equivariant stable model category if the adjunction

$$S^V \wedge -: \mathcal{C} \rightleftarrows \mathcal{C} : \Omega^V(-) = (-)^{S^V}$$

is a Quillen equivalence for any finite dimensional orthogonal G -representation V .

Examples of G -equivariant stable model categories are the model category Sp_G^O of G -equivariant orthogonal spectra [MM02, II-III], the model category of G -equivariant orthogonal spectra equipped with the \mathbb{S} -model structure [Sto11], the model category of S_G -modules [MM02, IV.2], the model category of G -equivariant continuous functors [Blu06] and the model categories of G -equivariant topological symmetric spectra ([Man04], [Hau13]).

The following proposition is an equivariant version of [SS03, 3.8].

Proposition 3.1.2. *Let \mathcal{C} be a cofibrantly generated (Definition 3.3.2), proper, G -equivariant stable model category. Then the category $\mathrm{Sp}_G^O(\mathcal{C})$ of internal orthogonal G -spectra in \mathcal{C} (Definition 3.2.1) possesses a G -equivariant stable model structure and the G - \mathbf{Top}_* -adjunction*

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathrm{Sp}_G^O(\mathcal{C}) : \mathrm{Ev}_0$$

is a Quillen equivalence.

The proof of this proposition is a straightforward equivariant generalization of the arguments in [SS03, 3.8]. However, we still decided to provide details here as they don't seem to appear in the literature. The proof of Proposition 3.1.2 will occupy a significant part of this section.

The point of Proposition 3.1.2 is that one can replace (under some technical assumptions) any G -equivariant stable model category by a G -spectral one (Definition 3.5.1), i.e., by an Sp_G^O -model category. This in particular implies that $\mathrm{Ho}(\mathcal{C})$ is tensored over the G -equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_G^O)$.

To stress the importance of Proposition 3.1.2, we will now sketch a general strategy how one should try to prove Conjecture 1.1.2. Recall that we are given a triangulated equivalence

$$\Psi : \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C})$$

with certain properties. By Proposition 3.1.2, there is a G - \mathbf{Top}_* -Quillen equivalence

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathrm{Sp}_G^O(\mathcal{C}) : \mathrm{Ev}_0.$$

Let X be a cofibrant replacement of $(\mathbf{L}\Sigma^\infty \circ \Psi)(\mathbb{S})$. Since $\mathrm{Sp}_G^O(\mathcal{C})$ is G -spectral (Definition 3.5.1), there is a G - \mathbf{Top}_* -Quillen adjunction

$$- \wedge X : \mathrm{Sp}_G^O \rightleftarrows \mathrm{Sp}_G^O(\mathcal{C}) : \mathrm{Hom}(X, -).$$

Hence, in order to prove Conjecture 1.1.2, it suffices to show that the latter Quillen adjunction is a Quillen equivalence. Next, it follows from the properties of Ψ that we have isomorphisms

$$\Psi(\Sigma_+^\infty G/H) \cong \mathbf{R}\mathrm{Ev}_0(\Sigma_+^\infty G/H \wedge^{\mathbf{L}} X)$$

which are natural with respect to transfers, conjugations, and restrictions. Using these isomorphisms, we can choose an inverse of Ψ

$$\Psi^{-1} : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

such that $\Psi^{-1}(\mathbf{R}\mathrm{Ev}_0(\Sigma_+^\infty G/H \wedge^{\mathbf{L}} X)) = \Sigma_+^\infty G/H$. Moreover, since the isomorphisms above are natural with respect to the stable Burnside category (Subsection 2.7), we get the identities

$$\Psi^{-1}(\mathbf{R}\mathrm{Ev}_0(g \wedge^{\mathbf{L}} X)) = g, \quad \Psi^{-1}(\mathbf{R}\mathrm{Ev}_0(\mathrm{res}_K^H \wedge^{\mathbf{L}} X)) = \mathrm{res}_K^H, \quad \Psi^{-1}(\mathbf{R}\mathrm{Ev}_0(\mathrm{tr}_K^H \wedge^{\mathbf{L}} X)) = \mathrm{tr}_K^H,$$

where $g \in G$ and $K \leq H \leq G$. Now let us consider the composite

$$F : \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{- \wedge^{\mathbf{L}} X} \mathrm{Ho}(\mathrm{Sp}_G^O(\mathcal{C})) \xrightarrow{\mathbf{R}\mathrm{Ev}_0} \mathrm{Ho}(\mathcal{C}) \xrightarrow{\Psi^{-1}} \mathrm{Ho}(\mathrm{Sp}_G^O).$$

Since the functors $\mathbf{R}\mathrm{Ev}_0$ and Ψ^{-1} are equivalences, to prove that $(- \wedge X, \mathrm{Hom}(X, -))$ is a Quillen equivalence is equivalent to showing that the endofunctor

$$F : \mathrm{Ho}(\mathrm{Sp}_G^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

is an equivalence of categories. By the assumptions of Conjecture 1.1.2 and the properties of Ψ^{-1} , we see that F enjoys the following properties:

- (i) $F(\Sigma_+^\infty G/H) = \Sigma_+^\infty G/H$, $H \leq G$;
- (ii) F preserves transfers, conjugations, and restrictions (and hence the stable Burnside category);
- (iii) F is an exact functor of triangulated categories and preserves infinite coproducts.

Similarly, if we start with the 2-localized genuine G -equivariant stable homotopy category $\mathrm{Ho}(\mathrm{Sp}_{G,(2)}^O)$ and an equivalence $\mathrm{Ho}(\mathrm{Sp}_{G,(2)}^O) \sim \mathrm{Ho}(\mathcal{C})$ as in the formulation of Theorem 1.1.1, we obtain an endofunctor $\mathrm{Ho}(\mathrm{Sp}_{G,(2)}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{G,(2)}^O)$ which also satisfies the properties (i), (ii) and (iii) above (see Subsection 3.7 for more details). The following proposition which is one of the central results of this thesis, immediately implies Theorem 1.1.1:

Theorem 3.1.3. *Let G be a finite group and $F: \text{Ho}(\text{Sp}_{G,(2)}^O) \longrightarrow \text{Ho}(\text{Sp}_{G,(2)}^O)$ an exact functor of triangulated categories that preserves arbitrary coproducts and such that*

$$F(\Sigma_+^\infty G/H) = \Sigma_+^\infty G/H, \quad H \leq G,$$

and

$$F(g) = g, \quad F(\text{res}_K^H) = \text{res}_K^H, \quad F(\text{tr}_K^H) = \text{tr}_K^H, \quad g \in G, \quad K \leq H \leq G.$$

Then F is an equivalence of categories.

The proof of this proposition will be completed at the very end of this thesis. In this section we will concentrate on the proof of Proposition 3.1.2 and on the p -localization of the stable model structure of [MM02] on the category of G -equivariant orthogonal spectra.

Before starting the preparation for the proof of Proposition 3.1.2, let us outline the plan that will lead to the proof of Proposition 3.1.2. We first define the category $\text{Sp}_G^O(\mathcal{C})$ of orthogonal G -spectra internal to a G -**Top** $_*$ -model category \mathcal{C} and discuss its categorical properties. Next, for any cofibrantly generated G -**Top** $_*$ -model category \mathcal{C} we construct the level model structure on $\text{Sp}_G^O(\mathcal{C})$. Finally, using the same strategy as in [SS03], we establish the G -equivariant stable model structure on $\text{Sp}_G^O(\mathcal{C})$ for any proper, cofibrantly generated, G -**Top** $_*$ -model category \mathcal{C} that is stable as an underlying model category.

3.2 Orthogonal G -spectra in equivariant model categories

Recall from Subsection 2.4 the G -**Top** $_*$ -category O_G . The objects of O_G are finite dimensional orthogonal G -representations. For any finite dimensional orthogonal G -representations V and W , the pointed morphism G -space from V to W is the Thom space $O_G(V, W)$. Recall also that the category Sp_G^O is equivalent to the category of O_G -spaces (which is the category of G -**Top** $_*$ -enriched functors from O_G to G -**Top** $_*$).

Now suppose that \mathcal{C} is a G -**Top** $_*$ -model category (in particular, \mathcal{C} is pointed). We remind the reader that this means that we have tensors $K \wedge X$, cotensors X^K and pointed mapping G -spaces $\text{Map}(X, Y)$ for $K \in G$ -**Top** $_*$ and $X, Y \in \mathcal{C}$, which are related by adjunctions and satisfy certain properties (Definition 2.1.4). In particular, the pushout-product axiom holds: Let $i: K \longrightarrow L$ be a cofibration in the model category G -**Top** $_*$ and $j: A \longrightarrow B$ a cofibration in the model category \mathcal{C} . Then the induced map

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \longrightarrow L \wedge B$$

is a cofibration in \mathcal{C} . Furthermore, if either i or j is an acyclic cofibration, then so is $i \square j$.

Definition 3.2.1. Let \mathcal{C} be a G -**Top** $_*$ -model category. An *orthogonal G -spectrum* in \mathcal{C} is a G -**Top** $_*$ -enriched functor ([Kel05, 1.2]) from the category O_G to \mathcal{C} .

The category of orthogonal G -spectra in \mathcal{C} will be denoted by $\mathrm{Sp}_G^O(\mathcal{C})$. Note that by [MM02, II.4.3] (see also Subsection 2.4), the category $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$ is equivalent to Sp_G^O . Next, since \mathcal{C} is complete and cocomplete, so is the category $\mathrm{Sp}_G^O(\mathcal{C})$ (see [Kel05, 3.3]) and limits and colimits are constructed levelwise.

Remark 3.2.2. The category O_G is skeletally small. We can fix once and for all a small skeleton of O_G . In particular, when talking about ends and coends over O_G and using the notations $\int_{V \in O_G}$ and $\int^{V \in O_G}$, we will always implicitly mean that the indexing category is the chosen small skeleton of O_G .

Next, we want to check that $\mathrm{Sp}_G^O(\mathcal{C})$ is enriched, tensored and cotensored over Sp_G^O . For this we first review the closed symmetric monoidal structure on Sp_G^O . The main reference here is [MM02, II]. Recall, that the category O_G has a symmetric monoidal product \oplus given by the direct sum of orthogonal representations on objects and by the continuous G -map

$$O_G(V, W) \wedge O_G(V', W') \longrightarrow O_G(V \oplus V', W \oplus W'), \quad (\alpha, w) \wedge (\beta, w') \mapsto (\alpha \oplus \beta, (w, w'))$$

on morphisms. In fact, the product \oplus can be interpreted as a $G\text{-}\mathbf{Top}_*$ -functor

$$\oplus: O_G \wedge O_G \longrightarrow O_G.$$

Here the category $O_G \wedge O_G$ has pairs of finite dimensional orthogonal G -representations as its objects. The morphisms in $O_G \wedge O_G$ are given by

$$O_G \wedge O_G((V, V'), (W, W')) = O_G(V, W) \wedge O_G(V', W').$$

The product \oplus on O_G together with the smash product on $G\text{-}\mathbf{Top}_*$ gives a symmetric monoidal product on $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$. This construction is a special case of the Day convolution product [Day70]. More precisely, let X and Y be objects of $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$. Then we have the external smash product

$$X \bar{\wedge} Y: O_G \wedge O_G \longrightarrow G\text{-}\mathbf{Top}_*,$$

defined by $(X \bar{\wedge} Y)(V, W) = X(V) \wedge Y(W)$. The $G\text{-}\mathbf{Top}_*$ -enriched left Kan extension [Kel05, Section 4.1, Proposition 4.33] of $X \bar{\wedge} Y$ along \oplus

$$\begin{array}{ccc} O_G \wedge O_G & \xrightarrow{X \bar{\wedge} Y} & G\text{-}\mathbf{Top}_* \\ \oplus \downarrow & \nearrow \mathrm{Lan}_{\oplus}(X \bar{\wedge} Y) & \\ O_G & & \end{array}$$

is called the smash product of X and Y and is denoted by $X \wedge Y$. This is a symmetric monoidal product [Day70] (see also [MM02, II.3.7-3.8]). It follows from [Kel05, Section 4.2, (4.25)] that one can describe this smash product as a $G\text{-}\mathbf{Top}_*$ -enriched coend (see Remark 3.2.2)

$$X \wedge Y \cong \int^{V, W \in O_G} O_G(V \oplus W, -) \wedge X(V) \wedge Y(W).$$

In fact, $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$ is a closed symmetric monoidal category and the internal Hom-objects are given by the $G\text{-}\mathbf{Top}_*$ -enriched end construction

$$\mathrm{Hom}(X, Y)(V) = \int_{W \in O_G} \mathrm{Map}(X(W), Y(W \oplus V)).$$

Note that the equivalence of [MM02, II.4.3] is an equivalence of closed symmetric monoidal categories and in particular, the latter Day convolution product corresponds to the smash product of [MM02, II.3]. As noted in Subsection 2.4, we will once and for all identify the symmetric monoidal category $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$ with the symmetric monoidal category Sp_G^O having [MM02, Theorem II.4.3] in mind.

The generality of constructions we recalled here allows us to prove the following Proposition:

Proposition 3.2.3. *Let \mathcal{C} be a $G\text{-}\mathbf{Top}_*$ -model category. The category $\mathrm{Sp}_G^O(\mathcal{C})$ is enriched, tensored and cotensored over the symmetric monoidal category Sp_G^O of equivariant orthogonal G -spectra.*

Proof. Let K be an object of $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$ and X an object of $\mathrm{Sp}_G^O(\mathcal{C})$. Mimicking the construction of the smash product on $\mathrm{Sp}_G^O(G\text{-}\mathbf{Top}_*)$, we define an object in $\mathrm{Sp}_G^O(\mathcal{C})$

$$K \wedge X = \int^{V, W \in O_G} O_G(V \oplus W, -) \wedge K(V) \wedge X(W).$$

This product is unital and coherently associative. The proof uses the enriched Yoneda Lemma [Kel05, Section 3.10, (3.71)] and the Fubini theorem [Kel05, Section 3.10, (3.63)]. We do not provide the details here as they are standard and well-known. Next, one defines cotensors by a $G\text{-}\mathbf{Top}_*$ -enriched end

$$X^K(V) = \int_{W \in O_G} X(W \oplus V)^{K(W)}.$$

Finally, for any $X, Y \in \mathrm{Sp}_G^O(\mathcal{C})$, one can define Hom- G -spectra by a $G\text{-}\mathbf{Top}_*$ -enriched end

$$\mathrm{Hom}(X, Y)(V) = \int_{W \in O_G} \mathrm{Map}(X(W), Y(W \oplus V)).$$

It is an immediate consequence of [Kel05, Section 3.10, (3.71)] that these functors satisfy all the necessary adjointness properties:

$$\mathrm{Hom}(K \wedge X, Y) \cong \mathrm{Hom}(X, Y^K) \cong \mathrm{Hom}(K, \mathrm{Hom}(X, Y)). \quad \square$$

3.3 The level model structure on $\mathrm{Sp}_G^O(\mathcal{C})$

We start with the following well-known lemma which is an important technical ingredient for establishing the level model structure on $\mathrm{Sp}_G^O(\mathcal{C})$. The author was unable to find a reference for this lemma and decided to provide a proof.

Lemma 3.3.1. *Let G be a finite group and V and W finite dimensional orthogonal G -representations. Then the pointed G -space $O_G(V, W)$ admits a structure of a pointed G -CW complex.*

Proof. Assume that $\mathbf{L}(V, W) \neq \emptyset$. Nonequivariantly the space $\mathbf{L}(V, W)$ is a Stiefel manifold and the trivial vector bundle

$$\mathbf{L}(V, W) \times W \longrightarrow \mathbf{L}(V, W)$$

is smooth. Moreover, this bundle has a structure of a smooth Euclidean vector bundle induced from the scalar product on W . Let $\zeta(V, W) \subset \mathbf{L}(V, W) \times W$ denote the subbundle defined by

$$\zeta(V, W) = \{(\alpha, w) \in \mathbf{L}(V, W) \times W \mid w \in \text{Im } \alpha\}.$$

In fact, $\zeta(V, W)$ is a smooth trivial subbundle of $\mathbf{L}(V, W) \times W$. This follows from the fact that any fixed basis of V gives us a set of everywhere linearly independent smooth sections of $\zeta(V, W)$ that fiberwise span $\zeta(V, W)$ (see [Lee13, Lemma 10.32]). The bundle

$$\xi(V, W) = \{(f, x) \in \mathbf{L}(V, W) \times W \mid x \perp f(V)\}$$

is the orthogonal complement of $\zeta(V, W)$. Since the metric on $\mathbf{L}(V, W) \times W$ is smooth it follows that $\xi(V, W)$ is a smooth Euclidean vector bundle. Next, $O(V)$ and $O(W)$ act smoothly on $\mathbf{L}(V, W)$ and this implies that the action of G on $\mathbf{L}(V, W)$ and hence on $\xi(V, W)$ is smooth. Consequently, the disc bundle $D\xi(V, W)$ is a smooth G -manifold with boundary the sphere bundle $S\xi(V, W)$. Now Illman's results [Ill83, Theorem 7.1, Corollary 7.2] imply that $(D\xi(V, W), S\xi(V, W))$ is a G -CW pair and hence

$$O_G(V, W) = D\xi(V, W)/S\xi(V, W)$$

admits a structure of a pointed G -CW complex. □

In order to establish the stable model structure on $\text{Sp}_G^O(\mathcal{C})$, one needs an additional assumption on \mathcal{C} . We have to assume that \mathcal{C} is a cofibrantly generated G -**Top** $_*$ -model category.

Definition 3.3.2. Let \mathcal{C} be a G -**Top** $_*$ -model category. We say that \mathcal{C} is a *cofibrantly generated G -**Top** $_*$ -model category*, if there are sets I and J of maps in \mathcal{C} such that the following hold:

(i) Let A be the domain or codomain of a morphism from I . Then for any subgroup $H \leq G$ and any $n \geq 0$, the object

$$(G/H \times D^n)_+ \wedge A$$

is small relative to I -cell (and hence relative to I -cof by [Hov99, 2.1.16]).

(ii) Domains of morphisms in J are small relative to J -cell and I -cell.

(iii) The class of fibrations is J -inj.

(iv) The class of acyclic fibrations is I -inj.

The model category $G\text{-Top}_*$ is a cofibrantly generated $G\text{-Top}_*$ -model category [MM02, Theorem III.1.8]. Other important examples of cofibrantly generated $G\text{-Top}_*$ -model categories are the model category Sp_G^O of G -equivariant orthogonal spectra [MM02, Theorem III.4.2], the model category of G -equivariant orthogonal spectra equipped with the \mathbb{S} -model structure [Sto11, Theorem 2.3.27], the model category of S_G -modules [MM02, Theorem IV.2.8], the model category of G -equivariant continuous functors [Blu06, Theorem 1.3] and the model categories of G -equivariant topological symmetric spectra ([Man04], [Hau13]).

Remark 3.3.3. If a $G\text{-Top}_*$ -model category \mathcal{C} is cofibrantly generated as an underlying model category (Definition 2.1.1), then it doesn't necessarily follow that \mathcal{C} is a cofibrantly generated $G\text{-Top}_*$ -model category in the sense of Definition 3.3.2.

The conditions in Definition 3.3.2 are essentially needed at the end of the proof of Proposition 3.4.10. In fact, all the claims in this section that come before Proposition 3.4.10 do not really use that \mathcal{C} satisfies all the conditions of Definition 3.3.2. They still hold if we only assume that \mathcal{C} is a $G\text{-Top}_*$ -model category and cofibrantly generated as an underlying model category. However, for the rest of the thesis, we decided to concentrate only on cofibrantly generated $G\text{-Top}_*$ -model categories in the sense of Definition 3.3.2 since more general model categories are irrelevant here.

Remark 3.3.4. Note that some conditions are redundant in Definition 3.3.2 if the group G is finite. Indeed, Condition (i) of Definition 3.3.2 can be simplified. If we assume that for any A which is the domain or codomain of a morphism from I , the object $D_+^n \wedge A$ is small relative to I -cell, then Condition (i) automatically holds. This follows by adjunction and from the fact that fixed point functors commute with sequential colimits for finite groups. Since Condition (i) is essentially used in the proof of Proposition 3.4.10, for convenience, we decided to put it in the definition.

Now suppose that \mathcal{C} is a cofibrantly generated $G\text{-Top}_*$ -model category with I and J generating cofibrations and acyclic cofibrations

Definition 3.3.5. Let $f: X \rightarrow Y$ be a morphism in $\text{Sp}_G^O(\mathcal{C})$. The map f is called a *level equivalence* if $f(V): X(V) \rightarrow Y(V)$ is a weak equivalence in \mathcal{C} for any $V \in O_G$. It is called a *level fibration* if $f(V): X(V) \rightarrow Y(V)$ is a fibration in \mathcal{C} for any $V \in O_G$. A map in $\text{Sp}_G^O(\mathcal{C})$ is called a *cofibration* if it has the left lifting property with respect to all maps that are level fibrations and level equivalences (i.e., level acyclic fibrations).

The level model structure on $\text{Sp}_G^O(\mathcal{C})$ which we will construct now is a cofibrantly generated model structure. Before stating the main proposition of this subsection we would like to introduce the set of morphisms that will serve as generators of (acyclic) cofibrations in the level model structure on $\text{Sp}_G^O(\mathcal{C})$.

The evaluation functor $\text{Ev}_V: \text{Sp}_G^O(\mathcal{C}) \rightarrow \mathcal{C}$, given by $X \mapsto X(V)$, has a left adjoint $G\text{-Top}_*$ -functor

$$F_V: \mathcal{C} \rightarrow \text{Sp}_G^O(\mathcal{C})$$

which is defined by

$$F_V A = O_G(V, -) \wedge A.$$

For any finite dimensional orthogonal G -representation V , consider the following sets of morphisms

$$F_V I = \{F_V i \mid i \in I\} \quad \text{and} \quad F_V J = \{F_V j \mid j \in J\}.$$

Next, fix (once and for all) a small skeleton $\text{sk } O_G$ of the category O_G . We finally define

$$FI = \bigcup_{V \in \text{sk } O_G} F_V I \quad \text{and} \quad FJ = \bigcup_{V \in \text{sk } O_G} F_V J.$$

The following proposition is a special case of [GM11, Theorem 2.12]. For the sake of completeness, we provide a proof.

Proposition 3.3.6. *Suppose \mathcal{C} is a cofibrantly generated G - \mathbf{Top}_* -model category. Then the category $\text{Sp}_G^O(\mathcal{C})$ of orthogonal G -spectra in \mathcal{C} together with the level equivalences, cofibrations and level fibrations described in Definition 3.3.5 forms a cofibrantly generated model category. The set FI generates cofibrations and the set FJ generates acyclic cofibrations.*

Proof. The strategy is to apply Proposition 2.1.2. If $f(V): X(V) \rightarrow Y(V)$ is a weak equivalence (resp. fibration) for any $V \in \text{sk } O_G$, then f is a level equivalence (resp. fibration) (see Definition 3.3.5). Hence, by adjunction, the class FJ -inj coincides with the class of level fibrations and the class FI -inj with the class of level acyclic fibrations. In particular, if we let \mathscr{W}^{lv} denote the class of level equivalences, then FJ -inj $\cap \mathscr{W}^{lv} = FI$ -inj. Further, recall that for any finite dimensional orthogonal G -representations V and W and any object A in \mathcal{C} ,

$$F_V A(W) = O_G(V, W) \wedge A.$$

Since the evaluation functors preserve colimits, Lemma 3.3.1 together with the pushout-product axiom implies that any morphism in FI -cell is a levelwise cofibration and any morphism in FJ -cell is a levelwise acyclic cofibration. By [Hov99, Proposition 2.1.16], the domains of maps in I are small relative to the class of cofibrations in \mathcal{C} . Hence, by adjunction, the domains of morphisms in FI are small relative to FI -cell. The same argument also applies to FJ . It remains to check that

$$FJ\text{-cell} \subset \mathscr{W}^{lv} \cap FI\text{-cof}.$$

We have already verified that $FJ\text{-cell} \subset \mathscr{W}^{lv}$. To see the second inclusion, note that $J \subset I\text{-cell}$. Since F_V is a left adjoint for any $V \in O_G$, one gets the inclusion

$$F_V J \subset F_V I\text{-cell}$$

which implies that $FJ \subset FI\text{-cof}$. As $FI\text{-cof}$ is closed under coproducts, pushouts and transfinite compositions, we conclude that the class $FJ\text{-cell}$ is included in $FI\text{-cof}$. By Proposition 2.1.2 this completes the proof. \square

Recall that Proposition 3.2.3 tells us that $\mathrm{Sp}_G^O(\mathcal{C})$ is enriched, tensored and cotensored over the category Sp_G^O . We conclude this section with the following compatibility result:

Proposition 3.3.7. *Let \mathcal{C} be a cofibrantly generated G - \mathbf{Top}_* -model category. Suppose that $i: K \rightarrow L$ is a cofibration in Sp_G^O and $j: A \rightarrow B$ a cofibration in $\mathrm{Sp}_G^O(\mathcal{C})$. Then the pushout-product*

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \rightarrow L \wedge B$$

is a cofibration in $\mathrm{Sp}_G^O(\mathcal{C})$. Moreover, if in addition i or j is a level equivalence, then so is $i \square j$.

Proof. By [Hov99, Corollary 4.2.5], it suffices to prove the claim for generating cofibrations and generating acyclic cofibrations. For any finite dimensional orthogonal G -representations V and W , there is a natural isomorphism

$$F_V K \wedge F_W A \cong F_{V \oplus W}(K \wedge A)$$

which follows from the Fubini theorem and the enriched Yoneda lemma [Kel05, Section 3.10, (3.63), (3.71)]. Now the functors F_V preserve colimits and cofibrations. Furthermore, they send acyclic cofibrations to cofibrations which are additionally level equivalences. Combined with the latter isomorphism this implies the desired result. \square

Proposition 3.3.7 together with Proposition 3.2.3 yields

Corollary 3.3.8. *Let \mathcal{C} be a cofibrantly generated G - \mathbf{Top}_* -model category. Suppose that $i: K \rightarrow L$ is a cofibration in Sp_G^O and $p: X \rightarrow Y$ a level fibration in $\mathrm{Sp}_G^O(\mathcal{C})$. Then the following hold:*

(i) *The induced map*

$$\mathrm{Hom}(L, X) \rightarrow \mathrm{Hom}(K, X) \times_{\mathrm{Hom}(K, Y)} \mathrm{Hom}(L, Y)$$

is a level fibration of G -equivariant orthogonal spectra. Moreover, this map is also a level equivalence if in addition i or p is a level equivalence.

(ii) *The induced map*

$$X^L \rightarrow X^K \times_{Y^K} Y^L$$

is a level fibration in $\mathrm{Sp}_G^O(\mathcal{C})$. Moreover, this map is also a level equivalence if in addition i or p is a level equivalence.

3.4 The stable model structure on $\mathrm{Sp}_G^O(\mathcal{C})$

This subsection establishes the stable model structure on $\mathrm{Sp}_G^O(\mathcal{C})$. For this one needs more assumptions than in Proposition 3.3.6. More precisely, we have to assume that the cofibrantly generated G - \mathbf{Top}_* -model category \mathcal{C} is proper and stable as an underlying model category. The strategy is to generalize the arguments given in [SS03, 3.8].

Let W be a finite dimensional orthogonal G -representation and

$$\lambda_W = \lambda_{0,W}: F_W S^W \longrightarrow F_0 S^0 = \mathbb{S}$$

denote the stable equivalence of G -equivariant orthogonal spectra that is adjoint to the identity map

$$\text{id}: S^W \longrightarrow \text{Ev}_W(\mathbb{S}) = S^W$$

(see [MM02, III.4.3, III.4.5] or Subsection 2.6).

Definition 3.4.1. Let \mathcal{C} be a G -**Top** $_*$ -model category. An object Z of $\text{Sp}_G^O(\mathcal{C})$ is called an Ω -spectrum if it is level fibrant and for any finite dimensional orthogonal G -representation W , the induced map

$$\lambda_W^*: Z \cong Z^{F_0 S^0} \longrightarrow Z^{F_W S^W}$$

is a level equivalence.

Since $Z^{F_W S^W} \cong Z(W \oplus -)^{S^W}$, this definition recovers the definition of a G - Ω -spectrum in the sense of [MM02, Definition III.3.1] when $\mathcal{C} = G$ -**Top** $_*$ (see also Definition 2.6.1).

Now suppose again that \mathcal{C} is a cofibrantly generated G -**Top** $_*$ -model category. By Proposition 3.3.6, the level model structure on $\text{Sp}_G^O(\mathcal{C})$ is cofibrantly generated. Hence we can choose (and fix once and for all) a cofibrant replacement functor

$$(-)^c: \text{Sp}_G^O(\mathcal{C}) \longrightarrow \text{Sp}_G^O(\mathcal{C}).$$

Definition 3.4.2. A morphism $f: A \longrightarrow B$ in $\text{Sp}_G^O(\mathcal{C})$ is a stable equivalence if for any Ω -spectrum Z , the map

$$\text{Hom}(f^c, Z): \text{Hom}(B^c, Z) \longrightarrow \text{Hom}(A^c, Z)$$

is a level equivalence of G -equivariant orthogonal spectra.

Lemma 3.4.3. *Let K be a cofibrant G -equivariant orthogonal spectrum, A a cofibrant object in $\text{Sp}_G^O(\mathcal{C})$ and Z an Ω -spectrum in $\text{Sp}_G^O(\mathcal{C})$. Then $\text{Hom}(A, Z)$ is a G - Ω -spectrum and Z^K is an Ω -spectrum in $\text{Sp}_G^O(\mathcal{C})$.*

Proof. Corollary 3.3.8 (i) implies that $\text{Hom}(A, Z)$ is level fibrant. Next, it follows from Proposition 3.2.3 that there is a natural isomorphism

$$\text{Hom}(A, Z)^L \cong \text{Hom}(A, Z^L)$$

of G -equivariant orthogonal spectra. Hence, the map

$$\lambda_W^*: \text{Hom}(A, Z)^{F_0 S^0} \longrightarrow \text{Hom}(A, Z)^{F_W S^W}$$

is isomorphic to the map

$$\text{Hom}(A, \lambda_W^*): \text{Hom}(A, Z^{F_0 S^0}) \longrightarrow \text{Hom}(A, Z^{F_W S^W}).$$

By Corollary 3.3.8 (ii), the objects $Z^{F_0 S^0}$ and $Z^{F_W S^W}$ are level fibrant as Z is. Since Z is an Ω -spectrum, the morphism

$$\lambda_W^*: Z \cong Z^{F_0 S^0} \longrightarrow Z^{F_W S^W}$$

is a level equivalence. Consequently, the latter morphism is a level equivalence between level fibrant objects. Now using Corollary 3.3.8 (i), Ken Brown's Lemma (see e.g. [Hov99, Lemma 1.1.12]) and that A is cofibrant, we conclude that $\text{Hom}(A, \lambda_W^*)$ and hence

$$\lambda_W^*: \text{Hom}(A, Z)^{F_0 S^0} \longrightarrow \text{Hom}(A, Z)^{F_W S^W}$$

is a level equivalence. This implies that $\text{Hom}(A, Z)$ is an Ω -spectrum.

The proof for Z^K is similar. We use the natural isomorphism $(Z^L)^K \cong (Z^K)^L$, Corollary 3.3.8 (ii) and Ken Brown's Lemma. \square

Lemma 3.4.4. *Let \mathcal{C} be a left proper and cofibrantly generated G - \mathbf{Top}_* -model category. Then a cofibration $i: A \longrightarrow B$ is a stable equivalence if and only if for any Ω -spectrum Z in $\text{Sp}_G^O(\mathcal{C})$, the orthogonal G -spectrum $\text{Hom}(B/A, Z)$ is level weakly G -contractible.*

Proof. Recall that $(-)^c$ denotes the cofibrant replacement functor on $\text{Sp}_G^O(\mathcal{C})$. By Proposition 3.3.6, the map $i^c: A^c \longrightarrow B^c$ admits a factorization into a cofibration $j: A^c \longrightarrow \overline{B}$ followed by a level acyclic fibration $p: \overline{B} \longrightarrow B^c$. The morphism p is a level equivalence between cofibrant objects. Therefore, by Corollary 3.3.8 (i) and Ken Brown's Lemma, the map $\text{Hom}(p, Z)$ is a level equivalence of G -equivariant orthogonal spectra. Hence, $i: A \longrightarrow B$ is stable equivalence if and only if the map

$$\text{Hom}(j, Z): \text{Hom}(\overline{B}, Z) \longrightarrow \text{Hom}(A^c, Z)$$

is a level equivalence for any Ω -spectrum Z . Next, for every Ω -spectrum Z , the cofiber sequence

$$A^c \xrightarrow{j} \overline{B} \longrightarrow \overline{B}/A^c$$

induces a level fiber sequence

$$\text{Hom}(\overline{B}/A^c, Z) \longrightarrow \text{Hom}(\overline{B}, Z) \xrightarrow{\text{Hom}(j, Z)} \text{Hom}(A^c, Z),$$

where all G -spectra are G - Ω -spectra by Lemma 3.4.3. Consequently, $i: A \longrightarrow B$ is a stable equivalence if and only if the G -spectrum $\text{Hom}(\overline{B}/A^c, Z)$ is level weakly G -contractible. Thus, to complete the proof it suffices to check that $\text{Hom}(\overline{B}/A^c, Z)$ and $\text{Hom}(B/A, Z)$ are level equivalent. Since \mathcal{C} is left proper and the maps $A^c \longrightarrow A$ and $\overline{B} \longrightarrow B$ are level equivalences, the induced map on the cofibers

$$\overline{B}/A^c \longrightarrow B/A$$

is a level equivalence. The objects \overline{B}/A^c and B/A are cofibrant in $\text{Sp}_G^O(\mathcal{C})$ and therefore, Corollary 3.3.8 (i) and Ken Brown's Lemma imply that the latter morphism induces

a level equivalence between $\text{Hom}(\overline{B}/A^c, Z)$ and $\text{Hom}(B/A, Z)$. Hence i is a stable equivalence if and only if the G -spectrum $\text{Hom}(B/A, Z)$ is level weakly G -contractible for any Ω -spectrum Z . \square

Corollary 3.4.5. *Let \mathcal{C} be a left proper and cofibrantly generated G - \mathbf{Top}_* -model category. Then a cofibration $i: A \rightarrow B$ is a stable equivalence if and only if for any Ω -spectrum Z in $\text{Sp}_G^O(\mathcal{C})$, the G -spectrum $\text{Hom}(B/A, \Omega Z)$ is level weakly G -contractible.*

Proof. By Lemma 3.4.3, for any Ω -spectrum Z in $\text{Sp}_G^O(\mathcal{C})$, the object

$$\Omega Z = Z^{S^1} = Z^{F_0 S^1}$$

is an Ω -spectrum. Hence, Lemma 3.4.4 implies that if the cofibration $i: A \rightarrow B$ is a stable equivalence, then the G -spectrum $\text{Hom}(B/A, \Omega Z)$ is level weakly G -contractible.

Conversely, suppose that for any Ω -spectrum Z , the G -spectrum $\text{Hom}(B/A, \Omega Z)$ is level weakly G -contractible. By Lemma 3.4.3, the cotensor $Z^{F_1 S^0}$ is an Ω -spectrum and Definition 3.4.1 tells us that the map

$$Z \rightarrow \Omega(Z^{F_1 S^0}) \cong Z^{F_1 S^1}$$

is a level equivalence between level fibrant objects. Since B/A is cofibrant, Corollary 3.3.8 (i) and Ken Brown's Lemma imply that $\text{Hom}(B/A, Z)$ and $\text{Hom}(B/A, \Omega(Z^{F_1 S^0}))$ are level equivalent. According to the assumption, $\text{Hom}(B/A, \Omega(Z^{F_1 S^0}))$ is level weakly G -contractible and by Lemma 3.4.4 this finishes the proof. \square

The next proposition is very important. After showing that under certain assumptions the category $\text{Sp}_G^O(\mathcal{C})$ has a stable model structure, the proposition will imply that $\text{Sp}_G^O(\mathcal{C})$ is an Sp_G^O -model category (i.e., a G -spectral category see Definition 3.5.1).

Proposition 3.4.6. *Let \mathcal{C} be a left proper and cofibrantly generated G - \mathbf{Top}_* -model category. Suppose that $i: K \rightarrow L$ is a cofibration in Sp_G^O and $j: A \rightarrow B$ a cofibration in $\text{Sp}_G^O(\mathcal{C})$. Then the pushout-product*

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \rightarrow L \wedge B$$

is a cofibration in $\text{Sp}_G^O(\mathcal{C})$. The map $i \square j$ is also a stable equivalence if in addition i or j is a stable equivalence.

Proof. The morphism $i \square j$ is a cofibration by Proposition 3.3.7. The cofiber of $i \square j$ is isomorphic to

$$(L/K) \wedge (B/A).$$

Hence, by Lemma 3.4.4, it suffices to show that the G -equivariant orthogonal spectrum

$$\text{Hom}((L/K) \wedge (B/A), Z)$$

is level weakly G -contractible for any Ω -spectrum Z if i or j is a stable equivalence. First suppose that the morphism $j: A \rightarrow B$ is a stable equivalence in $\mathrm{Sp}_G^O(\mathcal{C})$. According to Proposition 3.2.3, we have an isomorphism

$$\mathrm{Hom}((L/K) \wedge (B/A), Z) \cong \mathrm{Hom}(B/A, Z^{(L/K)}).$$

By Lemma 3.4.3, for any Ω -spectrum Z , the spectrum $Z^{(L/K)}$ is an Ω -spectrum as L/K is cofibrant. Since $j: A \rightarrow B$ is a stable equivalence, it follows from Lemma 3.4.4 that $\mathrm{Hom}(B/A, Z^{(L/K)})$ is level weakly G -contractible for any Ω -spectrum Z . This completes the proof of the first case.

Now suppose that $i: K \rightarrow L$ is a stable equivalence in Sp_G^O . Proposition 3.2.3 yields an isomorphism

$$\mathrm{Hom}((L/K) \wedge (A/B), Z) \cong \mathrm{Hom}(L/K, \mathrm{Hom}(A/B, Z)).$$

Lemma 3.4.3 implies that $\mathrm{Hom}(A/B, Z)$ is a G - Ω -spectrum. On the other hand, since $i: K \rightarrow L$ is a stable equivalence and the model category Sp_G^O is left proper [MM02, III.4.2], the orthogonal G -spectrum L/K is G -stably contractible (i.e., $\pi_*^H(L/K) = 0$, $H \leq G$). Hence, by [MM02, Proposition III.7.5], the G -spectrum

$$\mathrm{Hom}(L/K, \mathrm{Hom}(A/B, Z))$$

is a G -stably contractible G - Ω -spectrum. Since any G -stably contractible G - Ω -spectrum is level weakly G -contractible [MM02, Lemma III.9.1], it follows that the orthogonal G -spectrum $\mathrm{Hom}(L/K, \mathrm{Hom}(A/B, Z))$ is level weakly G -contractible and this completes the proof. \square

Next, we introduce the set J_{st} which will serve as a set of generating acyclic cofibrations for the stable model structure on $\mathrm{Sp}_G^O(\mathcal{C})$ that we are going to establish. Let W be a finite dimensional orthogonal G -representation. Consider the levelwise mapping cylinder $M\lambda_W$ of the map $\lambda_W: F_W S^W \rightarrow F_0 S^0$. The map λ_W factors as a composite

$$F_W S^W \xrightarrow{\kappa_W} M\lambda_W \xrightarrow{r_W} F_0 S^0,$$

where r_W is a G -equivariant homotopy equivalence and κ_W a cofibration and a stable equivalence [MM02, III.4.5-4.6] (see also Subsection 2.6). Define

$$K = \{\kappa_W \square F_V i \mid i \in I, V, W \in \mathrm{sk} O_G\},$$

where \square is the pushout-product, I is the fixed set of generating cofibrations in \mathcal{C} (see Definition 3.3.2) and $\mathrm{sk} O_G$ the fixed small skeleton of O_G as in Subsection 3.3. Next, recall from Proposition 3.3.6 that we have sets $F I$ and $F J$, generating cofibrations and acyclic cofibrations, respectively, in the level model structure. Define

$$J_{\mathrm{st}} = F J \cup K.$$

For the convenience we will denote the set FI by I_{st} . The cofibrations in the stable model structure on $\text{Sp}_G^O(\mathcal{C})$ will be the same as in the level model structure and thus $I_{\text{st}} = FI$ will serve as a set of generating cofibrations for the stable model structure.

The following proposition is an important technical statement used to establish the stable model structure on $\text{Sp}_G^O(\mathcal{C})$.

Proposition 3.4.7. *Let \mathcal{C} be a left proper and cofibrantly generated G - \mathbf{Top}_* -model category. Then any morphism in $J_{\text{st}}\text{-cell}$ is an I_{st} -cofibration (i.e., a cofibration) and a stable equivalence.*

Proof. By Proposition 3.3.7, the morphisms in K are cofibrations. Next, Proposition 3.3.6 implies that $FJ \subset FI\text{-cof}$. Hence, one has

$$J_{\text{st}} = FJ \cup K \subset FI\text{-cof} = I_{\text{st}}\text{-cof}.$$

Since $I_{\text{st}}\text{-cof}$ is closed under coproducts, pushouts and transfinite compositions, we get

$$J_{\text{st}}\text{-cell} \subset I_{\text{st}}\text{-cof}.$$

It remains to check that all maps in $J_{\text{st}}\text{-cell}$ are stable equivalences in $\text{Sp}_G^O(\mathcal{C})$. We claim that for any J_{st} -cofibration $A \rightarrow B$ and any Ω -spectrum Z , the induced map

$$\text{Hom}(B, \Omega Z) \rightarrow \text{Hom}(A, \Omega Z)$$

is a level acyclic fibration of orthogonal G -spectra. Before proving the claim, let us show how it completes the proof. Indeed, if the latter map is a level acyclic fibration, then its fiber $\text{Hom}(B/A, \Omega Z)$ is level weakly G -contractible by the long exact sequence of homotopy groups of a fibration. Hence, Corollary 3.4.5 applies and we conclude that $A \rightarrow B$ is a stable equivalence.

We will now show that the above claim holds. The property of inducing a level acyclic fibration after applying $\text{Hom}(-, \Omega Z)$ is obviously closed under coproducts, pushouts, transfinite compositions and retracts. Thus, it suffices to show that for any Ω -spectrum Z , the functor $\text{Hom}(-, \Omega Z)$ gives level acyclic fibrations when applied to morphisms from $J_{\text{st}} = FJ \cup K$. For morphisms from FJ this follows immediately from Proposition 3.3.6 and Corollary 3.3.8 (i). For K we proceed as follows. By Proposition 3.4.6, for any $V, W \in \text{sk } O_G$ and $i \in I$, the morphism $\kappa_W \square F_V i$ is a stable equivalence and a cofibration in $\text{Sp}_G^O(\mathcal{C})$. Hence, Corollary 3.3.8 (i) implies that $\text{Hom}(\kappa_W \square F_V i, Z)$ is a level fibration and Lemma 3.4.4 implies that the fiber of $\text{Hom}(\kappa_W \square F_V i, Z)$ is level weakly G -contractible. Together with the long exact sequence of homotopy groups of a fibration this allows us to conclude that $\Omega \text{Hom}(\kappa_W \square F_V i, Z)$ is a level acyclic fibration. The desired result now follows from the isomorphism

$$\Omega \text{Hom}(\kappa_W \square F_V i, Z) \cong \text{Hom}(\kappa_W \square F_V i, \Omega Z). \quad \square$$

The next Lemma provides a lifting property characterization of Ω -spectra in $\text{Sp}_G^O(\mathcal{C})$. After establishing the stable model structure on $\text{Sp}_G^O(\mathcal{C})$, this lemma will describe the fibrant objects in the stable model structure.

Lemma 3.4.8. *Let \mathcal{C} be a cofibrantly generated G - \mathbf{Top}_* -model category and X an object of $\mathrm{Sp}_G^O(\mathcal{C})$. Then the map $X \rightarrow *$ is J_{st} -injective if and only if X is an Ω -spectrum.*

Proof. By Proposition 3.3.6, the morphism $X \rightarrow *$ is FJ -injective if and only if X is level fibrant. Now we assume that X is level fibrant and show that $X \rightarrow *$ is K -injective if and only if X is an Ω -spectrum. By adjunction the map $X \rightarrow *$ is K -injective if and only if for any W , the map $X^{\kappa_W} : X^{M\lambda_W} \rightarrow X^{F_W S^W}$ has the right lifting property with respect to FI , i.e., if and only if it is a level acyclic fibration. Since the map X^{κ_W} is always a level fibration for any W and a level fibrant X (according to Corollary 3.3.8 (ii)), it follows that $X \rightarrow *$ is K -injective if and only if $X^{\kappa_W} : X^{M\lambda_W} \rightarrow X^{F_W S^W}$ is a level equivalence in $\mathrm{Sp}_G^O(\mathcal{C})$. Further, the map $r_W : M\lambda_W \rightarrow F_0 S^0 = \mathbb{S}$ is a G -equivariant homotopy equivalence [MM02, III.4.6] and hence the induced map $X^{r_W} : X \rightarrow X^{M\lambda_W}$ is a level equivalence. On the other hand, we have the equality $\lambda_W = r_W \circ \kappa_W$ and thus, for a level fibrant X , the map $X \rightarrow *$ is K -injective if and only if $X^{\lambda_W} : X \rightarrow X^{F_W S^W}$ is a level equivalence. That is, for a level fibrant X , the map $X \rightarrow *$ is K -injective if and only if X is an Ω -spectrum. \square

The following proposition is the last major technical statement used for constructing the stable model structure. Here we have to assume that our cofibrantly generated G - \mathbf{Top}_* -model category \mathcal{C} is right proper and stable as an underlying model category.

Proposition 3.4.9. *Let \mathcal{C} be a right proper and cofibrantly generated G - \mathbf{Top}_* -model category which is stable as an underlying model category. Then a map in $\mathrm{Sp}_G^O(\mathcal{C})$ is J_{st} -injective and a stable equivalence if and only if it is a level acyclic fibration.*

Proof. By Corollary 3.3.8 (i), every level equivalence in $\mathrm{Sp}_G^O(\mathcal{C})$ is a stable equivalence. Hence, every level acyclic fibration is a stable equivalence. Further, the class J_{st} is contained in $I_{\mathrm{st}}\text{-cof}$ and hence,

$$I_{\mathrm{st}}\text{-inj} \subset J_{\mathrm{st}}\text{-inj}.$$

The class $I_{\mathrm{st}}\text{-inj} = FI\text{-inj}$ coincides with the class of level acyclic fibrations (Proposition 3.3.6). Thus, every level acyclic fibration is J_{st} -injective and a stable equivalence. The converse statement needs much more work.

Suppose a morphism $E \rightarrow B$ in $\mathrm{Sp}_G^O(\mathcal{C})$ is J_{st} -injective and a stable equivalence. Since $FJ \subset J_{\mathrm{st}}$, it follows from Proposition 3.3.6 that the map $E \rightarrow B$ is a level fibration. Next, let F denote the fiber of $E \rightarrow B$. Choose a cofibrant replacement $F^{\mathrm{cof}} \xrightarrow{\simeq_{lv}} F$ in the level model category and factor the composite $F^{\mathrm{cof}} \xrightarrow{\simeq_{lv}} F \rightarrow E$ in the level model structure

$$\begin{array}{ccc} F^{\mathrm{cof}} & \xrightarrow{\quad} & E \\ & \searrow & \nearrow \simeq_{lv} \\ & E^{\mathrm{cof}} & \end{array}$$

as a cofibration followed by a level equivalence. We get a commutative diagram

$$\begin{array}{ccccc}
F^{\text{cof}} & \xrightarrow{\quad} & E^{\text{cof}} & \longrightarrow & E^{\text{cof}}/F^{\text{cof}} \\
\downarrow \simeq_{lv} & & \downarrow \simeq_{lv} & & \downarrow \\
F & \longrightarrow & E & \longrightarrow & B.
\end{array}$$

The upper sequence in this diagram is a homotopy cofiber sequence. Since the model category \mathcal{C} is right proper, the lower sequence is a levelwise homotopy fiber sequence. The stability of \mathcal{C} implies that homotopy cartesian and cocartesian squares are the same in \mathcal{C} (see e.g. [Hov99, Remark 7.1.12]). Therefore, the induced map $E^{\text{cof}}/F^{\text{cof}} \rightarrow B$ is a level equivalence. As the morphism $E \rightarrow B$ is a stable equivalence, it follows that the map $E^{\text{cof}} \rightarrow E^{\text{cof}}/F^{\text{cof}}$ is a stable equivalence as well. Now let Z be any Ω -spectrum in $\text{Sp}_G^O(\mathcal{C})$ and consider the sequence

$$\text{Hom}(E^{\text{cof}}/F^{\text{cof}}, Z) \longrightarrow \text{Hom}(E^{\text{cof}}, Z) \longrightarrow \text{Hom}(F^{\text{cof}}, Z).$$

Corollary 3.3.8 (i) implies that the second map in this sequence is a level fibration and hence this sequence is a level fiber sequence of orthogonal G - Ω -spectra. Further, the first map in this fiber sequence is a level equivalence since $E^{\text{cof}} \rightarrow E^{\text{cof}}/F^{\text{cof}}$ is a stable equivalence. This implies that $\text{Hom}(F^{\text{cof}}, Z)$ is level weakly G -contractible for any Ω -spectrum Z . Since $F \rightarrow *$ is a pullback of $E \rightarrow B$ along the map $* \rightarrow B$, we see that the morphism $F \rightarrow *$ is J_{st} -injective and thus F is an Ω -spectrum according to Lemma 3.4.8. Consequently, one concludes that the G - Ω -spectrum $\text{Hom}(F^{\text{cof}}, F)$ is level weakly G -contractible. This in particular implies that $\pi_0^G(\text{Hom}(F^{\text{cof}}, F)(0)) = 0$. But the set $\pi_0^G(\text{Hom}(F^{\text{cof}}, F)(0))$ is isomorphic to the set of endomorphisms of F in the homotopy category of the level model structure on $\text{Sp}_G^O(\mathcal{C})$, yielding that F is level equivalent to a point. Thus we conclude that $E \rightarrow B$ is a level equivalence and this completes the proof. \square

Finally, we are ready to establish the stable model structure. The following proposition constructs the desired model structure. The proof of the fact that this model structure is stable is postponed to the next subsection.

Proposition 3.4.10. *Let \mathcal{C} be a proper and cofibrantly generated G - \mathbf{Top}_* -model category which is stable as an underlying model category. Then the category $\text{Sp}_G^O(\mathcal{C})$ admits a cofibrantly generated model structure with stable equivalences as weak equivalences. Moreover, the sets I_{st} and J_{st} generate cofibrations and acyclic cofibrations, respectively.*

Proof. The strategy of the proof is to verify the conditions of Proposition 2.1.2. Let \mathcal{W}^{st} denote the class of stable equivalences in $\text{Sp}_G^O(\mathcal{C})$. It immediately follows from Definition 3.4.2 that the class \mathcal{W}^{st} satisfies the two out of three property and is closed under retracts. Next, Proposition 3.4.7 implies that we have an inclusion

$$J_{\text{st}}\text{-cell} \subset \mathcal{W}^{\text{st}} \cap I_{\text{st}}\text{-cof}$$

(this uses that \mathcal{C} is left proper) and Proposition 3.4.9 tells us that the following holds (here we use that \mathcal{C} is right proper and stable as an underlying model category):

$$J_{\text{st-inj}} \cap \mathcal{W}^{\text{st}} = I_{\text{st-inj}}.$$

Hence, the only things that still have to be checked are the smallness conditions. That the domains of morphisms from I_{st} are small relative to $I_{\text{st-cell}}$ follows from the equality $I_{\text{st}} = FI$ and Proposition 3.3.6. Next, recall that $J_{\text{st}} = FJ \cup K$. We will now verify that the domains of morphisms from J_{st} are small relative to levelwise cofibrations. This will immediately imply that the domains of morphisms in J_{st} are small relative to $J_{\text{st-cell}}$ since

$$J_{\text{st-cell}} \subset I_{\text{st-cof}}$$

by Proposition 3.4.7 and any morphism in $I_{\text{st-cof}}$ is a levelwise cofibration as we saw in the proof of Proposition 3.3.6. That the domains of morphisms in FJ are small relative to levelwise cofibrations follows from an adjunction argument, Definition 3.3.2 (ii) and [Hov99, 2.1.16]. It remains to show that the domains of morphisms from K are small relative to levelwise cofibrations. Any morphism in K is a pushout-product of the form

$$\kappa_W \square F_V i: (M\lambda_W \wedge F_V A) \bigvee_{F_W S^W \wedge F_V A} (F_W S^W \wedge F_V B) \longrightarrow M\lambda_W \wedge F_V B.$$

where the morphism $i: A \longrightarrow B$ is from the set I and V and W are finite dimensional orthogonal G -representations. For any finite G -CW complex L and any object D which is the domain or codomain of a map from I , the spectrum $F_W L \wedge F_V D$ is small relative to levelwise cofibrations. Indeed, we have an isomorphism

$$\text{Hom}(F_W L \wedge F_V D, X) \cong \text{Map}(L \wedge D, X(V \oplus W)).$$

Since a pushout of small objects is small, Definition 3.3.2 (i) implies that $L \wedge D$ is small with respect to I -cof and hence $F_W L \wedge F_V D$ is small relative to levelwise cofibrations. Now we use twice that pushouts of small objects are small. First we conclude that $M\lambda_W \wedge F_V A$ is small relative to levelwise cofibrations and then we also see that

$$(M\lambda_W \wedge F_V A) \bigvee_{F_W S^W \wedge F_V A} (F_W S^W \wedge F_V B)$$

is small relative to levelwise cofibrations. \square

3.5 G -equivariant stable model categories and the proof of Proposition 3.1.2

We start with the following

Definition 3.5.1. An Sp_G^O -model category is called G -spectral. In other words, a model \mathcal{C} category is G -spectral if it is enriched, tensored and cotensored over the model category Sp_G^O and the pushout-out product axiom for tensors holds (see Definition 2.1.4).

By Proposition 2.6.5 the model category Sp_G^O is G -spectral. Next, Proposition 3.4.6 shows that the model structure of Proposition 3.4.10 on $\mathrm{Sp}_G^O(\mathcal{C})$ is G -spectral.

Recall from Definition 3.1.1 that a G -equivariant stable model category is a G - \mathbf{Top}_* -model category such that the Quillen adjunction

$$S^V \wedge - : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega^V(-)$$

is a Quillen equivalence for any finite dimensional orthogonal G -representation V . Before stating the next proposition, note that every G -spectral model category is obviously a G - \mathbf{Top}_* -model category.

Proposition 3.5.2. *Let \mathcal{C} be a G -spectral model category. Then \mathcal{C} is a G -equivariant stable model category.*

Proof. Consider the left Quillen functors

$$S^V \wedge - : \mathcal{C} \longrightarrow \mathcal{C} \quad \text{and} \quad F_V S^0 \wedge - : \mathcal{C} \longrightarrow \mathcal{C}$$

and their derived functors

$$S^V \wedge^{\mathbf{L}} - : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C}) \quad \text{and} \quad F_V S^0 \wedge^{\mathbf{L}} - : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

Since the map $\lambda_V : F_V S^V \longrightarrow \mathbb{S}$ is a stable equivalence [MM02, III.4.5], for every cofibrant X in \mathcal{C} , one has the following weak equivalences

$$S^V \wedge F_V S^0 \wedge X \cong F_V S^V \wedge X \xrightarrow[\simeq]{\lambda_V \wedge 1} X$$

and

$$F_V S^0 \wedge S^V \wedge X \cong F_V S^V \wedge X \xrightarrow[\simeq]{\lambda_V \wedge 1} X.$$

This implies that the functors $S^V \wedge^{\mathbf{L}} -$ and $F_V S^0 \wedge^{\mathbf{L}} -$ are mutually inverse equivalences of categories. \square

Corollary 3.5.3. *Let \mathcal{C} be a proper and cofibrantly generated G - \mathbf{Top}_* -model category which is stable as an underlying model category. Then the category $\mathrm{Sp}_G^O(\mathcal{C})$ together with the model structure of Proposition 3.4.10 is a G -equivariant stable model category.*

From this point on, the model structure of Proposition 3.4.10 will be referred to as the stable model structure on $\mathrm{Sp}_G^O(\mathcal{C})$.

Finally, we are ready to prove Proposition 3.1.2:

Proof of Proposition 3.1.2. We want to show that the G - \mathbf{Top}_* -adjunction

$$F_0 = \Sigma^\infty : \mathcal{C} \rightleftarrows \mathrm{Sp}^O(\mathcal{C}) : \mathrm{Ev}_0$$

is a Quillen equivalence for every cofibrantly generated (in the sense of Definition 3.3.2) and proper G -equivariant stable model category \mathcal{C} . This adjunction is a Quillen adjunction since $F_0 I \subset F I = I_{\mathrm{st}}$ and $F_0 J \subset F J \subset J_{\mathrm{st}}$. By [Hov99, 1.3.16], in order to

show that this Quillen adjunction is a Quillen equivalence, it suffices to check that the functor Ev_0 reflects stable equivalences between stably fibrant objects (i.e., between Ω -spectra according to Lemma 3.4.8) and that for any cofibrant A in \mathcal{C} , the composite

$$A \longrightarrow \text{Ev}_0(\Sigma^\infty A) \longrightarrow \text{Ev}_0(R(\Sigma^\infty A)),$$

is a weak equivalence. Here R is a fibrant replacement in the stable model structure on $\text{Sp}_G^O(\mathcal{C})$, the first map is the adjunction unit and the second map is induced by the stable equivalence $\Sigma^\infty A \longrightarrow R(\Sigma^\infty A)$. So suppose that $f: X \longrightarrow Y$ is a map of Ω -spectra in $\text{Sp}_G^O(\mathcal{C})$ with $f_0: X_0 \longrightarrow Y_0$ a weak equivalence in \mathcal{C} . Since X and Y are Ω -spectra, it follows that for any finite dimensional orthogonal G -representation V , the map

$$\Omega^V f(V): \Omega^V X(V) \longrightarrow \Omega^V Y(V)$$

is a weak equivalence in \mathcal{C} and $X(V)$ and $Y(V)$ are fibrant in \mathcal{C} . The G -equivariant stability of \mathcal{C} implies that the functor $\Omega^V: \mathcal{C} \longrightarrow \mathcal{C}$ induces a self-equivalence of the homotopy category $\text{Ho}(\mathcal{C})$. Hence, the map $f(V): X(V) \longrightarrow Y(V)$ is a weak equivalence for any finite dimensional orthogonal G -representation V . In other words, $f: X \longrightarrow Y$ is a level equivalence. As every level equivalence is a stable equivalence we conclude that f is a stable equivalence.

We now check the second condition. Let A be a cofibrant object of \mathcal{C} and $\Sigma_{\text{fib}}^\infty A$ denote a fibrant replacement of $\Sigma^\infty A$ in the level model structure. For any finite dimensional orthogonal G -representation V , we have

$$\Sigma^\infty A(V) = F_0 A(V) = O_G(0, V) \wedge A = S^V \wedge A.$$

Therefore, $\Sigma_{\text{fib}}^\infty A(V) = (S^V \wedge A)^{\text{fib}}$, where $(-)^{\text{fib}}$ is a fibrant replacement in \mathcal{C} . The fibrant replacement $\Sigma_{\text{fib}}^\infty A$ comes with a morphism of spectra $g: \Sigma^\infty A \longrightarrow \Sigma_{\text{fib}}^\infty A$ which is a level equivalence. Therefore, for V and W finite dimensional orthogonal G -representations, we get a commutative diagram

$$\begin{array}{ccc} S^V \wedge A & \longrightarrow & \Omega^W(S^W \wedge S^V \wedge A) \\ g(V) \downarrow & & \downarrow \Omega^W(g(V \oplus W)) \\ (S^V \wedge A)^{\text{fib}} & \longrightarrow & \Omega^W(S^W \wedge S^V \wedge A)^{\text{fib}}, \end{array}$$

where the left vertical map is a weak equivalence and horizontal maps are the adjoint structure maps. By the G -equivariant stability of \mathcal{C} , the adjunction $(S^W \wedge -, \Omega^W(-))$ is a Quillen equivalence. Since the map

$$S^V \wedge A \longrightarrow \Omega^W(S^W \wedge S^V \wedge A) \xrightarrow{\Omega^W(g_{V \oplus W})} \Omega^W(S^W \wedge S^V \wedge A)^{\text{fib}}$$

is the evaluation of the derived unit of this adjunction on $S^V \wedge A$, it follows that it is a weak equivalence [Hov99, 1.3.13]. Hence, this together with the latter commutative square tells us that the composite

$$S^V \wedge A \xrightarrow{g_V} (S^V \wedge A)^{\text{fib}} \longrightarrow \Omega^W(S^W \wedge S^V \wedge A)^{\text{fib}}$$

is a weak equivalence. Now as the map g_V is a weak equivalence, by the two out of three property, the morphism

$$(S^V \wedge A)^{\text{fib}} \longrightarrow \Omega^W(S^W \wedge S^V \wedge A)^{\text{fib}}$$

is a weak equivalence for any finite dimensional orthogonal G -representations V and W . This means that $\Sigma_{\text{fib}}^\infty A$ is an Ω -spectrum and that the map $g: \Sigma^\infty A \longrightarrow \Sigma_{\text{fib}}^\infty A$ is a model for the stably fibrant replacement $\Sigma^\infty A \longrightarrow R(\Sigma^\infty A)$. This completes the proof, since g is a level equivalence and hence the map

$$A \xrightarrow{\cong} \text{Ev}_0(\Sigma^\infty A) \xrightarrow{g_0} \text{Ev}_0(\Sigma_{\text{fib}}^\infty A)$$

is a weak equivalence. □

Remark 3.5.4. The Quillen equivalence

$$\Sigma^\infty: \mathcal{C} \rightleftarrows \text{Sp}^O(\mathcal{C}) : \text{Ev}_0$$

is in fact a G -**Top** $_*$ -Quillen equivalence. Indeed, $(\Sigma^\infty, \text{Ev}_0)$ is a G -**Top** $_*$ -enriched adjunction and an enriched adjunction which is an underlying Quillen equivalence is an enriched Quillen equivalence by definition. Next, since enriched left adjoints preserve tensors [Kel05, Sections 3.2 and 3.7], the functor Σ^∞ preserves tensors. Similarly, the right adjoint Ev_0 preserves cotensors. Further, the equivalence

$$\mathbf{L}\Sigma^\infty: \text{Ho}(\mathcal{C}) \rightleftarrows \text{Ho}(\text{Sp}^O(\mathcal{C})) : \mathbf{R}\text{Ev}_0$$

is a $\text{Ho}(G$ -**Top** $_*)$ -enriched equivalence. Finally, we note that the functor $\mathbf{L}\Sigma^\infty$ preserves derived tensors and since $\mathbf{R}\text{Ev}_0$ is an inverse of $\mathbf{L}\Sigma^\infty$, it is also compatible with derived tensors.

3.6 The p -local model structure on G -equivariant orthogonal spectra

This subsection discusses the p -localization of the stable model structure on Sp_G^O for any prime p . Although this model structure is well-known, we give a detailed proof here since we were unable to find a reference. Note that one can construct the p -local model structure on Sp_G^O by using general localization techniques of [Hir03] or [Bou01]. We will not use any of these machineries here and give a direct proof by generalizing the arguments of [SS02, Section 4] to the equivariant context.

Definition 3.6.1. (i) A map $f: X \longrightarrow Y$ of orthogonal G -spectra is called a p -local equivalence if the induced map

$$\pi_*^H(f) \otimes_{\mathbb{Z}_{(p)}}: \pi_*^H X \otimes_{\mathbb{Z}_{(p)}} \longrightarrow \pi_*^H Y \otimes_{\mathbb{Z}_{(p)}}$$

is an isomorphism for any subgroup H of G .

(ii) A map $p: X \longrightarrow Y$ of orthogonal G -spectra is called a p -local fibration if it has the right lifting property with respect to all maps that are cofibrations and p -local equivalences.

Proposition 3.6.2. *Let G be a finite group and p a prime. Then the category Sp_G^O of G -equivariant orthogonal spectra together with p -local equivalences, cofibrations and p -local fibrations forms a cofibrantly generated model category.*

We need some technical preparation before proving this proposition.

Recall from Section 2 that the stable model structure on Sp_G^O is cofibrantly generated with $I_{\mathrm{st}}^G = I_{\mathrm{lv}}^G$ and $J_{\mathrm{st}}^G = K^G \cup J_{\mathrm{lv}}^G$ generating cofibrations and acyclic cofibrations, respectively. Further, we also recall that the mod l Moore space $M(l)$ is defined by the following pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{\cdot l} & S^1 \\ \downarrow & & \downarrow \\ CS^1 & \longrightarrow & M(l). \end{array}$$

($C(-) = (I, 0) \wedge -$ is the pointed cone functor.) Let $\iota: M(l) \rightarrow CM(l)$ denote the inclusion of $M(l)$ into the cone $CM(l)$. Define $J_{(p)}^G$ to be the set of maps of orthogonal G -spectra

$$F_n(G/H_+ \wedge \Sigma^m \iota): F_n(G/H_+ \wedge \Sigma^m M(l)) \longrightarrow F_n(G/H_+ \wedge \Sigma^m CM(l)),$$

where $n, m \geq 0$, $H \leq G$ and l is prime to p , i.e., invertible in $\mathbb{Z}_{(p)}$. We let J_{loc}^G denote the union $J_{\mathrm{st}}^G \cup J_{(p)}^G$. This set will serve as a set of generating acyclic cofibrations for the p -local model structure on Sp_G^O .

Lemma 3.6.3. *Let X be a G -equivariant orthogonal spectrum. Then the map $X \rightarrow *$ is J_{loc}^G -injective if and only if X is a G - Ω -spectrum and the H -equivariant homotopy groups $\pi_*^H X$ are p -local for any $H \leq G$.*

Proof. By definition, $X \rightarrow *$ is J_{loc}^G -injective if and only if $X \rightarrow *$ is J_{st}^G -injective and $J_{(p)}^G$ -injective. It follows from [MM02, III.4.10] that the map $X \rightarrow *$ is J_{loc}^G -injective if and only if X is a G - Ω -spectrum and $X \rightarrow *$ is $J_{(p)}^G$ -injective. Now for a G - Ω -spectrum X , having the right lifting property with respect to $J_{(p)}^G$ means that

$$[F_n(G/H_+ \wedge \Sigma^m M(l)), X]^G = 0,$$

for any $m, n \geq 0$, $H \leq G$ and any l which is prime to p . The distinguished triangles

$$F_n(G/H_+ \wedge S^{m+1}) \xrightarrow{\cdot l} F_n(G/H_+ \wedge S^{m+1}) \longrightarrow F_n(G/H_+ \wedge \Sigma^m M(l)) \longrightarrow \Sigma F_n(G/H_+ \wedge S^{m+1})$$

in $\mathrm{Ho}(\mathrm{Sp}_G^O)$ imply that the latter amounts to the fact that the maps

$$[F_n(G/H_+ \wedge S^{m+1}), X]^G \xrightarrow{\cdot l} [F_n(G/H_+ \wedge S^{m+1}), X]^G$$

are isomorphisms for any $m, n \geq 0$, $H \leq G$ and any l which is prime to p . Since $[F_n(G/H_+ \wedge S^{m+1}), X]^G \cong \pi_{m+1-n}^H X$ the desired result follows. \square

Lemma 3.6.4. *Let $f: X \rightarrow Y$ be a morphism of orthogonal G -spectra which is J_{loc}^G -injective and a p -local equivalence. Then f is a stable equivalence and a stable fibration.*

Proof. Since f is J_{loc}^G -injective, it is in particular J_{st}^G -injective and hence a stable fibration in Sp_G^O . Let F denote the fiber of f . Since any stable fibration is a level fibration, the level fiber sequence

$$F \longrightarrow X \xrightarrow{f} Y$$

induces a long exact sequence of equivariant stable homotopy groups

$$\cdots \longrightarrow \pi_*^H F \longrightarrow \pi_*^H X \xrightarrow{\pi_*^H f} \pi_*^H Y \longrightarrow \cdots,$$

for any $H \leq G$. The \mathbb{Z} -module $\mathbb{Z}_{(p)}$ is flat over \mathbb{Z} . Hence if we tensor the latter long exact sequence with $\mathbb{Z}_{(p)}$, we get a long exact sequence of p -local homotopy groups. By the assumptions, the morphism f induces an isomorphism on $\pi_*^H(-) \otimes \mathbb{Z}_{(p)}$ and thus it follows that

$$\pi_*^H F \otimes \mathbb{Z}_{(p)} = 0$$

for any subgroup $H \leq G$. On the other hand, the map $F \rightarrow *$ is J_{loc}^G -injective as a pullback of $f: X \rightarrow Y$ and hence, by Lemma 3.6.3, the G -spectrum F has p -local H -equivariant homotopy groups for all $H \leq G$. Combining the last two facts, we conclude that $\pi_*^H F = 0$, for any subgroup H of G . Finally, using the above long exact sequence of integral equivariant homotopy groups, one sees that the maps

$$\pi_*^H f: \pi_*^H X \longrightarrow \pi_*^H Y, \quad H \leq G$$

are isomorphisms, i.e., f is a stable equivalence. \square

The following lemma is the main technical statement needed for establishing the p -local model structure:

Lemma 3.6.5. *Every morphism of G -equivariant orthogonal spectra can be factored as a composite $q \circ i$, where q is J_{loc}^G -injective and i is a cofibration and a p -local equivalence.*

Proof. Since compact topological spaces are sequentially small with respect to closed T_1 -inclusions, it follows that the domains of morphisms from J_{loc}^G are sequentially small relative to levelwise cofibrations. By definition, every map in J_{loc}^G is a cofibration in Sp_G^O . This implies that every map in J_{loc}^G -cell is a cofibration. On the other hand, by Lemma 3.3.1, every cofibration in Sp_G^O is a levelwise cofibration. Consequently, the domains of morphisms from J_{loc}^G are sequentially small with respect to J_{loc}^G -cell. Hence we can use the countable version of Quillen's small object argument (see e.g. [Qui67, II.3] or [Hov99, Theorem 2.1.14]) to factor a given map as a composite $q \circ i$, where q is J_{loc}^G -injective and i is a possibly countable composition of pushouts of coproducts of morphisms from J_{loc}^G . In particular, the morphism i is in J_{loc}^G -cell and thus a cofibration. It remains to show that i is a p -local equivalence. We first check that the morphisms

in J_{loc}^G are p -local equivalences. Recall that $J_{\text{loc}}^G = J_{\text{st}}^G \cup J_{(p)}^G$. The morphisms in J_{st}^G are stable equivalences and thus p -local equivalences. Further, for any l which is prime to p , the map

$$\pi_*^L(F_n(G/H_+ \wedge S^{m+1})) \otimes \mathbb{Z}_{(p)} \xrightarrow{\cdot l} \pi_*^L(F_n(G/H_+ \wedge S^{m+1})) \otimes \mathbb{Z}_{(p)}, \quad n, m \geq 0, \quad H, L \leq G$$

is an isomorphism. The distinguished triangle

$$F_n(G/H_+ \wedge S^{m+1}) \xrightarrow{\cdot l} F_n(G/H_+ \wedge S^{m+1}) \longrightarrow F_n(G/H_+ \wedge \Sigma^m M(l)) \longrightarrow \Sigma F_n(G/H_+ \wedge S^{m+1})$$

then implies that

$$\pi_*^L(F_n(G/H_+ \wedge \Sigma^m M(l))) \otimes \mathbb{Z}_{(p)} = 0, \quad n, m \geq 0, \quad H, L \leq G.$$

This tells us that the maps from $J_{(p)}^G$ are p -local equivalences. Now since equivariant homotopy groups commute with coproducts, the coproducts of maps from J_{loc}^G are p -local equivalences as well. Next, using that any cofibration induces a long exact sequence of equivariant homotopy groups and that $\mathbb{Z}_{(p)}$ is flat, we see that the pushouts of coproducts of maps from J_{loc}^G are p -local equivalences. Hence every map in the countable composite defining the map i is a p -local equivalence. Equivariant stable homotopy groups commute with sequential colimits of cofibrations and the tensor product preserves colimits. Since every morphism in the latter countable composite is a cofibration and a p -local equivalence, we conclude that their composite i is also a p -local equivalence. This finishes the proof. \square

The next lemma provides a lifting property characterization of p -local fibrations.

Lemma 3.6.6. *A map of G -equivariant orthogonal spectra is a p -local fibration if and only if it has the right lifting property with respect to J_{loc}^G (i.e., if and only if it is J_{loc}^G -injective).*

Proof. Every map in J_{loc}^G is a cofibration and a p -local equivalence according to proof of the previous lemma. Hence, by definition, every p -local fibration has the right lifting property with respect to J_{loc}^G . To show the converse statement it suffices to check that every morphism j which is a cofibration and a p -local equivalence is contained in J_{loc}^G -cof. For this we use the retract argument (see e.g. [Hov99, 1.1.9]). Factor $j = q \circ i$ as in the proof of Lemma 3.6.5. The map i is in J_{loc}^G -cell and a p -local equivalence and q is J_{loc}^G -injective. Since i and j are both p -local equivalences, so is q . Hence, by Lemma 3.6.4, the morphism q is stable equivalence and a stable fibration and thus has the right lifting property with respect to any cofibration. In particular, it has the right lifting property with respect to j . This implies that j is a retract of i which is in J_{loc}^G -cell. Consequently, j is a J_{loc}^G -cofibration. \square

Finally, we are ready to prove Proposition 3.6.2.

Proof of Proposition 3.6.2. We check that all the properties from [DS95, Definition 3.3] are satisfied. In addition, we verify that the conditions from Definition 2.1.1 are also fulfilled to see that the p -local model structure is cofibrantly generated.

The category of G -equivariant orthogonal spectra has all small limits and colimits. The p -local equivalences satisfy the two out of three property and the classes of cofibrations, p -local equivalences and p -local fibrations are closed under retracts. Further, Lemma 3.6.4 and Lemma 3.6.6 imply that every map that is a p -local fibration and a p -local equivalence is a stable equivalence and a stable fibration and thus has the right lifting property with respect to cofibrations. The second lifting axiom is just the definition of p -local fibrations. Next, the stable model structure provides a factorization of every map into a cofibration followed by a map which is a stable equivalence and a stable fibration, i.e., stably acyclic fibration. Stably acyclic fibrations are p -local equivalences and p -local fibrations (they have the right lifting property with respect to any cofibration and in particular, with respect to J_{loc}^G). Hence we obtain one of the desired factorizations in the factorization axiom. The second part of the factorization axiom immediately follows from Lemma 3.6.5 and Lemma 3.6.6. This completes the construction of the p -local model structure for orthogonal G -spectra.

Now we prove that the established p -local model structure is cofibrantly generated. The set I_{st}^G will serve as a set of generating cofibrations and the set J_{loc}^G will be the set of generating acyclic cofibrations. The smallness conditions from Definition 2.1.1 for the set I_{st}^G follow from [MM02, III.4.2] and the smallness conditions for J_{loc}^G were discussed in the proof of Lemma 3.6.5. Further, the class J_{loc}^G -inj coincides with the class of p -local fibrations according to Lemma 3.6.6. Finally, the class of p -local fibrations which are additionally p -local equivalences coincides with the class of stably acyclic fibrations and hence with the class I_{st}^G -inj. \square

From this point on we will denote the category of orthogonal G -spectra equipped with the p -local model structure of Proposition 3.6.2 by $\text{Sp}_{G,(p)}^O$. The following proposition shows that the model category $\text{Sp}_{G,(p)}^O$ is a monoidal model category:

Proposition 3.6.7. *Suppose that $i: K \rightarrow L$ and $j: A \rightarrow B$ are cofibrations in $\text{Sp}_{G,(p)}^O$. Then the pushout-product*

$$i \square j: K \wedge B \bigvee_{K \wedge A} L \wedge A \rightarrow L \wedge B$$

is a cofibration in $\text{Sp}_{G,(p)}^O$. Moreover, if in addition i or j is a p -local equivalence (i.e., a weak equivalence in $\text{Sp}_{G,(p)}^O$), then so is $i \square j$.

Proof. The fact that $i \square j$ is a cofibration follows from the monoidality of Sp_G^O [MM02, III.7.5] (see also Proposition 2.6.5). Next, by [Hov99, Corollary 4.2.5] it suffices to prove the statement for generating cofibrations and acyclic cofibrations. So suppose that i is a generating cofibration and j is a generating acyclic cofibration. Recall that the set $J_{\text{loc}}^G = J_{(p)}^G \cup J_{\text{st}}^G$ is a set of generating acyclic cofibrations for $\text{Sp}_{G,(p)}^O$. If j is in

J_{st}^G , then $i \square j$ is a stable equivalence and hence, in particular, a p -local equivalence. Now let j be a map

$$F_n(G/H_+ \wedge \Sigma^m M(l)) \longrightarrow F_n(G/H_+ \wedge \Sigma^m CM(l))$$

from $J_{(p)}^G$ and i a map

$$F_V(G/K_+ \wedge S_+^{t-1}) \longrightarrow F_V(G/K_+ \wedge D_+^t)$$

from I_{st}^G . Then the target of the pushout-product $i \square j$

$$F_V(G/K_+ \wedge D_+^t) \wedge F_n(G/H_+ \wedge \Sigma^m CM(l)) \cong C(F_{V \oplus \mathbb{R}^n}(G/K_+ \wedge D_+^t \wedge G/H_+ \wedge \Sigma^m M(l)))$$

is zero in $\text{Ho}(\text{Sp}_G^O)$ and thus has trivial equivariant stable homotopy groups. Consequently, in order to show that $i \square j$ is a p -local equivalence, it suffices to check that the equivariant stable homotopy groups of the source of the pushout-product $i \square j$ become trivial after tensoring with $\mathbb{Z}_{(p)}$. Since the pushout square defining the source of $i \square j$ is a homotopy pushout square in Sp_G^O and

$$F_V(G/K_+ \wedge S_+^{t-1}) \wedge F_n(G/H_+ \wedge \Sigma^m CM(l))$$

is stably contractible, it follows that the source of $i \square j$ is stably equivalent to the orthogonal G -spectrum

$$F_V(G/K_+ \wedge S^t) \wedge F_n(G/H_+ \wedge \Sigma^m M(l)).$$

This spectrum is a mapping cone in $\text{Ho}(\text{Sp}_G^O)$ of the map

$$F_V(G/K_+ \wedge S^t) \wedge F_n(G/H_+ \wedge S^{m+1}) \xrightarrow{-l} F_V(G/K_+ \wedge S^t) \wedge F_n(G/H_+ \wedge S^{m+1})$$

which induces an isomorphism on $\pi_*(-) \otimes \mathbb{Z}_{(p)}$ (by definition of $J_{(p)}^G$, the integer l is prime to p). Using the long exact sequence of equivariant stable homotopy groups and flatness of $\mathbb{Z}_{(p)}$ we conclude that

$$\pi_*^L(F_V(G/K_+ \wedge S^t) \wedge F_n(G/H_+ \wedge \Sigma^m M(l))) \otimes \mathbb{Z}_{(p)} = 0$$

for any subgroup L of G . Hence the p -localized equivariant stable homotopy groups of the source of $i \square j$ are trivial and this completes the proof. \square

Since every stable equivalence of G -equivariant orthogonal spectra is a p -local equivalence, one obtains the following corollary:

Corollary 3.6.8. *The model category $\text{Sp}_{G,(p)}^O$ is G -spectral, i.e., an Sp_G^O -model category (see Definition 3.5.1).*

In view of Proposition 3.5.2, we also obtain

Corollary 3.6.9. *The model category $\mathrm{Sp}_{G,(p)}^O$ is a G -equivariant stable model category (see Definition 3.1.1).*

We end this subsection with some useful comments and remarks about the homotopy category $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$. Since the model category $\mathrm{Sp}_{G,(p)}^O$ is stable, the homotopy category $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$ is naturally triangulated. Further, the set

$$\{\Sigma_+^\infty G/H \mid H \leq G\}$$

is a set of compact generators for $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$. Indeed, let X be an orthogonal G -spectrum and let X^f denote a fibrant replacement of X in $\mathrm{Sp}_{G,(p)}^O$. Then, by Lemma 3.6.3, the spectrum X^f is a G - Ω -spectrum and has p -local equivariant homotopy groups. This gives us the following chain of natural isomorphisms:

$$\begin{aligned} [\Sigma_+^\infty G/H, X]_*^{\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)} &\cong [\Sigma_+^\infty G/H, X^f]_*^{\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)} \cong [\Sigma_+^\infty G/H, X^f]_*^{\mathrm{Ho}(\mathrm{Sp}_G^O)} \cong \\ &\pi_*^H(X^f) \cong \pi_*^H(X^f) \otimes \mathbb{Z}_{(p)} \cong \pi_*^H X \otimes \mathbb{Z}_{(p)} \end{aligned}$$

Hence, the object $\Sigma_+^\infty G/H$ in $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$ represents the p -localized H -equivariant homotopy group functor and therefore, the set $\{\Sigma_+^\infty G/H \mid H \leq G\}$ is a set of compact generators for $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$.

Finally, we note that for any G -equivariant orthogonal spectra X and Y , the abelian group of morphisms $[X, Y]_*^{\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)}$ in $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$ is p -local. This follows from the fact that for any integer l which is prime to p , the map $l \cdot \mathrm{id}: X \rightarrow X$ is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$.

3.7 Reduction to Theorem 3.1.3

In this subsection we will explain precisely how the proof of Theorem 1.1.1 reduces to the proof of Theorem 3.1.3. The main idea was already indicated in Subsection 3.1. The arguments in this subsection work at any prime p . So there is no point in restricting ourselves to prime 2 here.

Let \mathcal{C} be a cofibrantly generated (in the sense of Definition 3.3.2), proper, G -equivariant stable model category. Suppose that

$$\Psi: \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C})$$

is an equivalence of triangulated categories such that

$$\Psi(\Sigma_+^\infty G/H) \cong G/H_+ \wedge^{\mathbf{L}} \Psi(\mathbb{S}),$$

for any $H \leq G$. Suppose further that the latter isomorphisms are natural with respect to the restrictions, conjugations and transfers. Then by Proposition 3.1.2, there is a G - \mathbf{Top}_* -Quillen equivalence

$$\Sigma^\infty: \mathcal{C} \rightleftarrows \mathrm{Sp}^O(\mathcal{C}) : \mathrm{Ev}_0.$$

Next, as in Subsection 3.1, let X be a cofibrant replacement of $(\mathbf{L}\Sigma^\infty \circ \Psi)(\mathbb{S})$. Since $\mathrm{Sp}_G^O(\mathcal{C})$ is G -spectral (Proposition 3.4.6), there is a G - \mathbf{Top}_* -Quillen adjunction

$$- \wedge X : \mathrm{Sp}_G^O \rightleftarrows \mathrm{Sp}_G^O(\mathcal{C}) : \mathrm{Hom}(X, -).$$

We will now show that this Quillen adjunction passes through the p -localization $\mathrm{Sp}_{G,(p)}^O$.

Proposition 3.7.1. *The adjunction*

$$- \wedge X : \mathrm{Sp}_{G,(p)}^O \rightleftarrows \mathrm{Sp}_G^O(\mathcal{C}) : \mathrm{Hom}(X, -).$$

is a G - \mathbf{Top}_ -Quillen adjunction.*

Proof. The adjunction is a G - \mathbf{Top}_* -adjunction and hence it suffices to verify that it is an underlying Quillen adjunction. The functor $- \wedge X$ preserves cofibrations, since the cofibrations in $\mathrm{Sp}_{G,(p)}^O$ and Sp_G^O coincide. Now suppose $i : K \rightarrow L$ is a cofibration and a p -local equivalence. We want to show that the cofibration $i \wedge 1 : K \wedge X \rightarrow L \wedge X$ is a stable equivalence. By Lemma 3.4.4, it suffices to check that for any Ω -spectrum Z , the orthogonal G -spectrum $\mathrm{Hom}(L/K \wedge X, Z)$ is level weakly G -contractible. By adjunction, there is an isomorphism

$$\mathrm{Hom}(L/K \wedge X, Z) \cong \mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$$

and we will now verify that the G -spectrum $\mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$ is level weakly G -contractible.

The spectrum $\mathrm{Hom}(X, Z)$ is a G - Ω -spectrum according to Lemma 3.4.3. Furthermore, for any subgroup $H \leq G$, the H -equivariant homotopy of $\mathrm{Hom}(X, Z)$ is isomorphic to

$$[\Sigma_+^\infty G/H \wedge X, Z]_*^{\mathrm{Ho}(\mathrm{Sp}_G^O(\mathcal{C}))}.$$

The latter groups are p -local since $\mathrm{Ho}(\mathrm{Sp}_G^O(\mathcal{C}))$ is equivalent to $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$ by the assumptions and Proposition 3.1.2 and the Hom-groups in $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$ are p -local by the last remark in the previous subsection. Hence, Lemma 3.6.3 tells us that $\mathrm{Hom}(X, Z)$ is fibrant in $\mathrm{Sp}_{G,(p)}^O$. Next, Proposition 3.6.7 implies that the orthogonal G -spectrum $\mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$ is fibrant in $\mathrm{Sp}_{G,(p)}^O$. Therefore, $\mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$ is a G - Ω -spectrum and the H -equivariant homotopy of it is isomorphic to

$$[\Sigma_+^\infty G/H \wedge L/K, \mathrm{Hom}(X, Z)]_*^{\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)}.$$

On the other hand, by the assumptions, the cofibration $i : K \rightarrow L$ is a p -local equivalence and therefore, L/K is zero in $\mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$. Hence, the equivariant homotopy groups of the G - Ω -spectrum $\mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$ vanish, i.e., it is stably contractible. Any stably contractible G - Ω spectrum is level weakly G -contractible ([MM02, Lemma III.9.1]), implying that $\mathrm{Hom}(L/K, \mathrm{Hom}(X, Z))$ is level weakly G -contractible. This finishes the proof. \square

Now we continue as in Subsection 3.1. In order to prove the p -local version of Conjecture 1.1.2, it suffices to show that the Quillen adjunction of Proposition 3.7.1 is a Quillen equivalence. Next, by the properties of Ψ and Remark 3.5.4, we have the following chain of isomorphisms

$$\Psi(\Sigma_+^\infty G/H) \cong G/H_+ \wedge^{\mathbf{L}} \Psi(\mathbb{S}) \cong G/H_+ \wedge^{\mathbf{L}} \mathbf{R} \mathrm{Ev}_0 X \cong \mathbf{R} \mathrm{Ev}_0(\Sigma_+^\infty G/H \wedge^{\mathbf{L}} X)$$

which is natural with respect to transfers, conjugations, and restrictions. Using these isomorphisms, we can choose an inverse of Ψ

$$\Psi^{-1}: \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$$

such that $\Psi^{-1}(\mathbf{R} \mathrm{Ev}_0(\Sigma_+^\infty G/H \wedge^{\mathbf{L}} X)) = \Sigma_+^\infty G/H$. Moreover, since the isomorphisms above are natural with respect to the stable Burnside category (Subsection 2.7), we get the identities

$$\Psi^{-1}(\mathbf{R} \mathrm{Ev}_0(g \wedge^{\mathbf{L}} X)) = g, \quad \Psi^{-1}(\mathbf{R} \mathrm{Ev}_0(\mathrm{res}_K^H \wedge^{\mathbf{L}} X)) = \mathrm{res}_K^H, \quad \Psi^{-1}(\mathbf{R} \mathrm{Ev}_0(\mathrm{tr}_K^H \wedge^{\mathbf{L}} X)) = \mathrm{tr}_K^H,$$

where $g \in G$ and $K \leq H \leq G$. Now let us consider the composite

$$F: \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O) \xrightarrow{-\wedge^{\mathbf{L}} X} \mathrm{Ho}(\mathrm{Sp}_G^O(\mathcal{C})) \xrightarrow{\mathbf{R} \mathrm{Ev}_0} \mathrm{Ho}(\mathcal{C}) \xrightarrow{\Psi^{-1}} \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O).$$

Since the functors $\mathbf{R} \mathrm{Ev}_0$ and Ψ^{-1} are equivalences, to prove that $(-\wedge X, \mathrm{Hom}(X, -))$ is a Quillen equivalence is equivalent to showing that the endofunctor

$$F: \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{G,(p)}^O)$$

is an equivalence of categories. By the assumptions and the properties of Ψ^{-1} , we see that F enjoys the following properties:

- (i) $F(\Sigma_+^\infty G/H) = \Sigma_+^\infty G/H$, $H \leq G$;
- (ii) $F(g) = g$, $F(\mathrm{res}_K^H) = \mathrm{res}_K^H$, $F(\mathrm{tr}_K^H) = \mathrm{tr}_K^H$, $g \in G$, $K \leq H \leq G$;
- (iii) F is an exact functor of triangulated categories and preserves infinite coproducts.

So finally, we see that in order to prove Theorem 1.1.1, it suffices to prove Theorem 3.1.3. Note that we do not expect that an odd primary version of Theorem 3.1.3 is true. However, we still think that Conjecture 1.1.2 holds. Schwede's paper [Sch07] suggests that the proof in the odd primary case should use the explicit construction and the whole content behind the endofunctor F , whereas in the 2-local case certain axiomatic properties of F are enough to get the desired result as Theorem 3.1.3 shows. This is a generic difference between the 2-local case and the p -local case for p an odd prime.

4 Free G-spectra

Since the set $\{\Sigma_+^\infty G/H \mid H \leq G\}$ is a set of compact generators for the triangulated category $\mathrm{Ho}(\mathrm{Sp}_G^O, (2))$, to prove Theorem 3.1.3 it suffices to show that for any subgroups H and K of G , the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/K)]_*^G = [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G$$

induced by F is an isomorphism.

In this section we show that under the assumptions of 3.1.3 the map

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \longrightarrow [F(\Sigma_+^\infty G), F(\Sigma_+^\infty G)]_*^G = [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism. Note that the graded endomorphism ring $[\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$ is isomorphic to the graded group algebra $\pi_*\mathbb{S}[G]$ and the localizing subcategory generated by $\Sigma_+^\infty G$ in $\mathrm{Ho}(\mathrm{Sp}_G^O)$ is equivalent to $\mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$, where $\Sigma_+^\infty G$ is considered as the group ring spectrum of G .

We say that an object $X \in \mathrm{Ho}(\mathrm{Sp}_G^O)$ is a *free G-spectrum* if it is contained in the localizing subcategory generated by $\Sigma_+^\infty G$.

In what follows, everything will be 2-localized and hence we will mostly omit the subscript 2. The arguments here are just equivariant generalizations of those in [Sch01].

4.1 Cellular structures

We start with the following

Definition 4.1.1. Let R be an orthogonal ring spectrum, X an R -module and n and m integers such that $n \leq m$. We say that X admits a *finite (n, m) -cell structure* if there are sequences of distinguished triangles

$$\bigvee_{I_k} \Sigma^{k-1} R \longrightarrow \mathrm{sk}_{k-1} X \longrightarrow \mathrm{sk}_k X \longrightarrow \bigvee_{I_k} \Sigma^k R$$

in $\mathrm{Ho}(\mathbf{Mod}\text{-}R)$, $k = n, \dots, m$, such that the sets I_k are finite and $\mathrm{sk}_{n-1} X = *$ and $\mathrm{sk}_m X = X$.

In other words, an R -module X admits a finite (n, m) -cell structure if and only if it admits a structure of a finite R -cell complex with all possible cells in dimensions between n and m .

Recall that there is a Quillen adjunction

$$G_+ \wedge : \mathbf{Mod}\text{-}\mathbb{S} \rightleftarrows \mathbf{Mod}\text{-}\Sigma_+^\infty G : U$$

and that $[\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \cong \pi_*\mathbb{S}[G]$. The following proposition can be considered as a 2-local naive equivariant version of [Sch01, Lemma 4.1] (cf. [Coh68, 4.2]).

Proposition 4.1.2. *Any $\alpha \in [\Sigma_+^\infty G, \Sigma_+^\infty G]_n^{\text{Ho}(\text{Sp}_G^{O(2)})}$, $n \geq 8$, factors over an $\Sigma_+^\infty G$ -module that admits a finite $(1, n-1)$ -cell structure.*

Proof. We will omit the subscript 2. Under the derived adjunction

$$G_+ \wedge^{\mathbf{L}} -: \text{Ho}(\mathbf{Mod}\text{-}\mathbb{S}) \rightleftarrows \text{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G) : \mathbf{R}U,$$

the element α corresponds to some map $\tilde{\alpha}: \mathbb{S}^n \rightarrow \mathbf{R}U(\Sigma_+^\infty G) \cong \bigvee_G \mathbb{S}$. By the proof of [Sch01, Lemma 4.1], for any $g \in G$, we have a factorization

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\tilde{\alpha}} & \mathbf{R}U(\Sigma_+^\infty G) \cong \bigvee_G \mathbb{S} \xrightarrow{\text{proj}_g} \mathbb{S} \\ & \searrow & \nearrow \\ & & Z_g \end{array}$$

in the stable homotopy category, where Z_g has \mathbb{S} -cells in dimensions between 1 and $n-1$. This uses essentially that $n \geq 8$. Indeed, since $n \geq 8$, for any $g \in G$, the morphism $\text{proj}_g \circ \tilde{\alpha}$ has \mathbb{F}_2 -Adams filtration at least 2 by the Hopf invariant one Theorem [Ada60] and hence, one of the implications of [Sch01, Lemma 4.1] applies to $\text{proj}_g \circ \tilde{\alpha}$. Assembling these factorizations together, we get a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\tilde{\alpha}} & \mathbf{R}U(\Sigma_+^\infty G) \\ & \searrow & \nearrow \\ & & \bigvee_{g \in G} Z_g \end{array}$$

Finally, by adjunction, one obtains the desired factorization

$$\begin{array}{ccc} \Sigma^n \Sigma_+^\infty G & \xrightarrow{\alpha} & \Sigma_+^\infty G \\ & \searrow & \nearrow \\ & & G_+ \wedge^{\mathbf{L}} (\bigvee_{g \in G} Z_g). \end{array}$$

□

Next, we use Proposition 4.1.2 to prove the following important

Lemma 4.1.3. *Suppose that the map of graded rings*

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \longrightarrow [F(\Sigma_+^\infty G), F(\Sigma_+^\infty G)]_*^G = [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism below and including dimension n for some $n \geq 0$.

(i) *Let K and L be $\Sigma_+^\infty G$ -modules that admit finite (β_K, τ_K) and (β_L, τ_L) -cell structures, respectively, and assume that $\tau_K - \beta_L \leq n$. Then the map*

$$F: [K, L]^G \longrightarrow [F(K), F(L)]^G$$

is an isomorphism.

(ii) Let K be an $\Sigma_+^\infty G$ -module admitting a finite (β_K, τ_K) -cell structure with $\tau_K - \beta_K \leq n + 1$. Then there is an $\Sigma_+^\infty G$ -module K' with a finite $(\beta_{K'}, \tau_{K'})$ -cell structure such that $\beta_K \leq \beta_{K'}$, $\tau_{K'} \leq \tau_K$ and $F(K') \cong K$.

(iii) If $n + 1 \geq 8$, then the map

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_{n+1}^G \longrightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_{n+1}^G$$

is an isomorphism.

Proof. (i) When K and L are both finite wedges of type $\bigvee \Sigma^{l_0} \Sigma_+^\infty G$ (l_0 is fixed), then the claim holds.

We start with the case when L is a finite wedge of copies of $\Sigma^{l_0} \Sigma_+^\infty G$, for some integer l_0 , and proceed by induction on $\tau_K - \beta_K$. As already noted, the claim holds when $\tau_K - \beta_K = 0$. Now suppose we are given K with $\tau_K - \beta_K = r$, $r \geq 1$, and assume that the claim holds for all $\Sigma_+^\infty G$ -modules M that have a finite (β_M, τ_M) -cell structure with $\tau_M - \beta_M < r$. Consider the distinguished triangle

$$\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G \longrightarrow \text{sk}_{\tau_K-1} K \longrightarrow K \longrightarrow \bigvee_{I_{\tau_K}} \Sigma^{\tau_K} \Sigma_+^\infty G.$$

The $\Sigma_+^\infty G$ -module $\text{sk}_{\tau_K-1} K$ has a finite $(\beta_K, \tau_K - 1)$ -cell structure. For convenience, let P denote the wedge $\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G$. The latter distinguished triangle induces a commutative diagram

$$\begin{array}{ccccccccc} [\Sigma \text{sk}_{\tau_K-1} K, L]^G & \longrightarrow & [\Sigma P, L]^G & \longrightarrow & [K, L]^G & \longrightarrow & [\text{sk}_{\tau_K-1} K, L]^G & \longrightarrow & [P, L]^G \\ \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\ [F(\Sigma \text{sk}_{\tau_K-1} K), F(L)]^G & \longrightarrow & [F(\Sigma P), F(L)]^G & \longrightarrow & [F(K), F(L)]^G & \longrightarrow & [F(\text{sk}_{\tau_K-1} K), F(L)]^G & \longrightarrow & [F(P), F(L)]^G \end{array}$$

with exact rows (The functor F is exact.). By the induction basis, the second and the last vertical morphisms in this diagram are isomorphisms. The fourth morphism is an isomorphism by the induction assumption. Finally, since $\Sigma \text{sk}_{\tau_K-1} K$ has a finite $(\beta_K + 1, \tau_K)$ -cell structure, the first vertical map is also an isomorphism by the induction assumption. Hence, the claim follows by the Five lemma.

Next, we do a similar induction with respect to $\tau_L - \beta_L$. The case $\tau_L - \beta_L = 0$ is the previous paragraph. For the inductive step we choose a distinguished triangle

$$\text{sk}_{\beta_L} L \longrightarrow L \longrightarrow L' \longrightarrow \Sigma \text{sk}_{\beta_L} L.$$

The $\Sigma_+^\infty G$ -module L' admits a finite $(\beta_L + 1, \tau_L)$ -cell structure. To see this one uses the octahedral axiom. Define $\text{sk}_i L'$ by the distinguished triangle

$$\text{sk}_{\beta_L} L \longrightarrow \text{sk}_i L \longrightarrow \text{sk}_i L' \longrightarrow \Sigma \text{sk}_{\beta_L} L.$$

Then $\text{sk}_{\beta_L} L' = *$ and $\text{sk}_{\tau_L} L' = L'$. By the octahedral axiom, for any $i \in \{\beta_L, \dots, \tau_L\}$,

there is a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{sk}_{\beta_L} L & \longrightarrow & \mathrm{sk}_i L & \longrightarrow & \mathrm{sk}_i L' & \longrightarrow & \Sigma \mathrm{sk}_{\beta_L} L \\
\parallel & & \downarrow & & \downarrow & & \parallel \\
\mathrm{sk}_{\beta_L} L & \longrightarrow & \mathrm{sk}_{i+1} L & \longrightarrow & \mathrm{sk}_{i+1} L' & \longrightarrow & \Sigma \mathrm{sk}_{\beta_L} L \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \bigvee_{J_i} \Sigma^{i+1} \Sigma_+^\infty G & \xlongequal{\quad} & \bigvee_{J_i} \Sigma^{i+1} \Sigma_+^\infty G & \longrightarrow & \Sigma \mathrm{sk}_i L \\
& & \downarrow & & \downarrow & & \\
& & \Sigma \mathrm{sk}_i L & \longrightarrow & \Sigma \mathrm{sk}_i L' & &
\end{array}$$

where the triangle

$$\mathrm{sk}_i L' \longrightarrow \mathrm{sk}_{i+1} L' \longrightarrow \bigvee_{J_i} \Sigma^{i+1} \Sigma_+^\infty G \longrightarrow \Sigma \mathrm{sk}_i L'$$

is distinguished. Now, as in the previous case, a five lemma argument finishes the proof.

(ii) We do induction on $\tau_K - \beta_K$. If $\tau_K - \beta_K = 0$, then K is stably equivalent to a finite wedge $\bigvee \Sigma^{l_0} \Sigma_+^\infty G$, for a fixed integer l_0 , and the claim holds since $F(\Sigma_+^\infty G) = \Sigma_+^\infty G$. For the induction step, choose a distinguished triangle

$$\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G \xrightarrow{\alpha} \mathrm{sk}_{\tau_K-1} K \longrightarrow K \longrightarrow \bigvee_{I_{\tau_K}} \Sigma^{\tau_K} \Sigma_+^\infty G.$$

as above. By the induction assumption, there is an $\Sigma_+^\infty G$ -module M with a finite (β_M, τ_M) -cell structure such that $\beta_K \leq \beta_M$, $\tau_M \leq \tau_K - 1$ and $F(M) \cong \mathrm{sk}_{\tau_K-1} K$. Consider the composite

$$F(\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G) \xrightarrow{\cong} \bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G \xrightarrow{\alpha} \mathrm{sk}_{\tau_K-1} K \xrightarrow{\cong} F(M).$$

Since $\tau_K - 1 - \beta_M \leq \tau_K - 1 - \beta_K \leq n$, part (i) yields that there exists

$$\alpha' \in \left[\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G, M \right]^G$$

such that $F(\alpha')$ equals the latter composition. Next, choose a distinguished triangle

$$\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G \xrightarrow{\alpha'} M \longrightarrow K' \longrightarrow \bigvee_{I_{\tau_K}} \Sigma^{\tau_K} \Sigma_+^\infty G.$$

The $\Sigma_+^\infty G$ -module K' has a finite (β_M, τ_K) -cell structure. On the other hand, since F is exact, one of the axioms for triangulated categories implies that there is a morphism $K \longrightarrow F(K')$ which makes the diagram

$$\begin{array}{ccccccc}
\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G & \xrightarrow{\alpha} & \mathrm{sk}_{\tau_K-1} K & \longrightarrow & K & \longrightarrow & \bigvee_{I_{\tau_K}} \Sigma^{\tau_K} \Sigma_+^\infty G \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong \\
F(\bigvee_{I_{\tau_K}} \Sigma^{\tau_K-1} \Sigma_+^\infty G) & \xrightarrow{F(\alpha')} & F(M) & \longrightarrow & F(K') & \longrightarrow & F(\bigvee_{I_{\tau_K}} \Sigma^{\tau_K} \Sigma_+^\infty G).
\end{array}$$

commute. Now another five lemma argument shows that in fact the map $K \rightarrow F(K')$ is an isomorphism in $\text{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$ and thus the proof of part (ii) is completed.

(iii) By Proposition 4.1.2, any morphism $\alpha \in [\Sigma_+^{n+1}\Sigma_+^\infty G, \Sigma_+^\infty G]^G$ factors over some $\Sigma_+^\infty G$ -module K which has a finite $(1, n)$ -cell structure. By part (ii), there exists an $\Sigma_+^\infty G$ -module K' admitting a finite $(\beta_{K'}, \tau_{K'})$ -cell structure and such that $1 \leq \beta_{K'}$, $\tau_{K'} \leq n$ and $F(K') \cong K$. Hence we get a commutative diagram

$$\begin{array}{ccc} F(\Sigma_+^{n+1}\Sigma_+^\infty G) \cong \Sigma_+^{n+1}\Sigma_+^\infty G & \xrightarrow{\alpha} & \Sigma_+^\infty G = F(\Sigma_+^\infty G) \\ & \searrow & \nearrow \\ & & F(K'). \end{array}$$

Since $n + 1 - \beta_{K'} \leq n + 1 - 1 = n$ and $\tau_{K'} - 0 = \tau_{K'} \leq n$, part (i) implies that both maps in the latter factorization are in the image of F . Hence, the map α is also in the image of the functor F yielding that

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_{n+1}^G \rightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_{n+1}^G$$

is surjective. As the source and target of this morphism are finite of the same cardinality, we conclude that it is an isomorphism. \square

Corollary 4.1.4. *Let F be as in 3.1.3. If the morphism*

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \rightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism for $ \leq 7$, then the functor F restricts to an equivalence on the full subcategory of free G -spectra.*

Proof. It suffices to show that the map

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \rightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism of graded rings. Suppose this is not the case. Then we choose the minimal n for which

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_n^G \rightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_n^G$$

is not an isomorphism. Now since by the assumption $n - 1 + 1 = n \geq 8$, we get a contradiction by Lemma 4.1.3 (iii). \square

4.2 Taking care of the dimensions ≤ 7

In this subsection we show that the map

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \rightarrow [F(\Sigma_+^\infty G), F(\Sigma_+^\infty G)]_*^G = [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism for $* \leq 7$. By Corollary 4.1.4 this will imply that the functor F restricts to an equivalence on the full subcategory of free G -spectra.

Recall that we have a preferred isomorphism $[\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \cong \pi_* \mathbb{S}[G]$. Next, since the functor F is compatible with the stable Burnside (orbit) category, $F(g) = g$ for any $g \in G$. On the other hand, the map $F: \pi_* \mathbb{S}[G] \rightarrow \pi_* \mathbb{S}[G]$ is a ring homomorphism and thus we conclude that it is an isomorphism for $* = 0$. Note that $\pi_* \mathbb{S}[G]$ is finite for $* > 0$ and the Hopf maps η, ν and σ multiplicatively generate $\pi_{* \leq 7} \mathbb{S}$. Hence, it remains to show that the Hopf maps (considered as elements of $\pi_* \mathbb{S}[G]$ via the unit map $\mathbb{S} \rightarrow \Sigma_+^\infty G$) are in the image of F .

We start by showing that $F(\eta) = \eta$. Recall that the mod 2 Moore spectrum $M(2)$ in the 2-localized (non-equivariant) stable homotopy category is defined by the distinguished triangle

$$\mathbb{S} \xrightarrow{2} \mathbb{S} \xrightarrow{\iota} M(2) \xrightarrow{\partial} \mathbb{S}^1$$

and the map $2: M(2) \rightarrow M(2)$ factors as a composite

$$M(2) \xrightarrow{\partial} \mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \xrightarrow{\iota} M(2).$$

Applying the functor $G_+ \wedge^{\mathbf{L}} -: \mathbf{Ho}(\mathbf{Mod}\text{-}\mathbb{S}) \rightarrow \mathbf{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$ to the distinguished triangle gives a distinguished triangle

$$\Sigma_+^\infty G \xrightarrow{2} \Sigma_+^\infty G \xrightarrow{1 \wedge \iota} G_+ \wedge M(2) \xrightarrow{1 \wedge \partial} \Sigma \Sigma_+^\infty G$$

in $\mathbf{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$. Further, the map $2: G_+ \wedge M(2) \rightarrow G_+ \wedge M(2)$ factors as

$$G_+ \wedge M(2) \xrightarrow{1 \wedge \partial} \Sigma \Sigma_+^\infty G \xrightarrow{\eta} \Sigma_+^\infty G \xrightarrow{1 \wedge \iota} G_+ \wedge M(2).$$

One of the axioms for triangulated categories implies that we can choose an isomorphism

$$F(G_+ \wedge M(2)) \cong G_+ \wedge M(2)$$

so that the diagram

$$\begin{array}{ccccccc} \Sigma_+^\infty G & \xrightarrow{2} & \Sigma_+^\infty G & \xrightarrow{1 \wedge \iota} & G_+ \wedge M(2) & \xrightarrow{1 \wedge \partial} & \Sigma \Sigma_+^\infty G \\ \parallel & & \parallel & & \downarrow \cong & & \downarrow \cong \\ F(\Sigma_+^\infty G) & \xrightarrow{2} & F(\Sigma_+^\infty G) & \xrightarrow{F(1 \wedge \iota)} & F(G_+ \wedge M(2)) & \xrightarrow{F(1 \wedge \partial)} & F(\Sigma \Sigma_+^\infty G). \end{array}$$

commutes. We fix the latter isomorphism once and for all and identify $F(G_+ \wedge M(2))$ with $G_+ \wedge M(2)$. Note that under this identification the morphisms $F(1 \wedge \iota)$ and $F(1 \wedge \partial)$ correspond to $1 \wedge \iota$ and $1 \wedge \partial$, respectively. Next, since $F(2) = 2$ and $2 = (1 \wedge \iota)\eta(1 \wedge \partial)$, one gets the identity

$$(1 \wedge \iota)F(\eta)(1 \wedge \partial) = 2.$$

It is well known that the map $2: M(2) \rightarrow M(2)$ is non-zero (In fact, $[M(2), M(2)] \cong \mathbb{Z}/4$) (see e.g. [Sch10, Proposition 4]). Hence, $2: G_+ \wedge M(2) \rightarrow G_+ \wedge M(2)$ is non-zero as there is a preferred ring isomorphism

$$[G_+ \wedge M(2), G_+ \wedge M(2)]_*^G \cong [M(2), M(2)]_* \otimes \mathbb{Z}[G].$$

Now it follows that $F(\eta) \neq 0$. Suppose $F(\eta) = \sum_{g \in A} \eta g$, where A is a non-empty subset of G . We want to show that $A = \{1\}$. The identity $(1 \wedge \iota)F(\eta)(1 \wedge \partial) = 2$ yields

$$2 = (1 \wedge \iota)\left(\sum_{g \in A} \eta g\right)(1 \wedge \partial) = \sum_{g \in A} (1 \wedge \iota)\eta(1 \wedge \partial)g = \sum_{g \in A} 2g.$$

Once again using the isomorphism $[G_+ \wedge M(2), G_+ \wedge M(2)]_*^G \cong [M(2), M(2)]_* \otimes \mathbb{Z}[G]$ and the fact that $2 \neq 0$, we conclude that $A = \{1\}$ and hence, $F(\eta) = \eta$.

Next, we show that ν is in the image of F . Let

$$F(\nu) = m\nu + \sum_{g \in G \setminus \{1\}} n_g g \nu.$$

Recall that 2-locally we have an identity (see e.g. [Tod62, 14.1 (i)])

$$\eta^3 = 4\nu.$$

Since $F(\eta) = \eta$, after applying F to this identity one obtains

$$4\nu = \eta^3 = F(\eta^3) = F(4\nu) = 4m\nu + \sum_{g \in G \setminus \{1\}} 4n_g g \nu.$$

As the element ν is a generator of the group $\pi_3\mathbb{S}_{(2)} \cong \mathbb{Z}/8$, we conclude that $m = 2k + 1$, for some $k \in \mathbb{Z}$, and for any $g \in G \setminus \{1\}$, $n_g = 2l_g$, $l_g \in \mathbb{Z}$. Hence

$$F(\nu) = (2k + 1)\nu + \sum_{g \in G \setminus \{1\}} 2l_g g \nu.$$

Using that $F(g) = g$, we also deduce that

$$F(g_0\nu) = (2k + 1)g_0\nu + \sum_{g \in G \setminus \{1\}} 2l_g g_0 g \nu,$$

for any fixed $g_0 \in G \setminus \{1\}$. Thus the image of F in $\pi_3\mathbb{S}_{(2)}[G] \cong \bigoplus_G \mathbb{Z}/8$ is additively generated by the rows a $G \times G$ -matrix of the form

$$\begin{pmatrix} 2k + 1 & & & & \\ & 2k + 1 & & \text{even} & \\ & \text{even} & \ddots & & \\ & & & 2k + 1 & \\ & & & & 2k + 1 \end{pmatrix},$$

where each diagonal entry is equal to $2k + 1$ and all the other entries are even. Since the determinant of this matrix is odd and hence a unit in $\mathbb{Z}/8$, the homomorphism $F: \pi_3\mathbb{S}_{(2)}[G] \rightarrow \pi_3\mathbb{S}_{(2)}[G]$ is an isomorphism and hence the element ν is in the image of F .

Finally, it remains to show that $\sigma \in \pi_7\mathbb{S} \subset \pi_7\mathbb{S}[G]$ is in the image of F . In order to do so, we will need certain Toda bracket relations in $\pi_*\mathbb{S}[G]$. First, we recall the definition of a Toda bracket.

Suppose that

$$X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} W$$

is a sequence in a triangulated category such that $ba = 0$ and $cb = 0$. Choose a distinguished triangle

$$X \xrightarrow{a} Y \xrightarrow{\iota} C(a) \xrightarrow{\partial} \Sigma X.$$

Since $ba = 0$, there exists $\lambda: C(a) \rightarrow Z$ such that $\lambda\iota = b$. Further, the identity $c\lambda = cb = 0$ implies that there exists $t: \Sigma X \rightarrow W$ such that the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & W \\ & & \downarrow \iota & \nearrow \lambda & & & \\ & & C(a) & & & & \\ & & \downarrow \partial & \nearrow t & & & \\ & & \Sigma X & & & & \end{array}$$

commutes. The set of all morphisms $t: \Sigma X \rightarrow W$ obtained in this way is called the *Toda bracket* of the sequence above and is denoted by $\langle a, b, c \rangle$. In fact, $\langle a, b, c \rangle$ coincides with a well-defined coset from

$$\mathrm{Hom}(\Sigma X, W) / (c_* \mathrm{Hom}(\Sigma X, Z) + a^* \mathrm{Hom}(\Sigma Y, W)).$$

The abelian group

$$c_* \mathrm{Hom}(\Sigma X, Z) + a^* \mathrm{Hom}(\Sigma Y, W)$$

is called the *indeterminacy* of the Toda bracket $\langle a, b, c \rangle$.

Next, let us recall the following technical

Lemma 4.2.1. *Suppose we are given a diagram*

$$X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{a'} \end{array} Y \begin{array}{c} \xrightarrow{b} \\ \xrightarrow{b'} \end{array} Z \begin{array}{c} \xrightarrow{c} \\ \xrightarrow{c'} \end{array} W$$

in a triangulated category.

(i) *If $ba = 0$ and $cb = c'b = 0$, then*

$$\langle a, b, c + c' \rangle \subset \langle a, b, c \rangle + \langle a, b, c' \rangle.$$

(as subsets of $[\Sigma X, W]$).

(ii) *If $ba = b'a = 0$ and $cb = cb' = 0$, then*

$$\langle a, b + b', c \rangle = \langle a, b, c \rangle + \langle a, b', c \rangle.$$

(iii) *If $ba = ba' = 0$ and $cb = 0$, then*

$$\langle a + a', b, c \rangle \subset \langle a, b, c \rangle + \langle a', b, c \rangle$$

Proof. (i) Take any $t \in \langle a, b, c + c' \rangle$. By definition of a Toda bracket, there is a commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c+c'} & W \\
 & & \downarrow \iota & \nearrow \lambda & & \nearrow t & \\
 & & C(a) & & & & \\
 & & \downarrow \partial & & & & \\
 & & \Sigma X & & & &
 \end{array}$$

where

$$X \xrightarrow{a} Y \xrightarrow{\iota} C(a) \xrightarrow{\partial} \Sigma X$$

is a distinguished triangle. Since $c\lambda = cb = 0$, we can choose $\tau: \Sigma X \rightarrow W$, such that $\tau\partial = c\lambda$. Then $\tau \in \langle a, b, c \rangle$. One has

$$(t - \tau)\partial = (c + c')\lambda - c\lambda = c'\lambda.$$

Hence, $t - \tau \in \langle a, b, c' \rangle$ and we conclude that

$$t = \tau + (t - \tau) \in \langle a, b, c \rangle + \langle a, b, c' \rangle.$$

(ii) Take any $t \in \langle a, b, c \rangle$ and $t' \in \langle a, b', c \rangle$. By definition of a Toda bracket, we can choose maps $\lambda, \lambda': C(a) \rightarrow Z$ such that

$$\begin{aligned}
 \lambda\iota &= b, & t\partial &= c\lambda \\
 \lambda'\iota &= b', & t'\partial &= c\lambda'.
 \end{aligned}$$

These identities imply that $(\lambda + \lambda')\iota = b + b'$ and $(t + t')\partial = c(\lambda + \lambda')$. Hence, we see that $t + t' \in \langle a, b + b', c \rangle$.

Conversely, suppose $s \in \langle a, b + b', c \rangle$. Choose any $t \in \langle a, b, c \rangle$. Then there are maps $\tilde{\lambda}, \lambda: C(a) \rightarrow Z$ such that

$$\begin{aligned}
 \tilde{\lambda}\iota &= b + b' & s\partial &= c\tilde{\lambda} \\
 \lambda\iota &= b, & t\partial &= c\lambda.
 \end{aligned}$$

It follows that $(\tilde{\lambda} - \lambda)\iota = b + b' - b = b'$ and $(s - t)\partial = c\tilde{\lambda} - c\lambda = c(\tilde{\lambda} - \lambda)$. Hence $s = t + s - t \in \langle a, b, c \rangle + \langle a, b', c \rangle$.

(iii) The proof of this claim is dual to that of (i). \square

Next we prove the following simple lemma about Toda brackets in $\pi_*\mathbb{S}[G]$:

Lemma 4.2.2. *Let $a \in \pi_m\mathbb{S}$, $b \in \pi_n\mathbb{S}$ and $c \in \pi_l\mathbb{S}$ and suppose $ba = 0$ and $cb = 0$. Further, assume that $\pi_{m+n+1}\mathbb{S} = 0$ and $\pi_{n+l+1}\mathbb{S} = 0$ (i.e., the Toda bracket $\langle a, b, c \rangle$ has no indeterminacy). Then*

$$\langle ag, bh, cu \rangle = \langle a, b, c \rangle uhg$$

in $\pi_*\mathbb{S}[G]$, for any $g, h, u \in G$.

Proof. Let $t = \langle a, b, c \rangle$. Then there is a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{S}^{m+n+l} & \xrightarrow{a} & \mathbb{S}^{n+l} & \xrightarrow{b} & \mathbb{S}^l & \xrightarrow{c} & \mathbb{S} \\
& & \downarrow \iota & \nearrow \lambda & & & \\
& & C(a) & & & & \\
& & \downarrow \partial & \nearrow t & & & \\
& & \mathbb{S}^{m+n+l+1} & & & &
\end{array}$$

in the stable homotopy category, where the triangle

$$\mathbb{S}^{m+n+l} \xrightarrow{a} \mathbb{S}^{n+l} \xrightarrow{\iota} C(a) \xrightarrow{\partial} \mathbb{S}^{m+n+l+1}$$

is distinguished. (Here we slightly abuse notation by writing a and b , instead of their shifts $\Sigma^{n+l}a$ and $\Sigma^l b$, respectively.) By applying the functor $G_+ \wedge^{\mathbf{L}} -$ to the commutative diagram, we see that $t = \langle a, b, c \rangle$ in $\pi_*\mathbb{S}[G]$. Next, for any $g \in G$, the isomorphism of triangles

$$\begin{array}{ccccccc}
\Sigma^{m+n+l}\Sigma_+^\infty G & \xrightarrow{ag} & \Sigma^{n+l}\Sigma_+^\infty G & \xrightarrow{1\wedge\iota} & G_+ \wedge C(a) & \xrightarrow{g^{-1}(1\wedge\partial)} & \Sigma^{m+n+l+1}\Sigma_+^\infty G \\
\cong \downarrow g & & \parallel & & \parallel & & \cong \downarrow g \\
\Sigma^{m+n+l}\Sigma_+^\infty G & \xrightarrow{a} & \Sigma^{n+l}\Sigma_+^\infty G & \xrightarrow{1\wedge\iota} & G_+ \wedge C(a) & \xrightarrow{1\wedge\partial} & \Sigma^{m+n+l+1}\Sigma_+^\infty G
\end{array}$$

in $\mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$ implies that the triangle

$$\Sigma^{m+n+l}\Sigma_+^\infty G \xrightarrow{ag} \Sigma^{n+l}\Sigma_+^\infty G \xrightarrow{1\wedge\iota} G_+ \wedge C(a) \xrightarrow{g^{-1}(1\wedge\partial)} \Sigma^{m+n+l+1}\Sigma_+^\infty G$$

is distinguished in $\mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$. Finally, the commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{m+n+l}\Sigma_+^\infty G & \xrightarrow{ag} & \Sigma^{n+l}\Sigma_+^\infty G & \xrightarrow{bh} & \Sigma^l\Sigma_+^\infty G & \xrightarrow{cu} & \Sigma_+^\infty G \\
& & \downarrow 1\wedge\iota & \nearrow h(1\wedge\lambda) & & & \\
& & G_+ \wedge C(a) & & & & \\
& & \downarrow g^{-1}(1\wedge\partial) & \nearrow tuhg & & & \\
& & \Sigma^{m+n+l+1}\Sigma_+^\infty G & & & &
\end{array}$$

in $\mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty G)$ completes the proof. \square

Now we are ready to show that the last Hopf element $\sigma \in \pi_7\mathbb{S} \subset \pi_7\mathbb{S}[G]$ is in the image of F . Recall that σ is a generator of $\pi_7\mathbb{S}_{(2)} \cong \mathbb{Z}/16$. We use the Toda bracket relation

$$8\sigma = \langle \nu, 8, \nu \rangle$$

(see e.g. [Tod62, 5.13-14]) in $\pi_*\mathbb{S}_{(2)}$ that holds without indeterminacy as $\pi_4\mathbb{S} = 0$. This implies that

$$8\sigma = \langle \nu, 8, \nu \rangle$$

in $\pi_*\mathbb{S}[G]$. Now since F is a triangulated functor, one obtains

$$8F(\sigma) = \langle F(\nu), 8, F(\nu) \rangle.$$

Recall that

$$F(\nu) = (2k+1)\nu + \sum_{g \in G \setminus \{1\}} 2l_g g \nu.$$

Let $F(\sigma) = m\sigma + \sum_{g \in G \setminus \{1\}} n_g g \sigma$. By Lemma 4.2.1 and Lemma 4.2.2 and the relation $16\sigma = 0$, we get

$$8(m\sigma + \sum_{g \in G \setminus \{1\}} n_g g \sigma) = \langle (2k+1)\nu + \sum_{g \in G \setminus \{1\}} 2l_g g \nu, 8, (2k+1)\nu + \sum_{g \in G \setminus \{1\}} 2l_g g \nu \rangle = 8(2k+1)^2 \sigma.$$

Hence we see that m is odd and the numbers n_g are even. Now a similar argument as in the case of ν implies that $F: \pi_7\mathbb{S}[G] \rightarrow \pi_7\mathbb{S}[G]$ is surjective and hence σ is in the image of F .

By combining the results of this subsection with Corollary 4.1.4 we conclude that under the assumptions of 3.1.3, the functor $F: \text{Ho}(\text{Sp}_{G,(2)}^O) \rightarrow \text{Ho}(\text{Sp}_{G,(2)}^O)$ becomes an equivalence when restricted to the full subcategory of free G -spectra, or equivalently, when restricted to $\text{Ho}((\mathbf{Mod}\text{-}\Sigma_+^\infty G)_{(2)})$. In fact, we have proved the following more general

Proposition 4.2.3. *Let G be any finite group and*

$$F: \text{Ho}((\mathbf{Mod}\text{-}\Sigma_+^\infty G)_{(2)}) \rightarrow \text{Ho}((\mathbf{Mod}\text{-}\Sigma_+^\infty G)_{(2)})$$

an exact endofunctor which preserves arbitrary coproducts and such that

$$F(\Sigma_+^\infty G) = \Sigma_+^\infty G,$$

and $F(g) = g$ for any $g \in G$. Then F is an equivalence of categories.

5 Reduction to endomorphisms

In this section we will show that in order to prove Theorem 3.1.3 (and hence Theorem 1.1.1), it suffices to check that for any subgroup $L \leq G$, the map of graded endomorphism rings

$$F: [\Sigma_+^\infty G/L, \Sigma_+^\infty G/L]_*^G \longrightarrow [F(\Sigma_+^\infty G/L), F(\Sigma_+^\infty G/L)]_*^G = [\Sigma_+^\infty G/L, \Sigma_+^\infty G/L]_*^G$$

is an isomorphism.

5.1 Formulation

Let G be a finite group and H and K subgroups of G . For the rest of this section we fix once and for all a set $\{g\}$ of double coset representatives for $K \backslash G/H$. Recall that for any $g \in G$, the conjugated subgroup gHg^{-1} is denoted by gH . Further,

$$\kappa_g: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/({}^gH \cap K), \Sigma_+^\infty G/({}^gH \cap K)]_*^G$$

will stand for the map which is defined by the following commutative diagram:

$$\begin{array}{ccc} [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{\kappa_g} & [\Sigma_+^\infty G/({}^gH \cap K), \Sigma_+^\infty G/({}^gH \cap K)]_*^G \\ g^* \downarrow & & \uparrow (\text{res}_{{}^gH \cap K}^{{}^gH})^* \\ [\Sigma_+^\infty G/{}^gH, \Sigma_+^\infty G/K]_*^G & \xrightarrow{(\text{tr}_{{}^gH \cap K}^K)^*} & [\Sigma_+^\infty G/{}^gH, \Sigma_+^\infty G/({}^gH \cap K)]_*^G \end{array}$$

(see Subsection 2.7). The aim of this section is to prove

Proposition 5.1.1. *The map*

$$[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \xrightarrow{(\kappa_g)_{[g] \in K \backslash G/H}} \bigoplus_{[g] \in K \backslash G/H} [\Sigma_+^\infty G/({}^gH \cap K), \Sigma_+^\infty G/({}^gH \cap K)]_*^G$$

is a split monomorphism.

The author thinks that this statement should be known to experts. However, since we were unable to find a reference, we decided to provide a detailed proof here. The proof is mainly based on the equivariant Spanier-Whitehead duality ([LMSM86, III.2, V.9], [May96, XVI.7]) and on a combinatorial analysis of certain pointed G -sets.

Before starting to prove Proposition 5.1.1, we explain how it reduces the proof of Theorem 3.1.3 to endomorphisms. Indeed, there is a commutative diagram

$$\begin{array}{ccc}
[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{F} & [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \\
\downarrow g^* & & \downarrow g^* \\
[\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{F} & [\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/K]_*^G \\
\downarrow (\mathrm{tr}_{{}^g H \cap K}^K)_* & & \downarrow (\mathrm{tr}_{{}^g H \cap K}^K)_* \\
[\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/({}^g H \cap K)]_*^G & \xrightarrow{F} & [\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/({}^g H \cap K)]_*^G \\
\downarrow (\mathrm{res}_{{}^g H \cap K}^g)_* & & \downarrow (\mathrm{res}_{{}^g H \cap K}^g)_* \\
[\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G & \xrightarrow{F} & [\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G.
\end{array}$$

for any $g \in G$, which implies that the diagram

$$\begin{array}{ccc}
[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{(\kappa_g)_{[g] \in K \backslash G/H}} & \bigoplus_{[g] \in K \backslash G/H} [\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G \\
\downarrow F & & \downarrow \bigoplus_{[g] \in K \backslash G/H} F \\
[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{(\kappa_g)_{[g] \in K \backslash G/H}} & \bigoplus_{[g] \in K \backslash G/H} [\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G
\end{array}$$

commutes. If we now assume that for any subgroup $L \leq G$, the map

$$F: [\Sigma_+^\infty G/L, \Sigma_+^\infty G/L]_*^G \longrightarrow [\Sigma_+^\infty G/L, \Sigma_+^\infty G/L]_*^G$$

is an isomorphism, then the right vertical map in the latter commutative square is an isomorphism. Proposition 5.1.1 implies that the upper horizontal map is injective. Hence, by a simple diagram chase, it follows that the left vertical morphism is injective as well. But now we know that for $* = 0$ the morphism

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G$$

is the identity and for $* > 0$ it has the same finite source and target (Subsection 2.7). Combining this with the latter injectivity result allows us to conclude that the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G$$

is indeed an isomorphism for any integer $*$.

The rest of this section is devoted to the proof of Proposition 5.1.1.

5.2 Induction and coinduction

Let H be a subgroup of G and $i: H \hookrightarrow G$ denote the inclusion. The class of all finite dimensional orthogonal H -representations of the form i^*V , where V is a finite dimensional orthogonal G -representation, contains the trivial representation and is closed under direct sums. Hence, according to [MM02, II.2.2, III.4.2], there is a stable model

category $\mathrm{Sp}_{H \leq G}^O$ of H -equivariant orthogonal spectra indexed on the class of such representations (cf. Subsection 2.6). Since finite dimensional orthogonal H -representations which come from G -representations are cofinal in the class of all finite dimensional orthogonal H -representations, [MM02, V.1.10] implies that the Quillen adjunction

$$\mathrm{id}: \mathrm{Sp}_{H \leq G}^O \rightleftarrows \mathrm{Sp}_H^O : \mathrm{id}$$

is a Quillen equivalence. Next, recall that there is a Quillen adjunction

$$G \times_H -: \mathrm{Sp}_{H \leq G}^O \rightleftarrows \mathrm{Sp}_G^O : \mathrm{Res}_H^G,$$

where $(G \times_H X)(V) = G_+ \wedge_H X(i^*V)$, for any $X \in \mathrm{Sp}_{H \leq G}^O$ and any finite dimensional orthogonal G -representation V . The functor Res_H^G is just the restriction along the map $i: H \hookrightarrow G$. In fact, the functor Res_H^G preserves weak equivalences and moreover, it is also a left Quillen functor as we see from the Quillen adjunction

$$\mathrm{Res}_H^G: \mathrm{Sp}_G^O \rightleftarrows \mathrm{Sp}_{H \leq G}^O : \mathrm{Map}_H(G_+, -).$$

The right adjoint $\mathrm{Map}_H(G_+, -)$ is defined by $\mathrm{Map}_H(G_+, X)(V) = \mathrm{Map}_H(G_+, X(i^*V))$. Now since the functor $\mathrm{id}: \mathrm{Sp}_{H \leq G}^O \rightarrow \mathrm{Sp}_H^O$ is a left Quillen functor, we also get a Quillen adjunction

$$\mathrm{Res}_H^G: \mathrm{Sp}_G^O \rightleftarrows \mathrm{Sp}_H^O : \mathrm{Map}_H(G_+, -).$$

These Quillen adjunctions induce corresponding adjunctions on the derived level:

$$G \times_H -: \mathrm{Ho}(\mathrm{Sp}_H^O) \sim \mathrm{Ho}(\mathrm{Sp}_{H \leq G}^O) \rightleftarrows \mathrm{Ho}(\mathrm{Sp}_G^O) : \mathrm{Res}_H^G,$$

and

$$\mathrm{Res}_H^G: \mathrm{Ho}(\mathrm{Sp}_G^O) \rightleftarrows \mathrm{Ho}(\mathrm{Sp}_H^O) : \mathrm{Map}_H(G_+, -).$$

Here we slightly abuse notation by denoting point-set level functors and their associated derived functors with same symbols. Next, note that the equivalence

$$\mathrm{Ho}(\mathrm{Sp}_H^O) \sim \mathrm{Ho}(\mathrm{Sp}_{H \leq G}^O)$$

is a preferred one and is induced from the Quillen equivalence at the very beginning of this subsection.

The adjunctions recalled here are in fact special instances of the “change of groups” and “change of universe” functors of [MM02, V]. The functor $G \times_H -$ is usually called the *induction* and the functor $\mathrm{Map}_H(G_+, -)$ is called the *coinduction*.

Let now $G \times_H X$ denote the balanced product $G_+ \wedge_H X$ for any pointed G -set (space) X . Consider the the following natural point-set level map:

$$w_H: G \times_H X \longrightarrow \mathrm{Map}_H(G_+, X)$$

given by

$$w_H([g, x])(\gamma) = \begin{cases} \gamma g x & \text{if } \gamma g \in H \\ * & \text{if } \gamma g \notin H. \end{cases}$$

We remind the reader of the following result due to Wirthmüller:

Proposition 5.2.1 (Wirthmüller Isomorphism, see e.g. [May03]). *The map w_H induces a natural isomorphism between the derived functors*

$$G \times_H -: \mathrm{Ho}(\mathrm{Sp}_H^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

and

$$\mathrm{Map}_H(G_+, -): \mathrm{Ho}(\mathrm{Sp}_H^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O).$$

That is, the left and right adjoint functors of

$$\mathrm{Res}_H^G: \mathrm{Ho}(\mathrm{Sp}_G^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_H^O)$$

are naturally isomorphic.

As a consequence of the Wirthmüller isomorphism, one gets that for any subgroup $L \leq G$, the equivariant spectrum $\Sigma_+^\infty G/L$ is self-dual. Indeed, the map

$$\Sigma_+^\infty G/L \cong G \times_L \mathbb{S} \xrightarrow{w_L} \mathrm{Map}_L(G_+, \mathbb{S}) \cong \mathrm{Map}(\Sigma_+^\infty G/L, \mathbb{S}) \cong D(\Sigma_+^\infty G/L)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_G^O)$, where D is the equivariant Spanier-Whitehead duality functor ([LMSM86, II.6, III.2, V.9], [May96, XVI.7]).

We conclude the subsection with the following well-known lemma and its corollaries.

Lemma 5.2.2. *Suppose G is a finite group and H and K arbitrary subgroups of G . Let $c_g: {}^g H \rightarrow H$ denote the map $c_g(x) = g^{-1}xg$, $g \in G$. Then for any pointed H -set X , the K -equivariant maps*

$$K \times_{gH \cap K} \mathrm{Res}_{gH \cap K}^{gH}(c_g^* X) \longrightarrow \mathrm{Res}_K^G(G \times_H X), \quad [k, x] \mapsto [kg, x],$$

induce a natural splitting

$$\bigvee_{[g] \in K \backslash G/H} K \times_{gH \cap K} \mathrm{Res}_{gH \cap K}^{gH}(c_g^* X) \cong \mathrm{Res}_K^G(G \times_H X).$$

Corollary 5.2.3. *Suppose G is a finite group and H and K subgroups of G . Then for any $Y \in \mathrm{Ho}(\mathrm{Sp}_H^O)$, there is a natural splitting*

$$\bigvee_{[g] \in K \backslash G/H} K \times_{gH \cap K} \mathrm{Res}_{gH \cap K}^{gH}(c_g^* Y) \simeq \mathrm{Res}_K^G(G \times_H Y).$$

Note that if X is a pointed G -set, then there is a natural isomorphism

$$G \times_H \mathrm{Res}_H^G X \cong G/H_+ \wedge X$$

given by $[g, x] \mapsto ([g] \wedge gx)$.

Corollary 5.2.4. *The maps*

$$G/({}^g H \cap K)_+ \longrightarrow G/H_+ \wedge G/K_+, \quad [x] \mapsto [xg] \wedge [x]$$

of pointed G -sets induce a natural splitting

$$\bigvee_{[g] \in K \backslash G/H} G/({}^g H \cap K)_+ \xrightarrow{\cong} G/H_+ \wedge G/K_+.$$

Proof. By the last observation and Lemma 5.2.2, we have a chain of isomorphisms of pointed G -sets:

$$\begin{aligned} \bigvee_{[g] \in K \backslash G/H} G/({}^g H \cap K)_+ &\cong \bigvee_{[g] \in K \backslash G/H} G \times_K (K/({}^g H \cap K))_+ \cong \\ G \times_K \left(\bigvee_{[g] \in K \backslash G/H} K/({}^g H \cap K)_+ \right) &\cong G \times_K \left(\bigvee_{[g] \in K \backslash G/H} K \times_{gH \cap K} S^0 \right) \cong \\ G \times_K \operatorname{Res}_K^G(G \times_H S^0) &\cong G \times_K \operatorname{Res}_K^G(G/H_+) \cong G/K_+ \wedge G/H_+ \cong G/H_+ \wedge G/K_+. \end{aligned}$$

Here the last isomorphism is the twist. Going through these explicit isomorphisms we see that any $[x] \in G/({}^g H \cap K)_+$ is sent to $[xg] \wedge [x] \in G/H_+ \wedge G/K_+$. \square

5.3 Proof of Proposition 5.1.1

As we already mentioned after Proposition 5.2.1, we have the isomorphisms

$$D(\Sigma_+^\infty G/L) \cong \Sigma_+^\infty G/L, \quad L \leq G,$$

in $\operatorname{Ho}(\operatorname{Sp}_G^O)$, where D is the equivariant Spanier-Whitehead duality. It follows from [LMSM86, II.6, III.2, V.9] (see also [Lew98]) that under these isomorphisms the transfer maps correspond to restrictions. In particular, for any $g \in G$, the diagram

$$\begin{array}{ccc} D(\Sigma_+^\infty G/({}^g H \cap K)) & \xrightarrow{D(\operatorname{tr}_{gH \cap K}^K)} & D(\Sigma_+^\infty G/K) \\ \cong \uparrow & & \uparrow \cong \\ \Sigma_+^\infty G/({}^g H \cap K) & \xrightarrow{\operatorname{res}_{gH \cap K}^K} & \Sigma_+^\infty G/K. \end{array}$$

commutes. Combining this with the Spanier-Whitehead duality, for any $g \in G$, one gets the following commutative diagram with all vertical maps isomorphisms:

$$\begin{array}{ccc} [\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{(\operatorname{tr}_{gH \cap K}^K)_*} & [\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/({}^g H \cap K)]_*^G \\ \cong \downarrow & & \downarrow \cong \\ [\Sigma_+^\infty G/{}^g H \wedge D(\Sigma_+^\infty G/K), \mathbb{S}]_*^G & \xrightarrow{(1 \wedge D(\operatorname{tr}_{gH \cap K}^K))^*} & [\Sigma_+^\infty G/{}^g H \wedge D(\Sigma_+^\infty G/({}^g H \cap K)), \mathbb{S}]_*^G \\ \cong \downarrow & & \downarrow \cong \\ [\Sigma_+^\infty G/{}^g H \wedge \Sigma_+^\infty G/K, \mathbb{S}]_*^G & \xrightarrow{(1 \wedge \operatorname{res}_{gH \cap K}^K)^*} & [\Sigma_+^\infty G/{}^g H \wedge \Sigma_+^\infty G/({}^g H \cap K), \mathbb{S}]_*^G \\ \cong \downarrow & & \downarrow \cong \\ [\Sigma^\infty(G/{}^g H_+ \wedge G/K_+), \mathbb{S}]_*^G & \xrightarrow{(1 \wedge \operatorname{res}_{gH \cap K}^K)^*} & [\Sigma^\infty(G/{}^g H_+ \wedge G/({}^g H \cap K)_+), \mathbb{S}]_*^G \end{array}$$

Using again the Spanier-Whitehead duality and that $\Sigma_+^\infty G/L$, $L \leq G$, is self-dual, we also have commutative diagrams for every $g \in G$:

$$\begin{array}{ccc}
[\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G & \xrightarrow{g^*} & [\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/K]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma_+^\infty G/H \wedge D(\Sigma_+^\infty G/K), \mathbb{S}]_*^G & \xrightarrow{(g \wedge 1)^*} & [\Sigma_+^\infty G/{}^g H \wedge D(\Sigma_+^\infty G/K), \mathbb{S}]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma_+^\infty G/H \wedge \Sigma_+^\infty G/K, \mathbb{S}]_*^G & \xrightarrow{(g \wedge 1)^*} & [\Sigma_+^\infty G/{}^g H \wedge \Sigma_+^\infty G/K, \mathbb{S}]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma^\infty(G/H_+ \wedge G/K_+), \mathbb{S}]_*^G & \xrightarrow{(g \wedge 1)^*} & [\Sigma^\infty(G/{}^g H_+ \wedge G/K_+), \mathbb{S}]_*^G
\end{array}$$

and

$$\begin{array}{ccc}
[\Sigma_+^\infty G/{}^g H, \Sigma_+^\infty G/({}^g H \cap K)]_*^G & \xrightarrow{(\text{res}_{{}^g H \cap K}^g)^*} & [\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma_+^\infty G/{}^g H \wedge D(\Sigma_+^\infty G/({}^g H \cap K)), \mathbb{S}]_*^G & \xrightarrow{(\text{res}_{{}^g H \cap K}^g \wedge 1)^*} & [\Sigma_+^\infty G/({}^g H \cap K) \wedge D(\Sigma_+^\infty G/({}^g H \cap K)), \mathbb{S}]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma_+^\infty G/{}^g H \wedge \Sigma_+^\infty G/({}^g H \cap K), \mathbb{S}]_*^G & \xrightarrow{(\text{res}_{{}^g H \cap K}^g \wedge 1)^*} & [\Sigma_+^\infty G/({}^g H \cap K) \wedge \Sigma_+^\infty G/({}^g H \cap K), \mathbb{S}]_*^G \\
\cong \downarrow & & \downarrow \cong \\
[\Sigma^\infty(G/{}^g H_+ \wedge G/({}^g H \cap K)_+), \mathbb{S}]_*^G & \xrightarrow{(\text{res}_{{}^g H \cap K}^g \wedge 1)^*} & [\Sigma^\infty(G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+), \mathbb{S}]_*^G.
\end{array}$$

Hence by definition, for any $g \in G$, the morphism

$$\kappa_g: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/({}^g H \cap K), \Sigma_+^\infty G/({}^g H \cap K)]_*^G$$

is isomorphic to the morphism induced by the composite

$$\begin{array}{ccc}
G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+ & \longrightarrow & G/H_+ \wedge G/K_+ \\
\text{res}_{{}^g H \cap K}^g \wedge 1 \downarrow & & \uparrow g \wedge 1 \\
G/{}^g H_+ \wedge G/({}^g H \cap K)_+ & \xrightarrow{1 \wedge \text{res}_{{}^g H \cap K}^g} & G/{}^g H_+ \wedge G/K_+
\end{array}$$

after applying the functor $[\Sigma^\infty(-), \mathbb{S}]_*^G$. To simplify notations let us denote this composite of maps of pointed G -sets by $\omega_g: G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+ \longrightarrow G/H_+ \wedge G/K_+$. Thus, in order to prove Proposition 5.1.1, it suffices to check that the map of pointed G -sets

$$\bigvee_{[g] \in K \backslash G/H} (G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+) \xrightarrow{(\omega_g)_{[g] \in K \backslash G/H}} G/H_+ \wedge G/K_+$$

has a G -equivariant section. This follows from the commutative diagram of pointed G -sets

$$\begin{array}{ccc}
 \bigvee_{[g] \in K \backslash G/H} (G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+) & \xrightarrow{(\omega_g)_{[g] \in K \backslash G/H}} & G/H_+ \wedge G/K_+ \\
 & \swarrow \bigvee_{[g] \in K \backslash G/H} \Delta_g & \uparrow \cong \\
 & & \bigvee_{[g] \in K \backslash G/H} G/({}^g H \cap K)_+,
 \end{array}$$

where the vertical map is the isomorphism from Corollary 5.2.4 and

$$\Delta_g : G/({}^g H \cap K)_+ \longrightarrow G/({}^g H \cap K)_+ \wedge G/({}^g H \cap K)_+$$

is the diagonal defined by $[x] \mapsto [x] \wedge [x]$ for any g . □

6 Geometric fixed points and inflation

This section is devoted to a review of the geometric fixed point functor $\Phi^N: \mathrm{Sp}_G^O \longrightarrow \mathrm{Sp}_J^O$ for any extension of finite groups

$$E: 1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varepsilon} J \longrightarrow 1.$$

The main reference here is [MM02, V.4]. The eventual goal of this section is to establish (recall) several useful statements needed for the proof of Theorem 3.1.3. The first half of this section shows that the composite

$$\Phi^N \circ \varepsilon^*: \mathrm{Ho}(\mathrm{Sp}_J^O) \xrightarrow{\varepsilon^*} \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{\Phi^N} \mathrm{Ho}(\mathrm{Sp}_J^O),$$

where $\varepsilon^*: \mathrm{Ho}(\mathrm{Sp}_J^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$ is the inflation (pull-back of scalars) functor, is naturally isomorphic to the identity functor. This result goes back to the classical reference [LMSM86, II.9.10]. Here we provide the details of the proof based on the more recent language of [MM02].

The second half of this section constructs a split short exact sequence that will play a fundamental role in the inductive proof of Theorem 3.1.3.

6.1 Geometric fixed points: Definition and basic properties

We start by briefly reviewing the definitions of some indexing categories needed in the construction of the geometric fixed point functor.

Recall from Subsection 2.4 the G -**Top**_{*}-category O_G . The objects of O_G are finite dimensional orthogonal G -representations. For any finite dimensional orthogonal G -representations V and W , the pointed morphism G -space from V to W is the Thom space $O_G(V, W)$. We also remind the reader that the category Sp_G^O is equivalent to the category of O_G -spaces (which is the category of G -**Top**_{*}-enriched functors from O_G to G -**Top**_{*}). Further, let

$$E: 1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varepsilon} J \longrightarrow 1$$

be an extension of finite groups. The next category we need is O_E which is a J -**Top**_{*}-category. The objects of O_E are again finite dimensional orthogonal G -representations. The pointed J -space $O_E(V, W)$ is defined to be $O_G(V, W)^N$, for any $V, W \in O_G$. Note that $O_G(V, W)^N$ has a natural J -action induced from the extension E . See more details in [MM02, V.4].

Next, we define two functors

$$\phi: O_E \longrightarrow O_J, \quad \nu: O_J \longrightarrow O_E.$$

The functor ϕ sends an orthogonal G -representation V to an orthogonal J -representation V^N and a morphism $(f, x) \in O_E(V, W) = O_G(V, W)^N$ to $(f^N, x) \in O_J(V^N, W^N)$. The

functor ν maps a J -representation V to the G -representation ε^*V . On morphisms, ν is given by the identity

$$\text{id}: O_J(V, W) \longrightarrow O_J(V, W) = O_G(\varepsilon^*V, \varepsilon^*W)^N = O_E(\varepsilon^*V, \varepsilon^*W).$$

To simplify the exposition, we introduce the following notation. For any finite group Γ and any Γ -**Top** $_*$ -categories \mathcal{A} and \mathcal{B} , let $\text{Fun}_\Gamma(\mathcal{A}, \mathcal{B})$ denote the category of Γ -**Top** $_*$ -enriched functors. As already noted, we have a preferred equivalence of categories

$$\text{Sp}_G^O \sim \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*)$$

for any finite group G [MM02, II.4.3] (Subsection 2.4).

Now we are finally ready to recall the definition of the geometric fixed point functor. The functors ϕ and ν induce adjunctions

$$P_\phi: \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*) : \rightleftarrows \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*) : U_\phi.$$

and

$$P_\nu: \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*) : \rightleftarrows \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*) : U_\nu,$$

where the right adjoints U_ϕ and U_ν are precompositions with ϕ and ν , respectively, and P_ϕ and P_ν are J -**Top** $_*$ -enriched left Kan extensions [Kel05, Sections 4.1-2]. Further, we have a fixed point functor

$$\text{Fix}^N: \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \longrightarrow \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*).$$

It is defined by $\text{Fix}^N X(V) = X(V)^N$ on objects. On morphisms Fix^N is given by the adjoint of

$$X(V)^N \wedge O_G(V, W)^N \longrightarrow X(W)^N$$

which is the N -fixed point of the structure map

$$X(V) \wedge O_G(V, W) \longrightarrow X(W)$$

of the spectrum X .

Definition 6.1.1 (see [MM02, V.4.3]). The composite

$$P_\phi \circ \text{Fix}^N: \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \xrightarrow{\text{Fix}^N} \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*) \xrightarrow{P_\phi} \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*)$$

is called the *geometric fixed point functor* and is denoted by

$$\Phi^N: \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \longrightarrow \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*).$$

Having the preferred equivalences

$$\text{Sp}_G^O \sim \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \quad \text{and} \quad \text{Sp}_J^O \sim \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*)$$

in mind we see that one also gets a functor $\Phi^N: \text{Sp}_G^O \longrightarrow \text{Sp}_J^O$.

The geometric fixed point functor enjoys the following important properties:

Proposition 6.1.2 ([MM02, V.4.5]). *Let V be a finite dimensional orthogonal G -representation and A a pointed G -space. Then there is a natural isomorphism of J -spectra*

$$\Phi^N(F_V A) \cong F_{V^N} A^N.$$

Furthermore, the functor $\Phi^N: \mathrm{Sp}_G^O \rightarrow \mathrm{Sp}_J^O$ preserves cofibrations and acyclic cofibrations.

Corollary 6.1.3. *For any based G -space A , there is a natural isomorphism of J -spectra*

$$\Phi^N(\Sigma^\infty A) \cong \Sigma^\infty(A^N).$$

6.2 The inflation functor

As in the previous subsection we start with an extension of groups

$$E: 1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\varepsilon} J \longrightarrow 1.$$

The homomorphism $\varepsilon: G \rightarrow J$ induces a Quillen adjunction

$$\varepsilon^*: \mathrm{Sp}_J^O \rightleftarrows \mathrm{Sp}_{G, N\text{-fixed}}^O : (-)^N,$$

where $\mathrm{Sp}_{G, N\text{-fixed}}^O$ is the stable model category of equivariant orthogonal G -spectra indexed on the N -fixed universe. The left adjoint ε^* is defined by

$$(\varepsilon^* X)(V) = \varepsilon^*(X(V))$$

for any G -representation V which is N -fixed and hence can be regarded as a J -representation by means of ε . The right adjoint $(-)^N$ is just the point-set level (categorical) fixed point functor. The fact that ε^* is a left Quillen functor follows from the following isomorphisms

$$\varepsilon^*(F_V A) \cong F_{\varepsilon^* V} \varepsilon^* A, \quad \varepsilon^*(J/K) \cong G/\varepsilon^{-1}(K).$$

Next, we have the change of universe Quillen adjunction

$$\mathrm{id}: \mathrm{Sp}_{G, N\text{-fixed}}^O \rightleftarrows \mathrm{Sp}_G^O : \mathrm{id}.$$

Combining this Quillen adjunction with the previous one gives a Quillen adjunction

$$\varepsilon^*: \mathrm{Sp}_J^O \rightleftarrows \mathrm{Sp}_G^O : (-)^N.$$

The left Quillen functor $\varepsilon^*: \mathrm{Sp}_J^O \rightarrow \mathrm{Sp}_G^O$ is referred to as the *inflation functor*.

The rest of this subsection is devoted to study the relation between the inflation functor and the geometric fixed point functor. By Proposition 6.1.2 and Ken Brown's Lemma one can derive the functor Φ^N and get the functor

$$\Phi^N: \mathrm{Ho}(\mathrm{Sp}_G^O) \rightarrow \mathrm{Ho}(\mathrm{Sp}_J^O).$$

We can also derive the left Quillen functor ε^* and obtain the derived inflation

$$\varepsilon^*: \text{Ho}(\text{Sp}_J^O) \longrightarrow \text{Ho}(\text{Sp}_G^O).$$

Our aim in this subsection is to show that the composite

$$\Phi^N \circ \varepsilon^*: \text{Ho}(\text{Sp}_J^O) \xrightarrow{\varepsilon^*} \text{Ho}(\text{Sp}_G^O) \xrightarrow{\Phi^N} \text{Ho}(\text{Sp}_J^O)$$

is isomorphic to the identity functor. We start by defining a point-set level natural transformation

$$\text{id} \longrightarrow \Phi^N \varepsilon^*.$$

We follow [MM02, V.4]. By definition of the functor

$$\text{Fix}^N: \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \longrightarrow \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*),$$

one gets

$$(-)^N = U_\nu \text{Fix}^N: \text{Fun}_G(O_G, G\text{-}\mathbf{Top}_*) \longrightarrow \text{Fun}_J(O_J, J\text{-}\mathbf{Top}_*).$$

Next, since $\phi\nu = \text{id}$, we see that $U_\nu U_\phi = \text{id}$ holds. Now the unit $\eta: \text{id} \longrightarrow U_\phi P_\phi$ of the adjunction (P_ϕ, U_ϕ) induces a natural transformation

$$X^N = U_\nu \text{Fix}^N X \xrightarrow{U_\nu \eta \text{Fix}^N} U_\nu U_\phi P_\phi \text{Fix}^N X = P_\phi \text{Fix}^N X = \Phi^N X.$$

As there is an evident point-set level natural isomorphism $(\varepsilon^* X)^N \cong X$ (which does not necessarily hold on the derived level!), we get a natural transformation

$$\omega: \text{id} \cong U_\nu \text{Fix}^N \varepsilon^* \xrightarrow{U_\nu \eta \text{Fix}^N \varepsilon^*} \Phi^N \varepsilon^*.$$

If we precompose this natural transformation with the functor

$$F_0 = \Sigma^\infty: J\text{-}\mathbf{Top}_* \longrightarrow \text{Sp}_J^O$$

we get a chain of isomorphisms of natural transformations:

$$\begin{array}{ccc} F_0 & \xrightarrow{\omega_{F_0}} & \Phi^N \varepsilon^* F_0 \\ \downarrow \cong & & \downarrow \cong \\ U_\nu \text{Fix}^N \varepsilon^* F_0 & \xrightarrow{U_\nu \eta \text{Fix}^N \varepsilon^* F_0} & U_\nu U_\phi P_\phi \text{Fix}^N \varepsilon^* F_0 \\ \downarrow \cong & & \downarrow \cong \\ U_\nu \text{Fix}^N F_0 \varepsilon^* & \xrightarrow{U_\nu \eta \text{Fix}^N F_0 \varepsilon^*} & U_\nu U_\phi P_\phi \text{Fix}^N F_0 \varepsilon^* \\ \downarrow \cong & & \downarrow \cong \\ U_\nu F_0 (-)^N \varepsilon^* & \xrightarrow{U_\nu \eta F_0 (-)^N \varepsilon^*} & U_\nu U_\phi P_\phi F_0 (-)^N \varepsilon^* \\ \downarrow \cong & & \downarrow \cong \\ U_\nu F_0 & \xrightarrow{U_\nu \eta F_0} & U_\nu U_\phi P_\phi F_0, \end{array}$$

where the functor $F_0: J\text{-}\mathbf{Top}_* \rightarrow \text{Fun}_J(O_E, J\text{-}\mathbf{Top}_*)$ is defined by

$$F_0A(V) = O_E(0, V) \wedge A = S^{V^N} \wedge A.$$

(Here we decided to abuse notations by denoting two different functors by F_0 in order not to make the notations even more complicated.) We want to deduce that the upper horizontal map in this diagram is an isomorphism of functors. For this we will now show that the map

$$F_0 \xrightarrow{\eta^{F_0}} U_\phi P_\phi F_0$$

is an isomorphism. The following lemma from enriched category theory will help us:

Lemma 6.2.1. *Suppose \mathcal{V} is a closed symmetric monoidal category, \mathcal{A} and \mathcal{B} skeletally small \mathcal{V} -categories, \mathcal{C} a bicomplete \mathcal{V} -category and $K: \mathcal{A} \rightarrow \mathcal{B}$ a \mathcal{V} -functor. Let $F_A C: \mathcal{A} \rightarrow \mathcal{C}$ denote the \mathcal{V} -functor*

$$\mathcal{A}(A, -) \otimes C: \mathcal{A} \rightarrow \mathcal{C}$$

for $A \in \mathcal{A}$ and $C \in \mathcal{C}$. The evaluation at $F_A C$ of the unit

$$id \rightarrow K^* \text{Lan}_K$$

of the adjunction

$$\text{Lan}_K: \text{Fun}_{\mathcal{V}}(\mathcal{A}, \mathcal{C}) \rightleftarrows \text{Fun}_{\mathcal{V}}(\mathcal{B}, \mathcal{C}) : K^*$$

is isomorphic to the natural transformation

$$\mathcal{A}(A, -) \otimes C \xrightarrow{K \otimes C} \mathcal{B}(K(A), K(-)) \otimes C.$$

Proof. By the enriched Yoneda lemma [Kel05, (3.71)], for any \mathcal{V} -functor $G: \mathcal{A} \rightarrow \mathcal{C}$, there is a natural isomorphism

$$GA'' \cong \int^{A' \in \mathcal{A}} \mathcal{A}(A', A'') \otimes GA'.$$

It follows from [Kel05, (3.71), (4.25)] and the proof of [Kel05, Proposition 4.23] that the unit

$$G \rightarrow \text{Lan}_K G \circ K$$

is given by

$$G \cong \int^{A' \in \mathcal{A}} \mathcal{A}(A', -) \otimes GA' \xrightarrow{\int^{A' \in \mathcal{A}} K \otimes GA'} \int^{A' \in \mathcal{A}} \mathcal{B}(KA', K(-)) \otimes GA' \cong \text{Lan}_K G \circ K.$$

The latter morphism specializes to

$$\int^{A' \in \mathcal{A}} \mathcal{A}(A', -) \otimes \mathcal{A}(A, A') \otimes C \xrightarrow{\int^{A' \in \mathcal{A}} K \otimes \mathcal{A}(A, A') \otimes C} \int^{A' \in \mathcal{A}} \mathcal{B}(K(A'), K(-)) \otimes \mathcal{A}(A, A') \otimes C$$

if $G = F_A C$. But now again the enriched Yoneda lemma [Kel05, (3.71)] implies that this map is isomorphic to

$$\mathcal{A}(A, -) \otimes C \xrightarrow{K \otimes C} \mathcal{B}(K(A), K(-)) \otimes C. \quad \square$$

Corollary 6.2.2. *The natural transformation*

$$F_0 \xrightarrow{\eta^{F_0}} U_\phi P_\phi F_0$$

is an isomorphism of functors.

Proof. Since $F_0 A = O_E(0, -) \wedge A$, it follows from Lemma 6.2.1 that the evaluation of η^{F_0} at $A \in J\text{-Top}_*$ is isomorphic to

$$\phi \wedge A: O_E(0, -) \wedge A \longrightarrow O_J(\phi(0), \phi(-)) \wedge A$$

which is an isomorphism by definition of ϕ . \square

Corollary 6.2.3. *The natural transformation*

$$\omega F_0: F_0 \longrightarrow \Phi^N \varepsilon^* F_0$$

is an isomorphism of functors.

The natural transformation $\omega: \text{id} \longrightarrow \Phi^N \varepsilon^*$ descends to a natural transformation on the homotopy level. Further, note that the functor $\Phi^N: \text{Ho}(\text{Sp}_G^O) \longrightarrow \text{Ho}(\text{Sp}_J^O)$ is triangulated. This follows either by an explicit verification or by [MM02, V.4.17]. Next, the general description of the unit $G \longrightarrow K^*(\text{Lan}_K G)$ in the proof of 6.2.1 implies that the diagram

$$\begin{array}{ccc} S^1 \wedge (-) & \xrightarrow{S^1 \wedge \eta} & S^1 \wedge U_\phi P_\phi \\ & \searrow \eta^{(S^1 \wedge (-))} & \downarrow \cong \\ & & U_\phi P_\phi(S^1 \wedge (-)) \end{array}$$

commutes. Hence the diagram

$$\begin{array}{ccc} S^1 \wedge (-) & \xrightarrow{S^1 \wedge \omega} & S^1 \wedge \Phi^N \varepsilon^* \\ & \searrow \omega^{(S^1 \wedge (-))} & \downarrow \cong \\ & & \Phi^N \varepsilon^*(S^1 \wedge (-)) \end{array}$$

commutes, yielding that the derived natural transformation $\text{id} \longrightarrow \Phi^N \varepsilon^*$ of endofunctors on $\text{Ho}(\text{Sp}_J^O)$ is triangulated.

Theorem 6.2.4 (cf. [LMSM86, II.9.10]). *The derived natural transformation*

$$\omega: \text{id} \longrightarrow \Phi^N \varepsilon^*$$

of endofunctors on $\text{Ho}(\text{Sp}_J^O)$ is an isomorphism.

Proof. By Corollary 6.2.3, the derived natural transformation $\omega: \text{id} \rightarrow \Phi^N \varepsilon^*$ is an isomorphism when evaluated on $\Sigma_+^\infty J/K$ for any subgroup $K \leq J$. Next, note that Φ^N commutes with infinite coproducts. Since the set of compact objects $\{\Sigma_+^\infty J/K \mid K \leq J\}$ generates the triangulated category $\text{Ho}(\text{Sp}_J^O)$ and $\omega: \text{id} \rightarrow \Phi^N \varepsilon^*$ is triangulated, the desired result follows. \square

Now we start the preparation for the construction of the short exact sequence that we advertised at the beginning of this section.

6.3 Weyl groups

Let G be a finite group and H a subgroup of G . Then H is a normal subgroup of its normalizer $N(H) = \{g \in G \mid gH = Hg\}$ and the quotient group $W(H) = N(H)/H$ is called the *Weyl group* of H . According to the previous subsection, the short exact sequence

$$1 \longrightarrow H \xrightarrow{\iota} N(H) \xrightarrow{\varepsilon} W(H) \longrightarrow 1$$

gives us the geometric fixed point functor

$$\Phi^H: \text{Ho}(\text{Sp}_{N(H)}^O) \longrightarrow \text{Ho}(\text{Sp}_{W(H)}^O)$$

and the inflation functor

$$\varepsilon^*: \text{Ho}(\text{Sp}_{W(H)}^O) \longrightarrow \text{Ho}(\text{Sp}_{N(H)}^O).$$

By slightly abusing notations, we will denote the composite functor

$$\Phi^H \circ \text{Res}_{N(H)}^G: \text{Ho}(\text{Sp}_G^O) \longrightarrow \text{Ho}(\text{Sp}_{W(H)}^O)$$

also by Φ^H . It then follows from Corollary 6.1.3 that there is an isomorphism

$$\Phi^H(\Sigma_+^\infty G/H) \cong \Sigma_+^\infty (G/H)^H$$

in $\text{Ho}(\text{Sp}_{W(H)}^O)$. (This holds already on the point-set level.) Since $(G/H)^H = W(H)$ as $W(H)$ -sets, one in fact gets an isomorphism

$$\Phi^H(\Sigma_+^\infty G/H) \cong \Sigma_+^\infty W(H)$$

in $\text{Ho}(\text{Sp}_{W(H)}^O)$. Further, by definition, one has $\varepsilon^*(\Sigma_+^\infty W(H)) = \Sigma_+^\infty N(H)/H$ in $\text{Ho}(\text{Sp}_{N(H)}^O)$ and hence we get

$$G \rtimes_{N(H)} \varepsilon^*(\Sigma_+^\infty W(H)) = \Sigma_+^\infty G/H.$$

Having in mind these identifications, we are now ready to prove the main technical result of this subsection:

Proposition 6.3.1. *The composite*

$$[\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)} \xrightarrow{G \times_{N(H)} \varepsilon^*} [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \xrightarrow{\Phi^H} [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}$$

is an isomorphism.

Proof. By Theorem 6.2.4 we have the natural isomorphism

$$\omega: \text{id} \xrightarrow{\cong} \Phi^H \varepsilon^*$$

which is a triangulated transformation. Since $(G \times_{N(H)} -, \text{Res}_{N(H)}^G)$ is a Quillen adjunction (Subsection 5.2), the derived unit map

$$\text{id} \xrightarrow{\eta} \text{Res}_{N(H)}^G(G \times_{N(H)} -)$$

is a triangulated transformation as well. Hence, the composite

$$\Phi^H \eta \varepsilon^* \circ \omega: \text{id} \xrightarrow{\omega} \Phi^H \varepsilon^* \xrightarrow{\Phi^H \eta \varepsilon^*} \Phi^H(\text{Res}_{N(H)}^G(G \times_{N(H)} \varepsilon^*))$$

is a triangulated transformation. Moreover, it follows from Theorem 6.2.4 and the identifications we did before Proposition 6.3.1 that this natural transformation is an isomorphism in $\text{Ho}(\text{Sp}_{W(H)}^O)$ when applied to $\Sigma_+^\infty W(H)$. Consequently, the restriction of the functor $\Phi^H(\text{Res}_{N(H)}^G(G \times_{N(H)} \varepsilon^*))$ to the localizing subcategory of $\text{Ho}(\text{Sp}_{W(H)}^O)$ generated by $\Sigma_+^\infty W(H)$ is isomorphic to the identity functor and thus an equivalence of categories. This implies the desired result. \square

6.4 The short exact sequence

Suppose G is a finite group and \mathcal{F} a set of subgroups of G . The set \mathcal{F} is said to be a *family of subgroups of G* if it is closed under conjugation and taking subgroups.

Recall that for any finite group G and any family \mathcal{F} , there is a classifying space (G -CW complex) $E\mathcal{F}$ characterized up to G -homotopy equivalence by the property that $E\mathcal{F}^H$ is contractible if $H \in \mathcal{F}$ and $E\mathcal{F}^H = \emptyset$ if $H \notin \mathcal{F}$ (see e.g. [Elm83]).

Let \mathcal{P} denote the family of proper subgroups of G . Consider the equivariant map $E\mathcal{P}_+ \xrightarrow{\text{proj}} S^0$ which sends the elements of $E\mathcal{P}$ to the non-base point of S^0 . The mapping cone sequence of this map (called the isotropy separation sequence) combined with the tom Dieck splitting [tD87, II.7.7] gives the following well-known:

Proposition 6.4.1. *Suppose G is a finite group. Then there is a split short exact sequence*

$$0 \longrightarrow [\mathbb{S}, \Sigma_+^\infty E\mathcal{P}]_*^G \xrightarrow{\text{proj}_*} [\mathbb{S}, \mathbb{S}]_*^G \xrightarrow{\Phi^G} [\mathbb{S}, \mathbb{S}]_* \longrightarrow 0.$$

Now suppose H is a subgroup of G . Then for any $X \in \text{Ho}(\text{Sp}_G^O)$, there is a natural isomorphism

$$G \times_H \text{Res}_H^G X \cong G/H_+ \wedge^{\mathbf{L}} X$$

given on the point-set level by $[g, x] \mapsto ([g] \wedge gx)$. In particular,

$$G \times_H \mathbb{S} \cong \Sigma_+^\infty G/H.$$

Having in mind this preferred isomorphism, we will once and for all identify $G \times_H \mathbb{S}$ with $\Sigma_+^\infty G/H$. Next, let $\mathcal{P}[H]$ denote the family of proper subgroups of H . Note that this is a family with respect to H and not necessarily with respect to the whole group G . Here is the main result of this section which is an important tool in the proof of Theorem 3.1.3:

Proposition 6.4.2. *Let G be a finite group and H its subgroup. Then there is a split short exact sequence*

$$[\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G \xrightarrow{\text{proj}_*} [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \xrightarrow{\Phi^H} [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)},$$

where the morphism $\text{proj}: G \times_H \Sigma_+^\infty E\mathcal{P}[H] \rightarrow \Sigma_+^\infty G/H$ is defined as the composite

$$G \times_H \Sigma_+^\infty E\mathcal{P}[H] \xrightarrow{G \times_H \text{proj}} G \times_H \mathbb{S} \cong \Sigma_+^\infty G/H.$$

Before proving this proposition we have to recall some important technical facts.

6.5 Technical preparation

It immediately follows from the definition that for any $K \in \mathcal{P}[H]$, the set of H -fixed points of G/K is empty. Together with Corollary 6.1.3 this implies that $\Phi^H(\Sigma_+^\infty G/K) = *$ in $\text{Ho}(\text{Sp}_{W(H)}^O)$. Since the classifying space $E\mathcal{P}[H]$ is built out of H -cells of orbit type H/K with $K \leq H$ and $K \neq H$ one obtains:

Proposition 6.5.1. *Let G be a finite group. For any subgroup $H \leq G$, the G -CW complex $G \times_H E\mathcal{P}[H]$ is built out of G -cells of orbit type G/K with $K \leq H$ and $K \neq H$. Furthermore, $\Phi^H(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) = *$ in $\text{Ho}(\text{Sp}_{W(H)}^O)$.*

Next, we recall that the functor $\text{Map}_H(G_+, -): \text{Ho}(\text{Sp}_H^O) \rightarrow \text{Ho}(\text{Sp}_G^O)$ is right adjoint to Res_H^G . Recall also the map $w_H: G \times_H (-) \rightarrow \text{Map}_H(G_+, -)$ inducing the Wirthmüller isomorphism (Proposition 5.2.1). The following proposition is an immediate consequence of the Wirthmüller isomorphism.

Proposition 6.5.2. *For any Y in $\text{Ho}(\text{Sp}_H^O)$, the natural map*

$$[\mathbb{S}, G \times_H Y]_*^G \xrightarrow{w_H^*} [\mathbb{S}, \text{Map}_H(G_+, Y)]_*^G \cong [\text{Res}_H^G(\mathbb{S}), Y]_*^H = [\mathbb{S}, Y]_*^H$$

is an isomorphism.

Corollary 6.5.3. *Let G be a finite group and H and K subgroups of G . Then for any spectrum $Y \in \text{Ho}(\text{Sp}_H^O)$, there is a natural isomorphism*

$$[\Sigma_+^\infty G/K, G \times_H Y]_*^G \cong \bigoplus_{[g] \in K \backslash G/H} [\mathbb{S}, \text{Res}_{K \cap^g H}^{gH}(c_g^*(Y))]_*^{K \cap^g H}.$$

Proof. By adjunction, Corollary 5.2.3 and Proposition 6.5.2, one has the following chain of isomorphisms:

$$\begin{aligned} [\Sigma_+^\infty G/K, G \times_H Y]_*^G &\cong [G \times_K \mathbb{S}, G \times_H Y]_*^G \cong [\mathbb{S}, \text{Res}_K^G(G \times_H Y)]_*^K \cong \\ &[\mathbb{S}, \bigvee_{[g] \in K \backslash G/H} K \times_{K \cap^g H} \text{Res}_{K \cap^g H}^{gH}(c_g^*(Y))]_*^K \cong \bigoplus_{[g] \in K \backslash G/H} [\mathbb{S}, K \times_{K \cap^g H} \text{Res}_{K \cap^g H}^{gH}(c_g^*(Y))]_*^K \cong \\ &\bigoplus_{[g] \in K \backslash G/H} [\mathbb{S}, \text{Res}_{K \cap^g H}^{gH}(c_g^*(Y))]_*^{K \cap^g H}. \end{aligned}$$

□

6.6 Proof of Proposition 6.4.2

It follows from Proposition 6.3.1 that

$$\Phi^H: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}$$

is a retraction and thus in particular surjective. Further, Proposition 6.5.1 implies that

$$\Phi^H \circ \text{proj}_* = 0.$$

Hence, it remains to show that the map

$$\text{proj}_*: [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is injective and $\text{Ker } \Phi^H \subset \text{Im}(\text{proj}_*)$. For this we choose a set $\{g\}$ of double coset representatives for $H \backslash G/H$. By Corollary 6.5.3, there is a commutative diagram with all vertical arrows isomorphisms

$$\begin{array}{ccc} [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{\text{proj}_*} & [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \\ \cong \downarrow & & \downarrow \cong \\ [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{(G \times_H \text{proj})_*} & [\Sigma_+^\infty G/H, G \times_H \mathbb{S}]_*^G \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_{[g] \in H \backslash G/H} [\mathbb{S}, \Sigma_+^\infty \text{Res}_{H \cap^g H}^{gH}(c_g^*(E\mathcal{P}[H]))]_*^{H \cap^g H} & \xrightarrow{\bigoplus_{[g] \in H \backslash G/H} (\text{proj})_*} & \bigoplus_{[g] \in H \backslash G/H} [\mathbb{S}, \mathbb{S}]_*^{H \cap^g H}. \end{array}$$

We will now identify the summands of the lower horizontal map. For this one has to consider two cases:

Case 1. $H \cap {}^g H = H$: In this case $\text{Res}_{H \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H])) = c_g^*(E\mathcal{P}[H])$ is a model for classifying space of $\mathcal{P}[H]$ (and hence G -homotopy equivalent to $E\mathcal{P}[H]$). By Proposition 6.4.1, we get a short exact sequence

$$0 \longrightarrow [\mathbb{S}, \Sigma_+^\infty \text{Res}_{H \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H]))_*]^{H \cap {}^g H} \xrightarrow{\text{proj}_*} [\mathbb{S}, \mathbb{S}]_*^{H \cap {}^g H} \xrightarrow{\Phi^{H \cap {}^g H}} [\mathbb{S}, \mathbb{S}]_* \longrightarrow 0.$$

Case 2. $H \cap {}^g H$ is a proper subgroup of H : In this case $\text{Res}_{H \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H]))$ is an $(H \cap {}^g H)$ -contractible cofibrant $(H \cap {}^g H)$ -space and hence the map

$$[\mathbb{S}, \Sigma_+^\infty \text{Res}_{H \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H]))_*]^{H \cap {}^g H} \xrightarrow{\text{proj}_*} [\mathbb{S}, \mathbb{S}]_*^{H \cap {}^g H}$$

is an isomorphism.

Altogether, after combining the latter diagram with Case 1. and Case 2., we see that the map

$$\text{proj}_* : [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is injective. It still remains to check that $\text{Ker } \Phi^H \subset \text{Im}(\text{proj}_*)$. For this, first note that $H \cap {}^g H = H$ if and only if $g \in N(H)$. Further, if $g \in N(H)$, then the double coset class HgH is equal to gH . Hence, the set of those double cosets $[g] \in H \backslash G/H$ for which the equality $H \cap {}^g H = H$ holds is in bijection with the Weyl group $W(H)$. Consequently, using the latter diagram and Case 1. and Case 2., one gets an isomorphism

$$[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G / \text{Im}(\text{proj}_*) \cong \bigoplus_{W(H)} [\mathbb{S}, \mathbb{S}]_* \cong [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}.$$

On the other hand, we have already checked that

$$\Phi^H : [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}$$

is surjective and this yields an isomorphism

$$[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G / \text{Ker } \Phi^H \cong [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}.$$

Combining this with the previous isomorphism implies that the graded abelian group $[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G / \text{Im}(\text{proj}_*)$ is isomorphic to $[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G / \text{Ker } \Phi^H$. Now if the grading $* > 0$, then $[\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$ is finite and it follows that $\text{Im}(\text{proj}_*)$ and $\text{Ker } \Phi^H$ are finite groups of the same cardinality (Subsection 2.7). Since we already know that $\text{Im}(\text{proj}_*) \subset \text{Ker } \Phi^H$ (We have already observed that this is a consequence of Proposition 6.5.1.), one finally gets the equality $\text{Im}(\text{proj}_*) = \text{Ker } \Phi^H$. For $* = 0$ a Five lemma argument completes the proof. We do not give here the details of the case $* = 0$ as it is irrelevant for our proof of Theorem 3.1.3. \square

7 Proof of the main theorem

In this section we complete the proof of Theorem 3.1.3 and hence of Theorem 1.1.1.

We start by recalling from [MM02, IV.6] the \mathcal{F} -model structure on the category of G -equivariant orthogonal spectra, where \mathcal{F} is a family of subgroups of a finite group G .

7.1 The \mathcal{F} -model structure and localizing subcategory determined by \mathcal{F}

Let G be a finite group and \mathcal{F} a family of subgroups of G .

Definition 7.1.1. A morphism $f: X \rightarrow Y$ of G -equivariant orthogonal spectra is called an \mathcal{F} -equivalence if it induces isomorphisms

$$f_*: \pi_*^H X \xrightarrow{\cong} \pi_*^H Y$$

on H -equivariant homotopy groups for any $H \in \mathcal{F}$. Similarly, a morphism $g: X \rightarrow Y$ in $\text{Ho}(\text{Sp}_G^O)$ is called an \mathcal{F} -equivalence if it induces an isomorphism on π_*^H for any $H \in \mathcal{F}$.

The category of G -equivariant orthogonal spectra has a stable model structure with weak equivalences the \mathcal{F} -equivalences and with cofibrations the \mathcal{F} -cofibrations [MM02, IV.6.5]. By restricting our attention to those orbits G/H which satisfy $H \in \mathcal{F}$, we can obtain the generating \mathcal{F} -cofibrations and acyclic \mathcal{F} -cofibrations in a similar way as for the absolute case of Sp_G^O [MM02, III.4] (see Subsection 2.5 and Subsection 2.6). We will denote this model category by $\text{Sp}_{G, \mathcal{F}}^O$.

Any \mathcal{F} -equivalence can be detected in terms of geometric fixed points. To see this we need the following proposition which relates the classifying space $E\mathcal{F}$ with the concept of an \mathcal{F} -equivalence:

Proposition 7.1.2 ([MM02, IV.6.7]). *A morphism $f: X \rightarrow Y$ of G -equivariant orthogonal spectra is an \mathcal{F} -equivalence if and only if $1 \wedge f: E\mathcal{F}_+ \wedge X \rightarrow E\mathcal{F}_+ \wedge Y$ is a G -equivalence, i.e., a stable equivalence of orthogonal G -spectra.*

Corollary 7.1.3. *A morphism $f: X \rightarrow Y$ of G -equivariant orthogonal spectra is an \mathcal{F} -equivalence if and only if for any $H \in \mathcal{F}$, the induced map*

$$\Phi^H(\text{Res}_H^G(f)): \Phi^H(\text{Res}_H^G(X)) \rightarrow \Phi^H(\text{Res}_H^G(Y))$$

on H -geometric fixed points is a stable equivalence of (non-equivariant) spectra.

Proof. By Proposition 7.1.2, $f: X \rightarrow Y$ is an \mathcal{F} -equivalence if and only if

$$1 \wedge f: E\mathcal{F}_+ \wedge X \rightarrow E\mathcal{F}_+ \wedge Y$$

is a stable equivalence of orthogonal G -spectra. But the latter is the case if and only if

$$\Phi^H(\text{Res}_H^G(1 \wedge f)): \Phi^H(\text{Res}_H^G(E\mathcal{F}_+ \wedge X)) \rightarrow \Phi^H(\text{Res}_H^G(E\mathcal{F}_+ \wedge Y))$$

is a stable equivalence of spectra for any subgroup $H \leq G$ ([May96, XVI.6.4]). Now using that the restriction and geometric fixed points commute with smash products as well as the defining properties of $E\mathcal{F}$, we obtain the desired result. \square

By definition of \mathcal{F} -equivalences and \mathcal{F} -cofibrations we get a Quillen adjunction

$$\mathrm{id}: \mathrm{Sp}_{G,\mathcal{F}}^O \rightleftarrows \mathrm{Sp}_G^O : \mathrm{id}.$$

After deriving this Quillen adjunction one obtains an adjunction

$$\mathbf{L}: \mathrm{Ho}(\mathrm{Sp}_{G,\mathcal{F}}^O) \rightleftarrows \mathrm{Ho}(\mathrm{Sp}_G^O) : \mathbf{R}$$

on the homotopy level. We now examine the essential image of the left adjoint functor \mathbf{L} . Since a weak equivalence in Sp_G^O is also a weak equivalence in $\mathrm{Sp}_{G,\mathcal{F}}^O$, the unit

$$\mathrm{id} \longrightarrow \mathbf{RL}$$

of the adjunction (\mathbf{L}, \mathbf{R}) is an isomorphism of functors. Hence the functor

$$\mathbf{L}: \mathrm{Ho}(\mathrm{Sp}_{G,\mathcal{F}}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$$

is fully faithful.

Proposition 7.1.4. *For any $X \in \mathrm{Sp}_{G,\mathcal{F}}^O$, there are natural isomorphisms*

$$\mathbf{L}(X) \cong E\mathcal{F}_+ \wedge^{\mathbf{L}} X \cong E\mathcal{F}_+ \wedge X.$$

Proof. Let $\lambda_X: X^c \rightarrow X$ be a (functorial) cofibrant replacement of X in $\mathrm{Sp}_{G,\mathcal{F}}^O$. By [MM02, Theorem IV.6.10], the projection map $E\mathcal{F}_+ \wedge X^c \rightarrow X^c$ is a weak equivalence in Sp_G^O . On the other hand, Proposition 7.1.2 implies that the morphism of G -spectra $1 \wedge \lambda_X: E\mathcal{F}_+ \wedge X^c \rightarrow E\mathcal{F}_+ \wedge X$ is a weak equivalence in Sp_G^O . This completes the proof. \square

Next, note that the triangulated category $\mathrm{Ho}(\mathrm{Sp}_{G,\mathcal{F}}^O)$ is compactly generated with

$$\{\Sigma_+^\infty G/H \mid H \in \mathcal{F}\}$$

as a set of compact generators. Indeed, this follows from the following chain of isomorphisms:

$$\begin{aligned} [\Sigma_+^\infty G/H, X]_*^{\mathrm{Ho}(\mathrm{Sp}_{G,\mathcal{F}}^O)} &\cong [E\mathcal{F}_+ \wedge \Sigma_+^\infty G/H, E\mathcal{F}_+ \wedge X]_*^G \cong \\ &[\Sigma_+^\infty G/H, E\mathcal{F}_+ \wedge X]_*^G \cong \pi_*^H(E\mathcal{F}_+ \wedge X) \cong \pi_*^H X. \end{aligned}$$

The first isomorphism in this chain follows from Proposition 7.1.4 and from the fact that \mathbf{L} is fully faithful. The second isomorphism holds since $H \in \mathcal{F}$. Finally, the last isomorphism is an immediate consequence of Corollary 7.1.3.

Proposition 7.1.5. *The essential image of the functor $\mathbf{L}: \text{Ho}(\text{Sp}_{G, \mathcal{F}}^O) \rightarrow \text{Ho}(\text{Sp}_G^O)$ is exactly the localizing subcategory generated by $\{\Sigma_+^\infty G/H \mid H \in \mathcal{F}\}$.*

Proof. The functor \mathbf{L} is exact and as we already noted, $\text{Ho}(\text{Sp}_{G, \mathcal{F}}^O)$ is generated by the set $\{\Sigma_+^\infty G/H \mid H \in \mathcal{F}\}$. Next, by Proposition 7.1.4, for any $H \in \mathcal{F}$,

$$\mathbf{L}(\Sigma_+^\infty G/H) \cong E\mathcal{F}_+ \wedge \Sigma_+^\infty G/H.$$

The projection map $E\mathcal{F}_+ \wedge \Sigma_+^\infty G/H \rightarrow \Sigma_+^\infty G/H$ is a weak equivalence in Sp_G^O . The rest follows from the fact that \mathbf{L} is full. \square

Next, we need the following simple lemma from category theory.

Lemma 7.1.6. *Let*

$$\mathbf{L}: \mathcal{D} \rightleftarrows \mathcal{E} : \mathbf{R}.$$

be an adjunction and assume that the unit

$$id \rightarrow \mathbf{RL}$$

is an isomorphism (or, equivalently, \mathbf{L} is fully faithful). Further, suppose we are given morphisms

$$X \xrightarrow{\alpha} Z \xleftarrow{\beta} Y$$

in \mathcal{E} such that X and Y are in the essential image of \mathbf{L} and $\mathbf{R}(\alpha)$ and $\mathbf{R}(\beta)$ are isomorphisms in \mathcal{D} . Then there is an isomorphism $\gamma: X \xrightarrow{\cong} Y$ in \mathcal{E} such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Y \\ & \searrow \alpha & \swarrow \beta \\ & Z & \end{array}$$

commutes.

Proof. One has the commutative diagram

$$\begin{array}{ccccc} \mathbf{LR}(X) & \xrightarrow[\cong]{\mathbf{LR}(\alpha)} & \mathbf{LR}(Z) & \xleftarrow[\cong]{\mathbf{LR}(\beta)} & \mathbf{LR}(Y) \\ \cong \downarrow \text{counit} & & \downarrow \text{counit} & & \cong \downarrow \text{counit} \\ X & \xrightarrow{\alpha} & Z & \xleftarrow{\beta} & Y, \end{array}$$

where the left and right vertical arrows are isomorphisms since X and Y are in the essential image of \mathbf{L} and the functor \mathbf{L} is fully faithful. We can choose $\gamma: X \xrightarrow{\cong} Y$ to be the composite

$$X \xrightarrow{\text{counit}^{-1}} \mathbf{LR}(X) \xrightarrow{\mathbf{LR}(\alpha)} \mathbf{LR}(Z) \xrightarrow{(\mathbf{LR}(\beta))^{-1}} \mathbf{LR}(Y) \xrightarrow{\text{counit}} Y. \quad \square$$

Corollary 7.1.7. *Let \mathcal{F} be a family of subgroups of G and suppose X and Y are in the essential image of $\mathbf{L}: \text{Ho}(\text{Sp}_{G, \mathcal{F}}^O) \rightarrow \text{Ho}(\text{Sp}_G^O)$ (which is the localizing subcategory generated by $\{\Sigma_+^\infty G/H \mid H \in \mathcal{F}\}$ according to 7.1.5). Further assume that we have maps*

$$X \xrightarrow{\alpha} Z \xleftarrow{\beta} Y$$

such that $\pi_^H \alpha$ and $\pi_*^H \beta$ are isomorphisms for any $H \in \mathcal{F}$ (Or, in other words, α and β are \mathcal{F} -equivalences.). Then there is an isomorphism $\gamma: X \xrightarrow{\cong} Y$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Y \\ & \searrow \alpha & \swarrow \beta \\ & Z & \end{array}$$

commutes.

Proof. We apply the previous lemma to the adjunction

$$\mathbf{L}: \text{Ho}(\text{Sp}_{G, \mathcal{F}}^O) \rightleftarrows \text{Ho}(\text{Sp}_G^O) : \mathbf{R}$$

and use the isomorphism $\pi_*^H \mathbf{R}(T) \cong \pi_*^H T$, $H \in \mathcal{F}$. □

7.2 Inductive strategy and preservation of induced classifying spaces

Recall that we are given an exact functor of triangulated categories

$$F: \text{Ho}(\text{Sp}_{G, (2)}^O) \longrightarrow \text{Ho}(\text{Sp}_{G, (2)}^O)$$

that preserves arbitrary coproducts and such that

$$F(\Sigma_+^\infty G/H) = \Sigma_+^\infty G/H, \quad H \leq G,$$

and

$$F(g) = g, \quad F(\text{res}_K^H) = \text{res}_K^H, \quad F(\text{tr}_K^H) = \text{tr}_K^H, \quad g \in G, \quad K \leq H \leq G.$$

We want to show that F is an equivalence of categories. Proposition 5.1.1 implies that in order to prove that F is an equivalence, it suffices to check that for any subgroup $H \leq G$, the map between graded endomorphism rings

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G = [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism. The strategy is to do this inductively. We proceed by induction on the cardinality of H . The induction starts with the case $H = e$. Proposition 4.2.3 tells us that the map

$$F: [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G \longrightarrow [\Sigma_+^\infty G, \Sigma_+^\infty G]_*^G$$

is an isomorphism and hence the basis step is proved. The induction step follows from the next proposition which is one of the main technical results of this thesis:

Proposition 7.2.1. *Let G be a finite group and H a subgroup of G . Assume that for any subgroup K of G which is proper subconjugate to H , the map*

$$F: [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G \longrightarrow [F(\Sigma_+^\infty G/K), F(\Sigma_+^\infty G/K)]_*^G = [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G$$

is an isomorphism. Then the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G = [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism.

Before starting to prove this proposition, one has to show that under its assumptions the functor F preserves a certain induced classifying space. More precisely, let $\mathcal{P}[H]$ denote the family of proper subgroups of H . This is a family with respect to H and not necessarily with respect to the whole group G . Next, let

$$\text{proj}: G \times_H \Sigma_+^\infty E\mathcal{P}[H] \longrightarrow \Sigma_+^\infty G/H$$

be the projection (as in Subsection 6.4). The following is valid:

Lemma 7.2.2. *Suppose G is a finite group and H a subgroup of G . Assume that for any subgroup K of G which is proper subconjugate to H , the map*

$$F: [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G$$

is an isomorphism. Then there is an isomorphism

$$\gamma: F(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) \xrightarrow{\cong} G \times_H \Sigma_+^\infty E\mathcal{P}[H]$$

such that the diagram

$$\begin{array}{ccc} F(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) & \xrightarrow[\gamma]{\cong} & G \times_H \Sigma_+^\infty E\mathcal{P}[H] \\ F(\text{proj}) \downarrow & & \downarrow \text{proj} \\ F(\Sigma_+^\infty G/H) & \xlongequal{\quad} & \Sigma_+^\infty G/H, \end{array}$$

commutes.

Proof. Let $\mathcal{P}[H|G]$ denote the family of subgroups of G which are proper subconjugate to H . By Proposition 7.1.5, the essential image of the fully faithful embedding

$$\mathbf{L}: \text{Ho}(\text{Sp}_{G, \mathcal{P}[H|G]}^O) \longrightarrow \text{Ho}(\text{Sp}_G^O).$$

is the localizing subcategory generated by the set $\{\Sigma_+^\infty G/K \mid K \in \mathcal{P}[H|G]\}$. Obviously, the spectrum $G \times_H \Sigma_+^\infty E\mathcal{P}[H]$ is an object of this localizing subcategory as the H -CW complex $E\mathcal{P}[H]$ is built out of H -cells of orbit type H/K with $K \leq H$ and $K \neq H$. Next, since the endofunctor $F: \text{Ho}(\text{Sp}_G^O) \longrightarrow \text{Ho}(\text{Sp}_G^O)$ is triangulated

and $F(\Sigma_+^\infty G/L) = \Sigma_+^\infty G/L$ for any $L \leq G$, the spectrum $F(G \times_H \Sigma_+^\infty E\mathcal{P}[H])$ is contained in the essential image of $\mathbf{L}: \mathrm{Ho}(\mathrm{Sp}_{G, \mathcal{P}[H|G]}^O) \rightarrow \mathrm{Ho}(\mathrm{Sp}_G^O)$ as well. Hence by Corollary 7.1.7, it suffices to show the maps in the zigzag

$$F(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) \xrightarrow{F(\mathrm{proj})} F(\Sigma_+^\infty G/H) = \Sigma_+^\infty G/H \xleftarrow{\mathrm{proj}} G \times_H \Sigma_+^\infty E\mathcal{P}[H]$$

are $\mathcal{P}[H|G]$ -equivalences (which means that they induce isomorphisms on $\pi_*^K(-)$ for any subgroup $K \in \mathcal{P}[H|G]$). It is easy to see that the map

$$\mathrm{proj}: G \times_H \Sigma_+^\infty E\mathcal{P}[H] \rightarrow \Sigma_+^\infty G/H$$

is $\mathcal{P}[H|G]$ -equivalence. Indeed, by Corollary 6.5.3, for any $K \in \mathcal{P}[H|G]$, one has a commutative diagram

$$\begin{array}{ccc} \pi_*^K(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) & \xrightarrow{\mathrm{proj}_*} & \pi_*^K(\Sigma_+^\infty G/H) \\ \cong \downarrow & & \downarrow \cong \\ [\Sigma_+^\infty G/K, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{\mathrm{proj}_*} & [\Sigma_+^\infty G/K, G \times_H \mathbb{S}]_*^G \\ \cong \downarrow & & \downarrow \cong \\ \bigoplus_{[g] \in K \backslash G/H} [\mathbb{S}, \Sigma_+^\infty \mathrm{Res}_{K \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H]))]_*^{K \cap {}^g H} & \xrightarrow{\bigoplus_{[g] \in K \backslash G/H} (\mathrm{proj})_*} & \bigoplus_{[g] \in K \backslash G/H} [\mathbb{S}, \mathbb{S}]_*^{K \cap {}^g H}. \end{array}$$

If L is a subgroup of $K \cap {}^g H$, then $g^{-1}Lg$ is a subgroup of H . In fact, $g^{-1}Lg$ is a proper subgroup of H since $K \in \mathcal{P}[H|G]$. This implies that for any subgroup $L \leq K \cap {}^g H$ the space $(\mathrm{Res}_{K \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H])))^L = (E\mathcal{P}[H])^{g^{-1}Lg}$ is contractible. Hence, $\mathrm{Res}_{K \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H]))$ is a $(K \cap {}^g H)$ -contractible cofibrant $(K \cap {}^g H)$ -space and we see that the map

$$\mathrm{proj}: \Sigma_+^\infty \mathrm{Res}_{K \cap {}^g H}^{{}^g H}(c_g^*(E\mathcal{P}[H])) \rightarrow \mathbb{S}$$

is a $(K \cap {}^g H)$ -equivalence. This allows us to conclude that the lower horizontal map in the latter commutative diagram is an isomorphism. Hence, the upper horizontal map is an isomorphism for any subgroup $K \in \mathcal{P}[H|G]$ and one concludes that the map

$$\mathrm{proj}: G \times_H \Sigma_+^\infty E\mathcal{P}[H] \rightarrow \Sigma_+^\infty G/H$$

is $\mathcal{P}[H|G]$ -equivalence.

It remains to show that the morphism $F(\mathrm{proj}): F(G \times_H \Sigma_+^\infty E\mathcal{P}[H]) \rightarrow F(\Sigma_+^\infty G/H)$ is a $\mathcal{P}[H|G]$ -equivalence as well. We first note that the assumptions imply that for any $K \in \mathcal{P}[H|G]$ and any (not necessarily proper) subgroup $L \leq H$, the map

$$F: [\Sigma_+^\infty G/K, \Sigma_+^\infty G/L]_*^G \rightarrow [F(\Sigma_+^\infty G/K), F(\Sigma_+^\infty G/L)]_*^G = [\Sigma_+^\infty G/K, \Sigma_+^\infty G/L]_*^G$$

is an isomorphism. Indeed, this follows from Proposition 5.1.1 as well as from the commutative diagram

$$\begin{array}{ccc} [\Sigma_+^\infty G/K, \Sigma_+^\infty G/L]_*^G & \xrightarrow{(\kappa_\lambda)_{[\lambda] \in K \backslash G/L}} & \bigoplus_{[\lambda] \in K \backslash G/L} [\Sigma_+^\infty G/(\lambda L \cap K), \Sigma_+^\infty G/(\lambda L \cap K)]_*^G \\ \downarrow F & & \downarrow \bigoplus_{[\lambda] \in K \backslash G/L} F \\ [\Sigma_+^\infty G/K, \Sigma_+^\infty G/L]_*^G & \xrightarrow{(\kappa_\lambda)_{[\lambda] \in K \backslash G/L}} & \bigoplus_{[\lambda] \in K \backslash G/L} [\Sigma_+^\infty G/(\lambda L \cap K), \Sigma_+^\infty G/(\lambda L \cap K)]_*^G \end{array}$$

where the right vertical map is an isomorphism since $\lambda L \cap K$ is proper subconjugate to H for any λ . In particular, the map

$$F: [\Sigma_+^\infty G/K, \Sigma_+^\infty G/H]_*^G \longrightarrow [\Sigma_+^\infty G/K, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism. Next, using a standard argument on triangulated categories, we see that for any $K \in \mathcal{P}[H|G]$ and any X from the localizing subcategory of $\text{Ho}(\text{Sp}_G^O)$ generated by $\{\Sigma_+^\infty G/L \mid L \leq H\}$, the map

$$F: [\Sigma_+^\infty G/K, X]_*^G \longrightarrow [F(\Sigma_+^\infty G/K), F(X)]_*^G$$

is an isomorphism (recall $F(\Sigma_+^\infty G/L) = \Sigma_+^\infty G/L$ for any $L \leq G$). As a consequence, we see that the morphism

$$F: [\Sigma_+^\infty G/K, G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G \longrightarrow [F(\Sigma_+^\infty G/K), F(G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H])]_*^G$$

is an isomorphism. Finally, for any $K \in \mathcal{P}[H|G]$, consider the commutative diagram

$$\begin{array}{ccc} [\Sigma_+^\infty G/K, G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{\text{proj}_*} & [\Sigma_+^\infty G/K, \Sigma_+^\infty G/H]_*^G \\ \cong \downarrow F & & \cong \downarrow F \\ [F(\Sigma_+^\infty G/K), F(G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H])]_*^G & \xrightarrow{F(\text{proj})_*} & [F(\Sigma_+^\infty G/K), F(\Sigma_+^\infty G/H)]_*^G \\ \parallel & & \parallel \\ [\Sigma_+^\infty G/K, F(G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H])]_*^G & \xrightarrow{F(\text{proj})_*} & [\Sigma_+^\infty G/K, F(\Sigma_+^\infty G/H)]_*^G. \end{array}$$

As we already explained, the upper horizontal map is an isomorphism. Thus the lower horizontal map in this diagram is an isomorphism as well and therefore, the map

$$F(\text{proj}): F(G \rtimes_H \Sigma_+^\infty E\mathcal{P}[H]) \longrightarrow F(\Sigma_+^\infty G/H)$$

is a $\mathcal{P}[H|G]$ -equivalence. \square

7.3 Completing the proof of Theorem 3.1.3

In this subsection we continue the induction started in the previous subsection and prove Proposition 7.2.1. Finally, at the end, we complete the proof of Theorem 3.1.3 and hence prove the main Theorem 1.1.1.

Proof of Proposition 7.2.1. Recall (Section 6) that the extension

$$1 \longrightarrow H \xrightarrow{\iota} N(H) \xrightarrow{\varepsilon} W(H) \longrightarrow 1.$$

determines the inflation functor

$$\varepsilon^*: \mathrm{Ho}(\mathrm{Sp}_{W(H)}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{N(H)}^O)$$

and the geometric fixed point functor

$$\Phi^H: \mathrm{Ho}(\mathrm{Sp}_{N(H)}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{W(H)}^O).$$

Let $\hat{F}: \mathrm{Ho}(\mathrm{Sp}_{W(H)}^O) \longrightarrow \mathrm{Ho}(\mathrm{Sp}_{W(H)}^O)$ denote the composite

$$\mathrm{Ho}(\mathrm{Sp}_{W(H)}^O) \xrightarrow{\varepsilon^*} \mathrm{Ho}(\mathrm{Sp}_{N(H)}^O) \xrightarrow{G \times_{N(H)}^-} \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{F} \mathrm{Ho}(\mathrm{Sp}_G^O) \xrightarrow{\mathrm{Res}_{N(H)}^G} \mathrm{Ho}(\mathrm{Sp}_{N(H)}^O) \xrightarrow{\Phi^H} \mathrm{Ho}(\mathrm{Sp}_{W(H)}^O).$$

It follows from the identifications we did in Subsection 6.3 and from the properties of F that the functor \hat{F} is triangulated, preserves infinite coproducts and sends $\Sigma_+^\infty W(H)$ to itself. Moreover, it also follows that the restriction

$$\hat{F}|_{\mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty W(H))}: \mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty W(H)) \longrightarrow \mathrm{Ho}(\mathbf{Mod}\text{-}\Sigma_+^\infty W(H))$$

of \hat{F} on the localizing subcategory of $\mathrm{Ho}(\mathrm{Sp}_{W(H)}^O)$ generated by $\Sigma_+^\infty W(H)$ satisfies the assumptions of Proposition 4.2.3. Hence, the map

$$\hat{F}: [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)} \longrightarrow [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}$$

is an isomorphism. Next, by the assumptions and Proposition 5.1.1 (like in the proof of Lemma 7.2.2), we see that for any proper subgroup L of H , the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/L]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/L]_*^G$$

is an isomorphism. This, using a standard argument on triangulated categories, implies that for any X which is contained in the localizing subcategory of $\mathrm{Ho}(\mathrm{Sp}_G^O)$ generated by $\{\Sigma_+^\infty G/L \mid L \in \mathcal{P}[H]\}$, the map

$$F: [\Sigma_+^\infty G/H, X]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(X)]_*^G$$

is an isomorphism and hence, in particular, so is the morphism

$$F: [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(G \times_H \Sigma_+^\infty E\mathcal{P}[H])]_*^G.$$

Finally, we have the following important commutative diagram

$$\begin{array}{ccccc} [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{\mathrm{proj}_*} & [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G & \xleftarrow{G \times_{N(H)} \varepsilon^*} & [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)} \\ \cong \downarrow F & & \downarrow F & & \cong \downarrow \hat{F} \\ [F(\Sigma_+^\infty G/H), F(G \times_H \Sigma_+^\infty E\mathcal{P}[H])]_*^G & \xrightarrow{F(\mathrm{proj})_*} & [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G & \xrightarrow{\Phi^H} & [\hat{F}(\Sigma_+^\infty W(H)), \hat{F}(\Sigma_+^\infty W(H))]_*^{W(H)} \\ \cong \downarrow & & \parallel & & \parallel \\ [\Sigma_+^\infty G/H, G \times_H \Sigma_+^\infty E\mathcal{P}[H]]_*^G & \xrightarrow{\mathrm{proj}_*} & [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G & \xrightarrow{\Phi^H} & [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}. \end{array}$$

Lemma 7.2.2 implies that the lower left square commutes and the lower left vertical map is an isomorphism. Other squares commute by definitions. Further, according to Proposition 6.4.2, the lower row in this diagram is a short exact sequence and hence so is the middle one.

Now a simple diagram chase shows that the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G = [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism. Indeed, assume that $*$ $>$ 0 (the case $*$ $= 0$ is obvious by the assumptions on F). Then the latter map has the same finite source and target and hence it suffices to show that it is surjective. Fix $*$ $>$ 0 and take any $\alpha \in [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G$. Since the map

$$\hat{F}: [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)} \longrightarrow [\hat{F}(\Sigma_+^\infty W(H)), \hat{F}(\Sigma_+^\infty W(H))]_*^{W(H)}$$

is an isomorphism, there exists $\beta \in [\Sigma_+^\infty W(H), \Sigma_+^\infty W(H)]_*^{W(H)}$ such that

$$\hat{F}(\beta) = \Phi^H(\alpha).$$

By definition of the functor \hat{F} , the element

$$F(G \times_{N(H)} \varepsilon^*(\beta)) - \alpha \in [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G$$

is in the kernel of

$$\Phi^H: [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G \longrightarrow [\hat{F}(\Sigma_+^\infty W(H)), \hat{F}(\Sigma_+^\infty W(H))]_*^{W(H)}.$$

But the kernel of this map is contained in the image of

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [F(\Sigma_+^\infty G/H), F(\Sigma_+^\infty G/H)]_*^G$$

since the middle row in the commutative diagram above is exact and the upper left vertical map is an isomorphism. Consequently, $F(G \times_{N(H)} \varepsilon^*(\beta)) - \alpha$ is in the image of F and this completes the proof. \square

Proof of Theorem 3.1.3. Now we continue with the induction. Recall, that our aim is to show that for any subgroup $H \in G$, the map

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism. The strategy that was indicated at the beginning of Subsection 7.2 is to proceed by induction on the cardinality of H . The induction basis follows from Proposition 4.2.3 as we already explained. Now suppose $n > 1$, and assume that the claim holds for all subgroups of G with cardinality less than or equal to $n - 1$. Let H be any subgroup of G that has cardinality equal to n . Then, by the induction assumption, for any subgroup K which is proper subconjugate to H , the map

$$F: [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G \longrightarrow [\Sigma_+^\infty G/K, \Sigma_+^\infty G/K]_*^G$$

is an isomorphism. Proposition 7.2.1 now implies that

$$F: [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G \longrightarrow [\Sigma_+^\infty G/H, \Sigma_+^\infty G/H]_*^G$$

is an isomorphism and this completes the proof of the claim.

The rest follows from Proposition 5.1.1 as already explained in Section 5. \square

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Summary

This thesis establishes a uniqueness result about the models for the 2-local G -equivariant stable homotopy category for a finite group G . Let $Sp_{G,(2)}^O$ denote the 2-local stable model category of G -equivariant orthogonal spectra indexed on a complete G -universe in the sense of Mandell and May. Further, suppose that \mathcal{C} is a cofibrantly generated, proper, G -equivariant stable model category. We show that if we are given a triangulated equivalence

$$\mathrm{Ho}(\mathcal{C}) \sim \mathrm{Ho}(Sp_{G,(2)}^O)$$

which respects the stable Burnside (orbit) category, then the model categories \mathcal{C} and $Sp_{G,(2)}^O$ are G -Quillen equivalent. In other words, the model category $Sp_{G,(2)}^O$ has a unique equivariant model up to G -Quillen equivalence. This means that the suspension functor, homotopy cofiber sequences and the basic π_0 -information of $\mathrm{Ho}(Sp_G^O)$ (the stable Burnside (orbit) category) determine all “higher order structure” of the 2-local G -equivariant stable homotopy category, such as the equivariant homotopy types of function G -spaces.

There are several other models for the G -equivariant stable homotopy category which are equivalent to the model category of G -equivariant orthogonal spectra. These include the \mathbb{S} -model structure of Stolz, the model category of S_G -modules of Mandell and May, the model category of G -equivariant continuous functors of Blumberg and the model categories of G -equivariant topological symmetric spectra due to Mandell and Hausmann. All these models are cofibrantly generated, proper, G -equivariant stable model categories.

Our theorem can be seen as an equivariant version of Schwede’s rigidity theorem at the prime 2. Schwede’s theorem asserts that the classical stable homotopy category has a unique model up to Quillen equivalence.

The proof of the main theorem is divided into two main parts: The first is categorical and the second is computational. In the categorical part, for any cofibrantly generated, proper, G -equivariant stable model category \mathcal{C} , we construct a stable model structure on the category $\mathrm{Sp}_G^O(\mathcal{C})$ of internal orthogonal G -spectra in \mathcal{C} . We also show that the model categories \mathcal{C} and $\mathrm{Sp}_G^O(\mathcal{C})$ are G -Quillen equivalent and using this reduce the proof of the main result to showing that a certain exact endofunctor F of the homotopy category $\mathrm{Ho}(Sp_{G,(2)}^O)$ is an equivalence of categories. The computational part shows that this endofunctor is indeed an equivalence of categories. The proof starts by generalizing Schwede’s arguments to free (naive) G -spectra. From this point on, classical techniques of equivariant stable homotopy theory enter the proof. These include the Wirthmüller isomorphism, geometric fixed points, isotropy separation and the tom Dieck splitting. The central idea is to do induction on the order of subgroups and use the case of free G -spectra as the induction basis.

We believe that the integral version of our main theorem should also be true. We formulate this as a conjecture in this thesis. In fact, most of the arguments here work integrally. The only part of the proof of the main result which uses that we are working 2-locally is the part about free G -spectra. The essential fact one needs here is that the

self map $2 \cdot \text{id}: M(2) \rightarrow M(2)$ of the mod 2 Moore spectrum is not zero in the stable homotopy category. For p an odd prime, the map $p \cdot \text{id}: M(p) \rightarrow M(p)$ is equal to zero and this makes a big difference between the 2-primary and odd primary cases.