Geometric Structures arising from Partial Differential Equations

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Wir bestehen nur aus Ideen, die in uns aufgetaucht sind und die wir verwirklichen wollen, die wir verwirklichen müssen, weil wir sonst tot sind, so Roithamer. Jede Idee und jede Verfolgung einer Idee in uns ist das Leben, so Roithamer, Ideenlosigkeit ist der Tod.

Thomas Bernhard, *Korrektur*

Contents

CHAPTER 1

Acknowledgements and Summary

1. Acknowledgements

No man is an Iland, intire of it selfe
John Donne, 1624

It is a great pleasure to acknowledge and thank a profound number of people and events that contributed to the creation of this thesis in a variety of different ways. First and foremost, I am grateful to my advisor Prof. Dr. Herbert Koch for his infinite patience with my ever-changing obsessions, his willingness to contribute some piece of heuristic insight and his unrelenting attention to detail even in subjects removed from his main line of research. His standards in mathematical research and his taste in what constitutes a good statement have had a surprising impact on me – and so has his approach to some of the more meta-mathematical issues regarding the mathematical community at large.

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2. Summary

This thesis is concerned with a variety of different topics and any summarizing title is bound to be a relatively crude approximation of the content. The grand unifying theme, however, is the connection between analysis, especially the analysis of differential operators, and elementary geometric concepts.

The content of **Chapter 1** is concerned with one particular dispersive equation, the defocusing generalized Korteweg - de Vries equation

$$u_t + u_{xxx} - (|u|^{p-1}u)_x = 0,$$

where $u : \mathbb{R} \to \mathbb{R}$ is real-valued, $p > 1$ and we assume the initial datum to be in the energy space $u_0 \in H^1(\mathbb{R})$. Since there is not the slightest reason to assume otherwise, the defocusing structure of the

equation should imply scattering. We are aware of four papers on the subject: Tao [**Tao07**] gave a monotonicity formula and a dispersion estimate ruling out strong spatial concentration in space for all time. Quite recently, in 2012, Kwon & Shao [**KS**] managed to give a simple adaption of Tao's argument excluding a wider notion of concentration. We give slightly improved results incorporating time dependency – our approach uses a different notion of dispersion and bootstrapping to get slightly improved results; a particularly interesting aspect of the argument is that the conditions of some of its arguments seem in perfect alignment with basic heuristics concerning the equation (giving rise to the hope of having captured some aspects of the true dynamics). The particular case $p = 5$ has recently been solved completely by Dodson [**Dod13**] (relying on particular structures having been established earlier by Killip, Kwon, Vishan & Shao [**KKSV12**]). The full problem is still wide open and – as one of the simplest possible example of a truly generic nonlinear PDE – of great interest.

Chapter 2 is concerned with Laplacian eigenvalues

$$-\Delta u = \lambda u$$

in the general setting of a smooth, compact manifold (M, g). The problem is extremely foundational (aspects of it dating back to the 1850s) and its study has motivated and enrichened entire subjects (such as, for example, early spectral theory). Our question dates back to S.-T. Yau [**Yau82**], who asked for bounds on the nodal set of such an eigenfunction and conjectured that

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \sim \lambda^{\frac{1}{2}}.$$

The currently best lower bounds in dimensions $n \geq 3$ were derived in the last three years by Sogge & Zelditch [**SZ11**, **SZ**] and Colding & Minicozzi [**CM11**] in two different approaches. We give a third approach and reach the same results with a completely new type of argument that is based on an essentially combinatorial viewpoint of stochastic processes associated to diffusion equations: the counting of Brownian motion particles being absorbed/reflected by the boundary of a nodal set. The approach has a series of consequences and suggests a long list of connections between *heat content*, a quantity studied (primarily in an asymptotic sense) in global analysis, and traditional elliptic estimates.

The **third Chapter** is much shorter – it is inspired by the second chapter insofar as Laplacian eigenfunctions behave essentially as harmonic functions on small length scales. We study the regularity of level sets of harmonic functions defined in the unit disk and prove assuming a certain (necessary) condition the sharp upper bound on their mean curvature in the origin:

$$\kappa \leq 8.$$

Perhaps surprisingly, there is a unique extremizing function for the problem, whose closed form expression we are able to derive. We conjecture this family of explicit functions to also extremize a related geometric problem in the set of harmonic functions. Partial inspiration is an earlier paper by De Carli & Hudson [**DCH10**], who proved $\kappa \leq 24$ under the same assumptions.

Chapter 4 is certainly the odd one out as it barely contains a single differential operator (however, there is one derivative with respect to time). We stumbled across it by accident and were enchanted by its geometric subtlety: given a Banach space B and a L-Lipschitz continuous functions $f : B \to B$, what is the smallest possible period P of a periodic solution to the ordinary differential equation

$$\dot{y}(t) = f(y(t))?$$

The question originates in a 1969 paper of Yorke, who derived the optimal lower bound $PL \geq 2\pi$ in Hilbert spaces. The question for Banach spaces is more subtle and there was series of improvements from $PL \geq 4$ [**LY71**] over $PL \geq 4.5$ [**BM87**] to $PL \geq 6$ [**BFM86**]. Subsequently, this last bound was shown to be sharp by means of an explicit construction in $B = L^1([0,1]^2)$, which remains the only known example today. Certainly, this example should be rather special. Our contribution is to show that the constant 6 is not sharp in strictly convex Banach spaces and to derive explicit quantitative lower bounds for a particular class of strictly convex Banach spaces, those being $B = \ell_p^n$ and $B = L^p(M, d\mu)$ for fairly general sets M and measures μ and p close to 2. These results are joint work with James Robinson (Mathematics Institute, University of Warwick) and Michaela Nieuwenhuis (Mathematical Institute, Oxford).

Chapter 5 is maybe the most purely geometric chapter – although it is an entirely inspired by a classical problem in elliptic PDEs and lives up to its initial purpose in being applicable to said problem. Given an open, bounded domain $\Omega \subset \mathbb{R}^2$, consider a Laplacian eigenfunction

$$-\Delta u = \lambda u$$

with Dirichlet data: what bounds can be proven on the number of connected subsets of

$$\Omega \setminus \{x \in \Omega : u(x) = 0\}?$$

This *number of nodal domains* is classical problem with the first result going back to Courant [**Cou23**] in 1923. Around thirty years later, Pleijel [**Ple56**] noticed a simple argument that allowed for an improvement. His argument rests on a geometrically unreasonable assumption (as is mentioned by Polterovich [**Pol09**]): this was then exploited in 2013 by Bourgain [**Bou13**], who gave a minor improvement. We are interested in the abstract geometric principle at work and prove an elementary uncertainty principle: a partition of a set Ω into many small sets requires that either some sets are not discs or that some sets are bigger than others. Indeed, this uncertainty principle comes with a *universal* constant, which we show to be bigger than $6 \cdot 10^{-5}$ and which we consider likely to be between 0.01 and 0.1. We consider improving the constant a highly natural and challenging problem, which would require additional insight into the structure of sets in the plane.

CHAPTER 2

Dispersion Properties of the Defocusing Generalized Korteweg-de Vries equation

1. Introduction

No mar tanta tormenta, e tanto dano,
Tantas vezes a morte apercebida!
In the sea, such tempest and damage,
So often a glimpse of Death before your eyes!
Luís de Camões, *Os Lusíadas*

1.1. Historical aspects. In this chapter we will study certain properties of an equation, whose 'ancestor equation' enjoys a rich history and had quite an impact on mathematics on a large scale. The story begins in 1834, when the Englishman John Scott Russell observed a curious phenomenon.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped–not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation. (Russell [**Rus44**])

The observation was met with opposition by people like Airy and Stokes – it was only explained in 1895, when the *Korteweg-de Vries (KdV) equation* [**KdV95**]

$$\partial_t u + \partial_{xxx} u + \partial_x (u^2) = 0 \quad \text{with} \quad u(t, x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$$

was put forth as a model: among its solutions are solitary waves of the type described by Russell. These solitary waves were shown by Russell himself to have the additional surprising property of being able to pass through one another undisturbed.

A major impetus was the discovery of a Lax pair [**Lax68**] and the fact that there are infinitely many quantities invariant under the evolution [**MGK68**]. Using the so-called inverse scattering machinery, the long-time behavior of KdV is understood: for sufficiently smooth initial data, the solution decouples into a radiation term moving to the left and independent solitons moving to the right (with different speeds depending on their height).

1.2. gKdV. The abundance of structure in KdV has allowed breakthroughs, which heavily rely on particular structure being present and are not stable under perturbations. A natural generalisation is the *generalized Korteweg - de Vries equation* (gKdV)

$$\partial_t u + \partial_{xxx} u + \partial_x(|u|^{p-1}u) = 0 \quad \text{with} \quad u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R},\ p \in \mathbb{R}_{>1}.$$

It has conservation laws for mass and energy

$$\begin{aligned} M(u) &= \int_{\mathbb{R}} u(t,x)^2 dx \\ E(u) &= \int_{\mathbb{R}} \frac{1}{2} u_x(t,x)^2 - \frac{1}{p+1} |u(t,x)|^{p+1} dx \end{aligned}$$

as well as a scaling symmetry

$$u(t,x) \to \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right).$$

The gKdV equation continues to have solitary waves – their existence can be easily proven: the problem of minimizing the energy under the constraint of fixed mass admits a unique minimizer (up to translation). As it turns out, the case $p = 3$ (*modified Korteweg - de Vries equation*, mKdV) is intimately linked with the KdV via the *Miura transform* [**Miu68**]

$$M(v) = \partial_x v + v^2,$$

which maps solution of mKdV to solutions of KdV. It is not invertible but things are not entirely hopeless [**KPST05**] and this connection has been fruitfully exploited (see e.g. Buckmaster & Koch [**BK11**] or Merle & Vega [**MV03**]). The equation describes a balance between dispersion and nonlinear effects: the critical exponent is $p = 5$. For $p < 5$ (and small initial mass for $p = 5$), mass and energy imply a bound on the H^1 norm. For $p = 5$ we can consider the ground state, i.e. the function $Q \in H^1(\mathbb{R})$ satisfying

$$Q_{xx} + Q^5 = Q,$$

which can be explicitely written as

$$Q(x) = \frac{3^{\frac{1}{4}}}{\cosh^{\frac{1}{2}} 2x}.$$

It was then shown by Merle [**MV03**] that there exists a small $\alpha_0 > 0$ such that any initial datum $u_0 \in H^1(\mathbb{R})$ with

$$\|u_0\|_{L^2}^2 < \|Q\|_{L^2}^2 + \alpha_0$$

and negative energy blows up in $H^1(\mathbb{R})$ (in finite or infinite time). The literature on this subject is *extensive.*

The issue of dispersion in the focusing case has also produced very interesting results of a different kind. We sketch a Liouville-type result due Martel & Merle [**MM00**] for the L^2-critical case $p = 5$: there is an $\alpha_0 > 0$ such that if

$$\|u(0) - Q\|_{H^1} \le \alpha_0,$$

where Q is a soliton and u is assumed to satisfy $c_1 \le \|u\|_{H^1} \le c_2$ for all times, then the property of L^2-compactness

$$\exists\ x(t) : \mathbb{R} \to \mathbb{R}\ \forall \varepsilon > 0\ \exists R > 0\ \forall t > 0 \qquad \int_{|x-x(t)|>R} u(t,x)^2 dx \le \varepsilon$$

implies that $u(0)$ is a soliton itself. This gives a very nice characterization of solutions whose L^2-norm lives concentrated around a fixed point in space for all time.

1.3. Defocusing gKdV. The developement for gKdV mirrors that of KdV: the developement of a very refined and highly attuned machinery, which does not seem to easily translate to other problems. We mirror previous developements by considering yet another equation, the *defocusing generalized Korteweg - de Vries equation*

$$\partial_t u + \partial_{xxx} u - \partial_x(|u|^{p-1}u) = 0$$

with $u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $p > 1$. Again, there are conservation laws for mass and energy

$$M(u) = \int_{\mathbb{R}} u(t,x)^2 dx$$
$$E(u) = \int_{\mathbb{R}} \frac{1}{2} u_x(t,x)^2 + \frac{1}{p+1}|u(t,x)|^{p+1} dx$$

and the scaling symmetry

$$u(t,x) \to \lambda^{-\frac{2}{p-1}} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right).$$

Note that the sign in the energy is reversed: local existence in $H^1(\mathbb{R})$ follows from the work of Kenig-Ponce-Vega [**KPV93**] and the conservation laws then imply global existence in that space. The different sign destroys the variational structure of the energy seen as a functional: the problem of minimizing the energy for fixed mass has no solution. It seems extremely reasonable to suspect scattering: vaguely put, the dispersion effect cannot be counterbalanced by the nonlinearity because of the sign. In the L^2-case $p = 5$ scattering has been proven by Dodson [**Dod13**], however, the general case is still wide open.

1.4. Main question. Our main question now is the following: what can be said about the long-time behavior of the defocusing generalized Korteweg – de Vries equation? In particular, is it possible to exclude behavior resembling that of solitary waves? While we have arrived at this questions from a historical perspective, it seems worthwhile to remark that – at least superficially – the equation certainly looks like it should be one of the simpler 'generic' nonlinear PDEs: 'simple' because it is real-valued, one-dimensional and has an algebraic nonlinearity and 'generic' because there is no additional overarching structure to assist us. Certainly, fewer derivatives would be even simpler but this is the first case, where truly dispersive phenomena appear (∂_x = transport, ∂_{xx} = heat).

2. Previous Results

2.1. Tao's monotonicity. The first result is a monotonicity formula by Tao, which – at the moment – is the cornerstone of the entire theory. The monotonicity formula is based on the notion of normalized centers of mass and energy, which are easily defined via

$$\langle x \rangle_M := \frac{1}{M}\int_{\mathbb{R}} x u^2 dx \quad \text{and} \quad \langle x \rangle_E := \frac{1}{E}\int_{\mathbb{R}} x \left(\frac{1}{2}u_x^2 + \frac{1}{p+1}|u|^{p+1}\right) dx,$$

where M and E denote mass and energy of the solution, respectively. A heuristic approach is given by assuming the linear dynamics to be dominant.

> From stationary phase one expects that portions of the solution u at frequency ξ should move left with velocity $-3\xi^2$, thus high frequency components should move left extremely fast compared to low frequency components. The energy density is more weighted towards high frequencies than the mass density [...] (Tao, [**Tao07**])

The monotonicity formula itself compares the speed of movement of the (normalized) centers of mass and energy, respectively.

Theorem (Tao's monotonicity formula, [**Tao07**])**.** *Let $p \geq \sqrt{3}$ and let u be a global-in-time Schwartz solution. Then*

$$\partial_t \langle x \rangle_E < \partial_t \langle x \rangle_M .$$

The condition $p \geq \sqrt{3}$ is required in the proof but might be an artifact. The statement itself (and all subsequent statements based on it) are formal in nature, however, it is expected that they all remain true for rougher solutions.

Proof summarized from [**Tao07**]. The proof is algebraic but not at all obvious. We introduce strictly positive quantities a, b, q, r, s by solving

$$a^2 M = \int_{\mathbb{R}} u_{xx}^2 dx \qquad b^2 M = \int_{\mathbb{R}} |u|^{2p} dx \qquad aqM = \int_{\mathbb{R}} u_x^2 dx$$
$$brM = \int_{\mathbb{R}} |u|^{p+1} dx \qquad absM = p \int_{\mathbb{R}} |u|^{p-1} u_x^2 dx.$$

The first step is showing that the matrix

$$\begin{pmatrix} 1 & q & r \\ q & 1 & s \\ r & s & 1 \end{pmatrix}$$

is positive semi-definite; this is done as follows. For any $\alpha, \beta, \gamma \in \mathbb{R}$

$$\int_{\mathbb{R}} |\gamma u(t,x) - \frac{\alpha}{a} u_{xx}(t,x) + \frac{\beta}{b} |u(t,x)|^{p-1} u(t,x)|^2 dx \geq 0.$$

Using partial integration, the integrand can be expanded into

$$M\left(\gamma^2 + \alpha^2 + \beta^2 - 2\gamma\alpha r - 2\gamma\beta s - 2\alpha\beta q\right)$$

and this implies the positivity of the matrix. In turn, positive of the matrix implies (via determinants and minors)

$$0 < q, r, s < 1 \qquad \text{and} \qquad 1 - q^2 - r^2 - s^2 + 2qrs \geq 0.$$

The claimed monotonicity can then be equivalently rewritten as

$$EM(\partial_t \langle x \rangle_M - \partial_t \langle x \rangle_E) = \frac{3}{2}(1-q^2)a^2 + \left(2s - \frac{p+3}{p+1} qr\right) ab + \frac{1}{2}\left(1 - \frac{4p}{(p+1)^2} r^2\right) b^2 > 0.$$

Following Tao, $p > 1$ implies

$$\frac{4p}{(p+1)^2} < 1$$

and it remains to show that

$$\frac{3}{2}(1-q^2)a^2 + \left(2s - \frac{p+3}{p+1} qr\right) ab + \frac{1}{2}\left(1 - r^2\right) b^2 \geq 0.$$

The quadratic formula (and positivity of a and b) reduce that problem to showing

$$\frac{p+3}{p+1} qr - 2s \leq \sqrt{3(1-q^2)(1-r^2)}.$$

Now, for $p \geq \sqrt{3}$ (this is the only place where this assumption enters)

$$\frac{p+3}{p+1} \leq \sqrt{3},$$

it suffices to show

$$s \geq \sqrt{\frac{3}{4}} \left(qr - \sqrt{(1-q^2)(1-r^2)}\right).$$

Completion of squares in $1 - q^2 - r^2 - s^2 + 2qrs \geq 0$ gives

$$(s - qr)^2 \leq (1-q^2)(1-r^2)$$

and therefore

$$s \geq qr - \sqrt{(1-q^2)(1-r^2)}.$$

s is nonnegative and this yields the statement. $\square$

We see that the assumption of global-in-time-Schwartz is required to ensure that a, b and s are finite (which, by conservation laws, is enough to ensure that all variables are finite). The second observation is that $p \geq \sqrt{3}$ comes from purely algebraic restrictions: we are interested in the set

$$\left\{(q,r,s) \in [0,1]^3 : \frac{p+3}{p+1}qr - 2s \leq \sqrt{3(1-q^2)(1-r^2)}\right\}$$

and need to ensure that any sufficiently smooth function gives rise to q, r, s in that set.

2.2. Tao's dispersion statement. Tao uses the monotonicity formula to derive a dispersion estimate.

Theorem (Tao, [**Tao07**]). *Let $p \geq \sqrt{3}$ and let u be a global-in-time Schwartz solution. Then*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} |x - x(t)| \left(u(t,x)^2 + \frac{1}{2}u_x(t,x)^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\right) dx = \infty$$

for any function $x : \mathbb{R} \to \mathbb{R}$.

This excludes solutions having their L^2- and $\dot{H}^1-$mass strongly concentrated around some center $x(t)$. The proof relies on a (non-explicit) quantitative refinement of the monotonicity formula. We follow its presentation, which uses Landau symbols $f = O(g)$ (meaning $f \leq cg$ for some constant $c > 0$), $f = \Omega(g)$ (meaning $f \geq cg$) and $f = \Theta(g)$ (meaning $f = O(g)$ and $f = \Omega(g)$).

Proof summarized from [**Tao07**]. The solution is Schwartz and not zero, therefore

$$M, E = \Theta(1).$$

The Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}} |u(t,x)|^{p+1} dx \lesssim \left(\int_{\mathbb{R}} |u(t,x)|^2 dx\right)^{\frac{p+3}{4}} \left(\int_{\mathbb{R}} u_x(t,x)^2 dx\right)^{\frac{p-1}{4}}$$

implies

$$\int_{\mathbb{R}} u_x^2 dx = \Theta(1).$$

Sobolev embedding

$$\|u\|_{L^\infty} \lesssim M^{\frac{1}{4}} E^{\frac{1}{4}}$$

implies $\|u\|_{L^\infty_{t,x}} = O(1)$ and thus, combined with mass conservation,

$$\left(\int_{\mathbb{R}} |u(t,x)|^q dx\right)^{\frac{1}{q}} = O(1)$$

for all $2 \leq q \leq \infty$. Suppose now that

$$\int_{\mathbb{R}} |x - x(t)| \left(u(t,x)^2 + \frac{1}{2}u_x(t,x)^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\right) dx = O(1).$$

This implies that almost all the mass needs to be concentrated at some fixed scale around $x(t)$. At the same time mass is conserved and the bound on the gradient implies that it cannot be too tightly concentrated around $x(t)$, thus

$$\int_{x=x(t)+O(1)} u(t,x)^2 dx = \Theta(1).$$

The uniform bound on u and Hölder's inequality yield the stronger statement

$$\left(\int_{\mathbb{R}} |u(t,x)|^q dx\right)^{\frac{1}{q}} = \Theta(1)$$

for all $2 \leq q \leq \infty$. Recall, from the previous proof, the definition

$$br = \frac{1}{M}\int_{\mathbb{R}} |u|^{p+1} dx.$$

Therefore $br = \Theta(1)$. Reviewing the previous algebraic proof, this gives

$$\partial_t \langle x \rangle_E < \partial_t \langle x \rangle_M - \Omega(1).$$

This certainly contradicts the assumption

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |x - x(t)| \left(u(t,x)^2 + \frac{1}{2} u_x(t,x)^2 + \frac{1}{p+1} |u(t,x)|^{p+1} \right) dx < \infty.$$

□

As Tao remarks, the proof could be motivated quantitative to yield something of the type

$$\sup_{t \in I} \int_{\mathbb{R}} |x - x(t)| \left(u(t,x)^2 + \frac{1}{2} u_x(t,x)^2 + \frac{1}{p+1} |u(t,x)|^{p+1} \right) dx = \Omega(|I|^c)$$

for all compact time intervals I with $|I| \geq 1$ and some explicitely computable c depending on p.

2.3. Kwon-Shao dispersion estimate. Tao's dispersion estimate only excludes very strong notions of concentration as it puts a weight not only u but also u_x. A recent observation due to Kwon & Shao [**KS**] circumvents this difficulty at the cost of increasing the weight on u.

Theorem (Kwon & Shao, [**KS**])**.** *Let $p \geq \sqrt{3}$ and let u be a global-in-time Schwartz solution. Then, for any function $x : \mathbb{R} \to \mathbb{R}$,*

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}} (x - x(t))^2 u(t,x)^2 dx = \infty.$$

In particular, if a solution satisfies

$$|u(t,x)| \lesssim \frac{1}{|x - x(t)|^{3/2+\varepsilon}}$$

for some $x(t) : \mathbb{R} \to \mathbb{R}$, then $u \equiv 0$. The key notion of the proof is that the variance-type functional is minimized for $x(t) = \langle x \rangle_M$ and it is sufficient to prove the statement for that particular expression.

The published proof in the preprint [**KS**] has a (very) minor gap. We informed the authors about this and are in agreement that it can be filled by more closely emulating the compactness considerations in Tao's argument – the journal version of the paper (not yet appeared) will surely fill out the details on this. We note that the gap can also be filled using our version of a refined monotonicity statement, which is presented further below.

Argument summarized from [**KS**]. A simple computation yields

$$\frac{d}{dt} \int_{\mathbb{R}} (x - \langle x \rangle_M)^2 u(t,x)^2 dx = -12E(\langle x \rangle_E - \langle x \rangle_M) - \frac{4p-12}{p+1} \int_{\mathbb{R}} |u(x)|^{p+1} (x - \langle x \rangle_M) dx.$$

Under the assumption that the statement were incorrect, the second quantity can be shown to be bounded using Sobolev embedding and conservation laws as above. The first quantity is monotonically increasing by Tao's monotonicity. □

However, and as such the argument is incomplete, monotonically increasing does not necessary imply that the quantity will ever be positive.

2.4. Related results. There are several results of a related nature, which should be mentioned. de Bouard & Martel [**dBM04**] as well as Laurent & Martel [**LM03**] study the focusing case. Their arguments heavily employ the structure of the focusing nonlinearity and are entirely different in nature (the most obvious example being not even existence of solitons but the fact that mass moves to the right instead of the left). The failure of scattering in the defocusing L^2-critical case $p = 5$ has been investigated by Killip, Kwon, Shao & Visan [**KKSV12**]: they, following the induction on energy approach pioneered by Bourgain [**Bou99**] and relying on a theorem of Dodson [**Dod12**] describe three possible obstructions to scattering, one of which is a soliton-type solution. Their description does not include any decay properties of that soliton-type solution but it is quite possible [**Vis**] that even very weak decay bounds can be bootstrapped and, in combination with the Kwon-Shao or our dispersion estimate, exclude this scenario. In the special case $p = 5$, this has recently been carried out by Dodson [**Dod13**] relying on the structure provided by Killip, Kwon, Shao & Visan [**KKSV12**].

3. Our results

3.1. Summary. Our approach rests on three ingredients: a refined version of Tao's monotonicity formula

$$\partial_t \langle x \rangle_M - \partial_t \langle x \rangle_E \gtrsim_p \frac{1}{EM^3} \left(\int_{\mathbb{R}} |u(x)|^{p+1} dx \right)^2,$$

a new dispersion functional $I : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ given by

$$I(f) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)^2 (x-y)^2 f(y)^2 dx dy$$

and the insight that these quantities allow for bootstrapping. This allows us to derive some improved nonexistence results and some additional statements about the dynamics of the equation. Along the way, we encounter a very enjoyable elementary inequality: for $p > 1$, there exists $c_p > 0$ such that for any function $u \in L^2(\mathbb{R})$

$$\left(\int_{\mathbb{R}} u^2 x^2 dx \right) \left(\int_{\mathbb{R}} |u|^{p+1} dx \right)^{\frac{4}{p-1}} \geq c_p \left(\int_{\mathbb{R}} u^2 dx \right)^{\frac{3p+1}{p-1}}.$$

Additionally, there exists a minimizer. This inequality is very basic. We provide three different proofs and prove the existence of exrtremizers. As was pointed out to us by Soonsik Kwon (personal communication), the existence of extremizers combined with a Lagrange multiplier approach immediately yields their closed form expression and the sharp constant. This inequality – in some sense made precise further below – nicely complements Gagliardo-Nirenberg and the uncertainty principle.

3.2. Results on the dynamics. Our main result is a sublevel estimate formulated in terms of the dispersion functional $I : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ given by

$$I(f) := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x)^2 (x-y)^2 f(y)^2 dx dy.$$

Similar dispersion functionals have been also considered by Colliander, Grillakis & Tzirakis [**CGT07**] as well as Planchon & Vega [**PV09**] for the NLS. Our statement relies on the following observation.

Lemma (Refined monotonicity formula)**.** *Let $u(t,x)$ be a global-in-time Schwartz solution to the defocusing gKdV for some $p \geq \sqrt{3}$. Then*

$$\partial_t \langle x \rangle_M - \partial_t \langle x \rangle_E \gtrsim_p \frac{1}{EM^3} \left(\int_{\mathbb{R}} |u(x)|^{p+1} dx \right)^2.$$

This refined monotonicity also immediately implies Tao's dispersion estimate and fills the gap in the Kwon-Shao argument. As it relies on Tao's monotonicity, it inherits all its requirements. It seems exceedingly likely that all these statements hold for a much rougher class of solutions.

Theorem (Sublevel estimate)**.** *Let $p \geq \sqrt{3}$ and let u be a global-in-time Schwartz solution. Then*

$$|\{t > 0 : I(u(t)) \leq z\}| \lesssim_{u(0)} z^{\frac{p}{2}}.$$

The precise dependence of the implicit constant on the initial data follows from the proof and is related to mass, energy, the centers of mass and energy but nothing else (and could be made fairly explicit). This result yields the following improved results on the nonexistence of solitons.

Corollary. *Let $\varepsilon > 0$ be fixed. Under the assumptions of Theorem 1, if*

$$|u(t,x)| \lesssim \frac{(1+t)^{\frac{2}{p}-\varepsilon}}{|x - x(t)|^{\frac{3}{2}+\varepsilon}}$$

for some $x : \mathbb{R} \to \mathbb{R}$*, then* $u \equiv 0$.

For the particular case of the mKdV ($p = 3$), some algebraic simplifications immediately imply that the functional $I(u(t))$ is convex – this has already been observed by Tao [**Tao07**, Remark 1.6.] and can also be observed in the Kwon & Shao result, where the error term cancels for $p = 3$. At the core of the proof for the general case is a differential inequality for $I(u(t))$, which has subtle implications for the dynamics of the equation.

Corollary. *Let* $p \geq \sqrt{3}$ *and let* u *be a global-in-time Schwartz solution. There exists a constant* $c > 0$ *depending only on* p *such that if*

$$\langle x \rangle_E \big|_{t=0} \leq \langle x \rangle_M \big|_{t=0},$$

and

$$I(u(0)) \leq cM^{4-\frac{p+3}{p-1}}E^{-1}$$

then

$$\inf_{t>0} I(u(t)) \geq \frac{1}{4} I(u(0)).$$

Remark. As we will show below, the assumption $I(u(0)) \leq cM^{4-\frac{p+3}{p-1}}E^{-1}$ does imply $M < c'$ for some constant $c' = c'(c,p)$. The entire statement is thus only applicable to initial data with small mass.

Interpretation. If the function is sufficiently localized and very smooth, then at least some part of it needs to break off and go away quickly never to return. The assumption on the centers is not at all unreasonable: assume $u(0)$ is some L^2-normalized bump function localized in space $x \sim 0$ and Fourier space $\xi \sim 1$ and add a small perturbation w localized around $x \sim x_0 \gg 1$ and $\xi \sim N$. Then, for $x_0 \gg \|w\|_{L^2}^{-1}$,

$$I(u(0)) \sim x_0^2 \|w\|_{L^2}^2.$$

Assuming w to be small in L^∞, we expect linear dynamics to be dominating. This means that the perturbation w moves with speed $-3N^2$ and noticeably decreases the functional (if $N \gg \sqrt{x_0}$) while the big bump function u_0 barely moves at all during that time. We need to exclude this scenario and indeed, for centers of mass and energy, we have

$$\langle x \rangle_M \big|_{t=0} \sim x_0 \|w\|_{L^2}^2 \qquad \langle x \rangle_E \big|_{t=0} \sim x_0 \|w_x\|_{L^2}^2 \sim x_0 N^2 \|w\|_{L^2}^2.$$

The condition $\langle x \rangle_E \big|_{t=0} \leq \langle x \rangle_M \big|_{t=0}$ now implies $N \lesssim 1$ but in that case the perturbation actually moves slower than $u(0)$ and the problem cannot occur.

3.3. Elementary inequalities. We need to make sure that there is no inequality of the type $I(u(0)) \geq cM^{4-\frac{p+3}{p-1}}E^{-1}$, otherwise the statement would be a statement about the energy landscape of the functional I and not about the dynamics of the equation. We give a quick classification of all $(\alpha, \beta) \in \mathbb{R}^2$ for which $I(u) \gtrsim M(u)^\alpha E(u)^\beta$ holds true.

Proposition. *Let* $p > 1$,

$$3 \leq \alpha \leq \frac{4p}{p-1} \qquad \text{and} \qquad \beta = \frac{(4-\alpha)p + 5\alpha - 12}{p+3}.$$

Then there exists a constant $c_p > 0$ *such that for any function* $u \in H^1(\mathbb{R})$

$$\left(\int_{\mathbb{R}} \frac{u_x^2}{2} + \frac{|u|^{p+1}}{p+1} dx \right)^\beta \left[\int_{\mathbb{R}} \int_{\mathbb{R}} u(x)^2 (x-y)^2 u(y)^2 dx dy \right] \geq c_p \left(\int_{\mathbb{R}} u^2 dx \right)^\alpha.$$

A simple scaling argument shows that these are the only (α, β) for which such an inequality can possibly hold. In particular, setting $\alpha = 3$ gives $\beta = -1$ and therefore

$$I(0) \gtrsim M^3 E^{-1},$$

which renders Corollary 2 nontrivial but also shows that it is only applicable in the case of small mass. These inequalities seem fairly technical and of little intrinsic interest. That is why we were surprised about the following: Proposition 1 is implied by Sobolev embedding and interpolation with the following elementary inequality, which we couldn't find in the literature – we give three easy proofs (without the sharp constant) and a simple argument due to Soonsik Kwon (personal communication) on how to get the sharp constant as well as extremizers.

Proposition. *Let $p > 1$ and*

$$c_p = \frac{1}{2\pi} \frac{\Gamma\left(\frac{p+1}{p-1}\right)}{\Gamma\left(\frac{2p}{p-1}\right)} \left(\frac{\Gamma\left(\frac{2p}{p-1}\right)}{\Gamma\left(\frac{5p-1}{2p-2}\right)}\right)^{\frac{p+3}{p-1}} \left(\frac{\Gamma\left(\frac{3p+1}{2p-2}\right)}{\Gamma\left(\frac{p+1}{p-1}\right)}\right)^{\frac{3p+1}{p-1}} \geq \frac{1}{2\pi e}.$$

Then, for every $u \in L^2(\mathbb{R})$,

$$\left(\int_{\mathbb{R}} u^2 x^2 dx\right) \left(\int_{\mathbb{R}} |u|^{p+1} dx\right)^{\frac{4}{p-1}} \geq c_p \left(\int_{\mathbb{R}} u^2 dx\right)^{\frac{3p+1}{p-1}}.$$

Additionally, all minimizers are given by translation, scaling and dilation of the compactly supported function

$$u(x) = \begin{cases} (1 - x^2)^{\frac{1}{p-1}} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

As already mentioned above, the inequality has some connections with the classical uncertainty principle and the Gagliardo-Nirenberg inequality. This connection is as follows: Suppose $u \in H^1(\mathbb{R})$. Then there is the uncertainty principle in the form

$$\|ux\|_{L^2} \|u_x\|_{L^2} \geq \frac{1}{2} \|u\|_{L^2}^2.$$

Combining our inequality with the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq G_p \|u\|_{L^2}^{\frac{p+3}{2}} \|u_x\|_{L^2}^{\frac{p-1}{2}}$$

yields a version of the uncertainty principle with different constants and an additional term sandwiched in the middle

$$\|ux\|_{L^2} \|u_x\|_{L^2} \geq \frac{1}{G_p^{\frac{2}{p-1}}} \frac{\|ux\|_{L^2} \|u\|_{L^{p+1}}^{\frac{2p+2}{p-1}}}{\|u\|_{L^2}^{\frac{p+3}{p-1}}} \geq \frac{\sqrt{c_p}}{G_p^{\frac{2}{p-1}}} \|u\|_{L^2}^2.$$

This chain of inequalities implies that if a function is close to extremizing the classical uncertainty principle, then it needs to be of a highly particular shape and one has rather explicit bounds on its higher L^p-norms.

Our proof is based on the derivation of a differential inequality for $I(u(t))$. In contrast to standard virial arguments in PDEs (for example [**Gla77**]), this differential inequality is not strong enough to prove convexity of the functional (except in the case $p = 3$) – however, it it is strong enough to imply some sublevel estimates.

Lemma. *Let $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$ arbitrary. Assume that $f(x) > \delta > 0$ and*

$$f'(x) \geq \alpha \int_0^x f(y)^{\frac{1-p}{2}} dy - \beta \sqrt{f(x)} - \gamma,$$

then

$$|\{x > 0 : f(x) \leq z\}| \lesssim_{\alpha,\beta,\gamma,\delta} z^{\frac{p}{2}}.$$

4. Proofs

4.1. Proof of refined monotonicity. The proof of refined monotonicity simply picks up a term discarded by Tao: as such, it is neither intrinsically interesting nor truly novel – the statement, however, is useful.

Proof. Tao's proof is based on introducing strictly positive quantities a, b, q, r, s by solving

$$a^2M = \int_{\mathbb{R}} u_{xx}^2 dx \qquad b^2M = \int_{\mathbb{R}} |u|^{2p} dx \qquad aqM = \int_{\mathbb{R}} u_x^2 dx$$
$$brM = \int_{\mathbb{R}} |u|^{p+1} dx \qquad absM = p\int_{\mathbb{R}} |u|^{p-1}u_x^2 dx,$$

where partial integration implies

$$0 < q, r, s < 1 \qquad \text{and} \qquad 1 - q^2 - r^2 - s^2 + 2qrs \geq 0.$$

Then the proof can be finished algebraically by showing that for $p \geq \sqrt{3}$

$$EM(\partial_t \langle x \rangle_M - \partial_t \langle x \rangle_E) = \frac{3}{2}(1-q^2)a^2 + \left(2s - \frac{p+3}{p+1}qr\right)ab + \frac{1}{2}\left(1 - \frac{4p}{(p+1)^2}r^2\right)b^2 > 0.$$

However, Tao actually proves the stronger statement

$$\frac{3}{2}(1-q^2)a^2 + \left(2s - \frac{p+3}{p+1}qr\right)ab + \frac{1}{2}\left(1-r^2\right)b^2 > 0.$$

Hence

$$EM(\partial_t \langle x \rangle_M - \partial_t \langle x \rangle_E) \geq \left(\frac{1}{2} - \frac{2p}{(p+1)^2}\right)b^2r^2 \gtrsim_p \frac{1}{M^2}\left(\int_{\mathbb{R}} |u(x)|^{p+1} dx\right)^2.$$

□

4.1.1. *Application: Tao's dispersion.*

Proof. Suppose Tao's dispersion estimate

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} |x - x(t)| \left(u(t,x)^2 + \frac{1}{2}u_x(t,x)^2 + \frac{1}{p+1}|u(t,x)|^{p+1}\right)dx = \infty$$

was false. Then the difference between the centers of mass and energy would be bounded uniformly in time

$$\left|\langle x \rangle_E - \langle x \rangle_M\right| < \infty.$$

Looking at refined monotonicity, we see that this implies

$$\int_0^\infty \left(\int_{\mathbb{R}} |u(t,x)|^{p+1} dx\right)^2 dt < \infty.$$

This, however, implies that the function needs to spread its L^2-mass quickly over a large area and this is in direct contradiction to the assumption of the dispersion statement being incorrect. □

4.1.2. *Application: Kwon-Shao argument.* We present one particular way to fix the gap in the Kwon-Shao argument.

Proof. Under the assumption of

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}} (x - x(t))^2 u(t,x)^2 dx \leq c,$$

we need to show that for *some* $t > 0$

$$\langle x \rangle_E < \langle x \rangle_M.$$

The assumption of bounded variance and conservation of mass immediately imply

$$\int_{\mathbb{R}} |u(t,x)|^{p+1} dx \gtrsim_{M,c} 1$$

and refined monotonicity now yields the statement. ☐

4.2. Derivation of the differential inequality. The first step in the proof of the main statement is the derivation of the differential inequality

$$\frac{d}{dt}I(u(t)) \geq \alpha \int_0^t I(u(t))^{\frac{1-p}{2}}dz - \beta\sqrt{I(u(t))} - \gamma,$$

where α and β are positive numbers at the scales $\alpha \sim M^{2p-2}$ and $\beta \sim M(EM)^{\frac{p-1}{4}}$. The constant γ encodes the initial difference between the center of mass and the center of energy

$$\gamma = EM(\langle x\rangle_M - \langle x\rangle_E)\big|_{t=0}.$$

Proof. Repeated partial integration yields

$$\partial_t \int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2u(y)^2(x-y)^2dxdy = 4\int_{\mathbb{R}}\int_{\mathbb{R}} u(y)^2\left(\frac{p}{p+1}|u(x)|^{p+1} + \frac{3}{2}u_x(x)^2\right)(y-x)\,dxdy.$$

By introducing the normalized centers of mass and energy,

$$\langle x\rangle_M := \frac{1}{M}\int_{\mathbb{R}} xu^2dx \qquad \text{and} \qquad \langle x\rangle_E := \frac{1}{E}\int_{\mathbb{R}} x\left(\frac{1}{2}u_x^2 + \frac{1}{p+1}|u|^{p+1}\right)dx,$$

we can rewrite the first derivative as

$$(\diamondsuit) \qquad \partial_t \int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2u(y)^2(x-y)^2dxdy = 12EM\left(\langle x\rangle_M - \langle x\rangle_E\right) + \frac{4p-12}{p+1}\left(\langle x\rangle_M M\int_{\mathbb{R}}|u|^{p+1}dx - M\int_{\mathbb{R}}|u|^{p+1}xdx\right).$$

In the special case $p=3$, the monotonicity formula immediately implies convexity of $I(u(t))$.

The following simple argument will be used also in later proofs. If $I(u(t))$ is small, then there is a small interval containing a lot of the L^2-mass: for a fixed time t, let J be the unique interval such that

$$\int_{x<\inf J} u^2dx = \int_{x>\sup J} u^2dx = \frac{1}{4}\int_{\mathbb{R}} u^2dx,$$

then

$$\frac{M^2}{16}|J|^2 \leq \int_{x\in\mathbb{R}\setminus J}\int_{x\in\mathbb{R}\setminus J} u(t,x)^2u(t,y)^2(x-y)^2dxdy \leq I(u(t)),$$

which implies $|J| \leq 16\sqrt{I(u(t))}/M$. Hence, with Hölder,

$$\frac{M}{2} = \int_J u^2dx \leq \left(\int_J |u|^{p+1}dx\right)^{\frac{2}{p+1}}|J|^{\frac{p-1}{p+1}}$$

and thus, as a consequence,

$$\left(\int_{\mathbb{R}}|u|^{p+1}dx\right)^2 \geq \left(\int_J |u|^{p+1}dx\right)^2 \gtrsim M^{2p}I(u(t))^{1/2-p/2}.$$

The fundamental theorem of calculus and refined monotonicity yield

$$\begin{aligned}(\langle x\rangle_M - \langle x\rangle_E)\big|_t - (\langle x\rangle_M - \langle x\rangle_E)\big|_{t=0} &\gtrsim \int_0^t \frac{1}{EM^3}\left(\int_{\mathbb{R}}|u(z,x)|^{p+1}dx\right)^2dz\\ &\gtrsim \frac{M^{2p-3}}{E}\int_0^t I(u(z))^{\frac{1-p}{2}}dz.\end{aligned}$$

The remaining term on the right-hand side of $(\diamondsuit)$ can be easily controlled via

$$\left|\langle x\rangle_M M\int_{\mathbb{R}}|u|^{p+1}dx - M\int_{\mathbb{R}}|u|^{p+1}xdx\right| \leq \int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2|u(y)|^{p+1}|x-y|dxdy,$$

which, using Hölder and

$$\|u\|_{L^\infty} \lesssim \left(\int_{\mathbb{R}} u^2 dx\right)^{\frac{1}{4}} \left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{4}} \lesssim M^{1/4}E^{1/4}$$

can be bounded by

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2|u(y)|^{p+1}|x-y|dxdy \lesssim M(ME)^{\frac{p-1}{4}} I(u(t))^{1/2}.$$

Altogether, this yields that the real function $f(t) := I(u(t))$ satisfies the differential inequality

$$f'(t) \geq \alpha \int_0^t f(z)^{\frac{1-p}{2}} dz - \beta\sqrt{f(t)} - \gamma,$$

for positive constants $\alpha \sim M^{2p-2}$, $\beta \sim M(EM)^{\frac{p-1}{4}}$ and a constant γ which encodes the initial difference between the center of mass and the center of energy

$$\gamma = EM(\langle x\rangle_M - \langle x\rangle_E)\big|_{t=0}.$$

□

4.3. Differential inequalities. This section provides two elementary statements on differential inequalities of the type discussed above – the first implies the main statement, the second implies the corollary.

Lemma. *Let $f \in C^1(\mathbb{R},\mathbb{R}_+)$, $\alpha,\beta > 0$ and $\gamma \in \mathbb{R}$ arbitrary. Assume that $f(x) > \delta > 0$ and*

$$f'(x) \geq \alpha \int_0^x f(y)^{\frac{1-p}{2}} dy - \beta\sqrt{f(x)} - \gamma,$$

then

$$|\{x > 0 : f(x) \leq z\}| \lesssim_{\alpha,\beta,\gamma,\delta} z^{\frac{p}{2}}.$$

Proof. Let us quickly describe the argument: the lower bound on $f'(x)$ is comprised of one trivial component $-\beta\sqrt{f(x)}-\gamma$, which merely depends on the value $f(x)$ and one term with 'memory': whether the integral is large compared to the trivial component depends on whether or not the function has been small in the past. In particular, if the function has been small in the past for a long time, the integral will dominate the trivial component and force the function to grow. Since $f(x) > \delta > 0$, the statement we are trying to prove is trivially true for $z \leq \delta$ and we may assume $z \geq \delta$. Fix a $z > \delta$, allow c to be a large positive constant, consider $I = \{x > 0 : f(x) \leq z\}$ and take K sufficiently large such that

$$|[0,K] \cap I| = cz^{\frac{p}{2}}.$$

It remains to show that taking $c > 0$ sufficiently large yields a contradiction. Let us take a look at the derivative at $x = K$ (trivially, K can be chosen such that $K \in I$)

$$f'(K) \geq \alpha \int_0^K f(y)^{\frac{1-p}{2}} dy - \beta\sqrt{f(K)} - \gamma \geq \alpha \int_{I\cap[0,K]} f(y)^{\frac{1-p}{2}} dy - \beta\sqrt{z} - \gamma \geq \alpha c\sqrt{z} - \beta\sqrt{z} - \gamma$$

Then, for any c suffiently large depending on α,β,γ, this implies that at K the derivative is of order $f'(K) \gtrsim \sqrt{z}$. By the same reasoning, the same holds true for all points in $(K,\infty)\cap I$. This growth implies that $(\mathbb{R}\setminus I)\cap(K, K+C\sqrt{z}) \neq \emptyset$, where the constant C depends on α,β,γ but not z. We show now that $\sup I \leq K + C\sqrt{z}$. Suppose, this were not the case. Since $(\mathbb{R}\setminus I)\cap(K,K+C\sqrt{z}) \neq \emptyset$, there would then be a smallest point $y^* > k$ with $f(y^*) = z$, where the previous inequality implies $f'(y^*) > 0$ and this is a contradiction. Altogether, the final ingredient $f(x) > \delta > 0$ implies

$$|I| \leq cz^{\frac{p}{2}} + C\sqrt{z} \lesssim_\delta (c+C)cz^{\frac{p}{2}},$$

where c could be chosen depending only on α,β,γ and C was finite. □

Lemma. *Let $f \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha, \beta > 0, \gamma \leq 0$. If*

$$f'(x) \geq \alpha \int_0^x f(y)^{\frac{1-p}{2}} dy - \beta \sqrt{f(x)} - \gamma$$

and

$$f(0) < \left(\frac{\alpha}{\beta^2}\right)^{\frac{2}{p-1}},$$

then

$$\inf_{x>0} f(x) \geq \frac{f(0)}{4}.$$

Proof. It follows from the structure of the inequality that we can restrict ourselves to functions being monotonically decreasing until they reach a global minimum. The function

$$g(x) = \frac{1}{4}(2\sqrt{f(0)} - \beta x)^2$$

satisfies $g(0) = f(0)$ and $g'(x) = -\beta\sqrt{g(x)}$. At $x = \sqrt{f(0)}/\beta$, we have from monotonicity and the bound on $f(0)$ that

$$f'(x) \geq \alpha \int_0^x f(y)^{\frac{1-p}{2}} dy - \beta\sqrt{f(x)} \geq \frac{\alpha}{\beta}\sqrt{f(0)} f(0)^{\frac{1-p}{2}} - \beta\sqrt{f(0)} > 0$$

implying that the minimum is assumed before $x = \sqrt{f(0)}/\beta$. However,

$$g\left(\frac{\sqrt{f(0)}}{\beta}\right) = \frac{1}{4}(2\sqrt{f(0)} - \beta\frac{\sqrt{f(0)}}{\beta})^2 = \frac{f(0)}{4}.$$

□

4.4. The energy landscape of the functional. Here we give a complete proof of the following proposition.

Proposition. *Let $p > 1$,*

$$3 \leq \alpha \leq \frac{4p}{p-1} \qquad \text{and} \qquad \beta = \frac{(4-\alpha)p + 5\alpha - 12}{p+3}.$$

Then there exists a constant $c_p > 0$ such that for any function $u \in H^1(\mathbb{R})$

$$\left(\int_{\mathbb{R}} \frac{u_x^2}{2} + \frac{|u|^{p+1}}{p+1} dx\right)^{\beta} \left[\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2(x-y)^2 u(y)^2 dx dy\right] \geq c_p \left(\int_{\mathbb{R}} u^2 dx\right)^{\alpha}.$$

The product structure of the inequalities implies that it is sufficient to prove the two endpoints $\alpha = 3$ and $\alpha = 4p/(p-1)$. The endpoint $\alpha = 3$ follows quickly from Sobolev embedding. All further considerations are for general functions $u \in H^1(\mathbb{R})$, where we use M and E to denote their mass and energy, respectively. There are no time-dependent elements in the arguments nor does the gKdV equation play any role.

Lemma. *For any $u \in H^1(\mathbb{R})$*

$$I(u) \gtrsim \frac{M^3}{E^1}.$$

Proof. By Sobolev embedding

$$\|u\|_{L^\infty}^2 \leq \left(\int_{\mathbb{R}} u^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} u_x^2 dx\right)^{\frac{1}{2}} \lesssim M^{1/2}E^{1/2}.$$

At the same time, reusing an argument employed earlier, there is an interval J of length $|J| \lesssim \sqrt{I(u)}/M$ such that J contains half of the L^2-mass of u. Therefore

$$\|u\|_{L^\infty}^2 \geq \|u\|_{L^\infty(J)}^2 \geq \frac{1}{|J|}\int_J u(x)^2 dx \gtrsim \frac{M}{|J|} \gtrsim \frac{M^2}{\sqrt{I(u)}}.$$

Combining these two inequalities gives

$$\frac{M^2}{\sqrt{I(u)}} \lesssim M^{1/2}E^{1/2}$$

and this is what we were trying to prove. □

The proof of the second endpoint uses symmetric decreasing rearrangement to gain an additional symmetry, which then yields an algebraic simplification of the functional. The following statement will come as no surprise at all, it can certainly be founded in the literature in more general form (a good introduction to rearrangement inequalities is given by the book of Lieb & Loss [**LL01**]).

Lemma. *The functional I is decreasing under symmetrically decreasing rearrangement.*

PROOF. We use the familiar layer-cake decomposition

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(t,x)^2u(t,y)^2(x-y)^2dxdy = \int_0^\infty\int_0^\infty rs\int_{\{u(x)^2=r\}\times\{u(y)^2=s\}}(x-y)^2d\mathcal{H}^2drds,$$

where $\mathcal{H}^2$ is the 2−dimensional Hausdorff measure. The statement would then follow if it were the case that for fixed positive $a,b>0$ and subsets $A,B\subset\mathbb{R}$

$$\inf_{|A|=a,|B|=b}\int_A\int_B(x-y)^2dxdy$$

is assumed precisely when A,B are intervals with the same midpoint (potentially ignoring Lebesgue null sets in the process). Let A,B be hypothetical counterexamples, then there exist constants c_1,c_2 such that a neighbourhood of c_1 is not contained in A and both $A_1 := A\cap\{x : x>c_1\}$ and $A_2 := A\cap\{x : x<c_1\}$ are nonempty and likewise for c_2 and B. Let us then replace A_1 and B_1 by $A_1-\varepsilon$ and $B_1-\varepsilon$ for sufficiently small ε such that no overlap occurs. Then the integration between A_1 and B_1 as well as between A_2 and B_2 remains unchanged while it decreases between A_1 and B_2 as well as A_2 and B_1. This shows that the infimum can only be assumed by a pair of intervals (up to Lebesgue null sets) and an explicit calculation yields the midpoint property. □

For symmetric functions, the functional simplifies to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)^2(x-y)^2u(y)^2dxdy = 2\left(\int_{\mathbb{R}}u(x)^2x^2dx\right)\left(\int_{\mathbb{R}}u(x)^2dx\right).$$

Assuming a lower bound of the type $I \gtrsim M^\alpha E^\beta$, we may use the fact that the gKdV scaling acts nicely on M and E to derive the necessary condition

$$4-\frac{8}{p-1} = \alpha\left(1-\frac{4}{p-1}\right)+\beta\left(3-\frac{4p}{p-1}\right).$$

A standard scaling $u(\cdot)\to au(b\cdot)$ with $a,b>0$ implies $2\alpha+(p+1)\beta\le 4$ and $2\le\alpha+\beta\le 4$. These scaling considerations have the endpoint $(\alpha,\beta)=(3,-1)$, which is implied by the previous Lemma. The other endpoint remains to be proven, which we do in the subsequent section.

4.5. An elementary anti-concentration inequality. This section is devoted to a proof of the following inequality.

Proposition. *Let $p>1$ and*

$$c_p = \frac{1}{2\pi}\frac{\Gamma\left(\frac{p+1}{p-1}\right)}{\Gamma\left(\frac{2p}{p-1}\right)}\left(\frac{\Gamma\left(\frac{2p}{p-1}\right)}{\Gamma\left(\frac{5p-1}{2p-2}\right)}\right)^{\frac{p+3}{p-1}}\left(\frac{\Gamma\left(\frac{3p+1}{2p-2}\right)}{\Gamma\left(\frac{p+1}{p-1}\right)}\right)^{\frac{3p+1}{p-1}} \ge \frac{1}{2\pi e}.$$

Then, for every $u \in L^2(\mathbb{R})$,

$$\left(\int_{\mathbb{R}} u^2x^2dx\right)\left(\int_{\mathbb{R}} |u|^{p+1}dx\right)^{\frac{4}{p-1}} \geq c_p \left(\int_{\mathbb{R}} u^2dx\right)^{\frac{3p+1}{p-1}}.$$

Additionally, all minimizers are given by translation, scaling and dilation of the compactly supported function

$$u(x) = \begin{cases} (1-x^2)^{\frac{1}{p-1}} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We give different approaches. The first proof uses invariance under scaling and dilations and a characterization of compactness in L^2. It implies the existence of a minimizer but no numerical bounds on c_p. The second argument is merely a combination of the Markov and Hölder inequality, the third one relies on a simple decomposition.

First proof. It suffices to consider functions u invariant under symmetric decreasing rearrangement. We pick a minimizing sequence $u_n \in H^1(\mathbb{R})$ of the functional

$$\frac{\left(\int_{\mathbb{R}} u^2x^2dx\right)^{\frac{p-1}{4}} \int_{\mathbb{R}} u^{p+1}dx}{\left(\int_{\mathbb{R}} u^2dx\right)^{\frac{3p+1}{p-1}}}$$

and use invariance under scaling and dilations to prescribe $u(0) = 1$ and $\int_{\mathbb{R}} u^2x^2 = 1$. Trivially, the sequence is then bounded in L^2 by

$$\int_{\mathbb{R}} u^2dx \leq 2 + \int_{|x|\geq 1} u^2x^2dx = 3.$$

We use an observation that is usually ascribed to Feichtinger [**Fei84**] or Pego [**Peg85**] and has close connections with the Riesz-Kolmogorov theorem and variants thereof (we refer to Hanche-Olsen & Holden [**HOH10**] for the historical details). A bounded set $K \subset L^2$ is compact in L^2 if there exists a function $C : \mathbb{R}_+ \to \mathbb{R}_+$ giving uniform control of the form

$$\forall f \in K \qquad \int_{|x|>C(\varepsilon)} |f(x)|^2dx + \int_{|\xi|>C(\varepsilon)} |\hat{f}(\xi)|^2d\xi < \varepsilon.$$

The first part is easy and follows from the normalization of the variance. The symmetry of our functions implies that their Fourier transform is real-valued and thus

$$\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x)\cos\xi x dx\right| \lesssim \frac{1}{|\xi|},$$

due to the fact that the monotonicity gives rise to an alternating series. Hence the minimizing sequence is compact and there exists a minimizer. □

Second proof. We use the invariance under scaling to assume that

$$\int_{\mathbb{R}} u^2dx = 1 \quad \text{and abbreviate} \quad \alpha := \int_{\mathbb{R}} u^2x^2dx.$$

Markov's inequality implies for any $c > 0$

$$\int_{|x|\geq\sqrt{c\alpha}} u^2dx \leq \frac{1}{c},$$

which under the above normalization is only nontrivial for $c > 1$. An application of Hölder implies

$$1 - \frac{1}{c} \leq \int_{-\sqrt{c\alpha}}^{\sqrt{c\alpha}} u^2dx \leq \left(\int_{-\sqrt{c\alpha}}^{\sqrt{c\alpha}} |u|^{p+1}dx\right)^{\frac{2}{p+1}} (2\sqrt{c\alpha})^{\frac{p-1}{p+1}}$$

and therefore

$$\left(\int_{\mathbb{R}} u^2 x^2 dx\right)\left(\int_{\mathbb{R}} |u|^{p+1} dx\right)^{\frac{4}{p-1}} \geq \alpha \frac{\left(1-\frac{1}{c}\right)^{\frac{2p+2}{p-1}}}{4c\alpha}.$$

Any $c > 1$ now implies the statement. $\square$

Third proof. We use the invariance under scaling to assume that

$$\int_{\mathbb{R}} u^2 dx = 1 \quad \text{and define} \quad A := \{x \in \mathbb{R} : u(x) > c\},$$

where $c > 0$ is some constant to be chosen later. Clearly,

$$\left(\int_{\mathbb{R}} |u|^{p+1} dx\right)^{\frac{4}{p-1}} \geq \left(\int_{A} |u|^{p+1} dx\right)^{\frac{4}{p-1}} \geq \left(\int_{A} c^{p-1} u^2 dx\right)^{\frac{4}{p-1}} \geq c^4 \left(\int_{A} u^2 dx\right)^{\frac{4}{p-1}}.$$

All the L^2-mass that is supported outside of A is spread out over a large set I of size at least

$$|I| \geq \frac{\int_{\mathbb{R}\setminus A} u^2 dx}{c^2} =: 2\alpha$$

and therefore

$$\int_{\mathbb{R}} u^2 x^2 dx \geq \int_{-\alpha}^{\alpha} x^2 c^2 dx = \frac{2}{3} c^2 \alpha^3 = \frac{1}{12}\left(\int_{\mathbb{R}\setminus A} u^2 dx\right)^3 \frac{1}{c^4}.$$

Altogether,

$$\left(\int_{\mathbb{R}} u^2 x^2 dx\right)\left(\int_{\mathbb{R}} |u|^{p+1} dx\right)^{\frac{4}{p-1}} \geq \frac{1}{12}\left(\int_{A} u^2 dx\right)^{\frac{4}{p-1}}\left(\int_{\mathbb{R}\setminus A} u^2 dx\right)^3.$$

Pick c such that

$$\int_{\mathbb{R}\setminus A} u^2 dx = \frac{1}{2}$$

and this implies the statement. $\square$

Sharp constant, Soonsik Kwon. We fix

$$\int_{\mathbb{R}} u^2 x^2 dx = 1 = \int_{\mathbb{R}} u^2$$

and want to minimize

$$\int_{\mathbb{R}} |u|^{p+1} dx$$

under these two constraints. An extremizer is known to exist. The Lagrange multiplier theorem implies that

$$u^p = \lambda_1 x^2 u + \lambda_2 u$$

for some constants λ_1, λ_2 from which the statement follows. $\square$

CHAPTER 3

Laplacian Eigenfunctions and Heat Flow

Be not affeard, the Isle is full of noyses,
Sounds, and sweet aires, that give delight and hurt not:
Sometimes a thousand twangling Instruments
Will hum about mine eares; and sometime voices,
That if I then had wak'd after long sleepe,
Will make me sleepe againe, and then in dreaming,
The clouds methought would open, and shew riches
Ready to drop vpon me, that when I wak'd
I cri'de to dreame againe.
William Shakespeare, *The Tempest*

This chapter is devoted to the exposition of a particular new local approach to the structure of nodal sets of eigenfunctions of the Beltrami-Laplacian on smooth, compact Riemannian manifolds. This approach is based on exploiting the action of the heat flow on a Laplacian eigenfunction and has some advantages compared to other methods. We aim to give an essentially complete presentation of recent developements in the study of bounds on nodal sets in the last five years – this influx of new arguments has its main protagonists in Colding & Minicozzi [**CM11**], Hezari & Sogge [**HS**], Mangoubi [**Man08a**, **Man10**] and Sogge & Zelditch [**SZ11**, **SZ**] and is quite different in nature from the Carleman approach of Donnelly & Fefferman [**DF88**, **DF90**]. These new developements are all firmly embedded in a more classical framework: any Laplacian eigenfunction, when restricted to a nodal domain, corresponds to the ground state of that domain. The Rayleigh-Ritz characterization immediately implies a lot of connection between the ground state and the geometry of the domain. As for higher eigenfunctions, a recent survey of Zelditch [**Zel08**] is particularly comprehensive on these matters.

1. Introduction

We always consider a compact $n-$dimensional $C^\infty-$manifold (M, g) without boundary and write Δ_g for the Laplace-Beltrami operator. Our object of study are Laplacian eigenfunctions

$$-\Delta_g u = \lambda u.$$

Ultimately, we are interested in its nodal set

$$Z = \{x \in M : u(x) = 0\}$$

and, in particular, estimates on its $(n-1)-$dimensional measure. A profoundly basic guiding principle is that eigenfunctions corresponding to large eigenvalues are increasingly oscillatory in nature – this follows essentially from the orthogonality required in the variational characterization. On a compact $C^\infty-$manifold, nodal sets of highly oscillatory functions reduces the setting to a smooth perturbation of a bounded subset Ω of the flat Euclidean space with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

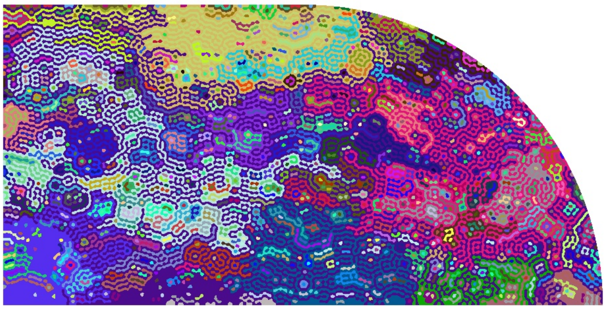

FIGURE 1. Color coded nodal domains taken from [**Bar**]

The smallest Laplacian eigenvalue has the variational characterization

$$\lambda_1(\Omega) = \inf_{\substack{f \neq 0 \\ f \in H_0^1(\Omega)}} \frac{\int_\Omega |\nabla f|^2 dx}{\int_\Omega f^2 dx}.$$

Higher eigenvalues and eigenfunctions can be iteratively constructed by prescribing orthogonality to the already existing set. We are not treating this in detail but refer to Reed & Simon [**RS78**].

A word of warning. We just introduced a crucial reduction, which will play a fundamental role in all subsequent chapters and whose disadvantages need to be clearly emphasized: given a Laplacian eigenfunction, we can restrict it to any nodal domain on which it will then coincides with the ground state of the domain allowing the application of classical elliptic theory. However, this reduction *does not incorporate* how those nodal domains came to be in the first place, which is a seperate set of questions, see for example the survey by Nonnenmacher [**Non**]. This information is difficult to incorporate. Thanks to the isoperimetric inequality, lower bounds can be achieved without using advanced global information: if a set has a certain measure, it certainly has a certain boundary. In particular, one can take any nodal domain and via perturbing its boundary create an arbitarily large perimeter without affecting the first Laplacian eigenvalue in any noticeable degree. Upper bounds require an understanding as to why this doesn't happen.

1.1. Faber-Krahn inequality. It was first conjectured by Rayleigh [**Ray96**] and later proven by Faber [**Fab23**] and Krahn [**Kra25**,**Kra26**] that among all given domains $\Omega \subset \mathbb{R}^n$ the circle has the smallest principial frequency $\lambda_1(\Omega)$. Nowadays, this proof is a standard consequence of the monotonicity properties of the Polya-Szegö rearrangement, which provides a very satisfactory connection to the isoperimetric properties of the ball. As one particular appliation, Faber-Krahn gives an elementary upper bound on the spatial dimensions of a nodal domain – as for lower bounds, that problem is much harder and the current state of the art is summarized in the following theorem.

Theorem (Mangoubi, [**Man08a**]). *Let Ω_λ be a nodal domain of a Laplacian eigenfunction. Then we have the following bound on the inradius.*

$$\frac{1}{\lambda^{\frac{n-1}{4}+\frac{1}{2n}}} \lesssim inrad(\Omega_\lambda) \lesssim \frac{1}{\lambda^{\frac{1}{2}}}.$$

The ease with which symmetrically decreasing rearrangement yields a proof of the Faber-Krahn inequality has perhaps been an impediment to a more profound understanding of its stability. There is a stability statement due to Melas [**Mel92**], which states that if a convex domain Ω saturates the Faber-Krahn inequality up to a factor of $1+\varepsilon$, then there are balls $B_1 \subset \Omega \subset B_2$, where $|B_1| \geq (1 - c_n \varepsilon^{\frac{1}{2n}})|\Omega|$ and $\Omega \geq (1 - c_n \varepsilon^{\frac{1}{2n}})|B_2|$. A stability result in terms of Frankel asymmetry is given by Fusco, Maggi & Pratelli [**FMP09a**].

Hardy inequalities. There is a connection to Hardy inequalities that we cannot not mention. There is some intuition that the energy of the gradient of a generic Laplacian eigenfunction is used to grow away from the boundary of the nodal domain *and not* for anything complicated in the interior of the domain. This behavior is captured by Hardy inequalities, the literature on which is quite extensive. As an example, Ancona [**Anc86**] proved that if $\Omega \subset \mathbb{R}^2$ is simply connected, then

$$\frac{1}{16} \int_{\Omega} \frac{u(x)^2}{\delta(x)^2} dx \leq \int_{\Omega} |\nabla u|^2,$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$ is the distance to the boundary. Of course, choosing a Laplacian eigenfunction u gives

$$\frac{1}{16} \int_{\Omega} \frac{u(x)^2}{\delta(x)^2} dx \leq \int_{\Omega} |\nabla u|^2 = \lambda \int_{\Omega} u(x)^2 dx,$$

which can be read as a 'cheap Faber-Krahn inequality': the bulk of the L^2-norm of an eigenfunction is located at distance $\sim \lambda^{-1/2}$ from the boundary. Combined with the upper bound on the inradius of a nodal domain, it can also be read as an inverse statement: given a Laplacian eigenfunction, it is true that its restriction onto each nodal domain yields a function which is within a constant factor of saturating Hardy's inequality.

1.2. Inradius, Hayman's theorem and Brüning's bound. If we restrict ourselves to $n = 2$, things simplify. Mangoubi's bound [**Man08a**] has already been mentioned. In two dimensions, it merely asserts that

$$\text{inrad}(\Omega_\lambda) \sim \lambda^{-1/2}$$

and has a much longer history. Clearly, one direction is Faber-Krahn. The other direction is not immediately obvious and was conjectured by Polya & Szegő [**PS51**] in 1951 and proven by Hayman [**Hay78**] in 1978 with an explicit constant

$$\text{inrad}(\Omega) \geq \frac{1}{30\lambda(\Omega)^{1/2}}$$

for any simply connected $\Omega \subset \mathbb{R}^2$. We give a generalization of this statement to higher dimensions further below. A generalization in two dimensions to domains with multiplicativity $k \geq 2$ is due to Croke [**Cro81**]. Using this result, it is simple to derive the only known result for general $C^\infty-$manifolds that is suspected to be sharp.

Theorem (Brüning [**Brü78**] - Yau (unpublished))**.** *Let (M, g) be a compact C^∞-surface and let u be a Laplacian eigenfunction with eigenvalue λ. Then, for implicit constants depending only on (M, g),*

$$\mathcal{H}^1\left(\{x \in M : \phi(x) = 0\}\right) \gtrsim \lambda^{1/2}.$$

Sketch of a proof. Cover the manifold with balls of size $c\lambda^{-1/2}$

$$M = \bigcup_{i=1}^{N} B_i.$$

For a suitable choice of c, the Faber-Krahn inequality implies that each ball B_i contains an element of the nodal set x_i. For each x_i there is either a distinct x_j at distance $\sim \lambda^{-1/2}$ connected by the nodal set or there isn't. For each of the λ balls of radius $\lambda^{-1/2}$ in the first case, we get a curve of length $\sim \lambda^{-1/2}$, which implies the bound. In the second case, we are dealing with a nodal set that is entirely contained in a small ball, however, Hayman's theorem immediately implies a lower bound on the measure of that nodal domain and the isoperimetric inequality implies the statement. □

This is drastically different in dimensions $n \geq 3$. Indeed, given an arbitrary domain $\Omega \subset \mathbb{R}^n$, it is possible to introduce thin spikes into the domain without severely affecting the first Laplacian eigenvalue: the function needs to have large growth away from the spikes but the $\varepsilon-$neighbourhood (on which the gradient will be large) of such a spike has small measures.

A truly remarkable theorem by Lieb [**Lie83**] gives, however, that these spikes are – in a certain sense – the only obstruction. We describe the theorem in full because we conjecture an analogous statement for the the heat content, a concept we describe further below.

Theorem (Lieb, [**Lie83**]). *Let $A, B \subset \mathbb{R}^n$ be bounded domains. Then there exists a $x \in \mathbb{R}^n$ such that*

$$\lambda_1\left((A+x) \cap B\right) \leq \lambda_1(A) + \lambda_1(B).$$

If A is now within a small factor of saturating the Faber-Krahn inequality, pick B to be the ball with the same measure as A and the theorem implies that A cannot be too spread out (this is elaborated further in Lieb [**Lie83**]).

1.3. Aside: more results. This section can be skipped entirely as it collects some results, which play no further role in the subsequent argument – strictly speaking, this is also true for some of the results outside of this section, however, we consider those to have a closer connection in structure; the results collected here are listed because we consider them suitable for furthering one's understanding of Laplacian eigenfunctions (and because it would be a crime to write an introductory text about Laplacian eigenfunctions without mentioning Cheeger's inequality).

1.3.1. *Cheeger's inequality.*

> This trend in Riemannian geometry started with the work of Jeff Cheeger. Earlier, up to some point, people were thinking about manifolds in very abstract terms. There were many indices and you could not take the subject into your hand. [...] And then there was the work by Jeff Cheeger, formally a very different subject but with the same attitude, realizing that things got quite simple when formalized, if that was done properly. So I was just following in the steps of these people. (Mikhail Gromov, *Interview* [**RS10**])

Cheeger's inequality [**Che70**] is a comparatively simple (the original paper spans only 4 pages) inequality relating the size of the smallest Laplacian eigenvalue on a manifold with its geometric structure. Define an isoperimetric quantity via

$$h(M) = \inf_C \frac{\mathcal{H}^{n-1}(C)}{\min(\mathcal{H}^n(M_1), \mathcal{H}^n(M_2))},$$

where the infimum is taken over all compact codimension one submanifolds, which divide M into two manifolds M_1 and M_2. Cheeger's inequality gives that

$$\lambda_1(M) \geq \frac{h^2}{4}.$$

The geometry behind the inequality is astonishing simple: the first eigenfunction $u(x)$ is orthogonal to constants, as such it needs to change its size somewhere between $\sup_M u(x)$ and $\inf_M u(x)$. Assuming L^2-normalization on the compact manifold, we get a lower bound on $\sup_M u(x)$. The function can't change its size globally very quickly as this would lead to a large $\dot{H}^1$ norm. Therefore, the change needs to happen on a global scale. Cheeger's inequality simply makes sure that the geometry of the manifold does

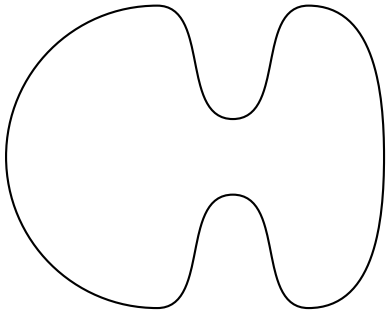

FIGURE 2. A domain whose rotation around an axis gives rise to a dumbbell-type two-dimensional manifold embedded in $\mathbb{R}^3$.

not have a 'dumbbell'-type structure, where the function could rapidly change its size without contributing to a large first eigenvalue because the Hausdorff measure is locally small. Cheeger's inequality had a profound impact and is of paramount importance in the extension of the theory to the discrete setting of graphs [**Chu97**].

1.3.2. *Bers' scaling.* Consider the equation

$$-\Delta u = \lambda u$$

close to x_0 with $u(x_0) = 0$. If u does not vanish to infinite order, then it behaves asymptotically like a harmonic polynomial (first results in this direction are very old, we refer to Lipman Bers' 1955 paper [**Ber55**]).

Theorem (Bers' scaling, formulation from [**Zel08**])**.** *Assume that ϕ_λ vanishes to order k at x_0. Let $\phi_\lambda(x) = \phi_k^{x_0} + \phi_{k+1}^{x_0} + \dots$ denote the C^∞ Taylor expansion of ϕ_λ into homogeneous terms in normal coordinates x centered at x_0. Then $\phi_k^{x_0}(x)$ is a Euclidean harmonic homogeneous polynomial of degree k.*

This theorem serves as a description of the vanishing of Laplacian eigenfunctions in critical points, where they behave in a fashion that is not more complicated than harmonic functions. When it comes to deriving upper bounds for the measure of the nodal set, these are the problematic points: the nodal set around a nodal point with a nondegenerate gradient look locally like a hyperplane for trivial reasons (though the quality of the approximation depends on the norm of the gradient, which is again a complicated quantity).

In the special case $n = 2$, S.-Y. Cheng [**Che76**] remarks that nodal lines in a critical point form an equiangular system, which makes it all the more surprising that the problem has proven so difficult even in two dimensions.

1.4. Rays of sunshine: Croke-Dzérdinski.

τον ηλιον ανατελλει επι πονηρους και αγαθους και βρεχει επι δικαιους και αδικους
(*For He maketh His sun to rise on the evil and on the good, and sendeth rain on the just and on the unjust.*)
Matt. 5:45

There is an intimate connection between the eigenfunctions and the behavior of the geodesic flow, cf. Zelditch [**Zel08**] – something that current (unconditional) lower bounds do not exploit and which we will not even describe in the roughest outline. However, there is a nice theorem due to Croke & Dzérdinski

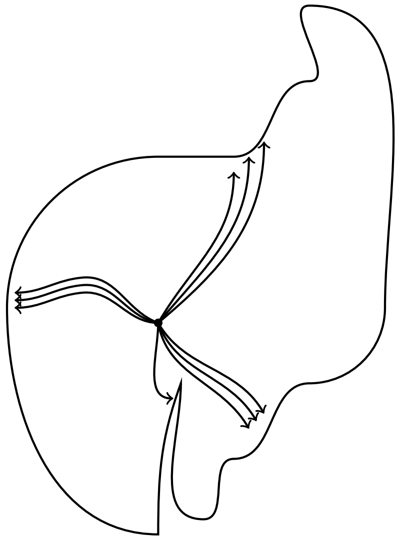

FIGURE 3. A domain with a spike in the interior: the geodesics in the Croke-Dzérdinski inequality take notice but are not as sensitive to its presence as other quantities (say, the inradius).

[**CD87**], which is maybe not quite as well-known as it deserves: it gives an elementary connection between $\lambda_1(\Omega)$ and the behavior of geodesics.

Theorem (Croke & Dzérdinski, [**CD87**]). *Let (M,g) be a compact manifold with boundary and let $\ell_x(v)$ be the (possibly infinite) distance between x and the boundary in direction v. Then*

$$\lambda_1(M) \geq \frac{n\pi^2}{\omega_{n-1}} \inf_{x \in int(M)} \int_{U_x} \frac{1}{\ell_x(v)^2} dv_x,$$

where ω_{n-1} is the volume of the unit $(n-1)-$dimensional sphere and U_x is the unit tangent bundle in x.

A graphical description would be that there is a point x inside a nodal domain such that 'in most directions' the boundary seems to be at a distance of at least $\lambda^{-1/2}$, which makes it a stable analogue of Hayman's theorem that survives the transition to higher dimensions: spikes may be present, however, they are very thin and barely visible.

This inequality seems to have no possible use in trying to derive upper bounds on the measure of nodal sets (because of the infimum), however, it seems not entirely inconceivable that it could produce *some* lower bound, which would then be certainly interesting in terms of the structure of the underlying argument alone. Let us note a beautiful corollary, which can be found at the very end of [**CD87**]. If Ω is a domain contained in the $n-$unit sphere and the interior of Ω does not contain a great circle, then

$$\lambda_1(\Omega) \geq \frac{\pi}{4}.$$

2. Bounds, the early years (1978 - 2005)

> I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures of what a complete analysis of their asymptotic behaviour should aim at but, since for more than 35 years I have made no serious effort to prove them, I think I had better keep them to myself.

(Hermann Weyl, [**Wey50**])

We should not even attempt a rough sketch of the history of Laplacian eigenfunctions: clear landmarks are Chladni figures, the developement of Fourier series, Lorentz' conjecture and its resolution by Weyl as well as Courant's minimax representation. Our question is usually attributed to Yau, who in 1982 wrote in a collection of open problems

> 74. Let M be a compact surface. Let $\lambda_1 \leq \lambda_2 \leq \dots$ be the spectrum of M and let $\{\phi_i\}$ be the corresponding eigenfunctions. For each i, the set $\{x|\phi_i(x) = 0\}$ is a one-dimensional rectifiable simplicial complex. Let L_i be the length of such a set. It is not difficult to prove that $\liminf_{i\to\infty} \sqrt{\lambda_i}^{-1}(L_i)$ has a positive lower bound depending only on the area of M. (This was independently observed by Bruning [B]). It seems more difficult to find an upper bound if $\limsup_{i\to\infty} \sqrt{\lambda_i}^{-1}(L_i)$ (Yau, [**Yau82**])

The conjecture is now understood in a more general context: given a compact smooth manifold (M, g) and let ϕ be a Laplacian eigenfunction, then it is conjectured that

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \sim \lambda^{\frac{1}{2}}.$$

As has been said of the Kepler conjecture [**Rog58**], *many mathematicians believe, and all physicists know* this conjecture to be true – it is implied by very basic heuristic ideas and supported by numerics. For real-analytic (M, g) the conjecture has been proven by Donnelly & Fefferman [**DF88**]. In two dimensions they also derived an upper bound [**DF90**] for $C^{\infty}-$manifolds complementing Brüning's [**Brü78**] lower bound and yielding the state of the art

$$\lambda^{\frac{1}{2}} \lesssim \mathcal{H}^{1}\left(\{x \in M : u(x) = 0\}\right) \lesssim \lambda^{\frac{3}{4}}.$$

Another proof of a completely different nature has been given by Dong [**Don92**]: while Donnelly & Fefferman used Carleman inequalities, Dong's approach is based on an integral identity which has proven influental insofar as it inspired the approach of Sogge & Zelditch [**SZ11**]. In higher dimensions, the best upper (and, originally, lower) bounds date back to a frequency function approach by Hardt & Simon [**HS89**] and are not polynomial in λ.

2.1. Donnelly-Fefferman, Dong. We will not describe the arguments of Donnelly & Fefferman [**DF88, DF90**] who rely on Carleman estimates, however, we consider it worthwhile to discuss their derivation of the upper bound. As has been mentioned earlier, upper bounds in the generic $C^{\infty}-$case seem to require an understanding of how nodal domains arise from the manifold, which is a global problem. In the analytic case the eigenfunctions behave essentially like polynomials. Summarized in a rough picture,

> Let us indicate the idea for obtaining the upper bound, $\mathcal{H}^{n-1}(N) \leq c_2\sqrt{\lambda}$, of Theorem 1.2. First suppose that $P(x)$ is a non-zero polynomial of degree $c_3\sqrt{\lambda}$, defined for $x \in R^n$. Let $V = \left\{|x| < 1 \middle| P(x) = 0\right\}$. If $\mathcal{L}$ denotes the set of lines in R^n that intersect $|x| < 1$, then integral geometry gives
>
> $$\mathcal{H}^{n-1}(V) \leq \int_{\mathcal{L}} |L \cap V| d\mu(L)$$
>
> Here $L \in \mathcal{L}$ and $d\mu$ is a measure on $\mathcal{L}$. Moreover, $|L \cap V|$ denotes the cardinality of $L \cap V$. Clearly, $|L \cap V| \leq c_3\sqrt{\lambda}$ almost everywhere. So $\mathcal{H}^{n-1}(V)$ is bounded by a multiple of $\sqrt{\lambda}$. Our eigenfunction $F(x)$ need not be a polynomial but it does extend to an analytic function satisfying (1.8). We shall show that integral geometry methods carry over [...] (Donnelly & Fefferman, [**DF88**])

Dong's argument is centered around a new integral identity (he cites a paper by Alt, Caffarelli & Friedman [**ACF84**] as inspiration)

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) = \frac{1}{2}\int_M \frac{\Delta|u| + \lambda|u|}{\sqrt{|\nabla u|^2 + \frac{\lambda}{n}u^2}}.$$

He continues to bound the expression from above: abbreviating

$$q = |\nabla u|^2 + \frac{\lambda}{n}u^2,$$

he shows that for any subset of the manifold $\Omega \subset M$

$$2\mathcal{H}^{n-1}\left(\{x \in \Omega : u(x) = 0\}\right) \leq \frac{1}{2}\int_\Omega |\nabla \log q| + \sqrt{n\lambda}|\Omega| + |\partial\Omega|.$$

The bound is then implied by vanishing order estimates, which give control over the integral.

2.2. Almgren's monotonicity formula and Hardt-Simon.

> After all, this was a man whose bedside table when he died contained a gyroscope, two magnets, four colored balls from Cheerios boxes, two magnifying glasses, a Star Trek communicator button, and five pretty rocks.
> (Dana Mackenzie on Fred Almgren [**Mac97**])

The work of Hardt-Simon deals with the general smooth case; before describing their result, it seems most appropriate to quickly describe Almgren's monotonicity formula [**Alm79**], which is a key insight into the behavior of harmonic functions and provides the theoretical underpinnings in the Hardt-Simon approach.

Theorem (Almgren)**.** *If u is harmonic in the unit ball $B(0,1)$, then the function*

$$\frac{r\int_{B_r} |\nabla u|^2 dx}{\int_{\partial B_r} u^2 d\mathcal{H}^{n-1}}$$

is monotonically increasing in r on $(0,1)$.

An immediate consequence is unique continuation for the Laplacian: if a harmonic function vanishes in an open set, it vanishes everywhere. This monotonicity formula (and variations thereof) has been crucial in many different applications – one of them being the work of Hardt-Simon, whose work is nicely summarized in their introduction.

> Here we study, on a connected domain $\Omega \subset \mathbb{R}^n$, the zero set $u^{-1}\{0\}$ of a solution u of an elliptic equation
>
> $$a_{ij}D_iD_ju + b_jD_ju + cu,$$
>
> where a_{ij}, b_j, c are bounded and a_{ij} is continuous. Our principial result [...] is that the $(n-1)-$dimensional Hausdorff measure of $u^{-1}\{0\}$ is finite in a neighbourhood of any point $x_0 \in \Omega$ at which u has finite order of vanishing. [...] We actually obtain an explicit bound on the Hausdorff measure of $u^{-1}\{0\}$ in terms of the order of vanishing of u, the modulus of continuity of a_{ij}, and the bounds on a_{ij}, b_j, c. (Hardt & Simon, [**HS89**])

The generality of their result renders Laplacian eigenfunctions a mere special case, their result

$$\mathcal{H}^{n-1}\left(\{x \in \Omega : u(x) = 0\}\right) \lesssim \lambda^{c\sqrt{\lambda}}$$

for some $c > 0$ follows indeed within a few lines from the general scenario (they quote Donnelly & Fefferman on the order of vanishing). At the same time, naturally, the generality of their result does not quite incorporate the very particular structure of the problem of Laplacian eigenfunctions – despite this, it remains the only known bound for $n \geq 3$.

3. Bounds, the current philosophy (2005 - present)

3.1. Mangoubi and local geometry. A new surge of activity started in 2005 with the preprint of Mangoubi [**Man08b**] and the work on asymmetry due to Nazarov, Polterovich & Sodin [**NPS05**]. We quickly comment on the former. Mangoubi [**Man08b**] gave polynomial lower bounds on the inradius of a nodal domain; as we hinted at before, it was not immediately obvious to the community at that time that the isoperimetric inequality would imply polynomial lower bounds on the Hausdorff measure of the nodal domain. The core of Mangoubi's argument are local arguments based on Poincaré-type inequalities due to Maz'ya.

Theorem (Maz'ya, [**Maz63**])**.** *Let $Q \subset \mathbb{R}^n$ be a cube whose edge is of length a. Let $F \subset Q$. Then*

$$\int_Q |u|^2 dx \leq \frac{C_1 a^n}{cap(F, 2Q)} \int_Q |\nabla u|^2 dx$$

for all Lipschitz functions u on Q, which vanish on F.

Here, $2Q$ is simply the cube with twice the length and cap denotes the capacity, i.e.

$$\text{cap}(F, \Omega) = \inf_{u \in \mathcal{F}} \int_\Omega |\nabla u|^2 dx,$$

where $\mathcal{F} = \{u \in C^\infty(\Omega), u \equiv 1 \text{ on } F, \text{supp}(u) \subset \Omega\}$. This statement becomes useful in conjunction with a capacity-volume inequality.

Theorem (Maz'ya, [**Maz85**])**.** *For $n \geq 3$*

$$cap(F, \Omega) \geq C_3 |F|^{\frac{n-2}{n}}.$$

For details we refer to [**Man08b**]. Three years later, Mangoubi [**Man08a**] gave a quantitative improvement of these results. The techniques are once again 'classically elliptic' local statements. In particular, he uses quantitative versions of the statement that a Laplacian eigenfunction is essentially harmonic at length scales below $\lambda^{-1/2}$, the fact that harmonic functions within long and narrow domains subjected to Dirichlet boundary conditions must exhibit exponential growth (cf. Landis [**Lan63**]) and a propagation of smallness principle. One particular statement mentioned in [**Man08a**] is that in two dimensions, two nodal lines cannot be closer than $\lambda^{-1/2}$ for the entire duration of their existence – this is but a version of Hayman's theorem and will be generalized by us to an optimal version of this result in higher dimensions.

3.2. September 2010. Two completely new approaches were detailed in September 2010, when first Sogge & Zelditch [**SZ11**] and then Colding & Minicozzi [**CM11**] uploaded preprints giving new polynomial lower bounds for the nodal set. Their techniques are completely different.

Sogge & Zelditch prove the new integral formula

$$\lambda \int_M |u| d\mathcal{H}^n = 2 \int_{\{x : u(x) = 0\}} |\nabla u| d\mathcal{H}^{n-1}.$$

The left-hand side has an easy estimate from above

$$\int_{\{x : u(x) = 0\}} |\nabla u| d\mathcal{H}^{n-1} \lesssim \|\nabla u\|_{L^\infty} \mathcal{H}^{n-1} \left(\{x \in M : u(x) = 0\}\right),$$

where $\|\nabla u\|_{L^\infty} \lesssim \lambda^{\frac{n+1}{4}}$ is well-known. We require a lower bound for the L^1-norm. This is comparatively easy, Hölder

$$1 = \|u\|_{L^2}^2 \leq \|u\|_{L^1}^{\frac{p-2}{p-1}} \|u\|_{L^p}^{\frac{p}{p-1}}$$

for $p = 2(n+1)/(n-1)$ in conjunction with Sogge's L^p-estimates yields

$$\|u\|_{L^1} \geq \lambda^{\frac{1-n}{8}}$$

and this implies

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \geq \lambda^{\frac{7-3n}{8}}.$$

Colding & Minicozzi employ a geometric argument. They place a number of disjoint balls of size $\lambda^{-1/2}$ on the manifold such that

- the L^2-norm of the eigenfunction on their union is at least 3/4,
- the eigenfunction vanishes in the center of each ball and
- the doubling size of the balls is controlled, i.e.

  $$\int_{2B} u^2 \leq 2^d \int_B u^2$$

  for some parameter d.

Elliptic theory yields that for such a ball the measure of the nodal set within such a ball cannot be too small. Simultaneously, there cannot be too few of these balls: if there were only few, the L^2-mass would be contained in a small area and the L^p-mass would be too big and contradict Sogge's L^p-estimates. This argument gives

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \geq \lambda^{\frac{3-n}{4}}.$$

3.3. Sogge-Zelditch II. Finally, in August 2012, Sogge & Zelditch [**SZ**] described a modification of their earlier proof, which gives the same exponent as the Colding-Minicozzi argument. Their observation is that

$$\|\nabla u\|_{L^\infty} \lesssim \lambda^{\frac{n+1}{4}}$$

can be improved to

$$\|\nabla u\|_{L^\infty} \lesssim \lambda^{\frac{n+1}{4}} \|u\|_{L^1},$$

which leads to cancellation of the L^1-norm on both sides. Variations of these types of arguments were also given by Hezari & Wang [**HW12**] and Hezari & Sogge [**HS**]. Ariturk [**Ari**] discusses manifolds with boundaries.

4. Statement of results

Les solutions que j'ai données de ces questions principales
sont aujourd'hui généralement connues; elles ont été
confirmées par les recherches de plusieurs géomètres.
(Joseph Fourier, *Mémoire sur la Théorie analytique de la chaleur*)

4.1. Main result. Our main result is a new proof of the currently optimal bound. We should emphasize that the proof is not just a variation on existing arguments but a completely new argument relying on hithereto unused structures.

Theorem. *The volume of nodal sets satisfies*

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{3-n}{4}}.$$

The main thrust of the argument is that a Laplacian eigenfunction has a trivial evolution under the heat equation – at the same time, the heat equation is very well understood and has a multitude of interpretations, one of them being the Feynman-Kac formula and expectation under a diffusion process given by Brownian motion. Different boundary conditions give rise to different types of diffusion processes, however, the entire difference being in *their behavior at the boundary.* If the boundary was too small, these processes would barely differ at all.

A technical analysis of arguments of this type end up in quantities that are understood: our approach is fully self-contained with the exception of our using a previously quoted global inequality due to Sogge & Zelditch [**SZ**]

$$\lambda \frac{\|u\|_{L^1(M)}}{\|\nabla u\|_{L^\infty(M)}} \gtrsim \lambda^{\frac{3-n}{4}},$$

which is known to be sharp on spherical harmonics. We also sketch a variant of our proof that comes to rely on $\lambda^{\frac{1}{2}}\|u\|_{L^1(M)} \gtrsim \lambda^{\frac{3-n}{4}}\|u\|_{L^\infty(M)}$ (also used by Sogge & Zelditch [**SZ11**]), which is easily seen to be equivalent because of $\|\nabla u\|_{L^\infty(M)} \sim \lambda^{1/2}\|u\|_{L^\infty(M)}$.

4.2. A nonsqueezing result. We expect the argument to be applicable to more general diffusion processes and possibly other questions about Laplacian eigenfunctions. One example is the shape of nodal domains, where we briefly describe a simple geometrical result. It deals with the question whether nodal domains can be contained in a small neighbourhood of a 'flat' surface of codimension 1 – as it turns out, it cannot be squeezed to tightly along any single codimension. In two dimensions, the statement reduces to a statement mentioned by Mangoubi [**Man10**] and the theorem of Hayman [**Hay78**], however, in this generality it seems to be new.

Let $\Sigma \subset M$ be an arbitrary smooth $(n-1)-$dimensional surface. We ask whether a nodal domain can be contained in a small neighbourhood of Σ. The $\varepsilon-$neighbourhood of a generic geodesic (being itself as 'flat' as possible) on the torus $\mathbb{T}^2$ already coincides with the entire torus - we therefore need to place some restrictions on Σ for the question to be meaningful. Using $d_g(\cdot,\cdot)$ to denote the geodesic distance, we call Σ admissible up to distance r if

$$\forall x \in M : d_g(x,\Sigma) \leq r \quad \Longrightarrow \quad \#\left\{y \in \Sigma : d(x,y) = d(x,\Sigma)\right\} = 1.$$

This precludes the scenario of dense geodesics and implies that Σ is essentially flat at length scales smaller than r. Our next theorem states that a nodal domain cannot be much flatter than the wavelength $\lambda^{-1/2}$ in any direction.

Theorem. *There is a constant $c > 0$ depending only on (M,g) such that if $\Sigma \subset M$ is admissible up to distance $\lambda^{-1/2}$, then no nodal domain can be a subset of the $c\lambda^{-1/2}-$neighbourhood of Σ.*

As mentioned above, the function $u(x) = \mathrm{Re}\exp(i\sqrt{\lambda}x)$ on $\mathbb{T}^2$ endowed with the flat metric has all its nodal domains contained in a $0.5\lambda^{-1/2}$ neighbourhood of a geodesic of length 1 (being admissible up to $r = 0.5$) and the example easily generalizes to higher dimensions.

4.3. Miscellaneous. We also sketch a different kind of non-squeezing result in two dimensions. If two line segments contained in the nodal set are contained in a thin rectangle, then the rectangle has bounded eccentricity. Let (M, g) be as above and, additionally, two-dimensional. Fix some $\alpha > 1/2$. We define an *avoided crossing* as follows: let T be a geodesic line segment between two points $a, b \in M$ and D be a nodal domain. We say that D avoids a crossing if there exists a $\lambda^{-\alpha}$-neighbourhood of T such that

$$x \in \partial D \cap \left\{y \in M : d(y, T) = \lambda^{-\alpha}\right\} \implies \min\left(d(x, a), d(x, b)\right) \leq 2\lambda^{-\alpha}.$$

Proposition. *If D avoids a crossing, then*

$$d(a, b) \leq C\lambda^{1/2-\alpha} \log \lambda$$

for some constant $C < \infty$ depending only on (M, g).

This result is at the same level of quality as predicted by the best known general asymmetry results (cf. [**Man08a**]) but is a factor $\sqrt{\log \lambda}$ weaker than the currently best bound (cf. [**NPS05**], where this gain comes from using complex analysis techniques.) The tools we use (the notion of heat content in particular) have quite natural geometric implications and seem to have been under-studied. We formulate two (very hard) conjectures in the spirit of our approach and in that terminology whose resolution would imply a slightly sharper version of Yau's conjecture.

5. Some relevant tools

This section gives a short overview of tools. Its purpose is to provide a brief introduction into some of the notions that will make an appearance later on – we are well aware that this particular confluence of topics is very beautiful and are conscious of our sacrificing beauty at the altar of brevity.

5.1. Heat kernel. The theory of the heat equation and its interplay with the geometry of a manifold is well understood and it is a pleasure to refer to a remarkable recent book by Grigor'yan [**Gri09**]. Given a compact Riemannian manifold M, one way of studying the diffusion process

$$u_t - \Delta u = 0$$

is in terms of the heat Kernel

$$(e^{t\Delta} f)(x) = \int_M p(t, x, y) f(y) d\mathcal{H}^n(y),$$

where $p : \mathbb{R}_+ \times M \times M \to \mathbb{R}$ is called the heat kernel. The Laplacian is a compact, self-adjoint operator and gives rise to a sequence of eigenvalues λ_i and eigenfunctions ϕ_i, which form an orthonormal basis of $L^2(M)$. As such, one can rewrite

$$f = \sum_{i=1}^{\infty} a_i \phi_i,$$

note that

$$e^{t\Delta}\phi_i = e^{-\lambda_i t}\phi_i$$

and that hence

$$(e^{t\Delta} f)(x) = \sum_{i=1}^{\infty} a_i e^{-\lambda_i t}\phi_i.$$

As such, eigenfunctions of the Laplacian diagonalize the heat kernel. The maximum principle implies

$$p(t, x, y) \geq 0.$$

The heat kernel can be regarded as the probability distribution of a particle starting in x after time t. This interpretation implies in particular

$$\int_M p(t,x,y)d\mathcal{H}^n(y) = 1,$$

which is in essence the conservation law $\int_M u(x)dx$.[1] Our proof will require Gaussian estimates for the heat kernel, which in our context of a compact, smooth manifold is classical. We will require only these estimates on smaller and smaller length scales and thus essentially inherit them from the Euclidean space: this statement is due to Varadhan [**Var67**] and can be concisely written as

$$\lim_{t\to 0} -4t \log p(t,x,y) = d(x,y)^2,$$

where $d(\cdot,\cdot)$ is the distance introduced by the manifold. Gaussian estimates for large times mirror the local geometry, see [**Gri09**, Chapter 16].

5.2. Brownian motion.

contemplator enim, cum solis lumina cumque
inserti fundunt radii per opaca domorum:
multa minuta modis multis per inane videbis
corpora misceri radiorum lumine in ipso
(*for behold whenever*
The sun's light and the rays, let in, pour down
Across dark halls of houses: thou wilt see
The many mites in many a manner mixed)
Lucretius, *De rerum naturae*

Brownian motion is classical, see e.g. [**KS91**]. The most natural version of our main argument, however, runs into some interesting difficulties and is not carried out. Consider the setting of a nodal domain on a Riemannian manifold and start a Brownian motion at some point in its interior: the stopping time of hitting the boundary has all mathematical properties one could wish for because the path of a Brownian motion is almost surely continuous (analytically: the Dirichlet problem is not overly sensitive with regards to regularity of the boundary).

A natural analogue would be considering *reflected* Brownian motion, a stochastic process where a Brownian motion particle is orthogonally reflected upon impacting the boundary. The construction of this stochastic process is not trivial at all, see e.g. Bass & Hsu [**BH00**], and obviously requires the boundary to have some regularity.

The boundary of a nodal set on a Riemannian manifold can have singularities in which the normal vector is not even defined. It is known that the critical set

$$\{x \in D : u(x) = |\nabla u(x)| = 0\}$$

has $(n-1)-$dimensional *Minkowski* measure 0 – recent results by Cheeger, Naber & Valtorta [**CNV**] give upper bounds on its $(n-2)-$dimensional Minkowski measure. A celebrated result of Dahlberg [**Dah77**] states that in Lipschitz domains the harmonic measure and the Hausdorff measure $\mathcal{H}^{n-1}$ are absolutely continuous. This suggests that the rigorous construction of reflected Brownian motion in a nodal domain should be possible, however, it is certainly outside of the scope of this thesis: we will give a different proof

[1]This is a simplification: the statement is true in our setting of compact manifolds but false in general. Its truth depends on whether the heat equation has a unique solution on the manifold, which need not be the case if the manifold exhibits strong volume growth. Manifolds, for which the statement is true, are called *stochastically complete.* [**Gri09**, Section 8.4.]

not relying on reflected Brownian motion. Nonetheless, we do considered it the most natural embodiment of the main idea and will sketch a proof under the assumption of existence.

5.3. Feynman-Kac.

> I prefer concrete things and I don't like to learn more about abstract stuff than I absolutely have to. (Mark Kac, [**Kac85**])

The Feynman-Kac formula for the Dirichlet problem is also classical (see e.g. Taylor [**Tay96**, Section 11.3]). Given an open domain $\Omega \in \mathbb{R}^n$, $f \in C_0^\infty(\Omega)$, $x \in \Omega$, then

$$(e^{t\Delta_D} f)(x) = \mathbb{E}_x(f(\omega(t))\psi_\Omega(\omega,t)),$$

where $t > 0$ is arbitrary, $\omega(t)$ denotes an element of the probability space of Brownian motions starting in x, $\mathbb{E}_x$ is to be understood with regards to the measure of that probability space and

$$\psi_\Omega(\omega,t) = \begin{cases} 1 & \text{if } \omega([0,t]) \subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

We give a toy application.

Theorem (A compact Liouville-type theorem)**.** *Let (M,g) be a smooth, compact $n-$dimensional manifold without boundary and let $u \in C^\infty(\Omega)$ be a real harmonic function. Then u is constant.*

PROOF. We note that a harmonic function is invariant under the heat equation. Applying the Feynman-Kac formula, we get

$$u(x) = (e^{t\Delta_D}u)(x) = \mathbb{E}_x(u(\omega(t)))$$

for every $t \geq 0$ because $\psi_\Omega(\omega,t) \equiv 1$. However, as time becomes large

$$\lim_{t\to\infty} \mathbb{E}_x(u(\omega(t))) = \frac{1}{|M|}\int_M u(x)dx.$$

Since x was arbitrary, this gives the result. □

5.4. Heat content. Given an open, bounded set $\Omega \subset \mathbb{R}^n$, we use $p_t(x) : \mathbb{R}_+ \times \Omega \to [0,1]$ to denote the solution to the following heat equation

$$\begin{aligned} (\partial_t - \Delta_g)p_t(x) &= 0 \qquad && x \in \Omega \\ p_t(x) &= 1 && x \in \partial\Omega \\ p_0(x) &= 0 && x \in \Omega. \end{aligned}$$

The Feynman-Kac formula implies that this can be understood as the probability that a Brownian motion particle started in x will hit the boundary within t units of time.[2] The quantity

$$\int_\Omega p_t(x)dx$$

is called the heat content of Ω at time t. It can be seen as a 'soft' measure of boundary size – for large times the function will be roughly of size 1 in the entire domain and all information on the size of the boundary will be lost. Within t units of time for t small, however, a typical Brownian motion particle travels a distance of $\sim t^{1/2}$, which implies that $p_t(x)$ should be have the bulk of its L^1-mass close to the boundary (to make this notion precise, we refer to the book of Grigoryan [**Gri09**] for the Gaussian heat kernel estimates).

[2] It seems to be convention to define the heat content with initial datum 1 and Dirichlet boundary data; as such, it measures the cooling process. Since we do not actually rely on known results for the heat content, we choose this definition as measuring the probability of hitting the boundary is conceptually more natural.

This simple heuristics has been intensively studied. Around 1970, Greiner [**Gre70**] and Seeley [**See69**] independently showed that as $t \to 0^+$, there exists an asymptotic series

$$\int_{\Omega} p_t(x)dx \sim \sum_{n=1}^{\infty} a_n(\Omega) t^{\frac{n}{2}}.$$

There has been some interested in expressing the initial coefficients in terms of geometric quantities of Ω: this can indeed be done in the smooth concept and yields frightenting expressions (see, for example, the survey of Gilkey [**Gil08**]). Of course, however, in perfect correspondence with the Feynman-Kac formula and Varadhan's principle,

$$a_1(\Omega) = c|\partial\Omega|$$

for some normalizing constant $c > 0$. In 1994, van den Berg & Le Gall [**vdBLG94**] derived that $a_2(\Omega)$ can be written as the integral over the mean curvature of the boundary if $\partial\Omega$ is C^3. A recent 2013 paper [**AMM13**] seems to be among the first to adress the issue of rougher sets of finite perimeter (again in the asymtotic regime $t \to 0^+$). The heat content has an obvious and natural connection with the Feynman-Kac formula: for any point $x \in \Omega$ and any $t > 0$

$$\mathbb{E}_x(\psi_\Omega(\omega, t)) = 1 - p_t(x).$$

5.5. Harmonic measure. We will not appeal to the theory of harmonic measure at all. It should be mentioned nonetheless as it is, in some sense, the inverse problem to our approach (in a sense discussed below). Given a bounded, open domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$, any continuous function $f : \partial\Omega \to \mathbb{R}$ gives rise to a Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega \\ u &= f \qquad \text{on } \partial\Omega. \end{aligned}$$

If we fix a point $x \in \Omega$, the Riesz representation theorem implies the existence of a measure μ_x on $\partial\Omega$ such that

$$u(x) = \int_{\partial\Omega} f(y) d\mu_x(y).$$

This is the *harmonic measure* and one would like to understand its properties and how they depend on the regularity of the boundary. Kellogg proved in 1931 [**Kel31**] that if $\partial\Omega$ is $C^{1+\alpha}$, then the Radon-Nikodym derivative $\omega = d\mu/d\mathcal{H}$ with respect to the Hausdorff measure satisfies $\log \mu \in C^{1+\alpha}$ (an inverse was given by Alt-Caffarelli [**AC81**]). We mention the book by Capogna, Kenig & Lanzani [**CKL05**] on the subject.

Having already introduced the heat content, its connection to the harmonic measure is apparent: for a fixed point $x \in \Omega$, let $\omega(t)$ denote again an element of the probability space of Brownian motions starting in x and denote the random variable ω_∂

$$\omega_\partial = \omega\left(\sup\left\{t : \omega([0,t]) \in \Omega \setminus \partial\Omega\right\}\right).$$

Then, for any set $\mathcal{A} \in \partial\Omega$,

$$\mathbb{P}\left(\omega_\partial \in \mathcal{A}\right) = \omega_x(\mathcal{A}).$$

This gives a first indication of in what way our problem can be considered dual to the notion of harmonic measure: whereas here the problem is providing the domain and asks for properties of the harmonic measure, we are given pointwise estimates of a Feynman-Kac nature (from which rough bounds on the probability of hitting the boundary follow) and ask for properties of the boundary. This will be made precise in the proof.

6. A Proof of the Main Theorem

This section gives a complete proof of the main statement with a focus on brevity. Other applications of the argument are given in subsequent chapters – as will other proofs (formally different but all of them flowing from the same underlying idea).

6.1. Definitions. Let (M, g) be a smooth, compact manifold, let $u(x)$ be a Laplacian eigenfunction with eigenvalue λ and let D be an arbitary nodal domain of that eigenfunction. Without loss of generality, we assume the eigenfunction $u(x)$ to be positive within D: otherwise consider $-u(x)$. Given $u(x)$, we define a one-parameter functions $v(t,x)$ as solutions to the heat equation with $u(x)\big|_D$ as initial data and Dirichlet conditions. We set

$$v(t,x) := e^{-\lambda t}u(x)$$

and note that $v(t,x)$ then solves

$$\begin{aligned}(\partial_t - \Delta_g)v(t,x) &= 0 \quad \text{on } D \setminus \{u(x) = 0\}\\ v(t,x) &= 0 \quad \text{on } \{u(x) = 0\}\\ v(0,x) &= u(x) \quad \text{on } D.\end{aligned}$$

The heat equation with Dirichlet boundary conditions is a smoothing process for which the Feynman-Kac formula gives a natural stochastic interpretation. Note the connection with the heat content: for any point $x \in \Omega$ and any $t > 0$

$$\mathbb{E}_x(\psi_\Omega(\omega, t)) = 1 - p_t(x).$$

6.2. Ξ-evolution. Ideally, we would like to be in a position to not write this section entirely and simply take the heat equation with Neumann boundary conditions as our second diffusion operator and use reflected Brownian motion as its associated stochastic process. As has been pointed out in the previous discussion, a rigorous justification of these steps seems to not be provided by the literature (though we are confident it can be carried out).

Somewhat to our surprise, we were able to come up with a suitable substitute, which is entirely defined in admissible terms and has all the necessary properties.

Definition (Ξ-diffusion.)**.** *Let $f \in C_0^\infty(\Omega)$, $\omega(t)$ be an element from the probability space of Brownian motions starting in x and let ψ be their survival probability*

$$\psi_\Omega(\omega, t) = \begin{cases} 1 & \text{if } \omega[0,t]) \subset \Omega \\ 0 & \text{otherwise.}\end{cases}$$

Then we define

$$(e^{t\Xi}f)(x) := \mathbb{E}_x(f(\omega(t))\psi_\Omega(\omega,t)) + \mathbb{E}_x(1 - \psi_\Omega(\omega,t))f(x).$$

As is easily seen from the definition,

$$(e^{t\Xi}f)(x) = e^{t\Delta_D}f + \mathbb{E}_x(1 - \psi_\Omega(\omega,t))f(x),$$

where Δ_D denotes the Laplacian with Dirichlet boundary conditions. Geometrically, we take an average over the function at the points where Brownian motion ends up being (just like the heat equation) except in cases, where we hit the boundary and just take the value of the initial datum f in x (whereas for Δ_D these points are discarded). This operator is initially smoothing but ceases being so as time progresses.

It is to easy see (and motivating our notation) that

$$\lim_{t\to 0} \frac{e^{t\Xi}f - f}{t} = \Delta f.$$

Furthermore (in our setting where everything is compact)

$$\lim_{t\to\infty} e^{t\Xi} f = f.$$

A crucial property is conservation of L^1-norm

$$\int_\Omega e^{t\Xi} f dx = \int_\Omega f dx.$$

This statement is equivalent to

$$\int_\Omega p_t(x)u(x)dx = \int_\Omega (1 - e^{t\Delta_D})u(x)dx.$$

A stochastic argument would be to say that among paths not leaving the domain, it is equally likely to start in a point x and end in a point y than the other way around – a statement that follows from the symmetry $p(t,x,y) = p(t,y,x)$ of the heat kernel.

6.3. A Comparison Lemma. We are interested in comparing the behavior of the Dirichlet solution $e^{t\Delta_D}u$ with the behavior of $e^{t\Xi}u$ on a fixed nodal domain D, where we assume without loss of generality that $u\big|_D \geq 0$ (otherwise: consider $-u(x)$). It is obvious from the definition that

$$e^{t\Xi}u \geq e^{t\Delta_D}u.$$

Lemma. *There exists a constant $C > 0$ depending only on (M,g) such that*

$$e^{t\Xi}u - e^{t\Delta_D}u \leq Ct^{1/2}p_t(x)\|\nabla u\|_{L^\infty}.$$

Proof. The definition yields that

$$e^{t\Xi}u - e^{t\Delta_D}u = p_t(x)u(x).$$

It follows from heat kernel asymptotics that the quantity $p_t(x)$ is localized in a $\sim t^{1/2}-$neighbourhood of the nodal set and has an exponentially decaying tail at larger distances (or, alternatively via Feynman-Kac, a Brownian motion particle travels only a distance of $\sim t^{1/2}$ within time t and the probability of traveling farther has a (super-)exponentially decaying tail). The statement then follows from a consequence of the mean-value theorem

$$u(x) \leq d(x,\partial D)\|\nabla u\|_{L^\infty}.$$

□

Remark. The manifold is compact: therefore the estimate of $p_t(x)$ being localized within a $t^{1/2}-$neighbourhood of the nodal set is extremely rough for large time and only really accurate for $t \lesssim \lambda^{-1/2}$, which is precisely the time-scale on which the Lemma will be ultimately applied.

6.4. Boundary size via heat content. This section gives a proof of the limit relation

$$\lim_{t\to 0^+} \frac{1}{\sqrt{t}} \int_D p_t(x)dx \sim \mathcal{H}^{n-1}(\partial D).$$

A limit relation of this type for sets of finite perimeter has also been published by Miranda, Pallara, Paronetto & Preunkert [**MPPP07**] in 2005: their result is concerned with precise constants at the cost of being applicable only in $\mathbb{R}^n$. We do not care about constants and give a rougher proof on manifolds. The result now follows once we show that up to constants depending on the manifold

Proof. The involved quantities are essentially local and so is our argument: it is known that the critical set

$$\{x \in D : u(x) = |\nabla u(x)| = 0\}$$

has $(n-1)-$dimensional *Minkowski* measure 0 - recent results by Cheeger, Naber & Valtorta [**CNV**] even give bounds on its $(n-2)-$dimensional Minkowski measure. Fix a small $t > 0$ and cover the ∂D with cubes of side length $t^{-1/2}$. We call a cube *regular* if it does not contain an element of the singular set and *singular* otherwise. From the result above, it follows that the number of singular cubes is of order $o(t^{(-(n-1)/2)})$.

Regular cubes. Let $t > 0$ be fixed and let Q be a regular cube. Once a cube is regular, there is no need to further refine it as time tends to 0: the nodal set is given as a $C^\infty-$hypersurface.

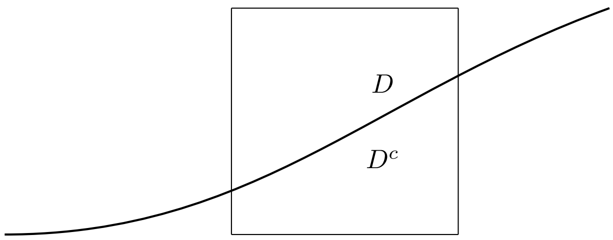

FIGURE 4. A regular cube with its nodal set.

To see this, we fix a second time-parameter $z > 0$ and study the behavior of

$$\lim_{z \to 0^+} \frac{1}{\sqrt{z}} \int_{D \cap Q} p_z(x) dx$$

as $z \to 0^+$. As $z \to 0^+$, the function $p_z(x)$ becomes concentrated in smaller and smaller neighbourhoods of the surface. Since the surface is locally C^∞, we may treat it as a flat hyperplane of codimension 1 embedded in $\mathbb{R}^n$ and use the explicit heat kernel in $\mathbb{R}^n$ to compute the relevant quantity. The fact that this argument is actually stable under small perturbations of the surface follows immediately from Varadhan's large deviation formula [**Var67**]

$$\lim_{z \to 0^+} -4z \log K(z, x, y) = d(x, y)^2.$$

In turn this implies that as $z \to 0$, we have that $p_z(x)$ is of size $p_z(x) \sim 1$ for x in a $z^{1/2}-$neighbourhood of ∂D and vanishes superexponentially at larger distances. From the (local) $C^\infty-$regularity of the set $\partial D \cap Q$, we get that

$$\lim_{z \to 0^+} \frac{1}{\sqrt{z}} \int_{D \cap Q} p_z(x) dx \geq c\mathcal{H}^{n-1}(\partial D \cap Q)$$

for some constant $c > 0$ depending only (M, g).

Singular cubes. It remains to show that the error introduced by those cubes containing an element of the singular set is small: it is not enough to note that their relative proportion is small because they are weighted with a factor $t^{-1/2}$, which becomes singular for small times. Using $0 \leq p_t(x) \leq 1$ gives

$$\left| \frac{1}{\sqrt{t}} \int_{D_{\text{sing}}} p_t(x) dx \right| \leq \frac{1}{\sqrt{t}} |D_{\text{sing}}|.$$

D_{sing} consists of $o(t^{(-(n-1)/2)})$ cubes of side-length $t^{1/2}$, therefore

$$\frac{1}{\sqrt{t}} |D_{\text{sing}}| \leq \frac{1}{\sqrt{t}} t^{n/2} o(t^{(-(n-1)/2)}) = o(1).$$

Improved estimates on the Minkowski dimension actually imply a faster rate of decay but these are not necessary for the conclusion of the argument. $\square$

6.5. Conclusion. Fix again an arbitrary nodal domain D and we assume again without loss of generality that $u(x)\big|_D \geq 0$. The heat equation, the comparison lemma and the L^1-conservation of Ξ give

$$\begin{aligned} e^{-\lambda t}\int_D u(x)dx &= \int_D e^{t\Delta_D}u(x)dx \\ &\geq \int_D e^{t\Xi}u(x)dx - Ct^{1/2}\|\nabla u\|_{L^\infty}\int_D p_t(x)dx \\ &= \int_D u(x)dx - Ct^{1/2}\|\nabla u\|_{L^\infty}\int_D p_t(x)dx \end{aligned}$$

Therefore

$$\int_D p_t(x)dx \gtrsim \frac{1-e^{-\lambda t}}{t^{1/2}}\frac{\|u\|_{L^1(D)}}{\|\nabla u\|_{L^\infty(D)}}.$$

The result now follows from

$$\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\int_D p_t(x)dx \sim \mathcal{H}^{n-1}(\partial D),$$

summation over all nodal domains and the inequality (cf. [**SZ**])

$$\|\nabla u\|_{L^\infty} \lesssim \lambda^{\frac{n+1}{4}}\|u\|_{L^1}.$$

7. An application: thin nodal sets

In this section, we give a proof of the 'non-squeezing result'. It is based on the fact that at scale $\sim \lambda^{-1/2}$ the neighbourhood of a surface admissible up to $\lambda^{-1/2}$ behaves like the neighbourhood of a hyperplane in $\mathbb{R}^n$, which allows for problems to be reduced to well-known one-dimensional facts (indeed, our notion of 'admissible' is chosen such that this is true). It is perhaps easiest to understand the proof first for $n=2$, where all key elements are already present: for $n=2$ an admissible surface is merely a curve with curvature $\kappa \leq \lambda^{-1/2}$. The main idea is that a Brownian motion particle is equally likely to wander in every direction: in a a 'squeezed nodal domain', it will hit the boundary too often.

Proof of Theorem 2. We consider the heat flow with Dirichlet conditions on the nodal domain D

$$\begin{aligned} (\partial_t - \Delta_g)v(t,x) &= 0 \quad \text{on } D \setminus \{u(x)=0\} \\ v(t,x) &= 0 \quad \text{on } \{u(x)=0\} \\ v(0,x) &= u(x). \end{aligned}$$

The other case being identical, we assume $u(x)$ to be positive in D. We start by proving a statement showing the existence of some point $x \in D$ such that Brownian motions starting in x are not very likely to hit the boundary: from physical intuition it is not surprising that these points should be close to those points, where the eigenfunction assumes its maximum and this guides our argument; for some fascinating results in that direction, we refer to Grieser & Jerison [**?**]. We will prove that

$$\forall\ t > 0 \qquad \inf_{x\in D} p_t(x) \leq 1 - e^{-\lambda t}$$

by showing the following slightly stronger statement

$$\forall x \in D \qquad u(x) = \|u\|_{L^\infty(D)} \implies p_t(x) \leq 1 - e^{-\lambda t}.$$

Given a $x \in D$ with $u(x) = \|u\|_{L^\infty(D)}$, we see using the heat equation and Feynman-Kac

$$\begin{aligned} e^{-\lambda t}\|u\|_{L^\infty(D)} &= e^{-\lambda t}u(x) = \mathbb{E}_x(u(\omega(t))\psi_D(\omega(t))) \\ &\leq \|u\|_{L^\infty(D)}\mathbb{E}_x(\psi_D(\omega(t))) = \|u\|_{L^\infty(D)}(1-p_t(x)). \end{aligned}$$

This proves the claim.

We now set the time to be $t = \lambda^{-1}$. It remains to show that choosing c small enough, we can derive a contradiction to this bound on $p_t(x)$. Take a small $c > 0$ and a $c\lambda^{-1/2}-$neighbourhood of the admissible surface Σ. Assume D to be a nodal domain fully contained in that set and let $x \in D$ be such that $u(x) = \|u\|_{L^\infty(D)}$. The statement we need to contradict is $p_{\lambda^{-1}}(x) \leq 1 - e^{-1}$ and we will do so but suitably bounding the probability of leaving the $c\lambda^{-1/2}-$neighbourhood of Σ from below to achieve a contradiction: if the nodal domain was contained in a small neighbourhood of Σ, it will hit the boundary pretty definitely and we expect $p_{\lambda^{-1}}(x)$ to be arbitrarily close to 1 as c becomes small. The remainder of the proof consists in making this precise.

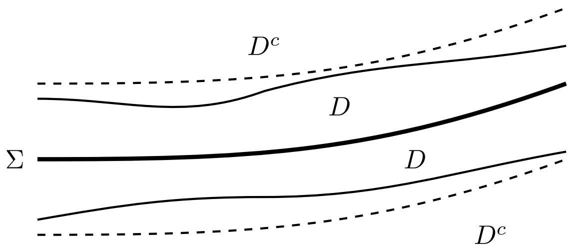

FIGURE 5. An example in two dimensions: the surface Σ (thick), its $c\lambda^{-1/2}-$neighbourhood (dashed) and the boundary of the nodal domain D.

Let now $x \in D$. A Brownian motion starting in x has, at any point inside D, at most $n-1$ 'good' directions in which it can wander unhindered and at least 1 'bad' direction: it is only allowed to wander in direction of the normal of Σ for a very short distance before impacting on the boundary. For $0 < c \ll 1$, the curvature of the surface plays hardly any role: we can assume the surface to be a flat hyperplane.

If $\Sigma = \mathbb{R}^{n-1}$, then $\operatorname{dist}(\omega(t), \Sigma)$ behaves like a one-dimensional Brownian motion $B(t)$ and we have

$$p_{\lambda^{-1}(x)} \geq \mathbb{P}\left(\sup_{0<s<\lambda^{-1}} B(t) > c\lambda^{-1/2}\right).$$

This quantity, however, is well-understood and the reflection principle (see e.g. [**KS91**]) implies

$$\mathbb{P}\left(\sup_{0<s<\lambda^{-1}} B(t) > c\lambda^{-1/2}\right) = 2\mathbb{P}\left(B(\lambda^{-1}) > c\lambda^{-1/2}\right).$$

However, $B(\lambda^{-1})$ is just a random variable following a normal distribution with mean $\mu = 0$ and variance $\sigma = \lambda^{-1}$. By symmetry

$$2\mathbb{P}\left(B(\lambda^{-1}) > c\lambda^{-1/2}\right) = \mathbb{P}\left(\left|B(\lambda^{-1})\right| > c\lambda^{-1/2}\right)$$

and by bounding the normal distribution by its maximal value

$$\mathbb{P}\left(\left|B(\lambda^{-1})\right| > c\lambda^{-1/2}\right) \geq 1 - \int_{-c\lambda^{-1/2}}^{c\lambda^{1/2}} \frac{1}{\sqrt{2\pi}}\frac{1}{\lambda^{1/2}}dx = 1 - \sqrt{\frac{2}{\pi}}c.$$

This yields a contradiction for $c < \sqrt{\pi}/(\sqrt{2}e)$ in the case of $\Sigma = \mathbb{R}^{n-1}$. A perturbative version of this argument applies to more general curved surfaces (with a potentially smaller c). $\square$

Question. It could be interesting to study isoperimetric principles for these types of problems. Let $\Sigma \subset \mathbb{R}^n$ be a $C^\infty-$surface, $x \in \Sigma$ and $\varepsilon > 0$. Is the probability of a Brownian motion leaving a $\varepsilon-$neighbourhood of Σ minimized in the flat case $\Sigma = \mathbb{R}^{n-1}$ and the Brownian motion starting in a point $x \in \Sigma$? Do minimal surfaces play a distinguished role?

8. An application: avoided crossings

Let (M, g) be as above and, additionally, two-dimensional. Fix some $\alpha > 1/2$. We define an *avoided crossing* as follows: let T be a geodesic connecting two points $a, b \in M$ and D be a nodal domain containing T. We say that D avoids a crossing if there exists a $\lambda^{-\alpha}$-neighbourhood of T such that if a point in the nodal domain $x \in D$ is at distance $\lambda^{-\alpha}$ from the geodesic $d(x,T) = \lambda^{-\alpha}$, then it must be close to one of the of the endpoints a or b

$$\min(d(a,x), d(b,x)) \leq \lambda^{-\alpha}.$$

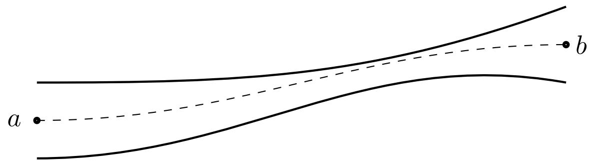

FIGURE 6. Two nodal lines almost crossing: a nodal domain contained in a small neighbourhood of a geodesic (dashed).

Alternatively, between a and b every point x of the nodal domain is at distance at most $\lambda^{-\alpha}$ from the geodesic line segment T. If α is big, then for this to be possible, a and b need to be very close together.

Proposition. *If D avoids a crossing, then*

$$d(a,b) \leq C\lambda^{1/2-\alpha} \log \lambda$$

for some constant $C < \infty$ depending only on (M, g).

We have a nontrivial statement precisely if $\alpha > 1/2$ – as we have seen above, a nodal line may well be contained in the $\lambda^{-1/2}$ neighbourhood of a geodesic. The result could be optimal up to the logarithmic factor.

PROOF. We consider the set

$$D \cap \left\{ y \in M : d(y,T) \leq \lambda^{-\alpha} \right\}$$

and cover it with N squares of scale $\lambda^{-\alpha} \times \lambda^{-\alpha}$, which we call $R_1, R_2, \dots, R_N$ and where the enumeration is such that R_i borders on R_{i-1} and R_{i+1}. Our goal is to prove the upper bound $N \lesssim \sqrt{\lambda} \log \lambda$ on the number of squares. This will then imply the result since $d(a,b) \sim \lambda^{-\alpha} N$.

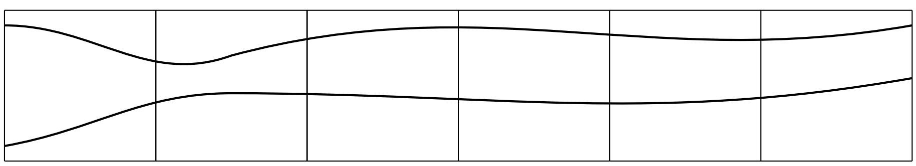

FIGURE 7. An almost crossing and a covering with squares.

Let us quickly illustrate the main idea: we consider the evolution of the heat equation with Dirichlet boundary with the eigenfunction as initial data for very short time $t = \lambda^{-2\alpha}$. The explicit solution implies that this time is too short for any real change to happen, the function is almost static on that time scale: we write

$$\begin{aligned}(\partial_t - \Delta_g)v(t,x) &= 0 \quad \text{on } D \setminus \{u(x)=0\}\\ v(t,x) &= 0 \quad \text{on } \{u(x)=0\}\\ v(0,x) &= u(x).\end{aligned}$$

Since $\alpha > 1/2$ and $t = \lambda^{-2\alpha}$, we have

$$v\left(\lambda^{-2\alpha}, x\right) = e^{-\lambda^{1-2\alpha}}u(x) \sim u(x).$$

Let us now consider a square in the covering and a Brownian motion particle: it moves a distance of $\sim \lambda^{-\alpha}$: it will thus likely either enter another square or impact on the boundary of the nodal domain (and both events will happen with a probability uniformly bounded away from 0). However, the effect of particles impacting on the boundary implies a loss of the L^1-norm, which we know is not there – therefore this loss is being counterbalanced by the surviving particles carrying back larger mass.

A Brownian motion particle started in R_i for time $t = e^{-\lambda^{\alpha}}$ can either impact on the boundary with probability $p_b > 0$ (bounded away uniformly from 0), can end up in any of the other squares with probability p_{ij} or exit the entire covered domain entirely with probability p_{ie}. Note that

$$p_b + \sum_{j=1}^{N} p_{ij} + p_{ie} = 1.$$

Using the Feynman-Kac formula, this implies

$$e^{-\lambda^{1-2\alpha}} \sup_{x\in R_i} |u(x)| \leq p_{ie}\|u\|_{L^\infty(M)} + \sum_{j=1}^{N} p_{ij} \sup_{x\in R_j} |u(x)| \qquad (\diamond)$$

Note that the decay of the heat kernel implies

$$\begin{aligned}p_{ij} &\lesssim \exp\left(-|i-j|^2\right)\\ p_{ie} &\lesssim \exp\left(\min\left(i^2, (N-i)^2\right)\right).\end{aligned}$$

Let us now prove the statement by contradiction: we assume from now on that $N \gtrsim \sqrt{\lambda}\log\lambda$.

Pick some $N/3 \leq i \leq 2N/3$. Then the contribution gained from exiting the entire domain is negligible since

$$p_{ie}\|u\|_{L^\infty(M)} \lesssim \lambda^{\frac{n-1}{4}} e^{-N^2/100} \leq \lambda^{\frac{n-1}{4}} \left(\frac{c}{\lambda}\right)^{\lambda\log\lambda}.$$

Let us now imply the inequality $(\diamond)$ for $i = \lfloor N/2 \rfloor$. It now implies that there is a rectangle j such that

$$\exp\left(c_1 |\lfloor N/2 \rfloor - j|^2\right) \sup_{x\in R_{\lfloor N/2 \rfloor}} |u(x)| \leq \sup_{x\in R_j} |u(x)|$$

for some universal constant c_1 depending only on (M,g). We want to iterate this inequality several times to show that u has to be much bigger than $\sup_{x\in R_{\lfloor N/2\rfloor}} |u(x)|$ at some other place. If $j \leq N/3$ or $j \geq 2N/3$, we quit, otherwise we reiterate the procedure until the index leaves the range $\{N/3, N/3+1, \dots, 2N/3\}$. The worst case is that for each i the inequality holds true with $j = i+1$ in which case we still have

$$(1+c_2)^{N/6} \sup_{x\in R_{N/2}} |u(x)| \leq \|u\|_{L^\infty(M)} \lesssim \lambda^{\frac{n-1}{4}}.$$

At the same time we have the vanishing order estimate due to Donnelly & Fefferman and thus

$$\sup_{x\in R_{\lfloor \frac{N}{2} \rfloor}} |u(x)| \gtrsim \inf_i \sup_{x\in R_i} |u(x)| \gtrsim \left(\frac{1}{\lambda^{c_2\alpha}}\right)^{\sqrt{\lambda}}$$

for some $c_2 > 0$ depending only on the manifold, which combined implies

$$N \lesssim \sqrt{\lambda} \log \lambda.$$

$\square$

Remark. This example gives a heat-flow approach to the phenomenon that elliptic equations in narrow domains exhibit rapid growth – a classical elliptic version of this principle also appears in the work of Mangoubi [**Man10**].

9. Conjectures

9.1. A geometric conjecture. The heat content is a very stable notion and well-defined even for very irregular domains with rough boundaries. We consider the following statement to be highly plausible and, at least compared to other conjectures in this chapter, probably not overly difficult.

Conjecture (Heat content isoperimetry)**.** *Let (M, g) be a compact $C^\infty-$manifold without boundary. There exists a constant $c > 0$ depending only on (M, g) such that for any open subset $N \subset M$ and all times $t > 0$*

$$\int_N p_t(x)dx \le c\mathcal{H}^{n-1}(\partial N)\sqrt{t},$$

where we define $\mathcal{H}^{n-1}(\partial N) := \infty$ if it is undefined.

Extremizers of the inequality need to have a smooth boundary: small irregularities in the boundary increase the surface measure but have very limited impact on the left-hand side. It would be interesting to understand the relation between the nature of extremizers and geometric properties of the manifold.

If the domain N has the property that there is a real number $r > 0$ such that each point $x \in N$ is contained in a ball of radius r itself entirely contained in N (but possibly centered around another point), then the two quantities should be comparable up to $t \sim r^2$. If N is a nodal domain of the Laplacian, the Faber-Krahn inequality implies that the inradius is at most $\sim \lambda^{-1/2}$ and therefore $t \sim \lambda^{-1}$ is the maximum time up to which we expect the quantities to be comparable.

9.2. A Local Yau conjecture. However, it does seem very likely that this heuristics is sharp and that both quantities are indeed comparable up to $t = \lambda^{-1}$.

Conjecture (Local Yau conjecture.)**.** *Let D be a nodal domain, then*

$$\lambda^{\frac{1}{2}} \int_D p_{\lambda^{-1}}(x)dx \sim \mathcal{H}^{n-1}(\partial D).$$

Of course, the trivial estimate

$$\int_D p_{\lambda^{-1}}(x)dx \le |D|$$

would then immediately imply

$$\mathcal{H}^{n-1}\left(x \in M : u(x) = 0\right) \lesssim \lambda^{\frac{1}{2}}.$$

We consider this conjecture interesting and natural because it mirrors a previous developement regarding the inradius of a nodal domain. We expect the inradius to be of size $\lambda^{-1/2}$, or – for emphasis – we expect every point to be at a distance of at most $\lambda^{-1/2}$ to the boundary. We know from the Croke-Dzerdinski theorem that this is true in the sense of an average over geodesics eminating from a point. At the same time, we expect the distance function $d(\cdot, \partial D)$ to be locally C^2 up to some distance $\lambda^{-1/2}$ from the boundary. Speaking in verbal puzzles,

inradius: Croke-Dzerdinski = regularity of the distance function : Local Yau conjecture,

where we should remark that one can at most hope for the regularity of the distance function in some averaged sense (of which the Local Yau conjecture may serve as a possible quantization).

9.3. Geometric structure of nodal sets. The quantity $p_t(x)$ can be seen as a local measure of the closeness and size of the boundary.

Conjecture. *Let (M,g) be a compact $C^\infty-$manifold without boundary. There exists a constant $c > 0$ depending only on (M,g) such that if $p_t(x)$ is globally defined with respect to the nodal set of a Laplacian eigenfunction with eigenvalue λ, then*

$$p_{\lambda^{-1}}(x) > c \qquad \text{for all } x \in M.$$

This conjecture, while seeming likely, should be extremely difficult. In particular, combining it with heat content isoperimetry at time $t = \lambda^{-1}$ immediately gives

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{1}{2}}.$$

9.4. Heat content, Laplacian eigenvalues and the inradius. Given a domain $\Omega \in \mathbb{R}^n$, we can define the first eigenvalue of the domain as

$$\lambda_1(\Omega) = \inf_{f \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla f(x)|^2 \, dx}{\int_\Omega f(x)^2 dx}.$$

An inequality of the form

$$\lambda_1(\Omega) \gtrsim (\mathrm{inrad}(\Omega))^{-2}$$

is trivial if $n = 1$, true for simply connected domains in $n = 2$ (Hayman's theorem) and false for $n \geq 3$. Indeed, in dimensions $n \geq 3$ it is possible to introduce very thin spikes making the inradius small but having little overall influence on the eigenvalue. However, Lieb [**Lie83**] in a celebrated paper has shown that Hayman's theorem 'essentially' generalizes to higher dimensions: for any domain $\Omega \in \mathbb{R}^n$, there is a ball B of radius $r \sim \lambda^{-1/2}$ such that $|\Omega \cap B| \sim |B|$ (the theorem also gives a precise relationship between the implicit constants). We conjecture that this phenomenon persists for the heat content.

Conjecture. *Let $\Omega \in \mathbb{R}^n$ be a bounded open set. If $c_1 > 0$ and some $t > 0$*

$$\int_\Omega p_t(x) dx \geq c_1 \mathcal{H}^{n-1}(\partial\Omega)\sqrt{t},$$

then there is a ball B of radius $\sqrt{t}$ such that $|B \cap \Omega| \geq c_2 |B|$, where c_2 depends only on the dimension and c_1.

Of course, via Lieb's theorem, this would establish a mutual equivalence between the first Laplacian eigenvalue, the size of balls having a large intersection with the domain and the time up to which heat content isoperimetry is sharp ($t = \lambda_1(\Omega)^{-1}$).

10. A variation of the proof

In this short section we quickly record what we consider the cleanest embodiment of the main idea. It is technically not the simplest as it relies on the heat content isoperimetry conjecture as well as the construction of reflected Brownian motion in the nodal domain but it is certainly the most transparent.

Theorem. *Assuming heat content isoperimetry and existence of reflected Brownian motion, we have*

$$\mathcal{H}^{n-1}\left(\{x \in M : u(x) = 0\}\right) \gtrsim \lambda^{\frac{3-n}{4}}.$$

Proof. The proof has strong similarities to our previous argument. Again, without loss of generality, we assume $u(x) > 0$ on D and write $e^{t\Delta_D}$ and $e^{t\Delta_N}$ for evolution under Dirichlet and Neumann boundary conditions, respectively. Our new comparison estimate is even simpler and states we have on the nodal domain D that

$$e^{t\Delta_N} u - e^{t\Delta_D} u \leq p_t(x) \|u\|_{L^\infty(D)}.$$

The proof for this comparison statement is easy to sketch: the difference between Dirichlet and Neumann solutions arises from those Brownian motions hitting the boundary. The difference is maximized if all those particles hitting the boundary arrive in a maximum of u after having been reflected. Integrating the comparison at time $t = \lambda^{-1}$ yields

$$\begin{aligned} e^{-\lambda t} \int_D u(x)dx &= \int_D e^{t\Delta_D} u(x)dx \\ &\geq \int_D e^{t\Delta_N} u(x) - p_t(x)\|u\|_{L^\infty(D)} dx \\ &= \int_D u(x)dx - \|u\|_{L^\infty(D)} \int_D p_t(x)dx. \end{aligned}$$

Therefore, assuming heat content isoperimetry and using the Sogge-Zelditch inequality

$$\mathcal{H}^{n-1}(\partial D) \gtrsim \lambda^{1/2} \frac{\|u\|_{L^1}}{\|u\|_{L^\infty}} \gtrsim \lambda^{\frac{3-n}{4}}.$$

□

CHAPTER 4

Curvature of Level Sets of Harmonic Functions

The miller sees not all the water that goes by his mill.
(Robert Burton, *Anatomy of Melancholy*)

1. Introduction

Motivation. Here we study the local structure of harmonic functions. The problem is of independent interest but obviously motivated by our study of Laplacian eigenfunctions. Given a Laplacian eigenfunction $-\Delta u = \lambda u$, one can locally rescale by a factor of $\lambda^{-1/2}$ to transform its solution into the solution of the equation $-\Delta v = v$ and further rescaling additionally diminishes the right hand side. This is a heuristic justification for the guiding principle that a Laplacian eigenfunction behaves like a harmonic function at spatial scales below the wavelength $\lambda^{-1/2}$ and is at the heart of Bers scaling, see e.g. the survey of Zelditch [**Zel08**, Section 3.11]. In particular, there is some hope that purely local properties of eigenfunctions can be treated as perturbations of harmonic functions at the cost of statements being restricted to a small scale. Our particular object of study is the (mean) curvature of level sets of harmonic functions.

Known results. Level sets of solutions of partial differential equations in general are very well studied – the literature is *vast*, an excellent starting point is the book of Kawohl [**Kaw85**]. We summarize the development of a particular question: one of the first results is given in Ahlfors [**Ahl10**] and states that the level curves of the Green function on a simply connected convex domain in the plane are convex Jordan curves. This result is a cornerstone and has been extended in various directions, for example to higher dimensions and different operators (the $p-$Laplacian), see for instance [**CS82**, **Gab57**, **Lew77**]. These results are purely qualitative. Results of a more qualitative nature usually state that *provided* a level set is convex, some notion of curvature – Gauss-curvature, principial curvature or a product expression containing one of these and some power of the norm of the gradient – assumes its minimum at the boundary. Early results in this direction are due to Ortel-Schneider [**OS83**] and Longelli [**Lon83**, **Lon87**], a recent result of this type was given by Chang-Ma-Yang [**CMY10**].

Our question. We are interested in the *opposite* problem: showing upper bounds on the curvature in the interior. It is not obvious that this can be done at all: indeed, further below we derive a class of harmonic function in three dimensions, whose principial curvatures are arbitrarily large. We learned of the possibility of giving suitable bounds through a recent paper of De Carli & Hudson [**DCH10**], who proved an upper bound.
The statement of De Carli & Hudson is as follows.

Theorem 1.1 Suppose that $\Delta u = 0$ on $B_r(0) = \{|x| \leq r\} \subset \mathbb{R}^2$ with continuous boundary values $f(\theta)$. Assume that $f(\theta) > 0$ on an interval $I = (\alpha, \beta) \subset [-\pi, \pi], f(\alpha) = f(\beta) = 0$ and $f(\theta) < 0$ otherwise. Suppose

$$u(0,0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta = 0.$$

> Then Z [the zero set] is a curve and its curvature at $(0,0)$ is bounded in absolute value by $\frac{C}{r}$, where $C \leq 24$ does not depend on u. [De Carli & Hudson, [**DCH10**]]

Our theorem gives the sharp result and completely classifies extremizers.

Theorem. *Let $u : B(0,1) \to \mathbb{R}$ be harmonic. If*

$$\gamma := \{x : u(x) = u(0)\}$$

is diffeomorphic to an interval, then the curvature of γ in the origin satisfies

$$\kappa \leq 8.$$

The bound is sharp and attained for the unique extremizer (up to symmetries)

$$w(x,y) = \frac{(x^2+y^2-1)(x-2x^2+x^3-4y^2+xy^2)}{(1-2x+x^2+y^2)^3}.$$

The bound $\kappa \leq 24$ was recently proven by De Carli & Hudson [**DCH10**]. There are many natural related questions (general differential operators of second order, right hand side, L^p-norms of the curvature integrated over the entire ball, other geometric quantities, ...). We would like to draw attention to two elementary questions in particular. Suppose for $B(0,1) \subset \mathbb{R}^n$ that $u : B(0,1) \to \mathbb{R}$ is harmonic and $\{x : u(x) > u(0)\}$ is simply connected. What bounds can be proven on the $(n-1)-$dimensional measure of the level set going through the origin? The question already seems difficult for $n = 2$. The way we construct the extremizing function for our problem seems to make the following conjecture exceedingly likely, however, the question seems again difficult.

Conjecture. *Suppose $u : B(0,1) \subset \mathbb{R}^2$ is harmonic. Then the best constant $c > 0$ in the inequality*

$$\frac{|\{x \in B(0,1) : u(x) \geq u(0)\}|}{|\{x \in B(0,1) : u(x) \leq u(0)\}|} \geq c$$

is assumed if and only if $u = w$.

2. Proof

Outline. We study harmonic functions inside the unit disc which might not have a continuous extension to the boundary. Indeed, the function that will turn out to be the extremizer

$$\frac{(x^2+y^2-1)(x-2x^2+x^3-4y^2+xy^2)}{(1-2x+x^2+y^2)^3},$$

has a singularity in $(1,0)$. This motivates a proof by contradiction: assume $u : B(0,1) \to \mathbb{R}$ to be a function harmonic inside the unit disc with $\{x : u(x) = u(0)\}$ diffeomorphic to an interval and the curvature κ of that curve in the origin satisfying $\kappa > 8$. Let $r = 1 - \varepsilon$ and

$$\tilde{u}(x,y) := u(rx, ry).$$

Then, for $\varepsilon > 0$, this yields a counterexample $\tilde{u}$ to the inequality $\kappa \leq 8$ for a function $\tilde{u}$ having a continuous extension to the boundary. We use this function and ideas from mass transportation to prove the statement: we will, for any given harmonic function with a continuous extension to the boundary, create a series of rearrangements and symmetrizations such that the curvature at the origin increases in every single step. The precise form of the Poisson kernel plays an important role.

Relevant transports. We assume w.l.o.g. that $u(0,0) = 0$. Furthermore, we can assume after possibly rotating the disk that $\nabla u(0,0) = (\sigma, 0)$ for some $\sigma > 0$. We will now study a one-parameter and a two-parameter family of harmonic functions inside the unit disk: h_s and $g_{s,t}$. These functions will be defined as the Poisson extension of certain measures in such a way that adding weighted integrals to an existing

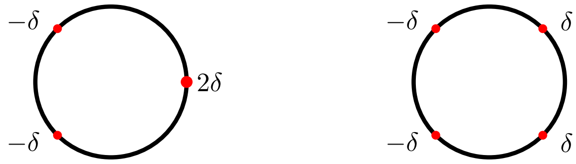

FIGURE 1. Measures on the boundary whose Poisson extension creates h_s and $g_{s,t}$, respectively.

harmonic function, for example,

$$u(x,y) \to u(x,y) + \int_0^{\pi} h_s(x,y)\phi(s)ds$$

corresponds to rearranging the measure given by restricting u to the boundary (which, by our assumption above, is absolutely continuous with respect to the Lebesgue measure) in a way described by the function ϕ (in all our applications, ϕ will have a closed-form expression depending on nothing except the measure on the boundary itself).

1. The functions h_s. For $0 \leq s < \pi$, we define a family of harmonic functions on the open unit disk $h_s : D \to \mathbb{R}$ via

$$h_s(x,y) = (1-x^2-y^2)\left(\frac{2}{(x-1)^2+y^2} - \frac{1}{(x-\cos s)^2+(y-\sin s)^2} - \frac{1}{(x-\cos s)^2+(y+\sin s)^2}\right).$$

These functions are (up to an irrelevant scalar) given as the Poisson extension of the measure

$$\mu = 2\delta_{(1,0)} - \delta_{(\cos s, \sin s)} - \delta_{(\cos s, -\sin s)}.$$

By direct computation, we see that for the function h_s, we have that $\nabla h_s(0,0)$ is contained in the $x-$axis, that $h_s(0,0) = 0$ and that the curvature of the level set $\{(x,y) : h_s(x,y) = 0\}$ through the origin is given by

$$0 \leq 4(1+\cos s) \leq 8.$$

A typical application of this function family will be the following: let $I \subset \partial B$ be some set such that $u\big|_I > 0$ and additionally symmetric

$$(x,y) \in I \Rightarrow (x,-y) \in I,$$

then by direct computation (which may be instructive but not necessary; a simpler argument is presented below) the function $\tilde{u}$

$$\tilde{u}(x,y) = u(x,y) + \int_I u(z)h_z(x,y)d\lambda_1,$$

where λ_1 is the arclength-measure on the boundary, has at least as large a curvature of the level set in the origin as u and satisfies all other properties of u except the existence of a continuous extension to the boundary as well. While we lose the continuous extension, this is no real loss as we know explicitely the measure whose Poisson extension yields $\tilde{u}$.

2. The functions $g_{s,t}$. For $0 \leq s < t < \pi$, we define the harmonic function on the unit disk $g_{s,t} : D \to \mathbb{R}$ via

$$\frac{g_{s,t}(x,y)}{1-x^2-y^2} = \frac{1}{(x-\cos s)^2+(y-\sin s)^2} + \frac{1}{(x-\cos s)^2+(-y-\sin s)^2} - \frac{1}{(x-\cos t)^2+(y-\sin t)^2} - \frac{1}{(x-\cos t)^2+(-y-\sin t)^2}.$$

These functions are (up to an irrelevant scalar) given as the Poisson extension of the measure

$$\mu = \delta_{(\cos s, \sin s)} + \delta_{(\cos s, -\sin s)} - \delta_{(\cos t, \sin t)} - \delta_{(\cos t, -\sin t)}.$$

Again, by direct computation, we see that for the function $g_{s,t}$, we have that $\nabla g_{s,t}(0,0)$ is contained in the $x-$axis, that $g_{s,t}(0,0) = 0$ and that the curvature of the level set $\{(x,y) : g_{s,t}(x,y) = 0\}$ through the origin is given by

$$0 \leq 4(\cos s + \cos t) \leq 8.$$

It is easily seen that

$$\lim_{s \to 0} g_{s,t}(x,y) = h_t(x,y)$$

for any point in the interior of the disk. In particular, as before, they have a dynamic interpretation in terms of rearranging mass on the boundary with the property of increasing the curvature while doing so.

The precise properties of these functions will not be important until the end: in the meanwhile, merely the following two facts are required. Using our assumptions on u, the curvature of the level set in the origin can be written as

$$\kappa = 2 \sup \left\{ a \in \mathbb{R} : \lim_{y \to 0} \frac{u(ay^2, y)}{y^2} \leq 0 \right\}$$

and, secondly, that

$$\forall\ 0 < s < t < \pi\ \forall a > 0 : \lim_{y \to 0} \frac{h_s(ay^2, y)}{y^2} > 0 \wedge \lim_{y \to 0} \frac{g_{s,t}(ay^2, y)}{y^2} > 0.$$

The proof will, given an initial harmonic function u, create a (finite) sequence of harmonic functions $u_1, u_2, \dots, u_k$ where each u_i can be reconstructed from u_{i+1} by subtracting a suitable weighted average over either h_s or $g_{s,t}$. From these two statements, it follows that for any such sequence the curvature of the level set in the origin necessarily increases.

Proof. Let now $u(x,y)$ be a counterexample to the statement: summarizing, we assume w.l.o.g. that $u(0,0) = 0$, that $\{(x,y) : u(x,y) = 0\}$ is diffeomorphic to an interval, that the curvature of $\{(x,y) : u(x,y) = 0\}$ exceeds 8 in the origin, that $u(x,y)$ has a continuous extension to the boundary and that $\nabla u(0,0) = (\sigma, 0)$ for some $\sigma > 0$. We write

$$I = \{(x,y) \in \partial B(0,1) : u(x,y) > 0\}.$$

We want to show that I is an open interval. From the assumption of γ being diffeomorphic to an interval, it follows that $I = \{(x,y) \in \partial B(0,1) : u(x,y) \geq 0\}$ is an interval. Since we argue by contradiction and work with functions that are harmonic on a small neighbourhood of the disk and satisfy the assumption of γ being diffeomorphic to an interval on that larger set, the maximum principle implies that u cannot vanish in the interior of I. Thus I is open. Using the Poisson kernel and $\nabla u(0,0) = (\sigma, 0)$ for some $\sigma > 0$ implies $(1,0) \in I$. We assume w.l.o.g. that $|I| < \pi$ (otherwise consider the function $-u$).

We identify the boundary of the disk $\partial B(0,1)$ with the torus of length 2π, i.e. $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with the convention that $(1,0)$ is mapped to 0 and denote $f := u\big|_{\partial B}$ for a function $f : \mathbb{T} \to \mathbb{R}$. The interval I can be written as some interval on the torus

$$I = (a, b) \qquad \text{with} \quad a < 0 < b.$$

We assume w.l.o.g. that $|a| \leq |b|$ (otherwise consider $u(x, -y)$) and write the interval as

$$I = (a, -a) \cup [-a, b),$$

where we will refer to $(a, -a)$ as the 'interior interval'. Our first step consists of moving all positive mass from $[-a, b)$ into the interior interval using the h_s functions (of course, if I is symmetric around the origin,

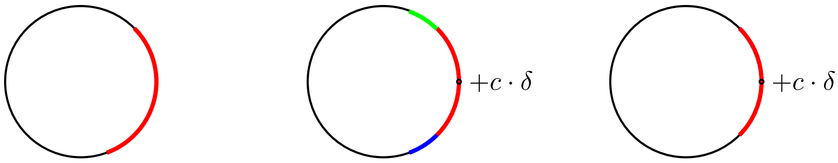

FIGURE 2. First rearrangement procedure: We have an (asymmetric) red interval I and move the entire positive mass from its asymmetric part into the point $(1, 0)$. Using the h_s-functions, this also requires us to move positive mass from the green interval into the origin making it more negative there.

nothing needs to be done in the first step). This yields a new function with the same properties as u except that the curvature in the origin has possibly increased and that in addition it now longer has a continuous extension to the boundary but is now given by the Poisson extension of a measure which is the sum of weighted Lebesgue and Dirac measures

$$\mu_1 = \left(\begin{cases} 0 & \text{if} - a < x < b \\ f(x) - f(-x) & \text{if} - a < -x < b \\ f(x) & \text{otherwise} \end{cases} \right) dx + \left(\int_{-a}^{b} f(x) dx \right) \delta_0.$$

The measure μ_1 assigns a positive weight only within the interior interval $(a, -a)$.

The second symmetrization procedures moves all symmetrically aligned negative mass outside of the interior interval to the boundary of the interior interval using the function $g_{a,t}$. This creates a measure μ_2 given by

$$\mu_2 = \left(\begin{cases} 0 & \text{if} - a < x < b \\ f(x) - f(-x) & \text{if} - a < -x < b \\ f(x) & \text{if} - a < x < a \\ f(x) - \max(f(x), f(\pi - x)) & \text{if } x \notin (-b, b) \end{cases} \right) dx$$
$$+ \left(\int_{-a}^{b} f(x) dx \right) \delta_0 + \frac{\delta_a + \delta_{-a}}{2} \left(\int_{x \notin (-b,b)} \max(f(x), f(\pi - x)) dx \right)$$

Note first that this rearrangement procedure only introduces Dirac measures at the points $-a$ and a, the density function remains continuous outside of that interval (and, of course, continuous in the interior interval with the exception of 0 as well). In particular, the density has gained an asymmetry around the $y-$axis: written in terms of Radon-Nikodym derivatives, we have for any point $x \notin [-a, a]$

$$\frac{d\mu_2}{dx}(x) < 0 \Rightarrow \frac{d\mu_2}{dx}(\pi - x) = 0.$$

This asymmetry property is useful and our next step will be aimed towards establishing it for the interior interval as well.

The third rearrangement procedure will operate entirely within the interior interval $(-a, a)$ and will use the h_s functions (in the range $0 < s < a$) to transport positive mass to 0 just like in the previous case

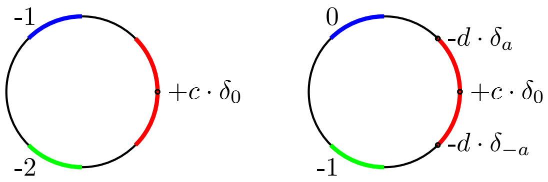

FIGURE 3. Second rearrangement procedure: suppose we have two intervals with values -1 (blue interval) and -2 (green interval). Using the $g_{a,t}$ functions, we shift the negative mass to the points $-a$ and a, where $d > 0$ is some number. The actual rearrangement procedure is a continuous version of this.

with negative mass. The arising measure is

$$\mu_3 = \left(\begin{cases} 0 & \text{if} -a < x < b \\ f(x) - f(-x) & \text{if} -a < -x < b \\ f(x) - \min(f(x), f(-x)) & \text{if} -a < x < a \\ f(x) + \max(f(x), f(\pi - x)) & \text{if } x \notin (-b, b) \end{cases} \right) dx$$
$$+ \left(\int_{-a}^{b} f(x)dx + \int_{-a}^{a} \min(f(x), f(-x))dx \right) \delta_0 + \frac{\delta_a + \delta_{-a}}{2} \left(\int_{x \notin (-b,b)} \max(f(x), f(\pi - x))dx \right)$$

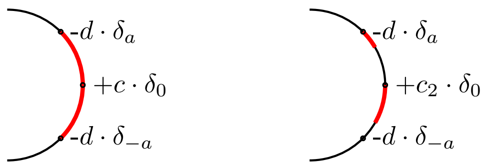

FIGURE 4. Third rearrangement procedure: using the h_s-functions we transport positive measure from the interior interval to the origin.

We have now total asymmetry: if $|x| \notin \{0, a\}$ (i.e. away from the Dirac measures), then

$$\frac{d\mu_3}{dx}(x) \neq 0 \Rightarrow \frac{d\mu_3}{dx}(-x) = 0.$$

Now we can finally symmetrize: trivially,

$$\Delta \left(u(x, y) + u(x, -y) \right) = 0,$$

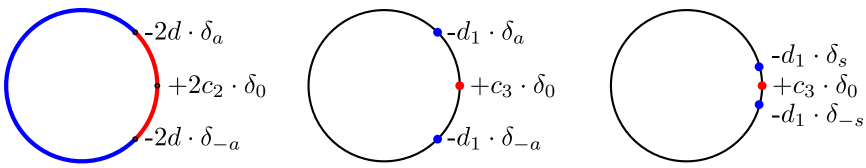

FIGURE 5. Final rearrangement procedure for symmetric functions: all the negative mass from the outside to the boundary of the interior interval (using $g_{a,t}$), all the interior mass to the origin (using h_s) and then the remaining negative mass closer to the origin (using $g_{s,a}$).

and it is obvious from the definition that the level set of the function $u(x,y)+u(x,-y)$ going through the origin has the same curvature as the one for $u(x,y)$.

In this totally symmetric framework, we can use the h_s functions for $0 < s < a$ to transport all the remaining positive mass from the interior interval to the origin and then use the $g_{a,t}$ functions to transport all the remaining negativ mass symmetrically to $-a$ and a. Finally, we can use the $g_{s,a}$ function one last time to transport the negative mass at $-a$ and a to $-s$ and s for s arbitrarily small. We can track the value of the function at the origin through all these rearrangements and still find it to satisfy $u(0,0) = 0$. Therefore, the final function is merely a scalar multiple of h_s, where s can be arbitrary small. The largest curvature assumed by h_s at the origin is bounded from above by 8. At the same time, we can explicitely calculate

$$\lim_{s\to 0} \frac{h_s(x,y)}{s^2} = \frac{(x^2+y^2-1)(x-2x^2+x^3-4y^2+xy^2)}{(1-2x+x^2+y^2)^3}$$

and this an extremizer – its uniqueness up to symmetries follows immediately from the proof. □

CHAPTER 5

Periodic ODEs in Banach spaces

> At that time, however, the theory seemed to me to contain for the immediate future nothing but some decades of rather formal and thin work. By this I do not mean to reproach the work of Banach himself but rather that of many inferior writers, hungry for easy doctors' theses, who were drawn to it. (Norbert Wiener, [**Wie56**])

1. Introduction

This chapter is concerned with the very basic property of periodicity of ODEs in the very general framework of Banach spaces and all of it is joint work with James Robinson (Mathematics Institute, University of Warwick) and Michaela Nieuwenhuis (Mathematical Institute, Oxford).

Given a Banach space B and a Lipschitz continuous functions $f : B \to B$, what can be said about solutions to the ordinary differential equation

$$\dot{y} = f(y)?$$

In particular, can it have a periodic solution with an arbitrarily small period? The natural invariant quantity is the product of the period P and the Lipschitz constant L of the function f.

All our results and arguments will be on the basis of one particular flow curve $y(t)$ with a function f defined on the orbit and not necessarily anywhere else (but satisfying Lipschitz conditions along the orbit, of course). Literature on the problem does mention extension theorems (see e.g. [**BFM89**]) to extend a particular orbit to the orbit of a globally defined ODE but this is not what we are really interested in.

1.1. Early results.

The earliest published result dates back to a paper of Yorke.

Theorem (Yorke, [**Yor69**]). *If $B = \mathbb{R}^n$ and f satisfies*

$$\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|,$$

then

$$PL \geq 2\pi.$$

We should remark that he himself ascribes an earlier result to Yasutaka Sibuya.

> The fact that p is bounded below by c/L for some c is surprising. This was first proved by Y. Sibuya (unpublished), who showed $p \geq 2/L$. (Yorke, [**Yor69**])

We will not sketch Yorke's proof as it is based on results, which do not translate to more general Banach spaces: Fenchel [**Fen29**] proved in 1929 that a closed curve in $\mathbb{R}^3$ has total curvature at least 2π, a result that was generalized to all $\mathbb{R}^n$ by Borsuk [**Bor47**]. In particular, if the curve $y([0, P]) \subset \mathbb{R}^n$, Yorke's approach combined with a theorem of Milnor [**Mil50**] implies

$$PL > 4\pi.$$

Only two years later, Lasota & Yorke proved the first lower bound in Banach spaces while also remarking that the proof of the previous bound can be extended to all Hilbert spaces.

Theorem (Lasota & Yorke, [**LY71**]). *If B is a Banach space and f has Lipschitz constant L then*

$$PL \geq 4.$$

If B is a Hilbert space, then

$$PL \geq 2\pi.$$

There is a shorter proof for the case of Hilbert spaces due to Busenberg, Martelli & Fisher, which we quote below. The main idea of the proof for Banach spaces is interesting: one searches for the largest value of $\|x'(t)\|$ along the flow line, to find the linear functional associated to the direction at the maximum and use this to reduce it to a problem for one-dimensional functions.

Proof, taken from [**LY71**]. Let $y : [0, P] \to B$ be a periodic solution and set

$$\lambda = \max_{0 \leq t \leq P} \|x'(t)|.$$

Using periodicity, we may assume the maximum to be attained at $t = 0$. It follows from Lipschitz continuity that for all $0 \leq t_1 < t_2 \leq P$

$$\|x'(t_2) - x'(t_1)\| = \|f(x(t_2)) - f(x(t_1))\| \leq L(t_2 - t_1)\lambda.$$

Let r be the linear functional on B such that

$$\|r\| = 1 \qquad \text{and} \qquad r(x'(0)) = \lambda.$$

We define the function $\xi(t) = r(x'(t))$. By linearity of the functional and periodicity of the curve

$$\int_0^P \xi(t)dt = r\left(\int_0^P x'(t)dt\right) = 0.$$

By normalization $\xi(0) = \xi(P) = \lambda$ and certainly $|\xi(t)| \leq \lambda$ for all $0 \leq t \leq P$. The normalization $\|r\| = 1$ implies with the previous argument that

$$\frac{|\xi(t) - \xi(s)|}{|t - s|} \leq L\lambda.$$

All these facts combined imply that there has to be a point $0 < t_0 < P$ such that

$$\int_0^{t_0} \xi(t)dt = 0.$$

Using the reflection symmetry $t \to -t$ (going through the curve backwards), we may assume that $t_0 \leq P/2$. Let $h : \mathbb{R} \to \mathbb{R}$ be the affine function satisfying $h(0) = \lambda$ and $h(t_0) = -\lambda$. Trivially,

$$\int_0^{t_0} h(t)dt = 0.$$

Then there assuredly is a point $t_1 \in (0, t_0]$ such that $\xi(t_1) = h(t_1)$, in which case

$$L\lambda \geq \frac{|\xi(t_1) - \xi(0)|}{|t_1|} = \frac{|h(t_1) - h(0)|}{|t_1|} \geq \frac{2\lambda}{t_0}.$$

This and $t_0 \leq P/2$ imply the statement. □

1.2. The optimal constant. Lasota & Yorke mention that they do not know whether the constant 4 was sharp. It took more than 15 years to see that it was not.

Theorem (Busenberg & Martelli, [**BM87**]). *If B is a Banach space and and f has Lipschitz constant L, then*

$$PL \geq 4.5.$$

We will not describe this proof in details because at almost the same time, Busenberg & Martelli & Fisher [**BFM86**] obtained a stronger theorem with what turned out to be the optimal constant. Their approach is based on the derivation of a very impressive Wirtinger-type inequality in Banach spaces.

Theorem (Busenberg, Martelli & Fisher, [**BFM86**])**.** *If B is a Banach space and and f has Lipschitz constant L then*

$$PL \geq 6.$$

At the time of the publication, it was not known (but added in proof) that the constant 6 is indeed optimal. Indeed, their example was only published later [**BFM89**] and remains the only known example today: the setting is $B = L^1([0,1]^2)$. Consider the function

$$\phi(x,y) = \begin{cases} y(1-x) & x \geq y \\ x(1-y) & y \geq x. \end{cases}$$

We can extend it to a periodic function in $\mathbb{R}^2$. Consider the curve $z : [0,1] \to L^1([0,1]^2)$ given by

$$z(t)(x,y) = \phi(x+t, y).$$

To quote Busenberg, Martelli & Fisher, *some tedious but easy* computations show that

$$\begin{aligned} \|z(t) - z(s)\|_{L^1([0,1]^2)} &= \|z(t-s) - z(0)\|_{L^1([0,1]^2)} \\ \|z(t) - z(0)\|_{L^1([0,1]^2)} &= \frac{1}{3} t(1-t) \\ \|z'(t) - z'(0)\|_{L^1([0,1]^2)} &= 2t(1-t). \end{aligned}$$

Therefore

$$\|z'(t) - z'(s)\|_{L^1([0,1]^2)} = 6\|z(t) - z(s)\|_{L^1([0,1]^2)}$$

and this completes the construction. Note that this is but one orbit and one needs to use extension properties to create a fully defined ODE on the entire spaces but, as mentioned above, this is not of any real interest.

1.3. Wirtinger inequalities. This section describes the proof of the Busenberg-Martelli-Fisher theorem for both Banach and Hilbert space. The crucial ingredient is a Wirtinger-type inequality.

Theorem (Wirtinger)**.** *Let $f : [0,\pi] \to \mathbb{R}$ be a continuous function satisfying $f(0) = f(\pi) = 0$, then*

$$\int_0^\pi f(x)^2 dx \leq \int_0^\pi f'(x)^2 dx.$$

The inequality is elementary and easily proven via Fourier series but certainly foundational: it provides the simplest example of a ground state of the Laplacian and (with a modicum of imagination) can be seen as a stepping stone to spectral theory. Its history is convolved: the inequality seems to have been first given by Scheeffer [**Sch85**] in 1885, whose result is mentioned in a paper by Kneser in 1900 [**Kne00**]. Emilé Picard [**Pic96**] studied expressions of Rayleigh-Ritz-type in 1896; a seemingly independent proof is due to Almansi [**Alm05**] in 1905 – the conditions in his result were weakened by Eugenio Levi (the younger brother of Beppo) in 1911 [**Lev11**] and Tonelli in 1914 [**Ton14**]. Wirtinger himself never published 'his' inequality but did communicate it to his student Blaschke, who mentions it in his 1916 classic *Kreis und Kugel* [**Bla56**] as a Lemma and writes in (Blaschke, [**Bla56**]) '[...] den folgenden noch durchsichtigeren Beweis verdanke ich Herrn W. Wirtinger'. The wide spread of *Kreis and Kugel* has assuredly contributed to the now prevalent naming convention, which is not without its enemies.

> In the literature, (1.2) is known as Wirtinger's inequality (see [**HLP05**] and [**BB65**]). We will also accept this name. The French and Italian mathematical literature do not mention the name of Wirtinger. ([**MPF91**])

Let us now sketch a relevant application. The following theorem was already proven by Lasota & Yorke, here we give a proof following the argument of Busenberg, Martelli & Fisher.

Theorem (Lasota & Yorke, [**LY71**]). *If B is a Hilbert space and and f has Lipschitz constant L then*

$$PL \geq 2\pi.$$

Proof taken from [**BFM86**]. Let $w : \mathbb{R} \to B$ be continuous, $P-$periodic with

$$\int_0^P w(t)dt = 0.$$

If $\|w'(t)\|^2$ is integrable, then there is the Wirtinger inequality

$$\int_0^P \|w(t)\|^2 dt \leq \frac{P^2}{4\pi^2} \int_0^P \|w'(t)\|^2 dt.$$

Since the image of y is compact, there is orthonormal expansion

$$w(t) = \sum_{i=1}^{\infty} a_i(t) e_i$$

and one can apply Wirtinger's inequality[1] to each coefficient function $a_i(t)$. The proof follows from noting that

$$w(t) = y(t+h) - y(t)$$

for some $h \neq 0$ satisfies the conditions of the Wirtinger inequality. Thus,

$$\begin{aligned} \int_0^P \|y(t+h) - y(t)\|^2 &\leq \frac{P^2}{4\pi^2} \int_0^P \|y'(t+h) - y'(t)\|^2 dt \\ &\leq \frac{P^2 L^2}{4\pi^2} \int_0^P \|y(t+h) - y(t)\|^2 dt. \end{aligned}$$

This proves the statement (provided y is nonconstant). □

We conclude the section with the main statement, a Wirtinger-type inequality for Banach spaces.

Theorem (Busenberg, Martelli & Fisher, [**BFM86**]). *Let B be a Banach space and let $y : \mathbb{R} \to B$ be continuous and $P-$periodic with $\|y'(t)\|$ integrable. Then*

$$\int_0^P \int_0^P \|y(t) - y(s)\| ds dt \leq \frac{P}{6} \int_0^P \int_0^P \|y'(s) - y'(t)\| ds dt.$$

[1] Technically, we apply a slightly different version of Wirtinger's inequality where Dirichlet boundary conditions are replaced by the condition of the function having average 0. All these things are very easily seen to be equivalent, we do not go into details.

Proof taken from [**BFM86**].

$$\begin{aligned}
\int_0^P \int_0^P \|y(t)-y(s)\| dsdt &= \int_0^P \int_0^P \|y(s+t)-y(s)\| dsdt \\
&= \int_0^P \int_0^P \frac{P-t)t}{P} \left\| \frac{y(s+t)-y(s)}{t} - \frac{y(s)-y(s+t-P)}{P-t} \right\| dsdt \\
&= \int_0^P \int_0^P \frac{P-t)t}{P} \left\| \int_0^P \left(y'\left(s+\frac{tr}{P}\right) - y'\left(s+\frac{tr}{P}-r\right)\right) dr \right\| dsdt \\
&\leq \int_0^P \int_0^P \frac{P-t)t}{P} \int_0^P \left\| \left(y'\left(s+\frac{tr}{P}\right) - y'\left(s+\frac{tr}{P}-r\right)\right) \right\| drdsdt \\
&= \int_0^P \frac{P-t)t}{P} \int_0^P \int_0^P \left\| \left(y'\left(s+\frac{tr}{P}\right) - y'\left(s+\frac{tr}{P}-r\right)\right) \right\| drdsdt \\
&= \int_0^P \frac{P-t)t}{P} \int_0^P \int_0^P \left\| \left(y'(s+r) - y'(s)\right) \right\| drdsdt \\
&= \frac{P}{6} \int_0^P \int_0^P \left\| \left(y'(s+r) - y'(s)\right) \right\| drds.
\end{aligned}$$

□

This inequality is sharp in every Banach space: pick an arbitrary vector $\mathbf{0} \neq \mathbf{v} \in X$ and consider a 2-periodic extension of

$$y(t) := t\mathbf{v} - (2t-2)\mathbf{v}\chi_{1 \leq t \leq 2} \qquad \text{for } 0 \leq t < 2$$

where χ is the indicator function. Any such $y(t)$ achieves equality. In this example, however, there is a rapid change of direction at $t \in \mathbb{N}$ which is something that solutions to Lipschitz ODEs cannot do. This observation will be the main idea behind our approach: we characterize extremisers of the inequality in strictly convex Banach spaces and show that these paths cannot arise as the solution of an ODE.

The lower bound $PL \geq 6$ follows as a trivial consequence

$$\begin{aligned}
\int_0^P \int_0^P \|y(t)-y(s)\| dsdt &\leq \frac{P}{6} \int_0^P \int_0^P \|y'(s)-y'(t)\| dsdt \\
&\leq \frac{PL}{6} \int_0^P \int_0^P \|y(s)-y(t)\| dsdt.
\end{aligned}$$

2. Statement of Results

In this section we describe our two results, both of which (as mentioned above) are joint work with James Robinson (Mathematics Institute, University of Warwick) and Michaela Nieuwenhuis (Mathematical Institute, Oxford).

All previous results can be quickly summarized as follows: if B is a Banach space, $y : \mathbb{R} \to B$ is $P-$periodic and satisfies

$$\dot{y}(t) = f(y)$$

with a Lipschitz-continious $f : B \to B$

$$\|f(x) - f(y)\| \leq L\|x-y\|,$$

then

$$PL \geq \begin{cases} 6 & \text{for general Banach spaces } B \\ 2\pi & \text{if } B \text{ is a Hilbert space} \end{cases}$$

and both these bounds are optimal. However, on the Banach space side of things, further understanding is desirable: what characterizes Banach spaces in which the lower bound 6 is assumed? Can the lower bound

6 be improved for finite-dimensional Banach spaces? Is it maybe the case that every number in $[6, 2\pi)$ is attained by some Banach space? What can be said about the existence of paths satisfying $P \leq c/L$ for some $c < \infty$ in general Banach spaces?

2.1. Some intuition. In finite dimensions, a Banach space B can be identified with a centrally symmetric convex body K in $\mathbb{R}^n$ (i.e. its unit ball, which uniquely defines the space). If K coincides with the unit ball, the Banach space coincides with the regular Euclidean space and is Hilbert – if K is close to the ball in the sense of

$$(1-\varepsilon)\|x\|_{\ell^2} \leq \|x\|_B \leq (1+\varepsilon)\|x\|_{\ell^2},$$

then there is a simple continuity argument: Suppose $f : B \to B$ has Lipschitz constant L and suppose y is a $P-$periodic solution of

$$\dot{y} = f(y).$$

Then we can equally regard it as a curve in the Hilbert space and estimate

$$\|f(x) - f(y)\|_{\ell^2} \leq \frac{1}{1-\varepsilon}\|f(x) - f(y)\|_B \leq \frac{L}{1-\varepsilon}\|x-y\|_B \leq L\frac{1+\varepsilon}{1-\varepsilon}\|x-y\|_{\ell^2}.$$

At the same time, the length of the curve $y([0, P])$ when measured in the Hilbert space is bounded by $(1+\varepsilon)P$. The Hilbert space bound of Lasota & Yorke implies

$$PL \geq \frac{1-\varepsilon}{(1+\varepsilon)^2}2\pi.$$

In particular, finite-dimensional Banach spaces which are 'close' to the Euclidean space in this sense are close with regards to lower bounds on minimal periods. Note that this is very partial information as we do not even know, whether there is a finite-dimensional Banach space with an example of $PL \leq 2\pi$. A second observation is that the only known example of a Banach space, where $PL = 6$ is possible, is $L^1([0, 1]^2)$, which has a series of particular proprties – one of them is its fairly of strict convexity: the unit ball contains line segments. These two observation motivate our main results.

2.2. Main results. The first result gives improved bounds for a natural family of Banach spaces close to the Hilbert case. It is based on a recent Wirtinger inequality due to Croce & Dacorogna [**CD03**]: assume $p > 1$, $u \in W^{1,p}([0, 1])$, $u(0) = u(1)$ and

$$\int_0^1 u(t)dt = 0.$$

Then

$$\|u\|_{L^p} \leq \frac{p}{4(p-1)^{\frac{1}{p}}}\left(\int_0^1 t^{-\frac{1}{p}}(1-t)^{\frac{1}{p}-1}dt\right)^{-1}\|u_x\|_{L^p}.$$

Theorem (Nieuwenhuis, Robinson & S.). *Let $B = \ell^p(\mathbb{R}^n)$ or $B = L^p(M, \mu)$ (for some set M and some measure μ). Then*

$$PL \geq \frac{4(p-1)^{\frac{1}{p}}}{p}\int_0^1 t^{-\frac{1}{p}}(1-t)^{\frac{1}{p}-1}dt \geq 4.$$

This bound is bigger than 6 for $1.44 \leq p \leq 3.35$ – this result will be nicely complemented by our second result, which shows that one can improve the lower bound 6 for all $1 < p < \infty$. However, unfortunately, this second result is non-quantitative in nature.

Theorem (Nieuwenhuis, Robinson & S.). *If B is a strictly convex Banach space, then*

$$PL > 6.$$

The proof of this theorem consists of a characterization of extremizers for the Busenberg-Martelli-Fisher-Wirtinger inequality in strictly convex Banach spaces – as we will see, such extremizers do exist but can never arise as the flow curve of an ODE.

2.3. Wild, unfounded speculation.

Human beings can lose their lives in libraries. They ought to be warned. (Saul Bellow)

The abundance of strange examples in the theory of Banach spaces makes it difficult to not see connections where none exist. As its designation suggests, this paragraph can be skipped by the somber-minded. Regarding the lower bound 6, we note a related result by Lasota & Yorke [**LY71**]: given a Banach space B and a periodic curve $x : \mathbb{R} \to B$ with period P satisfying

$$\|\dot{x}\| \leq L\|x(t)\|,$$

then $PL \geq 4$ and this constant is sharp. An example is given in $L^1([0,4])$, where

$$x(t)(s) = \phi(t+s)$$

with

$$\phi(s) = \frac{1}{2}|s-2| - \frac{1}{2}.$$

It can be checked that

$$\|x'\| = \|x\| = 1$$

and the period length is obviously 4. Four years earlier, in 1967, Schaeffer [**Sch67**] introduced the notion of a *flat* Banach space. A Banach space B is said to be flat if there exists a curve $\gamma : [0,2] \to B$ such that γ has Lipschitz constant 1, $\|\gamma(t)\| = 1$ and $\gamma(0) + \gamma(2) = 0$. Such a curve connects two antipodal points on the unit sphere *while being itself contained in the unit sphere*: despite this, it is not any longer than the direct straight line through the origin. These curves can be extended to centrally symmetric period curves of length 4. Examples of such flat Banach spaces are somewhere between and L^1 and C^0 (see [**vDP84**]). Slightly later, Schaeffer [**Sch70**] proved that such examples do not exist in finite dimensions by proving a lower bound on the length ℓ of a curve connecting antipodal points

$$\ell \geq 2 + \frac{1}{n}$$

in $2n$ and $2n+1$ dimensions, respectively. One cannot help but suspect (or maybe hope?) that all these concepts are connected.

3. Proofs

The proofs of the two theorems are not related and based on different insights, however, both employ a Wirtinger(-type) inequality as its crucial ingredient.

3.1. Lower bounds for ℓ^p and $L^p(M,\mu)$. We denote the sharp constant coming from Dacorogna-Croce [**CD03**] via

$$C_p := \frac{p}{4(p-1)^{\frac{1}{p}}} \left(\int_0^1 t^{-\frac{1}{p}} (1-t)^{\frac{1}{p}-1} dt \right)^{-1} \|u_x\|_{L^p}.$$

Lemma. *Let $u \in W^{1,p}_{\mathrm{per}}([0,T], X)$ where X is either $\ell^p(\mathbb{R}^n)$ or $L^p(M,\mu)$ and assume that $\int_0^T u(t)\,dt = 0$. Then*

$$\int_0^T \|u(t)\|_X^p \,dt \leq C_p^p T^p \int_0^T \|\dot{u}(t)\|_X^p \,dt,$$

where C_p is optimal.

PROOF. By a simple change of variables it suffices to prove the result for $T = 1$. When $X = \ell^p(\mathbb{R}^n)$ we have

$$\int_0^1 \sum_{j=1}^n |u_j(t)|^p \, dt = \sum_{j=1}^n \int_0^1 |u_j(t)|^p \, dt \leq C_p^p \sum_{j=1}^n \int_0^1 |\dot{u}_j(t)|^p \, dt,$$

from which the statement is immediate. One can see that the constant is optimal by considering $u = (u_1, \ldots, u_n)$ with $u_1 \in W^{1,p}_{per}(0,1)$ and $u_j = 0$ for $j = 2, \ldots, n$. Similarly, for $X = L^p(M, \mu)$ we have

$$\begin{aligned}\int_0^1 \int_U |u(x,t)|^p \, d\mu \, dt &= \int_U \int_0^1 |u(x,t)|^p \, dt \, d\mu \\ &\leq C_p^p \int_U \int_0^1 |\dot{u}(x,t)|^p \, dt \, d\mu = C_p^p \int_0^1 \int_U |\dot{u}(x,t)|^p \, d\mu, t,\end{aligned}$$

and the statement follows once more. Optimality of the constant follows by taking $f(t,x) = f(t)\mathbf{1}_A$ for some $f \in W^{1,p}_{per}(0,1)$ and $A \subset U$ with $\mu(A) > 0$. □

The rest of the proof mimicks the earlier approaches very closely.

Theorem. *Let x be a non-constant T-periodic solution to $\dot{x} = f(x)$ in either $X = \ell^p(\mathbb{R}^n)$ or $X = L^p(M, \mu)$. Further, suppose that f is Lipschitz continuous from X into X with Lipschitz constant L. Then*

$$TL \geq C_p^{-1}.$$

PROOF. As the function x is a solution to the ODE, it is differentiable by definition. Moreover, a simple calculation shows that

$$\int_0^T x(t+h) - x(t) \, dt = 0.$$

Hence Wirtinger's inequality for $W^{1,p}_{per}((0,T), X)$ is applicable to $x(t+h) - x(t)$ and thus

$$\begin{aligned}\int_0^T \|x(t+h) - x(t)\|_X^p \, dt &\leq C_p^p T^p \int_0^T \|\dot{x}(t+h) - \dot{x}(t)\|_X^p \, dt \\ &= C_p^p T^p \int_0^T \|f(x(t+h)) - f(x(t))\|_X^p \, dt \\ &\leq L^p C_p^p T^p \int_0^T \|x(t+h) - x(t)\|_X^p \, dt.\end{aligned}$$

Dividing both sides by $\int_0^T \|x(t+h) - x(t)\|_X^p \, dt$, which is non-zero as x is non-constant, yields the statement. □

3.2. Improved lower bounds for strictly convex Banach spaces. As already mentioned above, this proof is based on a characterization of extremizers of the Busenberg-Martelli-Fisher-Wirtinger inequality in strictly convex Banach spaces.

We prove a refinement of an inequality originally due to Busenberg, Martelli & Fisher.

Theorem. *Let X be a strictly convex Banach space and let $y : \mathbb{R} \to X$ be nonconstant and $T-$periodic with $\dot{y}(t)$ Lipschitz continuous. Then*

$$\int_0^T \int_0^T \|y(s) - y(t)\| ds dt > \frac{T}{6} \int_0^T \int_0^T \|\dot{y}(s) - \dot{y}(t)\| ds dt.$$

In particular, this implies the following corollary.

Corollary. *Let X be a uniformly convex Banach space. Then*

$$LT > 6.$$

PROOF. The following calculations were carried out by Busenberg, Martelli & Fisher.

$$\begin{aligned}
\int_0^T \int_0^T \|y(s) - y(t)\| ds dt &= \int_0^T \int_0^T \|y(s+t) - y(t)\| ds dt \\
&= \int_0^T \int_0^T \frac{(T-t)t}{T} \left\| \frac{y(s+t) - y(s)}{t} - \frac{y(s) - y(s+t-T)}{T-t} \right\| ds dt \\
&= \int_0^T \int_0^T \frac{(T-t)t}{T^2} \left\| \int_0^T \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) dr \right\| ds dt \\
&\le \int_0^T \int_0^T \frac{(T-t)t}{T^2} \int_0^T \left\| \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) \right\| dr ds dt \\
&= \int_0^T \frac{(T-t)t}{T^2} dt \int_0^T \int_0^T \|\dot{y}(s+r) - \dot{y}(s)\| ds dr \\
&= \frac{T}{6} \int_0^T \int_0^T \|\dot{y}(r) - \dot{y}(s)\| ds dr,
\end{aligned}$$

where the inner integral could be shifted by $\frac{tr}{T} - r$ because it is taken over one period. The statement would follow if we can show that the application of the triangle inequality cannot be sharp for some s and $0 < t < T$ because it is the only inequality in the entire proof of Busenberg, Martelli & Fisher. Note that from the continuity of $\dot{y}(t)$ we obtain that the functions

$$(s,t) \to \left\| \int_0^T \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) dr \right\|$$

$$(s,t) \to \int_0^T \left\| \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) dr \right\|$$

are continuous as well. Fix s and $0 < t < T$, fix an arbitrarily fine decomposition $0 = a_0 < a_1 < \cdots < a_n = T$ and abbreviate

$$b_i := \int_{a_i}^{a_{i+1}} \dot{y}\left(s + \frac{tr}{T}\right) \qquad \text{and} \qquad c_i := \int_{a_i}^{a_{i+1}} \dot{y}\left(s + \frac{tr}{T} - r\right).$$

If the inequality was not strict, then we need to have equality in every step of iteratively applying the triangle inequality and thus

$$\begin{aligned}
\left\| \sum_{i=0}^{n-1} b_i - c_i \right\| &= \|b_0 - c_0\| + \left\| \sum_{i=1}^{n-1} b_i - c_i \right\| \\
&= \|b_0 - c_0\| + \|b_1 - c_1\| + \left\| \sum_{i=2}^{n-1} b_i - c_i \right\| \\
&= \ldots \\
&= \sum_{i=0}^{n-1} \|b_i - c_i\|.
\end{aligned}$$

We assume first that all the terms satisfy $b_i - c_i \neq \mathbf{0}$. Strict convexity implies in the last line of this argument that $b_{n-2} - c_{n-2}$ and $b_{n-1} - c_{n-1}$ are collinear. By the same reasoning $b_{n-3} - c_{n-3}$ and $(b_{n-2} - c_{n-2}) + (b_{n-1} - c_{n-1})$ are collinear, however, the last expression itself is collinear to $b_{n-2} - c_{n-2}$ as well as $b_{n-1} - c_{n-1}$. Iterating this argument shows that all $b_i - c_i$ are necessarily collinear. Using the continuity of $\dot{y}(t)$, making the partition sufficiently small and applying the fundamental theorem of calculus, we can deduce that for every fixed s and $0 < t < T$ there exists a vector $\mathbf{v} \in B$ and a function $f : [0,T] \to \mathbb{R}$ such that for all $0 \le r \le T$

$$(\diamond) \qquad \dot{y}\left(s + \frac{tr}{T}\right) - \dot{y}\left(s + \frac{tr}{T} - r\right) = f(r)\mathbf{v}.$$

The general case $b_i - c_i = \mathbf{0}$ for some (or all) $i \in \{0, 1, \ldots, n-1\}$ can be treated the same way. Note, however, that both f and $\mathbf{v}$ depend on the previously fixed s, t.

Since y is not constant, it is possible to find and fix a s such that

$$\dot{y}(s) \neq \mathbf{0}.$$

We now claim that this already implies

$$\dot{y}(s+r) = g(r)\mathbf{v} + \dot{y}(s).$$

Suppose this was false, i.e. there is a r such that

$$\dot{y}(s+r) \notin \{\dot{y}(s) + \lambda \mathbf{v} | \lambda \in \mathbb{R}\}.$$

In particular,

$$\min_{\lambda \in \mathbb{R}} \|\dot{y}(s+r) - \dot{y}(s) + \lambda \mathbf{v}\| > 0.$$

This, however, can be seen to contradict ($\diamond$) by taking t sufficiently small.

Since y is periodic with period T,

$$\int_0^T \dot{y}(s+r)dr = \mathbf{0} = \left(\int_0^T g(r)dr\right)\mathbf{v} + T\dot{y}(s).$$

This implies that $\dot{y}(s)$ is a scalar multiple of $\mathbf{w}$, in which case

$$\dot{y}(s+r) = \left(g(r) - \frac{1}{T}\int_0^T g(r)dr\right)\mathbf{v}.$$

This establishes that $\dot{y}(t)$ is one-dimensional, i.e.

$$\dot{y}(t) = f(t)\mathbf{v}$$

for some $\mathbf{v} \neq \mathbf{0}$ and a continuous function f.

However, now we can again play through the same argument with this additional piece of information: $\|\mathbf{v}\|$ cancels on both sides. We get that for any fixed s, t with $0 < t < T$, we require that

$$\left|\int_0^T f(s+\frac{tr}{T}) - f(s+\frac{tr}{T} - r)dr\right| = \int_0^T \left|f(s+\frac{tr}{T}) - f(s+\frac{tr}{T} - r)\right| dr.$$

However, since f is continuous and

$$\int_0^T f(z)dz = 0,$$

f has to vanish in a point, say $f(s) = 0$. By taking t to be small, we derive that $f \equiv 0$. □

Remark. It is interesting to remark that there are functions f for which the last property holds. Take $T = 1$ and a Haar function

$$f(s) = \chi_{[0,0.5]}(s) - \chi_{[0.5,1]}(s).$$

This corresponds to a curve

$$\dot{y}(s) = f(s)\mathbf{v},$$

which does not satisfy the requirement that $\dot{y}(s)$ be Lipschitz continuous.

3.3. Remarks. Short periods. Virtually nothing seems to be known about the construction of short periodic solutions in general Banach spaces. It seems conceivable that one can always construct an ODE having a solution with period P satisfying $P \leq 2\pi/L$ (or, in any case, the opposite statement would be magnitudes more surprising).

Dvoretzky's theorem [**Dvo61**] guarantees that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ sufficiently large such that any Banach space with $\dim X \geq n$ contains a two-dimensional subspace with Banach-Mazur distance to

ℓ_2^2 at most $1+\varepsilon$. The example of a simple circle in ℓ_2^2 realizes $TL = 2\pi$. This means that in any Banach space X it is possible to construct an ODE satisfying $TL \leq 2\pi + \varepsilon$, where ε depends only on the dimension of X. We do not know whether there is always an ODE for which $TL \leq 2\pi$.

An explicit construction. For $1 \leq p < \infty$ we can construct such an example in $L^p(M,\mu)$ for fairly general measures μ. Suppose there are two sets $A \cap B = \emptyset$ such that

$$0 < \mu(A) = \mu(B).$$

Consider the ODE

$$\dot{z} = f(z)$$

with $f : L^p(M,\mu) \to L^p(M,\mu)$ given by

$$f(z) = -\frac{\chi_B}{\mu(A)} \int_A z d\mu + \frac{\chi_A}{\mu(B)} \int_B z d\mu.$$

Hölder's inequality gives that $L = 1$

$$\begin{aligned}
\|f(z) - f(w)\|_{L^p(M,\mu)}^p &= \left\| -\chi_B \frac{1}{\mu(A)} \int_A z - w d\mu + \chi_A \frac{1}{\mu(A)} \int_B z - w d\mu \right\|_{L^p}^p \\
&= \left(\frac{1}{\mu(A)} \int_A z - w d\mu \right)^p \mu(B) + \left(\frac{1}{\mu(B)} \int_B z - w d\mu \right)^p \mu(A) \\
&\leq \|(z-w)\left(\chi_A + \chi_B\right)\|_{L^p}^p \\
&\leq \|z - w\|_{L^p}^p
\end{aligned}$$

and one explicit $2\pi-$periodic solution is given by

$$z(t) = -(\cos t)\chi_A + (\sin t)\chi_B.$$

CHAPTER 6

A Geometric Uncertainty Principle

If you choose to represent the various parts in life by holes upon a table, of different shapes,–some circular, some triangular, some square, some oblong,–and the person acting these parts by bits of wood of similar shapes, we shall generally find that the triangular person has got into the square hole, the oblong into the triangular, and a square person has squeezed himself into the round hole. The officer and the office, the doer and the thing done, seldom fit so exactly, that we can say they were almost made for each other. (Sydney Smith, [**Smi50**])

1. Introduction

This Chapter deals with a particular question about the structure of nodal domains of Laplacian eigenfunctions associated to a bounded, open domain $\Omega \subset \mathbb{R}^2$. As we have already seen before, the Laplacian operator with Dirichlet conditions gives rise to a sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ and associated eigenfunctions $(\phi_n)_{n \in \mathbb{N}}$, where

$$\begin{aligned} -\Delta \phi_n &= \lambda_n \phi_n \quad \text{in } \Omega \\ \phi_n &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

One natural question is to find bounds on the number of connected components of

$$\Omega \setminus \{x \in \Omega : \phi_n(x) = 0\}.$$

Let us denote this quantity by $N(\phi_n)$. Eremenko, Jakobson & Nadirashvili [**EJN07**] have shown that there are no nontrivial lower bounds on $N(\phi_n)$ in general. Denoting the smallest positive zero of the Bessel function by $j \sim 2.40\dots$, the known upper bounds are as follows

$$\begin{aligned} N(\phi_n) &\leq n && \text{(Courant, 1924)} \\ \limsup_{n \to \infty} \frac{N(\phi_n)}{n} &\leq \left(\frac{2}{j}\right)^2 && \text{(Pleijel, 1956)} \\ \limsup_{n \to \infty} \frac{N(\phi_n)}{n} &\leq \left(\frac{2}{j}\right)^2 - 3 \cdot 10^{-9} && \text{(Bourgain, 2013),} \end{aligned}$$

where $(2/j)^2 \leq 7/10$. Polterovich [**Pol09**] suggests that the optimal constant might be $2/\pi \sim 0.63$ with equality for a rectangle.

1.1. Pleijel's argument. Pleijel's argument is quickly summarized. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain and consider an eigenvalue λ_n and a Laplacian eigenfunction ϕ_n which decomposes Ω into N nodal domains

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$

The variational characterization yields that

$$\lambda_1(\Omega_i) \leq \lambda_n(\Omega).$$

Weyl's law implies that asymptotically

$$\lambda_n(\Omega) = \frac{4\pi n}{|\Omega|} + o(n).$$

Finally, we have the Faber-Krahn inequality telling us that

$$\lambda_1(\Omega_i) \geq \lambda_1(B),$$

where B is the disk satisfying $|B| = |\Omega_i|$. It is explicitely known that

$$\lambda_1(B) = \frac{\pi j^2}{|B|},$$

where $j \sim 2.4$ is the smallest positive zero of the first Bessel function. Altogether, we have the chain of inequalities

$$N \leq \frac{|\Omega|}{|B|} = \frac{\lambda_1(B)|\Omega|}{\pi j^2} \leq \frac{\lambda_n(\Omega)|\Omega|}{\pi j^2} = \frac{4\pi n}{\pi j^2} + o(n).$$

This proves the statement.

Note that Pleijel's argument can only be sharp in the case of Ω being the union of disjoint disks of equal radius – and, as Polterovich [**Pol09**] remarks, never in the limit.

1.2. Bourgain's improvement. Bourgain employs a spectral stability estimate due to Hansen & Nadirashvili, which is formulated in terms of the inradius of a domain: for a nonempty, bounded domain $\Omega \subset \mathbb{R}^2$, we have

$$\lambda_1(\Omega) \geq \left[1 + \frac{1}{250}\left(1 - \frac{r_i(\Omega)}{r_o(\Omega)}\right)^3\right]\lambda_1(\Omega_0),$$

where Ω_0 is the ball with $|\Omega_0| = |\Omega|$, $r_0(\Omega)$ is the radius of Ω_0 and r_i the inradius of Ω. The second ingredient is a packing result due to Blind [**Bli69**]: the packing density of a collection of disks in the plane with radii $a_1, a_2, \dots$ satisfying $a_i \geq (3/4)a_j$ for all i, j is bounded from above by $\pi/\sqrt{12}$. These two results imply the improvement.

2. Main statement

2.1. Introduction. Bourgain's argument implicitely relies on the following simple fact: a partition

$$\bigcup_{i=1}^{\infty} \Omega_i = \mathbb{R}^2$$

implies that not all the Ω_i are disks with identical radius. However, it is easy to decompose $\mathbb{R}^2$ into sets of equal measure that are 'almost' disks (the hexagonal packing, for example) and it is also possible to decompose $\mathbb{R}^2$ into disks of different size – but obviously not both at the same time.

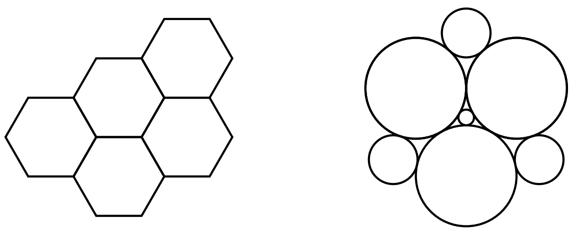

FIGURE 1. A partition into sets of equal measure and a partition into disks (only large disks visible).

We are interested in a quantitative descriptions of this phenomenon – finding quantitative formulations is a problem which we consider to be of independent interest. Our quantitative study in terms of Fraenkel asymmetry and size, however, is very much motivated by the applicability to nodal domain estimates – it could be of interest to capture the same phenomenon in other geometrically natural quantities.

2.2. Geometric notions. Consider an open, bounded domain $\Omega \subset \mathbb{R}^2$ with a given decomposition

$$\Omega = \bigcup_{i=1}^{N} \Omega_i.$$

We require two quantities to measure

(1) the deviation of Ω_i from a disk
(2) the deviation of $|\Omega_i|$ from

$$\min_{1 \le j \le N} |\Omega_j|.$$

In measuring how much a set deviates from a circle, Fraenkel asymmetry has recently become an increasingly central notion (i.e. [**FMP08**]): given a domain $\Omega \subset \mathbb{R}^2$, its Fraenkel asymmetry is defined via

$$\mathcal{A}(\Omega) := \inf_{B} \frac{|\Omega \triangle B|}{|\Omega|},$$

where the infimum ranges over all disks $B \subset \mathbb{R}^2$ with $|B| = |\Omega|$ and $\triangle$ is the symmetric difference

$$\Omega \triangle B = (\Omega \setminus B) \cup (B \setminus \Omega).$$

Fraenkel asymmetry is scale-invariant

$$0 \le \mathcal{A}(\Omega) \le 2.$$

As for deviation in size, we define the (scale-invariant) deviation from the smallest element in the partition via

$$D(\Omega_i) := \frac{|\Omega_i| - \min_{1 \le j \le N} |\Omega_j|}{|\Omega_i|},$$

which is scale invariant as well and satisfies

$$0 \le D(\Omega_i) \le 1.$$

2.3. Main result. Our main result states that for partitions with $\min_{1 \le j \le N} |\Omega_j|$ tending to 0, an average element of the partition needs to have either its Fraenkel asymmetry $\mathcal{A}(\Omega_i)$ or its deviation from the smallest element $D(\Omega_i)$ bounded away from 0 by a *universal* constant. This statement obviously fails if we only pick one term: any set can be decomposed into N sets of measure $|\Omega|/N$ each or each set can be decomposed into disks of different radii with an arbitrarily small measure of different shape (packings of Apollonian type).

Theorem. *Suppose $\Omega \subset \mathbb{R}^n$ is an open and bounded domain and*

$$\Omega = \bigcup_{i=1}^{N} \Omega_i$$

with measurable sets Ω_i satisfying

$$\Omega_i \cap \Omega_j = \emptyset \qquad \text{for} \quad i \neq j.$$

There exists a universal constant $c_n > 0$ depending only on the dimension and a constant $N_0 \in \mathbb{N}$ depending only on Ω such that for $N \ge N_0$

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \ge c_n.$$

In particular,

$$c_2 \ge \frac{1}{60000}.$$

There are no assumptions whatsoever on the shape of Ω_j (in particular, it does not need to be connected). Taking Ω to be the union of disjoint balls of equal radius shows that such a statement can only hold for N sufficiently large (depending on Ω). The proof for two dimensions immediately generalizes to higher dimensions. Fraenkel asymmetry turns the problem into a non-local one as the 'missing' measure $\Omega \triangle B$ can be arbitrarily spread over the plane: this is why we believe that any argument yielding a substantially improved constant will need to be based on significantly new ideas.

What is the optimal constant c_n? A natural candidate for an extremizer in $\mathbb{R}^2$ is the hexagonal tiling, which would suggest that

$$c_2 \sim 0.074465754\dots$$

As packing density of spheres decreases in higher dimensions, we consider it extremely natural to conjecture that

$$c_2 \leq c_3 \leq \dots$$

2.4. Variants and extensions. There are many possible variations and extensions. We can write Fraenkel asymmetry as

$$\mathcal{A}(\Omega) = \inf_{x \in \mathbb{R}^n} \frac{|\Omega \triangle (B+x)|}{|\Omega|},$$

where B is the ball centered at the origin scaled in such a way that $|B| = |\Omega|$. However, this definition can be easily generalized by considering other sets K instead of the ball if one corrects for the arising lack of rotational symmetry, i.e.

$$\mathcal{A}_K(\Omega) := \inf_{x \in \mathbb{R}^n} \inf_{R \in \mathcal{R}} \frac{|\Omega \triangle (RK+x)|}{|\Omega|},$$

where K is scaled in such a way that $|K| = |\Omega|$ and $\mathcal{R}$ is the set of all rotations. The proof of our main statement is quite robust: it immediately allows to prove the following variant.

Theorem. *Let $K \subset \mathbb{R}^n$ be a bounded, convex set with a smooth boundary which contains no line segment. Then there exists a constant $c(K) > 0$ and a geometric uncertainty principle*

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i) \right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \geq c(K).$$

This is certainly not the most general form of the theorem. Let $\mathcal{S}$ be the set of bounded sets in $\mathbb{R}^n$ such that $\mathbb{R}^n$ can be partitioned into translations and rotations of $\mathcal{S}$. Suppose K is a bounded set satisfying

$$\inf_{S \in \mathcal{S}} \mathcal{A}_S(K) > \varepsilon$$

for some $\varepsilon > 0$. Does this imply a geometric uncertainty principle for $\mathcal{A}_K$ with a constant depending only on ε?

2.5. The optimal constant. We have no real understanding of how the optimal constant behaves. Let us give two particular constructions, where we would consider it surprising if they were too far removed from the truth.

1. Hexagonal disc packing. Consider the hexagonal packing of discs of radius 1 in $\mathbb{R}^2$. The packing density is

$$d = \frac{\pi}{\sqrt{12}} \sim 0.90\dots.$$

This packing leaves small gaps: each disc is adjacent to 6 gaps and each gap is surrounded by 3 circles. Thus, there are twice as many gaps as discs meaning that the ratio of the size of a disc to the size g of a gap is

$$g : \pi = \frac{1}{2}\left(\frac{\sqrt{12}}{\pi} - 1 \right) \sim \frac{1}{19.48\dots}.$$

This determines our construction. We take every disc to be an element of the partition and the remaining elements are chosen in a greedy way from the gaps such that the measure of all elements in the partition coincide. A typical element of the partition is thus either a disc or a collection of 19 gaps and roughly one half of another gap.

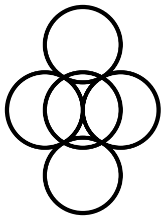

FIGURE 2. Covering two gaps with a Fraenkel ball.

The figure shows that a ball can cover two gaps, therefore we have for the Fraenkel asymmetry of our exceptional sets

$$\mathcal{A} = 2 - 2g = 1 + \frac{\pi}{\sqrt{12}}.$$

At the same time, the probability p of hitting an exceptional set is merely

$$p = \left(1 - \frac{\pi}{\sqrt{12}}\right).$$

Thus the optimal constant satisfies

$$c \leq \left(1 - \frac{\pi}{\sqrt{12}}\right)\left(1 + \frac{\pi}{\sqrt{12}}\right) \sim 0.177\ldots$$

2. Hexagonal packing. A simpler and more efficient construction is given by the hexagonal packing: we simply cover the entire plane with hexagons. They are all of the same size, relevant is therefore its Fraenkel asymmetry. A simple calculation yields

$$c \sim 0.07446\ldots$$

It remains an open problem whether this could actually be optimal.

2.6. The connection to Pleijel estimates. Exploiting stability estimates for the Faber-Krahn inequality in terms of Fraenkel asymmetry, we are able to prove the following result.

Corollary. *There exists a constant $\varepsilon_0 > 0$ such that*

$$\limsup_{n \to \infty} \frac{N(\phi_n)}{n} \leq \left(\frac{2}{j}\right)^2 - \varepsilon_0.$$

An explicit value for ε_0 would follow from an explicit constant in a Faber-Krahn stability result involving Fraenkel asymmetry (these constants are known to exist but have not yet been determined explicitely). Given the general interest in this question, we are confident that such a result will be eventually obtained. Much like Bourgain, however, we consider the underlying geometry more interesting than the actual numerical value – particularly in light of the following obstruction.

2.7. Limits of the method. Take $\Omega = [0,1]^2$ of unit measure and cover it using again the hexagonal covering (with obvious modifications at the boundary). Numerical computations (e.g. [**Sun04**]) give that the first Laplacian eigenvalue of a hexagon H satisfies

$$\lambda_1(H) \sim \frac{18.5762}{|H|}.$$

Weyl Law gives

$$\lambda_n(\Omega) \sim 4\pi n.$$

We can place N hexagons of size $|H|$ in Ω, where

$$N|H| = 1.$$

Since we need to have $\lambda_n(\Omega) \geq \lambda_1(H)$, this implies

$$4\pi n \sim \frac{18.5762}{|H|}$$

and thus

$$N = \frac{1}{|H|} \sim \frac{4\pi}{18.5762} n \sim 0.676\ldots n.$$

As a consequence, any type of argument that leads to an improved Pleijel inequality with a constant smaller than $0.67\ldots$ will need to employ completely different arguments: the arguments given by Pleijel, Bourgain and this paper argue based on the *assumption* that a partition of Ω into nodal domains is given. However, such a partition could very well be the hexagonal partition given above. Arguments leading to a better constant than $0.67\ldots$ will need to *explain* why, say, an eigenfunction on a domain will not have eigenfunctions corresponding to a partition into hexagons. It seems natural to assume that a domain $\Omega \subset \mathbb{R}^2$ giving rise to a large number of nodal domains needs to have a completely integrable geodesic flow. Some numerical experiments in this direction have been carried out by Blum, Gnutzmann & Smilansky [**BGS02**].

3. Proofs

3.1. Two possible strategies. There is a very natural strategy of proof: the inequality can be regarded as a probabilistic statement. Pick a random domain weighted according to size (i.e. the probability of picking Ω_i is Ω_i/Ω. Our statement can be read as a lower bound on the expectation of the random variable

$$\mathcal{A}(\Omega_i) + D(\Omega_i).$$

This motivates the following argument. Pick a random domain: either it already has large Fraenkel asymmetry (in which case we are done) or it does not and behaves quite disk-like. In the second case, we look at its neighbouring domains. If there are few adjacent domains, at least one of them touches along a long arc of the boundary meaning that the neighbouring domain has large Fraenkel asymmetry (two disks touch in at most one point). If there are many neighbours, either most are significantly smaller (making our randomly chosen domain big in comparison and giving the statement) or some will need to get squeezed together because there is not enough room (creating a large Fraenkel asymmetry). We believe that such a strategy, properly implemented, could give a relatively sharp constant – however, making all these steps quantitative seems complicated.

Our proof. We chose a different approach of a more global nature: given a decomposition, we immediately switch to a collection of N disks by taking disks realizing the Fraenkel asymmetry for each partition. Then, we show that

- there are few very large elements: the size of neighbourhood of the union of all disks whose size is bounded away from the smallest element in the partition by a constant factor can be bounded from above.
- ignoring the large sets (of which there are few), the Fraenkel balls of small sets usually do not overlap too much; the exceptional set is small.

Removing all large disks and all overlapping disks, we may shrink the remaining disks such that no two of them overlap: the resulting disk packing cannot have too high a density.

3.2. Proof for $n = 2$. The limes inferior in the statement guarantees that boundary effects coming from $\partial\Omega$ become neglible and we will ignore the boundary throughout the proof (equivalently, we could have phrased the statement for periodic partitions of $\mathbb{R}^2$).

PROOF. We assume w.l.o.g. that $|\Omega| = 1$. For a point $x \in \mathbb{R}^2$ and a set $A \subset \mathbb{R}^2$, we abbreviate

$$\|x - A\| := \inf_{y \in A} \|x - y\|.$$

The numbers $c_1, c_2 > 0$ denote fixed positive constants to be determined later: we call Ω_i 'big', if

$$|\Omega_i| \geq (1 + c_1) \min_{1 \leq j \leq N} |\Omega_j|.$$

The constant c_2 will serve as a measure of overlap: two disks with centers in $x, y \in \mathbb{R}^2$ and radii r_1, r_2 will be considered to have 'large' overlap if

$$|x - y| \leq (1 - c_2)(r_1 + r_2).$$

We define the length scale η_0 of the smallest set via

$$\pi \eta_0^2 = \min_{1 \leq i \leq N} |\Omega_i|.$$

Everything here is scale-invariant and, correspondingly, the actual size of η_0 is irrelevant throughout the proof: the variable cancels in the end. However, we consider it helpful to imagine a fixed length scale η_0 at which everything plays out and will phrase all arising quantities in terms of η_0. The proof is by contradiction, we assume

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \right) \leq c$$

for some small constant c. More precisely, for $d_1, d_2 \geq 0$ and $c = d_1 + d_2$, we assume

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \right) \leq d_1 \qquad \text{and} \qquad \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{D}(\Omega_i) \right) \leq d_2.$$

We assign to each of the sets $\Omega_1, \dots, \Omega_n$ a disk $B_1, B_2, \dots, B_n$ such that $|B_i| = |\Omega_i|$ and

$$\mathcal{A}(\Omega_i) = \frac{|\Omega_i \triangle B_i|}{|\Omega_i|}.$$

Note that a disk B_i need not be uniquely determined by Ω_i (if there is more than one possible choice, we pick an arbitrary one and fix it for the rest of the proof). Each of these disks B_i has a center x_i and a radius $r_i \geq \eta_0$.

1. Large sets have small measure in total. The first step is to prove that the measure of the union of sets having large measure is small, i.e. we derive an upper bound on the set

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\pi\eta_0^2} \Omega_i \right|.$$

If the constant d_2 is to be small, there need to be many elements in the partition:

$$d_2 \geq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) = \sum_{i=1}^{N} (|\Omega_i| - \pi\eta_0^2) = 1 - N\pi\eta_0^2$$

and therefore

$$N \geq \frac{1 - d_2}{\pi \eta_0^2}.$$

Now, let us suppose that $0 \leq M \leq N$ elements of the partition satisfy $|\Omega_i| \leq (1+c_1)\pi\eta_0^2$. We wish to show that M itself has to be big. Trivially,

$$\left| \bigcup_{|\Omega_i| \leq (1+c_1)\pi\eta_0^2} \Omega_i \right| \geq M\pi\eta_0^2.$$

The remaining measure is divided among big sets, hence the number of 'big' elements is at most

$$N - M \leq \frac{1 - M\pi\eta_0^2}{(1+c_1)\pi\eta_0^2}$$

and thus, in total,

$$\frac{1-d_2}{\pi\eta_0^2} \leq N = M + (N-M) \leq M + \frac{1-M\pi\eta_0^2}{(1+c_1)\pi\eta_0^2}.$$

Rewriting gives

$$M \geq \frac{c_1 - d_2 - d_2c_1}{c_1\pi\eta_0^2},$$

which implies

$$\left| \bigcup_{|\Omega_i| \leq (1+c_1)\pi\eta_0^2} \Omega_i \right| \geq \frac{c_1 - d_2 - d_2c_1}{c_1}$$

and therefore, since $|\Omega| = 1$,

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\pi\eta_0^2} \Omega_i \right| \leq \frac{d_2}{c_1} + d_2.$$

We define the index set I of partition elements with 'big' measure

$$I = \left\{ i \in \{1, \dots, N\} : |\Omega_i| \geq (1+c_1)\pi\eta_0^2 \right\}$$

and claim that a $2\eta_0-$neighbourhood of

$$\bigcup_{i \in I} B_i$$

also has small measure. The worst case is if the entire measure of large sets (being bounded from above by $d_2/c_1 + d_2$) consists of isolated disks of radius $\sqrt{1+c_1}\eta_0$ such that the $2\eta_0-$neighbourhoods of different disks do not overlap, in which case, the total measure gets amplified by factor

$$\frac{(\sqrt{1+c_1}+2)^2\eta_0^2\pi}{(1+c_1)\eta_0^2\pi} \leq 9$$

and

$$\left| \left\{ x \in \Omega : \left\| x - \bigcup_{i \in I} B_i \right\| \leq 2\eta_0 \right\} \right| \leq \frac{9d_2}{c_1} + 9d_2.$$

2. Most small sets have well-separated balls. Let us look at the set of small balls B_i (i.e. where $i \notin I$) and show that the measure of the union of elements from that set, where two centers are too close to each other, is also small: we prove an upper bound on

$$\left| \bigcup_{i \notin I} \{B_i : \exists_j i \neq j \notin I : \ |x_i - x_j| \leq (1-c_2)(r_i + r_j)\} \right|.$$

For simplicity, we introduce the index set

$$J = \{ i \notin I : \ \exists_{j \notin I} \ i \neq j : \ |x_i - x_j| \leq (1-c_2)(r_i + r_j) \}.$$

We derive an upper bound on the measure of the set, which we now can abbreviate as $\cup_{j\in J} B_j$, using nothing but

$$\sum_{i=1}^{N} |\Omega_i| \mathcal{A}(\Omega_i) \leq d_1.$$

Suppose $i \in J$. Then there exists a $j \in J$ such that the balls B_i, B_j have controlled radius

$$\eta_0 \leq r_i, r_j \leq \sqrt{1+c_1}\eta_0$$

and intersect in a quantitatively controlled way

$$|x_i - x_j| \leq (1-c_2)(r_i + r_j).$$

Then the intersection $B_i \cap B_j$ is of interest: if the Fraenkel asymmetry of Ω_i is to be small, then almost all of its measure should be contained in B_i but the very same reasoning also holds for Ω_j and B_j. In particular, since every point in the intersection can only belong to one of the two sets, we have

$$|\Omega_i|\mathcal{A}(\Omega_i) + |\Omega_j|\mathcal{A}(\Omega_j) \geq |B_i \cap B_j| \qquad (\diamond).$$

An elementary computation yields (assuming $c_2 \leq 0.1$)

$$|B_i \cap B_j| \geq \frac{37}{10} c_2^{\frac{3}{2}} \eta_0^2.$$

A priori, the intersection patterns of $\{B_i : i \in J\}$ can be very complicated. However, there is a very simple monotonicity: we can remove areas, where three or more balls intersect and arrange the balls in (possibly more than one) chain. This increases the area and decreases the area of intersection and thus the lower bound on the sum over the Fraenkel asymmetry of the Ω_i that comes from $(\diamond)$.

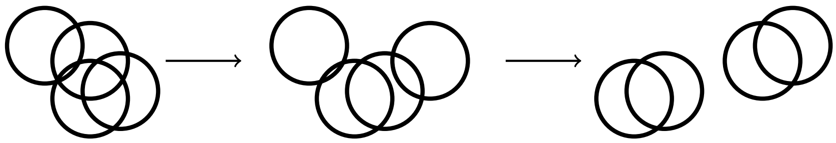

FIGURE 3. Increasing area while decreasing average Fraenkel asymmetry

By the same argument, the area further increases if we cut the chain into pairs of two. A pair of balls contributes less than $2(1+c_1)\pi\eta_0^2$ to the measure and adds at least $37c_2^{\frac{3}{2}}\eta_0^2/10$ to the sum

$$\sum_{i=1}^{N} |\Omega_i| \mathcal{A}(\Omega_i) \leq d_1.$$

Thus

$$\left| \bigcup_{j\in J} B_j \right| \leq \frac{2(1+c_1)\pi\eta_0^2}{\frac{37}{10}c_2^{3/2}\eta_0^2} d_1 = \frac{20\pi}{37} \frac{1+c_1}{c_2^{3/2}} d_1.$$

By applying the same reasoning as before, we could argue that by considering an entire $2\eta_0$-neighbourhood the measure gets amplified by a factor of at most 9. However, any disk is touching at least one other disk and this automatically implies the existence of some overlap. By doing elementary computations, we can bound the multiplication factor from above by $32/5$. Then

$$\left| \left\{ x \in \Omega : \left\| x - \bigcup_{j \in J} B_j \right\| \leq 2\eta_0 \right\} \right| \leq \frac{128\pi}{37} \frac{(1+c_1)}{c_2^{3/2}} d_1.$$

3. Finding a dense disk packing. We conclude our argument by deriving the existence of a disk packing in the plane with impossible properties. Here, we employ a result of Blind [**Bli69**] that also played a role in Bourgain's argument and was mentioned before: the packing density of a collection of disks in the plane with radii $a_1, a_2, \dots$ satisfying $a_i \geq (3/4)a_j$ for all i, j is bounded from above by $\pi/\sqrt{12}$. Taking Ω and removing

$$\left\{x \in \Omega : \left\| x - \bigcup_{i \in I} B_i \right\| \leq 2\eta_0 \right\} \qquad \text{and} \qquad \left\{x \in \Omega : \left\| x - \bigcup_{j \in J} B_j \right\| \leq 2\eta_0 \right\}$$

leaves us with a subset of Ω, in which we find disks with radii satisfying

$$\eta_0 \leq r_i \leq \sqrt{1+c_1}\eta_0$$

and with the additional property that the centers of any two disks are well-seperated

$$|x_i - x_j| \geq (1-c_2)(r_i + r_j).$$

By shrinking all these balls by a factor of $1-c_2$, they become disjoint. Note that the application of Blind's result in this form is not valid: by removing a subset from a set, we could increase the packing density of disk in the remaining set. However, this fact is a boundary effect coming from a neighbourhood of the set we removed and can be counteracted by removing (as we did) an entire neighbourhood of the set as well (alternatively, one could interpret our algebra as the assumption that we are able achieve perfect packing density in that neighbourhood). Altogether,

$$\frac{\pi}{\sqrt{12}} \geq (1-c_2)^2 \left[1 - \left(\frac{9d_2}{c_1} + 9d_2 + \frac{128\pi}{37}\frac{(1+c_1)}{c_2^{3/2}}d_1\right)\right].$$

We need to find a set of parameters, for which the inequality fails. Indeed, setting

$$c_1 = \frac{1}{250} \qquad \text{and} \qquad c_2 = \frac{7}{250},$$

we get for any $d_1, d_2 \geq 0$ with

$$d_1 + d_2 = \frac{1}{60000},$$

that

$$(1-c_2)^2 \left[1 - \left(\frac{9d_2}{c_1} + 9d_2 + \frac{128\pi}{37}\frac{(1+c_1)}{c_2^{3/2}}d_1\right)\right] \geq \frac{\pi}{\sqrt{12}} + \frac{1}{1000}.$$

This contradiction proves the statement. □

3.3. The general case. Here we give a proof of the uniform version of the statement in general dimensions, which obviously implies the main result in higher dimensions as a special case. This section essentially recapitulates the previous argument without caring about the actual numerical values at all. The new ingredients are all trivial.

Proof. The argument is again by contradiction. η_0 plays the same role as before, we define it via

$$\eta_0 = \left(\min_{1 \leq j \leq N} |\Omega_j|\right)^{1/n}.$$

The constant c_1 again determines whether a domain is 'big', which we define to be the case if

$$|\Omega_i| \geq (1+c_1) \min_{1 \leq j \leq N} |\Omega_j|.$$

The precise meaning of c_2 is introduced further below. Suppose now that

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|}\mathcal{A}_K(\Omega_i)\right) + \left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|}D(\Omega_i)\right) \leq c.$$

Following the same argument as before, we again get a bound on the number of nodal sets

$$\left| \bigcup_{|\Omega_i| > (1+c_1)\eta_0^n} \Omega_i \right| \leq \frac{c}{c_1} + c.$$

Switching again to the Fraenkel bodies $K_1, \dots, K_N$, we wish to remove a $c_3\eta_0$ neighbourhood of any 'large' Fraenkel body K_i, where $c_3 < \infty$ is chosen such that $c_3\eta_0$ is many multiples of the diameter of a 'small' K_i having measure at most $(1+c_1)\eta_0^n$. This allows us to bound the size of a $c_3\eta_0$ neighbourhood of

$$\bigcup_{|\Omega_i| > (1+c_1)\eta_0^n} K_i$$

by $c_4(c/c_1 + c)$ for some finite constant c_4. The constant c_2 now measures whether two 'small' Fraenkel bodies have large intersection, writing again

$$I = \{i \in \{1, \dots, N\} : |\Omega_i| \geq (1+c_1)\eta_0^n\},$$

we consider

$$\bigcup_{i \notin I} \{K_i : \exists_{j \notin I} i \neq j : \ |(K_i \cap K_j)| \geq c_2 \eta_0^n\}.$$

The same argument as before implies that for any two elements in the set, we get

$$\mathcal{A}_K(K_i)|K_i| + \mathcal{A}_K(K_j)|K_j| \geq c_2 \eta_0^n.$$

Since

$$\left(\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}_K(\Omega_i) \right) \leq c,$$

this implies a bound on the measure of the set

$$\left| \bigcup_{i \notin I} \{K_i : \exists_{j \notin I} i \neq j : \ |(K_i \cap K_j)| \geq c_2 \eta_0^n\} \right| \leq c_5 c$$

for some constant $c_5 < \infty$ and a bound of the form $c_6 c$ on the measure of its $c_3\eta_0$ neighbourhood. Finally, since the boundary of the convex body K contains no line segment, we get that for every $\varepsilon_1 > 0$ there is a $\varepsilon_2 > 0$ such that any collection $K_1, K_2, \dots$ of nonoverlapping rotated and scaled translates of K in the plane with volumes $v_1, v_2, \dots$ satisfying

$$\inf_{i,j} \frac{v_i}{v_j} \geq 1 - \varepsilon_1$$

has packing density at most $1 - \varepsilon_2$. Finally, there exists a constant c_7 such that for any two scaled, translated copies K_1, K_2 of K with

$$|(K_i \cap K_j)| \leq c_2 \eta_0^n,$$

the rescaled bodies $c_7 K_1, c_7 K_2$ (rescaling being done in a way to fix, say, their center of mass) satisfy

$$(c_7 K_1) \cap (c_7 K_2) = \emptyset.$$

Note that the optimal c_7 depends continuously on c_2 and tends to 1 as c_2 tends to 0. Now, following the same argument as before, we can derive the inequality

$$1 - \varepsilon_2 \geq c_7^n \left(1 - \frac{c_4 c}{c_1} - c_4 c - c_6 c \right).$$

The dependence is easy: pick some $0 < \varepsilon_1 \ll 1$. This yields $\varepsilon_2 > 0$. Given ε_1, pick $c_1 \ll \varepsilon_1$. We pick c_2 so small that $c_7^n > 1 - \varepsilon_2$. c_4 and c_6 are again externally given but the inequality can now be seen to be false if $c = 0$. By continuity $c > 0$. □

3.4. Proof of the Corollary. The Corollary has a very simple proof: as in the proof of Pleijel, we get a lower bound on

$$\min_{1 \leq i \leq N} |\Omega_i|$$

from the Faber-Krahn inequality. Geometric uncertainty now implies that either not all elements in the partition are of that size (in which case some need to be bigger and their requirement for more spaces allows for a smaller number of nodal domains) or that some deviate from the disk in a controlled way (in which case stability estimates require them to have a larger measure).

PROOF. Let
$$\Omega = \bigcup_{i=1}^{N} \Omega_i$$
be the decomposition introduced by a Laplacian eigenfunction with eigenvalue $\lambda \gg 1$ and let η_0 be chosen in such a way that $\pi\eta_0^2 = |B|$, where B is the disk such that $\lambda_1(B) = \lambda$. Suppose
$$\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \left(\mathcal{A}(\Omega_i) + D(\Omega_i)\right) \geq c,$$
then either
$$\sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) \geq \frac{c}{2} \qquad \text{or} \qquad \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} \mathcal{A}(\Omega_i) \geq \frac{c}{2}.$$
Suppose the first inequality holds. Then
$$\frac{c}{2} \leq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|} D(\Omega_i) = \frac{1}{|\Omega|}\left(|\Omega| - N\pi\eta_0^2\right)$$
in which case
$$N \leq \left(1 - \frac{c}{2}\right)\frac{|\Omega|}{\pi\eta_0^2}.$$
The fact that Pleijel's argument is sharp for a partition into equally sized disks (or, equivalently, Weyl's law) implies
$$\lim_{\lambda\to\infty} \frac{|\Omega|}{\pi\eta_0^2} = \left(\frac{2}{j}\right)^2$$
and this yields the result. Suppose the second inequality holds. We recall some stability estimates for the Faber-Krahn inequality in terms of Fraenkel asymmetry. Brasco, De Philippis & Velichkov [**BDPV13**] (improving an earlier result of Fusco, Maggi & Pratelli [**FMP09b**]) have shown that
$$\frac{\lambda_1(\Omega) - \lambda_1(\Omega_0)}{\lambda_1(\Omega_0)} \gtrsim \mathcal{A}(\Omega)^2,$$
where Ω_0 is again the disk with $|\Omega_0| = |\Omega|$.

Claim. We have
$$\left| \bigcup_{\mathcal{A}(\Omega_i)\geq \frac{c}{6}} \Omega_i \right| \geq \frac{c}{6}|\Omega|.$$
Suppose the statement was false. Then, using $\mathcal{A}(\Omega_i) \leq 2$,
$$\begin{aligned} \frac{c}{2} &\leq \sum_{i=1}^{N} \frac{|\Omega_i|}{|\Omega|}\mathcal{A}(\Omega_i) \\ &\leq \frac{2}{|\Omega|}\left| \bigcup_{\mathcal{A}(\Omega_i)\geq \frac{c}{6}} \Omega_i \right| + \frac{c}{6}\frac{1}{|\Omega|}\left| \bigcup_{\mathcal{A}(\Omega_i)\leq \frac{c}{6}} \Omega_i \right| \\ &\leq \frac{c}{3} + \frac{c}{6} = \frac{c}{2}. \end{aligned}$$
Pick any domain Ω_i with $\mathcal{A}(\Omega_i) > c/6$ and use B to denote the disk such that $|B| = |\Omega_i|$. The stability estimate
$$\frac{\lambda_1(\Omega) - \lambda_1(\Omega_0)}{\lambda_1(\Omega_0)} \geq C \cdot \mathcal{A}(\Omega)^2$$
can be rewritten as
$$\lambda_1(\Omega_i) \geq \left(1 + C\frac{c^2}{36}\right)\lambda_1(B),$$

which implies that

$$\frac{|B|}{\pi\eta_0^2} \geq 1 + C\frac{c^2}{36}.$$

In conclusion: we cannot improve on Pleijel's estimate for most domains, however, on a set of measure $c/6$ all nodal domains are a quantitative factor bigger than what is predicted by Faber-Krahn and thus

$$\begin{aligned} N &\leq \left[\left(1-\frac{c}{6}\right)\left(\frac{2}{j}\right)^2 + \frac{c}{6}\frac{(2/j)^2}{1+C\frac{c^2}{36}}\right] n \\ &= \left(1 - \frac{c^3 C}{216+6c^2C}\right)\left(\frac{2}{j}\right)^2 n. \end{aligned}$$

□

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