# Uniqueness and Stability near Stationary Solutions for the Thin-Film Equation in Multiple Space Dimensions with Small Initial Lipschitz Perturbations

## Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn


vorgelegt von

Dominik John

aus

Gelsenkirchen



Bonn, April 2013

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms Universität Bonn

1. Gutachter: Prof. Dr. Herbert Koch

2. Gutachter: Prof. Dr. Felix Otto

Tag der Promotion: 16. September 2013

Erscheinungsjahr: 2013

*"Le silence éternel des ces espaces infinis m'effraie."* –
*The eternal silence of these infinite spaces frightens me.*

---

Blaise Pascal (1623 – 1662), *Pensées*

# Acknowledgments

Good things come to those who wait but one thing is for sure, it was really getting quite late! The final phase of my doctoral thesis is now drawing to a close. I am already looking forward to placing my heavy head on my pillow to savor the deepest of slumbers and return to the once relaxed state of mind I enjoyed before I embarked on this daunting and challenging journey. However, before I finally rest my weary head, I would like to take the opportunity to express my deepest gratitude to those people whose support, understanding and encouragement enabled me to bring my thesis to completion.

First and foremost, my deepest thanks go to Professor Herbert Koch for supervising this thesis with enthusiasm, encouragement and infinite patience. His guidance and academic expertise consistently helped me throughout the duration of my endeavor. Rather like a good wine, many ideas were able to mature and develop, through the rich and fruitful discussions of the numerous research challenges with which I was confronted.

My studies have been supported by the Bonn International Graduate School in Mathematics (BIGS), and the DFG through the Collaborative Research Center (SFB) 611 *Singular Phenomena and Scaling in Mathematical Models*. Besides supporting me financially, they offered me a pleasant work environment and made the University of Bonn such an excellent place to study mathematics.

Whilst it is not possible to include each and every one, a heartfelt thanks goes out to all of my fellow students. I however, am particularly deeply grateful to Clemens Kienzler with whom I had the pleasure to work, not merely a colleague, but more accurately, a real friend to me. Needless to say that this body of work would not have been possible without his integrity and diligent tenacity. Thanks to the other members of the workgroup Stefan Steinerberger, Tobias Schottdorf, Angkana Rüland, Christian Zillinger and Habiba Kalantarova for proofreading parts of my thesis and for providing all of those crucial moments of distraction. On the subject of distraction, I would also like to offer sincerest of thanks to the "Mensa group" for tolerating my presence in countless discussions about all and sundry, helping to free my mind and inspiring me to keep writing.

And last, but by no means least, though they possibly are not aware of how much of a help they have been, I would like to thank my family, particularly my father and my brother Mathias, and friends for their love and support during my odyssey in Bonn. They provided me with an indispensable safe haven and consequently rescued me from peril more times than I can recall ensuring my sanity in all adversity. To you all... I salute you.

Bonn, April 2013

# Contents

# Chapter 1

# Introduction

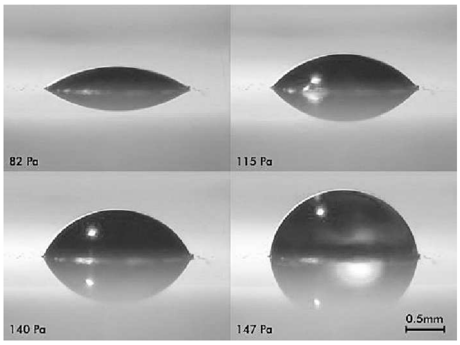


*Figure 1.1: Pressure-induced evolution of a liquid interface, taken from [24].*

The motion of a fluid undergoing environmental influences is something we experience in many real life situations. Common examples include condensing water on the bathroom mirror after taking a hot shower, rain patterns forming on window panes as well as sweating while doing a workout. It is an ongoing challenge to translate the observed phenomena into models that use mathematical concepts and language allowing for precise treatment of the problem. In this dissertation we discuss the model problem of a purely surface tension driven evolution of a *thin film*.

Thin films of liquid appear in a variety of physical applications in which a thin layer of fluid is deposited onto a bulk material (substrate) in order to improve the substrate's properties. This process is ubiquitous in the manufacture of optics, where thin films are used to create optical surface modifications such as reflective and anti-reflective coatings. Further applications may also be found in other branches of physical science, prominent examples being heat sinks and cooling systems, corrosion and oxidation protection, adhesion, gas/liquid sensors and diffusion barriers which can only be manufactured using special chemical properties of various liquids.

All of these examples can be described by the interaction between the fluid, the surrounding vapor and the adjacent solid materials. A prevailing trend in modern material science is to make use of the possibility to control these interactions and, accordingly, it may come as no surprise that specific surface properties gain increasing importance to physicists, engineers and chemists.

## 1.1 On Degenerate Diffusion

Working within a certain physical regime, we will derive (see section 1.2 below) the following governing equation from the Navier-Stokes equations:

$$\eta\,\partial_s h \,+\, \gamma\,\nabla_y\cdot\Big(\big(\frac{1}{3}\,h^3 + b^{\,3-m}\,h^m\big)\,\nabla_y\Delta_y\,h\Big) \;=\; 0\,, \tag{LE}$$

the so-called *lubrication equation.* Here, $s$ represents time, $y$ is the $n$-dimensional space variable and $h = h(s, y)$ models the thickness of a liquid film on a plain substrate. The constants $\eta$ and $\gamma$ stand for the viscosity and for surface tension, respectively. At the liquid-solid interface we obtain various slip conditions depending on the slip length $b \geq 0$ and the parameter $m$ ranging over all numbers in the interval $(0, 3)$. The case $b = 0$ entails a no-slip condition. Equations of this type can be seen as describing the dynamics of a liquid film that spreads along a solid surface, a phenomenon we refer to as diffusion.

Diffusion equations are partial differential equations that often serve as the basis for an introduction to the area of (nonlinear) PDEs. The heat equation $\partial_s h = \Delta_y h$ is certainly the most important representative in the classical linear theory. It describes the temperature distribution in a given medium as time progresses. Before moving on to fourth order problems, it is instructive to take a look at second-order degenerate diffusion equations first. Here a class of equations that has recently come to be fairly well understood goes by the name *porous medium equation* and takes the form

$$\partial_s h \;=\; \nabla_y \cdot \big(h^m \, \nabla_y \, h\big) \qquad \text{in} \qquad (0,\infty) \times \mathbb{R}^n\,, \tag{PME}$$

where $m > 0$.

There are a number of striking features known about this equation including the following:

1. In regions where $h$ is strictly positive and hence the evolution is uniformly parabolic a solution becomes instantaneously smooth,
2. there exists a comparison principle for (PME),
3. compactly supported initial data generate solutions that are compactly supported for any fixed time (finite speed of propagation), in particular, there exists a moving interface, and
4. the Cauchy problem has a unique solution for a wide range of initial data including $L^1(\mathbb{R}^n)$.

Now we consider the fourth order analogue of (PME),

$$\partial_s h \;+\; \nabla_y \cdot \big(h^m \, \nabla_y \Delta_y \, h\big) \;=\; 0\,. \tag{TFE}$$

In the context of thin films, one can derive this equation from the lubrication equation (LE) as follows. We may suppose that the film thickness $h$ is small or even $h \ll b$. Then the second term in the mobility, that is $b^{3-m}\, h^m$, is the dominant one such that we choose to neglect the term $h^3$. In order to pass to a parameter-free form of the fourth order diffusion equation, we non-dimensionalize horizontal length $y$, vertical length $h$ and time $s$ in such a way that $\eta = 1$ and $\gamma\, b^{3-m} = 1$, although now one is constrained to measure the vertical length $h$ in terms of the slip length $b > 0$.
Assuming $h$ models the height of the liquid film, the wetted region at time $s$ is given by the positivity set $P_s(h) = \{y \in \mathbb{R}^n \mid h(s,y) > 0\}$. The equation (TFE) is parabolic in this region, while the parabolicity degenerates at points at which $h$ vanishes. From now on we will refer to (TFE) simply as *thin-film equation*.

First, note that both equations (PME) and (TFE) can be written in the form of the conservation law

$$\partial_s h \;+\; \nabla_y \cdot q(h) \;=\; 0\,, \tag{1.1.1}$$

where by $q(h)$ we denote the vector-valued flux of either an ideal gas in a porous medium or a thin layer of fluid on a flat surface. In the absence of flows across the boundary of the support of $h$, that is

$$q(h)\big|_{\partial\{h>0\}} \cdot \vec{\nu} \;=\; 0 \tag{1.1.2}$$

with $\vec{\nu}$ being the outer normal to $\partial\{h > 0\}$, we expect conservation of mass:

$$\int_{\mathbb{R}^n} h(s,y)\, dy \;\equiv\; const \qquad \forall\, s$$

Moreover, we assume

$$h\big|_{\partial\{h>0\}} \;=\; 0\,. \tag{1.1.3}$$

Thus, with $q(h) = h^m\, \nabla_y \Delta_y\, h$, we implicitly deal with a free boundary problem for a fourth-oder equation. Note that solutions to the initial-boundary-value-problem (1.1.1)–(1.1.3) are not necessarily unique, for a counterexample see [16]. Accordingly, we need to impose an additional boundary condition at the moving contact line. Typically, one prescribes either a zero contact angle (complete wetting) or a fixed positive contact angle $\theta_e$ (partial wetting). In case of both complete wetting and partial wetting there exists quite a large variety of literature offering a substantial body of work on existence and qualitative properties of weak solutions of (TFE). We will discuss some of the results in the section after next.

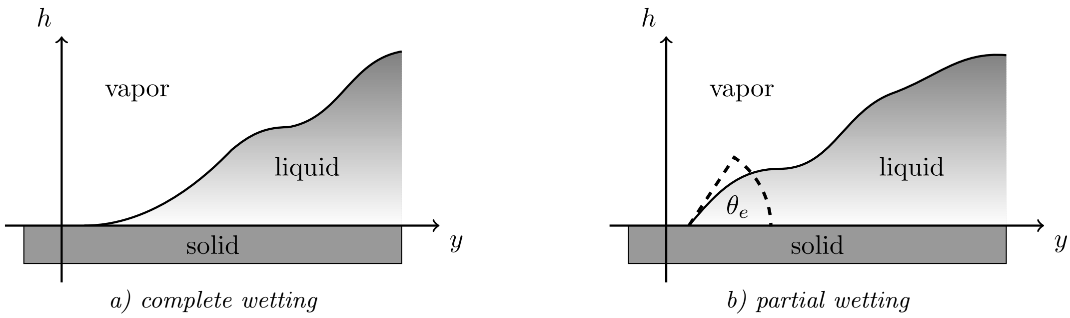


*Figure 1.2: Wetting regimes*

Now the question arises which of the above properties (1)–(4) still hold true in the context of the fourth order equation (TFE). While the fact that both equations are diffusive guarantees that property (1) is preserved, a remarkable difference between (PME) and (TFE) is the lack of a comparison (or maximum) principle for the fourth order equation. In order to see this we consider the non-degenerate case of $m = 0$ and observe that the fourth order heat kernel has an oscillating tail that changes sign, [7].
At the current state of research it is not entirely clear how to answer the remaining questions (3)–(4). In this thesis we address some of the issues related to these properties:

- What can be said about the regularity of the moving contact line?
- Which assumptions on the initial data guarantee uniqueness of solutions to the Cauchy problem?

## 1.2 From Navier-Stokes to Lubrication

The terminology "thin film" originates from the fact that the evolving wetted region is thin and only slightly sloping. The lubrication approximation uses this separation of vertical and horizontal length scales to reduce the complexity of the Navier-Stokes equations that describe the motion of liquid substances. Now, consider a viscous thin film that moves slowly on a horizontal substrate. The film is bounded above by a boundary layer between the liquid and the encompassing vapor or vacuum. We examine the $n$-dimensional case, not only the physically relevant cases $n = 2, 3$, and aim for an equation for the film thickness $h = h(s, y)$. The complete fluid is described by the set

$$F_s := \left\{(y, z) \in \mathbb{R}^{n+1} \mid y \in \Omega \ \wedge \ 0 < z < h(s, y)\right\}$$

for some $\Omega \in \mathbb{R}^n$, and the free boundary by

$$\Sigma_h \ := \ \left\{(y, z) \in \mathbb{R}^{n+1} \mid y \in \Omega \ \wedge \ z = h(s, y)\right\}.$$

We may assume that the ratio between the average horizontal length $Y$ and the typical vertical length $Z$ is small:

$$\varepsilon \ = \ \frac{Z}{Y} \ \ll \ 1\,.$$

It remains to find an appropriate time scale $S$ for the evolution and, in the limit $\varepsilon \to 0$, equation (LE) appears from the Navier-Stokes equations. The $(n+1)$-dimensional Navier-Stokes equations are

$$\partial_s \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix} \cdot \nabla \begin{pmatrix} v \\ w \end{pmatrix} = -\nabla p + \eta \Delta \begin{pmatrix} v \\ w \end{pmatrix}, \tag{1.2.1}$$

where $v \in \mathbb{R}^n$ is the flow velocity in the plane, $w$ the velocity of the fluid in vertical direction and $p$ stands for the pressure. We further suppose that the fluid is incompressible, so it satisfies the condition

$$\nabla_y \cdot v + \partial_z w = 0 . \tag{1.2.2}$$

In the model case we only take the highest order derivatives into account meaning that we can ignore the inertia terms, that is the left hand side of the Navier-Stokes equations (1.2.1). Moreover, we assume that the pressure is constant in the direction perpendicular to the substrate and that its variation in horizontal direction compensates for the viscosity stress $\eta\, \partial_z^2 v$. Altogether, this turns (1.2.1) into

$$0 = -\nabla_y p + \eta\, \partial_z^2 v . \tag{1.2.3}$$

Denoting the external pressure by $p_e$ we have $p_e - p \sim \gamma\, \kappa$ (Laplace's law), see [21, pp. 6–7], where $\kappa$ is the mean curvature of the liquid-vapor interface. In the lubrication regime, the pressure becomes

$$p = p_e - \gamma\, \Delta_y\, h .$$

Assuming that $p_e$ is constant, (1.2.3) reads as

$$\eta\, \partial_z^2 v = -\gamma\, \nabla_y \Delta_y\, h . \tag{1.2.4}$$

As this is a system of second-order PDEs, it requires to impose two conditions on the boundary. We choose

$$v = k(h)\, \partial_z v \qquad \text{if } z = 0 , \tag{1.2.5}$$

and

$$\partial_z v = 0 \qquad \text{if } z = h . \tag{1.2.6}$$

Here, the first condition represents a weighted slip condition near the liquid-solid interface. It demands that the velocity in the horizontal directions behaves proportionally to its vertical derivative. In case of the classical Navier slip condition the proportionality factor is equal to the slip length $b \geq 0$, while otherwise the velocity depends on the film thickness $h$.
On the other hand, the second assumption (1.2.6) appears far more natural to us. It states that the shear stress generated by the normal derivative of the tangential velocity field $\partial_z v$ is continuous near the liquid-vapor interface. The physical interpretation of this condition is that there are no cross currents vertical to the fluid's flow direction $v$, that is, we have a so-called laminar flow.

Integrating (1.2.4) component-wise over $z \in (0, h)$ and using (1.2.5)–(1.2.6) yields the horizontal profile of $v$,

$$v = \frac{\gamma}{\eta} \left( h\, z - \frac{1}{2}\, z^2 + h\, k(h) \right) \nabla_y \Delta_y\, h . \tag{1.2.7}$$

Since the liquid and the solid are in contact, a condition of no penetration must be specified for the contact zone. We assume that the vertical velocity $w$ is equal to 0 if $z = 0$. A kinematic relation for $v$ can then be obtained by integrating the incompressibility property (1.2.2) along $z \in (0, h)$. We arrive at

$$\partial_s h + \nabla_y \cdot \Big( \int_0^h v\, dz \Big) = 0 . \tag{1.2.8}$$

Plugging (1.2.7) in (1.2.8) we get

$$\partial_s h + \frac{\gamma}{\eta}\,\nabla_y \cdot \Big( \big( \frac{1}{3}\, h^3 + h^2\, k(h) \big)\, \nabla_y \Delta_y\, h \Big) \;=\; 0\,.$$

In particular, for $k(h) := b^{3-m}\, h^{m-2}$, this yields equation (LE).

**Remark:** The purpose of this heuristic derivation is to model the evolution of a thin film that is only driven by surface tension and viscosity. Other external effects such as molecular (Van der Waals) interaction, gravity, shear forces and rotational forces are neglected. In the thin-film regime, this approach is commonly refered to as *lubrication approximation*. The details appeared in [68].

## 1.3 Related Previous Works and Open Problems

The underlying problem (LE) or rather (TFE) is to be understood as a subclass of the more general equation

$$\eta\, \partial_s h \,+\, \nabla_y \cdot \big(\phi_b(h)\, \vec{\tau}\big) \,+\, \gamma\, \nabla_y \cdot \Big( \big( \frac{1}{3}\, h^3 + b^{\,3-m}\, h^m \big)\, \nabla_y \Delta_y\, h \Big) \;=\; Q(h)\,, \tag{1.3.1}$$

where $\vec{\tau}$ denotes the shear stress that might occur at the liquid-vapor interface, and $\phi_b(h)$ is to model the convection that affects the propagation of the fluid. Finally, $Q(h)$ is a source term that models effects of vaporization or condensation. For a derivation of (1.3.1) we refer the reader to [68] and the references therein. A good review appeared in [7].

In the existing literature, one can typically find only parts of the parameters in this equation considered. For example, let us take again equation (1.3.1) with vanishing surface tension $\gamma$ and constant shear stress $\vec{\tau}$. Under the classical Navier slip condition ($m = 2$) we then have $\phi_b(h) = \frac{1}{2}\, h^2 + bh$ such that the second term reads as $(h + b)\, \vec{\tau} \cdot \nabla_y h$. Hence, equation (1.3.1) turns into a nonlinear wave equation of first order whose solutions describe waves that propagate in direction of the "wind stress" $\vec{\tau}$; and as there is no surface tension present, their amplitude increases with the elapse of time. See [63, 72] for more details. But rather than considering (1.3.1), we will completely neglect both intermolecular ($\phi_b$) and external forces ($\vec{\tau}$ and $Q$), and restrict ourselves to the model problem (LE) or, as a matter of fact, to (TFE).

The mathematical analysis of the Cauchy problem for (TFE) on some domain $(0, T) \times \Omega$ in the case of complete wetting started with the paper [5] by Bernis and Friedman. In one space dimension they succeed to prove existence and positivity of weak solutions for (LE) for all values $m \geq 1$. Aside from conservation of mass, their findings are essentially built upon the so-called energy identity given by

$$\partial_s \int_0^T \|\nabla h(s)\|^2_{L^2(\Omega)}\, ds \;=\; -\int_0^T \|u^{\frac{m}{2}}\, \nabla \Delta h(s)\|^2_{L^2(\Omega)}\, ds\,,$$

and nonlinear integral estimates, called entropy estimates, which they derive from an identity of the form

$$c(m)\, \partial_s \int_0^T \int_\Omega u(s, y)^{2-m}\, dy\, ds \;=\; \int_0^T \|\Delta u(s)\|^2_{L^2(\Omega)}\,. \tag{1.3.2}$$

Integral estimates from a local version of this entropy identity in space dimensions $n = 2, 3$ are derived in [9]. In this paper, the authors basically follow the ideas presented in [16], where a global version of (1.3.2) for arbitrary $n$ is attained. In both papers these results are used to show existence of weak solutions in the parameter regime $m \in (\frac{1}{8}\,, 3)$. However, they fail to prove that solutions have a finite speed of propagation for $m \geq 2$. In [38] one can find a different approach in which an interpolation inequality helps to generalize the energy inequality in such a way that the Cauchy problem for compactly supported nonnegative initial data can be solved for $m \in [2, 3)$. Moreover, the new energy estimate is the key to proving other results on the qualitative behavior of weak solutions in that regime such as finite speed of propagation. But even though there has been tremendous progress in the study of equations of the form of (LE) during recent years, uniqueness of weak solutions still remains an open problem.

The regime of partial wetting has been studied by Bertsch, Giacomelli and Karali [10]. They reach for a similar existence result by slightly modifying the notion of solution: When the system of liquid, solid and vapor comes to rest, the three interfacial energies balance and an equilibrium contact angle $\theta_e$ appears. This angle is determined by the well known Young's law (for a full explanation see [61]),

$$\cos\theta_e \;=\; \frac{\gamma_{sv}-\gamma_{ls}}{\gamma_{lv}}\,.$$

Here $\gamma_{lv}$ is the surface energy/tension between the liquid and the vapor, $\gamma_{ls}$ and $\gamma_{sv}$ are the energies at the liquid-solid interface and the solid-vapor interface, respectively. These three energies tend to concentrate at the triple junction where liquid, solid and vapor meet, which in turn justifies to fix the dynamic contact angle to be equal to the equilibrium contact angle $\theta_e$. In the lubrication regime, this leads to an energy functional $E_{\theta_e}(h)$ and then to a notion of a weak solution of (LE). It is possible to show existence if such a solution satisfies the dissipation (or entropy) inequality

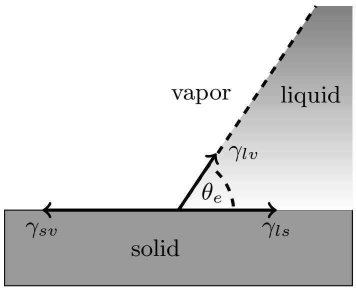


*Figure 1.3: Young's diagram*

$$\partial_s\!\int E_{\theta_e}\big(h(s)\big)\,ds \;\le\; -\int \|\sqrt{h^3+h^m}\;\nabla\Delta h(s)\|^2_{L^2(\Omega)}\,ds\,.$$

The theory for classical solutions is restricted to the 1-dimensional case so far. This problem is addressed in [37, 49], where the authors pursue the problem of obtaining unique solutions to (TFE) of maximal regularity, exclusively for $m = 1$ (flow in the Hele-Shaw cell). Well-posedness under the classical Navier slip condition in the case of partial wetting is discussed in [48]. As the qualitative behavior of solutions is strongly dependent on the mobility parameter $m \in (0,3)$, it might be interesting as well to consider other values of $m$. We expect the results to be the same, but well-posedness in the context of classical solutions has not yet been proven.

Special solutions play an important role in the study of parabolic equations of the type (1.1.1), including both (PME) and (TFE). A prominent example of such solutions are *source type solutions*. These are explicit nonnegative solutions that converge to a multiple of the delta function as time approaches initial time 0 and exhibit a self-similar behavior while conserving their mass. In [4] the authors investigate such solutions of (TFE) in dimension $n \ge 2$. They use Green functions and a regularization to show that, for all values of $m \in (0,3)$, there exists a unique nonnegative solution $h \in C^\infty$ at $y = 0$.[1] More precisely, by making the ansatz that $h(s,y)$ is of the form $s^{-n\alpha}u\big(|y|/s^\alpha\big)$ the source solution appears as the unique solution of a third order ODE for $u(r)$. There is no explicit formula for this solution known for other values of $m$ than 1.[2] However, its asymptotic behavior at the interface can be established in terms of the mobility parameter $m$. Indeed, if $a > 0$ is the number for which $u(r) > 0$ on $(-a,a)$ and 0 elsewhere, then there exists a positive constant $c$ such that

$$u(r) \;\sim\; c\,(a-r)^\beta \qquad \text{as} \qquad r \to a^-$$

with $\beta = 2$ for $0 < m < \frac{3}{2}$ and $\beta = \frac{3}{m}$ for $\frac{3}{2} < m < 3$. Thus the source solution exhibits an interface at $r = r(s) = a\,s^\alpha$, where $\alpha = \frac{1}{4+mn}$, which also implies a finite speed of propagation. The 1-dimensional case is treated in [6, 44]. There also exist *traveling wave solutions* of the form $h(s,y) = H(y - vs)$ with

[1] They also proved that such solutions cease to exist for $m \ge 3$.
[2] In dimension $n = 1$ the explicit formula has been found by Hill and Smyth [42].

constant velocity $v$ (see e.g. [11]). For $\frac{3}{2} < m < 3$, these have the simple form

$$h(s,y) \;=\; \begin{cases} c\,(y - vs)^{\frac{3}{m}} & y > vs \\ 0 & \text{otherwise}\,, \end{cases} \qquad v \;=\; \Big(\frac{3}{m} - 2\Big)\Big(\frac{3}{m} - 1\Big)\,\frac{3}{m}\,c^m\,.$$

Finally, we note that there are also *stationary solutions*

$$h(s,y) \;=\; h_{st}(y) \;=\; \begin{cases} c_1 - c_2\,|y|^2 & |y| < \frac{c_1}{\sqrt{c_2}} \\ 0 & \text{otherwise} \end{cases}$$

for all values of $m$.

## 1.4 Organization and Results of the Thesis

In chapter 2 we deal with weighted Sobolev spaces, which will be denoted by $W^{k,p}(\Omega,\omega)$, where $k$ is a nonnegative integer, $p \geq 1$ is a real number, $\Omega \subseteq \mathbb{R}^n$ is an open set, and $\omega$ is a nonnegative (continuous) weight function defined on $\Omega$. We say that the weight degenerates at the boundary $\partial\Omega$ if $\omega(x) \to 0$ for $x \to x_0 \in \partial\Omega$.

Using the notation

$$\langle u, v\rangle_{L^2(\Omega,\omega)} \;=\; \int_\Omega u(x)\,v(x)\,\omega(x)\,dx\,,$$

the space $W^{k,2}(\Omega,\omega)$ consists of all real-valued functions $u$ whose distributional derivatives of orders $|\alpha| \leq k$ satisfy

$$\|\partial^\alpha u\|^2_{L^2(\Omega,\omega)} \;=\; \langle \partial^\alpha u, \partial^\alpha u\rangle_{L^2(\Omega,\omega)} \;<\; \infty\,.$$

Weighted Sobolev spaces provide a wide range of applications in the theory of partial differential equations. We illustrate their usefulness by a simple example. Let us consider the elliptic linear equation

$$Lu \;=\; -\,\nabla\cdot(\omega\,\nabla u) \;=\; f \qquad \text{on} \quad \Omega\,, \tag{1.4.1}$$

and let us examine the homogeneous Dirichlet problem for this equation, that is, we impose the boundary condition

$$u\big|_{\partial\Omega} \;=\; 0\,.$$

Testing the equation with $u$ and applying integration by parts, we obtain the so-called energy identity

$$\|\nabla u\|^2_{L^2(\Omega,\omega)} \;=\; \langle f, u\rangle_{L^2(\Omega)} \;=\; \int_\Omega f(x)\,u(x)\,dx\,,$$

which is associated with the elliptic boundary value problem for the equation $Lu = f$. It is this identity which often serves as a starting point of the theory of weak (variational) solutions. Therefore, the weighted spaces provide an opportunity to investigate equations of the above kind by functional-analytical methods.

In chapter 3 we consider the Cauchy problem for the thin-film equation (TFE). We search for solutions with prescribed $|\nabla_y h|$ at the free contact line. Much in the spirit of [51], we transform and linearize both the equation and the geometry resulting in a linear equation for a perturbation $u$ of a stationary solution. More precisely, fixing a point $(s_0, y_0)$ in $spt\,h$ we assume that $\partial_{y_n} h(s_0, y_0) > 0$. Then we can solve the equation $z = h(s,y)$ locally with respect to $y_n$ giving rise to a function

$$y_n \;=\; v(s, y', z)\,.$$

We suppose that $h(0) \sim \frac{1}{4}\,(y_n)_+^2$ and set $\tilde{h} = \sqrt{h}$, where we consider $h$ as a solution of (TFE). The equation then reads

$$\partial_s \tilde{h}^2 + \nabla\cdot\big(\tilde{h}^{2m}\,\nabla\Delta\tilde{h}^2\big) = 0\,.$$

Now we set

$$(s,y) \mapsto \big(s,y',\tilde{h}(s,y)\big) =: (t,x)$$

interchanging the roles of the independent variable $y_n$ and the dependent variable $\tilde{h}$. This leads to an equation for the now dependent variable $v = v(t,x)$. At the same time, the stationary solution $\tilde{h}_{st}(y) = \frac{1}{2}\,(y_n)_+$, defined on its positivity set $P(\tilde{h}_{st}) = P_0(\tilde{h}_{st})$, is transformed into $v_{st}(x) = \frac{1}{2}\,x_n$ on a subset of the fixed domain $H = \{x_n > 0\}$. We assume that any solution $v$ is a small perturbation of $v_{st}$ by $u$. This implies that $v = v_{st} + u$ solves the transformed equation if and only if $u$ satisfies the equation

$$x_n\,\partial_t u + \nabla\cdot\big(x_n^{2m+1}\,\nabla\Delta u\big) + 3\,x_n^{2m}\,\partial_{x_n}\Delta u + 2\,m\,x_n^{2m-1}\big(\Delta u + 2\,\partial^2_{x_n} u\big) = x_n\,f[u]\,.$$

All the nonlinear terms are collected in the inhomogeneity $f[u]$ - its precise form will be discussed in section 3.3. For simplicity we only consider the model which undergoes a "linear slip" ($m = 1$) and the equation simplifies to

$$x_n\,(\partial_t + L_0)u = x_n\,f[u] \tag{1.4.2}$$

with linear spatial part

$$x_n\,L_0 u := \Delta\big(x_n^3\,\Delta u\big) - 4\,x_n\,\Delta' u\,.$$

The spatial part of the linear operator induces a Riemannian metric on the fixed domain $H$, giving us the ability to measure the length of curves. The geodesics can be computed. Extended to the boundary $\partial H$, this gives rise to an intrinsic metric (Carnot-Caratheodory metric), denoted by $d$, on the closed half space $\overline{H}$. The natural measure is $\mu_\sigma := x_n^\sigma\,dx$ for $\sigma > -1$. It satisfies a doubling condition with respect to $d$, i.e. for each intrinsic ball $B_R(x)$ there exist constants $c, b \geq 1$ such that

$$0 < \mu_\sigma\big(B_{cR}(x)\big) \leq b\,\mu_\sigma\big(B_R(x)\big) < \infty\,.$$

Hence the metric measure spaces $\big(\overline{H}, d, \mu_\sigma\big)$ and, taking into account time, $\big(\mathbb{R}\times\overline{H}, d^{(t)}, \mathcal{L}\times\mu_\sigma\big)$, where

$$d^{(t)}\big((t,x),(s,y)\big) = \sqrt[4]{|t-s| + d(x,y)^4}\,,$$

define spaces of homogeneous type. In this setting we search for local, in particular pointwise estimates for solutions of the linear equation and from this we can derive a Gaussian estimate for a general solution, called Green function, and its derivatives in terms of the intrinsic metric.

Chapter 4 is dedicated to the analysis of the linear equation $x_n(\partial_t + L_0)u = x_n f$ and addresses issues regarding existence and uniqueness of weak solutions for this problem. To formulate a definition of a weak solution, we follow the example (1.4.1) and multiply the equation by a suitable test function $\varphi \in C_c^\infty(\Omega)$, $\Omega \subset \overline{H}$ relatively open, integrate the result over $\Omega$ and then integrate by parts to get

$$\big\langle \partial_t u(t), \varphi\big\rangle_{L^2(\Omega,x_n)} + a\big(u(t),\varphi\big) = \big\langle f(t),\varphi\big\rangle_{L^2(\Omega,x_n)}$$

for any $0 \leq t \leq T$, where

$$a\big(u(t),\varphi\big) = \int_\Omega \Delta u(t)\,\Delta\varphi\,x_n^3\,dx + 4\int_\Omega \nabla' u(t)\cdot\nabla'\varphi\,x_n\,dx$$

is the bilinear form associated with $x_n\,L_0$. In order to achieve meaningful use of these expressions, there are two points about it that deserve further comment. First, all the derivatives are to be understood as regular distributional derivatives. Second, as our particular interest is the behavior of $u$ towards the boundary of $H$, we need to adjust the test function space in such a way as to allow values on $\partial H$.

A solution is constructed by a Galerkin approximation. The basic idea of such an existence proof is to approximate $u : [0,T] \to X$ (here $X$ is a weighted Hilbert space) by functions $u_N \in C\big([0,T]; V_N\big)$ which take values in finite dimensional spaces $V_N$. To obtain the functions $u_N$, we project the PDE onto $V_N$. The problem reduces to an ODE which can be solved by standard ODE theory. Each of the $u_N$ satisfies an a priori estimate for solutions of the PDE, the so-called energy estimate, and so we can pass to the limit $N \to \infty$ to obtain a general solution. Uniqueness of weak solutions of the Cauchy problem is now a direct consequence of the continuity in time. Moreover, other energy estimates follow from an integration of the linear equation by using different test functions.

In the final section of chapter 4 we establish a variety of local results culminating in the pointwise estimate

$$\big|u(t,x)\big| \;\lesssim\; \mu_1\big(B_{\sqrt[4]{t}}(x)\big)^{-\frac{1}{2}}\, e^{\,c_n c_L^{\,4} t - \Psi(x)}\, \|e^{\Psi}\, u(0)\|_{L^2(H,\mu_1)}\,.$$

Here, $\Psi$ is a Lipschitz function with Lipschitz constant $c_L$. Optimizing the estimate with respect to $\Psi$ and using duality, now leads to one of our main results, the following Gaussian estimate: Let the kernel distribution $G(t,x,s,y)$ represent the Green function for the Cauchy problem on $[0,T] \times \overline{H}$, that is,

$$u(t,x) \;=\; \int_H G(t,x,0,y)\, g(y)\, dy$$

satisfies $x_n(\partial_t + L_0)u = 0$ on $[0,T] \times \overline{H}$ subject to $u(0) = g$. Then there exists a constant $c(n) > 0$ such that

$$\big|\partial_t^l \partial_x^\alpha\, G(t,x,s,y)\big| \;\le\; c(n,l,\alpha)\, \frac{\delta_{l,\alpha}\big(\sqrt[4]{t-s},x\big)}{\mu_1\big(B_{\sqrt[4]{t-s}}(x)\big)}\; y_n\, e^{-c_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \tag{1.4.3}$$

for any $0 \le s < t$, where $\delta_{l,\alpha}(R,x) \,=\, R^{-4l-|\alpha|}\,(R+\sqrt{x_n}\,)^{-|\alpha|}$ denotes the derivative coefficient. The approach to (1.4.3) follows an idea of Fabes and Stroock [28], but now for a degenerate parabolic problem instead of the non-degenerate case with measurable coefficients. A full proof is provided in chapter 5. In the further course of this chapter, we discuss several useful consequences of the Gaussian estimate. For example, it can be used to show that the maps with kernel

$$y_n^{-1}\,\partial_t G(t,x,s,y)\,, \quad y_n^{-1}\, D_x^2 G(t,x,s,y)\,, \quad y_n^{-1}\, x_n\, D_x^3 G(t,x,s,y) \quad \text{and} \quad y_n^{-1}\, x_n^{\,2}\, D_x^4 G(t,x,s,y)\,,$$

respectively, define singular integral operators from $L^2(x_n)$ into $L^2(x_n)$. Due to Calderón-Zygmund theory (see appendix A), the operator is then also bounded on all $L^p(x_n)$, $p \in (1,\infty)$. Now, the theory of Muckenhoupt weights provides us with a tool to dispense with the weight. We get boundedness of the operators on unweighted $L^p$-spaces, that is, the inequality

$$\|\partial_t u\|_{L^p} \,+\, \|D_x^2 u\|_{L^p} \,+\, \|x_n\, D_x^3 u\|_{L^p} \,+\, \|x_n^{\,2}\, D_x^4 u\|_{L^p} \;\lesssim\; \|f\|_{L^p} \tag{1.4.4}$$

holds for all $p \in (1,\infty)$ provided $u$ solves the inhomogeneous equation with vanishing Cauchy data.

Now consider the equation (1.4.2) with nonlinearity $f[u]$ and initial data in an appropriate space: We assume that the initial data has finite Lipschitz norm $\|g\|_{\dot C^{0,1}(H)} = \|\nabla g\|_{L^\infty(H)}$. As a byproduct of the linear theory, we can also produce a local estimate of the form

$$\|\nabla u(t)\|_{L^\infty(H)} \,+\, \big|Q_R(x)\big|^{-\frac{1}{p}}\, R^{\,4l+|\alpha|-1}\,\big(R+\sqrt{x_n}\big)^{|\alpha|-2j-1}\, \|x_n^{\,j}\,\partial_t^l\partial_x^\alpha\, u\|_{L^p(Q_R(x))} \;\lesssim\; \|g\|_{\dot C^{0,1}(H)}\,.$$

In view of inequality (1.4.4), this suggests to take $\big(j,l,|\alpha|\big) \in \big\{(0,1,0),(0,0,2),(1,0,3),(2,0,4)\big\}$ and to introduce a new norm, denoted by $X_p$, based on time-space cylinders $Q_R(x) := \big(\frac{R^4}{2}, R^4\big] \times B_R(x)$ which are bounded away from initial time 0. Defining the $Y_p$ norm as the set of all functions $f$ for which

$$\sup_{R^4\in(0,T)} \sup_{x\in H} \left|Q_R(x)\right|^{-\frac{1}{p}} R^3 \left(R+\sqrt{x_n}\right)^{-1} \|f\|_{L^p(Q_R(x))} < \infty\,,$$

we prove as intermediate step

$$\|u\|_{X_p} \lesssim \|f[u]\|_{Y_p}\,.$$

Consequently,

$$\|u\|_{X_p} \lesssim \|f[u]\|_{Y_p} + \|g\|_{\dot{C}^{0,1}(H)}$$

by Duhamel's principle. The next step is to apply a contraction mapping argument. In order to do so, one needs to impose additional requirements concerning the nonlinearity $f[u]$, the maximal time of existence $T$ or the initial data $g$, at least one of which needs to be small. However, if $\|u\|_{X_p}$ is small enough, then

$$\|f[u]\|_{Y_p} \lesssim \|u\|_{X_p}^2\,,$$

and a similar estimate holds for the difference $f[u_1]-f[u_2]$ provided $\|u_1\|_{X_p}$ and $\|u_2\|_{X_p}$ are sufficiently small. We conclude that there exist $\varepsilon, c>0$ such that for every $g\in \dot{C}^{0,1}(H)$ satisfying $\|g\|_{\dot{C}^{0,1}(H)}<\varepsilon$ there exists a solution $u^*\in X_p$ of the perturbation equation (1.4.2). Moreover, this solution is unique in the ball $B^X_{c\varepsilon}=\left\{u\in X_p \mid \|u\|_{X_p}\le c\,\varepsilon\right\}$. This implies large time stability of solutions $v$ that are initially close to the stationary solution $v_{st}=x_n$.
This approach originates in work by Koch and Tataru [55] and was subsequently developed further by Koch and Lamm in their 2012 paper [54] on geometric flows with rough initial data. In the spirit of these works we reach global existence and uniqueness for the perturbation of the stationary solution, a possibly optimal result in terms of the regularity of the initial data. Moreover, we use an idea of Koch and Lamm [54] to obtain analyticity of solutions in time and all tangential directions up to the boundary of its support. Analyticity in vertical direction, i.e. the $x_n$-direction, is still an open problem.

In a last step, we can use the unique solution $u^*$, or rather $v^*=v_{st}+u^*$, to generate a solution $h$ of the thin-film equation on its positivity set $P(h)=\left\{(s,y)\in(0,T)\times\mathbb{R}^n \mid h(s,y)>0\right\}$. For this $h$, the identity

$$\int_I\int_{\mathbb{R}^n} h\,\partial_s\varphi + h\,\nabla\Delta h\cdot\nabla\varphi\,dyds = 0 \tag{1.4.5}$$

holds for all $\varphi\in C_c^\infty((0,T)\times\mathbb{R}^n)$, that is, it is a weak solution of (TFE). In fact, the solution obtained in this manner is unique and we have that

$$[\sqrt{h}]_{X_p^1} = \sup_{\substack{s\in(0,T)\\ y\in P_s(h)}} \left|Q_{\sqrt[4]{s}}(y)\right|^{-\frac{1}{p}} \sum_{(j,l,\alpha)\in\mathcal{CZ}} \sqrt[4]{s}^{\,4l+|\alpha|-1}\left(\sqrt[4]{s}+\sqrt[4]{h_0(y)}\right)^{|\alpha|-2j-1} \|\sqrt{h}^{\,j}\partial_s^l\partial_y^\alpha\sqrt{h}\|_{L^p(Q_{\sqrt[4]{s}}(y))}$$

is finite, where $(j,l,\alpha)\in\mathcal{CZ}$ means that $j,l$ and $|\alpha|$ are admissible in the above sense.

**Main Result:** *(See theorem 5.2.15.) Suppose $T>0$ and $\varepsilon>0$ small. Given an initial datum $h(0)=h_0$ with*

$$\left|\nabla_y\sqrt{h_0(y)}-e_n\right| < \varepsilon\,,$$

*there exist a constant $c>0$ and a unique weak solution $h^*\in C((0,T)\times\mathbb{R}^n)$ of (TFE) with initial value $h_0$,*

$$\|\nabla_y\sqrt{h^*}-e_n\|_{L^\infty(P(h))} + \left[\sqrt{h^*}\right]_{X_p^1} \le c\,\varepsilon\,,$$

*and $h^*$ satisfies the equation (1.4.5). Moreover, the level sets of $h^*$ are analytic.*

# Chapter 2

# Weighted Sobolev Spaces

Here we introduce the ideas of weighted Sobolev spaces and establish the basic notation and terminology that is needed in the chapters to come. One of the central theorems will be the approximation theorem at the end of section 2.5. It will enable us to extend several embedding results to weighted function spaces. Such embeddings will occupy the remainder of this chapter.

## 2.1 The Half Space and Topology

First of all we fix an integer $n \geq 1$. By $\mathbb{R}^n$ we denote the $n$-dimensional real Euclidean space equipped with the usual topology. A typical element in $\mathbb{R}^n$ is the point $x = (x_1, \ldots, x_n)$ and sometimes we write $x = (x', x_n)$ for $x' \in \mathbb{R}^{n-1}$. The (open) upper half space $H$ is then the upper part into which the hyperplane $\{x \in \mathbb{R}^n \mid x_n = 0\}$ divides $\mathbb{R}^n$, i.e.

$$H \;=\; \{x_n > 0\} \;:=\; \{x \in \mathbb{R}^n \mid x_n > 0\}\,.$$

In particular, $\overline{H}$ denotes the closed upper half space $\{x_n \geq 0\}$. Their boundary is given by the hyperplane $\partial H = \partial \overline{H} = \{x_n = 0\}$. Note that the set $\overline{H}$ as a subset of $\mathbb{R}^n$ is closed with respect to the Euclidean topology, whereas taken as a subspace by itself it defines a new topological space equipped with the induced topology. For example, by saying $M$ is open as a subset of $\overline{H}$, we mean $M$ is relatively open in $\overline{H}$ but not necessarily open in $\mathbb{R}^n$. In particular, the (relatively) open subsets of $\overline{H}$ can have a nonempty intersection with $\partial H$.

Now suppose $X$ is some vector space provided with a metric $d : X \times X \to [0, \infty)$. The open set

$$B_R(x) \;=\; B_R(x; d) \;:=\; \big\{y \in X \;\big|\; d(x, y) < R\big\}$$

is called the $d$-ball with radius $R > 0$ and center $x \in X$. If $X \subseteq \mathbb{R}^n$ and $d_{eu}(x, y) = |x - y|$, then we write $B_R^{eu}(x) := B_R(x; d_{eu})$ to mean the Euclidean ball of radius $R$ and center at $x$. Moreover, let $C_R(x)$ be the cube of edge length $R$ centered at $x$ whose edges are parallel to the coordinate axes, that is

$$C_R(x) \;=\; \big\{y \in \mathbb{R}^n \mid \max_{i=1,\ldots,n} |x_i - y_i| < R\big\}\,.$$

We note further that we have the following inclusion relation between the cube and the Euclidean ball:

$$C_{\frac{R}{\sqrt{n}}}(x) \;\subset\; B_R^{eu}(x) \;\subset\; C_R(x)\,.$$

Throughout $x$ represents space, while $t$ always denotes time. A time interval $I$ is determined by its endpoints $t_1 < t_2$, where we also allow $t_1 = -\infty$ and $t_2 = \infty$. Then the closure is to be understood in the Euclidean topology, i.e. $\bar{I} = [t_1\,, t_2]$ if $I$ is bounded, $\bar{I} = (-\infty, t_2]$ if $t_1 = -\infty$ and $t_2$ is finite, $\bar{I} = [t_1\,, \infty)$ if $-\infty < t_1 < t_2 = \infty$ and $\bar{I} = \mathbb{R}$ if $I = \mathbb{R}$.

## 2.2 Spaces of Continuous Functions

For $k \in \mathbb{N}_0 \cup \{\infty\}$ and an arbitrary subset $M$ of $\mathbb{R}^n$ endowed with its induced topology, we refer to

$$C_c^k(M) \;:=\; \big\{\varphi : M \to \mathbb{R} \,\big|\, spt\,\varphi \text{ is compact}, \varphi \text{ can be extended to } \breve{\varphi} \in C_c^k(\mathbb{R}^n)\big\}$$

as the space of $k$ times continuously differentiable functions with compact support in $M$. This definition is made such that it is consistent with the usual notion of the function space $C_c^k(M)$ whenever it is defined, that is, if $k = 0$ and $M$ is either open or closed, or if $k \geq 1$ and $M$ is open. To see this, let first $k \in \mathbb{N}_0 \cup \{\infty\}$ be arbitrary, $M \subseteq \mathbb{R}^n$ open and $\varphi \in C_c^k(M)$. Then, $spt\,\varphi$ is a proper subset of $M$ and $\varphi$ can be extended by 0 to all of $\mathbb{R}^n$. On the other hand, if $k = 0$ and $M$ is closed, we can use the Tietze extension theorem to obtain a continuous mapping $\breve{\varphi} \in C(\mathbb{R}^n)$ such that $\breve{\varphi}\big|_M = \varphi$. Note that in this case, the support of $\breve{\varphi}$ possibly exceeds $M$, whereas it still can be chosen as a compact set in $\mathbb{R}^n$. These considerations show that the above definition generalizes the notion of continuous differentiability to functions defined on arbitrary sets in the Euclidean space $\mathbb{R}^n$.

We can interpret $C_c^k(M)$ as a subspace of the function space $C^k(M)$ consisting of all functions $\varphi$ which, together with all their derivatives up to order $k$, are continuous on $M$. Here differentiation on non-open sets is to be carried out by means of extension according to the above definition. Moreover, let

$$C^\infty(M) \;=\; \bigcap_{k=0}^{\infty} C^k(M)\,.$$

Let $\Omega$ be an open subset of $\mathbb{R}^n$. By equipping the linear space $C_c^\infty(\Omega)$ with a suitable family of seminorms it is topologized to become the locally convex topological vector space $\mathcal{D}(\Omega)$ called the space of test functions. Its dual space is then, accordingly, denoted by $\mathcal{D}'(\Omega)$ and its elements are referred to as distributions or generalized functions. The theory of distributions makes it possible to differentiate any $u \in \mathcal{D}'(\Omega)$ even if the derivative does not exist in the classical sense. Let us illustrate this with the help of a standard example. Suppose that $u : \Omega \to \mathbb{R}$ is a locally integrable function with respect to the $n$-dimensional Lebesgue measure $\mathcal{L}^n$. The corresponding distribution is then defined by the linear mapping

$$T_u(\varphi) \;:=\; \int_\Omega u\,\varphi\,d\mathcal{L}^n \qquad \forall\,\varphi \in C_c^\infty(\Omega)\,.$$

Since $u \mapsto T_u$ is injective, we may identify $u \in L^1_{loc}(\Omega, \mathcal{L}^n)$ and $T_u \in \mathcal{D}'(\Omega)$. In particular, we can understand any $u \in L^p(\Omega, \mathcal{L}^n) \subset L^1_{loc}(\Omega, \mathcal{L}^n)$ for $1 \leq p \leq \infty$ as a distribution and therefore differentiate it infinitely many times. The distributional derivatives are given by

$$\partial_x^\alpha\, u(\varphi) \;=\; (-1)^{|\alpha|} \int_\Omega u\,\partial_x^\alpha \varphi\,d\mathcal{L}^n\,.$$

When defining general Sobolev spaces, this weak notion of differentiability will play a central role.

## 2.3 Spaces of $\mu$-Integrable Functions

Let $(X, \Sigma, \mu)$ be a measure space and $p$ be a real number satisfying $1 \le p < \infty$. Consider the set of all $\mu$-measurable functions $u$ whose modulus to the power of $p$ has finite integral, or equivalently, that

$$\|u\|_{L^p(X,\mu)} \;=\; \Big(\int_X |u|\, d\mu\Big)^{\frac{1}{p}} \;<\; \infty\,.$$

Minkowski's inequality tells us the triangle inequality holds for $\|\cdot\|_{L^p}$ endowing the set of $p$-th power integrable functions with a seminorm. In these spaces we generally deal with equivalence classes of functions, rather than with individual functions, where two functions are identified if they differ only on a set of measure zero. The reason to regard functions that are almost everywhere equal is so that $\|u\|_{L^p} = 0$ implies that $u = 0$. This makes the seminormed spaces into normed vector spaces denoted by $L^p(X, \mu)$. We write $u \in L^\infty(X, \mu)$ if $u$ is essentially bounded on $X$, that is for $\mu$-almost all $x \in X$. The space $L^p(X, \mu)$ is complete in the $L^p$-norm for all $1 \le p \le \infty$ and hence a Banach space (Riesz-Fischer theorem). If $1 < p < \infty$, then the dual space of $L^p(X, \mu)$ has an isomorphism with $L^{\frac{p}{p-1}}(X, \mu)$ which associates a function $v \in L^{\frac{p}{p-1}}(X, \mu)$ with the linear functional

$$J_g:\; L^p(X,\mu) \ni u \;\mapsto\; \int_X u\, v\, d\mu\,.$$

This allows us to identify those two spaces in a natural way. If $X$ is the countable union of sets with finite $\mu$-measure, then the dual space of $L^1(X, \mu)$ is naturally identified with $L^\infty(X, \mu)$.

By $L^p_{loc}(X, \mu)$ we mean the set of all locally integrable functions, i.e. the $\mu$-integral is finite on all compact subsets $K$ of $X$. An example that is of particular interest for us is provided by the function $\omega : x \mapsto |x_n|^\sigma$. We have $\omega \in L^1_{loc}(\mathbb{R}^n, \mathcal{L}^n)$ if and only if $\sigma > -1$, and hence by

$$|M|_\sigma \;=\; \mu_\sigma(M) \;=\; \int_M |x_n|^\sigma\, d\mathcal{L}^n(x) \;=\; \int_M |x_n|^\sigma\, dx$$

a Radon measure is canonically associated with the weight $\omega$, $d\mu_\sigma = \omega\, d\mathcal{L}^n$. We write $|M| = |M|_0$ to denote the Lebesgue measure of the set $M$.

**Lemma 2.3.1** *If $\sigma > -1$, then $L^\infty(\mathbb{R}^n, \mu_\sigma) = L^\infty(\mathbb{R}^n, \mathcal{L}^n)$. Moreover, the dual space of $L^1(\mathbb{R}^n, \mu_\sigma)$ has a natural isomorphism with $L^\infty(\mathbb{R}^n)$.*

**Proof:** We show that $\mu_\sigma$ and the Lebesgue measure $\mathcal{L}^n$ are mutually absolutely continuous. We say $\mu$ is absolutely continuous with respect to $\nu$, and write $\mu \ll \nu$, if every $\nu$-null set is also $\mu$-null.

$\mu_\sigma \ll \mathcal{L}^n$ : This is an immediate consequence of the fact that $|x_n|^\sigma \in L^1_{loc}(\mathbb{R}^n, \mathcal{L}^n)$.

$\mathcal{L}^n \ll \mu_\sigma$ : Let $|M|_\sigma = 0$ and suppose that $|M| > 0$. Then either $|M \cap H| > 0$ or $\big|M \cap (-H)\big| > 0$. Assuming, without loss of generality, the former case to hold, there exist $x_0 \in H$ and $R > 0$ such that $B^{eu}_R(x_0) \subset H$ and $\big|M \cap B^{eu}_R(x_0)\big| > 0$. This implies

$$|M|_\sigma \;\ge\; \big|M \cap B^{eu}_R(x_0)\big|_\sigma \;\ge\; \min\big\{(x_{0,n} - R)^\sigma, (x_{0,n} + R)^\sigma\big\}\, \big|M \cap B^{eu}_R(x_0)\big| \;>\; 0\,.$$

Thus, by contradiction, $M$ is a $\mathcal{L}^n$-null set.

Given any lattice point $x_0 \in \mathbb{Z}^n$, the volume of the cube $C_1(x_0)$ with respect to the measure $\mu_\sigma$ becomes

$$\big|C_1(x_0)\big|_\sigma \;=\; \frac{2^{n-1}}{\sigma+1} \begin{cases} 2 & \text{if } x_{0,n} = 0 \\ \Big(\big(|x_{0,n}| + 1\big)^{\sigma+1} - \big(|x_{0,n}| - 1\big)^{\sigma+1}\Big) & \text{otherwise}\,. \end{cases}$$

Moreover,

$$\mathbb{R}^n = \bigcup_{x_0 \in \mathbb{Z}^n} C_1(x_0)\,, \qquad \text{and} \qquad \big|C_1(x_0)\big|_\sigma = c(n, \sigma, x_{0,n}) < \infty$$

for all $x_0 \in \mathbb{Z}^n$. But this means that $\mathbb{R}^n$ is the countable union of measurable sets with finite $\mu_\sigma$-measure and the second part of the lemma follows. ∎

Now if $M \subseteq X = \mathbb{R}^n$ satisfies

$$\overline{\mathring{M}} = \overline{M}\,,$$

then $M$ is either open or closed. Its boundary is a $\mathcal{L}^n$-null set and accordingly also a $\mu_\sigma$-null set. Thus

$$L^p(M, \mu_\sigma) = L^p(\mathring{M}, \mu_\sigma) = L^p(\overline{M}, \mu_\sigma) \qquad \forall\, 1 \le p \le \infty\,.$$

Also, note that $L^p_{loc}(\overline{M}, \mu_\sigma) \subsetneq L^p_{loc}(\mathring{M}, \mu_\sigma)$ because the compact subsets of $\mathring{M}$ do not cover all the compact subsets of $\overline{M}$. In fact, for any Radon measure $\mu$ and any $\mu$-measurable set $M \subseteq \mathbb{R}^n$, the following embeddings are continuous:

$$L^p(M, \mu) \subset L^1_{loc}(\overline{M}, \mu) \subset L^1_{loc}(M, \mu) \qquad \forall\, 1 \le p \le \infty\,.$$

Mostly, we will consider the case $M = H$ or $M = \overline{H}$. Then we sometimes use the notation $L^p(\mu)$ instead of $L^p(H, \mu)$, and accordingly we write $\|\cdot\|_{L^p(\mu)}$ for $\|\cdot\|_{L^p(H,\,\mu)}$. On the other hand, whenever we leave the "global" setting to achieve local results, we emphasize the set that is underlying our analysis.

Another important function space includes a time component $t \in I \subseteq \mathbb{R}$. The assigned measure on $I \times M$ will be the product measure $\mathcal{L} \times \mu_\sigma$. A measurable function $u : I \times M \to \mathbb{R}$ is said to belong to $L^q\big(I; L^p(M, \mu_\sigma)\big)$ if

$$\int_I \|u(t, \cdot)\|^q_{L^p(M,\,\mu_\sigma)}\, dt < \infty\,,$$

where $1 \le q, p < \infty$, with the common alterations for $q, p = \infty$. This extends the definition of Lebesgue integrability to functions that take values in the Banach space $L^p(M, \mu_\sigma)$ (see Bochner integrability).

## 2.4 Classical versus Weak Derivatives

An $n$-dimensional vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each entry is a nonnegative integer, is called a multi-index. Given a multi-index $\alpha$, denote by $x^\alpha$ the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Similarly, $\partial^\alpha_x = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_n}_{x_n}$ defines a partial differential operator of order $|\alpha|$. Sometimes we drop the subscripted $x$ from the notation and merely write $\partial^\alpha = \partial^\alpha_x$. The factorial of a multi-index $\alpha$ is defined by $\alpha! = \alpha_1! \ldots \alpha_n!$. If $\beta$ is another multi-index, we write $\beta \le \alpha$ provided $\beta_i \le \alpha_i$ for $i = 1, \ldots, n$. In this case $\alpha - \beta$ is also a multi-index and we can define the binomial coefficient

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!\,(\alpha - \beta)!}\,.$$

With the multi-index notation there is a Leibniz formula available,

$$\partial^\alpha_x \big(u\, v\big)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta_x u(x)\, \partial^{\alpha-\beta}_x v(x)$$

valid for all functions $u, v$ that are at least $|\alpha|$ times differentiable at $x$. If $k$ is a nonnegative integer, we set

$$D^k_x u(x) := \big\{ \partial^\alpha_x u(x) \,\big|\, |\alpha| = k \big\}\,.$$

The gradient of $u$ is denoted

$$\nabla_x u = \begin{pmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_n} u \end{pmatrix}$$

and its length is given by $|\nabla_x u| = \Big(\sum_{i=1}^n (\partial_{x_i} u)^2\Big)^{\frac{1}{2}}$. For the Laplacian of a scalar function $u(x_1, \ldots, x_n)$ we write

$$\Delta u = \sum_{i=1}^n \partial_{x_i}^2 u \,.$$

In order to define weighted Sobolev spaces of integer order $k$, we need to develop a notion of how we can define distributional derivatives in general $L^p(M, \mu)$-spaces. In this context we restrict ourselves to the case $M = \Omega$, where $\Omega$ is an open subset of $H$, and $\mu = \mu_\sigma$ for $\sigma > -1$. We first claim that

$$L^1_{loc}(\Omega, \mu_\sigma) \subset L^1_{loc}(\Omega, \mathcal{L}^n)\,.$$

Then, due to the considerations in section 2.2, every $u \in L^p(\Omega, \mu_\sigma)$ has a distributional derivative. Let us prove the claim: Suppose $K$ is a compact set in $H$. Then $K \cap \partial H$ is empty, or more precisely, there exists a constant $c > 1$ such that $c^{-1} < x_n < c$ for all $x \in K$. But this implies the assertion since

$$\int_K |u|\, d\mathcal{L}^n \le \sup_K x_n^{-\sigma}\, \|u\|_{L^1(K,\mu_\sigma)} < c^{|\sigma|}\, \|u\|_{L^1(K,\mu_\sigma)} < \infty$$

for all $u \in L^1_{loc}(\Omega, \mu_\sigma)$ and all compact $K \subset \Omega$. In light of these results, one can define the Sobolev space

$$W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k}) := \big\{u \in L^p(\Omega, \mu_{\sigma_0}) \,\big|\, \partial_x^\alpha u \in L^p(\Omega, \mu_{\sigma_{|\alpha|}})\ \forall\, |\alpha| \le k\big\}$$

for any $k \in \mathbb{N}$ and all $p \ge 1$, where the differentiation is understood in a distributional sense. Using the norm

$$\|u\|_{W^{k,p}(\Omega,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} = \begin{cases} \Big(\sum_{|\alpha|\le k} \|\partial_x^\alpha u\|^p_{L^p(\Omega,\mu_{\sigma_{|\alpha|}})}\Big)^{\frac{1}{p}} & \text{if} \quad p \in [1,\infty)\,, \\ \sum_{|\alpha|\le k} \operatorname*{ess\,sup}_\Omega \big|\partial_x^\alpha u\big| & \text{if} \quad p = \infty\,, \end{cases}$$

these vector spaces become Banach spaces for all $p \ge 1$ and for all positive integers $k$. We summarize this property in a lemma.

**Lemma 2.4.1** *Let $k$ be a positive integer, $1 \le p \le \infty$ and $\sigma_0, \ldots, \sigma_k > -1$. Then $W^{k,p}(\Omega,\, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$ endowed with the norm $\|\cdot\|_{W^{k,p}(\Omega,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})}$ is a Banach space.*

**Proof:** Let $(u_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in the space $W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$. Then $(\partial^\alpha u_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega, \mu_{|\alpha|})$ for $0 \le |\alpha| \le k$. Since $L^p(\Omega, \mu_{|\alpha|})$ is a complete space there exist functions $v_0 = u$ and $v_\alpha$ such that $\partial^\alpha u_i \to v_\alpha$ in $L^p(\Omega, \mu_{\sigma_{|\alpha|}})$. Now $L^p(\Omega, \mu_{\sigma_{|\alpha|}}) \subset L^1_{loc}(\Omega, \mathcal{L}^n)$ and so $\partial^\alpha u_i$ determines the regular distribution $T_{\partial^\alpha u_i} \in \mathcal{D}'(\Omega)$ as above. Now for any $0 \le |\alpha| \le k$ we have

$$\big|T_{\partial^\alpha u_i}(\varphi) - T_{v_\alpha}(\varphi)\big| \le c^{\frac{|\sigma_{|\alpha|}|}{p}}\, \|\varphi\|_{L^{p'}(\Omega)}\, \|\partial^\alpha u_i - v_\alpha\|_{L^p(\Omega,\mu_{\sigma_{|\alpha|}})}$$

for all $\varphi \in C_c^\infty(\Omega)$ by Hölder's inequality, where $c > 1$ is the constant such that $c^{-1} < x_n < c$ for all $x \in spt\,\varphi$ and $p' = \frac{p}{p-1}$ is the conjugate Hölder exponent of $p$. Since $\partial^\alpha u_i \to v_\alpha$ in $L^p(\Omega, \mu_{\sigma_{|\alpha|}})$, we get

$$T_{v_\alpha}(\varphi) = \lim_{i\to\infty} T_{\partial^\alpha u_i}(\varphi) = (-1)^{|\alpha|} \lim_{i\to\infty} T_{u_i}(\partial^\alpha \varphi) = (-1)^{|\alpha|}\, T_u(\partial^\alpha \varphi)$$

for all $\varphi \in C_c^\infty(\Omega)$. Thus $v_\alpha = \partial^\alpha u$ in the sense of distributions on $\Omega$ for all $0 \le |\alpha| \le k$, whence it follows $u \in W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$. In particular, $\partial^\alpha u_i \to \partial^\alpha u$ in $L^p(\Omega, \mu_{\sigma_{|\alpha|}})$ for all $0 \le |\alpha| \le k$, and hence $u_i$

converges to $u$ in $W^{k,p}(\Omega, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$. ■

Also note that

$$\|u\|^2_{W^{k,2}(\Omega,\, \mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;=\; \big(u \mid u\big)_{W^{k,2}(\Omega,\, \mu_{\sigma_0},\dots,\mu_{\sigma_k})}\,,$$

where

$$\big(u \mid v\big)_{W^{k,2}(\Omega,\, \mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;=\; \sum_{|\alpha|\le k} \int_\Omega (\partial_x^\alpha\, u)\,(\partial_x^\alpha\, v)\, d\mu_{\sigma_{|\alpha|}}$$

defines a symmetric bilinear form. Thus $W^{k,2}(\Omega, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$ together with this inner product is a Hilbert space. A remarkable property is that every Hilbert space has a Schauder basis: Let $X$ be a topological vector space over $\mathbb{R}$. A sequence $\{v_i\}_{i\in\mathbb{N}} \subset X$ is called Schauder basis if for any $x \in X$ there exists a unique sequence $\{\lambda_i\}_{i\in\mathbb{N}} \in \mathbb{R}$ such that $x = \sum_i \lambda_i\, v_i$, where the convergence is understood with respect to the norm topology. This definition is equivalent to say that the set of all finite linear combinations of $v_i$ is dense in $X$ and all finite subsets of $\{v_i\}_{i\in\mathbb{N}}$ are linearly independent.

**Lemma 2.4.2** *Suppose $\big(X, (\cdot|\cdot)_X\big)$ and $\big(Y, (\cdot|\cdot)_Y\big)$ are Hilbert spaces with $X \subseteq Y$. Moreover, let $\{v_i\}_{i\in\mathbb{N}}$ be a Schauder basis of $X$. Then the matrix $A_m := \big((v_i \mid v_j)_Y\big)_{i,j=1}^m$ is invertible for any $m \in \mathbb{N}$.*

**Proof:** Let $\vec{\lambda} \in \mathbb{R}^m \backslash \{0\}$. Then we have

$$\vec{\lambda}^T A_m\, \vec{\lambda} \;=\; \sum_{i,j=1}^m \lambda_i\, \big(v_i \mid v_j\big)_Y\, \lambda_j \;=\; \Big(\sum_{i=1}^m \lambda_i v_i \mid \sum_{j=1}^m \lambda_j v_j\Big)_Y \;=:\; (w \mid w)_Y$$

with $w = \sum_{i=1}^m \lambda_i\, v_i \in X \subseteq Y$. Thanks to the fact that all finite subsets $\{v_i\}_{i=1}^m$ are linearly independent we get $w \neq 0$. Thus, since $Y$ is also a Hilbert space, we find $(w \mid w)_Y > 0$. This means $A_m$ is positive definite and therefore invertible for any positive integer $m$. ■

## 2.5 Approximation by Continuous Functions

A key scheme in real analysis is that of studying quite general functions by first approximating them by functions that are somewhat "nicer". In the context of PDEs, for example, one naturally deals with weak solutions which can usually be found in Sobolev spaces rather than in spaces of continuous functions. A technique that is called "mollification" provides a particularly powerful tool to produce a function that behaves better than the original one while still remaining close to it. Before proving such approximations in weighted Sobolev spaces as defined above, we recall the usual regularization procedure on the measure space $(\mathbb{R}^n, \mathcal{L}^n)$. Let $j$ be function in $C_c^\infty(\mathbb{R}^n)$ such that

$$spt\, j \;\subset\; B_1^{eu}(0) \qquad \text{and} \qquad \int_{\mathbb{R}^n} j\, d\mathcal{L}^n \;=\; 1\,.$$

For example, take

$$j(x) \;=\; \begin{cases} c\, \exp\!\Big(\frac{1}{|x|^2-1}\Big) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1\,, \end{cases}$$

where $c > 0$ is selected so that the second property is satisfied. If $u \in L^1_{loc}(\mathbb{R}^n)$ and $\varepsilon > 0$, then the convolution

$$j_\varepsilon * u(x) \;=\; \int_{\mathbb{R}^n} j_\varepsilon(x-y)\, u(y)\, d\mathcal{L}^n(y) \;\in\; C^\infty(\mathbb{R}^n)\,,$$

where $j_\varepsilon(x) := \varepsilon^{-n}\, j(\varepsilon^{-1}x)$, is called a mollifier. To be more precise, for any multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$\partial_x^\alpha \big(j_\varepsilon * u\big)(x) \;=\; \partial_x^\alpha\, j_\varepsilon * u(x) \;=\; j_\varepsilon * \partial_x^\alpha u(x)\,.$$

Since $j_\varepsilon * u$ smoothes out irregularities of $u$, it is called regularization or mollification of $u$.

In the unweighted case one usually uses the standard mollification to prove a series of results concerning the approximation of Sobolev functions by smooth functions. One of the most famous discoveries in this context is certainly the "$H = W$" theorem presented in a two-sided paper by Meyers and Serrin [62]. However, by making only minor modifications, we can adopt their methods to achieve the same result in the weighted case.

**Proposition 2.5.1** *Suppose $\Omega \subseteq H$ is open and $\sigma_k \geq \cdots \geq \sigma_0 > -1$. If $1 \leq p < \infty$, then we have*

$$H^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k}) \;=\; W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k}) \,,$$

*where the left hand side is to be understood as the completion of $C^\infty(\Omega)$ in $\|\cdot\|_{W^{k,p}(\Omega,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})}$.*

**Proof:** (See [62].) Since "$H \subset W$" is clear from the very definition, it suffices to show that $C^k(\Omega) \cap W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$ is dense in $W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$. Let $u \in W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$. If $\psi \in C_c^\infty(\Omega)$, then

$$\|\partial_x^\alpha(\psi\, u)\|^p_{L^p(\Omega,\mu_{\sigma_{|\alpha|}})} \;\lesssim\; \sum_{\beta\leq\alpha} \sup_{x\in spt\,\psi} \big|\partial_x^{\alpha-\beta}\,\psi(x)\big|^p\, x_n^{\sigma_{|\alpha|}-\sigma_{|\beta|}}\; \|\partial_x^\beta u\|^p_{L^p(\Omega,\mu_{\sigma_{|\beta|}})} \lesssim \|u\|^p_{W^{k,p}(\Omega,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})}$$

for $|\alpha| \leq k$, since $\sigma_{|\beta|} \leq \sigma_{|\alpha|}$ and $x_n > c > 0$ for all $x \in spt\,\psi$. Thus $\psi\, u \in W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$. Also, $spt\,(\psi\, u)$ is a compact subset of $\Omega$, and hence stays away from its boundary. This implies $j_\varepsilon * (\psi\, u) \in C_c^\infty(\Omega)$ and $\partial_x^\alpha\big(j_\varepsilon * (\psi\, u)\big) = j_\varepsilon * \partial_x^\alpha(\psi\, u)$ on any open and bounded $U$ with $\overline{U} \subset \Omega \subseteq H$ as long as $\varepsilon > 0$ is suitably small. If we let $\varepsilon \to 0^+$, this yields

$$\|\psi\, u - j_\varepsilon * (\psi\, u)\|_{W^{k,p}(U,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \;\to\; 0\,, \tag{$*$}$$

where $U$ is so chosen that $spt\,(\psi\, u) \subset U$.

Now let $\Omega_N$ be an open and bounded set such that $\overline{\Omega}_N \subset \Omega$ and $\overline{\Omega}_N \subset \Omega_{N+1}$ for any positive integer $N$, and

$$\bigcup_{N\in\mathbb{N}} \Omega_N \;=\; \Omega\,.$$

(For example, take $\Omega_N = \Omega \cap B_N^{eu}(0) \cap \big\{x \in H \;\big|\; dist(x, \partial\Omega > \tfrac{1}{N})\big\}$.) Furthermore, set

$$\Omega_{-1} \;:=\; \Omega_0 \;:=\; \emptyset\,, \qquad U_i \;:=\; \Omega_{i+2} \setminus \overline{\Omega}_{i-2} \qquad \text{and} \qquad V_i \;:=\; \Omega_{i+1} \setminus \overline{\Omega}_{i-1} \qquad (i \in \mathbb{N})\,.$$

Clearly, we have

$$\bigcup_{i\in\mathbb{N}} U_i\,, \bigcup_{i\in\mathbb{N}} V_i \;=\; \Omega \qquad \text{and} \qquad \overline{V}_i \;\subset\; U_i$$

for $i \in \mathbb{N}$. Additionally,

$$\overline{U}_i \;\subset\; \overline{\Omega}_N^{\complement} \quad\Longrightarrow\quad \overline{\Omega}_N \;\subset\; \overline{U}_i^{\complement} \;\subset\; \overline{V}_i^{\complement}$$

whenever $i > N+2$. Let $\{\psi_i\}_{i\in\mathbb{N}}$ be a partition of unity subordinate to $\{V_i\}_{i\in\mathbb{N}}$. Then for $x \in \Omega_N$, it follows

$$u(x) \;=\; \sum_{i\in\mathbb{N}} \psi_i(x)\, u(x) \;=\; \sum_{i=1}^{N+2} \psi_i(x)\, u(x)\,,$$

as well as

$$\sum_{i\in\mathbb{N}} j_{\varepsilon_i} * \big(\psi_i\, u\big)(x) \;=\; \sum_{i=1}^{N+2} j_{\varepsilon_i} * \big(\psi_i\, u\big)(x)$$

for $0 < \varepsilon_i < dist(V_i, \partial U_i)$. Now fix $\delta > 0$. Then

$$\begin{aligned} \|u - \sum_{i\in\mathbb{N}} j_{\varepsilon_i} * (\psi_i\, u)\|_{W^{k,p}(\Omega_N,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} &\le \sum_{i=1}^{N+2} \|\psi_i\, u - j_{\varepsilon_i} * (\psi_i\, u)\|_{W^{k,p}(U_i,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \\ &< \sum_{i=1}^{N+2} 2^{-i}\,\delta \;=\; (1 - 2^{-N-2})\,\delta\,. \end{aligned}$$

by virtue of $(*)$. Finally, by the dominant convergence theorem we infer that

$$\begin{aligned} \|u - \sum_{i\in\mathbb{N}} j_{\varepsilon_i} * (\psi_i\, u)\|_{W^{k,p}(\Omega,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} &= \lim_{N\to\infty} \|u - \sum_{i\in\mathbb{N}} j_{\varepsilon_i} * (\psi_i\, u)\|_{W^{k,p}(\Omega_N,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \\ &< \lim_{N\to\infty} (1 - 2^{-N-2})\,\delta \;=\; \delta\,, \end{aligned}$$

as stated. ∎

An essential ingredient in the proof of the previous theorem is that of keeping the function $u$ away from $\partial\Omega$ (and thus $\partial H$) by introducing a cut-off function that has compact support in $\Omega$. This is important because on the boundary of $H$ the weight $x_n^\sigma$ changes its nature so as to become either unbounded ($\sigma < 0$) or zero ($\sigma > 0$). This suggests to adjust the mollification in such a way that it qualifies for our weighted setting, even when approaching the critical area near $\partial H$. Suppose $u \in L^1_{loc}(\Omega, \mathcal{L}^n)$, where $\Omega$ is again an arbitrary open subset of $H$. By abuse of notation we define

$$j_{\varepsilon x_n} * u(x) \;:=\; \int_{\mathbb{R}^n} j_{\varepsilon x_n}(x-y)\,\bar{u}(y)\,d\mathcal{L}^n(y) \tag{2.5.1}$$

for all $\varepsilon > 0$ and any $x \in H$, where $\bar{u}$ denotes the zero extension of $u$ outside of $\Omega$ to all of $\mathbb{R}^n$. In the following we summarize some useful properties of this "pseudo convolution".

i) Using a change of coordinates we readily see that

$$\int_{\mathbb{R}^n} j_{\varepsilon x_n}(x-y)\,dy \;=\; \int_{B^{eu}_{\varepsilon x_n}(x)} j_{\varepsilon x_n}(x-y)\,dy \;=\; 1$$

for all $\varepsilon > 0$ and for all $x \in H$. Also note that $B^{eu}_{\varepsilon x_n}(x) \subset H$ if $\varepsilon < 1$.

ii) Another transformation of the integral shows that

$$j_{\varepsilon x_n} * u(x) \;=\; \int_{B^{eu}_{\varepsilon}(0)} j_\varepsilon(y)\,\bar{u}(x - x_n\, y)\,d\mathcal{L}^n(y) \qquad \forall\,\varepsilon > 0\,,\ \forall\, x \in H\,.$$

iii) Since $j_{\varepsilon x_n} \in C^\infty(H)$, one can differentiate under the integral in (2.5.1) on the first factor and get that $j_{\varepsilon x_n} * u$ is infinitely many times differentiable in $x \in H$. If additionally $spt\, u$ is compact in $\Omega$, then $j_{\varepsilon x_n} * u \in C^\infty_c(\Omega)$ for suitably small $\varepsilon > 0$.

With all necessary preparations made, it is then crucial to ensure that the next lemma is available.

**Lemma 2.5.2** *Let $p, \sigma$ be any numbers satisfying $1 \le p < \infty$ and $\sigma > -1$. Then there exists a constant $c = c(\sigma, p)$ such that*

$$\|j_{\varepsilon x_n} * u\|_{L^p(\Omega,\mu_\sigma)} \;\le\; c\,\|u\|_{L^p(\Omega,\mu_\sigma)}$$

*whenever $0 < \varepsilon \le \frac{1}{2}$ and $u \in L^p(\Omega, \mu_\sigma)$ with $\Omega \subseteq H$.*

**Proof:** We decompose the upper half space $H$ into

$$A_i := \{x \in \mathbb{R}^n \mid 2^i < x_n \le 2^{i+1}\}\,, \qquad i \in \mathbb{Z}\,.$$

With $u_j := \bar{u}\,\chi_{A_j}$, we thus have

$$\|j_{\varepsilon x_n} * u\|^p_{L^p(H,\mu_\sigma)} \;=\; \sum_{i\in\mathbb{Z}} \|j_{\varepsilon x_n} * \sum_{j\in\mathbb{Z}} u_j\|^p_{L^p(A_i\,,\mu_\sigma)}\,.$$

Now fix a point $x \in H$. Then there exists an integer $i$ such that $x \in A_i$ and, for any $y \in B^{eu}_{\varepsilon x_n}(x)$, we get

$$(1-\varepsilon)\,x_n \;<\; y_n \;<\; (1+\varepsilon)\,x_n\,.$$

Assuming that $\varepsilon \le \frac{1}{2}$ implies

$$B^{eu}_{\varepsilon x_n}(x) \;\subset\; \bigcup_{j=i-1}^{i+1} A_j\,.$$

For $i \in \mathbb{Z}$ this gives

$$\|j_{\varepsilon x_n} * \sum_{j\in\mathbb{Z}} u_j\|^p_{L^p(A_i\,,\mu_\sigma)} \overset{(i)}{\le} \sum_{j=i-1}^{i+1} \int_{A_i} \underbrace{\Big(\int_{\mathbb{R}^n} j_{\varepsilon x_n}(x-y)\,dy\Big)^{p-1}}_{=1 \text{ by } (i)} \Big(\int_{A_j} j_{\varepsilon x_n}(x-y)\,\big|u_j(y)\big|^p\,dy\Big)\,x_n^{\,\sigma}\,dx\,,$$

where we also applied Hölder's inequality to $j_{\varepsilon x_n}^{1-\frac{1}{p}}\, j_{\varepsilon x_n}^{\frac{1}{p}}\, |u_j|$. Since $x_n^{\,\sigma} \le 2^{|\sigma|}\,2^{i\sigma}$ for all $x \in A_i$, we get

$$\sum_{i\in\mathbb{Z}} \|j_{\varepsilon x_n} * u\|^p_{L^p(A_i\,,\mu_\sigma)} \;\lesssim\; \sum_{i\in\mathbb{Z}} 2^{i\sigma} \sum_{j=i-1}^{i+1} \int_{A_i}\int_{A_j} j_{\varepsilon x_n}(x-y)\,\big|u_j(y)\big|^p\,dy\,dx$$

and using Fubini's theorem this is equal to

$$\sum_{i\in\mathbb{Z}} 2^{i\sigma} \sum_{j=i-1}^{i+1} \int_{A_j} \big|u_j(y)\big|^p \int_{A_i} j_{\varepsilon x_n}(x-y)\,dx\,dy\,.$$

For the inner integral, we substitute $\frac{x-y}{x_n}$ by $z$ to find that

$$\int_{A_i} j_{\varepsilon x_n}(x-y)\,dx \;=\; \int_{A_i} \frac{x_n}{y_n}\, j_\varepsilon\big(x_n^{-1}\,(x-y)\big)\, \frac{y_n}{x_n^{\,n+1}}\,dx \;\le\; 4\int_{\mathbb{R}^n} j_\varepsilon(z)\,dz \;=\; 4$$

for any $y \in A_j$ with $j \in \{i-1, i, i+1\}$. It follows

$$\|j_{\varepsilon x_n} * u\|^p_{L^p(H,\mu_\sigma)} \;\lesssim\; \sum_{i\in\mathbb{Z}} 2^{i\sigma} \sum_{j=i-1}^{i+1} \|u_j\|^p_{L^p(A_j)} \;\lesssim\; \sum_{i\in\mathbb{Z}} \sum_{j=i-1}^{i+1} \|\bar{u}\|^p_{L^p(A_j\,,\mu_\sigma)} \;\lesssim\; \sum_{i\in\mathbb{Z}} \|\bar{u}\|^p_{L^p(A_i\,,\mu_\sigma)}$$

as required. This, combined with the identity

$$\sum_{i\in\mathbb{Z}} \|\bar{u}\|^p_{L^p(A_i\,,\mu_\sigma)} \;=\; \|\bar{u}\|^p_{L^p(H,\mu_\sigma)} \;=\; \|u\|^p_{L^p(\Omega,\mu_\sigma)}\,,$$

closes the argument. ∎

**Corollary 2.5.3** *Suppose $\Omega$ is an open subset of $H$. If $u \in L^p(\Omega, \mu_\sigma)$ for some $1 \le p < \infty$ and $\sigma > -1$, then*

$$\|u - j_{\varepsilon x_n} * u\|_{L^p(\Omega,\mu_\sigma)} \;\to\; 0$$

*as $\varepsilon \to 0^+$. In particular, the set $C_c^\infty(\Omega)$ is dense in $L^p(\Omega, \mu_\sigma)$.*

**Proof:** We note that the assertion is true if $u \in C_c(\Omega)$: Using (ii) and the fact that $\int_{\mathbb{R}^n} j_\varepsilon(y)dy = 1$ we find

$$\big|u(x) - j_{\varepsilon x_n} * u(x)\big| \;\le\; \sup_{y\in B^{eu}_\varepsilon(0)} \big|u(x) - u(x - x_n\,y)\big| \;\to\; 0$$

uniformly for all $x \in \Omega$ as $\varepsilon \to 0$. Thus, for any $\delta > 0$ there exists an $\varepsilon > 0$ such that the following holds:

$$\|u - j_{\varepsilon x_n} * u\|_{L^p(\Omega,\mu_\sigma)} \;\le\; |K|_\sigma^{\frac{1}{p}} \sup_{x\in\Omega} \sup_{y\in B_\varepsilon^{eu}(0)} \big|u(x) - u(x - x_n\, y)\big| \;<\; \delta\,.$$

Here, $K = spt\,(j_{\varepsilon x_n} * u)$ is a compact set and, since $\mu_\sigma$ is a Radon measure, $|K|_\sigma$ is in fact finite.

Suppose now $u \in L^p(\Omega, \mu_\sigma)$. To complete the proof, we fix some $\delta > 0$ and let $\varphi \in C_c(\Omega)$ be a function with $\|u - \varphi\|_{L^p(\Omega,\mu_\sigma)} < \delta$. Such a function exists because $C_c(\Omega)$ is dense in $L^p(\Omega, \mu_\sigma)$.[3] Choosing $\varepsilon > 0$ small enough we can ensure that

$$\|u - j_{\varepsilon x_n} * u\|_{L^p(\mu_\sigma)} \;\le\; \|u - \varphi\|_{L^p(\mu_\sigma)} + \|\varphi - j_{\varepsilon x_n} * \varphi\|_{L^p(\mu_\sigma)} + \|j_{\varepsilon x_n} * (\varphi - u)\|_{L^p(\mu_\sigma)} \;<\; (c+2)\delta\,,$$

where we have used lemma 2.5.2 applied to $u - \varphi$. Taking only the first two terms of the right hand side, we immediately see that $u$ is arbitrarily close to $j_{\varepsilon x_n} * \varphi$ which, by property (iii), belongs to $C_c^\infty(\Omega)$. The corollary follows. ■

Next we will show that the mollification $j_{\varepsilon x_n} * u$ converges to $u$ in any weighted Sobolev space.

**Lemma 2.5.4** *Let $1 \le p < \infty$ and $\sigma_0, \ldots, \sigma_k > -1$. If $\Omega'$ is an open and bounded subset of $H$ with $\overline{\Omega'} \subset \Omega \cup \partial H$, then*

$$\lim_{\varepsilon\to 0^+} \|u - j_{\varepsilon x_n} * u\|_{W^{k,p}(\Omega',\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \;=\; 0$$

*for all $u \in W^{k,p}(\Omega, \mu_{\sigma_0}, \ldots, \mu_{\sigma_k})$.*

**Proof:** Fix $x \in \Omega'$. Choosing $\varepsilon > 0$ suitably small, we can guarantee that $x - x_n\, y \in \Omega$ for all $y \in B_\varepsilon^{eu}(0)$. Setting $T : x \mapsto x - x_n\, y =: z$, we use the representation of $j_{\varepsilon x_n} * u$ from (ii) and the fact that

$$\partial_{x_i}\big(u \circ T\big)(x) \;=\; \sum_{j=1}^{n} \partial_{z_j} u(z)\big|_{z=T(x)}\,(\delta_{ij} - y_j\,\delta_{in})\,,$$

where $\delta_{ij}$ is the Kronecker delta, that is $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise, to establish the formula

$$\partial_x^\alpha\big(j_{\varepsilon x_n} * u\big)(x) = \sum_{j=0}^{\alpha_n} (-1)^j \binom{\alpha_n}{j} \sum_{|\beta|=j} \int_{B_\varepsilon^{eu}(0)} j_\varepsilon(y)\, \partial_z^{\alpha+\beta-je_n} u(z)\Big|_{z=T(x)} y^\beta\, dy$$

valid for all $|\alpha| \le k$. Note that the first summand ($j = 0$) gives the expression $j_{\varepsilon x_n} * \partial_x^\alpha u(x)$, and so we write

$$\partial_x^\alpha\big(j_{\varepsilon x_n} * u\big)(x) \;=:\; j_{\varepsilon x_n} * \partial_x^\alpha u(x) \;+\; r_{\varepsilon,\alpha}(x)\,.$$

If $\varepsilon \le 1$, then

$$\big|r_{\varepsilon,\alpha}(x)\big| \;\le\; \sum_{j=1}^{\alpha_n} \binom{\alpha_n}{j} \varepsilon^j \sum_{|\beta|=j} \Big(j_{\varepsilon x_n} * \big|\partial_x^{\alpha+\beta-je_n} u\big|\Big)(x) \;\le\; c(\alpha_n)\,\varepsilon \sum_{|\beta|=|\alpha|} \big(j_{\varepsilon x_n} * |\partial_x^\beta u|\big)(x)\,.$$

By lemma 2.5.2,

$$\|r_{\varepsilon,\alpha}\|_{L^p(\Omega',\,\mu_{\sigma_{|\alpha|}})} \;\lesssim\; \varepsilon\, \|\partial_x^\alpha u\|_{L^p(\Omega',\,\mu_{\sigma_{|\alpha|}})}$$

for all multi-indices $\alpha$ with $0 \le |\alpha| \le k$, and the right hand side approaches zero as $\varepsilon \to 0^+$. This together with corollary 2.5.3 completes the proof. ■

[3] In fact this is true for $L^p(X, \mu)$, where $1 \le p < \infty$, $X$ is a locally compact Hausdorff space and $\mu$ any Radon measure.

We have now finally reached the crucial Sobolev embedding theorem we actually started out for.

**Theorem 2.5.5** *Suppose $1 \le p < \infty$ and $\{\sigma_j\}_{j=0}^k$ is a sequence of nonnegative numbers satisfying $0 \le \sigma_{j+1} - \sigma_j \le p$. Then the set of restrictions to $H$ of functions in $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$, that is,*

$$\overline{C_c^\infty(\overline{H})} \;=\; W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$$

*holds, where the closure of $C_c^\infty(\overline{H})$ refers to the norm $\|\cdot\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})}$.*

We mimic the proof of the corresponding statement in the unweighted case in [1]. For weights very much similar to ours, the result is discussed in greater detail in Kufner's book [56].

**Proof:** Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function satisfying

i) $\eta \equiv 1$ on $\overline{B_1^{eu}(0)}$ and $spt\,\eta \subset B_2^{eu}(0)$. ii) $\left|\partial_x^\alpha \eta(x)\right| \le c$ for all $x \in \mathbb{R}^n$ and $0 \le |\alpha| \le k$.

Moreover, define $\eta^\varepsilon(x) := \eta(\varepsilon x)$ for $\varepsilon > 0$. Then $\eta^\varepsilon(x) = 1$ if $|x| \le \frac{1}{\varepsilon}$, $\eta^\varepsilon(x) = 0$ if $|x| \ge \frac{2}{\varepsilon}$ and we have

$$\left|\partial_x^\alpha \eta^\varepsilon(x)\right| \;\le\; c\,\varepsilon^{|\alpha|} \qquad \forall\, x \quad \text{and} \quad 0 \le |\alpha| \le k\,.$$

With $u^\varepsilon := \eta^\varepsilon u$, we get for all $x \in H$ and $0 \le |\alpha| \le k$ that

$$\left|\partial_x^\alpha u^\varepsilon(x)\right|^p x_n^{\sigma_{|\alpha|}} \;\le\; c(n,k,p) \sum_{\beta \le \alpha} \varepsilon^{p\left(|\alpha|-|\beta|\right)} \Big( \sup_{x \,\in\, spt\,\eta^\varepsilon \,\cap\, H} x_n^{\sigma_{|\alpha|}-\sigma_{|\beta|}} \Big) \left|\partial_x^\beta u(x)\right|^p x_n^{\sigma_{|\beta|}}\,.$$

From the assumptions on $\sigma_0, \dots, \sigma_k$, we deduce that $\sigma_{|\alpha|} - \sigma_{|\beta|} \le p\big(|\alpha| - |\beta|\big)$ for $\beta \le \alpha$, such that $x_n^{\sigma_{|\alpha|}-\sigma_{|\beta|}} \le \left(\frac{\varepsilon}{2}\right)^{p(|\beta|-|\alpha|)}$ for all $x \in H$ with $|x| < \frac{2}{\varepsilon}$. Thus there exists a constant $c$ (independent of $\varepsilon$) such that

$$\|u^\varepsilon\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;\le\; c\,\|u\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;<\; \infty\,,$$

that is to say $u^\varepsilon$ belongs to $W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$. Moreover, setting $H^\varepsilon = \left\{x \in H \;\middle|\; |x| > \frac{1}{\varepsilon}\right\}$, we find

$$\begin{aligned}
\|u - u^\varepsilon\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} &\overset{(i)}{=} \|u - u^\varepsilon\|_{W^{k,p}(H^\varepsilon,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \\
&\le \|u\|_{W^{k,p}(H^\varepsilon,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} + \|u^\varepsilon\|_{W^{k,p}(H^\varepsilon,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \\
&\overset{(ii)}{\le} c\,\|u\|_{W^{k,p}(H^\varepsilon,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;\to\; 0
\end{aligned}$$

as $\varepsilon \to 0^+$, since then $H^\varepsilon \to \emptyset$. Consequently, as $u^\varepsilon$ has compact support in $H$ and approximates $u$, we may assume that $u \in W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$ has compact support, i.e. $K = \{x \in H \mid u(x) \ne 0\}$ is bounded. We explicitly do not exclude the possibility that $\partial K$ intersects with $\partial H$. Since $\overline{K}$ is compact, there exist finite covers $U_0, \dots, U_N$ and $V_0, \dots, V_N$ of bounded sets such that $\overline{V}_i \subset U_i$ for $0 \le i \le N$,

$$\overline{K} \;\subset\; \bigcup_{i=0}^N U_i \qquad \text{but still} \qquad \overline{K} \;\subset\; \bigcup_{i=0}^N V_i\,,$$

where $\overline{U}_0, \overline{V}_0 \subset H$ are compact. Now consider a partition of unity $\{\psi_i\}_{i=0}^N$ subordinate to $\{V_i\}_{i=0}^N$, that is

$$\psi_i \in C_c^\infty(V_i) \qquad \text{with} \qquad \sum_{i=0}^N \psi_i(x) \;=\; 1 \quad \forall\, x \in K\,.$$

Let $u_i := \psi_i\, u$, where we extend $u$ by 0 outside $H$. Now we claim that for every $\delta > 0$ there exist $\varphi_i \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|u_i - \varphi_i\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;<\; \frac{\delta}{N+1}\,. \tag{$*$}$$

Since $spt\, u_0 \subset V_0$ and $\overline{V}_0 \subset U_0 \subset \overline{U}_0 \subset H$, for any $\delta > 0$ there exists an $\varepsilon_0 < dist(V_0, \partial U_0)$ such that

$$\|u_0 - \varphi_0\|_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;=\; \|u_0 - \varphi_0\|_{W^{k,p}(U_0,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \;<\; \frac{\delta}{N+1}$$

by means of lemma 2.5.4, where $\varphi_0 := j_{\varepsilon_0\, x_n} * u_0$.

For $i = 1, \dots, N$ the situation is different. Here it is not clear whether or not $u_i$ is (weakly) differentiable on $\partial H$. We only know that $u_i \in W^{k,p}\big(\mathbb{R}^n \setminus (\overline{V_i} \cap \partial H), \mu_{\sigma_0}, \dots, \mu_{\sigma_k}\big)$. To see this, let $x \in V_i \cap H$. Then

$$\big|\partial_x^\alpha u_i(x)\big| \;\le\; c(\alpha) \sum_{\beta \le \alpha} \big|\partial_x^{\alpha-\beta} \psi_i(x)\big|\, \big|\partial_x^\beta u(x)\big|$$

for any $0 \le |\alpha| \le k$ by the Leibniz formula. This implies

$$\|\partial_x^\alpha u_i\|^p_{W^{k,p}(V_i \cap H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})} \lesssim \sum_{\beta \le \alpha} \sup_{x \in V_i \cap H} \big|\partial_x^{\alpha-\beta}\psi_i(x)\big|^p\, x_n^{\sigma_{|\alpha|}-\sigma_{|\beta|}}\, \|\partial_x^\beta u\|^p_{L^p(H,\mu_{\sigma_{|\beta|}})} \lesssim \|u\|^p_{W^{k,p}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})}$$

as above. Moreover, by construction of $u_i$, we have $u_i \equiv 0$ on $-H \cap V_i$ and $V_i^{\complement}$. On $\overline{V_i} \cap \partial H$, on the other hand, there is no such statement possible, and so an application of lemma 2.5.4 would not lead to the desired result. Our strategy will be as follows: We apply lemma 2.5.4 to $u_i$ and then translate it by a small amount in direction of the vector $-e_n$. Now we can exploit the property that the translation of the mollified $u_i$ is differentiable on $\partial H$.

For $i = 1, \dots, N$, consider the function $j_{\varepsilon_i x_n} * u_i$ extended to be identically zero outside of $H$. First we observe that $j_{\varepsilon_i x_n} * u_i \in W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k})$. Since $u_i \equiv 0$ outside $V_i$, we can find an $\varepsilon > 0$ so that

$$spt\,(j_{\varepsilon_i x_n} * u_i) \;\subset\; U_i \cap \overline{H}\,.$$

Let us define $\Omega' = U_i \cap H$. For this choice of $\Omega'$, the assumptions of lemma 2.5.4 are fulfilled and so for any $\delta' > 0$ there exists an $\varepsilon_i > 0$ such that the following inequality holds:

$$\|\partial_x^\alpha u_i - \partial_x^\alpha (j_{\varepsilon_i x_n} * u_i)\|_{L^p(U_i \cap H, \mu_{\sigma_{|\alpha|}})} \;<\; \delta'$$

for each $1 \le i \le N$ and any $|\alpha| \le k$, that is

$$j_{\varepsilon_i x_n} * u_i \;\to\; u_i \qquad \text{in} \quad W^{k,p}(H, \mu_{\sigma_0}, \dots, \mu_{\sigma_k}) \quad \text{as} \quad \varepsilon_i \to 0^+\,.$$

Let us turn to the translation. Since $\overline{V}_i \subset U_i$, there exists $h_i > 0$ such that $x \pm h\, e_n \in U_i$ for all $x \in \overline{V}_i$ and all $h \in (0, h_i)$. If $\varepsilon_i, h_i > 0$ are sufficiently small, then we have $spt\,\varphi_i \subset U_i \cap (\overline{H} - h\, e_n)$, where $\varphi_i(x) := \big(j_{\varepsilon_i x_n} * u_i\big)(x + h\, e_n)$ is a smooth function on $U_i \cap (H - h\, e_n)$. Restricted to $H$, this means $\varphi_i \in C^\infty(H)$ and $spt\,\varphi_i \subset U_i \cap \overline{H}$. The important thing is that $\varphi_i$ can be extended continuously across the boundary of $H$ and therefore is defines an eligible function.

To finish the proof, we need continuity of the translation operator. Unfortunately, in general this fails to be true in the weighted $L^p$-classes, but happens to be true, if $\sigma_{|\alpha|} \ge 0$ and the translation is of the above kind. That is why we need to impose the positivity condition on $\sigma_0, \dots, \sigma_k$.

**Lemma:** *Suppose $u \in L^p(H, \mu_\sigma)$ for $1 \le p < \infty$ and some nonnegative $\sigma$. Then for any number $\delta' > 0$ we have*

$$\|u - T_h\, u\|_{L^p(H,\mu_\sigma)} \;<\; 2\,\delta'$$

*for sufficiently small $h > 0$, where $T_h\, u(x) := u(x + h\, e_n)$.*

**Proof:** Fix $\delta' > 0$ and let $\varphi$ be a function in $C_c(H)$ with $\|u-\varphi\|_{L^p(\mu_\sigma)} < \frac{\delta'}{2}$. Similarly as in the proof of corollary 2.5.3, we see that $\|\varphi - T_h\,\varphi\|_{L^p(\mu_\sigma)} < \delta'$ provided $h > 0$ is small. Finally, we consider

$$\int_H |T_h\,u(x) - T_h\,\varphi(x)|^p\,x_n^{\,\sigma}\,dx = \int_{H+h\,e_n} |u(x)-\varphi(x)|^p\,(x_n-h)^\sigma\,dx \le \int_H |u(x)-\varphi(x)|^p\,x_n^{\,\sigma}\,dx\,.$$

Herein, the inequality follows from $\sigma \ge 0$ and the fact that $x_n - h\,e_n > 0$ on $H + h\,e_n$. Altogether, this gives

$$\|u - T_h\,u\|_{L^p(\mu_\sigma)} \;\le\; 2\,\|u-\varphi\|_{L^p(\mu_\sigma)} \;+\; \|\varphi - T_h\,\varphi\|_{L^p(\mu_\sigma)} \;<\; 2\,\delta'$$

as stated. □

With the auxiliary lemma we are provided the missing piece to complete the proof of $(*)$. We obtain

$$\|\partial_x^\alpha(j_{\varepsilon_i x_n} * u_i) - \partial_x^\alpha\varphi_i\|_{L^p(H,\mu_{\sigma_{|\alpha|}})} \;<\; 2\,\delta'$$

for any $0 \le |\alpha| \le k$, since the translation operator $T_h$ commutes with derivatives. We arrive at

$$\begin{aligned}\|u_i - \varphi_i\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} &= \|u_i - j_{\varepsilon_i x_n} * u_i\|_{W^{k,p}(H\cap U_i,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} + \|j_{\varepsilon_i x_n} * u_i - \varphi_i\|_{W^{k,p}(H\cap U_i,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})}\\ &< \; c(n,k,p)\,\delta' \;\le\; \frac{\delta}{N+1}\end{aligned}$$

provided $0 < h < h_i$ and $\varepsilon_i$ are suitably small. This proves $(*)$ for $0 \le i \le N$. Putting $\varphi = \sum_{i=0}^N \varphi_i$, we get

$$\|u-\varphi\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \;\le\; \sum_{i=0}^N \|u_i - \varphi_i\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \;<\; \delta\,,$$

which amounts to the statement of the theorem. ■

## 2.6 Hardy Inequality

In [39] Hardy formulated the following fundamental result: If $f$ is a nonnegative and integrable function, then

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x u(z)\,dz\Big)^p\,dx \;<\; c_p\int_0^\infty u(x)^p\,dx$$

unless $u \equiv 0$. The constant $c_p = \big(\frac{p}{p-1}\big)^p$ is the best possible. The usefulness of this inequality can be exemplified in the following way. Setting $v(x) := \int_0^x u(z)\,dz$, the Hardy inequality basically states that the weighted $L^p$-norm of $v$ is controlled by the (unweighted) $L^p$-norm of $v'$ subject to boundary conditions. The inequality can then be generalized in such a way that weights are included on both sides of the inequality, and therefore it is well suited for our case of weighted $L^p$-classes.

**Lemma 2.6.1 (global Hardy inequality)** *Let $\sigma > -1$ and $1 \le p < \infty$. If $\nabla u \in L^p(H,\mu_{\sigma+p})$, then we have*

$$\|u - c_0\|_{L^p(H,\,\mu_\sigma)} \;\le\; c\,\|\nabla u\|_{L^p(H,\,\mu_{\sigma+p})}$$

*for some $c_0 \in \mathbb{R}$ and a positive constant $c = c(n,\sigma,p)$.*

**Proof:** Assume first that $p = 1$. Then, by the assumptions, $\partial_{x_n}u(x',\cdot) \in L^1\big(\mathbb{R}_+,\mu_{\sigma+1}\big)$ for almost all $x' \in \mathbb{R}^{n-1}$, and hence $\partial_{x_n}u(x',\cdot) \in L^1_{loc}(\mathbb{R}_+)$. For any $b > 0$, the fundamental theorem of calculus implies that

$$v(x) \;:=\; -\int_{x_n}^b \partial_z u(x',z)\,dz$$

is locally absolutely continuous and we have $v'(x) = \partial_{x_n} u(x)$ almost everywhere in $(0, b)$. Moreover, we obtain

$$u(x) \;-\; u(x', b) \;=\; -\int_{x_n}^{b} \partial_z u(x', z)\, dz\,, \qquad x \in (0, b)\,.$$

Thus

$$\lim_{b\to\infty} \big|u(x', b)\big| \;\le\; \big|u(x)\big| \;+\; \lim_{b\to\infty} x_n^{-(\sigma+1)} \int_{x_n}^{b} \big|\partial_z u(x', z)\big|\, z^{\sigma+1}\, dz \;<\; \infty\,,$$

since $\sigma + 1 > 0$ and $\partial_{x_n} u(x', \cdot) \in L^1(\mathbb{R}_+, \mu_{\sigma+1})$. It follows that there exists a $c(x')$ that is independent of $x_n$ such that

$$u(x) \;-\; c(x') \;=\; -\int_{x_n}^{\infty} \partial_z u(x', z)\, dz\,.$$

A calculation then gives

$$\begin{aligned} \|u - c(x')\|_{L^1(H\,,\mu_\sigma)} \;&=\; \|\int_{x_n}^{\infty} \partial_z u(x', z)\, dz\|_{L^1(H\,,\mu_\sigma)} \;\le\; \int_0^{\infty} \|\partial_z u(\cdot\,, z)\|_{L^1(\mathbb{R}^{n-1})} \Big(\int_0^{z} x_n^{\sigma}\, dx_n\Big)\, dz \\ &=\; \frac{1}{\sigma+1}\, \|\partial_{x_n} u\|_{L^1(H\,,\mu_{\sigma+1})}\,, \end{aligned} \tag{$*$}$$

where we used Fubini's theorem in the first inequality. Now suppose $p \in (1, \infty)$. The assumption on $\nabla u$ then assures that $u(x', \cdot) \in AC_{loc}(\mathbb{R}_+) \subset L^p_{loc}(\mathbb{R}_+)$ for almost every $x' \in \mathbb{R}^{n-1}$, and consequently

$$\|\partial_{x_n} u(x', \cdot)^p\|_{L^1(K)} \;\le\; p\, \|u(x', \cdot)\|^{p-1}_{L^p(K)}\, \|\partial_{x_n} u(x', \cdot)\|_{L^p(K)}$$

is finite for any compact set $K \subset \mathbb{R}_+$.
Now applying $(*)$ to $\big(u - c(x')\big)^p$ and using Hölder's inequality, we get

$$\begin{aligned} \|u - c(x')\|^p_{L^p(\mu_\sigma)} \;&=\; \|\big(u - c(x')\big)^p\|_{L^1(\mu_\sigma)} \;\le\; c(\sigma)\, \|\partial_{x_n} \big(u - c(x')\big)^p\|_{L^1(\mu_{\sigma+1})} \\ &\le\; c(\sigma)\, p\, \|u - c(x')\|^{p-1}_{L^p(\mu_\sigma)}\, \|\partial_{x_n} u\|_{L^p(\mu_{\sigma+p})} \end{aligned} \tag{$**$}$$

with $\|\cdot\|_{L^p(\mu_\sigma)} = \|\cdot\|_{L^p(H, \mu_\sigma)}$, and it remains to show that $c(x')$ is independent of $x' \in \mathbb{R}^{n-1}$. To this end, let $v \in \mathbb{R}^{n-1}$ and define $u_v(x) = u(x' - x_n\, v, x_n)$. We apply $(**)$ to the function $u_v$ and obtain

$$\|u - c_v(x' + x_n\, v)\|_{L^p(\mu_\sigma)} \;=\; \|u_v - c_v(x')\|_{L^p(\mu_\sigma)} \;\lesssim\; \|\partial_{x_n} u_v\|_{L^p(\mu_{\sigma+p})} \;\lesssim\; \|\nabla u\|_{L^p(\mu_{\sigma+p})}\,.$$

It follows that

$$\|c(x') - c_v(x' + x_n\, v)\|_{L^p(\mu_\sigma)} \;\lesssim\; \|\nabla u\|_{L^p(\mu_{\sigma+p})} \;<\; \infty$$

by Minkowski's inequality. Putting $v = e_1' = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$, we find that

$$\|c(x_1) - c_v(x_1 + \cdot)\|_{L^p(\mathbb{R}_+, \mu_\sigma)} \;<\; \infty$$

for almost every $x'' = (x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-2}$ and almost every $x_1$. We fix $x''$ in the set of full measure as well as two Lebesgue points $s$ and $t$ of $c(x_1)$, where we suppress $x''$ in the notation. We have

$$\|c(s) - c_v(s + \cdot)\|_{L^p(\mathbb{R}_+, \mu_\sigma)} \;+\; \|c(t) - c_v(t + \cdot)\|_{L^p(\mathbb{R}_+, \mu_\sigma)} \;<\; \infty\,.$$

Let $s < t$ without loss of generality. Then

$$\|c(s) - c_v(s + \cdot)\|_{L^p([2(t-s), \infty), \mu_\sigma)} \;+\; \|c(t) - c_v(t + \cdot)\|_{L^p([t-s, \infty), \mu_\sigma)} \;<\; \infty\,.$$

We shift the integration by $t - s$ in the first norm to find

$$\int_{t-s}^{\infty} \big|c(s) - c_v(t + x_n)\big|^p\, (x_n + t - s)^{\sigma}\, dx_n \;=\; \int_{2(t-s)}^{\infty} \big|c(s) - c_v(s + x_n)\big|^p\, x_n^{\sigma}\, dx_n \;<\; \infty\,.$$

Now we claim that

$$(x_n + t - s)^\sigma \;\sim\; x_n^\sigma$$

for $x_n > t - s$. This follows directly from

$$0 \;\le\; \big|\ln(x_n + t - s) \,-\, \ln(x_n)\big| \;\le\; \frac{t-s}{x_n}$$

by exponentiation: If $x_n > t - s$, then $1 \le \frac{x_n+t-s}{x_n} \le e^{\frac{t-s}{x_n}} < e$, i.e. $x_n + t - s \sim x_n$. But now we also have

$$\big|c(s) - c(t)\big|^p \int_{t-s}^{\infty} x_n^\sigma \, dx_n \;\lesssim\; \int_{t-s}^{\infty} \Big(\big|c(s) - c_v(t + x_n)\big|^p \,+\, \big|c(t) - c_v(t + x_n)\big|^p\Big)\, x_n^\sigma \, dx_n \;<\; \infty$$

which is only possible if $c(s) = c(t)$, that is, $c(x')$ is independent of $x_1$. Proceeding iteratively for all the other variables $x_i$ with $i = 2, \ldots, n-1$, we get successively $c(x') = c(x'') = \cdots = c_0$. This concludes the proof of the lemma. ■

For compactly supported $u$, Hardy's inequality has the following immediate consequence.

**Corollary 2.6.2** *If $\nabla u \in L^p(H, \mu_{\sigma+p})$ and $spt\, u \subset \overline{H}$ is compact, then $c_0$ in lemma 2.6.1 can be chosen to be* 0.

**Proof:** Let $K \subset \overline{H}$ denote the support of $u$. Then, using lemma 2.6.1, we know that

$$\|\nabla u\|^p_{L^p(H,\mu_{\sigma+p})} \;\gtrsim\; \|u - c_0\|^p_{L^p(H,\mu_\sigma)} \;\ge\; \int_{H\setminus K} |c_0|^p \, d\mu_\sigma \;=\; |c_0|^p \, \mu_\sigma(H \setminus K)\,.$$

Since $K$ is compact, we have $\mu_\sigma(H \setminus K) = \infty$ and hence $|c_0|^p = 0$. ■

Next we use Hardy's inequality to relax the conditions on $\sigma_j$ in the density result 2.5.5. More precisely, the positivity assumption (except for $\sigma_k$) can be dropped from the statement of this theorem.

**Remark 2.6.3** *Let $1 \le p < \infty$, $k$ be a nonnegative integer, and $\sigma_k \ge \cdots \ge \sigma_0 > -1$ with $\sigma_{j+1} - \sigma_j \le p$ and $\sigma_k \ge 0$. Then the statement of theorem 2.5.5 remains valid.*

**Proof:** Let us return to the setting of the proof of theorem 2.5.5. We are given two functions $j_{\varepsilon_i x_n} * u_i$, $\varphi_i \in C^\infty(H)$ with $spt\,(j_{\varepsilon_i x_n} * u_i - \varphi_i) \subset U_i$ and $U_i$ is bounded. Applying Hardy's inequality from lemma 2.6.1 to this difference, we obtain

$$\|j_{\varepsilon_i x_n} * u_i - \varphi_i\|_{L^p(H,\mu_{\sigma_0})} \;\lesssim\; \|D_x(j_{\varepsilon_i x_n} * u_i - \varphi_i)\|_{L^p(H,\mu_{\sigma_0+p})} \;=\; \|D_x(j_{\varepsilon_i x_n} * u_i - \varphi_i)\|_{L^p(U_i\cap H,\mu_{\sigma_0+p})}\,.$$

Now the assumptions on $\sigma_0, \sigma_1$ and $p$ ensure that $0 \le \sigma_0 - \sigma_1 + p \le p$ such that on $U_i \cap H$ we have the estimate

$$x_n^{\sigma_0+p} \;=\; x_n^{\sigma_0-\sigma_1+p}\, x_n^{\sigma_1} \;\le\; c(p)\, x_n^{\sigma_1}\,.$$

Hence

$$\|j_{\varepsilon_i x_n} * u_i - \varphi_i\|_{L^p(H,\mu_{\sigma_0})} \;\lesssim\; \|D_x(j_{\varepsilon_i x_n} * u_i) - D_x\varphi_i\|_{L^p(H,\mu_{\sigma_1})}\,.$$

Repeating this procedure until the upper bound contains a weight with nonnegative exponent $\sigma_j$, this procedure eventually stops after at most $k$ steps since by assumption $\sigma_k \ge 0$. But in this new setting we can use the fact that the translation by $h > 0$ in the inverse direction of the $n$-th unit vector $e_n$ is continuous. Therefore

$$\|u_i - \varphi_i\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \;\le\; \|u_i - j_{\varepsilon_i x_n} * u_i\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})} \,+\, \|j_{\varepsilon_i x_n} * u_i - \varphi_i\|_{W^{k,p}(H,\mu_{\sigma_0},\ldots,\mu_{\sigma_k})}$$

and the right hand side gets arbitrarily small for $h$ and $\varepsilon_i$ small enough. The rest follows as in the proof of theorem 2.5.5. ∎

## 2.7 Sobolev Inequalities

The term Sobolev inequalities is used for a collection of various embeddings of certain Sobolev spaces into others. Contributions of mathematicians other than Sobolev such as Morrey, Poincaré, Gagliardo and Nirenberg are significant here. In order to make them compatible with our needs, we also include weighted versions of such inequalities.

The classical Sobolev inequality is certainly the following one:

**Lemma 2.7.1 (Sobolev inequality)** *Let $p \in [1,n)$. Then there exists a constant $c$ depending on $n$ and $p$ such that*

$$\|\varphi\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \;\le\; c\,\|\nabla\varphi\|_{L^p(\mathbb{R}^n)}$$

*for all $\varphi \in C_c^1(\mathbb{R}^n)$.*

**Proof:** Since $\varphi$ has compact support, for all $x \in \mathbb{R}^n$ and $i = 1,\dots,n$ we have

$$\begin{aligned} 2\left|\varphi(x)\right| &= \Big|\int_{-\infty}^{x_i} \partial_{x_i}\varphi(x_1,\dots,y_i\,,\dots,x_n)\,dy_i \;-\; \int_{x_i}^{\infty} \partial_{x_i}\varphi(x_1,\dots,y_i\,,\dots,x_n)\,dy_i\Big| \\ &\le \int_{\mathbb{R}} \left|\partial_{x_i}\varphi(x_1,\dots,y_i\,,\dots,x_n)\right| dy_i\,, \end{aligned}$$

and hence

$$\left|\varphi(x)\right|^{\frac{n}{n-1}} \;\le\; \Big(2^{-n}\prod_{i=1}^{n}\int_{\mathbb{R}}\left|\partial_{x_i}\varphi(x_1,\dots,y_i\,,\dots,x_n)\right| dy_i\Big)^{\frac{1}{n-1}}.$$

We integrate this inequality with respect to $x_1,\dots,x_n$ and apply Hölder's inequality repeatedly to get

$$\int_{\mathbb{R}^n}\left|\varphi(x)\right|^{\frac{n}{n-1}} dx \;\le\; \Big(2^{-n}\prod_{i=1}^{n}\|\partial_{x_i}\varphi\|_{L^1(\mathbb{R}^n)}\Big)^{\frac{1}{n-1}}.$$

By Young's inequality,

$$\prod_{i=1}^{n}|a_i| \;\le\; \frac{1}{n}\sum_{i=1}^{n}|a_i|^n\,,$$

followed by $\sum|a_i| \le \left(n\sum a_i^{\,2}\right)^{\frac12}$, we infer that

$$\|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \;\le\; \frac{1}{2\sqrt{n}}\,\|\nabla\varphi\|_{L^1(\mathbb{R}^n)}\,.$$

This implies the statement of the lemma for $p = 1$. In case of $p \in (1,n)$ we apply the inequality to $|\varphi|^\gamma$ with $\gamma := \frac{(n-1)p}{n-p} > 1$ to find

$$\begin{aligned} \Big(\int_{\mathbb{R}^n}\left|\varphi(x)\right|^{\frac{np}{n-p}} dx\Big)^{\frac{n-1}{n}} &\le \frac{\gamma}{2\sqrt{n}}\int_{\mathbb{R}^n}\left|\varphi(x)\right|^{\gamma-1}\left|\nabla\varphi(x)\right| dx \\ &\le \frac{\gamma}{2\sqrt{n}}\Big(\int_{\mathbb{R}^n}\left|\varphi(x)\right|^{\frac{(\gamma-1)p}{p-1}} dx\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^n}\left|\nabla\varphi(x)\right|^p dx\Big)^{\frac{1}{p}} \end{aligned}$$

by virtue of Hölder's inequality. Eventually, the choice of $\gamma$ makes for a simplification of the involved exponents, that is, we have $\frac{(\gamma-1)p}{p-1} = \frac{np}{n-p} = \left(\frac{n-1}{n} - \frac{p-1}{p}\right)^{-1}$. This turns the inequality into the Sobolev inequality as stated in the lemma. ∎

As we can clearly see, lemma 2.7.1 only considers the case $p < n$. Under the opposite condition $p > n \geq 1$ the preceding inequality fails. We shall need another embedding lemma to cover this case, usually known as Morrey inequality. Here we only formulate a version of this inequality that is somewhat weaker than the original one but entirely sufficient for our purposes.

**Lemma 2.7.2 (Morrey-type inequality)** *Let $k$ be a positive integer and $p \geq 1$ be any number satisfying the condition $kp > n$. Then there exists a positive constant $c = c(n,k,p)$ such that*

$$\big|\varphi(x)\big| \leq c\,\|\varphi\|_{W^{k,p}(\mathbb{R}^n)}$$

*for all $x \in \mathbb{R}^n$ and all $\varphi \in C^k(\mathbb{R}^n)$.*

**Proof:** Assume first that $k = 1$. We fix $x \in \mathbb{R}^n$ and apply Hölder's inequality to get

$$\begin{aligned} \big|\varphi(x)\big| &\leq \fint_{B_1^{eu}(x)} \big|\varphi(y)\big|\,dy + \fint_{B_1^{eu}(x)} \big|\varphi(x)-\varphi(y)\big|\,dy \\ &\leq \big|B_1^{eu}(x)\big|^{-\frac{1}{p}}\,\|\varphi\|_{L^p(\mathbb{R}^n)} + \fint_{B_1^{eu}(x)} \big|\varphi(x)-\varphi(y)\big|\,dy \end{aligned}$$

with $B_1^{eu}(x) = \big\{y \in \mathbb{R}^n \,\big|\, |x-y| < 1\big\}$ being the Euclidean unit ball centered at $x$. This combined with the inequality

$$\fint_{B_1^{eu}(x)} \big|\varphi(y)-\varphi(x)\big|\,dy \;\leq\; c(n) \int_{B_1^{eu}(x)} \frac{\big|\nabla\varphi(y)\big|}{|x-y|^{n-1}}\,dy$$

yields

$$\big|\varphi(x)\big| \lesssim \|\varphi\|_{L^p(\mathbb{R}^n)} + \Big(\int_{B_1^{eu}(x)} |x-y|^{-\frac{p(n-1)}{p-1}}\,dy\Big)^{\frac{p-1}{p}} \|\nabla\varphi\|_{L^p(\mathbb{R}^n)}\,,$$

after another application of Hölder's inequality. Now, the assumptions on $n,p$ imply $\frac{p(n-1)}{p-1} < n$, whence it follows that

$$\widetilde{\varphi}(y) := |x-y|^{-\frac{n-1}{p-1}} \in L^p\big(B_1^{eu}(x)\big)$$

for any $x \in \mathbb{R}^n$. Putting this together, we obtain $\big|\varphi(x)\big| \leq c(n,p)\,\|\varphi\|_{W^{1,p}(\mathbb{R}^n)}$.

Suppose now $k$ is any positive integer. Then one can show that there exists a constant $c(n,k)$ such that

$$\big|\varphi(x)\big| \leq \sum_{j=0}^{k-1} \frac{1}{j!} \fint_{B_1^{eu}(x)} \big|D_x^j\varphi(y)\big|\,dy + c(n,k) \int_{B_1^{eu}(x)} \frac{\big|D_x^k\varphi(y)\big|}{|x-y|^{n-k}}\,dy$$ [4]

for arbitrary points $x \in \mathbb{R}^n$. Similarly as above, we use Hölder's inequality together with $kp > n$ to find

$$\big|\varphi(x)\big| \lesssim \sum_{j=0}^{k} \|D_x^j\varphi\|_{L^p(\mathbb{R}^n)}\,.$$

This completes the proof of the lemma. ■

**Corollary 2.7.3** *Suppose $\Omega \subseteq H$ is open and satisfies the cone condition. Further let $p \geq 1$ and $k \in \mathbb{N}$ with $kp > n$. If $u \in W^{k,p}(\Omega)$, then the assertion of lemma 2.7.2 holds for almost every $x \in \Omega$, where this time the constant $c$ depends on $n,k,p$ and $\Omega$.*

[4] In [27] a proof is provided for $k = 1$. The general case may be found in [1]. Indeed, this inequality holds on arbitrary finite cones having their vertex at $x \in \Omega$. Hence, lemma 2.7.3 remains valid for all domains that satisfy the cone condition.

**Proof:** Let $u \in W^{k,p}(\Omega)$, then by proposition 2.5.1 one can find a sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ in $C^\infty(\Omega)$ with $\varphi_i \to u$ in $W^{k,p}(\Omega)$. For $i_1, i_2 \in \mathbb{N}$, we find

$$\big|\varphi_{i_1}(x) - \varphi_{i_2}(x)\big| \lesssim \|\varphi_{i_1} - \varphi_{i_2}\|_{W^{k,p}(\Omega)}$$

for all $x \in \Omega$. Since $\{\varphi_i\}_{i\in\mathbb{N}}$ is in particular a Cauchy sequence in $W^{k,p}(\Omega)$, $\big\{\varphi_i(x)\big\}_{i\in\mathbb{N}}$ is also a Cauchy sequence and therefore converges for every $x \in \Omega$. By uniqueness of the distributional limit, this implies $\varphi(x) \to u(x)$ for almost every $x$. But now

$$\big|u(x)\big| \le \big|u(x) - \varphi_i(x)\big| + \big|\varphi_i(x)\big| \lesssim \big|u(x) - \varphi_i(x)\big| + \|\varphi_i\|_{W^{k,p}(\Omega)}$$

by lemma 2.7.2, and the right hand side approaches $\|u\|_{W^{k,p}(\Omega)}$ for almost every $x \in \Omega$ as $i \to \infty$. The proof is complete. ■

As a next step we present a version of lemma 2.7.1 that includes weights of the form $x_n^\sigma$. The presented inequality interpolates between two well-known inequalities: There is on the one hand the classical Sobolev inequality in a weighted setting, and on the other hand Hardy's inequality.

**Proposition 2.7.4** *Let $\sigma > -1$ and $1 \le q \le p < \infty$ be any numbers satisfying the conditions $\sigma > -\frac{1}{p}$ and*

$$\frac{1}{p} = \frac{1}{q} - \frac{1-\theta}{n}, \qquad \theta \in [0,1]\,.$$

*Then we have*

$$\|x_n^\sigma \varphi\|_{L^p(H)} \le c(n,\sigma,p,q)\, \|x_n^{\sigma+\theta} \nabla\varphi\|_{L^q(H)}$$

*for all $\varphi \in C_c^1(\overline{H})$.*

Note that we consider the more general case $\varphi \in C_c^1(\overline{H})$. Such functions, in contrast to $\varphi \in C_c^1(H)$, may take values towards the boundary of $H$.

**Proof:** Assume first that $q = 1$. For $\psi(x) := \varphi(x', 2x_n) - \varphi(x)$, $A_i := \{x \in \mathbb{R}^n \mid 2^i < x_n \le 2^{i+1}\}$ and $B_i := A_i \cup A_{i+1} = \{x \in \mathbb{R}^n \mid 2^i < x_n \le 2^{i+2}\}$, the fundamental theorem of calculus shows that

$$\begin{aligned}\|\psi\|_{L^1(A_i,\mu_\sigma)} &= \|x_n^\sigma \int_{x_n}^{2x_n} \partial_z\varphi(\,\cdot\,,z)\,dz\|_{L^1(A_i)} \le \int_{2^i}^{2^{i+2}} \|\partial_z\varphi(\,\cdot\,,z)\|_{L^1(\mathbb{R}^{n-1})} \Big(\int_{\frac{z}{2}}^{z} x_n^\sigma\,dx_n\Big)\,dz \\ &\le \frac{1}{\sigma+1}\,\|\partial_{x_n}\varphi\|_{L^1(B_i,\mu_{\sigma+1})}\,, \end{aligned} \tag{$*$}$$

where we used Fubini's theorem in the first inequality. Now once we have established the inequality

$$\|x_n^\sigma \psi\|_{L^{\frac{n}{n-1}}(A_i)} \lesssim \|x_n^\sigma \nabla\varphi\|_{L^1(B_i)}\,, \tag{$**$}$$

we recover the statement of the proposition by interpolation between the spaces $L^{\frac{n}{n-1}}$ and $L^1$. Indeed,

$$\|x_n^\sigma \psi\|_{L^p(H)} = \sum_{i\in\mathbb{Z}} \|x_n^\sigma \psi\|_{L^p(A_i)} \le \sum_{i\in\mathbb{Z}} \|x_n^\sigma \psi\|_{L^r(A_i)}^{1-\theta}\, \|x_n^\sigma \psi\|_{L^q(A_i)}^{\theta}\,,$$

where $\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{q}$, and $0 \le \theta \le 1$ is to be selected. In order to be able to apply $(*)$ and $(**)$, we choose $r = \frac{n}{n-1}$ and $q = 1$ such that $\theta$ has to be $1 + \frac{n}{p} - n$. This gives

$$\|x_n^\sigma \psi\|_{L^p(H)} \lesssim \sum_{i\in\mathbb{Z}} \|x_n^\sigma \nabla\varphi\|_{L^1(B_i)}^{1-\theta}\, \|x_n^{\sigma+1} \nabla\varphi\|_{L^1(B_i)}^{\theta}\,.$$

Restricted to $B_i$ we have $2^i < x_n \le 2^{i+2}$. This implies that the right hand side is (up to a constant) bounded by

$$\sum_{i\in\mathbb{Z}} 2^{i(\sigma+\theta)}\, \|\nabla\varphi\|_{L^1(B_i)} \;\le\; c(n,\sigma,p)\, \|x_n^{\sigma+\theta}\, \nabla\varphi\|_{L^1(H)}\,.$$

We recall the definition of $\psi$ and arrive at

$$\begin{aligned}\|x_n^{\sigma}\, \varphi\|_{L^p(H)} \;&\le\; \|x_n^{\sigma}\, \varphi(\cdot, 2x_n)\|_{L^p(H)} \,+\, \|x_n^{\sigma}\, \psi\|_{L^p(H)}\\ &\le\; 2^{-\sigma-\frac{1}{p}}\, \|x_n^{\sigma}\, \varphi\|_{L^p(H)} \,+\, c(n,\sigma,p)\, \|x_n^{\sigma+\theta}\, \nabla\varphi\|_{L^1(H)}\end{aligned}$$

by the transformation formula and the above estimate. Now the assumption $\sigma > -\frac{1}{p}$ allows us to subtract $2^{-\sigma-1/p}\, \|x_n^{\sigma}\, \varphi\|_{L^p(H)}$ from both sides and then divide by $1 - 2^{-\sigma-1/p} > 0$. The proposition follows for $q = 1$ and $\theta = 1 + n\big(\frac{1}{p} - 1\big)$. As by assumption $0 \le \theta \le 1$, this requires $1 \le p \le \frac{n}{n-1}$.

The general case can now be reduced to the case $q = 1$. To see this, let $1 \le r \le \frac{n}{n-1}$ be the number satisfying $\frac{1}{r} = \theta + (1-\theta)\frac{n-1}{n}$ with $0 \le \theta \le 1$. Using the above estimate and Hölder's inequality we see

$$\begin{aligned}\|x_n^{\sigma}\, \varphi\|_{L^p(H)}^{\frac{p}{r}} \;&=\; \|x_n^{\frac{\sigma p}{r}}\, |\varphi|^{\frac{p}{r}}\|_{L^r(H)} \;\lesssim\; \|x_n^{\frac{\sigma p}{r}+\theta}\, \nabla\big(|\varphi|^{\frac{p}{r}}\big)\|_{L^1(H)}\\ &=\; \|\big(x_n^{\sigma}\, |\varphi|\big)^{\frac{p}{r}-1}\, x_n^{\sigma+\theta}\, \nabla\varphi\|_{L^1(H)} \;\le\; \|x_n^{\sigma}\, \varphi\|_{L^p(H)}^{\frac{p}{r}-1}\, \|x_n^{\sigma+\theta}\, \nabla\varphi\|_{L^q(H)}\,,\end{aligned}$$

if $\big(\frac{p}{r} - 1\big)\frac{q}{q-1} = p$ and $\frac{q-1}{q} = \frac{1}{p}\big(\frac{p}{r} - 1\big)$. These equalities and the condition on $r$ then demand to take $\theta \in [0,1]$ such that

$$\frac{1}{p} \;=\; \frac{1}{q} \;-\; \frac{1-\theta}{n}\,.$$

It remains to prove $(**)$. First we assume that $n \ge 2$. If $\psi$, $A_i$ and $B_i$ are as above, then we have

$$\begin{aligned}\|x_n^{\sigma}\, \psi\|_{L^{\frac{n}{n-1}}(A_i)}^{\frac{n}{n-1}} \;&=\; \int_{2^i}^{2^{i+1}} \big\|\big|x_n^{\sigma}\, \psi(\cdot\,, x_n)\big|\, \big|x_n^{\sigma}\, \psi(\cdot\,, x_n)\big|^{\frac{1}{n-1}}\big\|_{L^1(\mathbb{R}^{n-1})}\, dx_n\\ &\le\; \int_{2^i}^{2^{i+1}} \|x_n^{\sigma}\, \psi(\cdot\,, x_n)\|_{L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})}\, \|x_n^{\sigma}\, \psi(\cdot\,, x_n)\|_{L^1(\mathbb{R}^{n-1})}^{\frac{1}{n-1}}\, dx_n\\ &\le\; \int_{2^i}^{2^{i+1}} x_n^{\sigma}\, \|\psi(\cdot\,, x_n)\|_{L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})}\, dx_n \Big(\sup_{2^i<x_n<2^{i+1}} x_n^{\sigma}\, \|\psi(\cdot\,, x_n)\|_{L^1(\mathbb{R}^{n-1})}\Big)^{\frac{1}{n-1}}\end{aligned}$$

by Hölder's inequality with $\frac{n-2}{n-1} + \frac{1}{n-1} = 1$. Now, lemma 2.7.1 on $\mathbb{R}^{n-1}$ yields

$$\begin{aligned}\int_{2^i}^{2^{i+1}} x_n^{\sigma}\, \|\psi(\cdot\,, x_n)\|_{L^{\frac{n-1}{n-2}}(\mathbb{R}^{n-1})}\, dx_n \;&\lesssim\; \int_{2^i}^{2^{i+1}} x_n^{\sigma}\, \|\nabla'\psi(\cdot\,, x_n)\|_{L^1(\mathbb{R}^{n-1})}\, dx_n\\ &\lesssim\; \int_{2^i}^{2^{i+2}} x_n^{\sigma}\, \|\nabla'\varphi(\cdot\,, x_n)\|_{L^1(\mathbb{R}^{n-1})}\, dx_n\end{aligned}$$

by the triangle inequality and a substitution in the term $\psi(\cdot\,, x_n) = \varphi(\cdot\,, 2x_n) - \varphi(\cdot\,, x_n)$. For the second term we proceed as follows:

$$\begin{aligned}\sup_{2^i<x_n<2^{i+1}} x_n^{\sigma}\, \|\psi(\cdot\,, x_n)\|_{L^1(\mathbb{R}^{n-1})} \;&\le\; \sup_{2^i<x_n<2^{i+1}} x_n^{\sigma} \int_{x_n}^{2x_n} \|\partial_z\varphi(\cdot\,, z)\|_{L^1(\mathbb{R}^{n-1})}\, dz\\ &\le\; 4^{|\sigma|} \int_{2^i}^{2^{i+2}} z^{\sigma}\, \|\partial_z\varphi(\cdot\,, z)\|_{L^1(\mathbb{R}^{n-1})}\, dz\,.\end{aligned}$$

(Similarly, if $n = 1$ we get $\|x^{\sigma}\, \psi\|_{L^\infty(A_i)} \le c(\sigma)\|x^{\sigma}\, \varphi'\|_{L^1(B_i)}$ and hence the desired estimate). Altogether, this gives

$$\|x_n^{\sigma}\, \psi\|_{L^{\frac{n}{n-1}}(A_i)}^{\frac{n}{n-1}} \;\lesssim\; \|x_n^{\sigma}\, \nabla'\varphi\|_{L^1(B_i)}\, \|x_n^{\sigma}\, \partial_{x_n}\varphi\|_{L^1(B_i)}^{\frac{1}{n-1}} \;\le\; \frac{n-1}{n}\, \|x_n^{\sigma}\, \nabla'\varphi\|_{L^1(B_i)}^{\frac{n}{n-1}} \,+\, \frac{1}{n}\, \|x_n^{\sigma}\, \partial_{x_n}\varphi\|_{L^1(B_i)}^{\frac{n}{n-1}}$$

because $\frac{n-1}{n} + \frac{1}{n} = 1$ (Young exponents). Finally, we realize that any two norms in finite dimensions are equivalent so that

$$\|x_n^\sigma\,\nabla'\varphi\|_{L^1(B_i)}^{\frac{n}{n-1}} + \|x_n^\sigma\,\partial_{x_n}\varphi\|_{L^1(B_i)}^{\frac{n}{n-1}} \;\le\; c_n\,\|x_n^\sigma\,\big(|\nabla'\varphi| + |\partial_{x_n}\varphi|\big)\|_{L^1(B_i)}^{\frac{n}{n-1}} \;\le\; 2\,c_n\,\|x_n^\sigma\,\nabla\varphi\|_{L^1(B_i)}^{\frac{n}{n-1}}\,.$$

This proves the full statement. ■

**Corollary 2.7.5** *Let $k \in \mathbb{N}$ and $\theta \in [0,1]$. If $1 \le q \le p < \infty$ are any numbers satisfying $\frac{1}{p} = \frac{1}{q} - \frac{k(1-\theta)}{n}$, $\sigma > -1$ as well as $\sigma \ge -k\theta p$, and $\sigma_k \ge \dots \ge \sigma_0 > -1$ such that $\sigma_{j+1} - \sigma_j \le q$ and $\sigma_k = \big(\frac{\sigma}{p} + k\,\theta\big)q$ hold. Then we have*

$$\|u\|_{L^p(H,\mu_\sigma)} \;\le\; c(n,\sigma,p,q)\,\|D_x^k u\|_{L^q(H,\mu_{\sigma_k})} \tag{2.7.1}$$

*for all $u \in W^{k,q}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})$.*

**Proof:** Suppose $\varphi \in C_c^\infty(\overline{H})$. Taking proposition 2.7.4 as a starting point for an iteration on the derivatives we attain

$$\|\varphi\|_{L^p(\mu_\sigma)} \;\lesssim\; \|D_x^k\varphi\|_{L^q(\mu_{\sigma_k})}\,, \tag{$*$}$$

where $p, q, \sigma, \sigma_k$ are as in the current corollary.

The assumptions on $\sigma_0,\dots,\sigma_k$ imply the assumptions of theorem 2.5.5 or rather remark 2.6.3. This means the closure of $C_c^\infty(\overline{H})$ in the norm $\|\cdot\|_{W^{1,q}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})}$ is the space $W^{1,q}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})$ and we can find a sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ in $C_c^\infty(\overline{H})$ that converges to $u \in W^{1,q}(H,\mu_{\sigma_0},\dots,\mu_{\sigma_k})$. By $(*)$ we get

$$\|u\|_{L^p(\mu_\sigma)} \;\lesssim\; \|u - \varphi_i\|_{L^p(\mu_\sigma)} + \|D_x^k\varphi_i\|_{L^q(\mu_{\sigma_k})}\,. \tag{$**$}$$

We apply the iterated Hardy-Sobolev inequality $(*)$ yet another time which leads us to the estimate

$$\|\varphi_{i_1} - \varphi_{i_2}\|_{L^p(\mu_\sigma)} \;\lesssim\; \|D_x^k(\varphi_{i_1} - \varphi_{i_2})\|_{L^q(\mu_{\sigma_k})}\,.$$

Since $\{D_x^k\varphi_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mu_{\sigma_k})$, this shows that $\{\varphi_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu_\sigma)$. It follows that $\varphi_i \to u$ in $L^p(\mu_\sigma)$. To see this, note that $L^p(\mu_\sigma)$ is a Banach space, and hence the Cauchy sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ is convergent to some function $v$ in that space. But we also have $\varphi_i \to u$ in $L^q(\mu_{\sigma_0})$, and since any distributional limit is unique, we have $v = u$. We finish the proof by sending $i \to \infty$ in $(**)$. ■

**Remark 2.7.6** *The previous estimate not only combines the Hardy inequality and the classical Sobolev inequality into one, but also generalizes them to weighted Sobolev spaces. Take for example $\theta = 0$, then $p = q^* = \frac{nq}{n-kq}$ with $kq < n$ and and we obtain an iterated version of lemma 2.7.1 in the weighted Sobolev classes. On the other hand, for the second endpoint $\theta = 1$, we get $q = p$ and $\sigma_k = \sigma + kp$, such that (2.7.1) becomes the Hardy inequality (cf. lemma 2.6.1).*

## 2.8 Interpolation Inequalities

Suppose we have $q \le r \le p$ and $\varphi \in L^q \cap L^p$. Then it is $\frac{1}{p} \le \frac{1}{r} \le \frac{1}{q}$ and there exists a $\theta \in [0,1]$ such that

$$\frac{1}{r} \;=\; \frac{1-\theta}{q} + \frac{\theta}{p}\,. \tag{2.8.1}$$

Therefore, $\frac{q}{(1-\theta)r}$ and $\frac{p}{\theta r}$ are conjugate exponents, and $|\varphi|^{(1-\theta)r} \in L^{\frac{q}{(1-\theta)r}}$ and $|\varphi|^{\theta r} \in L^{\frac{p}{\theta r}}$. Hence, we can apply Hölder's inequality from which follows that $\varphi$ is also in $L^r$. To be more precise, we have

$$\|\varphi\|_{L^r} \;\le\; \|\varphi\|_{L^q}^{1-\theta}\,\|\varphi\|_{L^p}^{\theta}\,.$$

Such an inequality is called interpolation inequality as it interpolates between the $L^q$-norm and the $L^p$-norm of $\varphi$. Now we are interested in a generalization of this simple result in as much as we would like to additionally involve an interpolation between certain derivatives of $\varphi$. The unweighted interpolation is usually known as Gagliardo-Nirenberg interpolation - the proof is largely taken from [33]. In a second step, we formulate this result for weighted Sobolev spaces.

**Lemma 2.8.1 (Gagliardo-Nirenberg interpolation)** *Let $j, k \in \mathbb{N}$ be such that $0 \leq \theta := \frac{j}{k} < 1$, and $1 \leq r, p, q \leq \infty$ satisfying (2.8.1). Then there exists a positive constant $c$ that depends only on $n$ and $r$ such that*

$$\|D_x^j \varphi\|_{L^r(H_a)} \;\leq\; c\, \|\varphi\|_{L^q(H_a)}^{1-\theta}\, \|D_x^k \varphi\|_{L^p(H_a)}^{\theta}$$

*for all $\varphi \in C_c^k(\overline{H}_a)$, where $H_a = (a, \infty) \times \mathbb{R}^{n-1}$ is the upper half space beginning at $a \in \mathbb{R}$.*

**Proof:** Suppose first that $j + 1 = 2 = k$, $p > 1$, $q < \infty$ and $n = 1$. The statement then follows from the inequality

$$\int_I |\varphi'(x)|^r\, dx \;\leq\; c(r)\, |I|^{1-r\left(1+\frac{1}{q}\right)} \left(\int_I |\varphi(x)|^q\, dx\right)^{\frac{r}{q}} + c(r)\, |I|^{1+r\left(1-\frac{1}{p}\right)} \left(\int_I |\varphi''(x)|^p\, dx\right)^{\frac{r}{p}}, \qquad (*)$$

where $\frac{2}{r} = \frac{1}{q} + \frac{1}{p}$ and $I = (a, b)$ is any bounded interval in $\mathbb{R}$.

Take any integer $N \in \mathbb{N}$ and define $\widetilde{I}_1 := (a, a + \frac{|I|}{N})$. If the second summand in $(*)$ is larger than the first one, we set $I_1 = \widetilde{I}_1 =: (a_1, b_1)$. This implies

$$\int_{I_1} |\varphi'(x)|^r\, dx \;\leq\; 2\, c(r) \left(\frac{|I|}{N}\right)^{1+r\left(1-\frac{1}{p}\right)} \left(\int_I |\varphi''(x)|^p\, dx\right)^{\frac{r}{p}}.$$

If the first term is larger, we take $I_1 = (a_1, b_1)$ where $a_1 = a$ and $b_1 > a + \frac{|I|}{N}$ such that both terms on the right hand side of $(*)$ are equal. Such an interval exists since, by assumption, we have $\varphi \in C_c^2(\overline{H}_a)$: $\varphi'' \equiv 0$ implies that $\varphi$ is linear and hence constant to 0. Additionally, we have that the factor

$$|I_1|^{1-r\left(1+\frac{1}{q}\right)}$$

is monotonically decreasing for $q < \infty$, while the other prefactor increases as $|I_1|$ gets larger. Consequently, we can find such an interval for which

$$\int_{I_1} |\varphi'(x)|^r\, dx \;\leq\; 2\, c(r)\, |I_1|^{1-\frac{r}{2}\left(\frac{1}{q}+\frac{1}{p}\right)} \left(\int_{I_1} |\varphi(x)|^q\, dx\right)^{\frac{r}{2q}} \left(\int_{I_1} |\varphi''(x)|^p\, dx\right)^{\frac{r}{2p}}.$$

Here we also used the simple equality $a + b = 2(ab)^{\frac{1}{2}}$ provided we have $a = b$. Therefore with $\frac{2}{r} = \frac{1}{q} + \frac{1}{p}$, we arrive at

$$\int_{I_1} |\varphi'(x)|^r\, dx \;\lesssim\; \left(\frac{|I|}{N}\right)^{1+r\left(1-\frac{1}{p}\right)} \left(\int_I |\varphi''(x)|^p\, dx\right)^{\frac{r}{p}} + \left(\int_{I_1} |\varphi(x)|^q\, dx\right)^{\frac{r}{2q}} \left(\int_{I_1} |\varphi''(x)|^p\, dx\right)^{\frac{r}{2p}}.$$

Repeating the procedure $N_0$ times until $b_{N_0-1} < b \leq b_{N_0}$, this process must eventually stop since $a + N\frac{|I|}{N} = b$, i.e. $N_0 \leq N$. Summing over the disjoint intervals $I_l$ ($1 \leq l \leq N_0$) and using Hölder's inequality then yields

$$\begin{aligned}
\int_I |\varphi'(x)|^r\, dx \;&\leq\; \int_a^{b_{N_0}} |\varphi'(x)|^r\, dx \;=\; \sum_{l=1}^{N_0} \int_{I_l} |\varphi'(x)|^r\, dx \\
&\lesssim\; |I| \left(\frac{|I|}{N}\right)^{r\left(1-\frac{1}{p}\right)} \left(\int_I |\varphi''(x)|^p\, dx\right)^{\frac{r}{p}} + \left(\int_a^{\infty} |\varphi(x)|^q\, dx\right)^{\frac{r}{2q}} \left(\int_a^{\infty} |\varphi''(x)|^p\, dx\right)^{\frac{r}{2p}}
\end{aligned}$$

Since we assume that $p > 1$, the first term tends to zero as $N \to \infty$ and the desired inequality follows for $n = 1$, $j = 1$ and $k = 2$, $q < \infty$ and $p > 1$. The cases $q = \infty$ and $p = 1$ are attained by sending $q \to \infty$ and $p \to 1$ in the above inequalities.

For $n \geq 2$ consider $\varphi$ as a function of $x_{i_0}$ treating all the other variables as parameters. Integrating with respect to the other variables $x_i$ with $i \neq i_0$ and using Hölder's inequality allows us to prove the $n$-dimensional case.

Finally, if we assume the assertion holds for $j = 1$ and $k = 2$, we can perform an induction on $k$ to show that the general statement is true for any $0 \leq j < k \in \mathbb{N}$.

**Proof of inequality (∗):** Suppose $I = (a, b)$ is an interval in $\mathbb{R}$ with $|I| < \infty$ and $\varphi \in C^2(I)$. Moreover, let $x_1, x_2 \in I$ be such that

$$a \;<\; x_1 \;<\; a + \frac{|I|}{4} \;<\; a + \frac{3\,|I|}{4} \;=\; b - \frac{|I|}{4} \;<\; x_2 \;<\; b\,.$$

Then by the intermediate value theorem, there exists a $\xi \in (x_1, x_2)$ such that

$$\varphi'(\xi) \;=\; \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1}\,.$$

By the fundamental theorem of calculus, we thus have

$$\big|\varphi'(x)\big| \;=\; \Big|\varphi'(\xi) + \int_\xi^x \varphi''(y)\,dy\Big| \;\leq\; 2\,\frac{\big|\varphi(x_1)\big| + \big|\varphi(x_2)\big|}{|I|} \;+\; \int_0^b \big|\varphi''(y)\big|\,dy$$

for all $x \in I$. We then integrate with respect to $x_1, x_2$ in their respective intervals to get

$$\begin{aligned}\frac{|I|^2}{16}\,\big|\varphi'(x)\big| \;&\leq\; \frac{1}{2}\int_a^{a+\frac{|I|}{4}} \big|\varphi(x_1)\big|\,dx_1 \;+\; \frac{1}{2}\int_{b-\frac{|I|}{4}}^b \big|\varphi(x_2)\big|\,dx_2 \;+\; \frac{|I|^2}{16}\int_I \big|\varphi''(y)\big|\,dy \\ &\leq\; \frac{1}{2}\int_I \big|\varphi(y)\big|\,dy \;+\; \frac{|I|^2}{16}\int_I \big|\varphi''(y)\big|\,dy\,.\end{aligned}$$

Taking the $r$-th power and applying Hölder's inequality to both summands on the right hand side yields

$$\big|\varphi'(x)\big|^p \;\leq\; c\bigg(|I|^{-r-\frac{r}{q}}\Big(\int_I \big|\varphi(y)\big|^q\,dy\Big)^{\frac{r}{q}} \;+\; |I|^{r-\frac{r}{p}}\Big(\int_I \big|\varphi(y)\big|^p\,dy\Big)^{\frac{r}{p}}\bigg)\,.$$

Now integration of $x$ over $I$ proves (∗) for any bounded interval $I \subset \mathbb{R}$, $\varphi \in C^2(I)$ and for all $1 \leq r, q, p \leq \infty$. Also note that the constant $c$ is independent of $I$, $q$ and $p$, and so we have $c = c(r)$ as required. ■

As a sample application of lemma 2.8.1, let us present a weighted version of this inequality. For special - albeit similar to ours - power weights such as $|x|^\sigma$ the corresponding embedding is treated in [60]. A generalization to weighted Sobolev spaces may be found in [67], where the authors prove a Gagliardo-Nirenberg interpolation for a broader class of weights.

**Proposition 2.8.2** *Let $j, k$ be any integers satisfying $0 \leq \theta := \frac{j}{k} < 1$, and $r, p, q, \sigma$ be any positive numbers with $\sigma > 0$ and $1 \leq r, q, p \leq \infty$, respectively. If $u$ is a function in $L^q(H, \mu_\sigma) \cap W^{k,p}(H, \mu_\sigma)$, then we have*

$$\|D_x^j u\|_{L^r(H,\,\mu_\sigma)} \;\leq\; c\,\|u\|_{L^q(H,\,\mu_\sigma)}^{1-\theta}\,\|D_x^k u\|_{L^p(H,\,\mu_\sigma)}^{\theta}\,,$$

*where*

$$\frac{1}{r} \;=\; \frac{1-\theta}{q} \;+\; \frac{\theta}{p}$$

*and $c$ is the constant from lemma 2.8.1.*

**Proof:** Applying the Gagliardo-Nirenberg inequality 2.8.1 to $\varphi \in C_c^\infty(\overline{H})$ we find

$$\begin{aligned}\int_H \left|D_x^j\varphi(x)\right|^r x_n^\sigma\,dx &= \sigma\int_0^\infty\Big(\int_{\mathbb{R}^{n-1}}\left|D_x^j\varphi(x)\right|^r dx'\Big)\Big(\int_0^{x_n} y^{\sigma-1}\,dy\Big)\,dx_n\\ &= \sigma\int_0^\infty\Big(\int_y^\infty\int_{\mathbb{R}^{n-1}}\left|D_x^j\varphi(x)\right|^r dx'\,dx_n\Big)\,y^{\sigma-1}\,dy\\ &\lesssim \sigma\int_0^\infty\Big(\int_y^\infty\int_{\mathbb{R}^{n-1}}\left|\varphi(x)\right|^q dx\Big)^{\frac{(1-\theta)r}{q}}\Big(\int_y^\infty\int_{\mathbb{R}^{n-1}}\left|D_x^k\varphi(x)\right|^p dx\Big)^{\frac{\theta r}{p}} y^{\sigma-1}\,dy\,,\end{aligned}$$

where $\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p}$ with $\theta = \frac{j}{k}$. In the second line we used Fubini's theorem. For $\big(\frac{q}{(1-\theta)r}\,,\,\frac{p}{\theta r}\big)$ is a Hölder pair, we get

$$\int_H \left|D_x^j\varphi(x)\right|^r x_n^\sigma\,dx \;\lesssim\; \sigma\int_0^\infty\Big(\int_{H_y}\left|\varphi(x)\right|^q dx\,y^{\sigma-1}\Big)^{\frac{(1-\theta)r}{q}}\Big(\int_{H_y}\left|D_x^k\varphi(x)\right|^p dx\,y^{\sigma-1}\Big)^{\frac{\theta r}{p}} dy\,,$$

with $H_y := (y,\infty)\times\mathbb{R}^{n-1}$ for $y\in(0,\infty)$. Hölder's inequality and a further application of Fubini's theorem then give the upper bound

$$\begin{aligned}&\sigma\Big(\int_H\left|\varphi(x)\right|^q\int_0^{x_n} y^{\sigma-1}\,dy\,dx\Big)^{\frac{(1-\theta)r}{q}}\Big(\int_H\left|D_x^k\varphi(x)\right|^p\int_0^{x_n} y^{\sigma-1}\,dy\,dx\Big)^{\frac{\theta r}{p}}\\ &= \Big(\int_H\left|\varphi(x)\right|^q x_n^\sigma\,dx\Big)^{\frac{(1-\theta)r}{q}}\Big(\int_H\left|D_x^k\varphi(x)\right|^p x_n^\sigma\,dx\Big)^{\frac{\theta r}{p}}\end{aligned}$$

with $\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{p}$ and $1\le r,q,p\le\infty$. Since all the estimates but the Gagliardo-Nirenberg interpolation are sharp we have that the constant appearing on the right hand side of the inequality is the the same one as in lemma 2.8.1. This implies the proposition for smooth functions. Since these are dense in $L^q(\mu_\sigma)\cap W^{k,p}(\mu_\sigma)$ for nonnegative $\sigma$, one can find a sequence $\{\varphi_i\}_{i\in\mathbb{N}}\subset C_c^\infty(\overline{H})$ that converges to $u$ in $L^q(\mu_\sigma)\cap W^{k,p}(\mu_\sigma)$. As before, we conclude that also $D_x^j\varphi_i \to D_x^j u$ in $L^r(\mu_\sigma)$ such that

$$\|D_x^j u\|_{L^r(\mu_\sigma)} \;\lesssim\; \|D_x^j u - D_x^j\varphi_i\|_{L^r(\mu_\sigma)} \;+\; \|\varphi_i\|_{L^q(\mu_\sigma)}^{1-\theta}\,\|D_x^k\varphi_i\|_{L^p(\mu_\sigma)}^{\theta} \;\to\; \|u\|_{L^q(\mu_\sigma)}^{1-\theta}\,\|D_x^k u\|_{L^p(\mu_\sigma)}^{\theta}$$

as $i$ approaches infinity. This finishes the proof. ■

Next we propose a lemma that also interpolates between the power weights on the right hand side. Fortunately, a proof only requires a slight modification in the previous proof.

**Lemma 2.8.3** *Let $p\ge 2$ and $\sigma>0$. Then for any $\varepsilon>0$ there exists a constant $c$ that depends on $n,\sigma,p$ such that*

$$\|\nabla u\|_{L^p(H,\mu_{\sigma+1})} \;\le\; c\,\varepsilon^{-1}\,\|u\|_{L^p(H,\mu_\sigma)} \;+\; \varepsilon\,\|D_x^2 u\|_{L^p(H,\mu_{\sigma+2})}$$

*for all $u\in W^{2,p}(H,\mu_\sigma,\mu_{\sigma_1},\mu_{\sigma+2})$ with $\sigma_1\in[\sigma,\sigma+2]$.*

**Proof:** Putting $j=1$, $k=2$ as well as $r=q=p$, we obtain

$$\|\nabla\varphi\|_{L^p(\mu_{\sigma+1})}^p \;\lesssim\; \int_0^\infty\Big(\int_y^\infty\int_{\mathbb{R}^{n-1}}\left|\varphi(x)\right|^p dx\Big)^{\frac{1}{2}}\Big(\int_y^\infty\int_{\mathbb{R}^{n-1}}\left|D_x^2\varphi(x)\right|^p dx\Big)^{\frac{1}{2}} y^\sigma\,dy\,,$$

just as above. Using

$$\sigma \;=\; \frac{\sigma-1}{2} + \frac{\sigma+1}{2}\,,$$

this amounts to

$$\begin{aligned}\|\nabla\varphi\|_{L^p(\mu_{\sigma+1})}^p &\lesssim \Big(\int_H\left|\varphi(x)\right|^p\int_0^{x_n} y^{\sigma-1}\,dy\,dx\Big)^{\frac{1}{2}}\Big(\int_H\left|D_x^2\varphi(x)\right|^p\int_0^{x_n} y^{\sigma+1}\,dy\,dx\Big)^{\frac{1}{2}}\\ &\lesssim \|\varphi\|_{L^p(\mu_\sigma)}^{\frac{p}{2}}\,\|D_x^2\varphi\|_{L^p(\mu_{\sigma+2})}^{\frac{p}{2}}\,.\end{aligned}$$

The prerequisite $\sigma > 0$ ensures boundedness of the factors $\sigma^{-1}$ and $(\sigma+1)^{-1}$ that come up by integrating the inner $y$-integrals. The assertion for $\varphi \in C_c^2(\overline{H})$ now follows from Young's inequality or rather $\sqrt{ab} \leq \frac{a}{4\varepsilon} + \varepsilon b$. By density, the same inequality also holds for all $u$ that satisfy the assumptions of the present lemma. ■

# Chapter 3

# Linearization of the Thin-Film Equation

Consider the equation

$$\partial_s h \;+\; \nabla_y \cdot \big(h^m \, \nabla_y \Delta_y \, h\big) \;=\; 0\,. \tag{TFE}$$

Interchanging the roles of the independent variable $y_n$ and the dependent variable $h$, a technique known as von Mises transformation, transforms our equation (TFE) near the free boundary into a nonlinear degenerate problem with fixed domain. In the next step, we linearize the transformed equation around some specific solution. The equation thus obtained serves as a basis for implementing a comprehensive linear theory, one of the main issues addressed in this thesis.

## 3.1 Transformation

Let $h(s, y)$ be a solution to (TFE) and assume that near $(s_0, y_0)$ it is a $C^1$-function with $h(s_0, y_0) = 0$ and $\nabla_y h(s_0, y_0) = \varepsilon_0 > 0$. Then there exists a small number $0 < \varepsilon < \varepsilon_0$ for which $\nabla_y h(s, y) \geq \varepsilon$ for all

$$(s, y) \;\in\; U_\varepsilon(s_0, y_0) \;=\; \big\{(s, y) \;\big|\; P_s(h) \cap B_\varepsilon^{eu}(y_0)\,,\; s \in (s_0 - \varepsilon, s_0]\big\}\,,$$

where by $P_s(h)$ we denote the set $P_s(h) = \{y \in \mathbb{R}^n \mid h(s, y) > 0\}$. In $U_\varepsilon(s_0, y_0)$ we can apply the implicit function theorem to solve the equation $z = h(s, y)$ with respect to $y_n$ giving rise to a function

$$y_n \;=\; v(s, y', z)\,.$$

Written in local coordinates, the graph has the form

$$\Gamma \;=\; \big\{(s, y, z) \mid z = h(s, y)\big\} \;=\; \big\{(s, y', y_n, z) \mid y_n = v(s, y', z)\big\}\,.$$

This suggests to introduce the new variables

$$t = s\,, \quad x' = y'\,, \quad x_n = z \qquad \text{and} \qquad y_n = v(t, x)\,,$$

and the graph reads $\Gamma = \big\{(t, x', y_n, x_n) \mid F(t, x', y_n, x_n) = 0\big\}$ with $F(t, x', y_n, x_n) = v(t, x) - y_n$.

**Remark:** As we suppose that $h$ is positive in its support the transformation ensures that the original problem becomes one on a fixed domain, namely the upper half plane $H := \{x_n > 0\}$.

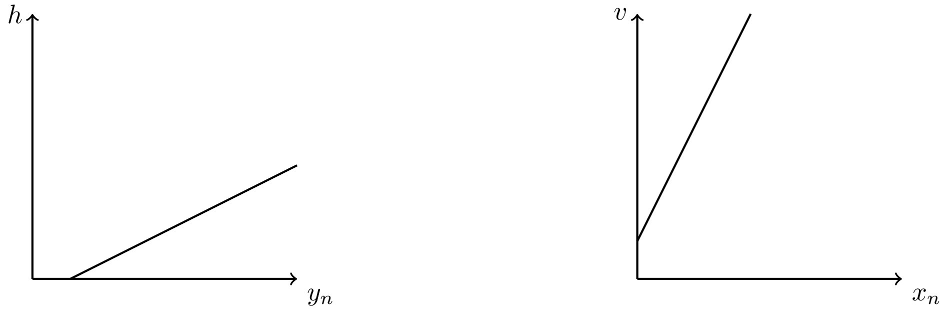


*Figure 3.1: Changing independent and dependent variables.*

In order to express derivatives of $h$ in terms of $v$ we apply the implicit function theorem to $F(t, x', y_n, x_n)$ with $x_n = h(t, x', y_n)$ to obtain

$$\partial_{(s,\,y)}\, h \;=\; \partial_{(t,\,x',\,y_n)}\, x_n \;=\; -\,\frac{\partial_{(t,\,x',\,y_n)} F}{\partial_{x_n} F}\;,$$

that is $\partial_s h = -\big(\frac{\partial_t v}{\partial_{x_n} v}\big)$, $\partial_{y_i} h = -\big(\frac{\partial_{x_i} v}{\partial_{x_n} v}\big)$, $i \neq n$, and $\partial_{y_n} h = \frac{1}{\partial_{x_n} v}$. Hence, the entries in the follwing matrix

$$\frac{\partial(t,x)}{\partial(s,y)} \;=\; \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial_t v}{\partial_{x_n} v} & -\frac{\partial_{x_1} v}{\partial_{x_n} v} & -\frac{\partial_{x_2} v}{\partial_{x_n} v} & \dots & \frac{1}{\partial_{x_n} v} \end{pmatrix} \tag{3.1.1}$$

are the partial derivatives of the coordinate change $(s, y) \mapsto (t, x)$. Using the chain rule we calculate

$$\begin{aligned} \Delta_y h \;&=\; \sum_{i=1}^{n} \partial_{y_i y_i} h \;\overset{(3.1.1)}{=}\; -\sum_{i=1}^{n-1} \partial_{y_i}\Big(\frac{\partial_{x_i} v}{\partial_{x_n} v}\Big) \;+\; \partial_{y_n}\big(\partial_{x_n} v\big)^{-1} \\ &=\; -\sum_{i=1}^{n-1}\sum_{k=1}^{n}\big(\partial_{y_i} x_k\big)\;\partial_{x_k}\Big(\frac{\partial_{x_i} v}{\partial_{x_n} v}\Big) \;+\; \sum_{k=1}^{n}\big(\partial_{y_n} x_k\big)\;\partial_{x_k}\big(\partial_{x_n} v\big)^{-1} \\ &\overset{(3.1.1)}{=}\; -\,\big(\partial_{x_n} v\big)^{-1}\Bigg[\sum_{i=1}^{n-1}\big(\partial_{x_n} v\big)\;\partial_{x_i}\Big(\frac{\partial_{x_i} v}{\partial_{x_n} v}\Big) \;-\; \sum_{i=1}^{n-1}\big(\partial_{x_i} v\big)\;\partial_{x_n}\Big(\frac{\partial_{x_i} v}{\partial_{x_n} v}\Big) \;-\; \partial_{x_n}\big(\partial_{x_n} v\big)^{-1}\Bigg] \\ &=\; -\,\big(\partial_{x_n} v\big)^{-1}\Big[\Delta_x v - \partial_{x_n}\Big(\frac{1 + |\nabla_x v|^2}{\partial_{x_n} v}\Big)\Big]\,. \end{aligned}$$

Similarly, we find

$$\nabla_y \;=\; \begin{pmatrix} \nabla_x' - (\partial_{x_n} v)^{-1}\,\nabla_x' v\;\partial_{x_n} \\ (\partial_{x_n} v)^{-1}\,\partial_{x_n} \end{pmatrix}$$

such that

$$\begin{aligned} \nabla_y\cdot\big(h^m\,\nabla_y\big) \;&= x_n^{\,m}\,\Delta_x' \;-\; x_n^{\,m}\;\nabla_x'\cdot\Big(\frac{\nabla_x' v}{\partial_{x_n} v}\;\partial_{x_n}\Big) \;+\; \big(\partial_{x_n} v\big)^{-1}\partial_{x_n}\Big(\frac{x_n^{\,m}}{\partial_{x_n} v}\;\partial_{x_n}\Big) \,- \\ &\quad -\;\sum_{i=1}^{n-1}\Big[\frac{\partial_{x_i} v}{\partial_{x_n} v}\;\partial_{x_n}\big(x_n^{\,m}\;\partial_{x_i}\big) \;-\; \frac{\partial_{x_i} v}{\partial_{x_n} v}\;\partial_{x_n}\Big(x_n^{\,m}\;\frac{\partial_{x_i} v}{\partial_{x_n} v}\;\partial_{x_n}\Big)\Big]\,. \end{aligned}$$

Combining these results we get

$$0 = \partial_t v + \Bigg( (\partial_{x_n} v)\, \nabla' \cdot \Big[ x_n^{\,m} \Big( \nabla' - \frac{\nabla' v}{\partial_{x_n} v}\, \partial_{x_n} \Big) \Big] - (\nabla' v) \cdot \partial_{x_n} \Big[ x_n^{\,m} \Big( \nabla' - \frac{\nabla' v}{\partial_{x_n} v}\, \partial_{x_n} \Big) \Big] + \\ + \partial_{x_n} \Big[ \frac{x_n^{\,m}}{\partial_{x_n} v}\, \partial_{x_n} \Big] \Bigg) \ \Bigg( (\partial_{x_n} v)^{-1} \Big[ \Delta v - \partial_{x_n} \Big( \frac{1 + |\nabla v|^2}{\partial_{x_n} v} \Big) \Big] \Bigg) . \tag{3.1.2}$$

Note that by $\nabla' = \nabla_{x'}$ we mean the gradient of dimension $n-1$ leaving aside the $x_n$-direction. A corresponding notation applies to $\Delta'$.

## 3.2 Perturbation of Stationary Solutions

From other equations it is known that solutions close to a special solution can give both insight and detailed information about general solutions and their qualitative behavior. Therefore, we assume that any initial datum $h_0$ is a small perturbation of the stationary solution

$$h_{st}(y) \ = \ y_n^{\,a}\ \chi_{\{y_n > 0\}}(y) \, .$$

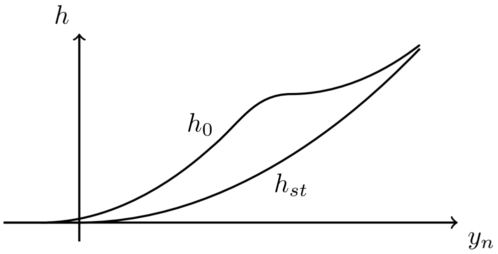


*Figure 3.2: Initial datum $h_0$ as a small perturbation of the stationary solution $h_{st}(y) = (y_n)_+^2$*

Our particular interest will be the case of quadratic growth $a = 2$. For this purpose we set $\tilde{h} = \sqrt{h}$, where we consider $h$ as a solution of (TFE). Using this as an ansatz we rewrite the problem in the form

$$\partial_s \tilde{h}^2 + \nabla \cdot \big( \tilde{h}^{2m}\ \nabla \Delta \tilde{h}^2 \big) = 0 \, ,$$

or equivalently

$$0 \ = \ \partial_s \tilde{h} \ + \ \nabla \cdot \big( \tilde{h}^{2m}\, \nabla \Delta \tilde{h} \big) \ + \ 4\, \tilde{h}^{2m-1} \nabla \tilde{h} \cdot \nabla \Delta \tilde{h} \ + \ \tilde{h}^{2m-1} (\Delta \tilde{h})^2 \ + \ 2\, \tilde{h}^{2m-1} |D_y^2 \tilde{h}|^2 \ + \\ + \ 2\, m\, \tilde{h}^{2m-2} |\nabla \tilde{h}|^2 \Delta \tilde{h} \ + \ 4\, m\, \tilde{h}^{2m-2} \sum_{i,j=1}^{n} (\partial_{y_i} \tilde{h})(\partial_{y_j} \tilde{h})\, \partial_{y_i y_j} \tilde{h} \, . \tag{3.2.1}$$

Since $h_{st}$ is a solution to (TFE), $\tilde{h}_{st}(y) = \big( y_n \big)_+$ is a solution to (3.2.1). Now, similar as above we transform the equation term by term followed by a linearization around $v_{st}(x) = x_n$ for $x \in H$. To put it plainly, we set $v(t,x) = v_{st}(x) + u(t,x)$, where $u$ can be regarded as a small perturbation of the steady state. Then $v$ solves the transformed equation if and only if $u$ satisfies the linear equation

$$f[u] \ = \ \partial_t u \ + \ \nabla \cdot \big( x_n^{\,2m}\, \nabla \Delta u \big) \ + \ 4\, x_n^{\,2m-1}\, \partial_{x_n} \Delta u \ + \ 2\, m\, x_n^{\,2m-2} \big( \Delta u \ + \ 2\, \partial_{x_n}^2 u \big)$$

for a nonlinearity $f[u]$ whose precise form will be discussed in lemma 3.3.1. For simplicity we only consider the case $m = 1$. The equation now reads

$$f[u] \ = \ \partial_t u \ + \ \nabla \cdot \big( x_n^{\,2}\, \nabla \Delta u \big) \ + \ 4\, x_n\, \partial_{x_n} \Delta u \ + \ 2\, \Delta u \ + \ 4\, \partial_{x_n}^2 u \ =: \ \partial_t u \ + \ L_0 u \, .$$

**Remark:** In one space dimension the linear spatial part $L_0$ becomes the (formally) self-adjoint operator

$$L_0 u = \frac{1}{x}\, \partial_x^2 \big(x^3\, \partial_x^2 u\big)$$

on the Hilbert space $L^2(\mathbb{R}_+, x)$, that is, $L_0$ is associated with the quadratic form

$$\langle u, v\rangle_{L^2(\mathbb{R}_+,x)} \;=\; \int_{\mathbb{R}_+} \partial_x^2 u(x)\, \partial_x^2 v(x)\, x^3\, dx\,.$$

In [36, 37, 35] Giacomelli et al. investigated this equation in one space dimension and perceived that such a symmetric structure is really needed to carry out the analysis presented in these papers.

## 3.3 The Nonlinearity

In order to give $f[u]$ a more suitable form we first introduce some notations to be used in the following. We employ the $\star$ notation to mean an arbitrary linear combination of products of indices for derivatives of $u$. For example, $\nabla u \star D_x^2 u$ denotes $\partial_{x_n} u\, \Delta u$ and $\nabla' u \cdot \nabla' \partial_{x_n} u$, as well as their sum. Moreover, we abbreviate the iterated application of $\star$ to derivatives of $k$-th order, with $k \in \mathbb{N}_0$, by

$$P_j(D_x^k u) \;:=\; \underbrace{D_x^k u \,\star\, \ldots \,\star\, D_x^k u}_{j-\text{times}}\,, \qquad j \in \mathbb{N}_0\,.$$

Utilizing these schematic notations enables us to rewrite $f[u]$ in a more convenient form.

**Lemma 3.3.1** *The thin-film equation (TFE) can be transformed into*

$$\partial_t u \,+\, L_0 u \;=\; f_0[u] \,+\, x_n\, f_1[u] \,+\, x_n^{\,2}\, f_2[u] \qquad on \quad I \times H\,,$$

*with*

$$\begin{aligned} f_0[u] \;&=\; f_0^1(\nabla u) \,\star\, \nabla u \,\star\, D_x^2 u\,,\\ f_1[u] \;&=\; f_1^1(\nabla u) \,\star\, \nabla u \,\star\, D_x^3 u \,+\, f_1^2(\nabla u) \,\star\, P_2(D_x^2 u) \quad and\\ f_2[u] \;&=\; f_2^1(\nabla u) \,\star\, \nabla u \,\star\, D_x^4 u \,+\, f_2^2(\nabla u) \,\star\, D_x^2 u \,\star\, D_x^3 u \,+\, f_2^3(\nabla u) \,\star\, P_3(D_x^2 u)\,, \end{aligned}$$

*and* $L_0 u = \nabla\cdot\big(x_n^{\,2}\, \nabla\Delta u\big) + 4\, x_n \partial_{x_n} \Delta u + 2\,\Delta u + 4\,\partial_{x_n}^2 u$.

**Proof:** Considering equation (3.2.1) we first transform each summand separately and then linearize the transformed terms around the steady state $v_{st}(x) = x_n$.

For the temporal part we get $\partial_s \tilde h = -\frac{\partial_t u}{v_n}$, where $v_n := \partial_{x_n} v = 1 + \partial_{x_n} u$. For simplicity, from now on we multiply the equation by $-v_n$ in all of the transformations. Next, by formula (3.1.2) we discover

$$-\, v_n\, \nabla\cdot\big(\tilde h^2\, \nabla\Delta\tilde h\big) \;=\; \nabla\cdot\big(x_n^{\,2}\, \nabla\Delta u\big) \,-\, R_1(u)\,.$$

With $v_n$ as above and $q[u] := \partial_{x_n}\Big(\frac{1+|\nabla v|^2}{v_n}\Big) = \partial_{x_n}\Big(\frac{|\nabla u|^2}{v_n}\Big)$, we calculate term by term which gives

$$\begin{aligned} R_1(u) \;&=\; 2\,x_n^{\,2}\Big(\partial_{x_n}\nabla'\cdot\Big(\frac{\nabla' u}{v_n}\,(\Delta u - q[u])\Big) \;+\; \partial_{x_n}\Big(\nabla' u\cdot\frac{\nabla'\partial_{x_n}u}{v_n^2}\,(\Delta u - q[u])\Big)\Big) - \\ &\quad - x_n^{\,2}\Big(\partial_{x_n}\Big(\Big(\frac{\Delta' u}{v_n} \;+\; \frac{|\nabla' u|^2}{v_n^3}\,\partial_{x_n}^2 u\Big)(\Delta u - q[u])\Big) \;+\; \partial_{x_n}^2\Big(\frac{|\nabla' u|^2}{v_n^2}\,(\Delta u - q[u])\Big)\Big) + \\ &\quad + 2\,x_n\Big(\nabla'\cdot\Big(\frac{\nabla' u}{v_n}\,(\Delta u - q[u])\Big) \;-\; \partial_{x_n}\Big(\frac{|\nabla' u|^2}{v_n^2}\,(\Delta u - q[u])\Big)\Big) + \\ &\quad + 2\,x_n\Big(\Big(2\,\nabla' u\cdot\frac{\nabla'\partial_{x_n}u}{v_n^2} \;-\; \frac{\Delta' u}{v_n} \;-\; \frac{|\nabla' u|^2}{v_n^3}\,\partial_{x_n}^2 u\Big)(\Delta u - q[u])\Big) - \\ &\quad - x_n^{\,2}\,\partial_{x_n}\Big((v_n^{-2}-1)\partial_{x_n}\Delta u\Big) \;+\; x_n^{\,2}\,\Delta' q[u] \;-\; 2\,x_n\,(v_n^{-2}-1)\partial_{x_n}\Delta u + \\ &\quad + \partial_{x_n}\Big(\frac{x_n^2}{v_n^2}\Big(\partial_{x_n}q[u] \;+\; \frac{\partial_{x_n}^2 u}{v_n}\,(\Delta u - q[u])\Big)\Big) \quad=\quad x_n\,f_1[u] \;+\; x_n^{\,2}\,f_2[u]\,. \end{aligned}$$

Since $v_n^{-2} - 1 = -v_n^{-2}(2+\partial_{x_n}u)\partial_{x_n}u$ we have $f_i[u] = f_i^1[\nabla u]\star\nabla u\star D_x^{i+2}u + \ldots$ for $i = 1, 2$.

In a similar fashion, we treat the other expressions in equation (3.2.1). For instance, we get

$$-\,4\,v_n\,\tilde h\,\nabla\tilde h\cdot\nabla\Delta\tilde h \;=\; 4\,x_n\,\partial_{x_n}\Delta u \;-\; R_2(u)\,,$$

where the remainder is given by

$$\begin{aligned} R_2(u) \;&=\; 4\,x_n\Big(\nabla' u\cdot\nabla'\Big(v_n^{-1}(\Delta u - q[u])\Big) \;-\; \frac{|\nabla' u|^2}{v_n}\,\partial_{x_n}\Big(v_n^{-1}(\Delta u - q[u])\Big) - \\ &\qquad - (v_n^{-2}-1)\partial_{x_n}\Delta u \;+\; \frac{\partial_{x_n}^2 u}{v_n^3}\,(\Delta u - q[u]) \;+\; v_n^{-2}\,\partial_{x_n}q[u]\Big) \\ &=\; x_n\,f_1[u]\,. \end{aligned}$$

Again, only expressions of the form $v_n^{-k}\star P_j(\nabla u)\star P_2(D_x^2 u)$ and $v_n^{-k}\star P_j(\nabla u)\star D_x^3 u$ appear. Moreover,

$$\begin{aligned} -\,2\,v_n\,|\nabla\tilde h|^2\,\Delta\tilde h \;&=\; 2\,\Delta u \;+\; 2\Big(\frac{|\nabla u|^2}{v_n^2} \;-\; 2\,\frac{\partial_{x_n}u}{v_n}\Big)\Delta u \;-\; 2\,\frac{1+|\nabla' u|^2}{v_n^2}\,q[u] \\ &=\; 2\,\Delta u \;-\; f_0[u]\,. \end{aligned}$$

Also note that

$$q[u] \;=\; D_x^2 u\,\star\,\sum_{k=1}^{2} v_n^{-k}\,P_k(\nabla u)\,.$$

The last term of identity (3.2.1) transforms to

$$\begin{aligned} -\,4\,v_n\sum_{i,j=1}^{n}(\partial_{y_i}\tilde h)(\partial_{y_j}\tilde h)\,\partial_{y_iy_j}\tilde h \;&=\; 4\,\partial_{x_n}^2 u \;+\; 4\,\Big(v_n^{-4}\big(|\nabla' u|^4 \;+\; 2\,|\nabla' u|^2\big) \;+\; \big(v_n^{-4}-1\big)\Big)\partial_{x_n}^2 u - \\ &\quad - 8\,v_n^{-3}\big(|\nabla' u|^2\,\nabla' u\cdot\nabla'\partial_{x_n}u \;+\; \nabla' u\cdot\nabla'\partial_{x_n}u\big) \;+\; 4\,v_n^{-2}\sum_{i,j=1}^{n-1}(\partial_{x_i}u)(\partial_{x_j}u)\,\partial_{x_ix_j}u\,, \end{aligned}$$

which is equal to $4\,\partial_{x_n}^2 u - f_0[u]$. It remains to check the quadratic expressions in (3.2.1). These terms, however, do not contain any linear parts such that both are completely absorbed by the inhomogeneity. Indeed, one can prove that

$$-\,v_n\Big(\tilde h\,(\Delta\tilde h)^2 \;+\; 2\,\tilde h\,|D_y^2\tilde h|^2\Big) \;=\; -\,x_n\,f_1[u]$$

in the same manner as above. Altogether this finishes the proof of the lemma. ■

## 3.4 Distributional Solution

As a starting point we consider the linearized thin-film equation

$$\partial_t u \,+\, L_0 u \,=\, f \qquad \text{on} \quad I \times H$$

for some open interval $I \subseteq \mathbb{R}$ and $H$ being the upper half plane. In this, the spatial part is given by

$$L_0 u \;=\; x_n^2\, \Delta^2 u \,+\, 6\, x_n\, \Delta \partial_{x_n} u \,+\, 2\, \Delta u \,+\, 4\, \partial_{x_n}^2 u\,.$$

In the context of distributions $\mathcal{D}'(I \times \Omega)$, for some open $\Omega \subseteq H$, there are equivalent expressions for $L_0 u$, namely

$$L_0 u \;=\; LLu \;=\; x_n^{-1}\, \Delta\big(x_n^3\, \Delta u\big) \,-\, 4\, \Delta' u\,, \tag{3.4.1}$$

where $L$ is the second-order differential operator $Lu = -\, x_n^{-1}\, \nabla\cdot(x_n^2\, \nabla u)$.

Note that we have $x_n \in C^\infty(I \times \Omega)$, and therefore $u \in \mathcal{D}'(I \times \Omega)$ satisfies the equation $\partial_t u + L_0 u = f$ if and only if it also satisfies $x_n \partial_t u + x_n L_0 u = x_n f$. A distributional solution is then characterized by the equation

$$-\, u\big(x_n\, \partial_t \varphi\big) \,+\, \Delta u\big(x_n^3\, \Delta\varphi\big) \,+\, 4\, \nabla' u\big(x_n\, \nabla'\varphi\big) \;=\; f\big(x_n\, \varphi\big) \qquad \forall\, \varphi \in C_c^\infty(I \times \Omega)\,.$$

Here we have used the second identity in (3.4.1) and that $x_n^3$ is also a smooth weight over $I \times \Omega$. If now $u, \nabla' u, \Delta u, f \in L^1_{loc}(I \times \Omega)$, then this characterization translates into

$$-\int\limits_{I\times\Omega} x_n\, u\, \partial_t\varphi\, dxdt \,+\, \int\limits_{I\times\Omega} x_n^3\, \Delta u\, \Delta\varphi\, dxdt \,+\, 4 \int\limits_{I\times\Omega} x_n\, \nabla' u \cdot \nabla'\varphi\, dxdt \;=\; \int\limits_{I\times\Omega} x_n\, f\, \varphi\, dxdt \tag{3.4.2}$$

for all $\varphi \in C_c^\infty(I \times \Omega)$. As in that way solutions need not to be necessarily four times differentiable (at the boundary) we can relax the requirements on the differentiability of solutions. However, this definition has two weak points: For one thing, it is restricted to $H$ and its open subsets and, for another thing, the weak assumptions on the solution and the inhomogeneity do not allow for energy techniques. Thus, we shall need both an adjustment of the test function space and stronger assumptions concerning the regularity of $u$ and $f$.

### 3.4.1 Basic Properties of the Linear Operator

From the distributional point of view it is obvious that both temporal and tangential derivatives commute with the operator $\partial_t + L_0$ , i.e.

$$\partial_t\big(\partial_t^l \partial_x^{\alpha'} u\big) + L_0\big(\partial_t^l \partial_x^{\alpha'} u\big) = \partial_t^l \partial_x^{\alpha'} f \tag{3.4.3}$$

for all multi-indices $\alpha' = \big(\alpha_i\big)_{1\le i\le n-1}$ and $l \in \mathbb{N}_0$. For simplicity, we write $\partial_x^{\alpha'}$ instead of $\partial_{x'}^{\alpha'}$. Vertical derivatives (derivatives in $x_n$-direction), however, can not be treated as straightforward. For this let $u^{(k)} := \partial_{x_n}^k u$ with $k \in \mathbb{N}_0$ and $u$ be as above. It satisfies the more general linear equation

$$\partial_t u^{(k)} \,+\, L_k u^{(k)} \;:=\; \partial_t u^{(k)} \,+\, x_n^{-k-1}\, \Delta\Big(x_n^{k+3}\, \Delta u^{(k)}\Big) \,-\, 4\, \Delta' u^{(k)} \;=\; \widetilde{f}_k$$

in the sense of definition (3.4.2) with $x_n$ and $x_n^3$ replaced by $x_n^{k+1}$ and $x_n^{k+3}$, respectively. The inhomogeneity $\widetilde{f}_k$ can be derived iteratively provided equality holds for $k = 0$.

**Lemma 3.4.1** *Let $u \in L^1_{loc}(I \times \Omega)$ satisfy (3.4.2) and $f \in C^\infty(I \times \Omega)$. Then for all $k \in \mathbb{N}_0$ we have*

$$\widetilde{f}_k \;=\; \partial_{x_n}^k f \,-\, 2k\, x_n\, \Delta'\Delta u^{(k-1)} \,-\, k(k-1)\, \Delta'\Delta u^{(k-2)}\,. \tag{3.4.4}$$

**Proof:** Identity (3.4.4) follows by induction over $k$. The induction basis is already given by the prerequisite that $u$ is a solution and the fact that $\widetilde{f}_0 = f$. Hence, it remains to verify the formula for $k+1$ under the hypothesis that (3.4.4) is valid for $k \in \mathbb{N}_0$. At first we observe that

$$\begin{aligned} \partial_t u^{(k+1)} \;+\; L_{k+1} u^{(k+1)} &=: \partial_t u^{(k+1)} \;+\; A_{k+1}\big[u^{(k+1)}\big] \;-\; 4\,\Delta' u^{(k+1)} \\ &= \partial_{x_n}\Big(\partial_t u^{(k)} \;-\; 4\,\Delta' u^{(k)}\Big) \;+\; A_{k+1}[u^{(k+1)}]\,, \end{aligned}$$

where we always write $u$ instead of $u(\varphi)$ for $\varphi \in C_c^\infty(I\times\Omega)$. By simple computations, we obtain successively

$$\begin{aligned} A_{k+1}[u^{(k+1)}] &= A_k[u^{(k+1)}] \;+\; 2\,x_n\,\Delta u^{(k+2)} \;+\; 2(k+3)\,\Delta u^{(k+1)} \\ &= \partial_{x_n} A_k[u^{(k)}] \;+\; (k+1)\,x_n^{-k-2}\,\Delta\big(x_n^{k+3}\,\Delta u^{(k)}\big) \;-\; (k+3)\,x_n^{-k-1}\,\Delta\big(x_n^{k+2}\,\Delta u^{(k)}\big) \;+ \\ &\quad +\; 2\,x_n\,\Delta u^{(k+2)} \;+\; 2(k+3)\,\Delta u^{(k+1)} \\ &= \partial_{x_n} A_k[u^{(k)}] \;-\; 2\,x_n\,\Delta'\Delta u^{(k)}\,. \end{aligned}$$

Using this and the induction hypothesis yields

$$\begin{aligned} \partial_t u^{(k+1)} \;+\; L_{k+1} u^{(k+1)} &= \partial_{x_n}\big(\partial_t u^{(k)} \;+\; L_k u^{(k)}\big) \;-\; 2\,x_n\,\Delta'\Delta u^{(k)} \\ &= \partial_{x_n}\big(\partial_{x_n}^k f \;-\; 2k\,x_n\,\Delta'\Delta u^{(k-1)} \;-\; k(k-1)\,\Delta'\Delta u^{(k-2)}\big) \;-\; 2\,x_n\,\Delta'\Delta u^{(k)} \\ &= \partial_{x_n}^{k+1} f \;-\; 2\,(k+1)\,x_n\,\Delta'\Delta u^{(k)} \;-\; (k+1)k\,\Delta'\Delta u^{(k-1)}\,, \end{aligned}$$

as required. ■

**Remark:** Rewriting the terms on the right-hand side as

$$2k\,x_n^{-k-1}\,\Delta'\big(x_n^{k+2}\,\Delta u^{(k-1)}\big) \qquad \text{and} \qquad k(k-1)\,x_n^{-k-1}\,\Delta'\big(x_n^{k+1}\,\Delta u^{(k-2)}\big)$$

points out that these terms also have a structure similar to the one of the leading term of $L_k$, but with lower vertical derivatives of $u$. This enables us to iterate certain estimates on vertical derivatives as well.

**Translation invariance:** The linear equation $\partial_t u + L_0 u = f$ is invariant under translation in any direction except the $x_n$-direction. The corresponding translation operators are defined by

$$T_0 : (t,x) \mapsto (t_0 + t, x) \qquad \text{and} \qquad T_j : (t,x) \mapsto (t, x + \kappa\, e_j) \quad 1 \le j \le n-1\,, \tag{3.4.5}$$

where $e_j$ denotes the $j$-th unit vector, $t_0 \in I$ and $\kappa \in \mathbb{R}$. Then the translation invariance means that

$$\big(\partial_t + L_0\big)(u \circ T_j) \;=\; f \circ T_j \qquad \text{on} \quad T_j^{-1}(I\times\Omega) \qquad \text{for} \quad j = 0,\dots,n-1\,,$$

whenever $u$ is a solution in the sense of integral identity (3.4.2).

**Scaling invariance:** The equation $\partial_t u + L_0 u = f$ is invariant under the sacling

$$T_\lambda :\; (t,x) \mapsto (\lambda^2 t, \lambda x) \;=:\; (\hat{t}, \hat{x}) \qquad (\lambda > 0)\,. \tag{3.4.6}$$

In order to see this let $u$ be a solution to $\partial_t u + L_0 u = f$. Then $u_{\lambda,2}(t,x) := \lambda^{-2}\, u(\hat{t},\hat{x})$ is a solution on $T_\lambda^{-1}(I\times\Omega)$,

$$\partial_t u_{\lambda,2}(t,x) \;+\; L_0 u_{\lambda,2}(t,x) \;=\; \partial_{\hat{t}} u(\hat{t},\hat{x}) \;+\; \widehat{L}_0\, u(\hat{t},\hat{x}) \;=\; f(\hat{t},\hat{x})\,.$$

Here $\widehat{L}_0$ denotes the spatial linear operator with respect to $\hat{x}$. From this calculation one can also read off that $u_{\lambda,\gamma}(t,x) := \lambda^{-\gamma}\,(u \circ T_\lambda)(t,x)$ solves the homogeneous problem for any $\gamma \in \mathbb{R}$.

In view of the explicit representation of $f[u]$, given in lemma 3.3.1, we observe that the nonlinear equation has the solution $u_{\lambda,1}$ provided $u$ is also a solution. Indeed, we have

$$(\partial_t + L_0)u_{\lambda,1} \;=\; \lambda\,(\partial_{\hat{t}} + \widehat{L}_0)u \;=\; \lambda\, f[u] \;=\; f[u_{\lambda,1}]\,,$$

where the last equality can be verified by means of the identity $\partial_x^\alpha\, u_{\lambda,1} = \lambda^{|\alpha|-1}\,\partial_{\hat{x}}^\alpha\, u$.

## 3.5 Geometry

In this section we generate a distance on the upper half plane that arises intrinsically from the geometry of the differential operator $L_0$. The structure of subsections 3.5.1 and 3.5.2 is essentially based on works by Koch [51], and Daskalopoulos and Hamilton [17].
In the last portion, subsection 3.5.3, we make a brief digression into the history of the calculus of variations. A more detailed overview of this area can be found in [31, 70], to name but two examples.

### 3.5.1 The Carnot-Caratheodory Metric

The spatial part of the linear operator gives rise to the Riemannian metric

$$g_x(v,w) \;=\; x_n^{-1}\, v \cdot w$$

on the tangent space $T_xH \cong \mathbb{R}^n$ that is attached to $x \in H = \{x_n > 0\}$, where by $v \cdot w$ we denote the standard scalar product on $\mathbb{R}^n$. Its scaling behavior is given by $g_{\lambda x}(\lambda v, \lambda w) = \lambda g_x(v,w)$. The Riemannian structure allows us to measure the length of parametrized curves $\gamma : \mathbb{R} \supset [a,b] \to H$ by

$$\ell_g(\gamma) \;=\; \int_a^b |\gamma'(s)|_g\, ds \;:=\; \int_a^b \sqrt{g_{\gamma(s)}\big(\gamma'(s),\gamma'(s)\big)}\, ds\,,$$

and hence induces an intrinsic metric called the metric of Carnot–Caratheodory.

**Definition 3.5.1** *Let $x, y \in H$. The Carnot-Caratheodory metric or intrinsic distance between $x$ and $y$ is given by*

$$d(x,y) := \inf\Big\{\ell_g(\gamma)\;\big|\;\gamma : [a,b] \to H \text{ is piecewise smooth, } \gamma(a) = x \text{ and } \gamma(b) = y\Big\}\,.$$

*The shortest curve in $H$ joining $x$ and $y$, i.e. the curve that realizes the distance, is termed geodesic.*

It is easy to check that $d$ indeed fulfills the defining conditions of a metric.

**Remark 3.5.2** *A curve $\gamma$ of finite length is said to be of unit speed or parametrized by arc length if for any $s_1 < s_2 \in [a,b]$, we have $\ell_g(\gamma|_{[s_1,s_2]}) = s_2 - s_1$. In particular, if $\gamma$ is continuously differentiable and non-zero everywhere, then this definition is equivalent to asking for $|\gamma'(s)|_g = 1$ for all $s \in [a,b]$. This means that the velocity of the corresponding motion of a point is constant by 1 at all times.*

In order to find the shortest connection between two given points one traditionally uses techniques of the calculus of variations. Typically, one considers the energy functional of $\gamma$,

$$E_g(\gamma) \;:=\; \frac{1}{2}\int_a^b \big|\gamma'(s)\big|_g^2\, ds \;=\; \frac{1}{2}\int_a^b \gamma_n(s)^{-1}\big|\gamma'(s)\big|^2\, ds\,,$$

on an arbitrary but fixed interval $(a, b)$, and minimizes this quantity over all piecewise smooth curves joining $x$ and $y$. As opposed to length, energy is dependent on the parametrization of $\gamma$. However, owing to the Cauchy-Schwarz inequality, we always have

$$\ell_g(\gamma)^2 \;\leq\; 2\,(b-a)\,E_g(\gamma)\,,$$

and equality holds if and only if $\gamma$ is parametrized by a multiple of the arc length, i.e. $|\gamma|_g$ is constant. So we search not only for the energy minimizing curve between $x$ and $y$ but also for a curve of unit speed. From calculus of variations we know that $E_g$ has an extremum only if the Euler-Lagrange equations are satisfied, that is

$$0 \;=\; \left(\gamma_n^{-1}\,\gamma_i'\right)' \;+\; \frac{1}{2}\,\gamma_n^{-2}\,|\gamma'|^2\,\delta_{in} \;=\; \left(\gamma_n^{-1}\,\gamma_i'\right)' \;+\; \frac{1}{2}\,\gamma_n^{-1}\,\delta_{in}$$

for $1 \leq i \leq n$. First we observe that $\gamma_n^{-1}\,\gamma_i' \equiv$ *const.* for $i = 1, \ldots, n-1$ which is equivalent to: There exists an $\vec{\eta} = (\eta_i) \in \mathbb{R}^{n-1}$ such that for $\gamma_n \neq 0$,

$$\gamma_i' = \eta_i\,\gamma_n\,. \tag{3.5.1}$$

Now we define $\eta := |\vec{\eta}| = \Big(\sum_{i<n} \eta_i{}^2\Big)^{\frac{1}{2}}$. Then, assuming that $|\gamma'|_g^2 = 1$, we obtain

$$1 \;=\; \gamma_n^{-1}\Big[\left(\gamma_n'\right)^2 + \sum_{i=1}^{n-1}\left(\gamma_i'\right)^2\Big] \;=\; \gamma_n^{-1}\Big[\left(\gamma_n'\right)^2 + \sum_{i=1}^{n-1}\left(\eta_i\gamma_n\right)^2\Big] \;=\; \gamma_n^{-1}\Big[\left(\gamma_n'\right)^2 + \eta^2{\gamma_n}^2\Big]\,,$$

and hence $(\gamma_n')^2 = \gamma_n \;-\; \eta^2\gamma_n^2$. The general solution to this ordinary differential equation is

$$\gamma_n(s) \;=\; \frac{1}{2\eta^2}\Big(1 \;-\; \cos\big(\eta\,(s-k)\big)\Big) \tag{3.5.2}$$

for a $k \in \mathbb{R}$. If we substitute this into (3.5.1) and then integrate in time we get

$$\begin{aligned} \gamma_i(s) \;&=\; \gamma_i(a) \;+\; \eta_i \int_a^s \gamma_n(t)\,dt \\ &=\; \gamma_i(a) \;+\; \frac{\eta_i}{2\eta^2}\bigg(s \;-\; a \;+\; \frac{1}{\eta}\Big(\sin\big(\eta\,(a-k)\big) \;-\; \sin\big(\eta\,(s-k)\big)\Big)\bigg) \end{aligned} \tag{3.5.3}$$

for all $s \in [a, b]$ and $i = 1, \ldots, n-1$. All such curves are the critical points for our minimization problem, and thus only satisfy a necessary condition for minimality. Using the second variation one checks that these curves in fact realize the minimal energy $E_g$ among all curves from $x$ to $y$ on the at first fixed interval $(a, b)$. The intrinsic distance $d(x, y)$ is then defined as the minimal value of $b - a$. In addition, these calculations show that $\ell_g(\gamma|_{[s_1,s_2]}) = s_2 - s_1$ for any $a \leq s_1 < s_2 \leq b$, and hence $\gamma$ is a geodesic of unit velocity.
If $k = a$, we have $x_n = \gamma_n(a) = 0$, that is, $x$ is on the boundary. On the other hand, we can always choose $\eta_i$ and $k$ in such a way that $y_n = \gamma_n(b) = 0$. In any of these cases there exists a geodesic of the above kind that joins $x$ to $y$ and has finite length in the intrinsic metric. As before the distance is then given by the smallest $b - a$. With these considerations at hand, it makes sense to extend the metric space $(H, d)$ to all of $\overline{H}$.

In what follows, we approach a better understanding of the Carnot-Caratheodory metric. We begin by discussing some special cases: If $x$ and $y$ are vertically aligned, that is $x - y \in \mathbb{R}\,e_n$, we can give an explicit formula to calculate their distance. On the opposite end, we examine the case that $x$ and $y$ have identical $n$-th coordinates.

**Lemma 3.5.3** *Suppose a geodesic $\gamma : [a, b] \to \overline{H}$ with $\gamma(a) = x$ and $\gamma(b) = y$ is given.*

*i) Let $y = (x', y_n)$. Then we have*

$$d(x, y) \;=\; 2\big|\sqrt{x_n} - \sqrt{y_n}\big|\,.$$

*ii) For $x_n = y_n = 0$, it holds*

$$d(x,y) \;=\; 2\,\sqrt{\pi\,|x-y|}\,.$$

*iii) If $x_n = y_n$, then*

$$d(x,y) \;\le\; x_n^{-\frac{1}{2}}\,|x-y|\,.$$

**Proof:**

**(i)** First suppose that $x_i = y_i$ for $i < n$, i.e. for $x = (x', x_n) \in \overline{H}$ we have $y = (x', y_n)$. Then by (3.5.3) it follows

$$x_i \;=\; y_i \;=\; \gamma_i(b) \;=\; \gamma_i(a) \,+\, \eta_i \int_a^b \gamma_n(t)\,dt \;=\; x_i \,+\, \eta_i \int_a^b \gamma_n(t)\,dt\,.$$

Since $\gamma_n > 0$, this yields $\eta_i = 0$ for any $1 \le i \le n-1$ such that $\gamma_i(s) \equiv x_i$ for all $s \in [a,b]$. Furthermore the differential equation for $\gamma_n$ simplifies to $\gamma_n' \;=\; \pm\sqrt{\gamma_n}$, which has the solution

$$\gamma_n(s) \;=\; \frac{1}{4}\,\big(s-k\big)^2$$

for $k \in \mathbb{R}$ such that $(a-k)^2 = 4\,x_n$ and $(b-k)^2 = 4\,y_n$. Combining these identities we get

$$0 \;\le\; d(x,y) \;=\; b-a \;=\; k \,\pm\, 2\sqrt{y_n} \;-\; \big(k \,\pm\, 2\sqrt{x_n}\big) \;=\; \pm 2\,\big(\sqrt{y_n} - \sqrt{x_n}\big)\,,$$

which is equivalent to 3.5.3 (i).

**(ii)** Now we assume that $\gamma_n(a) = \gamma_n(b) = 0$, that is, $x$ and $y$ are both on the the boundary. By formula (3.5.2) we get

$$\cos\big(\eta(b-k)\big) \;=\; 1 \;=\; \cos\big(\eta(a-k)\big)\,,$$

and hence $\eta(b-k) = \eta(a-k) + 2\pi$. Rearranging the terms gives $b = a + \frac{2\pi}{\eta}$. Then, an evaluation of (3.5.3) shows that

$$\gamma_i(b) \;=\; \gamma_i(a) \,+\, \frac{\eta_i}{2\eta^2}\left(\frac{2\pi}{\eta} \,+\, \frac{1}{\eta}\Big(\sin\big(\eta\,(a-k)\big) \;-\; \sin\big(\eta\,(a-k)+2\pi\big)\Big)\right) \;=\; \gamma_i(a) \,+\, \frac{\pi}{\eta^3}\,\eta_i\,.$$

This implies

$$\big|x \,-\, y\big| \;=\; \big|\gamma(a) \,-\, \gamma(b)\big| \;=\; \frac{\pi}{\eta^2}\,,$$

and thus the intrinsic distance is

$$d(x,y) \;=\; b-a \;=\; \frac{2\pi}{\eta} \;=\; 2\,\sqrt{\pi\,|x-y|}$$

as claimed.

**(iii)** Suppose $\overline{\gamma} : [a,b] \to \overline{H}$ is the linear path from $x$ to $y$ and that $x_n = y_n$. Then we obviously have $\overline{\gamma}_n(s) \equiv x_n$ for all $s \in [a,b]$, and thus

$$d(x,y) \;\le\; \ell_g(\overline{\gamma}) \;=\; x_n^{-\frac{1}{2}} \int_a^b \big|\overline{\gamma}'(t)\big|\,dt \;=\; x_n^{-\frac{1}{2}}\,|x-y|\,,$$

since $\int |\overline{\gamma}'|\,dt$ is the Euclidean length of $\overline{\gamma}$. ■

In the subsequent section we will make use of these (exact) formulas in order to deduce convenient estimates for the distance $d(x,y)$ between two arbitrary points.

### 3.5.2 Properties of the Intrinsic Geometry

**Lemma 3.5.4** *For any two points $x, y \in \overline{H}$, let $\gamma : [a,b] \to \overline{H}$ be the geodesic curve such that $\gamma(a) = x$ and $\gamma(b) = y$. We set $z_+ := \max\{x_n, y_n\}$, $z_- := \min\{x_n, y_n\}$ and $z := \max_s \gamma_n(s)$.*

*i) Then we have the following estimate from above,*

$$d(x,y) \;\leq\; \min\left\{2\sqrt{x_n} + 2\sqrt{y_n} + 2\sqrt{\pi|x'-y'|}\,,\; 2\left|\sqrt{x_n} - \sqrt{y_n}\right| \,+\, \frac{|x'-y'|}{\sqrt{z_+}}\right\}.$$

*ii) The distance can be estimated from below as*

$$d(x,y) \;\geq\; \max\left\{\frac{|x-y|}{\sqrt{z}},\; 2\left(\sqrt{z} - \sqrt{z_-}\right)\right\}.$$

**Proof:**

**(i)** By virtue of the triangle inequality we find

$$d(x,y) \;\leq\; d\big(x,(x',0)\big) \,+\, d\big((x',0)(y',0),\big) \,+\, d\big((y',0),y\big) \;=\; 2\sqrt{x_n} + 2\sqrt{\pi|x'-y'|} + 2\sqrt{y_n}\,,$$

where we have used lemma 3.5.3 (i) in the first and third summand and (ii) in the second one. For the second estimate we use the triangle inequality once more to see

$$d(x,y) \;\leq\; d\big(x,(x',y_n)\big) \,+\, d\big(y,(x',y_n)\big) \qquad \text{and} \qquad d(x,y) \;\leq\; d\big(x,(y',x_n)\big) \,+\, d\big(y,(y',x_n)\big)\,.$$

The estimate now follows from an application of lemma 3.5.3 (i) and (iii) to the corresponding terms.

**(ii)** Obviously, $\gamma_n^{-\frac{1}{2}}\,|\gamma'(s)| \geq \frac{|\gamma'(s)|}{\sup_s\left(\gamma_n(s)^{\frac{1}{2}}\right)}$ , and thus

$$d(x,y) \;=\; \ell_g(\gamma) \;=\; \int_a^b \left|\gamma'(s)\right| \gamma_n(s)^{-\frac{1}{2}}\, ds \;\geq\; z^{-\frac{1}{2}} \underbrace{\int_a^b \left|\gamma'(s)\right| ds}_{\text{Euclidean length of } \gamma} \;\geq\; \frac{|x-y|}{\sqrt{z}}\,.$$

Now let $\gamma_z$ the part of $\gamma$ that joins $x$ to the point that realizes the maximal $\gamma_n$-value $z$. Since the respective linear paths from $x$ and $y$ to $(x',z)$ and $(y',z)$ describe geodesic curves it follows directly that

$$\begin{aligned} d(x,y) \;&=\; \ell_g(\gamma) \;\geq\; \ell_g(\gamma_z) \;\geq\; \max\left\{d\big(x,(x',z)\big),\, d\big(y,(y',z)\big)\right\} \\ &=\; 2\,\max\left\{\sqrt{z} - \sqrt{x_n},\, \sqrt{z} - \sqrt{y_n}\right\} \;=\; 2\left(\sqrt{z} - \sqrt{z_-}\right), \end{aligned}$$

where the second line is a consequence of lemma 3.5.3 (i). ■

Using the second estimate from below we can show that the intrinsic distance from some given point $x \in \overline{H}$ to the level set $\{z \in \overline{H} \mid z_n = c\}$ is realized by the vertically projected point $(x', c)$.

**Corollary 3.5.5** *Let $x$ be an arbitrary point in $\overline{H}$ and $\Gamma_c := \{z \in \overline{H} \mid z_n = c\}$ for some constant $c \geq 0$. Then we have*

$$\inf_{y \in \Gamma_c} d(x,y) \;=\; 2\left|\sqrt{x_n} - \sqrt{c}\right|,$$

*and the infimum is attained at $y^* = (x', c) \in \Gamma_c$.*

**Proof:** For a fixed point $x \in \overline{H}$ we set $y^* = (x', c)$. It suffices to consider the case $x_n \neq c$, as otherwise there is nothing to prove. Clearly, $y^* \in \Gamma_c$ and we have

$$d(x, \Gamma_c) \;\leq\; d(x, y^*) \;=\; 2\left|\sqrt{x_n} - \sqrt{c}\right| \qquad (*)$$

by lemma 3.5.3 (i).

For the other direction, let $z(x, y) = z$, $z_-$ and $z_+$ as in proposition 3.5.4. From part (ii) we know that

$$d(x, y) \;\geq\; 2\left(\sqrt{z(x, y)} - \sqrt{z_-}\right)$$

holds for all $y \in \overline{H}$. Now we claim that $z_+ = z(x, \tilde{y})$, where $\tilde{y}$ is a point on the level set $\Gamma_c$ that realizes the distance to $x$. As a direct consequence we get that $d(x, \tilde{y}) \geq 2\big(\sqrt{z_+} - \sqrt{z_-}\big)$ which implies

$$d(x, \Gamma_c) \;=\; d(x, \tilde{y}) \;\geq\; 2\left|\sqrt{x_n} - \sqrt{c}\right| .$$

This, combined with $(*)$, yields the desired identity and, as $d(x, y^*) = 2\big|\sqrt{x_n} - \sqrt{c}\big|$, the infimum is in fact attained at $y^*$.

In order to prove the claim, we employ inequality $(*)$ to see that $z(x, \tilde{y}) \leq z_+$, for if not this would mean $d(x, \tilde{y}) > d(x, y^*)$. However, this contradicts the assumption that $\tilde{y}$ minimizes the distance to $x$. On the other hand, $z(x, \tilde{y}) \geq z_+$ because the geodesic joins $x$ to $\tilde{y}$. This gives the claim and the corollary follows.
■

We now obtain an equivalent function that combines all the estimates from lemma 3.5.4 into a single expression. We set

$$\rho(x, y) := \frac{|x - y|}{x_n^{\frac{1}{2}} + y_n^{\frac{1}{2}} + |x - y|^{\frac{1}{2}}}$$

for all $x, y \in \overline{H}$, and by convention $\rho(x, y) = 0$ if $x = y \in \partial H$. Obviously, $\rho$ is positive definite and symmetric. Also take note of the fact that the boundary $\partial H$ is nothing but the level set $\Gamma_0$.

**Proposition 3.5.6** *There exists a constant $c_d > 1$ such that*

$$c_d^{-1}\, d(x, y) \;\leq\; \rho(x, y) \;\leq\; d(x, y)$$

*for all $x, y \in \overline{H}$. We say $\rho$ and $d$ are equivalent and write $\rho \sim d$.*

**Remarks 3.5.7**

1) *Proposition 3.5.6 implies that $\rho(x, z) \leq c_d\left(\rho(x, y) + \rho(y, z)\right)$, and hence $\rho$ satisfies a "$c_d$-relaxed triangle inequality". Such functions are referred to as quasimetrics.*

2) *An immediate consequence of the above proposition is that the two expressions $d$ and $\rho$ induce the same topology on $\overline{H}$. This implies, in particular, that for every point $x \in \overline{H}$ each intrinsic ball $B_{r_1}(x; d)$ contains a ball $B_{r_2}(x; \rho)$ for some $r_2 > 0$ and vice versa. To see this, we define $r_2 := c_d^{-1} r_1$ and then apply proposition 3.5.6 to discover that for $y \in B_{r_2}(x; \rho)$ we have*

$$d(x, y) \;\leq\; c_d\, \rho(x, y) \;<\; c_d\, r_2 \;=\; r_1 .$$

*This implies $B_{r_1}(x; d) \supset B_{r_2}(x; \rho)$. Conversely, set $r_1 := r_2$ to find $B_{r_2}(x; \rho) \supset B_{r_1}(x; d)$.*

3) *The expression*

$$\widetilde{\rho}(x, y) := \frac{|x - y|}{\left(x_n^2 + y_n^2 + |x - y|^2\right)^{\frac{1}{4}}} \qquad \forall\, x, y \in \overline{H}$$

*defines another equivalent quasimetric. More precisely, we can show that*

$$\rho(x,y) \;\le\; \widetilde{\rho}(x,y) \;\le\; 3\,\rho(x,y) \qquad \textit{for all}\, x,y \in \overline{H}\,,$$

*and, by transitivity, we also have* $\widetilde{\rho} \sim d$.

**Proof (of proposition 3.5.6):** Let $\sqrt{z} \ge \sqrt{x_n} + \sqrt{y_n} + \sqrt{|x-y|}$. Then it follows

$$\rho(x,y) \;\le\; \sqrt{|x-y|} \;\le\; \sqrt{z} \,-\, \sqrt{x_n} \,-\, \sqrt{y_n} \;\le\; 2\left(\sqrt{z} - \sqrt{z_-}\right) \;\overset{3.5.4\,(ii)}{\le}\; d(x,y)\,.$$

Otherwise we may estimate

$$\rho(x,y) \;<\; \frac{|x-y|}{\sqrt{z}} \;\overset{3.5.4\,(ii)}{\le}\; d(x,y)\,,$$

which proves the right hand side inequality.

Now suppose $\sqrt{x_n} + \sqrt{y_n} \ge \sqrt{|x-y|}$. By lemma 3.5.4 (i), we then get

$$\begin{aligned} d(x,y) \le\; & \frac{|x'-y'|}{\sqrt{z_+}} + 2\left|\sqrt{x_n} \,-\, \sqrt{y_n}\right| \;\le\; \frac{|x'-y'|}{\frac{1}{2}(\sqrt{x_n} + \sqrt{y_n})} + 2\left|\sqrt{x_n} - \sqrt{y_n}\right| \\ =\; & 4\,\frac{|x'-y'| \,+\, |x_n - y_n|}{2\,(\sqrt{x_n} + \sqrt{y_n})} \;\le\; 6\,\rho(x,y)\,. \end{aligned}$$

In case of $\sqrt{x_n} + \sqrt{y_n} < \sqrt{|x-y|}$ we obtain

$$d(x,y) \;\overset{3.5.4\,(i)}{\le}\; 2\left(\sqrt{x_n} \,+\, \sqrt{y_n} \,+\, \sqrt{\pi|x-y|}\right) \;<\; 6\sqrt{|x-y|} \;=\; 6\,\frac{|x-y|}{\sqrt{|x-y|}} \;\le\; 12\,\rho(x,y)\,,$$

which proves the first inequality with $c_d = 12$ and therefore completes the proof. ∎

**Lemma 3.5.8** *If a continuous function* $\Psi : \overline{H} \to \mathbb{R}$ *is in* $C^1(H)$*, then* $\Psi$ *is Lipschitz continuous on* $\overline{H}$ *if and only if there exists a positive constant* $c_L$ *such that* $x_n \left|\nabla\,\Psi(x)\right|^2 \le {c_L}^2$ *for all* $x \in H$.

**Proof:** Suppose the indicated condition $x_n^{\frac{1}{2}} \left|\nabla\Psi(x)\right| \le \bar{c}_L$ holds for any $x \in H$ and some $\bar{c}_L > 0$. Further let $\gamma : [a,b] \to H$ be the geodesic between the two points $x$ and $y$ which is parameterized by arc length, i.e. we may assume that $b = a + d(x,y)$ and $\gamma_n^{-1}|\gamma'|^2 \equiv 1$ (see remark 3.5.2). By the fundamental theorem of calculus we then get

$$\begin{aligned} \left|\Psi(x) - \Psi(y)\right| \;&=\; \left|\Psi\big(\gamma(a)\big) - \Psi\big(\gamma(b)\big)\right| \;=\; \Big|\int_a^b \nabla\,\Psi\big(\gamma(s)\big)\cdot\gamma'(s)\,ds\Big| \\ &\le\; \Big(\sup \gamma_n^{\frac{1}{2}}\left|\nabla\,\Psi(\gamma)\right|\Big)\,\Big(\sup \gamma_n^{-\frac{1}{2}}\,|\gamma'|\Big)\,(b-a) \\ &\le\; \bar{c}_L\,d(x,y)\,. \end{aligned}$$

This means that $\Psi$ is $\bar{c}_L$-Lipschitz on $H$, and since both the metric $d(\cdot,y)$ and $\Psi$ are continuous on $\overline{H}$, it is possible to extend $\Psi$ to a function that satisfies the Lipschitz property on all of $\overline{H}$ .

Conversely, let $\Psi$ be Lipschitz continuous on $\overline{H}$. Then there exists a constant $\tilde{c}_L > 0$ such that

$$\left|\Psi(x) - \Psi(y)\right| \;\le\; \tilde{c}_L\,d(x,y) \;\le\; \tilde{c}_L\,c_d\,\sqrt{n}\,x_n^{-\frac{1}{2}}\,|x-y| \qquad \forall\, x,y \in H\,,$$

where the second inequality follows from proposition 3.5.6. Now we use the prerequisite that $\Psi$ is differentiable on its domain $H$, and therefore we have

$$\sqrt{x_n}\,\big|\nabla\,\Psi(x)\big| \;\le\; \tilde{c}_L\,c_d\,\sqrt{n} \qquad \text{for a.e. } x \in H\,,$$

and hence the claim follows with $c_L = \max\{\bar{c}_L, \tilde{c}_L\,c_d\,\sqrt{n}\}$. ■

In the following example we construct a Lipschitz function that has bounded second derivatives.

**Example 3.5.9** *Let $y \in \overline{H}$ be some fixed point and $c > 0$ a constant. Then we define*

$$\widetilde{\Psi}(z) \;:=\; f\big(\widetilde{\rho}(z,y)\big) \;:=\; \frac{\widetilde{\rho}(z,y)^2}{\big(c^2+\widetilde{\rho}(z,y)^2\big)^{\frac{1}{2}}}$$

*with $\widetilde{\rho}$ as in remark 3.5.7 (3). Note that $f \in C^\infty(\mathbb{R})$ is Lipschitz continuous. A calculation then gives that*

$$\begin{aligned} z_n\,\big|\nabla_z\,\widetilde{\Psi}(z)\big|^2 \;&=\; f'\big(\widetilde{\rho}(z,y)\big)^2\,z_n\,\big|\nabla_z\,\widetilde{\rho}(z,y)\big|^2 \;\lesssim\; z_n\,\big|\nabla_z\,\widetilde{\rho}(z,y)\big|^2 \\ &\lesssim\; z_n\left(\frac{1}{\big(z_n^2+y_n^2+|z-y|^2\big)^{\frac{1}{2}}} \;+\; \frac{|z-y|^2\big(|z-y|+z_n\big)^2}{\big(z_n^2+y_n^2+|z-y|^2\big)^{\frac{5}{2}}}\right) \\ &\lesssim\; 1 \;+\; \frac{z_n\,|z-y|^4 + z_n^{\,3}\,|z-y|^2}{\big(z_n+|z-y|\big)^5} \;\lesssim\; 1\,, \end{aligned}$$

*such that, with $\Psi(z) := c_L\,\varepsilon\,\widetilde{\Psi}(z)$ for some positive constant $c_L$ and $\varepsilon > 0$ sufficiently small,*

$$z_n\,\big|\nabla_z\,\Psi(z)\big|^2 \;\le\; {c_L}^2\,.$$

*Since $\Psi$ is obviously continuously differentiable, we thus get from lemma 3.5.8 that $\Psi$ is Lipschitz continuous on $\overline{H}$ with Lipschitz constant $c_L$.*

*The Laplacian is given by $\Delta_z\widetilde{\Psi}(z) = f''\big(\widetilde{\rho}(z,y)\big)\,\big|\nabla_z\,\widetilde{\rho}(z,y)\big|^2 + f'\big(\widetilde{\rho}(z,y)\big)\,\Delta_z\,\widetilde{\rho}(z,y)$. Similarly as for the gradient, we compute*

$$\Big|f''\big(\widetilde{\rho}(z,y)\big)\Big| \;\lesssim\; \big(c+\widetilde{\rho}(z,y)\big)^{-1} \qquad \textit{and} \qquad z_n\,\big|\Delta_z\,\widetilde{\rho}(z,y)\big| \;\lesssim\; \widetilde{\rho}(z,y)^{-1}\,.$$

*Furthermore, we observe*

$$0 \;\le\; \frac{2\,\widetilde{\rho}(z,y)}{\big(c^2+\widetilde{\rho}(z,y)\big)^{\frac{1}{2}}}\left(1-\Big(\frac{\widetilde{\rho}(z,y)}{c+\widetilde{\rho}(z,y)}\Big)^2\right) \;\le\; f'\big(\widetilde{\rho}(z,y)\big) \;\le\; \frac{2\sqrt{2}\,\widetilde{\rho}(z,y)}{c+\widetilde{\rho}(z,y)}\,.$$

*Collecting all these results we obtain $z_n\,\big|\Delta_z\Psi(z)\big| = c_L\,\varepsilon\,z_n\,\big|\Delta_z\widetilde{\Psi}(z)\big| \le c_L\,c\,\big(c+\widetilde{\Psi}(z)\big)^{-1} \le c_L$.*

*Now if we set $c := \widetilde{\rho}(x,y) > 0$ for some $x \ne y \in \overline{H}$ we notice that $\sqrt{2}\,\widetilde{\Psi}(x) = \widetilde{\rho}(x,y)$ which is by remark 3.5.7 (3) equivalent to $d(x,y)$. Using an approximation argument we can also achieve that $\Psi$ is bounded on $\overline{H}$.*

In later considerations, the following class of functions will play a role, and specifically that it contains a nontrivial function $\Psi$ will be of importance.

**Definition 3.5.10** *We say $\Psi : (H,d) \to \mathbb{R}$ is in the class of two times differentiable Lipschitz functions $Lip_2(H)$ if it is bounded and satisfies the conditions*

$$\sqrt{x_n}\,\big|\nabla\Psi(x)\big| \;\le\; c_L \qquad \textit{and} \qquad x_n\,\big|\Delta\Psi(x)\big| \;\le\; c_L$$

*for all $x \in H$ and some constant $c_L > 0$.*

Through lemma 3.5.8 we know that the first condition ensures that $\Psi$ is $c_L$-Lipschitz continuous.

Let us recall the definitions of the intrinsic ball of radius $R > 0$ centered at $x \in \overline{H}$,

$$B_R(x) \;=\; B_R(x;d) \;=\; \big\{ y \in \overline{H} \mid d(x,y) < R \big\}\,.$$

By $B_R^{eu}(x)$ we denote the Euclidean ball (intersected with $\overline{H}$), i.e. $B_R^{eu}(x) = \big\{ y \in \overline{H} \;\big|\; |x-y| < R \big\}$. Some useful properties of such balls are expressed in the next lemma.

**Lemma 3.5.11** *Let $R > 0$. Then the following inclusions hold for all $x \in \overline{H}$:*

$$B^{eu}_{c_d^{-2}\,R\left(R+\sqrt{x_n}\right)}(x) \;\subset\; B_R(x) \;\subset\; B^{eu}_{2\,R\left(R+2\sqrt{x_n}\right)}(x)$$

*As a consequence we obtain*

$$\big|B_R(x)\big|_\sigma \;\sim\; R^n \left(R + \sqrt{x_n}\right)^{n+2\sigma}. \tag{3.5.4}$$

**Proof:** We need to prove two parts, the inclusions and (3.5.4).

**Part 1:** Let $y \in B_R(x)$. If $y_n \geq x_n$ we have

$$R \;>\; d(y,x) \;\geq\; d\left((x', y_n), x\right) \;\overset{3.5.3\,(i)}{=}\; 2\big(\sqrt{y_n} - \sqrt{x_n}\big)\,,$$

which is equivalent to $\sqrt{y_n} < \sqrt{x_n} + \frac{R}{2}$. For $0 \leq y_n < x_n$ this estimate is quite obvious. Using this and the standard inequality $2ab \leq a^2 + b^2$ yields

$$|y-x| \;\leq\; d(y,x)\left(\sqrt{y_n} + \sqrt{x_n} + \sqrt{|y-x|}\right) \;<\; R\left(\frac{R}{2} + 2\sqrt{x_n}\right) + \frac{1}{2}\Big(R^2 + \big|y-x\big|\Big)\,,$$

from which follows the assertion, namely $|y-x| < 2R\big(R + 2\sqrt{x_n}\big)$.
For the first inclusion let $|y-x| < c_d^{-2}\,R\big(R + \sqrt{x_n}\big)$. Since $\rho$ is monotone in $|x-y|$ we get

$$d(y,x) \;\leq\; c_d\,\rho(y,x) \;\leq\; c_d^{-1}\,\frac{R\big(R+\sqrt{x_n}\big)}{\sqrt{x_n} + c_d^{-1}\sqrt{R\big(R+\sqrt{x_n}\big)}} \;<\; \frac{R\big(R+\sqrt{x_n}\big)}{\sqrt{x_n} + \sqrt{R^2}} \;=\; R\,.$$

This concludes the first part of the proof.

**Part 2:** First we consider the case that the radius $R$ is small compared to $x_n$. To be more precise, we assume that $6R < \sqrt{x_n}$ such that $B_R(x) \subset B^{eu}_{\frac{3}{4}x_n}(x)$ by virtue of part 1. But this in turn implies that $y_n \sim x_n$ whenever $y \in B_R(x)$. Moreover, again by the inclusions taken from the first part of the lemma, we know that $\big|B_R(x)\big| \sim R^n\big(R + \sqrt{x_n}\big)^n$. Hence

$$\big|B_R(x)\big|_\sigma \;=\; \int_{B_R(x)} {y_n}^\sigma\,dy \;\sim\; x_n^\sigma\,\big|B_R(x)\big| \;\sim\; R^n\left(R + \sqrt{x_n}\right)^{n+2\sigma}\,,$$

since also $\sqrt{x_n} \sim R + \sqrt{x_n}$.

If $C_r(x) \subset H$ denotes the cube of side length $2r > 0$ centered at $x \in H$, then the $\mu_\sigma$-volume is given by

$$\big|C_r(x)\big|_\sigma \;=\; (2r)^{n-1}\int_0^{x_n+r} y_n^\sigma\,dy_n \;=\; \frac{(2r)^{n-1}}{\sigma+1}\left(x_n + r\right)^{\sigma+1} \tag{$*$}$$

if $x_n \le r$, and otherwise by

$$\big|C_r(x)\big|_\sigma \;=\; (2r)^{n-1}\int_{x_n-r}^{x_n+r} y_n^{\,\sigma}\,dy_n \;=\; \frac{(2r)^{n-1}}{\sigma+1}\Big(\big(x_n+r\big)^{\sigma+1} - \big(x_n-r\big)^{\sigma+1}\Big)\,. \qquad (*')$$

Now suppose $\sqrt{x_n} \le 6R$. Then $B_R(x) \subset C_{36R^2}(x)$ and we can apply $(*)$ with $r = (6R)^2 \ge x_n$ ,

$$\big|B_R(x)\big|_\sigma \;<\; \big|C_{36R^2}(x)\big|_\sigma \;\lesssim\; R^{2n-2}\left(R+\sqrt{x_n}\right)^{2\sigma+2} \;\lesssim\; R^n\left(R+\sqrt{x_n}\right)^{n+2\sigma}.$$

The last estimate follows from $R \sim R+\sqrt{x_n}$.

The other direction can be derived in a similar manner. Here we distinguish between the situation $\sqrt{x_n} \ll R$, say $\sqrt{x_n} \le \varepsilon R$ where $\varepsilon := (\sqrt[4]{n}\,c_d)^{-1}$, and $\varepsilon R < \sqrt{x_n} \le 6R$. In the first case we apply formula $(*)$ once more, this time with $r := \varepsilon^2 R^2 \ge x_n$, to get

$$\big|B_R(x)\big|_\sigma \;>\; \big|C_{\varepsilon^2R^2}(x)\big|_\sigma \;\gtrsim\; R^{2n-2}\left(R+\sqrt{x_n}\right)^{2\sigma+2} \;\gtrsim\; R^n\left(R+\sqrt{x_n}\right)^{n+2\sigma}.$$

It remains to study the case in which $0 < \varepsilon R < \sqrt{x_n} \le 6R$. If we set $r := \varepsilon^2\sqrt{x_n}R$, then $0 < \frac{\varepsilon^2}{6}\,x_n \le r < x_n$ and $B_R(x)$ contains $C_r(x)$. All this combined with $R \sim \sqrt{x_n} \sim R+\sqrt{x_n}$, allows us to infer that

$$\begin{aligned}
\big|B_R(x)\big|_\sigma &\overset{(*')}{>} \frac{(2\varepsilon^2)^{n-1}}{\sigma+1}\left(\sqrt{x_n}R\right)^{n-1}\Big((x_n+r)^{\sigma+1}-(x_n-r)^{\sigma+1}\Big)\\
&\ge \frac{(2\varepsilon^2)^{n-1}}{\sigma+1}\left(\sqrt{x_n}R\right)^{n-1}\left(\Big(1+\frac{\varepsilon^2}{6}\Big)^{\sigma+1}-\Big(1-\frac{\varepsilon^2}{6}\Big)^{\sigma+1}\right)x_n^{\,\sigma+1}\\
&\ge c(n,\sigma)^{-1}\,R^n\,\sqrt{x_n}^{\,n+2\sigma} \;\sim\; R^n\left(R+\sqrt{x_n}\right)^{n+2\sigma}.
\end{aligned}$$

Altogether, this proves (3.5.4) and hence the lemma. ■

For $B_R(x)$ given, let

$$\partial B_R(x) \;=\; \big\{y \in \overline{H} \;\big|\; d(x,y) = R\big\}$$

denote the corresponding sphere. If the intrinsic ball intersects with the boundary of $H$, then $B_R(x) \cap \partial H$ consists only of the two points on $\partial H$ that have the largest distance between them. This is important to note because otherwise $dist\big(\partial B_{\lambda R}(x), \partial B_R(x)\big) = 0$ as long as $2\sqrt{x_n} \le \lambda R$. However, we want this distance to be positive for any scaling factor $\lambda \ne 1$.

**Corollary 3.5.12** *The measure $\mu_\sigma$ is doubling with respect to the intrinsic metric $d$ if and only if $\sigma > -1$. In particular, for any scaling factor $\lambda > 0$ and for every point $x \in \overline{H}$ there exists a constant $c_{n,\sigma} > 0$ such that*

$$\big|B_{\lambda R}(x)\big|_\sigma \;\le\; c_{n,\sigma}\begin{cases}\lambda^{\,n}\;\big|B_R(x)\big|_\sigma & \text{if}\quad \lambda \in (0,1)\\ \lambda^{\,2(n+\sigma)}\,\big|B_R(x)\big|_\sigma & \text{if}\quad \lambda \ge 1\end{cases}$$

*provided we have $\sigma \ge -\frac{n}{2}$ .*

**Proof:** Using (3.5.4), with $\sigma \ge -\frac{n}{2}$ and $\lambda \ge 1$, yields

$$\big|B_{\lambda R}(x)\big|_\sigma \;\lesssim\; \lambda^{2n+2\sigma}\,R^n\left(R+\sqrt{x_n}\right)^{n+2\sigma} \;\lesssim\; \lambda^{2n+2\sigma}\,\big|B_R(x)\big|_\sigma\,,$$

which corresponds to the doubling condition. The estimate for $\lambda \in (0,1)$ is similar.
In case of $\sigma \le -1$ the volume of a ball containing 0 is infinite from which the necessity of $\sigma > -1$ becomes clear. ■

Remark that the case $-1 < \sigma < -\frac{n}{2}$ only occurs if $n = 1$ and $-\sigma \in (\frac{1}{2}, 1)$. Then, in case of $\lambda \ge 1$, an upper bound is given by $c_\sigma\,\lambda\,\big|B_R(x)\big|_\sigma$. Otherwise we have $\big|B_{\lambda R}(x)\big|_\sigma \le c_\sigma\,\lambda^{\,2(\sigma+1)}\,\big|B_R(x)\big|_\sigma$. The only relevant case, however, is when $\sigma \geq -\frac{n}{2}$.

We combine the previous two results into another handy estimate.

**Lemma 3.5.13** *Let $\sigma > -1$, $R > 0$ and $x, y \in \overline{H}$. If $\sigma \geq -\frac{n}{2}$, then*

$$\frac{\big|B_R(x)\big|_\sigma}{\big|B_R(y)\big|_\sigma} \;\leq\; c(n,\sigma)\,\Big(1 + \frac{d(x,y)}{R}\Big)^{2n+2\sigma}\,.$$

*Consequently, we get*

$$\frac{R + \sqrt{x_n}}{R + \sqrt{y_n}} \;\leq\; c(n)\,\Big(1 + \frac{d(x,y)}{R}\Big)^{2}\,.$$

**Proof:** Using the triangle inequality shows that $B_R(x) \subset B_{R+d(x,y)}(y)$, and consequently we have

$$\big|B_R(x)\big|_\sigma \;\leq\; \big|B_{R+d(x,y)}(y)\big|_\sigma \;\lesssim\; \Big(1 + \frac{d(x,y)}{R}\Big)^{2n+2\sigma}\,\big|B_R(y)\big|_\sigma$$

by corollary 3.5.12. For the second inequality we choose $\sigma = 0$ and apply the first one to get

$$R + \sqrt{x_n} \;\sim\; R^{-1}\,\big|B_R(x)\big|^{\frac{1}{n}} \;\lesssim\; \Big(1 + \frac{d(x,y)}{R}\Big)^{2}\,R^{-1}\,\big|B_R(y)\big|^{\frac{1}{n}} \;\sim\; \Big(1 + \frac{d(x,y)}{R}\Big)^{2}\,\big(R + \sqrt{y_n}\big)\,,$$

where we also make use of formula (3.5.4). ■

Up to this point we only deal with the geometry of the elliptic problem leaving aside the time component. In order to incorporate the time we observe that the principal symbol for the parabolic equation is

$$p(t,x,\tau,\xi) \;=\; i\,\tau \,+\, x_n^2\,|\xi|^4\,.$$

Following [30] we extend $d$ to a metric on $\mathbb{R} \times \overline{H}$. The newly obtained metric is canonically attached to the differential operator $\partial_t + L_0$.

**Definition 3.5.14** *Let $(t,x),(s,y) \in \mathbb{R} \times H$ and $A$ be a set in $\mathbb{R} \times \overline{H}$. Then the intrinsic distance between $(t,x)$ and $(s,y)$ is given by*

$$d^{(t)}\big((t,x),(s,y)\big) \;:=\; \sqrt[4]{|t-s| + d(x,y)^4}\,.$$

*We denote by $|A|_\sigma$ the measure of $A$ with respect to the measure $\mathcal{L} \times \mu_\sigma$, i.e. $|A|_\sigma = \int_A x_n^\sigma\, dxdt$.*

**Lemma 3.5.15** *Let $Q_R(t,x) := I_R(t) \times B_R(x)$, where we use the notation $I_R(t) = (t - R^4, t]$. Then we have*

$$\big|Q_R(t,x)\big|_\sigma \;\sim\; R^{n+4}\,\Big(R + \sqrt{x_n}\Big)^{n+2\sigma}\,.$$

**Proof:** This follows immediately from lemma 3.5.11. ■

**Remark 3.5.16** *Accordingly, the metric measure spaces $(\overline{H}, d, \mu_\sigma)$ and $\big(\mathbb{R} \times \overline{H}, d^{(t)}, \mathcal{L} \times \mu_\sigma\big)$ are spaces of homogeneous type if and only if $\sigma > -1$. If it is clear from the context, we will drop the $\mathcal{L}$ in the notation of the "time-space measure" and simply write $\mu_\sigma$.*

In this general setting, one may ask which weight functions are in the Muckenhoupt class $A_p(\mu_\sigma)$, see definition A.11.

**Lemma 3.5.17** *Let $1 < p < \infty$ and $\sigma > -1$. Then $x_n^{s-\sigma} \in A_p(\mu_\sigma)$ if and only if $s \in \mathbb{R}$ satisfies $-1 < s < p(\sigma+1)-1$.*

**Proof:**

"⇒" If $x_n^{s-\sigma} \in A_p(\mu_\sigma)$, then $x_n^{-\frac{s-\sigma}{p-1}} \in A_{\frac{p}{p-1}}(\mu_\sigma)$ by lemma A.14. But then lemma A.13 implies that

$$x_n^{s-\sigma}\, d\mu_\sigma \;=\; x_n^{s}\, d\mathcal{L}^n \qquad \text{as well as} \qquad x_n^{-\frac{s-\sigma}{p-1}}\, d\mu_\sigma \;=\; x_n^{\sigma-\frac{s-\sigma}{p-1}}\, d\mathcal{L}^n$$

satisfy the doubling condition which is by corollary 3.5.12 equivalent to $-1 < s < p(\sigma+1)-1$.

"⇐" Conversely, let $-1 < s < p(\sigma+1)-1$. Then $\mu_s$ and $\mu_{\sigma-\frac{s-\sigma}{p-1}}$ are doubling and

$$⨍_B x_n^{s-\sigma}\, d\mu_\sigma \Big[⨍_B x_n^{-\frac{s-\sigma}{p-1}}\, d\mu_\sigma\Big]^{p-1} \;=\; \frac{|B|_s}{|B|_\sigma}\,\Big[\,\frac{|B|_{\sigma-\frac{s-\sigma}{p-1}}}{|B|_\sigma}\,\Big]^{p-1} \;\le\; c(n,s)\, c\Big(n,\sigma-\frac{s-\sigma}{p-1}\Big)^{p-1} c(n,\sigma)^p$$

for all $d$-balls $B$ by (3.5.4). We set $[\omega]_{A_p} = c(n,s)\,c\big(n,\sigma-\frac{s-\sigma}{p-1}\big)^{p-1} c(n,\sigma)^p < \infty$ and the claim follows. ■

### 3.5.3 The Brachistochrone Problem

The *brachistochrone* problem may be considered as the starting point of the theory of the calculus of variations. In June 1696 Johann Bernoulli (1667-1748) addressed an open challenge to "the most brilliant mathematicians in the world" in *Acta Eruditorum* with the following problem:

> Given two points $A$ and $B$ situated at different distances from the horizontal and not in the same vertical line, determine the curved path of most rapid descent of a particle sliding from $A$ to $B$ exclusively under the influence of gravity.

This problem in fact traces back to a similar problem formulated by Galileo Galilei (1564-1642) in 1638, even though he did not solve it explicitly.

As seemingly no one was able to solve the problem within the period agreed, it was Gottfried Wilhelm Leibniz (1646-1716) who persuaded Johann Bernoulli to allow more time for solutions to be handed in than the originally intended six months. Leibniz also suggested to call the problem *tachistoptotam*, derived from the Greek words tachistos (swiftest) and piptein (to fall). Bernoulli, however, decided to name it brachistochrone which originates from brachistos (the shortest) and chronos (time). After the expiration of the second deadline, five mathematicians were able to produce a solution to the problem, Newton (1642-1727), Leibniz, de l'Hôspital (1661-1704) and Jacob Bernoulli (1654-1705), and Johann Bernoulli. According to legend, Newton solved the problem within one evening: On 29 January 1697 he came home from work at 4pm and found a letter containing the problem that Bernoulli had sent to him directly. At 4am, only twelve hours later, he communicated his solution anonymously to the Royal Society. But as Bernoulli examined the work he could easily spot its author and coined the phrase "ex ungue leonem", to recognize "the lion by his claw".

By this time the personal relations between the great mathematical minds of these days were often damaged by publicly fought disputes, mainly about priority in the discovery of scientific results. Particularly fierce was the rivalry in the Bernoulli family; their controversy over the brachistochrone was just another episode in what has been later called a "bitter feud". Reportedly, one of Johann Bernoulli's main motivations to pose the problem was to prove his superiority over elder brother Jacob.

Eventually, four solutions (all but de l'Hôspital's) were published in the May 1697 edition of Acta Eruditorum, but Johann could not refrain from highlighting his version when summarizing the results:

> "I have with one blow solved two fundamental problems, one optical and the other mechanical and have accomplished more than I have asked of others [...]"

This refers to the fact that Fermat's least time principle and the brachistochrone problem basically describe the same phenomena. It should not go unmentioned that all solutions reached the same result: There is just one upside down cycloid that passes through $A$ and $B$ and does not have maximum points. In parametric form, this is written

$$x(s) \;=\; x_0 \,+\, c\,(s - \sin s) \qquad \text{and} \qquad y(s) \;=\; c\,(\cos s - 1)$$

for some constant $c$. These equations of a cycloid are generated by a fixed point on the circle line which rolls on the underside of the $x$-axis. There is only one inverted cycloid through the points $A$ and $B$ determined by a suitable choice of $x_0$ and $c$. All the proofs, however, were based on geometrical arguments. It was not until a couple of years later when Leonhard Euler (1707-1783), supported by the ideas of Joseph Louis Lagrange (1736-1813), developed a very strong instrument to solve a rather general class of problems including a variety of geodesic issues. In his opus he introduced the famous Euler-Lagrange differential equation. This marks the real birth of the calculus of variations as we know it today.

Indeed, such a brachistochrone curve (inverted cycloid) is closely related to the geometry that we have introduced in the preceding pages. Daskalopoulos and Hamilton [17] studied the degenerate parabolic equation

$$\partial_t u \;-\; y\,(\partial_x^2 u + \partial_y^2 u) \;+\; (\sigma + 1)\,\partial_y u \;=\; f$$

with $\sigma > -1$ on the half space $\overline{H} \subset \mathbb{R}^2$. Given a point $A = (x, y) \in \overline{H}$, diffusion is governed by the Riemannian metric

$$g_A(v, w) \;=\; \frac{1}{2y}\; v \cdot w$$

for $v, w \in T_A H$. As above, this leads to an ODE which can be solved explicitly for the geodesics.

**Proposition 3.5.18 (Daskalopoulos and Hamilton, '98)**
*The cycloid curve*

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} \;=\; \begin{pmatrix} s - \sin s \\ 1 - \cos s \end{pmatrix}$$

*describes a geodesic that is parametrized by arc length $s$ for the metric $g$. All the other geodesic curves are obtained by translation $x \mapsto x_0 + x$ and dilation $x \mapsto cx$, $y \mapsto cy$, or are vertical line segments.*

### 3.5.4 Notes

Distance functions that are derived from a partial differential operator's coefficients turned out to be a powerful tool for proving local results for solutions of the corresponding equation. Given an operator of order $m$, say

$$L \;=\; \sum_{|\alpha| \le m} a_\alpha(x)\,\partial_x^\alpha\,,$$

we replace $\partial_{x_i}$ by $\xi_i$ (formally this is done by a Fourier transform) to convert the PDE into a polynomial of the same degree, with the top degree being a homogeneous polynomial that is most significant for its classification. More precisely, we obtain a $m$-homogeneous map

$$T^*M \;\ni\; (x, \xi) \;\mapsto\; p(x, \xi) \;:=\; \sum_{|\alpha| = m} a_\alpha(x)\,\xi^\alpha \;\in\; \mathbb{R}\,,$$

where we interpret $(x, \xi)$ as variables in the cotangent bundle, i.e. $x \in M$ is any local coordinate chart, then $\xi$ is the linear coordinate in each tangent space $T_xM$, and the principal symbol is an invariantly defined function on $T^*M$. If we further impose the usual ellipticity condition on the coefficients $a_\alpha(x)$, that is $p(x, \xi) \neq 0$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then the principle symbol is invertible which in turn gives rise to a positive definite bilinear form on the tangent space. Once this is given, one can develop an entire intrinsic geometry, as implemented in subsection 3.5.1, that is attached to the differential operator $L$ in a natural way.

This illustration shows that metric structures and elliptic equations are intimately related to one another. A mutual relation between intrinsic distances and differential operators has been also found in the class of subelliptic equations (for a definition see Hörmander [43] or Egorov [26]): Any collection of vector fields induces a linear partial differential operator and vice versa. Now think of $X_1(x), \ldots, X_m(x)$ as the vector fields at $x \in M$ that are in some sense associated with a differential operator. They are said to satisfy Hörmander's condition if the vector fields, along with all their commutators (called distribution), span $T_xM$ for every $x \in M$. This means that the corresponding PDE is subelliptic. If the vector fields satisfy Hörmander's condition for every $x$ in the manifold $M$ on which the operator is initially defined, then any two points can be joined by a curve $\gamma$ for which

$$\gamma'(t) \ \in \ span\Big\{X_1\big(\gamma(t)\big), \ldots, X_m\big(\gamma(t)\big)\Big\}$$

for any $t$. Such a curve on $M$ is called admissible or horizontal. This generates a natural intrinsic metric, termed Carnot–Carathéodory metric, given by $d(x, y) = \inf \ell(\gamma)$, where the infimum extends over all horizontal curves $\gamma$ connecting $x \in M$ with $y \in M$. There is a sequence of papers by different combinations of the authors Fefferman, Phong and Sanchez-Cálle [29, 71, 30] who obtained pointwise estimates for the Green function in terms of the intrinsic geometry that arises from its associated differential operator. Let us also mention the work by Nagel, Stein and Wainger [66] for further results in that direction.

Rather than merely from a differential equations point of view, we find strong connections to the much more general framework of sub-Riemannian manifolds: By a sub-Riemannian manifold we mean a Riemannian manifold together with a constraint that governs the admissible directions of movement, expressed by the vector fields $X_i : M \to TM$ $(i = 1, \ldots, m)$, such that the image of $x$, denoted by $X_i(x)$, lies in $T_xM$. Hörmander's condition then implies that the distance that is generated by these vector fields is finite. For an introduction to sub-Riemannian geometry see [58].

# Chapter 4

# The Linear Equation

Considering the equation $\partial_t u + L_0 u = f$ one can in general not expect to have classical solutions. Indeed, not even the solution concept of a distributional or weak solution that we have introduced in section 3.4 satisfies our requirements. However, we take the integral representation (3.4.2) as a starting point (or motivation) and search for admissible extensions to *relatively open* subsets of $\overline{H}$. We will call a solution to be legitimate if it retains a certain behavior towards the boundary in the sense that it admits values at $\{x_n = 0\}$. For this it is important to choose the test function space properly. More precisely, a test function is sometimes supposed to cut off at initial time depending on whether we consider the initial value problem or not.
Before giving the definition of an energy solution it has to be said that, under the assumption in place $u \in L^1_{loc}(I \times \overline{H})$, one can not get the subsequent methods and techniques to work. Indeed, the following solution concept asks for minimum requirements which are needed to set up a rigorous and complete energy theory.

**Definition 4.0.1 (energy solution)** *Let $I = (t_1, t_2) \subseteq \mathbb{R}$ be open, $\Omega \subseteq \overline{H}$ relatively open and $k \in \mathbb{N}_0$.*

*i) Suppose $f \in L^1_{loc}\big(I; L^2(\Omega, \mu_1)\big)$. We call $u$ an energy solution to $\partial_t u + L_k u = f$ on $I \times \Omega$, or a $L_k$-solution to $f$ on $I \times \Omega$, if and only if $u \in L^2_{loc}\big(I; L^2(\Omega, \mu_{k+1})\big)$, $\nabla u \in L^2\big(I; L^2(\Omega, \mu_{k+1})\big)$, $D^2_x u \in L^2\big(I; L^2(\Omega, \mu_{k+3})\big)$ and*

$$-\int\limits_{I\times\Omega} u\,\partial_t\varphi\,d\mu_{k+1} + \int\limits_{I\times\Omega} \Delta u\,\Delta\varphi\,d\mu_{k+3} + 4\int\limits_{I\times\Omega} \nabla' u \cdot \nabla'\varphi\,d\mu_{k+1} = \int\limits_{I\times\Omega} f\,\varphi\,d\mu_{k+1}$$

*for all $\varphi \in C^\infty_c(I \times \Omega)$.*

*ii) Let $t_1 > -\infty$, $f \in L^1_{loc}\big([t_1, t_2); L^2(\Omega, \mu_1)\big)$ and $g \in L^2(\Omega, \mu_{k+1})$. We call $u$ an energy solution to $\partial_t u + L_k u = f$ on $[t_1, t_2) \times \Omega$ with initial value $u(t_1) = g$, if $u \in L^2_{loc}\big([t_1, t_2); L^2(\Omega, \mu_{k+1})\big)$, $\nabla u$ and $D^2_x u$ satisfy the assumptions of (i), and the identity*

$$-\int\limits_{I\times\Omega} u\,\partial_t\varphi\,d\mu_{k+1} + \int\limits_{I\times\Omega} \Delta u\,\Delta\varphi\,d\mu_{k+3} + 4\int\limits_{I\times\Omega} \nabla' u \cdot \nabla'\varphi\,d\mu_{k+1} = \int\limits_{I\times\Omega} f\,\varphi\,d\mu_{k+1} + \int\limits_{\Omega} g\,\varphi(t_1)\,d\mu_{k+1}$$

*holds for all $\varphi \in C^\infty_c\big([t_1, t_2) \times \Omega\big)$.*

If $u$ is a $L_0$-solution to $f = 0$ on $I \times \Omega$, then for any $\tilde{t}_1 \in I$, $u$ can be regarded as initial value solution on $[\tilde{t}_1, t_2) \times \Omega$ subject to the initial condition $u(\tilde{t}_1) = g$. Conversely, any solution in the sense of definition 4.0.1 (ii) gives rise to a solution in the sense of definition 4.0.1 (i) on $(\tilde{t}_1, t_2) \times \Omega$ for any $t_1 \leq \tilde{t}_1 < t_2$. This implies that any statement obtained for a solution of either the initial value problem or the general problem can thus be applied to a solution on $[\tilde{t}_1, t_2) \times \Omega$.

It is clear from the definition of $C_c^\infty$ (see section 2.2) that in (i) the test functions in question vanish towards the endpoints of $I$, while in (ii) they allow values at initial time $t_1$. On the other hand, the test functions vanish towards the boundary of $\Omega$ unless it coincides with $\partial H$. For now, however, we only consider the case $\Omega = \overline{H}$.

As a first observation, we state the following result.

**Lemma 4.0.2** *If $u$ is an energy solution to $\partial_t u + L_k u = f$ on $I \times \overline{H}$, then $\nabla u(t) \in L^2(H, \mu_{k+2})$ for almost every $t \in I$.*

**Proof:** The regularity gain is a simple consequence of the weighted interpolation inequality 2.8.3. It implies

$$\|\nabla u\|_{L^2(\mu_{k+2})} \lesssim \|u\|_{L^2(\mu_{k+1})} + \|D_x^2 u\|_{L^2(\mu_{k+3})},$$

and the assertion follows. ∎

## 4.1 Existence and Uniqueness of Energy Solutions

We start our analysis by considering the case $k = 0$ only, but we remark that all the arguments can be applied to the general operator $L_k$, $k \geq 1$.

**Proposition 4.1.1 (existence)** *Let $I$ and $f$ be as in definition 4.0.1. Then for any time $\tilde{t}_1 \in I$ and $g \in L^2(H, \mu_1)$ there exists an energy solution to $\partial_t u + L_0 u = f$ on $[\tilde{t}_1, t_2) \times \overline{H}$ with initial value $u(\tilde{t}_1) = g$.*

**Proof:** We prove existence of an energy solution by a method called Galerkin approximation. For this we propose to construct solutions of certain projections of the original problem onto problems in finite dimensional spaces which approximate the initial value problems. Then we pass to the limit and show that this limit is a solution to the original problem.

**Step 1:** Let $\{v_i\}_{i \in \mathbb{N}}$ be a Schauder basis of the Hilbert space $X := W^{2,2}(\mu_1, \mu_1, \mu_3) \subset L^2(\mu_1)$. Here we can choose the basis in such a way that it is orthonormal in $L^2(\mu_1)$. We set

$$\begin{aligned} A_m &:= \Big( (v_i \,|\, v_j)_{L^2(\mu_1)} \Big)_{i,j=1}^m &&=: (a^{i,j})_{i,j=1}^m, \\ B_m &:= \Big( (\nabla' v_i \,|\, \nabla' v_j)_{L^2(\mu_1)} \Big)_{i,j=1}^m &&=: (b^{i,j})_{i,j=1}^m, \\ C_m &:= \Big( (\Delta v_i \,|\, \Delta v_j)_{L^2(\mu_3)} \Big)_{i,j=1}^m &&=: (c^{i,j})_{i,j=1}^m \end{aligned}$$

and $\vec{f}_m(t) := \Big( (f(t) \,|\, v_i)_{L^2(\mu_1)} \Big)_{i=1}^m$ for any $m \in \mathbb{N}$.
By lemma 2.4.2 we know that the matrix $A_m$ is invertible. Consequently, $A_m^{-1} B_m$, $A_m^{-1} C_m$ and $A_m^{-1} \vec{f}_m(t)$ exist and all entries are in $L^1_{loc}(I)$. According to standard existence theory for ordinary differential equations, then for any $\tilde{t}_1 \in \bar{I}$ and $\vec{g}_m = (g_m^1, \ldots, g_m^m) \in \mathbb{R}^m$ there exists a unique solution $\vec{d}_m = (d_m^1, \ldots, d_m^m) \in W^{1,1}_{loc}(\bar{I}) = AC_{loc}(\bar{I})$ such that

$$\partial_t \vec{d}_m(t) + A_m^{-1} C_m \, \vec{d}_m(t) + 4 \, A_m^{-1} B_m \, \vec{d}_m(t) = A_m^{-1} \vec{f}_m(t) \qquad \text{a.e. in } I$$

subject to $\vec{d}_m(\tilde{t}_1) = \vec{g}_m$. But this is equivalent to saying that $d_m^i(t)$ satisfies the equation

$$\sum_{j=1}^m \Big( a^{i,j} \, \partial_t d_m^j(t) + c^{i,j} \, d_m^j(t) + 4 \, b^{i,j} \, d_m^j(t) \Big) = \big( f(t) \,|\, v_i \big)_{L^2(\mu_1)} \qquad (i = 1, \ldots, m)$$

for almost every $t \in I$ with $d_m^i(\tilde{t}_1) = g_m^i$. We thus have proved that for each integer $m = 1, 2, \dots$ there exists a unique function $u_m \in W^{1,1}_{loc}(I; X)$ of the form

$$u_m(t) \;:=\; \sum_{i=1}^{m} d_m^i(t)\, v_i \qquad \text{such that} \qquad \lim_{t\to\tilde{t}_1} u_m(t) =: u_m(\tilde{t}_1) = \sum_{i=1}^{m} g_m^i\, v_i\,. \tag{4.1.1}$$

Moreover, for $i = 1, \dots, m$ we find

$$\big(\partial_t u_m(t)\big|v_i\big)_{L^2(\mu_1)} + \big(\Delta u_m(t)\big|\Delta v_i\big)_{L^2(\mu_3)} + 4\big(\nabla' u_m(t)\big|\nabla' v_i\big)_{L^2(\mu_1)} = \big(f(t)\big|v_i\big)_{L^2(\mu_1)} \tag{$*$}$$

holds for almost every $t \in I$. This means $u_m(t)$ satisfies the projection of the initial value problem onto the finite dimensional subspace spanned by $\{v_i\}_{i=1}^m$.

**Step 2:** We intend to send $m$ to infinity and to prove that a subsequence of our solution $u_m$ of the approximate equation converges to an energy solution. But first we shall need some preliminary estimates.

We multiply equation $(*)$ by $d_m^i(t)$ and then sum over $i$ to get

$$\frac{1}{2}\,\partial_t \int_H u_m(t)^2\, d\mu_1 \,+\, \int_H \big(\Delta u_m(t)\big)^2\, d\mu_3 \,+\, 4\int_H \big|\nabla' u_m(t)\big|^2\, d\mu_1 \;=\; \int_H f(t)\, u_m(t)\, d\mu_1$$

for almost every $t \in I$. Integrating this equality over $(\tilde{t}_1, \tilde{t}_2) \subset \bar{I}$ and using the fundamental theorem of calculus gives

$$\frac{1}{2}\,\|u_m(\tilde{t}_2)\|^2_{L^2(\mu_1)} \,+\, \int_{\tilde{t}_1}^{\tilde{t}_2} \|\Delta u_m(t)\|^2_{L^2(\mu_3)}\, dt \,+\, 4\int_{\tilde{t}_1}^{\tilde{t}_2} \|\nabla' u_m(t)\|^2_{L^2(\mu_1)}\, dt$$

$$=\; \int_{\tilde{t}_1}^{\tilde{t}_2} \big(f(t) \mid u_m(t)\big)_{L^2(\mu_1)}\, dt \,+\, \frac{1}{2}\,\|u_m(\tilde{t}_1)\|^2_{L^2(\mu_1)}\,,$$

By Hölder's inequality we readily check

$$\begin{aligned}\int_{\tilde{t}_1}^{\tilde{t}_2} \big(f(t) \mid u_m(t)\big)_{L^2(\mu_1)}\, dt \;&\le\; \int_{\tilde{t}_1}^{\tilde{t}_2} \|f(t)\|_{L^2(\mu_1)}\, \|u_m(t)\|_{L^2(\mu_1)}\, dt\\ &\le\; \|f\|^2_{L^1(I;L^2(\mu_1))} \,+\, \frac{1}{4}\, \sup_{t\in\overline{[\tilde{t}_1, t_2)}} \|u_m(t)\|^2_{L^2(\mu_1)}\end{aligned}$$

Taking the supremum over $\tilde{t}_2 \in \overline{J}$ with $J := (\tilde{t}_1, t_2)$ yields

$$\begin{aligned}&\sup_{t\in\overline{J}} \|u_m(t)\|^2_{L^2(\mu_1)} \,+\, \int_J \|\nabla' u_m(t)\|^2_{L^2(\mu_1)}\, dt \,+\, \int_J \|\Delta u_m(t)\|^2_{L^2(\mu_3)}\, dt\\ &\lesssim\; \|f\|^2_{L^1(I;L^2(\mu_1))} \,+\, \|u_m(\tilde{t}_1)\|^2_{L^2(\mu_1)} \;\le\; \|f\|^2_{L^1(I;L^2(\mu_1))} \,+\, \|g\|^2_{L^2(\mu_1)}\,.\end{aligned} \tag{4.1.2}$$

For the last inequality we set $g_m^i := (g \,|\, v_i)_{L^2(\mu_1)}$ such that

$$\|u_m(\tilde{t}_1)\|_{L^2(\mu_1)} \;\overset{(4.1.1)}{=}\; \|\sum_{i=1}^{m} \big(g \mid v_i\big)_{L^2(\mu_1)}\, v_i\|_{L^2(\mu_1)} \;\le\; \|g\|_{L^2(\mu_1)}\,.$$

This estimate gives a uniform upper bound independent of $m$.

**Step 3:** Now we are ready to pass to the limit. According to the estimate (4.1.2), we see that the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in $L^\infty\big(J; L^2(\mu_1)\big)$, $\{\nabla' u_m\}_{m\in\mathbb{N}}$ is bounded in $L^2\big(J; L^2(H, \mu_1)\big)$, and $\{\Delta u_m\}_{m\in\mathbb{N}}$ is bounded in $L^2\big(J; L^2(H, \mu_3)\big)$. Consequently, we can find a subsequence $\{u_{m_l}\}_{l\in\mathbb{N}}$ of $\{u_m\}_{m\in\mathbb{N}}$, such that for $l \to \infty$ we have

$$\left.\begin{aligned} u_{m_l} &\rightharpoonup^* u \;\in\; L^\infty\big(J; L^2(\mu_1)\big)\,, \\ \nabla' u_{m_l} &\rightharpoonup \vec{w} \;\in\; L^2\big(J; L^2(H,\mu_1)\big)\,, \\ \Delta u_{m_l} &\rightharpoonup w \;\in\; L^2\big(J; L^2(H,\mu_3)\big)\,. \end{aligned}\right\} \tag{4.1.3}$$

This means

$$\int_J \big(u_{m_l}(t) \mid \zeta(t)\big)_{L^2(\mu_1)}\,dt \;\to\; \int_J \big(u(t) \mid \zeta(t)\big)_{L^2(\mu_1)}\,dt \qquad \forall\, \zeta \in L^1\big(J; L^2(\mu_1)\big)\,,$$

$$\int_J \big(\nabla' u_{m_l}(t) \mid \vec{\zeta}(t)\big)_{L^2(\mu_1)}\,dt \;\to\; \int_J \big(\vec{w}(t) \mid \vec{\zeta}(t)\big)_{L^2(\mu_1)}\,dt \qquad \forall\, \vec{\zeta} \in L^2\big(J; L^2(H,\mu_1)\big)$$

and

$$\int_J \big(\Delta u_{m_l}(t) \mid \zeta(t)\big)_{L^2(\mu_3)}\,dt \;\to\; \int_J \big(w(t) \mid \zeta(t)\big)_{L^2(\mu_3)}\,dt \qquad \forall\, \zeta \in L^2\big(J; L^2(H,\mu_3)\big)$$

as $l \to \infty$. From this follows in fact that $\vec{w} = \nabla' u$ and $w = \Delta u$. Now for some fixed $N \in \mathbb{N}$, we define

$$v(t) \;:=\; \sum_{i=1}^{N} d^i(t)\, v_i\,, \tag{4.1.4}$$

where $d^i \in C_c^\infty\big([\tilde{t}_1, t_2)\big)$ for $i = 1, \ldots, N$. Let $m_l > N$. We multiply equation $(*)$ by $d^i(t)$, sum up over $i = 1, \ldots, N$ and integrate with respect to time $t \in J$ to obtain

$$\int_J \big(\partial_t u_{m_l}(t) \mid v(t)\big)_{L^2(\mu_1)}\,dt + \int_J \big(\Delta u_{m_l}(t) \mid \Delta v(t)\big)_{L^2(\mu_3)}\,dt + 4\int_J \big(\nabla' u_{m_l}(t) \mid \nabla' v(t)\big)_{L^2(\mu_1)}\,dt$$
$$= \int_J \big(f(t) \mid v(t)\big)_{L^2(\mu_1)}\,dt\,.$$

Thanks to the Sobolev embedding 2.5.5 we know that $C_c^\infty\big([\tilde{t}_1, t_2) \times \overline{H}\big)$ is dense in the space spanned by functions of the form (4.1.4), and hence the equality holds in particular for all such test functions $\varphi$.

Next we apply an integration by parts to the first term to get

$$-\int_J \big(u_{m_l}(t) \mid \partial_t \varphi(t)\big)_{L^2(\mu_1)}\,dt + \int_J \big(\Delta u_{m_l}(t) \mid \Delta \varphi(t)\big)_{L^2(\mu_3)}\,dt + 4\int_J \big(\nabla' u_{m_l}(t) \mid \nabla' \varphi(t)\big)_{L^2(\mu_1)}\,dt$$
$$= \int_J \big(f(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\,dt + \big(u_{m_l}(\tilde{t}_1) \mid \varphi(\tilde{t}_1)\big)_{L^2(\mu_1)}\,.$$

We recall (4.1.1) and the fact that $\{v_i\}_{i\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mu_1)$ to deduce that

$$\Big(\sum_{i=1}^{m_l} (g \mid v_i)_{L^2(\mu_1)} v_i \,\Big|\, \varphi(\tilde{t}_1)\Big)_{L^2(\mu_1)} \;\longrightarrow\; \big(g \mid \varphi(\tilde{t}_1)\big)_{L^2(\mu_1)} \qquad (\text{for} \quad l \to \infty)\,.$$

Now we employ the weak convergence (4.1.3), with test functions $\partial_t \varphi, \nabla' \varphi$ and $\Delta \varphi$, to discover that

$$-\int_J \big(u(t) \mid \partial_t \varphi(t)\big)_{L^2(\mu_1)}\,dt + \int_J \big(\Delta u(t) \mid \Delta \varphi(t)\big)_{L^2(\mu_3)}\,dt + 4\int_J \big(\nabla' u(t) \mid \nabla' \varphi(t)\big)_{L^2(\mu_1)}\,dt$$
$$= \int_J \big(f(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\,dt + \big(g \mid \varphi(\tilde{t}_1)\big)_{L^2(\mu_1)}$$

as $l \to \infty$. Altogether, given $\tilde{t}_1 \in \bar{I}$ and $g \in L^2(\mu_1)$, this construction gives us an energy solution $u \in L^\infty\big(J; L^2(\mu_1)\big) \subset L^2\big(J; L^2(\mu_1)\big)$ to $\partial_t u + L_0 u = f$ on $J \times \overline{H}$ subject to $u(\tilde{t}_1) = g$. ■

**Remark 4.1.2** *In view of (4.1.1), we find that the spatial part of the approximate solution is in the Hilbert space $W^{2,2}(\mu_1, \mu_1, \mu_3)$, and so the same is true for the solution obtained by the Galerkin approximation. This shows that a Galerkin solution is also an energy solution, but with the additional feature that $u \in$*

$L^\infty L^2(\mu_1)$. *Now one may ask the question whether the assumptions on $u$ in definition 4.0.1 are the weakest possible. The answer is yes and no at the same time. On the one hand, demanding that $\nabla' u \in L^2(\mu_1)$ and $\Delta u \in L^2(\mu_3)$ is enough to make sense of the expressions contained in the definition of an energy solution, and a Galerkin solution would still be an energy solution in the sense of the definition. On the other hand, all the following statements would be of no significance for energy solutions, but rather for Galerkin solutions exclusively. Indeed, this is why we need to place additional conditions upon $u$ in our present definition of solution. Later we will see that under these regularity assumptions the solution turns out to be even more regular.*

Before we move on to the next results, we need to make some preparations. Let $I = (t_1, t_2) \subseteq \mathbb{R}$ be any interval. For $\varepsilon > 0$ small enough, we consider the intervals

$$I_\varepsilon := (t_1\,, t_2 - \varepsilon) \qquad \text{and} \qquad I_{-\varepsilon} := (t_1 + \varepsilon\,, t_2)$$

with the usual conventions if $t_1 = -\infty$ or $t_2 = \infty$. Moreover, for $u \in L^1_{loc}(I)$ we refer to

$$u_\varepsilon(t) := \begin{cases} \varepsilon^{-1} \int_t^{t+\varepsilon} u(\tau)\, d\tau & \text{if } t \in I_\varepsilon \\ 0 & \text{if } t \in I \backslash I_\varepsilon\,. \end{cases}$$

as the regularization of $u$ in time, since $u_\varepsilon$ is differentiable on $I_\varepsilon$ and $u_\varepsilon \in L^1_{loc}(I)$. Consequently, $u_\varepsilon(t) \to u(t)$ as $\varepsilon \to 0$ pointwise in $I_\varepsilon$. Appealing to Fubini's theorem we get

$$\begin{aligned} \int_I u_\varepsilon(t)\, \varphi(t)\, dt \;&=\; \varepsilon^{-1} \int_{I_\varepsilon \cap I_{-\varepsilon}} \int_t^{t+\varepsilon} u(\tau)\, d\tau\, \varphi(t)\, dt \;=\; \varepsilon^{-1} \int_{I_{-\varepsilon}} u(\tau) \int_{\tau-\varepsilon}^{\tau} \varphi(t)\, dt\, d\tau \\ &=: \int_I u(\tau)\, \varphi_{-\varepsilon}(\tau)\, d\tau \end{aligned}$$

for all $\varphi \in C_c^\infty(I)$ with $spt\,\varphi \subset \bar{I}_\varepsilon \cap \bar{I}_{-\varepsilon}$. Imposing the requirement that $\partial_t u \in L^1_{loc}(I)$ we define the temporal difference quotient by $D^\varepsilon u := (\partial_t u)_\varepsilon = \partial_t(u_\varepsilon) \in L^1_{loc}(I)$ and, likewise, $D^{-\varepsilon} u := \partial_t(u_{-\varepsilon})$. For this an integration by parts is available, given by

$$\int_I D^\varepsilon u(t)\, \varphi(t)\, dt \;=\; -\int_I u(t)\; D^{-\varepsilon}\, \varphi(t)\, dt$$

for all $\varphi \in C_c^\infty(I)$ with $spt\,\varphi \subset \bar{I}_\varepsilon \cap \bar{I}_{-\varepsilon}$.
Note also that if $u$ is as in definition 4.0.1, then we have $u_\varepsilon \in C\big(I_\varepsilon\,; L^2(\mu_1)\big)$, $\nabla' u_\varepsilon \in C\big(I_\varepsilon\,; L^2(\mu_1)\big)$ and $\Delta u_\varepsilon \in C\big(I_\varepsilon\,; L^2(\mu_3)\big)$, as well as $\partial_t u_\varepsilon \in L^2_{loc}\big(I_\varepsilon\,; L^2(\mu_1)\big)$.

**Remark 4.1.3** *Since $C_c^\infty(\overline{H})$ is densely contained in $W^{2,2}(H, \mu_1, \mu_1, \mu_3)$ it suffices to consider test functions $\varphi \in L^2\big(I; L^2(H, \mu_1)\big)$ that have a compact time support in $I$ and $[t_1, t_2)$, respectively, with additionally $\partial_t \varphi \in L^2\big(I; L^2(\mu_1)\big)$, $\nabla\varphi \in L^2\big(I; L^2(H, \mu_1)\big)$ and $D_x^2 \varphi \in L^2\big(I; L^2(H, \mu_3)\big)$. Then the space of test functions is a dense subspace of such a class of functions. Alongside this, we see that a solution itself almost qualifies as a test function.*

**Proposition 4.1.4** *Suppose $t_1$ is finite, $I = (t_1, t_2) \subset \mathbb{R}$ is some open interval and $f \in L^1\big(I; L^2(H, \mu_1)\big)$.*

- *i) If $u$ is a $L_0$-solution on $I \times \overline{H}$ to $f$, then $u \in C\big([\tilde{t}_1, t_2]; L^2(H, \mu_1)\big)$ for all $\tilde{t}_1 \in I$.*
- *ii) If $u$ is an energy solution on $[t_1, t_2) \times \overline{H}$ to $f$ with initial value $g \in L^2(H, \mu_1)$, then we have $u \in C\big(\bar{I}; L^2(H, \mu_1)\big)$ with $u(t_1) = g$.*

**Proof:** Formally, we use $\varphi = \chi_I u$ as a test function. To make this rigorous, we replace $u$ by $u_\varepsilon$ in both the equation and the test function, and approximate $\chi_I$ by an appropriate cut-off function, and let then $\varepsilon \to 0$.

**(i)** We define the approximate characteristic function for $t_1 < \tilde{t}_1 < \tilde{t}_1 + \delta_0 < \tilde{t}_2 < \tilde{t}_2 + \delta_1 < t_2$ by

$$\chi_\delta(t) \;:=\; \chi_{\delta_0,\delta_1}^{\tilde{t}_1,\tilde{t}_2}(t) \;:=\begin{cases} 0 & \text{if} \quad t_1 < t \leq \tilde{t}_1 \\ \frac{t-\tilde{t}_1}{\delta_0} & \text{if} \quad \tilde{t}_1 < t \leq \tilde{t}_1 + \delta_0 \\ 1 & \text{if} \quad \tilde{t}_1 + \delta_0 < t \leq \tilde{t}_2 \\ \frac{\tilde{t}_2 - t}{\delta_1} + 1 & \text{if} \quad \tilde{t}_2 < t \leq \tilde{t}_2 + \delta_1 \\ 0 & \text{if} \quad \tilde{t}_2 + \delta_1 < t < t_2 \,. \end{cases}$$

Now for sufficiently small $\varepsilon > 0$ we have $spt\, \chi_\delta \subseteq [\tilde{t}_1, \tilde{t}_2 + \delta_1] \subset I_\varepsilon \cap I_{-\varepsilon}$ and thus $\varphi \;:=\; (\chi_\delta\, u_\varepsilon)_{-\varepsilon}$ is an admissible test function (see remark 4.1.3). In particular, $\partial_t u_\varepsilon = D^\varepsilon u \in L^2_{loc}\big(I; L^2(\mu_1)\big)$.

Plugging the test function into the equation yields

$$\begin{aligned} &\int_I \big(f(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\, dt \;-\; \int_I \int_H \Delta u(t)\, \Delta\varphi(t)\, d\mu_3\, dt \;-\; 4 \int_I \int_H \nabla' u(t) \cdot \nabla'\varphi(t)\, d\mu_1\, dt \\ &\quad= \int_I \chi_\delta(t) \Big( \big(f_\varepsilon(t) \mid u_\varepsilon(t)\big)_{L^2(\mu_1)} - \|\Delta u_\varepsilon(t)\|^2_{L^2(\mu_3)} - 4\, \|\nabla' u_\varepsilon(t)\|^2_{L^2(\mu_1)} \Big)\, dt \\ &\overset{(\varepsilon\to 0)}{\longrightarrow} \int_I \chi_\delta(t) \Big( \big(f(t) \mid u(t)\big)_{L^2(\mu_1)} - \|\Delta u(t)\|^2_{L^2(\mu_3)} - 4 \|\nabla' u(t)\|^2_{L^2(\mu_1)} \Big) dt =: \int_I \chi_\delta(t)\, \eta(t)\, dt\,, \end{aligned}$$

and the limit $\varepsilon \to 0$ poses no difficulty. Moreover, we readily compute

$$\begin{aligned} -\int_I \int_H u(t)\, \partial_t \varphi(t)\, d\mu_1\, dt \;&=\; -\frac{1}{2} \int_I \int_H \chi_\delta(t)\, \partial_t \big(u_\varepsilon{}^2(t)\big)\, d\mu_1\, dt \;-\; \int_I \int_H \partial_t \chi_\delta(t)\, u_\varepsilon{}^2(t)\, d\mu_1\, dt \\ &=\; -\frac{1}{2} \int_I \partial_t \chi_\delta(t)\, \|u_\varepsilon(t)\|^2_{L^2(\mu_1)}\, dt \\ &=\; \frac{1}{2} \Big( \|u_\varepsilon(\tilde{t}_2)\|^2_{L^2(\mu_1)} \Big)_{\delta_1} \;-\; \frac{1}{2} \Big( \|u_\varepsilon(\tilde{t}_1)\|^2_{L^2(\mu_1)} \Big)_{\delta_0}\,. \end{aligned}$$

Now we replace $\tilde{t}_2 \in J := (\tilde{t}_1, t_2)$ by $t$, send then $\varepsilon \to 0$ and rearrange the terms to get

$$\Big( \|u(t)\|^2_{L^2(\mu_1)} \Big)_{\delta_1} \;=\; 2 \int_{t_1}^{t+\delta_1} \chi_\delta(\tau)\, \eta(\tau)\, d\tau \;+\; \Big( \|u(\tilde{t}_1)\|^2_{L^2(\mu_1)} \Big)_{\delta_0}\,.$$

with $\chi_\delta = \chi_{\delta_0,\delta_1}^{\tilde{t}_1,t}$. An identity of this kind is called regularized energy equation. From this we get directly that $t \mapsto \big(\|u(t)\|^2_{L^2(\mu_1)}\big)_{\delta_1}$ is a sequence in $C(J)$. Furthermore, for any $\delta_1, \delta_1' > 0$ we find

$$\begin{aligned} \sup_{t\in J} \left| \Big( \|u(t)\|^2_{L^2(\mu_1)} \Big)_{\delta_1} - \Big( \|u(t)\|^2_{L^2(\mu_1)} \Big)_{\delta_1'} \right| \;&=\; 2 \sup_{t\in J} \left| \int_I \Big( \chi_{\delta_0,\delta_1}^{\tilde{t}_1,t}(\tau) - \chi_{\delta_0,\delta_1'}^{\tilde{t}_1,t}(\tau) \Big)\, \eta(\tau)\, d\tau \right| \\ &\leq\; 2 \sup_{t\in J} \int_t^{t+\max\{\delta_1,\delta_1'\}} \big|\eta(\tau)\big|\, d\tau \;\to\; 0\,, \end{aligned}$$

as $\max\{\delta_1, \delta_1'\} \to 0$. We also used that $\eta \in L^1(I)$ and $\chi_\delta$ is bounded from above by 1. But this implies that $\big(\|u(t)\|^2_{L^2(\mu_1)}\big)_{\delta_1}$ is in fact a Cauchy sequence in $C(J)$ and thus converges uniformly to a continuous limit.

From the fact that in every Lebesgue point we have $\big(\|u(t)\|^2_{L^2(\mu_1)}\big)_{\delta_1} \to \|u(t)\|^2_{L^2(\mu_1)}$ as $\delta_1 \to 0$ since $t \mapsto \int_H u(t,\cdot)^2\, d\mu_1 \in L^2(I)$, follows $t \mapsto \|u(t)\|_{L^2(\mu_1)} \in C(I)$. At this point we turn back to the regularized energy equation and take the supremum over all $t \in \overline{J}$ to discover that $t \mapsto \|u(t)\|_{L^2(\mu_1)}$ is bounded and therefore in $C_b(\overline{J})$.

Now for $\breve{\varphi} \in C_c^\infty(\overline{H})$ we use $\varphi(t,x) := \chi_\delta(t)\breve{\varphi}(x)$ as test function to discover in a similar manner as above that $t \mapsto \big(u(t) \mid \breve{\varphi}\big)_{L^2(\mu_1)} \in C_b(\overline{J})$. By density we thus get that $t \mapsto u(t) \in L^2(\mu_1)$ is weakly continuous. But weak continuity and norm continuity in $L^2(\mu_1)$ imply continuity on $\overline{J}$ for any $\tilde{t}_1 \in I$, and hence in particular on $\bar{I}\backslash\{t_1\}$ as stated.

**(ii)** Regarding definition 4.0.1 (ii) we need to adjust the test function in such a way as to enable it to take a value at initial time $t_1$. To this end we fix $\bar{t} \in I$ and define

$$\chi_{\bar{t},\delta}(t) := \begin{cases} 1 & \text{if } t_1 \le t < \bar{t}\,, \\ \frac{\bar{t}-t}{\delta} + 1 & \text{if } \bar{t} \le t < \bar{t}+\delta\,, \\ 0 & \text{if } \bar{t}+\delta \le t < t_2\,. \end{cases}$$

for sufficiently small $\delta > 0$. Then $\varphi = \chi_{\bar{t},\delta}\,\breve{\varphi}$ with $\breve{\varphi} \in C_c^\infty(\overline{H})$ is an legitimate test function and we proceed as above to get

$$\begin{aligned} \big(u_\delta(\bar{t}) \mid \breve{\varphi}\big)_{L^2(\mu_1)} \;=\; & -\int_I \chi_{\bar{t},\delta}(t)\Big(\big(\Delta u(t) \mid \Delta\breve{\varphi}\big)_{L^2(\mu_3)} + 4\big(\nabla' u(t) \mid \nabla'\breve{\varphi}\big)_{L^2(\mu_1)}\Big)\,dt + \\ & + \int_I \chi_{\bar{t},\delta}(t)\,\big(f(t) \mid \breve{\varphi}\big)_{L^2(\mu_1)}\,dt + \big(g \mid \breve{\varphi}\big)_{L^2(\mu_1)}\,. \end{aligned}$$

In view of part (i) we now let $\delta \to 0$ to see that $\bar{t} \mapsto \int_I \big(u(\bar{t}) \mid \breve{\varphi}\big)_{L^2(\mu_1)} \in C(I)$ for all $\breve{\varphi} \in C_c^\infty(\overline{H})$. Indeed, passing to the limit $\bar{t} \to t_1$ gives

$$\lim_{\bar{t}\to t_1} \big(u(\bar{t}) \mid \breve{\varphi}\big)_{L^2(\mu_1)} \;=\; \big(g \mid \breve{\varphi}\big)_{L^2(\mu_1)} \qquad \forall\, \breve{\varphi} \in C_c^\infty(\overline{H})\,.$$

This shows that, in contrast to part (i), weak continuity can be extended down to $t_1$. Using once more the uniformly boundedness of $\|u(t)\|_{L^2(\mu_1)}$ on $I$ yields strong continuity on all of $\bar{I}$ with $u(t_1) = g$. ■

A consequence of the continuity is that any solution $u$, regardless if an initial value is given or not, can be evaluated at any time $t \in I$.

**Remark 4.1.5 (energy identity)** *The proof of proposition 4.1.4 (i) also reveals that the energy identity*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mu_1)} + \int_{\tilde{t}_1}^t \|\Delta u\|^2_{L^2(\mu_3)} + 4\,\|\nabla' u\|^2_{L^2(\mu_1)}\,d\mathcal{L} \;=\; \int_{\tilde{t}_1}^t \big(f \mid u\big)_{L^2(\mu_1)}\,d\mathcal{L} + \frac{1}{2}\,\|u(\tilde{t}_1)\|^2_{L^2(\mu_1)}$$

*holds for any $t \ge \tilde{t}_1 \in \bar{I}\backslash\{t_1\}$. If $u$ solves the initial value problem we even obtain*

$$\frac{1}{2}\,\|u(t)\|^2_{L^2(\mu_1)} + \int_{t_1}^t \|\Delta u\|^2_{L^2(\mu_3)} + 4\,\|\nabla' u\|^2_{L^2(\mu_1)}\,d\mathcal{L} \;=\; \int_{t_1}^t \big(f \mid u\big)_{L^2(\mu_1)}\,d\mathcal{L} + \frac{1}{2}\,\|g\|^2_{L^2(\mu_1)}$$

*for any $t \ge t_1$. In case of $f = 0$ these equalities have a simple implication: Then $t \mapsto \|u(t)\|_{L^2(\mu_1)}$ is monotonically decreasing.*

Uniqueness is now an immediate consequence.

**Corollary 4.1.6 (uniqueness)** *An energy solution of the initial value problem is unique.*

**Proof:** Let $u_1$ and $u_2$ be two energy solutions with $u_1(t_1) = u_2(t_1) = g$. Setting $u := u_1 - u_2$ we find that $\partial_t u + L_0 u = 0$ on $[t_1, t_2) \times \overline{H}$ in the energy sense with $u(t_1) = 0$. But then by the previous remark 4.1.5 it follows that $\|u(t)\|_{L^2(\mu_1)} \le 0$, and hence we have $u \equiv 0$. ■

**Corollary 4.1.7** *Let $I = (-\infty, t_2)$ and $f$ be as in definition 4.0.1. Then there exists a uniquely determined energy solution $u$ with $\lim_{t\to-\infty} \|u(t)\|_{L^2(H,\mu_1)} = 0$. Moreover, the energy identity holds for any $t \ge \tilde{t}_1 \in \bar{I}$.*

**Proof:** Let $(t^i)_{i\in\mathbb{N}} \subset I$ be a monotonously decreasing sequence with $\lim_{i\to-\infty} t^i = -\infty$. Then, according to proposition 4.1.1 and 4.1.4 (ii), there exist time-continuous energy solutions $u_i$ on $[t^i, t_2) \times \overline{H}$ with $u_i(t^i) = 0$. These can be continuously extended by 0 such that in fact $u_i \in C_b\big(\bar{I}; L^2(\mu_1)\big)$. Now we observe that for $j > i$ the function $w_{i,j} := u_j - u_i$ satisfies $\partial_t w_{i,j} + L_0 w_{i,j} = \chi_{(t^j,t^i)}\, f$ on $I$ and we know that $w_{i,j}(t) = 0$ for $t \le t^j$. In this situation we can apply the energy identity to discover

$$\sup_{t\in I} \|w_{i,j}(t)\|_{L^2(\mu_1)} \;\le\; 2\int_{t^j}^{t^i} \|f(t)\|_{L^2(\mu_1)}\, dt\,.$$

Now since $f \in L^1\big(I; L^2(\mu_1)\big)$ there exists for any $\varepsilon > 0$ an integer $N = N(\varepsilon, f) \in \mathbb{N}_0$ such that $\int_{t^j}^{t^i} \|f(t)\|_{L^2(\mu_1)}\, dt < \varepsilon$ for all $j > i \ge N$. Consequently, $u_i$ is a Cauchy sequence in $I$ and hence the limit $\lim_{i\to\infty} u_i = u$ exists. Moreover, $u \in C_b\big(I; L^2(\mu_1)\big)$ satisfies $\partial_t u + L_0 u = f$ on $I \times \overline{H}$ and

$$\lim_{i\to\infty} \|u(t^i)\|_{L^2(\mu_1)} \;=\; \lim_{i\to\infty} \|u(t^i) - u_i(t^i)\|_{L^2(\mu_1)} \;\le\; \lim_{i\to\infty}\, \sup_{t\in I} \|u(t) - u_i(t)\|_{L^2(\mu_1)} \;=\; 0\,,$$

as required. Uniqueness can then be derived as in corollary 4.1.6. ■

**Remark 4.1.8** *We have seen that in case $t_1 > -\infty$ the energy solution is uniquely determined by $f$ and $g$. Moreover, if $t_1 = -\infty$ there is exactly one function $u$ which satisfies $\partial_t u + L_0 u = f$ in the energy sense and which tends to zero in $L^2(\mu_1)$ as $t \to -\infty$. Henceforward, we will always mean this special solution when we talk about an energy solution of the initial value problem on $(-\infty, t_2) \times \overline{H}$. To put it another way, given an initial datum $g$ there exists exactly one energy solution with $u(t_1) = g$ and, if $t_1 = -\infty$, we only consider the case $g = 0$.*

## 4.2 Energy Estimates

In this section we shall use similar regularization techniques as above to derive $L^2$- estimates for derivatives of an energy solution. Here it is often crucial to choose the right test function and, eventually, this also justifies the minimal choice of requirements we have made in definition 4.0.1.

**Proposition 4.2.1 (temporal energy estimates)** *Let $I = (t_1, t_2) \subseteq \mathbb{R}$ and $f \in L^2\big(I; L^2(H, \mu_1)\big)$. If $u$ satisfies the equation $\partial_t + L_0 u = f$ on $I \times \overline{H}$ in the energy sense, then for any $\tilde{t}_1 \in I$ we have $t \mapsto \|\Delta u(t)\|_{L^2(H,\mu_3)} + \|\nabla' u(t)\|_{L^2(H,\mu_1)} \in C_b\big(\overline{J}\big)$ with $J := (\tilde{t}_1, t_2)$ and*

$$\int_J \|\partial_t u(t)\|^2_{L^2(H,\mu_1)}\, dt \;\le\; \int_J \|f(t)\|^2_{L^2(H,\mu_1)}\, dt \;+\; \|\Delta u(\tilde{t}_1)\|^2_{L^2(H,\mu_3)} \;+\; 4\,\|\nabla' u(\tilde{t}_1)\|^2_{L^2(H,\mu_1)}\,.$$

**Proof:** We would like to use $\varphi = \chi_I\, \partial_t u$ as a test function. But this $\varphi$ does not exhibit the required regularity of a test function, and therefore we approximate the temporal derivative by finite differences $D^\varepsilon u$ or, equivalently, $\partial_t u_\varepsilon$. Indeed, we have seen that $\partial_t u_\varepsilon = D^\varepsilon u$.

Let $u$ be an energy solution, then $u_\varepsilon$ satisfies the equality

$$-\int_I \big(u_\varepsilon \mid \partial_t\varphi\big)_{L^2(\mu_1)}\, d\mathcal{L} + \int_I \big(\Delta u_\varepsilon \mid \Delta\varphi\big)_{L^2(\mu_3)}\, d\mathcal{L} \,+\, 4\int_I \big(\nabla' u_\varepsilon \mid \nabla'\varphi\big)_{L^2(\mu_1)}\, d\mathcal{L} \,=\, \int_I \big(f_\varepsilon \mid \varphi\big)_{L^2(\mu_1)}\, d\mathcal{L},$$

for all test functions defined as in definition 4.0.1 (i). An integration by parts applied to the first term yields

$$-\int_I \big(u_\varepsilon(t) \mid \partial_t\varphi(t)\big)_{L^2(\mu_1)}\, dt \;=\; \int_I \big(D^\varepsilon u(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\, dt\,.$$

Here $\varphi = \chi_\delta \, D^\varepsilon u$, with $\chi_\delta$ defined as above, is an admissible test function. Similar calculations as in the proof of proposition 4.1.4 (i) then lead to

$$\int_I \big(D^\varepsilon u(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\, dt \;=\; \int_I \chi_\delta(t)\, \|D^\varepsilon u(t)\|^2_{L^2(\mu_1)}\, dt$$

and, by Hölder's inequality,

$$\begin{aligned}\int_I \big(f_\varepsilon(t) \mid \varphi(t)\big)_{L^2(\mu_1)}\, dt \;&=\; \int_I \chi_\delta(t)\, \big(f_\varepsilon(t) \mid D^\varepsilon u(t)\big)_{L^2(\mu_1)}\, dt \\ &\le\; \frac{1}{2}\int_I \chi_\delta(t)\, \|f(t)\|^2_{L^2(\mu_1)}\, dt \,+\, \frac{1}{2}\int_I \chi_\delta(t)\, \|\partial_t u(t)\|^2_{L^2(\mu_1)}\, dt\,.\end{aligned}$$

Moreover, for $t_1 < \tilde t_1 < \tilde t_1 + \delta_0 < \tilde t_2 < \tilde t_2 + \delta_1 < t_2$ with $\delta_0, \delta_1 > 0$ sufficiently small, we get

$$\int_I \big(\Delta u_\varepsilon(t) \mid \Delta\varphi(t)\big)_{L^2(\mu_3)}\, dt \;=\; \frac{1}{2}\Big(\|\Delta u_\varepsilon(\tilde t_2)\|^2_{L^2(\mu_3)}\Big)_{\delta_1} \,-\, \frac{1}{2}\Big(\|\Delta u_\varepsilon(\tilde t_1)\|^2_{L^2(\mu_3)}\Big)_{\delta_0}$$

and

$$4\int_I \big(\nabla' u_\varepsilon(t) \mid \nabla'\varphi(t)\big)_{L^2(\mu_1)}\, dt \;=\; 2\Big(\|\nabla' u_\varepsilon(\tilde t_2)\|^2_{L^2(\mu_1)}\Big)_{\delta_1} \,-\, 2\Big(\|\nabla' u_\varepsilon(\tilde t_1)\|^2_{L^2(\mu_1)}\Big)_{\delta_0}.$$

In these equalities we pass to the limit $\varepsilon \to 0$ to find

$$\begin{aligned}&\Big(\|\Delta u(\tilde t_2)\|^2_{L^2(\mu_3)}\Big)_{\delta_1} \,+\, 4\Big(\|\nabla' u(\tilde t_2)\|^2_{L^2(\mu_1)}\Big)_{\delta_1} \\ &=\; 2\int_I \chi_\delta(t)\,\eta(t)\, dt \,+\, \Big(\|\Delta u(\tilde t_1)\|^2_{L^2(\mu_3)}\Big)_{\delta_0} \,+\, 4\Big(\|\nabla' u(\tilde t_1)\|^2_{L^2(\mu_1)}\Big)_{\delta_0}\end{aligned}$$

for all $\tilde t_2 > \tilde t_1 \in I$, where $\eta := (f \,|\, \partial_t u)_{L^2(\mu_1)} - \|\partial_t u\|^2_{L^2(\mu_1)}$. Moreover,

$$\int_I \chi_\delta(t)\, \|\partial_t u(t)\|^2_{L^2(\mu_1)}\, dt \;\le\; \int_J \|f(t)\|^2_{L^2(\mu_1)}\, dt \,+\, \Big(\|\Delta u(\tilde t_1)\|^2_{L^2(\mu_3)}\Big)_{\delta_0} \,+\, 4\Big(\|\nabla' u(\tilde t_1)\|^2_{L^2(\mu_1)}\Big)_{\delta_0}.$$

Now for $\delta_0, \delta_1 \to 0$, we use the same arguments as in the proof of proposition 4.1.4 (i) to verify continuity and boundedness of $t \mapsto \|\Delta u(t)\|_{L^2(\mu_3)} + \|\nabla' u(t)\|_{L^2(\mu_1)}$ on $\overline{(\tilde t_1, t_2)}$, and hence the energy estimate of the present proposition. ■

**Remark 4.2.2** *If $u$ is an energy solution of the parabolic problem, then proposition 4.2.1 enables us to control one temporal derivative. Treating $t$ as a parameter, it therefore suffices to consider the elliptic equation $L_0 u = f$ on $\overline{H}$. We say $u \in W^{2,2}(H, \mu_1, \mu_1, \mu_3)$ is a solution if it satisfies the identity*

$$\int_H \Delta u\, \Delta\breve\varphi\, d\mu_3 \,+\, 4\int_H \nabla' u \cdot \nabla'\breve\varphi\, d\mu_1 \;=\; \int_H f\, \breve\varphi\, d\mu_1$$

*for all $\breve\varphi \in W^{2,2}(H, \mu_1, \mu_1, \mu_3)$.*

**Lemma 4.2.3** *Suppose $I = (t_1, t_2) \subseteq \mathbb{R}$ and $f \in L^1\big(I; L^2(H, \mu_1)\big)$. Let $u$ satisfy $\partial_t u + L_0 u = f$ on $I \times \overline{H}$ in the energy sense. For $\tilde t_1 \in I$ we define $J := (\tilde t_1, t_2)$. Then*

$$\int_J \|\nabla u(t)\|^2_{L^2(H,\mu_1)}\, dt \,+\, \int_J \|D_x^2 u(t)\|^2_{L^2(H,\mu_3)}\, dt \;\lesssim\; \Big(\int_J \|f(t)\|_{L^2(H,\mu_1)}\, dt\Big)^2 \,+\, \|u(\tilde t_1)\|^2_{L^2(H,\mu_1)}\,.$$

**Proof:** From the energy identity (see remark 4.1.5) it follows

$$\frac{1}{2}\sup_{t\in\overline{J}}\|u(t)\|^2_{L^2(\mu_1)} + \int_J \|\nabla' u(t)\|^2_{L^2(\mu_1)}\,dt + \int_J \|\Delta u(t)\|^2_{L^2(\mu_3)}\,dt$$
$$\leq\; 2\int_J \big(f(t)\mid u(t)\big)_{L^2(\mu_1)}\,dt \;+\; \|u(\tilde{t}_1)\|^2_{L^2(\mu_1)}\,.$$

Using Hölder's inequality as well as $ab \leq 2a^2 + \frac{b^2}{8}$ we get

$$2\int_J \big(f(t)\mid u(t)\big)_{L^2(\mu_1)}\,dt \;\leq\; 4\,\|f\|^2_{L^1(J;L^2(\mu_1))} \;+\; \frac{1}{4}\,\sup_{t\in\overline{J}}\,\|u(t)\|^2_{L^2(\mu_1)}$$

such that

$$\sup_{t\in\overline{J}}\,\|u(t)\|^2_{L^2(\mu_1)} + \int_J \|\nabla' u(t)\|^2_{L^2(\mu_1)}\,dt + \int_J \|\Delta u(t)\|^2_{L^2(\mu_3)}\,dt \;\lesssim\; \|f\|^2_{L^1(J;L^2(\mu_1))} + \|u(\tilde{t}_1)\|^2_{L^2(\mu_1)}\,.^{5}$$

Let us apply integration by parts repeatedly, now to calculate

$$\begin{aligned}
\|\Delta\varphi\|^2_{L^2(\mu_3)} \;&=\; \sum_{i,j=1}^{n}\int_H \big(\partial^2_{x_i}\varphi\big)\,\big(\partial^2_{x_j}\varphi\big)\,d\mu_3\\
&= -\sum_{i,j=1}^{n}\int_H \big(\partial_{x_ix_ix_j}\varphi\big)\,\big(\partial_{x_j}\varphi\big)\,d\mu_3 \;+\; 3\,\big(\partial^2_{x_i}\varphi\big)\,\big(\partial_{x_n}\varphi\big)\,d\mu_2\\
&= \int_H \big|D^2_x\varphi\big|^2\,d\mu_3 \;+\; 3\sum_{i=1}^{n}\int_H \big(\partial_{x_nx_i}\varphi\big)\,\big(\partial_{x_i}\varphi\big) \;-\; \big(\partial_{x_ix_i}\varphi\big)\,\big(\partial_{x_n}\varphi\big)\,d\mu_2\\
&= \|D^2_x\varphi\|^2_{L^2(\mu_3)} \;-\; 6\,\|\nabla'\varphi\|^2_{L^2(\mu_1)}\,.
\end{aligned}$$

Note that due to the weight the possibly existing contributions of $u$ at $\{x_n = 0\}$ vanish such that these computations hold for all $\varphi \in C_c^\infty(\overline{H})$ and hence, by density, as well for $u \in W^{2,2}(\mu_1,\mu_1,\mu_3)$. Furthermore,

$$\|\partial_{x_n}u\|^2_{L^2(\mu_1)} \;\lesssim\; \|D^2_x u\|^2_{L^2(\mu_3)} \;=\; \|\Delta u\|^2_{L^2(\mu_3)} \;+\; 6\,\|\nabla' u\|^2_{L^2(\mu_1)} \tag{4.2.1}$$

which follows from the Hardy-Sobolev inequality 2.7.5. But now we have finally reached that

$$\int_J \|\nabla u(t)\|^2_{L^2(\mu_1)}dt \;+\; \int_J \|D^2_x u(t)\|^2_{L^2(\mu_3)}dt \;\lesssim\; \|f\|^2_{L^1(J;L^2(\mu_1))} \;+\; \|u(\tilde{t}_1)\|^2_{L^2(\mu_1)}$$

as required. ■

**Proposition 4.2.4 (spatial energy estimates)** *Suppose $f \in L^2(H,\mu_1)$ and $u$ satisfies the equation $L_0u = f$ on $\overline{H}$ in the sense of remark 4.2.2. Then there exists a positive constant $c = c(n)$ such that*

$$\|D^2_x u\|_{W^{2,2}(H,\,\mu_1,\mu_3,\mu_5)} \;\leq\; c\,\|f\|_{L^2(H,\,\mu_1)}\,.$$

**Proof:** Formally, one can prove the energy estimate by testing the elliptic equation with the function $L_0u$. A rigorous justification of this result requires a precise treatment of certain commutators.

We take a different approach, exploiting the fact that the fourth-order operator $L_0$ can be factorized as $L_0 = L\,L$, where

$$Lu \;=\; -\,x_n^{-1}\,\nabla\!\cdot\!\big(x_n^2\,\nabla u\big)\,,$$

[5] Boundedness of the first term on the left hand side and proposition 4.1.4 imply that $u \in C_b$ in its corresponding time interval.

see (3.4.1). We begin with the analysis of $L$ and consider a weak form of the second order elliptic equation

$$Lu \;=\; -\,x_n\,\Delta u \,-\, 2\,\partial_{x_n} u \;=\; w$$

on $H$ and perform a Fourier transformation in the tangential directions $x_1, \ldots, x_{n-1}$ to get the equation

$$x_n\,\partial^2_{x_n}\hat{u} \,+\, 2\,\partial_{x_n}\hat{u} \,-\, x_n\,|\xi|^2\,\hat{u} \;=\; -\,\widehat{w}\,.$$

Taking the Fourier variable $\xi \in \mathbb{R}^{n-1}$ as a parameter and putting $z = |\xi| x_n$, this becomes an ODE of the form

$$\hat{L}\hat{u} \;=\; z\,\partial_z^2\hat{u} \,+\, 2\,\partial_z\hat{u} \,-\, z\,\hat{u} \;=\; -\,|\xi|^{-1}\,\widehat{w}\,,$$

with $\hat{u} = \hat{u}(\xi, z)$ and $\widehat{w} = \widehat{w}(\xi, z)$. Renaming $\hat{L} = L$, $\hat{u} = u$ and $-|\xi|^{-1}\widehat{w} = w$, we obtain

$$Lu \;=\; z\,\partial_z^2 u \,+\, 2\,\partial_z u \,-\, z\,u \;=\; w\,, \tag{$*$}$$

an equation of one independent variable $z \in \mathbb{R}_+$. Now if we substitute $u = z^{-\frac{1}{2}}\,v$ into $(*)$, we recover the modified Bessel differential equation (B.1) of order $\nu = \frac{1}{2}$ for which $I_\nu$ and $K_\nu$ are a fundamental system (see appendix B), and hence a fundamental system for $(*)$ is given by $\varphi(z) = z^{-\frac{1}{2}}\,I_{\frac{1}{2}}(z)$ and $\psi(z) = z^{-\frac{1}{2}}\,K_{\frac{1}{2}}(z)$. The Wronskian is $\mathcal{W}\big(\varphi(z), \psi(z)\big) = z^{-2}$ and the operator $T : w \mapsto z^j\,u$ has the kernel

$$k(z,y) \;=\; \begin{cases} -\,y\,\varphi(z)\,\psi(y) & \text{if } z < y \\ \;\;\,y\,\varphi(y)\,\psi(z) & \text{if } z > y\,. \end{cases}$$

Note that the first order derivative has a jump discontinuity of height $y^{-1}$ at $z = y$. From standard ODE theory we know that any solution of $(*)$ can be written

$$z^j\,u(z) \;=\; \int_0^\infty z^j\,k(z,y)\,w(y)\,dy$$

for almost every $z \in \mathbb{R}_+$. Now we would like to find conditions on $j$ which ensure that

$$i)\;\sup_{z\in\mathbb{R}_+}\int_0^\infty z^j\,\big|k(z,y)\big|\,dy \;<\; \infty \qquad\text{and}\qquad ii)\;\sup_{y\in\mathbb{R}_+}\int_0^\infty z^j\,\big|k(z,y)\big|\,\frac{z}{y}\,dz \;<\; \infty\,.$$

Then, (i) implies that $T : L^\infty\big(\mathbb{R}_+\big) \to L^\infty\big(\mathbb{R}_+\big)$, and by (ii) it follows that $T$ maps $L^1\big(\mathbb{R}_+, \mu_1\big)$ into itself. Using the Marcinkiewicz interpolation theorem applied to the operator $x_n\,T\,x_n^{-1}$, we thus have

$$\|u\|_{L^2(\mathbb{R}_+,\mu_{2j+1})} \;\le\; c\,\|w\|_{L^2(\mathbb{R}_+,\mu_1)}\,.$$

We fix $0 < r \ll 1 \ll R < \infty$. In the range of $(0,r)$ we know that $\varphi(z) \sim 1$ and $\psi(z) \sim z^{-1}$, while for large $z$ we have $\varphi(z) \sim z^{-1}\,e^z$ and $\psi(z) \sim z^{-1}\,e^{-z}$. This follows from the corresponding asymptotics of the modified Bessel functions $I_{\frac{1}{2}}(z)$ and $K_{\frac{1}{2}}(z)$ which are discussed in appendix B. In order to check the validity of conditions (i) and (ii), we first observe that

$$\sup_{\mathbb{R}_+}\int_0^\infty \ldots \;\le\; \sup_{(0,r)}\int_0^\infty \ldots \,+\, \sup_{(r,R)}\int_0^\infty \ldots \,+\, \sup_{(R,\infty)}\int_0^\infty \ldots\,,$$

and hence it suffices to show that each of the suprema on the right hand side is finite. We estimate term by term starting with

$$\begin{aligned}\sup_{z\in(0,r)}\int_0^\infty z^j\,\big|k(z,y)\big|\,dy &= \sup_{z\in(0,r)} z^j\,\Big(\big|\psi(z)\big|\int_0^z y\,\big|\varphi(y)\big|\,dy + \big|\varphi(z)\big|\int_z^r y\,\big|\psi(y)\big|\,dy + \\ &\qquad + \big|\varphi(z)\big|\int_r^R y\,\big|\psi(y)\big|\,dy + \big|\varphi(z)\big|\int_R^\infty y\,\big|\psi(y)\big|\,dy\Big) \\ &\lesssim \sup_{z\in(0,r)} z^j\,\Big(z + (r-z) + \int_r^R y\,\big|\psi(y)\big|\,dy + e^{-R}\Big).\end{aligned}$$

Since $[r,R]\subset\mathbb{R}_+$ is a compact set, the remaining integral is bounded by a constant depending on $r$ and $R$. Thus

$$\sup_{z\in(0,r)}\int_0^\infty z^j\,\big|k(z,y)\big|\,dy \lesssim \sup_{z\in(0,r)}\big(z^{j+1} + z^j\big)$$

which is bounded for all $j\geq 0$. The supremum over $z\in(r,R)$ poses no difficulties. Here we find

$$\begin{aligned}\sup_{z\in(r,R)}\int_0^\infty z^j\,\big|k(z,y)\big|\,dy &= \sup_{z\in(r,R)} z^j\,\Big(\big|\psi(z)\big|\int_0^r y\,\big|\varphi(y)\big|\,dy + \big|\psi(z)\big|\int_r^z y\,\big|\varphi(y)\big|\,dy + \\ &\qquad + \big|\varphi(z)\big|\int_z^R y\,\big|\psi(y)\big|\,dy + \big|\varphi(z)\big|\int_R^\infty y\,\big|\psi(y)\big|\,dy\Big),\end{aligned}$$

and since both $\varphi$ and $\psi$ are bounded on $(r,R)$, the supremum is obviously finite for all such $z$. On the unbounded interval $(R,\infty)$ we have

$$\begin{aligned}\sup_{z\in(R,\infty)}\int_0^\infty z^j\,\big|k(z,y)\big|\,dy &= \sup_{z\in(R,\infty)} z^j\,\Big(\big|\psi(z)\big|\int_0^r y\,\big|\varphi(y)\big|\,dy + \big|\psi(z)\big|\int_r^R y\,\big|\varphi(y)\big|\,dy + \\ &\qquad + \big|\psi(z)\big|\int_R^z y\,\big|\varphi(y)\big|\,dy + \big|\varphi(z)\big|\int_z^\infty y\,\big|\psi(y)\big|\,dy\Big) \\ \lesssim \sup_{z\in(R,\infty)}&\Big(z^{j-1}\,e^{-z}\,r^2 + z^{j-1}\,e^{-z}\int_r^R y\,\big|\varphi(y)\big|\,dy + z^{j-1}\,e^{-z}\int_R^z e^{\,y}\,dy + z^{j-1}\,e^{\,z}\int_z^\infty e^{-y}\,dy\Big).\end{aligned}$$

The first two terms do not cause any problems. The remaining two are bounded for all $z\in(R,\infty)$ if $j\leq 1$. This means condition (i) is satisfied whenever $j\in[0,1]$ which includes as limiting cases both $j=0$ and $j=1$.

We proceed similarly for the second kernel condition. Here we have

$$\begin{aligned}\sup_{y\in(0,r)}\int_0^\infty z^j\,\big|k(z,y)\big|\,\frac{z}{y}\,dz &= \sup_{y\in(0,r)}\Big(\big|\psi(y)\big|\int_0^y z^{j+1}\,\big|\varphi(z)\big|\,dz + \big|\varphi(y)\big|\int_y^\infty z^{j+1}\,\big|\psi(z)\big|\,dz\Big) \\ &\lesssim \sup_{y\in(0,r)}\Big(y^{j+1} + \big(r^{j+1} - y^{j+1}\big) + \int_r^R z^{j+1}\,\big|\psi(z)\big|\,dz + \int_R^\infty z^j\,e^{-z}\,dz\Big).\end{aligned}$$

Now suppose $j+1\geq 0$, then this is bounded by a constant depending only on $r$ and $R$. The interval $(r,R)$ is straightforward. Finally, we compute

$$\begin{aligned}\sup_{y\in(R,\infty)}\int_0^\infty z^j\,\big|k(z,y)\big|\,\frac{z}{y}\,dz &= \sup_{y\in(R,\infty)}\Big(\big|\psi(y)\big|\int_0^y z^{j+1}\,\big|\varphi(z)\big|\,dz + \big|\varphi(y)\big|\int_y^\infty z^{j+1}\,\big|\psi(z)\big|\,dz\Big) \\ \lesssim \sup_{y\in(R,\infty)} y^{-1}\,e^{-y}&\Big(\int_0^r z^{j+1}\,dz + \int_r^R z^{j+1}\,\big|\varphi(z)\big|\,dz + \int_R^y z^j\,e^{\,z}\,dz + e^{\,2y}\int_y^\infty z^j\,e^{-z}\,dz\Big).\end{aligned}$$

The first two integrals can be treated as above. For the third and fourth term, we write

$$y^{-1}\,e^{-y}\int_R^y z^j\,e^{\,z}\,dz + y^{-1}\,e^{\,y}\int_y^\infty z^j\,e^{-z}\,dz = y^{j-1}\Big(y^{-j}\,e^{-y}\int_R^y z^j\,e^{\,z}\,dz + y^{-j}\,e^{\,y}\int_y^\infty z^j\,e^{-z}\,dz\Big)$$

which is bounded on $(R,\infty)$ if and only if $j\leq 1$. To summarize, the kernel conditions (i) and (ii) hold provided that $j \in [0,1]$. In particular, $j = 0$ and $j = 1$ are admissible and we have

$$\|u\|_{L^2(\mathbb{R}_+,\mu_1)} + \|u\|_{L^2(\mathbb{R}_+,\mu_3)} \lesssim \|w\|_{L^2(\mathbb{R}_+,\mu_1)}\,.$$

Once this has been established we can put $z\,u$ onto the right hand side of the equation $(*)$. With $v = \partial_z u$, we obtain

$$z\,\partial_z v + 2\,v = f + z\,u =: \widetilde{w}\,.$$

A solution of the homogeneous equation is given by $z^{-2}$ such that

$$\widetilde{k}(z,y) = y \begin{cases} z^{-2} & \text{if } z > y \\ 0 & \text{otherwise} \end{cases}$$

defines the corresponding integral kernel. Now consider the operator $z^\delta\,\widetilde{w} \mapsto z^\delta\,v$ for some $\delta > 0$. Since

$$\sup_{z\in\mathbb{R}_+} \int_0^\infty \left(\frac{z}{y}\right)^\delta \left|\widetilde{k}(z,y)\right| dy = \sup_{z\in\mathbb{R}_+} z^{\delta-2} \int_0^z y^{1-\delta}\,dy = \frac{1}{2-\delta}\,,$$

it follows

$$\|z^\delta\,v\|_{L^\infty(\mathbb{R}_+)} \lesssim \|z^\delta\,\widetilde{w}\|_{L^\infty(\mathbb{R}_+)}$$

for $\delta < 2$. On the $L^1$-side of the estimate we obtain

$$\sup_{y\in\mathbb{R}_+} \int_0^\infty \left(\frac{z}{y}\right)^\delta \left|\widetilde{k}(z,y)\right| dz = \sup_{y\in\mathbb{R}_+} y^{1-\delta} \int_y^\infty z^{\delta-2}\,dz = -\frac{1}{\delta-1}\,.$$

If $\delta < 1$, then

$$\|z^\delta\,v\|_{L^1(\mathbb{R}_+)} \lesssim \|z^\delta\,\widetilde{w}\|_{L^1(\mathbb{R}_+)}\,.$$

Choosing $\delta = \frac{1}{2}$, we get

$$\|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_1)} = \|v\|_{L^2(\mathbb{R}_+,\mu_1)} \lesssim \|\widetilde{w}\|_{L^2(\mathbb{R}_+,\mu_1)} = \|w + z\,u\|_{L^2(\mathbb{R}_+,\mu_1)} \lesssim \|w\|_{L^2(\mathbb{R}_+,\mu_1)}$$

by interpolation. Using $(*)$, it follows immediately that

$$\|\partial_z^2 u\|_{L^2(\mathbb{R}_+,\mu_3)} = \|w - 2\partial_z u + z\,u\|_{L^2(\mathbb{R}_+,\mu_1)} \lesssim \|w\|_{L^2(\mathbb{R}_+,\mu_1)}\,.$$

Now recall the notation $u = \hat{u}$ and $|\xi|\,w = -\widehat{w}$, where $\widehat{\cdot}$ denotes the Fourier-transformation in $\xi$. A retransformation from $z$ to $x_n$ and an integration in $\xi \in \mathbb{R}^{n-1}$ yield

$$\||\xi|\,\hat{u}\|_{L^2(H,\mu_1)} + \||\xi|^2\,\hat{u}\|_{L^2(H,\mu_3)} + \|\partial_{x_n}\hat{u}\|_{L^2(H,\mu_1)} + \|\partial_{x_n}^2\hat{u}\|_{L^2(H,\mu_3)} \lesssim \|\widehat{w}\|_{L^2(H,\mu_1)}\,.$$

By another application of the estimate, we get

$$\|\widehat{w}\|_{L^2(H,\mu_1)} \lesssim \||\xi|^{-1}\,\hat{f}\|_{L^2(H,\mu_1)}$$

which is possible since $\hat{L}\widehat{w} = |\xi|^{-1}\,\widehat{f}$. Next, we carry out an inverse Fourier transformation to convert the inequality into

$$\|\nabla'\nabla' u\|_{L^2(H,\mu_1)} + \|\nabla'\Delta' u\|_{L^2(H,\mu_3)} + \|\nabla'\partial_{x_n} u\|_{L^2(H,\mu_1)} + \|\nabla'\partial_{x_n}^2 u\|_{L^2(H,\mu_3)} \lesssim \|f\|_{L^2(H,\mu_1)}\,.$$

Note that Plancherel's theorem ensures that the Fourier transform preserves the $L^2$-norm. Eventually, we use the auxiliary identity (4.2.1) to control not only the Laplacian but all second derivatives of $u$ reaching

$$\|\nabla'\nabla u\|_{L^2(H,\mu_1)} + \|\nabla' D_x^2 u\|_{L^2(H,\mu_3)} \leq c\,\|f\|_{L^2(H,\mu_1)}\,.$$

Before we proceed to estimate higher derivatives let us make some further preparations. Following the same line of argument, we conclude that

$$i')\ \sup_{z\in\mathbb{R}_+}\int_0^\infty z^j\,\big|\partial_z k(z,y)\big|\,dy\ <\ \infty \qquad\text{and}\qquad ii')\ \sup_{y\in\mathbb{R}_+}\int_0^\infty z^j\,\big|\partial_z k(z,y)\big|\,\frac{z}{y}\,dz\ <\ \infty\,,$$

if $j \in (0,1]$. Then

$$\|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_{2j+1})}\ \lesssim\ \|w\|_{L^2(\mathbb{R}_+,\mu_1)}$$

for all $j \in [0,1]$ whenever $Lu = w$. Now differentiating $(*)$ gives the equation

$$z\,\partial_z^2(\partial_z u)\ +\ 3\,\partial_z(\partial_z u)\ -\ z\,\partial_z u\ =\ \partial_z w\ +\ u\,.$$

This can be transformed into the modified Bessel equation of order $\nu = \frac{\sigma}{2} = 1$. Repeating this procedure we get

$$z\,\partial_z^2(\partial_z^2 u)\ +\ 4\,\partial_z(\partial_z^2 u)\ -\ z\,\partial_z^2 u\ =\ \partial_z^2 w\ +\ 2\,\partial_z u\,,$$

and hence

$$\begin{aligned}\|\partial_z^2 u\|_{L^2(\mathbb{R}_+,\mu_3)}\ +\ \|\partial_z^2 u\|_{L^2(\mathbb{R}_+,\mu_5)}\ +\ \|\partial_z^3 u\|_{L^2(\mathbb{R}_+,\mu_3)}\ +\ \|\partial_z^4 u\|_{L^2(\mathbb{R}_+,\mu_5)}\\ \lesssim\ \|\partial_z^2 w\|_{L^2(\mathbb{R}_+,\mu_3)}\ +\ \|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_3)}\ \lesssim\ \|f\|_{L^2(\mathbb{R}_+,\mu_1)}\,,\end{aligned}$$

provided we have $LLu = Lw = f$ with $f = |\xi|^{-2}\widehat{f}$. Finally, a retransformation from $z$ to $x_n$, an integration with respect to the variable $\xi \in \mathbb{R}^{n-1}$, an inverse Fourier transformation and formula (4.2.1) give

$$\|\nabla\partial_{x_n}^2 u\|_{L^2(H,\mu_3)}\ +\ \|D_x^2\partial_{x_n}^2 u\|_{L^2(H,\mu_5)}\ \lesssim\ \|f\|_{L^2(H,\mu_1)}\,.$$

The missing third derivative follows from

$$\|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_3)}\ \lesssim\ \|w\|_{L^2(\mathbb{R}_+,\mu_1)}\ \lesssim\ \|f\|_{L^2(\mathbb{R}_+,\mu_1)}\,,$$

because then

$$\|\partial_{x_n}\Delta' u\|_{L^2(H,\mu_3)}\ \lesssim\ \|f\|_{L^2(H,\mu_1)}$$

after a retransformation. In order to estimate the tangential derivatives of order four we need to show that

$$\|u\|_{L^2(\mathbb{R}_+,\mu_5)}\ \lesssim\ \|f\|_{L^2(\mathbb{R}_+,\mu_1)}\,. \tag{$**$}$$

Then

$$\|D_{x'}^4 u\|_{L^2(H,\mu_5)}\ \lesssim\ \|f\|_{L^2(H,\mu_1)}$$

as required. To check the validity of $(**)$, consider the equation $z\,\partial_z^2 u + 4\,\partial_z u - z\,u = w + 2\,\partial_z u$ which follows directly from $(*)$ by adding $2\,\partial_z u$ to both sides of the equation. The left hand side can be transformed into a Bessel equation of order $\nu = \frac{3}{2}$ by plugging $u = z^{-\frac{3}{2}}v$ into the equation. However, this allows us to perform the same calculations as for $(*)$, only with $\sigma = 3$ instead of $\sigma = 1$, to get

$$\|u\|_{L^2(\mathbb{R}_+,\mu_{2j+3})}\ \lesssim\ \|w\|_{L^2(\mathbb{R}_+,\mu_3)}\ +\ \|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_3)}$$

for all $j \in [0,1]$. To prove this, one has to verify that conditions (i) and (ii) hold, where $k(z,y)$ is the kernel associated to the operator $T : w \mapsto z^j u$. Using $Lu = w$ and $Lw = f$ generates

$$\|w\|_{L^2(\mathbb{R}_+,\mu_3)}\ +\ \|\partial_z u\|_{L^2(\mathbb{R}_+,\mu_3)}\ \lesssim\ \|f\|_{L^2(\mathbb{R}_+,\mu_1)}$$

as desired. This concludes the proof of $(**)$ and hence the proof of proposition 4.2.4. ■

**Remark 4.2.5** *Suppose $t_1$ is finite and $u$ is an energy solution to the initial value problem on $[t_1, t_2) \times \overline{H}$ with $u(t_1) = 0$. Then by proposition 4.1.4 (ii), we can extend this solution continuously to $J = (\tilde{t}_1, t_2)$, for some $\tilde{t}_1 < t_1$, by zero. Applying the propositions 4.2.1, 4.2.3 and 4.2.4, with the roles of $I$ and $J$ interchanged, delivers the corresponding energy estimates for this $u$. Since $g = 0$, the initial value terms disappear from the estimates.*
*In case $t_1 = -\infty$ we repeat the proof of corollary 4.1.7 to conclude, together with the results for $t_1 > -\infty$, that the inequalities also hold for the unique solution which vanishes as $t \to -\infty$.*

**Corollary 4.2.6** *Suppose $I = (t_1, t_2) \subseteq \mathbb{R}$ is an open interval, $l \in \mathbb{N}_0$, $\alpha \in {\mathbb{N}_0}^n$ and $j \in \mathbb{N}_0$ is some non-negative integer which satisfies $j = 2l + |\alpha| - 2$ and $2j \leq |\alpha|$. Then the operator*

$$L^2\big(I; L^2(H, \mu_1)\big) \ni f \;\mapsto\; x_n^j\, \partial_t^l\, \partial_x^\alpha\, u \in L^2\big(I; L^2(H, \mu_1)\big)$$

*is bounded, where $u$ is is the unique energy solution on $I \times \overline{H}$ to $f$ with $u(t_1) = 0$.*

The corollary specifically applies to $I = \mathbb{R}$.

**Proof:** The indicated conditions we put on $j, l$ and $\alpha$ imply $j \leq 2$, $l \leq 1$ and $|\alpha| \in \{0, 2, 3, 4\}$. The statement for all admissible combinations of such parameters is then an immediate consequence of the propositions 4.2.4 and 4.2.1. ■

We conclude this section with a duality result, which we formulate for later use. More precisely, the following lemma can be used to extend $L^2$- $L^\infty$- estimates to $L^1$- $L^\infty$- estimates.

**Lemma 4.2.7 (duality)** *Suppose $I = (t_1, t_2) \subset \mathbb{R}$ is an open interval and $g_1$, $g_2 \in L^2(H, \mu_1)$. Further let $u_1$ and $u_2$ be $L_0$-solutions to $f = 0$ on $[t_1, t_2) \times \overline{H}$ with initial value $g_1$ and $g_2$, respectively. Then the duality identity $\big(u_1(\tilde{t}_1) \mid u_2(\tilde{t}_2)\big)_{L^2(H,\mu_1)} = \big(u_1(\tilde{t}_2) \mid u_2(\tilde{t}_1)\big)_{L^2(H,\mu_1)}$ holds for all $\tilde{t}_2 \geq \tilde{t}_1 \in \bar{I}$, that is, the solution operator that assigns an initial datum to its corresponding solution is self-adjoint*

**Proof:** Let $\tilde{t}_2 \geq \tilde{t}_1 \in \bar{I}$ be fixed and $J := (\tilde{t}_1, \tilde{t}_2)$. Then the time-inverting operator

$$T: \; I \ni t \mapsto \tilde{t}_1 + \tilde{t}_2 - t =: \tau \in \mathbb{R}$$

is bijective on $J$. Note that we only give a formal proof here. However, an approach to a rigorous justification of the statement follows the same line of argument as the proof of proposition 4.1.4, where we have seen that an energy solution is continuous in time.
We consider the equation for $u_2$, where $\chi_J(u_1 \circ T)$ plays the role of the test function. This gives

$$\begin{aligned} \int_I \Big(u_2 \,\Big|\, \partial_t\big(\chi_J(u_1 \circ T)\big)\Big)_{L^2(\mu_1)} d\mathcal{L} &= \int_I \partial_t\, \chi_J\, \big(u_2 \mid u_1 \circ T\big)_{L^2(\mu_1)} + \chi_J\, \big(u_2 \mid \partial_t(u_1 \circ T)\big)_{L^2(\mu_1)}\, d\mathcal{L} \\ &= \int_I \chi_J \Big(\big(\Delta\, u_2 | \Delta(u_1 \circ T)\big)_{L^2(\mu_3)} + 4\,\big(\nabla' u_2 | \nabla'(u_1 \circ T)\big)_{L^2(\mu_1)}\Big)\, d\mathcal{L}. \end{aligned}$$

Now, according to the definition of the operator $T$, we substitute $t = T^{-1}(\tau)$ to calculate

$$\begin{aligned} \int_I \chi_J(t)\, \Big(u_2(t) \,\Big|\, \partial_t\big(u_1 \circ T\big)(t)\Big)_{L^2(\mu_1)} dt \;&=\; -\int_I \Big(\partial_t\big(\chi_J(t) u_2(t)\big) \,\Big|\, \big(u_1 \circ T\big)(t)\Big)_{L^2(\mu_1)} dt \\ &=\; -\int_{T(I)} \Big(\partial_\tau\big[\big(\chi_J\, u_2\big) \circ T^{-1}(\tau)\big] \,\Big|\, u_1(\tau)\Big)_{L^2(\mu_1)} d\tau\,, \end{aligned}$$

where the first equality is a consequence of an integration by parts in time. We notice that $T(J) = J$ and that the integrals above vanish outside of $J$. Consequently, we can replace $T(I)$ by $I$ again. But then the last term can be understood as the result of taking $(\chi_J\, u_2) \circ T^{-1}$ as test function in the equation for $u_1$. There we have

$$\begin{aligned}
&- \int_I \Big( u_1(\tau) \,\Big|\, \partial_\tau \big[ (\chi_J\, u_2) \circ T^{-1}(\tau) \big] \Big)_{L^2(\mu_1)} \, d\tau \\
&\quad = - \int_I \Big( \Delta u_1(\tau) \,\Big|\, \Delta \big[ (\chi_J\, u_2) \circ T^{-1}(\tau) \big] \Big)_{L^2(\mu_3)} + 4 \Big( \nabla' u_1(\tau) \,\Big|\, \nabla' \big[ (\chi_J\, u_2) \circ T^{-1}(\tau) \big] \Big)_{L^2(\mu_1)} \, d\tau \\
&\quad = \int_{T^{-1}(I)} \chi_J(t) \, \Big( \big( \Delta \big[ u_1 \circ T(t) \big] \mid \Delta u_2(t) \big)_{L^2(\mu_3)} + 4 \, \big( \nabla' \big[ u_1 \circ T(t) \big] \mid \nabla' u_2(t) \big)_{L^2(\mu_1)} \Big) \, dt \\
&\quad = \int_I \chi_J \Big( \big( \Delta (u_1 \circ T) \mid \Delta\, u_2 \big)_{L^2(\mu_3)} + 4 \, \big( \nabla' (u_1 \circ T) \mid \nabla' u_2 \big)_{L^2(\mu_1)} \Big) \, d\mathcal{L} \, .
\end{aligned}$$

In the second equality we substituted back for $\tau = T(t)$. Altogether, we subsume all the achieved results to gain

$$\Big( u_2(\tilde t_2) \,\Big|\, (u_1 \circ T)(\tilde t_2) \Big)_{L^2(\mu_1)} - \Big( u_2(\tilde t_1) \,\Big|\, (u_1 \circ T)(\tilde t_1) \Big)_{L^2(\mu_1)} = \int_I \partial_t \, \chi_J \, \big( u_2 \mid u_1 \circ T \big)_{L^2(\mu_1)} \, d\mathcal{L} = 0 \, .$$

But since $(u_1 \circ T)(\tilde t_2) = u_1(\tilde t_1)$ and $(u_1 \circ T)(\tilde t_1) = u_1(\tilde t_2)$, this identity takes the desired form. ■

## 4.3 Local Estimates

In this chapter we show how to obtain local estimates for solutions of the linear initial value problem. Here, we use the same arguments as before with $u$ multiplied by an appropriate cut-off function, where $u$ solves the equation $\partial_t u + L_0 u = f$ on some relatively open subset of $\mathbb{R} \times \overline{H}$ in the sense of definition 4.0.1. A temporal cut-off ensures that the temporal initial terms disappear from the inequalities. The spatial cut-off, on the other hand, helps to avoid boundary values. It is implemented in terms of the intrinsic geometry on the closed upper half space which has been discussed in detail in section 3.5.

### 4.3.1 Local Energy Estimates

In this section we set out for localized versions of the energy estimates that we have collected in paragraph 4.2. With the preparations made in section 3.5, we are in a position to construct the cut-off functions that are necessary to "localize" our solutions to parabolic cylinders. One of the main insights that emerges from the geometry section is the following: If an intrinsic ball is located "near" the boundary, then we have $B_R(x) \sim B^{eu}_{R^2}(x)$,[6] while "far away" from there a ball behaves more like $B^{eu}_{R\sqrt{x_n}}(x)$ (cf. lemma 3.5.11). This particular behavior suggests to consider different treatments depending on the ball's position relative to $\partial H$. Indeed, scaling reduces the energy estimates to $Q_1(0,0)$ or $Q_r\big(0,(0,\ldots,0,1)\big)$ with $r \ll 1$. We start with the derivation of such an estimate in the latter case.

**Lemma 4.3.1** *Let $l$ be any nonnegative integer and $\alpha$ any multi-index. If $u$ is a an energy solution of $\partial_t u + L_0 u = 0$ on $Q_r(0,e_n)$, with $r \ll 1$ and $e_n = (0,\ldots,0,1) \in H$, then there exists a small $\delta > 0$ such that*

$$\|\partial_t^l \partial_x^\alpha \, u\|_{L^2(Q_{\delta r}(0,\,e_n))} \;\le\; c\, r^{-4l-|\alpha|} \, \|u\|_{L^2(Q_r(0,\,e_n))}$$

*for some positive constant $c = c(n,l,\alpha)$.*

**Proof:** Throughout this proof, think of $Q_\rho = I_\rho \times B_\rho$ as the parabolic cylinder of radius $\rho > 0$ and center at $(0,e_n)$, that is, $I_\rho = I_\rho(0)$ and $B_\rho = B_\rho(e_n)$.

[6] This relation is to be understood as follows: There exists a $c \ge 1$ such that $B^{eu}_{c^{-1}R^2}(x) \subset B_R(x) \subset B^{eu}_{cR^2}(x)$.

i) First we choose a suitable "bump function" $\eta \in C_c^\infty(\mathbb{R} \times \overline{H})$ such that $\eta \equiv 1$ on $Q_{\tilde{\delta}r}$, for some $\tilde{\delta} \in (0,1)$, and $spt\,\eta \subset \widetilde{I} \times B_r$ with $\widetilde{I} \supset I_r$. By taking a product ansatz, we can additionally achieve that

$$\left|\partial_t^l \partial_x^\alpha \eta(t,x)\right| \lesssim r^{-4l-|\alpha|}. \qquad (*)$$

Then for $\tilde{\delta} = \frac{1}{12{c_d}^2}$, we infer that $|\partial_t^l \partial_x^\alpha \eta| \le dist(I_{\tilde{\delta}r}, \partial\widetilde{I})^{-l}\, dist(B_{\tilde{\delta}r}, \partial B_r)^{-|\alpha|}$. Now using the inclusions

$$I_{\tilde{\delta}r} \subset (-r^4, r^4) =: \widetilde{I} \qquad \text{and} \qquad B_{\tilde{\delta}r} \subset B^{eu}_{6\tilde{\delta}r} \subset B^{eu}_{12\tilde{\delta}r} \subset B_r$$

(cf. lemma 3.5.11) implies $(*)$.

ii) The next step is to find an equation which is solved by $\eta u^{(k)} = \eta\, \partial_{x_n}^k u$. Using integration by parts, we recover the identity

$$\begin{aligned}\big(\partial_t + L_k\big)(\eta u^{(k)}) &= \eta\, \widetilde{f}_k + \big(\partial_t \eta + L_k \eta\big) u^{(k)} - 8\, \nabla'\eta \cdot \nabla' u^{(k)} + 2\, {x_n}^2\, \Delta\eta\, \Delta u^{(k)} + \\ &\quad + 2 {x_n}^{-k-1} \Big(\nabla({x_n}^{k+3}\Delta\eta) \cdot \nabla u^{(k)} + \nabla\eta \cdot \nabla({x_n}^{k+3}\Delta u^{(k)}) + \Delta({x_n}^{k+3}\nabla\eta \cdot \nabla u^{(k)})\Big) \\ &=: \eta\, \widetilde{f}_k + \omega_k, \qquad \text{with } k \in \{0, \dots, \alpha_n\}, \qquad \text{on } \mathbb{R} \times \overline{H}\end{aligned}$$

in the sense of distributions, where $\widetilde{f}_k = \partial_{x_n}^k f - 2k\, x_n\, \Delta\Delta' u^{(k-1)} - k(k-1)\Delta\Delta' u^{(k-2)}$. Note also that $\widetilde{f}_0 = f$.

iii) Finally, we observe that $\frac{1}{4} < x_n < \frac{7}{4}$ for all $x \in B_r$, if $r \le \frac{1}{6}$ is sufficiently small, i.e. $x_n \sim 1$.

**Step 1:** For the moment let $f \in L^2\big(I_r; L^2(B_r, \mu_1)\big)$. By (iii) this is equivalent to saying that $f \in L^2(Q_r)$. Now by construction, the evaluation $\eta(\tilde{t}_1)$ and all its derivatives vanish for $\tilde{t}_1 = -r^4$. By the energy identity 4.1.5 together with (4.2.1) and $\eta$ replaced by $\eta^2$, we therefore have

$$\|\nabla(\eta^2 u^{(k)})\|^2_{L^2(Q_r)} + \|D_x^2(\eta^2 u^{(k)})\|^2_{L^2(Q_r)} \lesssim \|\eta^2 \widetilde{f}_k\|_{L^2(Q_r)}\, \|\eta^2 u^{(k)}\|_{L^2(Q_r)} + \big(\omega_0 \mid \eta^2 u^{(k)}\big)_{L^2(Q_r)},$$

where we also applied the Cauchy-Schwarz inequality to the term $(\eta^2 \widetilde{f}_k \mid \eta^2 u^{(k)})_{L^2(Q_r)}$. Now we claim that

$$\big(\omega_k \mid \eta^2 u^{(k)}\big)_{L^2(Q_r)} \le c\, r^{-4}\, \|u^{(k)}\|^2_{L^2(Q_r)} + \frac{1}{2}\, \|D_x^2(\eta^2 u^{(k)})\|^2_{L^2(Q_r)}.$$

Indeed, repeated application of spatial integration by parts leads to the upper bound

$$\begin{aligned}&\int_{Q_r} \big(\partial_t(\eta^2) + 16\, |\nabla'\eta|^2\big)\, (\eta u^{(k)})^2\, d\mathcal{L}^{n+1} - \int_{Q_r} \nabla\big(x_n^3\, \Delta(\eta^2)\big) \cdot \nabla(\eta^2)\, (u^{(k)})^2 d\mathcal{L}^{n+1} + \\ &+ 2 \int_{Q_r} \nabla(\eta^2) \cdot \nabla u^{(k)}\, \Delta(\eta^2)\, u^{(k)} - \big|\nabla(\eta^2)\big|^2\, \Delta u^{(k)}\, u^{(k)} + 2\, \big|\nabla(\eta^2)\big|^2\, |\nabla u^{(k)}|^2\, d\mathcal{L}^{n+1},\end{aligned}$$

where the boundary terms vanish since $spt\,\eta(t,\cdot) \subset B_r$ for all $t \in I_r$. The first line already has the right form, whereas the second line requires another series of integration by parts to bring the integrals into a suitable form. Now we add property $(*)$ from (i) into the estimate and the desired claim follows. We subtract $\frac{1}{2}\, \|D_x^2(\eta^2 u^{(k)})\|^2_{L^2(Q_r)}$ from both sides and then multiply by 2 to get

$$\|\nabla(\eta^2 u^{(k)})\|^2_{L^2(Q_r)} + \|D_x^2(\eta^2 u^{(k)})\|^2_{L^2(Q_r)} \lesssim r^{-4}\, \|u^{(k)}\|^2_{L^2(Q_r)} + r^4\, \|\widetilde{f}_k\|^2_{L^2(Q_r)}.$$

Eventually, we would like to optimize the estimate with respect to the first summand on the left hand side. To this end, we apply the Poincaré inequality to the function $\nabla(\eta^2 u^{(k)}) \in W^{1,2}(B_r)$ to find

$$\|\nabla(\eta^2 u^{(k)})\|^2_{L^2(B_r)} \le c(n)\, r^2\, \|D_x^2(\eta^2 u^{(k)})\|^2_{L^2(B_r)},$$

since also $B_r \sim B_r^{eu}$. This is possible because $\eta(t) = 0$ on $\partial B_r$ for all $t \in I_r$. Altogether this amounts to

$$r^2\, \|\nabla u^{(k)}\|^2_{L^2(Q_{\tilde{\delta}r})} + r^4\, \|D_x^2 u^{(k)}\|^2_{L^2(Q_{\tilde{\delta}r})} \lesssim \|u^{(k)}\|^2_{L^2(Q_r)} + r^8\, \|\widetilde{f}_k\|^2_{L^2(Q_r)}.$$

The left hand side of the inequality is a consequence of the fact that on the smaller set $Q_{\tilde\delta r}$ the cut-off function is constant to 1.

Taking $k = 0$ and testing the equation for $\eta u$ with $\partial_t(\eta u)$ give rise to boundedness of one temporal derivative:

$$\|\partial_t u\|_{L^2(Q_{\delta r})} \lesssim r^{-4}\,\|u\|_{L^2(Q_r)} + r^{-1}\,\|D_x^3 u\|_{L^2(Q_{\tilde\delta r})} + \|f\|_{L^2(Q_r)}\,.$$

For this we choose $\eta$ so that $spt\,\eta \subset \big(-(\tilde\delta r)^4, (\tilde\delta r)^4\big) \times B_{\tilde\delta r}$ and $\eta \equiv 1$ on $Q_{\delta r}$ for some $\delta < \tilde\delta < 1$. In the next step we derive a bound on the second norm on the right hand side.

**Step 2:** Now formally taking $\chi_{(\tilde t_1,t_2)}\big(x_n^{-1}\,\Delta(x_n^{\,3}\,\Delta u^{(k)}) - \Delta' u^{(k)}\big)$ as test function, for $\tilde t_1 \in I_r$, leads to

$$\int_{(\tilde t_1,t_2)} \|\Delta' u^{(k)}\|^2_{L^2(\mu_1)} + \|\nabla'\Delta u^{(k)}\|^2_{L^2(\mu_3)} + \|x_n^{-1}\Delta(x_n^{\,3}\,\Delta u^{(k)})\|^2_{L^2(\mu_1)}\,d\mathcal{L}$$

$$\lesssim \int_{(\tilde t_1,t_2)} \big(\widetilde{f}_k \mid x_n^{-1}\,\Delta(x_n^{\,3}\,\Delta u^{(k)}) - \Delta' u^{(k)}\big)_{L^2(\mu_1)}\,d\mathcal{L} + \|\nabla' u^{(k)}(\tilde t_1)\|^2_{L^2(\mu_1)} + \|\Delta u^{(k)}(\tilde t_1)\|^2_{L^2(\mu_3)}\,.$$

For a rigorous justification of this result we refer to the proof of proposition 4.2.4. We apply this inequality to $\eta^4 u^{(k)}$ so that, similar to the proof of the global result, we get

$$\begin{aligned}&\|\Delta'(\eta^4 u^{(k)})\|^2_{L^2(Q_r)} + \|\nabla'\Delta(\eta^4 u^{(k)})\|^2_{L^2(Q_r)} + \|x_n^{-1}\Delta\big(x_n^{\,3}\,\Delta(\eta^4 u^{(k)})\big)\|^2_{L^2(Q_r)}\\ &\lesssim r^{-8}\,\|u^{(k)}\|^2_{L^2(Q_r)} + r^{-6}\,\|\nabla u^{(k)}\|^2_{L^2(Q_{\tilde\delta r})} + r^{-4}\,\|D_x^2 u^{(k)}\|^2_{L^2(Q_{\tilde\delta r})} + \|\widetilde{f}_k\|^2_{L^2(Q_r)}\,.\end{aligned}$$

Recall that the localized solution is a $L_k$-solution to the inhomogeneity given by (ii). Then again, as in step 1, we discover that the left hand side is bounded below by the expression $\|D_x^2 u^{(k)}\|^2_{W^{2,2}(Q_{\delta r})}$ for some $0 < \delta < \tilde\delta < 1$. Now, employing the local energy inequality from step 1 to the second and third term on the right side of the inequality we also get rid of these terms. Eventually, we apply the same argument as used in step 1 to find

$$r^3\,\|D_x^3 u^{(k)}\|_{L^2(Q_{\delta r})} \lesssim r^4\,\|D_x^4 u^{(k)}\|_{L^2(Q_{\delta r})} \lesssim \|u^{(k)}\|_{L^2(Q_{\tilde\delta r})} + r^4\,\|\widetilde{f}_k\|_{L^2(Q_r)}\,.$$

Note as well that one can use this estimate with $k = 0$ to optimize the local estimate on $\partial_t u$ from step 1 to

$$r^4\,\|\partial_t u\|_{L^2(Q_{\delta r})} \lesssim \|u\|_{L^2(Q_r)} + r^4\,\|f\|_{L^2(Q_r)}$$

as desired.

**Step 3:** Let $1 \le i \le n-1$. As in the assumptions of our lemma, we now assume that $f = 0$ and we immediately get

$$\|\partial_{x_i} u\|_{L^2(Q_{\delta_1 r})} \lesssim r^{-1}\,\|u\|_{L^2(Q_r)}$$

by virtue of step 1. Since tangential derivatives commute with the operator $\partial_t + L_0$, we need to check that $\partial_{x_i} u$ has the required regularity to be a local solution in the sense of definition 4.0.1. This, however, is already derived in steps 1 and 2 with $k = 0$. An iteration of these arguments then leads to

$$\|\partial_x^{\alpha'} u\|_{L^2(Q_{\delta_{|\alpha'|} r})} \lesssim r^{-|\alpha'|}\,\|u\|_{L^2(Q_r)}$$

for some $\delta' < \cdots < \delta_1 < 1$. Note that a simultaneous iteration of the energy estimates ensures the regularity needed to perform the subsequent step. Next, we investigate the vertical direction. We show that

$$r\,\|\nabla u^{(k)}\|_{L^2(Q_{\delta_{k+1} r})} + r^4\,\|D_x^2 u^{(k)}\|_{W^{1,2}(Q_{\delta_{k+1} r})} \lesssim r^{-k}\,\|u\|_{L^2(Q_r)}$$

by induction over $k \in \{0, \dots, \alpha_n\}$. The induction basis, i.e. the estimate for $k = 0$, follows from steps 1 and 2 with $\delta_1 = \tilde{\delta}$. Hence we need to verify the statement for $k+1$ provided it holds true for $0, \dots, k$. Once more we use steps 1 and 2, now to establish

$$r \,\|\nabla u^{(k+1)}\|_{L^2(Q_{\delta_{k+2} r})} + r^4 \,\|D_x^2 u^{(k+1)}\|_{W^{1,2}(Q_{\delta_{k+2} r})} \lesssim \|u^{(k+1)}\|_{L^2(Q_{\delta_{k+1} r})} + r^4 \,\|\widetilde{f}_{k+1}\|_{L^2(Q_{\delta_{k+1} r})}$$

for some suitable scaling factors $\delta_{k+2} < \delta_{k+1}$. We apply the induction hypothesis to the first term to find this bounded by

$$\|\nabla u^{(k)}\|_{L^2(Q_{\delta_{k+1} r})} \lesssim r^{-(k+1)} \,\|u\|_{L^2(Q_r)} \,.$$

To bound the norm containing $\widetilde{f}_{k+1}$, we estimate as follows:

$$\|\widetilde{f}_{k+1}\|_{L^2(Q_{\delta_{k+1} r})} \overset{(ii)}{\lesssim} \|D_x^2 (\Delta' u)^{(k-1)}\|_{W^{1,2}(Q_{\delta_{k+1} r})} \,.$$

Now by the first part of step 3, we recall that $\Delta' u$ is a local energy solution to $f = 0$, and hence the same local estimates (steps 1 and 2) also hold for $(\Delta' u)^{(k-1)}$, possibly with a smaller scaling factor $\delta$. This brings us in a position where we can apply the induction hypothesis and so we eventually reach

$$r^4 \,\|\widetilde{f}_{k+1}\|_{L^2(Q_{\delta_{k+1} r})} \lesssim r^{-(k-1)} \,\|\Delta' u\|_{L^2(Q_{\tilde{\delta} r})} \lesssim r^{-(k+1)} \,\|u\|_{L^2(Q_r)}$$

for some suitably chosen $0 < \delta_{k+1} < \tilde{\delta} < 1$. This completes the induction step.

Finally, solving inductively $\partial_t^j (\partial_t u + L_0 u) = 0$ for $\partial_t^{j+1} u$ and using the bounds for spatial derivatives we get

$$\|\partial_t^l u\|_{L^2(Q_{\delta_l r})} \lesssim \dots \lesssim r^{-4l} \,\|u\|_{L^2(Q_{\delta_l r})}$$

for some $\delta_l < \dots < \delta_1 < 1$. Once again, the needed regularity gain is obtained by a simultaneous induction of the estimates derived in steps 1 and 2.

**Conclusion:** We combine the results from step 3 into a consistent form. To this end, we first apply the estimate in $x_n$-direction to $(\partial_t^l \partial_x^{\alpha'} u)^{(\alpha_n)}$, followed by the first part of step 3 applied to $\partial_t^l \partial_x^{\alpha'} u$. This yields

$$\|\nabla (\partial_t^l \partial_x^\alpha u)\|_{L^2(Q_{\delta r})} \lesssim r^{-4l-|\alpha|-1} \|u\|_{L^2(Q_r)}$$

for a sufficiently small $\delta > 0$. The lemma follows immediately. ■

**Remark 4.3.2** *The property $x_n \sim 1$ in $B_r(e_n)$ allows us to replace any weighted measure by the Lebesgue measure and vice versa. In such balls, $\partial_t u + L_0 u = f$ is a (locally) uniformly parabolic equation of fourth order, a fact that is also reflected in the coefficient appearing on the right hand side of the energy estimate.*

The situation at the boundary is covered by the following lemma.

**Lemma 4.3.3** *Let $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$. If $u$ is a $L_0$-solution on $Q_1(0,0)$ to $f = 0$, then there exists a $\delta > 0$ such that*

$$\|\partial_t^l \partial_x^\alpha \, u\|_{L^2(Q_\delta(0,0),\, \mu_1)} \le c \,\|u\|_{L^2(Q_1(0,0),\, \mu_1)}$$

*for a positive constant $c$ depending on $n, l$ and $\alpha$.*

**Proof:** As before we keep the right endpoint of the time interval and the center of the ball stationary, and we merely write $Q_\rho$ to mean $Q_\rho(0,0)$. Both the proof of lemma 4.3.1 and the present one show basically the same pattern, therefore we only highlight the differences between them. The major change concerns the cut-off function, that is property (i). Via a product ansatz and the inclusions

$$I_{\frac{1}{2c_d}} \subset (-1,1) \qquad \text{and} \qquad B_{\frac{1}{2c_d}} \subset B^{eu}_{\frac{1}{2c_d{}^2}} \subset B^{eu}_{c_d^{-2}} \subset B_1$$

we obtain

$$\left|\partial_t^l \partial_x^\alpha \eta(t,x)\right| \lesssim 1$$

for any $l \in \mathbb{N}_0$ and any multi-index $\alpha$, as well as $\eta \equiv 1$ on $Q_{\frac{1}{2c_d}}$ while $spt\,\eta \subset (-1,1) \times B_1$. Another difference is the behavior of the measure. Near the boundary we have no control of the weight from below, but still from above. More precisely, we know that $x_n < 2$ for $x \in B_1$.

By means of these preliminary considerations, we proceed the same way as in steps 1–3 of the previous proof to find

$$\|\nabla u^{(k)}\|_{L^2(Q_{\frac{1}{2c_d}},\mu_{k+1})} + \|D_x^2 u^{(k)}\|_{L^2(Q_{\frac{1}{2c_d}},\mu_{k+3})} \lesssim \|u^{(k)}\|_{L^2(Q_1,\mu_{k+1})} + \|\widetilde{f}_k\|_{L^2(Q_1,\mu_{k+1})},$$

followed by

$$\Big(\int_{I_{\delta_1}} \|D_x^2 u(t)\|^2_{W^{2,2}(Q_{\delta_1},\mu_1,\mu_3,\mu_5)}\,dt\Big)^{\frac{1}{2}} \lesssim \|u\|_{L^2(Q_1,\mu_1)} + \|f\|_{L^2(Q_1,\mu_1)}$$

for some scaling factors $0 < \delta_1 < \frac{1}{2c_d} < 1$. Now let $f = 0$, then there exist $0 < \delta', \delta_{k+1} < 1$ such that

$$\|\partial_t^l \partial_x^{\alpha'} u\|_{L^2(Q_{\delta'},\mu_1)} \lesssim \|u\|_{L^2(Q_1,\mu_1)} \tag{$*$}$$

and

$$\|\nabla u^{(k)}\|_{L^2(Q_{\delta_{k+1}},\mu_{k+1})} \lesssim \|u\|_{L^2(Q_1,\mu_1)} \qquad (k \in \mathbb{N}_0)\,. \tag{$**$}$$

If $\alpha_n = 0$, the statement already follows from $(*)$ with $\delta = \delta'$. Suppose now $\alpha_n \geq 1$. With the Hardy inequality applied $\alpha_n$ times to $\partial_{x_n}^{\alpha_n}(\psi u)$, where $\psi$ is a spatial cut-off function obeying the above estimate, we obtain

$$\begin{aligned}\|\partial_{x_n}^{\alpha_n} u\|_{L^2(B_{\delta_n},\mu_1)} &\leq \|\partial_{x_n}^{\alpha_n}(\psi u)\|_{L^2(H,\mu_1)} \lesssim \|\partial_{x_n}^{2\alpha_n}(\psi u)\|_{L^2(H,\mu_{2\alpha_n+1})} \\ &\lesssim \|u\|_{L^2(Q_1,\mu_1)} + \sum_{1\leq\beta_n\leq 2\alpha_n} \|\underbrace{(\partial_{x_n}^{2\alpha_n-\beta_n}\psi)}_{|\cdot|\lesssim 1}(\nabla\partial_{x_n}^{\beta_n-1}u)\|_{L^2(H,\mu_{\beta_n})} \lesssim \|u\|_{L^2(Q_1,\mu_1)}\end{aligned}$$

for a small $\delta_n < \delta_{2\alpha_n+1} < 1$. The last estimate follows from $(**)$ with $k = \beta_n - 1 \in \mathbb{N}_0$. We integrate in time over the interval $I_1$ to get the iterated local energy estimate in the vertical direction. We combine this estimate with $(*)$ and thus finish the proof of the lemma. ■

A rescaled version of the preceding two lemmas is given in the next proposition.

**Proposition 4.3.4 (local energy estimate)** *Let $t_0 \in \mathbb{R}$, $x_0 \in \overline{H}$, $l \in \mathbb{N}_0$ and $\alpha$ be a multi-index. If $u$ is a $L_0$-solution on $Q_R(t_0,x_0)$ for $R > 0$ to $f = 0$, then there exist an $\varepsilon < 1$ and a constant $c = c(n,l,\alpha)$ such that*

$$\|\partial_t^l \partial_x^\alpha u\|_{L^2(Q_{\varepsilon R}(t_0,x_0),\mu_1)} \leq c\,R^{-4l-|\alpha|}\left(R + \sqrt{x_{0,n}}\right)^{-|\alpha|}\|u\|_{L^2(Q_R(t_0,x_0),\mu_1)}\,.$$

**Proof:** By the translation invariance it suffices to consider $t_0 = 0$ and $x_0 = (0,\ldots,0,x_{0,n}) \in \overline{H}$. Now remember that solutions are invariant under the scaling $T_\lambda : (t,x) \mapsto (\lambda^2 t, \lambda x) =: (\hat{t},\hat{x})$ (cf. section 3.4.1), that is, $u \circ T_\lambda$ is a $L_0$-solution on $T_\lambda^{-1}\big(Q_R(0,x_0)\big) = \frac{1}{\lambda^2} I_R(0) \times \frac{1}{\lambda} B_R(x_0)$ whenever $u$ is a solution on $Q_R(0,x_0)$. Note also that derivatives transform as

$$\partial_{\hat{t}}^l \partial_{\hat{x}}^\alpha u(\hat{t},\hat{x}) = \lambda^{-2l-|\alpha|}\,\partial_t^l \partial_x^\alpha \big(u \circ T_\lambda\big)(t,x)\,.$$

By the transformation formula we have

$$\|\partial_t^l \partial_{\hat{x}}^\alpha\, u\|_{L^2(Q_\rho(0,x_0),\,\mu_1)} \;=\; \lambda^{-2l-|\alpha|} \Big( \int_{\frac{1}{\lambda^2} I_\rho(0)} \|\partial_t^l \partial_x^\alpha\, (u\circ T_\lambda)\|^2_{L^2(\frac{1}{\lambda} B_\rho(x_0),\,\mu_1)}\, \lambda^{n+3}\, d\mathcal{L} \Big)^{\frac{1}{2}} .$$

We would like to set $\rho = \varepsilon\, R$ with $\varepsilon > 0$ so small that the left hand side can be bounded above by $u$ in the weighted $L^2$-norm over $Q_R(0,x_0)$. To this end, we define $\widetilde{\varepsilon} = \frac{\delta_1}{16{c_d}^4}$ and $C_\delta = \frac{4{c_d}^2}{\delta_0} + 1 \gg 1$, where by $\delta_1$ we denote the $\delta$ from lemma 4.3.1 and by $\delta_0$ the one from lemma 4.3.3. Then, the desired estimate follows with $\varepsilon = C_\delta^{-1}\widetilde{\varepsilon}$. Let us prove this by cases on the relation between $x_{0,n}$ and $R$, or more precisely, we consider the case $2\, C_\delta \sqrt{x_{0,n}} < R$ as well as $0 < R \le 2\, C_\delta \sqrt{x_{0,n}}$.

i) If the latter is true, we choose $\lambda = x_{0,n}$ as scaling factor. Using the inclusion

$$T_\lambda^{-1}\big(Q_\rho(0,x_0)\big) \;\subset\; Q_{\delta_1 r}(0,e_n)$$

with $r = \frac{R}{4{c_d}^2 C_\delta \sqrt{x_{0,n}}} \ll 1$, we apply lemma 4.3.1 to $\partial_t^l \partial_x^\alpha\, (u\circ T_\lambda)$. After a retransformation, this yields

$$\|\partial_t^l \partial_{\hat{x}}^\alpha\, u\|_{L^2(Q_{\varepsilon R}(0,x_0),\,\mu_1)} \;\lesssim\; R^{-4l-|\alpha|} \sqrt{x_{0,n}}^{\;-|\alpha|}\, \|u\|_{L^2(Q_R(0,x_0),\,\mu_1)} \,,$$

since also

$$T_{x_{0,n}}\big(Q_r(0,e_n)\big) \;\subset\; Q_{\frac{R}{C_\delta}}(0,x_0) \;\subset\; Q_R(0,x_0)\,.$$

The stated estimate is now a direct consequence of $\sqrt{x_{0,n}} \ge \frac{1}{2\,C_\delta+1}\big(R+\sqrt{x_{0,n}}\big)$.

ii) Now suppose $2\, C_\delta \sqrt{x_{0,n}} < R$. First we observe that $\varepsilon\, R < \frac{R}{C_\delta} =: \rho$ such that $2\sqrt{x_{0,n}} < \rho$. By the triangle inequality and corollary 3.5.5 we then find $B_\rho(x_0) \subset B_{2\rho}(0)$. With $\lambda = \big(\frac{2\sqrt{2}\, c_d}{\delta_0}\, \rho\big)^2$, this implies that

$$T_\lambda^{-1}\big(Q_\rho(0,x_0)\big) \;\subset\; Q_{\delta_0}(0,0)\,.$$

Thus, we can apply lemma 4.3.3 to get

$$\|\partial_t^l \partial_{\hat{x}}^\alpha\, u\|_{L^2(Q_\rho(0,x_0),\,\mu_1)} \;\lesssim\; \rho^{-4l-2|\alpha|}\, \|u\|_{L^2(Q_{C_\delta\,\rho}(0,x_0),\,\mu_1)}\,.$$

Here we also used that

$$T_\lambda\big(Q_1(0,0)\big) \;\subset\; Q_{C_\delta\,\rho}(0,x_0)\,.$$

The statement follows with $\rho > \varepsilon R$, $C_\delta\, \rho = R$ and $\rho > \frac{2}{2\,C_\delta+1}\big(R+\sqrt{x_{0,n}}\big)$. ∎

For a simpler presentation we write

$$\delta_{l,\alpha}(R,x) \;:=\; R^{-4l-|\alpha|}\big(R+\sqrt{x_n}\big)^{-|\alpha|}\,. \tag{4.3.1}$$

### 4.3.2 Pointwise Estimates

Our goal here is to prove a pointwise estimate for solutions of the linear equation $\partial_t u + L_0 u = 0$ on the cylinder $Q_R(t_0,x_0)$ that serves as a starting point for further investigation. On the other hand, it captures the fact that any local solution is indeed smooth, at least on a smaller cylinder.

**Proposition 4.3.5** *Suppose $u$ satisfies the equation $\partial_t u + L_0 u = 0$ on $Q_R(t_0,x_0)$ in the energy sense for some $(t_0,x_0) \in \mathbb{R}\times\overline{H}$. Then for any $l \in \mathbb{N}_0$ and $\alpha \in {\mathbb{N}_0}^n$ there exist $\varepsilon > 0$ and a constant $c = c(n,l,\alpha)$ such that*

$$\big|\partial_t^l \partial_x^\alpha\, u(t,x)\big| \;\le\; c\; \frac{\delta_{l,\alpha}(R,x_0)}{R^2\,\big|B_R(x_0)\big|_1^{\frac{1}{2}}}\; \|u\|_{L^2(Q_R(t_0,x_0),\,\mu_1)}$$

*for almost all $(t,x) \in Q_{\varepsilon R}(t_0,x_0)$.*

This statement will be a consequence of the two results stated in lemma 4.3.1 and lemma 4.3.3. For the situation away from the boundary we shall also need the Morrey-type inequality 2.7.3 localized to $B_r\big((0,\dots,0,1)\big)$ with $r \le 1$ so chosen that the ball does not touch $\partial H$.

**Lemma 4.3.6** *Let $k$ be a positive integer satisfying the condition $2k > n+1$. If $u$ belongs to the Sobolev space $W^{k,2}\big(Q_r(0,e_n)\big)$, with $r \ll 1$ and $e_n = (0,\dots,0,1) \in H$, then there exists $\delta > 0$ such that*

$$\big|u(t,x)\big| \;\le\; c(n) \sum_{j+|\beta|\le k} r^{4j+|\beta|-\frac{n+4}{2}}\, \|\partial_t^j \partial_x^\beta u\|_{L^2(Q_r(0,e_n))}$$

*for almost every $(t,x) \in Q_{\delta r}(0,e_n)$.*

**Proof:** We set $u_r(t,x) = \big(u\circ T\big)(t,x)$ with $T:(t,x)\mapsto (r^4 t, rx) =: (\hat t,\hat x)$. Moreover, with $\delta = \frac{1}{6{c_d}^2}$, let $\widetilde Q$ be the Euclidean cylinder defined by $\widetilde Q = I_\delta(0)\times B^{eu}_{6\delta}\big(\frac{e_n}{r}\big)$. Then the following inclusions hold:

$$T^{-1}\big(Q_{\delta r}(0,e_n)\big) \;\subset\; \widetilde Q \qquad \text{and} \qquad T(\widetilde Q) \;\subset\; Q_r(0,e_n)\,. \tag{$*$}$$

In particular, $u_r \in W^{k,2}(\widetilde Q)$. Therefore we can apply corollary 2.7.3 with $\Omega = \widetilde Q$ to the function $u_r$ to conclude

$$\big|u(\hat t,\hat x)\big| \;=\; \big|u_r(t,x)\big| \;\le\; c(n,k,\delta)\,\|u_r\|_{W^{k,2}(\widetilde Q)}$$

for almost all $(t,x)\in\widetilde Q$, and hence in particular for almost all $(\hat t,\hat x)\in Q_{\delta r}(0,e_n)$, if $2k>n+1$. Choosing

$$k = \frac12 \begin{cases} n+3 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even} \end{cases}$$

and noting that $\delta$ is an independent constant, then leads to

$$\big|u(\hat t,\hat x)\big| \;\le\; c(n) \sum_{j+|\beta|\le k} \|\partial_t^j\partial_x^\beta\, u_r\|_{L^2(\widetilde Q)} \;=\; c(n) \sum_{j+|\beta|\le k} r^{4j+|\beta|-\frac{n+4}{2}}\, \|\partial_{\hat t}^j \partial_{\hat x}^\beta\, u\|_{L^2(T(\widetilde Q))}$$

for almost all $(\hat t,\hat x)\in Q_{\delta r}(0,e_n)$. In the equality we have used a change of coordinates together with the identity

$$\partial_t^j\partial_x^\beta\, u_r(t,x) \;=\; r^{4j+|\beta|}\;\partial_{\hat t}^j\partial_{\hat x}^\beta\, u(\hat t,\hat x)\,.$$

The second inclusion in $(*)$ then completes the proof of the lemma. ∎

**Proof (of proposition 4.3.5):** As before, it suffices to consider the case $(t_0,x_0) = \big(0,(0,\dots,x_{0,n})\big)$ for some $x_{0,n}\ge 0$. Moreover, we recall the scaling invariance of solutions under the mapping $T_\lambda : (t,x)\mapsto (\lambda^2 t,\lambda x) = (\hat t,\hat x)$ and that derivatives transform as

$$\big|\partial_{\hat t}^l\,\partial_{\hat x}^\alpha\, u(\hat t,\hat x)\big| \;=\; \lambda^{-2l-|\alpha|}\,\big|\partial_t^l\partial_x^\alpha\,\big(u\circ T_\lambda\big)(t,x)\big|\,.$$

As in the proof of proposition 4.3.4, by $\delta_0$ and $\delta_1$ we denote the $\delta$ from lemma 4.3.3 and lemma 4.3.1, respectively. Moreover, let $\tilde\delta_1$ be the $\delta$ in lemma 4.3.6 and define $\varepsilon = C_\delta^{-1}\widetilde\varepsilon$, where $C_\delta = \frac{8{c_d}^3}{\delta_0} + 1 \gg 1$ and $\widetilde\varepsilon = \frac{\tilde\delta_1\,\delta_1}{(2c_d)^4} \ll 1$.

i) First we consider the case $R \le 2\,C_\delta\sqrt{x_{0,n}}$ and take $\lambda = x_{0,n}$. In this case $r = \frac{R}{4{c_d}^2 C_\delta\sqrt{x_{0,n}}}$ is a legitimate radius in lemma 4.3.1, and so we get for almost every $(\hat t,\hat x)\in Q_{\varepsilon R}(0,x_0)$ that

$$\begin{aligned}
\big|\partial_{\hat t}^l\,\partial_{\hat x}^\alpha\, u(\hat t,\hat x)\big| &\lesssim x_{0,n}^{-2l-|\alpha|} \sum_{j+|\beta|\le k} r^{4j+|\beta|-\frac{n+4}{2}}\,\|\partial_t^{l+j}\partial_x^{\alpha+\beta}\big(u\circ T_{x_{0,n}}\big)\|_{L^2(Q_{\delta_1 r}(0,e_n))} \\
&\lesssim x_{0,n}^{-2l-|\alpha|}\, r^{-4l-|\alpha|-\frac{n+4}{2}}\,\|u\circ T_{x_{0,n}}\|_{L^2(Q_r(0,e_n)),\mu_1)} \\
&\le x_{0,n}^{-2l-|\alpha|-\frac{n+3}{2}}\, r^{-4l-|\alpha|-\frac{n+4}{2}}\,\|u\|_{L^2(Q_R(0,x_0)),\mu_1)}\,.
\end{aligned}$$

In the first estimate we applied lemma 4.3.6 to $\partial_t^l\partial_x^\alpha\big(u\circ T_{x_{0,n}}\big)$ with $r$ replaced by $\delta_1 r$, and in the second one we used lemma 4.3.1 applied to $\partial_t^{l+j}\partial_x^{\alpha+\beta}\big(u\circ T_{x_{0,n}}\big)$ and the fact that $x_n\sim 1$ in $B_r(e_n)$. The last line follows from $|\det\nabla_{t,x}T_{x_{0,n}}|^{-1}\,d\mu_1(x)=x_{0,n}^{-n-3}\,d\mu_1(\hat x)$ and the cylinder enclosures

$$T_{x_{0,n}}\big(Q_r(0,e_n)\big)\;\subset\;Q_{\frac{R}{C_\delta}}(0,x_0)\;\subset\;Q_R(0,x_0)\,.$$

Finally, the assumption $R\le 2\,C_\delta\sqrt{x_{0,n}}$ ensures that the present coefficient can be estimated as

$$\lambda^{-2l-|\alpha|-\frac{n+3}{2}}\,r^{-4l-|\alpha|-\frac{n+4}{2}}\;\lesssim\;R^{-4l-|\alpha|}\sqrt{x_{0,n}}^{\;-|\alpha|}\,\Big(R^{n+4}\sqrt{x_{0,n}}^{\;n+2}\Big)^{-\frac12}\,,$$

which is bounded above (up to some constant) by $\delta_{l,\alpha}(R,x_0)\,R^{-2}\,\big|B_R(x_0)\big|_1^{-\frac12}$ as stated.

ii) In order to prepare the situation at $\partial H$, we shall need an analogue of lemma 4.3.6. We claim that

$$\big|u(t,x)\big|\;\le\;c(n)\sum_{j+|\beta|\le k}\|\partial_t^j\partial_x^\beta\,u\|_{L^2(Q_{\delta_0}(0,0),\mu_1)}\;+\;\|\nabla_x\partial_t^j\partial_x^\beta\,u\|_{L^2(Q_{\delta_0}(0,0),\mu_1)}\qquad(*)$$

for almost all $(t,x)\in Q_{\delta_0/2c_d}(0,0)$. Now let $0\le 2\,C_\delta\sqrt{x_{0,n}}\;<R$. As in the proof of proposition 4.3.4, we put $\rho=\frac{R}{C_\delta}$; then this relation reads as $0\le 2\sqrt{x_{0,n}}\;<\rho$. Taking $\lambda=2\,\big(\frac{4c_d^2}{\delta_0}\,\rho\big)^2$ we achieve

$$T_\lambda^{-1}\big(Q_\rho(0,x_0)\big)\;\subset\;Q_{\frac{\delta_0}{2c_d}}(0,0)\,.$$

We apply $(*)$ to $\partial_t^l\partial_x^\alpha\big(u\circ T_\lambda\big)$, followed by lemma 4.3.3, to get for almost all $(\hat t,\hat x)\in Q_\rho(0,x_0)$ that

$$\big|\partial_{\hat t}^l\,\partial_{\hat x}^\alpha\,u(\hat t,\hat x)\big|\;\lesssim\;\rho^{-4l-2|\alpha|-n-3}\,\|u\|_{L^2(T_\lambda(Q_1(0,0)),\mu_1)}\,,$$

where the factor $\rho^{-n-3}$ appears due to the reverse transformation of $u\circ T_\lambda$. Finally, we use that

$$T_\lambda\big(Q_1(0,0)\big)\;\subset\;Q_{C_\delta\rho}(0,x_0)\;=\;Q_R(0,x_0)$$

and follow the line of argument in paragraph (ii) of the proof of proposition 4.3.4, combined with

$$\rho^{-n-3}\;\lesssim\;\Big(R^{n+4}\,\big(R+\sqrt{x_{0,n}}\,\big)^{n+2}\Big)^{-\frac12}\;\sim\;\big|Q_R(0,x_0)\big|_1^{-\frac12}\,,$$

to give the local estimate the desired form.

It remains to check that $(*)$ holds. To this end, let $\tilde\delta_0=\frac{\delta_0}{2c_d}$. Similar to the proof of lemma 4.3.6, with $r=\delta_0\ll 1$, we see that

$$\big|u(t,x)\big|\;\lesssim\;\sum_{j+|\beta|\le k}\|\partial_t^j\partial_x^\beta\,u\|_{L^2(\widetilde Q)}$$

for a.e. $(t,x)\in\widetilde Q=I_{\delta_0}(0)\times B^{eu}_{2\tilde\delta_0^2}(0)\supset Q_{\tilde\delta_0}(0,0)$. By choosing a cut-off function $\psi\in C_c^\infty(\overline H)$, for which we have $\psi\equiv 1$ on $B^{eu}_{2\tilde\delta_0^2}(0)$ and $spt\,\psi\subset B^{eu}_{4\tilde\delta_0^2}(0)\subset B_{\delta_0}(0)$, we can adjust the weight in as much as

$$\big|u(t,x)\big|\;\lesssim\;\sum_{j+|\beta|\le k}\|\nabla_x\big(\psi\,\partial_t^j\partial_x^\beta\,u\big)\|_{L^2(Q_{\delta_0}(0,0),\mu_2)}\,,$$

where we also used the Hardy-Sobolev inequality 2.7.5 with $p=q=2$, $k=\theta=1$ and $\sigma=0$. The claim follows since derivatives of $\psi$ are bounded above by some independent constant and $x_n\lesssim 1$ in $B_{\delta_0}(0)$. ■

**Corollary 4.3.7** *Since $\|u\|_{L^2(Q_R,\mu_1)}\le|Q_R|_1^{\frac12}\,\|u\|_{L^\infty(Q_R)}$, the pointwise estimate from proposition 4.3.5 reduces to*

$$\big|\partial_t^l\partial_x^\alpha\,u(t_0,x_0)\big|\;\lesssim\;\delta_{l,\alpha}(R,x_0)\,\|u\|_{L^\infty(Q_R(t_0,x_0))}$$

*for all $l\in\mathbb N_0$, all multi-indices $\alpha$ and any solution of the homogeneous equation in $Q_R(t_0,x_0)$.*

**Remark 4.3.8** *Suppose $u$ is a $L_0$-solution on $(t_1,t_2)\times\overline{H}$. If $t_1 \le s < t \le t_2$, we set $R = \sqrt[4]{t-s} > 0$. Then for any $x \in \overline{H}$, $u$ is also a solution on $Q_R(t,x)$ and we can use proposition 4.3.5 to obtain the pointwise estimate*

$$\left|\partial_t^l \partial_x^\alpha\, u(t,x)\right| \;\lesssim\; \delta_{l,\alpha}(R,x)\left(⨍_{Q_R(t,x)} u^2\,d\mu_1\right)^{\frac{1}{2}}.$$

### 4.3.3 Pointwise Estimates by Initial Values

From now on we return to global solutions, that is, we consider energy solutions on $I\times\overline{H}$ again. Proposition 4.3.5 in conjunction with the property that the $L^2(H,\mu_1)$-norm of solutions decreases in time (compare with the energy identity 4.1.5) provides the following proposition.

**Proposition 4.3.9** *Let $I=(t_1,t_2)\subseteq\mathbb{R}$ be an open interval, $l\in\mathbb{N}_0$ and $\alpha\in{\mathbb{N}_0}^n$. If $g\in L^2(H,\mu_1)$ and $u$ is a $L_0$-solution on $[t_1,t_2)\times\overline{H}$ to $f=0$ with $u(t_1)=g$, then there exists a constant $c=c(n,l,\alpha)$ such that*

$$\left|\partial_t^l \partial_x^\alpha\, u(t,x)\right| \;\le\; c\,\delta_{l,\alpha}\big(\sqrt[4]{t-t_1},x\big)\left|B_{\sqrt[4]{t-t_1}}(x)\right|_1^{-\frac{1}{2}}\|g\|_{L^2(H,\mu_1)}$$

*for all $(t,x)\in\bar{I}\setminus\{t_1\}\times\overline{H}$.*

**Proof:** We fix $t>t_1$ and $x\in\overline{H}$. Since $u$ is an initial value solution on $[t_1,t_2)\times\overline{H}$, it is also one on the smaller set $[t_1,t)\times\overline{H}$. By proposition 4.3.5, or rather remark 4.3.8, we obtain

$$\left|\partial_t^l \partial_x^\alpha\, u(t,x)\right| \;\lesssim\; \delta_{l,\alpha}\big(\sqrt[4]{t-t_1}\,,x\big)\left|Q_{\sqrt[4]{t-t_1}}(x)\right|_1^{-\frac{1}{2}}(t-t_1)^{\frac{1}{2}}\sup_{\tau\in(t_1,t)}\|u(\tau)\|_{L^2(H,\mu_1)}.$$

The estimate now follows from remark 4.1.5. ∎

Our main focus is to derive an exponential decay. A refinement of the argument above allows us to include an exponential function in our estimate. At this point we can not use the property anymore that $\|u(t)\|_{L^2(\mu_1)}$ decreases in $t$. This, however, has been the crucial ingredient in the proof of proposition 4.3.9, and hence we require some sort of compensation for that loss.

**Lemma 4.3.10** *Suppose $g\in L^2(H,\mu_1)$ and $I=(t_1,t_2)\subseteq\mathbb{R}$ is open. Further let $\Psi:(H,d)\to\mathbb{R}$ be in the $Lip_2$-class with Lipschitz constant $c_L\ge 1$, i.e. $\sqrt{x_n}\left|\nabla\Psi(x)\right|\le c_L$ and $x_n\left|\Delta\Psi(x)\right|\le c_L$ for all $x\in H$. If $u$ is a $L_0$-solution to $f=0$ on $[t_1,t_2)\times\overline{H}$ with $u(t_1)=g$, then there exist constants $c,c_n>0$ such that*

$$\|e^\Psi u(t)\|_{L^2(H,\mu_1)} \;\le\; c\,e^{c_n {c_L}^4 (t-t_1)}\,\|e^\Psi g\|_{L^2(H,\mu_1)}$$

*for all $t\in\bar{I}$.*

**Proof:** Existence of a function $\Psi$ with the required properties has been proven in example 3.5.9. Now suppose $\Psi\in Lip_2(H)$. We set $v:=e^\Psi u$ and perform some elementary calculations to discover

- $\Delta(e^\Psi v)=e^\Psi\big(\Delta v+|\nabla\Psi|^2\,v+2\nabla\Psi\cdot\nabla v+(\Delta\Psi)v\big)$,
- $e^\Psi\Delta u=\Delta v+|\nabla\Psi|^2\,v-2\nabla\Psi\cdot\nabla v-(\Delta\Psi)v$,
- $\nabla'(e^\Psi v)=e^\Psi\big(\nabla' v+(\nabla'\Psi)v\big)$ and
- $e^\Psi\nabla' u=\nabla' v-(\nabla'\Psi)v$.

With this we compute

$$\begin{aligned}\partial_t \|v\|^2_{L^2(\mu_1)} &= 2\int_H e^{\Psi} v\, \partial_t u\, d\mu_1 \;=\; -2\int_H \Delta(e^{\Psi}v)\,\Delta u\, d\mu_3 \;-\; 8\int_H \nabla'(e^{\Psi}v)\cdot\nabla' u\, d\mu_1 \\ &= -2\int_H \Big((\Delta v)^2 + 2\,|\nabla\Psi|^2\, v\,\Delta v + |\nabla\Psi|^4\, v^2 - 4\,|\nabla\Psi|^2\,|\nabla v|^2 - (\Delta\Psi)^2\, v^2 - \\ &\qquad - 2\,\nabla\Psi\cdot\nabla v(\Delta\Psi) v\Big)\, d\mu_3 \;-\; 8\int_H \Big(|\nabla' v|^2 - |\nabla'\Psi|^2\, v^2\Big)\, d\mu_1 \,.\end{aligned}$$

Then, using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}\partial_t \|v\|^2_{L^2(\mu_1)} &\lesssim -\,\|\Delta v\|^2_{L^2(\mu_3)} - \|\nabla' v\|^2_{L^2(\mu_1)} + \|(x_n|\Psi|^2)\, v\|^2_{L^2(\mu_1)} + \|(x_n\,\Delta\Psi)\, v\|^2_{L^2(\mu_1)} + \\ &\quad + \|(\sqrt{x_n}\;\Psi)\,\nabla v\|^2_{L^2(\mu_2)} + \|(\sqrt{x_n}\;\Psi)\, v\|^2_{L^2} \\ &\lesssim -\,\|D^2 v\|^2_{L^2(\mu_3)} + ({c_L}^2 + {c_L}^4)\, \|v\|^2_{L^2(\mu_1)} + {c_L}^2\, \|\nabla v\|^2_{L^2(\mu_2)} + {c_L}^2\, \|v\|^2_{L^2} \\ &\lesssim -\,\|D^2 v\|^2_{L^2(\mu_3)} + {c_L}^4\, \|v\|^2_{L^2(\mu_1)} + {c_L}^2\, \|\nabla v\|^2_{L^2(\mu_2)}\,.\end{aligned}$$

The second inequality follows from the auxiliary equation (4.2.1) and the properties of $\Psi$. In the last line we simply applied the Hardy-Sobolev inequality (stated in corollary 2.7.5) to the fourth summand of the previous line and that $c_L \geq 1$. By the weighted interpolation lemma 2.8.3, we thus have

$$\partial_t \|v\|^2_{L^2(\mu_1)} \lesssim -\big(1 - {c_L}^2\,\varepsilon\big)\, \|D^2 v\|^2_{L^2(\mu_3)} + c_n\Big({c_L}^4 + \frac{{c_L}^2}{\varepsilon}\Big)\, \|v\|^2_{L^2(\mu_1)}\,.$$

We choose $\varepsilon \sim c_L^{-2}$ sufficiently small to get

$$\partial_t \|v(t)\|^2_{L^2(\mu_1)} \lesssim -\,\|D^2 v(t)\|^2_{L^2(\mu_3)} + c_n\, {c_L}^4\, \|v(t)\|^2_{L^2(\mu_1)}\,.$$

Now let $t \in I$ and define

$$F(t) := e^{-c_n\, {c_L}^4\,(t-t_1)}\, \|v(t)\|^2_{L^2(\mu_1)} + \int_{t_1}^t e^{-c_n\, {c_L}^4\,(\tau - t_1)}\, \|D^2 v(\tau)\|^2_{L^2(\mu_3)}\, d\tau\,.$$

Then the above calculations imply that

$$\partial_t F(t) = e^{-c_n\, {c_L}^4\,(t-t_1)}\Big(\partial_t \|v(t)\|^2_{L^2(\mu_1)} - c_n\, {c_L}^4\, \|v(t)\|^2_{L^2(\mu_1)} + \|D^2 v(t)\|^2_{L^2(\mu_3)}\Big) \leq 0\,.$$

Hence we have $F(t) \leq F(t_1) = \|e^{\Psi} g\|^2_{L^2(\mu_1)}$ for any $t_1 \leq t \leq t_2$ and the claim follows. ■

Now we are in a position to prove the following result.

**Proposition 4.3.11** *Let $I = (t_1, t_2) \subseteq \mathbb{R}$, $l \in \mathbb{N}_0$, $\alpha \in {\mathbb{N}_0}^n$, $\Psi \in Lip_2(H)$ with Lipschitz constant $c_L \geq 1$ and $u$ be a $L_0$-solution to $f = 0$ on $[t_1, t_2) \times \overline{H}$ with $u(t_1) = g \in L^2(H, \mu_1)$. Then there exist $c, c_n > 0$ such that*

$$\big|\partial_t^l \partial_x^\alpha\, u(t,x)\big| \leq c\, \frac{\delta_{l,\alpha}\big(\sqrt[4]{t - t_1}, x\big)}{\big|B_{\sqrt[4]{t-t_1}}(x)\big|_1^{\frac{1}{2}}}\; e^{\,c_n {c_L}^4 (t - t_1) - \Psi(x)}\, \|e^{\Psi}\, g\|_{L^2(H,\,\mu_1)}$$

*for any $(t,x) \in \bar{I} \setminus \{t_1\} \times \overline{H}$.*

**Proof:** We argue as in the proof of proposition 4.3.9 to get

$$\begin{aligned}\big|\partial_t^l \partial_x^\alpha\, u(t,x)\big| &\lesssim \delta_{l,\alpha}(\sqrt[4]{t - t_1}\,, x)\, \big|Q_{\sqrt[4]{t-t_1}}(t,x)\big|_1^{-\frac{1}{2}} \Big(\int_{(t_1,t)} \|e^{\Psi - \Psi}\, u(\tau)\|^2_{L^2(B_{\sqrt[4]{t-t_1}}(x),\,\mu_1)}\, d\tau\Big)^{\frac{1}{2}} \\ &\leq \delta_{l,\alpha}(\sqrt[4]{t - t_1}\,, x)\, \big|B_{\sqrt[4]{t-t_1}}(x)\big|_1^{-\frac{1}{2}} \sup_{z \in B_{\sqrt[4]{t-t_1}}(x)} e^{-\Psi(z)} \sup_{\tau \in (t_1, t)} \|e^{\Psi}\, u(\tau)\|_{L^2(H,\,\mu_1)}\,,\end{aligned}$$

since $\left|Q_{\sqrt[4]{t-t_1}}(t,x)\right|_1^{-\frac{1}{2}} \sqrt{t-t_1} = \left|B_{\sqrt[4]{t-t_1}}(x)\right|_1^{-\frac{1}{2}}$. We arrive at

$$\left|\partial_t^l \partial_x^\alpha\, u(t,x)\right| \;\lesssim\; \frac{\delta_{l,\alpha}\big(\sqrt[4]{t-t_1}\,,x\big)}{\left|B_{\sqrt[4]{t-t_1}}(x)\right|_1^{\frac{1}{2}}} \left(e^{\,c_n {c_L}^4(t-t_1)-\Psi(x)} \sup_{z\in B_{\sqrt[4]{t-t_1}}(x)} e^{\Psi(x)-\Psi(z)}\right) \|e^{\Psi}\, g\|_{L^2(\mu_1)}$$

by virtue of lemma 4.3.10. Now since $\Psi$ is Lipschitz continuous on $H$ and therefore in particular on $B_{\sqrt[4]{t-t_1}}(x)$, we have $\Psi(x)-\Psi(z) \le c_L \sqrt[4]{t-t_1}$ for all $z \in B_R(x)$. Moreover, we have that $e^{\,c_L\sqrt[4]{t-t_1}+c_n\, {c_L}^4(t-t_1)} \lesssim e^{\,c_n {c_L}^4(t-t_1)}$ and the estimate appears as stated in the proposition. ■

**Remark 4.3.12** *In case of $c_L \in [0,1)$ lemma 4.3.10 has to be modified to*

$$\|e^{\Psi} u(t)\|_{L^2(H,\mu_1)} \;\lesssim\; e^{\,c_n {c_L}^2(t-t_1)}\, \|e^{\Psi} g\|_{L^2(H,\mu_1)} \qquad \forall\, t\in \bar I\,.$$

*But then we can repeat the proof of proposition 4.3.11 with ${c_L}^4$ replaced by ${c_L}^2$. This has the following implication: Proposition 4.3.9 follows from proposition 4.3.11 with $\Psi = c_L = 0$.*

As an important consequence of the pointwise estimate 4.3.11, we derive the following result for solutions of the homogeneous initial value problem with initial datum $g \in \dot C^{\,0,1}(H)$, that is, we assume

$$\|g\|_{\dot C^{\,0,1}(H)} \;=\; \|\nabla g\|_{L^\infty(H)} \;=\; \sup_{x\neq y\in H} \frac{\left|g(x)-g(y)\right|}{|x-y|} \;<\; \infty\,.$$

**Proposition 4.3.13** *Let $I=(t_1,t_2)\subseteq \mathbb{R}$, $j\ge 0$, $l\in\mathbb{N}_0$ and $\alpha\in{\mathbb{N}_0}^n$ with either $l\neq 0$ or $\alpha\neq 0$. Further suppose that $u$ is an energy solution to $\partial_t u + L_0 u = 0$ on $[t_1,t_2)\times \overline{H}$ with $u(t_1)=g\in \dot C^{\,0,1}(H)$. Then there exists a constant $c=c(n,j,l,\alpha)>0$ such that*

$$x_n^{\,j}\left|\partial_t^l\partial_x^\alpha\, u(t\,,x)\right| \;\le\; c\,\sqrt[4]{t-t_1}^{\,1-4l-|\alpha|}\left(\sqrt[4]{t-t_1}+\sqrt{x_n}\,\right)^{2j+1-|\alpha|}\|g\|_{\dot C^{\,0,1}(H)}$$

*for all $(t,x)\in \bar I\setminus\{t_1\}\times\overline{H}$.*

**Proof:** Let $x\in\overline{H}$ and $t\in I$, with $t-t_1\le 1$, be fixed and $C$ be some constant. Since either $l$ or $\alpha$ is nonzero, we conclude

$$\partial_t^l\partial_x^\alpha\left(u(t,x)-C\right) \;=\; \partial_t^l\partial_x^\alpha\, u(t,x)\,.$$

Proposition 4.3.11 together with the fact that $u-C$ is a $L_0$-solution to 0 with $(u-C)(t_1)=g-C$ then implies

$$\left|\partial_t^l\partial_x^\alpha\, u(t,x)\right| \;\lesssim\; \frac{\delta_{l,\alpha}\big(\sqrt[4]{t-t_1},x\big)}{\left|B_{\sqrt[4]{t-t_1}}(x)\right|_1^{\frac{1}{2}}}\; e^{\,c_n\, {c_L}^4(t-t_1)-\Psi(x)}\,\|e^{\Psi}(g-C)\|_{L^2(\mu_1)}\,.$$

For $C=g(x)$, we have $\left|g(y)-C\right| \le |x-y|\,\|\nabla g\|_{L^\infty(H)} = |x-y|\,\|g\|_{\dot C^{\,0,1}(H)}$. Now for a fixed radius $R\in(0,1]$, we decompose $H$ into the annuli $A_i(x)=B_{iR}(x)\setminus B_{(i-1)R}(x)$ with $i\in\mathbb{N}$. This gives the estimate

$$\|e^{\Psi}(g-C)\|_{L^2(\mu_1)} \;\le\; \Big(\sum_{i\in\mathbb{N}}\int_{A_i(x)} e^{\,2\Psi(y)}\,|x-y|^2\,d\mu_1(y)\Big)^{\frac{1}{2}}\,\|g\|_{\dot C^{\,0,1}(H)}\,.$$

Example 3.5.9 allows us to choose $\Psi$ so that $\Psi\sim -\frac{1}{R}\,d(x,\cdot)$. Consequently, $\Psi(x)=0$ and the Lipschitz constant $c_L=\frac{1}{R}\ge 1$. Then we refer to the properties of the intrinsic metric (see lemma 3.5.11) and the doubling condition (see corollary 3.5.12) to find

$$\begin{aligned}\Big(\sum_{i\in\mathbb{N}}\int_{A_i(x)} e^{\,2\Psi(y)}\,|x-y|^2\,d\mu_1(y)\Big)^{\frac{1}{2}} \;&\lesssim\; R\left(R+\sqrt{x_n}\,\right)\sum_{i\in\mathbb{N}} e^{-i+1}\,i^{\,2}\left|B_{iR}(x)\right|_1^{\frac{1}{2}}\\ &\lesssim\; R\left(R+\sqrt{x_n}\,\right)\left|B_R(x)\right|_1^{\frac{1}{2}}\sum_{i\in\mathbb{N}} e^{-i}\,i^{\,n+3}\,.\end{aligned}$$

Since $e^{-i}$ goes to zero as $i \to \infty$ faster than any polynomial, the sum is bounded above. Now, as usual, we set $R = \sqrt[4]{t-t_1}$ such that $e^{\,c_n\, c_L^{\,4}(t-t_1)-\Psi(x)} = e^{\,c_n}$ is constant. Altogether, we obtain

$$\big|\partial_t^l \partial_x^\alpha\, u(t,x)\big| \;\lesssim\; \sqrt[4]{t-t_1}^{\,1-4l-|\alpha|}\,\big(\sqrt[4]{t-t_1}+\sqrt{x_n}\,\big)^{1-|\alpha|}\,\|g\|_{\dot{C}^{0,1}(H)} \qquad (*)$$

for all $(t,x) \in \big(t_1\,, \min\{t_1\!+\!1, t_2\}\big] \times \overline{H}$. The fact that $x_n^{\,j} \le \big(\sqrt[4]{t-t_1}+\sqrt{x_n}\,\big)^{2j}$ proves the estimate for all such times $t$ and all $x \in \overline{H}$.

In order to prove the general case, i.e. also for $t > t_1\!+\!1$, we implement the same strategy used in the proof of proposition 4.3.5. To this end, we remember the scaling $T_\lambda : (t,x) \to (\lambda^2 t, \lambda x)$ and the time shift $T_0 : (t,x) \mapsto (t_1+t, x)$, see section 3.4.1, under which solutions are invariant and define

$$T \;:=\; T_0 \circ T_\lambda :\; (t,x) \overset{T_\lambda}{\mapsto} (\lambda^2 t, \lambda x) \overset{T_0}{\mapsto} (t_1+\lambda^2 t, \lambda x) \;=:\; (\hat{t}, \hat{x})\,.$$

We wish to apply $(*)$ to $u \circ T$ which is a $L_0$-solution on $[0,1) \times \overline{H}$ with initial value $\big(u\circ T\big)(0) = g$. This is possible if we choose $\lambda > 0$ sufficiently small. But then we get

$$\begin{aligned}
\hat{x}_n^{\,j}\,\big|\partial_{\hat{t}}^l\, \partial_{\hat{x}}^\alpha\, u(t_1\!+\!\lambda^2, \hat{x})\big| \;&=\; \lambda^{-2l-|\alpha|}\,\hat{x}_n^{\,j}\,\big|\partial_t^l\, \partial_x^\alpha\,\big(u\circ T\big)(1,x)\big| \\
&\lesssim\; \lambda^{-2l-|\alpha|}\,\hat{x}_n^{\,j}\,\Big(1+\sqrt{\frac{\hat{x}_n}{\lambda}}\,\Big)^{1-|\alpha|}\,\|\nabla_x\big(u\circ T\big)(0)\|_{L^\infty(H)} \\
&<\; \sqrt{\lambda}^{\,-1-4l-|\alpha|}\,\big(\sqrt{\lambda}+\sqrt{\hat{x}_n}\,\big)^{2j+1-|\alpha|}\,\|\nabla_x\, g(\lambda\,\cdot)\|_{L^\infty(H)} \\
&=\; \sqrt{\lambda}^{\,1-4l-|\alpha|}\,\big(\sqrt{\lambda}+\sqrt{\hat{x}_n}\,\big)^{2j+1-|\alpha|}\,\|\nabla_{\hat{x}}\, g\|_{L^\infty(H)}
\end{aligned}$$

with $0 < \lambda \le \sqrt{t_2-t_1}$. Now for arbitrary but fixed $(\hat{t},\hat{x}) \in (t_1, t_2] \times \overline{H}$, we set $\lambda = \sqrt{\hat{t}-t_1}$ and the estimate takes the form stated in the proposition. ∎

Using this we are able to control the $L^p$-norm on the cylinder bounded away from initial time $t_1$. We define $Q_R(x) := \big(t_1+\frac{R^4}{2}, t_1+R^4\big] \times B_R(x)$ for $R>0$ and $x \in \overline{H}$.

**Corollary 4.3.14** *Suppose $I, j, l$ and $\alpha$ as well as $g$ and $u$ are as in proposition 4.3.13.*

*i) Then we have*

$$\big|Q_R(x)\big|^{-\frac{1}{p}}\,R^{\,4l+|\alpha|-1}\,\big(R+\sqrt{x_n}\,\big)^{|\alpha|-2j-1}\,\|\partial_t^l\partial_x^\alpha\, u\|_{L^p(Q_R(x),\,\mu_{jp})} \;\le\; c\,\|g\|_{\dot{C}^{0,1}(H)}$$

*for all $p \in [1,\infty)$, $R>0$ and $x \in \overline{H}$.*

*ii) The estimate $\|\nabla u(t)\|_{L^\infty(H)} \le c\,\|g\|_{\dot{C}^{0,1}(H)}$ holds for all $t \in I$.*

**Proof:** Using proposition 4.3.13 we obtain

$$\|\partial_t^l\partial_x^\alpha\, u\|_{L^p(Q_R(x),\,\mu_{jp})} \;\lesssim\; \sup_{(t,\,y)\in Q_R(x)} \Big(\sqrt[4]{t-t_1}^{\,1-4l-|\alpha|}\,\big(\sqrt[4]{t-t_1}+\sqrt{y_n}\,\big)^{2j+1-|\alpha|}\Big)\,\|g\|_{\dot{C}^{0,1}(H)}\,\big|Q_R(x)\big|^{\frac{1}{p}}\,.$$

Now since $\sqrt{y_n} \lesssim R+\sqrt{x_n}$ as well as $\sqrt{x_n} \lesssim R+\sqrt{y_n}$ if $y \in B_R(x)$, we have

$$\sup_{(t,\,y)\in Q_R(x)} \sqrt[4]{t-t_1}^{\,1-4l-|\alpha|}\,\big(\sqrt[4]{t-t_1}+\sqrt{y_n}\,\big)^{2j+1-|\alpha|} \;\lesssim\; R^{1-4l-|\alpha|}\,\big(R+\sqrt{x_n}\,\big)^{2j+1-|\alpha|}\,.$$

At this point it is also crucial that the supremum is taken over all $t \in \big(t_1+\frac{R^4}{2}, t_1+R^4\big]$ which guarantees that $\sqrt[4]{t-t_1} \sim R$, and part (i) of the corollary follows. For the estimate in (ii) we simply apply proposition 4.3.13 with $j=l=0$ and $|\alpha|=1$. ∎

## Notes

A guideline for the weak type of solution that we consider in definition 4.0.1, i.e. we do not assume any (even weak) differentiability in time, has been pointed out in [51], chapter 4. For the Galerkin approximation in subsection 4.1 we adhere to strict standard techniques, such as can be found in many text books on linear partial differential equations (e.g. [27]). The regularization $u_\varepsilon$ that we use is sometimes referred to as "Steklov averaging", compare §4 of chapter II in [57]. Here, the energy estimates are derived in [57], chapter III, §2, and regularity in time is proven in §4. Also, see the regularization methods, such as the definitions of the different test functions, in [50]. For the local results we follow standard cut-off arguments such as those used in [51, 36, 47, 35].

# Chapter 5

# Gaussian Estimates and Consequences

Solutions of parabolic equations are often given by their corresponding kernels which in turn can be estimated by Gaussian functions. For example, Koch and Lamm [54] show that the biharmonic heat kernel $G(t,x,y)$ that is associated to the equation $\partial_t u + \Delta^2 u = 0$ has a pointwise control of the type

$$\big|G(t,x,y)\big| \;\leq\; c\,t^{-\frac{n}{4}}\, e^{-\varepsilon\left(\frac{|x-y|^4}{t}\right)^{\frac{1}{3}}} . \tag{5.0.1}$$

The power of $t$ in front of the Gaussian factor appears in situations in which the volume of a ball is comparable to its radius - here a Euclidean setting is considered with $\big|B^{eu}_{\sqrt[4]{t}}(x)\big| \sim t^{\frac{n}{4}}$ for every $x \in \mathbb{R}^n$. In non-Euclidean situations, on the other hand, one has to replace this factor by an expression of the form

$$\mu\big(B_{\sqrt[4]{t}}(x)\big)^{-\frac{1}{2}}\,\mu\big(B_{\sqrt[4]{t}}(y)\big)^{-\frac{1}{2}} ,$$

where $\mu$ denotes the underlying measure and $B_{\sqrt[4]{t}}(\cdot)$ denotes the ball of radius $\sqrt[4]{t}$ with respect to the intrinsic metric. This illustrates that both analytic and geometric properties are combined by the kernel $G$.

As a consequence of such an estimate, one obtains that the semigroup generated by the parabolic equation satisfies certain $L^p$-estimates such as those presented in the previous section 4.3.3 containing the Gaussian factor $e^{\Psi}$. The objective here is to derive a Gaussian estimate in terms of the intrinsic metric $d$ and the measure $\mu_1$.

## 5.1 The Green Function

Now we turn to the crucial Gaussian estimate for the Green function. Our notion of such a type of function is that it can be employed to fashion a solution to the homogeneous Cauchy problem. In the modern study of linear partial differential equations, a Green function is therefore often referred to as general or representative solution. The Green function $G$ is defined as the integral kernel such that

$$u(t,x) \;=\; \int_H G(t,x,t_1,y)\, g(y)\, dy \tag{5.1.1}$$

satisfies the equation $\partial_t u + L_0 u = 0$ on $I \times \overline{H}$ subject to the initial condition $u(t_1) = g$. Remark that in general we may not assume that such a kernel exists. However, in our situation we can apply the Riesz representation theorem to ensure its existence in $L^2(\mu_1)$. This occupies the first part of our key theorem

5.1.1. In addition, we show that $G$ is in fact essentially bounded and satisfies an estimate of the form (5.0.1).

**Theorem 5.1.1 (Gaussian estimate)** *Let $I = (t_1, t_2) \subseteq \mathbb{R}$ be open, $l$ any nonnegative integer and $\alpha$ any multi-index. Then there exists a Green function $G : I \times \overline{H} \times I \times \overline{H} \to \mathbb{R}$ with $G(t,x,s,y) = 0$ for $t < s \in [t_1, t_2)$, and*

$$\partial_t^l \partial_x^\alpha\, u(t,x) \;=\; \int_H \partial_t^l \partial_x^\alpha\, G(t,x,s,y)\, u(s,y)\, dy$$

*for all $t > s \in \bar{I}$, $x \in \overline{H}$ and any $L_0$-solution $u$ on $[t_1,t_2) \times \overline{H}$ to $f = 0$ with $u(t_1) = g \in L^2(H,\mu_1)$. In particular, there exist positive constants $c = c(n,l,\alpha)$ and $c_n = c(n)$ such that*

$$\left|\partial_t^l \partial_x^\alpha\, G(t,x,s,y)\right| \;\leq\; c\, \delta_{l,\alpha}\big(\sqrt[4]{t-s}, x\big) \left|B_{\sqrt[4]{t-s}}(x)\right|_1^{-\frac{1}{2}} \left|B_{\sqrt[4]{t-s}}(y)\right|_1^{-\frac{1}{2}} y_n\, e^{-c_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \qquad \text{(ge)}$$

*for almost every $x \neq y$.*

**Proof:** Suppose $u$ is an energy solution on $[t_1,t_2) \times \overline{H}$ to $f = 0$ with $u(t_1) = g$. Then proposition 4.3.9 ensures that the linear functional that assigns $u(s,\cdot)$ to the evaluation $\partial_t^l \partial_x^\alpha\, u(t,x)$ is continuous. Thus we can apply the Riesz representation theorem to find a kernel $k_{l,\alpha}(t,x,s;\cdot) \in L^2(\mu_1)$ such that

$$\partial_t^l \partial_x^\alpha\, u(t,x) \;=\; \int_H k_{l,\alpha}(t,x,s;y)\, u(s,y)\, d\mu_1(y)\,.$$

Putting $G_{l,\alpha}(t,x,s,y) := y_n\, k_{l,\alpha}(t,x,s;y)$, this reads

$$\partial_t^l \partial_x^\alpha\, u(t,x) \;=\; \int_H G_{l,\alpha}(t,x,s,y)\, u(s,y)\, dy$$

which already proves the existence of $G = G_{0,0}$. The desired identity for derivatives follows from Lebesgue's dominated convergence theorem. Indeed, we have

$$\partial_t^l \partial_x^\alpha\, u(t,x) \;=\; \partial_t^l \partial_x^\alpha \int_H G(t,x,s,y)\, u(s,y)\, dy \;=\; \int_H \partial_t^l \partial_x^\alpha\, G(t,x,s,y)\, u(s,y)\, dy$$

which means $G_{l,\alpha} = \partial_t^l \partial_x^\alpha\, G$.

Now fix $t > s \in [t_1,t_2)$ and $x \in \overline{H}$. Moreover, let $\Psi \in Lip_2(H)$ (see definition 3.5.10). By duality we get

$$\|e^{-\Psi} \left|B_{\sqrt[4]{t-s}}(\cdot)\right|_1^{\frac{1}{2}} k_{l,\alpha}\|_{L^\infty(H)} \;=\; \sup_{\|g\|_{L^1(\mu_1)} \leq 1} \left| \int_H k_{l,\alpha}\, e^{-\Psi} \left|B_{\sqrt[4]{t-s}}(\cdot)\right|_1^{\frac{1}{2}} g\, d\mu_1 \right|. \qquad (*)$$

Using the notation $k_{l,\alpha}(y) = y_n^{-1}\, \partial_t^l \partial_x^\alpha\, G(t,x,s,y)$, this reads

$$\sup_{\|g\|_{L^1(\mu_1)} \leq 1} \left\{ \left|\partial_t^l \partial_x^\alpha\, v(t,x)\right| \;\middle|\; v \text{ is } L_0\text{-solution with } v(s) = e^{-\Psi} \left|B_{\sqrt[4]{t-s}}(\cdot)\right|_1^{\frac{1}{2}} g \right\}.$$

Next we appeal to proposition 4.3.11 applied to $v$ in the points $\frac{s+t}{2} < t$ to get

$$e^{\Psi(x)} \left|B_{\sqrt[4]{t-s}}(x)\right|_1^{\frac{1}{2}} \left|\partial_t^l \partial_x^\alpha v(t,x)\right| \;\lesssim\; \delta_{l,\alpha}\big(\sqrt[4]{t-s}, x\big)\, e^{c_n c_L^4 \frac{t-s}{2}}\, \|e^{\Psi}\, v\Big(\frac{s+t}{2}\Big)\|_{L^2(\mu_1)}\,.$$

For simplicity we introduce the following operators: By $M$ we denote the multiplication operator that assigns to a function in $L^2(\mu_1)$ its multiplication by $\left|B_{\sqrt[4]{t-s}}(\cdot)\right|_1^{\frac{1}{2}}$. The modified solution operator is denoted by $\widetilde{S}_{\tilde{s}}(\tilde{t}) : L^2(\mu_1) \ni e^{-\Psi} v(\tilde{s}) \mapsto e^{-\Psi} v(\tilde{t}) \in L^2(\mu_1)$, where $v$ is any energy solution to $\partial_t v + L_0 v = 0$ on $I \times \overline{H}$. In these notations, we apply proposition 4.3.11 once more, but now in the points $s < \frac{s+t}{2}$, for

$l = \alpha = 0$ and with $\Psi$ replaced by $-\Psi$ to find

$$\|M\, \widetilde{S}_s\Big(\frac{s+t}{2}\Big)\, e^{\Psi}\, v(s)\|_{L^\infty(H)} \,\lesssim\, e^{\,c_n\, c_L^4\, \frac{t-s}{2}}\, \|e^{-\Psi}\, v(s)\|_{L^2(\mu_1)}\,,$$

and consequently $M\,\widetilde{S}_s$ is also an operator from $L^2(\mu_1)$ to $L^\infty(H)$ with operator norm bounded by $c(n)\, e^{\,c_n\, c_L^4\, \frac{t-s}{2}}$.

Now let $u_1$ and $u_2$ be as in lemma 4.2.7. Then obviously the identity

$$\big(e^{\Psi} u_1(\tilde{s}) \mid e^{-\Psi}\, u_2(\tilde{t})\big)_{L^2(H,\,\mu_1)} = \big(e^{\Psi} u_1(\tilde{t}) \mid e^{-\Psi}\, u_2(\tilde{s})\big)_{L^2(H,\,\mu_1)}$$

holds for all $\tilde{t} > \tilde{s} \in \bar{I}$. Adopting the terminology of the modified solution operator we interpret this as follows:

$$\widetilde{S}_{\tilde{s}}(\tilde{t})^* : L^2(\mu_1) \ni e^{\Psi} v(\tilde{s}) \mapsto e^{\Psi} v(\tilde{t}) \in L^2(\mu_1)$$

is the dual operator to $\widetilde{S}_{\tilde{s}}(\tilde{t}) : e^{-\Psi} v(\tilde{s}) \mapsto e^{-\Psi} v(\tilde{t})$. The multiplication operator, on the other hand, is self-adjoint. But this implies that

$$(M\,\widetilde{S}_s)^* :\ (L^\infty)' \supset L^1(\mu_1) \ni \tilde{g} \mapsto \widetilde{S}_s{}^* M\, \tilde{g} \in L^2(\mu_1)$$

and the operator norms coincide. Now choosing $\tilde{g} = e^{\Psi}\, \big|B_{\sqrt[4]{t-s}}(\cdot)\big|_1^{-\frac{1}{2}}\, v(s) \in L^1(\mu_1)$ this amounts to

$$\|e^{\Psi}\, v\Big(\frac{s+t}{2}\Big)\|_{L^2(\mu_1)} \;=\; \|\widetilde{S}_s{}^*\Big(\frac{s+t}{2}\Big) M\, \tilde{g}\|_{L^2(\mu_1)} \,\lesssim\, e^{\,c_n\, c_L^4\, \frac{t-s}{2}}\, \|\tilde{g}\|_{L^1(\mu_1)}\,.$$

The two estimates combined give

$$\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{\frac{1}{2}}\, e^{\Psi(x)}\, \big|\partial_t^l \partial_x^\alpha\, v(t,x)\big| \,\lesssim\, \delta_{l,\alpha}\big(\sqrt[4]{t-s}, x\big)\, e^{\,c_n\, c_L^4 (t-s)}\, \|e^{\Psi}\, \big|B_{\sqrt[4]{t-s}}(\cdot)\big|_1^{-\frac{1}{2}}\, v(s)\|_{L^1(\mu_1)}\,.$$

Plugging this inequality into $(*)$ now yields

$$\|e^{-\Psi}\, \big|B_{\sqrt[4]{t-s}}(\cdot)\big|_1^{\frac{1}{2}}\, k_{l,\alpha}\|_{L^\infty(H)} \,\lesssim\, \frac{\delta_{l,\alpha}(\sqrt[4]{t-s}, x)}{\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{\frac{1}{2}}}\, e^{-\Psi(x)+c_n\, c_L^4(t-s)} \sup_{\|g\|_{L^1(\mu_1)}\le 1} \|g\|_{L^1(\mu_1)}\,,$$

where we also used that $v(s) = e^{-\Psi}\, \big|B_{\sqrt[4]{t-s}}(\cdot)\big|_1^{\frac{1}{2}}\, g$. But this implies that for almost every $y \in \overline{H}$

$$\big|\partial_t^l \partial_x^\alpha\, G(t,x,s,y)\big| \,\lesssim\, \frac{\delta_{l,\alpha}\big(\sqrt[4]{t-s}, x\big)\, y_n}{\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{\frac{1}{2}}\, \big|B_{\sqrt[4]{t-s}}(y)\big|_1^{\frac{1}{2}}}\; e^{-\big(\Psi(x)-\Psi(y)-c_n\, c_L{}^4(t-s)\big)}\,.$$

This is where we specify the choice of the Lipschitz function and define $\Psi(x) = c_L d(x,y)$ (see example 3.5.9). Now we optimize the estimate with respect to $\Psi$ or rather $c_L$. Fixing all the other variables the Gaussian function attains its minimum if $c_L = \big(\frac{d(x,y)}{4\, c_n(t-s)}\big)^{\frac{1}{3}}$. Indeed,

$$-\big(c_L\, d(x,y) - c_n\, c_L{}^4(t-s)\big) \,\ge\, -\,\frac{d(x,y)^{\frac{4}{3}}}{\big(c_n(t-s)\big)^{\frac{1}{3}}}\, \underbrace{\big(4^{-\frac{1}{3}} - 4^{-\frac{4}{3}}\big)}_{>0} \;=\; -\,c_n^{-1}\, \Big(\frac{d(x,y)}{\sqrt[4]{t-s}}\Big)^{\frac{4}{3}}\,,$$

and the pointwise estimate as stated in the theorem follows. This completes the proof. ■

**Corollary 5.1.2** *The Gaussian estimate allows us to solve the initial value problem also for other data than those in $L^2(H, \mu_1)$.*

**Sketch of proof:** Given an initial datum $g$ in either $L^1(H, \mu_1)$ or $\dot{C}^{0,1}(H)$, one can truncate $g$ to become a function in $L^2(H, \mu_1)$. A solution is then obtained by the representation formula (5.1.1) and the exponential decay ensures convergence of the truncated solution. ■

**Remarks 5.1.3**

1) *For the proof of the Gaussian estimate we require that* $c_L \geq 1$. *In the opposite case* $c_L < 1$, *we use the inequality*

$$\big|B_{\sqrt[4]{t-s}}(y)\big|_1^{\frac{1}{2}}\,\big|G_{l,\alpha}(t,x,s,y)\big| \;\lesssim\; \frac{\delta_{l,\alpha}(\sqrt[4]{t-s},x)\,y_n}{\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{\frac{1}{2}}}$$

*instead. Now if* $c_L = \Big(\frac{d(x,y)}{c_n(t-s)}\Big)^{\frac{1}{3}} < 1$ *with* $\sqrt[4]{t-s} \leq 1$, *we conclude that* $\frac{d(x,y)}{\sqrt[4]{t-s}} \leq \frac{d(x,y)}{t-s}$ *implying that*

$$1 \;\leq\; e^{1-\varepsilon\big(\frac{d(x,y)^4}{t-s}\big)^{\frac{1}{3}}}\,.$$

*This shows that (ge) remains valid if* $d(x,y) < c_n(t-s) \leq c_n$.

*On the other hand, if* $c_L < 1$ *and* $\sqrt[4]{t-s} > 1$, *we repeat the proof of the Gaussian estimate, but this time for* $u \circ T$ *instead of* $u$. *The operator* $T$ *is defined as in the proof of proposition 4.3.13, that is,* $T$ *is a bijection of* $[0,1] \times \overline{H}$ *onto* $[s,t] \times \overline{H}$, *with* $[s,t] \subset I$, *and* $u \circ T$ *is again a* $L_0$*-solution with* $\big(u \circ T\big)(0) = u(s)$. *Note also that* $G \circ T^2$ *is the corresponding Green function on* $(0,1) \times \overline{H}$. *Hence, it suffices to consider the case* $s = 0$ *and* $t = 1$ *such that the condition* $\sqrt[4]{t-s} \leq 1$ *is always fulfilled. This proves, regardless of the relation between* $d(x,y)$ *and* $\sqrt[4]{t-s}$, *that the assumption* $c_L \geq 1$ *is not a limiting condition.*

2) *Note that*

$$\big|\partial_t^l \partial_x^\alpha\, G(t,x,s,y)\big| \;\lesssim\; \frac{\delta_{l,\alpha}\big(\sqrt[4]{t-s},x\big)}{\big|B_{\sqrt[4]{t-s}}(x)\big|_1}\;y_n\,e^{-c_n^{-1}\big(\frac{d(x,y)^4}{t-s}\big)^{\frac{1}{3}}}$$

*and (ge) are comparable up to changing the constants* $c, c_n > 0$. *To see this, note that*

$$\big|B_{\sqrt[4]{t-s}}(y)\big|_1^{-\frac{1}{2}} \;\lesssim\; \Big(1+\frac{d(x,y)}{\sqrt[4]{t-s}}\Big)^{n+1}\,\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{-\frac{1}{2}}$$

*by lemma 3.5.13. However, due to the exponential decay the emerging expression* $\big(1+\frac{d(x,y)}{R}\big)^{n+1}$ *can be controlled by the Gaussian function: For any number* $m \geq 0$ *there exists a positive constant* $c(m)$ *such that* $e^{-z} \leq c(m)\,\big(1+z\big)^{-m}$ *holds for all* $z \geq 0$. *Thus, with* $m = n+1$, *we find*

$$\Big(1+\frac{d(x,y)}{\sqrt[4]{t-s}}\Big)^{n+1}\,e^{-\,c_n^{-1}\,\big(\frac{d(x,y)^4}{t-s}\big)^{\frac{1}{3}}} \;\leq\; c(n+1)\,e^{\,1-\,(2\,c_n)^{-1}\,\big(\frac{d(x,y)^4}{t-s}\big)^{\frac{1}{3}}}\,.$$

*In the same manner, we can replace* $\delta_{l,\alpha}\big(\sqrt[4]{t-s},x\big)$ *by* $\delta_{l,\alpha}\big(\sqrt[4]{t-s},y\big)$. *Indeed,*

$$\delta_{l,\alpha}\big(R,x\big) \;=\; R^{-4l-|\alpha|}\,\big(R+\sqrt{x_n}\big)^{-|\alpha|} \;\lesssim\; \Big(1+\frac{d(x,y)}{R}\Big)^{2|\alpha|}\,\delta_{l,\alpha}\big(R,y\big)\,,$$

*again by lemma 3.5.13. Throughout the rest of this work we choose the combination of* $x$ *and* $y$ *in the factor* $\delta_{l,\alpha}\big(\sqrt[4]{t-s},\cdot\big)\,\big|B_{\sqrt[4]{t-s}}(\cdot)\big|_1^{-1}$ *that turns out to be suitable and refer to an estimate of the form of (ge) always as "Gaussian estimate".*

In the next lemma we also allow for $s$- and $y$-derivatives to enter into the Gaussian estimate.

**Lemma 5.1.4** *Let* $I = (t_1,t_2)$ *be an open interval,* $l, m \in \mathbb{N}_0$ *and* $\alpha, \beta \in {\mathbb{N}_0}^n$. *If* $G$ *is the Green function associated to the homogeneous initial value problem, then there exist* $\varepsilon$ *and a constant* $c = c(n,l,m,\alpha,\beta)$ *such that*

$$\big|\partial_s^m \partial_y^\beta\big(y_n^{-1}\,\partial_t^l \partial_x^\alpha\, G(t,x,s,y)\big)\big| \;\leq\; c\;\frac{\delta_{l+m,\alpha+\beta}\big(\sqrt[4]{t-s},x\big)}{\big|B_{\sqrt[4]{t-s}}(x)\big|_1}\;e^{-\varepsilon\big(\frac{d(x,y)^4}{t-s}\big)^{\frac{1}{3}}}$$

*for any* $t > s \in I$ *and* $x, y \in \overline{H}$.

**Proof:** First, one may check that $y_n^{-1}\,G(t,x,s,y)$ is a solution with respect to $(s,y)$ on $I \times \overline{H}$ in the sense of definition 4.0.1, and hence the same holds true for $y_n^{-1}\,\partial_t^l\partial_x^\alpha\,G(t,x,s,y)$. To see this, first note that $x_n^{-1}\,G(s,y,t,x)$ is an energy solution with respect to $(s,y)$ and then use that $G$ satisfies the symmetry property

$$G(t,x,s,y) \;=\; \frac{y_n}{x_n}\;G(s,y,t,x)$$

for almost every $x,y \in \overline{H}$. Now for $s < t \in I$, there always exists a positive constant $c$ such that $I_R(s) = (s - R^4, s]$ is contained in $I$, where the radius is defined as $R = c\sqrt[4]{t-s}$. Moreover, we have

$$I_R(s) \;=\; \left\{\tau \;\middle|\; t-s \le t-\tau < (c^4+1)(t-s)\right\}, \tag{$*$}$$

i.e. $\sqrt[4]{t-\tau} \sim R$ for all $\tau \in I_R(s)$. We now apply corollary 4.3.7 to $\xi_n^{-1}\,\partial_t^l\partial_x^\alpha\,G(t,x,\tau,\xi)$ in $Q_R(s,y)$ to find

$$\begin{aligned}\left|\partial_s^m\partial_y^\beta\left(y_n^{-1}\,\partial_t^l\partial_x^\alpha\,G(t,x,s,y)\right)\right| \;&\lesssim\; \delta_{m,\beta}(R,y)\,\|\xi_n^{-1}\,\partial_t^l\partial_x^\alpha\,G(t,x,\cdot,\cdot)\|_{L^\infty(Q_R(s,y))}\\ &\lesssim\; \delta_{m,\beta}(R,y)\,\delta_{l,\alpha}(R,x)\,\left|B_R(x)\right|_1^{-1}\,\|e^{-\varepsilon\left(\frac{d(x\,,\,\cdot)}{R}\right)^{\frac{4}{3}}}\|_{L^\infty(Q_R(s,y))}\end{aligned}$$

by virtue of (ge) and $(*)$. Finally, we use the triangle inequality and Young's inequality to show that $d(x,\xi)^{\frac{4}{3}} \ge \frac{1}{2}\,d(x,y)^{\frac{4}{3}} - d(\xi,y)^{\frac{4}{3}}$ and hence

$$e^{-\varepsilon\left(\frac{d(x,\xi)}{R}\right)^{\frac{4}{3}}} \;\le\; e^{-\frac{\varepsilon}{2}\left(\frac{d(x,y)}{R}\right)^{\frac{4}{3}}}\;e^{\varepsilon\left(\frac{d(\xi,y)}{R}\right)^{\frac{4}{3}}}\,.$$

The lemma follows with $d(\xi,y) < R$ for $\xi \in B_R(y)$. ■

In the next lemma we rephrase the Gaussian estimate (ge) on the Green function and its derivatives in a more convenient form, but at the cost of a reduction of its application range. Indeed, as we will presently see, the following estimate is limited to $(s,y)$ outside of a certain cylinder $Q$.

**Lemma 5.1.5** *Suppose $\delta \in [0,\frac{1}{2}]$ and $\rho \ge 1$ are fixed parameters. Let $G$ be the Green function to $\partial_t u + L_0 u = 0$ on $(0,1)\times\overline{H}$ and $(t,x) \in (2\delta,1]\times\overline{H}$. Then, for every $j \ge 0$, $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0{}^n$ with $|\alpha| \ge 2j$, we have*

$$x_n^{\,j}\,\left|\partial_t^l\partial_x^\alpha\,G(t,x,s,y)\right| \;\le\; c\left(1+\sqrt{y_n}\right)^{2j-|\alpha|}\left|B_1(y)\right|^{-1}\,e^{-\frac{d(x,y)}{4\,c_n}}$$

*for almost all $(s,y) \in \left((0,t]\times\overline{H}\right)\setminus\left((\delta,t]\times B_\rho(x)\right)$. Here, $c_n$ is the constant from the Gaussian estimate (ge) and $c$ depends on $n, j, l, \alpha$ and the choices of $\delta$ and $\rho$.*

**Proof:** Let $0 \le \delta \le \frac{1}{2}$ and $\rho \ge 1$ be arbitrary, but fixed. With $Q := (\delta,t]\times B_\rho(x)$ for $t \in [2\delta,1]$, we decompose

$$\left((0,t]\times\overline{H}\right)\setminus Q \;=\; (0,\delta]\times B_\rho(x) \,\cup\, (0,\delta]\times B_\rho(x)^c \,\cup\, (\delta,t]\times B_\rho(x)^c \;=:\; \bigcup_{i=1}^{3} M_i\,.$$

We observe that $y_n\left|B_{\sqrt[4]{t-s}}(y)\right|_1^{-1} \lesssim \left|B_{\sqrt[4]{t-s}}(y)\right|^{-1}$ such that

$$x_n^{\,j}\,\left|\partial_t^l\partial_x^\alpha\,G(t,x,s,y)\right| \;\lesssim\; \sqrt[4]{t-s}^{\;-4l-|\alpha|}\left(\sqrt[4]{t-s}+\sqrt{y_n}\right)^{2j-|\alpha|}\left|B_{\sqrt[4]{t-s}}(y)\right|^{-1}e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}}\,.$$

Here we have used the condition $2j-|\alpha| \le 0$. For the next step we require that the time interval, and consequently also the time difference $t-s$, is bounded. This and the doubling property imply

$$\begin{aligned}\left(1+\sqrt{y_n}\right)^{|\alpha|-2j}\left|B_1(y)\right| \;&\le\; \sqrt[4]{t-s}^{\;2j-|\alpha|}\left(\sqrt[4]{t-s}+\sqrt{y_n}\right)^{|\alpha|-2j}\left|B_1(y)\right|\\ &\lesssim\; \sqrt[4]{t-s}^{\;-2n-|\alpha|+2j}\left(\sqrt[4]{t-s}+\sqrt{y_n}\right)^{|\alpha|-2j}\left|B_{\sqrt[4]{t-s}}(y)\right|.\end{aligned}$$

Thus it is sufficient to show that

$$\sqrt[4]{t-s}^{\,-2n-4l-2|\alpha|+2j}\; e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \;\le\; c\, e^{-\frac{d(x,y)}{4\,c_n}} \qquad \forall\,(s,y)\in M_1\cup M_2\cup M_3\,.$$

Note also that $\gamma := 2\big(n+2l+|\alpha|-j\big)$ is strictly positive which is due to the assumption that $|\alpha|-2j\ge 0$.

First we consider $(s,y)\in M_1$. Then

$$\sqrt[4]{t-s}^{\,-\gamma}\; e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \;\le\; \delta^{-\frac{\gamma}{4}} \;<\; \delta^{-\frac{\gamma}{4}}\; e^{\rho-d(x,y)}\,,$$

since $t-s\ge 2\delta-\delta$ and $d(x,y)<\rho$.
Now let $(s,y)$ be in $M_2$. Then $\left(\frac{d(x,y)}{t-s}\right)^{\frac{1}{3}} > 1$ such that

$$\sqrt[4]{t-s}^{\,-\gamma}\; e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \;<\; \delta^{-\frac{\gamma}{4}}\; e^{-\frac{d(x,y)}{2\,c_n}}\,.$$

Eventually, if $(s,y)\in M_3$ we use the exponential decay which makes the increase in the prefactor controllable: $e^{-(4c_nR)^{-1}}\le c(m)R^m$ for any $m\ge 0$. Choosing $R=\sqrt[4]{t-s}$ and $m=\gamma$ gives

$$\sqrt[4]{t-s}^{\,-\gamma}\; e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \;=\; R^{-m}\, e^{-\left(\frac{d(x,y)^4}{R}\right)^{\frac{1}{3}}(4c_nR)^{-1}}\; e^{-\left(\frac{d(x,y)}{R^4}\right)^{\frac{1}{3}}\frac{d(x,y)}{4\,c_n}} \;\lesssim\; e^{-\frac{d(x,y)}{4\,c_n}}\,.$$

Here we also use that $\frac{d(x,y)^4}{R}>1$ and $\frac{d(x,y)}{R^4}>1$ since $t-s<1-\delta$ and $d(x,y)\ge\rho$ for $(s,y)\in M_3$ and $t\le 1$. The claimed estimate then follows with $c=\max\big\{\delta^{-\frac{\gamma}{4}}\,e^{\rho},c(\gamma)\big\}$. ■

We conclude this section with another immediate consequence of the Gaussian estimate, namely that, for a certain range of $q\ge 1$, the Green function and its weighted derivatives (leaving temporal derivatives aside) are in the space $L^q$, where the integral is taken with respect to $\mathcal{L}^{n+1}$.

**Lemma 5.1.6** *Let $G$ be the Green function on $(0,1)\times\overline{H}$, $j\ge 0$ and $\alpha\in\mathbb{N}_0{}^n$ with $2j\le|\alpha|<j+2$, then*

$$\|x_n^{\,j}\,\partial_x^\alpha\, G(t,x,\cdot,\cdot)\|_{L^q((0,t)\times H)} \;\le\; c(n,j,\alpha,q)\,\big(1+\sqrt{x_n}\,\big)^{2j-|\alpha|}\,\big|B_1(x)\big|^{\frac{1}{q}-1}$$

*for all $t\in(0,1]$ and almost all $x\in\overline{H}$, and for any $1\le q<\frac{n+2}{n-j+|\alpha|}$.*

**Proof:** Applying the Gaussian estimate (ge) we find that

$$\int_H x_n^{\,jq}\,\big|\partial_x^\alpha\, G(t,x,s,y)\big|^q\,dy \;\lesssim\; x_n^{\,jq}\,\delta_{0,\alpha}\big(\sqrt[4]{t-s},x\big)^q\,\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{-q}\int_H e^{-qc_n^{-1}\left(\frac{d(x,y)^4}{1-s}\right)^{\frac{1}{3}}}\,d\mu_q(y)$$

for $t>s$. Next, with $A_i(x):=B_{i\,\sqrt[4]{t-s}}(x)\setminus B_{(i-1)\,\sqrt[4]{t-s}}(x)$, we decompose the half plane into the annuli,

$$H \;=\; \bigcup_{i\in\mathbb{N}} A_i(x)\,.$$

This yields

$$\begin{aligned}\int_H x_n^{\,jq}\,\big|\partial_x^\alpha\, G(t,x,s,y)\big|^q\,dy \;&\lesssim\; x_n^{\,jq}\,\delta_{0,\alpha}\big(\sqrt[4]{t-s},x\big)^q\,\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{-q}\sum_{i\in\mathbb{N}}\int_{A_i(x)} e^{-qc_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}}\,d\mu_q(y)\\ &\le\; x_n^{\,jq}\,\delta_{0,\alpha}\big(\sqrt[4]{t-s},x\big)^q\,\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{-q}\sum_{i\in\mathbb{N}} e^{-qc_n^{-1}(i-1)^{\frac{4}{3}}}\,\big|B_{i\,\sqrt[4]{t-s}}(x)\big|_q\,.\end{aligned}$$

In view of the doubling condition (see corollary 3.5.12) we get $\big|B_{i\,\sqrt[4]{t-s}}(x)\big|_q\lesssim i^{\,2n+2q}\,\big|B_{\sqrt[4]{t-s}}(x)\big|_q$ and hence

$$\big|B_{\sqrt[4]{t-s}}(x)\big|_1^{-q}\,\big|B_{\sqrt[4]{t-s}}(x)\big|_q \;\sim\; \big|B_{\sqrt[4]{t-s}}(x)\big|^{1-q} \;\lesssim\; \sqrt[4]{t-s}^{\,2n(1-q)}\,\big|B_1(x)\big|^{1-q}\,.$$

Moreover,

$$x_n^{\,j}\,\delta_{0,\alpha}\big(\sqrt[4]{t-s},x\big) \;\le\; \sqrt[4]{t-s}^{\;2j-2|\alpha|}\,\big(1+\sqrt{x_n}\,\big)^{2j-|\alpha|}$$

if $|\alpha| \ge 2j$. We subsume the convergent series $\sum_{i\in\mathbb{N}} i^{\,2(n+q)}\,e^{-qc_n^{-1}(i-1)^{\frac{4}{3}}}$ into the constant and then integrate the estimate in $s \in (0,t)$ to realize,

$$\begin{aligned}\int_0^t \|x_n^{\,j}\,\partial_x^\alpha\,G(t,x,s,\cdot)\|_{L^q(H)}^q\,ds \;&\lesssim\; \big(1+\sqrt{x_n}\,\big)^{(2j-|\alpha|)q}\,\big|B_1(x)\big|^{1-q}\int_0^t \tau^{\,\frac{n}{2}-\frac{q}{2}(n-j+|\alpha|)}\,d\tau\\ &=\; 2\,\big(1+\sqrt{x_n}\,\big)^{(2j-|\alpha|)q}\,\big|B_1(x)\big|^{1-q}\;\frac{\tau^{\,\frac{n+2}{2}-\frac{q}{2}(n-j+|\alpha|)}}{n+2-q\big(n-j+|\alpha|\big)}\,\Bigg|_{\tau=0}^{\tau=t}.\end{aligned}$$

If both the denominator and the exponent are strictly positive, that is for $q < \frac{n+2}{n-j+|\alpha|}$ , this last expression is bounded above by a constant depending on $n, j, \alpha$ and $q$. But since also $q \ge 1$, this condition is satisfied as long as $-j + |\alpha| < 2$. ■

**Remark 5.1.7** *Using similar arguments we can also show that, for any $2j \le |\alpha| < j+2$, we have*

$$\Big(\int_s^1 \|\partial_x^\alpha\,G(t,\cdot,s,y)\|_{L^q(H,\,\mu_{jq})}^q\,dt\Big)^{\frac{1}{q}} \;\le\; c(n,j,\alpha,q)\,\big(1+\sqrt{y_n}\,\big)^{2j-|\alpha|}\,\big|B_1(y)\big|^{\frac{1}{q}-1}$$

*for all $s \in (0,1]$, almost all $y \in \overline{H}$ and for any $1 \le q < \frac{n+2}{n-j+|\alpha|}$ .*

## 5.2 The Inhomogeneous Problem

In this section we present the main results of this work. We would like to apply a fixed point argument in certain function spaces to obtain well-posedness for the nonlinear initial value problem

$$\partial_t u \,+\, L_0 u \;=\, f_0[u] \,+\, x_n\,f_1[u] \,+\, x_n^{\,2}\,f_2[u] \;=\; f[u]\,,\quad u(t_1) \;=\; g\,, \tag{5.2.1}$$

where $f_0[u], f_1[u]$ and $f_2[u]$ are as in lemma 3.3.1. In the previous chapter we have already seen that there exists a unique solution $u$ of the homogeneous linear initial value problem, that is, $u$ satisfies the equation

$$\partial_t u \,+\, L_0 u \;=0\,,\quad u(t_1) \;=\; g\,. \tag{5.2.2}$$

Let $S_{t_1}(t)$ denote the solution operator such that $S_{t_1}(t)g(x) = u(t,x)$ for $t \in \bar{I} = [t_1,t_2] \subset \mathbb{R}$. Hence $S_{t_1}(t_1)g = g$ as the initial condition holds and $(\partial_t + L_0)S_{t_1}(t)g = 0$. Now we apply Duhamel's principle which states that one can start with such a solution to build a solution to the inhomogeneous problem by thinking of it as a set of initial value problems each beginning anew at the starting time $f(s,\cdot)$ instead of $g$. Integrating trough time then gives the desired solution. This means that Duhamel's formula,

$$u(t,x) \;=\; S_{t_1}(t)g(x) \,+\, \int_{t_1}^t S_s(t)f(s,x)\,ds \qquad \text{for } (t,x) \in I\times\overline{H}\,,$$

is the unique solution of (5.2.1). To confirm this, first realize that the initial condition is certainly satisfied because the integral vanishes at time $t = t_1$. Next, applying $\partial_t + L_0$ to Duhamel's formula yields

$$\big(\partial_t + L_0\big)u(t,x) \;=\; S_t(t)f(t,x) \,+\, \int_{t_1}^t \big(\partial_t + L_0\big)S_s(t)f(s,x)\,ds \;=\; f(t,x)\,,$$

where the first summand comes up by differentiating the upper limit of the integral. We also used that $S_t(t)f(t) = f(t)$ and $(\partial_t + L_0)S_s(t)f(s) = 0$. In the following proposition we combine these considerations with the fact that the solution operator can be expressed in terms of the Green function.

**Proposition 5.2.1 (Duhamel's principle)** *Suppose $I=(t_1,t_2)\subset\mathbb{R}$ is an open interval, $g\in L^2(H,\mu_1)$ and $f\in L^2\big(I;L^2(H,\mu_1)\big)$. Further let $G$ denote the Green function on $I\times\overline{H}$ associated with (5.2.2). Then,*

$$u(t,x) \;=\; \int_H G(t,x,t_1,y)\,g(y)\,dy \;+\; \int_{t_1}^t\int_H G(t,x,s,y)\,f(s,y)\,dy\,ds$$

*satisfies the equation $\partial_t u + L_0 u = f$ on $\bar{I}\times\overline{H}$ with initial condition $u(t_1)=g$.*

The fact that we can write the solution of the inhomogeneous problem with zero initial value as integral operator allows us to treat $f\mapsto x_n^{\,j}\,\partial_t^l\partial_x^\alpha\,u$ as a kernel operator. In particular, we will see that

$$y_n^{-1}\,\partial_t G\,,\qquad y_n^{-1}\,D_x^2 G\,,\qquad y_n^{-1}\,x_n\,D_x^3 G\qquad\text{and}\qquad y_n^{-1}\,x_n^{\,2}\,D_x^4 G$$

define integral kernels which satisfy certain cancellation properties. From the Calderón-Zygmund theory in spaces of homogeneous type we have learned that this implies that the corresponding operators are bounded on $L^p\big(I;L^p(H,\mu_1)\big)$ for any $p\in(1,\infty)$.

When it comes to integral kernel operators Schur's lemma is surely one of the most basic facts. It states that

$$f\;\mapsto\;\int_Y K(x,y)\,f(y)\,d\nu(y)$$

is a bounded operator from $L^p(Y,\nu)$ to $L^p(X,\mu)$, $1\le p\le\infty$, if for almost every $x\in X$,

$$\int_Y \big|K(x,y)\big|\,d\nu(y)\;\le\;C$$

and, for almost every $y\in Y$,

$$\int_X \big|K(x,y)\big|\,d\mu(x)\;\le\;C\,.$$

A proof of this standard result may be found in [32]. Utilizing Schur's lemma we can establish $L^p$-boundedness for additional integral operators without using Calderón-Zygmund theory.

**Lemma 5.2.2** *Let $u$ be the $L_0$-solution to $f\in L^2\big((0,1);L^2(H,\mu_1)\big)$ on $[0,1)\times\overline{H}$ with $u(0)=0$. Then we have*

$$\|x_n^{\,j}\,\partial_x^\alpha\,u\|_{L^p((0,1)\times H)}\;\le\;c\,\|f\|_{L^p((0,1)\times H)}$$

*for any $j\ge 0$ and any multi-index $\alpha$ with $2j\le|\alpha|<j+2$, and especially for $j=\frac{|\alpha|}{2}$ if $|\alpha|<4$.*

### 5.2.1 Kernel Estimates

Let $V(t,x,s,y)$ be the volume of the "smallest" ball centered at $(t,x)$ that contains $(s,y)$. As the volume function $V$ is essentially symmetric, i.e. we have $V(t,x,s,y)\sim V(s,y,t,x)$, it is equivalent to say $V$ is given by

$$V(t,x,s,y)\;:=\;\big|B_{d_0}(t,x)\big|_1\;+\;\big|B_{d_0}(s,y)\big|_1\,,$$

where $d_0:=d^{(t)}\big((t,x),(s,y)\big)=\sqrt[4]{|t-s|+d(x,y)^4}\,$.

**Proposition 5.2.3** *Suppose $G$ is the Green function and the kernel $K(t,x,s,y)$ is given by any of the following expressions:*

$$y_n^{-1}\,\partial_t G(t,x,s,y),\;\; y_n^{-1}\,D_x^2 G(t,x,s,y),\;\; y_n^{-1}\,x_n\,D_x^3 G(t,x,s,y),\quad\text{or}\quad y_n^{-1}\,x_n^{\,2}\,D_x^4 G(t,x,s,y)\,.$$

*Then there exists a positive constant $C=C(n)$ such that $\big|K(t,x,s,y)\big|\le C\,V(t,x,s,y)^{-1}$.*

For the stronger kernel estimate we set

$$D \;:=\; \frac{d^{(t)}\big((t,x),(\bar t,\bar x)\big) + d^{(t)}\big((s,y),(\bar s,\bar y)\big)}{d^{(t)}\big((t,x),(s,y)\big) + d^{(t)}\big((\bar t,\bar x),(\bar s,\bar y)\big)}\,.$$

**Proposition 5.2.4** *Under the assumptions of proposition 5.2.3 we have*

$$\big|K(t,x,s,y) - K(\bar t,\bar x,\bar s,\bar y)\big| \;\le\; C(n)\,\frac{D}{V(t,x,s,y)}$$

*if* $D \le \frac{1}{10}$.

### 5.2.2 Weighted $L^p$-Estimates

The kernel estimates and corollary 4.2.6 provide all that is needed to apply the theory of singular integral operators. We obtain that, for $j,l$ and $\alpha$ admissible, the operator that maps the inhomogeneity $f$ to $x_n^{\,j}\,\partial_t^l\partial_x^\alpha\,u$ is a Calderón-Zygmund operator on a homogeneous-type metric space.

**Definition 5.2.5** *We say the triple $(j,l,\alpha)$ is of Calderón-Zygmund type if it belongs to the set*

$$\mathcal{CZ} \;:=\; \big\{(j,l,\alpha)\in[0,\infty)\times\mathbb{N}_0\times\mathbb{N}_0^{\,n}\mid j=2l+|\alpha|-2 \quad\text{and}\quad 2j\le|\alpha|\big\}\,,$$

*and observe that $(j,l,\alpha)\in\mathcal{CZ}$ if and only if $\big(j,l,|\alpha|\big)\in\big\{(0,1,0),(0,0,2),(1,0,3),(2,0,4)\big\}$.*

**Corollary 5.2.6** *Suppose $I=(t_1,t_2)\subset\mathbb{R}$ is an open interval and $u$ a $L_0$-solution on $[t_1,t_2)\times\overline{H}$ to $f\in L^2\big(I;L^2(H,\mu_1)\big)$ with $g=0$. Further let*

$$T: L^2\big(I;L^2(H,\mu_1)\big)\ni f\;\mapsto\; x_n^{\,j}\,\partial_t^l\partial_x^\alpha\,u\;\in\;L^2\big(I;L^2(H,\mu_1)\big)\,.$$

*Then $T$ is a Calderón-Zygmund operator on $\big(I\times\overline{H},d_0\,,\mathcal{L}\times\mu_1\big)$ if and only if $(j,l,\alpha)\in\mathcal{CZ}$, i.e. if $T$ assigns $f$ to either*

$$\partial_t u\,,\quad D_x^2 u\,,\quad x_n\,D_x^3 u\quad\text{or}\quad x_n^{\,2}\,D_x^4 u\,.$$

Hence we can apply the theory of Muckenhoupt weights to formulate the following result.

**Proposition 5.2.7 (weighted $L^p$-estimate)** *Let $I=(t_1,t_2)\subset\mathbb{R}$ be open, $f\in L^2\big(I;L^2(H,\mu_1)\big)$ and $p\in(1,\infty)$. If $u$ is a $L_0$-solution on $[t_1,t_2)\times\overline{H}$ to $f$ with $g=0$, then*

$$\int_I \|x_n^{\,\sigma}\partial_t\,u\|^p_{L^p(H)}+\|x_n^{\,\sigma}D_x^2u\|^p_{L^p(H)}+\|x_n^{\,\sigma+1}D_x^3u\|^p_{L^p(H)}+\|x_n^{\,\sigma+2}D_x^4u\|^p_{L^p(H)}\,d\mathcal{L}\lesssim\int_I\|x_n^{\,\sigma}f\|^p_{L^p(H)}\,d\mathcal{L}$$

*for all $-\frac{1}{p}<\sigma<2-\frac{1}{p}$. In particular, this holds true for $\sigma\in[0,1]$.*

**Proof:** According to lemma 3.5.17, it is $x_n^{\,\sigma p-1}\in A_p(\mu_1)$ if and only if $-1<\sigma p<2p-1$. But then the statement follows from corollary 5.2.6 in conjunction with theorem A.15. ■

### 5.2.3 Setting and Main Results

The first step consists in defining appropriate function spaces. Throughout this section we assume that the initial value $g$ is contained in the homogeneous Lipschitz space $\dot{C}^{0,1}(H)$, that is $\|g\|_{\dot{C}^{0,1}(H)} = \|\nabla g\|_{L^\infty(H)} < \infty$. By proposition 4.3.13 this is a natural bound on the solution of the homogeneous initial value problem and hence motivates the following definition.

**Definition 5.2.8 (function spaces)** *Let $I = (t_1, t_2)$ for some $t_1 < t_2 \leq \infty$ be an open interval in $\mathbb{R}$, $Q_R(x) := \big(t_1 + \frac{R^4}{2}, t_1 + R^4\big] \times B_R(x)$ and $p \in [1, \infty)$. Let $X_p$ denote the space of all functions with finite norm*

$$\|u\|_{X_p} \;:=\; \|\nabla u\|_{L^\infty(I\times H)} \;+\; \|u\|_{X_p^1}\,,$$

*where*

$$\|u\|_{X_p^1} \;:=\; \sup_{\substack{R^4\in(0,t_2-t_1)\\ x\in H}} \big|Q_R(x)\big|^{-\frac{1}{p}} \sum_{(j,l,\alpha)\in\mathcal{CZ}} R^{\,4l+|\alpha|-1}\,\big(R+\sqrt{x_n}\,\big)^{|\alpha|-2j-1}\,\|\partial_t^l\partial_x^\alpha\, u\|_{L^p(Q_R(x),\mu_{jp})}\,.$$

*By $B_\varepsilon^X := \{u \in X_p \mid \|u\|_{X_p} \leq \varepsilon\}$ we denote an $\varepsilon$-ball in $X_p$. The function space $Y_p$ is defined by*

$$Y_p \;:=\; \Big\{f \;\Big|\; \|f\|_{Y_p} \;:=\; \sup_{\substack{R^4\in(0,t_2-t_1)\\ x\in H}} \big|Q_R(x)\big|^{-\frac{1}{p}}\,R^3\,\big(R+\sqrt{x_n}\,\big)^{-1}\,\|f\|_{L^p(Q_R(x))} \;<\; \infty\Big\}\,.$$

Note that $X_p$ and $Y_p$ are Banach spaces because they are constructed as the intersection of complete function spaces.

We see at once that the $X_p$-norm is bounded by the homogeneous Lipschitz-norm of the initial datum. This observation follows directly from corollary 4.3.14.

**Lemma 5.2.9** *Given an initial datum $g \in \dot{C}^{0,1}(H)$, let $u$ be the solution of (5.2.2) on $[t_1, t_2) \times \overline{H}$. Then we have*

$$\|u\|_{X_p} \;\leq\; c(n,p)\,\|g\|_{\dot{C}^{0,1}(H)}$$

*for any $p \in [1, \infty)$.*

It turns out that the scaling behavior of a solution has a significant impact on the upcoming analysis. If $u$ is a $L_0$-solution to $f$ with initial condition $u(t_1) = g$, then the rescaled function $u\circ T_\lambda$ is one to $\lambda^2(f\circ T_\lambda)$, where by $T_\lambda$ we denote the coordinate transformation from (3.4.6) under which solutions are invariant. Moreover, both the solution itself as well as the initial datum exhibit the same scaling behavior in their respective norms, i.e.

$$\lambda\,\|u\|_{X_p} \;\sim\; \|u\circ T_\lambda\|_{X_p} \qquad \text{and} \qquad \lambda\,\|\nabla_{\hat{x}}\, g\|_{L^\infty(H)} \;=\; \|\nabla_x\, g(\lambda\,\cdot)\|_{L^\infty(H)}\,.$$

As opposed to this, the scaling of the $Y_p$-norm is characterized by the estimate

$$\|f\circ T_\lambda\|_{Y_p} \;\lesssim\; \lambda^{-1}\,\|f\|_{Y_p}\,. \tag{5.2.3}$$

(A proof is provided in lemma 5.2.18 in the proof section 5.2.5 hereafter). Our goal now is to show that

$$\|u\circ T_\lambda\|_{X_p} \;\lesssim\; \lambda^2\,\|f\circ T_\lambda\|_{Y_p}\,,$$

which in return allows us to conclude the following crucial result.

**Proposition 5.2.10** *Suppose $I = (t_1, t_2) \subset \mathbb{R}$ for some $t_1 < t_2 \leq \infty$ and $f \in L^2\big(I; L^2(H, \mu_1)\big)$. Further let $u$ be a $L_0$-solution on $[t_1, t_2) \times \overline{H}$ to $f$ with $u(t_1) = 0$. Then there exists a positive constant $c = c(n, p)$ such that*

$$\|u\|_{X_p} \;\leq\; c\,\|f\|_{Y_p}$$

*for any $p > n + 2$.*

From the definition of the function spaces $X_p$ and $Y_p$, and the special structure of $f[u]$ we get the next lemma.

**Lemma 5.2.11** *Let $1 \leq p < \infty$, $I = (t_1, t_2) \subset \mathbb{R}$ be an open interval with $t_1 > -\infty$, $\varepsilon < \frac{1}{2}$ and $f : X_p \to Y_p$ be defined as in lemma 3.3.1. Then the operator $f : B_\varepsilon^X \to Y_p$ is analytic and we have the estimates*

$$\|f[u]\|_{Y_p} \;\leq\; c\,\big(\,\|u\|_{X_p}^2 \,+\, \|u\|_{X_p}^3\,\big)$$

*for all $u \in B_\varepsilon^X$ and*

$$\|f[u_1] - f[u_2]\|_{Y_p} \;\leq\; c\,\big(\|u_1\|_{X_p} + \|u_2\|_{X_p}\big)\,\|u_1 - u_2\|_{X_p}$$

*for all $u_1, u_2 \in B_\varepsilon^X$, where the constant $c$ depends only on $n$ and $p$.*

Now we combine the results from lemma 5.2.9, proposition 5.2.10 and lemma 5.2.11 to prove the main theorem of this thesis.

**Theorem 5.2.12** *Let $t_1 > -\infty$, $I = (t_1, t_2)$ and $p > n + 2$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that for every $g \in \dot{C}^{0,1}(H)$ satisfying $\|g\|_{\dot{C}^{0,1}(H)} < \varepsilon_1$ there exists a unique solution $u^* \in B_{\varepsilon_2}^X$ of (5.2.1) for which*

$$\|u^*\|_{X_p} \;\leq\; c(n, p)\,\|g\|_{\dot{C}^{0,1}(H)}$$

*holds.*

**Proof:** Let $S : g \mapsto u$ denote the solution operator for (5.2.2) and $\Psi : f \mapsto u$ the parametrix for the inhomogeneous equation with zero Cauchy data. Furthermore, for every $g \in \dot{C}^{0,1}(H)$ define the operator $F_g : X_p \to X_p$ by

$$F_g(u) \;:=\; F(g, u) \;:=\; Sg \,+\, \Psi f[u]\,,$$

where $f[u]$ is given by lemma 3.3.1. With $\tilde{u} := F_g(u)$, we then have

$$\partial_t \tilde{u} \,+\, L_0 \tilde{u} \;=\; f[u]\,, \quad \tilde{u}(t_1) \;=\; g\,. \tag{$*$}$$

Via the results 5.2.9 – 5.2.11 we achieve

$$\|F_g(u)\|_{X_p} \;\lesssim\; \|g\|_{\dot{C}^{0,1}(H)} \,+\, \|u\|_{X_p}^2 \,+\, \|u\|_{X_p}^3 \qquad \forall\, u \in X_p\,,$$

that is, $F_g$ is bounded in $B_\varepsilon^X$ for any $\varepsilon \in (0, \frac{1}{2})$. Using the second inequality in lemma 5.2.11 we see that $F_g$ is a contraction map within $B_{\varepsilon_2}^X$ provided $\|g\|_{\dot{C}^{0,1}(H)}$ and $\varepsilon_2$ are chosen sufficiently small. Indeed,

$$\|F_g(u_1) - F_g(u_2)\|_{X_p} \;\leq\; c_L\,\|u_1 - u_2\|_{X_p}$$

for all $u_1, u_2 \in B_{\varepsilon_2}^X$ and some $c_L \in (0, 1)$. Then by the Banach fixed point theorem, $F_g$ has a unique fixed point $u^* \in B_{\varepsilon_2}^X$ that depends Lipschitz continuously on the initial condition. In view of ($*$), this turns out to be the unique global solution of (5.2.1) we were looking for. ∎

We call the unique solution $u^*$, obtained in theorem 5.2.12 by variation of constants, a *mild solution.*

Using an argument introduced by Angenent [2], and later improved by Koch and Lamm [54], we show next that the unique solution $u^*$ obtained in theorem 5.2.12 is analytic in temporal and all tangential directions.

**Proposition 5.2.13** *Let $u^*$ be the unique solution of (5.2.1) in $B^X_{\varepsilon_2}$. This solution depends analytically on the initial data $g \in \dot{C}^{0,1}(H)$. Moreover, $u^*$ is analytic in temporal and all tangential directions, and there exists a number $R > 0$ such that for any $l \in \mathbb{N}_0$ and for any $\alpha' \in {\mathbb{N}_0}^{n-1}$ the estimate*

$$\sup_{t\in I}\sup_{x\in H}\left|(t-t_1)^{l+\frac{1}{2}|\alpha'|}\,\partial_t^l\partial_x^{\alpha'}\nabla_x u^*(t,x)\right| \;\le\; c\,R^{-l-|\alpha'|}\,l!\,\alpha'!\,\|g\|_{\dot{C}^{0,1}(H)} \tag{5.2.4}$$

*holds with a constant $c > 0$ depending only on $n$ and $R$.*

In order to obtain analyticity in the $x_n$-direction, a different approach is needed. For the related equation

$$\partial_t u \,+\, Lu \;=\; \partial_t u \,-\, x_n^{-\sigma}\,\nabla\cdot\left(x_n^{\sigma+1}\,\nabla u\right) \;=\; f[u] \quad \text{on} \quad [t_1,t_2)\times\overline{H}$$

with $u(t_1) = g$ and

$$f[u] \;=\; -\,x_n^{-\sigma}\,\partial_{x_n}\left(x_n^{\sigma+1}\,\frac{|\nabla u|^2}{1+\partial_{x_n}u}\right),$$

this was proven by Koch [51]. Since $L_0 = LL$, cf. (3.4.1), there is good reason to believe that solutions of (5.2.1) are still analytic in space up to the boundary of its support. This is stated in the following conjecture.

**Conjecture 5.2.14** *If $u^*$ is an energy solution of (5.2.1), then this solution is analytic in time and space for all $t \in I$ and $x \in H$. Moreover, estimate (5.2.4) remains valid for every $l \in \mathbb{N}_0$ and for every multi-index $\alpha \in {\mathbb{N}_0}^n$.*

### 5.2.4 Conclusion

In order to conclude this work, we reformulate the above results for the thin-film equation (TFE) on $I\times\mathbb{R}^n$. We will show that given any initial datum near the stationary solution $(y_n)_+^2$, there exists a unique weak solution $h$ satisfying the equation $\partial_s h + \nabla_y\cdot(h\,\nabla_y\Delta_y h) = 0$ in the following sense:

$$\int_I\int_{\mathbb{R}^n} h\,\partial_s\varphi \,+\, h\,\nabla\Delta h\cdot\nabla\varphi\,dyds \;=\; 0 \tag{5.2.5}$$

for every $\varphi \in C_c^\infty(I\times\mathbb{R}^n)$.

Next we introduce a new expression which corresponds to the $X_p^1$-norm (see definition 5.2.8) under the transformation $(t,x)\mapsto(s,y)$. Let $\phi(X_p^1)$ denote the set of all functions $h : I\times\mathbb{R}^n \to \mathbb{R}$ for which

$$[h]_{X_p^1} \;:=\; \sup_{\substack{R^4\in(0,t_2-t_1)\\ y\in P_s(h)}} \left|Q_R(y)\right|^{-\frac{1}{p}} \sum_{(j,l,\alpha)\in\mathcal{CZ}} R^{4l+|\alpha|-1}\left(R+\sqrt{h(0,y)}\right)^{|\alpha|-2j-1}\|h^j\,\partial_s^l\partial_y^\alpha\,h\|_{L^p(Q_R(y))}$$

is finite. Here

$$P_s(h) \;=\; P_{t_1+R^4}(h) \;=\; \left\{y\in\mathbb{R}^n \;\middle|\; h(s,y)>0\right\},$$

and $\phi : x \mapsto \left(x', v(x)\right) := y$ is the inverted transformation that has been applied in section 3.1 to motivate the consideration of the transformed equation on $I\times H$. We will see below (lemma 5.2.19) that $\phi$ defines a bijection, or rather a quasi-isometry, through $(t,H)\mapsto spt\,h(t)$.

**Theorem 5.2.15** *Let $I = (t_1, t_2) \subset \mathbb{R}$ be an open interval and $\varepsilon > 0$ small. Given a nonnegative initial datum $h(t_1) = h_0$ with*

$$\left|\nabla_y \sqrt{h_0(y)} - e_n\right| \;<\; \varepsilon\,,$$

*there exist a constant $c > 0$ and a unique weak solution $h^* \in C(I \times \mathbb{R}^n)$ of $\partial_s h + \nabla_y \cdot (h\, \nabla_y \Delta_y h) = 0$ with initial value $h_0$,*

$$\|\nabla_y \sqrt{h^*} - e_n\|_{L^\infty(P(h))} \;+\; \left[\sqrt{h^*}\,\right]_{X_p^1} \;\leq\; c\,\varepsilon\,,$$

*and $h^*$ satisfies the equation in the sense of identity (5.2.5).*

Moreover, the level sets at a fixed level $\lambda$ are analytic. To see this, we fix $x_n = \lambda \geq 0$ and note that

$$graph\left(\lambda + u^*(t, x', \lambda)\right) \;=\; \left\{(s, y) \mid h^*(s, y) = \lambda^2\right\}.$$

Now the analyticity of $u^*$ (see proposition 5.2.13) immediately implies the following result.

**Corollary 5.2.16** *The level sets of $h^*$ are analytic.*

### 5.2.5 The Proofs

**Proof (of lemma 5.2.2):** From Duhamel's principle we infer that for all $(t, x) \in (0, 1] \times \overline{H}$,

$$x_n^{\,j}\, \partial_x^\alpha\, u(t, x) \;=\; \int_0^t \int_H x_n^{\,j}\, \partial_x^\alpha\, G(t, x, s, y)\, f(s, y)\, dy\, ds\,,$$

where $G$ is the Green function on $(0, 1) \times \overline{H}$. Since $2j \leq |\alpha| < j + 2$, we can apply lemma 5.1.6 to find

$$\|x_n^{\,j}\, \partial_x^\alpha\, G(t, x, \cdot, \cdot)\|_{L^1((0,t)\times H)} \;\lesssim\; \left(1 + \sqrt{x_n}\,\right)^{2j-|\alpha|} \;\leq\; 1$$

for all $t \in (0, 1]$ and almost all $x \in \overline{H}$, and by remark 5.1.7,

$$\int_s^1 \|\partial_x^\alpha\, G(t, \cdot, s, y)\|_{L^1(H,\mu_j)}\, dt \;\lesssim\; \left(1 + \sqrt{y_n}\,\right)^{2j-|\alpha|} \;\leq\; 1$$

for all $s \in [0, 1)$ and almost all $y \in \overline{H}$. This, however, verifies the assumptions of Schur's lemma (see section 5.2) for $K(t, x, s, y) = x_n^{\,j}\, \partial_x^\alpha\, G(t, x, s, y)$, and hence the assertion. ■

Let us now prove the two kernel estimates.

**Proof (of proposition 5.2.3):** First we fix some $t > s \in \bar{I}$. Further let $j \geq 0$, $l \in \mathbb{N}_0$ and $\alpha$ be a multi-index. The Gaussian estimate (ge) then gives

$$\begin{aligned} y_n^{-1}\, x_n^{\,j}\, \left|\partial_t^l \partial_x^\alpha\, G(t, x, s, y)\right| \;&\lesssim\; \sqrt[4]{t-s}^{\,-4l-|\alpha|}\, \left(\sqrt[4]{t-s} + \sqrt{x_n}\,\right)^{2j-|\alpha|}\, \left|B_{\sqrt[4]{t-s}}(x)\right|_1^{-1}\, e^{-c_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \\ &\lesssim\; \sqrt[4]{t-s}^{\,2j-4l-2|\alpha|}\, \left|B_{\sqrt[4]{t-s}}(x)\right|_1^{-1}\, e^{-c_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}}\,, \end{aligned}$$

where the second line only holds if $|\alpha| \geq 2j$. In order to exchange the ball center $x$ by $y$ we now apply lemma 3.5.13, that is, we get

$$\left|B_{\sqrt[4]{t-s}}(x)\right|_1^{-1} \;\lesssim\; \left(\left|B_{\sqrt[4]{t-s}}(x)\right|_1 + \left|B_{\sqrt[4]{t-s}}(y)\right|_1\right)^{-1} \left(1 + \frac{d(x, y)}{\sqrt[4]{t-s}}\right)^{2n+2}.$$

The doubling property in corollary 3.5.12 then allows us to replace the ball radius by $d_0 = \lambda \sqrt[4]{t-s}$, with $1 \leq \lambda = \sqrt[4]{1 + \frac{d(x,y)^4}{t-s}} \leq 1 + \frac{d(x,y)}{\sqrt[4]{t-s}}$, such that $\left|B_{d_0}(\cdot)\right|_1 \lesssim \lambda^{2n+2} \left|B_{\sqrt[4]{t-s}}(\cdot)\right|_1$. Thus, due to the exponential decay of the Gaussian function, we discover that

$$\left|B_{\sqrt[4]{t-s}}(x)\right|_1^{-1} e^{-c_n^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \lesssim \left(\left|B_{d_0}(x)\right|_1 + \left|B_{d_0}(y)\right|_1\right)^{-1} e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}}.$$

Moreover, since $R^{-m}\big(R+d(x,y)\big)^m e^{-\varepsilon\left(\frac{d(x,y)}{R}\right)^{\frac{4}{3}}} \leq c(m)$ for all $m \geq 0$ and all $\varepsilon > 0$, we have

$$\sqrt[4]{t-s}^{\,2j-4l-2|\alpha|} e^{-(2c_n)^{-1}\left(\frac{d(x,y)^4}{t-s}\right)^{\frac{1}{3}}} \lesssim \left(\sqrt[4]{t-s}+d(x,y)\right)^{2j-4l-2|\alpha|} \leq d_0^{2j-4l-2|\alpha|} = d_0^{-4}$$

if $\frac{m}{2} := 2l + |\alpha| - j = 2$. Combining all these estimates leads to

$$\left|K(t,x,s,y)\right| \lesssim d_0^{-4}\left(\left|B_{d_0}(x)\right|_1 + \left|B_{d_0}(y)\right|_1\right)^{-1} = \left(\left|Q_{d_0}(t,x)\right|_1 + \left|Q_{d_0}(s,y)\right|_1\right)^{-1} \lesssim V(t,x,s,y)^{-1}.$$

This corresponds to the desired estimate since both conditions, $|\alpha| \geq 2j$ and $2l + |\alpha| - j = 2$, are satisfied if and only if $(j,l,\alpha) \in \mathcal{CZ}$ (see definition 5.2.5). ∎

**Proof (of proposition 5.2.4):** Let $d_0 = \sqrt[4]{|t-s| + d(x,y)^4}$ and $\bar{d}_0 = \sqrt[4]{|\bar{t}-\bar{s}| + d(\bar{x},\bar{y})^4}$. With $K(t,x,s,y)$ as in proposition 5.2.3, we have

$$\begin{aligned}\left|K(t,x,s,y) - K(\bar{t},\bar{x},\bar{s},\bar{y})\right| &\leq \left|K(t,x,s,y) - K(\bar{t},x,s,y)\right| + \left|K(\bar{t},x,s,y) - K(\bar{t},\bar{x},s,y)\right| + \\ &+ \left|K(\bar{t},\bar{x},s,y) - K(\bar{t},\bar{x},\bar{s},y)\right| + \left|K(\bar{t},\bar{x},\bar{s},y) - K(\bar{t},\bar{x},\bar{s},\bar{y})\right| =: (I) + (II) + (III) + (IV).\end{aligned}$$

If $t \leq \bar{t}$, then

$$(I) = \left|\int_t^{\bar{t}} \partial_\tau K(\tau)\, d\tau\right| \leq |t-\bar{t}| \sup_{\tau\in(t,\bar{t})} \left|\partial_\tau K(\tau)\right|.$$

Applying the Gaussian estimate (ge) to $\partial_\tau K(\tau) = y_n\, x_n^j\, \partial_\tau^{l+1}\partial_x^\alpha\, G(\tau,x,s,y)$ yields the pointwise estimate

$$\left|\partial_\tau K(\tau)\right| \lesssim \left(|\tau-s| + d(x,y)^4\right)^{-1} V(\tau,x,s,y)^{-1},$$

if $(j,l,\alpha) \in \mathcal{CZ}$. The calculation here is essentially the same as in the preceding proof. All this amounts to

$$(I) \lesssim \frac{\sqrt[4]{|t-\bar{t}|}}{d_0} V(t,x,s,y)^{-1} \leq \frac{D}{V(t,x,s,y)},$$

since also $d_0 \sim \bar{d}_0$ and $|t-\bar{t}| \lesssim d_0^4$, which both follows from the assumption that $D$ is small. If $t > \bar{t}$, we additionally require that $d_0 \lesssim \sqrt[4]{|\bar{t}-s| + d(x,y)^4}$.

Now suppose $\gamma : [a,b] \to \overline{H}$ is the geodesic between $x$ and $\bar{x}$, i.e. $\gamma(a) = x$, $\gamma(b) = \bar{x}$ and the length of $\gamma$ is $d(x,\bar{x})$. Then by the fundamental theorem of calculus, followed by definition 3.5.1, $(II)$ equals

$$\left|\int_a^b \nabla_{\gamma(\tau)} K\big(\gamma(\tau)\big)\, \gamma'(\tau)\, d\tau\right| \leq d(x,\bar{x}) \sup_{z\in(\gamma)} \sqrt{z_n}\, \left|\nabla_z K(z)\right|.$$

As above, we get

$$\sqrt{z_n}\, \left|\nabla_z K(z)\right| \lesssim |\bar{t}-s|^{-\frac{5}{4}} \left|B_{\sqrt{\bar{t}-s}}(y)\right|_1^{-1} e^{-\varepsilon\left(\frac{d(z,y)^4}{\bar{t}-s}\right)^{\frac{1}{3}}}.$$

If $z \in \overline{B}_{d(x,\bar{x})}(x)$, then

$$d(x,y) \lesssim \sqrt[4]{|\bar{t}-s| + d(z,y)^4}.$$

Now note that $\gamma \subset \overline{B}_{d(x,\bar{x})}(x)$, and consequently it appears that

$$\left|K(x) - K(\bar{x})\right| = \left|K(\bar{t},x,s,y) - K(\bar{t},\bar{x},s,y)\right| \lesssim \frac{d(x,\bar{x})}{\sqrt[4]{|\bar{t}-s| + d(x,y)^4}} V(\bar{t},x,s,y)^{-1}.$$

Eventually, the assumption on $D$ implies that $d_0 \lesssim \sqrt[4]{|\bar{t}-s| + d(x,y)^4}$, and hence we arrive at the estimate

$$(II) \lesssim \frac{d(x,\bar{x})}{d_0 + \bar{d}_0} \, V(t,x,s,y)^{-1} .$$

Similarly,

$$(III) \lesssim |s-\bar{s}| \sup_{\tau \in (s,\bar{s})} \big(|\bar{t}-\tau| + d(x,y)^4\big)^{-1} \, V(\bar{t},x,\tau,y)^{-1}$$

which follows from the Gaussian estimate 5.1.4 and $d(x,y) \lesssim \sqrt[4]{|\bar{t}-\tau| + d(\bar{x},y)^4}$ for $D \le \frac{1}{10}$ and $\tau \in [s,\bar{s}]$. Employing the condition on $D$ once more, we also find $d_0 \lesssim \sqrt[4]{\bar{t} - \min\{s,\bar{s}\} + d(x,y)^4}$ which gives the estimate the required form.

Only an estimate for $(IV)$ is left for which we choose a length-minimizing curve $\gamma$ from $y$ to $\bar{y}$ such that

$$(IV) \lesssim d(y,\bar{y}) \sup_{z \in (\gamma)} \sqrt{z_n} \, \big|\nabla_z K(z)\big| .$$

We know that $\gamma$ is contained in the set $\overline{B}_{d(y,\bar{y})}(y) \cap \overline{B}_{d(y,\bar{y})}(\bar{y})$ and in particular, if $z \in (\gamma)$, we have that

$$d(x,z) \lesssim \sqrt[4]{\bar{t} - \bar{s} + d(\bar{x},z)^4} \qquad \text{and} \qquad d(x,y) \lesssim \sqrt[4]{\bar{t} - \bar{s} + d(x,z)^4} ,$$

each of which involves a series of calculations using $D \le \frac{1}{10}$. By lemma 5.1.4 applied to $K(z)$, we therefore get

$$(IV) \lesssim \frac{d(y,\bar{y})}{\sqrt[4]{\bar{t} - \bar{s} + d(x,y)^4}} \, V(\bar{t},x,\bar{s},y)^{-1} ,$$

and the statement follows with $d_0 \lesssim \sqrt[4]{\bar{t} - \bar{s} + d(x,y)^4}$. ■

For the proof of proposition 5.2.10 we distinguish between the situation on and off the diagonal. The following lemma deals with the off-diagonal part.

**Lemma 5.2.17** *Let $x_0 \in \overline{H}$, $\delta \in [0,\frac{1}{2}]$ and $\rho \ge 1$ be fixed, and $f \in L^2\big((0,1); L^2(H,\mu_1)\big)$ with $spt\, f \subseteq \big([0,1] \times \overline{H}\big) \setminus \big((\delta,1] \times B_{2\rho}(x_0)\big)$. Suppose further that $j \ge 0$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0{}^n$ and $u$ is a $L_0$-solution to $f$ on $[0,1) \times \overline{H}$ with initial condition $u(0) = 0$. Then, if $|\alpha| \ge 2j$, we have*

$$x_n^{\,j} \, \big|\partial_t^l \partial_x^\alpha \, u(t,x)\big| \le c(n,j,l,\alpha,\delta,\rho) \, \big(1 + \sqrt{x_{0,n}}\big)^{2j+1-|\alpha|} \, \|f\|_{Y_p}$$

*for all $(t,x) \in (2\delta,1] \times B_\rho(x_0) \subseteq (0,1] \times \overline{H}$ and all $p \ge 1$.*

**Proof:** Suppressing the parameters $\delta$ and $\rho$, we let $Q(x_0) := (\delta,t] \times B_{2\rho}(x_0)$ be the cylinder for which $Q(x_0) \cap spt\, f = \emptyset$. Using Duhamel's formula (proposition 5.2.1) we get

$$x_n^{\,j} \, \big|\partial_t^l \partial_x^\alpha \, u(t,x)\big| \le \int\limits_{\big((0,t) \times H\big) \setminus Q(x_0)} x_n^{\,j} \, \big|\partial_t^l \partial_x^\alpha \, G(t,x,s,y)\big| \, \big|f(s,y)\big| \, dyds .$$

Now, since $x \in B_\rho(x_0)$ implies $B_\rho(x) \subset B_{2\rho}(x_0)$ we find

$$\big((0,t] \times H\big) \setminus Q(x_0) \subset \big((0,t] \times H\big) \setminus \big((\delta,t] \times B_\rho(x)\big) ,$$

and an application of lemma 5.1.5 under the integral is possible. This yields

$$x_n^{\,j} \, \big|\partial_t^l \partial_x^\alpha \, u(t,x)\big| \lesssim \int\limits_{(0,1) \times H} \big(1 + \sqrt{y_n}\big)^{2j+1-|\alpha|} \, \big(1 + \sqrt{y_n}\big)^{-1} \, \big|B_1(y)\big|^{-1} \, e^{-\frac{d(x,y)}{4\,c_n}} \, \big|f(s,y)\big| \, dyds$$

for all $(t,x) \in (2\delta,1] \times B_\rho(x_0)$. Thanks to the exponential decay in $d(x,y)$ we may replace $y_n$ by $x_n$ in the first factor, regardless of the sign of $2j+1-|\alpha|$. Hence we obtain the upper bound

$$\big(1+\sqrt{x_n}\big)^{2j+1-|\alpha|} \int\limits_{(0,1)\times H} \big(1+\sqrt{y_n}\big)^{-1} \big|B_1(y)\big|^{-1} e^{-\frac{d(x,y)}{8\,c_n}} \, \big|f(s,y)\big| \, dyds\,.$$

From lemma 3.5.13 it follows that

$$\big(1+\sqrt{x_n}\big)^{\gamma} \;\lesssim\; \big(1+d(x,x_0)\big)^{2|\gamma|} \big(1+\sqrt{x_{0,n}}\big)^{\gamma} \;<\; \big(1+\rho\big)^{2|\gamma|} \big(1+\sqrt{x_{0,n}}\big)^{\gamma}$$

for all $x \in B_\rho(x_0)$, and hence it remains to estimate the integral. For this we first cover $\overline{H}$ by countably many balls $B_1(y_0)$. By the triangle inequality $d(x,y) > d(x,y_0) - 1$ for all $y \in B_1(y_0)$. Then the above integral is (up to a constant) bounded by

$$\begin{aligned} &\sum_{y_0} e^{-\frac{d(x,y_0)}{8\,c_n}} \int_0^1 \int_{B_1(y_0)} \big(1+\sqrt{y_n}\big)^{-1} \big|B_1(y)\big|^{-1} \big|f(s,y)\big| \, dyds \\ \le\; &\left(\sup_{y_0} \int_0^1 \int_{B_1(y_0)} \big(1+\sqrt{y_n}\big)^{-1} \big|B_1(y)\big|^{-1} \big|f(s,y)\big| \, dyds\right) \sum_{y_0} e^{-\frac{d(x,y_0)}{8\,c_n}}\,, \end{aligned}$$

where the series is uniformly convergent in $x$ since

$$\sum_{y_0} e^{-\frac{d(x,y_0)}{8\,c_n}} \;\le\; \sum_{k\in\mathbb{N}} \sum_{y_0 \in B_k(x)} e^{-\frac{k-1}{8\,c_n}} \;=\; e^{\frac{1}{8\,c_n}} \sum_{k\in\mathbb{N}} e^{-\frac{k}{8\,c_n}} \,\#\big\{y_0 \mid y_0 \in B_k(x)\big\}^7\,,$$

and the number of lattice points in a ball grows at most polynomially in $k$. As for the time interval $(0,1]$, we choose the cover $(\frac{1}{2}R_m^4\,, R_m^4]$, $m \in N_0$, where $R_m := 2^{-\frac{m}{4}}$. In the next step we want to use Vitali's covering lemma (see lemma A.1), which tells us that, for every $m \in \mathbb{N}_0$, there exists a number $N = N(m) \in \mathbb{N}$ such that $\{B_{\frac{1}{3}R_m}(z_i)\}_{i=1}^N$ is a disjoint collection of balls in $B_1(y_0)$ and we have

$$\bigcup_{i=1}^N B_{R_m}(z_i) \;\supset\; B_1(y_0)\,.$$

Using the fact that $\mu_0 = \mathcal{L}^n$ satisfies the doubling condition with respect to the intrinsic balls we also get

$$\sum_{i=1}^N \big|B_{R_m}(z_i)\big| \;\le\; c \sum_{i=1}^N \big|B_{\frac{1}{3}R_m}(z_i)\big| \;=\; c\,\Big|\bigcup_{i=1}^N B_{\frac{1}{3}R_m}(z_i)\Big| \;\le\; c\,\big|B_1(y_0)\big|\,, \tag{$*$}$$

and the constant $c$ is independent of $m$. The preceding integral then reads as

$$\sup_{y_0} \sum_{m\in\mathbb{N}_0} \sum_{i=1}^N \int_{Q_{R_m}(z_i)} \big(1+\sqrt{y_n}\big)^{-1} \big|B_1(y)\big|^{-1} \big|f(s,y)\big| \, dyds\,.$$

The idea now is to consider the boundary-case and the situation away from the boundary separately. To this end we write

$$\{y_0\} \;=\; \big\{\sqrt{y_{0,n}} \ll 1\big\} \cup \big\{1 \ll \sqrt{y_{0,n}}\,\big\} \;=:\; A \cup A'\,.$$

Clearly, we have

$$\sup_{y_0} \;\le\; \sup_A + \sup_{A'}\,.$$

We start by discussing the latter case, that is $y_0 \in A'$. If $y_{0,n}$ is sufficiently large in comparison with 1, we

[7] This is related to the *Gauss circle problem* that asks how many lattice points are inside a given ball of radius $k$. In the 2-dimensional Euclidean setting there are about $N(k) = \pi k^2 + \mathcal{O}(k^{0.5+\varepsilon})$, with $0 < \varepsilon \le 0.1298\ldots$, integer lattice points in $B_k(0)$. The lower limit 0 was obtained independently by Hardy and Landau in 1915, and the upper bound by Huxley [45].

already know that $y_{0,n} \sim y_n$ for all $y \in B_1(y_0)$ and hence, in particular, for $y = z_i$. Thus by transitivity,

$$\sum_{m\in\mathbb{N}_0} \sum_{i=1}^{N} \big|B_1(y_0)\big|^{-1} \big(R_m + \sqrt{z_{i,n}}\,\big)^{-1} \int_{Q_{R_m}(z_i)} \big|f(s,y)\big|\, dyds \tag{$**$}$$

serves as upper bound for the off-boundary part. Furthermore, by Hölder's inequality and $(*)$, we conclude

$$\begin{aligned} (**) \;&\le\; \big|B_1(y_0)\big|^{-1} \sum_{m\in\mathbb{N}_0} \sum_{i=1}^{N} \big(R_m + \sqrt{z_{i,n}}\,\big)^{-1} \big|Q_{R_m}(z_i)\big|^{\frac{p-1}{p}} \, \|f\|_{L^p(Q_{R_m}(z_i))} \\ &\le\; \big|B_1(y_0)\big|^{-1} \sum_{m\in\mathbb{N}_0} R_m \sum_{i=1}^{N} \big|B_{R_m}(z_i)\big| \, \|f\|_{Y_p} \;\lesssim\; \sum_{m\in\mathbb{N}_0} R_m \, \|f\|_{Y_p} \end{aligned}$$

for all $p \ge 1$.

Turning to the case $y_0 \in A$, we first apply the rather rough estimate $\big(1 + \sqrt{y_n}\,\big)^{-1} \big|B_1(y)\big|^{-1} \lesssim 1$ to find the boundary part to be less than

$$\begin{aligned} \sum_{m\in\mathbb{N}_0} \sum_{i=1}^{N} \int_{Q_{R_m}(z_i)} \big|f(s,y)\big|\, dyds \;&\le\; \sum_{m\in\mathbb{N}_0} \sum_{i=1}^{N} R_m \big(R_m + \sqrt{z_{i,n}}\,\big) \big|B_{R_m}(z_i)\big| \, \|f\|_{Y_p} \\ &\overset{(*)}{\lesssim}\; \big|B_1(y_0)\big| \sum_{m\in\mathbb{N}_0} R_m \, \|f\|_{Y_p} \,. \end{aligned}$$

In the last line we also used the estimate

$$R_m + \sqrt{z_{i,n}} \;\le\; 1 + \sqrt{z_{i,n}} \;\overset{(lemma\ 3.5.13)}{\lesssim}\; \big(1 + d(z_i, y_0)\big)^2 \big(1 + \sqrt{y_{0,n}}\,\big) \;\lesssim\; 1$$

for any $z_i \in B_1(y_0)$ and $y_{0,n} \lesssim 1$. Now, near the boundary we can always embed $B_1(y_0)$ into a ball centered on the boundary such that $\big|B_1(y_0)\big|$ is bounded by some finite number which does not depend on the location of $y_0$. Summation over $m \in \mathbb{N}_0$ is possible in any of the cases and so we have

$$x_n^{\,j} \,\big|\partial_t^l \partial_x^\alpha\, u(t,x)\big| \;\lesssim\; \big(1 + \sqrt{x_{0,n}}\,\big)^{2j+1-|\alpha|} \, \|f\|_{Y_p} \sum_{m\in\mathbb{N}_0} R_m$$

for any $(t,x) \in (2\delta, 1] \times B_\rho(x_0)$. This amounts to the assertion of the lemma. ∎

In order to allow the inhomogeneity to have a larger (or possibly smaller) time-support it requires a rescaling argument of the form (5.2.3). In the next step we formulate this scaling behavior (supplemented by a time shift) in an independent lemma.

**Lemma 5.2.18 (scaling of the $Y_p$-norm)** *Let $I = (t_1, t_2)$ be an open interval, spt $f \subseteq [t_1, t_2] \times \overline{H}$ and*

$$T :\, [0,1] \times \overline{H} \,\ni\, (t,x) \;\mapsto\; (t_1 + \lambda^2 t, \lambda x) \;=:\; (\hat{t}, \hat{x}) \,\in\, \bar{I} \times \overline{H}$$

*for $0 < \lambda \le \sqrt{t_2 - t_1}$. Then*

$$\lambda \, \|f \circ T\|_{Y_p} \;\le\; c_{n,p} \, \|f\|_{Y_p}$$

*for all $1 \le p \le \infty$.*

**Proof:** We fix a radius $0 < R \le 1$ and a point $x \in \overline{H}$. Applying the transformation formula, we find

$$\|f \circ T\|_{L^p((\frac{R^4}{2}, R^4] \times B_R(x))} \;=\; \lambda^{-\frac{n+2}{p}} \, \|f\|_{L^p(T(\left(\frac{R^4}{2}, R^4\right] \times B_R(x)))} \,.$$

(Note: $T = T_0 \circ T_\lambda$ with $T_0$ and $T_\lambda$ as in (3.4.5)–(3.4.6), and the Jacobian determinant satisfies $J_T = J_{T_\lambda}$ ).

In view of lemma 3.5.11, we obtain

$$T\Big(\Big(\frac{R^4}{2}, R^4\Big] \times B_R(x)\Big) \;\subset\; \Big(t_1 + \frac{(\sqrt{\lambda}\,R)^4}{2}, t_1 + (\sqrt{\lambda}\,R)^4\Big] \times B_{4\,c_d^2\sqrt{\lambda}\,R}(\lambda x)\,.$$

Now we cover $B_{4\,c_d^2\sqrt{\lambda}\,R}(\lambda x)$ by $N(n) \in \mathbb{N}$ balls of radius $\sqrt{\lambda}\,R$ centered at $\lambda z_i \in B_{4\,c_d^2\sqrt{\lambda}\,R}(\lambda x)$, and thus have

$$T\Big(\Big(\frac{R^4}{2}, R^4\Big] \times B_R(x)\Big) \;\subset\; \bigcup_{i=1}^{N(n)} Q_{\sqrt{\lambda}\,R}(\lambda z_i)\,.$$

This implies

$$\|f \circ T\|_{L^p((\frac{R^4}{2}, R^4] \times B_R(x))} \;\lesssim\; \lambda^{-\frac{n+2}{p}} \sum_{i=1}^{N(n)} \|f\|_{L^p(Q_{\sqrt{\lambda}\,R}(\lambda z_i))}\,.$$

Moreover,

$$\Big|\Big(\frac{R^4}{2}, R^4\Big] \times B_R(x)\Big|^{-\frac{1}{p}} \;\sim\; \lambda^{\frac{n+2}{p}} \left|Q_{\sqrt{\lambda}\,R}(\lambda x)\right|^{-\frac{1}{p}} \;\lesssim\; \lambda^{\frac{n+2}{p}} \left|Q_{\sqrt{\lambda}\,R}(\lambda z_i)\right|^{-\frac{1}{p}},$$

where the first estimate follows from lemma 3.5.15 while the second one is a consequence of the following: By the triangle inequality we get $B_{\sqrt{\lambda}\,R}(\lambda z_i) \subset B_{(1+4\,c_d^2)\sqrt{\lambda}\,R}(\lambda x)$ and hence, by the doubling property,

$$\left|B_{\sqrt{\lambda}\,R}(\lambda x)\right|^{-\frac{1}{p}} \;\lesssim\; (1 + 4\,c_d^2)^{\frac{2n}{p}} \left|B_{\sqrt{\lambda}\,R}(\lambda z_i)\right|^{-\frac{1}{p}}.$$

Finally,

$$R^3 \left(R + \sqrt{x_n}\,\right)^{-1} \;\lesssim\; \lambda^{-1} (\sqrt{\lambda}\,R)^3 \left(1 + \frac{d(\lambda x, \lambda z_i)}{\sqrt{\lambda}\,R}\right)^2 \left(\sqrt{\lambda}\,R + \sqrt{\lambda\,z_{i,n}}\,\right)^{-1}$$

by virtue of lemma 3.5.13. But regarding in which ball the $\lambda z_i$ lie gives $d(\lambda x, \lambda z_i) < 4\,c_d^2\sqrt{\lambda}\,R$.

We arrive at

$$\begin{aligned} &\lambda \left|Q_R(x)\right|^{-\frac{1}{p}} R^3 \left(R + \sqrt{x_n}\,\right)^{-1} \|f \circ T\|_{L^p((\frac{R^4}{2}, R^4] \times B_R(x))} \\ \lesssim\; &\sum_{i=1}^{N(n)} \left|Q_{\sqrt{\lambda}\,R}(\lambda z_i)\right|^{-\frac{1}{p}} (\sqrt{\lambda}\,R)^3 \left(\sqrt{\lambda}\,R + \sqrt{\lambda\,z_{i,n}}\,\right)^{-1} \|f\|_{L^p(Q_{\sqrt{\lambda}\,R}(\lambda z_i))} \end{aligned}$$

for all $p \geq 1$. Taking the supremum over $R \in (0,1]$ and $x \in \overline{H}$ then leads to

$$\lambda\, \|f \circ T\|_{Y_p} \;\lesssim\; N(n) \sup_{0<R^4<\lambda^2}\; \sup_{z_i \in H} \left|Q_R(z_i)\right|^{-\frac{1}{p}} R^3 \left(R + \sqrt{z_{i,n}}\,\right)^{-1} \|f\|_{L^p(Q_R(z_i))}\,,$$

from which the right scaling follows since $\lambda \leq \sqrt{t_2 - t_1}\,$. ■

Proposition 5.2.7, and the lemmas 5.2.17 and 5.2.18 are the key components for the next proof.

**Proof (of proposition 5.2.10):** We first consider the case that $spt\, f \subseteq [0,1] \times \overline{H}$ and write

$$f \;=\; \chi_{Q(x_0)}\, f \;+\; \big(1 - \chi_{Q(x_0)}\big)\, f \;=:\; f_1 \;+\; f_2\,, \qquad \text{where} \quad Q(x_0) \;:=\; (\frac{1}{4}, 1] \times B_2(x_0)$$

for any fixed $x_0 \in \overline{H}$. This decomposition in turn splits $u$ into a sum of $u_1$ and $u_2$ with $u_1$ being a solution to $f_1$ while $u_2$ is one to inhomogeneity $f_2$. Moreover, we have

$$\|\partial_t^l \partial_x^\alpha\, u\|_{L^p(Q_1(x_0),\, \mu_{jp})} \;\leq\; \|\partial_t^l \partial_x^\alpha\, u_1\|_{L^p(Q_1(x_0),\, \mu_{jp})} \;+\; \|\partial_t^l \partial_x^\alpha\, u_2\|_{L^p(Q_1(x_0),\, \mu_{jp})} \;=:\; (I) \;+\; (II)\,.$$

Now we estimate each term separately and start with $(I)$. From the global $L^p$-estimates, proposition 5.2.7 with $\sigma = 0$, it follows immediately that

$$\|\partial_t^l \partial_x^\alpha\, u_1\|_{L^p(Q_1(x_0),\, \mu_{jp})} \;\lesssim\; \|f_1\|_{L^p((0,1) \times H)} \;=\; \|f\|_{L^p(Q(x_0))} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\,\right) \left|Q_1(x_0)\right|^{\frac{1}{p}} \|f\|_{Y_p}$$

for all $p \in (1,\infty)$. For the first estimate we require $j, l$ and $\alpha$ to be admissible, that is such that $(j,l,\alpha) \in \mathcal{CZ}$. In these cases $f_1 \mapsto x_n^j\, \partial_t^l \partial_x^\alpha\, u_1$ is a Calderón-Zygmund operator and proposition 5.2.7 can be applied. In the second estimate we cover $Q(x_0)$ by $Q_{r_k}(x_i)$, with $r_k \in \{2^{-\frac{1}{4}}, 1\}$ and $x_i \in B_2(x_0)$ for $1 \le i \le c(n)$, and appeal to lemma 3.5.13 leading directly to

$$\|f\|_{L^p(Q(x_0))} \;\le\; \sum_{i=1}^{c(n)} \sum_{k=1}^{2} r_k^{-3} \left(r_k + \sqrt{x_{i,n}}\right) \left|Q_{r_k}(x_i)\right|^{\frac{1}{p}} \|f\|_{Y_p} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\right) \left|Q_1(x_0)\right|^{\frac{1}{p}} \|f\|_{Y_p}\,.$$

Now suppose $\sqrt{x_{0,n}} \lesssim 1$ and $(j,l,\alpha) \in \mathcal{CZ}$ such that $|\alpha| - 2j \ge 0$. Then, through the obvious inequality

$$1 + \sqrt{x_{0,n}} \;\lesssim\; 2^{|\alpha|-2j} \left(1 + \sqrt{x_{0,n}}\right)^{2j+1-|\alpha|},$$

the above estimate takes on the form

$$\|\partial_t^l \partial_x^\alpha\, u_1\|_{L^p(Q_1(x_0),\,\mu_{jp})} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\right)^{2j+1-|\alpha|} \left|Q_1(x_0)\right|^{\frac{1}{p}} \|f\|_{Y_p}\,.$$

For $1 \lesssim \sqrt{x_{0,n}}$, it requires a different ansatz to close the gap between the factor at hand, $1 + \sqrt{x_{0,n}}$, and the one to the power $2j + 1 - |\alpha|$. As mentioned previously, $x_n \sim x_{0,n}$ for all $x \in B_2(x_0)$, and thus

$$\left(1 + \sqrt{x_{0,n}}\right)^{|\alpha|-2j} \|\partial_x^\alpha\, u_1\|_{L^p(Q_1(x_0),\,\mu_{jp})} \;\lesssim\; \|x_n^{\frac{|\alpha|}{2}}\, \partial_x^\alpha\, u_1\|_{L^p((0,1)\times H)} \;\lesssim\; \|f\|_{L^p(Q(x_0))}$$

by virtue of lemma 5.2.2.

To bound $(II)$ we apply lemma 5.2.17 with $\delta = \frac{1}{4}$ and $\rho = 1$ to obtain that

$$\|\partial_t^l \partial_x^\alpha\, u_2\|_{L^q(Q_1(x_0),\,\mu_{jq})} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\right)^{2j+1-|\alpha|} \left|Q_1(x_0)\right|^{\frac{1}{q}} \|f\|_{Y_p} \qquad (p, q \ge 1)\,,$$

if $|\alpha| \ge 2j$. This last condition is, in particular, satisfied if $(j,l,\alpha)$ is of Calderón-Zygmund type. Also observe that it is sufficient to only consider the case $q = p$. As we will see later on, this allows us to merge the estimates for $(I)$ and $(II)$ into a single one.

In addition, with $|\alpha| = 1$ and $j = l = 0$, lemma 5.2.17 yields

$$\left|\nabla u_2(t,x)\right| \;\lesssim\; \|f\|_{Y_p}$$

for all $(t,x) \in Q_1(x_0) = (\frac{1}{2}, 1] \times B_1(x_0)$ and hence, in particular, for $(t,x) = (1, x_0)$. It remains to show that the same pointwise bound holds for $\nabla u_1$. By proposition 5.2.1 and Hölder's inequality we get

$$\left|\nabla u_1(1,x_0)\right| \le \int_{Q(x_0)} \left|\nabla_x G(1,x_0,s,y)\, f(s,y)\right| dyds \;\le\; \|\nabla_x G(1,x_0,\cdot,\cdot)\|_{L^{\frac{p}{p-1}}((0,1)\times H)} \|f\|_{L^p(Q(x_0))}.$$

Then applying lemma 5.1.6 to the first norm, the one that contains the Green function, gives

$$\|\nabla_x G(1,x_0,\cdot,\cdot)\|_{L^{\frac{p}{p-1}}((0,1)\times H)} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\right)^{-1} \left|B_1(x_0)\right|^{-\frac{1}{p}} \;=\; 2^{-\frac{1}{p}} \left(1 + \sqrt{x_{0,n}}\right)^{-1} \left|Q_1(x_0)\right|^{-\frac{1}{p}}\,.$$

This is possible if $\frac{p}{p-1} < \frac{n+2}{n+1}$, that is for $p > n+2$, and the desired bound follows.

Altogether we have seen that for all $p > n+2$ and $(j,l,\alpha) \in \mathcal{CZ}$ we have

$$\|\partial_t^l \partial_x^\alpha\, u\|_{L^p(Q_1(x_0),\,\mu_{jp})} \;\lesssim\; \left(1 + \sqrt{x_{0,n}}\right)^{2j+1-|\alpha|} \left|Q_1(x_0)\right|^{\frac{1}{p}} \|f\|_{Y_p}\,, \tag{$*$}$$

as well as

$$\left|\nabla u(1,x_0)\right| \;\lesssim\; \|f\|_{Y_p}\,, \tag{$**$}$$

whenever $f$ is supported in $[0,1]\times\overline{H}$. Now let $f$ be as in the assumptions of the proposition. Then $u$ is a $L_0$-solution on $[t_1,t_1+\lambda^2)\times\overline{H}$, for any $0<\lambda\le\sqrt{t_2-t_1}$, to $f$ with $u(t_1)=0$. If we define

$$T:\ (t,x)\ \mapsto\ (t_1+\lambda^2 t,\lambda x)\ =:\ (\hat t,\hat x)\,,$$

we know that $u\circ T$ is again a solution on $[0,1)\times\overline{H}$ to $\lambda^2(f\circ T)$ with $(u\circ T)(0)=u(t_1)=0$ and we have

$$\|\partial_{\hat t}^l\partial_{\hat x}^\alpha\, u\|_{L^p(Q_{\sqrt{\lambda}}(\hat x_0),\,\mu_{jp})}\ \lesssim\ \lambda^{\frac{n+2}{p}+j-2l-|\alpha|}\,\|\partial_t^l\partial_x^\alpha\,(u\circ T)\|_{L^p(Q_1(x_0),\,\mu_{jp})}\,.$$

(Proceed as in the proof of lemma 5.2.18).

Applying $(*)$, the right hand side can be bounded by

$$\lambda^{-2l-\frac{|\alpha|}{2}-\frac{1}{2}}\left(\sqrt{\lambda}+\sqrt{\hat x_{0,n}}\right)^{2j+1-|\alpha|}\left|Q_{\sqrt{\lambda}}(\hat x_0)\right|^{\frac1p}\|\lambda^2(f\circ T)\|_{Y_p}$$

since also

$$\left|Q_1(x_0)\right|^{\frac1p}\ =\ \left|Q_1\Big(\frac{\hat x_0}{\lambda}\Big)\right|^{\frac1p}\ \sim\ \lambda^{-\frac{n+2}{p}}\left|Q_{\sqrt{\lambda}}(\hat x_0)\right|^{\frac1p}.$$

Then, due to lemma 5.2.18, we see that

$$\lambda^{2l+\frac{|\alpha|}{2}-\frac12}\left(\sqrt{\lambda}+\sqrt{\hat x_{0,n}}\right)^{|\alpha|-2j-1}\left|Q_{\sqrt{\lambda}}(\hat x_0)\right|^{-\frac1p}\|\partial_{\hat t}^l\partial_{\hat x}^\alpha\, u\|_{L^p(Q_{\sqrt{\lambda}}(\hat x_0),\,\mu_{jp})}\ \lesssim\ \|f\|_{Y_p}\,.$$

Now choosing $\lambda=r^2$ and taking the supremum over $r$ and $\hat x_0$ gives us $\|u\|_{X_p^1}\lesssim\|f\|_{Y_p}$.

Likewise,

$$\left|\nabla_{\hat x}\,u(t_1+\lambda^2,\lambda x_0)\right|\ =\ \lambda^{-1}\left|\nabla_x\big(u\circ T\big)(1,x_0)\right|\ \lesssim\ \lambda\,\|f\circ T\|_{Y_p}\ \overset{(5.2.3)}{\lesssim}\ \|f\|_{Y_p}$$

for any $0<\lambda\le\sqrt{t_2-t_1}$ and for almost every $x_0\in\overline{H}$. This yields the complete statement. ■

We finally turn to the nonlinear problem and search for an estimate for $f[u]$ by the solution of the linear equation. This closes the circle and allows for the fixed point argument as used in the proof of the main theorem 5.2.12.

**Proof (of lemma 5.2.11):** The proof requires a careful examination of the inhomogeneity coupled with the definition of the considered function spaces $Y_p$ and $X_p$.

**Part 1:** We first notice that

$$R^3\left(R+\sqrt{x_n}\right)^{-1}\ \le\ R^{4l+|\alpha|-1}\left(R+\sqrt{x_n}\right)^{|\alpha|-2j-1}\tag{$*$}$$

for any $(j,l,\alpha)\in\mathcal{CZ}$ such that the factor in the $Y_p$-norm can be bounded by each of the factors appearing in the $X_p^1$-norm (cf. definition 5.2.8). Now from lemma 3.3.1 we know that the inhomogeneity can be written as

$$f[u]\ =\ f_0[u]\ +\ x_n\,f_1[u]\ +\ {x_n}^2\,f_2[u]$$

with

$$\begin{aligned} f_0[u]\ &=\ f_0^1(\nabla u)\star\nabla u\star D_x^2u\,,\\ f_1[u]\ &=\ f_1^1(\nabla u)\star\nabla u\star D_x^3u\ +\ f_1^2(\nabla u)\star P_2(D_x^2u)\quad\text{and}\\ f_2[u]\ &=\ f_2^1(\nabla u)\star\nabla u\star D_x^4u\ +\ f_2^2(\nabla u)\star D_x^2u\star D_x^3u\ +\ f_2^3(\nabla u)\star P_3(D_x^2u)\,.\end{aligned}$$

The functions $f_i^k(\nabla u)$ always contain factors of the form $(1+\partial_{x_n}u)^{-m}$ for some integer $1\le m\le 6$. By assumption we have $u\in B_\varepsilon^X$ with $0<\varepsilon<\frac12$ and hence, in particular, $\|\nabla u\|_{L^\infty}\le\varepsilon$ such that

$|1+\partial_{x_n}u|^{-m} \le (1-\varepsilon)^{-m} < 2^m$. The remaining parts of $f_i^k(\nabla u)$ are "$\star$-polynomials" of $\nabla u$ from which follows

$$\|f_i^k(\nabla u)\|_{L^\infty(I\times H)} \ < \ c(\varepsilon) \tag{$**$}$$

for every $i=0,1,2$ and $1\le k\le i+1$. It is important to know that this constant can be chosen independently of $\varepsilon$ if $\varepsilon$ is bounded above by some number smaller than 1.

Moreover, this allows us to expand each of the $f_i^k(\nabla u)$ into a power series. Since also any polynomial is an analytic function, we can write $f[u]$ as a convergent power series which is convergent for every $u\in B_\varepsilon^X$.

We now consider each of the summands in the expression of $f[u]$ separately. For the $f_i^1$ we observe that

$$\|f_i^1(\nabla u)\,\nabla u\,D_x^{|\alpha|}u\|_{L^p(Q_R(x),\mu_{jp})} \ \le \ \|f_i^k(\nabla u)\|_{L^\infty(I\times H)}\,\|\nabla u\|_{L^\infty(I\times H)}\,\|D_x^{|\alpha|}u\|_{L^p(Q_R(x),\,\mu_{jp})}\,,$$

where $j=0,1,2$ and $|\alpha|=j+2$. Using $(*)$ and $(**)$ we arrive at the estimate

$$\big|Q_R(x)\big|^{-\frac{1}{p}}\,R^3\left(R+\sqrt{x_n}\right)^{-1}\|x_n^{\,j}\,f_i^1(\nabla u)\,\nabla u\,D_x^{|\alpha|}u\|_{L^p(Q_R(x))} \ \lesssim \ \|u\|_{X_p}^2\,.$$

In order to bound the other parts of $f[u]$, we appeal to the weighted Gagliardo-Nirenberg interpolation in its local version[8] to get

$$\||D_x^2u|^2\|_{L^p(Q_R(x),\,\mu_p)} \ = \ \|D_x^2u\|^2_{L^{2p}(Q_R(x),\,\mu_p)} \ \lesssim \ \|\nabla u\|_{L^\infty(I\times H)}\,\|D_x^3u\|_{L^p(Q_R(x),\,\mu_p)}$$

and

$$\||D_x^2u|^3\|_{L^p(Q_R(x),\,\mu_{2p})} \ = \ \|D_x^2u\|^3_{L^{3p}(Q_R(x),\,\mu_{2p})} \ \lesssim \ \|\nabla u\|^2_{L^\infty(I\times H)}\,\|D_x^4u\|_{L^p(Q_R(x),\,\mu_{2p})}\,.$$

Moreover,

$$\begin{aligned}\|D_x^2u\,D_x^3u\|_{L^p(Q_R(x),\,\mu_{2p})} \ &\le \|D_x^2u\|_{L^{3p}(Q_R(x),\,\mu_{2p})}\,\|D_x^3u\|_{L^{\frac{3}{2}p}(Q_R(x),\,\mu_{2p})}\\ &\lesssim \|\nabla u\|^{\frac{2}{3}}_{L^\infty(I\times H)}\,\|D_x^4u\|^{\frac{1}{3}}_{L^p(Q_R(x),\,\mu_{2p})}\,\|\nabla u\|^{\frac{1}{3}}_{L^\infty(I\times H)}\,\|D_x^4u\|^{\frac{2}{3}}_{L^p(Q_R(x),\,\mu_{2p})}\,,\end{aligned}$$

where in the first line we have used Hölder's inequality. Then, again by $(*)$ and $(**)$, we obtain

$$\big|Q_R(x)\big|^{-\frac{1}{p}}\,R^3\left(R+\sqrt{x_n}\right)^{-1}\Big(\|f_1[u]\|_{L^p(Q_R(x),\,\mu_p)} \ + \ \|f_2[u]\|_{L^p(Q_R(x),\,\mu_{2p})}\Big) \ \lesssim \ \|u\|^2_{X_p} \ + \ \|u\|^3_{X_p}\,.$$

**Part 2:** Next we address the second part of the lemma, namely the one that includes the estimate for

$$f[u_1]-f[u_2] \ = \ \big(f_0[u_1]-f_0[u_2]\big) \ + \ x_n\left(f_1[u_1]-f_1[u_2]\right) \ + \ x_n^{\,2}\left(f_2[u_1]-f_2[u_2]\right)$$

for $u_1,u_2\in B_\varepsilon^X$ with $0<\varepsilon<\frac{1}{2}$. This difference expressed as a telescoping sum reads

$$\begin{aligned}f_0[u_1]-f_0[u_2] \ &= \big(f_0^1(\nabla u_1)-f_0^1(\nabla u_2)\big)\,\star\,\nabla u_1\,\star\,D_x^2u_1 \ + \ f_0^1(\nabla u_2)\,\star\,\nabla(u_1-u_2)\,\star\,D_x^2u_1 \,+\\ &\quad + \ f_0^1(\nabla u_2)\,\star\,\nabla u_2\,\star\,D_x^2(u_1-u_2)\,,\end{aligned}$$

and similarly we rewrite all the other terms except the last one. Here we have

$$\begin{aligned}&f_2^3(\nabla u_1)\,\star\,P_3(D_x^2u_1) \ - \ f_2^3(\nabla u_2)\,\star\,P_3(D_x^2u_2)\\ &= \ \big(f_2^3(\nabla u_1)-f_2^3(\nabla u_2)\big)\,\star\,P_3(D_x^2u_1) \ + \ f_2^3(\nabla u_2)\,\star\sum_{i=0}^{2}P_{2-i}(D_x^2u_1)\star D_x^2(u_1-u_2)\,\star\,P_i(D_x^2u_2)\,.\end{aligned}$$

[8] Apply proposition 2.8.2 to $D_x^2u$ multiplied by a suitable cut-off function $\eta$ which localizes the inequality to a time-space cylinder.

By means of the estimates from the first part of the proof applied to each of these summands we obtain

$$\|f[u_1]-f[u_2]\|_{Y_p} \lesssim \left(\|u_1\|_{X_p}+\|u_2\|_{X_p}\right)\|u_1-u_2\|_{X_p} + \|u_1\|_{X_p}^2 \sum_{i=0}^{2} \|f_i^1(\nabla u_1)-f_i^1(\nabla u_2)\|_{L^\infty(I\times H)}$$

for every $u_1, u_2 \in B_\varepsilon^X$ with $0<\varepsilon<\frac{1}{2}$. Finally, in order to estimate the sum, we obtain through the identity

$$\begin{aligned}&(1+\partial_{x_n}u_1)^{-m}-(1+\partial_{x_n}u_2)^{-m}\\ =\;& \frac{(\partial_{x_n}u_2-\partial_{x_n}u_1)}{(1+\partial_{x_n}u_1)^m(1+\partial_{x_n}u_2)^m}\sum_{i=1}^{m}\sum_{k=0}^{i-1}\binom{m}{i}(\partial_{x_n}u_1)^{i-1-k}(\partial_{x_n}u_2)^k\end{aligned}$$

that

$$\|f_i^1(\nabla u_1)-f_i^1(\nabla u_2)\|_{L^\infty(I\times H)} \lesssim \|u_1-u_2\|_{X_p}$$

for all $u_1,u_2\in B_\varepsilon^X$. The complete statement follows with $\|u_1\|_{X_p}^2 \le \|u_1\|_{X_p}\left(\|u_1\|_{X_p}+\|u_2\|_{X_p}\right)$. ■

In the proof of the analyticity result 5.2.13 we follow the same arguments as used in [54, Thm. 3.1] to obtain analyticity for solutions of the non-degenerate analogue of the equation (TFE).

**Proof (of proposition 5.2.13):** First we note that

$$F(g,u) \;=\; Sg \;+\; \Psi f[u]\,,$$

and hence $G(g,u)=u-F(g,u)$, is analytic on $\dot C^{0,1}(H)\times B_\varepsilon^X$ for $\varepsilon<\frac{1}{2}$. This, combined with the fact that $G(0,0)=0$ and $D_uG(0,0)=id$, allows us to apply the analytic implicit function theorem (see e.g. [22]): There exist positive numbers $\delta_0,\varepsilon_0$ and a unique analytic mapping $A: B_{\delta_0}^{Lip}\to B_{\varepsilon_0}^X$, with

$$B_{\delta_0}^{Lip} \;=\; \left\{g\in \dot C^{0,1}(H)\mid \|g\|_{\dot C^{0,1}(H)}<\delta_0\right\}$$

and $B_{\varepsilon_0}^X\subset B_\varepsilon^X$, such that $A(0)=0$ and $G(g,u)=0$ for every $g\in B_{\delta_0}^{Lip}$ and $u\in B_{\varepsilon_0}^X$ if and only if $u=A(g)$. However, since by theorem 5.2.12, there exists exactly one solution $u^*\in B_\gamma^X$ of (5.2.1), where $\gamma=\min\{\varepsilon_2,\varepsilon_0\}$, we thus have that $u^*$ depends analytically on $g$.

In the next part we prove that $u^*$ is an analytical function in $t\in I$ and $x'$. To this end, let $t_2<\infty$, $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}^{n-1}$ be parameters satisfying $(\tau,\xi)\in(1-\bar\rho,1+\bar\rho)\times\widetilde B_{\bar r}^{eu}(0)$, for $\bar\rho,\bar r>0$ small, and define

$$\tilde f_{\tau,\xi}[u] \;:=\; \tau\, f[u] \;+\; (1-\tau)\,L_0u \;-\; \xi\cdot\nabla_x' u\,.$$

Clearly, $\tilde f_{1,0}[u]=f[u]$. Moreover, we define $\widetilde F,\widetilde G:(1-\bar\rho,1+\bar\rho)\times\widetilde B_{\bar r}^{eu}(0)\times\dot C^{0,1}(H)\times B_\varepsilon^X\to X_p$ by

$$\widetilde F(\tau,\xi,g,u) \;:=\; Sg \;+\; \Psi\tilde f_{\tau,\xi}[u]$$

and $\widetilde G(\tau,\xi,g,u):=u-\widetilde F(\tau,\xi,g,u)$, just as above. Now we use lemma 5.2.9 and proposition 5.2.10 to conclude that

$$\|\widetilde G(\tau,\xi,g,u)\|_{X_p} \;\lesssim\; \|u\|_{X_p} \;+\; \|g\|_{\dot C^{0,1}(H)} \;+\; \|\tilde f_{\tau,\xi}[u]\|_{Y_p}$$

provided $p>n+2$. With the help of lemma 5.2.11 and definition 5.2.8, we estimate the last norm to get

$$\|\tilde f_{\tau,\xi}[u]\|_{Y_p} \;\lesssim\; \left(\tau\,\|u\|_{X_p} \;+\; \tau\,\|u\|_{X_p}^2 \;+\; |1-\tau| \;+\; |\xi|\,\sqrt{t_2-t_1}\right)\|u\|_{X_p}\,,$$

since also

$$\|\nabla_x' u\|_{Y_p} \;\le\; \|\nabla_x u\|_{L^\infty(I\times H)}\,\sup_{t\in I}\sqrt{t-t_1}\,.$$

Since $\widetilde G(1,0,0,0)=0$ and $D_u\widetilde G(1,0,0,0)=id$, there exist positive numbers $\rho<\bar\rho, r<\bar r, \varepsilon_3<\varepsilon, \delta_1$ and a uniquely determined analytic mapping $\widetilde{A} : (1-\rho, 1+\rho) \times \widetilde{B}_r^{eu}(0) \times B_{\delta_1}^{Lip} \times B_{\varepsilon_3}^{X} \to X_p$ that satisfies

$$\widetilde{G}\big(\tau, \xi, g, \widetilde{A}(\tau, \xi, g)\big) \;=\; 0$$

by yet another application of the analytic implicit function theorem, and therefore we have the identity

$$\widetilde{A}(\tau, \xi, g) \;=\; Sg \,+\, \Psi \tilde{f}_{\tau,\xi}\big[\widetilde{A}(\tau, \xi, g)\big]\,.$$

Now let $g \in B_\delta^{Lip}$, where $\delta = \min\{\delta_0, \delta_1\}$. Then we observe that $A(g)(t_1, \cdot) = g = \widetilde{A}(\tau, \xi, g)(t_1, \cdot)$ and

$$\widetilde{G}\Big(\tau, \xi, g, A(g)\big(\tau\,(t-t_1), x' - (t-t_1)\,\xi, x_n\big)\Big) \;=\; 0\,.$$

Thus, by the above uniqueness results, we obtain $A(g)\big(\tau\,(t-t_1), x' - (t-t_1)\,\xi, x_n\big) = \widetilde{A}(\tau, \xi, g)$, and in particular $u^*\big(\tau\,(t-t_1), x' - (t-t_1)\,\xi, x_n\big)$ is analytic in $\tau \in (1-\rho, 1+\rho)$ and $\xi \in \widetilde{B}_r^{eu}(0) \subset \mathbb{R}^{n-1}$. For $t \le t_2 < \infty$, we moreover get

$$\begin{aligned} \partial_\tau u^*\big(\tau\,(t-t_1), x' - (t-t_1)\,\xi, x_n\big)\Big|_{(\tau,\xi)=(1,0)} &\;=\; (t-t_1)\,\partial_t u^*(t,x)\,, \\ \nabla_\xi' u^*\big(\tau\,(t-t_1), x' - (t-t_1)\,\xi, x_n\big)\Big|_{(\tau,\xi)=(1,0)} &\;=\; (t_1-t)\,\nabla_x' u^*(t,x) \end{aligned}$$

and similar formulas for (mixed) derivatives of higher order. This proves the analyticity of $u^*$ in $(t, x') \in I \times \mathbb{R}^{n-1}$. The estimate (5.2.4) now follows from these formulas coupled with a scaling argument: Let $T : (t,x) \mapsto (t_1 + \lambda^2 t, \lambda x) =: (\hat{t}, \hat{x})$ with $0 < \lambda < \sqrt{t_2 - t_1}$ . Applying the analyticity estimate

$$\big|\partial_t^l \partial_{x'}^{\alpha'} u^*(1,x)\big| \;\lesssim\; R^{-l-|\alpha'|}\, l!\, \alpha'!\, \|g\|_{\dot{C}^{0,1}(H)}$$

yields

$$\big|\partial_{\hat{t}}^l \partial_{\hat{x}'}^{\alpha'} u^*(t_1 + \lambda^2, \hat{x})\big| \;\lesssim\; \lambda^{1-2l-|\alpha'|}\, R^{-l-|\alpha'|}\, l!\, \alpha'!\, \|g\|_{\dot{C}^{0,1}(H)}\,,$$

since also $\|\nabla_x\, g(\lambda\,\cdot)\|_{L^\infty(H)} = \lambda\, \|\nabla_{\hat{x}}\, g\|_{L^\infty(H)}$. Now for $(\hat{t}, \hat{x}) \in (t_1, t_2) \times \overline{H}$, we set $\lambda = \sqrt{\hat{t} - t_1}$ and the estimate takes the desired form. ■

Before we can prove the uniqueness result for the original problem, we shall need the fact that the change of coordinates $(t,x) \mapsto (s,y)$ is a quasi-isometry. This is formalized in the following auxiliary lemma.

**Lemma 5.2.19** *Let $\phi : x \mapsto \big(x', v(x)\big)$ with $v : \mathbb{R}^n \to \mathbb{R}$ satisfying $|\nabla_x v - e_n| < \varepsilon$ for an $\varepsilon < 1$. Then we have*

$$(1-\varepsilon)\,|x - \bar{x}| \;<\; \big|\phi(x) - \phi(\bar{x})\big| \;<\; (1-\varepsilon)^{-1}\,|x - \bar{x}|$$

*for all $x, \bar{x} \in \mathbb{R}^n$.*

**Proof:** We may assume that $x_n > \bar{x}_n$ without loss of generality. By the mean value theorem, only applied in vertical direction, there exists a number $\bar{x}_n < z < x_n$ such that

$$\begin{aligned} \big|\phi(x) - \phi(\bar{x}) - (x - \bar{x})\big| &\;=\; \big|v(x) - v(\bar{x}) - (x_n - \bar{x}_n)\big| \;=\; \big|\partial_{x_n} v(z)(x_n - \bar{x}_n) - (x_n - \bar{x}_n)\big| \\ &\;\le\; \big|\nabla_x v(z) - e_n\big|\,(x_n - \bar{x}_n) \;<\; \varepsilon\,(x_n - \bar{x}_n) \\ &\;\le\; \varepsilon\,\Big(\big|\phi(x) - \phi(\bar{x})\big| \,+\, \big|\phi(x) - \phi(\bar{x}) - (x - \bar{x})\big|\Big)\,. \end{aligned}$$

We subtract $\varepsilon$ times the left hand side from both sides of the inequality, divide by $(1-\varepsilon)$ and arrive at

$$\big|\phi(x) - \phi(\bar{x}) - (x - \bar{x})\big| \;<\; \frac{\varepsilon}{1-\varepsilon}\,\min\Big\{\big|\phi(x) - \phi(\bar{x})\big|\,, |x - \bar{x}|\Big\}\,.$$

The assertion follows by the triangle inequality. ■

**Corollary 5.2.20** *For $y \in \mathbb{R}^n$, let $B^i_R(y) := \phi\big(B_R(x)\big)$ with $x = \phi^{-1}(y)$, where $\phi^{-1} : y \mapsto \big(y', \tilde h(y)\big)$. Then we have*

$$B^i_R(y) \;\sim\; B^{eu}_{R\left(R+\sqrt{\tilde h}\,\right)}(y) \cap spt\, \tilde h$$

*This relation reads as follows: There exists $c = c(\varepsilon) > 1$ such that $c^{-1}\, B^{eu}_r(y) \subset B^i_R(y) \subset c\, B^{eu}_r(y)$, where $c\, B^{eu}_r(x)$ denotes the Euclidean ball with the same center and radius dilated by the factor $c$, i.e. $B^{eu}_{cr}(x)$ with $r = R\big(R + \sqrt{\tilde h}\,\big)$.*

**Proof:** First suppose that $R^2 \ll x_n$, i.e. $R^2 \ll \tilde h$ after changing variables. In this situation we have

$$B_R(x) \;\sim\; B^{eu}_{R\sqrt{x_n}}(x)$$

by virtue of lemma 3.5.11, whilst for $R^2 \gtrsim x_n$, this relation becomes $B_R(x) \sim B^{eu}_{R^2}(x) \cap \overline{H}$. However, this implies

$$B^{eu}_{(1-\varepsilon)R\,\sqrt{x_n}}\big(\phi(x)\big) \;\subset\; \phi\big(B^{eu}_{R\sqrt{x_n}}(x)\big) \;\subset\; B^{eu}_{\frac{R}{1-\varepsilon}\,\sqrt{x_n}}\big(\phi(x)\big)\,,$$

where we used lemma 5.2.19, and the assertion follows with $y = \phi(x)$ and $\phi(\overline{H}) = spt\, \tilde h$. ■

**Proof (of theorem 5.2.15):** Assume that there exist numbers $\delta \in (0, 1)$ and $C > 1$ such that $v : I \times \overline{H} \to \mathbb{R}$ satisfies

$$\delta \;\le\; \big|\nabla_x v(t, x)\big| \;\le\; C\,. \qquad (*)$$

Then by lemma 5.2.19, we can make the change of variables $(t, x) \mapsto (s, y)$ to globally transform the equation $\partial_t v + L_0 v = f[v]$, with $v = x_n + u$. Now using $\tilde h = x_n$ as the new dependent variable, we obtain

$$\nabla_y \tilde h \;=\; -\, v_n^{-1} \begin{pmatrix} \nabla'_x v \\ -1 \end{pmatrix}$$

and thus

$$0 \;<\; \frac{\delta + 1}{2C} \;\le\; \big|\nabla_y \tilde h\big| \;\le\; \frac{C + 1}{\delta} \;<\; \infty\,.$$

If additionally $|\partial_t v| < \infty$, then

$$|\partial_s \tilde h| \;=\; \left|\frac{\partial_t v}{v_n}\right| \;<\; \infty\,,$$

and the function $\tilde h$ is Lipschitz in $s$ and $y$ up to the boundary of its support and has bounded first derivatives. Now let $v = x_n + u^*$, where $u^*$ is the unique solution of (5.2.1) given by theorem 5.2.12. Then we have

$$\big|\nabla_{t,x} u^*(t, x)\big| \;\lesssim\; \|g\|_{\dot C^{\,0,1}(H)}$$

by means of (5.2.4), and the required bounds follow for sufficiently small $\|g\|_{\dot C^{\,0,1}(H)}$.

It remains to prove two parts. First, we show that a mild solution of (5.2.1) yields a weak solution of the thin-film equation in the sense of definition (5.2.5). In a second step, we prove uniqueness of this solution by imposing additional conditions on $h$, or rather $\tilde h$, in terms of the transformed intrinsic cylinders $Q_R(x)$.

**Existence:** Putting $\tilde h = \sqrt{h}$, we observe that

$$\int_I \int_{\mathbb{R}^n} \tilde h^2\, \partial_s \varphi\, dy ds \;=\; -\int_I \int_{\mathbb{R}^n} \partial_s \tilde h^2\, \varphi\, dy ds$$

for all test functions $\varphi \in C^\infty_c(I \times \mathbb{R}^n)$ by integration by parts. It therefore suffices to show that for all $s \in I$,

$$\int_{\mathbb{R}^n} \partial_s \tilde h^2\, \varphi\, dy \;=\; \int_{\mathbb{R}^n} \tilde h^2\, \nabla_y \Delta_y \tilde h^2 \cdot \nabla_y \varphi\, dy\,.$$

Under a change of coordinates $(s, y) \mapsto (t, x)$ (cf. section 3.1), the left hand side integral transforms to

$$-2 \int_H x_n \frac{\partial_t u}{v_n} \varphi \frac{\partial y_n}{\partial x_n} dx = -2 \int_H \partial_t u \, \varphi \, d\mu_1 ,$$

where $v_n = \partial_{x_n} v = 1 + \partial_{x_n} u$. For the second integral, we proceed as in the proof lemma 3.3.1 to get

$$2 \int_H \Big[ \begin{pmatrix} v_n^{-1} \nabla_x' \\ \partial_{x_n} \end{pmatrix} (x_n^3 \Delta_x u) + 2 x_n^2 \begin{pmatrix} v_n^{-1} \nabla_x' \\ \partial_{x_n} \end{pmatrix} \partial_{x_n} u - 2 x_n^2 e_n \Delta_x u + R(u) \Big] \nabla_y \varphi \, v_n \, dx .$$

Next we employ the $\star$-notation, as introduced in section 3.3, to rewrite the remainder as follows:

$$R(u) = x_n^2 \tilde{f}_2(\nabla_x u) \star \nabla_x u \star D_x^2 u + x_n^3 \big( \tilde{f}_3^1(\nabla_x u) \star \nabla_x u \star D_x^3 u + \tilde{f}_3^2(\nabla_x u) \star P_2(D_x^2 u) \big)$$

Again, the functions $\tilde{f}_2$ and $\tilde{f}_3^k$ contain factors of the form $v_n^{-m}$ for some $2 \leq m \leq 5$. Now recall that

$$\nabla_y = \begin{pmatrix} \nabla_x' - v_n^{-1} \nabla_x' v \, \partial_{x_n} \\ v_n^{-1} \partial_{x_n} \end{pmatrix} .$$

An integration by parts is possible giving us the identity

$$2 \int_H x_n \big( \partial_t u + L_0 u - f[u] \big) \varphi \, dx = 0 ,$$

and $2x_n\varphi$ is an admissible test function. To see this, we calculate that

$$\begin{aligned} \nabla_y \Big[ \begin{pmatrix} \nabla_x' \\ v_n \partial_{x_n} \end{pmatrix} (x_n^3 \Delta_x u) + 2 x_n^2 \begin{pmatrix} \nabla_x' \\ v_n \partial_{x_n} \end{pmatrix} \partial_{x_n} u - 2 x_n^2 v_n e_n \Delta_x u + v_n R(u) \Big] \\ = \Delta_x \big( x_n^3 \Delta_x u \big) - 4 x_n \Delta_x' u - f[u] . \end{aligned}$$

Now if $u^*$ is the unique solution of (5.2.1) given by theorem 5.2.12, then each of the single terms $\partial_t u^*$, $D_x^2 u^*$, $x_n D_x^3 u^*$, $x_n^2 D_x^4 u^*$ and $f[u^*]$ is bounded above by $c\,\varepsilon_1 > 0$. Reversing the transformation from above yields the existence of a solution $h$ of $\partial_s h + \nabla_y \cdot (h \nabla_y \Delta_y h) = 0$ on its positivity set $P(h)$. Finally, extending $h$ by 0 outside of *spt* $h$ we conclude that $h$ is a weak solution in the sense of (5.2.5). To see this, we calculate that

$$\int_I \int_{\mathbb{R}^n} h \, \partial_s \varphi + h \nabla \Delta h \cdot \nabla \varphi \, dy ds = - \int_I \int_{\mathbb{R}^n} \big( \partial_s h + \nabla \cdot (h \nabla \Delta h) \big) \varphi \, dy ds = 0$$

using integration by parts. Note that the boundary terms vanish since $h$ vanishes on $\partial P(h)$.

**Uniqueness:** Given $g_v$ satisfying $|\nabla_x g_v - e_n| < \varepsilon$, then by theorem 5.2.12 there exists a unique solution $v^*$ of the transformed thin-film equation and we have $v^*(t_1) = g_v$. Moreover, we know that

$$\|v^* - x_n\|_{X_p} \lesssim \varepsilon .$$

This implies $|\nabla_x v^* - e_n| \lesssim \varepsilon$, cf. $(*)$, and $|\nabla_y \tilde{h} - e_n| \lesssim \varepsilon$ after the transformation $(t, x) \mapsto (s, y)$. Under this transformation applied to cylinders of the form $Q_R(x) = \big(t_1 + \frac{R^4}{2}, t_1 + R^4\big] \times B_R(x)$ we get

$$Q_R(x) \sim \Big( t_1 + \frac{R^4}{2}, t_1 + R^4 \Big] \times B_R(y) = Q_R(y)$$

where we have used corollary 5.2.20. Now let $(j, l, \alpha) \in \mathcal{CZ}$, for example take $(j, l, |\alpha|) = (0, 0, 2)$. Then

$$\big| Q_R(x) \big|^{-\frac{1}{p}} R \big( R + \sqrt{x_n} \big) \, \|D_x^2 v\|_{L^p(Q_R(x))} \sim \big| Q_R(y) \big|^{-\frac{1}{p}} R \big( R + \sqrt{\tilde{h}} \big) \, \|D_y^2 \tilde{h}\|_{L^p(Q_R(y))} ,$$

with similar transforms for the other combinations of $j, l$ and $|\alpha|$. The supremum is now taken over all

$R^4 \in (0, t_2 - t_1)$ and all $y \in P_s(\tilde{h}) = \{y \in \mathbb{R}^n \mid \tilde{h}(s,y) > 0\}$, $s \in I$. Also note that $\tilde{h}$ is controlled by

$$(1-\tilde{\varepsilon})\,\tilde{h}(s,y) \;<\; dist\big(y, \mathbb{R}^n \setminus spt\,\tilde{h}(s)\big) \;<\; (1-\tilde{\varepsilon})^{-1}\,\tilde{h}(s,y)$$

which follows from a transformation of the statement in lemma 5.2.19. Using $|\partial_s \tilde{h}| \lesssim \varepsilon$, this amounts to

$$R + \sqrt{\tilde{h}(s,y)} \;\sim\; R + \sqrt{\tilde{h}_0(y)}$$

in $Q_R(y)$. All these calculations show that $v^*$ generates a solution $\tilde{h}_1$ via $(t,x) \mapsto (s,y)$ which satisfies

$$\sup_{P(\tilde{h})} \big|\nabla_y \tilde{h}_1 - e_n\big| \,+\, \big[\tilde{h}_1\big]_{X^1_p} \;\lesssim\; \varepsilon\,.$$

Let $\tilde{h}_2$ be another weak solution. Then, inverting the transformation, we obtain a second mild solution, say $v^{**}$, of the transformed problem. Thus, by uniqueness of such a solution, $\tilde{h}_1 = \tilde{h}_2$ is a unique solution of

$$\partial_s \tilde{h}^2 + \nabla_y \cdot \big(\tilde{h}^2\; \nabla_y \Delta_y \tilde{h}^2\big) = 0\,.$$

Finally, we substitute back for $\tilde{h} = \sqrt{h}$ to see that the initial value problem for the equation (TFE) has a unique weak solution, denoted by $h^*$. ■

# Chapter 6

# Appendix

## A Singular Integrals

To study partial differential equations is often intimately connected with the study of singular integrals. A singular integral is defined as an operator $T : L^{p_0}(X, \mu) \to L^{p_0}(X, \mu)$, for $p_0 > 1$, that is expressible in the form

$$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y)\,.$$

The corresponding kernel $K : X \times X \to \mathbb{R}$ is singular along the diagonal $\{(x, x) \mid x \in X\}$, smooth off the diagonal and approximately translation invariant. The theory of singular integrals provides some useful tools for estimating these operators.

This theory has developed into various directions, as for example the theory of weights. To be more precise, we will study the class of positive functions $\omega$, called weights, for which we can estimate as follows:

$$\int_X |Tf|^p\, \omega\, d\mu \;\le\; c \int_X |f|^p\, \omega\, d\mu\,. \tag{A.1}$$

One has to characterize the class of functions $\omega$ in such a way that this estimate holds true. A necessary and sufficient condition is that $\omega$ belongs to a class of weights called the Muckenhoupt class which is denoted by $A_p$.

This appendix is organized as follows. The harmonic analysis only requires little structure on the underlying space. Hardy-Littlewood maximal functions and functions of bounded mean oscillation still make sense on spaces of homogeneous type, a setting in which a Calderón-Zygmund theory can be established. Following the standard outline, we rediscover all the relevant $L^p$-estimates on maximal functions and the sharp function, and then prove the Calderón-Zygmund inequality which is an unweighted version of (A.1). Surprisingly, all these estimates are closely related. A detailed exposition of the material presented in this section as well as the missing proofs in subsections A.1 and A.1 may be found in [73, 74, 52, 53]. Eventually, we survey the theory of Muckenhoupt weights leading to a proof of the weighted estimate above.

### A.1 Harmonic Analysis in Spaces of Homogeneous Type

We consider a metric space $(X, d)$ with metric $d$, and equip it with a Borel regular measure $\mu$ on $X$. This means that $d(\,.\,, x)$ is measurable and for every open set $M \subseteq X$ we have $\mu(M) = \sup \mu(K)$, where $K \subset M$ is compact. In addition, we assume that $\mu$ and $d$ are compatible in the following sense: There exist constants $c, b \geq 1$ such that

$$0 \;<\; \mu\big(B_{cr}(x)\big) \;\leq\; b\,\mu\big(B_r(x)\big) \;<\; \infty\,. \tag{A.2}$$

Such a measure is called "doubling" with doubling constant $b$. This terminology originates from the fact that it is equivalent to formulate inequality (A.2) with $c = 2$. The triple $(X, d, \mu)$ is termed space of homogeneous type. These spaces are locally compact and separable. The measure $\mu$ is a Radon measure and $X$ is $\sigma$-finite.

Examples for such metric measure spaces include the following:

1. $X = \mathbb{R}^n$, $d$ is the Euclidean distance and $\mu = \mathcal{L}^n$ is the Lebesgue measure.
2. $X \subset \mathbb{R}^n$ is a bounded domain and $d, \mu$ are as above.
3. $X = \mathbb{Z}^n$, $d$ is the Euclidean distance together with the counting measure.

The first example is the standard setting for the theory of singular integrals while the second one is related to elliptic boundary value problems. The last example, on the other hand, exposes a discrete setting. One should also take note of the fact that Coifman and Weiss generalized the definition of spaces of homogeneous type by replacing the metric $d$ by a quasi-metric $\rho$. In this context a quasi-metric is to be understood as a mapping $\rho : X \times X \to [0, \infty)$ which is positive definite and symmetric, but may violate the triangle inequality. Instead it is assumed that the weaker form,

$$\rho(x, z) \;\leq\; c\,\big(\rho(x, y) + \rho(y, z)\big)\,,$$

holds for all $x, y, z \in X$ and a constant $c \geq 1$. In this more general framework the proofs are slightly more complicated. It was shown in [69] that for a given quasi-metric $\rho$ there is a related metric, and then one may use this metric in place of $\rho$. For our purposes, however, it is entirely sufficient to work in a metric space of homogeneous type.

### The Spaces $\mathcal{H}^1$ and $BMO$

First let us introduce the Banach space $\mathcal{H}^1$ as a subspace of $L^1(\mu)$. Suppose $f \in L^1_{loc}(\mu)$. The maximal function of Hardy and Littlewood is then defined by

$$Mf(x) \;:=\; \sup_{B \ni x} \mu(B)^{-1} \int_B |f|\, d\mu\,, \tag{A.3}$$

where the supremum is taken over all balls $B = B_r(y)$ that contain $x$. To prepare another definition we fix a ball $B = B_r(x_0)$ and set

$$\mathcal{L}(B) \;:=\; \left\{ \phi \in C(X) \;\middle|\; |\phi(x)| \leq \frac{\max\{r - d(x, x_0), 0\}}{r\,\mu(B)}\,,\; |\phi(x) - \phi(y)| \leq \frac{d(x, y)}{r\,\mu(B)} \right\}.$$

If $\phi \in \mathcal{L}(B)$, then $-\phi \in \mathcal{L}(B)$, $\|\phi\|_{sup} \leq \mu(B)^{-1}$ and $spt\,\phi \subset B$. In addition to (A.3), we define the second Hardy-Littlewood maximal function by

$$\widetilde{M}f(x) \;:=\; \sup_{B \ni x,\, \phi \in \mathcal{L}(B)} \int_B f\,\phi\, d\mu \tag{A.4}$$

and observe that $\widetilde{M}f(x) \leq Mf(x)$. Hence $\big\{x \in X \mid \widetilde{M}f(x) > \lambda\big\} \supseteq \big\{x \in X \mid Mf(x) > \lambda\big\}$ and both sets are open. Consequently, $Mf$ and $\widetilde{M}f$ are $\mu$-measurable. A basic estimate for maximal functions is

$$\|Mf\|_{L^p(\mu)} \;\leq\; 2\left(\frac{bp}{p-1}\right)^{\frac{1}{p}} \|f\|_{L^p(\mu)}\,. \tag{A.5}$$

To prove (A.5) one relies on the weak type $(1, 1)$ estimate

$$\mu\big(\{x \in X \mid M(f) > \lambda\}\big) \;\leq\; \frac{b}{\lambda} \int_X |f|\, d\mu$$

that holds for any $\lambda > 0$. Behind this is the fundamental covering lemma of Vitali valid in homogeneous-type metric spaces.

**Lemma A.1 (Vitali)** *Let $(X, d, \mu)$ be a metric space of homogeneous type and $K \subset X$ a compact set. If $\{B_{r_i}\}_{i\in I}$, labeled by means of an index set $I = \{1, \dots, N\}$ for some $1 \leq N \leq \infty$, is a collection of balls that covers $K$, then there is a finite subset $J$ of $I$ such that $\{B_{r_j}\}_{j\in J}$ is pairwise disjoint and $\{B_{3\,r_j}\}_{j\in J}$ still covers $K$.*

The proof is as follows: Since $K$ is compact, we may suppose that $N < \infty$. Moreover, let $\{B_{r_i}\}_{i\in I}$ be ordered by the size of their radii, i.e $r_1 \geq r_2 \geq \cdots \geq r_N$. Then $B_{r_1}$ is a ball of greatest radius. Among the other balls we pick the ball that has the greatest possible radius $r_k$ for which $B_{r_1} \cap B_{r_k}$ is empty. We repeat this process until there are no more balls to choose. Since $I$ is a finite index set, this procedure eventually stops after $M \leq N$ steps. We claim that the so obtained sub-collection satisfies the requirements of Vitali's lemma. By construction, all the balls are pairwise disjoint. Now let $x \in K$. Then $x$ is contained in a ball $B_{r_i}$. This ball is either in the sub-collection, or there is a ball $B_{r_j}$ with $r_j \geq r_i$ for which $B_{r_i} \cap B_{r_j}$ is nonempty. By the triangle inequality $B_{r_i} \subset B_{3\,r_j}$ which shows that

$$K \;\subset\; \bigcup_i B_{r_i} \;\subset\; \bigcup_j B_{3\,r_j}\,.$$

A standard consequence of the estimates for the maximal functions is the following one. If $f \in L^1_{loc}(\mu)$, then

$$f(x) \;=\; \lim_{r\to 0} \inf_{B_r(y)\ni x} \mu\big(B_r(y)\big)^{-1} \int_{B_r(y)} f\, d\mu = \lim_{r\to 0} \sup_{B_r(y)\ni x} \mu\big(B_r(y)\big)^{-1} \int_{B_r(y)} f\, d\mu$$

for $\mu$-almost every $x$. Therefore we can pick a canonical representative in each equivalence class for which both limits exist and which vanishes otherwise.

**Definition A.2 (Hardy space)** *A function $f \in L^1_{loc}(\mu)$ belongs to $\mathcal{H}^1(\mu)$, called Hardy space, if $Mf \in L^1(\mu)$. If $\mu(X) = 0$ we require in addition that $\int f\, d\mu = 0$. We define the Hardy-norm of $f$ by*

$$\|f\|_{\mathcal{H}^1(\mu)} \;:=\; \|\widetilde{M}f\|_{L^1(\mu)}\,.$$

One particular class of functions in $\mathcal{H}^1(\mu)$ is the class of atoms.

**Definition A.3 (atom)** *An atom is a function $a$ for which there exists a ball $B$ such that*

$$\textit{i) } \operatorname{spt} a \subseteq \overline{B}, \qquad \textit{ii) } |a| \leq \mu(B)^{-1} \ a.e. \qquad \textit{and} \qquad \textit{iii) } \int a\, d\mu = 0\,.$$

These functions, in turn, characterize any Hardy function.

**Theorem A.4 (atomic decomposition)** *Every $f \in \mathcal{H}^1(\mu)$ can be written as a sum, $f = \sum \lambda_k a_k$, with $\{a_k\}$ being a collection of $\mathcal{H}^1(\mu)$-atoms and $\{\lambda_k\}$ an absolutely summable sequence in $\mathbb{R}$ with*

$$\sum_{k\in\mathbb{N}} |\lambda_k| \;\leq\; c\, \|f\|_{\mathcal{H}^1(\mu)} \quad \textit{for some constant } c = c(b) > 0\,.$$

**Definition A.5 (BMO)** *We say a locally integrable function $f$ is of bounded mean oscillation if, for any ball $B$, we have*

$$\mu(B)^{-1}\int_B \big|f(x)-f_B\big|\,d\mu(x) \;\le\; C\,.$$

*Here*

$$f_M \;=\; \mu(M)^{-1}\int_M f\,d\mu$$

*is the average of $f$ over the measurable set $M$ with $0<\mu(M)<\infty$.*

This means that any function in BMO has its average oscillation bounded. The smallest upper bound $C$ is denoted by $\|f\|_{BMO}$. Notably, the use of the mean value $f_B$ in definition A.5 is not mandatory. Instead, it can be replaced by arbitrary constants $c_B$ and a perfectly equivalent definition arises. Then the inequality $|c_B - f_B| \le C$ holds and, indeed, $\|f\|_{BMO} \le 2C$.

It is obvious that any bounded function is of bounded mean oscillation, while the converse is false. For this, $f(x) = \log d(x, x_0)$ sometimes serves as a prime example for an unbounded function in $BMO$.

The sharp function defines a dual object to the Hardy-Littlewood maximal functions.

**Definition A.6 (sharp maximal function)** *Let $f\in L^1_{loc}(\mu)$. The sharp maximal function $f^\#$ is given by*

$$f^\#(x) \;:=\; \sup_{B\ni x}\mu(B)^{-1}\int_B \big|f-f_B\big|\,d\mu\,.$$

Clearly, $f^\#$ is also $\mu$-measurable. Furthermore, we notice that a function $f$ is in $BMO$ if and only if the sharp function $f^\#$ is bounded. Indeed, we have

$$\|f\|_{BMO} \;=\; \|f^\#\|_{L^\infty(\mu)}\,.$$

This expression becomes a norm on the $BMO$ functions after quotienting out by the constant functions for which $\|\cdot\|_{BMO}$ is equal to 0. With this convention $BMO$ is a Banach space.

Now there is a series of inequalities that sets $f$, $f^\#$, $Mf$ and $\widetilde{M}f$ into relation to each other. We begin with a result that is attributed to Fefferman.

**Theorem A.7** *Let $1<p,p'<\infty$ with $\frac{1}{p}+\frac{1}{p'}=1$, $f\in L^p(\mu)$ and $g\in L^{p'}(\mu)$. Then there exists a constant $c=c(b)$, see inequality (A.2), such that*

$$\int_X f\,g\,d\mu \;\le\; c\int_X f^\#\,\widetilde{M}g\,d\mu\,.$$

*The same inequality holds for $f\in BMO$ and $g\in\mathcal{H}^1(\mu)$.*

Since the sharp function is pointwise dominated by the larger maximal function $Mf$, or more specifically $f^\#(x)\le 2Mf(x)$, we have

$$\|f^\#\|_{L^p(\mu)} \;\le\; 4\Big(\frac{bp}{p-1}\Big)^{\frac{1}{p}}\,\|f\|_{L^p(\mu)} \qquad \forall\, p\in(1,\infty)\,.$$

The converse is also true: Indeed, theorem A.7 and estimate (A.5) imply

$$\begin{aligned}\|f\|_{L^p(\mu)} &= \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \int_X f\, g\, d\mu \;\le\; c(b) \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \int_X f^{\#}\, Mg\, d\mu \\ &\le c(b)\, \|f^{\#}\|_{L^p(\mu)} \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \|Mg\|_{L^{p'}(\mu)} \\ &\le 2\, c(b) bp\, \|f^{\#}\|_{L^p(\mu)}\,.\end{aligned}$$

### Estimates for Singular Integral Operators

After having collected some basic estimates for maximal functions and the sharp maximal function we turn our attention to singular integral operators and verify under weak assumptions on the kernel that the associated linear operator $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$ is, in fact, continuous on $L^p(\mu)$ for all $1 < p < \infty$.

**Definition A.8** *A continuous function $K : (X \times X) \setminus \{(x,x) \mid x \in X\} \to \mathbb{R}$ is said to be a Calderón-Zygmund singular integral kernel if there exist $0 < \gamma \le 1$ and $C < \infty$ such that*

$$\big|K(x,y)\big| \;\le\; C\,\Big(\mu\big(B_{d(x,y)}(x)\big) + \mu\big(B_{d(x,y)}(y)\big)\Big)^{-1} \;=:\; C\,V(x,y)^{-1}$$

*for $\mu$-almost every $x \ne y \in X$, and*

$$\big|K(x,y) - K(\bar{x},\bar{y})\big| \;\le\; C\,V(x,y)^{-1}\left(\frac{d(x,\bar{x}) + d(y,\bar{y})}{d(x,y) + d(\bar{x},\bar{y})}\right)^{\gamma}$$

*for $\mu$-almost every $x \ne y, \bar{x} \ne \bar{y} \in X$ with $\frac{d(x,\bar{x})+d(y,\bar{y})}{d(x,y)+d(\bar{x},\bar{y})} \le \delta$ for some $\delta \in (0,1)$.*

We are now ready to give the definition of a Calderón-Zygmund operator.

**Definition A.9 (Calderón-Zygmund operator)** *Let $1 < p_0 < \infty$. A continuous and linear operator $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$ is said to be a Calderón-Zygmund singular integral operator if associated to $T$ there is a Calderón-Zygmund integral kernel $K$ (in the sense of definition A.8) such that*

$$Tf(x) \;=\; \int_X K(x,y)\, f(y)\, d\mu(y)$$

*for all $f \in L^{p_0}(\mu)$ with compact support and $x \notin spt\, f$.*

The first non-trivial result for such an integral operator is this pointwise estimate.

**Lemma A.10** *Suppose $1 < p_0 < \infty$ and $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$ is a Calderón-Zygmund operator. Then we have*

$$\big(Tf\big)^{\#}(x) \;\le\; c(b,p_0,T)\,\Big(M\big(|f|^{p_0}\big)(x)\Big)^{\frac{1}{p_0}}$$

*for all $f \in L^{p_0}(\mu)$ and all $x \in X$.*

Armed with these estimates we can now approach the crucial Calderón-Zygmund estimate. It is proven in two steps. First let $p > p_0$. Then,

$$\begin{aligned}\|Tf\|_{L^p(\mu)} &\le c\,\|(Tf)^{\#}\|_{L^p(\mu)} \;\le\; c\,\Big\|\Big(M\big(|f|^{p_0}\big)\Big)^{\frac{1}{p_0}}\Big\|_{L^p(\mu)} \;=\; c\,\big\|M\big(|f|^{p_0}\big)\big\|^{\frac{1}{p_0}}_{L^{p/p_0}(\mu)} \\ &\le c\,\|f^{p_0}\|^{\frac{1}{p_0}}_{L^{p/p_0}(\mu)} \;=\; c\,\|f\|_{L^p(\mu)}\end{aligned} \tag{A.6}$$

for all $f \in L^p(\mu)$. It is easy to check that the assumptions on $T$ for $p_0$ imply the assumptions on $T^*$ for the conjugate Hölder exponent $p_0' = \frac{p_0}{p_0-1}$. In case of $p < p_0$ we have $p' > p_0'$, and hence we are in a situation where we can apply estimate (A.6) to $T^* : L^{p_0'}(\mu) \to L^{p_0'}(\mu)$. This results in

$$\begin{aligned}\|Tf\|_{L^p(\mu)} &= \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \int_X (Tf)\, g\, d\mu = \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \int_X f\,(T^*g)\, d\mu \\ &\le \sup_{\|g\|_{L^{p'}(\mu)}\le 1} \|f\|_{L^p(\mu)}\, \|T^*g\|_{L^{p'}(\mu)} \le c\,\|f\|_{L^p(\mu)}\,,\end{aligned}$$

and therefore shows that the Calderón-Zygmund estimate (A.6) is valid for all $p \in (1,\infty)$.

## A.2 Weighted Norm Estimates for Singular Integral Operators

In this section we introduce the class of Muckenhoupt weights $A_p$. Moreover, given a $\mu$-measurable set $M \subset X$, let

$$\omega(M) = \int_M \omega\, d\mu\,. \tag{A.7}$$

Then, Muckenhoupt's class $A_p$ consists of those weights $\omega$ for which the Hardy–Littlewood maximal operator is bounded on $L^p(\omega)$. More precisely, we wish to characterize $\omega$ in such a way that

$$\|\widetilde{M}f\|_{L^p(\omega)} \le c\,\|f\|_{L^p(\omega)} \tag{A.8}$$

for all $f \in L^p(\mu)$. As an immediate consequence we get that any Calderón-Zygmund singular integral operator is also bounded on these weighted $L^p$-spaces.

**Definition A.11 (Muckenhoupt weight)** *Let $p \in (1,\infty)$. We say $\omega$ is a weight in Muckenhoupt's $A_p(\mu)$-class, or an $A_p(\mu)$-weight, if $\omega \ge 0$ is a locally $\mu$-integrable function in $X$ such that*

$$\sup_B \mu(B)^{-1} \int_B \omega\, d\mu \Big[\mu(B)^{-1} \int_B \omega^{-\frac{1}{p-1}}\, d\mu\Big]^{p-1} \le c(p,\omega) < \infty\,, \tag{A.9}$$

*where the supremum is taken with respect to all d-balls $B$. The best $A_p$ constant of $\omega$ is denoted by $[\omega]_{A_p}$.*

**Remark A.12** *In the sequel we sometimes identify $\omega$ with the measure $\omega\, d\mu$ in the sense of (A.7).*

First we annotate that there is another related inequality:

$$f_B \le \left(\frac{[\omega]_{A_p}}{\omega(B)} \int_B |f|^p\, d\omega\right)^{\frac{1}{p}} \tag{A.10}$$

for all balls $B$ and any locally $\mu$-integrable function $f$. In fact, inequality (A.10) is equivalent to the Muckenhoupt condition (A.9). In order to see this we first suppose that (A.9) holds true. But then

$$\mu(B)^{-p}\Big(\int_B \omega^{-\frac{1}{p-1}}\, d\mu\Big)^{p-1} \le \frac{[\omega]_{A_p}}{\omega(B)}\,,$$

and consequently we have

$$\begin{aligned}(f_B)^p &= \mu(B)^{-p}\Big(\int_B f\,\omega^{\frac{1}{p}}\omega^{-\frac{1}{p}}\, d\mu\Big)^p \le \mu(B)^{-p}\Big(\int_B |f|^p\,\omega\, d\mu\Big)\Big(\int_B \omega^{-\frac{1}{p-1}}\, d\mu\Big)^{p-1} \\ &\le \frac{[\omega]_{A_p}}{\omega(B)} \int_B |f|^p\, d\omega\end{aligned}$$

by Hölder's inequality. Conversely, we set $f_\varepsilon := (\omega+\varepsilon)^{-\frac{1}{p-1}}$ and let $\varepsilon \to 0$. An immediate but important consequence of characterization (A.10) is this result.

**Lemma A.13** *Let $\omega \in A_p(\mu)$ for some fixed $1 < p < \infty$. Then $\omega$ satisfies the doubling condition. Moreover, if $\mu = \mathcal{L}^n$, then $\omega$ and the Lebesgue measure are mutually absolutely continous.*

**Proof:** Choosing $f = \chi_{B_R}$ we obtain by (A.10) that

$$\left(f_{B_{2R}}\right)^p \;\leq\; \frac{[\omega]_{A_p}}{\omega(B_{2R})} \int_{B_{2R}} |f|^p \, d\omega \;=\; [\omega]_{A_p} \, \frac{\omega(B_R)}{\omega(B_{2R})} \, .$$

Moreover, we have

$$b^{-1} \;\leq\; \frac{\mu(B_R)}{\mu(B_{2R})} \;=\; f_{B_{2R}} \, .$$

by virtue of (A.2). But this implies $\omega(B_{2R}) \leq b^p [\omega]_{A_p} \omega(B_R)$ and hence the claim. ■

Definition A.11 also reveals the following simple property of the class $A_p$.

**Lemma A.14** *Let $p, p' \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\omega \in A_p(\mu)$ if and only if $\omega^{-\frac{p'}{p}} \in A_{p'}(\mu)$.*

**Proof:** Let $\omega$ be as in definition A.11. Then

$$\begin{aligned} &\mu(B)^{-1} \int_B \omega^{-\frac{p'}{p}} \, d\mu \Big[\mu(B)^{-1} \int_B \Big(\omega^{-\frac{p'}{p}}\Big)^{-\frac{1}{p'-1}} \, d\mu\Big]^{p'-1} \\ = \;& \bigg(\Big[\mu(B)^{-1} \int_B \omega^{-\frac{1}{p-1}} \, d\mu\Big]^{p-1} \mu(B)^{-1} \int_B \omega \, d\mu\bigg)^{\frac{1}{p-1}} \;\leq\; [\omega]_{A_p}^{\frac{1}{p-1}} \;<\; \infty \, . \end{aligned}$$

Conversely, given $\omega^{-\frac{p'}{p}} \in A_{p'}(\mu)$, we see that (A.9) is satisfied with $c(p,\omega) = [\omega^{-\frac{p'}{p}}]_{A_{p'}}^{p-1}$. ■

The next step is to derive a reverse Hölder inequality from which follows that $\omega \in A_{p-\varepsilon}(\mu)$ for some $\varepsilon > 0$ if $\omega \in A_p(\mu)$. This in turn implies that the Muckenhoupt condition (A.9) is equivalent to inequality (A.8). For the details of these results see e.g. [52, 53, 59]. We should also note that we still have

$$\|f\|_{L^p(\omega)} \;\leq\; c(b,p) \, \|f^\#\|_{L^p(\omega)} \, .$$

The proof is exactly the same as in the unweighted case.

With all these weighted $L^p$-estimates we can finally state the main result of this section.

**Theorem A.15** *Let $1 < p < \infty$, $\omega \in A_p(\mu)$ and $T : L^{p_0}(\mu) \to L^{p_0}(\mu)$, for some $p_0 > 1$, be a Calderón-Zygmund operator. Then there exists a positive constant $c = c(b, p_0, p, \omega)$ such that*

$$\|Tf\|_{L^p(\omega)} \;\leq\; c \, \|f\|_{L^p(\omega)}$$

*for all $f \in L^p(\omega)$.*

In order to check the validity of theorem A.15 we simply follow the same line of argument as in the unweighted case (see subsection A.1).

## A.3 Historical Background

In the 1950s, Zygmund and his doctoral student Calderón came up with an entirely new strategy for proving $L^p$-estimates, [13]. They found out that, for every $f \in L^1(\mathbb{R}^n)$, $Tf$ belongs to weak-$L^1$ if the operator $T$ is bounded on $L^2(\mathbb{R}^n)$ and if its distributional kernel satisfies some weak assumptions. The crucial step in deriving this result is the Calderón-Zygmund decomposition. It states that an arbitrary integrable function $f$ can be split into the sum $u + v$ of a "small" and a "large" function where $|u|$ is pointwise bounded by a given threshold $\lambda$ and belongs to $L^2(\mathbb{R}^n)$, while $v$ is oscillating and supported in a set of small measure. The Marcinkiewicz interpolation theorem then implies the desired $L^p$-estimate for $1 < p \leq 2$. Applying the same arguments to $T^*$ instead yields the same bound for $2 \leq p < \infty$.

Then by the 1970s, Coifman and Weiss [15] discovered that the theory of singular integral operators does not depend on the Euclidean structure and, therefore, they introduced spaces of homogeneous type. These are spaces to which the Calderón-Zygmund theory extends in a natural way. Indeed, a doubling property of the measure and the covering results of Vitali and Whitney are the essential ingredients which are needed to get all the mechanisms to work.
The Hardy space $\mathcal{H}^1$ and the space $BMO$ are borderline cases of this theory and closely connected to the theory of integral operators, [14]. As an example we would like to mention the celebrated $T(1)$ theorem which gives a criterion for $L^2$-continuity, and hence $L^p$-boundedness, of certain singular integral operators. It was first proven on $\mathbb{R}^n$ by David and Journé [19], and later generalized to homogeneous-type spaces by these authors and Semmes [20]. It is also remarkable that the theory of the Muckenhoupt class $A_p$, which was introduced by Muckenhoupt [64], carries over almost word for word to this general geometric framework. This issue has been addressed by a variety of authors, as for example by Coifman and Fefferman [14], Muckenhoupt [65] and Stein [74]. An extensive treatment of the Muckenhoupt class from a different point of view may be found in [41, 46].

# B Bessel Functions

The Bessel functions of first kind are defined as the complex functions represented by the power series

$$J_\nu(z) \;=\; \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)} \left(\frac{1}{2}\,z\right)^{2j+\nu}.$$

Here $\nu$ is an arbitrary real or complex number called the order of the Bessel function; the most common cases are Bessel functions in the form of integer or half-integer order. The notation $\Gamma$ denotes the gamma function defined by

$$\Gamma(z) \;=\; \int_0^\infty t^{z-1}\, e^{-t}\, dt\,.$$

$J_\nu$ is convergent everywhere in the complex plane $\mathbb{C}$. The Bessel function of second kind $Y_\nu$, called Weber function, is generated by a special linear combination of $J_\nu$: For noninteger order $\nu$, $Y_\nu$ is related to $J_\nu$ by

$$Y_\nu(z) \;=\; \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}\,.$$

When $\nu \in \mathbb{Z}$,

$$Y_\nu(z) \;=\; \lim_{\mu\to\nu} Y_\mu(z)\,.$$

It can be shown that $J_\nu$ and $Y_\nu$ form a fundamental system of solutions for the Bessel differential equation

$$z^2\,\partial_z^2 v \;+\; z\,\partial_z v \;+\; (z^2+\nu^2)\,v \;=\; 0\,.$$

In a similar way, A. B. Basset (1888) and H. M. MacDonald (1899) introduced the modified Bessel functions

$$I_\nu(z) \;=\; \sum_{j=0}^{\infty} \frac{1}{j!\,\Gamma(j+\nu+1)} \left(\frac{1}{2}\,z\right)^{2j+\nu}$$

and

$$K_\nu(z) \;=\; \frac{1}{2}\,\pi\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}\,, \qquad \nu \notin \mathbb{Z}\,,$$

(or as the limit $K_\nu(z) = \lim_{\mu\to\nu} K_\mu(z)$ if $\nu$ is an integer) which satisfy the modified Bessel differential equation

$$z^2\,\partial_z^2 v \;+\; z\,\partial_z v \;-\; (z^2+\nu^2)\,v \;=\; 0\,. \tag{B.1}$$

The Wronskian of $I_\nu$ and $K_\nu$ is $\mathcal{W}\big(I_\nu(z), K_\nu(z)\big) \;=\; \frac{1}{z}$, and hence $I_\nu$ and $K_\nu$ are linearly independent

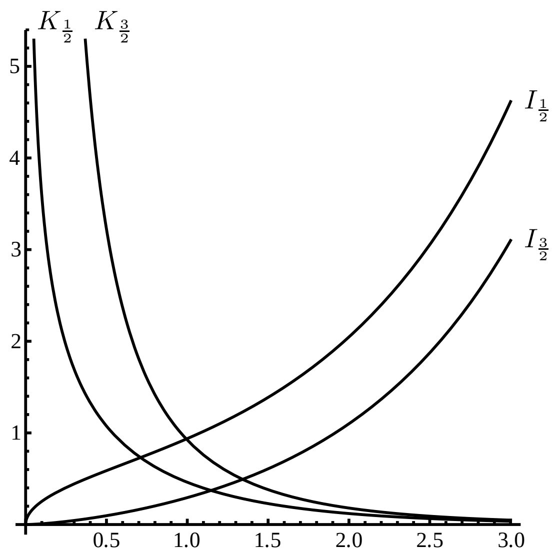


*Figure B.1: Modified Bessel functions*

solutions of (B.1). Their asymptotic behavior is described by the following formulas: For $0 < |z| \ll \sqrt{\nu+1}$, we have

$$I_\nu(z) \;\approx\; \frac{1}{\Gamma(\nu+1)} \left(\frac{1}{2}\, z\right)^\nu, \quad \nu \notin -\mathbb{N},$$

and

$$K_\nu(z) \;\approx\; \begin{cases} \frac{1}{2}\,\Gamma(\nu)\big(\frac{1}{2}\, z\big)^{-\nu} & \text{if } \mathfrak{Re}(\nu) > 0 \\ -\ln\big(\frac{z}{2}\big) - 0.5772... & \text{if } \nu = 0\,, \end{cases}$$

while for large arguments $z \gg |\nu^2 - \frac{1}{4}|$, the modified Bessel functions behave like

$$I_\nu(z) \;\approx\; \frac{e^z}{\sqrt{2\pi z}} \left(1 \,-\, \frac{4\nu^2 - 1^2}{1(8z)} \left(1 \,-\, \frac{4\nu^2 - 3^2}{2(8z)} \left(1 \,-\, \frac{4\nu^2 - 5^2}{3(8z)} \left(1 \,-\, \ldots\right)\right)\right)\right)$$

and

$$K_\nu(z) \;\approx\; \sqrt{\frac{\pi}{2z}}\, e^{-z} \left(1 \,-\, \frac{4\nu^2 - 1^2}{1(8z)} \left(1 \,-\, \frac{4\nu^2 - 3^2}{2(8z)} \left(1 \,-\, \frac{4\nu^2 - 5^2}{3(8z)} \left(1 \,-\, \ldots\right)\right)\right)\right).$$

Note that all the terms except the first drop out when $\nu = \frac{1}{2}$. In fact, these approximations then become

$$I_{\frac{1}{2}}(z) \;=\; \sqrt{\frac{2}{\pi z}}\, \sinh(z) \qquad \text{and} \qquad K_{\frac{1}{2}}(z) \;=\; \sqrt{\frac{\pi}{2z}}\, e^{-z}\,.$$

The functions $I_\nu$ and $K_\nu$ satisfy the recurrence relations

$$Z_{\nu-1}(z) \,-\, Z_{\nu+1}(z) \;=\; \frac{2\nu}{z}\, Z_\nu(z)$$

and

$$Z_{\nu-1}(z) \,-\, Z_{\nu+1}(z) \;=\; 2\,\frac{d}{dx}\, Z_\nu(z)\,,$$

where $Z_\nu$ denotes either $I_\nu$ or $e^{\nu\pi i}\, K_\nu$. That way, one can derive Bessel functions of higher orders (or higher derivatives) from Bessel functions of lower orders for all real values of $\nu$. In particular, it follows

$$\left(\frac{1}{z}\,\frac{d}{dz}\right)^k \big(z^\nu\, Z_\nu(z)\big) \;=\; z^{\nu-k}\, Z_{\nu-k}(z)\,,$$

and

$$\left(\frac{1}{z}\,\frac{d}{dz}\right)^k \big(z^{-\nu}\, Z_\nu(z)\big) \;=\; z^{-\nu-k}\, Z_{\nu+k}(z)\,.$$

For more details on (modified) Bessel functions and their properties we refer the reader to [12] and the references therein.

# C Notation

| | |
|---|---|
| $\overline{A}$ | the closure of the set $A$ |
| $\mathring{A}$ | the interior of the set $A$ |
| $A^{\complement}$ | the complement of the set $A$ |
| $\partial A$ | the boundary of $A$ |
| $\mathbb{N}$ | the set of all natural numbers $\{1, 2, \dots\}$ |
| $\mathbb{N}_0$ | $\mathbb{N} \cup \{0\}$ |
| $\mathbb{Z}$ | the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$ |
| $\mathbb{R}^n$ | $n$-dimensional real Euclidean space, $\mathbb{R}^1 = \mathbb{R}$ |
| $\mathbb{R}_+$ | the set of positive real numbers $(0, \infty)$ |
| $H$ | the upper half space $\{x \in \mathbb{R}^n \mid x_n > 0\}$ |
| $\Omega$ | usually used to denote an open subset in $\mathbb{R}^n$ |
| $B_R(x)$ | the open ball around $x$ with radius $R > 0$, often with respect to a metric $d$ |
| $Q_R(t, x)$ | the parabolic cylinder $(t - R^4, t] \times B_R(x)$ |
| $dist(A, B)$ | the infimum of the distances between any two of their respective points, $dist(x, A) = \inf\big\{d(x, y) \mid y \in A\big\}$ |
| $e_i$ | the standard unit vector $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with only the $i$-th entry holding a value of 1 |
| $\lvert v\rvert$ | the Euclidean norm of $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, i.e. $\sqrt{{v_1}^2 + \cdots + v_n^2}$ |
| $v \cdot w$ | the standard inner product on $\mathbb{R}^n$, i.e. $v \cdot w = v_1 w_1 + \cdots + v_n w_n$ |
| $\mathcal{L}^n$ | the $n$-dimensional Lebesgue measure, $\lvert A\rvert = \int_A d\mathcal{L}^n = \int_A dx$ |
| $\mu_\sigma$ | the measure ${x_n}^\sigma dx$ for some $\sigma > -1$, $\lvert A\rvert_\sigma = \int_A d\mu_\sigma$ |
| $c, C$ | (generic) constants which may vary from line to line |
| $f \lesssim g$ | $f \leq c\, g$ for some constant $c$ |
| $f \gtrsim g$ | $g \leq c\, f$ form some constant $c$ |
| $f \sim g$ | $f \lesssim g \lesssim f$ |
| $f \ll g$ | $f \leq C\, g$ for a given constant $C$ much larger than 1 |
| $f \gg g$ | $g \ll f$ |
| $\chi_A$ | the indicator function of the set $A$ |
| $spt\, f$ | the support of the function $f$, that is the closure of the set of points where the function is not zero-valued |
| $P(f)$ | the positivity set of the function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $P_t(f) = P(f)$ at time $t \in \mathbb{R}$ |

# Bibliography


[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, vol. 140, Academic Press, 2003.

[2] S. B. Angenent, *Nonlinear analytic semiflows*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 115(1-2): pp. 91–107, 1990.

[3] J. Becker and G. Grün, *The thin-film equation: recent advances and some new perspectives*, Journal of Physics: Condensed Matter, 17: p. S291, 2005.

[4] F. Bernis and R. Ferreira, *Source-type solutions to thin-film equations in higher dimensions*, European Journal of Applied Mathematics, 8(05): pp. 507–524, 1997.

[5] F. Bernis and A. Friedman, *Higher order nonlinear degenerate parabolic equations*, Journal of Differential Equations, 83(1): pp. 179–206, 1990.

[6] F. Bernis, L. A. Peletier and S. M. Williams, *Source type solutions of a fourth order nonlinear degenerate parabolic equation*, Nonlinear Analysis, 18(3): pp. 217–234, 1992.

[7] A. L. Bertozzi, *The mathematics of moving contact lines in thin liquid films*, Notices of the American Mathematical Society, 45: pp. 689–697, 1998.

[8] A. L. Bertozzi and M. Pugh, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Communications on Pure and Applied Mathematics, 49(2): pp. 85–123, 1996.

[9] M. Bertsch, R. Dal Passo, H. Garcke and G. Grün, *The thin viscous flow equation in higher space dimensions*, Advances in Differential Equations, 3(3): pp. 417–440, 1998.

[10] M. Bertsch, L. Giacomelli and G. Karali, *Thin-film equations with "partial wetting" energy: existence of weak solutions*, Physica D: Nonlinear Phenomena, 209(1): pp. 17–27, 2005.

[11] S. Boatto, L. P. Kadanoff and P. Olla, *Traveling-wave solutions to thin-film equations*, Physical Review E, 48(6): p. 4423, 1993.

[12] R. F. Boisvert, C. W. Clark, D. W. Lozier and F. W. J. Olver (editors), *NIST handbook of mathematical functions*, Cambridge University Press, 2010.

[13] A. P. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Mathematica, 88(1): pp. 85–139, 1952.

[14] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, 51(3): pp. 241–250, 1974.

[15] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogenes*, Lecture Notes in Mathematics, 242, 1971.

[16] R. Dal Passo, H. Garcke and G. Grün, *On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions*, SIAM Journal on Mathematical Analysis, 29(2): pp. 321–342, 1998.

[17] P. Daskalopoulos and R. Hamilton, *Regularity of the free boundary for the porous medium equation*, Journal of the American Mathematical Society, 11(4): pp. 899–966, 1998.

[18] P. Daskalopoulos, R. Hamilton and K. Lee, *All time $C^\infty$-regularity of the interface in degenerate diffusion: a geometric approach*, Duke Mathematical Journal, 108(2): pp. 295–327, 2001.

[19] G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, The Annals of Mathematics, 120(2): pp. 371–397, 1984.

[20] G. David, J. L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Revista Matemática Iberoamericana, 1(4): pp. 1–56, 1985.

[21] P. G. De Gennes, F. Brochard-Wyart and D. Quéré, *Capillarity and wetting phenomena: drops, bubbles, pearls, waves*, Springer Verlag, 2004.

[22] K. Deimling, *Nonlinear functional analysis*, Springler-Verlag, Berlin, 1985.

[23] D. G. Deng and Y. Han, *Harmonic analysis on spaces of homogeneous type*, vol. 1966, Springer Verlag, 2009.

[24] S. Dharmatilleke, A. H. Khaw, P. M. Moran, K. W. Tan, M. L. Chan and I. Rodriguez, *Fluidic lenses with variable focal length*, Applied Physics Letters, 88(4): pp. 041120–041120, 2006.

[25] P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*, vol. 5, Walter de Gruyter, 1997.

[26] Y. B. Egorov, *Subelliptic operators*, Russian Mathematical Surveys, 30(2): pp. 59–118, 1975.

[27] L. C. Evans, *Partial differential equations*, vol. 19, American Mathematical Society, Providence, RI, 1998.

[28] E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Archive for Rational Mechanics and Analysis, 96(4): pp. 327–338, 1986.

[29] C. L. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, Conference on Harmonic Analysis in Honor of Antoni Zygmund, 1: pp. 590–606, 1981.

[30] C. L. Fefferman and A. Sánchez-Calle, *Fundamental solutions for second order subelliptic operators*, The Annals of Mathematics, 124(2): pp. 247–272, 1986.

[31] J. Ferguson, *A brief survey of the history of the calculus of variations and its applications*, arXiv preprint arXiv:math/0402357v1, 2004.

[32] G. B. Folland, *Real analysis: modern techniques and their applications*, John Wiley & Sons, NY, 1984.

[33] A. Friedman, *Partial differential equations*, Dover Publications Inc., NJ, 2008.

[34] L. Giacomelli and H. Knüpfer, *Flat data solutions to the thin-film equation do not rupture*, SFB 611, 2006.

[35] L. Giacomelli and H. Knüpfer, *A free boundary problem of fourth order: Classical solutions in weighted Hölder spaces*, Communications in Partial Differential Equations, 35(11): pp. 2059–2091, 2010.

[36] L. Giacomelli, H. Knüpfer and F. Otto, *Maximal regularity for a degenerate operator of fourth order*, SFB 611, 2004.

[37] L. Giacomelli, H. Knüpfer and F. Otto, *Smooth zero-contact-angle solutions to a thin-film equation around the steady state*, Journal of Differential Equations, 245(6): pp. 1454–1506, 2008.

[38] G. Grün, *Droplet spreading under weak slippage—existence for the Cauchy problem*, Communications in Partial Differential Equations, 29(11-12): pp. 1697–1744, 2005.

[39] G. H. Hardy, *Note on a theorem of Hilbert*, Mathematische Zeitschrift, 6(3): pp. 314–317, 1920.

[40] J. Heinonen, *Nonsmooth calculus*, Bulletin of the American Mathematical Society, 44(2): pp. 163–232, 2007.

[41] J. Heinonen, T. Kilpelèainen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, 1993.

[42] J. M. Hill and N. F. Smyth, *High-order nonlinear diffusion*, IMA Journal of Applied Mathematics, 40(2): pp. 73–86, 1988.

[43] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Mathematica, 119(1): pp. 147–171, 1967.

[44] J. Hulshof, *Some aspects of the thin film equation*, in *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, vol. 202 of Progr. Math., pp. 291–302, Birkenhäuser, Basel, 2001.

[45] M. N. Huxley, *Exponential sums and lattice points III*, Proceedings of the London Mathematical Society, 87(3): pp. 591–609, 2003.

[46] T. Kilpeläinen, *Weighted Sobolev spaces and capacity*, Annales Academiæ Scientiarum Fennicæ Series A. I. Mathematica, 19(1): pp. 95–113, 1994.

[47] H. Knüpfer, *Classical solutions for a thin-film equation*, Ph.D. thesis, Universitäts- und Landesbibliothek Bonn, 2007.

[48] H. Knüpfer, *Well-posedness for the Navier slip thin-film equation in the case of partial wetting*, Communications on Pure and Applied Mathematics, 64(9): pp. 1263–1296, 2011.

[49] H. Knüpfer and N. Masmoudi, *Darcy flow on a plate with prescribed contact angle—well-posedness and lubrication approximation*, Preprint, 2010.

[50] H. Koch, *Mixed problems for fully nonlinear hyperbolic equations*, Mathematische Zeitschrift, 214(1): pp. 9–42, 1993.

[51] H. Koch, *Non-Euclidean singular integrals and the porous medium equation*, Habilitation thesis, Universität Heidelberg, 1999.

[52] H. Koch, *Partial differential equations and singular integrals*, in *Dispersive nonlinear problems in mathematical physics*, vol. 15, pp. 59–122, Dipartimento di Matematica, Seconda Università di Napoli, Caserta, 2004.

[53] H. Koch, *Partial differential equations with non-Euclidean geometries*, Discrete and Continuous Dynamical Systems Series S, 1(3): pp. 481–504, 2008.

[54] H. Koch and T. Lamm, *Geometric flows with rough initial data*, Asian Journal of Mathematics, 16(2): pp. 209–235, 2012.

[55] H. Koch and D. Tataru, *Well-posedness for the Navier–Stokes equations*, Advances in Mathematics, 157(1): pp. 22–35, 2001.

[56] A. Kufner, *Weighted Sobolev spaces*, John Wiley & Sons, NY, 1985.

[57] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, vol. 23, American Mathematical Society, Providence, RI, 1968.

[58] E. Le Donne, *Lecture notes on sub-Riemannian geometry*, Preprint, 2010.

[59] A. K. Lerner, *An elementary approach to several results on the Hardy-Littlewood maximal operator*, Proceedings of the American Mathematical Society, 136(8): pp. 2829–2834, 2008.

[60] C. S. Lin, *Interpolation inequalities with weights*, Communications in Partial Differential Equations, 11(14): pp. 1515–1538, 1986.

[61] I. Lunati, *Young's law and the effects of interfacial energy on the pressure at the solid-fluid interface*, Physics of Fluids, 19(11): pp. 118105–1–118105–4, 2007.

[62] N. G. Meyers and J. B. Serrin, $H = W$, Proceedings of the National Academy of Sciences USA, 51: pp. 1055–1056, 1964.

[63] J. W. Miles, *The hydrodynamic stability of a thin film of liquid in uniform shearing motion*, Journal of Fluid Mechanics, 8(4): pp. 593–610, 1960.

[64] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Transactions of the American Mathematical Society, 165, 1972.

[65] B. Muckenhoupt, *The equivalence of two conditions for weight functions*, Studia Mathematica, 49: pp. 101–106, 1974.

[66] A. Nagel, E. Stein and S. Wainger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Mathematica, 155(1): pp. 103–147, 1985.

[67] F. Nicolosi and I. V. Skrypnik, *Nirenberg-Gagliardo interpolation inequality and regularity of solutions of nonlinear higher order equations*, Topological Methods in Nonlinear Analysis, 7: pp. 327–347, 1996.

[68] A. Oron, S. H. Davis and S. G. Bankoff, *Long-scale evolution of thin liquid films*, Reviews of Modern Physics, 69(3): pp. 931–980, 1997.

[69] M. Paluszynski and K. Stempak, *On quasi-metric and metric spaces*, Proceedings of the American Mathematical Society, 137(12): pp. 4307–4312, 2009.

[70] V. F. Rickey, *Build a brachistochrone and captivate your class*, Hands On History: A Resource for Teaching Mathematics, 2008.

[71] A. Sánchez-Calle, *Fundamental solutions and geometry of the sum of squares of vector fields*, Inventiones Mathematicae, 78(1): pp. 143–160, 1984.

[72] M. K. Smith and S. H. Davis, *The instability of sheared liquid layers*, Journal of Fluid Mechanics, 121: pp. 187–206, 1982.

[73] E. M. Stein, *Singular integrals and differentiability properties of functions*, vol. 30, Princeton University Press, Princeton, NJ, 1970.

[74] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, vol. 43, Princeton University Press, Princeton, NJ, 1993.

## Zusammenfassung der Arbeit

Betrachtet man im freien Randwertproblem für den Stokes-Fluss mit Oberflächenspannung den Grenzübergang für dünne Schichten, so ergibt sich formal aus den Navier-Stokes-Gleichungen für inkompressible Flüssigkeiten die Dünne-Film-Gleichung. Diese ist eine partielle Differentialgleichung vierter Ordnung mit degenerierter Parabolizität. Sie hat die Form

$$\partial_s h \, + \, \nabla_y \cdot (h^m \, \nabla_y \Delta_y h) \; = \; 0 \, .$$

In der vorliegenden Arbeit beschäftigen wir uns mit der Frage, ob schwache Lösungen des dazugehörigen Anfangswertproblems (mit linearer Mobilität $m = 1$) existieren, unter welchen Bedingungen an die Anfangswerte diese eindeutig sind und wie regulär sie sind.

Im ersten Schritt fixieren wir den freien Rand, indem wir auf der Positivitätsmenge die unabhängige Koordinate $y_n$ mit der abhängigen $h = h(s, y)$ vertauschen (von Mises-Transformation), und anschließend die resultierende Gleichung um die stationäre Lösung $y_n^2$ linearisieren. Somit gelangen wir zu der Gleichung

$$\partial_t u \, + \, x_n^{-1} \, \Delta(x_n^3 \, \Delta u) \, - \, 4 \, \Delta_{\mathbb{R}^{n-1}} u \; = \; f[u]$$

für die Störung der (transformierten) stationären Lösung. Sämtliche nichtlinearen Ausdrücke sammeln wir auf der rechten Seite. Wir ignorieren für den Moment die Abhängigkeit von $f[u]$ von $u$ und arbeiten eine umfassende Energietheorie für schwache Lösungen der linearen Gleichung aus. Die erzielten Resultate basieren maßgeblich auf gleichmäßigen (Energie-)Abschätzungen in gewichteten Normen. Eine wichtige Rolle spielt hierbei auch die intrinsische Geometrie, die sich auf natürliche Weise aus dem linearen Operator ergibt. Das spiegelt sich unter anderem in der Tatsache wider, dass die Green'sche Funktion mit all ihren Ableitungen einer für unsere weitere Analyse entscheidenden Gauß'schen Abschätzung genügt. Eine solche Abschätzung besagt, dass der Green'sche Kern exponentiell abfallende Ausläufer hat, was uns nun den Weg bereitet, unter Zuhilfenahme der Calderón-Zygmund-Theorie für singuläre Integrale lineare Normabschätzungen von $u$ gegen die Inhomogenität in geeigneten Normen zu generieren. Eine Abschätzung der Nichtlinearität gegen $u$ ist ebenfalls möglich und wir können damit ein Fixpunkt-Argument durchführen. Entscheidend bei diesen Abschätzungen ist die Konstruktion von skalierungsinvarianten Normen basierend auf parabolischen Zeit-Raum-Zylindern. Wir erhalten dann eine eindeutige Lösung der Störungsgleichung, und somit der transformierten Gleichung, ein möglicherweise optimales Resultat hinsichtlich der Regularität (Lipschitz) der Anfangswerte. Letztendlich gelingt es uns diese Ergebnisse auf das Ursprungsproblem zurückzuspielen und somit die Existenz einer eindeutigen (schwachen) Lösung nachzuweisen, sofern der Anfangswert $h_0$ nah genug an der stationären Lösung lag. Als Nebenprodukt dieser Schritte zeigen wir auch noch, dass die der eindeutigen Lösung zugeordneten Niveaulinien analytisch sind. Insbesondere bedeutet dies, dass die bewegte Kontaktlinie (Niveaulinie zur Höhe 0) maximale Regularität aufweist.

Die Arbeit gliedert sich nun wie folgt: Nach einigen einleitenden Worten in Kapitel 1 schaffen wir im Folgekapitel das für unsere weitere Analyse notwendige Rüstzeug und untersuchen gewichtete Sobolevräume. Kapitel 3 motiviert die im weiteren angestellten Betrachtungen der transformierten Gleichung und widmet sich der intrinsische Geometrie. In Kapitel 4 befassen wir uns mit der linearen Gleichung. Im letzten Kapitel beweisen wir die Gauß'sche Abschätzung mit all ihren Konsequenzen, betrachten die Nichtlinearität und diskutieren, was das erzielte Eindeutigkeitsresultat für unser Ursprungsproblem bedeutet.