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Contents

Chapter 1

Introduction and statement of results

On any length scale, partial differential equations (PDEs) are an invaluable tool for modeling the behavior of the physical reality that surrounds us. It is impossible to name them all since they can be found anywhere one looks, across all areas and length scales: From the Gross-Pitaevskii equation in the quantum world of dilute boson gases to the curvature of our spacetime universe as described by the Einstein field equation, or from biological processes on the level of cells in the context of chemotaxis and the Keller-Siegel model to Navier-Stokes equations (and its countless brethren) describing the motion of fluids and thus, in some variations, the effect of a tsunami on the coast of Japan on the sea levels on Hawaii.

With the ubiquity of PDEs comes the need to study them, from on a variety of vantage points ranging from theoretical to applied. Relevant questions abound: When does a given equation have solutions? Can they develop singularities, and if so, what form do they take? How well can solutions be approximated numerically? And how well does the model correspond to reality?

In this dissertation, we treat problems related to the global existence theory of some *dispersive PDEs*. That is, we try to abstractly construct solutions which exist globally in time and have "good" properties, under conditions on the data.

Loosely speaking, dispersive PDEs exhibit wave-like properties and interact well with the Fourier transform, so that solutions can be viewed as being a superposition of different frequency waves. It is this viewpoint we focus on, and we are motivated less by equations modeling a concrete physical process, but by the interplay of nonlinear interactions in a dispersive setting and its effects on the existence of global solutions, especially concerning the assumptions on the initial data. Assumptions typically imposed include (strong or weak) differentiability or spatial decay. Especially the latter is an assumption one may wish to avoid when trying to produce a result for "natural" initial data: A dispersive (linear) equation usually disperses over time, in the sense that initially localized data gets spread out over larger and larger spatial regions. On the other hand, $L^2(\mathbb{R}^n)$ is conserved, which suggests that one should aim for results for initial data in $L^2(\mathbb{R}^n)$ based Sobolev spaces such as $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$ and try to avoid requiring "strong" decay conditions (like weighted Lebesgue spaces or polynomial decay).

The outline of this dissertation is as follows. In section 1.1, we collect the basic notation, definitions and conventions. We then continue in chapter 2 to describe in a colloquial manner what a dispersive PDE is, present the geometry associated to such an equation - that is, the characteristic hypersurface Σ_h - and derive some heuristics to describe the behavior of linear solutions.

In particular, we describe how to quickly guess the $L^1 \to L^\infty$ decay of a linear dispersive PDE, the outcome of which we use in section 2.1 to derive the fundamental Strichartz estimates in an abstract setting. This allows us to treat all the Strichartz estimates occuring in this work in a unified and transparent manner. As a last point in that chapter, we outline briefly in section 2.2 the typical notion of a solution which applies in our context as well as the concept of well-posedness.

Moving on, in section 3.1 we define the Bourgain spaces $X^{s,b}$, briefly lay out some of their key properties and discuss their shortcomings with respect to global existence results. This leads to section 3.2 and the introduction and presentation of the spaces U^2 and V^2, upon which much of the techniques in this dissertation are based.

This concludes the expository part of this work, and in chapter 4 we prove the first theorem. Namely, we consider the Klein-Gordon equation with mass $m > 0$ in spatial dimensions $n \geq 2$,

$$\begin{aligned}(\Box + m^2)u &= Q(u)\\ u(0,x) &= u_0(x)\\ \partial_t u(0,x) &= u_1(x)\end{aligned}$$

where $\Box = \partial_{tt} - \Delta$, Q is a polynomial of terms of order at least two, $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and the inital data u_0 and u_1 are in Sobolev spaces $H^s(\mathbb{R}^n)$ and $H^{s-1}(\mathbb{R}^n)$, respectively, where $s \geq s_0$ depends on the degree of Q. The main result (formulated more generally for systems with masses satisfying a certain nondegeneracy condition) is that for small enough initial data, global solutions exist, become asymptotically free and depend on the initial data in a smooth way. This is most relevant in two dimensions, $n = 2$, when a quadratic nonlinearity is by far too weak to justify global existence based on just the decay in time of solutions of the linear equation ($Q = 0$).

Instead, we exploit a "non-resonance" condition inherent in the problem. This condition is always satisfied in the scalar case and manifests itself through a simple inequality involving the different masses in the case of systems of above type. This inequality implies, in a quantitative way, that points from two characteristic hypersurfaces cannot sum up to a point on the third hypersurface, and this allows for nonlinear estimates using U^2 and V^2 spaces, bilinear refinements of Strichartz' inequality, and a contraction mapping argument.

A similar approach is used in chapter 5, where we derive a similar result for the nonlinear Schrödinger equation

$$iu_t - \Delta u = \bar{u}\partial_{x_1}\bar{u}$$

with initial data in the scaling critical space $\dot{H}^{\frac{n-2}{2}}$. Again, we obtain small data global existence, scattering and good dependence on the initial data, and it is for similar reasons: A non-resonance condition holds except at a single point, where the derivative in the nonlinearity actually acts as an improvement in the estimates, hence allowing estimates along lines similar to chapter 4. This demonstrates that such an argument can also work if the non-resonance condition is violated; however, the nonlinearity needs to compensate when this happens. In a way, this effect here can be seen as a trivial kind of "null condition", akin to that of Klainerman-Machedon (see [KM93]). It's important to note that in this specific example, the placement of complex conjugates is critical, as any other combination induces a lot of resonance which our methods cannot handle, as will be discussed later in section 7.3.

As a last item in the series of small data global existence results, we treat in chapter 6 a problem related to the Novikov-Veselov (NV) equation

$$u_t + (\partial^3 + \bar{\partial}^3)u = N_{NV}(u) \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}$$

where $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$. This equation arises as a natural two-dimensional analogue to the Korteweg-de Vries equation; it is also completely integrable, and related to the - also completely integrable - modified Novikov-Veselov (mNV) equation through a Miura-type transform, which formally maps (mNV) solutions to solutions of (NV). Both equations share the same linear structure, but the nonlinearity is quadratic for (NV) and cubic for (mNV), making (mNV) the easier problem to treat with our methods. In fact, we can do so relying only on bilinear refinements of Strichartz' inequality and not on a non-resonance condition. Such a condition would be needed to treat (NV); however, it doesn't hold. Consequently, we derive only a small data global existence result for the modified Novikov-Veselov equation.

Yet, we return to (NV), among others, in the last chapter. Now, instead of looking at non-resonant situations, we explore the implications of resonance on global solutions. At the heart of this is the observation that if we have a quadratic nonlinearity, a smooth solution operator and a well-defined scattering operator, then we can compute the second derivative of the scattering operator and obtain a so-called convolution estimate of three characteristic hypersurfaces, taking the form

$$\|fd\mathcal{H}_{\Sigma_1} * gd\mathcal{H}_{\Sigma_2}\|_{L^2(\Sigma_3)} \leq C\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}$$

for three (regular) two-dimensional hypersurfaces $\Sigma_i \subset \mathbb{R}^3$. Such a convolution estimate lives on the set of resonant points (x, y, z) for which $x \in \Sigma_1$, $y \in \Sigma_2$ and $z = x + y \in \Sigma_3$. This is the same set that vanishes for the Klein-Gordon equations we treat and is an isolated point in the case of the nonlinear Schrödinger equation above.

Hence, for an equation with resonance, we can try to use the above to arrive at a contradiction. The reason is that the optimal constant in such a convolution estimate depends on the local transversality of the three surfaces, as measured by the determinant of the unit normals. Thus one way to obtain a negative result is to find a point around which this local transversality criterion degenerates. We show that at such a point, one can localize and show that no convolution estimate can hold, which in turn contradicts the assumptions on the solution and scattering operator of the dispersive PDE we are investigating.

We use this to show that the techniques used in this paper in chapters 4 to 6 cannot be adapted to deal with some more resonant situations. This includes the Klein-Gordon systems for which the mass condition $m_1+m_2 > m_3$ is violated, a quadratic Schrödinger equation, and the Novikov-Veselov equation.

To put these negative results into context, we finish our work by a brief summary of the different scenarios one can face when analyzing a quadratic dispersive PDE. Roughly speaking, the more resonance and non-transversality there is, the more difficult it becomes to treat an equation, and we discuss up to which point our techniques could possibly be adapted and extended without introducing decay on the initial data, which plays a part in most results dealing with certain amounts of nontransversality.

1.1 Notation and Preliminaries

In this section, we collect some definitions and constructions which will universally be used in what follows.

For positive numbers f and g, we write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$, wherever this expression makes sense. In a similar way, we say

$$f \gtrsim g \iff g \lesssim f, \qquad f \sim g \iff f \lesssim g \lesssim f.$$

If, for a small constant c we have$f \leq cg$, then we say $f \ll g$ and again $f \gg g$ is to mean that $g \ll f$.

We denote the spatial Fourier transform by $\hat{\cdot}$ or $\mathcal{F}_x$ and the Fourier transform in time by $\mathcal{F}_t$. Even though this is of little importance in the sequel, we use the L^2 normalized variants initially defined on the Schwartz space $S(\mathbb{R}^n)$,

$$\hat{f}(\xi) = (\mathcal{F}_x f)(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\cdot\xi} f(x)dx \qquad f \in \mathcal{S}(\mathbb{R}^n)$$
$$(\mathcal{F}_t g)(\tau) = (2\pi)^{-\frac{1}{2}} \int e^{-it\tau} g(t)dt \qquad g \in \mathcal{S}(\mathbb{R}),$$

and we may occasionally denote by $\mathcal{F}_{tx}$ the Fourier transform in both space and time. With the above definitions, the Fourier transforms extend to the spaces of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ and $S'(\mathbb{R})$; they define isometries on $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R})$, respectively. We usually use ξ, η and γ on the spatial Fourier side, and τ on the temporal Fourier side.

The letters M, N, O, H and L will be reserved for use as dyadic numbers denoting a localization in frequency space. A (homogeneous) dyadic number $N \in 2^{\mathbb{Z}}$ is simply a rational number of the form

$$N = 2^k \qquad k \in \mathbb{Z}.$$

For dyadic sums, we define

$$\sum_N a_N := \sum_{k\in\mathbb{Z}} a_{2^k}, \qquad \sum_{N\geq M} a_N := \sum_{k\in\mathbb{Z}:\ 2^k > M} a_{2^k}.$$

Other expressions of this type such as $\sum_{A\gg B} a_N$ are interpreted accordingly, even if A and B are not dyadic.

For settings in which small frequencies are treated equally, in particular in chapter 4, we will vary the definition of a dyadic number slightly. Precisely, as a (inhomogeneous) dyadic number in that context we take any number of the form 2^k for some $k \in \mathbb{N} \cup \{0\}$, where 2^0 is associated to frequencies less than one. The definition of dyadic sums is adapted accordingly,

$$\sum_N a_N := a_1 + \sum_{n\in\mathbb{N}} a_{2^n}, \qquad \sum_{N\geq M} a_N := \sum_{n\in\mathbb{N}:\ 2^n \geq M} a_{2^n} \quad (M > 0).$$

To avoid confusion between the two definitions, we use the convention that by default, the former notation is used, while the latter will be pointed out explicitly. In particular, all dyadic sums in the remainder of this section are homogeneous.

Now we define the usual Littlewood-Paley projection operators, which will be heavily used in everything that follows. We denote the spatial dimension by $n \in \mathbb{N}$. In our applications, we will always have $n \geq 2$.

Let $\chi \in C^\infty[-2, 2]$ an even, non-negative function such that $\chi(t) = 1$ for $|t| \leq 1$. Its precise form is not important, and one should think simply of the function $\mathbf{1}_{[-1,1]}(t)$.

We define $\psi(t) := \chi(t) - \chi(2t)$ and $\psi_N := \psi(N^{-1}\cdot)$ for $N > 0$. Then,

$$\sum_{N\in 2^{\mathbb{Z}}} \psi_N(t) = 1 \qquad t \neq 0.$$

We have the following

Definition 1.1 (Paley-Littlewood decomposition)**.** Let M, N dyadic, $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R})$. We define the *Littlewood-Paley projection operator* P_N by

$$P_N f := \mathcal{F}_x^{-1}\left(\psi_N(|\cdot|)\mathcal{F}_x f\right)$$

Similarly, we define the *temporal Littlewood-Paley projection operator* as

$$Q_M g := \mathcal{F}_t^{-1}\left(\psi_M \mathcal{F}_t g\right).$$

For later use with inhomogeneous dyadic decompositions, we also define

$$\psi_0 = 1 - \sum_{N\geq 1} \psi_N$$

and the corresponding projector

$$P_0 f := \mathcal{F}_x^{-1}\left(\psi_0 \mathcal{F}_t f\right).$$

We extend the previous notation for dyadic sums to the above operators. Thus, for instance $Q_{\geq M} = \sum_{N\in 2^{\mathbb{Z}}:N\geq M} Q_N$ and $Q_{<M} = I - Q_{\geq M}$.
Using the Littlewood-Paley decomposition, we now define the usual L^2 based Sobolev (or Bessel potential) and Besov spaces in the form best suited to our purposes.

Definition 1.2 (Japanese Bracket)**.** Let $\xi \in \mathbb{R}^n$. Then we write

$$\langle\xi\rangle := \sqrt{1+|\xi|^2}, \qquad \langle\xi\rangle_m := \sqrt{m^2+|\xi|^2} \qquad (m>0).$$

Definition 1.3 (Sobolev space)**.** Let $s \in \mathbb{R}$. We define the homogeneous Sobolev space $\dot{H}^s$ as the subspace of $\mathcal{S}'(\mathbb{R}^n)$ for which the seminorm

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} = \|\mathcal{F}_x^{-1}\left(|\xi|^s \hat{u}\right)\|_{L^2(\mathbb{R}^n)}$$

is finite. The inhomogeneous Sobolev space $\dot{H}^s$ is defined analogously, using instead the norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}_x^{-1}\left(\langle\xi\rangle^s \hat{u}\right)\|_{L^2(\mathbb{R}^n)}$$

The Littlewood-Paley projections partition a function into their dyadic frequency components. In particular, we have

Lemma 1.4 (Orthogonality)**.** *We have*

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \sum_N \|P_N u\|^2_{L^2(\mathbb{R}^n)}$$
$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} \sim \sum_N N^{2s} \|P_N u\|^2_{L^2(\mathbb{R}^n)}$$
$$\|u\|^2_{H^s(\mathbb{R}^n)} \sim \sum_{N\leq 1} \|P_N u\|^2_{L^2(\mathbb{R}^n)} + \sum_{N>1} N^{2s} \|P_N u\|^2_{L^2(\mathbb{R}^n)}.$$

Definition 1.5 (Besov spaces)**.** Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is the set of all functions $u \in \mathcal{S}'(\mathbb{R}, L^2(\mathbb{R}^n))$ such that the seminorm

$$\|u\|_{\dot{B}^s_{p,q}} = \begin{cases} \left(\sum_M M^{sq} \|Q_M u\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} & q < \infty \\ \sup_M M^s \|Q_M u\|_{L^p(\mathbb{R}^n)} & q = \infty \end{cases}$$

is finite. Similarly we define the inhomogeneous Besov space $B^s_{p,q}$.

Definition 1.6. Let $h : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Then the *free propagator* or *free evolution* associated to h is the unitary operator on $L^2(\mathbb{R}^n)$ defined by

$$e^{ith(D)} f = \mathcal{F}_x^{-1} \left(e^{ith(\xi)} \hat{f} \right) \qquad f \in L^2(\mathbb{R}^n)$$

for each $t \in \mathbb{R}$. The function h is usually called *dispersion relation.*

In later chapters, the temporal Paley-Littlewood decomposition will be adapted to a free evolution in the following way.

Definition 1.7. Let h as in Definition 1.6 and M dyadic. We define the modulation cut-off operator Q^h_M by

$$Q^h_M g := \mathcal{F}^{-1}_{\tau\xi} \left(\psi_M(\tau - h(\xi)) \mathcal{F}_{tx} g \right).$$

In other words, Q^h_M selects the Fourier spacetime region $|\tau - h(\xi)| \sim M$.

Chapter 2

Dispersive equations

We describe in this section some fundamental characteristics of constant coefficient dispersive equations. Since we are interested most in the case of equations which are first order in time, we sacrifice generality in favor of ease of exposition. The level of generality here will suffice to cover the intended purposes and the goal in this chapter is to convey first the heuristics and then rigorous results necessary for an intuitive understanding of the behavior of solutions related to their respective dispersive effects. See [Tao06] for a more exhaustive introduction, which has certainly influenced the exposition below.

We consider the Cauchy problem for a linear partial differential equation of the form

$$\begin{aligned} i\partial_t u + h(D)u &= 0 \\ u(0) &= u_0, \end{aligned} \tag{2.0.1}$$

where, say, $u_0 \in L^2(\mathbb{R}^n)$ and $h(D)$ is simply the Fourier multiplier with symbol h,

$$h(D)f = \mathcal{F}_x^{-1}\left(h(\xi)\hat{f}\right).$$

Definition 2.1 (Dispersion relation)**.** Given an equation of type (2.0.1), we refer to the function $h(\xi)$ as **dispersion relation**.

If we assume that for each $t \in \mathbb{R}$ we have $u(t) \in L^2(\mathbb{R}^n)$, then we can take a Fourier transform in space in (2.0.1) and quickly see that each Fourier mode evolves independently through an ODE,

$$\begin{aligned} i\partial_t \hat{u} &= -h(\xi)\hat{u} \\ u(0) &= \widehat{u_0}, \end{aligned}$$

and thus

$$\hat{u}(t,\xi) = \widehat{u_0}(\xi)e^{ith(\xi)}.$$

Consequently we write $e^{ith(D)}f = \mathcal{F}_x^{-1}\left(e^{ith(\xi)}\hat{f}\right)$ for the fundamental solution of (2.0.1), which is for each $t \in \mathbb{R}$ an isometry on any of the spaces $L^2(\mathbb{R}^n)$, $H^s(\mathbb{R}^n)$

or $\dot{H}^s(\mathbb{R}^n)$ simply because it is given as a Fourier multiplier of modulus one.
Consequently, if we choose to work with initial data in $L^2(\mathbb{R}^n)$ (or $H^s(\mathbb{R}^n)$ or $\dot{H}^2(\mathbb{R}^n)$, respectively) in the natural choice of a solution space $C(\mathbb{R}, L^s)$ (or $C(\mathbb{R}, H^s)$ or $C(\mathbb{R}, \dot{H}^s)$ for any $s \in \mathbb{R}$, respectively), we see that the solution is bounded in terms of its initial data,

$$\|e^{ith(D)}u_0\|_{C(\mathbb{R},L^2(\mathbb{R}^n))} \leq \|u_0\|_{L^2(\mathbb{R}^n)}$$

and similarly one may replace $L^2(\mathbb{R}^n)$ by $H^s(\mathbb{R}^n)$ or $\dot{H}^s(\mathbb{R}^n)$.
If instead of a Fourier transform in space we take a spacetime Fourier transform, we see that

$$(-\tau + h(\xi))\mathcal{F}_{tx}u = 0$$

and using the initial condition, we infer that

$$\mathcal{F}_{tx}u = \delta(\tau - h(\xi))\widehat{u_0}(\xi) \tag{2.0.2}$$

in the sense of $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^n)$. This motivates the following

Definition 2.2 (Characteristic surface)**.** The **characteristic surface** associated to an equation of type (2.0.1) is the smooth hypersurface of $\mathbb{R} \times \mathbb{R}^n$ defined by

$$\Sigma = \Sigma_h = \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : \ \tau = h(\xi)\},$$

endowed with the surface measure $\mu_\Sigma = \mathcal{H}^n\Big|_\Sigma$.
Using the Coarea formula (Theorem A.1), we can rewrite (2.0.2) in terms of the surface measure μ_Σ, which results in the formula

$$\mathcal{F}_{tx}e^{ith(D)}f = \frac{\hat{f}}{\langle \nabla h \rangle}\mu_\Sigma.$$

We will see soon that geometric properties of Σ are related to dispersive properties of (2.0.1).

Example 2.3. Three basic prototypes of dispersive equations are given by

- the Schrödinger equation $i\partial_t u - \Delta u = 0$, for which

$$h(\xi) = |\xi|^2 \text{ and } \Sigma = \mathbb{P} = \{\tau = |\xi|^2\},$$

- the (half) Klein-Gordon equation with mass $m > 0$, $i\partial_t u + \langle D \rangle_m u = 0$,

$$h(\xi) = \langle \xi \rangle_m = \sqrt{m^2 + |\xi|^2} \text{ and } \Sigma = \{\tau = \langle \xi \rangle_m\}, \quad \text{and}$$

- the (half) Wave equation $i\partial_t + |D|u = 0$, formally obtained as the case $m = 0$ of the previous equation with $h(\xi) = |\xi|$ and characteristic surface the cone $\{\tau = |\xi|\}$.

Furthermore, we will encounter in chapter 6

- the Novikov-Veselov equation, with linear part given by

$$h(\xi) = 2\xi_1^3 - 6\xi_1\xi_2^2, \quad \xi \in \mathbb{R}^2.$$

Remark 2.4. We drop the word "half" from the examples for reasons of brevity, since the full second-order Wave and Klein-Gordon equations are readily reduced to systems of the above type, and their study is largely equivalent; see section 4.2.1

Dispersion and its effects

In order to develop a good intuitive understanding of equations of dispersive type, we can try to guess the behavior of solutions emanating from localized data. For $\widehat{u_0} = \delta_0(\xi - \xi_0)$, we formally get

$$e^{ith(D)}u_0 = e^{ith(\xi_0)+ix\cdot\xi_0},$$

which is a wave oscillating in space at frequency $h(\xi_0)$. On the other hand, if we take the initial data localized in space around x_0 and around ξ_0 in frequency, that is, $u_0(x) = e^{ix\cdot\xi_0}\phi(x - x_0)$ for some bump function $\phi \in C_c^\infty(\mathbb{R}^n)$, then

$$\left|e^{ith(D)}u_0\right| = \left|\int e^{i((x-x_0)\cdot\xi+th(\xi+\xi_0))}\widehat{\phi}(\xi)d\xi\right|$$

and the principle of stationary phase (see Theorem A.2) would suggest that the solution in space at time t is largest where the phase has stationary points, which happens when $|\xi|$ is small such that

$$x = x_0 - t\nabla h(\xi + \xi_0) \sim x_0 - t\nabla h(\xi_0).$$

Summing up these heuristics, a solution which is concentrated around x_0 in space and ξ_0 in frequency (subject to limitations given by the Heisenberg uncertainty principle) should

- oscillate in space roughly at frequency $h(\xi_0)$ and
- move in direction $-\nabla h(\xi_0)$, with the speed given by the magnitude of that gradient.

From this description, we can identify the main mechanism for which this class of equations is called *dispersive*:

Different frequency components of a solution move at different velocities and/or in different directions, resulting in dispersion of the solution.

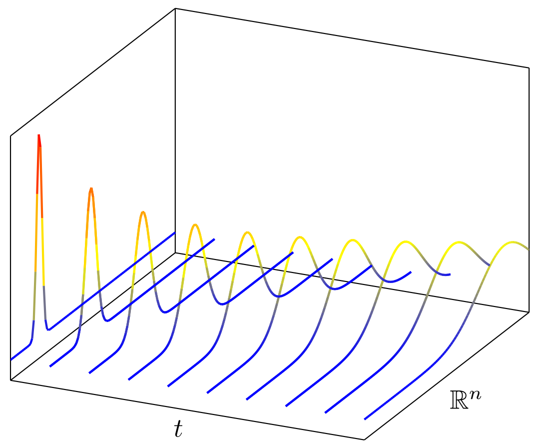

Figure 2.1: Spatial evolution in time of a bump function concentrated at some frequency ξ_0. Initially coherent, the bump moves in direction $-\nabla h(\xi_0)$, becoming smaller pointwise but conserving the $L^2(\mathbb{R}^n)$ norm due to the widening of the support.

Example 2.5. A solution at localized at frequency ξ_0, where $|\xi_0| \sim N$, should move at speed

- $2|\xi_0| \sim 2N$ for the Schrödinger equation,
- $\frac{|\xi_0|}{\langle \xi_0 \rangle_m} \sim_m \min(N, 1)$ for the Klein-Gordon equation and
- 1 for the Wave equation

in direction $\frac{\xi_0}{|\xi_0|}$ (as for any radial choice of h), and

- along the vector $3\begin{pmatrix} \xi_2^2 - \xi_1^2 \\ 2\xi_1\xi_2 \end{pmatrix}$ for the Novikov-Veselov equation.

To illustrate how local properties of the characteristic surface Σ_h come in, assume that we take as initial data bump function which is localized to a small region around the origin. If we take two parts of that solution located at nearby frequencies ξ_1 and ξ_2, then the difference between the velocity vectors $\nabla h(\xi_1)$ and $\nabla h(\xi_2)$ is approximately given through the Hessian of h by

$$\nabla h(\xi_1) - \nabla h(\xi_2) \approx D^2 h(\xi_1) \cdot (\xi_2 - \xi_1).$$

Thus, if the Hessian is nondegenerate, the parts of the solution belonging to ξ_1 and ξ_2 will move asynchronously against each other in time, which introduces cancellation and thus ultimately, decay in time of the solution.
One way to capture this phenomenon mathematically is through a *dispersive estimate*, which usually takes the form

$$\|e^{ith(D)} P_N f\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\sigma} N^\delta \|P_N f\|_{L^1(\mathbb{R}^n)} \qquad t \gg 1,\ f \in \mathcal{S}.$$

for some $\delta \in \mathbb{R}$, which is often nonnegative (corresponding to "losing" derivatives), and $\sigma > 0$. To see which values σ typically takes, we observe that formally, for $detD^2h \neq 0$ and a bump function f,

$$\begin{aligned} e^{ith(D)}f &= (2\pi)^{-n} \int e^{i(x\cdot\xi + th(\xi))}\hat{f}(\xi)d\xi \\ &\approx (2\pi)^{-\frac{n}{2}} \sum_{\xi:\ x+t\nabla h(\xi)=0} e^{i(x\cdot\xi + th(\xi) + \frac{\pi}{4}\operatorname{sgn} D^2h(\xi))}|\det tD^2h(\xi)|^{-\frac{1}{2}}\hat{f}(\xi) \end{aligned} \tag{2.0.3}$$

plus terms that decay faster as $t \to \infty$, by the principle of stationary phase (see Theorem A.2). Clearly, $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1}$. Furthermore,

$$|\det(tD^2h(\xi))| = t^n \prod_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the Eigenvalues of $D^2h(\xi)$.
Hence, in that case we expect $\sigma = \frac{n}{2}$, i.e. that the L^∞ norm of a solution decays at a rate of $t^{-\frac{n}{2}}$ as $|t| \to \infty$, and in fact this should be the best decay one can expect. For degenerate dispersion relations h, say, with a single zero eigenvalue at some ξ_0, one would instead integrate out this one direction (losing one derivative), and use a stationary phase argument in the remaining $n-1$ nondegenerate directions. Consequently, in this case one would expect only a decay of $t^{-\frac{n-1}{2}}$ as $|t| \to \infty$. Arguing similarly for multiple zero eigenvalues, we arrive at the following useful heuristic:

Heuristic 2.6 (Decay estimates)**.**

If at each critical point ξ_c of h with $|\xi_c| \sim N$ the Hessian $D^2h(\xi)$ has k vanishing Eigenvalues $\lambda_{n-k+1}, \ldots, \lambda_n$, then we expect the dispersive estimate

$$\|e^{ith(D)}P_N f\|_{L^\infty(\mathbb{R}^n)} \lesssim N^k t^{-\frac{n-k}{2}} \sup_{\xi_c} \Big| \prod_{i=1}^{n-k} \lambda_i(\xi_c) \Big|^{-\frac{1}{2}} \|P_N f\|_{L^1(\mathbb{R}^n)}$$

to hold true.

Remark 2.7. The dispersive estimate can also be interpreted in terms of the (inverse) Fourier transform of the measure $\nu_f := \frac{f(\xi)}{\langle\nabla h(\xi)\rangle}\mu_{\Sigma_h}$ on the characteristic hypersurface, defined by

$$\check{\nu}(t,x) = (2\pi)^{-n} \int_{\Sigma_h} e^{i(x,t)\cdot\zeta} d\nu(\zeta)$$

since

$$\text{LHS (2.0.3)} = (2\pi)^{-n} \int e^{i(x,t)\cdot(\xi,h(\xi))} \frac{\hat{f}(\xi)}{\langle\nabla h(\xi)\rangle} \langle\nabla h(\xi)\rangle d\xi = \check{\nu}_f(x,t).$$

This links dispersive estimates to the decay behavior of the inverse Fourier transform of the surface measure of the characteristic hypersurface, and the order of decay (in

$|(t,x)|$) is dictated by the number of nonvanishing principal curvatures, where the critical direction on the Fourier side is along the normal vector field.

Hence, in cases where ∇h remains bounded, the principal curvatures behave essentially like the eigenvalues of the Hessian $D^2 h$. This is the case for both the Wave and Klein-Gordon equations, but not for the Schrödinger and Novikov-Veselov equation.

Example 2.8. We apply this heuristic to the usual examples:

- for the Schrödinger equation, the situation is particularly simple, since $D^2|\xi|^2$ is twice the identity. Hence, we expect

$$\|e^{-it\Delta} P_N f\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \|P_N f\|_{L^1(\mathbb{R}^n)}$$

and in fact this is true, see for instance (2.22) in [Tao06].

- For the Klein-Gordon equation (without loss of generality) with $m = 1$, $h(\xi) = \langle \xi \rangle$ and we may assume after a rotation that $\xi = (|\xi|, 0, \ldots, 0)$ and thus

$$D^2 \langle \xi \rangle = \frac{1}{\langle \xi \rangle} \left(id - \frac{\xi \cdot \xi^t}{\langle \xi \rangle^2} \right) = \frac{1}{\langle \xi \rangle} \operatorname{diag}(\langle \xi \rangle^{-2}, 1, \ldots, 1).$$

whose determinant is comparable to $\langle N \rangle^{-(n+2)}$, therefore suggesting the validity of the estimate

$$\|e^{it\langle D \rangle} P_N f\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \langle N \rangle^{\frac{n+2}{2}} \|P_N f\|_{L^1(\mathbb{R}^n)}$$

which is derived rigorously in [DF08a], $(A.2)$.

- The case of the Wave equation is somewhat degenerate, since a zero eigenvalue appears:

$$D^2 |\xi| = \frac{1}{|\xi|} \operatorname{diag}(0, 1, \ldots, 1)$$

for $\xi = (|\xi|, 0, \ldots, 0)$; according to the above principle, we need to integrate out one direction, losing one factor of N, and use only the remaining $n-1$ eigenvalues, all of which are comparable to $\frac{1}{N}$. This results in

$$\|e^{it|D|} P_N f\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n-1}{2}} N^{\frac{n+1}{2}} \|P_N f\|_{L^1(\mathbb{R}^n)},$$

see [GV95].

- Finally, for the Novikov-Veselov equation, a quick computation gives a determinant of $-36|\xi|^2$, suggesting in fact a gain of one derivative and the dispersive estimate

$$\|e^{it\langle D \rangle} P_N f\|_{L^\infty(\mathbb{R}^2)} \lesssim t^{-1} N^{-1} \|P_N f\|_{L^1(\mathbb{R}^2)},$$

see [BAKS03].

2.1 Strichartz estimates

A Strichartz estimate for an equation (2.0.1) is an estimate which bounds a space-time $L^qL^r(\mathbb{R}\times\mathbb{R}^n)$ norm of a free solution in terms of the initial data in $L^2(\mathbb{R}^n)$, with a possible loss of derivatives. They originated in [Seg76] and [Str77], where a special case was derived; the theory has since been completed (see [KT98] and the references therein) and has long become a standard tool in the field. Bounds of Strichartz type can be interpreted as stating that even though at some fixed point in time the solution need not be in $L^r(\mathbb{R}^n)$, $r > 2$, it is still true that for *most times* this holds, in a quantitative way.

Abstract Strichartz estimates

We first give an abstract derivation of such estimates that, while being far from the most "pedestrian" proof available, highlights very clearly how Strichartz estimates are, up to endpoint cases, a consequence of a dispersive inequality, $L^2(\mathbb{R}^n)$ conservation of the free propagator $e^{ith(D)}$ and the Hardy-Littlewood-Sobolev inequality (Theorem A.3). The arguments here follow closely the abstract notation and arguments in [KT98]. Assume that we have a dispersive estimate of the form

$$\|(e^{ith(D)}P_Nu_0)(t)\|_{L^\infty(\mathbb{R}^n)} \lesssim N^\delta t^{-\sigma}\|P_Nu_0\|_{L^1(\mathbb{R}^n)}, \tag{2.1.1}$$

where either $N \in 2^{\mathbb{Z}}$ (homogeneous case) or $N \in 2^{\mathbb{N}}$ (inhomogeneous case). We fix $\sigma > 0$ and δ for the remainder of this subsection. After multiplying both sides by $N^{-\frac{\delta}{2}}$ and square summing over N, we obtain (in the inhomogeneous case) the estimate

$$\|(e^{ith(D)}u_0)(t)\|_{B^{-\frac{\delta}{2}}_{2,\infty}(\mathbb{R}^n)} \lesssim t^{-\sigma}\|u_0\|_{B^{\frac{\delta}{2}}_{2,1}(\mathbb{R}^n)}$$

and similarly for the homogeneous case using instead homogeneous Besov spaces. If we set $\mathcal{B}_1 = B^{\frac{\delta}{2}}_{2,1}$ in the inhomogeneous case (and similarly using $\dot{B}^{\frac{\delta}{2}}_{2,1}$ in the homogeneous case), then this last estimate is equivalent to

$$\|(e^{ith(D)}\cdot)(t)\|_{\mathcal{B}_1\to(\mathcal{B}_1)^*} \lesssim t^{-\sigma}$$

and combining this with the unitarity of $U(t)$ on $L^2(\mathbb{R}^n)$, we obtain for $0 \leq \theta \leq 1$ the interpolated estimates

$$\|(e^{ith(D)}\cdot)(t)\|_{\mathcal{B}_\theta\to(\mathcal{B}_\theta)^*} \lesssim t^{-\theta\sigma}. \tag{2.1.2}$$

where the interpolation spaces $\mathcal{B}_\theta$ are defined below.

Definition 2.9 (Interpolation space)**.** For $0 \leq \theta \leq 1$, we denote by $\mathcal{B}_\theta$ the real interpolation space $(L^2, \mathcal{B}_1)_{\theta,2}$ as defined in Definition A.7.

Lemma 2.10. *We have*

$$\mathcal{B}_\theta = B^{\theta\frac{\delta}{2}}_{2,\frac{2}{1+\theta}} \text{ or } \mathcal{B}_\theta = \dot{B}^{\theta\frac{\delta}{2}}_{2,\frac{2}{1+\theta}}$$

in the inhomogeneous and homogeneous cases, respectively.

Proof. see (2.4.2) in [Tri83]. □

Now we state the Strichartz estimate that can be proven under the above assumptions, namely

Proposition 2.11 (abstract Strichartz estimates)**.** *Let* $2 < q \leq \infty$ *and let* $\theta = \frac{2}{\sigma q}$*. Then, assuming* (2.1.1) *holds and with* $\frac{1}{q} + \frac{1}{q'} = 1$*, we have*

$$\|(e^{ith(D)}u_0)(x)\|_{L^q_t(\mathbb{R},(\mathcal{B}_\theta)^*)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}$$
$$\|\int e^{ish(D)}g(s,\cdot)ds\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta)}.$$

Remark 2.12. We can completely eliminate the interpolation spaces from the statement of Proposition 2.11 using Lemma 2.10 and the relationship between q and θ. In fact, denoting $r' = \frac{2}{1+\theta}$ and $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$(\mathcal{B}_\theta)^* = B_{2,r}^{-\frac{\delta}{q\sigma}}, \qquad \mathcal{B}_\theta = B_{2,r'}^{\frac{\delta}{q\sigma}}$$

where q and r are related by

$$\frac{1}{\sigma q} + \frac{1}{r} = \frac{1}{2}.$$

Furthermore, using the embedding $\dot{B}^0_{2,r}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ $(r \geq 2)$ (Corollary A.6) we can even replace the norms on the left hand side by $L^qL^r(\mathbb{R} \times \mathbb{R}^n)$ if we replace the $L^2(\mathbb{R}^n)$ norm on the right hand side by $\dot{H}^{\frac{\delta}{\sigma q}}$ or $H^{\frac{\delta}{\sigma q}}$, respectively.

Remark 2.13. The endpoint $r = 2$ can be treated when $\sigma \neq 1$, but the argument is much more delicate and represents the major new contribution in [KT98].

Proof. We are trying to prove the bound

$$\|e^{ith(D)} \cdot \|_{L^2(\mathbb{R}^n) \to L^q(\mathbb{R},(\mathcal{B}_\theta)^*)} \lesssim 1 \tag{2.1.3}$$

where θ will be determined. The second bound in Proposition 2.11,

$$\|\int e^{ish(D)} \cdot ds\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta) \to L^2(\mathbb{R}^n)} \lesssim 1, \tag{2.1.4}$$

is just the dual of (2.1.3) since

$$\left(e^{ish(D)}\cdot\right)^* g = \int e^{-ish(D)}g(s,x)ds.$$

By duality and the TT^* method[1], T is bounded if and only if TT^* is bounded, in the respective spaces. With T the operator in (2.1.3), the desired estimate for TT^* takes the form

$$\|e^{ith(D)}\left(e^{ish(D)}\cdot\right)^* \|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta) \to L^q(\mathbb{R},(\mathcal{B}_\theta)^*)} \lesssim 1.$$

[1] see, for instance, section 2.3 in [Tao06]

and, using once more duality, we arrive at the equivalent bilinear bound

$$\Big| \iint \langle e^{-ith(D)} f(t,x), e^{-ish(D)} g(s,x) \rangle ds dt \Big| \lesssim \|f\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta)} \|g\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta)}. \tag{2.1.5}$$

What we know from (2.1.2) is that

$$\Big| \langle e^{-ith(D)} f(t,x), e^{-ish(D)} g(s,x) \rangle \Big| \lesssim |t-s|^{-\theta\sigma} \|f\|_{\mathcal{B}_\theta}(t) \|g\|_{\mathcal{B}_\theta}(s),$$

and we can estimate, if we chose $\theta = \frac{2}{\sigma q}$,

$$\begin{aligned} \iint |t-s|^{-\theta\sigma} \|f\|_{\mathcal{B}_\theta}(t) \|g\|_{\mathcal{B}_\theta}(s) ds dt &\leq \Big\| \Big(|t|^{-\frac{2}{q}} * \|f\|_{\mathcal{B}_\theta}(t) \Big)(\cdot) \Big\|_{L^q(\mathbb{R})} \|g\|_{L^{q'}(\mathcal{B}_\theta)} \\ &\lesssim \|f\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta)} \|g\|_{L^{q'}(\mathbb{R},\mathcal{B}_\theta)} \end{aligned}$$

by the Hardy-Littlewood-Sobolev inequality (see Theorem A.3), since $0 \leq \frac{2}{q} < 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. □

Application to Schrödinger, Klein-Gordon and Wave equations

Armed with the general estimate, let us apply it to our three favorite examples.

Proposition 2.14 (Strichartz estimates)**.** *We have, for the Schrödinger and Klein-Gordon equations*[2]*,*

$$\begin{aligned} \|e^{it|D|^2} u_0\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} &\lesssim \|u_0\|_{L^2(\mathbb{R}^n)} \\ \|e^{it\langle D\rangle_m} u_0\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} &\lesssim \|\langle D\rangle^{\frac{1}{2}+\frac{1}{q}-\frac{1}{r}} u_0\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

where $m > 0$ and $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, $q > 2$. For the Wave equation, we have instead

$$\|e^{it|D|} u_0\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \||D|^{\frac{1}{2}+\frac{1}{q}-\frac{1}{r}} u_0\|_{L^2(\mathbb{R}^n)}$$

where $\frac{2}{q} + \frac{n-1}{r} = \frac{n-1}{2}$, $q > 2$.

Proof. For the Schrödinger propagator $h(D) = |D|^2$, we have the dispersive estimate (2.1.1) with $\sigma = \frac{n}{2}$ and $\delta = 0$. Consequently, the relation between r and q becomes

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$$

as desired, and the loss of derivatives $\frac{\delta}{q\sigma}$ is equal to zero. For the Klein-Gordon equation, similarly $\sigma = \frac{n}{2}$, but now $\sigma = \frac{n+2}{2}$ according to Example 2.8. Consequently we have a loss of derivatives of size

$$\frac{\theta\delta}{2} = \frac{\delta}{\sigma q} = \frac{n+2}{n}\frac{1}{q} = \frac{1}{q} + \frac{1}{2} - \frac{1}{r}$$

[2]the implicit constant in the second equation depends on $m > 0$

which we can move from the Besov space to the right hand side in $L^2(\mathbb{R}^n)$ as outlined in Remark 2.12.
Finally, for the wave equation, $h(D) = |D|$, $\sigma = \frac{n-1}{2}$ and $\delta = \frac{n+1}{2}$; hence the algebra is the same as in the case of the Klein-Gordon equation (but applying the homogeneous case of Proposition 2.11 this time), but with $n-1$ replacing n. This results in the claimed estimates.

□

2.2 Well-posedness and Solutions

The term "well-posedness" refers to the satisfactory solvability of a given problem, such as (2.0.1) or a nonlinear version thereof. Which properties one asks for specifically depends on the problem and the physical situation it may model. Since there is a plethora of settings to consider, there is also a corresponding wide range of notions of well-posedness. For that reason, a simple definition of what it means for a problem to be well-posed appears futile. However, informally speaking, in many standard situations, one asks at least for

- existence: for given initial data, there exists a solution,
- uniqueness: this solution is unique in a given class
- continuous dependence: the solution depends on its initial data in a continuous way.

We make no attempt here to formalize this further at this point, but make sure to state very precisely the notion of well-posedness used in chapters chapters 4 and 5 when it becomes relevant.
The main results in this work will deal with special cases of the nonlinear dispersive equation

$$\begin{aligned} i\partial_t u + h(D)u &= N(u) \\ u(0) &= u_0, \end{aligned} \tag{2.2.1}$$

and we will focus for the remainder of this section on this class.
Now we will address the notion of a solution, which is a delicate issue. More often than not, one is interested in solutions of (2.2.1) which do not possess enough strong derivatives in order to satisfy the equation in a classical sense, especially when low-regularity data are considered. Certainly, one will want at least a distributional solution, but at the level of distributions, it is difficult to derive a lot of desired properties, and one will try to find solutions in smaller spaces.
We are interested in perturbative settings, that is in situations in which one of the parameters (typically the initial data or the existence time of the solution) is "small", invoking hopes that a solution of (2.2.1) should inherit many properties of the linear flow $e^{ith(D)}$. To this end, one regards the nonlinear term $N(u)$ in (2.2.1) as a perturbation of (2.0.1). In this setting, it seems natural at first to work directly with spaces of the type $C(\mathbb{R}, \mathbb{L})$, where $\mathbb{L} = H^s$ or $\dot{H}^s$ for some $s \in \mathbb{R}$, since these

spaces interact very well with the linear evolution $e^{ith(D)}$. This would allow using Duhamel's principle to rewrite (2.2.1) as an operator equation,

$$u(t) = e^{ith(D)}u_0 - i\int_0^t e^{(t-s)h(D)}N(u(s))ds$$

and in turn define a solution to (2.2.1) as a solution to the above operator integral equation.

In practice however, working with all of $C(\mathbb{R}, \mathbb{L})$ can be unwieldy since that space does not capture a phenomenon that is at the heart of a Fourier analysis approach to existence problems.

More precisely, recall that linear waves, i.e. solutions to (2.0.1), are supported in Fourier spacetime on the characteristic hypersurface Σ_h. It turns out that for corresponding solutions to the nonlinear problem (2.2.1) the support is still concentrated around Σ_h, and to exploit this phenomenon, it is advisable to look for solutions in a smaller subspace $X \subset C(\mathbb{R}, \mathbb{L})$ which penalizes a function off Σ_h in Fourier spacetime, and only then try to solve the operator equation above, using for instance the Banach fixed point theorem. Since the symbol of $i\partial_t + h(D)$, $\tau - h(\xi)$, vanishes precisely on Σ_h, one should gain away from this surface (similar to elliptic regularity estimates) and see the most complicated phenomena close to Σ_h.

In the next section we introduce and describe the adapted function spaces $X^{s,b}$ (attributed chiefly to Bourgain [Bou93] but defined earlier, for instance in [RR82]) as well as the more recent spaces U^2 and V^2 (introduced in this setting by Tataru [KT05, KT07]), which have contributed much to the current state of affairs.

Chapter 3

Adapted function spaces

3.1 $X^{s,b}$ spaces

Definition 3.1 ($X^{s,b}$ spaces)**.** Let $s, b \in \mathbb{R}$ and let $h(\cdot)$ the dispersion relation in (2.2.1). Then the space $X_h^{s,b} = X^{s,b}$ is defined as the closure of all Schwartz functions $S(\mathbb{R} \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau - h(\xi)\rangle^b \mathcal{F}_{tx} u(\tau, \xi)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Similarly, we define $\dot{X}^{s,b}$ using instead the seminorm

$$\|u\|_{\dot{X}^{s,b}} = \||\xi|^s |\tau - h(\xi)|^b \mathcal{F}_{tx} u(\tau, \xi)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Example 3.2.

- for $b = 0$ and any h and s, $X^{s,b} = L^2(\mathbb{R}, H^s)$.
- for $h = 0$, we have $X^{s,b} = H^b(\mathbb{R}, H^s)$.
- in fact, for any h we have $\|\cdot\|_{X^{s,b}} = \|e^{-ith(D)} \cdot \|_{H^b H^s(\mathbb{R}\times\mathbb{R}^n)}$.

This last example suggests that $X^{s,b}$ spaces are well adapted to free solutions $e^{ith(D)}u_0$. This is true, however there are some caveats (some of which the U^2 and V^2 spaces introduced in the next section address): Firstly, the behaviour depends crucially on the choice of b as will become apparent soon; secondly, a free solution does not have finite $X^{s,b}$ norm unless one truncates in time first, implying that $X^{s,b}$ spaces are potentially ill-suited for global existence problems. More precisely, we have the following

Lemma 3.3. *Let $b, s \in \mathbb{R}$ and $\phi \in C_0^\infty(\mathbb{R})$. Then, for any $T > 0$ and denoting $\phi_T(\cdot) = \phi(\cdot/T)$, it holds*

$$\|\phi_T(t)e^{ith(D)}u_0\|_{X^{s,b}} \lesssim T^{\frac{1}{2}} \langle 1/T\rangle^b \|u_0\|_{H^s(\mathbb{R}^n)}.$$

Proof. We have

$$\mathcal{F}_t\phi_T = T(\mathcal{F}_t\phi)(T\cdot)$$

and hence, since $\mathcal{F}_t\phi$ decays rapidly, for some $K \gg 1 + |b|$

$$\begin{aligned}\|\phi_T(t)e^{ith(D)}u_0\|_{X^{s,b}} &= T\|(\mathcal{F}_t\phi)(T(\tau - h(\xi))\langle\tau - h(\xi)\rangle^b\langle\xi\rangle^s\widehat{u_0}(\xi)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\ &\lesssim T\|\langle T\tau\rangle^{-K}\langle\tau\rangle^b\|_{L^2(\mathbb{R})}\|u_0\|_{H^s(\mathbb{R}^n)} \\ &= T^{\frac{1}{2}}\|\langle\tau\rangle^{-K}\langle\tau/T\rangle^b\|_{L^2(\mathbb{R})}\|u_0\|_{H^s(\mathbb{R}^n)} \\ &\lesssim T^{\frac{1}{2}}\langle 1/T\rangle^b\|u_0\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

□

Among the desirable properties of spaces adapted to the linear equation (2.0.1) would be that its elements obey estimates similar to those known for free waves. For $X^{s,b}$ spaces, such a "transfer principle" exists, but only for $b > \frac{1}{2}$.

Proposition 3.4 (Transfer principle)**.** *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$ and assume that for some Banach space Y of spacetime functions, we have the estimate*

$$\|e^{it\tau'}e^{ith(D)}u_0\|_Y \lesssim \|u_0\|_{H^s(\mathbb{R}^n)} \tag{3.1.1}$$

for any $u_0 \in H^s(\mathbb{R}^n)$ and $\tau' \in \mathbb{R}$. Then, $X^{s,b} \subset Y$, i.e.

$$\|u\|_Y \lesssim \|u\|_{X^{s,b}}.$$

Proof. The goal is to rewrite an arbitrary $X^{s,b}$ function as a superposition of free waves. To this end, we write

$$\begin{aligned}(2\pi)^{-\frac{n+1}{2}}u(t,x) &= \iint e^{it\tau + x\cdot\xi}\mathcal{F}_{tx}u(\tau,\xi)d\tau d\xi \\ &= \int e^{it\tau}e^{ith(D)}\left(e^{-ith(D)}\int \mathcal{F}_{tx}u(\tau,\xi)e^{ix\cdot\xi}d\xi\right)d\tau \\ &= \int e^{it\tau}e^{ith(D)}\left(\int \mathcal{F}_{tx}u(\tau,\xi)e^{-ith(\xi)+ix\cdot\xi}d\xi\right)d\tau \\ &= \int e^{it\tau'}e^{ith(D)}\left(\int \mathcal{F}_{tx}u(\tau' + h(\xi),\xi)e^{ix\cdot\xi}\right)d\tau'.\end{aligned}$$

Denoting

$$v_{\tau'}(x) = \int \mathcal{F}_{tx}(\tau' + h(\xi),\xi)e^{ix\cdot\xi} = \mathcal{F}_\xi^{-1}\left(\mathcal{F}_{tx}u(\tau' + h(\xi),\xi)\right),$$

we now use Minkowski's inequality (that is, the properties of the Bochner integral) and (3.1.1) to estimate

$$\begin{aligned}\|u\|_Y &\lesssim \int \|e^{it\tau'} e^{ith(D)} v_{\tau'}\|_Y d\tau' \lesssim \int \|v_{\tau'}\|_{H^s(\mathbb{R}^n)} d\tau' \\ &\lesssim \left(\int \langle\tau'\rangle^{-2b} d\tau'\right)^{\frac{1}{2}} \left(\int \langle\tau'\rangle^{2b} \|v_{\tau'}\|^2_{H^s(\mathbb{R}^n)} d\tau'\right)^{\frac{1}{2}} \\ &\lesssim \|\langle\tau'\rangle^b \mathcal{F}_{tx} u(\tau' + h(\xi), \xi)\langle\xi\rangle^s\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\ &= \|u\|_{X^{s,b}}.\end{aligned}$$

□

Remark 3.5. As an artefact of the proof, we see that the argument just barely fails at the "endpoint" $b = \frac{1}{2}$ as a consequence of the logarithmic divergence of

$$\int_{-C}^{C} \langle\tau'\rangle^{-1} d\tau' \sim \log(1 + C) \qquad \text{as } C \to \infty.$$

This divergence, synonymous to the failure of the embedding $H^b(\mathbb{R}) \subset L^\infty(\mathbb{R})$, $b > \frac{1}{2}$ at the endpoint $b = \frac{1}{2}$, is the reason for many problems that arise if one is forced (or chooses) to work with endpoint $X^{s,b}$ spaces. Later in this section we will see some refinements for such cases, and eventually the U^2 and V^2 spaces, which are much better behaved in this respect.

As a first and important application of the transfer principle, we can show that for $b > \frac{1}{2}$, $X^{s,b}$ functions are contained in $C(\mathbb{R}, H^s)$; this is an important property in light of the discussion in section 2.2.

Corollary 3.6. *Let $b > \frac{1}{2}$. Then we have $X^{s,b} \subset C(\mathbb{R}, H^s)$, that is,*

$$\|u\|_{C(\mathbb{R},H^s)} \lesssim \|u\|_{X^{s,b}}.$$

Proof. This is just Proposition 3.4, applied to the choice $Y = C(\mathbb{R}, H^s)$. □

We have mentioned earlier that the endpoint $X^{s,\frac{1}{2}}$ spaces are somewhat ill-behaved. Unfortunately, this space is also the most natural to use as far as scaling is concerned. To illustrate this, assume for a moment that we replace the factors $\langle\tau - h(\xi)\rangle^b$ and $\langle\xi\rangle^s$ in the definition of the $X^{s,b}$ norm by their homogeneous counterparts $|\tau - h(\xi)|^b$ and $|\xi|^s$. Denoting the altered norm by $\|\cdot\|_{\tilde{X}^{s,b}_h}$, we find that

$$\|u(\lambda t, x)\|_{\tilde{X}^{s,b}_h} = \lambda^{b-\frac{1}{2}} \|u\|_{\tilde{X}^{s,b}_{h/\lambda}}$$

and so it is precisely at the endpoint $b = \frac{1}{2}$ where $\tilde{X}^{s,b}$ scales in time like the larger space $C(\mathbb{R}, \dot{H}^s)$, and in fact in situations where one is forced to respect scaling, this poses a serious problem.

For instance, when looking to solve an equation globally in time, say for small initial data, then at least one would need the free solutions $e^{ith(D)}f$, $f \in L^2(\mathbb{R}^n)$ to be bounded in terms of $\|f\|_{L^2(\mathbb{R}^n)}$. This can only work when $b = \frac{1}{2}$ in light of the above scaling or Lemma 3.3. Thus arises the need to replace, or at least refine the space $X^{s,\frac{1}{2}}$.

Endpoint refinements

Some approaches to dealing with the dysfunct behavior of $X^{s,\frac{1}{2}}$ spaces exist, notably by introducing a Besov structure on the temporal portion of the space to recover at least some desirable properties. Setting in what follows $s = 0$ for brevity, we have the following

Definition 3.7 ($X^{0,b,q}$ and $\dot{X}^{0,b,q}$)**.** Let $1 \leq q \leq \infty$ and $b \in \mathbb{R}$. The spaces $X^{0,b,q}$ and $\dot{X}^{0,b,q}$ are defined through the seminorms

$$\|u\|_{X^{0,b,q}} = \|e^{-ith(D)}u\|_{B^b_{2,q}} = \left(\sum_{M \in 2^{\mathbb{N}}} M^{bq} \|Q^h_M u\|^q_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{q}}$$

and

$$\|u\|_{\dot{X}^{0,b,q}} = \|e^{-ith(D)}u\|_{\dot{B}^b_{2,q}} = \left(\sum_{M \in 2^{\mathbb{Z}}} M^{bq} \|Q^h_M u\|^q_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{q}},$$

acting on $S'(\mathbb{R}, L^2(\mathbb{R}^n))$, respectively. In particular, we have

$$X^{0,b,2} = X^{0,b}, \qquad \dot{X}^{0,b,2} = \dot{X}^{0,b}.$$

It follows directly from the embedding $l^p \subset l^q$ that

$$X^{0,b,1} \subset X^{0,b,p} \subset X^{0,b,q} \subset X^{0,b,\infty}$$

for $1 \leq p \leq q \leq \infty$; hence $X^{0,b,1} \subset X^{0,b}$ is the smallest space in this family, and in fact the only one that regains the embedding into $C(\mathbb{R}, L^2(\mathbb{R}^n))$ in the endpoint case $b = \frac{1}{2}$, as demonstrated in the next

Proposition 3.8. *We have $X^{0,\frac{1}{2},1} \subset C(\mathbb{R}, L^2(\mathbb{R}^n))$, that is*

$$\|u\|_{C(\mathbb{R},L^2(\mathbb{R}^n))} \lesssim \|u\|_{X^{0,\frac{1}{2},1}}.$$

The same holds true for $\dot{X}^{0,\frac{1}{2}}$.

Proof. As in the proof of Proposition 3.4, we write

$$u = e^{it\tau'} e^{ith(D)} v_{\tau'},$$

where

$$v_{\tau'}(x) = \mathcal{F}^{-1}_{\xi} \left(\mathcal{F}_{tx} u(\tau' + h(\xi), \xi) \right),$$

and estimate

$$\begin{aligned}\|u\|_{C(\mathbb{R},L^2(\mathbb{R}^n))} &\leq \int \|e^{it\tau'}e^{ith(D)}v_{\tau'}\|_{C(\mathbb{R},L^2(\mathbb{R}^n))}d\tau' \lesssim \int \|v_{\tau'}\|_{L^2(\mathbb{R}^n)}d\tau' \\ &= \sum_M \int_{|\tau'|\sim M} \|v_\tau'\|_{L^2(\mathbb{R}^n)}d\tau' \lesssim \sum_M M^{\frac{1}{2}} \left(\int_{|\tau'|\ \sim M} \|v_{\tau'}\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}} \\ &\leq \|u\|_{X^{0,\frac{1}{2},1}}\end{aligned}$$

□

3.2 U^p and V^p spaces

We present in this section the spaces U^p and V^p, along with their important duality relation. Very recently, a fairly complete treatment of these spaces was undertaken in [Koc12][1], so that giving an exhaustive description here seems redundant. Instead, we will briefly present the basic definitions and properties and subsequently compare the behavior of these spaces to that of the $X^{s,b}$ spaces introduced in the last section.
Before we begin, we would like to make a note of the discussion in the last section. It was demonstrated that the $X^{s,b}$ space is just the space $H^bH^s(\mathbb{R}\times\mathbb{R}^n)$, adapted to the propagator $e^{ith(D)}$. Also, we have seen that the choice $b = \frac{1}{2}$, corresponding to invariance under rescaling in the time variable, is most natural from a scaling point of view. Since $H^{\frac{1}{2}}$ is not a good space to work with in light of the non-embedding $H^{\frac{1}{2}}(\mathbb{R}) \not\subset L^\infty(\mathbb{R})$ and the resulting undesirable behavior of the spaces $X^{s,\frac{1}{2}}$, one direction one can take is to try to substitute the space $H^{\frac{1}{2}}$ in the definition of $X^{s,b}$ by another, more well-behaved space.
Such an alternative should be invariant under rescaling in time, embed into $L^\infty(\mathbb{R})$ and ideally have good duality properties. We will see that the U^p and V^p spaces satisfy these requirements, and indeed result in an efficient $X^{s,b}$-type machinery.

Notation. In what follows, we let $1 < p < \infty$ unless stated otherwise and, as usual, $\frac{1}{p} + \frac{1}{p'} = 1$. In this section, we generally deal with functions defined on an interval $I = [a, b)$ or $I = (a, b)$, where $-\infty \leq a < b \leq \infty$, taking values in a Hilbert or Banach space $\mathcal{B}$. A *partition* of I is a sequence

$$a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b$$

and a step function associated to a partition as above is any function[2] which is constant on the open intervals (a, t_1), (t_1, t_2), ..., (t_n, b), regardless of its values on the endpoints of those intervals. Given any interval (c, d), we refer to c and d as the endpoints of that interval, even if c or d are infinite.
We extend functions defined on I by setting $f(b) = 0$, even if $b = \infty$ or if $f(b)$ does not coincide with the left-sided limit at b. In particular, $f(\infty) = 0$, for any

[1] see also [HHK09] and the erratum [HHK10] for its predecessor
[2] in particular, a step function has, by definition, only finitely many steps

$f : (a, \infty) \to \mathcal{B}$.

Definition 3.9 (Ruled functions)**.** A function $f : I \to \mathcal{B}$ is *ruled* if for any $x \in I$, both one-sided limits exist. We denote by $\mathcal{R}$ the collection of such functions. Similarly, we define $\mathcal{R}_{rc} \subset \mathcal{R}$ as those functions $f \in \mathcal{R}$ which are right-continuous and have

$$\lim_{t \to a^+} f(t) = 0,$$

and $\mathcal{S}_{rc} \subset \mathcal{R}_{rc}$ as the right-continuous step functions vanishing at a, all equipped with the supremum norm as well.

It is easy to check that $\mathcal{R}$ and $\mathcal{R}_{rc}$ are Banach spaces.

We introduce first the spaces V^p of bounded p-variation, whose history dates well back into the 20th century[3].

Definition 3.10 (p-variation)**.** For a function $f : I \to \mathcal{B}$, the *p-variation of* f is defined by

$$\|f\|_{\dot{V}^p(I)} = \sup_{(t_i)_{i=0}^{n+1} \text{ partition}} \left(\sum_{i=1}^{n-1} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}}^p \right)^{\frac{1}{p}}$$

This expression has some simple properties.

Lemma 3.11.

1. *$\|\cdot\|_{\dot{V}^p}$, where finite, defines a seminorm which is invariant under continuous monotone changes of coordinates of $I \subset \mathbb{R}$*
2. *the estimate $\|f\|_{\dot{V}^p} \le (b-a)^{\frac{1}{p}} \|f\|_{\dot{C}^{\frac{1}{p}}(\mathcal{B})}$ holds. Hence, $\dot{C}^{\frac{1}{p}} \subset \dot{V}^p$.*
3. *if $\|f\|_{\dot{V}^p}$ is finite, then f has one-sided limits on I, including the endpoints.*
4. *for bounded, monotone and real-valued f, we have $\|f\|_{\dot{V}^p} = \sup f - \inf f$.*

Proof. The first claim is clear. The next statement follows by direct computation since

$$\sum_{i=1}^{n-1} \|f(t_{i+1}) - f(t_i)\|_{\mathcal{B}}^p \le \|f\|_{\dot{C}^{\frac{1}{p}}}^p \sum_{i=1}^{n-1} (t_{i+1} - t_i) \le |b-a| \|f\|_{\dot{C}^{\frac{1}{p}}}^p .$$

For the third claim, assume by contradiction that, say, the left limit at some $c \in (a, b)$ does not exist (the argument at the endpoint is similar). Consequently there is $\epsilon > 0$ such that for any $\delta > 0$, we can find $c - \delta < t_0 < t_1 < c$ such that

$$\|f(t_1) - f(t_0)\|_{\mathcal{B}} \ge \epsilon.$$

After chosing $\delta \ll c - t_1$, we can find t_2 and t_3 with similar properties, and iteratively, after K steps, we can bound from below

$$\|f\|_{\dot{V}^p} \ge K\epsilon$$

[3] see [Wie24], or [Lyo98] for more recent applications in probability theory

which contradicts $\|f\|_{\dot{V}^p} < \infty$ as $K \to \infty$. Finally, for the last claim, we note that for $a < b < c$, we have[4]

$$|c - a|^p \geq |c - b|^p + |b - a|^p$$

and thus, for increasing f,

$$|f(t_2) - f(t_0)|^p \geq |f(t_2) - f(t_1)|^p + |f(t_1) - f(t_0)|^p.$$

Hence, any candidate for maximizing the $\dot{V}^p$ seminorm is dominated by choosing t_0 such that $f(t_0) \sim \sup f$ and $f(t_1) \sim \inf f$. ☐

Now we are ready to define the space V^p.

Definition 3.12. Let $1 \leq p < \infty$. The space $V^p((a,b),\mathcal{B}) = V^p$ is defined as the set of functions $v : (a,b) \to \mathcal{B}$ for which the norm

$$\begin{aligned} \|v\|_{V^p((a,b),\mathcal{B})} = \|v\|_{V^p} = &\sup_{(t_i)_{i=0}^{n+1} \text{ partition}} \left(\sum_{i=1}^{n} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}}^p \right)^{\frac{1}{p}} \\ &\sim \max\{\|v\|_{L^\infty((a,b),\mathcal{B})}, \|v\|_{\dot{V}^p((a,b),\mathcal{B})}\} \end{aligned} \tag{3.2.1}$$

is finite. Similarly, we define the space $V^p_{rc}((a,b),\mathcal{B}) = V^p((a,b),\mathcal{B}) \cap \mathcal{S}_{rc}$ using the same norm. Finally, it is natural to denote $V^\infty = \mathcal{R}$.

As indicated by the definitions above, it will be convenient to omit from the notation the interval and the underlying Banach space when they are assumed to be fixed. Some elementary properties are collected below.

Lemma 3.13. *Let* $1 \leq p < q < \infty$.

1. $V^p \subset \mathcal{R}$ *and* $V^p_{rc} \subset \mathcal{R}_{rc}$ *are closed (and hence Banach) subspaces.*
2. $V^p \subset V^q$ *is a continuous embedding, that is* $\|f\|_{V^q} \leq \|f\|_{V^p}$.
3. $V^p(I) \subset V^p(\mathbb{R})$ *through extension by zero.*

Proof. In light of Lemma 3.11, for (1) only closedness needs to be shown. Hence let $v \in \mathbb{R}$ the limit of Cauchy sequence $v_k \in V^p$, where $v \in V^p$ needs to be shown, which reduces to $\|v\|_{\dot{V}^p} < \infty$. Given an arbitrary partition $(t_i)_{i=0}^{n+1}$ and $\epsilon > 0$, we can find $K > 0$ such that for all $k > K$, we have $\|v_k(t) - v(t)\|_{\mathcal{B}} < \epsilon$ on (a,b). Hence

$$\begin{aligned} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}} &\leq \|v(t_{i+1}) - v_k(t_{i+1})\|_{\mathcal{B}} + \|v_k(t_{i+1}) - v_k(t_i)\|_{\mathcal{B}} + \|v_k(t_i) - v(t_i)\|_{\mathcal{B}} \\ &\leq 2\epsilon + \|v_k(t_{i+1}) - v_k(t_i)\|_{\mathcal{B}}. \end{aligned}$$

Upon taking ϵ small enough depending on the partition (t_i) and estimating $\|v\|_{\dot{V}}$, the claim follows. The second claim follows from $l^q(\mathbb{N}) \subset l^p(\mathbb{N})$ and the third is obvious. ☐

[4]dividing by $|c - a|^p$, this is equivalent to $|x|^p + |y|^p \leq 1$ for $|x| + |y| = 1$, which is trivial

Now we introduce the companion space U^p. We will collect some basic properties and subsequently outline their connections.
The space U^p will be built from linear combinations of atoms as defined below.

Definition 3.14 (U^p atoms)**.** A step function $a \in \mathcal{S}_{rc}$,

$$a(t) = \sum_{i=0}^{n} \phi_i \mathbf{1}_{[t_i,t_{i+1})}(t) = \sum_{i=1}^{n} \phi_i \mathbf{1}_{[t_i,t_{i+1})}(t)$$

is a U^p*atom* (or *p-atom*) if its steps ϕ_i, $i = 1, \dots, n$ satisfy

$$\sum_i \|\phi_i\|_{\mathcal{B}}^p \leq 1.$$

Note that since $a \in \mathcal{S}_{rc}$, we always have $\phi_0 = 0$.

Definition 3.15 (U^p)**.** Let $u : I \to \mathcal{B}$ such that there exist $\lambda_i \in \mathbb{C}$ and p-atoms $a_i \in \mathcal{S}_{rc}$,

$$u = \sum_{i=1}^{\infty} \lambda_i a_i$$

where the sum converges in $\mathcal{R}$. Then $u \in U^p = U^p(I, \mathcal{B})$. We define the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \;\middle|\; u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ a_j\ U^p\text{-atom} \right\}. \tag{3.2.2}$$

Lemma 3.16. *Let* $1 \leq p < q < \infty$*.*

1. $\|a\|_{U^p} \leq 1$ *for any p-atom* a*.*
2. $U^p \subset \mathcal{R}_{rc}$ *is a Banach subspace. In particular,* U^p *functions are right continuous and vanish at the left endpoint* a*.*
3. $U^p(I) \subset U^p(\mathbb{R})$ *through extension by zero.*
4. *The embedding* $U^p \subset U^q \subset \mathcal{R}_{rc}$ *is continuous.*
5. $\lim_{t\to\infty} u(t) \in \mathcal{B}$ *exists.*
6. *Let* Y *a Banach space and* $T : \mathcal{S}_{rc} \mapsto Y$ *a linear operator satisfying*

$$\|Ta\|_Y \leq C \qquad \forall \text{ p-atoms } a.$$

Then T *extends uniquely to a linear operator* $T : U^p \to Y$ *bounded by the same constant* C*.*

Proof. The first and third claim are obvious. For (2), take a Cauchy sequence $u_k \in U^p$. Passing to a subsequence, we may assume that $\|u_{k+1} - u_k\|_{U^p} < 2^{-k}$ and hence $u_{k+1} - u_k = \sum_i \lambda_i^k a_i^k$, where $\sum_i |\lambda_i| \leq 2^{-k}$. Consequently, with summations converging in $\mathcal{R}$,

$$u(t) - u_1 = \sum_{k=1}^{\infty} (u_{k+1} - u_k) = \sum_{k,i} \lambda_i^k a_i^k$$

gives a representation $u - u_1 = \sum_j \tilde{\lambda}_j \tilde{a}_j$ with

$$\sum_j |\tilde{\lambda}_j| \le \sum_k \sum_i |\lambda_i| \le \sum_k 2^{-k} \lesssim 1.$$

and hence, $u \in U^p$. The right continuity on (a,b) is clear for atoms, and carries over to U^p effortlessly. For the endpoint a, we take $u = \sum_i \lambda_i a_i$ and $N \in \mathbb{N}$ such that $\sum_{i>N} |\lambda_i| < \epsilon$. Then there is a $t^- \in (a,b)$ such that $\sum_{i\le N} \lambda_i a_i$ vanishes on (a, t^-). Thus, $\|u(t)\|_{\mathcal{B}} \le \epsilon$ on (a,t^-) and therefore $u(t) \to 0$ as $t \to a^+$. A similar argument gives (5). (4) is an easy consequence of the embedding $l^p(\mathbb{N}) \subset l^q(\mathbb{N})$. Namely, let $a = \sum_{i=1}^n \phi_i \mathbf{1}_{[t_i,t_{i+1})}(t)$ a p-atom. Then

$$b = \frac{(\sum \|\phi_i\|_{\mathcal{B}}^p)^{\frac{1}{p}}}{(\sum \|\phi_i\|_{\mathcal{B}}^q)^{\frac{1}{q}}} a$$

is a q-atom, and we have

$$\|a\|_{U^q} \le \frac{(\sum \|\phi_i\|_{\mathcal{B}}^q)^{\frac{1}{q}}}{(\sum \|\phi_i\|_{\mathcal{B}}^p)^{\frac{1}{p}}} \|b\|_{U^q} \le \frac{(\sum \|\phi_i\|_{\mathcal{B}}^q)^{\frac{1}{q}}}{(\sum \|\phi_i\|_{\mathcal{B}}^p)^{\frac{1}{p}}}.$$

But since $\|\cdot\|_{l^q(\mathbb{N})} \le \|\cdot\|_{l^p(\mathbb{N})}$, we obtain $\|a\|_{U^q} \le 1$, and the claim follows by the atomic structure of U^q and the triangle inequality. Finally, for (6) we simply define $T(u)$, for a U^p function $u = \sum_i \lambda_i a_i$, by the absolutely convergent sum $T(u) = \sum_i \lambda_i T(a_i) \in Y$. This has the desired properties, and if T' were another such extension, then $T - T'$ would vanish on all atoms and consequently on U^p, as can be seen by approximating a U^p function by a finite linear combination of atoms up to a small error and using the boundedness of $T - T'$. □

Proposition 3.17 (Relations between U^p and V^p)**.**

1. *Let $1 \le p < \infty$. Then $U^p \subset V^p_{rc}$.*

2. *Let $1 < p < q < \infty$. Then $V^p_{rc} \subset U^q$.*

Proof. For the first claim, it suffices to treat a p-atom $a = \sum_{j=1}^m \mathbf{1}_{[t_j,t_{j+1})}\phi_j$. Take a partition $(t_i)_{i=0}^{n+1}$ which realizes $\|a\|_{V^p}$ up to a small error ϵ and denote $a(t_{i+1}) - a(t_i) = \phi_{j(i+1)} - \phi_{j(i)}$, for some strictly increasing j. Then a direct calculation gives

$$\|a\|_{V^p} \le \left(\sum_{i=1}^n \|\phi_{j(i+1)}\|_{\mathcal{B}}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n \|\phi_{j(i)}\|_{\mathcal{B}}^p\right)^{\frac{1}{p}}$$

and hence $\|a\|_{V^p} \le 2$, as desired. The second claim is less trivial and we will use the following lemma. Its proof, which we skip for the purpose of brevity, is based on approximating a V^p_{rc} function by a sum of step functions.

Lemma 3.18 ([HHK09])**.** *Let $1 < p < q < \infty$. There exist $C, \kappa > 0$ depending only on p and q, such that given $M \ge 1$, any $v \in V^p_{rc}$ can be written as the sum of a U^p*

function u and a U^q function w,

$$v = u + w,$$

satisfying

$$\frac{\kappa}{M}\|u\|_{U^p} + e^M\|w\|_{U^q} \leq \|v\|_{V^p}.$$

In other words, we can split up a V^p_{rc} function in a large chunk which lies in the smaller space $U^p \subset V^p$, but are left with a small remainder in the larger space $U^q \supset V^p$. We apply the lemma with $M = 1$ and use the embedding $U^p \subset U^q$, resulting in the claimed inequality $\|u\|_{U^q} \lesssim \|v\|_{V^p}$. □

Stieltjes integration and the duality $(U^p)^* = V^{p'}$

One interesting bit of the history of the V^p spaces due to Young [You36] is a generalization of the Riemann-Stieltjes integral. Recall that the (generalized) Riemann-Stieltjes integral of a real function $f : (a, b) \to \mathbb{R}$ with respect to another such function g is defined, loosely speaking, as the limit of the expressions

$$\sum_{i=0}^{n} f(c_i)(g(t_{i+1} - g(t_i)) \qquad c_i \in [t_i, t_{i+1}] \tag{3.2.3}$$

over finer and finer partitions $(t_i)_{i=0}^{n+1}$ of (a, b) and all choices of $c_i = c_i$ subject to the constraint $c_i \in [t_i, t_{i+1}]$. If that limit exists, its value is denoted $\int_a^b f dg$.

Among the simplest results is that $\int_a^b f dg$ exists for $f \in \mathcal{R}$ and and g of bounded variation (that is, $g \in V^1$), along with, for $f \in \mathcal{S}_{rc}$, the explicit formula

$$\int_a^b f dg = \sum_{i=1}^{n} f(t_i)(g(t_{i+1}) - g(t_i)).$$

Noting that $\mathcal{R} = V^\infty$ and using the above formula shows that the mapping

$$V^\infty(\mathbb{R}) \times V^1(\mathbb{R}) \to \mathbb{R}, \quad (f, g) \mapsto \int f dg$$

is a bounded bilinear form. Young's extension (for $\mathcal{B} = \mathbb{R}$) to this states that for $1 < p < \infty$, $\int_a^b f dg$ in fact exists when $f \in V^p$, $g \in V^{\tilde{p}}$ along with the corresponding bilinear bound, however under the somewhat unnatural condition $\frac{1}{p} + \frac{1}{\tilde{p}} > 1$. This last condition barely misses the dual index $\tilde{p} = p'$, and it turns out that for the sharp result, one has to place g in the smaller space $U^{p'}$ instead of $V^{p'}$. This leads to the following theorem, and in fact induces a duality between U^p and $V^{p'}$.

Theorem 3.19. *Let $1 < p < \infty$. We have*

$$(U^p(\mathcal{B}))^* = V^{p'}(\mathcal{B}^*)$$

in the sense that there is a bounded bilinear form B such that the mapping

$$T : V^{p'}(\mathcal{B}^*) \to (U^p(\mathcal{B}))^*,\ T(v) := B(\cdot, v) \tag{3.2.4}$$

is an isometric isomorphism.

Remark 3.20. The bilinear form B corresponds precisely to the integral $\int f dg$ and Young's result is easily recovered from Theorem 3.19 using the embedding $V^{p'-\epsilon} \subset U^{p'}$, $p' > 1$.

Proof. Similar to (3.2.3), we begin by defining for $u \in \mathcal{S}_{rc}((a,b),\mathcal{B})$ with associated partition $(t_i)_{i=0}^{n+1}$ and $v \in V^{p'}((a,b),\mathcal{B}^*)$ the functional

$$F_v(u) := \sum_{i=1}^{n+1} \langle v(t_i), u(t_i) - u(t_{i-1})\rangle_{\mathcal{B}^*,\mathcal{B}} = -\sum_{i=1}^{n} \langle v(t_{i+1}) - v(t_i), u(t_i)\rangle_{\mathcal{B}^*,\mathcal{B}}$$

where we have used that $u(a) = 0$ since $u \in \mathcal{S}_{rc}$ and $v(b) = 0$, $t_0 = a$, $t_{n+1} = b$ by definition. Clearly, this is a linear expression in u and v, and for any p-atom a with steps ϕ_i, $i = 1, \ldots, n$, we have

$$|F_v(a)| \le \|v\|_{V^{p'}(\mathcal{B}^*)} \left(\sum_{i=1}^{n} \|\phi_i\|_{\mathcal{B}}^p \right)^{\frac{1}{p}} \le \|v\|_{V^{p'}(\mathcal{B}^*)}.$$

Hence, by Definition 3.15, F_v extends to U^p with norm $\|F_v\|_{U^p(\mathcal{B})\to\mathcal{B}} \le \|v\|_{V^{p'}(\mathcal{B}^*)}$, and we set $B(u,v) = F_v(u)$. It remains to show that the mapping

$$V^{p'}(\mathcal{B}^*) \ni v \mapsto F_v \in (U^p(\mathcal{B}))^*$$

defines a surjective isometry. To see the isometry part, take $\epsilon > 0$ and $v \in V^{p'}(\mathcal{B}^*)$ along with a partition $(t_i)_{i=0}^{n+1}$ which has

$$\left(\sum_{i=1}^{n} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}^*}^{p'} \right)^{\frac{1}{p'}} \ge (1-\epsilon)\|v\|_{V^{p'}(\mathcal{B}^*)}.$$

We use this partition to build a U^p atom by choosing $V_i \in \mathcal{B}$, $\|V_i\|_{\mathcal{B}} = 1$, such that

$$\langle v(t_{i+1}) - v(t_i), V_i\rangle_{\mathcal{B}^*,\mathcal{B}} \ge (1-\epsilon)\|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}^*}$$

and setting

$$a = \sum_{i=1}^{n} \phi_i \mathbf{1}_{[t_i,t_{i+1})}(t), \quad \phi_i = \|v\|_{V^{p'}}^{1-p'} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}^*}^{p'-1} V_i.$$

Then a is a $U^p(\mathcal{B})$ atom and

$$|B(a,v)| \ge (1-\epsilon)\|v\|_{V^{p'}}^{1-p'} \sum_{i=1}^{n} \|v(t_{i+1}) - v(t_i)\|_{\mathcal{B}^*}^{p'} \ge (1-\epsilon)^{1+p'}\|v\|_{V^{p'}}.$$

It remains to show surjectivity. Let $F \in (U^p(\mathcal{B}))^*$. Then for each $t \in (a,b)$, the mapping

$$F_t : \mathcal{B} \to \mathbb{C}, \quad \mathfrak{b} \mapsto F(\mathbf{1}_{[t,b)}\mathfrak{b})$$

is an element of $\mathcal{B}^*$ of norm $\|F\|_{(U^2(\mathcal{B}))^*}$. Consequently we can define $v(t) = F_t$ and compute for a p-atom $a = \sum_{i=1}^n \phi_i 1_{[t_i,t_{i+1})}$

$$\begin{aligned} B(a,v) &= -\sum_{i=1}^n \langle F_{t+1} - F_t, \phi_i \rangle_{\mathcal{B}^*,\mathcal{B}} = F(\sum_{i=1}^n \mathbf{1}_{[t_i,t_{i+1})}\phi_i) \\ &= F(a). \end{aligned}$$

Since this determines $B(u,v)$ for all $u \in U^p(\mathcal{B})$, the proof is complete. □

The definition of $B(f,g)$ clearly mimics the expression $\int f dg = -\int g df = -\int g f' dt$ for regular functions. In fact, this can be made rigorous under certain assumptions, as stated below.

Proposition 3.21. *Let $1 < p < \infty$, $u \in V^1(\mathcal{B})$ absolutely continuous on compact intervals, $\lim_{t\to-\infty} u(t) = 0$, and $v \in V^{p'}(\mathcal{B}^*)$. Then,*

$$B(u,v) = -\int_{-\infty}^{\infty} \langle v(t), u'(t) \rangle_{\mathcal{B}^*,\mathcal{B}} dt. \tag{3.2.5}$$

In particular, $B(u,v) = B(u,\tilde{v})$ if $v(t) = \tilde{v}(t)$ almost everywhere. Consequently, v may be replaced by its right-continuous version.

Proof. We refer to [HHK09] for a full proof. □

Applications to $X^{s,b}$ spaces

In what follows, we will assume $\mathcal{B} = L^2(\mathbb{R}^n)$ and $(a,b) = (-\infty,\infty)$ without further comment.

We began this chapter describing the $X^{s,b}$ spaces and some of their shortcomings with respect to the endpoint $b = \frac{1}{2}$. Most of these issues can be traced back to the failure of the embedding $H^{\frac{1}{2}}(\mathbb{R}) \not\subset L^\infty(\mathbb{R})$ and we were trying to find a substitute for this space. In this subsection, we shall indeed find such a replacement: The space U^2. This can be motivated by the following

Proposition 3.22. *We have $\dot{V}^p \subset \dot{B}^{\frac{1}{p}}_{p,\infty}$. More precisely, for $f \in \dot{V}^p$, we have*

$$\sup_{h>0} h^{-\frac{1}{p}} \|v(\cdot + h) - v(\cdot)\|_{L^p(\mathbb{R},L^2(\mathbb{R}^n))} \lesssim \|v\|_{\dot{V}^p} \tag{3.2.6}$$

or, using a different (but equivalent) norm on $\dot{B}^{\frac{1}{p}}_{p,\infty}$,

$$\sup_{M\in 2^{\mathbb{Z}}} M^{\frac{1}{p}} \|Q_M v\|_{L^p(\mathbb{R},L^2(\mathbb{R}^n))} \lesssim \|v\|_{\dot{V}^p}.$$

Furthermore, by duality, we have also, for $u \in \dot{B}^{\frac{1}{p}}_{p,1}$ *vanishing at* $-\infty$*,*

$$\|u\|_{U^p} \lesssim \|u\|_{\dot{B}^{\frac{1}{p}}_{p,1}} \sim \sum_{M \in 2^{\mathbb{Z}}} M^{\frac{1}{p}} \|Q_M u\|_{L^p(\mathbb{R}, L^2(\mathbb{R}^n))}.$$

Proof. The key statement is (3.2.6), the rest follows by duality; see [HHK09]. Let v have finite $\dot{V}^p$ norm. We write $\mathbb{R} = \cup_{n \in \mathbb{Z}} I_n$, where $I_n = [nh, (n+1)h]$. For each $h > 0$ and $\epsilon > 0$, we choose a $t_n \in I_n$ such that

$$\|v(t+h) - v(t)\|_{L^2(\mathbb{R}^n)} \leq (1+\epsilon)\|v(t_n + h) - v(t_n)\|_{L^2(\mathbb{R}^n)} \qquad t \in I_n.$$

Then

$$\begin{aligned}
&\int \|v(t+h) - v(t)\|^p_{L^2(\mathbb{R}^n)} dt = \sum_{n \in \mathbb{Z}} \int_{I_n} \|v(t+h) - v(t)\|^p_{L^2(\mathbb{R}^n)} dt \\
&\leq h(1+\epsilon) \sum_{n \in \mathbb{Z}} \|v(t_n + h) - v(t_n)\|^p_{L^2(\mathbb{R}^n)} \\
&\leq h(1+\epsilon) \sum_{n \in \mathbb{Z}} \|v(t_n + h) - v(t_n)\|^p_{L^2(\mathbb{R}^n)} + \|v(t_{n+2}) - v(t_n + h)\|^p_{L^2(\mathbb{R}^n)} \\
&\leq 2h(1+\epsilon)\|v\|^p_{\dot{V}^p},
\end{aligned}$$

as claimed.

□

According to Proposition 3.22 and Proposition 3.17, if we chose $p = 2$, we have

$$\dot{B}^{\frac{1}{2}}_{2,1} \subset U^2 \subset V^2_{rc} \subset \dot{B}^{\frac{1}{2}}_{2,\infty}$$

but also $\dot{B}^{\frac{1}{2}}_{2,1} \subset \dot{B}^{\frac{1}{2}}_{2,2} = \dot{H}^{\frac{1}{2}} \subset \dot{B}^{\frac{1}{2}}_{2,\infty}$ and hence U^2 is very close to $\dot{H}^{\frac{1}{2}}$ but remains contained in L^∞. Additionally, U^2 and V^2 are very close[5] and we have the duality $(U^2)^* = V^2$, so that we can use the formula

$$\|u\|_{U^2} = \sup_{\|v\|_{V^2} \leq 1} |B(u,v)|$$

which will come in handy to estimate Duhamel terms, especially in combination with Proposition 3.21. In practice, it is most convenient to work with U^2 and V^2_{rc}, in light of Proposition 3.17.

Now we adapt the U^p and V^p spaces to a linear propagator $e^{ith(D)}$.

Definition 3.23. We define the space U^2_h adapted to the linear propagator $e^{ith(D)}$ as those functions $u : \mathbb{R} \to L^2(\mathbb{R}^n)$ for which $t \mapsto e^{-ith(D)}u$ is a U^2 function, equipped with the norm

$$\|u\|_{U^2_h} = \|e^{-ith(D)}u\|_{U^2}.$$

and similarly for other V^p or U^p spaces.

[5] but not equal, see [Koc12]

U^p_h is again an atomic space, a U^p_h atom being a function $\tilde{a} = e^{ith(D)}a$ for a U^p atom a.

With this definition and using the atomic structure of U^p, we can prove a nice transfer principle, i.e. a way of transferring linear and multilinear estimates for free solutions $e^{ith(D)}u_0$ to U^2_h functions, stated below.

Proposition 3.24. *Let*

$$T_0 : L^2 \times \cdots \times L^2 \to L^1_{loc}(\mathbb{R}^n, \mathbb{C})$$

a m-linear operator and $h_1, \ldots, h_m$ dispersion relations. Furthermore, let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and assume that we have

$$\|T_0(e^{ith_1(D)}\phi_1, \ldots, e^{ith_m(D)}\phi_m)\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} \lesssim \prod_{i=1}^{m} \|\phi_i\|_{L^2}.$$

for all $\phi_1, \ldots, \phi_m \in L^2(\mathbb{R}^n)$. Then, there exists $T : U^p_{h_1} \times \cdots \times U^p_{h_m} \to L^p_t(\mathbb{R}; L^q(\mathbb{R}^n))$ satisfying

$$\|T(u_1, \ldots, u_m)\|_{L^p(\mathbb{R},L^q(\mathbb{R}^n))} \lesssim \prod_{i=1}^{m} \|u_i\|_{U^p_h},$$

such that for a.e. $t \in \mathbb{R}$,

$$T(u_1, \ldots, u_m)(t)(x) = T_0(u_1(t), \ldots, u_m(t))(x).$$

Proof. It suffices to prove that $u_k = \sum_i \mathbf{1}_{[t^k_i, t^k_{i+1})} e^{ith(D)}\phi^k_i$ are U^p_h atoms. Then we compute

$$\begin{aligned}
&\|T_0(u_1, \ldots, u_m)\|_{L^pL^q(\mathbb{R}\times\mathbb{R}^n)} \leq \\
&\quad \left\| \sum_{i_1,\ldots,i_m} \prod_{k=1}^{m} \mathbf{1}_{[t^k_i, t^k_{i+1})} \|T_0(e^{ith(D)}\phi^1_i, \ldots, e^{ith(D)}\phi^m_i)\|_{L^q(\mathbb{R}^n)} \right\|_{L^p(\mathbb{R})} \\
&\quad = \left(\sum_{i_1,\ldots,i_m} \left\| \prod_{k=1}^{m} \mathbf{1}_{[t^k_i, t^k_{i+1})} \|T_0(e^{ith(D)}\phi^1_i, \ldots, e^{ith(D)}\phi^m_i)\|_{L^q(\mathbb{R}^n)} \right\|^p_{L^p(\mathbb{R})} \right)^{\frac{1}{p}} \\
&\quad \lesssim \left(\sum_{i_1,\ldots,i_m} \prod_{k=1}^{m} \|\phi^k_i\|^p_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{p}} \leq 1.
\end{aligned}$$

□

Next we collect some useful estimates which will be used extensively in chapter 4 and chapter 5.

Proposition 3.25. *We have, for $M = 2^k$, $k \in \mathbb{Z}$,*

$$\|Q_M^h u\|_{L^2(\mathbb{R}^n)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_h^2} \tag{3.2.7}$$

$$\|Q_{\geq M}^h u\|_{L^2(\mathbb{R}^n)} \lesssim M^{-\frac{1}{2}} \|u\|_{V_h^2} \tag{3.2.8}$$

$$\|Q_{<M}^h v\|_{V_h^p} \lesssim \|v\|_{V_h^p} \ , \quad \|Q_{\geq M}^h u\|_{V_h^p} \lesssim \|u\|_{V_h^p} \tag{3.2.9}$$

$$\|Q_{<M}^h u\|_{U_h^p} \lesssim \|u\|_{U_h^p} \ , \quad \|Q_{\geq M}^h u\|_{U_h^p} \lesssim \|u\|_{U_h^p} \tag{3.2.10}$$

Proof. The first two statements are just consequences of Proposition 3.22. For the rest, writing $Q_{\geq M} = 1 - Q_{<M}$ shows that only $Q_{<M}$ needs to be considered. Now let $v \in V_h^p$ and $(t_i)_{i=0}^{n+1}$ a partition. By scaling, we may reduce to the case $M = 1$ and upon replacing v by $e^{ith(D)}$, we may assume $h = 0$. Then, we estimate

$$\begin{aligned}
&\sum_{i=0}^{n} \|Q_{<1}v(t_{i+1}) - Q_{<1}v(t_i)\|_{L^2(\mathbb{R}^n)}^p \\
&\qquad \leq \sum_{i=0}^{n} \left(\int |\chi(\tau)| \|v(t_{i+1}+\tau) - v(t_i+\tau)\|_{L^2(\mathbb{R}^n)} d\tau \right)^p \\
&\qquad \leq \|\chi\|_{L^1(\mathbb{R})}^p \sum_{i=0}^{n} \int |\chi(\tau)| \|v(t_{i+1}+\tau) - v(t_i+\tau)\|_{L^2(\mathbb{R}^n)}^p d\tau \\
&\lesssim \|v\|_{V^p}^p
\end{aligned}$$

as claimed. The remaining claim (3.2.10) follows by duality and a similar computation. □

The transfer principle makes all the bounds for free solutions available to U_h^p functions as well. However, in applications one typically has at least one function which requires an estimate in a V^p space, which does not follow from the corresponding U^p bounds and Proposition 3.17. In such situations, the following proposition comes in handy: It provides a V^p estimate from a corresponding U^p estimate by interpolating with a (worse) U^q estimate, at a logarithmic loss.

Proposition 3.26. *Let $q > 1$, E be a Banach space and $T : U_h^q \to E$ be a bounded, linear operator with $\|Tu\|_E \leq C_q \|u\|_{U_h^q}$ for all $u \in U_h^q$. In addition, assume that for some $1 \leq p < q$ there exists $C_p \in (0, C_q]$ such that the estimate $\|Tu\|_E \leq C_p \|u\|_{U_h^p}$ holds true for all $u \in U_h^p$. Then, T satisfies the estimate*

$$\|Tu\|_E \lesssim C_p(1 + \ln \frac{C_q}{C_p}) \|u\|_{V_h^p}, \quad u \in V_{rc,h}^p.$$

Proof. This is another consequence of Lemma 3.18. Namely, we may decompose $V_{rc,h}^p \ni v = u + w$, where $u \in U_h^p$, $w \in U_h^q$,

$$\|u\|_{U_h^p} \lesssim M \|v\|_{V_h^p}, \qquad \|w\|_{U_h^p} \lesssim e^{-M} \|v\|_{V_h^p}.$$

Using this composition, we obtain

$$\|Tv\|_E \lesssim (C_p M + C_q e^{-M})\|v\|_{V_h^p},$$

and we optimize over M, leading to the choice $M = \ln \frac{C_q}{C_p}$ and the desired estimate. □

The following lemma demonstrates that when proving a multilinear U_h^2 estimate, one can replace all but one of the factors by free solutions $e^{ith(D)}u_0$. The corresponding statement is false when one replaces all factors by free solutions: One obtains only U_h^1 bounds.

Lemma 3.27. *Let $m \geq 2$. Then the estimate*

$$\left| \iint \prod_{j=1}^{m} u_j dx dt \right| \lesssim \prod_{j=1}^{m} \|u_j\|_{U_h^2} \tag{3.2.11}$$

is equivalent to

$$\left| \iint a \prod_{j=1}^{m-1} e^{ith(D)} f_j dx dt \right| \lesssim \prod_{j=1}^{m-1} \|f_j\|_{L^2(\mathbb{R}^n)} \tag{3.2.12}$$

where a is a U_h^2 atom.

Proof. Of course, it suffices to prove (3.2.11) in the case where $u_j = a_j$ are atoms, say with underlying partition $\{t_l^j\}_{l=1\ldots n_i}$ and steps ϕ_l^j; we may assume that $t_0^j = 0$ for all j. We now inductively split the time integration into intervals according to the following algorithm.

- Let $t^* = \max\{t_l^1\}_{l=1\ldots n_1}$ so that $[0, t^*)$ is the interval associated to the first step of the "slowest" atom, which we assume to be a_1 by symmetry.
- Split all other atoms whose first interval is not $[0, t^*]$ at t^* by duplicating the value to the left of t^*.
- Restart this process at t^*.

In effect, what we obtain is a new set of step functions which we denote again by a_j. These are still atoms (modulo a factor of $\frac{1}{2}$), since we cut in half at most once on each interval of the respective step function by the maximal choice of t^* in each step. We denote the corresponding set of cuts by $t_0^*, \ldots, t_N^*$, denote $I_k = [t_{k-1}^*, t_k^*)$ and decompose $\int_{\mathbb{R}} = \sum_{k=1}^{N} \int_{I_k}$. Relabeling the atoms on each I_k, we may assume that $a_1 = e^{ith(D)}\phi_k$ is a free wave there. Then, assuming that we can estimate for U_h^2 atoms s_j

$$\left| \int e^{ith(D)} \phi \prod_{j=2}^{m} s_j dx dt \right| \lesssim \|\phi\|_{L^2(\mathbb{R}^n)} \tag{3.2.13}$$

we get

$$
\begin{aligned}
(3.2.11) &\le \sum_{k=1}^{N} \left| \int_{I_k} \int e^{ith(D)} \phi_k \prod_{j=2}^{m} a_j dxdt \right| \lesssim \sum_{k=1}^{N} \|\phi_k\|_{L^2(\mathbb{R}^n)} \prod_{j=2}^{m} \|\mathbf{1}_{I_k} a_j\|_{U_h^2} \\
&\le \left(\sum_{k=1}^{N} \|\phi_k\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \max_{j=3,\dots,m} \|\mathbf{1}_{I_k} a_j\|_{U_h^2}^{n-2} \left(\sum_{k=1}^{m} \|\mathbf{1}_{I_k} a_2\|_{U_h^2}^2 \right)^{\frac{1}{2}} \\
&\le \left(\sum_{k=1}^{N} \|\phi_k\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}} \prod_{j=2}^{m} \left(\sum_{k=1}^{N} \|\mathbf{1}_{I_k} a_j\|_{U_h^2}^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

Now the ϕ_k are taken from a set of functions whose total $l^2 L^2$ sum is at most $2n \lesssim 1$, and hence the first factor is $O(1)$. Furthermore, for a "step function" $s = \sum \mathbf{1}_{[t_l, t_{l+1})} e^{ith(D)} s_l$, we have

$$
\|s\|_{U_h^2} \le \left(\sum_{l} \|s_l\|_{L^2(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.
$$

In particular, since our atoms are adapted to the partition $\{I_k^n\}_k$ by the above construction, we have

$$
\sum_{k=1}^{m} \|I_k a_j\|_{U_h^2}^2 \lesssim 1.
$$

This proves the claim under assumption (3.2.13), which by induction reduces to the case where a_1 through a_{m-2} are free waves and only a_{m-1} and a_m are left as atoms. But there, the above computation goes through just the same, using exactly (3.2.12). □

A critical issue with U^2 and V^2 often arises with the need to pass from a U^2 estimate to a (stronger) V^2 estimate. The following lemma suggests that when one is willing to restrict to a finite time interval, then such an estimate might be possible.

Proposition 3.28. *Let $\phi_T(t) = \phi(t/T)$ where ϕ is a smooth unit scale bump. Then*

$$
\|(\phi_T u)_{\le \Lambda}\|_{U^2} \lesssim |\log \Lambda T| \|u\|_{V^2}.
$$

Proof. We begin with the following

Lemma 3.29. *Let $|N - M| \gg 1$. Then, with $H = \max(M, N)$ and denoting dyadic frequency localizations by lowercase indices,*

$$
\begin{aligned}
\|(\phi_T u_M)_N\| &\sim \|((\phi_T)_H u_M)_N\|_{L^2} = \|\chi_N(\widehat{\phi_T}\chi_H * \hat{u}\chi_M)\|_{L^2} \le N^{\frac{1}{2}} \|\chi_H \widehat{\phi_T} * \hat{u}\chi_M\|_{L^\infty} \\
&\le N^{\frac{1}{2}} \|\chi_H \widehat{\phi_T}\|_{L^\infty} \|\hat{u}\chi_M\|_{L^1} \le (NT)^{\frac{1}{2}} \langle HT \rangle^{-K} \left((MT)^{\frac{1}{2}} \|u_M\|_{L^2} \right).
\end{aligned}
$$

With this lemma, we try to sum over all $M \ge 1/T$ and $N \le \Lambda$ the expression

$$
N^{\frac{1}{2}} \|(\phi_T u_M)_N\|_{L^2}.
$$

if $T^{-1} \leq M \leq N$, we need to control

$$\sum_{N} \sum_{1/T \leq M \leq N} (NT) \langle NT \rangle^{-K} \leq \sum_{N > 1/T} (NT)^{1+\varepsilon} \langle NT \rangle^{-K} \lesssim 1$$

whereas if $1/T, N \leq M$ we get

$$\begin{aligned} \sum_{M \geq 1/T} \sum_{N \leq M} (TN) \langle TM \rangle^{-K} &\lesssim \sum_{M \geq 1/T} \left(1 + \sum_{T^{-1} \leq N \leq M} (TN)^{1-K/2} \right) (TM)^{-K/2} \\ &\lesssim 1. \end{aligned}$$

The diagonal case $1/T \leq M \sim N$ is quickly handled using Young's inequality,

$$\|\phi_T u_N\|_{L^2} \leq \|\widehat{\phi_T}\|_{L^1} \|u_N\|_{L^2} \sim \|u_N\|_{L^2}$$

and

$$\sum_{1/T \leq N \leq \Lambda} N^{\frac{1}{2}} \|u_N\|_{L^2} \leq \log(\Lambda T) \|u\|_{B^{\frac{1}{2},2}_{\infty}}.$$

Hence matters are reduced to the case where $M \leq 1/T$ and (still) $N \leq \Lambda$. Since $\dot{B}^{s,p}_q$ sees constants (i.e. low frequencies), we switch back to U^2 and V^2 and compute, after rescaling $\tilde{T} = 1$, $\tilde{\Lambda} = \Lambda T$ (which turns ϕ_T into the unit bump $\phi \in C_0^\infty(\mathbb{R})$)

$$\begin{aligned} \|(\phi u_{\leq 1})_{\leq \Lambda}\|_{U^2} &\leq \|\mathcal{F}^{-1}\psi_{\leq \Lambda} * \nabla(f(\psi * u))\|_{L^1} \\ &\lesssim \|\mathcal{F}^{-1}\psi_{\leq \Lambda}\|_{L^1} \|\nabla(f(\psi * u))\|_{L^1} \\ &\lesssim \|(\nabla f)(\psi * u))\|_{L^1} + \|f(\nabla \psi * u)\|_{L^1} \\ &\leq \|\nabla f\|_{L^1} \|\psi * u\|_{L^\infty} + \|f\|_{L^1} \|(\nabla \psi) * u\|_{L^\infty} \\ &\lesssim \|u\|_{L^\infty} \lesssim \|u\|_{V^\infty} \lesssim \|u\|_{V^2} \end{aligned}$$

and hence, undoing the rescaling, uniformly in Λ we have

$$\|(\phi_T u_{\leq 1/T})_{\leq \Lambda}\|_{U^2} \lesssim \|u\|_{V^2}.$$

Combining the above, we obtain

$$\begin{aligned} \|(\phi_T u)_{\leq \Lambda}\|_{U^2} &\lesssim \|(\phi_T u_{\geq 1/T})_{\leq \Lambda}\|_{\dot{B}^{\frac{1}{2},2}_1} + \|(\phi_T u_{\leq 1/T})_{\leq \Lambda}\|_{U^2} \\ &\lesssim \log(\Lambda T) \|u\|_{\dot{B}^{\frac{1}{2},2}_\infty} + \|u\|_{V^\infty} \\ &\lesssim \log(\Lambda T) \|u\|_{V^2}. \end{aligned}$$

□

Chapter 4

Nonlinear Klein-Gordon equations

4.1 Introduction and main results

From the late 1970s on, there has been a lot of progress on questions of global existence and blow-up for equations and systems of the type

$$\begin{aligned} (\Box + m^2)u(t,x) &= F_p(u(t,x)), \qquad (t,x) \in [0,T) \times \mathbb{R}^n \\ u(0,x) &= f(x) \\ \partial_t u(0,x) &= g(x) \end{aligned} \tag{4.1.1}$$

where the initial data (f, g) are "small", $m \geq 0$, $\Box = \partial_{tt} - \Delta$, u is scalar or vector-valued, and F_p is a power-type nonlinearity of order $p > 0$, i.e. $|\partial^j F_p(s)| \sim |s|^{p-j}$ $(j \leq p)$ together with a similar condition for differences.

An optimistic energy heuristic based on the decay of free solutions leads to a first guess that global existence from small data could hold for

$$\begin{aligned} p &> 1 + \frac{2}{n} \qquad && \text{if } m > 0 \\ p &> 1 + \frac{2}{n-1} \qquad && \text{if } m = 0. \end{aligned}$$

We shall be interested primarily in the first case $m > 0$ but summarize the massless version $m = 0$ briefly. As it turns out, that case - where one is dealing with a nonlinear wave equation - is somewhat singular in the sense that the above heuristic is incorrect. Instead, the decisive role is played by a larger number commonly known as the *Strauss exponent*, the positive root $\gamma = \gamma(n)$ of

$$\frac{n}{2}\frac{\gamma - 1}{\gamma + 1} = \frac{\gamma}{2}. \tag{4.1.2}$$

Note $\gamma(1) \sim 3.56$, $\gamma(2) \sim 2.41$, $\gamma(3) = 2$, $\gamma(4) = 1.78$, $\gamma(\infty) = 1$ and

$$1 + \frac{2}{n} < \gamma < 1 + \frac{4}{n}.$$

More precisely, for $m = 0$, $\gamma(n-1)$ is a threshold power such that for (4.1.1) we have the following dichotomy: If $p > \gamma(n-1)$, then small, smooth and localized data lead to global solutions. In the other case $p \leq \gamma(n-1)$, one can find such data blowing up in finite time. This conjecture-turned-theorem[1] goes back to Strauss, who based his prediction on results by John in 3D [Joh79] and his own work. Hence, for the wave equation case $m = 0$, there is a very clear dichotomy between global solutions and finite time blow-up, indicated by $\gamma(n-1)$.

For $m > 0$, where one is dealing with a nonlinear Klein-Gordon equation, the picture is less clear, and in particular the role of $\gamma(n)$. This is somewhat curious since $\gamma(n)$ seems to first have arisen in Strauss' work [Str81] on scattering in the case $m > 0$. The spaces used in that work are based on the $t^{-\frac{n}{2}}$ time decay of free solutions, and the Strauss exponent $\gamma(n)$ occurs as a natural threshold below which the nonlinearity $|u|^p$ inherits too little decay in time to close the estimates[2].

From all of the above, it would seem reasonable to expect $\gamma(n)$ to play a the role of a threshold for global existence, scattering, or both[3] when $m > 0$. First insights were again made in three spatial dimensions first, for quadratic nonlinearities by Klainerman [Kla85] and Shatah[Sha85] independently. Noting that $\gamma(3) = 2$, this corresponds to the missing endpoint in Strauss' work [Str81].
However, for $n \leq 3$, advances far below the Strauss exponent all the way up to the energy prediction $p > 1 + \frac{2}{n}$ have been obtained by Lindblad and Sogge [LS96]; blow-up for $p < 1 + \frac{2}{n}$ in these dimensions is due to Keel and Tao in [KT99] at least when F_p is allowed to depend on first derivatives. Additionally, for $n = 2$, even in the critical case $p = 1 + \frac{2}{n}$ global existence is known [OTT96].This gives a fairly concise picture in low dimensions[4].

As far as scattering goes in this case, the Strauss exponent also doesn't seem to be a reliable indicator: Recent results by Hayashi and Naumkin [HN09], [HN08] show

[1] modulo the endpoint, Schaeffer [Sch85] confirmed this for $n = 2$ and Glassey [Gla81] subsequently proved finite time blow-up for the critical cases $p = \gamma(n-1)$, in two and three dimensions. For larger n, blow-up from small data below $\gamma(n-1)$, was subsequently confirmed by Sideris [Sid84], while the positive part is due to Zhou [Zho95], Lindblad and Sogge [LS96], Georgiev, Lindblad and Sogge [GLS97]. Finally, Yordanov and Zhang [YZ06] proved blow-up also when $p = \gamma(n)$ for the open cases $n \geq 4$

[2] the corresponding decay in L^{p+1} is t^{-d}, $d = \frac{n}{2}\frac{p-1}{p+1}$. These parameters follow directly from interpolation between the unitary $L^2 \to L^2$ and the dispersive $t^{-\frac{n}{2}}$ $L^1 \to L^\infty$ estimates and hence are reasonable to ask for if one expects the solution to behave like a free wave for large times t. Chosing to work with L^{p+1} roughly allows solving in $L^\infty(\mathbb{R}, (1+|t|)^{-d} L^{p+1})$ if $1 > d > \frac{1}{p}$. Now the equation $d = \frac{1}{p}$ is exactly (4.1.2).

[3] note that the relevant number is $\gamma(n)$ when $m \neq 0$

[4] however, Keel and Tao conjecture that in higher dimensions, $1 + \frac{2}{n}$ is not the correct threshold to global existence when $m > 0$

that there is small data scattering in the optimal range[5] $p > 1 + \frac{2}{n}$ when $n = 1, 2$ and for $1 + \frac{4}{n+2} < p < 1 + \frac{4}{n}$ when $n \geq 3$ with small initial data in weighted H^s norms. Both of these results show that scattering results exist below the Strauss exponent in any dimension with only some mild decay on the initial data.

Hence, the Strauss exponent appears to play a less central role, if any, when the masses are positive, at least if the data are sufficiently localized.

Since we are interested in systems, we mention only in passing the Hamiltonian theory around H^1 data (see, for instance, [Caz85]). Two consequences of this are global existence from small $H^1 \times L^2$ data for F_p, $p > 1$, and large data scattering in the energy space for $\gamma(n) < 1 + \frac{4}{n} < p < 1 + \frac{4}{n-1}$.

What all of the previous results have in common is that they impose at least some decay on the initial data, the mildest of which being weights, or higher L^p norms on the Fourier side. We consider it interesting to study the necessity of such conditions, a direction indicated by Delort and Fang in [DF00]. They consider data only in H^s with a reasonable number of derivatives and prove almost global existence for nonlinearities which are quadratic[6] in all dimensions $n \geq 2$. More recently, a global result [GS11] in this spirit has appeared in the most difficult case $n = 2$ for $H^{1+\varepsilon}$ initial data, which shows that decay is not needed in the endpoint case of the $1 + \frac{2}{n}$ heuristic when $n = 2$.

Our contribution is an improvement of this last result in the framework of U^2 and V^2 spaces (see [HHK09]), that is, two-dimensional quadratic Klein-Gordon equations, at low regularity. We obtain global existence, scattering and smooth dependence on the initial data for algebraic quadratic nonlinearities in u in dimensions two and higher. It turns out that a certain "non-resonance" condition connected to the applicability of the normal forms method allows for a conceptually clear and efficient proof using our setup.

The main result is the following

Theorem 4.1. *Let $n \geq 2$, $K \in \mathbb{N}$, $N_1, \dots, N_K \in \mathbb{C}[x_1, \overline{x_1}, \dots, x_K, \overline{x_K}]$ polynomials without linear or constant terms, and $m_1, \dots, m_K > 0$ such that*

$$m_i + m_j > m_l \quad (1 \leq i, j, l \leq K). \tag{R}$$

Let $k = \max_{i=1,\dots,K} \deg N_i$ and let $s \in \mathbb{R}$,

$$s \geq \begin{cases} \max(\frac{1}{2}, \frac{n-2}{2}) & k = 2 \\ \max(\frac{k-2}{k-1}, \frac{n}{2} - \frac{2}{k-1}) & k \geq 3. \end{cases}$$

[5] [Mat77], [Gla73]

[6] and may contain derivatives under the assumption of a null structure

Then there is $\epsilon > 0$ such that for initial data

$$(f_i, g_i) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n): \qquad \|(f_i, g_i)\|_{H^s \times H^{s-1}} \leq \epsilon,\ i = 1, \ldots, K$$

the system

$$\begin{aligned} (\Box + m_i^2) u_i &= N_i(u_1, \overline{u_1}, \ldots, u_K, \overline{u_K}) \qquad i = 1, \ldots, K \\ u_i(0) &= f_i \\ \partial_t u_i(0) &= g_i \end{aligned} \tag{4.1.3}$$

has a global solution in $C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^n))$. In addition, the solution depends in Lipschitz fashion on (f, g) and scatters asymptotically as $t \to \pm\infty$. Furthermore, it is unique in the smaller spaces $X^s([0, \pm\infty))$ introduced in the next section, and Duhamel's formula holds.

Remark 4.2. The result is most interesting in the case $n = 2, 3$ and $k = 2$, i.e. for quadratic nonlinearities in low dimensions. As the proofs show, only in this scenario is the condition (R) relevant: once the spatial dimension exceeds three or no quadratic terms are present, Strichartz inequalities suffice. The regularity is not always optimal for $k > 2$ but this is not the primary focus. We note that $\frac{n}{2}$ is always above the regularity threshold above, and we can state the following non optimal, but perhaps more legible version:

Corollary 4.3. *The system* (4.1.3) *under condition* (R) *has global solutions and scattering for small $H^s \times H^{s-1}$ data when $s \geq \frac{n}{2}$. If $n \geq 4$ or no quadratic terms are present,* (R) *can be omitted.*

For the sake of clarity, we will first prove the result in the scalar case $K = 1$, where we can assume $m_1 = 1$ and drop the index of u_i and N_i. For most arguments it is clear how they carry over to systems; we add the missing pieces in section 4.7.

4.2 Reformulation and function spaces

4.2.1 Function spaces

We rewrite equation (4.1.3) (in the scalar case $K = 1$, $m_1 = 1$) as a first order system, which we can more comfortably taylor our function spaces to. To this end, we note that

$$\Box + 1 = (\langle D \rangle + i\partial_t)(\langle D \rangle - i\partial_t).$$

Hence, given a sufficiently regular function u that satisfies

$$(\Box + 1)u = F, \qquad u(0) = f,\ \ u'(0) = g$$

we define

$$u^{\pm} = \frac{\langle D \rangle \mp i\partial_t}{2\langle D \rangle} u. \tag{4.2.1}$$

Then the $u^\pm$ solve[7]

$$(\langle D\rangle \pm i\partial_t)u^\pm = \frac{F}{2\langle D\rangle}, \qquad u^\pm(0) = \frac{1}{2}\left(f \mp i\frac{g}{\langle D\rangle}\right). \tag{4.2.2}$$

Since we have the identity $u^+ + u^- = u$, we may reconstruct u from this system, and we will in our estimates work exclusively on (4.2.1) and (4.2.2).
With this construction in mind, we define the function spaces which we are going to use.

Remark 4.4. Since we are dealing with an inhomogeneous setup, we use the convention that all spatial frequency decompositions are inhomogeneous, that is

$$id = \sum_N P_N = P_1 + \sum_{N>1} P_N$$

where P_1 selects the frequencies less than one.

Definition 4.5. We define the closed spaces $X^s_\pm \subset C(\mathbb{R}, H^s(\mathbb{R}^n))$ as the closure of $C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap U^2$ with respect to the norm

$$\|u\|_{X^s_\pm} = \left(\sum_N N^{2s}\|P_N u^\pm\|^2_{U^2_\pm}\right)^{\frac{1}{2}}$$

$$\text{where } \|f\|_{U^2_\pm} = \|e^{\mp it\langle D\rangle} f\|_{U^2(\mathbb{R},L^2(\mathbb{R}^n))}.$$

We also define by Y^s the corresponding space where U^2 is replaced by V^2_{rc} (which we denote by $V^2_\pm$ once it is adapted to the linear evolution).
Furthermore, we define

$$X^s = X^s_+ \times X^s_-, \qquad Y^s = Y^s_+ \times Y^s_-.$$

With these definitions, we have

$$X^s \subset Y^s.$$

We also define the restricted space $X^s([0,\infty))$

$$X^s([0,\infty)) = \left\{u \in C([0,\infty), H^s) \mid \tilde{u} = \mathbf{1}_{[0,\infty)}(t)u(t) \in X^s\right\}$$

with norm

$$\|u\|_{X^s([0.\infty))} = \|\mathbf{1}_{[0,\infty)}u\|_{X^s}.$$

and define $Y^s([0,\infty))$ analogously. They are again Banach spaces.
The strategy of the proof follows the standard approach using the contraction mapping principle. We briefly outline the procedure below.
By the equivalent formulation as a system (4.2.2), a solution of (4.1.3) is equivalent

[7] the expression $\frac{F}{\langle D\rangle}$ corresponds to the gain of one full derivative of the linear Klein-Gordon equation

to

$$\begin{aligned} (\langle D\rangle \pm i\partial_t)u^\pm &= \frac{1}{2\langle D\rangle}N(u^+ + u^-) \\ u^\pm(0) &= u_0^\pm \end{aligned} \tag{4.2.3}$$

where $u_0^\pm = \frac{1}{2}\left(f \mp i\frac{g}{\langle D\rangle}\right) \in B_\epsilon(0) \subset H^s(\mathbb{R}^n)$.
Hence, by a solution of the above equation, we will mean $(u^+, u^-) \in X^s([0,\infty))$ which on $[0,\infty)$ solve the operator equation

$$u^\pm(t) = e^{\pm it\langle D\rangle}u_0^\pm \mp iI^\pm(u) \tag{4.2.4}$$

where $u = u^+ + u^-$ and

$$I^\pm(u) = \int_0^t e^{\pm i(t-s)\langle D\rangle}\frac{N(u(s))}{2\langle D\rangle}ds. \tag{4.2.5}$$

This equation can be solved by a contraction mapping argument in X^s once we have the bounds[8]

$$\begin{aligned} \|e^{\pm it\langle D\rangle}u_0^\pm\|_{X^s_\pm([0,\infty))} &\lesssim \|u_0^\pm\|_{H^s(\mathbb{R}^n)}, \\ \|I^\pm(u)\|_{X^s_\pm([0,\infty))} &\lesssim \|(u_+, u_-)\|^2_{X^s_\pm([0,\infty))} \end{aligned}$$

The linear part of the estimate is straightforward, since

$$\begin{aligned} \|e^{\pm it\langle D\rangle}u_0^\pm\|^2_{X^s_\pm([0,\infty))} &= \sum N^{2s}\|\mathbf{1}_{[0,\infty)}e^{\pm it\langle D\rangle}P_N(u_0^\pm)\|^2_{U^2_\pm} \\ &= \sum N^{2s}\|P_N(u_0^\pm)\|^2_{U^2} \lesssim \|u_0^\pm\|^2_{H^s(\mathbb{R}^n)}. \end{aligned} \tag{4.2.6}$$

Hence, the focus of the sections to come is on the nonlinear estimate of $I^\pm$. In the next section, we derive some spacetime estimates that will be crucial for the nonlinear estimate.

4.3 Bilinear and Strichartz estimates

We will tacitly assume that $n \geq 2$ and mention again that we use an inhomogeneous frequency decomposition. By virtue of Proposition 3.24, bounds in $U^p_\pm$ type spaces follow from L^p bounds on free solutions $e^{\pm it\langle D\rangle}\phi$. For our estimates in dimension three or higher, we will use the key estimate below.

Proposition 4.6. *Let $n \geq 3$, let O, M, N dyadic numbers and ϕ_M, ψ_N functions in $L^2(\mathbb{R}^n)$ localized at frequencies M, N respectively. Define $u_M = e^{\pm_1 it\langle D\rangle}\phi_M$, $v_N = e^{\pm_2 it\langle D\rangle}\psi_N$. Denote $L = \min(O, M, N)$, $H = \max(O, M, N)$. Then,*

$$\|P_O(u_M v_N)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \begin{cases} H^{\frac{1}{2}}L^{\frac{n-2}{2}}\|\phi_M\|_{L^2(\mathbb{R}^n)}\|\psi_N\|_{L^2(\mathbb{R}^n)} & \text{if } M \sim N \\ L^{\frac{n-1}{2}}\|\phi_M\|_{L^2(\mathbb{R}^n)}\|\psi_N\|_{L^2(\mathbb{R}^n)} & \text{otherwise} \end{cases} \tag{4.3.1}$$

[8] along with a difference version of the nonlinear bound

Proof. see section 4.8. □

We also state the Strichartz estimates available for the Klein-Gordon equation. These will mainly be used when $n = 2$ as the above more powerful bilinear refinement is not available in that case. The estimates come in two main flavors, depending on whether one chooses to use the radial curvature of the characteristic hypersurface. The condition $r < \infty$ serves to exclude inconvient endpoint cases, which we will not need in what follows.

Proposition 4.7 (Strichartz estimates)**.** *Let* $2 \leq r < \infty$, $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, *and* $l = \frac{1}{q} - \frac{1}{r} + \frac{1}{2}$. *Then*

$$\|e^{it\langle D\rangle}u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|\langle D\rangle^l u_0\|_{L^2(\mathbb{R}^n)}. \tag{4.3.2}$$

Proof. see [DF08b]. □

Proposition 4.6 implies the following bilinear refinement in L^4 which will be useful in controlling the worst interactions:

Proposition 4.8 (L^4 estimate)**.** *Let* $n \geq 3$ *and for* $M \lesssim N$, *let* $\phi_{N,M}$ *be supported in a ball of radius* M *located at frequency* N. *Then*

$$\|e^{it\langle D\rangle}\phi_{N,M}\|_{L^4(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{1}{4}}M^{\frac{n-2}{4}}\|\phi_{N,M}\|_{L^2(\mathbb{R}^n)}. \tag{4.3.3}$$

Proof. We omit the indices M, N and rewrite the estimate in the equivalent bilinear fashion

$$\|e^{it\langle D\rangle}\phi e^{-it\langle D\rangle}\bar{\phi}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{1}{2}}M^{\frac{n-2}{2}}\|\phi\|^2_{L^2(\mathbb{R}^n)}$$

Now the Fourier supports of ϕ and $\bar{\phi}$ are symmetric through the origin, and hence the sum of the supports is contained in a ball of radius $\lesssim M$ centered at the origin. We may thus insert a projector P_M and it remains to estimate

$$\|P_M(e^{it\langle D\rangle}\phi e^{-it\langle D\rangle}\bar{\phi})\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{1}{2}}M^{\frac{n-2}{2}}\|\phi\|^2_{L^2(\mathbb{R}^n)}$$

but this is one of the bilinear estimates in Proposition 4.6. □

With these building blocks, we transfer the estimates over on the corresponding $U^p_\pm$ and $V^2_\pm$ spaces using Proposition 3.24 and Proposition 3.26.

Proposition 4.9 ($U^4 \to L^4$)**.** *Let* $n \geq 3$ *and let* $u_{M,N}$ *have Fourier support in a ball of radius* $\sim M$ *centered at frequency* $N \gtrsim M$. *Then*[9]

$$\|u_{M,N}\|_{L^4(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{1}{4}}M^{\frac{n-2}{4}}\|u_{M,N}\|_{U^4_\pm}. \tag{4.3.4}$$

Proof. This follows from (4.3.3) and Proposition 3.24. □

[9]in the case where $u_{M,N} = P_N u$, the estimate of course still holds with $M = N$

Proposition 4.10 ($U^2 \times U^2 \to L^2$). *Let $n \geq 3$ and let L (H) the lowest (highest) of the frequencies M, N, O. Let $u_M \in U^2_{\pm_1}$, $u_N \in U^2_{\pm_2}$. Then we have*

$$\|P_O(u_M v_N)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \begin{cases} L^{\frac{n-1}{2}} \|u_M\|_{U^2_{\pm_1}} \|u_N\|_{U^2_{\pm_2}} & \text{if } M \ll N \\ H^{\frac{1}{2}} L^{\frac{n-2}{2}} \|u_M\|_{U^4_{\pm_1}} \|u_N\|_{U^4_{\pm_2}} & \text{if } M \sim N \end{cases} \tag{4.3.5}$$

Furthermore, we may take u_M and u_N in $V^2_\pm$ at the expense of a factor[10] $\log^2 \frac{H}{L}$ *in the case $M \ll N$. When $M \sim N$, the same is true without an additional factor.*

Proof. We omit the $\pm$ index in the U^2 spaces. The estimates in U^2 follow from Proposition 4.10. In the case $M \sim N$, we improve to $U^4_\pm$ by orthogonality (as outlined the proof of Proposition 4.23 in the appendix) and Proposition 4.9. It remains to interpolate with V^2 when $M \ll N$. Define $Tv = P_O(u_M P_N v)$. Then we have by (4.3.4), (4.3.5) and $U^2 \subset U^4$

$$\|T\|_{U^4_{\pm_2} \to L^2(\mathbb{R}^n))} \lesssim (MN)^{\frac{n-1}{4}} \|u_M\|_{U^2_{\pm_1}}, \qquad \|T\|_{U^2_{\pm_2} \to L^2(\mathbb{R}^n)} \lesssim L^{\frac{n-1}{2}} \|u_M\|_{U^2_{\pm_1}}$$

where

$$(MN)^{\frac{n-1}{4}} = (HL)^{\frac{n-1}{4}}.$$

Since $\log \frac{(HL)^{\frac{n-1}{4}}}{L^{\frac{n-1}{2}}} \lesssim \log \frac{H}{L}$, interpolation using Proposition 3.26 yields

$$\|T\|_{V^2_{\pm_2} \to L^2(\mathbb{R}^n)} \lesssim L^{\frac{n-1}{2}} (\log \frac{H}{L}) \|u_M\|_{U^2_{\pm_1}}$$

Now we iterate the argument with $S : u \mapsto P_O(P_M u, v_N)$. This time we have, using $V^2 \subset U^4$,

$$\|S\|_{U^4_{\pm_1} \to L^2(\mathbb{R}^n)} \lesssim C_{M,N} \|v_N\|_{V^2_{\pm_2}}, \qquad \|S\|_{U^2_{\pm_1} \to L^2(\mathbb{R}^n)} \lesssim L^{\frac{n-1}{2}} (\log \frac{H}{L}) \|v_N\|_{V^2_{\pm_2}},$$

and hence, since $L^{\frac{n-1}{2}} \log \frac{H}{L} \gtrsim L^{\frac{n-1}{2}}$, as before

$$\|S\|_{V^2_{\pm_1} \to L^2(\mathbb{R}^n)} \lesssim L^{\frac{n-1}{2}} (\log \frac{H}{L})^2 \|u_M\|_{V^2_{\pm_2}}.$$

□

4.4 Trilinear estimates

In this section, we perform the estimates necessary to prove bounds for the Duhamel terms $I^{\pm}(u)$ associated to quadratic nonlinearities. The fact that we are dealing with the quadratic case in combination with the important duality between U^2 and V^2 - as induced by the bilinear form B from Theorem 3.19 - is why these estimates are trilinear in nature. To motivate the precise form of the proposition below, we

[10] of course $\max(1, \log(\cdot))$ is meant

compute with $f = \frac{N(u)}{2\langle D\rangle}$ for the Duhamel term (4.2.5)

$$\begin{aligned}\|P_N I^\pm(u)\|_{U^2_\pm} &= \|e^{\mp it\langle D\rangle} I^\pm(u)\|_{U^2_0} = \|P_N \int_0^t e^{\mp is\langle D\rangle} f(s)ds\|_{U^2_0} \\ &= \sup_{\|v\|_{V^2}=1} \left| B\left(P_N \int_0^t e^{\mp is\langle D\rangle} f(s)ds, v\right)\right| \\ &= \sup_{\|v\|_{V^2}=1} \left| \iint f(t)\, \overline{e^{\pm it\langle D\rangle} P_N v(t)}\, dxdt\right| \\ &= \sup_{\|P_N v\|_{V^2_\pm}=1} \left| \iint f(t)\, \overline{P_N v(t)}\, dxdt\right|\end{aligned} \tag{4.4.1}$$

It will become apparent shortly that (4.4.2) and (4.4.3) below are exactly the estimates needed to sum the high-low interactions and high-high interactions, respectively.

Theorem 4.11 (Trilinear estimates)**.** *Let $s \geq \max(\frac{1}{2}, \frac{n-2}{2})$, assume that the signs $\pm_i$ $(i = 1, 2, 3)$ are arbitrary and that $H \sim H'$. Then,*

$$\frac{1}{H}\Big| \sum_{L \lesssim H} \iint u_L v_{H'} w_H dxdt\Big| \lesssim \left(\sum_{L \lesssim H} L^{2s} \|u_L\|^2_{V^2_{\pm_1}}\right)^{\frac{1}{2}} \|v_{H'}\|_{V^2_{\pm_2}} \|w_H\|_{V^2_{\pm_3}}. \tag{4.4.2}$$

Also, we have

$$\left(\sum_{L \lesssim H} L^{-2} L^{2s} \sup_{\|w_L\|_{V^2_{\pm_3}}=1} \Big| \iint_0 u_{H'} v_H w_L dxdt\Big|^2\right)^{\frac{1}{2}} \lesssim H'^s \|u_{H'}\|_{V^2_{\pm_1}} H^s \|v_H\|_{V^2_{\pm_2}}. \tag{4.4.3}$$

Proof. The proof will use the following

Lemma 4.12 (Modulation bound)**.** *Let $\xi_1 + \xi_2 = \xi_3$. Then we have*

$$\langle \xi_1\rangle + \langle \xi_2\rangle - \langle \xi_3\rangle \gtrsim \langle \xi_{min}\rangle^{-1}. \tag{4.4.4}$$

The above lemma can be improved, but we will only need (4.4.4).
We decompose each function in a low and high modulation part, where the threshold between the two regimes is set at $\Lambda > 0$ which will be chosen immediately. Recall that we defined

$$Q^\pm_{>M} u = \mathcal{F}^{-1}_{tx}\left(\phi\left(\frac{\tau \mp \langle D\rangle}{M}\right)\mathcal{F}_{tx} u\right) \text{ and } Q^\pm_{\leq M} = 1 - Q^\pm_{>M}.$$

For the proof of (4.4.2), we compose $u_L = u^h_L + u^l_L$, where $u^h_L = Q^{\pm_1}_{>\Lambda} u_N$. Similarly we decompose $v_{H'}$ and w_H, using instead the signs $\pm_2$ and $\pm_3$, respectively. Then we have

$$\int u^l_L v^l_{H'} w^l_H dxdt = \left(\mathcal{F}_{tx} u^l_L * \mathcal{F}_{tx} v^l_{H'} * \mathcal{F}_{tx} w^l_H\right)(0, 0)$$

to which only frequencies $\tau_1 + \tau_2 + \tau_3 = 0$, $\xi_1 + \xi_2 + \xi_3 = 0$ contribute. Since from the definition of $Q^{\pm}_{\leq \Lambda}$ we also have $\left|\tau_i \mp_i \langle \xi_i \rangle\right| \leq \Lambda$, we see that on the contributing set

$$3\Lambda \geq \Big| \sum_{i=1}^{3} (\tau_i \mp_i \langle \xi_i \rangle) \Big| = \Big| \sum_{i=1}^{3} \pm_i \langle \xi_i \rangle \Big| \gtrsim L^{-1},$$

which is obvious when the three signs coincide and follows from Lemma 4.12 otherwise. Chosing $\Lambda = C^{-1}L^{-1}$ for C large enough will ensure that the above integral vanishes and hence in what follows, we always have high modulation on (at least) one factor. We will indicate high modulation on f by f^h and treat now (4.4.2) in the case where $u_L = u_L^h$. Namely, we estimate the term by

$$\begin{aligned}
\text{LHS (4.4.2)} &\lesssim H^{-1} \sum_{L \lesssim H} L^{\frac{1}{2}} \|u_L\|_{V^2_{\pm_1}} \|P_L(v_{H'} w_H)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&\leq H^{-1} \left(\sum_{L \lesssim H} L^{2s} \|u_L\|^2_{V^2_{\pm_1}} \right)^{\frac{1}{2}} \left(\sum_{L \lesssim H} L^{1-2s} \|P_L(v_{H'} w_H)\|^2_{L^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

When $n = 2$, we simply use $L^{1-2s} \leq 1$, orthogonality, and the $q = r = 4$ Strichartz estimate from Proposition 4.7, obtaining

$$\sum_{L \lesssim H} L^{1-2s} \|P_L(v_{H'} w_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|v_{H'} w_H\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq H^2 \|v_{H'}\|^2_{V^2_{\pm_2}} \|w_H\|^2_{V^2_{\pm_3}}$$

and the claim follows. When $n \geq 3$, we have by (4.3.5)

$$H^{-1} \left(\sum_{L \lesssim H} L^{1-2s} \|P_L(v_{H'} w_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \right)^{\frac{1}{2}} \lesssim H^{-1} \left(\sum_{L \lesssim H} L^{n-1-2s} H \right)^{\frac{1}{2}} \|v_{H'}\|_{V^2_{\pm_2}} \|w_H\|_{V^2_{\pm_3}}$$

and the claim follows since

$$\sum_{L \lesssim H} L^{n-1-2s} \lesssim 1 + H^{n-1-2s} \lesssim H.$$

whenever $s \geq \frac{n-2}{2}$.

Now we investigate the easier case $v_{H'} = v^h_{H'}$ (the case $w_H = w^h_H$ is the same) again by putting the high modulation term in $L^2(\mathbb{R}\times\mathbb{R}^n)$. For $n = 2$ we get the expression

$$\begin{aligned}
H^{-1} \sum_{L \lesssim H} \|v^h_{H'}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|u_L w_H\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} &\lesssim H^{-1} \sum_{L \lesssim H} L^{\frac{1}{2}} L^{\frac{1}{2}} H^{\frac{1}{2}} \|u_L\|_{V^2_{\pm_1}} \|v_{H'}\|_{V^2_{\pm_2}} \|w_H\|_{V^2_{\pm_3}} \\
&\lesssim H^{-\frac{1}{2}} \left(\sum L^{2s} \|u_L\|^2_{V^2_{\pm_1}} \right)^{\frac{1}{2}} \left(\sum_{L \lesssim H} L L^{1-2s} \right)^{\frac{1}{2}} \|v_{H'}\|_{V^2_{\pm_2}} \|w_H\|_{V^2_{\pm_3}}
\end{aligned}$$

which gives the claim since

$$\sum_{L\lesssim H} LL^{1-2s} \leq \sum_{L\lesssim H} L \lesssim H.$$

In higher dimensions, we estimate

$$H^{-1}\sum_{L\lesssim H}\|v^h_{H'}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|u_L w_H\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{-1}\sum_{L\lesssim H} L^{\frac{1}{2}}L^{\frac{n-1}{2}}\log^2\frac{H}{L}\|u_L\|_{V^2_{\pm_1}}\|v_{H'}\|_{V^2_{\pm_2}}\|w_H\|_{V^2_{\pm_3}}.$$

After Cauchy-Schwarz with $L^s\|u_L\|_{V^2}$ and the rest, using $\log^4\frac{H}{L}\lesssim\frac{H}{L}$,

$$H^{-2}\sum_{L\lesssim H} LL^{n-1}L^{-2s}\log^4\frac{H}{L}\lesssim H^{-1}\sum_{L\lesssim H}L^{n-1-2s}\lesssim 1.$$

We now turn to the proof of (4.4.3) and perform the same modulation decomposition as before, starting with the case $w_L = w^h_L$. Then, for $n = 2$, using the high modulation, orthogonality and finally Strichartz estimates,

$$\begin{aligned}(4.4.3)^2 &\leq \sum_{L\lesssim H} L^{2s-1}\|P_L(u_{H'}v_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{2s-1}\|u_{H'}v_H\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}\\ &\lesssim H^{2s+1}\|u_{H'}\|^2_{V^2_{\pm_1}}\|v_H\|^2_{V^2_{\pm_2}} \lesssim H'^{2s}\|u_{H'}\|^2_{V^2_{\pm_1}}H^{2s}\|v_H\|^2_{V^2_{\pm_2}}\end{aligned}$$

and for $n\geq 3$

$$(4.4.3)^2 \leq \sum_{L\lesssim H} L^{2s-1}\|P_L(u_{H'}v_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \sum_{L\lesssim H} L^{2s+n-3}H\|u_{H'}\|^2_{V^2_{\pm_1}}\|v_H\|^2_{V^2_{\pm_2}}.$$

Now

$$H\sum_{L\lesssim H}L^{2s+n-3}\lesssim H^{2s+n-2}\leq H'^{2s}H^{2s}$$

and the claim follows. Finally, we treat the last case $u_{H'} = u^h_{H'}$. For $n = 2$, we get

$$\begin{aligned}(4.4.3)^2 &\leq \sum_{L\lesssim H} L^{2s-1}\|v_H w_L\|^2_{L^2(\mathbb{R}\times\mathbb{R}^2)}\|u_{H'}\|^2_{V^2_{\pm_1}} \lesssim \sum_{L\lesssim H} L^{2s}H\|u_{H'}\|^2_{V^2_{\pm_1}}\|v_H\|^2_{V^2_{\pm_2}}\\ &\lesssim H'^{2s}\|u_{H'}\|^2_{V^2_{\pm_1}}H^{2s}\|v_H\|^2_{V^2_{\pm_2}}\end{aligned}$$

and for $n\geq 3$, we obtain

$$\begin{aligned}(4.4.3)^2 &\lesssim \sum_{L\lesssim H} L^{2s-2}\sup_{\|w\|_{V^2}=1} L\|u_{H'}\|^2_{V^2_{\pm_1}}\|v_H w_L\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}\\ &\lesssim \sum_{L\lesssim H} L^{2s-2+n}(\log^4\frac{H}{L})\|u_{H'}\|^2_{V^2_{\pm_1}}\|v_H\|^2_{V^2_{\pm_2}}\end{aligned}$$

but we can replace $\log^4\frac{H}{L}$ by $\frac{H}{L}$ and estimate as in the last case.

We finally check off the case $n \geq 4$, where we show a stronger result in the sense that resonance does not matter and that the minimal regularity is lowered (see the definition of s_0 below).

$$\int u_L v_H w_{H'} dxdt \lesssim \|v_H\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \|w_{H'}\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \|u_L\|_{L^{\tilde{q}} L^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)}$$

where (q, r) is a Strichartz pair and $(\tilde{q}, \tilde{r})$ is prescribed by Hölder's inequality,

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \qquad \frac{2}{q} + \frac{1}{\tilde{q}} = 1 = \frac{2}{r} + \frac{1}{\tilde{r}}.$$

Morally we would like to use the symmetric pair $q = r = \frac{2(n+2)}{n}$, but to avoid logarithms we shift the balance a little bit by using $r < q$ instead. By the Strichartz estimate (4.7) we have

$$\|v_H\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{\frac{1}{2}+\frac{1}{q}-\frac{1}{r}} \|v_H\|_{V^2_{\pm_2}}$$

and similarly for w_H. For the remaining term $\|u_L\|_{L^{\tilde{q}} L^{\tilde{r}}}$ we try to reach a Strichartz pair $(\tilde{q}, R)$ by Bernstein's inequality in space. This is possible since

$$\begin{aligned} \frac{n}{R} - \frac{n}{\tilde{r}} &= \frac{n}{2} - \frac{2}{\tilde{q}} - n(1 - \frac{2}{r}) = \frac{n}{2} - 2 + \frac{4}{q} - n(1 - \frac{2}{r}) \\ &= n(\frac{2}{r} - \frac{1}{2}) - 2 + n(1 - \frac{2}{r}) = \frac{n}{2} - 2 \geq 0. \end{aligned}$$

Thus

$$\|u_L\|_{L^{\tilde{q}} L^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim L^{\frac{n}{2}-2} \|u_L\|_{L^{\tilde{q}} L^R(\mathbb{R}\times\mathbb{R}^n)} \lesssim L^{\frac{n+2}{r}+\frac{2}{n}-2} \|u_L\|_{V^2_{\pm_1}}.$$

If we temporarily set $s_0 = \frac{n}{2} + \frac{2}{n} - 2$ and sum (4.4.2) over $L \lesssim H$, we get

$$(4.4.2) \lesssim H^{-1} H^{2(\frac{1}{2}+\frac{1}{q}-\frac{1}{r}-s)} H^{\frac{n+2}{r}+\frac{2}{n}-2-s} \left(\sum_L L^{2s} \|u_L\|^2_{V^2_{\pm_1}} \right)^{\frac{1}{2}} \|v_H\|^2_{V^2_{\pm_2}} \|w_H\|^2_{V^2_{\pm_3}}.$$

but

$$H^{-1} H^{2(\frac{1}{2}+\frac{1}{q}-\frac{1}{r}-s)} H^{\frac{n+2}{r}+\frac{2}{n}-2-s} = H^{\frac{n}{2}+\frac{2}{n}-2} \lesssim 1$$

if $s \geq s_0$. In particular, since $s_0 < \frac{n-2}{2}$, the claim holds.

For (4.4.3), we use the same strategy and place w_L in $L^{\tilde{q}} L^{\tilde{r}}$. Summing up over $L \lesssim H$, we obtain

$$(4.4.3) \lesssim H^{s-1+\frac{n+2}{r}+\frac{2}{n}-2} H^{-2s+(n+2)(\frac{1}{2}-\frac{1}{r})} H^s \|u_H\|_{V^2_{\pm_1}} (H')^s \|v_H\|_{V^2_{\pm_2}}$$

and the claim follows just as above for $s \geq s_0$. ∎

4.5 Higher-order multilinear estimates

Having dealt with the delicate quadratic terms, we turn now to treating the cubic or higher order terms in the nonlinearity. For the following arguments, Strichartz estimates suffice, and consequently the "resonance-free" condition in (4.1) is irrelevant. The main estimate is

Theorem 4.13. *Let $k \geq 3$, $s \geq \max(\frac{k-2}{k-1}, \frac{n}{2} - \frac{2}{k-1})$ and assume that the signs $\pm_i$ $(i = 1, \ldots, k+1)$ are arbitrary, and that $H \sim H'$. Then,*

$$\frac{1}{H}\Big| \sum_{\substack{L_i \lesssim H \\ i=1,\ldots,k-1}} \iint \prod_{i=1}^{k-1} u^i_{L_i} u^k_{H'} w_H \, dxdt \Big| \lesssim \prod_{i=1}^{k-1} \left(\sum_{L_i \lesssim H} L^{2s} \|u^i_{L_i}\|^2_{V^2_{\pm_i}} \right)^{\frac{1}{2}} \|u^k_{H'}\|_{V^2_{\pm_k}} \|w_H\|_{V^2_{\pm_{k+1}}}. \tag{4.5.1}$$

Also, we have

$$\Big(\sum_{L \lesssim H} L^{-2} L^{2s} \sup_{1=\|w_L\|_{V^2_{\pm_{k+1}}}} \Big\{ \sum_{\substack{L_i \lesssim H \\ i=1,\ldots,k-2}} \Big| \iint_0 \prod_{i=1}^{k-2} u^i_{L_i} u^{k-1}_{H'} u^k_H w_L \, dxdt \Big| \Big\}^2 \Big)^{\frac{1}{2}} \lesssim \prod_{i=1}^{k-2} \|u^i\|_{Y^s_{\pm_i}} H'^s \|u^k_{H'}\|_{V^2_{\pm_{k-1}}} H^s \|u^k_H\|_{V^2_{\pm_k}}. \tag{4.5.2}$$

We prove first the case $n = 2$, where the regularity condition is $s \geq \frac{k-2}{k-1}$.

Proof. Let $n = 2$ in what follows.
For the high output case (4.5.1), we estimate

$$\Big| \iint \prod_{i=1}^{k-1} u^i_{L_i} u^k_H v_H \, dxdt \Big| \lesssim \|u^k_H\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \|w_H\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \prod_{i=1}^{k-1} \|u^i_{L_i}\|_{L^{(k-1)\tilde{q}} L^{(k-1)\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)}$$

where (q, r) is a Strichartz pair, $2 < r \leq 4$, and $\tilde{q}$, $\tilde{r}$ are determined by Hölder:

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2} \qquad \frac{2}{q} + \frac{1}{\tilde{q}} = 1 = \frac{2}{r} + \frac{1}{\tilde{r}}$$

Unfortunately, the symmetric Strichartz pair $(r, q) = (4, 4)$ leads to some logarithmic divergence. However, we can choose (q, r) close to the lossless energy estimate $(\infty, 2)$ and shift the balance of derivatives towards the preferable low frequencies. That is, we assume $2 < r < 4$, $s = s_{crit} = \frac{k-2}{k-1}$ and compute

$$\|u_N\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{1}{2}+\frac{1}{q}-\frac{1}{r}} \|u_N\|_{V^2_{\pm_i}} = N^{1-\frac{2}{r}} \|u_N\|_{V^2_{\pm_i}}.$$

Now we want to use $Q = (k-1)\tilde{q}$ and the corresponding Strichartz pair $\frac{1}{Q} + \frac{1}{R} = \frac{1}{2}$. The pair (Q, R) clearly exists, since $2 < Q < \infty$, but we have to check that

$(k-1)\tilde{q} > R$; otherwise, we cannot reach this Strichartz pair through Bernstein from $(Q,(k-1)\tilde{r})$. We compute

$$\frac{1}{R}-\frac{1}{(k-1)\tilde{r}}=\frac{1}{2}-\frac{1}{k-1}\left(\frac{1}{\tilde{q}}+\frac{1}{\tilde{r}}\right)=\frac{1}{2}-\frac{2}{k-1}\left(1-\frac{1}{q}-\frac{1}{r}\right)=\frac{1}{2}-\frac{1}{k-1}\geq 0$$

since $k \geq 3$ (in fact, for $k=3$, $(Q,\frac{1}{(k-1)\tilde{r}})$ is a Strichartz pair). Hence we may proceed by computing

$$\begin{aligned}\|u_N^i\|_{L^QL^{(k-1)\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)} &\lesssim N^{2(\frac{1}{R}-\frac{1}{(k-1)\tilde{r}})}\|u_N^i\|_{L^QL^R(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{2}{R}-\frac{2}{(k-1)\tilde{r}}+\frac{1}{2}+\frac{1}{Q}-\frac{1}{R}}\|u_N^i\|_{V^2_{\pm_i}}\\ &= N^{1-\frac{2}{(k-1)\tilde{r}}}\|u_N^i\|_{V^2_{\pm_i}} = N^{1-\frac{2}{k-1}(1-\frac{2}{r})}\|u_N^i\|_{V^2_{\pm_i}}.\end{aligned}$$

This last computation is of significance since we want to estimate

$$\prod_{i=1}^{k-1}\|u_{L_i}^i\|_{L^QL^{(k-1)\tilde{r}}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \prod_{i=1}^{k-1}L_i^{1-\frac{2}{k-1}(1-\frac{2}{r})}\|u_{L_i}^i\|_{V^2_{\pm_i}}$$

For brevity, we denote

$$\alpha=\alpha(r)=1-\frac{2}{k-1}(1-\frac{2}{r})=\frac{1}{k-1}\left(k-3+\frac{4}{r}\right).$$

and note that $\alpha - s_{crit} = \alpha - \frac{k-2}{k-1} > 0$. Then

$$H^{-1}\left|\iint\prod_{i=1}^{k-1}u_{L_i}^i u_H^k v_H dxdt\right| \lesssim H^{1-\frac{4}{r}}\left(\prod_{i=1}^{k-1}L_i^{\alpha}\|u_{L_i}^i\|_{V^2_{\pm_i}}\right)\|u_H^k\|_{V^2_{\pm_k}}\|w_H\|_{V^2_{\pm_{k+1}}}. \tag{4.5.3}$$

We sum this over all $L_1,\dots,L_{k-1}$ such that $1 \leq L_i \lesssim H$ and apply Cauchy-Schwartz in L_i with $L_i^s\|u_{L_i}^j\|_{V^2_{\pm_j}}$ and the remainder, i.e. for each $i=1,\dots,k-1$, we estimate

$$\sum_{L_i\lesssim H}L_i^{\alpha}\|u_{L_i}^i\|_{V^2_{\pm_i}} \leq H^{\alpha-s}\left(L_i^{2s}\|u_{L_i}^i\|^2_{V^2_{\pm_i}}.\right)^{\frac{1}{2}}$$

This contributes $H^{(k-1)(\alpha-s)}$. In total, we need to bound

$$H^{1-\frac{4}{r}}H^{(k-1)(\alpha-s)}=H^{(k-2)-(k-1)s}\lesssim 1$$

since $s \geq \frac{k-2}{k-1}$.
Now we treat the case of low output. The building block is of the form

$$\left|\iint\prod_{i=1}^{k-2}u_{L_i}^i u_H^{k-1}u_{H'}^k w_L dxdt\right|$$

where $L_i \lesssim H$, and we apply the same general strategy: The low frequency factors w_L and $u_{L_i}^i$ go to $L^{(k-1)\tilde{q}}L^{(k-1)\tilde{r}}$ in order to lose slightly more than s_{crit} derivatives to avoid logarithms in the summation, while the high frequency factors lose less and compensate in the end. We obtain, since $s-1+\alpha>0$ for $2<r<4$,

$$
\begin{aligned}
(4.5.2) &\lesssim \sum_{L\lesssim H}\left(L^{s-1+\alpha}\prod_{i=1}^{k-2}L_i^{\alpha}\|u^i_{L^i}\|_{V^2_{\pm_i}}H^{1-\frac{2}{r}}\|u^{k-1}_H\|_{V^2_{\pm_{k-1}}}{H'}^{1-\frac{2}{r}}\|u^k_{H'}\|_{V^2_{\pm_k}}\right)^2\\
&\lesssim \left(H^{s-1+\alpha}H^{(k-2)(\alpha-s)}H^{2-\frac{4}{r}-2s}\prod_{i=1}^{k-2}\|u^i_{L^i}\|_{Y^s_{\pm_i}}H^s\|u^{k-1}_H\|_{V^2_{\pm_{k-1}}}{H'}^s\|u^k_{H'}\|_{V^2_{\pm_k}}\right)^2
\end{aligned}
$$

but we have

$$
s-1+\alpha+(k-2)(\alpha-s)+2-\frac{4}{r}-2s=(k-1)(\alpha-s)+1-\frac{4}{r}=(k-2)-(k-1)s\le 0
$$

and hence, the claim follows.

□

Now we turn to the remaining case $n\ge 3$, where $s\ge\frac{n}{2}-\frac{2}{k-1}$.

Proof. We assume in the proof that $s=\frac{n}{2}-\frac{2}{k-1}$, which is the most relevant case; it is easy to see that the estimates hold for higher s as well. For the high output case, we assume that $L_1\ge L_2\ge\ldots\ge L_{k-1}$ and compute

$$
\begin{aligned}
\Big|\iint\prod_{i=1}^{k-1}u^i_{L_i}u^k_H w_{H'}\,dxdt\Big| &\lesssim \|u^1_{L_1}w_{H'}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|u^2_{L_2}u^k_H\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\prod_{i=3}^{k-1}\|u^i_{L_i}\|_{L^\infty L^\infty(\mathbb{R}\times\mathbb{R}^n)}\\
&\lesssim\left(\frac{H^2}{L_1L_2}\right)^{\delta}(L_1L_2)^{-\frac{1}{2}}\|w_{H'}\|_{V^2_{\pm_{k+1}}}\|u^k_H\|_{V^2_{\pm_k}}\prod_{i=1}^{k-1}L_i^{\frac{n}{2}}\|u^i_{L_i}\|_{V^2_{\pm_i}},
\end{aligned}
$$

where the bilinear estimates (4.3.5) for the first two terms and Bernstein's inequality for the other terms were used to replace $L^\infty L^\infty$ by $L^\infty L^2$ and then by $V^2_\pm$. Now we sum this expression over all L_i, and first use Cauchy-Schwarz to estimate for $i=3,\ldots,k-1$

$$
\sum_{L_i\lesssim L_2}L_i^{\frac{n}{2}}\|u^i_{L_i}\|_{V^2_{\pm_i}}\lesssim L_2^{\frac{n}{2}-s}\left(\sum_{L_i}L_i^{2s}\|u^i_{L_i}\|^2_{V^2_{\pm_i}}\right)^{\frac{1}{2}}
$$

and similarly for $i=1,2$ with $L_i^{\frac{n-1-\delta}{2}}$ instead of $L_i^{\frac{n}{2}}$. Taking into account the extra factor H^{-1}, we need to control

$$
H^{-1+2\delta}\left(\sum_{L_1\le H}\sum_{L_2\le L_1}\left(L_2^{(k-3)\left(\frac{n}{2}-s\right)+\frac{n-1}{2}-s-\delta}L_1^{\frac{n-1}{2}-s-\delta}\right)^2\right)^{\frac{1}{2}}.
$$

the exponent of L_2 is positive (for δ small enough), and hence adds to the exponent of L_1 upon summation, resulting in another positive power. Summing up over $L_1\lesssim H$, we obtain

$$
H^{-1+(k-3)(\frac{n}{2}-s)+(n-1)-2s}=H^{(k-1)\left(\frac{n}{2}-s\right)-2}\lesssim 1
$$

precisely when $s \geq \frac{n}{2} - \frac{2}{k-1}$.
The low output estimate follows in similar spirit, but we switch the roles of $u_{L_2}^2$ and w_L. That is, we group into $\|u_{L_1}^1 u_H^{k-1}\|_{L^2}$, $\|u_{H'}^k w_L\|_{L^2}$ and the rest in L^∞. We can immediately evaluate the summation over $L \ll H$,

$$\sum_{L \lesssim H} L^{2s+n-3-\delta} H^\delta \lesssim H^{2s+n-3}.$$

This yields

$$\begin{aligned}
(4.5.1) &\lesssim L_1^{-\frac{1}{2}} H^{s+\frac{n-3}{2}} \prod_{i=1}^{k-2} L_i^{\frac{n}{2}} \|u_{L_i}^i\|_{V^2_{\pm_i}} \|u_H^{k-1}\|_{V^2_{\pm_{k-1}}} \|u_{H'}^k\|_{V^2_{\pm_k}} \\
&\lesssim H^{\frac{n-4}{2}-s+(k-2)(\frac{n}{2}-s)} \prod_{i=1}^{k-2} \|u^i\|_{Y^s_{\pm_i}} H^s \|u_H^{k-1}\|_{V^2_{\pm_{k-1}}} (H')^s \|u_{H'}^k\|_{V^2_{\pm_k}}
\end{aligned}$$

but

$$H^{\frac{n-4}{2}-s+(k-2)(\frac{n}{2}-s)} = H^{(k-1)(\frac{n}{2}-s)-2} \lesssim 1$$

when $s \geq \frac{n}{2} - \frac{2}{k-1}$. □

4.6 Proof of the main theorem

We provide here the necessary estimates on the Duhamel term to set up a contraction mapping argument. For purposes of clarity, we treat purely quadratic ($k = 2$) nonlinearities only; for the other terms, one simply uses the higher-order counterparts (4.5.1) and (4.5.2) of (4.4.2) and (4.4.3) and then copies the proofs below with straightforward adjustments.

Hence we are dealing now with a nonlinearity which is a finite sum of quadratic terms in u^+, u^- and their conjugates. For brevity, we will from now on restrict to one such term without conjugates. Since $\|\bar{v}\|_{X^s_\pm} = \|v\|_{X^s_\mp}$, the other cases follow in the same manner. The main result is

Theorem 4.14. *let $s \geq \max(\frac{1}{2}, \frac{n-2}{2})$. For any $\pm_1$, $\pm_2$, we have*

$$I_{\pm_1 \pm_2} := I : \quad Y^s \times Y^s \to X^s,$$

where

$$\begin{aligned}
I((u^+, u^-), (v^+, v^-)) &= (I^+(u^{\pm_1}, v^{\pm_2}), I^-(u^{\pm_1}, v^{\pm_2})) \\
I^\pm(f, g) &= \int_0^t e^{\pm i(t-s)\langle D \rangle} \frac{fg}{2\langle D \rangle} ds.
\end{aligned}$$

In other words, for a constant $C = C(n)$,

$$\|I(u, v)\|_{X^s} \leq C \|u\|_{Y^s} \|v\|_{Y^s}.$$

In particular, since $X^s \subset Y^s$, we also have

$$I : X^s \times X^s \to X^s$$

and

$$I : Y^s \times Y^s \to Y^s.$$

Proof. It suffices to consider the terms S_i, $i = 1, 2$, where

$$S_1 = \Big\| \sum_H \sum_{L \ll H} I(\vec{u}_L, \vec{v}_H) \Big\|_{X^s}, \qquad S_2 = \Big\| \sum_H \sum_{H \sim H'} I(\vec{u}_H, \vec{v}'_H) \Big\|_{X^s}.$$

We treat I^+ only since the other component follows in the same manner, denote by u_H and v_H the components of $\vec{u}_H$ and $\vec{v}_H$ as selected by the signs $\pm_1$ and $\pm_2$, and begin by estimating S_1. By duality and (4.4.2) from Theorem 4.11,

$$\begin{aligned} \Big\| P_H \sum_{L \ll H} I^+(u_L, v_H) \Big\|_{U^2_+} &= \frac{1}{H} \sup_{\|w_{H'}\|_{V^2_-}=1} \Big| \sum_{L \ll H} \iint u_L v_H w_{H'} \, dx dt \Big| \\ &\lesssim \left(\sum_{L \lesssim H} L^{2s} \|u_L\|^2_{V^2_{\pm_1}} \right)^{\frac{1}{2}} \|v_H\|_{V^2_{\pm_2}} \end{aligned}$$

and thus

$$\sum_H H^{2s} \Big\| P_H \sum_{L \ll H} I^+(u_L, v_H) \Big\|^2_{U^2_+} \lesssim \|\vec{u}\|^2_{Y^s} \|\vec{v}\|^2_{Y^s}.$$

For S_2, we instead estimate

$$S_2 \le \sum_H \sum_{H' \sim H} \|I^+(u_{H'}, v_H)\|_{X^s_+} \lesssim \sum_H \sum_{H' \sim H} \left(\sum_{L \lesssim H} L^{2s} \|P_L I^+(u_{H'}, v_H)\|^2_{U^2_+} \right)^{\frac{1}{2}}.$$

Using duality again, we arrive exactly at $\sum_H \sum_{H' \sim H}$ (4.4.3), and using this, we get

$$S_2 \lesssim \sum_H \sum_{H' \sim H} H'^s \|u_{H'}\|_{V^2_{\pm_1}} H^s \|v_H\|_{V^2_{\pm_2}} \lesssim \|\vec{u}\|_{Y^s} \|\vec{v}\|_{Y^s}.$$

□

We now solve (4.1.3) by contraction mapping techniques, e.g. we are going to construct a solution of the operator equation

$$u^\pm(t) = T^\pm u^\pm := e^{\pm it\langle D \rangle} u_0^\pm \mp i I^\pm(u) \tag{4.6.1}$$

where $u = u_+ + u_-$ and

$$I^\pm(u) = \int_0^t e^{\pm i(t-s)\langle D \rangle} \frac{N(u(s))}{2\langle D \rangle} ds. \tag{4.6.2}$$

We look for the solution in the set

$$D_\delta = \{u \in X^s([0,\infty)) : \ \|u\|_{X^s([0,\infty))} \le \delta\}.$$

For $u \in D_\delta$ and initial data $u_0 = (u_0^+, u_0^-)$ of size at most $\epsilon = \epsilon(\delta) \ll \delta$, we have

$$\|e^{\pm it\langle D\rangle} u_0^\pm \mp iI^\pm(u)\|_{X^s_\pm([0,\infty))} \lesssim \epsilon + \delta^2 \le \delta$$

for small enough δ, due to (4.2.6) and the fact that $I^\pm(u)$ is a sum of operators for which Theorem 4.14 holds. Since we can factor $a^2 - b^2 = a(a-b) + (a-b)b$, we also obtain

$$\begin{aligned}\|I^\pm(f) - I^\pm(g)\|_{X^s_\pm([0,\infty))} &\lesssim (\|f\|_{X^s([0,\infty))} + \|g\|_{X^s([0,\infty))})\|f-g\|_{X^s([0,\infty))} \\ &\lesssim \delta\|f-g\|_{X^s([0,\infty))}\end{aligned}$$

and hence T is a contraction on D_δ when $\delta \ll 1$, which implies the existence of a unique fixed point in D_δ solving the integral equation (4.6.1).
As for scattering, by Theorem 4.14 we have that for each N,

$$e^{\mp it\langle D\rangle} P_N I^\pm(u) \in V^2_{rc}$$

and hence, the limit as $t \to \infty$ exists for each piece. Together with

$$\sum_N N^{2s}\|P_N I^\pm(u)\|^2_{V^2_\pm} \lesssim 1,$$

it follows that $\lim_{t\to\infty} e^{\mp it\langle D\rangle} I^\pm(u) \in H^s(\mathbb{R}^n)$. Hence, for the solution $u = (u^+, u^-)$ we have that

$$e^{\mp it\langle D\rangle} u^\pm \to u_0^\pm \mp i \lim_{t\to\infty} e^{\mp it\langle D\rangle} I^\pm(u) \in H^s(\mathbb{R}^n).$$

4.7 Systems of different masses

Since the bilinear estimates easily tolerate interactions between waves with different masses when $n \ge 3$ and the case $n = 2$ relies on Strichartz estimates only, the only obstruction to carrying out the proof of the main result for a system of such type is the absence of resonances[11]. Recalling the notation $\langle\cdot\rangle_m = \sqrt{m^2 + |\cdot|^2}$, we have the following

Lemma 4.15. *Let positive masses $m_1, \dots, m_N$ be given such that for any triple*

$$(m,n,o) \in \left(\{m_i\}_{i=1}^N\right)^3$$

we have

$$m + n > o \tag{4.7.1}$$

[11] in the framework of space-time resonances (cf. [Ger11]), our notion describes the absence of time resonance

Then we have the modulation bound

$$\langle \xi \rangle_m + \langle \eta \rangle_n - \langle \xi + \eta \rangle_o \gtrsim \langle \min(|\xi|, |\eta|, |\xi + \eta|) \rangle^{-1}. \tag{4.7.2}$$

Remark 4.16. The condition $m + n > o$ is similar to (albeit more restrictive[12] than) the condition

$$|m_1 + m_2 - m_3| \neq 0,$$

which appears in numerous places, most recently in [IP12].
Of course (4.7.1) is equivalent to

$$2 \min\{m_i\} > \max\{m_i\}.$$

The statements of Theorem 4.1 follow by inspection of the main arguments if in analogy to Lemma 4.12 we have the modulation bound (4.7.2) by obvious adaption of the function spaces and estimates to systems. We omit the details; it remains to prove (4.7.2).

Proof. By symmetry, we may assume $|\eta| \leq |\xi|$. Expanding the left hand side of (4.7.2) with

$$\Lambda := \langle \xi \rangle_m + \langle \eta \rangle_n + \langle \xi + \eta \rangle_o \sim \langle \xi \rangle,$$

it remains to look at the expression

$$m^2 + n^2 - o^2 + 2\langle \xi \rangle_m \langle \eta \rangle_n - 2\xi \cdot \eta.$$

If $\xi \cdot \eta \leq 0$ then, since $m + n - o > 0$,

$$\begin{aligned} m^2 + n^2 - o^2 + 2\langle \xi \rangle_m \langle \eta \rangle_n &\geq m^2 + n^2 - o^2 + 2\max(mn, \langle \xi \rangle_m \langle \eta \rangle_n) \\ &\gtrsim \langle \xi \rangle \langle \eta \rangle, \end{aligned}$$

we have

$$\langle \xi \rangle_m + \langle \eta \rangle_n - \langle \xi + \eta \rangle_o \gtrsim \frac{\langle \xi \rangle \langle \eta \rangle}{\Lambda} \gtrsim \langle \eta \rangle$$

which implies the claim regardless of whether $|\xi + \eta|$ or $|\eta|$ is the smallest number. Hence it remains to deal with the case where $\xi \cdot \eta > 0$, in which $|\eta|$ is comparable to the minimum frequency, and we replace $\xi \cdot \eta$ by $|\xi||\eta|$ to deal directly with the worst case[13]. With some hindsight, we rewrite the resulting expression as

$$(m + n)^2 - o^2 - 2\epsilon mn + 2\left(\langle \xi \rangle_m \langle \eta \rangle_n - |\xi||\eta| - (1 - \epsilon)mn\right)$$

where we chose $\epsilon \ll 1$ such that

$$(m + n)^2 - o^2 - 2\epsilon mn > 0,$$

[12] the fact that we need positivity as opposed to nonvanishing of this expression seems related to the fact that our method does not take advantage of the absence of space resonances

[13] if one does not do this, one sees that the worst case happens for interactions along a line and the general case is much better, but we ignore this here

and we now prove that

$$\langle\xi\rangle_m\langle\eta\rangle_n - |\xi||\eta| - (1-\epsilon)mn$$

is nonnegative and has the correct growth. We rewrite as

$$\langle\xi\rangle_m\langle\eta\rangle_n - |\xi||\eta| - (1-\epsilon)mn = \frac{m^2n^2 + n^2|\xi|^2 + m^2|\eta|^2 - (1-\epsilon)mn\langle\xi\rangle_m\langle\eta\rangle_n - (1-\epsilon)mn|\eta||\xi|}{\langle\xi\rangle_m\langle\eta\rangle_n + |\xi||\eta|}$$

and estimate the nominator using $ab \leq \frac{1}{2}(a^2+b^2)$ from below by

$$\begin{aligned} &m^2n^2 + n^2|\xi|^2 + m^2|\eta|^2 - \frac{(1-\epsilon)}{2}\left(n^2\langle\xi\rangle_m^2 + m^2\langle\eta\rangle_n^2 + n^2|\xi|^2 + m^2|\eta|^2\right) \\ &= m^2n^2 + n^2|\xi|^2 + m^2|\eta|^2 - (1-\epsilon)\left(n^2m^2 + n^2|\xi|^2 + m^2|\eta|^2\right) \\ &\gtrsim \epsilon\langle\xi\rangle^2. \end{aligned}$$

Hence, we have bounded

$$\langle\xi\rangle_m + \langle\eta\rangle_n - \langle\xi+\eta\rangle_o \gtrsim \frac{\langle\xi\rangle^2}{\langle\xi\rangle\langle\eta\rangle\Lambda} \gtrsim \langle\eta\rangle^{-1}$$

as claimed. □

4.8 Proof of the bilinear estimates

In this sections we prove the bilinear estimates, which are essentially identical to those of the free wave equation. We assume $n \geq 3$ and also allow for different masses to be able to treat more general systems. For this, we define

Definition 4.17.

$$\langle\cdot\rangle_m = \sqrt{m^2 + |\cdot|^2}.$$

Proposition 4.18. *Let $n \geq 3$, let $O, M, N \geq 1$ dyadic numbers and ϕ_M, ψ_N functions in $L^2(\mathbb{R}^n)$ localized at frequencies M, N respectively. Define $u_M = e^{\pm_1 it\langle D\rangle_{m_1}}\phi_M$, $v_N = e^{\pm_2 it\langle D\rangle_{m_2}}\psi_N$. Denote $L = \min(O,M,N)$, $H = \max(O,M,N)$. Then,*

$$\|P_{\leq O}(u_M v_N)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \begin{cases} H^{\frac{1}{2}}L^{\frac{n-2}{2}}\|\phi_M\|_{L^2(\mathbb{R}^n)}\|\psi_N\|_{L^2(\mathbb{R}^n)} & \text{if } M \sim N \\ L^{\frac{n-1}{2}}\|\phi_M\|_{L^2(\mathbb{R}^n)}\|\psi_N\|_{L^2(\mathbb{R}^n)} & \text{otherwise} \end{cases} \tag{4.8.1}$$

Remark 4.19. One could easily improve the constant in the first case to L^{n-1} if the signs $\pm_1$ and $\pm_2$ coincide, but we do not pursue this here.

Proof. The statements follow from Proposition 4.23 below upon approximation of

$$\delta(\tau \pm \langle\xi\rangle_m)$$

by $\epsilon^{-1}\mathbf{1}_{|\tau-\langle\xi\rangle_m|\leq\epsilon}$ when $L \gg 1$. It remains to deal with the part where $L \lesssim 1 \ll H$.

This is a routine exercise due to the fact that in that case, uniformly transversal hypersurfaces interact on a region of diameter L. We omit the details. □

We will generally follow the strategy carried through for $n = 3$ in [Sel08] for the wave equation, relying solely on estimates of intersections of thickened spheres. At high frequencies, the characteristic surface resembles the cone, and we stay in this regime due to the condition $L \gg 1$.

Lemma 4.20. *Let $n \geq 3$, $0 < \delta, \Delta \ll 1 \lesssim \min(r, R, L)$ and define*

$$S_\delta(r) = \{\xi \in \mathbb{R}^n : \ r - \delta \leq |\xi| \leq r + \delta\}.$$

Then, for $|\xi_0| \gtrsim \max(r, R)$, and denoting by $T(\xi, L)$ the tube of radius L in the direction of ξ_0, we have

$$|T_L(\xi_0) \cap S_\delta(r) \cap (\xi_0 + S_\Delta(R))| \lesssim \frac{\min(r, R, L)^{n-3} r R \delta \Delta}{|\xi_0|}.$$

Remark 4.21. The statement is symmetric in (r, δ) and (R, Δ).

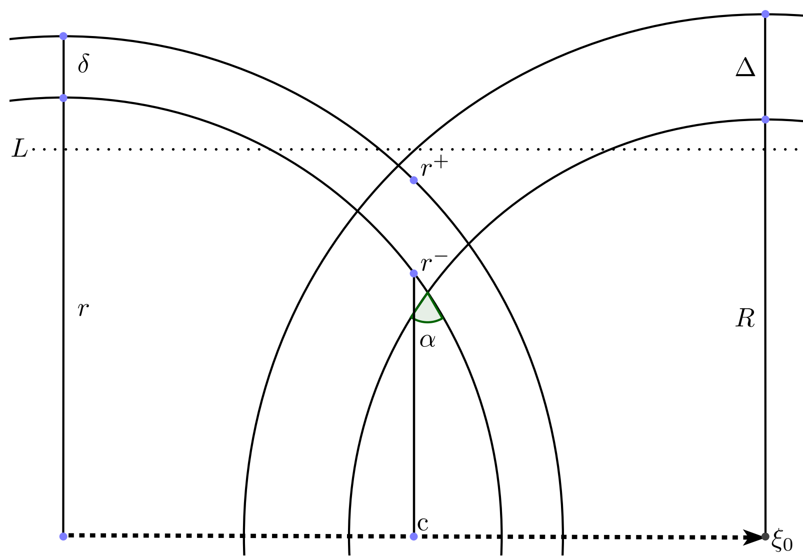

Figure 4.1: The worst case in the proof of Lemma 4.20 when $r \leq R$ in the case $n \geq 3$. Near height L, we have $\alpha \sim \frac{L}{r}$ and hence the intersection has volume $\alpha^{-1}\delta\Delta L^{n-2} \sim rL^{n-3}\delta\Delta$. We remark that in two dimensions, the critical intersection occurs as the circles touch tangentially, resulting in worse estimates.

Proof. Denote by A the above intersection. We may assume $L \lesssim r$, $|\xi_0| \sim \max(r, R)$ and

$$\xi_0 = (|\xi_0|, 0, \ldots, 0).$$

Hence $\xi \in S_\delta(r) \cap (\xi_0 + S_\Delta(R))$ if and only if

$$(r - \delta)^2 < (\xi^1)^2 + |\xi'|^2 < (r + \delta)^2$$

and

$$(R-\Delta)^2 < (\xi^1 - \xi_0^1)^2 + |\xi'|^2 < (R+\Delta)^2.$$

Subtracting these inequalities, we find that

$$(r-\delta)^2 - (R+\Delta)^2 < (\xi^1)^2 - (\xi^1 - |\xi_0|)^2 < (r+\delta)^2 - (R-\Delta)^2$$

and hence that $\xi^1 \in (a,b)$, where

$$\begin{aligned} a &= \frac{1}{2|\xi_0|}\left(|\xi_0|^2 + r^2 - R^2 + \Delta^2 - \delta^2 - 2(\delta r + \Delta R)\right) \\ b &= a + \frac{2}{|\xi_0|}(r\delta + R\Delta). \end{aligned}$$

In particular,

$$b - a \sim \frac{\max(r\delta, R\delta)}{|\xi_0|}$$

and hence it suffices to show, for $c \in (a,b)$,

$$v(c) := H^{n-1}(A \cap \{\xi^1 = c\}) \lesssim \min(r,R,L)^{n-3}\min(r\delta, R\Delta).$$

We now define the upper and lower radius over the slice $\{\xi^1 = c\}$ of $\xi_0 + S_\Delta(R)$ and $S_\delta(r)$ respectively by

$$\begin{aligned} R^{\pm}(c) &= \max(0, \sqrt{(R\pm\Delta)^2 - (c - |\xi_0|)^2}) \\ r^{\pm}(c) &= \max(0, \sqrt{(r\pm\Delta)^2 - c^2}) \end{aligned}$$

Ignoring for a second the intersection and only looking at $\xi_0 + S_\Delta(R)$, polar coordinates when $R^- \neq 0$ give

$$v(c) \lesssim R^+(c)^{n-2}(R^+(c) - R^-(c)) = R^+(c)^{n-2}\frac{R^+(c)^2 - R^-(c)^2}{R^+(c) + R^-(c)} \sim R^+(c)^{n-3}R\Delta,$$

whereas in the case $R^- = 0$ we get the better estimate

$$v(c) = R^+(c)^{n-1} \lesssim (R\Delta)^{\frac{n-1}{2}} = (R\Delta)(R\Delta)^{\frac{n-3}{2}}$$

since $R + \Delta > c > R - \Delta$.

Of course the same can be done for $S_\delta(r)$, resulting in

$$v(c) \lesssim r^+(c)^{n-3} r\delta.$$

Due to the tube $T_L(\xi_0) = T_L((1,0,\ldots,0))$ we also know that $\max(r^+, R^+) \leq L$, furthermore, of course, $\max(r^+, R^+) \lesssim \min(r,R)$ on A. In combination,

$$v(c) \lesssim \min(r,R,L)^{n-3}\min(r\delta, R\Delta)$$

as claimed, and we can estimate

$$|A| \leq \int_a^b v(c)dc \leq (b-a)\min(r,R,L)^{n-3}\min(r\delta, R\Delta) \sim \frac{r\delta R\Delta \min(r,R,L)^{n-3}}{|\xi_0|}.$$

□

Definition 4.22. For $M, m, \epsilon > 0$, we denote

$$K^{m,\pm}_{M,\epsilon} = \{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n : |\xi| \sim M, \, |\tau - \langle\xi\rangle_m| \leq \epsilon\}$$

Proposition 4.23. *Let $n \geq 3$, let $O, M, N > 1$ dyadic numbers, denote*

$$H = \max(O,M,N), \qquad L = \min(O,M,N)$$

and let

$$\operatorname{supp}(u) \subseteq K^{m_1,\pm_1}_{M,\epsilon_1}, \ \ \operatorname{supp}(v) \subseteq K^{m_2,\pm_2}_{N,\epsilon_2}.$$

Then we have

$$\|P_{\leq O}(uv)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \begin{cases} (\epsilon_1\epsilon_2)^{\frac{1}{2}} H^{\frac{1}{2}} L^{\frac{n-2}{2}} \|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} & \text{if } M \sim N \\ (\epsilon_1\epsilon_2)^{\frac{1}{2}} L^{\frac{n-1}{2}} \|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} & 1 \ll M \ll N \end{cases} \tag{4.8.2}$$

Remark 4.24. Note that for technical reasons, we do not treat $1 \sim M \ll N$ here.

Proof. If $H \lesssim 1$, losing derivatives does not matter, and the statement follows easily. Hence, in what follows, we may assume $H \gg 1$.

case 1

We begin with the case where $M \sim N \sim H \gg L \sim O$. Decomposing the spatial frequency supports of u and v in balls of radius L, we note that it suffices to prove the estimate

$$\|P_{\leq L}(u^B v^{B'})\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim (\epsilon_1\epsilon_2)^{\frac{1}{2}} H^{\frac{1}{2}} L^{\frac{n-2}{2}} \|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}, \tag{4.8.3}$$

where u^B and $v^{B'}$ are supported in balls B, B' of radius L located at frequency H. Indeed, denote by $\mathcal{B}$ a reasonable covering of $\{|\xi| \sim H\}$ with such balls. Then we can estimate

$$\|P_{\leq O}(uv)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \sum_{B,B'\in\mathcal{B}} \|P_{\leq O}(u^B v^{B'})\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \sim \sum_{B\sim B'} \|u^B v^{B'}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

where $B \sim B'$ if and only if $(B + B') \cap B(0, L) \neq \emptyset$. Since for fixed B there are only finitely many B' with $B \sim B'$, we can further estimate

$$\sum_{B\sim B'} \|u^B v^{B'}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim (\epsilon_1\epsilon_2)^{\frac{1}{2}} H^{\frac{1}{2}} L^{\frac{n-2}{2}} \sum_{B\sim B'} \|u^B\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|v^B\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$
$$\lesssim (\epsilon_1\epsilon_2)^{\frac{1}{2}} H^{\frac{1}{2}} L^{\frac{n-2}{2}} \|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Hence, we only need to prove (4.8.3), but we may even reduce to the case $u^B = v^{B'}$ since

$$\|u^B v^{B'}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq \|(u^B)^2\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|(v^{B'})^2\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Furthermore, we may assume $\pm_1 = +$, $m = 1$. We will need the following well-known

Lemma 4.25. *Let* $\operatorname{supp} \mathcal{F}_{tx} u \subseteq A$, $\operatorname{supp} F_{tx} v \subseteq B$. *Then*

$$\|uv\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq \left(\sup_{\tau,\xi} |A \cap ((\tau, \xi) - B)| \right)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Proof. We denote $\mathcal{F}_{tx}\cdot = \tilde{\cdot}$, $\zeta = (\tau, \xi)$, $\zeta' = (\tau', \xi')$ and estimate

$$\begin{aligned}
\|uv\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} &= \|\tilde{u} * \tilde{v}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \|\mathbf{1}_A \tilde{u} * \mathbf{1}_B \tilde{v}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&= \int \left(\int (\mathbf{1}_B \tilde{v})(\zeta - \zeta')(\mathbf{1}_A \tilde{u})(\zeta') d\zeta' \right)^2 d\zeta \\
&= \int \left(\int (\mathbf{1}_{A\cap(\zeta - B)})(\zeta') \tilde{v}(\zeta - \zeta') \tilde{u}(\zeta') d\zeta' \right)^2 d\zeta \\
&\leq \int \left(\int \mathbf{1}_{A\cap(\zeta - B)}(\zeta') d\zeta' \right) \left(\int |\tilde{v}(\zeta - \zeta')\tilde{u}(\zeta')|^2 d\zeta' \right) d\zeta \\
&\leq \left(\sup_\zeta |A \cap (\zeta - B)| \right) \|u\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|v\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}.
\end{aligned}$$

□

Applying the lemma to the present situation, where $A = B$ is a ball of radius L located at frequency H, we see that the constant in the estimate is $\sqrt{|E|}$, where

$$E = \Big\{ (\tau, \xi) : \ \xi \in B, \ \xi_0 - \xi \in B, \ \tau = \langle \xi \rangle + O(\epsilon_1), \\ \tau_0 - \tau = \langle \xi_0 - \xi \rangle + O(\epsilon_1) \Big\}$$

uniformly in (τ_0, ξ_0). We denote

$$E(\tau) = \{\xi : \ (\tau, \xi) \in E\}$$

and note that since $H \gg 1$, we have $\langle \xi \rangle \sim |\xi|$ and hence $E(\tau) = \emptyset$ unless

$$\tau = |\text{center(B)}| + O(L).$$

Thus, we have

$$|E| \lesssim L \sup_\tau |E(\tau)|.$$

For $\xi \in E(\tau)$ we have $\xi_0 = (\xi_0 - \xi) + \xi \in B + B \subset \{|\eta| \sim H\}$. Now we note that

$$|\tau - \langle\xi\rangle| \le \epsilon_1 \iff |\xi| \in \left[\sqrt{(\tau-\epsilon_1)^2 - 1}, \sqrt{(\tau+\epsilon_1)^2 - 1}\right].$$

This interval has length comparable to ϵ_1 and contains $\sqrt{\tau^2 - 1}$, hence it follows that

$$|\tau - \langle\xi\rangle| \le \epsilon_1 \Rightarrow \xi \in S_{C\epsilon_1}(\sqrt{\tau^2 - 1}).$$

In the same way,

$$|(\tau_0 - \tau) - \langle\xi_0 - \xi\rangle| \le \epsilon_1 \Rightarrow \xi_0 - \xi \in S_{C\epsilon_1}(\sqrt{(\tau_0-\tau)^2 - 1})$$

so that

$$E(\tau) \subset S_{C\epsilon_1}(\sqrt{\tau^2 - 1}) \cap \left(\xi_0 + S_{C\epsilon_1}(\sqrt{(\tau_0-\tau)^2 - 1})\right).$$

Remembering the additional restriction that the intersection happens in the ball B of radius L, and that $\xi_0 \in 2B$, we may intersect this last set with the tube of radius L along ξ_0. This puts us right in the situation of Lemma 4.20 about intersections of thin shells, and noting that $\sqrt{\tau^2-1} \sim H \sim \sqrt{(\tau_0-\tau)^2 - 1}$ together with $|\xi_0| \sim H$ gives

$$|E| \lesssim L \sup_\tau |E(\tau)| \lesssim L \frac{H^2 L^{n-3} \epsilon_1 \epsilon_2}{|\xi_0|} \sim L^{n-2} H \epsilon_1 \epsilon_2.$$

case 2

Now, without loss of generality, $L \sim M$, $H \sim N \sim O$ and $\pm_1 = +$. We may replace the projector $P_{\le O}$ by a projector on an annulus P_O (see [Sel08], 4.3.3). Again, we want to estimate $|E|$, where

$$E = K^{m_1,+}_{L,\epsilon_1} \cap \left((\tau_0, \xi_0) - K^{m_2,\pm_2}_{N,\epsilon_2}\right)$$

and we have $|\xi_0| \sim H$ due to the projector P_O. Going through the same procedure as before, we obtain

$$E = \Big\{(\tau,\xi): \ |\xi| \sim L, |\xi_0 - \xi| \sim H, \ \tau = \langle\xi\rangle_{m_1} + O(\epsilon_1), \\ \tau_0 - \tau = \pm\langle\xi_0 - \xi\rangle_{m_2} + O(\epsilon_2)\Big\}$$

and, recalling that $L \gg 1$,

$$E \subset S_{C\epsilon_1}(\sqrt{\tau^2 - m_1^2}) \cap \left(\xi_0 + S_{C\epsilon_2}(\sqrt{(\tau_0-\tau)^2 - m_2^2})\right).$$

Now we have

$$\sqrt{\tau^2 - m_1^2} \sim L, \quad \sqrt{(\tau_0-\tau)^2 - m_2^2} \sim H, \quad |\xi_0| \sim H$$

and thus

$$|E| \lesssim L \sup_{\tau} |E(\tau)| \lesssim L \frac{HLL^{n-3}\epsilon_1\epsilon_2}{H} = L^{n-1}\epsilon_1\epsilon_2$$

which gives the constant $L^{\frac{n-1}{2}}(\epsilon_1\epsilon_2)^{\frac{1}{2}}$ as claimed.

□

Chapter 5

Quadratic Schrödinger equations

We treat here a particular example of a nonlinear derivative Schrödinger equation and obtain global existence and scattering for small data. Generally speaking, such an equation is of Schrödinger type, with a quadratic nonlinearity containing one (or two) derivatives. Existence results for such equations are difficult to obtain in general and involve a high level of technical detail; a small data global result is not known and may not hold, but at least large data local existence is available under a necessary decay assumption on the initial data. We refer to [BT08] for details.

The example we treat is much simpler than the general case. It is special in the sense that it is the only representative of a quadratic derivative Schrödinger equation for which all interactions are nonresonant except at the origin (where instead the derivative in the nonlinearity smoothes out the otherwise negative impact of resonance). Applying to this the same techniques as in chapter 4 leads to a result which is certainly not new in spirit (for instance, [Coh94] also treats roughly the same example, albeit with different techniques) but represents a clean and transparent framework to understand this particular case. In particular, it demonstrates that the initial data can be taken from the natural scale invariant Sobolev space $\dot{H}^{s_c}$, $s_c = \frac{n-2}{2}$ $(n \geq 2)$. In contrast, a general quadratic derivative NLS appears to require $s \geq s_c + 1$ already for local well-posedness, on top of decay on the initial data (see [BT08, Sch10]).

5.1 Introduction

The equation

$$iu_t - \Delta u = Q(\bar{u}, \bar{u}) \tag{5.1.1}$$

where

$$Q(\bar{u}, \bar{u}) = \pm \bar{u} \partial_{x_i} \bar{u} \quad \text{or} \quad Q(\bar{u}, \bar{u}) = \pm \partial_{x_i} (\bar{u})^2$$

in $n \geq 2$ spatial dimensions can be treated quite easily in a scaling critical setup using U^2 and V^2 spaces. The basic reason is that the modulation is favorable,

$$|\xi|^2 + |\eta|^2 + |\xi + \eta|^2 \sim \max(|\xi|^2, |\eta|^2).$$

The scaling

$$u(x,t) \mapsto u_\lambda(x,t) = \lambda u(\lambda^2 t, \lambda x)$$

leaves the equation invariant, and the scaling critical Sobolev index is $\frac{n-2}{2}$ in the sense that

$$\|u_\lambda\|_{\dot{H}^{\frac{n-2}{2}}(\mathbb{R}^n)} = \|u\|_{\dot{H}^{\frac{n-2}{2}}(\mathbb{R}^n)}.$$

For this reason, we fix now

$$s = \frac{n-2}{2},$$

look for the solution in the U^2 version of the Besov space $\dot{B}_2^{s,2}$ adapted to the linear Schrödinger equation,

$$\|u\|_{\dot{X}^s} = \left(\sum_N N^{2s} \|u_N\|^2_{U^2_\Delta}\right)^{\frac{1}{2}} \qquad \text{where } U^2_\Delta = U^2_{|\xi|^2}$$

and define the space $\dot{Y}^s$ analogously by using $V^2_{-,rc}$ (henceforth abbreviated by V^2) instead of U^2. We will prove the following

Theorem 5.1. *Let $n \geq 2$ and $s = \frac{n-2}{2}$. Then for the nonlinear derivative Schrödinger equation* (5.1.1) *we have small data global well-posedness and scattering in $\dot{H}^s$ in the scaling critical space $\dot{X}^s$.*

Remark 5.2. We will focus on a nonlinearity $\bar{u}\partial_{x_i}\bar{u}$; the proof for $\partial_{x_i}(\bar{u}^2)$ is actually a bit easier. Of course, Theorem 5.1 generalizes to systems of equations of the above type effortlessly, requiring only trivial modifications of the arguments below.

For the proof, we denote by I the Duhamel term

$$I = I(x,t) = -i \int_0^t e^{(t-s)|D|^2} Q(\bar{u}, \bar{u})(s) ds$$

and estimate this expression in the $\dot{X}^s$ norm. What we need for a contraction argument is the boundedness

$$I : \dot{X}^s \times \dot{X}^s \to \dot{X}^s$$

but we can even prove the stronger version

Proposition 5.3.

$$I : \dot{Y}^s \times \dot{Y}^s \to \dot{X}^s$$

from which the former follows by the embedding $\dot{X}^s \subset \dot{Y}^s$. The difference estimate needed for the contraction argument follows trivially, since the nonlinearity is polynomial.

5.2 Bilinear and Strichartz estimates

As usual, we will use the known estimates for free solutions, the transfer principle and interpolation to estimate the nonlinearity. In this case, we need the bilinear estimates

$$\|u_H v_L\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{-\frac{1}{2}} L^{\frac{n-1}{2}} \|u\|_{U^2_{\pm_1\Delta}} \|v\|_{U^2_{\pm_2\Delta}}, \tag{5.2.1}$$

$$\|P_L(u_H v_H)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim L^{\frac{n-2}{2}} \|u\|_{V^2_{\pm_1\Delta}} \|v\|_{V^2_{\pm_2\Delta}}. \tag{5.2.2}$$

for any choice of signs $\pm_1, \pm_2$, which in turn imply (for fixed small δ, here $\delta = \frac{1}{4}$ will do)

$$\|u_H v_L\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{-\frac{1}{2}+\delta} L^{\frac{n-1}{2}-\delta} \|u\|_{V^2_{\pm_1\Delta}} \|v\|_{V^2_{\pm_2\Delta}}. \tag{5.2.3}$$

Proof. While (5.2.1) follows from the bilinear estimate for free solutions (see, for instance, Lemma 3.4 and Remark 3.5 in [CKS⁺08]) and the transfer principle, we carefully check (5.2.2), since it is crucial that it is valid in V^2_Δ as opposed to U^2_Δ to avoid a logarithmic divergence. By standard arguments, we may decompose $u_H \sim \sum_B u_B$ and $v_H \sim \sum_{B'} v_{B'}$ into balls of size L located at frequency H, such that

$$\|u_H v_H\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \sim \sum_{B\sim -B'} \|u_B v_{B'}\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}.$$

Using then for free solutions the estimate

$$\|e^{it\Delta}(u_0)_B\|_{L^4(\mathbb{R}\times\mathbb{R}^n)} \lesssim L^{\frac{n-2}{4}} \|(u_0)_B\|_{L^2(\mathbb{R}^n)}$$

which follows from the first bilinear estimate, we get

$$\|u_H v_H\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim L^{\frac{n-2}{2}} \|u_H\|_{U^4_\Delta} \|v_H\|_{U^4_\Delta} \lesssim L^{\frac{n-2}{2}} \|u_H\|_{V^2_\Delta} \|v_H\|_{V^2_\Delta}$$

as desired.
Lastly, we obtain (5.2.3) from (5.2.1) by interpolation. Noting that for $u_N = \overline{v_N}$ we have

$$\|u_N\|^2_{L^4(\mathbb{R}\times\mathbb{R}^n)} = \|u_N \overline{u_N}\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim N^{\frac{n-2}{2}} \|u_N\|^2_{U^4_\Delta},$$

we estimate

$$\|u_H v_L\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \leq \|u_H\|_{L^4(\mathbb{R}\times\mathbb{R}^n)} \|v_L\|_{L^4(\mathbb{R}\times\mathbb{R}^n)} \lesssim (HL)^{\frac{n-2}{4}} \|u_H\|_{U^4_\Delta} \|v_L\|_{U^4_\Delta}.$$

Interpolating this estimate with (5.2.1), we lose a factor

$$\log^2\left(1 + \frac{H}{L}\right) \lesssim \left(\frac{H}{L}\right)^\delta \qquad \delta \ll 1.$$

to transition into V^2_Δ on the right hand side of (5.2.1). It's easy to see that none of the arguments change for any other choice of signs $\pm_1$, $\pm_2$ as the estimates for free solutions remain valid. ∎

5.3 Trilinear estimates and proof of the main theorem

In this section, we prove the technical estimates needed for the proof of Proposition 5.3. To this end, we decompose

$$\|I\|_{\dot{X}^s}^2 = \sum_H H^{2s} \| \sum_{L \ll H} P_H I(u_L, u_H)\|_{U_\Delta^2}^2 + \sum_L L^{2s} \| \sum_{H \gtrsim L} \sum_{H' \sim H} P_L I(u_H, u_{H'})\|_{U_\Delta^2}^2$$

and obtain now estimates for each of the terms on the right.

The most difficult (and interesting) case for these estimates is $n = 2$ (where $s = 0$, corresponding to L^2 data) and we treat that case only, even though we keep the parameter n in the building block estimates whenever the argument applies to all n to indicate what they look like in general. Higher dimensions are easier to treat by the same methods; the only difference is that using orthogonality is less crucial. The bounds below will be used in the next section to prove (5.3).

Proposition 5.4. *Let $H \sim H'$. Then we have*

$$\sum_{L \lesssim H} \sup_{\|w\|_{V_\Delta^2} \le 1} \Big| \iint u_L v_H w_{H'} dxdt \Big| \lesssim H^{-1} \|u\|_{\dot{Y}^s} \|v_H\|_{V_\Delta^2} \tag{5.3.1}$$

for the first part of I above ("high-low to high interactions") and

$$\left(\sum_{L \lesssim H} L^{2s} \sup_{\|w\|_{V_\Delta^2} \le 1} \Big| \iint u_H v_{H'} w_L dxdt \Big|^2 \right)^{\frac{1}{2}} \lesssim H^{-1} H^s \|u_H\|_{V_\Delta^2} {H'}^s \|v_H\|_{V_\Delta^2} \tag{5.3.2}$$

for the second ("high-high to low interactions").

Remark 5.5. The above corresponds to a nonlinearity $\bar{u}\partial_{x_i}\bar{u}$, the (easier) nonlinearity $\partial_{x_i}(\bar{u})^2$ corresponds to replacing H^{-1} on the right hand side in (5.3.2) by an additional factor of L multiplied to the left hand side.

Proof. We assume from now on $s = 0$, $n = 2$ and we drop the distinction between H and H' (which is irrelevant in the arguments used). For both estimates, we decompose u, v and w according to high and low modulation, "low" in this context meaning that $|\tau - |\xi|^2| \ll H^2$ on the spacetime Fourier support. This is motivated by the modulation

$$|\xi|^2 + |\eta|^2 + |\xi + \eta|^2 \sim \max(|\xi|, |\eta|)^2,$$

due to the properties of which we may assume that at least one factor is at high modulation (which we denote by a superscript h) and we can use the estimate $\|f^h\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} = \|Q^\Delta_{\ge H^2} f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{-1}\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)}$, that is we gain a full derivative; the remaining bilinear product in $L^2(\mathbb{R}^n)$ is treated using bilinear and

Strichartz estimates (5.2.2), (5.2.3). Now we begin with the proof of Proposition 5.3. The estimates go along similar lines: The high modulation term goes into L^2 and then into V^2, gaining a factor H^{-1}. The remaining terms are estimated using the bilinear estimate, giving either a factor $H^{-\frac{1}{2}+\delta}L^{\frac{n-1-\delta}{2}}$ or $L^{\frac{n-2}{2}}$. Since the summations take place on the level of L^2, we need to use some orthogonality to close the estimate in the critical cases (this is less delicate when $s > 0$). We now begin with (5.3.1), in the easier case where $w_H = w_H^h$,

$$\Big|\int u_L v_H w_H^h dxdt\Big| \lesssim H^{-\frac{3}{2}+\delta} L^{\frac{n-1}{2}-\delta-s} L^s \|u_L\|_{V^2_\Delta} \|u_H\|_{V^2_\Delta}$$

where Cauchy-Schwarz, the high modulation estimate and (5.2.3) were used. We sum over $L \lesssim H$ and obtain

$$H^{-1} \sup_L L^s \|u_L\|_{V^2_\Delta} \|v_H\|_{V^2_\Delta}$$

which is stronger than what is needed. The case $v_H = v_H^h$ is identical.

The remaining case $u_L = u_L^h$ is more critical, but luckily we can use an orthogonality (note $s = 0$ and $\|w_H\|_{V^2_\Delta} \leq 1$)

$$\begin{aligned}
\sum_{L\lesssim H} H\Big|\int u_L^h v_H w_H dxdt\Big| &\lesssim \sum_{L\lesssim H} H\|u_L^h\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|P_L(v_H w_H)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&\lesssim \sum_{L\lesssim H} \|u_L\|_{V^2_\Delta} \|P_L(v_H w_H)\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \left(\sum_{L\lesssim H} \|u_L\|^2_{V^2_\Delta}\right)^{\frac{1}{2}} \|v_H w_H\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&\lesssim \|u\|_{\dot{Y}^s} \|v_H\|_{V^2_\Delta}
\end{aligned}$$

We turn to the second one, using the same strategy. In the case $w_L = w_L^h$, we get

$$\begin{aligned}
H^2 \sum_{L\lesssim H} L^{2s}\Big|\iint u_H v_H w_L^h dxdt\Big|^2 &\lesssim \sum_{L\lesssim H} L^{2s} \|P_L(u_H v_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&\lesssim H^{2s} \sum_{L\lesssim H} \|P_L(u_H v_H)\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim H^{2s} \|u_H v_H\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
&\lesssim H^{2s} H^{n-2} \|u_H\|^2_{V^2_\Delta} \|v_H\|^2_{V^2_\Delta} = H^{2s} \|u_H\|^2_{V^2_\Delta} H^{2s} \|v_H\|^2_{V^2_\Delta}
\end{aligned}$$

where we used again orthogonality and (5.2.2). In the remaining case $v_H = v_H^h$ we obtain

$$\begin{aligned}
H^2 \sum_{L\lesssim H} L^{2s}\Big|\iint u_H v_H^h w_L dxdt\Big|^2 &\lesssim \sum_{L\lesssim H} L^{2s} \|w_L v_H\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} \|u_H\|^2_{V^2_\Delta} \\
&\lesssim \sum_{L\lesssim H} L^{2s+n-1-\delta} H^{-1+\delta} \|u_H\|^2_{V^s_\Delta} \|v_H\|^2_{V^2_\Delta}
\end{aligned}$$

which sums just fine since

$$\sum_{L \lesssim H} L^{2s+n-1-\delta} H^{-1+\delta} \lesssim H^{n-2} H^{2s} = H^{2s} H^{2s}.$$

This concludes the proof of (5.3.2) and hence, Proposition 5.4.

Proof of the main theorem

Recall that we have decomposed

$$\|I\|_{\dot{X}^s}^2 = \sum_H H^{2s} \| \sum_{L \ll H} P_H I(u_L, u_H) \|_{U_\Delta^2}^2 + \sum_L L^{2s} \| \sum_{H \gtrsim L} \sum_{H' \sim H} P_L I(u_H, u_{H'}) \|_{U_\Delta^2}^2$$

Let us first treat the second term on the right hand side above, using the high-high estimate (5.3.2). Namely, we use the duality $(U^2)^* = V^2$ induced by the bilinear form B from Theorem 3.19 and write, for fixed $L \in 2^{\mathbb{Z}}$,

$$\begin{aligned} \| \sum_{H' \sim H \gtrsim L} P_L I(u_H, u_{H'}) \|_{U_\Delta^2} &= \sup_{\|v\|_{V^2_{-\Delta}}} \left| B\left(\sum_{H' \sim H \gtrsim L} P_L I(u_H, u_{H'}), v \right) \right| \\ &\leq \sup_{\|v_L\|_{V_\Delta^2}} \sum_{H' \sim H \gtrsim L} \left| \iint u_H \partial_{x_i} u_{H'} v_H dx dt \right| \end{aligned}$$

Hence, invoking (5.3.2) from Proposition 5.4 we find that

$$\begin{aligned} \sum_L L^{2s} \| \sum_{H \sim H' \gtrsim L} P_L I(u_H, \partial_{x_i} u_{H'}) \|_{U_\Delta^2}^2 &\lesssim \sum_H H^{-2} H^{2s} \|u_H\|_{V_\Delta^2}^2 {H'}^{2s} \|\partial_{x_i} u_{H'}\|_{V_\Delta^2}^2 \\ &\lesssim \|u\|_{\dot{Y}^s}^4. \end{aligned}$$

For the first term, we use the same argument, but using (5.3.1) this time. This leads to two cases, depending on whether the derivative ∂_{x_i} hits u_L or u_H, and we treat the (more difficult) latter case only. We estimate, with $H' \sim H$,

$$\begin{aligned} \sum_H H^{2s} \| \sum_{L \ll H} P_H I(u_L, u_H) \|_{U_\Delta^2}^2 &\leq \sum_H H^{2s} \left(\sum_{L \ll H} \sup_{\|v_{H'}\|_{V_\Delta^2} \leq 1} \left| \int u_L \partial_{x_i} u_H v_{H'} dx dt \right| \right)^2 \\ &\lesssim \sum_H H^{2s-2} \|u\|_{\dot{X}^s}^2 \|\partial_{x_i} u_H\|_{V_\Delta^2}^2 \\ &\lesssim \|u\|_{\dot{Y}^s}^4. \end{aligned}$$

□

In summary, we have proven that $I : \dot{Y}^s \times \dot{Y}^s \to \dot{X}^s$. This allows us to construct the solution following exactly the procedure carried out in section 4.6 for Klein-Gordon equations, to which we refer for details. This concludes the proof.

Chapter 6

The Novikov-Veselov equation

6.1 Introduction

The (zero energy) Novikov-Veselov (NV) and modified Novikov-Veselov (mNV) equations are dispersive equations in two-dimensional space. (NV) is a natural generalization of the Korteweg-de Vries (KdV) equation to two spatial dimensions: like (KdV), it is completely integrable with respect to the stationary Schrödinger equation (but in two spatial dimensions), and it reduces to (KdV) for solutions which do not depend on the second spatial variable. It takes the form

$$\begin{aligned} u_t + \left(\partial^3 + \bar{\partial}^3\right) u &= N_{NV}(u), \qquad u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \\ u(0) &= f \end{aligned} \tag{6.1.1}$$

where $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and the nonlinearity

$$N_{NV}(u) = \frac{3}{4}\partial\left(u\bar{\partial}^{-1}\partial u\right) + \frac{3}{4}\bar{\partial}\left(\bar{u}\partial^{-1}\bar{\partial}\bar{u}\right)$$

is quadratic. Its modified counterpart (mNV) is complex-valued but takes the same form, albeit with a cubic nonlinearity

$$\begin{aligned} \frac{4}{3}N_{mNV}(\cdot) = (\partial\bar{u})\left(\bar{\partial}\partial^{-1}|u|^2\right) + (\bar{\partial}u)\left(\bar{\partial}\partial^{-1}|u|^2\right) \\ +\bar{u}\bar{\partial}\partial^{-1}(\bar{u}\bar{\partial}u) + u\partial^{-1}\bar{\partial}(\bar{u}\bar{\partial}u). \end{aligned}$$

It is related to (NV) through the Miura- type transformation

$$\mathcal{M}(v) = 2\partial v + |v|^2 \tag{6.1.2}$$

and is also a completely integrable equation[1]. These last two facts have been used in [Per12] to construct solutions to (6.1.1): via the inverse scattering method solutions to (mNV) are obtained, and subsequently transferred to (NV) by virtue of the map $\mathcal{M}$. In effect, the precise structure of the nonlinearity will not play a crucial role, and the results we obtaine are weaker in some respects, but stronger in others.

Here, we complement these results somewhat by relying only on the dispersive nature of (mNV), and not its integrability. (mNV) has a natural scaling invariance, given for $\lambda > 0$ by

$$u_\lambda(t, x, y) = \lambda u(\lambda^3 t, \lambda x, \lambda y).$$

Since $\|u_\lambda(0, \cdot, \cdot)\|_{L^2(\mathbb{R}^2)} = \|u(0, \cdot, \cdot)\|_{L^2(\mathbb{R}^2)}$, the scaling critical space of initial data is $L^2(\mathbb{R}^2)$ and we will obtain the following result at this level of regularity.

Theorem 6.1. *The modified Novikov-Veselov equation is globally well-posed and scatters for small initial data in $L^2(\mathbb{R}^2)$, with bounds in the space $\dot{X}$ defined below.*

We solve (6.1.1) by a contraction mapping argument in the space defined by the norm

Definition 6.2.

$$\|u\|_{\dot{X}} = \left(\sum_{N \in 2^{\mathbb{Z}}} \|P_N u\|_{U_h^2}^2 \right)^{\frac{1}{2}} \tag{6.1.3}$$

where h is the dispersion relation associated to the linear propagator $h(D) = 2\operatorname{Re} \partial^3$. To make the required estimates more symmetric (and a little stronger than what is needed) we also introduce the corresponding space $\dot{Y}$, replacing U_h^2 above by $V_{rc,h}^2$

6.2 Bilinear and Strichartz estimates

Definition 6.3. We denote $z = \xi_1 + i\xi_2 \in \mathbb{C} = \mathbb{R}^2$ and

$$h(\xi) = h(z) = 2\operatorname{Re} z^3 = 2\xi_1^3 - 6\xi_1\xi_2^2.$$

As usual, we write $e^{ith(D)}$ for the linear propagator associated to the linear equation $u_t + \left(\partial^3 + \bar{\partial}^3\right) u = 0$.

Proposition 6.4 (Strichartz estimates)**.** *We have, for $N \in 2^{\mathbb{Z}}$ dyadic and*

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2} \qquad 2 < q \leq \infty$$

the Strichartz estimate

$$\|e^{ith(D)} P_N f\|_{L^q L^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim N^{-\frac{1}{q}} \|P_N f\|_{L^2(\mathbb{R}^2)}.$$

[1] with respect to the Davey-Stewartson II equation, see [Per12]

In particular, we have

$$\|e^{ith(D)}P_N f\|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \lesssim N^{-\frac{1}{4}}\|P_N f\|_{L^2(\mathbb{R}^2)}.$$

Proof. We give here a heuristic derivation of a suitable dispersive estimate only, based on Heuristic 2.6 and the abstract Strichartz estimate in Proposition 2.11. A rigorous derivation of the dispersive estimate below can be obtained from Theorem 5.6 in [BAKS03]. Recall that the dispersion relation is $h(\xi) = 2\xi_1^3 - 6\xi_1\xi_2^2$; according to Heuristic 2.6, we need to compute the determinant of the Hessian of h, which is the matrix

$$12\begin{pmatrix} \xi_1 & -\xi_2 \\ -\xi_2 & -\xi_1 \end{pmatrix}.$$

Consequently the determinant is $-12|\xi|^2$ and its absolute value is comparable to the frequency N. Since we have two spatial dimensions in this problem, we expect a time decay of t^{-1} and a gain of one derivative. In other words, we expect the estimate

$$\|e^{ith(D)}P_N f\|_{L^\infty(\mathbb{R}^2)} \lesssim t^{-1}N^{-1}\|P_N f\|_{L^1(\mathbb{R}^2)} \qquad t > 0.$$

Plugging this dispersive estimate into the machinery of (2.11) (with $\sigma = 1$, $\delta = -1$), we obtain that for (q, r) obeying $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ it holds

$$\|e^{ith(D)}f\|_{L^q(\mathbb{R}, B^{\frac{1}{q}}_{2,r}(\mathbb{R}^2))} \lesssim \|f\|_{L^2(\mathbb{R}^2)},$$

but this implies the claim upon replacing f by $P_N f$. □

As in many cases, Strichartz estimates do not give an optimal balance of frequencies when used to control bilinear interactions of frequencies of different size. Fortunately, the geometry here allows for good bilinear estimates, as stated below.

Proposition 6.5 (Bilinear estimates)**.** *Let $0 < L \lesssim H$ dyadic frequencies. Then we have*

$$\|e^{ith(D)}P_H f \cdot e^{ith(D)}P_L g\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim \frac{L^{\frac{1}{2}}}{H}\|P_H f\|_{L^2(\mathbb{R}^2)}\|P_L g\|_{L^2(\mathbb{R}^2)}.$$

Proof. If $H \sim L$, then the L^4 Strichartz estimate gives the claim; hence we assume $L \ll H$. After dualizing the $L^2(\mathbb{R}\times\mathbb{R}^2)$ norm with a function $g = \mathcal{F}_{tx}^{-1}w$ of unit L^2_{tx} norm and denoting $u = \mathcal{F}_x(P_H f)$, $v = \mathcal{F}_x(P_L g)$, we have to bound the expression

$$I := \iint w(\xi + \eta, h(\xi) + h(\eta))u(\xi)v(\eta)d\xi d\eta$$

in terms of the right hand side above, where w, u and v may be taken real and nonnegative. We introduce the new coordinates

$$\gamma = \xi + \eta, \qquad a = h(\xi) + h(\eta)$$

and compute

$$\frac{d(\gamma,a)}{d\eta_2 d\xi}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ \frac{dh(\eta)}{d\eta_2} & \frac{dh(\xi)}{d\xi_1} & \frac{dh(\xi)}{d\xi_2}\end{pmatrix}\qquad \frac{d(\gamma,a)}{d\eta_2 d\xi}=\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ \frac{dh(\eta)}{d\eta_1} & \frac{dh(\xi)}{d\xi_1} & \frac{dh(\xi)}{d\xi_2}\end{pmatrix}.$$

Hence, for $i=1,2$, we have

$$J_i := \left|\det\frac{d(\gamma,a)}{d\eta_i d\xi}\right| = \left|\det\begin{pmatrix}1 & 1\\ \frac{dh(\eta)}{d\eta_i} & \frac{dh(\xi)}{d\xi_1}\end{pmatrix}\right| = \left|\frac{dh(\eta)}{d\eta_i}-\frac{dh(\xi)}{\xi_1}\right|.$$

Modulo a common factor, this simplifies to $|\eta_1^2-\xi_1^2+\xi_2^2-\eta_2^2|$ in the former case $i=1$, and to $|\eta_1\eta_2-\xi_1\xi_2|$ in the latter. Now we split the integration in ξ in three parts, namely we decompose $\mathbb{R}^2=K_2\cup K_{1a}\cup K_{1b}$, where

$$K_{1a}=\{\xi\in\mathbb{R}^2\;\big|\;|\xi_1|\ll H,\ \xi_2\sim H\}\qquad K_{1b}=\{\xi\in\mathbb{R}^2\;\big|\;|\xi_2|\ll H,\ \xi_1\sim H\},$$

$$K_2=\{\xi\in\mathbb{R}^2\;\big|\;|\xi_1|\sim|\xi_2|\sim H\}.$$

Let's deal with K_2. The point is that for $\xi\in K^2$, we have $J_2=|\eta_1\eta_2-\xi_1\xi_2|\sim H^2$ and thus, changing coordinates in I, we see that

$$\begin{aligned}I&=\iiint g(\gamma,a)\left\{\frac{u(\xi)v(\eta)}{J_2}\right\}d\gamma da d\eta_1\le\|g\|_{L^2(\mathbb{R}\times\mathbb{R}^2)}\int\left\|\frac{u(\xi)v(\eta)}{J_2}\right\|_{L^2_{\gamma,a}(\mathbb{R}^2\times\mathbb{R})}d\eta_1\\&\le H^{-1}\int\left\|\frac{u(\xi)v(\eta)}{\sqrt{J_2}}\right\|_{L^2_{\gamma,a}(\mathbb{R}^2\times\mathbb{R})}d\eta_1=H^{-1}\int\|u(\xi)v(\eta)\|_{L^2_{\xi,\eta_2}(\mathbb{R}^2\times\mathbb{R})}d\eta_1\\&\lesssim\frac{L^{\frac12}}{H}\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}=\frac{L^{\frac12}}{H}\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

where we used the bound on J_2 as well as the fact that η_1 is localized to an interval of scale L. Hence we have profen the claim in that case. For the other two cases, the argument is almost identical: We simply switch the roles of η_1 and η_2 and now use the bound

$$J_1=|\eta_1^2-\xi_1^2+\xi_2^2-\eta_2^2|=H^2+O(L^2)\sim H^2,$$

resulting in the same estimate. □

As usual, we apply the transfer principle Proposition 3.24 to the Strichartz and Bilinear estimates, and also interpolate into V_h^2 using (3.26). This gives the following

Proposition 6.6 (Linear and bilinear estimates)**.** *Let $0<L\ll H$ dyadic frequencies. Then, for $2<q\le\infty$, $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$, we have the Strichartz estimate*

$$\|P_Hu\|_{L^qL^r(\mathbb{R}\times\mathbb{R}^2)}\lesssim H^{-\frac{1}{q}}\|P_Hu\|_{U_h^q}\lesssim H^{-\frac{1}{q}}\|P_Hu\|_{V_h^2},\tag{6.2.1}$$

and, for fixed $\epsilon > 0$, the (interpolated) bilinear estimate

$$\|P_H u \cdot P_L v\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim \frac{L^{\frac{1}{2}-\epsilon}}{H^{1-\epsilon}} \|u\|_{V_h^2} \|v\|_{V_h^2}. \tag{6.2.2}$$

The same estimate holds true after placing complex conjugates on u, v or both of them.

Remark 6.7. At interactions of equal frequencies, (6.2.1) implies (6.2.2) (even without the logarithmic loss). Thus we can freely make use of (6.2.2) for any type of interaction. Relying only on the $L^4(\mathbb{R}\times\mathbb{R}^2)$ Strichartz estimates in the following would not at all work: Compared to (6.2.2), they effectively lose a large factor $\frac{H^{\frac{3}{4}}}{L^{\frac{3}{4}}}$.

Proof. (6.2.1) and the U_h^2 version of (6.2.2) (with $\epsilon = 0$) follows directly from the corresponding estimate for free solutions in Proposition 6.4 and the transfer principle (3.24). Then, two consecutive applications of the interpolation lemma Proposition 3.26 lead to (6.2.2), similar to the proof of Proposition 4.10. The precise loss incurred in the process is

$$\log^2\left(1 + \frac{(HL)^{-\frac{1}{4}}}{L^{\frac{1}{2}}H^{-1}}\right) = \log^2\left(1 + \frac{H^{\frac{3}{4}}}{L^{\frac{3}{4}}}\right) \lesssim \frac{H^\epsilon}{L^\epsilon}.$$

□

6.3 Nonlinear estimates and proof of the main theorem

Now we can estimate the nonlinearity. To declutter the following argument, we first reduce the nonlinearity N_{mNV} to the more transparent nonlinearity

$$N(u) = u^2 \partial u. \tag{6.3.1}$$

Heuristically, this is self-evident. The symbol of the operators of type $\partial\bar{\partial}^{-1}$ is of unit size everywhere, and the bilinear estimate is invariant under taking absolute values on the Fourier side. In effect, for a bounded spatial Fourier multiplier $m(\xi)$ and suitably regular functions f_1, f_2 and $g = m(D)(f_3 f_4)$ we can estimate

$$\begin{aligned}
&\left|\int f_1 f_2 g dx dt\right| \leq \int (*_{i=1}^2 |\mathcal{F}_x f_i| * g)(\cdot,t) dt \lesssim \int *_{i=1}^4 |\mathcal{F}_x f_i|(\cdot,t) dt \\
&\leq \| |\mathcal{F}_x f_1| * |\mathcal{F}_x f_3| \|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \| |\mathcal{F}_x f_2| * |\mathcal{F}_x f_4| \|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \\
&= \|\mathcal{F}_\xi^{-1}(|\mathcal{F}_x f_1|) \cdot \mathcal{F}_\xi^{-1}(|\mathcal{F}_x f_3|)\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \|\mathcal{F}_\xi^{-1}(|\mathcal{F}_x f_2|) \cdot \mathcal{F}_\xi^{-1}(|\mathcal{F}_x f_4|)\|_{L^2(\mathbb{R}\times\mathbb{R}^2)}.
\end{aligned}$$

Applying this to each term of N_{mNV} and using that the dispersion relation $h(\cdot)$ is odd (and hence the spaces V_h^2 and $(U_h^2)^* = V_{-h(-\cdot)}^2$ coincide), we see that in effect we need to treat only the nonlinearity (6.3.1).

The relevant estimates are collected below in the most difficult case, in which the remaining derivative falls on a high frequency term.

Theorem 6.8 (Nonlinear estimates)**.** *Let $H \sim H'$ dyadic numbers. Then, for the high-low interactions, we have*

$$H\Big|\sum_{\substack{L_i \lesssim H\\ i=1,2}} \iint u^1_{L_1} u^2_{L_2} u^3_{H'} w_H \, dx dt\Big| \lesssim \prod_{i=1,2} \|u^i\|_{\dot{Y}} \|u^k_{H'}\|_{V^2_h} \|w_H\|_{V^2_h}. \tag{6.3.2}$$

For the high-high interactions, we can bound for any $L_1 \in 2^{\mathbb{Z}}$

$$\begin{aligned}\Big(\sum_{L \lesssim H} H^2 \sup_{\|w_L\|_{V^2_h}=1} \Big\{\sum_{L_1 \lesssim H} \Big|\iint u^1_{L_1} u^2_{H'} u^3_H w_L \, dx dt\Big|\Big\}^2\Big)^{\frac{1}{2}} \\ \lesssim \|u^1_{L_1}\|_{\dot{Y}} \|u^2_{H'}\|_{V^2_h} \|u^3_H\|_{V^2_h}.\end{aligned} \tag{6.3.3}$$

Proof. We treat (6.3.2) first and may assume that w_H has norm one. We estimate using Cauchy-Schwarz, (6.2.2) and $\|w_H\|_{V^2_h} \leq 1$

$$\begin{aligned}\sum_{L_1, L_2 \ll H} \Big|\iint u^1_{L_1} u^2_{L_2} u^3_{H'} w_H \, dx dt\Big| &\leq \sum_{L_1, L_2 \ll H} \|u^1_{L_2} u^3_{H'}\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \|u^2_{L_2} w_H\|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \\ &\lesssim \sum_{L_1, L_2 \ll H} \frac{(L_1 L_2)^{\frac{1-\epsilon}{2}}}{H^{2-\epsilon}} \|u^1_{L_1}\|_{V^2_h} \|u^2_{L_2}\|_{V^2_h} \|u^3_{H'}\|_{V^2_h} \\ &\leq H^{-1} \left(\sum_{L_1, L_2 \ll H} \frac{(L_1 L_2)^{1-\epsilon}}{(HH)^{1-\epsilon}}\right)^{\frac{1}{2}} \|u^1\|_{\dot{Y}} \|u^2\|_{\dot{Y}} \|u^3_{H'}\|_{V^2_h}\end{aligned}$$

and the claim follows since $\sum_{L_i \lesssim H} \big(\frac{L_i}{H}\big)^{1-\epsilon} \lesssim 1$. The second case (6.3.3) is very similar,

$$\begin{aligned}\text{LHS (6.3.3)}^2 &\leq \sum_{L \lesssim H} \Big\{\sum_{L_1 \lesssim H} \frac{(L_1 L)^{\frac{1-\epsilon}{2}}}{H^{1-\epsilon}} \|u^1_{L_1}\|_{V^2_h}\Big\}^2 \|u^2_{H'}\|^2_{V^2_h} \|u^3_H\|^2_{V^2_h} \\ &\lesssim \sum_{L \lesssim H} \frac{L^{1-\epsilon}}{H^{1-\epsilon}} \sum_{L_1 \lesssim H} \frac{L_1^{1-\epsilon}}{H^{1-\epsilon}} \|u^1_{L_1}\|^2_{\dot{Y}} \|u^2_{H'}\|^2_{V^2_h} \|u^3_H\|^2_{V^2_h} \\ &\lesssim \|u^1_{L_1}\|^2_{\dot{Y}} \|u^2_{H'}\|^2_{V^2_h} \|u^3_H\|^2_{V^2_h}\end{aligned}$$

as claimed. ◻

Just as in chapters 4 and 5, this gives that the Duhamel term is bounded as a map from $\dot{Y} \times \dot{Y}$ to $\dot{X}$, and by the usual fixed point argument, a unique global solution in $\dot{X}$ exists for small initial data in $L^2(\mathbb{R}^2)$ and scatters.

Chapter 7

Ill-posedness for degenerate interactions

7.1 From the Scattering operator to convolution inequalities

Assume that $n = 2$ and that we have constructed a solution operator $T_\pm$ for some (quadratic) nonlinear equation, say

$$i\partial_t + h(D)u = Q(u,u)$$

for small $H^s(\mathbb{R}^2)$ data on $[0, \pm\infty)$, and that this operator is $C^2(H^s(\mathbb{R}^2), X^s([0, \pm\infty))$ in a neighborhood of 0 (which typically follows from a fixed-point setup with smooth nonlinearity) for some Banach space $X^s([0, \pm\infty)) \subset C([0, \pm\infty), H^s(\mathbb{R}^2))$ which has the property that $\lim_{t\to\pm\infty} f \in H^s$ exists for all $f \in X^s([0, \pm\infty))$. Assume further that Duhamel's formula holds for the solutions given by $T_\pm$. Taylor expanding $T_\pm$ around 0, we find

$$T_\pm f = T_\pm(0) + dT_\pm(0)f + d^2T_\pm(0)(f,f) + o(\|f\|^2_{H^s(\mathbb{R}^2)})$$

where $T_\pm(0) = 0$, $dT_\pm(0)f = e^{ith(D)}f$ and $d^2T(f,g) = \phi_{f,g}$, where

$$(i\partial + h(D))\phi_{f,g} = Q(e^{ith(D)}f, e^{ith(D)}g), \qquad \phi_{f,g}(0) = 0.$$

Also assume that we have a well-defined inverse wave operator

$$V_\pm : B_{H^s(\mathbb{R}^2)}(0,\delta) \to H^s(\mathbb{R}^2),\ f \mapsto \lim_{t\to\pm\infty}(e^{-ith(D)}T_\pm f)(t).$$

Thus, differentiating twice,

$$d^2V_\pm : H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2) \ni (f,g) \mapsto \lim_{t\to\pm\infty} e^{-ith(D)}d^2T_\pm(f,g) = \lim_{t\to\pm\infty} e^{-ith(D)}\phi_{f,g}(t)$$

and Duhamels' formula for $\phi_{f,g}$ gives

$$i \cdot d^2 V_\pm(f,g) = \lim_{t\to\pm\infty} \int_0^t e^{-ish(D)} Q(e^{ish(D)} f, e^{ish(D)} g) ds.$$

Since $V_\pm(0) = 0$, $dV_\pm(0) = id$, by the inverse function theorem, we may set

$$W_\pm = (V_\pm)^{-1}$$

and observe that the scattering operator

$$S = V_+ \circ W_-$$

is defined in a neighborhood of 0. We compute using $V_\pm(0) = 0$, $dV_\pm(0) = id$ and $d^2(V_- \circ W_-)(0) = 0$

$$\begin{aligned} d^2 S(0)(f,g) &= d^2 V_+(W_-(0))(dW_-(0)f, dW_-(0)g) \\ &\quad + dV_+(W_-(0)) \circ d^2 W_-(0)(f,g) \\ &= d^2 V_+(0)(f,g) + d^2 W_-(0)(f,g) \\ &= d^2 V_+(0)(f,g) - d^2 V_-(0)(f,g) \end{aligned}$$

Expressing this using the previously obtained formuli, we obtain

$$\begin{aligned} i \cdot d^2 S(0)(f,g) &= \lim_{t\to\infty} \int_0^t e^{-ish(D)} Q(e^{ish(D)} f, e^{ish(D)} g) ds \\ &\quad - \lim_{t\to-\infty} \int_0^t e^{-ish(D)} Q(e^{ish(D)} f, e^{ish(D)} g) ds \\ &= \lim_{t\to\infty} \int_{-t}^t e^{-ish(D)} Q(e^{ish(D)} f, e^{ish(D)} g) ds. \end{aligned}$$

If $S \in C^2(B_{H^s(\mathbb{R}^2)}(0,\delta), H^s(\mathbb{R}^2))$, then we have

$$\|d^2 S(0)(f,g)\|_{H^s(\mathbb{R}^2)} \lesssim \|f\|_{H^s(\mathbb{R}^2)} \|g\|_{H^s(\mathbb{R}^2)}$$

or, if we dualize[1], for any $h \in H^{-s}$,

$$\left| \iint Q(e^{ish(D)} f, e^{ish(D)} g) \overline{e^{ish(D)} h} \; dxds \right| \lesssim \|f\|_{H^s(\mathbb{R}^2)} \|g\|_{H^s(\mathbb{R}^2)} \|h\|_{H^{-s}(\mathbb{R}^2)}. \tag{7.1.1}$$

The left hand side is in Fourier space simply a convolution estimate on the hypersurfaces

$$\Sigma_\pm = \{(\tau,\xi) : \tau = \pm h(\xi)\}$$

and for such an estimate it is fairly transparent that it cannot hold when the hypersurfaces don't interact fully transversally through the convolution, with frequency localized data, thus allowing to replace the $H^s(\mathbb{R}^2)$ norms on the right by $L^2(\mathbb{R}^2)$

[1] and carelessly interchange limits

at the expense of a (possibly large) constant.

In frequency, the free wave $e^{ith(D)}f$ corresponds via the spacetime Fourier transform to the measure

$$\delta(\tau - h(\xi))\hat{f}(\xi) = \frac{\hat{f}}{\langle \nabla h \rangle}\mathcal{H}^n\Big|_{\{\tau = h(\xi)\}} = \frac{\hat{f}}{\langle \nabla h \rangle}\mu_{\Sigma_h}.$$

Hence, we identify $\delta(\tau - h(\xi))\hat{f}(\xi)$ with the measure $\frac{\hat{f}(\xi)}{\langle \nabla h \rangle}\mu_{\Sigma_h}$ and we note that

$$\|\frac{\hat{f}(\xi)}{\langle \nabla h \rangle}\|_{L^2(\Sigma_h)} = \|f\|_{L^2(\mathbb{R}^2)}.$$

Thus, for frequency localized functions, eq. (7.1.1) can be rewritten as an honest convolution estimate. If one chooses a triple of resonant points (say, at frequencies λ_i) where the hypersurfaces are not transversal, one can localize $\hat{f}, \hat{g}$ and $\hat{h}$ around these points, and then contradict the above by suppling an arbitrarily large lower bound (with a constant depending on the frequencies λ_i). Carrying out this procedure is the subject of the next sections.

7.2 Convolution estimates and degeneracy

The convolution of three L^2 functions supported on two dimensional hypersurfaces in three-dimensional space is a bounded operation, as stated by the Loomis-Whitney inequality. Such inequalities are much harder to obtain when considering curved surfaces, but exist and depend on the transversality of the involved surfaces in a quantitative manner. This is an interesting circle of ideas which can at this point not be expanded upon as much as it would have deserved. Instead, we refer to [BHT10] for an introduction and state here only a version suited to our needs.

Definition 7.1 (Convolution on hypersurfaces)**.** Let Σ_1, $\Sigma_2 \subset \mathbb{R}^3$ smooth two-dimensional hypersurfaces, $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$. We associate compactly supported f and g with the distributions $f\mu_{\Sigma_1}$ and $g\mu_{\Sigma_2}$ and define their convolution by

$$(f\mu_{\Sigma_1} * g\mu_{\Sigma_2})(\psi) = \int_{\Sigma_1}\int_{\Sigma_2} f(x)g(y)\psi(x+y)d\mu_{\Sigma_2}(y)d\mu_{\Sigma_1}(x)$$

for test functions ψ.

A priori, it is not clear that $f * g$ can be evaluated pointwise on a hypersurfaces, as we desire. However, the following estimate, first proven for continuous functions and then extended by density, shows this can be done.

Theorem 7.2 (Convolution estimate, [BHT10])**.** *Let $\Sigma_i \subset \mathbb{R}^3$, $i = 1, 2, 3$ smooth two-dimensional hyperplanes of diameter at most one. Assume furthermore that the surfaces Σ_i are uniformly transversal in the sense that their respective unit normals*

n^i satisfy, uniformly in $x^i \in \Sigma_i$,

$$\left|\det\begin{pmatrix} n^1(x^1) & n^2(x^2) & n^3(x^3)\end{pmatrix}\right| \geq \theta > 0. \tag{7.2.1}$$

Then, we have the convolution estimate

$$\|f * g\|_{L^2(\Sigma_3)} \lesssim \theta^{-\frac{1}{2}}\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}.$$

Remark 7.3. Looking at Definition 7.1, we see that the interactions in the convolution estimate are restricted to the set

$$\mathcal{R} := \{(x, y, z) \in \Sigma_1 \times \Sigma_2 \times \Sigma_3 \mid x + y = z\}$$

only, which suggests that the transversality condition eq. (7.2.1) is only important there.

We show now that without condition (7.2.1), (7.2) fails.

Proposition 7.4. *Let Σ_i, $i = 1, 2, 3$ given as in Theorem 7.2, but instead of* (7.2.1), *assume that there exist $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \in \mathcal{R}$ such that*

$$\left|\det\begin{pmatrix} n^1(\tilde{x}^1) & n^2(\tilde{x}^2) & n^3(\tilde{x}^3)\end{pmatrix}\right| = 0.$$

Denote

$$\delta = \delta(\epsilon) := \max_{i=1,2,3} \sup\{|n^i(q) - n^i(\tilde{x}^i)| \mid |q - x_i| \leq \epsilon\} \to 0 \text{ as } \epsilon \to 0.$$

Then, for any $1 \gg \epsilon > 0$ there exist unit size $f_\epsilon \in L^2(\Sigma_1)$, $g_\epsilon \in L^2(\Sigma_2)$ supported in ϵ-neighborhoods of x and y respectively, such that

$$\|f * g\|_{L^2(\Sigma_3)} \gtrsim \delta^{-\frac{1}{2}}. \tag{7.2.2}$$

In other words, the convolution estimate fails.

Proof. Since the functions f, g and h are going to be localized on a small scale $\epsilon > 0$ around $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$, we may assume that after a linear transformation that $\tilde{x}^3 = 0$ and Σ_1 and Σ_2 are perturbations of the surfaces $\{x_1 = 0\}$ and $\{x_2 = 0\}$ respectively, that is

$$\Sigma_1 = \{(h_1(x_2, x_3), x_2, x_3)\}, \quad \Sigma_2 = \{(x_1, h(x_1, x_3), x_3)\}, \quad \Sigma_3 = \{(h_3(x_2, x_3), x_2, x_3)\}$$

where we have that h_1 and h_2 are Lipschitz with constant δ, $|\partial_3 h_3| \lesssim \delta$, and $|\partial_{1,2} h_3| \lesssim 1$. Now we define a map

$$p : (-\epsilon, \epsilon) \times \Sigma_3 \to \Sigma_1, \quad (t, z) \mapsto p(t, z) \in \{p_3 = t\} \cap \Sigma_1 \cap \{-z - \Sigma_3\},$$

mapping a tuple (t, z) to the unique element $p(t, z)$ in the intersection on the right. This map is constructed by a fixed point argument. More precisely, given a tuple

(t, z), the conditions $z \in \Sigma_3$ and $z_3 = t$ imply that $z = (h(z_2, t), z_2, t)$. We are looking for $p \in \mathbb{R}^3$ which is a fixed point of

$$T_{t,z} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad p \mapsto \begin{pmatrix} -h_1(p_2, t) \\ -z_2 - h_2(-z_1 - p_1, -t - p_3) \\ t \end{pmatrix}.$$

Obviously we have $|T_{t,z}(p) - T_{t,z}(\tilde{p})| \lesssim \delta|p - \tilde{p}|$ and hence for small δ, this map is a contraction and admits a unique fixed point, which is our desired $p(t, z)$.

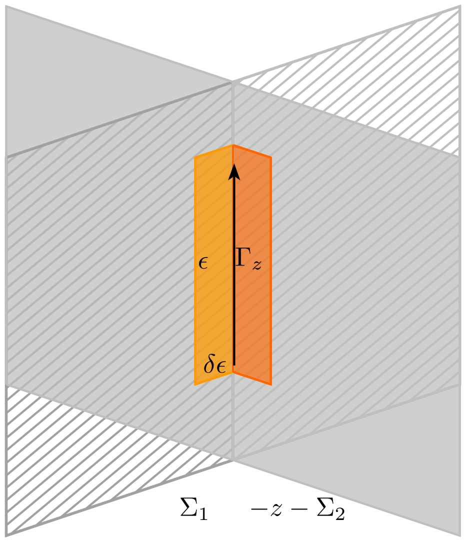

Figure 7.1: The path $\Gamma_z = \Sigma_1 \cap -z - \Sigma_2$ is oriented orthogonally to the normals n^1, n^2 and n^3, picking up a large contribution from the elongated supports of f and g uniformly over the z in the support of h, which is of similar shape.

Now, instead of (7.2.2), we can equivalently (up to flipping Σ_3) find a lower bound for the expression $(f * g * h)(0)$, with f, g and h unit size nonnegative L^2 functions on Σ_i, $i = 1, 2, 3$, respectively. Using the Coarea formula and neglecting geometric factors comparable to one, this can be rewritten as

$$\int_{\Sigma_3} h(z) \int_{\Sigma_1 \cap -z - \Sigma_2} f(x)g(-z - x)d\mathcal{H}^1(x)d\mathcal{H}^2(z)$$

and then, parametrizing $\Sigma_1 \cap -z - \Sigma_2$ along the e_3-direction,

$$\int_{\Sigma_3} h(z) \int_{-\epsilon}^{\epsilon} \int_{\{x_3 = t\} \cap \Sigma_1 \cap -z - \Sigma_2} f(x)g(-z - x)d\mathcal{H}^0(x)dtd\mathcal{H}^2(z).$$

Recalling the map p from before, we see that this last expression equals

$$\int_{-\epsilon}^{\epsilon} \int_{\Sigma_3} h(z)f(p(z, t))g(-z - p(z, t))d\mathcal{H}^2(z)dt. \tag{7.2.3}$$

We take $h = c_h \mathbf{1}_{\{z\in\Sigma_3:\ |(z_1,z_2)|\le\delta\epsilon,\ |z_3|\le\epsilon\}}$ and simply adapt the other two functions to that support,

$$f = c_f \mathbf{1}_{\{p(z,t):\ |t|<\epsilon, z\in \operatorname{supp} h\}}, \quad g = c_g \mathbf{1}_{\{-z-p(z,t):\ |t|<\epsilon, z\in \operatorname{supp} h\}}.$$

We choose c_f, c_g and c_h such that f, g and h have norm one. The degeneracy of the situation is important in what follows. Namely, the crucial observation is that f and g are again supported roughly in a region of size $[-\delta\epsilon,\delta\epsilon]\times[-\epsilon,\epsilon]$. While the support in the e_3-direction is clearly of size ϵ as p is a Lipschitz map, we have to be careful that changing z_3 by $O(\epsilon)$ can only introduce a $O(\delta\epsilon)$ variation in p_1 and p_2. Thus we estimate, for $p' = (p_1,p_2)$

$$\begin{aligned}|p'(z_1,z_2,z_3,t) - p'(z_1,z_2,0,t)| &= |T(p(z,t))' - T(p(z',0,t))'| \\ &\le |h_2(-z_1-p_1,-z_3-t) - h_2(-z_1-p_1,-t)| \lesssim \delta|z_3| \le \delta\epsilon\end{aligned}$$

as claimed. Hence f, g and h are contained roughly in rectangles of area $\delta\epsilon^2$, which gives $c_f \sim c_g \sim c_h = \delta^{-\frac{1}{2}}\epsilon^{-1}$. Evaluating (7.2.3) yields, with some notational freedom regarding f and g,

$$\begin{aligned}(f*g*h)(0) &\sim \int_{-\epsilon}^{\epsilon}\int_{-\delta\epsilon}^{\delta\epsilon}\int_{-\epsilon}^{\epsilon} h(h(z_1),z_2,z_3)f(h(z_1),z_2,z_3,t)g(h(z_1),z_2,z_3,t)dz_2z_3dt \\ &\sim (c_h)^3\epsilon^3\delta = \delta^{-\frac{1}{2}}\end{aligned}$$

which we can make arbitrarily large as $\epsilon\to 0$. Note that the functions f, g and h are essentially characteristic functions of an interval of size $\epsilon\times\epsilon\delta$, which is useful to keep in mind for what follows. ☐

7.3 Ill-posedness results

Degenerate quadratic Klein-Gordon equations

Let us assume that m,n and o are positive real numbers, and that we have $m+n\le o$. Then (4.1) does not apply, and we show in fact that this is not merely a technical artefact. More precisely, consider the quadratic Klein-Gordon system in two spatial dimensions,

$$\begin{aligned} iu_t - \langle D\rangle_m u &= vw \\ iv_t - \langle D\rangle_n v &= uw \\ iw_t - \langle D\rangle_o w &= uv \end{aligned} \tag{7.3.1}$$

with appropriate initial data.

Using convolution estimates, we now show that this equation can not be treated by our methods. To see this, it suffices to look, for instance, at the third equation. After going through the analysis outlined in the last chapter (the assumptions of which would be satisfied in any reasonable setup using U^2 and V^2), we need to find

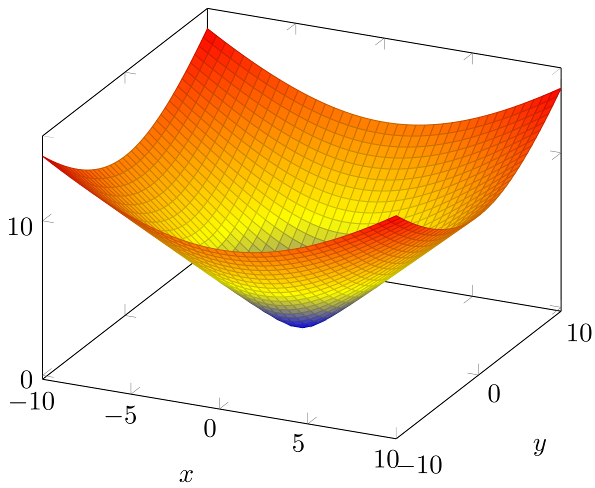

Figure 7.2: The characteristic hypersurface $\{\tau = \sqrt{1+|\xi|^2}\}$ in 2 + 1 dimensions.

$(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$\langle\xi\rangle_m + \langle\eta\rangle_n - \langle\xi+\eta\rangle_o = 0 = \det\begin{pmatrix} -1 & -1 & -1 \\ \frac{\xi}{\langle\xi\rangle_m} & \frac{\eta}{\langle\eta\rangle_n} & \frac{(\xi+\eta)}{\langle\xi+\eta\rangle_o} \end{pmatrix}$$

which gives rise to points on Σ_1 and Σ_2 associated to $\langle\cdot\rangle_m$ and $\langle\cdot\rangle_n$, which add up to a point on Σ_3 associated to $\langle\cdot\rangle_o$, for which the respective normals do not span the ambient space $\mathbb{R}^3$. A quick computation shows that for resonant points

$$\det\begin{pmatrix} -1 & -1 & -1 \\ \frac{\xi}{\langle\xi\rangle_m} & \frac{\eta}{\langle\eta\rangle_n} & \frac{\xi+\eta}{\langle\xi+\eta\rangle_o} \end{pmatrix} = \det\begin{pmatrix} -\langle\xi\rangle_m & -\langle\eta\rangle_n & \langle\xi+\eta\rangle_o \\ \xi & \eta & \xi+\eta \end{pmatrix} = 2\langle\xi+\eta\rangle_o \det\begin{pmatrix} \xi & \eta \end{pmatrix}$$

and hence ξ and η should be linearly dependent. Now we have

$$\langle\xi\rangle_m + \langle\eta\rangle_n - \langle\xi+\eta\rangle_o = 0 \iff (m+n)^2 - o^2 + 2(\langle\xi\rangle_m\langle\eta\rangle_n - \xi\cdot\eta - mn) = 0 \quad (7.3.2)$$

where $(m+n)^2 - o^2 \leq 0$. In the case $m + n = o$, we simply note that

$$2(\langle mt\rangle_m\langle nt\rangle_n - mt^2 - mn) = 0$$

and so we can pick our favorite tuple $\xi = mte_1$, $\eta = nte_1$, $t \geq 0$, to arrive at the conclusion. When $m + n < o$, then for $\xi = \eta = 0$, we have (7.3.2) < 0, and since

$$\langle mte_1\rangle_m\langle nse_1\rangle_n - mnts - mn = mn\left(\langle t\rangle\langle s\rangle - ts - 1\right) \to \infty$$

as $t, s \to \infty$, $\frac{t}{s} \to \infty$, by continuity we find ξ and η for which (7.3.2) vanishes, as desired. Hence, fixing ξ and η, we can invoke Proposition 7.4, let $\epsilon \to 0$ and obtain a contradiction to (7.1.1).

The Novikov-Veselov equation

The Novikov-Veselov equation

has the scaling given by

$$u_\lambda(t,x) = \lambda^2 u(\lambda^3 t, \lambda x),$$

which is critical in $\dot{H}^{-1}$. We contradict here a smooth scattering solution operator only in the homogeneous case; the same procedure is easier to carry out in the inhomogeneous setting.

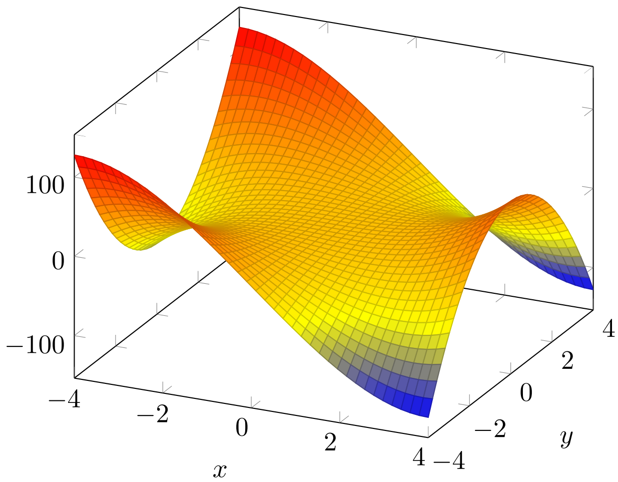

Figure 7.3: The characteristic hypersurface $\{\tau = \xi_1^3 - 3\xi_1\xi_2^2\}$ in $2+1$ dimensions.

Since the trilinear estimate (7.1.1) can only be valid when $s = -1$ due to the above scaling, we only need to treat that one case. Then, after a slightly modified derivation of (7.1.1) (using integration by parts and replacing h by $\partial^{-1}h$) we arrive at the inequality

$$\begin{aligned} &|(\mathcal{F}_{tx}e^{ith(D)}f * \mathcal{F}_{tx}e^{ith(D)}g * \mathcal{F}_{tx}e^{ith(D)}h)(0)| \\ &\qquad\qquad \lesssim \|f\|_{\dot{H}^{-1}(\mathbb{R}^2)}\|g\|_{\dot{H}^{-1}(\mathbb{R}^2)}\|h\|_{L^2(\mathbb{R}^2)} \end{aligned} \tag{7.3.3}$$

To find a degeneracy, we simply use the fact that the dispersion relation

$$h(\xi) = \xi_1^3 - 3\xi_1\xi_2^2 \qquad \xi \in \mathbb{R}^2$$

is odd in ξ_1 and even in ξ_2. Hence we can make our life simple and look for a resonant triple $(\xi, \eta, \xi+\eta)$ for which $\xi_2 = \eta_2 = 0$. Since then the second component of $\nabla h(\cdot)$ vanishes for ξ, η and $\xi+\eta$, we only need to make sure that $|\xi|, |\eta| > 0$ and that we have resonance. This is clear by symmetry when $\xi = -\eta \neq 0$ and in fact, this is the only case, as can be seen by checking that $h(\xi) + h(\eta) + h(-\xi-\eta) = 0$ is equivalent to

$$\eta_1(\xi_1^2 - \xi_2^2) + \xi_1(\eta_1^2 - \eta_2^2) = 2\xi_2\eta_2(\xi_1+\eta_1) \iff \eta_1\xi_1^2 + \xi_1\eta_1^2 = 0 \iff \xi_1 = -\eta_1$$

for $\xi, \eta \neq 0$.

We choose $\eta = (-1, 0) = -\xi$, and invoke the counterexamples of (7.4). This yields for (7.3.3) the lower bound

$$\epsilon^{-\frac{1}{2}} \|f_\epsilon\|_{L^2(\mathbb{R}^2)} \|g_\epsilon\|_{L^2(\mathbb{R}^2)} \|h_\epsilon\|_{L^2(\mathbb{R}^2)} \sim \epsilon^{-\frac{1}{2}}.$$

Since f_ϵ and g_ϵ are localized at frequency one and L^2 normalized, the $\dot{H}^{-1}$ norms on the right hand side of (7.3.3) are also comparable to one, leading to the contradiction $\epsilon^{-\frac{1}{2}} \lesssim 1$.

Quadratic Schrödinger equations

Now we consider the $2+1$ dimensional equation

$$iu_t - \Delta u = Q_\alpha(u)$$

where $Q_1(u) = |u|^2$ and $Q_2(u) = u^2$; we also show that chapter 5 can not be extended to cover nonlinearities of type Q_1 and Q_2 when a derivative is added somewhere. Q_1 appears to be the worse behaved nonlinearity: It is known that asymptotic scattering states basically have to vanish (see [ST06, IW12]) in two dimensions; also it is the only quadratic nonlinearity for which in 3D, global existence from small, localized data is not known (almost global existence holds). Hence it is not too surprising that we can arrive at a contradiction here if we assume that this equation fits into a nice fixed point setup. Q_2 is more delicate: global solutions and wave operators can be constructed under some assumptions on the initial data ([MTT03]). For that reason it would be more interesting to obtain a contradiction here; unfortunately, this doesn't work, and we see why that is first.

We deal with homogeneous settings, as a contradiction is easily obtained in an inhomogeneous setup. For Q_1 it will be possible to carry out the usual procedure; for Q_2 this fails, and we begin by investigating why.

The argument in section 7.1 naturally adapts to the homogeneous setting, with the same conclusions, but as we will see, some complications at frequency zero when closing the argument. This happens since resonance occurs only when zero is involved and the homogeneous Sobolev norms become very large as we test estimate (7.1.1).

Due to the scaling $u \mapsto u_\lambda = \lambda^2 u(\lambda^2 t, \lambda x)$ which is inherited by (7.1.1), we may assume that $s = -1$, as $\dot{H}^{-1}$ is the critical space associated to this scaling.

The case $Q_2(u) = u^2$. Here, $\Sigma_1 = \Sigma_2 = \mathbb{P}$, $\Sigma_3 = -\mathbb{P}$. For ξ, $\eta \in \mathbb{R}^2$ we have resonance if and only if

$$0 = |\xi|^2 + |\eta|^2 - |\xi + \eta|^2 = -2\xi \cdot \eta,$$

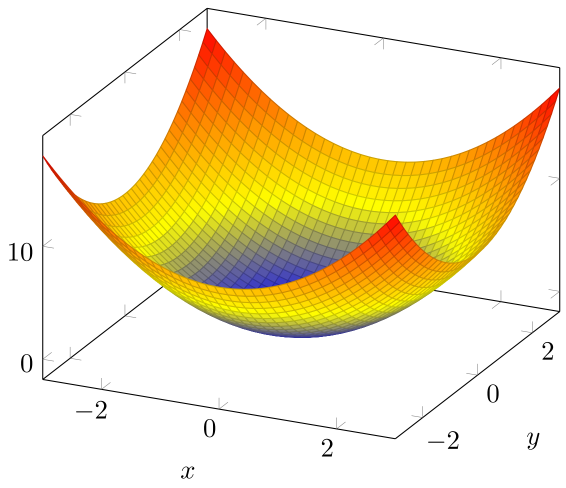

Figure 7.4: The characteristic hypersurface $\mathbb{P} = \{\tau = |\xi|^2\}$ in $2+1$ dimensions.

in other words, when ξ and η are orthogonal. We check the determinant

$$\left|\det\begin{pmatrix} -1 & -1 & 1 \\ 2\xi & 2\eta & -2(\xi+\eta) \end{pmatrix}\right| \sim |\xi||\eta| \sin\angle(\xi,\eta).$$

We see that this does not degenerate very often, only when part of the interaction comes from the origin. Hence, we can, for instance, take $\eta = 0$, $\xi = te_1$ for some $t > 0$, but since we are dealing with a homogeneous setup, we might as well set $\xi = e_1$ and check carefully the contributions of the Sobolev weights in eq. (7.1.1). The homogeneous version reads

$$\Big| \iint e^{ith_1(D)} f e^{ith_2(D)} g e^{ith_3(D)} h dx dt \Big| \\ \lesssim \|f\|_{\dot H^{-1}(\mathbb{R}^2)} \|g\|_{\dot H^{-1}(\mathbb{R}^2)} \|h\|_{\dot H^{1}(\mathbb{R}^2)}.$$

For the left hand side, (7.4) generically gives a behaviour of $\epsilon^{-\frac{1}{2}}$ when tested with the counterexamples belonging to the choice $\eta = 0$, $\xi = e_1$. On the right hand side however, the respective norms of f and h are reciprocal, and in effect $\|g\|_{\dot H^{-1}(\mathbb{R}^2)}$ remains. We can modify the counterexample a little bit to avoid a $2^{-10}([-\epsilon^2, \epsilon^2] \times [-\epsilon, \epsilon])$-neighborhood of zero, but still the norm grows too fast as $\epsilon \to 0$, at a rate of $\epsilon^{-\frac{3}{2}}$.

The case $Q_1(u) = u\bar{u}$. The analysis is very similar, but there is one crucial difference, namely that this time the roles of η and $\xi + \eta$ are switched: We will have $\xi = -\eta = e_1$, and thus f and g as given by the counterexample have $\dot H^{-1}$ norm comparable to one. The third function h on the other hand is localized in an ϵ-neighborhood of frequency zero, hence $\|h\|_{\dot H^1(\mathbb{R}^2)} \lesssim \epsilon$ and the contradiction $\epsilon^{-\frac{1}{2}} \lesssim \epsilon$ is obvious.

Quadratic nonlinearities with one derivative. A similar game can be played for the nonlinearities $Q_3(u) = \partial(u^2)$, $Q_4(u) = \partial(|u|^2)$, $Q_5(u) = u\partial\bar{u}$ or $\bar{u}\partial u$ and

$Q_6(u) = u\partial u$. These are scaling critical in $L^2(\mathbb{R}^2)$ and hence in the trilinear estimate, we have two $L^2(\mathbb{R}^2)$ norms and one $\dot{H}^{-1}(\mathbb{R}^2)$ norm for the factor which carries the derivative in the nonlinearity. Now we want to obtain lower bounds on the trilinear integral as before. Looking at the cases of Q_1 and Q_2, we need to localize precisely one factor around the origin. Since there are always two choices of such a factor, we may choose one that is not measured in $\dot{H}^{-1}$ and obtain an upper bound of order $\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}\|h\|_{L^2(\mathbb{R}^2)}$, compared to the divergent lower bound which has an additional factor of $\epsilon^{-\frac{1}{2}}$. This complements chapter 5: The techniques really do work only when both factors are conjugated.
Note however that when one splits the derivative according to the counterexamples so that they act as a null form on the resonant set, we can not close the argument. For instance, consider the nonlinearity $|\partial|^{\frac{1}{2}}u|\partial|^{\frac{1}{2}}u$; we would have to localize one of the factors near zero in frequency. Due to the half derivative, both choices introduce a growth of (more than) $\epsilon^{-\frac{1}{2}}$ in the upper bound, and no contradiction can be derived. This would suggest that this nonlinearity could effectly fall into category (2) below, even though this may be a naive guess.

7.4 Summary and outlook

We end this dissertation with an informal summary of the observed phenomena, and an outlook for further investigation. From the positive and negative results we have seen so far, the following loose categorization of a two-dimensional quadratic nonlinear equation with dispersion relation h has been obtained:

1. **no resonance.** By this we mean that the modulation equation $h(\xi) + h(\eta) \pm h(-\xi - \eta) = 0$ (where the choice of $\pm$ depends on the specific case) has no (or very few special) solutions. Depending on the quantitative behavior of that expression, it may then be possible to obtain global solutions for small data at low regularity. This was observed for admissible Klein-Gordon systems and the special derivative Schrödinger equation, as treated in chapters 4 and 5.

2. **only transversal resonance.** In the general case, the above resonance condition defines a $2n - 1 = 1$ dimensional set and hence, usually one has to expect resonance. As we have seen in the last section, the three surfaces involved need to be transversal on the set of resonant points. In that case, no counterexample to smoothness and scattering of the solution operator can be derived from eq. (7.2.1), and it may be possible to construct a smooth solution operator, albeit not with our techniques. See (7.4.1) below.

3. **degenerate resonance.** Finally, in the worst case, there is resonance and degeneracy in the transversality. Hence there can not be a smooth solution operator and scattering, but nevertheless global solutions may still exist. Recently, many such results were obtained using the method of space-time resonances, for which we refer to [Ger11] and the references therein.

Thus, it seems that one key point to understand is how the convolution estimate ties in with resonant behavior. One goal for which some hope may be justified is to treat, at least to some extent, a resonant but transversal situation. Unfortunately, there is no way to apply the convolution estimate directly: It will bound a trilinear interaction of free waves in terms of the initial data in $L^2(\mathbb{R}^2)$, but this implies only bounds in U^1. However, there is a toy problem below (due to Koch [Koc12]), where a variation on the function spaces gives a strong result.

The toy problem

We write (x, y) and (ξ, η) for the spatial and Fourier variables of $\mathbb{R}^2$ and consider the system

$$\begin{aligned} u_t + n_1 \cdot \nabla u &= \quad vw \\ v_t + n_2 \cdot \nabla v &= \quad uw \\ w_t + n_3 \cdot \nabla w &= \quad -uv, \end{aligned} \tag{7.4.1}$$

whose scaling critical space is $L^2(\mathbb{R}^2)$. The characteristic hypersurfaces are defined by $\tau + n_i \cdot \xi = 0$, with normals $(1, n_i)$ modulo a bounded factor. Hence the transversality condition for the three surfaces is

$$0 \neq \det \begin{pmatrix} 1 & 1 & 1 \\ n_1 & n_2 & n_3 \end{pmatrix} = \det(n_1 - n_3, n_2 - n_3).$$

Let $S(t) = (S_1(t), S_2(t), S_3(t))$ the linear evolution of the free system. Then we look for a solution in the space X with norm $\|f\|_X = \|S(-t)f(t)\|_{L^2_x L^\infty_t \times L^2_x L^\infty_t \times L^2_x L^\infty_t}$ to estimate (for instance, for the third equation) the Duhamel term $\int_0^t S_3(t - s)u(s)v(s)ds$. We switch coordinates $(x, y) \mapsto a(n_3 - n_1) + b(n_3 - n_2)$ (which is valid due to the transversality condition above) and obtain with $\tilde{u}(x, y) = \sup_s S_1(-s)u(x, y, s)$

$$\begin{aligned} S_3(-t) \int_0^t S_3(t-s)(uv)ds(a,b) &\leq \int S_3(-s)(|uv|) \leq \int S_3(-s)(S_1(s)\tilde{u}\ S_2(s)\tilde{v})ds \\ &\leq \|S_3(-s)S_1(s)\tilde{u}\|_{L^2_s} \|S_3(-s)S_2(s)\tilde{v}\|_{L^2_s} \\ &= \|\tilde{u}((a+s)(n_3 - n_1) + b(n_3 - n_2))\|_{L^2_s} \|\tilde{v}(a(n_3 - n_1) + (b+s)(n_3 - n_2))\|_{L^2_s} \\ &= \|\tilde{u}(s(n_3 - n_1) + b(n_3 - n_2))\|_{L^2_s} \|\tilde{v}(a(n_3 - n_1) + s(n_3 - n_2))\|_{L^2_s} \end{aligned}$$

and hence

$$\| \sup_t |S_3(-t) \int_0^t S_3(t-s)(uv)ds| \|_{L^2_{a,b}} \leq \|(u, v)\|_X.$$

This means that with the above, we can find global solutions for small data using the Banach fixed point theorem precisely when the three surfaces are transversal. While this is encouraging, it is not clear what more general strategy is behind this particular choice of function spaces in this special case. Rewriting the above nonlinear estimate in the form

$$\|uv\|_{L^2_x V^1_t} \lesssim \|u\|_{L^2_x V^\infty_t} \|v\|_{L^2_x V^\infty_t}$$

and using the embeddings $V^1 \subset U^2 \subset V^2 \subset V^\infty$ we get

$$\|uv\|_{L_x^2 U_t^2} \lesssim \|u\|_{L_x^2 V_t^2} \|v\|_{L_x^2 V_t^2}$$

which is weaker than the "usual" (but in this case, false - see Proposition 7.4) estimate

$$\|uv\|_{U_t^2 L_x^2} \lesssim \|u\|_{V_t^2 L_x^2} \|v\|_{V_t^2 L_x^2}$$

since $U_t^2 L_x^2 \subset L_x^2 U_t^2 \subset L_x^2 V_t^2 \subset V_t^2 L_x^2$. This puts the above estimates in a more familiar context, but it is not clear where to go from here.

Appendix A

Standard tools from Analysis

Theorem A.1 (Coarea formula)**.** *Let $k > n$, $U \subset \mathbb{R}^n$ open and $f : U \to \mathbb{R}^k$ Lipschitz. Then, for $g \in L^1_{loc}(U)$,*

$$\int_U g(x)|J(x)|dx = \int_{\mathbb{R}^k} \int_{f^{-1}(y)} g(x) dH^{n-k}(x) dy$$

where $J = \det_k(Df \cdot (Df)^t)$ is the k-dimensional Jacobian of f.

Proof. see [Fed69]. □

Theorem A.2 (Stationary phase)**.** *Let $a \in C_0^\infty(\mathbb{R}^n)$, $\phi \in C^\infty(\mathbb{R}^n)$ and let $x_0 \in$* supp *a a nondegenerate critical point of ϕ, i.e.*

$$D\phi(x_0) = 0, \qquad \det D^2\phi(x_0) \neq 0.$$

Furthermore, assume that there are no other critical points in supp *a. Then, for any $\lambda > 0$ and denoting*

$$I_\lambda = \int e^{i\lambda\phi(x)} a(x) dx$$

we have the estimate

$$\left| I_\lambda - \lambda^{-\frac{n}{2}} (2\pi)^{\frac{n}{2}} e^{i\lambda\phi(x_0)} e^{\frac{i\pi}{4} \operatorname{sgn} D^2\phi(x_0)} |\det D^2\phi(x_0)|^{-\frac{1}{2}} \right| \lesssim \lambda^{-\frac{n+2}{2}} \|a\|_{C^{n+3}(\mathbb{R}^n)}.$$

Proof. see [Zwo12] □

Theorem A.3 (Hardy-Littlewood-Sobolev inequality)**.** *Let $0 < \gamma < n$, $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = 1 - \frac{\gamma}{n}$. Then for all $f \in L^p(\mathbb{R}^n)$ we have*

$$\|f * |\cdot|^{-\gamma}\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. see [Ste93] □

Theorem A.4 (Minkowski's inequality)**.** *Let $1 \leq p \leq \infty$, (X_i, μ_i) measure spaces* *and* $F : X_1 \times X_2 \to \mathbb{R}$ *measurable. Then we have*

$$\left\| \int_{X_1} F(x,\cdot) d\mu_1(x) \right\|_{L^p(X_2)} \leq \int_{X_1} \|F(x,\cdot)\|_{L^p(X_2)}.$$

Theorem A.5 (Littlewood-Paley inequality)**.** *Let* $1 < p < \infty$*. Then*

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \left(\sum_N |P_N f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

Proof. see chapter VI, 7.14 in [Ste70]. □

As an immediate application of the last two statements, we obtain

Corollary A.6. *Let* $2 \leq r < \infty$*. Then we have*

$$\dot{B}^0_{2,r}(\mathbb{R}^n) \subset L^r(\mathbb{R}^n).$$

Proof. We have

$$\|f\|_{L^r(\mathbb{R}^n)} \sim \left\| \sum_N |P_N f|^2 \right\|^{\frac{1}{2}}_{L^{\frac{r}{2}}(\mathbb{R}^n)} \leq \left(\sum_N \| |P_N f|^2 \|_{L^{\frac{r}{2}}(\mathbb{R}^n)} \right)^{\frac{1}{2}} = \|f\|_{\dot{B}^0_{2,r}}.$$

□

Interpolation spaces

Since interpolation spaces are used only briefly and mainly to streamline an otherwise slightly more technical argument in this work, we give here only the most basic definition. An exhaustive treatment of the theory can be found in [Tri83].

Definition A.7 (Real interpolation space)**.** Let A_0 and A_1 Banach spaces contained in some larger Banach space A. Denote

$$K(t,a) = \inf_{a=a_0+a_1} \|a_0\|_{A_0} + t\|a_1\|_{A_1}.$$

Then, the real interpolation spaces $(A_0, A_1)_{\theta,q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$ are defined through the norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty t^{-\theta q} K(t,a)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

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Summary of this dissertation

In this dissertation, we treat problems related to the small data global existence theory of some *dispersive PDEs*. That is, we try to abstractly construct solutions which exist globally in time and have “good” properties, under a smallness condition on the initial data, typically given at time $t = 0$. What is special here is that we do not impose strong decay on this data, that is, we only assume that they are L^2 based Sobolev functions. More precisely, we treat

1. the Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = Q(u, \bar{u})$$

with a quadratic polynomial Q, mass $m > 0$ and initial data $(u_0, u_1) \in H^{s_0}(\mathbb{R}^n) \times H^{s_0-1}(\mathbb{R}^n)$ for some $s_0 = s_0(n)$, $n \geq 2$. We can also treat systems under a condition on the masses involved in the nonlinear interactions.

2. the quadratic nonlinear Schrödinger equation

$$iu_t - \Delta u = \bar{u}\partial_{x_1}\bar{u}$$

with initial data in the scaling critical space $\dot{H}^{\frac{n-2}{2}}(\mathbb{R}^n)$, $n \geq 2$.

3. the modified Novikov-Veselov equation in two space dimensions,

$$u_t + (\partial^3 + \bar{\partial}^3)u = N_{mNV}(u)$$

where $\partial = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$. The nonlinearity N_{mNV} is cubic and contains roughly one derivative. Again the initial data come from the scaling critical space, which in this case is $L^2(\mathbb{R}^2)$.

For each of the above equations and initial data from a sufficiently small ball around the origin, we construct global solutions which scatter and depend smoothly on the initial data, using a fixed point argument.

In the second part of this work, we turn towards negative results and start with the observation that a solution operator constructed by the techniques used in the proofs of the statements above imply that there is a smooth scattering operator, which in turn shows that a trilinear spacetime interaction of free waves can be bounded by their inital data.
Such an estimate is very close to so-called convolution estimates in Fourier spacetime, for which the behavior is known, and we can use this to derive contradictions in some cases. This is related to the concept of time resonance, and we can show that the results above for the Klein-Gordon and Schrödinger equations are sharp in some sense. Regarding the modified Novikov-Veselov equation, we show a negative result for the related Novikov-Veselov equation, for which the nonlinearity is replaced by a quadratic nonlinearity containing roughly one derivative.