

Affine Grassmannians and Geometric Satake Equivalences

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Zusammenfassung

AFFINE GRASSMANNIANS AND GEOMETRIC SATAKE EQUIVALENCES

BY TIMO RICHARZ

This thesis consists of two parts, cf. [11] and [12]. Each part can be read independently, but the results in both parts are closely related. In the first part I give a new proof of the geometric Satake equivalence in the unramified case. In the second part I extend the theory to the ramified case using as a black box the unramified Satake equivalence. Let me be more specific.

Part I. Split connected reductive groups are classified by their root data. These data come in pairs: for every root datum there is an associated dual root datum. Hence, for every split connected reductive group G , there is an associated dual group \hat{G} . Following Drinfeld's geometric interpretation of Langlands' philosophy, the representation theory of \hat{G} is encoded in the geometry of an infinite dimensional scheme canonically associated with G as follows, cf. Ginzburg [4], Mirković-Vilonen [8].

Let G be a connected reductive group over a separably closed field F . The *loop group* $L_z G$ is the group functor on the category of F -algebras

$$L_z G: R \longmapsto G(R((z))),$$

where z is an additional variable. The *positive loop group* $L_z^+ G$ is the group functor

$$L_z^+ G: R \longmapsto G(R[[z]]).$$

Then $L_z^+ G \subset L_z G$ is a subgroup functor, and the fpqc-quotient $\mathrm{Gr}_G = L_z G / L_z^+ G$ is called the *affine Grassmannian*. The fpqc-sheaf Gr_G is representable by an inductive limit of projective schemes over F . The positive loop group $L_z^+ G$ is representable by an infinite dimensional affine group scheme, and its left action on each $L_z^+ G$ -orbit on Gr_G factors through a smooth affine group scheme of finite type over F .

Fix a prime number ℓ different from the characteristic of F . The *unramified Satake category* Sat_G is the category

$$\mathrm{Sat}_G \stackrel{\mathrm{def}}{=} P_{L_z^+ G}(\mathrm{Gr}_G)$$

of $L_z^+ G$ -equivariant ℓ -adic perverse sheaves on Gr_G . This is a $\bar{\mathbb{Q}}_\ell$ -linear abelian category whose simple objects can be described as follows. Fix $T \subset B \subset G$ a maximal torus contained in a Borel subgroup. For every cocharacter $\mu \in X_*(T)$ there is an associated F -point $z^\mu \cdot e_0$ of Gr_G , where $z^\mu \in T(F((z)))$ and e_0 denotes the base point. Let Y_μ denote the reduced $L_z^+ G$ -orbit closure of $z^\mu \cdot e_0$ inside Gr_G . Then Y_μ is a projective variety over F which is in general not smooth. Let IC_μ be the intersection complex of Y_μ . The simple objects of Sat_G are the IC_μ 's where μ ranges over the set of dominant cocharacters $X_*(T)^+$.

Furthermore, the Satake category Sat_G is equipped with an inner product: with every $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{Sat}_G$ there is associated a perverse sheaf $\mathcal{A}_1 \star \mathcal{A}_2 \in \mathrm{Sat}_G$ called the *convolution product* of \mathcal{A}_1 and \mathcal{A}_2 , cf. Gaitsgory [3]. Denote by

$$\omega(-) \stackrel{\mathrm{def}}{=} \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_G, -): \mathrm{Sat}_G \longrightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$$

the global cohomology functor with values in the category of finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector spaces.

Let \hat{G} be the Langlands dual group over $\bar{\mathbb{Q}}_\ell$, i.e. the reductive group over $\bar{\mathbb{Q}}_\ell$ whose root datum is dual to the root datum of G . Denote by $\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$ the category of algebraic representations of \hat{G} . Then $\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$ is a semi-simple $\bar{\mathbb{Q}}_\ell$ -linear abelian tensor category with simple objects as follows. Let \hat{T} be the dual torus, i.e. the $\bar{\mathbb{Q}}_\ell$ -torus with $X^*(\hat{T}) = X_*(T)$. Then each dominant weight $\mu \in X^*(\hat{T})^+$ determines an irreducible representation of highest weight μ , and every simple object is isomorphic to a highest weight representation for a unique μ .

The following basic theorem describes Sat_G as a tensor category, and is called the (*unramified*) *geometric Satake equivalence*.

Theorem A.1. *i) The pair (Sat_G, \star) admits a unique symmetric monoidal structure such that the functor ω is symmetric monoidal.*

ii) The functor ω is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories

$$\begin{aligned} (\text{Sat}_G, \star) &\xrightarrow{\cong} (\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}), \otimes) \\ \mathcal{A} &\longmapsto \omega(\mathcal{A}), \end{aligned}$$

which is uniquely determined up to inner automorphisms of \hat{G} by elements in \hat{T} by the property that $\omega(IC_\mu)$ is the irreducible representation of highest weight μ .

In the case $F = \mathbb{C}$, this reduces to a theorem of Mirković and Vilonen [8] for coefficient fields of characteristic 0. However, for $F = \mathbb{C}$ their result is stronger: Mirković and Vilonen establish a geometric Satake equivalence with coefficients in any Noetherian ring of finite global dimension in the analytic topology. I give a proof of the theorem over any separably closed field F using ℓ -adic perverse sheaves. The method is different from the method of Mirković and Vilonen. My proof of Theorem A.1 proceeds in two main steps as follows.

In the first step I show that the pair (Sat_G, \star) is a symmetric monoidal category. This relies on the *BD-Grassmannians* [1] (BD = Beilinson-Drinfeld) and the comparison of the convolution product with the *fusion product* via Beilinson's construction of the nearby cycles functor. Here the fact that the convolution of two perverse sheaves is perverse is deduced from the fact that nearby cycles preserve perversity. The method is based on ideas of Gaitsgory [3] which were extended by Reich [10].

The second step is the identification of the group of tensor automorphisms $\text{Aut}^*(\omega)$ with the reductive group \hat{G} . Here, I use a theorem of Kazhdan, Larsen and Varshavsky [6] which states that the root datum of a split reductive group can be reconstructed from the Grothendieck semiring of its algebraic representations. The reconstruction of the root datum relies on the PRV-conjecture proven by Kumar [7].

The following result is a geometric analogue of the PRV-conjecture.

Theorem B.1. *Denote by $W = W(G, T)$ the Weyl group. Let $\mu_1, \dots, \mu_n \in X_*(T)^+$ be dominant coweights. Then, for every $\lambda \in X_*(T)^+$ of the form $\lambda = \nu_1 + \dots + \nu_k$ with $\nu_i \in W\mu_i$ for $i = 1, \dots, k$, the perverse sheaf IC_λ appears as a direct summand in the convolution product $IC_{\mu_1} \star \dots \star IC_{\mu_n}$.*

Using this theorem and the method in [6], I show that the Grothendieck semirings of Sat_G and $\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$ are isomorphic. Hence, the root data of $\text{Aut}^*(\omega)$ and \hat{G} are the same. This shows that there is an isomorphism $\text{Aut}^*(\omega) \simeq \hat{G}$, which is uniquely determined up to inner automorphisms of \hat{G} by elements in \hat{T} .

If F is any field, i.e. not necessarily separably closed, I apply Galois descent to reconstruct the full L -group, cf. [11, §5]: Let \bar{F} be a separable closure of F , and denote by $\Gamma = \text{Gal}(\bar{F}/F)$ the full Galois group. Then Γ acts on $\text{Sat}_{G_{\bar{F}}}$, and hence via Theorem A.1 above on \hat{G} . In

order to compare this Γ -action on \hat{G} with the usual action via outer automorphisms, the key fact is that \hat{G} is equipped with a canonical pinning via the unramified Satake category. This is based on joint work with Zhu [13, Appendix], and is used to recover the full L -group.

Part II. In the second part of the thesis, I generalize Theorem A.1 to the ramified case using the theory of Bruhat-Tits group schemes. The case of tamely ramified groups is treated by Zhu [13], and I extend his result to include wild ramification. As a prerequisite I prove basic results on the geometry of affine flag varieties as follows.

Specialize the field F to the case of a Laurent power series local field $k((t))$, where k is any separably closed field. As above let G be a connected reductive group over F . The *twisted loop group* LG is the group functor on the category of k -algebras

$$LG: R \longmapsto G(R((t))).$$

The twisted loop group is representable by a strict ind-affine ind-group scheme over k , cf. Pappas-Rapoport [9]. Let \mathcal{G} be a smooth affine model of G over $\mathcal{O}_F = k[[t]]$, i.e. a smooth affine group scheme over \mathcal{O}_F with generic fiber G . The *twisted positive loop group* $L^+\mathcal{G}$ is the group functor on the category of k -algebras

$$L^+\mathcal{G}: R \longmapsto \mathcal{G}(R[[t]]).$$

The twisted positive loop group $L^+\mathcal{G}$ is representable by a reduced affine subgroup scheme of LG of infinite type over k . In general, LG is neither reduced nor connected, whereas $L^+\mathcal{G}$ is connected if the special fiber of \mathcal{G} is connected.

The following result is a basic structure theorem.

Theorem A.2. *A smooth affine model of G with geometrically connected fibers \mathcal{G} over \mathcal{O}_F is parahoric in the sense of Bruhat-Tits [2] if and only if the fpqc-quotient $LG/L^+\mathcal{G}$ is representable by an ind-proper ind-scheme. In this case, $LG/L^+\mathcal{G}$ is ind-projective.*

Theorem A.2 should be viewed as the analogue of the characterization of parabolic subgroups in linear algebraic groups by the properness of their fppf-quotient. Note that the proof of the ind-projectivity of $LG/L^+\mathcal{G}$ for parahoric \mathcal{G} is implicitly contained in Pappas-Rapoport [9].

Let $\mathcal{B}(G, F)$ be the extended Bruhat-Tits building. Let $\mathfrak{a} \subset \mathcal{B}(G, F)$ be a facet, and let $\mathcal{G}_{\mathfrak{a}}$ be the corresponding parahoric group scheme. The fpqc-quotient $\mathcal{F}_{\mathfrak{a}} = LG/L^+\mathcal{G}_{\mathfrak{a}}$ is called the twisted affine flag variety associated with \mathfrak{a} , cf. [9]. As above the twisted positive loop group $L^+\mathcal{G}_{\mathfrak{a}}$ acts from the left on $\mathcal{F}_{\mathfrak{a}}$, and the action on each orbit factors through a smooth affine quotient of $L^+\mathcal{G}_{\mathfrak{a}}$ of finite type. This allows us to consider the category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}_{\mathfrak{a}})$ of $L^+\mathcal{G}_{\mathfrak{a}}$ -equivariant ℓ -adic perverse sheaves on $\mathcal{F}_{\mathfrak{a}}$. Recall that a facet $\mathfrak{a} \subset \mathcal{B}(G, F)$ is called *special* if it is contained in some apartment such that each wall is parallel to a wall passing through \mathfrak{a} .

The next result characterizes special facets \mathfrak{a} in terms of the category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}_{\mathfrak{a}})$.

Theorem B.2. *The following properties are equivalent.*

- i) The facet \mathfrak{a} is special.*
- ii) The stratification of $\mathcal{F}_{\mathfrak{a}}$ in $L^+\mathcal{G}_{\mathfrak{a}}$ -orbits satisfies the parity property, i.e. in each connected component all strata are either even or odd dimensional.*
- iii) The category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}_{\mathfrak{a}})$ is semi-simple.*

The implications $i) \Rightarrow ii) \Rightarrow iii)$ are due to Zhu [13] whereas the implication $iii) \Rightarrow i)$ seems to be new. In fact, the following properties are equivalent to Theorem B.2 $i)$ - $iii)$:

- vi) The special fiber of each global Schubert variety associated with \mathfrak{a} is irreducible.*
- v) The monodromy on Gaitsgory's nearby cycles functor associated with \mathfrak{a} vanishes.*
- vi) Each admissible set associated with \mathfrak{a} contains a unique maximal element.*

See [12, §2] for the definition of global Schubert varieties and admissible sets associated with a facet, and [12, §3] for the definition of Gaitsgory's nearby cycles functor in this context.

If the group G is split, then the choice of a special facet \mathfrak{a} is equivalent to the choice of an isomorphism $G \simeq G_0 \otimes_k F$, where G_0 is a connected reductive group defined over k . In this case, $\mathcal{G}_{\mathfrak{a}} = G_0 \otimes_k \mathcal{O}_F$, and hence $\mathcal{F}\ell_{\mathfrak{a}} \simeq \text{Gr}_{G_0}$ equivariantly for the action of $L^+\mathcal{G}_{\mathfrak{a}} \simeq L_z^+G_0$. Therefore, the category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}})$ is equivalent to the unramified Satake category for G_0 over k by transport of structure.

Now if the group G is not necessarily split, then we have the following description. Let \mathfrak{a} be a special facet. The *ramified Satake category* $\text{Sat}_{\mathfrak{a}}$ associated with \mathfrak{a} is the category

$$\text{Sat}_{\mathfrak{a}} \stackrel{\text{def}}{=} P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}}).$$

The ramified Satake category $\text{Sat}_{\mathfrak{a}}$ is semi-simple with simple objects as follows. Let A be a maximal F -split torus such that \mathfrak{a} lies in the apartment $\mathcal{A}(G, A, F)$ associated with A . Since k is separably closed, G is quasi-split by Steinberg's Theorem. The centralizer $T = Z_G(A)$ is a maximal torus. Let B be a Borel subgroup containing T . The Galois group Γ acts on the cocharacter group $X_*(T)$, and we let $X_*(T)_{\Gamma}$ be the group of coinvariants. With every $\bar{\mu} \in X_*(T)_{\Gamma}$, the Kottwitz morphism associates a k -point $t^{\bar{\mu}} \cdot e_0$ in $\mathcal{F}\ell_{\mathfrak{a}}$, where e_0 denotes the base point. Let $Y_{\bar{\mu}}$ be the reduced $L^+\mathcal{G}$ -orbit closure of $t^{\bar{\mu}} \cdot e_0$. The scheme $Y_{\bar{\mu}}$ is a projective variety over k which is not smooth in general. Let $X_*(T)_{\Gamma}^+$ be the image of the set of dominant cocharacters under the canonical projection $X_*(T) \rightarrow X_*(T)_{\Gamma}$. Then the simple objects of $\text{Sat}_{\mathfrak{a}}$ are the intersection complexes $\text{IC}_{\bar{\mu}}$ of $Y_{\bar{\mu}}$, as $\bar{\mu}$ ranges over $X_*(T)_{\Gamma}^+$.

Recall that in general, for every $\mathcal{A}_1, \mathcal{A}_2 \in \text{Sat}_{\mathfrak{a}}$, the convolution product $\mathcal{A}_1 \star \mathcal{A}_2$ is defined as an object in the bounded derived category of constructible ℓ -adic complexes, cf. [3].

The Galois group Γ acts on \hat{G} by pinning preserving automorphisms, and we let \hat{G}^{Γ} be the fixed points. Then \hat{G}^{Γ} is a reductive group over $\bar{\mathbb{Q}}_{\ell}$ which is not necessarily connected. Let $\text{Rep}_{\bar{\mathbb{Q}}_{\ell}}(\hat{G}^{\Gamma})$ be the category of algebraic representations of \hat{G}^{Γ} . Note that $X_*(T)_{\Gamma} = X^*(\hat{T}^{\Gamma})$, and that for every $\bar{\mu} \in X^*(\hat{T}^{\Gamma})^+$, there exists a unique irreducible representation of \hat{G}^{Γ} of highest weight $\bar{\mu}$, cf. [12, Appendix].

The last theorem describes $\text{Sat}_{\mathfrak{a}}$ as a tensor category, and is called the *ramified geometric Satake equivalence*.

Theorem C.2. *i) The category $\text{Sat}_{\mathfrak{a}}$ is stable under the convolution product \star , and the pair $(\text{Sat}_{\mathfrak{a}}, \star)$ admits a unique structure of a symmetric monoidal category such that the global cohomology functor*

$$\omega_{\mathfrak{a}}(-) \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathcal{F}\ell_{\mathfrak{a}}, -): \text{Sat}_{\mathfrak{a}} \longrightarrow \text{Vec}_{\bar{\mathbb{Q}}_{\ell}}$$

is symmetric monoidal.

ii) The functor $\omega_{\mathfrak{a}}$ is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories

$$\begin{aligned} (\text{Sat}_{\mathfrak{a}}, \star) &\xrightarrow{\cong} (\text{Rep}_{\bar{\mathbb{Q}}_{\ell}}(\hat{G}^{\Gamma}), \otimes), \\ \mathcal{A} &\longmapsto \omega_{\mathfrak{a}}(\mathcal{A}) \end{aligned}$$

which is uniquely determined up to inner automorphisms of \hat{G}^{Γ} by elements in \hat{T}^{Γ} by the property that $\omega_{\mathfrak{a}}(\text{IC}_{\bar{\mu}})$ is the irreducible representation of highest weight $\bar{\mu}$.

I also prove a variant of Theorem C.2 which includes Galois actions, and where k may be replaced by a finite field. If \mathfrak{a} is hyperspecial, then the Γ -action on \hat{G} is trivial, and Theorem C.2 reduces to Theorem A.1 above, cf. the remark below Theorem B.2.

Theorem C.2 is due to Zhu [13] in the case of tamely ramified groups. With Theorem B.2 at hand, my method follows the method of [13] with minor modifications. Let me outline the proof. Based on the unramified Satake equivalence for $G_{\bar{F}}$ as explained above, the main ingredient in the proof of Theorem C.2 is the BD-Grassmannian $\text{Gr}_{\mathfrak{a}}$ associated with the

group scheme \mathcal{G}_α : the BD-Grassmannian Gr_α is a strict ind-projective ind-scheme over $S = \text{Spec}(\mathcal{O}_F)$ such that there is a cartesian diagram of ind-schemes

$$\begin{array}{ccccc} \mathcal{F}_\alpha & \longrightarrow & \text{Gr}_\alpha & \longleftarrow & \text{Gr}_G \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \eta, \end{array}$$

where η (resp. s) denotes the generic (resp. special) point of S . Note that we used the additional formal variable z to define Gr_G as above. This allows us to consider *Gaitsgory's nearby cycles functor*

$$\Psi_\alpha : \text{Sat}_{G_{\bar{F}}} \longrightarrow \text{Sat}_\alpha$$

associated with $\text{Gr}_\alpha \rightarrow S$. The symmetric monoidal structure with respect to \star on the category $\text{Sat}_{G_{\bar{F}}}$ in the geometric generic fiber of Gr_α extends to the category Sat_α in the special fiber of Gr_α . This equips $(\text{Sat}_\alpha, \star)$ with a symmetric monoidal structure. Here, the key fact is the vanishing of the monodromy of Ψ_α for special facets α , cf. item *v*) in the list below Theorem B.2. It is then not difficult to exhibit $(\text{Sat}_\alpha, \star)$ as a Tannakian category with fiber functor ω_α . Theorem B.2 *iii*) implies that the neutral component $\text{Aut}^*(\omega_\alpha)^0$ of the \mathbb{Q}_ℓ -group of tensor automorphisms is reductive. In fact, the nearby cycles construction above realizes $\text{Aut}^*(\omega_\alpha)$ as a subgroup of \hat{G} via the unramified Satake equivalence. The group \hat{G} is equipped with a canonical pinning, and it is easy to identify $\text{Aut}^*(\omega_\alpha) = \hat{G}^\Gamma$ as the subgroup of \hat{G} where Γ acts by pinning preserving automorphisms. This concludes the proof Theorem C.2.

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A NEW APPROACH TO THE GEOMETRIC SATAKE EQUIVALENCE

BY TIMO RICHARZ

ABSTRACT. I give another proof of the geometric Satake equivalence from I. Mirković and K. Vilonen [16] over a separably closed field. Over a not necessarily separably closed field, I obtain a canonical construction of the Galois form of the full L -group.

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INTRODUCTION

Connected reductive groups over separably closed fields are classified by their root data. These come in pairs: to every root datum, there is associated its dual root datum and vice versa. Hence, to every connected reductive group G , there is associated its dual group \hat{G} . Following Drinfeld's geometric interpretation of Langlands' philosophy, Mirković and Vilonen [16] show that the representation theory of \hat{G} is encoded in the geometry of an ind-scheme canonically associated to G as follows.

Let G be a connected reductive group over a separably closed field F . The *loop group* LG is the group functor on the category of F -algebras

$$LG : R \longmapsto G(R((t))).$$

The *positive loop group* L^+G is the group functor

$$L^+G : R \longmapsto G(R[[t]]).$$

Then $L^+G \subset LG$ is a subgroup functor, and the fpqc-quotient $\mathrm{Gr}_G = LG/L^+G$ is called the *affine Grassmannian*. It is representable by an ind-projective ind-scheme (= inductive limit of projective schemes). Now fix a prime $\ell \neq \mathrm{char}(F)$, and consider the category $P_{L^+G}(\mathrm{Gr}_G)$ of L^+G -equivariant ℓ -adic perverse sheaves on Gr_G . This is a $\overline{\mathbb{Q}}_\ell$ -linear abelian category with simple objects as follows. Fix $T \subset B \subset G$ a maximal torus contained in a Borel. For every cocharacter μ , denote by

$$\overline{\mathcal{O}}_\mu \stackrel{\mathrm{def}}{=} \overline{L^+G \cdot t^\mu}$$

the reduced L^+G -orbit closure of $t^\mu \in T(F(\!(t)\!))$ inside Gr_G . Then $\overline{\mathcal{O}}_\mu$ is a projective variety over F . Let IC_μ be the intersection complex of $\overline{\mathcal{O}}_\mu$. The simple objects of $P_{L^+G}(\mathrm{Gr}_G)$ are the IC_μ 's where μ ranges over the set of dominant cocharacters X_+^\vee . Furthermore, the category $P_{L^+G}(\mathrm{Gr}_G)$ is equipped with an inner product: to every $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G)$, there is associated a perverse sheaf $\mathcal{A}_1 \star \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G)$ called the *convolution product* of \mathcal{A}_1 and \mathcal{A}_2 (cf. §2 below). Denote by

$$\omega(-) \stackrel{\mathrm{def}}{=} \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_G, -) : P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\overline{\mathbb{Q}}_\ell}$$

the global cohomology functor with values in the category of finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces. Fix a pinning of G , and let \hat{G} be the Langlands dual group over $\overline{\mathbb{Q}}_\ell$, i.e. the reductive group over $\overline{\mathbb{Q}}_\ell$ whose root datum is dual to the root datum of G .

Theorem 0.1. (i) *The pair $(P_{L^+G}(\mathrm{Gr}_G), \star)$ admits a unique symmetric monoidal structure such that the functor ω is symmetric monoidal.*

(ii) *The functor ω is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories*

$$\begin{aligned} (P_{L^+G}(\mathrm{Gr}_G), \star) &\xrightarrow{\simeq} (\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}), \otimes) \\ \mathcal{A} &\longmapsto \omega(\mathcal{A}), \end{aligned}$$

which is uniquely determined up to inner automorphisms of \hat{G} by the property that $\omega(\mathrm{IC}_\mu)$ is the irreducible representation of highest weight μ (for the dual torus \hat{T}).

In the case $F = \mathbb{C}$, this reduces to the theorem of Mirković and Vilonen [16] for coefficient fields of characteristic 0. The drawback of our method is the restriction to $\overline{\mathbb{Q}}_\ell$ -coefficients. Mirković and Vilonen are able to establish a geometric Satake equivalence with coefficients in any Noetherian ring of finite global dimension (in the analytic topology). I give a proof of the theorem over any separably closed field F using ℓ -adic perverse sheaves. My proof is different from the one of Mirković and Vilonen. It proceeds in two main steps as follows.

In the first step I show that the pair $(P_{L^+G}(\mathrm{Gr}_G), \star)$ is a symmetric monoidal category. This relies on the *Beilinson-Drinfeld Grassmannians* [2] and the comparison of the convolution product with the *fusion product* via Beilinson's construction of the nearby cycles functor. Here the fact that the convolution of two perverse sheaves is perverse is deduced from the fact that nearby cycles preserve perversity. The method is based on ideas of Gaitsgory [7] which were extended by Reich [19]. The constructions in this first step are essentially known, my purpose was to give a coherent account of these results.

The second step is the identification of the group of tensor automorphisms $\underline{\mathrm{Aut}}^*(\omega)$ with the reductive group \hat{G} . I use a theorem of Kazhdan, Larsen and Varshavsky [10] which states that the root datum of a split reductive group can be reconstructed from the Grothendieck semiring of its algebraic representations. The reconstruction of the root datum relies on the PRV-conjecture proven by Kumar [11]. I prove the following geometric analogue of the PRV-conjecture.

Theorem 0.2 (Geometric analogue of the PRV-Conjecture). *Denote by $W = W(G, T)$ the Weyl group. Let $\mu_1, \dots, \mu_n \in X_+^\vee$ be dominant coweights. Then, for every $\lambda \in X_+^\vee$ of the form $\lambda = \nu_1 + \dots + \nu_k$ with $\nu_i \in W\mu_i$ for $i = 1, \dots, k$, the perverse sheaf IC_λ appears as a direct summand in the convolution product $\mathrm{IC}_{\mu_1} \star \dots \star \mathrm{IC}_{\mu_n}$.*

Using this theorem and the method in [10], I show that the Grothendieck semirings of $P_{L^+G}(\mathrm{Gr}_G)$ and $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ are isomorphic. Hence, the root data of $\underline{\mathrm{Aut}}^*(\omega)$ and \hat{G} are the same. This shows that $\underline{\mathrm{Aut}}^*(\omega) \simeq \hat{G}$ uniquely up to inner automorphism of \hat{G} .

If F is not necessarily separably closed, we are able to apply Galois descent to reconstruct the full L -group. Fix a separable closure \bar{F} of F , and denote by $\Gamma = \mathrm{Gal}(\bar{F}/F)$ the absolute

Galois group. Let ${}^L G = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes \Gamma$ be the Galois form of the full L -group with respect to some pinning.

Theorem 0.3. *The functor $\mathcal{A} \mapsto \omega(\mathcal{A}_{\bar{F}})$ induces an equivalence of abelian tensor categories*

$$(P_{L+G}(\mathrm{Gr}_G), \star) \simeq (\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G), \otimes),$$

where $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G)$ is the full subcategory of the category of finite dimensional continuous ℓ -adic representations of ${}^L G$ such that the restriction to $\hat{G}(\bar{\mathbb{Q}}_\ell)$ is algebraic.

We outline the structure of the paper. In §1 we introduce the Satake category $P_{L+G}(\mathrm{Gr}_G)$. Appendix A supplements the definition of $P_{L+G}(\mathrm{Gr}_G)$ and explains some basic facts on perverse sheaves on ind-schemes as used in the paper. In §2-§3 we clarify the tensor structure of the tuple $(P_{L+G}(\mathrm{Gr}_G), \star)$, and show that it is neutralized Tannakian with fiber functor ω . Section 4 is devoted to the identification of the dual group. This section is supplemented by Appendix B on the reconstruction of root data from the Grothendieck semiring of algebraic representations. The reader who is just interested in the case of an algebraically closed ground field may assume F to be algebraically closed throughout §1-§4. The last section §5 is concerned with Galois descent and the reconstruction of the full L -group.

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1. THE SATAKE CATEGORY

Let G a connected reductive group over any field F . The *loop group* LG is the group functor on the category of F -algebras

$$LG : R \longmapsto G(R((t))).$$

The *positive loop group* L^+G is the group functor

$$L^+G : R \longmapsto G(R[[t]]).$$

Then $L^+G \subset LG$ is a subgroup functor, and the fpqc-quotient $\mathrm{Gr}_G = LG/L^+G$ is called the *affine Grassmannian* (associated to G over F).

Lemma 1.1. *The affine Grassmannian Gr_G is representable by an ind-projective strict ind-scheme over F . It represents the functor which assigns to every F -algebra R the set of isomorphism classes of pairs (\mathcal{F}, β) , where \mathcal{F} is a G -torsor over $\mathrm{Spec}(R[[t]])$ and β a trivialization of $\mathcal{F}[\frac{1}{t}]$ over $\mathrm{Spec}(R((t)))$.*

We postpone the proof of Lemma 1.1 to Section 2.1 below. For every $i \geq 0$, let G_i denote i -th jet group, given for any F -algebra R by $G_i : R \mapsto G(R[t]/t^{i+1})$. Then G_i is representable by a smooth connected affine group scheme over F and, as fpqc-sheaves,

$$L^+G \simeq \varprojlim_i G_i.$$

In particular, if G is non trivial, then L^+G is not of finite type over F . The positive loop group L^+G operates on Gr_G and, for every orbit \mathcal{O} , the L^+G -action factors through G_i for some i . Let $\bar{\mathcal{O}}$ denote the reduced closure of \mathcal{O} in Gr_G , a projective L^+G -stable subvariety. This presents the reduced locus as the direct limit of L^+G -stable subvarieties

$$(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_{\mathcal{O}} \bar{\mathcal{O}},$$

where the transition maps are closed immersions.

Fix a prime $\ell \neq \mathrm{char}(F)$, and denote by \mathbb{Q}_ℓ the field of ℓ -adic numbers with algebraic closure $\bar{\mathbb{Q}}_\ell$. For any separated scheme T of finite type over F , we consider the bounded derived category $D_c^b(T, \bar{\mathbb{Q}}_\ell)$ of constructible ℓ -adic complexes on T , and its abelian full subcategory

$P(T)$ of ℓ -adic perverse sheaves. If H is a connected smooth affine group scheme acting on T , then let $P_H(T)$ be the abelian subcategory of $P(T)$ of H -equivariant objects with H -equivariant morphisms. We refer to Appendix A for an explanation of these concepts.

The category of ℓ -adic perverse sheaves $P(\mathrm{Gr}_G)$ on the affine Grassmannian is the direct limit

$$P(\mathrm{Gr}_G) \stackrel{\mathrm{def}}{=} \varinjlim_{\mathcal{O}} P(\overline{\mathcal{O}}),$$

which is well-defined, since all transition maps are closed immersions, cf. Appendix A.

Definition 1.2. The *Satake category* is the category of L^+G -equivariant ℓ -adic perverse sheaves on the affine Grassmannian Gr_G

$$P_{L^+G}(\mathrm{Gr}_G) \stackrel{\mathrm{def}}{=} \varinjlim_{\mathcal{O}} P_{L^+G}(\overline{\mathcal{O}}),$$

where \mathcal{O} ranges over the L^+G -orbits.

The Satake category $P_{L^+G}(\mathrm{Gr}_G)$ is an abelian $\overline{\mathbb{Q}}_\ell$ -linear category, cf. Appendix A.

2. THE CONVOLUTION PRODUCT

We are going to equip the category $P_{L^+G}(\mathrm{Gr}_G)$ with a tensor structure. Let

$$-\star -: P(\mathrm{Gr}_G) \times P_{L^+G}(\mathrm{Gr}_G) \longrightarrow D_c^b(\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$$

be the convolution product with values in the derived category. We recall its definition [17, §2]. Consider the following diagram of ind-schemes

$$(2.1) \quad \mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G.$$

Here p (resp. q) is a right L^+G -torsor with respect to the L^+G -action on the left factor (resp. the diagonal action). The LG -action on Gr_G factors through q , giving rise to the morphism m .

For perverse sheaves $\mathcal{A}_1, \mathcal{A}_2$ on Gr_G , their box product $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is a perverse sheaf on $\mathrm{Gr}_G \times \mathrm{Gr}_G$. If \mathcal{A}_2 is L^+G -equivariant, then there is a unique perverse sheaf $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ on $LG \times^{L^+G} \mathrm{Gr}_G$ such that there is an isomorphism equivariant for the diagonal L^+G -action¹

$$p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \simeq q^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2).$$

Then the convolution is defined as $\mathcal{A}_1 \star \mathcal{A}_2 \stackrel{\mathrm{def}}{=} m_*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2)$.

Theorem 2.1. (i) *For perverse sheaves $\mathcal{A}_1, \mathcal{A}_2$ on Gr_G with \mathcal{A}_2 being L^+G -equivariant, their convolution $\mathcal{A}_1 \star \mathcal{A}_2$ is a perverse sheaf. If \mathcal{A}_1 is also L^+G -equivariant, then $\mathcal{A}_1 \star \mathcal{A}_2$ is L^+G -equivariant.*

(ii) *Let \overline{F} be a separable closure of F . The convolution product is a bifunctor*

$$-\star -: P_{L^+G}(\mathrm{Gr}_G) \times P_{L^+G}(\mathrm{Gr}_G) \longrightarrow P_{L^+G}(\mathrm{Gr}_G),$$

and $(P_{L^+G}(\mathrm{Gr}_G), \star)$ has a unique structure of a symmetric monoidal category such that the cohomology functor with values in finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces

$$\bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_{G, \overline{F}}, (-)_{\overline{F}}) : P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\overline{\mathbb{Q}}_\ell}$$

is symmetric monoidal.

Part (i) is due to Lusztig [12] and Gaitsgory [7]. Part (ii) is based on methods due to Reich [19]. Both parts of Theorem 2.1 are proved simultaneously in Subsection 2.3 below using universally locally acyclic perverse sheaves (cf. Subsection 2.2 below) and a global version of diagram (2.1) which we introduce in the next subsection.

¹Though LG is not of ind-finite type, we use Lemma 2.20 below to define $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$.

2.1. Beilinson-Drinfeld Grassmannians. Let X a smooth geometrically connected curve over F . For any F -algebra R , let $X_R = X \times \text{Spec}(R)$. Denote by Σ the moduli space of relative effective Cartier divisors on X , i.e. the fppf-sheaf associated with the functor on the category of F -algebras

$$R \longmapsto \{D \subset X_R \text{ relative effective Cartier divisor}\}.$$

Lemma 2.2. *The fppf-sheaf Σ is represented by the disjoint union of fppf-quotients*

$$\coprod_{n \geq 1} X^n / S_n,$$

where the symmetric group S_n acts on X^n by permuting its coordinates. □

Definition 2.3. The *Beilinson-Drinfeld Grassmannian (associated to G and X)* is the functor $\mathcal{G}r = \mathcal{G}r_{G,X}$ on the category of F -algebras which assigns to every R the set of isomorphism classes of triples (D, \mathcal{F}, β) with

$$\left\{ \begin{array}{l} D \in \Sigma(R) \text{ a relative effective Cartier divisor;} \\ \mathcal{F} \text{ a } G\text{-torsor on } X_R; \\ \beta : \mathcal{F}|_{X_R \setminus D} \xrightarrow{\cong} \mathcal{F}_0|_{X_R \setminus D} \text{ a trivialisation,} \end{array} \right.$$

where \mathcal{F}_0 denotes the trivial G -torsor. The projection $\mathcal{G}r \rightarrow \Sigma$, $(D, \mathcal{F}, \beta) \mapsto D$ is a morphism of functors.

Lemma 2.4. *The Beilinson-Drinfeld Grassmannian $\mathcal{G}r = \mathcal{G}r_{G,X}$ associated to a reductive group G and a smooth curve X is representable by an ind-proper strict ind-scheme over Σ .*

Proof. This is proven in [7, Appendix A.5.]. We sketch the argument. If $G = \text{GL}_n$, consider the functor $\mathcal{G}r_{(m)}$ parametrizing

$$J \subset \mathcal{O}_{X_R}^n(-m \cdot D) / \mathcal{O}_{X_R}^n(m \cdot D),$$

where J is a coherent \mathcal{O}_{X_R} -submodule such that $\mathcal{O}_{X_R}^n(-m \cdot D)/J$ is flat over R . By the theory of Hilbert schemes, the functor $\mathcal{G}r_{(m)}$ is representable by a proper scheme over Σ . For $m_1 < m_2$, there are closed immersions $\mathcal{G}r_{(m_1)} \hookrightarrow \mathcal{G}r_{(m_2)}$. Then as fpqc-sheaves

$$\varinjlim_m \mathcal{G}r_{(m)} \xrightarrow{\cong} \mathcal{G}r.$$

For general reductive G , choose an embedding $G \hookrightarrow \text{GL}_n$. Then the fppf-quotient GL_n/G is affine, and the natural morphism $\mathcal{G}r_G \rightarrow \mathcal{G}r_{\text{GL}_n}$ is a closed immersion. The ind-scheme structure of $\mathcal{G}r_G$ does not depend on the chosen embedding $G \hookrightarrow \text{GL}_n$. This proves the lemma. □

Now we define a global version of the loop group. For every $D \in \Sigma(R)$, the formal completion of X_R along D is a formal affine scheme. We denote by $\hat{\mathcal{O}}_{X,D}$ its underlying R -algebra. Let $\hat{D} = \text{Spec}(\hat{\mathcal{O}}_{X,D})$ be the associated affine scheme over R . Then D is a closed subscheme of \hat{D} , and we set $\hat{D}^\circ = \hat{D} \setminus D$. The *global loop group* is the functor on the category of F -algebras

$$\mathcal{L}G : R \mapsto \{(s, D) \mid D \in \Sigma(R), s \in G(\hat{D}^\circ)\}.$$

The *global positive loop group* is the functor

$$\mathcal{L}^+G : R \mapsto \{(s, D) \mid D \in \Sigma(R), s \in G(\hat{D})\}.$$

Then $\mathcal{L}^+G \subset \mathcal{L}G$ is a subgroup functor over Σ .

Lemma 2.5. (i) *The global loop group $\mathcal{L}G$ is representable by an ind-group scheme over Σ . It represents the functor on the category of F -algebras which assigns to every R the set of isomorphism classes of quadruples $(D, \mathcal{F}, \beta, \sigma)$, where $D \in \Sigma(R)$, \mathcal{F} is a G -torsor on X_R , $\beta : \mathcal{F} \xrightarrow{\cong} \mathcal{F}_0$ is a trivialisation over $X_R \setminus D$ and $\sigma : \mathcal{F}_0 \xrightarrow{\cong} \mathcal{F}|_{\hat{D}}$ is a trivialisation over \hat{D} .*

(ii) *The global positive loop group \mathcal{L}^+G is representable by an affine group scheme over Σ with geometrically connected fibers.*

(iii) *The projection $\mathcal{L}G \rightarrow \mathcal{G}r_G$, $(D, \mathcal{F}, \beta, \sigma) \rightarrow (D, \mathcal{F}, \beta)$ is a right \mathcal{L}^+G -torsor, and induces an isomorphism of fpqc-sheaves over Σ*

$$\mathcal{L}G/\mathcal{L}^+G \xrightarrow{\cong} \mathcal{G}r_G.$$

Proof. We reduce to the case that X is affine. Note that fppf-locally on R every $D \in \Sigma(R)$ is of the form $V(f)$. Then the moduli description in (i) follows from the descent lemma of Beauville-Laszlo [1] (cf. [14, Proposition 3.8]). The ind-representability follows from part (ii) and (iii). This proves (i).

For any $D \in \Sigma(R)$ denote by $D^{(i)}$ its i -th infinitesimal neighbourhood in X_R . Then $D^{(i)}$ is finite over R , and the Weil restriction $\text{Res}_{D^{(i)}/R}(G)$ is representable by a smooth affine group scheme with geometrically connected fibers. For $i \leq j$, there are affine transition maps $\text{Res}_{D^{(j)}/R}(G) \rightarrow \text{Res}_{D^{(i)}/R}(G)$ with geometrically connected fibers. Hence, $\varprojlim_i \text{Res}_{D^{(i)}/R}(G)$ is an affine scheme, and the canonical map

$$\mathcal{L}^+G \times_{\Sigma, D} \text{Spec}(R) \longrightarrow \varprojlim_i \text{Res}_{D^{(i)}/R}(G)$$

is an isomorphism of fpqc-sheaves. This proves (ii).

To prove (iii), the crucial point is that after a faithfully flat extension $R \rightarrow R'$ a G -torsor \mathcal{F} on \hat{D} admits a global section. Indeed, \mathcal{F} admits a R' -section which extends to $\hat{D}_{R'}$ by smoothness and Grothendieck's algebraization theorem. This finishes (iii). \square

Remark 2.6. The connection with the affine Grassmannian Gr_G is as follows. Lemma 2.2 identifies X with a connected component of Σ . Choose a point $x \in X(F)$ considered as an element $D_x \in \Sigma(F)$. Then $\hat{D}_x \simeq \text{Spec}(F[[t]])$, where t is a local parameter of X in x . Under this identification, there are isomorphisms of fpqc-sheaves

$$\begin{aligned} \mathcal{L}G_x &\simeq LG \\ \mathcal{L}^+G_x &\simeq L^+G \\ \mathcal{G}r_{G,x} &\simeq \text{Gr}_G. \end{aligned}$$

Using the theory of Hilbert schemes, the proof of Lemma 2.4 also implies that Gr_{GL_n} , and hence Gr_G is ind-projective. This proves Lemma 1.1 above.

By Lemma 2.5 (iii), the global positive loop group \mathcal{L}^+G acts on $\mathcal{G}r$ from the left. For $D \in \Sigma(R)$ and $(D, \mathcal{F}, \beta) \in \mathcal{G}r_G(R)$, denote the action by

$$((g, D), (\mathcal{F}, \beta, D)) \longmapsto (g\mathcal{F}, g\beta, D).$$

Corollary 2.7. *The \mathcal{L}^+G -orbits on $\mathcal{G}r$ are of finite type and smooth over Σ .*

Proof. Let $D \in \Sigma(R)$. It is enough to prove that the action of

$$\mathcal{L}^+G \times_{\Sigma, D} \text{Spec}(R) \simeq \varprojlim_i \text{Res}_{D^{(i)}/R}(G)$$

on $\mathcal{G}r \times_{\Sigma, D} \text{Spec}(R)$ factors over $\text{Res}_{D^{(i)}/R}(G)$ for some $i \gg 0$. Choose a faithful representation $\rho : G \rightarrow \text{GL}_n$, and consider the corresponding closed immersion $\mathcal{G}r_G \rightarrow \mathcal{G}r_{\text{GL}_n}$. This reduces us to the case $G = \text{GL}_n$. In this case, the $\mathcal{G}r_{(m)}$'s (cf. proof of Lemma 2.4) are $\mathcal{L}^+\text{GL}_n$ stable, and it is easy to see that the action on $\mathcal{G}r_{(m)}$ factors through $\text{Res}_{D^{(2m)}/R}(\text{GL}_n)$. This proves the corollary. \square

Now we globalize the convolution morphism m from diagram (2.1) above. The moduli space Σ of relative effective Cartier divisors has a natural monoid structure

$$\begin{aligned} - \cup - : \Sigma \times \Sigma &\longrightarrow \Sigma \\ (D_1, D_2) &\longmapsto D_1 \cup D_2. \end{aligned}$$

The key definition is the following.

Definition 2.8. For $k \geq 1$, the k -fold convolution Grassmannian $\tilde{\mathcal{G}}r_k$ is the functor on the category of F -algebras which associates to every R the set of isomorphism classes of tuples $((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k})$ with

$$\left\{ \begin{array}{l} D_i \in \Sigma(R) \text{ relative effective Cartier divisors, } i = 1, \dots, k; \\ \mathcal{F}_i \text{ are } G\text{-torsors on } X_R; \\ \beta_i : \mathcal{F}_i|_{X_R \setminus D_i} \xrightarrow{\cong} \mathcal{F}_{i-1}|_{X_R \setminus D_i} \text{ isomorphisms, } i = 1, \dots, k, \end{array} \right.$$

where \mathcal{F}_0 is the trivial G -torsor. The projection $\tilde{\mathcal{G}}r_k \rightarrow \Sigma^k$, $((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}) \mapsto ((D_i)_{i=1, \dots, k})$ is a morphism of functors.

Lemma 2.9. For $k \geq 1$, the k -fold convolution Grassmannian $\tilde{\mathcal{G}}r_k$ is representable by a strict ind-scheme which is ind-proper over Σ^k .

Proof. The lemma follows by induction on k . If $k = 1$, then $\tilde{\mathcal{G}}r_k = \mathcal{G}r$. For $k > 1$, consider the projection

$$\begin{aligned} p : \tilde{\mathcal{G}}r_k &\longrightarrow \tilde{\mathcal{G}}r_{k-1} \times \Sigma \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}) &\longmapsto ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k-1}, D_k). \end{aligned}$$

Then the fiber over a R -point $((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k-1}, D_k)$ is

$$p^{-1}(((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k-1}, D_k)) \simeq \mathcal{F}_{k-1} \times^G (\mathcal{G}r \times_{X_R} D_k),$$

which is ind-proper. This proves the lemma. \square

For $k \geq 1$, there is the k -fold global convolution morphism

$$\begin{aligned} m_k : \tilde{\mathcal{G}}r_k &\longrightarrow \mathcal{G}r \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}) &\longmapsto (D, \mathcal{F}_k, \beta_1|_{X_R \setminus D} \circ \dots \circ \beta_k|_{X_R \setminus D}), \end{aligned}$$

where $D = D_1 \cup \dots \cup D_k$. This yields a commutative diagram of ind-schemes

$$\begin{array}{ccc} \tilde{\mathcal{G}}r_k & \xrightarrow{m_k} & \mathcal{G}r \\ \downarrow & & \downarrow \\ \Sigma^k & \xrightarrow{\cup} & \Sigma, \end{array}$$

i.e., regarding $\tilde{\mathcal{G}}r_k$ as a Σ -scheme via $\Sigma^k \rightarrow \Sigma$, $(D_i)_i \mapsto \cup_i D_i$, the morphism m_k is a morphism of Σ -ind-schemes. The global positive loop group \mathcal{L}^+G acts on $\tilde{\mathcal{G}}r_k$ over Σ as follows: let $(D_i, \mathcal{F}_i, \beta_i)_i \in \tilde{\mathcal{G}}r_k(R)$ and $g \in G(\hat{D})$ with $D = \cup_i D_i$. Then the action is defined as

$$((g, D), (D_i, \mathcal{F}_i, \beta_i)_i) \longmapsto (D_i, g\mathcal{F}_i, g\beta_i g^{-1})_i.$$

Corollary 2.10. The morphism $m_k : \tilde{\mathcal{G}}r_k \rightarrow \mathcal{G}r$ is a \mathcal{L}^+G -equivariant morphism of ind-proper strict ind-schemes over Σ .

Proof. The \mathcal{L}^+G -equivariance is immediate from the definition of the action. Note that $\Sigma^k \xrightarrow{\cup} \Sigma$ is finite, and hence $\tilde{\mathcal{G}}r_k$ is an ind-proper strict ind-scheme over Σ . This proves the corollary. \square

Now we explain the global analogue of the L^+G -torsors p and q from (2.1). For $k \geq 1$, let $\tilde{\mathcal{L}}G_k$ be the functor on the category of F -algebras which associates to every R the set of isomorphism classes of tuples $((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}, (\sigma_i)_{i=2, \dots, k})$ with

$$\left\{ \begin{array}{l} D_i \in \Sigma(R), \quad i = 1, \dots, k; \\ \mathcal{F}_i \text{ are } G\text{-torsors on } X_R; \\ \beta_i : \mathcal{F}_i|_{X_R \setminus D_i} \xrightarrow{\cong} \mathcal{F}_0|_{X_R \setminus D_i} \text{ trivialisations, } i = 1, \dots, k; \\ \sigma_i : \mathcal{F}_0|_{\hat{D}_i} \xrightarrow{\cong} \mathcal{F}_{i-1}|_{\hat{D}_i}, \quad i = 2, \dots, k, \end{array} \right.$$

where \mathcal{F}_0 is the trivial G -torsor. There are two natural projections over Σ^k . Let

$$\mathcal{L}^+G_{\Sigma}^{k-1} = \Sigma^k \times_{\Sigma^{k-1}} \mathcal{L}^+G^{k-1}.$$

The first projection is given by

$$\begin{aligned} p_k : \tilde{\mathcal{L}}G_k &\longrightarrow \mathcal{G}r^k \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}, (\sigma_i)_{i=2, \dots, k}) &\longmapsto ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}). \end{aligned}$$

Then p_k is a right $\mathcal{L}^+G_{\Sigma}^{k-1}$ -torsor for the action on the σ_i 's. The second projection is given by

$$\begin{aligned} q_k : \tilde{\mathcal{L}}G_k &\longrightarrow \tilde{\mathcal{G}}r_k \\ ((D_i, \mathcal{F}_i, \beta_i)_{i=1, \dots, k}, (\sigma_i)_{i=2, \dots, k}) &\longmapsto ((D_i, \mathcal{F}'_i, \beta'_i)_{i=1, \dots, k}), \end{aligned}$$

where $\mathcal{F}'_1 = \mathcal{F}_1$ and for $i \geq 2$, the G -torsor \mathcal{F}'_i is defined successively by gluing $\mathcal{F}_i|_{X_R \setminus D_i}$ to $\mathcal{F}'_{i-1}|_{\hat{D}_i}$ along $\sigma_i|_{\hat{D}_i^\circ} \circ \beta_i|_{\hat{D}_i^\circ}$. Then q_k is a right $\mathcal{L}^+G_{\Sigma}^{k-1}$ -torsor for the action given by

$$(((D_i, \mathcal{F}_i, \beta_i)_{i \geq 1}, (\sigma_i)_{i \geq 2}), (D_1, (D_i, g_i)_{i \geq 2})) \longmapsto ((D_1, \mathcal{F}_1, \beta_1), (D_i, g_i^{-1} \mathcal{F}_i, g_i^{-1} \beta_i)_{i \geq 2}, (\sigma_i g_i)_{i \geq 2}).$$

In the following, we consider ind-schemes over Σ^k as ind-schemes over Σ via $\Sigma^k \rightarrow \Sigma$.

Definition 2.11. For every $k \geq 1$, the k -fold global convolution diagram is the diagram of ind-schemes over Σ

$$\mathcal{G}r^k \xleftarrow{p_k} \tilde{\mathcal{L}}G_k \xrightarrow{q_k} \tilde{\mathcal{G}}r_k \xrightarrow{m_k} \mathcal{G}r.$$

Remark 2.12. Fix $x \in X(F)$, and choose a local coordinate t at x . Taking the fiber over $\text{diag}(\{x\}) \in X^k(F)$ in the k -fold global convolution diagram, then

$$\text{Gr}_G^k \longleftarrow LG^{k-1} \times \text{Gr}_G \longrightarrow \underbrace{LG \times^{L^+G} \dots \times^{L^+G} \text{Gr}_G}_{k\text{-times}} \longrightarrow \text{Gr}_G.$$

For $k = 2$, we recover diagram (2.1).

2.2. Universal Local Acyclicity. The notion of universal local acyclicity (ULA) is used in Reich's thesis [19], cf. also the paper [3] by Braverman and Gaitsgory. We recall the definition. Let S be a smooth geometrically connected scheme over F , and $f : T \rightarrow S$ a separated morphism of finite type. For complexes $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$, $\mathcal{A}_S \in D_c^b(S, \bar{\mathbb{Q}}_\ell)$, there is a natural morphism

$$(2.2) \quad \mathcal{A}_T \otimes f^* \mathcal{A}_S \longrightarrow (\mathcal{A}_T \overset{\dagger}{\otimes} f^! \mathcal{A}_S)[2 \dim(S)],$$

where $\mathcal{A} \overset{\dagger}{\otimes} \mathcal{B} \stackrel{\text{def}}{=} \mathbb{D}(\mathbb{D}\mathcal{A} \otimes \mathbb{D}\mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$. The morphism (2.2) is constructed as follows. Let $\Gamma_f : T \rightarrow T \times S$ be the graph of f . The projection formula gives a map

$$\Gamma_{f,!}(\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_f^! \bar{\mathbb{Q}}_\ell) \simeq (\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_{f,!} \Gamma_f^! \bar{\mathbb{Q}}_\ell \longrightarrow \mathcal{A}_T \boxtimes \mathcal{A}_S,$$

and by adjunction a map $\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \otimes \Gamma_f^! \bar{\mathbb{Q}}_\ell \rightarrow \Gamma_f^!(\mathcal{A}_T \boxtimes \mathcal{A}_S)$. Note that

$$\Gamma_f^*(\mathcal{A}_T \boxtimes \mathcal{A}_S) \simeq \mathcal{A}_T \otimes f^* \mathcal{A}_S \quad \text{and} \quad \Gamma_f^!(\mathcal{A}_T \boxtimes \mathcal{A}_S) \simeq \mathcal{A}_T \overset{\dagger}{\otimes} f^! \mathcal{A}_S,$$

using $\mathbb{D}(\mathcal{A}_T \boxtimes \mathcal{A}_S) \simeq \mathbb{D}\mathcal{A}_T \boxtimes \mathbb{D}\mathcal{A}_S$. Since S is smooth, Γ_f is a regular embedding, and thus $\Gamma_f^! \bar{\mathbb{Q}}_\ell \simeq \bar{\mathbb{Q}}_\ell[-2 \dim(S)]$. This gives after shifting by $[2 \dim(S)]$ the map (2.2).

Definition 2.13. (i) A complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is called *locally acyclic with respect to f* (f -LA) if (2.2) is an isomorphism for all $\mathcal{A}_S \in D_c^b(S, \bar{\mathbb{Q}}_\ell)$.

(ii) A complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is called *universally locally acyclic with respect to f* (f -ULA) if $f_{S'}^* \mathcal{A}_T$ is $f_{S'}$ -LA for all $f_{S'} = f \times_S S'$ with $S' \rightarrow S$ smooth, S' geometrically connected.

Remark 2.14. (i) If f is smooth, then the trivial complex $\mathcal{A}_T = \bar{\mathbb{Q}}_\ell$ is f -ULA.

(ii) If $S = \text{Spec}(F)$ is a point, then every complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$ is f -ULA.

(iii) The ULA property is local in the smooth topology on T .

Lemma 2.15. *Let $g : T \rightarrow T'$ be a proper morphism of S -schemes of finite type. For every ULA complex $\mathcal{A}_T \in D_c^b(T, \bar{\mathbb{Q}}_\ell)$, the push forward $g_* \mathcal{A}_T$ is ULA.*

Proof. For any morphism of finite type $g : T \rightarrow T'$ and any two complexes $\mathcal{A}_T, \mathcal{A}_{T'}$, we have the projection formulas

$$g_!(\mathcal{A}_T \otimes g^* \mathcal{A}_{T'}) \simeq g_! \mathcal{A}_T \otimes \mathcal{A}_{T'} \quad \text{and} \quad g_*(\mathcal{A}_T \otimes g^! \mathcal{A}_{T'}) \simeq g_* \mathcal{A}_T \otimes \mathcal{A}_{T'}.$$

If g is proper, then $g_* = g_!$, and the lemma follows from an application of the projection formulas and proper base change. \square

Theorem 2.16 ([19]). *Let $D \subset S$ be a smooth Cartier divisor, and consider a cartesian diagram of morphisms of finite type*

$$\begin{array}{ccccc} E & \xrightarrow{i} & T & \xleftarrow{j} & U \\ \downarrow & & \downarrow f & & \downarrow \\ D & \longrightarrow & S & \longleftarrow & S \setminus D. \end{array}$$

Let \mathcal{A} be a f -ULA complex on T such that $\mathcal{A}|_U$ is perverse. Then:

(i) *There is a functorial isomorphism*

$$i^*[-1]\mathcal{A} \simeq i^![1]\mathcal{A},$$

and both complexes $i^[-1]\mathcal{A}, i^![1]\mathcal{A}$ are perverse. Furthermore, the complex \mathcal{A} is perverse and is the middle perverse extension $\mathcal{A} \simeq j_{!*}(\mathcal{A}|_U)$.*

(ii) *The complex $i^*[-1]\mathcal{A}$ is $f|_E$ -ULA.*

\square

Remark 2.17. The proof of Theorem 2.16 uses Beilinson's construction of the unipotent part of the tame nearby cycles as follows. Suppose the Cartier divisor D is principal, this gives a morphism $\varphi : S \rightarrow \mathbb{A}_F^1$ such that $\varphi^{-1}(\{0\}) = S \setminus D$. Let $g = \varphi \circ f$ be the composition. Fix a separable closure \bar{F} of F . In SGA VII, Deligne constructs the nearby cycles functor $\psi = \psi_g : P(U) \rightarrow P(E_{\bar{F}})$. Let ψ_{tame} be the tame nearby cycles, i.e. the invariants under the pro- p -part of $\pi_1(\mathbb{G}_{m, \bar{F}}, 1)$. Fix a topological generator T of the maximal prime- p -quotient of $\pi_1(\mathbb{G}_{m, \bar{F}}, 1)$. Then T acts on ψ_{tame} , and there is an exact triangle

$$\psi_{\text{tame}} \xrightarrow{T-1} \psi_{\text{tame}} \longrightarrow i^* j_* \xrightarrow{+1}$$

Under the action of $T - 1$ the nearby cycles decompose as $\psi_{\text{tame}} \simeq \psi_{\text{tame}}^{\text{u}} \oplus \psi_{\text{tame}}^{\text{nu}}$, where $T - 1$ acts nilpotently on $\psi_{\text{tame}}^{\text{u}}$ and invertibly on $\psi_{\text{tame}}^{\text{nu}}$. Let $N : \psi_{\text{tame}} \rightarrow \psi_{\text{tame}}(-1)$ be the logarithm of T , i.e. the unique nilpotent operator N such that $T = \exp(\bar{T}N)$ where \bar{T} is the image of T under $\pi_1(\mathbb{G}_{m, \bar{F}}, 1) \rightarrow \mathbb{Z}_\ell(1)$. Then for any $a \geq 0$, Beilinson constructs a local system \mathcal{L}_a on \mathbb{G}_m together with a nilpotent operator N_a such that for $\mathcal{A}_U \in P(U)$ and $a \geq 0$ with $N^{a+1}(\psi_{\text{tame}}^{\text{u}}(\mathcal{A}_U)) = 0$ there is an isomorphism

$$(\psi_{\text{tame}}^{\text{u}}(\mathcal{A}_U), N) \simeq (i^*[-1]j_{!*}(\mathcal{A}_U \otimes g^* \mathcal{L}_a)_{\bar{F}}, 1 \otimes N_a).$$

Set $\Psi_g^u(\mathcal{A}_U) \stackrel{\text{def}}{=} \lim_{a \rightarrow \infty} i^*[-1]j_{!*}(\mathcal{A}_U \otimes g^*\mathcal{L}_a)$. Then $\Psi_g^u : P(U) \rightarrow P(E)$ is a functor, and we obtain that N acts trivially on $\psi_{\text{tame}}^u(\mathcal{A}_U)$ if and only if $\Psi_g^u(\mathcal{A}_U) = i^*[-1]j_{!*}(\mathcal{A}_U)$. In this case, Ψ_g^u is also defined for non-principal Cartier divisors by the formula $\Psi_g^u = i^*[-1] \circ j_{!*}$. In the situation of Theorem 2.16 above Reich shows that the unipotent monodromy along E is trivial, and consequently

$$i^*[-1]\mathcal{A} \simeq \Psi_g^u \circ j^*(\mathcal{A}) \simeq i^![-1]\mathcal{A}.$$

Corollary 2.18 ([19]). *Let \mathcal{A} be a perverse sheaf on S whose support contains an open subset of S . Then the following are equivalent:*

- (i) *The perverse sheaf \mathcal{A} is ULA with respect to the identity $id : S \rightarrow S$.*
- (ii) *The complex $\mathcal{A}[-\dim(S)]$ is a locally constant system, i.e. a lisse sheaf.*

□

We use the universal local acyclicity to show the perversity of certain complexes on the Beilinson-Drinfeld Grassmannian. For every finite index set I , there is the quotient map $X^I \rightarrow \Sigma$ onto a connected component of Σ . Set

$$\mathcal{G}_I \stackrel{\text{def}}{=} \mathcal{G}_r \times_{\Sigma} X^I.$$

If $I = \{*\}$ has cardinality 1, we write $\mathcal{G}_X = \mathcal{G}_I$.

Remark 2.19. Let $X = \mathbb{A}_F^1$ with global coordinate t . Then \mathbb{G}_a acts on X via translations. We construct a \mathbb{G}_a -action on \mathcal{G}_r as follows. For every $x \in \mathbb{G}_a(R)$, let a_x be the associated automorphism of X_R . If $D \in \Sigma(R)$, then we get an isomorphism $a_{-x} : a_x D \rightarrow D$. Let $(D, \mathcal{F}, \beta) \in \mathcal{G}_G(R)$. Then the \mathbb{G}_a -action on $\mathcal{G}_G \rightarrow \Sigma$ is given as

$$(D, \mathcal{F}, \beta) \mapsto (a_{-x}^* \mathcal{F}, a_{-x}^* \beta, a_x D).$$

Let \mathbb{G}_a act diagonally on X^I , then the structure morphism $\mathcal{G}_I \rightarrow X^I$ is \mathbb{G}_a -equivariant. If $|I| = 1$, then by the transitivity of the \mathbb{G}_a -action on X , we get $\mathcal{G}_X = \text{Gr}_G \times X$. Let $p : \mathcal{G}_X \rightarrow \text{Gr}_G$ be the projection. Then for every perverse sheaf \mathcal{A} on Gr_G , the complex $p^*[1]\mathcal{A}$ is a ULA perverse sheaf on \mathcal{G}_X by Remark 2.14 (ii) and the smoothness of p .

Now fix a finite index set I of cardinality $k \geq 1$. Consider the base change along $X^I \rightarrow \Sigma$ of the k -fold convolution diagram from Definition 2.11,

$$(2.3) \quad \prod_{i \in I} \mathcal{G}_{r_{X,i}} \xleftarrow{p_I} \tilde{\mathcal{L}}G_I \xrightarrow{q_I} \tilde{\mathcal{G}}_I \xrightarrow{m_I} \mathcal{G}_I.$$

Now choose a total order $I = \{1, \dots, k\}$, and set $I^o = I \setminus \{1\}$. Then p_I (resp. q_I) is a $\mathcal{L}^+G_I^o$ -torsor, where $\mathcal{L}^+G_I^o = X^I \times_{X^{I^o}} \mathcal{L}^+G_{I^o}$.

Let $\mathcal{L}^+G_X = \mathcal{L}^+G \times_{\Sigma} X$, and denote by $P_{\mathcal{L}^+G_X}(\mathcal{G}_X)^{\text{ULA}}$ the category of \mathcal{L}^+G_X -equivariant ULA perverse sheaves on \mathcal{G}_X . For any $i \in I$, let $\mathcal{A}_{X,i} \in P(\mathcal{G}_X)^{\text{ULA}}$ such that $\mathcal{A}_{X,i}$ are \mathcal{L}^+G_X -equivariant for $i \geq 2$. We have the $\prod_{i \geq 2} \mathcal{L}^+G_{X,i}$ -equivariant ULA perverse sheaf $\boxtimes_{i \in I} \mathcal{A}_{X,i}$ on $\prod_{i \in I} \mathcal{G}_{r_{X,i}}$.

Lemma 2.20. *There is a unique ULA perverse sheaf $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$ on $\tilde{\mathcal{G}}_I$ such that there is a q_I -equivariant isomorphism²*

$$q_I^*(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}) \simeq p_I^*(\boxtimes_{i \in I} \mathcal{A}_{X,i}),$$

where q_I -equivariant means with respect to the action on the $\mathcal{L}^+G_I^o$ -torsor $q_I : \tilde{\mathcal{L}}G_I \rightarrow \tilde{\mathcal{G}}_I$. If $\mathcal{A}_{X,1}$ is also \mathcal{L}^+G_X -equivariant, then $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$ is \mathcal{L}^+G_I -equivariant

²See Remark 2.21 below.

Remark 2.21. The ind-scheme $\tilde{\mathcal{L}}G_I$ is not of ind-finite type. We explain how the pullback functors p_I^*, q_I^* should be understood. Let Y_1, \dots, Y_k be \mathcal{L}^+G -equivariant closed subschemes of \mathcal{G}_X containing the supports of $\mathcal{A}_1, \dots, \mathcal{A}_k$. Choose $N \gg 0$ such that the action of \mathcal{L}^+G_X on each Y_1, \dots, Y_k factors over the smooth affine group scheme $H_N = \text{Res}_{D^{(N)}/X}(G)$, where $D^{(N)}$ is the N -th infinitesimal neighbourhood of the universal Cartier divisor D over X . Let $K_N = \ker(\mathcal{L}^+G_X \rightarrow H_N)$, and $Y = Y_1 \times \dots \times Y_k$. Then the left K_N -action on each Y_i is trivial, and hence the restriction of the p_I -action resp. q_I -action on $p_I^{-1}(Y)$ to $\prod_{i \geq 2} K_N$ agree. Let $h_N : p_I^{-1}(Y) \rightarrow Y_N$ be the resulting $\prod_{i \geq 2} K_N$ -torsor. By Lemma A.4 below, we get a factorization

$$\begin{array}{ccccc} & & p_I^{-1}(Y) & & \\ & p_I \swarrow & \downarrow h_N & \searrow q_I & \\ Y & \xleftarrow{p_{I,N}} & Y_N & \xrightarrow{q_{I,N}} & q_I(p_I^{-1}(Y)), \end{array}$$

where $p_{I,N}, q_{I,N}$ are $\prod_{i \geq 2} H_N$ -torsors. In particular, Y_N is a separated scheme of finite type, and we can replace p_I^* (resp. q_I^*) by $p_{I,N}^*$ (resp. $q_{I,N}^*$).

Proof of Lemma 2.20. We use the notation from Remark 2.21 above. The sheaf $p_{I,N}^*(\boxtimes_{i \in I} \mathcal{A}_{X,i})$ is $\prod_{i \geq 2} H_N$ -equivariant for the $q_{I,N}$ -action. Using descent along smooth torsors (cf. Lemma A.2 below), we get the perverse sheaf $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$, which is ULA. Indeed, $p_{I,N}^*(\boxtimes_{i \in I} \mathcal{A}_{X,i})$ is ULA, and the ULA property is local in the smooth topology. Since the diagram (2.3) is \mathcal{L}^+G_I -equivariant, the sheaf $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$ is \mathcal{L}^+G_I -equivariant, if $\mathcal{A}_{X,1}$ is \mathcal{L}^+G_X -equivariant. This proves the lemma. \square

Let U_I be the open locus of pairwise distinct coordinates in X^I . There is a cartesian diagram

$$\begin{array}{ccc} \mathcal{G}_I & \xleftarrow{j_I} & (\mathcal{G}_X^I)|_{U_I} \\ \downarrow & & \downarrow \\ X^I & \longleftarrow & U_I. \end{array}$$

Proposition 2.22. *The complex $m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i})$ is a ULA perverse sheaf on \mathcal{G}_I , and there is a unique isomorphism of perverse sheaves*

$$m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}) \simeq j_{I,*}(\boxtimes_{i \in I} \mathcal{A}_{X,i}|_{U_I}),$$

which is \mathcal{L}^+G_I -equivariant, if $\mathcal{A}_{X,1}$ is \mathcal{L}^+G_X -equivariant.

Proof. The sheaf $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$ is by Lemma 2.20 a ULA perverse sheaf on $\tilde{\mathcal{G}}_I$. Now the restriction of the global convolution morphism m_I to the support of $\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}$ is a proper morphism, and hence $m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i})$ is a ULA complex by Lemma 2.15. Then $m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i}) \simeq j_{I,*}((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I})$, as follows from Theorem 2.16 (i) and the formula $u_{I*} \circ v_{I*} \simeq (u \circ v)_{I*}$ for open immersions $V \xrightarrow{v} U \xrightarrow{u} T$, because $m_I|_{U_I}$ is an isomorphism. In particular, $m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i})$ is perverse. Since m_I is \mathcal{L}^+G_I -equivariant, it follows from proper base change that $m_{I,*}(\tilde{\boxtimes}_{i \in I} \mathcal{A}_{X,i})$ is \mathcal{L}^+G_I -equivariant, if $\mathcal{A}_{X,1}$ is \mathcal{L}^+G_X -equivariant. This proves the proposition. \square

2.3. The Symmetric Monoidal Structure. First we equip $P_{\mathcal{L}^+G_X}(\mathcal{G}_X)^{\text{ULA}}$ with a symmetric monoidal structure \ast which allows us later to define a symmetric monoidal structure with respect to the usual convolution (2.1) of \mathcal{L}^+G -equivariant perverse sheaves on Gr_G .

Fix I , and let U_I be the open locus of pairwise distinct coordinates in X^I . Then the diagram

$$(2.4) \quad \begin{array}{ccccc} \mathcal{G}r_X & \xrightarrow{i_I} & \mathcal{G}r_I & \xleftarrow{j_I} & (\mathcal{G}r_X^I)|_{U_I} \\ \downarrow & \text{diag} & \downarrow & & \downarrow \\ X & \longrightarrow & X^I & \longleftarrow & U_I. \end{array}$$

is cartesian.

Definition 2.23. Fix some total order on I . For every tuple $(\mathcal{A}_{X,i})_{i \in I}$ with $\mathcal{A}_{X,i} \in P(\mathcal{G}r_X)^{ULA}$ for $i \in I$, the I -fold fusion product $*_{i \in I} \mathcal{A}_{X,i}$ is the complex

$$*_{i \in I} \mathcal{A}_{X,i} \stackrel{\text{def}}{=} i_I^*[-k+1]j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \in D_c^b(\mathcal{G}r_X, \bar{\mathbb{Q}}_\ell),$$

where $k = |I|$.

Now let $\pi : I \rightarrow J$ be a surjection of finite index sets. For $j \in J$, let $I_j = \pi^{-1}(j)$, and denote by U_π the open locus in X^I such that the I_j -coordinates are pairwise distinct from the $I_{j'}$ -coordinates for $j \neq j'$. Then the diagram

$$(2.5) \quad \begin{array}{ccccc} \mathcal{G}r_J & \xrightarrow{i_\pi} & \mathcal{G}r_I & \xleftarrow{j_\pi} & (\prod_j \mathcal{G}r_{I_j})|_{U_\pi} \\ \downarrow & & \downarrow & & \downarrow \\ X^J & \longrightarrow & X^I & \longleftarrow & U_\pi, \end{array}$$

is cartesian. The following theorem combined with Proposition 2.22 is the key to the symmetric monoidal structure:

Theorem 2.24. *Let I be a finite index set, and let $\mathcal{A}_{X,i} \in P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}$ for $i \in I$. Let $\pi : I \rightarrow J$ be a surjection of finite index sets, and set $k_\pi = |I| - |J|$.*

(i) *As complexes*

$$i_\pi^*[-k_\pi]j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \simeq i_\pi^![k_\pi]j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}),$$

*and both are \mathcal{L}^+G_J -equivariant ULA perverse sheaves on $\mathcal{G}r_J$. In particular, $*_{i \in I} \mathcal{A}_{X,i} \in P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}$.*

(ii) *There is an associativity and a commutativity constraint for the fusion product such that there is a canonical isomorphism*

$$*_{i \in I} \mathcal{A}_{X,i} \simeq *_{j \in J} (*_{i \in I_j} \mathcal{A}_{X,i}),$$

*where $I_j = \pi^{-1}(j)$ for $j \in J$. In particular, $(P_{\mathcal{L}^+G_X}(\mathcal{G}r_X)^{ULA}, *)$ is symmetric monoidal.*

Proof. Factor π as a chain of surjective maps $I = I_1 \rightarrow I_2 \rightarrow \dots \rightarrow I_{k_\pi} = J$ with $|I_{i+1}| = |I_i| + 1$, and consider the corresponding chain of smooth Cartier divisors

$$X^J = X^{I_{k_\pi}} \longrightarrow \dots \longrightarrow X^{I_2} \longrightarrow X^{I_1} = X^I.$$

By Proposition 2.22, the complex $j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I})$ is ULA. Then part (i) follows inductively from Theorem 2.16 (i) and (ii). This shows (i).

Let $\tau : I \rightarrow I$ be a bijection. Then τ acts on X^I by permutation of coordinates, and diagram (2.4) becomes equivariant for this action. Then

$$\tau^* j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \simeq j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,\tau^{-1}(i)})|_{U_I}).$$

Since the action on $\text{diag}(X) \subset X^I$ is trivial, we obtain

$$i_I^* j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \simeq i_I^* \tau^* j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \simeq i_I^* j_{I,!}*((\boxtimes_{i \in I} \mathcal{A}_{X,\tau^{-1}(i)})|_{U_I}),$$

and hence $*_{i \in I} \mathcal{A}_{X,i} \simeq *_{i \in I} \mathcal{A}_{X, \tau^{-1}(i)}$. It remains to give the isomorphism defining the symmetric monoidal structure. Since $j_I = j_\pi \circ \prod_j j_{I_j}$, diagram (2.5) gives

$$(j_{I,*}((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}))|_{U_\pi} \simeq \boxtimes_{j \in J} j_{I_j,*}((\boxtimes_{i \in I_j} \mathcal{A}_{X,i})|_{U_{I_j}}).$$

Applying $(i_\pi|_{U_\pi})^*[k_\pi]$ and using that $U_\pi \cap X^J = U_J$, we obtain

$$(i_\pi^*[k_\pi]j_{I,*}((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}))|_{U_J} \simeq \boxtimes_{j \in J} (*_{i \in I_j} \mathcal{A}_{X,i}).$$

But by (i), the perverse sheaf $i_\pi^*[k_\pi]j_{I,*}((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I})$ is ULA, thus

$$i_\pi^*[k_\pi]j_{I,*}((\boxtimes_{i \in I} \mathcal{A}_{X,i})|_{U_I}) \simeq j_{J,*}((\boxtimes_{j \in J} (*_{i \in I_j} \mathcal{A}_{X,i}))|_{U_J}),$$

and restriction along the diagonal in X^J gives the isomorphism $*_{i \in I} \mathcal{A}_{X,i} \simeq *_{j \in J} (*_{i \in I_j} \mathcal{A}_{X,i})$. This proves (ii). \square

Example 2.25. Let $G = \{e\}$ be the trivial group. Then $\mathcal{G}_X = X$. Let $\text{Loc}(X)$ be the category of ℓ -adic local systems on X . Using Corollary 2.18, we obtain an equivalence of symmetric monoidal categories

$$\mathcal{H}^0 \circ [-1] : (P(X)^{\text{ULA}}, *) \xrightarrow{\simeq} (\text{Loc}(X), \otimes),$$

where $\text{Loc}(X)$ is endowed with the usual symmetric monoidal structure with respect to the tensor product \otimes .

Corollary 2.26. Let $D_c^b(X, \bar{\mathbb{Q}}_\ell)^{\text{ULA}}$ be the category of ULA complexes on X . Denote by $f : \mathcal{G}_X \rightarrow X$ the structure morphism. Then the functor

$$f_*[-1] : (P(\mathcal{G}_X)^{\text{ULA}}, *) \longrightarrow (D_c^b(X, \bar{\mathbb{Q}}_\ell), \otimes)$$

is symmetric monoidal.

Proof. If $\mathcal{A}_X \in P(\mathcal{G}_X)^{\text{ULA}}$, then $f_*\mathcal{A}_X \in D_c^b(X, \bar{\mathbb{Q}}_\ell)^{\text{ULA}}$ by Lemma 2.15 and the ind-properness of f . Now apply f_* to the isomorphism in Theorem 2.24 (ii) defining the symmetric monoidal structure on $P(\mathcal{G}_X)^{\text{ULA}}$. Then by proper base change and going backwards through the arguments in the proof of Theorem 2.24 (ii), we get that $f_*[-1]$ is symmetric monoidal. \square

Corollary 2.27. Let $X = \mathbb{A}_F^1$. Let $p : \mathcal{G}_X \rightarrow \text{Gr}_G$ be the projection, cf. Remark 2.19.

(i) The functor

$$p^*[1] : P_{L+G}(\text{Gr}_G) \longrightarrow P_{\mathcal{L}^+G_X}(\mathcal{G}_X)^{\text{ULA}}$$

embeds $P_{L+G}(\text{Gr}_G)$ as a full subcategory and is an equivalence of categories with the subcategory of \mathbb{G}_a -equivariant objects in $P_{\mathcal{L}^+G_X}(\mathcal{G}_X)^{\text{ULA}}$.

(ii) For every I and $\mathcal{A}_i \in P_{L+G}(\text{Gr}_G)$, $i \in I$, there is a canonical \mathcal{L}^+G_X -equivariant isomorphism

$$p^*[1](\star_{i \in I} \mathcal{A}_i) \simeq *_{i \in I} (p^*[1]\mathcal{A}_i),$$

where the product is taken with respect to some total order on I .

Proof. Under the simply transitive action of \mathbb{G}_a on X , the isomorphism $\mathcal{G}_X \simeq \text{Gr}_G \times X$ is compatible with the action of L^+G under the zero section $L^+G \hookrightarrow \mathcal{L}^+G_X$. By Lemma 2.19, the complex $p^*[1]\mathcal{A}$ is a ULA perverse sheaf on \mathcal{G}_X . It is obvious that the functor $p^*[1]$ is fully faithful. Denote by $i_0 : \text{Gr}_G \rightarrow \mathcal{G}_X$ the zero section. If \mathcal{A}_X on \mathcal{G}_X is \mathbb{G}_a -equivariant, then $\mathcal{A}_X \simeq p^*[1]i_0^*[-1]\mathcal{A}_X$. This proves (i).

By Remark 2.12, the fiber over $\text{diag}(\{0\}) \in X^I(F)$ of (2.3) is the usual convolution diagram (2.1). Hence, by proper base change,

$$i_0^*[-1](\star_{i \in I} p^*[1]\mathcal{A}_i) \simeq *_{i \in I} i_0^*[-1]p^*[1]\mathcal{A}_i \simeq *_{i \in I} \mathcal{A}_i.$$

Since $*_{i \in I} p^*[1]\mathcal{A}_i$ is \mathbb{G}_a -equivariant, this proves (ii). \square

Now we are prepared for the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $X = \mathbb{A}_{\bar{F}}^1$. For every $\mathcal{A}_1, \mathcal{A}_2 \in P(\mathrm{Gr}_G)$ with \mathcal{A}_2 being L^+G -equivariant, we have to prove that $\mathcal{A}_1 \star \mathcal{A}_2 \in P(\mathrm{Gr}_G)$. By Theorem 2.24 (i), the \star -convolution is perverse. Then the perversity of $\mathcal{A}_1 \star \mathcal{A}_2$ follows from Corollary 2.27 (ii). Again by Corollary 2.27 (ii), the convolution $\mathcal{A}_1 \star \mathcal{A}_2$ is L^+G -equivariant, if \mathcal{A}_1 is L^+G -equivariant. This proves (i).

We have to equip $(P_{L^+G}(\mathrm{Gr}_G), \star)$ with a symmetric monoidal structure. By Corollary 2.27, the tuple $(P_{L^+G}(\mathrm{Gr}_G), \star)$ is a full subcategory of $(P_{\mathcal{L}^+G_X}(\mathcal{G}_X)^{\mathrm{ULA}}, \star)$, and the latter is symmetric monoidal by Theorem 2.24 (ii), hence so is $(P_{L^+G}(\mathrm{Gr}_G), \star)$. Since taking cohomology is only graded commutative, we need to modify the commutativity constraint of $(P_{L^+G}(\mathrm{Gr}_G), \star)$ by a sign as follows. Let \bar{F} be a separable closure of F . The $L^+G_{\bar{F}}$ -orbits in one connected component of $\mathrm{Gr}_{G, \bar{F}}$ are all either even or odd dimensional. Because the Galois action on $\mathrm{Gr}_{G, \bar{F}}$ commutes with the $L^+G_{\bar{F}}$ -action, the connected components of Gr_G are divided into those of even or odd parity. Consider the corresponding $\mathbb{Z}/2$ -grading on $P_{L^+G}(\mathrm{Gr}_G)$ given by the parity of the connected components of Gr_G . Then we equip $(P_{L^+G}(\mathrm{Gr}_G), \star)$ with the super commutativity constraint with respect to this $\mathbb{Z}/2$ -grading, i.e. if \mathcal{A} (resp. \mathcal{B}) is an L^+G -equivariant perverse sheaf supported on a connected component $X_{\mathcal{A}}$ (resp. $X_{\mathcal{B}}$) of Gr_G , then the modified commutativity constraint differs by the sign $(-1)^{p(X_{\mathcal{A}})p(X_{\mathcal{B}})}$, where $p(X) \in \mathbb{Z}/2$ denotes the parity of a connected component X of Gr_G .

Now consider the global cohomology functor

$$\omega(-) = \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_{G, \bar{F}}, (-)_{\bar{F}}) : P_{L^+G}(\mathrm{Gr}_G) \longrightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}.$$

Let $f : \mathcal{G}_X \rightarrow X$ be the structure morphism. Then the diagram

$$\begin{array}{ccc} P_{\mathcal{L}^+G_{X, \bar{F}}}(\mathcal{G}_{X, \bar{F}})^{\mathrm{ULA}} & \xrightarrow{f_*[-1]} & D_c^b(X_{\bar{F}}, \bar{\mathbb{Q}}_\ell) \\ p^*[1] \circ (-)_{\bar{F}} \uparrow & & \downarrow \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i \circ i_0^* \\ P_{L^+G}(\mathrm{Gr}_G) & \xrightarrow{\omega} & \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell} \end{array}$$

is commutative up to natural isomorphism. Now if \mathcal{A} is a perverse sheaf supported on a connected component X of Gr_G , then by a theorem of Lusztig [12, Theorem 11c],

$$R^i \Gamma(\mathrm{Gr}_{G, \bar{F}}, \mathcal{A}_{\bar{F}}) = 0, \quad i \not\equiv p(X) \pmod{2},$$

where $p(X) \in \mathbb{Z}/2$ denotes the parity of X . Hence, Corollary 2.26 shows that ω is symmetric monoidal with respect to the super commutativity constraint on $P_{L^+G}(\mathrm{Gr}_G)$. To prove uniqueness of the symmetric monoidal structure, it is enough to prove that ω is faithful, which follows from Lemma 3.4 below. This proves (ii). \square

3. THE TANNAKIAN STRUCTURE

In this section we assume that $F = \bar{F}$ is separably closed. Let X_+^\vee be a set of representatives of the L^+G -orbits on Gr_G . For $\mu \in X_+^\vee$ we denote by \mathcal{O}_μ the corresponding L^+G -orbit, and by $\bar{\mathcal{O}}_\mu$ its reduced closure with open embedding $j^\mu : \mathcal{O}_\mu \hookrightarrow \bar{\mathcal{O}}_\mu$. We equip X_+^\vee with the partial order defined as follows: for every $\lambda, \mu \in X_+^\vee$, we define $\lambda \leq \mu$ if and only if $\mathcal{O}_\lambda \subset \bar{\mathcal{O}}_\mu$.

Proposition 3.1. *The category $P_{L^+G}(\mathrm{Gr}_G)$ is semisimple with simple objects the intersection complexes*

$$\mathrm{IC}_\mu = j_{!*}^\mu \bar{\mathbb{Q}}_\ell[\dim(\mathcal{O}_\mu)], \quad \text{for } \mu \in X_+^\vee.$$

In particular, if ${}^p j_{}^\mu$ (resp. ${}^p j_!^\mu$) denotes the perverse push forward (resp. perverse extension by zero), then $j_{!*}^\mu \simeq {}^p j_!^\mu \simeq {}^p j_{*}^\mu$.*

Proof. For any $\mu \in X_+^\vee$, the étale fundamental group $\pi_1^{\acute{e}t}(\mathcal{O}_\mu)$ is trivial. Indeed, since $\bar{\mathcal{O}}_\mu \setminus \mathcal{O}_\mu$ is of codimension at least 2 in $\bar{\mathcal{O}}_\mu$, Grothendieck's purity theorem implies that $\pi_1^{\acute{e}t}(\mathcal{O}_\mu) = \pi_1^{\acute{e}t}(\bar{\mathcal{O}}_\mu)$. The latter group is trivial by [SGA1, XI.1 Corollaire 1.2], because

$\overline{\mathcal{O}}_\lambda$ is normal (cf. [6]), projective and rational. This shows the claim.

Since by [17, Lemme 2.3] the stabilizers of the L^+G -action are connected, any L^+G -equivariant irreducible local system supported on \mathcal{O}_μ is isomorphic to the constant sheaf $\overline{\mathbb{Q}}_\ell$. Hence, the simple objects in $P_{L^+G}(\text{Gr}_G)$ are the intersection complexes IC_μ for $\mu \in X_+^\vee$.

To show semisimplicity of the Satake category, it is enough to prove

$$\text{Ext}_{P(\text{Gr}_G)}^1(\text{IC}_\lambda, \text{IC}_\mu) = \text{Hom}_{D_c^b(\text{Gr}_G)}(\text{IC}_\lambda, \text{IC}_\mu[1]) \stackrel{!}{=} 0.$$

We distinguish several cases:

Case (i): $\lambda = \mu$.

Let $\mathcal{O}_\mu \xrightarrow{j} \overline{\mathcal{O}}_\mu \xleftarrow{i} \overline{\mathcal{O}}_\mu \setminus \mathcal{O}_\mu$, and consider the exact sequence of abelian groups

$$(3.1) \quad \text{Hom}(\text{IC}_\mu, i_! i^! \text{IC}_\mu[1]) \longrightarrow \text{Hom}(\text{IC}_\mu, \text{IC}_\mu[1]) \longrightarrow \text{Hom}(\text{IC}_\mu, j_* j^* \text{IC}_\mu[1])$$

associated to the distinguished triangle $i_! i^! \text{IC}_\mu \rightarrow \text{IC}_\mu \rightarrow j_* j^* \text{IC}_\mu$. We show that the outer groups in (3.1) are trivial. Indeed, the last group is trivial, since $j^* \text{IC}_\mu = \overline{\mathbb{Q}}_\ell[\dim(\mathcal{O}_\mu)]$ gives

$$\text{Hom}(\text{IC}_\mu, j_* j^* \text{IC}_\mu[1]) = \text{Hom}(j^* \text{IC}_\mu, j^* \text{IC}_\mu[1]) = \text{Ext}^1(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell).$$

And $\text{Ext}^1(\overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}}_\ell) = H_{\text{ét}}^1(\mathcal{O}_\mu, \overline{\mathbb{Q}}_\ell) = 0$, because \mathcal{O}_μ is simply connected. To show that the first group

$$\text{Hom}(\text{IC}_\mu, i_! i^! \text{IC}_\mu[1]) = \text{Hom}(i^* \text{IC}_\mu, i^! \text{IC}_\mu[1])$$

is trivial, note that $i^* \text{IC}_\mu$ lives in perverse degrees ≤ -1 because the 0th perverse cohomology vanishes, since IC_μ is a middle perverse extension along j . Hence, the Verdier dual $D(i^* \text{IC}_\mu)[1] = i^! \text{IC}_\mu[1]$ lives in perverse degrees ≥ 0 . This proves case (i).

Case (ii): $\lambda \neq \mu$ and either $\lambda \leq \mu$ or $\mu \leq \lambda$.

If $\lambda \leq \mu$, let $i : \overline{\mathcal{O}}_\lambda \hookrightarrow \overline{\mathcal{O}}_\mu$ be the closed embedding. Then

$$\text{Hom}(i_* \text{IC}_\lambda, \text{IC}_\mu[1]) = \text{Hom}(\text{IC}_\lambda, i^! \text{IC}_\mu[1]),$$

and this vanishes, since $i^! \text{IC}_\mu[1]$ lives in perverse degrees ≥ 1 or equivalently, the Verdier dual $D(i^! \text{IC}_\mu) = i^* \text{IC}_\mu$ lives in perverse degrees ≤ -2 . Indeed, by a theorem of Lusztig [12, Theorem 11c], $i^* \text{IC}_\mu$ is concentrated in even perverse degrees, and the 0th perverse cohomology vanishes, since IC_μ is a middle perverse extension. If $\mu \leq \lambda$, let $i : \overline{\mathcal{O}}_\mu \hookrightarrow \overline{\mathcal{O}}_\lambda$ be the closed embedding. Then

$$\text{Hom}(\text{IC}_\lambda, i_* \text{IC}_\mu[1]) = \text{Hom}(i^* \text{IC}_\lambda, \text{IC}_\mu[1])$$

vanishes, since $i^* \text{IC}_\lambda$ lives in perverse degrees ≤ -2 as before. This proves case (ii).

Case (iii): $\lambda \not\leq \mu$ and $\mu \not\leq \lambda$.

We may assume that λ and μ are contained in the same connected component of Gr_G . Choose some $\nu \in X_+^\vee$ with $\lambda, \mu \leq \nu$. Consider the cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{O}}_\lambda \times_{\overline{\mathcal{O}}_\nu} \overline{\mathcal{O}}_\mu & \xleftarrow{\iota_1} & \overline{\mathcal{O}}_\mu \\ \downarrow \iota_2 & & \downarrow i_2 \\ \overline{\mathcal{O}}_\lambda & \xleftarrow{i_1} & \overline{\mathcal{O}}_\nu. \end{array}$$

Then adjunction gives

$$(3.2) \quad \text{Hom}(i_{1,*} \text{IC}_\lambda, i_{2,*} \text{IC}_\mu[1]) = \text{Hom}(i_2^* i_{1,*} \text{IC}_\lambda, \text{IC}_\mu[1]),$$

and $i_2^* i_{1,*} \text{IC}_\lambda \simeq \iota_{1,*} \iota_2^* \text{IC}_\lambda$ by proper base change. Hence (3.2) equals $\text{Hom}(\iota_2^* \text{IC}_\lambda, \iota_1^! \text{IC}_\mu[1])$ which vanishes. This proves case (iii), hence the proposition. \square

The affine group scheme $L^+\mathbb{G}_m$ acts on Gr_G as follows. For $x \in L^+\mathbb{G}_m(R)$, denote by v_x the automorphism of $\mathrm{Spec}(R[[t]])$ induced by multiplication with x . If \mathcal{F} is a G -torsor over $\mathrm{Spec}(R[[t]])$, we denote by $v_x^*\mathcal{F}$ the pullback of \mathcal{F} along v_x . Let $(\mathcal{F}, \beta) \in \mathrm{Gr}_G(R)$. Then the action of $L^+\mathbb{G}_m$ on Gr_G is given by

$$(\mathcal{F}, \beta) \longmapsto (v_{x^{-1}}^*\mathcal{F}, v_{x^{-1}}^*\beta),$$

and is called the *Virasoro action*.

Note that every L^+G -orbit in Gr_G is stable under $L^+\mathbb{G}_m$. The semidirect product $L^+G \rtimes L^+\mathbb{G}_m$ acts on Gr_G , and the action on each orbit factors through a smooth connected affine group scheme. Hence, we may consider the category $P_{L^+G \rtimes L^+\mathbb{G}_m}(\mathrm{Gr}_G)$ of $L^+G \rtimes L^+\mathbb{G}_m$ -equivariant perverse sheaves on Gr_G .

Corollary 3.2. *The forgetful functor*

$$P_{L^+G \rtimes L^+\mathbb{G}_m}(\mathrm{Gr}_G) \longrightarrow P_{L^+G}(\mathrm{Gr}_G)$$

is an equivalence of categories. In particular, the category $P_{L^+G}(\mathrm{Gr}_G)$ does not depend on the choice of the parameter t .

Proof. By Proposition 3.1 above, every L^+G -equivariant perverse sheaf is a direct sum of intersection complexes, and these are $L^+\mathbb{G}_m$ -equivariant. \square

Remark 3.3. If $X = \mathbb{A}_F^1$ is the base curve, then the global affine Grassmannian \mathcal{G}_X splits as $\mathcal{G}_X \simeq \mathrm{Gr}_G \times X$. Corollary 3.2 shows that we can work over an arbitrary curve X as follows. Let \mathcal{X} be the functor on the category of F -algebras R parametrizing tuples (x, s) with

$$\begin{cases} x \in X(R) \text{ is a point;} \\ s \text{ is a continuous isomorphism of } R\text{-modules } \hat{\mathcal{O}}_{X_R, x} \xrightarrow{\cong} R[[t]], \end{cases}$$

where $\hat{\mathcal{O}}_{X_R, x}$ is the completion of the R -module $\mathcal{O}_{X_R, x}$ along the maximal ideal \mathfrak{m}_x at x . The affine group scheme $L^+\mathbb{G}_m$ operates from left on \mathcal{X} by $(g, (x, s)) \mapsto (x, gs)$. The projection $p : \mathcal{X} \rightarrow X, (x, s) \mapsto x$ gives \mathcal{X} the structure of a $L^+\mathbb{G}_m$ -torsor. Then $\mathcal{G}_X \simeq \mathrm{Gr}_G \times^{L^+\mathbb{G}_m} \mathcal{X}$, and we get a diagram of $L^+\mathbb{G}_m$ -torsors

$$\begin{array}{ccc} & \mathrm{Gr}_G \times \mathcal{X} & \\ p \swarrow & & \searrow q \\ \mathrm{Gr}_G \times X & & \mathcal{G}_X. \end{array}$$

For any $\mathcal{A} \in P_{L^+G}(\mathrm{Gr}_G)$, the perverse sheaf $\mathcal{A} \boxtimes \bar{\mathbb{Q}}_\ell[1]$ on $\mathrm{Gr}_G \times X$ is $L^+\mathbb{G}_m$ -equivariant by Corollary 3.2. Hence, $p^*(\mathcal{A} \boxtimes \bar{\mathbb{Q}}_\ell[1])$ descends along q to a perverse sheaf $\mathcal{A} \boxtimes \bar{\mathbb{Q}}_\ell[1]$ on \mathcal{G}_X .

We are going to define a fiber functor on $P_{L^+G}(\mathrm{Gr}_G)$. Denote by

$$(3.3) \quad \omega(-) = \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_G, -) : P_{L^+G}(\mathrm{Gr}_G) \rightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$$

the cohomology functor with values in the category of finite dimensional $\bar{\mathbb{Q}}_\ell$ -vector spaces.

Lemma 3.4. *The functor $\omega : P_{L^+G}(\mathrm{Gr}_G) \rightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_\ell}$ is additive, exact and faithful.*

Proof. Additivity is immediate. Exactness follows from Proposition 3.1, since every exact sequence splits, and ω is additive. To show faithfulness, it is enough, again by Proposition 3.1, to show that the intersection cohomology of the Schubert varieties is non-zero. Indeed, we claim that the intersection cohomology of any projective variety T is non-zero. Embedding T into projective space and projecting down on hyperplanes, we obtain a generically finite morphism $\pi : T \rightarrow \mathbb{P}^n$. Using the decomposition theorem, we see that the intersection complex of \mathbb{P}^n appears as a direct summand in $\pi_* \mathrm{IC}_T$. Hence, the intersection cohomology of T is non-zero. This proves the lemma. \square

Corollary 3.5. *The tuple $(P_{L+G}(\mathrm{Gr}_G), \star)$ is a neutralized Tannakian category with fiber functor $\omega : P_{L+G}(\mathrm{Gr}_G) \rightarrow \mathrm{Vec}_{\mathbb{Q}_\ell}$.*

Proof. We check the criterion in [5, Prop. 1.20]:

The category $(P_{L+G}(\mathrm{Gr}_G), \star)$ is abelian \mathbb{Q}_ℓ -linear (cf. Appendix A below) and by Theorem 2.1 (ii) above symmetric monoidal. To prove that ω is a fiber functor, we must show that ω is an additive exact faithful tensor functor. Lemma 3.4 shows that ω is additive exact and faithful, and Theorem 2.1 (ii) shows that ω is symmetric monoidal.

It remains to show that $(P_{L+G}(\mathrm{Gr}_G), \star)$ has a unit object and that any one dimensional object has an inverse. The unit object is the constant sheaf $\mathrm{IC}_0 = \bar{\mathbb{Q}}_\ell$ concentrated in the base point e_0 . We have $\mathrm{End}(\mathrm{IC}_0) = \mathbb{Q}_\ell$, and $\dim(\omega(\mathrm{IC}_0)) = 1$. Now, let $\mathcal{A} \in P_{L+G}(\mathrm{Gr}_G)$ with $\dim(\omega(\mathcal{A})) = 1$. Then \mathcal{A} is supported on a L^+G -invariant closed point $z_0 \in \mathrm{Gr}_G$. There exists z in the center of LG such that $z \cdot z_0 = e_0$ is the basepoint. If $z'_0 = z \cdot e_0$, then the intersection cohomology complex \mathcal{A}' supported on z'_0 satisfies $\mathcal{A} \star \mathcal{A}' = \mathrm{IC}_0$. This shows the corollary. \square

4. THE GEOMETRIC SATAKE EQUIVALENCE

In this section we assume that $F = \bar{F}$ is separably closed. Denote by $H = \mathrm{Aut}^*(\omega)$ the affine \mathbb{Q}_ℓ -group scheme of tensor automorphisms defined by Corollary 3.5.

Theorem 4.1. *The group scheme H is a connected reductive group over \mathbb{Q}_ℓ which is dual to G in the sense of Langlands, i.e. if we denote by \hat{G} the Langlands dual group with respect to some pinning of G , then there exists an isomorphism $H \simeq \hat{G}$ determined uniquely up to inner automorphisms.*

We fix some notation. Let T be a maximal split torus of G and B a Borel subgroup containing T with unipotent radical U . We denote by $\langle -, - \rangle$ the natural pairing between $X = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$. Let $R \subset X$ be the root system associated to (G, T) , and R_+ be the set of positive roots corresponding to B . Let $R^\vee \subset X^\vee$ the dual root system with the bijection $R \rightarrow R^\vee$, $\alpha \mapsto \alpha^\vee$. Denote by R_+^\vee the set of positive coroots. Let W the Weyl group of (G, T) . Consider the half sum of all positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Let Q_+^\vee (resp. Q_+^\vee) the subgroup (resp. submonoid) of X^\vee generated by R^\vee (resp. R_+^\vee). We denote by

$$X_+^\vee = \{\mu \in X^\vee \mid \langle \alpha, \mu \rangle \geq 0, \forall \alpha \in R_+\}$$

the cone of dominant cocharacters with the partial order on X^\vee defined as follows: $\lambda \leq \mu$ if and only if $\mu - \lambda \in Q_+^\vee$.

Note that (X_+^\vee, \leq) identifies with the partially ordered set of orbit representatives in Section 3 as follows: for every $\mu \in X_+^\vee$, let t^μ the corresponding element in $LT(F)$, and denote by $e_0 \in \mathrm{Gr}_G$ the base point. Then $\mu \mapsto t^\mu \cdot e_0$ gives the bijection of partial ordered sets, i.e. the orbit closures satisfy

$$\bar{\mathcal{O}}_\mu = \coprod_{\lambda \leq \mu} \mathcal{O}_\lambda, \quad (\text{Cartan stratification})$$

where \mathcal{O}_λ denotes the L^+G -orbit of $t^\lambda \cdot e_0$ (cf. [17, §2]).

For every $\nu \in X^\vee$, consider the LU -orbit $S_\nu = LU \cdot t^\nu e_0$ inside Gr_G (cf. [17, §3]). Then S_ν is a locally closed ind-subscheme of Gr_G , and for every $\mu \in X_+^\vee$, there is a locally closed stratification

$$\bar{\mathcal{O}}_\mu = \coprod_{\nu \in X^\vee} S_\nu \cap \bar{\mathcal{O}}_\mu. \quad (\text{Iwasawa stratification})$$

For $\mu \in X_+^\vee$, let

$$\Omega(\mu) \stackrel{\text{def}}{=} \{\nu \in X^\vee \mid w\nu \leq \mu, \forall w \in W\}.$$

Proposition 4.2. *For every $\nu \in X^\vee$ and $\mu \in X_+^\vee$ the stratum $S_\nu \cap \overline{\mathcal{O}}_\mu$ is non-empty if and only if $\nu \in \Omega(\mu)$, and in this case it is pure of dimension $\langle \rho, \mu + \nu \rangle$.*

Proof. The schemes G , B , T and all the associated data are already defined over a finitely generated \mathbb{Z} -algebra. By generic flatness, we reduce to the case where $F = \mathbb{F}_q$ is a finite field. The proposition is proven in [8, Proof of Lemma 2.17.4], which relies on [17, Theorem 3.1]. \square

For every sequence $\mu_\bullet = (\mu_1, \dots, \mu_k)$ of dominant cocharacters, consider the projective variety over F

$$\overline{\mathcal{O}}_{\mu_\bullet} \stackrel{\text{def}}{=} p^{-1}(\overline{\mathcal{O}}_{\mu_1}) \times^{L^+G} \dots \times^{L^+G} p^{-1}(\overline{\mathcal{O}}_{\mu_{k-1}}) \times^{L^+G} \overline{\mathcal{O}}_{\mu_k},$$

inside $LG \times^{L^+G} \dots \times^{L^+G} \text{Gr}_G$, where $p : LG \rightarrow \text{Gr}_G$ denotes the quotient map. The quotient exists, by the ind-properness of Gr_G and Lemma A.4 below.

Now let $|\mu_\bullet| = \mu_1 + \dots + \mu_k$. Then the restriction $m_{\mu_\bullet} = m|_{\overline{\mathcal{O}}_{\mu_\bullet}}$ of the k -fold convolution morphism factors as

$$m_{\mu_\bullet} : \overline{\mathcal{O}}_{\mu_\bullet} \longrightarrow \overline{\mathcal{O}}_{|\mu_\bullet|},$$

and is an isomorphism over $\mathcal{O}_{|\mu_\bullet|} \subset \overline{\mathcal{O}}_{|\mu_\bullet|}$.

Corollary 4.3. *For every $\lambda \in X_+^\vee$ with $\lambda \leq |\mu_\bullet|$ and $x \in \mathcal{O}_\lambda(F)$, one has*

$$\dim(m_{\mu_\bullet}^{-1}(x)) \leq \langle \rho, |\mu_\bullet| - \lambda \rangle,$$

i.e. the convolution morphism is semismall.

Proof. The proof of [17, Lemme 9.3] carries over word by word, and we obtain that

$$\dim(m_{\mu_\bullet}^{-1}(\mathcal{O}_\lambda)) \leq \langle \rho, |\mu_\bullet| + \lambda \rangle.$$

Since m_{μ_\bullet} is L^+G -equivariant and $\dim(\mathcal{O}_\lambda) = \langle 2\rho, \lambda \rangle$, the corollary follows. \square

The convolution $\text{IC}_{\mu_1} \star \dots \star \text{IC}_{\mu_n}$ is a L^+G -equivariant perverse sheaf, and by Proposition 3.1, we can write

$$(4.1) \quad \text{IC}_{\mu_1} \star \dots \star \text{IC}_{\mu_n} \simeq \bigoplus_{\lambda \leq |\mu_\bullet|} V_{\mu_\bullet}^\lambda \otimes \text{IC}_\lambda,$$

where $V_{\mu_\bullet}^\lambda$ are finite dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces.

Lemma 4.4. *For every $\lambda \in X_+^\vee$ with $\lambda \leq |\mu_\bullet|$ and $x \in \mathcal{O}_\lambda(F)$, the vector space $V_{\mu_\bullet}^\lambda$ has a canonical basis indexed by the irreducible components of the fiber $m_{\mu_\bullet}^{-1}(x)$ of exact dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.*

Proof. We follow the argument in Haines [9]. We claim that $\text{IC}_{\mu_\bullet} = \text{IC}_{\mu_1} \tilde{\boxtimes} \dots \tilde{\boxtimes} \text{IC}_{\mu_k}$ is the intersection complex on $\overline{\mathcal{O}}_{\mu_\bullet}$. Indeed, this can be checked locally in the smooth topology, and then easily follows from the definitions. Hence, the left hand side of (4.1) is equal to $m_{\mu_\bullet,*}(\text{IC}_{\mu_\bullet})$. If $d = -\dim(\mathcal{O}_\lambda)$, then taking the d -th stalk cohomology at x in (4.1) gives by proper base change

$$R^d \Gamma(m_{\mu_\bullet}^{-1}(x), \text{IC}_{\mu_\bullet}) \simeq V_{\mu_\bullet}^\lambda.$$

Since $m_{\mu_\bullet} : \overline{\mathcal{O}}_{\mu_\bullet} \rightarrow \overline{\mathcal{O}}_{|\mu_\bullet|}$ is semismall, the cohomology $R^d \Gamma(m_{\mu_\bullet}^{-1}(x), \text{IC}_{\mu_\bullet})$ admits by [9, Lemma 3.2] a canonical basis indexed by the top dimensional irreducible components. This proves the lemma. \square

In the following, we consider $\overline{\mathcal{O}}_{\mu_\bullet}$ as a closed projective subvariety of

$$\overline{\mathcal{O}}_{\mu_1} \times \overline{\mathcal{O}}_{\mu_1+\mu_2} \times \cdots \times \overline{\mathcal{O}}_{\mu_1+\dots+\mu_k},$$

via $(g_1, \dots, g_k) \mapsto (g_1, g_1 g_2, \dots, g_1 \dots g_k)$. The lemma below is the geometric analogue of the PRV-conjecture.

Lemma 4.5. *For every $\lambda \in X_+^\vee$ of the form $\lambda = \nu_1 + \dots + \nu_k$ with $\nu_i \in W\mu_i$ for $i = 1, \dots, k$, the perverse sheaf IC_λ appears as a direct summand in $\mathrm{IC}_{\mu_1} \star \dots \star \mathrm{IC}_{\mu_k}$.*

Proof. Let $\nu = w(\nu_2 + \dots + \nu_k)$ be the unique dominant element in the W -orbit of $\nu_2 + \dots + \nu_k$. Then $\lambda = \nu_1 + w^{-1}\nu$. Hence, by induction, we may assume $k = 2$. By Lemma 4.4, it is enough to show that there exists $x \in \mathcal{O}_\lambda(F)$ such that $m_{\mu_\bullet}^{-1}(x)$ is of exact dimension $\langle \rho, |\mu_\bullet| - \lambda \rangle$.

Let $w \in W$ such that $w\nu_1$ is dominant, and consider $w\lambda = w\nu_1 + w\nu_2$. We denote by $S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet}$ the intersection inside $\overline{\mathcal{O}}_{\mu_1} \times \overline{\mathcal{O}}_{\mu_1+\mu_2}$

$$S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet} \stackrel{\text{def}}{=} (S_{w\nu_1} \times S_{w\nu_1+w\nu_2}) \cap \overline{\mathcal{O}}_{\mu_\bullet}.$$

The convolution is then given by projection on the second factor. By [17, Lemme 9.1], we have a canonical isomorphism

$$S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet} \simeq (S_{w\nu_1} \cap \overline{\mathcal{O}}_{\mu_1}) \times (S_{w\nu_2} \cap \overline{\mathcal{O}}_{\mu_2}).$$

Let $y = (y_1, y_2)$ in $(S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet})(F)$. Since for $i = 1, 2$ the elements $w\nu_i$ are conjugate under W to μ_i , there exist by [17, Lemme 5.2] elements $u_1, u_2 \in L^+U(F)$ such that

$$\begin{aligned} y_1 &= u_1 t^{w\nu_1} \cdot e_0 \\ y_2 &= u_1 t^{w\nu_1} u_2 t^{w\nu_2} \cdot e_0. \end{aligned}$$

The dominance of $w\nu_1$ implies $t^{w\nu_1} u_2 t^{-w\nu_1} \in L^+U(F)$, and hence $Y = S_{w\nu_\bullet} \cap \overline{\mathcal{O}}_{\mu_\bullet}$ maps under the convolution morphism onto an open dense subset Y' in $S_{w\lambda} \cap \mathcal{O}_\lambda$. Denote by $h = m_{\mu_\bullet}|_Y$ the restriction to Y . Both Y, Y' are irreducible schemes (their reduced loci are isomorphic to affine space), thus by generic flatness, there exists $x \in Y'(F)$ such that

$$\dim(h^{-1}(x)) = \dim(Y) - \dim(Y') = \langle \rho, |\mu_\bullet| + w\lambda \rangle - \langle \rho, \lambda + w\lambda \rangle = \langle \rho, |\mu_\bullet| - \lambda \rangle.$$

In particular, $\dim(m_{\mu_\bullet}^{-1}(x)) \geq \langle \rho, |\mu_\bullet| - \lambda \rangle$, and hence equality by Corollary 4.3. \square

For the proof of Theorem 4.1, we introduce a weaker partial order \preceq on X_+^\vee defined as follows: $\lambda \preceq \mu$ if and only if $\mu - \lambda \in \mathbb{R}_+Q_+^\vee$. Then $\lambda \leq \mu$ if and only if $\lambda \preceq \mu$ and their images in X^\vee/Q^\vee coincide (cf. Lemma B.2 below).

Proof of Theorem 4.1. We proceed in several steps:

(1) *The affine group scheme H is of finite type over $\overline{\mathbb{Q}}_\ell$.*

By [5, Proposition 2.20 (b)] this is equivalent to the existence of a tensor generator in $P_{L+G}(\mathrm{Gr}_G)$. Now there exist $\mu_1, \dots, \mu_k \in X_+^\vee$ which generate X_+^\vee as semigroups. Then $\mathrm{IC}_{\mu_1} \oplus \dots \oplus \mathrm{IC}_{\mu_k}$ is a tensor generator.

(2) *The affine group scheme H is connected reductive.*

For every $\mu \in X_+^\vee$ and $k \in \mathbb{N}$, the sheaf $\mathrm{IC}_{k\mu}$ is a direct summand of $\mathrm{IC}_\mu^{\star k}$, hence the scheme H is connected by [5, Corollary 2.22]. By [5, Proposition 2.23], the connected algebraic group H is reductive if and only if $P_{L+G}(\mathrm{Gr}_G)$ is semisimple, and this is true by Proposition 3.1.

(3) *The root datum of H is dual to the root datum of G .*

Let $(X', R', \Delta', X'^\vee, R'^\vee, \Delta'^\vee)$ the based root datum of H constructed in Theorem B.1 below. By Lemma B.5 below it is enough to show that we have an isomorphism of partially ordered semigroups

$$(4.2) \quad (X_+^\vee, \leq) \xrightarrow{\simeq} (X'_+, \leq').$$

By Proposition 3.1, the map $X_+^\vee \rightarrow X'_+, \mu \mapsto [\mathrm{IC}_\mu]$, where $[\mathrm{IC}_\mu]$ is the class of IC_μ in $K_0^+ P_{L+G}(\mathrm{Gr}_G)$ is a bijection of sets.

For every $\lambda, \mu \in X_+^\vee$, we claim that $\lambda \preceq \mu$ if and only if $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. Assume $\lambda \preceq \mu$, and choose a finite subset $F \subset X_+^\vee$ satisfying Proposition B.3 (iii). Let $\mathcal{A} = \bigoplus_{\nu \in F} \mathrm{IC}_\nu$, and suppose IC_χ is a direct summand of IC_λ^{*k} for some $k \in \mathbb{N}$. In particular, $\chi \leq k\lambda$ and so $\chi \in WF + \sum_{i=1}^k W\mu$. By Lemma 4.5, the sheaf IC_χ is a direct summand of $\mathrm{IC}_\mu^{*k} \star \mathcal{A}$, which means $[\mathrm{IC}_\chi] \preceq' [\mathrm{IC}_\mu]$. Conversely, assume $[\mathrm{IC}_\lambda] \preceq' [\mathrm{IC}_\mu]$. Using Proposition B.3 (iv) below, this translates, by looking at the support, into the following condition: there exists $\nu \in X_+^\vee$ such that $\overline{\mathcal{O}}_{k\lambda} \subset \overline{\mathcal{O}}_{k\mu+\nu}$ holds for infinitely many $k \in \mathbb{N}$. Equivalently, $k\lambda \leq k\mu + \nu$ for infinitely many $k \in \mathbb{N}$ which implies $\lambda \preceq \mu$.

For every $\lambda, \mu \in X_+^\vee$, we claim that $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu] = [\mathrm{IC}_{\lambda+\mu}]$ in X'_+ : by the proof of Theorem B.1 below, $[\mathrm{IC}_\lambda] + [\mathrm{IC}_\mu]$ is the class of the maximal element appearing in $\mathrm{IC}_\lambda \star \mathrm{IC}_\mu$. Since the partial orders \preceq, \preceq' agree, this is $[\mathrm{IC}_{\lambda+\mu}]$.

It remains to show that the partial orders \leq, \leq' agree. The identification $X_+^\vee = X'_+$ prolongs to $X^\vee = X'$. We claim that $Q_+^\vee = Q'_+$ under this identification and hence $Q^\vee = Q'$, which is enough by Lemma B.2 below. Let $\alpha^\vee \in Q_+^\vee$ a simple coroot, and choose some $\mu \in X_+^\vee$ with $\langle \alpha, \mu \rangle = 2$. Then $\mu + s_\alpha(\mu) = 2\mu - \alpha^\vee$ is dominant, and hence $\mathrm{IC}_{2\mu - \alpha^\vee}$ appears by Lemma 4.5 as a direct summand in IC_μ^{*2} . By Lemma B.4 this means $\alpha^\vee \in Q'_+$, and thus $Q_+^\vee \subset Q'_+$. Conversely, assume $\alpha' \in Q'_+$ has the property that there exists $\mu \in X'_+$ with $2\mu - \alpha' \in X'_+$ and $\mathrm{IC}_{2\mu - \alpha'}$ appears as a direct summand in IC_μ^{*2} . Note that every element in Q'_+ is a sum of these elements. Then $2\mu - \alpha' \leq 2\mu$, and hence $\alpha' \in Q_+^\vee$. This shows $Q'_+ \subset Q_+^\vee$ and finishes the proof of (4.2). \square

5. GALOIS DESCENT

Let F be any field, and G a connected reductive group defined over F . Fix a separable closure \bar{F} , and let $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group. Let $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$ be the category of finite dimensional continuous ℓ -adic Galois representations. For any object defined over F , we denote by a subscript $(-)_{\bar{F}}$ its base change to \bar{F} . Consider the functor

$$\begin{aligned} \Omega : P_{L+G}(\mathrm{Gr}_G) &\longrightarrow \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F) \\ \mathcal{A} &\longmapsto \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathrm{Gr}_{G, \bar{F}}, \mathcal{A}_{\bar{F}}). \end{aligned}$$

There are canonical isomorphisms of fpqc-sheaves $(LG)_{\bar{F}} \simeq LG_{\bar{F}}$, $(L^+G)_{\bar{F}} \simeq L^+G_{\bar{F}}$ and $\mathrm{Gr}_{G, \bar{F}} \simeq \mathrm{Gr}_{G_{\bar{F}}}$. Hence, $\Omega \simeq \omega \circ (-)_{\bar{F}}$, cf. (3.3).

The absolute Galois group Γ_F operates on the Tannakian category $P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G_{\bar{F}}})$ by tensor equivalences compatible with the fiber functor ω . Hence, we may form the semidirect product ${}^L G = \mathrm{Aut}^*(\omega)(\bar{\mathbb{Q}}_\ell) \rtimes \Gamma_F$ considered as a topological group as follows. The group $\mathrm{Aut}^*(\omega)(\bar{\mathbb{Q}}_\ell)$ is equipped with the ℓ -adic topology, the Galois Γ_F group with the profinite topology and ${}^L G$ with the product topology. Note that Γ_F acts continuously on $\mathrm{Aut}^*(\omega)(\bar{\mathbb{Q}}_\ell)$ by Proposition 5.6 below. Let $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G)$ be the full subcategory of the category finite dimensional continuous ℓ -adic representations of ${}^L G$ such that the restriction to $\mathrm{Aut}^*(\omega)(\bar{\mathbb{Q}}_\ell)$ is algebraic.

Theorem 5.1. *The functor Ω is an equivalence of abelian tensor categories*

$$\begin{aligned} \Omega : P_{L+G}(\mathrm{Gr}_G) &\longrightarrow \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G) \\ \mathcal{A} &\longmapsto \Omega(\mathcal{A}). \end{aligned}$$

The proof of Theorem 5.1 proceeds in several steps.

Lemma 5.2. *Let H be an affine group scheme over a field k . Let $\mathrm{Rep}_k(H)$ be the category of algebraic representations of H , and let $\mathrm{Rep}_k(H(k))$ be the category of finite dimensional representations of the abstract group $H(k)$. Assume that H is reduced and that $H(k) \subset H$ is*

dense. Then the functor

$$\begin{aligned} \Psi : \text{Rep}_k(H) &\longrightarrow \text{Rep}_k(H(k)) \\ \rho &\longmapsto \rho(k) \end{aligned}$$

is a fully faithful embedding.

□

We recall some facts on the Tannakian formalism from the appendix in [20]. Let (\mathcal{C}, \otimes) be a neutralized Tannakian category over a field k with fiber functor v . We define a monoidal category $\text{Aut}^\otimes(\mathcal{C}, v)$ as follows. Objects are pairs (σ, α) , where $\sigma : \mathcal{C} \rightarrow \mathcal{C}$ is a tensor automorphism and $\alpha : v \circ \sigma \rightarrow v$ is a natural isomorphism of tensor functors. Morphisms between (σ, α) and (σ', α') are natural tensor isomorphisms between σ and σ' that are compatible with α, α' in an obvious way. The monoidal structure is given by compositions. Since v is faithful, $\text{Aut}^\otimes(\mathcal{C}, v)$ is equivalent to a set, and in fact is a group.

Let $H = \text{Aut}_{\mathcal{C}}^\otimes(v)$, the Tannakian group defined by (\mathcal{C}, v) . There is a canonical action of $\text{Aut}^\otimes(\mathcal{C}, v)$ on H by automorphisms as follows. Let (σ, α) be in $\text{Aut}^\otimes(\mathcal{C}, v)$. Let R be a k -algebra, and let $h : v_R \rightarrow v_R$ be a R -point of H . Then $(\sigma, \alpha) \cdot h$ is the following composition

$$v_R \xrightarrow{\alpha^{-1}} v_R \circ \sigma \xrightarrow{h \circ \text{id}} v_R \circ \sigma \xrightarrow{\alpha} v_R.$$

Let Γ be an abstract group. Then an action of Γ on (\mathcal{C}, v) is by definition a group homomorphism $\text{act} : \Gamma \rightarrow \text{Aut}^\otimes(\mathcal{C}, v)$.

Assume that Γ acts on (\mathcal{C}, v) . Then we define \mathcal{C}^Γ , the category of Γ -equivariant objects in \mathcal{C} as follows. Objects are $(X, \{c_\gamma\}_{\gamma \in \Gamma})$, where X is an object in \mathcal{C} and $c_\gamma : \text{act}_\gamma(X) \simeq X$ is an isomorphism, satisfying the natural cocycle condition, i.e. $c_{\gamma'\gamma} = c_{\gamma'} \circ \text{act}_{\gamma'}(c_\gamma)$. The morphisms between $(X, \{c_\gamma\}_{\gamma \in \Gamma})$ and $(X', \{c'_\gamma\}_{\gamma \in \Gamma})$ are morphisms between X and X' , compatible with c_γ, c'_γ in an obvious way.

Lemma 5.3. *Let Γ be a group acting on (\mathcal{C}, v) .*

- (i) *The category \mathcal{C}^Γ is an abelian tensor category.*
- (ii) *Assume that H is reduced and that k is algebraically closed. The functor v is an equivalence of abelian tensor categories*

$$\mathcal{C}^\Gamma \simeq \text{Rep}_k^o(H(k) \rtimes \Gamma)$$

where $\text{Rep}_k^o(H(k) \rtimes \Gamma)$ is the full subcategory of finite dimensional representations of the abstract group $H(k) \rtimes \Gamma$ such that the restriction to $H(k)$ is algebraic.

Remark 5.4. In fact, the category \mathcal{C}^Γ is neutralized Tannakian with fiber functor v . If Γ is finite, then $\text{Aut}_{\mathcal{C}^\Gamma}^\otimes(v) \simeq H \rtimes \Gamma$. However, if Γ is not finite, then $\text{Aut}_{\mathcal{C}^\Gamma}^\otimes(v)$ is in general not $H \rtimes \Gamma$, where the latter is regarded as an affine group scheme.

Proof of Lemma 5.3. The monoidal structure on \mathcal{C}^Γ is defined as

$$(X, \{c_\gamma\}_{\gamma \in \Gamma}) \otimes (X', \{c'_\gamma\}_{\gamma \in \Gamma}) = (X'', \{c''_\gamma\}_{\gamma \in \Gamma}),$$

where $X'' = X \otimes X'$ and $c''_\gamma : \text{act}_\gamma(X'') \rightarrow X''$ is the composition

$$\text{act}_\gamma(X \otimes X') \simeq \text{act}_\gamma(X) \otimes \text{act}_\gamma(X') \xrightarrow{c_\gamma \otimes c'_\gamma} X \otimes X'.$$

This gives \mathcal{C}^Γ the structure of an abelian tensor category.

Now assume that H is reduced and that k is algebraically closed. It is enough to show that as tensor categories

$$\Psi : \text{Rep}_k(H)^\Gamma \xrightarrow{\simeq} \text{Rep}_k^o(H(k) \rtimes \Gamma)$$

compatible with the forgetful functors. Let $((V, \rho), \{c_\gamma\}_{\gamma \in \Gamma}) \in \text{Rep}_k(H)^\Gamma$. Then we define $(V, \rho_\Gamma) \in \text{Rep}_k^o(H(k) \rtimes \Gamma)$ by

$$(h, \gamma) \longmapsto \rho(h) \circ \alpha_h(V) \circ v \circ c_\gamma^{-1} \in \text{GL}(V),$$

where $\alpha_h : v \circ \sigma_h \simeq v$ is induced by the action of Γ as above. Using the cocycle relation, one checks that this is indeed a representation. By Lemma 5.2, the natural map

$$\mathrm{Hom}_H(\rho, \rho') \longrightarrow \mathrm{Hom}_{H(k)}(\rho(k), \rho'(k))$$

is bijective. Taking Γ -invariants shows that the functor Ψ is fully faithful. Essential surjectivity is obvious. \square

Now we specialize to the case $(\mathcal{C}, \otimes) = (P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}}), \star)$ with fiber functor $v = \omega$. Then the absolute Galois group $\Gamma = \Gamma_{\bar{F}}$ acts on this Tannakian category (cf. Appendix A.1).

Proof of Theorem 5.1. The functor Ω is fully faithful.

Let $P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})^{\Gamma, c}$ be the full subcategory of $P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})^{\Gamma}$ consisting of perverse sheaves together with a continuous descent datum (cf. Appendix A.1). By Lemma A.6, the functor $\mathcal{A} \mapsto \mathcal{A}_{\bar{F}}$ is an equivalence of abelian categories $P_{L+G}(\mathrm{Gr}_G) \simeq P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})^{\Gamma, c}$. Hence, we get a commutative diagram

$$\begin{array}{ccc} P_{L+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})^{\Gamma} & \xrightarrow{\omega} & \mathrm{Rep}_{\bar{\mathbb{Q}}_{\ell}}^{\circ}({}^L G) \\ \mathcal{A} \mapsto \mathcal{A}_{\bar{F}} \uparrow & & \uparrow \\ P_{L+G}(\mathrm{Gr}_G) & \xrightarrow{\Omega} & \mathrm{Rep}_{\bar{\mathbb{Q}}_{\ell}}^{\varepsilon}({}^L G), \end{array}$$

where ω is an equivalence of categories by Lemma 5.3 (ii), and where the vertical arrows are fully faithful. Hence, Ω is fully faithful.

The functor Ω is essentially surjective.

Let ρ be in $\mathrm{Rep}_{\bar{\mathbb{Q}}_{\ell}}^{\varepsilon}({}^L G)$. Without loss of generality, we assume that ρ is indecomposable. Let $H = \mathrm{Aut}^*(\omega)$. By Proposition 3.1, the restriction $\rho|_H$ is semisimple. Denote by A the set of isotypic components of $\rho|_H$. Then Γ_F operates transitively on A , and for every $a \in A$ its stabilizer in Γ_F is the absolute Galois group Γ_E for some finite separable extension E/F . By Galois descent along finite extensions, we may assume that $E = F$, and hence that $\rho|_H$ has only one isotypic component. Let ρ_0 be the simple representation occurring in $\rho|_H$. Then $\mathrm{Hom}_H(\rho_0, \rho)$ is a continuous Γ -representation, and the natural morphism

$$\rho_0 \otimes \mathrm{Hom}_H(\rho_0, \rho) \longrightarrow \rho$$

given by $v \otimes f \mapsto f(v)$ is an isomorphism of ${}^L G$ -representations. Let IC_X be the simple perverse sheaf on $\mathrm{Gr}_{G, \bar{F}}$ with $\omega(\mathrm{IC}_X) \simeq \rho_0$. Since ρ has only one isotypic component, the support $X = \mathrm{supp}(\mathrm{IC}_X)$ is Γ -invariant, and hence defined over F . Denote by V the local system on $\mathrm{Spec}(F)$ given by the Γ -representation $\mathrm{Hom}_H(\rho_0, \rho)$. Then $\mathrm{IC}_X \otimes V$ is an object in $P_{L+G}(\mathrm{Gr}_G)$ such that $\Omega(\mathrm{IC}_X \otimes V) \simeq \rho_0 \otimes \mathrm{Hom}_H(\rho_0, \rho)$. This proves the theorem. \square

The proof of Theorem 5.1 also shows the following fact.

Corollary 5.5. *Let $\mathcal{A} \in P_{L+G}(\mathrm{Gr}_G)$ indecomposable. Let $\{X_i\}_{i \in I}$ be the set of irreducible components of $\mathrm{supp}(\mathcal{A}_{\bar{F}})$. Denote by E the minimal finite separable extension of F such that X_i is defined over E for all $i \in I$. Then as perverse sheaves on $\mathrm{Gr}_{G, E}$*

$$\mathcal{A}_E \simeq \bigoplus_{i \in I} \mathrm{IC}_{X_i} \otimes V_i,$$

where V_i are indecomposable local systems on $\mathrm{Spec}(E)$.

\square

We briefly explain the connection to the full L -group. For more details see the appendix in [20]. Let \hat{G} be the reductive group over $\bar{\mathbb{Q}}_{\ell}$ dual to $G_{\bar{F}}$ in the sense of Langlands, i.e. the

root datum of \hat{G} is dual to the root datum of $G_{\bar{F}}$. There are two natural actions of Γ_F on \hat{G} as follows. Up to the choice of a pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ of \hat{G} , we have an action act^{alg} via

$$(5.1) \quad \text{act}^{\text{alg}} : \Gamma_F \rightarrow \text{Out}(G_{\bar{F}}) \simeq \text{Out}(\hat{G}) \simeq \text{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{X}) \subset \text{Aut}(\hat{G}),$$

where $\text{Out}(-)$ denotes the outer automorphisms. On the other hand, we have an action $\text{act}^{\text{geo}} : \Gamma_F \rightarrow \text{Aut}(\hat{G})$ via the Tannakian equivalence from Theorem 4.1. The relation between act^{geo} and act^{alg} is as follows.

Let $\text{cycl} : \Gamma_F \rightarrow \mathbb{Z}_\ell^\times$ be the cyclotomic character of Γ_F defined by the action of Γ_F on the ℓ^∞ -roots of unity of \bar{F} . Let \hat{G}_{ad} be the adjoint group of \hat{G} . Let ρ be the half sum of positive coroots of \hat{G} , which gives rise to a one-parameter group $\rho : \mathbb{G}_m \rightarrow \hat{G}_{\text{ad}}$. We define a map

$$\chi : \Gamma_F \xrightarrow{\text{cycl}} \mathbb{Z}_\ell^\times \xrightarrow{\rho} \hat{G}_{\text{ad}}(\bar{\mathbb{Q}}_\ell),$$

which gives a map $\text{Ad}_\chi : \Gamma_F \rightarrow \text{Aut}(\hat{G})$ to the inner automorphism of \hat{G} .

Proposition 5.6 ([20] Proposition A.4). *For all $\gamma \in \Gamma_F$,*

$$\text{act}^{\text{geo}}(\gamma) = \text{act}^{\text{alg}}(\gamma) \circ \text{Ad}_\chi(\gamma).$$

□

Remark 5.7. Proposition 5.6 shows that act^{geo} only depends on the quasi-split form of G , since the same is true for act^{alg} . In particular, the Satake category $P_{L+G}(\text{Gr}_G)$ only depends on the quasi-split form of G whereas the ind-scheme Gr_G does depend on G .

Let ${}^L G^{\text{alg}} = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes_{\text{act}^{\text{alg}}} \Gamma_F$ be the full L -group. Set ${}^L G^{\text{geo}} = \hat{G}(\bar{\mathbb{Q}}_\ell) \rtimes_{\text{act}^{\text{geo}}} \Gamma_F$.

Corollary 5.8 ([20] Corollary A.5). *The map $(g, \gamma) \mapsto (\text{Ad}_{\chi(\gamma^{-1})}(g), \gamma)$ gives an isomorphism ${}^L G^{\text{alg}} \xrightarrow{\cong} {}^L G^{\text{geo}}$.*

□

Combining Corollary 5.8 with Theorem 5.1, we obtain the following corollary.

Corollary 5.9. *There is an equivalence of abelian tensor categories*

$$P_{L+G}(\text{Gr}_G) \simeq \text{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G^{\text{alg}}),$$

where $\text{Rep}_{\bar{\mathbb{Q}}_\ell}^c({}^L G^{\text{alg}})$ denotes the full subcategory of the category of finite dimensional continuous ℓ -adic representations of ${}^L G^{\text{alg}}$ such that the restriction to $\hat{G}(\bar{\mathbb{Q}}_\ell)$ is algebraic.

□

APPENDIX A. PERVERSE SHEAVES

For the construction of the category of ℓ -adic perverse sheaves, we refer to the work of Y. Laszlo and M. Olsson [13]. In this appendix we explain our conventions on perverse sheaves on ind-schemes.

Let F be an arbitrary field. Fix a prime $\ell \neq \text{char}(F)$, and denote by \mathbb{Q}_ℓ the field of ℓ -adic numbers with algebraic closure $\bar{\mathbb{Q}}_\ell$. For any separated scheme T of finite type over F , we consider the bounded derived category $D_c^b(T, \bar{\mathbb{Q}}_\ell)$ of constructible ℓ -adic sheaves on T . Let $P(T)$ be the abelian $\bar{\mathbb{Q}}_\ell$ -linear full subcategory of ℓ -adic perverse sheaves, i.e. the heart of the perverse t -structure on the triangulated category $D_c^b(T, \bar{\mathbb{Q}}_\ell)$.

Now let $(T)_{i \in I}$ be an inductive system of separated schemes of finite type over F with closed immersions as transition morphisms. A fpqc-sheaf \mathcal{T} on the category of F -algebras is called a *strict ind-scheme of ind-finite type over F* if there is an isomorphism of fpqc-sheaves $\mathcal{T} \simeq \varinjlim_i T_i$, for some system $(T)_{i \in I}$ as above. The inductive system $(T)_{i \in I}$ is called an *ind-presentation of \mathcal{T}* .

For $i \leq j$, push forward gives transition morphisms $D_c^b(T_i, \bar{\mathbb{Q}}_\ell) \rightarrow D_c^b(T_j, \bar{\mathbb{Q}}_\ell)$ which restrict to $P(T_i) \rightarrow P(T_j)$, because push forward along closed immersions is t -exact.

Definition A.1. Let \mathcal{T} be a strict ind-scheme of ind-finite type over F , and $(T_i)_{i \in I}$ be an ind-presentation.

(i) The *bounded derived category of constructible ℓ -adic complexes* $D_c^b(\mathcal{T}, \bar{\mathbb{Q}}_\ell)$ on \mathcal{T} is the inductive limit

$$D_c^b(\mathcal{T}, \bar{\mathbb{Q}}_\ell) \stackrel{\text{def}}{=} \varinjlim_i D_c^b(T_i, \bar{\mathbb{Q}}_\ell).$$

(ii) The *category of ℓ -adic perverse sheaves* $P(\mathcal{T})$ on \mathcal{T} is the inductive limit

$$P(\mathcal{T}) \stackrel{\text{def}}{=} \varinjlim_i P(T_i).$$

The definition is independent of the chosen ind-presentation of \mathcal{T} . The category $D_c^b(\mathcal{T}, \bar{\mathbb{Q}}_\ell)$ inherits a triangulation and a perverse t -structure from the $D_c^b(T_i, \bar{\mathbb{Q}}_\ell)$'s. The heart with respect to the perverse t -structure is the abelian $\bar{\mathbb{Q}}_\ell$ -linear full subcategory $P(\mathcal{T})$.

If $f : \mathcal{T} \rightarrow \mathcal{S}$ is a morphism of strict ind-schemes of ind-finite type over F , we have the Grothendieck operations f_* , $f_!$, f^* , $f^!$, and the usual constructions carry over after the choice of ind-presentations.

In Section 2.3 we work with equivariant objects in the category of perverse sheaves. The context is as follows. Let $f : T \rightarrow S$ be a morphism of separated schemes of finite type, and let H be a smooth affine group scheme over S with geometrically connected fibers acting on $f : T \rightarrow S$. Then a perverse sheaf \mathcal{A} on T is called *H -equivariant* if there is an isomorphism in the derived category

$$(A.1) \quad \theta : a^* \mathcal{A} \simeq p^* \mathcal{A},$$

where $a : H \times_S T \rightarrow T$ (resp. $p : H \times_S T \rightarrow T$) is the action (resp. projection on the second factor). A few remarks are in order: if the isomorphism (A.1) exists, then it can be rigidified such that $e_T^* \theta$ is the identity, where $e_T : T \rightarrow H \times_S T$ is the identity section. A rigidified isomorphism θ automatically satisfies the cocycle relation due to the fact that H has geometrically connected fibers.

The subcategory $P_H(T)$ of $P(T)$ of H -equivariant objects together with H -equivariant morphisms is called the *category of H -equivariant perverse sheaves* on T .

Lemma A.2 ([13] Remark 5.5). *Consider the stack quotient $H \backslash T$, an Artin stack of finite type over S . Let $p : T \rightarrow H \backslash T$ be the quotient map of relative dimension $d = \dim(T/S)$. Then the pull back functor*

$$p^*[d] : P(H \backslash T) \longrightarrow P_H(T),$$

is an equivalence of categories. In particular, $P_H(T)$ is abelian and $\bar{\mathbb{Q}}_\ell$ -linear.

□

Now let \mathcal{T} be a strict ind-scheme of ind-finite type, and $f : \mathcal{T} \rightarrow S$ a morphism to a separated scheme of finite type. Fix an ind-presentation $(T_i)_{i \in I}$ of \mathcal{T} . Let $(H_i)_{i \in I}$ be an inverse system of smooth affine group scheme with geometrically connected fibers. Let $\mathcal{H} = \varprojlim_i H_i$ be the inverse limit, an affine group scheme over S , because the transition morphisms are affine. Assume that \mathcal{H} acts on $f : \mathcal{T} \rightarrow S$ such that the action restricts to the inductive system $(f|_{T_i})_{i \in I}$. Assume that the \mathcal{H} -action factors through H_i on $f|_{T_i}$ for every $i \in I$.

Definition A.3. Let $f : \mathcal{T} \rightarrow S$, $(T_i)_{i \in I}$ and \mathcal{H} as above. The *category $P_{\mathcal{H}}(\mathcal{T})$ of \mathcal{H} -equivariant perverse sheaves* on \mathcal{T} is the inductive limit

$$P_{\mathcal{H}}(\mathcal{T}) \stackrel{\text{def}}{=} \varinjlim_i P_{H_i}(T_i).$$

It follows from Lemma A.2 that the category $P_{\mathcal{H}}(\mathcal{T})$ is an abelian $\bar{\mathbb{Q}}_\ell$ -linear category. The following lemma is used throughout the text.

Lemma A.4. *Let $T \rightarrow S$ be a \mathcal{H} -torsor, and let Y be a S -scheme with \mathcal{H} -action. Assume that the action of \mathcal{H} on Y factors over H_i for $i \gg 0$. Then there is a canonical isomorphism of fpqc-sheaves*

$$T \times^{\mathcal{H}} Y \xrightarrow{\cong} T^{(i)} \times^{H_i} Y,$$

where $T^{(i)} = T \times^{\mathcal{H}} H_i$.

□

Remark A.5. In particular, if $T^{(i)} \times^{H_i} Y$ is representable of finite type, then is $T \times^{\mathcal{H}} Y$ is representable of finite type.

A.1. Galois Descent of Perverse Sheaves. Fix a separable closure \bar{F} of F . Let $\Gamma = \text{Gal}(\bar{F}/F)$ be the absolute Galois group. For any complex of sheaves \mathcal{A} on T , we denote by $\mathcal{A}_{\bar{F}}$ its base change to $T_{\bar{F}} = T \otimes \bar{F}$. We define the category of *perverse sheaves with continuous Γ -descent datum* $P(T_{\bar{F}})^{\Gamma, c}$ as follows. The objects are pairs $(\mathcal{A}, \{c_\gamma\}_{\gamma \in \Gamma})$, where $\mathcal{A} \in P(T_{\bar{F}})$ and $\{c_\gamma\}_{\gamma \in \Gamma}$ is a family of isomorphisms

$$c_\gamma : \gamma_* \mathcal{A} \xrightarrow{\cong} \mathcal{A},$$

satisfying the cocycle condition $c_{\gamma'\gamma} = c_{\gamma'} \circ \gamma'_*(c_\gamma)$ such that the datum is continuous in the following sense. For every $i \in \mathbb{Z}$ and every locally closed subscheme $S \subset T$ such that the standard cohomology sheaf $\mathcal{H}^i(\mathcal{A})|_S$ is a local system, and for every $U \rightarrow S$ étale, with U separated quasi-compact, the induced ℓ -adic Galois representation on the $U_{\bar{F}}$ -sections

$$\Gamma \longrightarrow \text{GL}(\mathcal{H}^i(\mathcal{A})(U_{\bar{F}})),$$

is continuous. The morphisms in $P(T_{\bar{F}})^{\Gamma, c}$ are morphisms in $P(T_{\bar{F}})$ compatible with the c_γ 's. For every $\mathcal{A} \in P(T)$, its pullback $\mathcal{A}_{\bar{F}}$ admits a canonical continuous descent datum. Hence, we get a functor

$$\begin{aligned} \Phi : P(T) &\longrightarrow P(T_{\bar{F}})^{\Gamma, c} \\ \mathcal{A} &\longmapsto \mathcal{A}_{\bar{F}}. \end{aligned}$$

Lemma A.6 (SGA 7, XIII, 1.1). *The functor Φ is an equivalence of categories.*

□

APPENDIX B. RECONSTRUCTION OF ROOT DATA

Let G a split connected reductive group over an arbitrary field k . Denote by Rep_G the Tannakian category of algebraic representations of G . If k is algebraically closed of characteristic 0, then D. Kazhdan, M. Larsen and Y. Varshavsky [10, Corollary 2.5] show how to reconstruct the root datum of G from the Grothendieck semiring $K_0^+[G] = K_0^+ \text{Rep}_G$. In fact, their construction works over arbitrary fields. This relies on the conjecture of Parthasarathy, Ranga-Rao and Varadarajan (PRV-conjecture) proven by S. Kumar [11] ($\text{char}(k) = 0$) and O. Mathieu [15] ($\text{char}(k) > 0$).

Theorem B.1. *The root datum of G can be reconstructed from the Grothendieck semiring $K_0^+[G]$.*

This means, if H is another split connected reductive group over k , and if $\varphi : K_0^+[H] \rightarrow K_0^+[G]$ is an isomorphism of Grothendieck semirings, then there exists an isomorphism of group schemes $\phi : H \rightarrow G$ determined uniquely up to inner automorphism such that $\phi = K_0^+[\varphi]$.

Let T be a maximal split torus of G and B a Borel subgroup containing T . We denote by $\langle -, - \rangle$ the natural pairing between $X = \text{Hom}(T, \mathbb{G}_m)$ and $X^\vee = \text{Hom}(\mathbb{G}_m, T)$. Let $R \subset X$ be the root system associated to (G, T) , and R_+ be the set of positive roots corresponding to B . Let $R^\vee \subset X^\vee$ the dual root system with the bijection $R \rightarrow R^\vee$, $\alpha \mapsto \alpha^\vee$. Denote by R_+^\vee the

set of positive coroots. Let W the Weyl group of (G, T) . Consider the half sum of all positive roots

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Let Q (resp. Q_+) the subgroup (resp. submonoid) of X generated by R (resp. R_+). We denote by

$$X_+ = \{\mu \in X \mid \langle \mu, \alpha \rangle \geq 0, \forall \alpha \in R_+^\vee\}$$

the cone of dominant characters.

We consider partial orders \leq and \preceq on X defined as follows. For $\lambda, \mu \in X$, we define $\lambda \leq \mu$ if and only if $\mu - \lambda \in Q_+$, and we define $\lambda \preceq \mu$ if and only if $\mu - \lambda = \sum_{\alpha \in \Delta} x_\alpha \alpha$ with $x_\alpha \in \mathbb{R}_{\geq 0}$. The latter order is weaker than the former order in the sense that $\lambda \leq \mu$ implies $\lambda \preceq \mu$, but in general not conversely.

Lemma B.2 ([18]). *For every $\lambda, \mu \in X_+$, then $\lambda \leq \mu$ if and only if $\lambda \preceq \mu$ and the images of λ, μ in X/Q agree.*

Let

$$\text{Dom}_{\preceq \mu} = \{\nu \in X_+ \mid \nu \preceq \mu\}.$$

For a finite subset F of the euclidean vector space $E = X \otimes \mathbb{R}$, we denote by $\text{Conv}(F)$ its convex hull.

Proposition B.3. *For $\lambda, \mu \in X_+$, the following conditions are equivalent:*

- (i) $\lambda \preceq \mu$
- (ii) $\text{Conv}(W\lambda) \subset \text{Conv}(W\mu)$
- (iii) *There exists a finite subset $F \subset X_+$ such that for all $k \in \mathbb{N}$:*

$$\text{Dom}_{\preceq k\lambda} \subset WF + \sum_{i=1}^k W\mu$$

- (iv) *There exists a representation U such that for every $k \in \mathbb{N}$, every irreducible subquotient of $V_\lambda^{\otimes k}$ is a subquotient of $V_\mu^{\otimes k} \otimes U$.*

Proof. The equivalence of (i) and (ii) is well-known. The implication (ii) \Rightarrow (iii) follows from [10, Lemma 2.4]. Assume (iii), we show that (iv) holds: let $U = \bigoplus_{\nu \in F} V_\nu$, and suppose V_χ is a irreducible subquotient of $V_\lambda^{\otimes k}$, in particular $\chi \leq k\lambda$. By (iii), χ has the form $w\nu + \sum_{i=1}^k w_i \mu$ with $w, w_1, \dots, w_k \in W$ and $\nu \in F$. Using the PRV-conjecture [4, Theorem 4.3.2], we conclude that V_χ is a subquotient of $V_\mu^{\otimes k} \otimes V_\nu$, hence also of $V_\mu^{\otimes k} \otimes U$. This shows (iv). The implication (iv) \Rightarrow (i) is shown in [10, Proposition 2.2]. \square

For $\mu \in X_+$, let v_μ be the corresponding element in $K_0^+[G]$. Let $\mathcal{Q}_+ \subset X$ be the semigroup generated by the set

$$\{\alpha \in X \mid \exists \mu \in X_+ : 2\mu - \alpha \in X_+ \text{ and } v_\mu^2 - v_{2\mu - \alpha} \in K_0^+[G]\}.$$

Lemma B.4. *There is an equality of semigroups $\mathcal{Q}_+ = Q_+$.*

Proof. It is obvious that $\mathcal{Q}_+ \subset Q_+$, and we show that \mathcal{Q}_+ contains the simple roots. Let α be a simple root, and choose some $\mu \in X$ such that $\langle \mu, \alpha^\vee \rangle = 2$. Then $2\mu - \alpha$ paired with any simple root is positive, and hence $\mu + s_\alpha(\mu) = 2\mu - \alpha$ is dominant. By the PRV-conjecture [4, Theorem 4.3.2], the representation $V_{2\mu - \alpha}$ appears as an irreducible subquotient in $V_\mu^{\otimes 2}$, i.e. $v_\mu^2 - v_{2\mu - \alpha} \in K_0^+[G]$. \square

The proof of Theorem B.1 goes along the lines of [10, Corollary 2.5].

Proof of Theorem B.1. By Lemma B.5 below it is enough to construct the partially ordered semigroup (X_+, \leq) of dominant weights.

The underlying set of dominant weights X_+ is the set of irreducible objects in $K_0^+[G]$. Then the partial order \leq on X_+ is characterized by Proposition B.3 as follows: for $\lambda, \mu \in X_+$, one has $\lambda \leq \mu$ if and only if there exists a $u \in K_0^+[G]$ such that for all $k \in \mathbb{N}$ and $\nu \in X_+$,

$$v_\lambda^k - v_\nu \in K_0^+[G] \implies v_\mu^k \cdot u - v_\nu \in K_0^+[G].$$

The semigroup structure on X_+ is given by: for $\lambda, \mu \in X_+$, one has $\nu = \lambda + \mu$ if and only if ν is the unique dominant weight which is maximal (w.r.t. \leq) with the property that $v_\lambda \cdot v_\mu - v_\nu \in K_0^+[G]$.

Now X is the group completion of X_+ , and by Lemma B.4 we can reconstruct $Q_+ \subset X$. Then Q is the group completion of Q_+ , and by Lemma B.2 we can reconstruct \leq . This shows that the root datum of G can be reconstructed from $K_0^+[G]$.

Now if H is another split connected reductive group over k , and $\varphi : K_0^+[H] \rightarrow K_0^+[G]$ an isomorphism of Grothendieck semirings, then the argument above shows that there is an isomorphism of partially ordered semigroups

$$(B.1) \quad (X_+^H, \leq^H) \longrightarrow (X_+^G, \leq^G)$$

inducing φ on Grothendieck semirings. By Lemma B.5 below, the morphism B.1 prolongs to an isomorphism of the associated based root data. Hence, there exists an isomorphism of group schemes $\phi : H \rightarrow G$ inducing the isomorphism of based root data. In particular, $\varphi = K_0^+[\phi]$, and such an isomorphism ϕ is uniquely determined up to inner automorphism. This finishes the proof of Theorem B.1. \square

Lemma B.5. *Let $\mathcal{B} = (X, R, \Delta, X^\vee, R^\vee, \Delta^\vee)$ any based root datum. Denote by (X_+, \leq) the partially ordered semigroup of dominant weights. Then the root datum \mathcal{B} can be reconstructed from (X_+, \leq) , i.e. if $\mathcal{B}' = (X', R', \Delta', X'^\vee, R'^\vee, \Delta'^\vee)$ is another based root datum with associated dominant weights (X'_+, \leq') , then any isomorphism $(X, \leq) \rightarrow (X', \leq')$ of partially ordered semigroups prolongs to an isomorphism $\mathcal{B} \rightarrow \mathcal{B}'$ of based root data.*

Proof. The weight lattice X is the group completion of X_+ , a finite free \mathbb{Z} -module. The dominance order \leq extends uniquely to X , also denoted \leq . Then $X^\vee = \text{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ is the coweight lattice, and the natural pairing $X \times X^\vee \rightarrow \mathbb{Z}$ identifies with $\langle -, - \rangle$. The reconstruction of the roots and coroots proceeds in several steps:

(1) *The set of simple roots $\Delta \subset X$:*

A weight $\alpha \in X \setminus \{0\}$ is in Δ if and only if $0 \leq \alpha$, and α is minimal with this property.

(2) *The set of simple coroots $\Delta^\vee \subset X^\vee$:*

An element of X^\vee is uniquely determined by its value on X_+ . Fix $\alpha \in \Delta$ with corresponding simple coroot α^\vee . Then for any $\mu \in X_+$, the value $\langle \mu, \alpha^\vee \rangle$ is the unique number $m \in \mathbb{N}$ such that $2\mu - m\alpha$ is dominant, but $2\mu - (m+1)\alpha$ is not. Indeed, we have

$$\langle 2\mu - m\alpha, \alpha^\vee \rangle \geq 0 \iff \langle \mu, \alpha^\vee \rangle \geq m,$$

and, for every other simple coroot $\beta^\vee \neq \alpha^\vee$ and every $n \in \mathbb{N}$,

$$\langle 2\mu - n\alpha, \beta^\vee \rangle = 2\langle \mu, \beta^\vee \rangle - n\langle \alpha, \beta^\vee \rangle \geq 2\langle \mu, \beta^\vee \rangle \geq 0,$$

since $\langle \alpha, \beta^\vee \rangle \leq 0$. Hence, $\langle 2\mu - (m+1)\alpha, \alpha^\vee \rangle < 0$ and so $m = \langle \mu, \alpha^\vee \rangle$.

(3) *The sets of roots R and coroots R^\vee :*

The Weyl group $W \subset \text{Aut}_{\mathbb{Z}}(X)$ is the finite subgroup generated by the reflections s_{α, α^\vee} associated to the pair $(\alpha, \alpha^\vee) \in \Delta \times \Delta^\vee$. Then $R = W \cdot \Delta$, i.e., the roots are given by the translates of the simple roots under W . Since $\text{Aut}_{\mathbb{Z}}(X^\vee) = \text{Aut}_{\mathbb{Z}}(X)^{\text{op}}$, the Weyl group W acts on X^\vee and $R^\vee = W \cdot \Delta^\vee$. This proves the lemma. \square

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AFFINE GRASSMANNIANS AND GEOMETRIC SATAKE EQUIVALENCES

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ABSTRACT. I extend the ramified geometric Satake equivalence of Zhu [34] from tamely ramified groups to include the case of general connected reductive groups. As a prerequisite I prove basic results on the geometry of affine flag varieties.

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INTRODUCTION

Let k be a separably closed field. Let G be a connected reductive group over the Laurent power series local field $F = k((t))$. The (*twisted*) *loop group* LG is the functor on the category of k -algebras

$$LG: R \longmapsto G(R((t))).$$

The loop group is representable by a strict ind-affine ind-group scheme over k , cf. Pappas-Rapoport [24]. Let \mathcal{G} be a smooth affine model of G over $\mathcal{O}_F = k[[t]]$, i.e. a smooth affine group scheme over \mathcal{O}_F with generic fiber G . The (*twisted*) *positive loop group* $L^+\mathcal{G}$ is the functor on the category of k -algebras

$$L^+\mathcal{G}: R \longmapsto \mathcal{G}(R[[t]]).$$

The positive loop group $L^+\mathcal{G}$ is representable by a reduced affine subgroup scheme of LG of infinite type over k . In general, the loop group LG is neither reduced nor connected, whereas the positive loop group $L^+\mathcal{G}$ is connected if the special fiber of \mathcal{G} is connected.

Our first main result is a basic structure theorem.

Theorem A. *A smooth affine model of G with geometrically connected fibers \mathcal{G} over \mathcal{O}_F is parahoric in the sense of Bruhat-Tits [7] if and only if the fpqc-quotient $LG/L^+\mathcal{G}$ is representable by an ind-proper ind-scheme. In this case, $LG/L^+\mathcal{G}$ is ind-projective.*

Theorem A should be viewed as the analogue of the characterization of parabolic subgroups in linear algebraic groups by the properness of their fppf-quotient. Note that the proof of the ind-projectivity of $LG/L^+\mathcal{G}$ for parahoric \mathcal{G} is implicitly contained in Pappas-Rapoport [24].

Let $\mathcal{B}(G, F)$ be the extended Bruhat-Tits building. Let $\mathfrak{a} \subset \mathcal{B}(G, F)$ be a facet, and let $\mathcal{G}_{\mathfrak{a}}$ be the corresponding parahoric group scheme. The fpqc-quotient $\mathcal{F}\ell_{\mathfrak{a}} = LG/L^+\mathcal{G}_{\mathfrak{a}}$ is called the affine flag variety associated with \mathfrak{a} , cf. [24]. The positive loop group $L^+\mathcal{G}_{\mathfrak{a}}$ acts from the left on $\mathcal{F}\ell_{\mathfrak{a}}$, and the action on each orbit factors through a smooth affine quotient of $L^+\mathcal{G}_{\mathfrak{a}}$ of finite type. This allows us to consider the category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}})$ of $L^+\mathcal{G}_{\mathfrak{a}}$ -equivariant ℓ -adic perverse sheaves on $\mathcal{F}\ell_{\mathfrak{a}}$. Here ℓ is a prime number different from the characteristic of the ground field k . Recall that a facet $\mathfrak{a} \subset \mathcal{B}(G, F)$ is called *special* if it is contained in some apartment such that each wall is parallel to a wall passing through \mathfrak{a} .

Our second main theorem characterizes special facets \mathfrak{a} in terms of the category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}})$.

Theorem B. *The following properties are equivalent.*

- i) The facet \mathfrak{a} is special.*
- ii) The stratification of $\mathcal{F}\ell_{\mathfrak{a}}$ in $L^+\mathcal{G}_{\mathfrak{a}}$ -orbits satisfies the parity property, i.e. in each connected component all strata are either even or odd dimensional.*
- iii) The category $P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}})$ is semi-simple.*

The implications $i) \Rightarrow ii) \Rightarrow iii)$ are due to Zhu [34] whereas the implication $iii) \Rightarrow i)$ seems to be new. In fact, the following properties are equivalent to Theorem B $i)$ - $iii)$ (cf. §3 below):

- iv) The special fiber of each global Schubert variety associated with \mathfrak{a} is irreducible.*
- v) The monodromy on Gaitsgory's nearby cycles functor associated with \mathfrak{a} vanishes.*
- vi) Each admissible set associated with \mathfrak{a} contains a unique maximal element.*

See §2 for the definition of global Schubert varieties and admissible sets associated with a facet, and §3 for the definition of Gaitsgory's nearby cycles functor in this context.

Let \mathfrak{a} be a special facet. The *ramified Satake category* $\text{Sat}_{\mathfrak{a}}$ associated with \mathfrak{a} is the category

$$\text{Sat}_{\mathfrak{a}} \stackrel{\text{def}}{=} P_{L^+\mathcal{G}_{\mathfrak{a}}}(\mathcal{F}\ell_{\mathfrak{a}}).$$

The ramified Satake category $\text{Sat}_{\mathfrak{a}}$ is semi-simple with simple objects as follows. Let A be a maximal F -split torus such that $\mathfrak{a} \subset \mathcal{A}(G, A, F)$ lies in the corresponding apartment. Since k is separably closed, G is quasi-split by Steinberg's Theorem. The centralizer $T = Z_G(A)$ is a maximal torus. Let B be a Borel subgroup containing T . Let $I = \text{Gal}(\bar{F}/F)$ denote the absolute Galois group. The group I acts on the cocharacter group $X_*(T)$, and we let $X_*(T)_I$ be the group of coinvariants. To every $\mu \in X_*(T)_I$, the Kottwitz morphism associates a k -point $t^\mu \cdot e_0$ in $\mathcal{F}\ell_{\mathfrak{a}}$, where e_0 denotes the base point. Let Y_μ be the reduced $L^+\mathcal{G}$ -orbit closure of $t^\mu \cdot e_0$. The scheme Y_μ is a projective variety over k which is in general not smooth. The reduced locus of $\mathcal{F}\ell_{\mathfrak{a}}$ has an ind-presentation

$$(\mathcal{F}\ell_{\mathfrak{a}})_{\text{red}} = \varinjlim_{\mu \in X_*(T)_I^\dagger} Y_\mu,$$

where $X_*(T)_I^\dagger$ is the image of the set of dominant cocharacters under the canonical projection $X_*(T) \rightarrow X_*(T)_I$. Then the simple objects of $\text{Sat}_{\mathfrak{a}}$ are the intersection complexes IC_μ of Y_μ , as μ ranges over $X_*(T)_I^\dagger$.

Recall that for every $\mathcal{A}_1, \mathcal{A}_2 \in \text{Sat}_{\mathfrak{a}}$, the *convolution product* $\mathcal{A}_1 \star \mathcal{A}_2$ is defined as an object in the bounded derived category of constructible ℓ -adic complexes, cf. Gaitsgory [10], Pappas-Zhu [26].

Fix a pinning of G preserved by I , and denote by \hat{G} the Langlands dual group over $\bar{\mathbb{Q}}_{\ell}$, i.e. the connected reductive group over $\bar{\mathbb{Q}}_{\ell}$ whose root datum is dual to the root datum of G . The Galois group I acts on \hat{G} via outer automorphisms, and we let \hat{G}^I be the fixed points. Then \hat{G}^I is a not necessarily connected reductive group over $\bar{\mathbb{Q}}_{\ell}$. Note that $X_*(T)_I = X^*(\hat{T}^I)$, and that for every $\mu \in X^*(\hat{T}^I)^+$, there exists a unique irreducible representation of \hat{G}^I of highest weight μ , cf. Appendix A for the definition of highest weight representations of \hat{G}^I . Let $\text{Rep}_{\bar{\mathbb{Q}}_{\ell}}(\hat{G}^I)$ be the category of algebraic representations of \hat{G}^I .

Our third main result describes $\text{Sat}_{\mathfrak{a}}$ as a tensor category.

Theorem C. *i) The category $\text{Sat}_{\mathfrak{a}}$ is stable under the convolution product \star , and the pair $(\text{Sat}_{\mathfrak{a}}, \star)$ admits a unique structure of a symmetric monoidal category such that the global cohomology functor*

$$\omega(-) \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} R^i \Gamma(\mathcal{F}l_{\mathfrak{a}}, -) : \text{Sat}_{\mathfrak{a}} \longrightarrow \text{Vec}_{\bar{\mathbb{Q}}_{\ell}}$$

is symmetric monoidal.

ii) The functor ω is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories

$$\begin{aligned} (\text{Sat}_{\mathfrak{a}}, \star) &\xrightarrow{\simeq} (\text{Rep}_{\bar{\mathbb{Q}}_{\ell}}(\hat{G}^I), \otimes), \\ \mathcal{A} &\longmapsto \omega(\mathcal{A}) \end{aligned}$$

which is uniquely determined up to inner automorphisms of \hat{G}^I by elements in \hat{T}^I by the property that $\omega(\text{IC}_{\mu})$ is the irreducible representation of highest weight μ .

We also prove a variant of Theorem C which includes Galois actions, and where k may be replaced by a finite field, cf. Theorem 4.11 below.

Theorem C is due to Zhu [34] in the case of tamely ramified groups. With Theorem B at hand, our method follows the method of [34] with minor modifications. The proof uses the unramified Satake equivalence as a black box which we will now recall briefly.

The affine Grassmannian Gr_G is the fpqc-sheaf associated with the functor on the category of F -algebras $\text{Gr}_G : R \mapsto G(R((z)))/G(R[[z]])$ for an additional formal variable z . Denote by $L_z^+ G : R \mapsto G(R[[z]])$ the positive loop group formed with respect to z . Then $L_z^+ G$ acts on Gr_G from the left. Fix \bar{F} the completion of a separable closure of F . The unramified Satake category $\text{Sat}_{G, \bar{F}}$ is the category

$$\text{Sat}_{G, \bar{F}} \stackrel{\text{def}}{=} P_{L_z^+ G_{\bar{F}}}(\text{Gr}_{G, \bar{F}}),$$

cf. [29]. The category $\text{Sat}_{G, \bar{F}}$ is equipped with the structure of a neutralized Tannakian category with respect to the convolution product \star . The *unramified Satake equivalence* is an equivalence of abelian tensor categories

$$(\text{Sat}_{G, \bar{F}}, \star) \simeq (\text{Rep}_{\bar{\mathbb{Q}}_{\ell}}(\hat{G}), \star),$$

which is uniquely determined up to inner automorphism by elements in \hat{T} , cf. Mirković-Vilonen [22], Richarz [29].

The main ingredient in the proof of Theorem C is the *BD-Grassmannian* $\text{Gr}_{\mathfrak{a}}$ associated with \mathfrak{a} (BD = Beilinson-Drinfeld) which is a strict ind-projective ind-scheme over \mathcal{O}_F such that in the generic (resp. special) fiber

$$\text{Gr}_{\mathfrak{a}, \eta} \simeq \text{Gr}_G \quad (\text{resp. } \text{Gr}_{\mathfrak{a}, s} \simeq \mathcal{F}l_{\mathfrak{a}}).$$

This allows us to consider Gaitsgory's nearby cycles functor $\Psi_{\mathfrak{a}} : \text{Sat}_{G, \bar{F}} \rightarrow \text{Sat}_{\mathfrak{a}}$ associated with $\text{Gr}_{\mathfrak{a}} \rightarrow \text{Spec}(\mathcal{O}_F)$. The symmetric monoidal structure with respect to \star on the category

$\text{Sat}_{G, \bar{F}}$ in the generic fiber of $\text{Gr}_{\mathfrak{a}}$ extends to the category $\text{Sat}_{\mathfrak{a}}$ in the special fiber of $\text{Gr}_{\mathfrak{a}}$. This equips $(\text{Sat}_{\mathfrak{a}}, \star)$ with a symmetric monoidal structure. Here, the key fact is the vanishing of the monodromy of $\Psi_{\mathfrak{a}}$ for special facets \mathfrak{a} , cf. item *v*) in the list below Theorem B. It is then not difficult to exhibit $(\text{Sat}_{\mathfrak{a}}, \star)$ as a Tannakian category with fiber functor ω . Theorem B *iii*) implies that the neutral component $\text{Aut}^*(\omega)^0$ of the \mathbb{Q}_{ℓ} -group of tensor automorphisms is reductive. In fact, the nearby cycles construction above realizes $\text{Aut}^*(\omega)$ as a subgroup of \hat{G} via the unramified Satake equivalence. This equivalence equips \hat{G} with a canonical pinning, and it is easy to identify $\text{Aut}^*(\omega) = \hat{G}^I$ as the subgroup of \hat{G} where I acts by pinning preserving automorphisms.

In [14], Haines and Rostami establish the Satake isomorphism for Hecke algebras of special parahoric subgroups. Theorem C may be seen as a geometrization of this isomorphism in the case that the facet \mathfrak{a} is *very special*. Note that in Theorem C above the residue field k was assumed to be separably closed, and hence the notion of special facets and very special facets coincide (cf. Definition 3.7 below). However, some additional input is needed to identify Theorem C as a geometrization of the isomorphism given in [14] in this case. The case of quasi-split connected reductive groups and special facets which are not necessarily very special will be addressed on another occasion [30].

Let us briefly explain the structure of the paper. §1 is devoted to the proof of Theorem A. In §2, we introduce the global affine Grassmannian associated with a facet, and define the global Schubert varieties. In §3, we prove Theorem B. In §4, we collect some facts from the unramified and ramified geometric Satake equivalences, and explain the proof of Theorem C including the case of wild ramification. Appendix A complements §4. Here we introduce highest weight representations for the group of fixed points in a split connected reductive group under pinning preserving automorphisms.

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Notation. For a complete discretely valued field F , we denote by \mathcal{O}_F the ring of integers with residue field k . We let \bar{F} be the completion of a fixed separable closure, and $\Gamma = \text{Gal}(\bar{F}/F)$ the absolute Galois group. The completion of the maximal unramified subextension of F is denoted \check{F} with ring of integers $\mathcal{O}_{\check{F}}$ and residue field \check{k} .

1. AFFINE GRASSMANNIANS

In §1.1 and 1.2, we collect some facts on affine Grassmannians from the literature, cf. [15], [21], [24], [26], [33]. In §1.3-1.5, we prove Theorem A from the introduction.

1.1. Affine flag varieties. Let k be either a finite or a separably closed field, and let G be a connected reductive group over the Laurent series local field $F = k((t))$. The (*twisted*) *loop group* LG is the group functor on the category of k -algebras

$$LG: R \longrightarrow G(R((t))).$$

The loop group LG is representable by a strict ind-affine ind-group scheme, cf. [24, §1]. Let \mathfrak{a} be a facet in the enlarged Bruhat-Tits building $\mathcal{B}(G, F)$. Denote by $\mathcal{G}_{\mathfrak{a}}$ the associated parahoric group scheme over \mathcal{O}_F , i.e. the neutral component of the unique smooth affine group scheme over \mathcal{O}_F such that the generic fiber is G , and such that the \mathcal{O}_F -points are the pointwise fixer of \mathfrak{a} in $G(F)$. The (*twisted*) *positive loop group* $L^+\mathcal{G}_{\mathfrak{a}}$ is the group functor on the category of k -algebras

$$L^+\mathcal{G}_{\mathfrak{a}}: R \longrightarrow \mathcal{G}_{\mathfrak{a}}(R[[t]]).$$

The positive loop group $L^+\mathcal{G}_\mathfrak{a}$ is representable by a reduced affine connected group scheme of infinite type over k . Then $L^+\mathcal{G}_\mathfrak{a} \subset LG$ is a closed subgroup scheme. The (partial) affine flag variety $\mathcal{F}\ell_\mathfrak{a}$ is the fpqc-sheaf on the category of affine k -algebras associated with the functor

$$\mathcal{F}\ell_\mathfrak{a}: R \longrightarrow LG(R)/L^+\mathcal{G}_\mathfrak{a}(R).$$

The affine flag variety $\mathcal{F}\ell_\mathfrak{a}$ is a strict ind-scheme of ind-finite type and separated over k , cf. [24, Theorem 1.4]. We explain some of its basic structure. For the rest of this subsection, let k be a separably closed field.

Connected components of $\mathcal{F}\ell_\mathfrak{a}$: In general, $\mathcal{F}\ell_\mathfrak{a}$ is not connected. The connected components are given as follows, cf. [24].

Let $\pi_1(G)$ be the algebraic fundamental group of G , cf. Borovoi [5]. The group $\pi_1(G)$ is a finitely generated abelian group, and can be defined as the quotient of the coweight lattice by the coroot lattice of $G_{\bar{F}}$. Since k is separably closed, $I = \text{Gal}(\bar{F}/F)$ is the inertia group. Then I acts on $\pi_1(G)$, and we denote by $\pi_1(G)_I$ the group of coinvariants. There is a functorial surjective group morphism $\kappa_G: LG(k) \rightarrow \pi_1(G)_I$, cf. Kottwitz [19, §7]. By [24, §2.a.2], the morphism κ_G extends to a locally constant morphism of ind-group schemes

$$\kappa_G: LG \longrightarrow \pi_1(G)_I,$$

and induces an isomorphism on the groups of connected components $\pi_0(LG) \simeq \pi_1(G)_I$. Since $L^+\mathcal{G}_\mathfrak{a}$ is connected, the morphism κ_G also defines a bijection $\pi_0(\mathcal{F}\ell_\mathfrak{a}) \simeq \pi_1(G)_I$.

Schubert varieties in $\mathcal{F}\ell_\mathfrak{a}$: The reduced $L^+\mathcal{G}_\mathfrak{a}$ -orbits in $\mathcal{F}\ell_\mathfrak{a}$ are called Schubert varieties. Their basic structure is as follows.

Let $A \subset G$ be a maximal F -split torus such that \mathfrak{a} is contained in the apartment $\mathcal{A} = \mathcal{A}(G, A, F)$ of $\mathcal{B}(G, F)$. Let $N = N_G(A)$ be the normalizer of A , and denote by $W = N(F)/T_1$ the Iwahori-Weyl group where $T_1 \subset T(F)$ is the unique parahoric subgroup, cf. [13].

Definition 1.1. For $w \in W$, the *Schubert variety Y_w associated with w* is the reduced $L^+\mathcal{G}_\mathfrak{a}$ -orbit closure

$$Y_w \stackrel{\text{def}}{=} \overline{L^+\mathcal{G}_\mathfrak{a} \cdot n_w \cdot e_0} \subset \mathcal{F}\ell_\mathfrak{a},$$

where n_w is a representative of w in $LG(k)$, and e_0 is the base point in $\mathcal{F}\ell_\mathfrak{a}$.

Let us justify the definition. The orbit map $L^+\mathcal{G}_\mathfrak{a} \rightarrow \mathcal{F}\ell_\mathfrak{a}$, $g \mapsto g \cdot n_w e_0$ factors through some smooth affine quotient of $L^+\mathcal{G}_\mathfrak{a}$ of finite type. The Schubert variety $Y_w \subset \mathcal{F}\ell_\mathfrak{a}$ is the scheme theoretic closure of this morphism, and hence a reduced separated scheme of finite type over k . Let \hat{Y}_w denote the $L^+\mathcal{G}_\mathfrak{a}$ -orbit of $n_w \cdot e_0$ in Y_w . Then \hat{Y}_w is a smooth open dense subscheme of Y_w . Since $L^+\mathcal{G}_\mathfrak{a}$ is connected, \hat{Y}_w is irreducible and so is Y_w . It follows that Y_w is a variety over k .

The Iwahori-Weyl group W acts on the apartment \mathcal{A} by affine transformations. Let \mathfrak{a}_C be an alcove containing \mathfrak{a} in its closure. The choice of \mathfrak{a}_C equips W with a quasi-Coxeter structure (l, \leq) , i.e. the simple reflections are the reflections at the walls meeting the closure of \mathfrak{a}_C . Let $W_\mathfrak{a} = N(F) \cap \mathcal{G}_\mathfrak{a}(\mathcal{O}_F)/T_1$ the subgroup of W associated with \mathfrak{a} , cf. [13]. The group $W_\mathfrak{a}$ identifies with the subgroup generated by the reflections at the walls passing through \mathfrak{a} . For an element $w \in W$, denote by $w^\mathfrak{a}$ the representative of minimal length in $w \cdot W_\mathfrak{a}$. For every $w \in W$, there is a unique representative ${}_\mathfrak{a}w^\mathfrak{a}$ of maximal length in the set

$$\{(w'ww'')^\mathfrak{a} \mid w', w'' \in W_\mathfrak{a}\},$$

cf. [28, Lemma 2.15]. Let ${}_\mathfrak{a}W^\mathfrak{a} \subset W$ be the subset of all $w \in W$ such that $w = {}_\mathfrak{a}w^\mathfrak{a}$. Then the canonical map ${}_\mathfrak{a}W^\mathfrak{a} \rightarrow W_\mathfrak{a} \backslash W / W_\mathfrak{a}$ is bijective. The Bruhat decomposition, cf. [13], implies that there is a set-theoretically disjoint union into locally closed strata,

$$(1.1) \quad \mathcal{F}\ell_\mathfrak{a} = \coprod_{w \in {}_\mathfrak{a}W^\mathfrak{a}} \hat{Y}_w.$$

Proposition 1.2. *Let $w \in {}_a W^a$.*

i) *The scheme \mathring{Y}_w is of dimension $l(w)$.*

ii) *The Schubert variety Y_w is a proper k -variety, and*

$$Y_w = \coprod_{v \leq w} \mathring{Y}_v,$$

where $v \in {}_a W^a$ and \leq denotes the Bruhat order on W .

Proof. Part i), and the orbit stratification in part ii) is proven in [29, Proposition 2.8]. Let us show that Y_w is proper. Note that the Iwahori $L^+ \mathcal{G}_{a_C}$ is a closed subgroup scheme of $L^+ \mathcal{G}_a$. After multiplication with some $\tau \in LG(k)$ contained in the stabilizer of $L^+ \mathcal{G}_{a_C}$, we may assume that Y_w is the reduced $L^+ \mathcal{G}_{a_C}$ -orbit closure of $n_w \cdot e_0$ contained in the neutral component $(\mathcal{F}_a)^0$. Let \tilde{w} be a reduced decomposition of w as a product of simple reflections, and let $\pi_{\tilde{w}}: D_{\tilde{w}} \rightarrow Y_w$ be the associated Demazure resolution, [24, §8]. In [*loc. cit.*] full flag varieties are considered, but the composition $D_{\tilde{w}} \rightarrow \mathcal{F}_{a_C} \rightarrow \mathcal{F}_a$ factors through Y_w defining $\pi_{\tilde{w}}$. By [24, Proposition 8.8], the scheme $D_{\tilde{w}}$ is projective, hence proper. Because $\pi_{\tilde{w}}$ is surjective and Y_w separated, it follows that Y_w is proper. \square

Corollary 1.3. *The strict ind-scheme \mathcal{F}_a is ind-proper. The reduced locus admits an ind-presentation by Schubert varieties*

$$(\mathcal{F}_a)_{\text{red}} = \varinjlim_w Y_w.$$

Proof. Properness can be checked on the underlying reduced subscheme. The corollary follows from (1.1), and Proposition 1.2. \square

1.2. BD-Grassmannians over curves. Let S be a scheme, and let X be a separated S -scheme. Let \mathcal{G} be a fpqc-sheaf of groups over X . For a S -scheme T , denote by $X_T = X \times_S T$ the fiber product. If $x: T \rightarrow X$ is a morphism of S -schemes, let $\Gamma_x \subset X_T$ be the graph.

Definition 1.4. The *BD-Grassmannian* $\text{Gr}(\mathcal{G}, X)$ is the contravariant functor on the category of S -schemes parametrizing isomorphism classes of triples (x, \mathcal{F}, α) with

$$\begin{cases} x: T \rightarrow X \text{ is a morphism of } S\text{-schemes;} \\ \mathcal{F} \text{ a right } \mathcal{G}_T\text{-torsor on } X_T; \\ \alpha: \mathcal{F}_{X_T \setminus \Gamma_x} \xrightarrow{\cong} \mathcal{F}^0|_{X_T \setminus \Gamma_x} \text{ a trivialization,} \end{cases}$$

where \mathcal{F}^0 denotes the trivial torsor. Two triples (x, \mathcal{F}, α) and $(x', \mathcal{F}', \alpha')$ are isomorphic if $x = x'$, and as torsors $\mathcal{F} \simeq \mathcal{F}'$ compatible with the trivializations α and α' .

If $\mathcal{G} = G \times_k X$ is constant, then we write $\text{Gr}(G, X) = \text{Gr}(\mathcal{G}, X)$.

Remark 1.5. For constant groups the BD-Grassmannian (=Beilinson-Drinfeld) is defined by Beilinson and Drinfeld in [3]. Note that in general the BD-Grassmannian $\text{Gr}(\mathcal{G}, X)$ is a special case of Heinloth's construction [15, §2 Example (2)].

There is the canonical projection $\text{Gr}(\mathcal{G}, X) \rightarrow X$, $(x, \mathcal{F}, \alpha) \mapsto x$. The construction is functorial in the following sense. If $\tau: \mathcal{G}' \rightarrow \mathcal{G}$ is a morphism of fpqc-sheaves of groups over X , then there is a morphism of functors

$$(1.2) \quad \begin{aligned} \tau_*: \text{Gr}(\mathcal{G}', X) &\longrightarrow \text{Gr}(\mathcal{G}, X) \\ (x, \mathcal{F}, \alpha) &\longmapsto (x, \tau_* \mathcal{F}, \tau_* \alpha). \end{aligned}$$

If $f: Y \rightarrow X$ is a morphism of S -schemes, then there is a morphism of functors

$$(1.3) \quad \begin{aligned} f^*: \text{Gr}(\mathcal{G}, X) \times_X Y &\longrightarrow \text{Gr}(\mathcal{G}_Y, Y) \\ ((x, \mathcal{F}, \alpha), y) &\longmapsto (y, \mathcal{F}_Y, \alpha_Y). \end{aligned}$$

For the rest of the subsection, let $S = \text{Spec}(k)$ be the spectrum of a field k , and let X be a smooth connected curve over k .

Lemma 1.6. *Let \mathcal{E} be a vector bundle on X . Then $\text{Gr}(\text{GL}(\mathcal{E}), X) \rightarrow X$ is representable by an ind-proper strict ind-scheme which is, Zariski locally on X , ind-projective.*

Proof. This follows from [29, Lemma 2.4]. \square

Lemma 1.7. *Let \mathcal{G} be an affine group scheme of finite type over X .*

i) If $\iota: \mathcal{P} \hookrightarrow \mathcal{G}$ is a closed immersion of group schemes such that the fppf-quotient $\mathcal{G}/\iota(\mathcal{P})$ is affine (resp. quasi-affine) over X , then the morphism $\iota_: \text{Gr}(\mathcal{P}, X) \rightarrow \text{Gr}(\mathcal{G}, X)$ is relatively representable by a closed (resp. quasi-compact) immersion.*

ii) If \mathcal{G} is flat over X , then $\text{Gr}(\mathcal{G}, X)$ is strict ind-representable of ind-finite type and separated over X .

Proof. Part i) is analogous to the proof of Levin [21, Theorem 3.3.7]. For part ii) choose any vector bundle \mathcal{E} on X together with a faithful representation $\rho: \mathcal{G} \hookrightarrow \text{GL}(\mathcal{E})$ such that the fppf-quotient $\text{GL}(\mathcal{E})/\rho(\mathcal{G})$ is representable by a quasi-affine scheme. The existence of the pair (\mathcal{E}, ρ) is proven in [15, §2 Example (1)]. Now ii) follows from i) applied to $\rho: \mathcal{G} \hookrightarrow \text{GL}(\mathcal{E})$ and Lemma 1.6. \square

If $T = \text{Spec}(R)$ is affine and $x \in X(T)$, let $\hat{\Gamma}_x = \text{Spec}(\hat{\mathcal{O}}_{X_T, x})$ be the spectrum of the completion of X_T along Γ_x , i.e. if we choose a local parameter t_x at x , which exists Zariski locally on T , then $\hat{\mathcal{O}}_{X_T, x} \simeq R[[t_x]]$. The graph $\Gamma_x \hookrightarrow \hat{\Gamma}_x$ is a closed subscheme, and we let $\hat{\Gamma}_x^\circ = \hat{\Gamma}_x \setminus \Gamma_x$ be its open complement.

Definition 1.8. i) The *global loop group* \mathcal{LG} is the functor on the category of k -algebras

$$\mathcal{LG}: R \longmapsto \{(x, g) \mid x \in X(R), g \in \mathcal{G}(\hat{\Gamma}_x^\circ)\}.$$

ii) The *global positive loop group* $\mathcal{L}^+\mathcal{G}$ is the functor on the category of k -algebras

$$\mathcal{L}^+\mathcal{G}: R \longmapsto \{(x, g) \mid x \in X(R), g \in \mathcal{G}(\hat{\Gamma}_x)\}.$$

There is the canonical projection $\mathcal{LG} \rightarrow X$ (resp. $\mathcal{L}^+\mathcal{G} \rightarrow X$), and the construction is functorial in (\mathcal{G}, X) in the obvious sense.

Remark 1.9. The connection to the loop groups from §1.1 is as follows. Let $x \in X(K)$ for any field extension K of k , and choose a local parameter t_x in x . We let $F_x = K((t_x))$, with ring of integers $\mathcal{O}_{F_x} = K[[t_x]]$. Then as functors on the category of K -algebras

$$(\mathcal{LG})_x = L\mathcal{G}_{F_x} \quad \text{and} \quad (\mathcal{L}^+\mathcal{G})_x = L^+\mathcal{G}_{\mathcal{O}_x}.$$

Let \mathcal{G} be an affine group scheme over X . For $i \geq 0$ let \mathcal{G}_i be the functor on the category of k -algebras

$$R \longmapsto \{(x, g) \mid x \in X(R), g \in \mathcal{G}(\Gamma_{x,i})\},$$

where $\Gamma_{x,i}$ denotes the i -th infinitesimal neighbourhood of $\Gamma_x \hookrightarrow X_R$. If $x \in X(R)$, then $\mathcal{G}_i \otimes_X R$ is the Weil restriction of $\mathcal{G} \times_{X_R} \Gamma_{x,i}$ along the finite flat morphism $\Gamma_{x,i} \rightarrow \text{Spec}(R)$. Hence, $\mathcal{G}_i \rightarrow X$ is representable by an affine group scheme by [4, §7.6 Proof of Theorem 4]. As i varies, the \mathcal{G}_i form an inverse system with affine transition maps. In particular, $\varprojlim_i \mathcal{G}_i$ is representable by an affine group scheme over X , and the canonical morphism of group functors

$$(1.4) \quad \mathcal{L}^+\mathcal{G} \xrightarrow{\simeq} \varprojlim_i \mathcal{G}_i$$

is an isomorphism.

Lemma 1.10. *Let \mathcal{G} be a flat affine group scheme of finite type over X .*

- i) *The functor $\mathcal{L}\mathcal{G}$ is representable by an ind-affine ind-group scheme over X .*
- ii) *The functor $\mathcal{L}^+\mathcal{G}$ is representable by a closed affine subgroup scheme of $\mathcal{L}\mathcal{G}$. If \mathcal{G} is smooth (resp. has geometrically connected fibers) over X , then $\mathcal{L}^+\mathcal{G}$ is reduced and flat (resp. has geometrically connected fibers) over X .*

Proof. The representability assertions are due to Heinloth [15, Proposition 2]. First assume that \mathcal{G} is smooth. Then, for all $i \geq 0$, the group schemes $\mathcal{G}_i \rightarrow X$ are smooth by [4, §7.6 Proposition 5]. This implies that $\mathcal{L}^+\mathcal{G}$ is reduced and flat over X by (1.4). If \mathcal{G} has geometrically connected fibers, then it follows easily by induction on i that $\mathcal{G}_i \rightarrow X$ has geometrically connected fiber. Again by (1.4), this implies that $\mathcal{L}^+\mathcal{G} \rightarrow X$ has geometrically connected fibers. \square

Let \mathcal{G} be a smooth affine group scheme over X . Let $\mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G}$ be the fpqc-quotient. We construct a morphism of fpqc-sheaves

$$\text{ev}: \text{Gr}(\mathcal{G}, X) \longrightarrow \mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G}$$

as follows. Let $(x, \mathcal{F}, \beta) \in \text{Gr}(\mathcal{G}, X)(T)$ with T affine. First assume that $\mathcal{F}|_{\Gamma_x}$ is trivial, and choose some trivialization $\beta: \mathcal{F}|_{\Gamma_x} \simeq \mathcal{F}^0|_{\Gamma_x}$. Then the class of $(\alpha \circ \beta^{-1})(1)$ in $\mathcal{G}(\hat{\Gamma}_x^o)/\mathcal{G}(\hat{\Gamma}_x)$ is independent of β , and defines ev in this case. To construct ev in general, it is enough to observe that if \mathcal{F} is torsor on the Laurent series ring $R[[t]]$ under a smooth group scheme, then $\mathcal{F} \otimes_{R[[t]]} R'[[t]]$ is trivial for some $R \rightarrow R'$ fpqc. Indeed, $\mathcal{F}|_{t=0}$ is trivial over a faithfully flat extension $R \rightarrow R'$ giving rise to a section $s \in \mathcal{F}(R')$. By the smoothness, s extends to a formal section over $\text{Spf}(R'[[t]])$ which is algebraic by Grothendieck's algebraization theorem. Hence, $\mathcal{F}|_{R'[[t]]}$ is trivial.

Lemma 1.11. *Let \mathcal{G} be a smooth affine group scheme over X . The morphism $\text{ev}: \text{Gr}(\mathcal{G}, X) \rightarrow \mathcal{L}\mathcal{G}/\mathcal{L}^+\mathcal{G}$ is an isomorphism of fpqc-sheaves which is functorial in \mathcal{G} and X .*

Proof. Because $\text{Gr}(\mathcal{G}, X)$ is of ind-finite type, we may test on noetherian k -algebras that ev is an isomorphism. This follows from Proposition 4 in [15], cf. Remark 1.5. The functorialities are easy to check. \square

By Lemma 1.11, there is an action of the loop group

$$(1.5) \quad \mathcal{L}\mathcal{G} \times_X \text{Gr}(\mathcal{G}, X) \longrightarrow \text{Gr}(\mathcal{G}, X).$$

If we restrict the action to $\mathcal{L}^+\mathcal{G}$, then it factors on each orbit through a smooth affine quotient of $\mathcal{L}^+\mathcal{G}$ in the following sense.

Lemma 1.12. *Let \mathcal{G} be a smooth affine group scheme over X . Let $\mathcal{L}^+\mathcal{G} \simeq \varprojlim_i \mathcal{G}_i$ be as in (1.4). Let T be a quasi-compact X -scheme, and let $\mu: T \rightarrow \text{Gr}(\mathcal{G}, X)$ be a morphism over X . Then the T -morphism $\mathcal{L}^+\mathcal{G}_T \rightarrow \text{Gr}(\mathcal{G}, X)_T$, $g \mapsto g\mu$ factors through some $\mathcal{G}_{i,T}$.*

Proof. Since T is quasi-compact, the morphism μ factors through some closed subscheme of $\text{Gr}(\mathcal{G}, X)$. Now the lemma follows by reduction to the case of GL_n as in [29, Cor. 2.7]. \square

Corollary 1.13. *Let \mathcal{G} be a smooth affine group scheme over X .*

- i) *If $x \in X(K)$ for any extension K of k , let $F_x = K((t_x))$ and $\mathcal{O}_{F_x} = K[[t_x]]$ where t_x is a local parameter at x . The fiber of (1.5) over x is isomorphic to the action morphism*

$$L\mathcal{G}_{F_x} \times_{F_x} L\mathcal{G}_{F_x}/L^+\mathcal{G}_{\mathcal{O}_x} \longrightarrow L\mathcal{G}_{F_x}/L^+\mathcal{G}_{\mathcal{O}_x},$$

functorial in \mathcal{G} .

- ii) *If $f: Y \rightarrow X$ is étale, then $f^*: \text{Gr}(\mathcal{G}, X) \times_X Y \rightarrow \text{Gr}(\mathcal{G}_Y, Y)$ is an isomorphism.*

Proof. Part i) is a consequence of Lemma 1.11. Part ii) is proven in [33, Lemma 3.2]. \square

Remark 1.14. If \mathcal{G} is parahoric, then the fibers of $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ are affine flag varieties, cf. §1.1 and Remark 1.9.

Lemma 1.15. *Let \mathcal{G} be a smooth affine group scheme over X .*

- i) If $\mathcal{G} = G \times_k X$ is constant, then $\mathrm{Gr}(G, X) \rightarrow X$ is Zariski locally on X constant.*
- ii) If \mathcal{G} is connected reductive, then $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ is ind-proper.*

Proof. The Grassmannian $\mathrm{Gr}(G, X) \rightarrow X$ is constant for $X = \mathbb{A}_k^1$, cf. [29, Remark 2.19]. But Zariski locally on X , there exists a finite étale morphism $\tilde{X} \rightarrow \mathbb{A}_k^1$ which implies i) by Corollary 1.13 ii). For part ii), if \mathcal{G} is split reductive, then the existence of Chevalley models shows that there is a faithful representation $\mathcal{G} \hookrightarrow \mathrm{GL}_{n,X}$ such that the fppf-quotient $\mathrm{GL}_{n,X}/\mathcal{G}$ is affine. Use Lemma 1.7 i), and the ind-properness of $\mathrm{Gr}(\mathrm{GL}_n, X) \rightarrow X$ to conclude that $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ is ind-proper in this case. The lemma follows from the fact that every connected reductive group is split étale locally, and from Corollary 1.13 ii). \square

The following general lemma is needed in §1.4.

Lemma 1.16. *Let $\iota: \tilde{X} \rightarrow X$ be a finite flat surjection of smooth connected curves over k , and let \tilde{G} be an affine group scheme of finite type over \tilde{X} . Then $\mathcal{G} = \mathrm{Res}_{\tilde{X}/X}(\tilde{G})$ is an affine group scheme of finite type, and the canonical morphism of functors over X*

$$\mathrm{Res}_{\tilde{X}/X}(\mathrm{Gr}(\tilde{G}, \tilde{X})) \longrightarrow \mathrm{Gr}(\mathcal{G}, X)$$

is an isomorphism.

Proof. By [4, §7.6 Proof of Theorem 4], the sheaf of groups \mathcal{G} is affine and of finite type. The lemma follows from fpqc-descent for affine schemes, cf. [21, §2.6]. \square

1.3. Geometric characterization of parahoric groups. Let X be a smooth connected curve over a field k which is either finite or separably closed. Let $|X|$ be the set of closed points in X . For any $x \in |X|$, we denote by \mathcal{O}_x the completed local ring of X in x , and by F_x its field of fractions.

Definition 1.17. A smooth affine group scheme \mathcal{G} over X with geometrically connected fibers is called *parahoric* if its generic fiber is reductive and for all $x \in |X|$, the group $\mathcal{G}_{\mathcal{O}_x}$ is a parahoric group scheme in the sense of [7].

Theorem 1.18. *Let \mathcal{G} be a smooth affine group scheme over X with geometrically connected fibers.*

- i) Then \mathcal{G} is parahoric if and only if the fibers of $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ are ind-proper.*
- ii) In case i), $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ is ind-proper, and Zariski locally on X ind-projective.*

The proof of ii) is explained in §1.5 below, based on §1.4, in which we study the case of tori.

Proof of Theorem 1.18 i). Let \mathcal{G} be parahoric group scheme over the curve X . By Corollary 1.13, the generic fiber $\mathrm{Gr}(\mathcal{G}, X)_{\eta}$ is the affine Grassmannian associated with the reductive group \mathcal{G}_{η} , and hence ind-proper. Let $x \in |X|$. Then $\mathcal{F}\ell = \mathrm{Gr}(\mathcal{G}, X)_x$ is the affine flag variety associated with the parahoric group scheme $\mathcal{G}_{\mathcal{O}_x}$ which is ind-proper, cf. Corollary 1.3.

Conversely assume that the fibers of $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ are ind-proper. In particular, $\mathrm{Gr}(\mathcal{G}, X)_{\bar{\eta}}$ is ind-proper where $\bar{\eta}$ denotes a geometric generic point. But $\mathrm{Gr}(\mathcal{G}, X)_{\bar{\eta}}$ is the affine Grassmannian associated with the linear algebraic group $\mathcal{G}_{\bar{\eta}}$, and it follows from the argument in [10, Appendix] that $\mathrm{Gr}(\mathcal{G}, X)_{\bar{\eta}}$ is ind-proper if and only if $\mathcal{G}_{\bar{\eta}}$ is reductive. Fix $x \in |X|$. We need to show that $\mathcal{G}_{\mathcal{O}_x}$ is a parahoric group scheme. We may assume that k is separably closed. Let $\mathcal{O} = \mathcal{O}_x$, $F = F_x$, and choose a uniformizer $t = t_x$ in \mathcal{O} . Let $G = \mathcal{G}_F$ and $\mathcal{G} = \mathcal{G}_{\mathcal{O}}$. The subgroup $\mathcal{G}(\mathcal{O}) \subset G(F)$ is bounded for the t -adic topology, and hence is contained in some maximal bounded subgroup $\mathcal{G}(\mathcal{O})_{\max}$. By [32, §3.2], the group $\mathcal{G}(\mathcal{O})_{\max} = \mathcal{G}_a(\mathcal{O})$ is the

stabilizer of some point a of the Bruhat-Tits building. By [7, Proposition 1.7.6], the inclusion $\mathcal{G}(\mathcal{O}) \subset \mathcal{G}_a(\mathcal{O})$ prolongs into a morphism $\tau: \mathcal{G} \rightarrow \mathcal{G}_a$ of group schemes over \mathcal{O} . Since the special fiber of \mathcal{G} is connected, we may replace \mathcal{G}_a by its neutral component, and hence assume that \mathcal{G}_a is the parahoric group scheme associated with the facet containing a . Consider the induced morphism

$$\tau_*: LG/L^+\mathcal{G} \longrightarrow LG/L^+\mathcal{G}_a.$$

The fiber over the base point $e_{\mathcal{G}_a}$ in $LG/L^+\mathcal{G}_a$ is the fpqc-quotient $L^+\mathcal{G}_a/L^+\mathcal{G}$, and this is representable by an ind-proper ind-scheme since $LG/L^+\mathcal{G}$ is ind-proper by assumption. Let $\bar{\mathcal{G}}_a$ be the maximal reductive quotient of the special fiber $\mathcal{G}_a \otimes k$, and let $\bar{\mathcal{G}}'$ be the scheme theoretic image of $L^+\mathcal{G}$ under the morphism $L^+\mathcal{G}_a \rightarrow \mathcal{G}_a \otimes k \rightarrow \bar{\mathcal{G}}_a$, $t \mapsto 0$. The quotient $\bar{\mathcal{G}}_a/\bar{\mathcal{G}}'$ is representable by a separated scheme of finite type over k , and the morphism

$$L^+\mathcal{G}_a/L^+\mathcal{G} \longrightarrow \bar{\mathcal{G}}_a/\bar{\mathcal{G}}'$$

is surjective. Hence $\bar{\mathcal{G}}_a/\bar{\mathcal{G}}'$ is proper, i.e. $\bar{\mathcal{G}}'$ is a parabolic subgroup of $\bar{\mathcal{G}}_a$. The preimage of $\bar{\mathcal{G}}'(k)$ under the reduction $\mathcal{G}_a(\mathcal{O}) \rightarrow \bar{\mathcal{G}}_a(k)$ is by [32, 3.5.1] a parahoric subgroup of $G(F)$ associated with some facet \mathfrak{a} . Let $\mathcal{G}_{\mathfrak{a}}$ be the corresponding parahoric group scheme. The morphism $\mathcal{G} \rightarrow \mathcal{G}_a$ factorizes as $\mathcal{G} \rightarrow \mathcal{G}_{\mathfrak{a}} \rightarrow \mathcal{G}_a$. For $i \geq 0$, let

$$\mathcal{G}_i = \text{Res}_{(\mathcal{O}/t^{i+1})/k}(\mathcal{G} \otimes \mathcal{O}/t^{i+1})$$

be the Weil restriction. Then \mathcal{G}_i is a connected smooth affine group scheme over k , and $L^+\mathcal{G} \simeq \varprojlim_i \mathcal{G}_i$. Let $\mathcal{G}_{\mathfrak{a},i}$ be defined analogously. One shows as above that $\mathcal{G}_{\mathfrak{a},i}/\mathcal{G}'_i$ is representable by a connected proper scheme. Because the kernel of $L^+\mathcal{G}_{\mathfrak{a}} \rightarrow \bar{\mathcal{G}}_{\mathfrak{a}}$ is pro-unipotent, it follows that $\mathcal{G}_{\mathfrak{a},i} = \mathcal{G}'_i$ for all $i \geq 0$. Hence, the morphism $\mathcal{G}(\mathcal{O}/t^{i+1}) \rightarrow \mathcal{G}_{\mathfrak{a}}(\mathcal{O}/t^{i+1})$ is surjective, and the kernel is finite by dimension reasons. Using Mittag-Leffler we see that $\mathcal{G}(\mathcal{O}) = \mathcal{G}_{\mathfrak{a}}(\mathcal{O})$ is parahoric. \square

1.4. The case of tori. In this subsection, we assume that the field k is separably closed. Let \mathcal{T} be a smooth affine group scheme with geometrically connected fibers over X . If the generic fiber \mathcal{T}_{η} is a torus, then \mathcal{T} is a parahoric group scheme if and only if \mathcal{T} is the connected lft-Nerón model of \mathcal{T}_{η} , cf. [24].

Lemma 1.19. *Let \mathcal{T} be a parahoric group scheme over X . Then the generic fiber \mathcal{T}_{η} is a torus if and only if $\text{Gr}(\mathcal{T}, X) \rightarrow X$ is ind-finite.*

Proof. If \mathcal{T}_{η} is not a torus, then $\text{Gr}(\mathcal{T}, X)_{\bar{\eta}}$ is not ind-finite. Conversely assume that \mathcal{T}_{η} is a torus. The Kottwitz morphism implies that $\text{Gr}(\mathcal{T}, X) \rightarrow X$ is ind-quasi-finite, and it is enough to show the ind-properness.

There is a non-empty open subset $U \subset X$ such that $\mathcal{T} \times_X U$ is reductive which implies the ind-properness of $\text{Gr}(\mathcal{T}, X)|_U \rightarrow U$. For $x \in |X|$, let \mathcal{O}_x be the complete local ring at x with fraction field F_x . By fpqc-descent we are reduced to showing that $\text{Gr}(\mathcal{T}, X) \otimes \mathcal{O}_x$ is ind-finite for every $x \in |X|$. Fix $x \in |X|$.

The induced case. Assume first that $\mathcal{T} \simeq \prod_{i=1}^k \text{Res}_{X_i/X}(\mathbb{G}_m)^{n_i}$ where $X_i \rightarrow X$ are finite flat generically étale¹ surjections of smooth connected curves. By Lemma 1.16, we are reduced to considering the case of

$$\text{Res}_{\bar{X}/X}(\text{Gr}(\mathbb{G}_m, \tilde{X})).$$

Then on complete local rings

$$\mathcal{O}_{\bar{X}} \otimes \mathcal{O}_x \simeq \prod_{\bar{x} \rightarrow x} \mathcal{O}_{\bar{x}}$$

because the field extension on the generic points is separable. Since Weil restriction is compatible with base change, we are reduced to proving that $\text{Res}_{\mathcal{O}_{\bar{x}}/\mathcal{O}_x}(\text{Gr}(\mathbb{G}_m, \tilde{X}) \otimes \mathcal{O}_{\bar{x}})$ is

¹Otherwise, the generic fiber of \mathcal{T} is not reductive.

ind-finite. This follows from Lemma 1.15 i) using that the reduced locus of $\mathrm{Gr}(\mathbb{G}_m, \tilde{X}) \otimes \mathcal{O}_{\tilde{x}}$ is the constant scheme $\underline{\mathbb{Z}}$, and the tautological equality $\mathrm{Res}_{\mathcal{O}_{\tilde{x}}/\mathcal{O}_x}(\mathrm{Spec}(\mathcal{O}_{\tilde{x}})) = \mathrm{Spec}(\mathcal{O}_x)$.

In the general case, after replacing X by a Zariski neighbourhood of x there exists a morphism of parahoric group schemes

$$(1.6) \quad \pi: \mathcal{T}' \longrightarrow \mathcal{T},$$

such that $\mathcal{T}' \simeq \prod_{i=1}^k \mathrm{Res}_{X_i/X}(\mathbb{G}_m)^{n_i}$ is induced as above, and such that $\pi_*: \mathcal{Y}' \rightarrow \mathcal{Y}$ is surjective where $\mathcal{Y} = \mathrm{Gr}(\mathcal{T}, X) \otimes \mathcal{O}_x$ (resp. $\mathcal{Y}' = \mathrm{Gr}(\mathcal{T}', X) \otimes \mathcal{O}_x$). The morphism (1.6) is constructed as follows. By [8], there exists an exact sequence of tori $1 \rightarrow T'' \rightarrow T' \rightarrow \mathcal{T}'_\eta \rightarrow 1$ over η where T' is induced and T'' is flasque. As X_i we take the normalization of X in the field extensions η_i/η defining T' . This allows to define T' in a Zariski neighbourhood of x such that $\mathcal{T}'_\eta = T'$. Then \mathcal{T}' is the connected component of the lft-Nerón model of \mathcal{T}'_η , i.e. \mathcal{T}' is parahoric. Using the Nerón mapping property for \mathcal{T} , the group morphism on generic fibers $\mathcal{T}'_\eta \rightarrow \mathcal{T}_\eta$ extends to a group morphism $\pi: \mathcal{T}' \rightarrow \mathcal{T}$. Then $\pi_*: \mathcal{Y}' \rightarrow \mathcal{Y}$ is surjective over \bar{F}_x because $\mathcal{T}'_{\bar{F}_x} \rightarrow \mathcal{T}_{\bar{F}_x}$ is a surjection of split tori, and π_* is surjective over k by [19, §7 (7.2.5)]. Note that $\pi_* \otimes k$ identifies on reduced loci with the surjection in the lower row of [loc. cit.]. This shows the existence of (1.6). Since \mathcal{Y}' is ind-proper and \mathcal{Y} is separated, \mathcal{Y} is also ind-proper. The lemma follows. \square

Let \mathcal{T} be a parahoric group scheme whose generic fiber is a torus. Fix $x \in |X|$, and let $\mathrm{Gr}_{\mathcal{T}} = \mathrm{Gr}(\mathcal{T}, X) \otimes \mathcal{O}_x$. Since $\mathrm{Gr}_{\mathcal{T}}$ is ind-proper, there is a specialization map

$$(1.7) \quad \mathrm{sp}: \mathrm{Gr}_{\mathcal{T}}(\bar{F}) \longrightarrow \mathrm{Gr}_{\mathcal{T}}(k).$$

Note that $\mathrm{Gr}_{\mathcal{T}}$ is a sheaf of groups because \mathcal{T} is commutative, and that sp is a group morphism. The generic fiber $\mathrm{Gr}_{\mathcal{T}, \bar{F}}$ is by Corollary 2.2 equal to the affine Grassmannian associated with the split torus $\mathcal{T}_{\bar{F}}$, and hence $\mathrm{Gr}_{\mathcal{T}}(\bar{F}) \simeq X_*(T)$ as groups. The Kottwitz morphism $\kappa_{\mathcal{T}}: \mathrm{Gr}_{\mathcal{T}}(k) \rightarrow X_*(T)_I$ is an isomorphism, cf. [19, §7]. The following lemma is proven in [33, Proposition 3.4] in the tamely ramified case.

Lemma 1.20. *There is a commutative diagram of abelian groups*

$$\begin{array}{ccc} \mathrm{Gr}_{\mathcal{T}}(\bar{F}) & \xrightarrow{\mathrm{sp}} & \mathrm{Gr}_{\mathcal{T}}(k) \\ \downarrow \simeq & & \kappa_{\mathcal{T}} \downarrow \simeq \\ X_*(T) & \longrightarrow & X_*(T)_I \end{array}$$

where $X_*(T) \rightarrow X_*(T)_I$ is the canonical projection.

Proof. We show that $X_*(T) \simeq \mathrm{Gr}_{\mathcal{T}}(\bar{F}) \rightarrow \mathrm{Gr}_{\mathcal{T}}(k) \simeq X_*(T)_I$ is the canonical projection. Let $T'' \rightarrow T' \rightarrow T \rightarrow 1$ be a resolution of T by induced tori T', T'' as in [19, §7 (7.2.5)]. Using the argument in the proof of Lemma 1.19, we obtain an exact sequence $\mathrm{Gr}_{\mathcal{T}''} \rightarrow \mathrm{Gr}_{\mathcal{T}'} \rightarrow \mathrm{Gr}_{\mathcal{T}} \rightarrow 1$ of sheaf of groups over \mathcal{O}_x . This gives a commutative diagram of abelian groups

$$\begin{array}{ccccccc} \mathrm{Gr}_{\mathcal{T}''}(\bar{F}) & \longrightarrow & \mathrm{Gr}_{\mathcal{T}'}(\bar{F}) & \longrightarrow & \mathrm{Gr}_{\mathcal{T}}(\bar{F}) & \longrightarrow & 0 \\ \downarrow \mathrm{sp} & & \downarrow \mathrm{sp} & & \downarrow \mathrm{sp} & & \\ \mathrm{Gr}_{\mathcal{T}''}(k) & \longrightarrow & \mathrm{Gr}_{\mathcal{T}'}(k) & \longrightarrow & \mathrm{Gr}_{\mathcal{T}}(k) & \longrightarrow & 0 \end{array}$$

with exact rows, and we may reduce to the case that T is induced. Let $A \subset T$ be the maximal split subtorus. Since A is defined over k , we see that $\mathrm{Gr}_{A, \mathrm{red}} \simeq \underline{X_*(A)}$ is the constant group scheme over \mathcal{O}_x associated with $X_*(A)$, and hence sp is just the identity. Consider the

commutative diagram of abelian groups

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Gr}_A(\bar{F}) & \longrightarrow & \mathrm{Gr}_T(\bar{F}) \\ & & \downarrow \mathrm{sp} & & \downarrow \mathrm{sp} \\ 0 & \longrightarrow & \mathrm{Gr}_A(k) & \longrightarrow & \mathrm{Gr}_T(k). \end{array}$$

with exact rows. Note that $\mathrm{Gr}_A(k)$, $\mathrm{Gr}_T(k)$ have the same rank. Hence, the composition $X_*(T)_{\mathbb{Q}} \simeq \mathrm{Gr}_T(\bar{F})_{\mathbb{Q}} \rightarrow \mathrm{Gr}_T(k)_{\mathbb{Q}} \simeq X_*(T)_{\mathbb{Q},I}$ rationally is the canonical projection. The lemma follows from the fact that $\mathrm{Gr}_T(k)$ is torsionfree because T is induced. \square

1.5. Proof of the ind-projectivity. Let us prove Theorem 1.18 ii). Let \mathcal{G} be a parahoric group scheme over X . Then $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ is fiberwise ind-proper, and we need to show that it is ind-proper and, Zariski locally on X , ind-projective. Let $\iota: \mathcal{G} \rightarrow \mathrm{GL}(\mathcal{E})$ be a faithful representation such that $\mathrm{GL}(\mathcal{E})/\mathcal{G}$ is quasi-affine (cf. [15, §2 Example (1)]), where \mathcal{E} is some vector bundle on X . Then $\iota_*: \mathrm{Gr}(\mathcal{G}, X) \rightarrow \mathrm{Gr}(\mathrm{GL}(\mathcal{E}), X)$ is representable by an immersion, cf. Lemma 1.7 ii). It is enough to prove that $\mathrm{Gr}(\mathcal{G}, X) \rightarrow X$ is ind-proper. Since \mathcal{G}_η is reductive, there is a non-empty open subset $U \subset X$ such that \mathcal{G}_U is reductive. Lemma 1.15 ii) shows that $\mathrm{Gr}(\mathcal{G}, X)|_U \rightarrow U$ is ind-proper. By fpqc-descent, we are reduced to proving that $\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O}_x$ is ind-proper for every $x \in |X|$. We may assume that k is separably closed. Fix $x \in |X|$, and let $\mathcal{O} = \mathcal{O}_x$ and $F = F_x$. We claim that the reduced locus of $\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O}$ can be written as

$$(1.8) \quad (\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O})_{\mathrm{red}} = \varinjlim_{\gamma \in J} M_\gamma,$$

where M_γ are closed subschemes with the following properties: The M_γ are separated schemes of finite type over \mathcal{O} such that $M_\gamma \rightarrow \mathrm{Spec}(\mathcal{O})$ is surjective flat, and the generic fiber $M_\gamma \otimes F$ is connected. Since $\mathrm{Gr}(\mathcal{G}, X)$ is fiberwise ind-proper, Lemma 1.21 below reduces us to constructing the ind-presentation (1.8). Let J be the set of Galois orbits of $(\mathcal{L}^+\mathcal{G})_{\bar{F}}$ -orbits in $\mathrm{Gr}(\mathcal{G}, X) \otimes \bar{F}$. Then every $\gamma \in J$ defines a connected closed reduced subscheme $M_{\gamma,F}$ of $\mathrm{Gr}(\mathcal{G}, X) \otimes F$. Let M_γ be the scheme theoretic closure of $M_{\gamma,F}$ in $\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O}$. Then M_γ is a flat reduced $(\mathcal{L}^+\mathcal{G})_{\mathcal{O}}$ -equivariant closed subscheme of $\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O}$ (use Lemma 1.12). Let $\mathcal{T}_\eta \subset \mathcal{G}_\eta$ be a maximal torus, and denote by \mathcal{T} the scheme theoretic closure in \mathcal{G} . Then \mathcal{T} is a parahoric group scheme over X . By Lemma 1.19, $\mathrm{Gr}(\mathcal{T}, X)$ is ind-proper, and hence $\mathrm{Gr}(\mathcal{T}, X) \rightarrow \mathrm{Gr}(\mathcal{G}, X)$ is a closed immersion. The ind-presentation (1.8) follows from Lemma 1.20 noting that the affine flag variety in the special fiber of $\mathrm{Gr}(\mathcal{G}, X) \otimes \mathcal{O}$ is covered by the orbit closures of the translation elements. This proves Theorem 1.18 ii). \square

In the proof above we used the following lemma, a special case of [EGA IV, 15.7.10] under the hypothesis that the geometric fibers are connected. If the base is a complete discrete valuation ring, the hypothesis on the fibers can be weakened.

Lemma 1.21. *Let Y be a separated scheme of finite type over a complete discrete valuation ring \mathcal{O} . Assume that $Y \rightarrow \mathrm{Spec}(\mathcal{O})$ is surjective flat, and that the generic fiber Y_η is connected. Then Y is proper if and only if the fibers Y_η and Y_s are proper.*

Proof. Let Y_η and Y_s be proper. Note that by assumption both are non-empty. To prove properness we may assume that Y is reduced. Let $\iota: Y \rightarrow \bar{Y}$ be the Nagata compactification over \mathcal{O} , i.e. \bar{Y} is proper over \mathcal{O} , and ι is an open immersion. Now replace \bar{Y} by the scheme theoretic closure of $\iota(Y)$ in \bar{Y} . We claim that the open immersion $\iota: Y \rightarrow \bar{Y}$ is an isomorphism. Because Y_η is proper, ι_η is an isomorphism onto a connected component of \bar{Y}_η . But since Y is flat, the generic fiber Y_η is open dense in Y and hence, $\iota(Y_\eta)$ is open dense in \bar{Y} . It follows that ι_η is an isomorphism. Now since \bar{Y}_η is open dense in \bar{Y} , and \bar{Y} is reduced, it follows that $\pi: \bar{Y} \rightarrow \mathrm{Spec}(\mathcal{O})$ is flat. Hence, $\pi_*\mathcal{O}_{\bar{Y}}$ is a finite free \mathcal{O} -module of rank 1 because \bar{Y}_η is connected. By proper base change we have $\dim(H^0(\bar{Y}_s, \mathcal{O}_{\bar{Y}_s})) = 1$, and hence \bar{Y}_s is connected.

It follows that $\iota_s: Y_s \rightarrow \bar{Y}_s$ is an isomorphism because it is open and proper, $Y_s \neq \emptyset$ and \bar{Y}_s is connected. All in all, ι is a fiberwise isomorphism between flat schemes. The lemma follows. \square

2. THE BD-GRASSMANNIAN ASSOCIATED WITH A FACET

We define the global Schubert varieties which may be seen as analogues of local models in equal characteristic, cf. [25]. These are introduced by Zhu [33] in the tamely ramified case. The results of this paragraph are used in the proof of Theorem B of the introduction in the next section.

Let k be either a finite or separably closed field, and let G be a connected reductive group over the Laurent power series field $F = k((t))$. Let \mathfrak{a} be a facet in the extended Bruhat-Tits building $\mathcal{B}(G, F)$, and denote by $\mathcal{G} = \mathcal{G}_{\mathfrak{a}}$ the associated parahoric group scheme over $\mathcal{O}_F = k[[t]]$. Then \mathcal{G} is a smooth affine group scheme with geometrically connected fibers. The group scheme \mathcal{G} is already defined over some smooth connected pointed curve (X, x) with $\mathcal{O}_x = \mathcal{O}_F$, and we denote by \mathcal{G}_X the extension. Let \bar{F} be the completion of a separable closure of F with valuation subring $\mathcal{O}_{\bar{F}}$. Let $S = \text{Spec}(\mathcal{O}_F)$, $\bar{S} = \text{Spec}(\mathcal{O}_{\bar{F}})$ with generic points $\eta, \bar{\eta}$ and special points s, \bar{s} . This leads to the 6-tuple $(S, \bar{S}, \eta, \bar{\eta}, s, \bar{s})$.

Definition 2.1. i) The *global (resp. global positive) loop group $\mathcal{L}\mathcal{G}$ (resp. $\mathcal{L}^+\mathcal{G}$) associated with \mathfrak{a}* is the ind-group scheme (resp. group scheme) over S

$$\mathcal{L}\mathcal{G} \stackrel{\text{def}}{=} \mathcal{L}\mathcal{G}_X \times_X S \quad (\text{resp. } \mathcal{L}^+\mathcal{G} \stackrel{\text{def}}{=} \mathcal{L}^+\mathcal{G}_X \times_X S).$$

ii) The *BD-Grassmannian $\text{Gr}_{\mathfrak{a}}$ associated with \mathfrak{a}* is the ind-scheme over S

$$\text{Gr}_{\mathfrak{a}} \stackrel{\text{def}}{=} \text{Gr}(\mathcal{G}_X, X) \times_X S.$$

Note that the definitions do not depend on the choice of X . There is a left action

$$(2.1) \quad \mathcal{L}\mathcal{G} \times_S \text{Gr}_{\mathfrak{a}} \longrightarrow \text{Gr}_{\mathfrak{a}}.$$

Let us discuss the generic and the special fiber of (2.1). Let $L_z G$ (resp. $L_z^+ G$) be the functor on the category of F -algebras $L_z G: R \mapsto G(R((z)))$ (resp. $L_z^+ G: R \mapsto G(\llbracket z \rrbracket)$) where z is an additional variable. The *affine Grassmannian* Gr_G is the fpqc-quotient $\text{Gr}_G = L_z G / L_z^+ G$. There is a left action

$$L_z G \times_F \text{Gr}_G \longrightarrow \text{Gr}_G.$$

Recall from §1.1 the following objects: Let LG (resp. $L^+\mathcal{G}$) be the functor on the category of k -algebras $LG: R \mapsto G(R((t)))$ (resp. $L^+\mathcal{G}: R \mapsto G(R[[t]])$). The affine flag variety $\mathcal{F}\ell_{\mathfrak{a}}$ is the fpqc-quotient $\mathcal{F}\ell_{\mathfrak{a}} = LG / L^+\mathcal{G}$. There is a left action

$$LG \times_k \mathcal{F}\ell_{\mathfrak{a}} \longrightarrow \mathcal{F}\ell_{\mathfrak{a}}.$$

Lemma 2.2. *The ind-scheme $\text{Gr}_{\mathfrak{a}} \rightarrow S$ is ind-projective, and*

i) *the generic fiber $\text{Gr}_{\mathfrak{a}, \eta}$ is equivariantly isomorphic to Gr_G .*

ii) *the special fiber $\text{Gr}_{\mathfrak{a}, s}$ is equivariantly isomorphic to $\mathcal{F}\ell_{\mathfrak{a}}$.*

Proof. This follows from Theorem 1.18 and Corollary 1.13. \square

Next we introduce the global Schubert varieties which are reduced $\mathcal{L}^+\mathcal{G}$ -orbit closures in $\text{Gr}_{\mathfrak{a}}$, cf. [33]. Let $A \subset G$ be a maximal F -split torus such that $\mathfrak{a} \subset \mathcal{A}(G, A, F)$. Let \check{A} be a maximal \check{F} -split torus defined over F with $A \subset \check{A}$, cf. [7]. Let $T = Z_G(\check{A})$ be the centralizer which is a maximal torus since $G_{\check{F}}$ is quasi-split by Steinberg's Theorem. Let \mathcal{T} be the scheme theoretic closure of T in \mathcal{G} , which is a parahoric group scheme. Let $\text{Gr}_{\mathcal{T}}$ be the BD-Grassmannian over \mathcal{O}_F . Every $\mu \in X_*(T)$ determines a unique point

$$\tilde{\mu}: \bar{S} \longrightarrow \text{Gr}_{\mathfrak{a}}.$$

Indeed, we have $X_*(T) = \text{Gr}_{\mathcal{T}}(\bar{F}) = \text{Gr}_{\mathcal{T}}(\mathcal{O}_{\bar{F}})$ by the ind-properness, and the inclusion $\mathcal{T} \subset \mathcal{G}$ gives a closed immersion $\text{Gr}_{\mathcal{T}} \hookrightarrow \text{Gr}_{\mathfrak{a}}$.

Definition 2.3. Let $\mu \in X_*(T)$. The *global Schubert variety* M_μ is the scheme theoretic closure of the $\mathcal{L}^+\mathcal{G}_{\bar{S}}$ -orbit of $\tilde{\mu}$ in the BD-Grassmannian $\text{Gr}_{\mathfrak{a},\bar{S}}$.

Let us justify the definition. The morphism $\mathcal{L}^+\mathcal{G}_{\bar{S}} \rightarrow \text{Gr}_{\mathfrak{a},\bar{S}}$, $g \mapsto g\tilde{\mu}$ factors by Lemma 1.12 through some smooth affine quotient of $\mathcal{L}^+\mathcal{G}_{\bar{S}}$. Then M_μ is the scheme theoretic closure of this morphism, and hence a reduced flat projective \bar{S} -scheme, cf. Theorem 1.18. Since the fibers of $\mathcal{L}^+\mathcal{G}_{\bar{S}}$ are connected, M_μ has connected and equidimensional fibers.

Remark 2.4. Note that the global Schubert variety M_μ only depends on the $G(\bar{F})$ -conjugacy class of μ . In particular, M_μ is defined over \mathcal{O}_E where E is the Shimura field of μ , i.e. the finite extension of F defined by the stabilizer in the Galois group of the $G(\bar{F})$ -conjugacy class of μ .

In general, the special fiber of M_μ is not irreducible. It is related to the μ -admissible set (cf. Pappas-Rapoport-Smithling [25, §4.3]) as follows, see (2.3) below.

For the rest of this section, we assume that k is separably closed, i.e. $F = \check{F}$ and $A = \check{A}$. Let B be a Borel subgroup containing A . Let $R = R(G, A)$ be the set of relative roots, and $R^+ = R(B, A)$ the subset of positive roots. Let $W = W(G, A)$ be the Iwahori-Weyl group, cf. §1.1, and let $W_0 = W_0(G, A)$ be the finite Weyl group. There is a short exact sequence

$$1 \longrightarrow \Lambda_T \longrightarrow W \xrightarrow{\pi} W_0 \longrightarrow 1,$$

where $\pi: W \rightarrow W_0$ is the canonical projection and $\Lambda_T = \text{Gr}_{\mathcal{T}}(k)$, cf. [13]. Let $W_{\mathfrak{a}} \subset W$ be the subgroup associated with \mathfrak{a} , cf. §1.1. Then $\pi|_{W_{\mathfrak{a}}}: W_{\mathfrak{a}} \rightarrow W_0$ is injective, and hence, identifies $W_{\mathfrak{a}}$ with a reflection subgroup $W_{0,\mathfrak{a}} = \pi(W_{\mathfrak{a}})$ of W_0 . Consider

$$R_{\mathfrak{a}} \stackrel{\text{def}}{=} \{\alpha \in R \mid s_{\alpha} \in W_{0,\mathfrak{a}}\},$$

where $s_{\alpha} \in W_0$ denotes the reflection associated with the root α . Then $R_{\mathfrak{a}}$ is a root subsystem of R , and $R_{\mathfrak{a}}^+ = R_{\mathfrak{a}} \cap R^+$ is a system of positive roots in $R_{\mathfrak{a}}$.

Remark 2.5. Note that in general $W_{0,\mathfrak{a}}$ is not a parabolic subgroup of W_0 , i.e. the root subsystem $R_{\mathfrak{a}} \subset R$ is not the system associated with a standard Levi in G . In fact, let W'_0 be a proper maximal reflection subgroup of W_0 , i.e. W'_0 is a proper maximal subgroup, and is generated by the elements $w \in W'_0$ with $w^2 = 1$. If R is simple, then there exists a facet $\mathfrak{a} \subset \mathcal{A}$ such that $W_{0,\mathfrak{a}} = W'_0$, cf. [9, §2, Corollary 1]. In particular, all proper maximal root subsystems of R are of the form $R_{\mathfrak{a}}$ for some facet \mathfrak{a} .

The Kottwitz morphism gives an isomorphism $\Lambda_T \simeq X_*(T)_I$. There is a natural map $X_*(T)_I \rightarrow X_*(T)_I \otimes \mathbb{R} \simeq X_*(A)_{\mathbb{R}}$, $\bar{\mu} \mapsto \bar{\mu}_{\mathbb{R}}$. Define

$$X_*(T)_I^{\mathfrak{a}\text{-dom}} \stackrel{\text{def}}{=} \{\bar{\mu} \in X_*(T)_I \mid \langle \bar{\mu}_{\mathbb{R}}, \alpha \rangle \geq 0 \forall \alpha \in R_{\mathfrak{a}}^+\},$$

where $\langle -, - \rangle: X_*(A)_{\mathbb{R}} \times X^*(A)_{\mathbb{R}} \rightarrow \mathbb{R}$ denotes the canonical pairing. Then the canonical map $X_*(T)_I^{\mathfrak{a}\text{-dom}} \rightarrow W_{0,\mathfrak{a}} \backslash X_*(T)_I$ is bijective. For an element $\bar{\mu} \in X_*(T)_I$, we denote by $\bar{\mu}^{\mathfrak{a}\text{-dom}}$ the unique representative of $W_{0,\mathfrak{a}} \cdot \bar{\mu}$ in $X_*(T)_I^{\mathfrak{a}\text{-dom}}$.

Remark 2.6. At the one extreme, if \mathfrak{a} is a special facet, then $W_{0,\mathfrak{a}} = W_0$, and $X_*(T)_I^{\mathfrak{a}\text{-dom}}$ is the image of the B -dominant elements in $X_*(T)$ under the canonical projection $X_*(T) \rightarrow X_*(T)_I$. At the other extreme, if \mathfrak{a} is an alcove, then $W_{0,\mathfrak{a}}$ is trivial, and $X_*(T)_I^{\mathfrak{a}\text{-dom}} = X_*(T)_I$.

Now fix an alcove \mathfrak{a}_C which contains \mathfrak{a} in its closure, and fix a special vertex \mathfrak{a}_0 in the closure of \mathfrak{a}_C . By the choice of \mathfrak{a}_0 , we may identify $X_*(A)_{\mathbb{R}}$ with the apartment $\mathcal{A} = \mathcal{A}(G, A, F)$. Assume that the chamber in $X_*(A)_{\mathbb{R}}$ defined by B is opposite to the chamber which contains \mathfrak{a}_C . This can be arranged by possibly changing B . The choice is due to a sign convention in the Kottwitz morphism: The action of $t^\mu \in \Lambda_T$ on $X_*(A)_{\mathbb{R}}$ is given by $v \mapsto v + \mu_{\mathbb{R}}$.

Lemma 2.7. *The set ${}_{\mathfrak{a}}W^{\mathfrak{a}} \cap (W_{\mathfrak{a}}\Lambda_T W_{\mathfrak{a}})$ is contained in Λ_T , and is identified via the Kottwitz morphism with $X_*(T)_I^{\mathfrak{a}\text{-dom}}$.*

Proof. Let $\bar{\mu} \in X_*(T)_I$. From [29, Lemma 1.7, Equation (1.10)] one deduces the formula²

$$l((t^{\bar{\mu}})^{\mathfrak{a}}) = l(t^{\bar{\mu}}) - |\{\alpha \in R_{\mathfrak{a}}^+ \mid \langle \bar{\mu}, \alpha \rangle < 0\}|.$$

Note that the root systems $R_{\mathfrak{a}}$ and the subsystem of all affine roots α vanishing on \mathfrak{a} are elementwise proportional. Hence, $l((t^{\bar{\mu}})^{\mathfrak{a}})$ is maximal if and only if $\bar{\mu} = \bar{\mu}^{\mathfrak{a}\text{-dom}}$. The uniqueness of the element ${}_{\mathfrak{a}}(t^{\bar{\mu}})^{\mathfrak{a}}$ implies that

$$(2.2) \quad {}_{\mathfrak{a}}(t^{\bar{\mu}})^{\mathfrak{a}} = t^{\bar{\mu}^{\mathfrak{a}\text{-dom}}}$$

because both are contained in $(W_{\mathfrak{a}} t^{\bar{\mu}} W_{\mathfrak{a}}) \cap W^{\mathfrak{a}}$, and have the same length. This shows the lemma. \square

For $\bar{\mu} \in X_*(T)_I$, and $\rho \in X^*(T)^I$, define the integer

$$\langle \bar{\mu}, \rho \rangle \stackrel{\text{def}}{=} \langle \mu, \rho \rangle,$$

where μ is a representative of $\bar{\mu}$ in $X_*(T)$. Note that the number $\langle \bar{\mu}, \rho \rangle \in \mathbb{Z}$ does not depend on the choice of μ by the Galois equivariance of $\langle -, - \rangle: X_*(T) \times X^*(T) \rightarrow \mathbb{Z}$. For $\bar{\mu} \in X_*(T)_I$, we consider $\bar{\mu}^{\text{dom}} = \bar{\mu}^{\mathfrak{a}_0\text{-dom}}$.

Corollary 2.8. *Let $\bar{\mu} \in X_*(T)_I$, and denote by $t^{\bar{\mu}}$ the associated translation element in W . Then*

$$l({}_{\mathfrak{a}}(t^{\bar{\mu}})^{\mathfrak{a}}) = \langle \bar{\mu}^{\text{dom}}, 2\rho_B \rangle,$$

where $2\rho_B$ denotes the sum of the positive absolute roots of $B_{\bar{F}}$ with respect to $T_{\bar{F}}$.

Proof. By (2.2), we have ${}_{\mathfrak{a}}(t^{\bar{\mu}})^{\mathfrak{a}} = t^{\bar{\lambda}}$ with $\bar{\lambda} \in W_0 \cdot \bar{\mu}$. The corollary follows from [33, Lemma 9.1]. \square

Let us recall the definition of the μ -admissible set Adm_{μ} for $\mu \in X_*(T)$, cf. [25, §4.3]. Let $W_0^{\text{abs}} = W_0(G_{\bar{F}}, B_{\bar{F}})$ be the absolute Weyl group. For $\mu \in X_*(T)$ denote by $\tilde{\Lambda}_{\mu}$ the set of elements $\lambda \in W_0^{\text{abs}} \cdot \mu$ such that λ is dominant with respect to some F -rational Borel subgroup of G containing T . Let Λ_{μ} be the image of $\tilde{\Lambda}_{\mu}$ under the canonical projection $X_*(T) \rightarrow X_*(T)_I$. For $\mu \in X_*(T)$, the μ -admissible set Adm_{μ} is the partially ordered subset of the Iwahori-Weyl group

$$(2.3) \quad \text{Adm}_{\mu} \stackrel{\text{def}}{=} \{w \in W \mid \exists \bar{\lambda} \in \Lambda_{\mu} : w \leq t^{\bar{\lambda}}\},$$

where \leq is the Bruhat order of W . Note that the set Λ_{μ} , and hence Adm_{μ} only depends on the Weyl orbit $W_0^{\text{abs}} \cdot \mu$. Moreover, if μ is dominant with respect to some F -rational Borel subgroup containing T , then $\Lambda_{\mu} = W_0 \cdot \bar{\mu}$ where $\bar{\mu} \in X_*(T)_I$ is the image under the canonical projection.

We define the μ -admissible set $\text{Adm}_{\mu}^{\mathfrak{a}}$ relative to \mathfrak{a} as

$$\text{Adm}_{\mu}^{\mathfrak{a}} \stackrel{\text{def}}{=} {}_{\mathfrak{a}}W^{\mathfrak{a}} \cap (W_{\mathfrak{a}} \text{Adm}_{\mu} W_{\mathfrak{a}}).$$

Note that if $\mathfrak{a} = \mathfrak{a}_C$ is an alcove, then $\text{Adm}_{\mu} = \text{Adm}_{\mu}^{\mathfrak{a}}$.

Corollary 2.9. *Let $\mu \in X_*(T)$ be B -dominant, and denote by $\bar{\mu}$ the image in $X_*(T)_I$. Then the maximal elements in $\text{Adm}_{\mu}^{\mathfrak{a}}$ (wrt \leq) are the elements $(W_0 \cdot \bar{\mu})^{\mathfrak{a}\text{-dom}}$. In particular, each maximal element has length $\langle \mu, 2\rho_B \rangle$, and their number is*

$$|W_{0,\mathfrak{a}} \backslash W_0 / W_{0,\bar{\mu}}|,$$

where $W_{0,\bar{\mu}}$ is the stabilizer of $\bar{\mu}$ in W_0 .

²Note that the normalization of the Kottwitz morphism in [loc. cit.] differs by a sign!

Proof. The maximal elements in Adm_μ are $t^{\bar{\lambda}}$ where $\bar{\lambda} \in W_0 \cdot \bar{\mu}$. Hence, the maximal elements in $\text{Adm}_\mu^{\mathfrak{a}}$ are ${}_{\mathfrak{a}}(t^{\bar{\lambda}})^{\mathfrak{a}}$ for $\bar{\lambda} \in W_0 \cdot \bar{\mu}$. By Proposition 2.7, we have

$${}_{\mathfrak{a}}(t^{\bar{\lambda}})^{\mathfrak{a}} = t^{\bar{\lambda}^{\mathfrak{a}\text{-dom}}},$$

which implies the lemma using Corollary 2.8. \square

The combinatorial discussion above allows us to study the irreducible components of the special fiber of M_μ . In fact, the inclusion in Lemma 2.10 below is an equality on reduced loci, cf. [33] for tamely ramified groups, and [30] for the general case. Note that this implies Conjecture 4.3.1 of [25], cf. Remark 2.11 below.

Lemma 2.10. *Let $\mu \in X_*(T)$ be B -dominant, and denote by $\bar{\mu}$ the image in $X_*(T)_I$. The special fiber $M_{\mu,s}$ contains the union of Schubert varieties*

$$\bigcup_{w \in \text{Adm}_\mu^{\mathfrak{a}}} Y_w.$$

The Y_w of maximal dimension, for $w \in \text{Adm}_\mu^{\mathfrak{a}}$, are precisely the $Y_{t^{\bar{\lambda}}}$ with $\bar{\lambda} \in (W_0 \cdot \bar{\mu})^{\mathfrak{a}\text{-dom}}$. Each of them is an irreducible component of $M_{\mu,s}$ of dimension $\langle \mu, 2\rho_B \rangle$.

Proof. The geometric generic fiber $M_{\mu,\bar{\eta}}$ is the $L_z^+ G_{\bar{F}}$ -orbit of $z^\mu \cdot e_0$ in $\text{Gr}_{G,\bar{F}}$, and hence contains the \bar{F} -points $z^\lambda \cdot e_0$ with $\lambda \in W_0^{\text{abs}} \cdot \mu$. By Lemma 1.20, the special fiber $M_{\mu,s}$ contains the k -points $t^{\bar{\lambda}} \cdot e_0$ for $\lambda \in W_0^{\text{abs}} \cdot \mu$, where $\bar{\lambda}$ denotes the image in $X_*(T)_I$. Because G is quasi-split, the relative Weyl group W_0 is identified with the subgroup of I -invariant elements in W_0^{abs} , and the canonical projection $X_*(T) \rightarrow X_*(T)_I$ is W_0 -equivariant under this identification. This shows that $M_{\mu,s}$ contains the k -points $t^{\bar{\lambda}} \cdot e_0$ with $\bar{\lambda} \in W_0 \cdot \bar{\mu}$. The $L^+ \mathcal{G}$ -invariance of $M_{\mu,s}$ implies that $M_{\mu,s}$ contains the Schubert varieties Y_w for $w \in \text{Adm}_\mu^{\mathfrak{a}}$.

The rest of the lemma follows from Corollary 2.9 using that the fibers of M_μ are equidimensional, cf. Proposition 1.2. \square

Remark 2.11. Let us explain how the equality $M_{\mu,s} = \bigcup_{w \in \text{Adm}_\mu^{\mathfrak{a}}} Y_w$ on reduced loci implies Conjecture 4.3.1 of [25]. Specialize to the case that $\mathfrak{a} = \mathfrak{a}_C$ is an alcove, and assume μ to be B -dominant. By the proof of Lemma 2.10, the special fiber $M_{\mu,s}$ contains all Schubert varieties Y_w with $w \leq t^{\bar{\lambda}}$ for $\lambda \in W_0^{\text{abs}} \cdot \mu$, where $\bar{\lambda}$ denotes the image in $X_*(T)_I$. Hence,

$$(2.4) \quad \text{Adm}_\mu = \{w \in W \mid \exists \lambda \in W_0^{\text{abs}} \cdot \mu : w \leq t^{\bar{\lambda}}\}.$$

Indeed, Adm_μ is clearly contained in the right hand side of (2.4), and thus (2.4) follows from $M_{\mu,s} = \bigcup_{w \in \text{Adm}_\mu} Y_w$. Now Corollary 2.9 shows that the maximal elements in the image of $W_0^{\text{abs}} \cdot \mu$ in $X_*(T)_I$ are precisely the elements $W_0 \cdot \bar{\mu} = \Lambda_\mu$. This is Conjecture 4.3.1 of [25].

For $\mu \in X_*(T)$ let $\tau_\mu : \mathcal{L}^+ \mathcal{G}_{\bar{S}} \rightarrow M_\mu$, $g \mapsto g\tilde{\mu}$ be the orbit morphism, where $\tilde{\mu} \in M_\mu(\bar{S})$ as above. Let \mathring{M}_μ be the image of τ_μ in the sense of fppf-sheaves.

Corollary 2.12. *Let $\mu \in X_*(T)$ be dominant with respect to some F -rational Borel subgroup containing T . Then the fppf-sheaf \mathring{M}_μ is representable by a smooth open dense subscheme of M_μ .*

Proof. Write $\mathcal{L}^+ \mathcal{G} \simeq \varprojlim_i \mathcal{G}_i$ as in (1.4). The morphism τ_μ factors over some \mathcal{G}_i , and $\mathcal{G}_i / \mathcal{G}_{i,\mu} \simeq \mathring{M}_\mu$ where $\mathcal{G}_{i,\mu} \subset \mathcal{G}_i$ is the stabilizer of $\tilde{\mu}$. The the generic fiber and the special fiber of $\mathcal{G}_{i,\mu}$ are smooth and geometrically connected of the same dimension. The flat closure of $\mathcal{G}_{i,\mu,\eta}$ in \mathcal{G}_i stabilizes $\tilde{\mu}$, and hence, by counting dimensions, is equal to $\mathcal{G}_{i,\mu}$. This shows that $\mathcal{G}_{i,\mu}$ is flat and fiberwise smooth, and therefore smooth. Hence, the fppf-quotient $\mathcal{G}_i / \mathcal{G}_{i,\mu}$ is representable by a smooth scheme by the main result of [1]. This gives a quasi-finite separated monomorphism $\tau_\mu : \mathring{M}_\mu \rightarrow M_\mu$ which is open by Zariski's main theorem. The lemma follows. \square

3. SPECIALITY, PARITY AND MONODROMY

In §3.1 and 3.2, we give a list of characterizations for a facet of being very special (cf. Definition 3.7): geometric (cf. Theorem 3.2), combinatorial (cf. Corollary 3.6 ii)) and arithmetic (cf. Proposition 3.10). This implies Theorem B of the introduction.

Let F be an arbitrary field. Fix a prime ℓ different from the characteristic of F . Let $\bar{\mathbb{Q}}_\ell$ be an algebraic closure of the field of ℓ -adic numbers. For a separated scheme Y of finite type over F , we denote by $D_c^b(Y, \bar{\mathbb{Q}}_\ell)$ the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$ -complexes. Let $P(Y)$ be the core of the perverse t -structure on $D_c^b(Y, \bar{\mathbb{Q}}_\ell)$ which is an abelian $\bar{\mathbb{Q}}_\ell$ -linear full subcategory of $D_c^b(Y, \bar{\mathbb{Q}}_\ell)$. If \mathcal{Y} is a ind-scheme separated of finite type over F , and $\mathcal{Y} = (Y_\gamma)_{\gamma \in J}$ an ind-presentation, then let

$$D_c^b(\mathcal{Y}, \bar{\mathbb{Q}}_\ell) = \varinjlim_\gamma D_c^b(Y_\gamma, \bar{\mathbb{Q}}_\ell)$$

be the direct limit. Moreover, if $\mathcal{Y} = (Y_\gamma)_{\gamma \in J}$ is a strict ind-presentation, then let $P(\mathcal{Y}) = \varinjlim_\gamma P(Y_\gamma)$ be the abelian $\bar{\mathbb{Q}}_\ell$ -linear full subcategory of $D_c^b(\mathcal{Y}, \bar{\mathbb{Q}}_\ell)$ of perverse sheaves.

Let k be either a finite or a separably closed field, and specialize to the case that $F = k((t))$. Let G be a connected reductive group over F . Let \mathfrak{a} be a facet of the extended Bruhat-Tits building $\mathcal{B}(G, F)$, and let $\mathcal{G} = \mathcal{G}_\mathfrak{a}$ be the corresponding parahoric group scheme over $\mathcal{O}_F = k[[t]]$. Let \bar{F} be the completion of a separable closure of F , and denote by $\Gamma = \text{Gal}(\bar{F}/F)$ the absolute Galois group. Let $(S, \bar{S}, \eta, \bar{\eta}, s, \bar{s})$ be the 6-tuple as above, cf. §2. Let $\text{Gr}_\mathfrak{a} \rightarrow S$ be the BD-Grassmannian associated with the facet \mathfrak{a} , cf. §2. Then $\text{Gr}_\mathfrak{a}$ is an ind-projective strict ind-scheme, and there is the following cartesian diagram of ind-schemes

$$\begin{array}{ccccc} \mathcal{F}_\mathfrak{a} & \xrightarrow{i} & \text{Gr}_\mathfrak{a} & \xleftarrow{j} & \text{Gr}_G \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \eta, \end{array}$$

cf. Corollary 2.2.

Let $\bar{j}: \text{Gr}_{G, \bar{\eta}} \rightarrow \text{Gr}_{\mathfrak{a}, \bar{s}}$ (resp. $\bar{i}: \mathcal{F}_{\mathfrak{a}, \bar{s}} \rightarrow \text{Gr}_{\mathfrak{a}, \bar{s}}$) denote the base change of j (resp. i). The functor of nearby cycles $\Psi_\mathfrak{a}$ associated with \mathfrak{a} is

$$\Psi_\mathfrak{a}: D_c^b(\text{Gr}_G, \bar{\mathbb{Q}}_\ell) \rightarrow D_c^b(\mathcal{F}_\mathfrak{a} \times_s \eta, \bar{\mathbb{Q}}_\ell), \quad \Psi_\mathfrak{a}(\mathcal{A}) = \bar{i}^* \bar{j}_*(\mathcal{A}_{\bar{\eta}}).$$

Here $D_c^b(\mathcal{F}_\mathfrak{a} \times_s \eta, \bar{\mathbb{Q}}_\ell)$ denotes the bounded derived category of ℓ -adic complexes on $\mathcal{F}_{\mathfrak{a}, \bar{s}}$ together with a continuous Γ -action compatible with the base $\mathcal{F}_{\mathfrak{a}, \bar{s}}$, cf. [12, §5] and the discussion in the beginning of [26, §9]. See [SGA7 II, Exposé XIII] for the construction of the Galois action on the nearby cycles.

The global positive loop group $\mathcal{L}^+\mathcal{G}$ acts on $\text{Gr}_\mathfrak{a}$, and the action factors on each orbit through a smooth affine group scheme which is geometrically connected, cf. Lemma 1.10 and 1.12. Choosing a $\mathcal{L}^+\mathcal{G}$ -stable ind-presentation of $\text{Gr}_\mathfrak{a}$, this allows us to consider the category

$$P_{L_z^+G}(\text{Gr}_G) \quad (\text{resp. } P_{L^+G}(\mathcal{F}_\mathfrak{a}))$$

of L_z^+G -equivariant (resp. L^+G -equivariant) perverse sheaves on Gr_G (resp. $\mathcal{F}_\mathfrak{a}$) in the generic (resp. special) fiber of $\text{Gr}_\mathfrak{a}$. Let $P_{L^+G}(\mathcal{F}_\mathfrak{a} \times_s \eta)$ be the category of $L^+\mathcal{G}_s$ -equivariant perverse sheaves on $\mathcal{F}_{\mathfrak{a}, \bar{s}}$ compatible with the Galois action, cf. [26, Definition 9.3].

Lemma 3.1. *The nearby cycles restrict to a functor $\Psi_\mathfrak{a}: P_{L_z^+G}(\text{Gr}_G) \rightarrow P_{L^+G}(\mathcal{F}_\mathfrak{a} \times_s \eta)$.*

Proof. The functor $\Psi_\mathfrak{a}$ preserves perversity by [16, Appendice, Corollaire 4.2]. An application of the smooth base change theorem to the action morphism $\mathcal{L}^+\mathcal{G} \times_S \text{Gr}_\mathfrak{a} \rightarrow \text{Gr}_\mathfrak{a}$, cf. (2.1), implies the equivariance, and the compatibility of the $L^+\mathcal{G}_s$ -action with the Galois action. \square

3.1. Geometry of special facets. Recall the notion of a special facet (or vertex) in the Bruhat-Tits building $\mathcal{B}(G, F)$, cf. [6]. Let $A \subset G$ be a maximal F -split torus with associated apartment $\mathcal{A} = \mathcal{A}(G, A, F)$. A facet $\mathfrak{a} \subset \mathcal{A}$ is called *special* if for every affine hyperplane in \mathcal{A} there exist a parallel affine hyperplane containing \mathfrak{a} . A facet in the building $\mathfrak{a} \subset \mathcal{B}(G, F)$ is called *special* if \mathfrak{a} is special in one (hence every) apartment containing \mathfrak{a} .

For the rest of this subsection, we assume k to be separably closed. We consider the functor of nearby cycles $\bar{\Psi}_{\mathfrak{a}}$ over \bar{F}

$$\bar{\Psi}_{\mathfrak{a}}: P_{L_{\bar{z}}^+ G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}}) \rightarrow P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}}), \quad \bar{\Psi}_{\mathfrak{a}}(\mathcal{A}) = \bar{i}^* \bar{j}_*(\mathcal{A}).$$

Note that $P_{L_{\bar{z}}^+ G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})$ is semi-simple with simple objects the intersection complexes on the $L_{\bar{z}}^+ G_{\bar{F}}$ -orbit closures, cf. [29], and hence every object in $P_{L_{\bar{z}}^+ G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})$ is defined over some finite extension of F .

The following Theorem proves Theorem B of the introduction.

Theorem 3.2. *The following properties are equivalent.*

- i) *The facet \mathfrak{a} is special.*
- ii) *The stratification of $\mathcal{F}\ell_{\mathfrak{a}}$ in $L^+ \mathcal{G}$ -orbits satisfies the parity property, i.e. in each connected component of $\mathcal{F}\ell_{\mathfrak{a}}$ all orbits are either even or odd dimensional.*
- iii) *The category $P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}})$ is semi-simple.*
- iv) *The perverse sheaves $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A}) \in P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}})$ are semi-simple for all $\mathcal{A} \in P_{L_{\bar{z}}^+ G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})$.*

Let A be a maximal F -split torus, and T its centralizer. Note that T is a maximal torus because G is quasi-split by Steinberg's Theorem. For $\mu \in X_*(T)$, let M_{μ} be the corresponding global Schubert variety, cf. §2.

Lemma 3.3. *Let \mathfrak{a} be a facet which is not special. Then there exists $\mu \in X_*(T)$ such that the special fiber $M_{\mu, s}$ is not irreducible.*

Proof. Let $\bar{\mu} \in X_*(T)_I$ a strictly dominant element, and let μ be any preimage in $X_*(T)$ under the canonical projection. By Lemma 2.10, the special fiber $M_{\mu, s}$ contains at least

$$|W_{0, \mathfrak{a}} \setminus W_0 / W_{0, \bar{\mu}}|$$

irreducible components. This number is ≥ 2 because $W_{0, \bar{\mu}}$ is trivial, and $W_{0, \mathfrak{a}} \subset W_0$ is a proper subgroup if \mathfrak{a} is not very special. \square

The proof of Theorem 3.2 is based on the following geometric lemma.

Lemma 3.4. *Let Y be a separated scheme of finite type over k which is equidimensional of dimension d . Then for the compactly supported intersection cohomology*

$$\dim_{\mathbb{Q}_{\ell}} \mathbb{H}_c^d(Y, \mathrm{IC}) = \#\{\text{irreducible components in } Y\},$$

where IC denotes the intersection complex on Y .

Proof. We may assume that Y is reduced. Let $U \subset Y$ be an open dense smooth subscheme with reduced complement $\iota: Z \hookrightarrow Y$. Denote by ${}^p H^*$ the perverse cohomology functors. There is a cohomological spectral sequence

$$(3.1) \quad E_2^{i,j} \stackrel{\text{def}}{=} \mathbb{H}_c^i(Z, {}^p H^j(\iota^* \mathrm{IC})) \Rightarrow \mathbb{H}_c^{i+j}(Z, \iota^* \mathrm{IC}).$$

Then ${}^p H^j(\iota^* \mathrm{IC}) = 0$ for $j > 0$ because ι^* is t -right exact and ${}^p H^0(\iota^* \mathrm{IC}) = 0$ by the construction of IC . If \mathcal{A} is any perverse sheaf on Z , then $\mathbb{H}_c^i(Z, \mathcal{A}) = 0$ for $i \geq d$, as follows from $\dim(Z) \leq d - 1$ and the standard bounds on intersection cohomology. Hence, (3.1) implies that $\mathbb{H}_c^i(Z, \iota^* \mathrm{IC}) = 0$ for $i \geq d - 1$. The long exact cohomology sequence associated with

$$U \xrightarrow{j} Y \xleftarrow{\iota} Z$$

$$\mathbb{H}_c^d(U, j^* \mathrm{IC}) \xrightarrow{\simeq} \mathbb{H}_c^d(Y, \mathrm{IC}).$$

Since $j^* \mathrm{IC} = \bar{\mathbb{Q}}_{\ell}[-d]$, this implies the lemma. \square

Remark 3.5. If Y is not necessarily equidimensional, then a refinement of the argument in Lemma 3.4 shows that $\dim_{\mathbb{Q}_\ell} \mathbb{H}_c^d(Y, \mathbb{IC})$ is the number of topdimensional irreducible components, i.e. the irreducible components of dimension d .

Proof of Theorem 3.2. i) \Rightarrow ii) \Rightarrow iii): This is proven in [34, Lemma 1.1]. See also the discussion above [*loc. cit.*], and the displayed dimension formula. Note that the arguments in [*loc. cit.*] do not use the tamely ramified hypothesis.

iii) \Rightarrow iv): Trivial, since for $\mathcal{A} \in P_{L^+_{\mathbb{Z}} G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})$, the perverse sheaf $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is in $P_{L^+_{\mathcal{G}}}(\mathcal{F}\ell_{\mathfrak{a}})$, cf. Lemma 3.1.

iv) \Rightarrow i): Assume that \mathfrak{a} is not special. By Lemma 3.3, there exists $\mu \in X_*(T)$ such that the special fiber of the global Schubert variety M_μ is not irreducible. Let \mathcal{A} be the intersection complex on $M_{\mu, \bar{\eta}}$. We claim that $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is not semi-simple. Assume the contrary. The support of $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is equal to the whole special fiber $M_{\mu, s}$ by [33, Lemma 7.1] and, since $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is $L^+_{\mathcal{G}}$ -equivariant, the intersection complex on $M_{\mu, s}$ must be a direct summand of $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$. Let $d = \dim(M_{\mu, \bar{\eta}}) = \dim(M_{\mu, s})$. Taking cohomology

$$\mathbb{H}^d(M_{\mu, \bar{\eta}}, \mathcal{A}) \simeq \mathbb{H}^d(M_{\mu, s}, \bar{\Psi}_{\mathfrak{a}}(\mathcal{A}))$$

contradicts Lemma 3.4 because the left side is 1-dimensional, and the right side is at least 2-dimensional. This shows that $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is not semi-simple. \square

As a consequence of the proof, we obtain the following corollary which implies items iv) and vi) of Theorem B of the introduction.

Corollary 3.6. *The following properties are equivalent to properties i)-iv) of Theorem 3.2.*

v) The special fiber of the global Schubert M_μ in $\mathrm{Gr}_{\mathfrak{a}}$ is irreducible for all $\mu \in X_(T)$.*

vi) The admissible set $\mathrm{Adm}_{\mu}^{\mathfrak{a}}$ has a unique maximal element for all $\mu \in X_(T)_I$.* \square

3.2. Arithmetic of very special facets. In this subsection k is finite, so that $F = k((t))$ is a local non-archimedean field. We will show that the property of a facet of being very special (cf. Definition 3.7 below) is related to the vanishing of the monodromy operator on Gaitsgory's nearby cycles functor, and hence to the triviality of the weight filtration.

Let \check{F} be the completion of the maximal unramified subextension of \bar{F} , and let $\sigma \in \mathrm{Gal}(\check{F}/F)$ be the Frobenius. Note that there is a σ -equivariant embedding of buildings

$$\iota: \mathcal{B}(G, F) \longrightarrow \mathcal{B}(G, \check{F}),$$

which identifies $\mathcal{B}(G, F)$ with the σ -fixpoints in $\mathcal{B}(G, \check{F})$. In [34], Zhu defines the notion of very special facets as follows.

Definition 3.7. A facet $\mathfrak{a} \subset \mathcal{B}(G, F)$ is called *very special* if the unique facet $\mathfrak{a}^{\mathrm{nr}} \subset \mathcal{B}(G, \check{F})$ with $\iota(\mathfrak{a}) \subset \mathfrak{a}^{\mathrm{nr}}$ is special.

Remark 3.8. Every hyperspecial facet is very special. By [32] all hyperspecial facets are conjugate under the adjoint group, whereas this is *not* true for very special facets. In fact, the only case among all absolutely simple groups (up to central isogeny), where this is not true, is a ramified unitary group in odd dimensions, cf. [*loc. cit.*].

Lemma 3.9. *i) If \mathfrak{a} is a very special facet, then \mathfrak{a} is special.*

ii) The building $\mathcal{B}(G, F)$ contains very special facets if and only if the group G is quasi-split.

Proof. Part i) follows from [32, 1.10.1], and part ii) from [34, Lemma 6.1]. \square

Recall the construction of the monodromy operator, see [12, §5] for details. Let $I \subset \Gamma$ be the inertia subgroup, i.e. $\Gamma/I = \mathrm{Gal}(\check{F}/F)$. Let $P \subset I$ be the wild inertia group, so that

$$I/P = \prod_{\ell' \neq p} \mathbb{Z}_{\ell'}(1),$$

and denote by $t_\ell: I \rightarrow \mathbb{Z}_\ell(1)$ the composition of $I \rightarrow I/P$ with the projection on $\mathbb{Z}_\ell(1)$. If Y is a separated k -scheme of finite type, then let $D_c^b(Y \times_s \eta, \bar{\mathbb{Q}}_\ell)$ be the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$ -complexes together with a continuous Γ -action as above. Let $\mathcal{A} \in D_c^b(Y \times_s \eta, \bar{\mathbb{Q}}_\ell)$, and denote by $\rho: I \rightarrow \text{Aut}_{D_c^b}(\mathcal{A})$ the inertia action. Then $\rho(I)$ acts quasi-unipotently in the sense that there is an open subgroup $I_1 \subset I$ such that $\rho(g) - \text{id}_{\mathcal{A}}$ acts nilpotently for all $g \in I_1$. There is a unique nilpotent morphism

$$N_{\mathcal{A}}: \mathcal{A}(1) \longrightarrow \mathcal{A}$$

characterized by the equality $\rho(g) = \exp(t_\ell(g)N_{\mathcal{A}})$ for all $g \in I_1$, and $N_{\mathcal{A}}$ is independent of I_1 .

The choice of a Frobenius element in Γ defines a semi-direct product decomposition $\Gamma = I \rtimes \text{Gal}(\bar{k}/k)$. Recall that if $\mathcal{A} \in P(Y \times_s \eta)$ then, by restricting the Γ -action on \mathcal{A} to $\text{Gal}(\bar{k}/k)$, the underlying perverse sheaf is equipped with a continuous $\text{Gal}(\bar{k}/k)$ -descent datum, and hence defines an element $\mathcal{A}_0 \in P(Y)$. Then \mathcal{A} is called *mixed* (resp. *pure of weight w*) if \mathcal{A}_0 is mixed (resp. pure of weight w). Note that all Frobenius elements are conjugate under the inertia group I , and hence the notion of mixedness (resp. purity) does not depend on this choice, cf. [Weil2].

Let ω be the global cohomology functor with Tate twists included

$$(3.2) \quad \omega(-) \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} (R^i \Gamma(\text{Gr}_{G, \bar{F}}, (-)_{\bar{F}})(\frac{i}{2})): P_{L_z^+ G}(\text{Gr}_G) \longrightarrow \text{Vec}_{\bar{\mathbb{Q}}_\ell}.$$

Note that if \mathcal{A} is an intersection complex on a $L_z^+ G$ -stable closed subscheme of Gr_G , then the Galois action on $\omega(\mathcal{A})$ factors through a finite quotient of Γ , cf. [34, Appendix]. This explains the Tate twist in (3.2).

The following proposition together with Theorem 3.2 implies item v) of Theorem B of the introduction.

Proposition 3.10. *Let $\mathcal{A} \in P_{L_z^+ G}(\text{Gr}_G)$ such that the Γ -action on $\omega(\mathcal{A})$ factors through a finite quotient. Then the following properties are equivalent.*

- i) *The perverse sheaf $\Psi_{\mathfrak{a}}(\mathcal{A})_{\bar{s}} \in P_{L^+ \mathfrak{g}_{\bar{s}}}(\mathcal{F}_{\mathfrak{a}, \bar{s}})$ is semi-simple.*
- ii) *The nearby cycles complex $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure of weight 0.*
- iii) *The monodromy operator $N_{\Psi_{\mathfrak{a}}(\mathcal{A})} = 0$ vanishes.*

This proposition and Theorem 3.2 imply that the monodromy of $\Psi_{\mathfrak{a}}$ is non-trivial whenever \mathfrak{a} is not very special. Note that in Theorem 3.2 the residue field is assumed to be separably closed, and hence the notion of special facets and very special facets coincide. In fact, one can show that the monodromy of $\Psi_{\mathfrak{a}}$ is maximally non-trivial, cf. [30]. The equivalence ii) \Leftrightarrow iii) is a special case of the weight monodromy conjecture for perverse sheaves proven by Gabber [2]. Since the proof is easy using semi-continuity of weights, we explain it below.

Lemma 3.11. *Let \mathfrak{a} be a facet, and let $\mathcal{A} \in \text{Sat}_G$. Then $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure if and only if $N_{\Psi_{\mathfrak{a}}(\mathcal{A})} = 0$. In this case, $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure of weight 0.*

Proof. If $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure, then $N_{\Psi_{\mathfrak{a}}(\mathcal{A})}: \Psi_{\mathfrak{a}}(\mathcal{A})(1) \rightarrow \Psi_{\mathfrak{a}}(\mathcal{A})$ vanishes due to weight reasons. Conversely suppose that $N_{\Psi_{\mathfrak{a}}(\mathcal{A})} = 0$. By [16], there is a distinguished triangle

$$\bar{i}^* j_* \mathcal{A}[-1] \longrightarrow \Psi_{\mathfrak{a}}(\mathcal{A}) \xrightarrow{0} \Psi_{\mathfrak{a}}(\mathcal{A}) \longrightarrow$$

where $j: \text{Gr}_{\mathfrak{a}, \eta} \rightarrow \text{Gr}_{\mathfrak{a}}$ denotes the open embedding. Hence, on perverse cohomology

$$\Psi_{\mathfrak{a}}(\mathcal{A}) \simeq {}^p H^0(\bar{i}^* j_* \mathcal{A}[-1]) \simeq \bar{i}^* j_{!*}(\mathcal{A}).$$

This implies for the weights

$$w(\Psi_{\mathfrak{a}}(\mathcal{A})) \leq w(j_{!*}(\mathcal{A})) \leq w(\mathcal{A}) = 0,$$

and since $\Psi_{\mathfrak{a}}$ commutes with duality, we get $w(\Psi_{\mathfrak{a}}(\mathcal{A})) = 0$. \square

Proof of Proposition 3.10. The implication $ii) \Rightarrow i)$ is a consequence of Gabber's Decomposition Theorem (cf. [18, Chapter III.10]) because $\Psi_{\mathfrak{a}}(\mathcal{A})$ is defined over the ground field k . In view of Lemma 3.11, we are reduced to proving the implication $i) \Rightarrow ii)$: Let $\mathcal{A} \in P_{L_z^+G}(\mathrm{Gr}_G)$ such that the Γ -action on $\omega(\mathcal{A})$ factors through a finite quotient. Hence, after a finite base change $S' \rightarrow S$, we may assume that the Galois action on the global cohomology $\omega(\mathcal{A})$ is trivial. By Deligne [Weil2], the nearby cycles $\Psi_{\mathfrak{a}}(\mathcal{A})$ are mixed because $\mathrm{Gr}_{\mathfrak{a}}$ is already defined over a smooth curve over k . Let

$$\mathrm{gr}^{\bullet}\Psi_{\mathfrak{a}}(\mathcal{A}) \simeq \bigoplus_{\beta} \mathcal{A}_{\beta},$$

be the associated graded of the weight filtration, where $\mathcal{A}_{\beta} \in P_{L^+G}(\mathcal{F}\ell_{\mathfrak{a}})$ is pure of weight β . Let $\omega_s: P_{L^+G}(\mathcal{F}\ell_{\mathfrak{a}} \times_s \eta) \rightarrow \mathrm{Vec}_{\bar{\mathbb{Q}}_{\ell}}$ be the global cohomology with Tate twists included as in (3.2). If $\Psi_{\mathfrak{a}}(\mathcal{A})_{\bar{s}}$ is semi-simple, then $\omega_s(\Psi_{\mathfrak{a}}(\mathcal{A})) = \omega_s(\mathrm{gr}^{\bullet}\Psi_{\mathfrak{a}}(\mathcal{A}))$ as Galois representations. Because the Galois action on $\omega_s(\Psi_{\mathfrak{a}}(\mathcal{A})) \simeq \omega(\mathcal{A})$ is trivial, it follows that $\omega_s(\mathcal{A}_{\beta}) = 0$ for $\beta \neq 0$. But $\mathcal{A}_{\beta, \bar{s}}$ is the direct sum of intersection complexes, and hence $\omega_s(\mathcal{A}_{\beta}) = 0$ implies $\mathcal{A}_{\beta} = 0$, cf. Lemma 3.4. This shows that $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure of weight 0. \square

4. SATAKE CATEGORIES

In §4.1, we recall some facts from the unramified geometric Satake equivalence, cf. [11], [22] for complex coefficients, and [29], [34, Appendix] for the case of ℓ -adic coefficients. In §4.2, the ramified geometric Satake equivalence for ramified groups of Zhu [34] is explained. Zhu considers in [34] tamely ramified groups. We extend his results to include the wildly ramified case. The proof of Theorem C from the introduction is given at the end of §4.2.

4.1. The unramified Satake category. Let G be a connected reductive group over any field F . Let Gr_G be the affine Grassmannian over Gr_G with its left action by the positive loop group L_z^+G , cf. §2. Let \bar{F} be a separable closure of F , and denote by Γ the absolute Galois group. Let J be the set of Galois orbits on the set of $L_z^+G_{\bar{F}}$ -orbits in $\mathrm{Gr}_{G, \bar{F}}$. Each $\gamma \in J$ defines a connected smooth L_z^+G -invariant subscheme \mathcal{O}_{γ} over F . We have a L_z^+G -invariant ind-presentation of the reduced locus $(\mathrm{Gr}_G)_{\mathrm{red}} = \varinjlim_{\gamma} \bar{\mathcal{O}}_{\gamma}$ by the reduced closures $\bar{\mathcal{O}}_{\gamma}$.

Fix a prime ℓ different from the characteristic of F . Let

$$P_{L_z^+G}(\mathrm{Gr}_G) = \varinjlim_{\gamma} P_{L_z^+G}(\bar{\mathcal{O}}_{\gamma})$$

be the category of L_z^+G -equivariant ℓ -adic perverse sheaves on Gr_G , cf. §3.

Lemma 4.1. *The category $P_{L_z^+G}(\mathrm{Gr}_G)$ is abelian $\bar{\mathbb{Q}}_{\ell}$ -linear, and its simple objects are middle perverse extensions $i_*j_!(V[\dim(\mathcal{O}_{\gamma})])$, where $j: \mathcal{O}_{\gamma} \hookrightarrow \bar{\mathcal{O}}_{\gamma}$, $i: \bar{\mathcal{O}}_{\gamma} \hookrightarrow \mathrm{Gr}_G$, and V is a simple ℓ -adic local system on $\mathrm{Spec}(F)$.*

Proof. By [20], the simple objects in $P(\mathrm{Gr}_G)$ are of the form $\mathcal{A} = i_*j_!(\mathcal{A}_0)$ for $j: U \rightarrow \bar{U}$ a smooth irreducible open subscheme of a closed subscheme $i: \bar{U} \rightarrow \mathrm{Gr}_G$, and $\mathcal{A}_0[-\dim(U)]$ a simple ℓ -adic local system on U . If \mathcal{A} is L_z^+G -equivariant, then U is L_z^+G -invariant. In this case, $U_{\bar{F}}$ is a single Galois orbit of $L_z^+G_{\bar{F}}$ -orbits, and hence $U = \mathcal{O}_{\gamma}$ for some $\gamma \in J$. On the other hand, the stabilizers of the $L_z^+G_{\bar{F}}$ -action are connected by [23, Lemme 2.3], and thus $\mathcal{A}_0 = V[\dim(U)]$ where V is a simple ℓ -adic local system on $\mathrm{Spec}(F)$. \square

If F is separably closed, the category $P_{L_z^+G}(\mathrm{Gr}_G)$ is semi-simple with simple objects the intersection complexes on the L_z^+G -orbit closures, cf. [29].

Definition 4.2. The *unramified Satake category* $\mathrm{Sat}_{G, \bar{F}}$ over \bar{F} is the category $P_{L_z^+G_{\bar{F}}}(\mathrm{Gr}_{G, \bar{F}})$.

A version of $\mathrm{Sat}_{G, \bar{F}}$ over the ground field F is defined as follows. Fix $\sqrt{p} \in \bar{\mathbb{Q}}_{\ell}$ so that half-integral Tate twists are defined. For a complex $\mathcal{A} \in D_c^b(Y, \bar{\mathbb{Q}}_{\ell})$ on any separated scheme Y of finite type over F , we introduce the shifted and twisted version $\mathcal{A}\langle m \rangle = \mathcal{A}[m](\frac{m}{2})$ for $m \in \mathbb{Z}$.

Now let Y be a equidimensional smooth scheme over F . Let F'/F be a finite separable field extension. Then we say that a complex \mathcal{A}_0 in $D_c^b(Y, \bar{\mathbb{Q}}_\ell)$ is *constant on Y over F'* if $\mathcal{A}_{0, F'}$ is a direct sum of copies of $\bar{\mathbb{Q}}_\ell\langle \dim(Y) \rangle$.

For every $\gamma \in J$, let $\iota_\gamma: \mathcal{O}_\gamma \rightarrow \text{Gr}_G$ be the corresponding locally closed embedding.

Definition 4.3. The *unramified Satake category* Sat_G over F is the full subcategory of $P_{L_z^+ G}(\text{Gr}_G)$ of semi-simple objects \mathcal{A} such that there exists a finite separable extension F'/F with the property that the 0-th perverse cohomology ${}^p H^0(\iota_\gamma^* \mathcal{A})$ and ${}^p H^0(\iota_\gamma^! \mathcal{A})$ are constant on \mathcal{O}_γ over F' for each $\gamma \in J$.

For any $\gamma \in J$, we define $\text{IC}_\gamma = i_* j_{!*}(\bar{\mathbb{Q}}_\ell\langle \dim(\mathcal{O}_\gamma) \rangle)$ where $\mathcal{O}_\gamma \xrightarrow{j} \bar{\mathcal{O}}_\gamma \xrightarrow{i} \text{Gr}_G$ is the open embedding into the closure.

Lemma 4.4. *Let $\mathcal{A} \in P_{L_z^+ G}(\text{Gr}_G)$ be a simple object. Then $\mathcal{A} \in \text{Sat}_G$ if and only if there is an $\gamma \in J$ such that $\mathcal{A} \simeq \text{IC}_\gamma \otimes V$ where V is a local system on $\text{Spec}(F)$ that is trivial over some finite extension F'/F .*

Proof. Let $\mathcal{A} = i_* j_{!*}(V[\dim(\mathcal{O}_\gamma)])$ be simple for some $\gamma \in J$. Assume that $\mathcal{A} \in \text{Sat}_G$. Then there exists F'/F finite such that ${}^p H^0(\iota^* \mathcal{A}) = V[\dim(\mathcal{O}_\gamma)]$ is constant over F' for $\iota: \mathcal{O}_\gamma \rightarrow \text{Gr}_G$, i.e. $V[\dim(\mathcal{O}_\gamma)] = \bar{\mathbb{Q}}_\ell\langle \dim(\mathcal{O}_\gamma) \rangle \otimes V$ where V is a local system that is trivial over F' . Since the middle perverse extension commutes with smooth morphisms, we obtain $\mathcal{A} \simeq \text{IC}_\mu \otimes V$. The converse follows from the fact that ${}^p H^0(\iota_{\gamma'}^* \mathcal{A}) = 0$, unless $\gamma' = \gamma$ and in this case ${}^p H^0(\iota_{\gamma'}^* \mathcal{A}) = V_0[\dim(\mathcal{O}_\gamma)]$ for both restrictions $\iota_{\gamma'}^* = \iota_\gamma^*$ and $\iota_{\gamma'}^! = \iota_\gamma^!$. \square

We recall from [29] that the category $P_{L_z^+ G}(\text{Gr}_G)$ is equipped with a symmetric monoidal structure with respect to the convolution product \star uniquely determined by the property that the global cohomology functor $\omega: P_{L_z^+ G}(\text{Gr}_G) \rightarrow \text{Vec}_{\bar{\mathbb{Q}}_\ell}$ is symmetric monoidal, cf. (3.2).

Recall the classical geometric Satake isomorphism, first over \bar{F} . The tuple $(\text{Sat}_{G, \bar{F}}, \star)$ is a neutralized Tannakian category with fiber functor $\omega_{\bar{F}}$, and the group of tensor automorphisms $\hat{G} = \text{Aut}^*(\omega_{\bar{F}})$ is a connected reductive group over $\bar{\mathbb{Q}}_\ell$ whose root datum is dual to the root datum of $G_{\bar{F}}$ in the sense of Langlands.

Now for arbitrary F , it is shown in [34, Appendix] that for any object $\mathcal{A} \in \text{Sat}_G$ the Γ -action on $\omega(\mathcal{A})$ factors over a finite quotient of the Galois group. This explains the Tate twist in (3.2). Hence, Γ acts on \hat{G} via a finite quotient, and we may form ${}^L G = \hat{G} \rtimes \Gamma$ considered as a pro-algebraic group over $\bar{\mathbb{Q}}_\ell$ with neutral component \hat{G} . In this way, for every $\mathcal{A} \in \text{Sat}_G$, the cohomology $\omega(\mathcal{A})$ is an algebraic representation of the affine group scheme ${}^L G$. Denote by $\text{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L G)$ (resp. $\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G})$) the tensor category of algebraic representations of ${}^L G$ (resp. \hat{G}) over $\bar{\mathbb{Q}}_\ell$.

Theorem 4.5. *i) The category Sat_G is stable under the convolution product, and (Sat_G, \star) is a semi-simple abelian tensor subcategory of $(P_{L_z^+ G}(\text{Gr}_G), \star)$.*

ii) The base change to \bar{F} defines a tensor functor $(-)_\bar{F}: (\text{Sat}_G, \star) \rightarrow (\text{Sat}_{G, \bar{F}}, \star)$, and the following diagram of functors between abelian tensor categories

$$\begin{array}{ccc} (\text{Sat}_G, \star) & \xrightarrow{(-)_\bar{F}} & (\text{Sat}_{G, \bar{F}}, \star) \\ \downarrow \omega & & \downarrow \omega_{\bar{F}} \\ (\text{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L G), \otimes) & \xrightarrow{\text{res}} & (\text{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}), \otimes) \end{array}$$

is commutative up to natural isomorphism, where res denotes the restriction of representations along $\hat{G} \hookrightarrow {}^L G$.

Corollary 4.6. *Let $\mathcal{A} \in \text{Sat}_G$. Then the Galois group acts trivially on $\omega(\mathcal{A})$ if and only if \mathcal{A} is a direct sum of IC_γ for $\gamma \in J$ such that $\mathcal{O}_{\gamma, \bar{F}}$ is connected.*

Proof. We may assume that \mathcal{A} is simple, and hence $\mathcal{A} = \mathrm{IC}_\gamma \otimes V$ for some $\gamma \in J$ and some local system V on $\mathrm{Spec}(F)$ by Lemma 4.4. If Γ acts trivially on $\omega(\mathcal{A})$, then V is trivial, and $\mathcal{O}_{\gamma, \bar{F}}$ is connected. Conversely, if $\mathcal{A} = \mathrm{IC}_\gamma$ for $\gamma \in J$ with $\mathcal{O}_{\gamma, \bar{F}}$ connected, then Γ acts trivially on $\omega(\mathcal{A})$ by [34, Appendix], cf. the Tate twist in (3.2). \square

Remark 4.7. i) For an interpretation of the whole abelian tensor category $(P_{L^+G}(\mathrm{Gr}_G), \star)$ in terms of the dual group see [29, §5].

ii) The group of tensor automorphisms \hat{G} admits a canonical pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$, cf. Appendix A for the definition of a pinning. Moreover, the action of Γ on \hat{G} is via pinned automorphisms. As explained in §4 of [34], the canonical pinning is constructed as follows. The cohomological grading on ω defines a one parameter subgroup $\mathbb{G}_m \rightarrow \hat{G}$, and the centralizer \hat{T} is a maximal torus. Let \mathcal{L} be an ample line bundle on Gr_G . Then its isomorphism class $[\mathcal{L}] \in \mathrm{Pic}(\mathrm{Gr}_G)$ is unique. Cup product with the first Chern class $c_1([\mathcal{L}]) \in H^2(\mathrm{Gr}_{G, \bar{F}}, \mathbb{Z}_\ell(1))$ defines a principal nilpotent element $\hat{X} \in \mathrm{Lie}(\hat{G})$. This in turn determines the Borel subgroup \hat{B} with $\hat{T} \subset \hat{B}$ and $\hat{X} \in \mathrm{Lie}(\hat{B})$ uniquely. Since the Galois group Γ fixes the cohomological grading and $[\mathcal{L}]$, it acts on \hat{G} via pinned automorphisms.

4.2. The ramified Satake category. Let k be a finite field, and let G be a connected reductive group over the Laurent power series field $F = k((t))$. Let \mathfrak{a} be a facet in the Bruhat-Tits building $\mathcal{B}(G, F)$, and denote by $\mathcal{G} = \mathcal{G}_\mathfrak{a}$ the associated parahoric group scheme over \mathcal{O}_F .

There is the convolution product, cf. [10], [26]

$$- \star -: P(\mathcal{F}\ell_\mathfrak{a}) \times P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a}) \longrightarrow D_c^b(\mathcal{F}\ell_\mathfrak{a}, \bar{\mathbb{Q}}_\ell).$$

Note that $P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a})$ is not stable under \star in general, i.e. the convolution of two perverse sheaves need not to be perverse again. For the preservation of perversity we need a hypothesis on \mathfrak{a} .

For the rest of the section, let \mathfrak{a} be a very special facet, cf. Definition 3.7.

Definition 4.8. The *ramified Satake category* $\mathrm{Sat}_{\mathfrak{a}, \bar{s}}$ over \bar{s} is the category $P_{L+\mathcal{G}_{\bar{s}}}(\mathcal{F}\ell_{\mathfrak{a}, \bar{s}})$.

Remark 4.9. The connection with §2.1 is as follows. The choice of a hyperspecial facet \mathfrak{a} is equivalent to the choice of a Chevalley model of G over \mathcal{O}_F . In this case, the BD-Grassmannian $\mathrm{Gr}_\mathfrak{a}$ is constant over S , and the nearby cycles functor $\bar{\Psi}_\mathfrak{a}: \mathrm{Sat}_{G, \bar{\eta}} \rightarrow \mathrm{Sat}_{\mathfrak{a}, \bar{s}}$ is an equivalence of tensor categories, cf. the proof of Theorem 4.11 below.

A version of $\mathrm{Sat}_{\mathfrak{a}, \bar{s}}$ with Galois action is defined as follows. For a finite intermediate extension $F \subset F' \subset \bar{F}$, let $(S', \bar{S}, \eta', \bar{\eta}, s', \bar{s})$ be the associated 6-tuple with Galois group $\Gamma' = \mathrm{Gal}(\bar{F}/F')$. Then there is the functor

$$\mathrm{res}_{F'/F}: P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a} \times_s \eta) \longrightarrow P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a} \times_{s'} \eta')$$

given by restricting the Galois action from Γ to the subgroup Γ' . Furthermore, there is the functor

$$(-)_{\bar{s}}: P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a}) \longrightarrow P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a} \times_s \eta)$$

given by pullback along $\mathcal{F}\ell_{\mathfrak{a}, \bar{s}} \rightarrow \mathcal{F}\ell_\mathfrak{a}$. Note that $(-)_{\bar{s}}$ is fully faithful with essential image consisting of the objects $\mathcal{A} \in P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a} \times_s \eta)$ such that the inertia acts trivially.

Definition 4.10. The *ramified Satake category* $\mathrm{Sat}_\mathfrak{a}$ over s is the full subcategory of objects $\mathcal{A} \in P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a} \times_s \eta)$ with the property that there exists a finite separable extension F'/F such that

- a) the inertia $I' \subset \Gamma'$ acts trivially on $\mathrm{res}_{F'/F}(\mathcal{A})$, and
- b) the perverse sheaf $\mathrm{res}_{F'/F}(\mathcal{A}) \in P_{L+\mathcal{G}}(\mathcal{F}\ell_\mathfrak{a})$ is semi-simple and pure of weight 0.

We denote by $\omega_s: \text{Sat}_{\mathfrak{a}} \rightarrow \text{Vec}_{\overline{\mathbb{Q}}_\ell}$ the global cohomology, with Tate twists included, as in (3.2), and likewise $\omega_{\bar{s}}: \text{Sat}_{\mathfrak{a}, \bar{s}} \rightarrow \text{Vec}_{\overline{\mathbb{Q}}_\ell}$. Since the Galois group Γ acts via a finite quotient on $\hat{G} = \text{Aut}^*(\omega)$, we may consider the invariants \hat{G}^I under the inertia group. Then $\hat{G}^I \subset \hat{G}$ is a reductive subgroup which is not connected in general. The group Γ operates on \hat{G}^I , and we form the semi-direct product ${}^L G_r = \hat{G}^I \rtimes \Gamma$, considered as a pro-algebraic group over $\overline{\mathbb{Q}}_\ell$. Hence, ${}^L G_r \hookrightarrow {}^L G$ is a closed subgroup scheme.

Recall that there is the nearby cycles functor $\Psi_{\mathfrak{a}}: P_{L_z^+ G}(\text{Gr}_G) \rightarrow P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}})$ associated with \mathfrak{a} , cf. §3.

Theorem 4.11. *Let \mathfrak{a} be very special.*

- i) *The category $\text{Sat}_{\mathfrak{a}}$ is semi-simple and stable under the convolution product \star .*
- ii) *If $\mathcal{A} \in \text{Sat}_G$, then $\Psi_{\mathfrak{a}}(\mathcal{A}) \in \text{Sat}_{\mathfrak{a}}$, and the pair $(\text{Sat}_{\mathfrak{a}}, \star)$ admits a unique structure of a symmetric monoidal category such that $\Psi_{\mathfrak{a}}: (\text{Sat}_G, \star) \rightarrow (\text{Sat}_{\mathfrak{a}}, \star)$ is symmetric monoidal.*
- iii) *The following diagram of functors of abelian tensor categories*

$$\begin{array}{ccc} (\text{Sat}_G, \star) & \xrightarrow{\Psi_{\mathfrak{a}}} & (\text{Sat}_{\mathfrak{a}}, \star) \\ \downarrow \omega & & \downarrow \omega_s \\ (\text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G), \otimes) & \xrightarrow{\text{res}} & (\text{Rep}_{\overline{\mathbb{Q}}_\ell}({}^L G_r), \otimes) \end{array}$$

is commutative up to natural isomorphisms, and the vertical arrows are equivalences.

Proof. We explain the modifications in Zhu's proof of Theorem 4.11.

The geometric equivalence: Let $\bar{S} = \text{Spec}(\mathcal{O}_{\bar{F}})$, and consider the base change $\text{Gr}_{\mathfrak{a}, \bar{s}} = \text{Gr}_{\mathfrak{a}} \times_S \bar{S}$. Let $\text{Sat}_{G, \bar{\eta}} = P_{L_z^+ G_{\bar{F}}}(\text{Gr}_{G, \bar{F}})$ (resp. $\text{Sat}_{\mathfrak{a}, \bar{s}} = P_{L+\mathcal{G}_{\bar{s}}}(\mathcal{F}\ell_{\mathfrak{a}, \bar{s}})$) be the Satake category over $\bar{\eta}$ (resp. \bar{s}). Recall that there is the nearby cycles functor, cf. §3

$$\bar{\Psi}_{\mathfrak{a}}: \text{Sat}_{G, \bar{\eta}} \longrightarrow \text{Sat}_{\mathfrak{a}, \bar{s}}.$$

We go through the arguments in Zhu's paper [34].

- a) The category $\text{Sat}_{\mathfrak{a}, \bar{s}}$ is semi-simple and stable under the convolution product \star . Moreover, the pair $(\text{Sat}_{\mathfrak{a}, \bar{s}}, \star)$ is a monoidal category.

The category $P_{L+\mathcal{G}_{\bar{s}}}(\mathcal{F}\ell_{\mathfrak{a}, \bar{s}})$ is semi-simple by Theorem 3.2 iii) (Lemma 1.1 in [*loc. cit.*]). We show that it is stable under convolution. Let $\mathcal{A} \in \text{Sat}_{G, \bar{\eta}}$. The monodromy of $\bar{\Psi}_{\mathfrak{a}}(\mathcal{A})$ is trivial by Proposition 3.10 iii) (Lemma 2.3 in [*loc. cit.*]). As in the proof of Lemma 3.11 this implies the formula

$$(4.1) \quad \bar{\Psi}_{\mathfrak{a}}(\mathcal{A}) \simeq \bar{i}^* j_{!*}(\mathcal{A}),$$

which is Corollary 2.5 of [*loc. cit.*]. Hence, (4.1) holds for all $\mathcal{A} \in \text{Sat}_{G, \bar{\eta}}$. Let $\mu \in X_*(T)$ be dominant with respect to some F -rational Borel subgroup of G . Note that G is quasi-split by Lemma 3.9. Let M_μ be the global Schubert variety, and let IC_μ be the intersection complex on $M_{\mu, \bar{\eta}}$. Then the intersection complex on the Schubert variety $Y_{\mu, \bar{\mu}}$ in the special fiber appears with multiplicity 1 in $\bar{\Psi}_{\mathfrak{a}}(\text{IC}_\mu)$. This follows from the compatibility of nearby cycles along smooth morphisms applied to the open immersion $\hat{M}_\mu \hookrightarrow M_\mu$, cf. Corollary 2.12 and Lemma 2.10. This shows that Lemma 2.6 of [*loc. cit.*] holds. Proposition 2.7 [*loc. cit.*] carries over word by word in replacing $(\mathbb{A}_k^1, 0)$ by a pointed curve (X, x) . Corollary 2.8 in [*loc. cit.*] is a consequence of the above arguments. This proves a).

- b) The tuple $(\text{Sat}_{\mathfrak{a}, \bar{s}}, \star)$ has a unique structure of a neutral Tannakian category such that

$$\bar{\Psi}_{\mathfrak{a}}: (\text{Sat}_{G, \bar{\eta}}, \star) \longrightarrow (\text{Sat}_{\mathfrak{a}, \bar{s}}, \star)$$

is a tensor functor compatible with the fiber functors $\omega_{\bar{\eta}} \simeq \omega_{\bar{s}} \circ \bar{\Psi}_{\mathfrak{a}}$.

§3 in [*loc. cit.*] carries over literally: In Theorem-Definition 3.1 of [*loc. cit.*] one may replace \mathbb{A}_k^1 by any smooth curve X . This implies that $\bar{\Psi}_{\mathfrak{a}}: (\text{Sat}_{G, \bar{\eta}}, \star) \rightarrow (\text{Sat}_{\mathfrak{a}, \bar{s}}, \star)$ is a central functor,

and Proposition 3.2 of [loc. cit.] holds. Now as in [loc. cit.], we apply Lemma 3.3 of [loc. cit.] to deduce Corollary 3.5 of [loc. cit.]. In particular, $\text{Sat}_{\mathfrak{a}, \bar{s}}$ is a neutral Tannakian category. The uniqueness of the Tannakian structure follows from the uniqueness of the symmetric monoidal structure for $\omega_{\bar{\eta}}$, cf. the discussion above (3.2). This proves b).

c) There is a up to natural isomorphism commutative diagram of functors of abelian tensor categories

$$\begin{array}{ccc} (\text{Sat}_{G, \bar{\eta}}, \star) & \xrightarrow{\bar{\Psi}_{\mathfrak{a}}} & (\text{Sat}_{\mathfrak{a}, \bar{s}}, \star) \\ \downarrow \omega_{\bar{\eta}} & & \downarrow \omega_{\bar{s}} \\ (\text{Rep}_{\mathbb{Q}_{\ell}}(\hat{G}), \otimes) & \xrightarrow{\text{res}} & (\text{Rep}_{\mathbb{Q}_{\ell}}(\hat{G}^I), \otimes), \end{array}$$

where the vertical arrows are equivalences.

Let $H = \text{Aut}^*(\omega_{\bar{s}})$ be the affine \mathbb{Q}_{ℓ} -group scheme of tensor automorphisms defined by $(\text{Sat}_{\mathfrak{a}, \bar{s}}, \omega_{\bar{s}})$. Via the unramified Satake equivalence, the tensor functor $\bar{\Psi}_{\mathfrak{a}}$ defines a morphism $H \rightarrow \hat{G}$ which identifies H with a closed reductive subgroup of \hat{G} . Indeed, every object in $\text{Sat}_{\mathfrak{a}, \bar{s}}$ appears as a direct summand in the essential image of $\bar{\Psi}_{\mathfrak{a}}$, and since $\text{Sat}_{\mathfrak{a}, \bar{s}}$ is semi-simple, H is reductive. It remains to identify the subgroup $H \subset \hat{G}$. The inertia group I acts on $\text{Gr}_{G, \bar{\eta}} \rightarrow \text{Gr}_{G, \bar{\eta}}$ induced from the action on $\bar{\eta} \rightarrow \check{\eta}$ where $\check{\eta} = \text{Spec}(\check{F})$. As in the Appendix of [loc. cit.], this induces via $\text{Sat}_{G, \bar{\eta}} \times I \rightarrow \text{Sat}_{G, \bar{\eta}}$, $(\gamma, \mathcal{A}) \mapsto \gamma^* \mathcal{A}$, an action of I on the Tannakian category $(\text{Sat}_{G, \bar{\eta}}, \omega_{\bar{\eta}})$, and hence on $\text{Aut}^*(\omega_{\bar{\eta}}) = \hat{G}$. Since the tensor functor $\bar{\Psi}_{\mathfrak{a}}$ is invariant under this action, we get that $H \subset \hat{G}^I$ (cf. Lemma 4.5 in [loc. cit.]), and we need to show that equality holds. Recall that \hat{G} admits a canonical pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$, cf. Remark 4.7. The Galois action, and in particular I -action preserves the pinning, and we can apply Lemma A.1 below. This shows that the inclusion $\hat{T} \subset \hat{G}$ induces a bijection $\pi_0(\hat{T}^I) \simeq \pi_0(\hat{G}^I)$ on connected components. Now we may apply Corollary A.3 to conclude by the argument below Lemma 4.10 [loc. cit.]. This shows that $H = \hat{G}^I$, and finishes the proof of part c) and Theorem C from the introduction. The uniqueness of the equivalence in Theorem C is a consequence of the Isomorphism Theorem in the theory of reductive groups.

Galois descent: Based on the geometric equivalence above, one shows that

$$(\text{Sat}_{\mathfrak{a}}, \star) \simeq (\text{Rep}_{\mathbb{Q}_{\ell}}({}^L G_r), \otimes), \quad \mathcal{A} \mapsto \omega_s(\mathcal{A}),$$

as in [34, Appendix]. In particular, Theorem 4.11 i) holds, and part iii) follows from part ii).

For ii), let $\mathcal{A} \in \text{Sat}_G$. We claim that $\Psi_{\mathfrak{a}}(\mathcal{A}) \in \text{Sat}_{\mathfrak{a}}$. Indeed, $\Psi_{\mathfrak{a}}(\mathcal{A})$ is pure of weight 0, cf. Proposition 3.10, and it is enough to show that $\Psi_{\mathfrak{a}}(\text{IC}_{\mu}) \in P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}})$ is semi-simple for all $\mu \in X_*(T)$. By replacing k by a finite extension, we may assume that every $L^+\mathcal{G}$ -orbit is defined over k . The $L^+\mathcal{G}$ -equivariance implies that there is a finite direct sum decomposition

$$\Psi_{\mathfrak{a}}(\text{IC}_{\mu}) \simeq \bigoplus_w \text{IC}_w \otimes V_w,$$

where IC_w is the intersection complex of the Schubert variety $Y_w \subset \mathcal{F}\ell_{\mathfrak{a}}$, $w \in W$, and V_w is a local system on $\text{Spec}(k)$. In fact, V_w is constant because $\omega_s(\Psi_{\mathfrak{a}}(\text{IC}_{\mu})) \simeq \omega(\text{IC}_{\mu})$, cf. Corollary 4.6. This shows $\Psi_{\mathfrak{a}}(\mathcal{A}) \in \text{Sat}_{\mathfrak{a}}$.

It remains to show that $\Psi_{\mathfrak{a}}: \text{Sat}_G \rightarrow \text{Sat}_{\mathfrak{a}}$ is a tensor functor, i.e. that the isomorphism $\Psi_{\mathfrak{a}}(\mathcal{A} \star \mathcal{B}) \simeq \Psi_{\mathfrak{a}}(\mathcal{A}) \star \Psi_{\mathfrak{a}}(\mathcal{B})$ is Galois equivariantly compatible with the commutativity constraint, and defines a morphism in $P_{L+\mathcal{G}}(\mathcal{F}\ell_{\mathfrak{a}} \times_s \eta)$. This follows from the fact that the Beilinson-Drinfeld Grassmannians are defined over the ground field, cf. [26, §9.b]. The uniqueness is clear. This finishes the proof of the theorem. \square

APPENDIX A. THE GROUP OF FIXED POINTS UNDER A PINNING PRESERVING ACTION

Let G be a connected reductive group over an algebraically closed field C . Let I be a subgroup of the algebraic automorphisms of G , and assume that I fixes some pinning of G . Then I is finite, and we assume that the order $|I|$ is prime to the characteristic of C . The group of fixed points G^I is a reductive group which is not connected in general. In this appendix, we prove the existence and uniqueness of irreducible highest weight representations of G^I , and determine the group of connected components $\pi_0(G^I)$.

First recall the notion of a pinning. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup. Let $R = R(G, T)$ (resp. R^\vee) be the set of roots (resp. coroots), and let $R_+ = R(B, T)$ (resp. R_+^\vee) be the subset of positive roots (resp. coroots). There is a bijection $R \rightarrow R^\vee$, $a \mapsto a^\vee$ which preserves the subsets of positive roots. For $a \in R$, let $U_a \subset G$ be the root subgroup, and denote by $\mathfrak{u}_a \subset \text{Lie}(H)$ its Lie algebra. Denote by $\Delta \subset R^+$ (resp. $\Delta^\vee \subset R_+^\vee$) the set of simple roots (resp. coroots). For every $a \in \Delta$, choose a generator X_a of the 1-dimensional C -vector space \mathfrak{u}_a , and let $X = \sum_{a \in \Delta} X_a$ be the principal nilpotent element in $\text{Lie}(B)$. A *pinning* of G is a quadruple (G, B, T, X) where $T \subset B$ is a torus contained in a Borel subgroup, and $X \in \text{Lie}(B)$ is a principal nilpotent element. Note that there is a canonical isomorphism

$$(A.1) \quad \text{Aut}((G, B, T, X)) \simeq \text{Aut}((X^*(T), R, \Delta, X_*(T), R^\vee, \Delta^\vee))$$

between the pinning preserving automorphisms of G , and the automorphisms of the based root datum $(X^*(T), R, \Delta, X_*(T), R^\vee, \Delta^\vee)$.

Recall the following basic facts on the group of fixed points. Let H be any affine group scheme over C , and let $J \subset \text{Aut}_C(H)$ be a finite subgroup of algebraic automorphisms. Then the group of fixed points $H^J \subset H$ is a closed subgroup scheme. Assume that the order $|J|$ is prime to the characteristic of C . Then

- a) if H is smooth, then H^J is smooth, and
- b) if H is reductive, then H^J is reductive, cf. [27, Theorem 2.1].

Note that even if H is connected reductive, then H^J is in general not connected.

Lemma A.1. *Let G be a connected reductive group over an algebraically closed field C . Let (G, B, T, X) be a pinning of G , and let I be a subgroup of the pinning preserving automorphisms. Then I is finite, and we assume that the order $|I|$ is prime to the characteristic of C .*

- i) *The tuple $(G^{I,0}, B^{I,0}, T^{I,0}, X)$ is a pinning of the connected reductive group $G^{I,0}$.*
- ii) *The inclusion $T^I \subset G^I$ induces a bijection on connected components $\pi_0(T^I) \simeq \pi_0(G^I)$.*

Proof. i): Let $B = T \rtimes U$ be the Levi decomposition of B . Then U is I -invariant, and we claim that the fixed points U^I are connected. Indeed, the connectedness follows from the argument of Steinberg [31, Proof of Theorem 8.2]: Factoring $R = \prod_i R_i$ into a product of simple root systems, the group $U = \prod_i U_i$ factors accordingly, and I permutes the single factors. Hence, we may assume that the root system R is simple. The classification implies that I acts either through the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/3$ or S_3 . In case $I = S_3$, the system R is of type D_4 , and the S_3 -orbits on R coincide with the $\mathbb{Z}/3$ -orbits on R . Hence, we may replace I by $\mathbb{Z}/3$ in this case, and assume that I is cyclic. Now the argument in [loc. cit.] (2) shows that each I -orbit in R determines a 1-parameter subgroup in U^I , and their cartesian product is U^I . Note that for the elements $c_{\sigma_\alpha} = 1$ in the notation of [loc. cit.] because I acts via pinned automorphisms, and hence the equations in (2''') are automatically satisfied. This shows that U^I is connected. One checks that $G^{I,0}/B^{I,0}$ is proper, and hence $B^{I,0}$ is a Borel. Now $B^{I,0} = T^{I,0} \rtimes U^I$ by the connectedness of U^I . Thus, $T^{I,0} \subset G^{I,0}$ is a maximal torus. We have $X \in \text{Lie}(U)^I = \text{Lie}(U^I)$. The preceding argument shows that each I -orbit in $R(G, T)$ determines a root in $R(G^{I,0}, T^{I,0})$ preserving the positive roots and the basis. Hence, X is principally nilpotent in $\text{Lie}(G^{I,0})$.

ii): Let B^{op} be the unique Borel with $B \cap B^{\text{op}} = T$, and denote by $B = T \times U^{\text{op}}$ the Levi decomposition. It is enough to show that multiplication

$$(A.2) \quad U^{\text{op},I} \times T^I \times U^I \longrightarrow G^I,$$

is an open dense immersion because U^I (resp. $U^{\text{op},I}$) is connected, cf. i). The openness of (A.2) is clear, and we need to show that it is dense. Let $N = N_G(T)$, and $W_0 = N(C)/T(C)$ be the Weyl group. Choose a system $n_w \in N(C)$ of representatives of $w \in W_0$. Let $U_w = U \cap (n_w^{-1}U^{\text{op}}n_w)$. By the Bruhat decomposition, there is a set theoretically disjoint union

$$(A.3) \quad G = \coprod_{w \in W} U_w n_w B,$$

and every element $g \in G(C)$ can be written uniquely as a product $G = u_w n_w b$ with $u_w \in U_w$, $b \in B$. Since I preserves the pinning, the morphism $N^I \rightarrow W_0^I$ is surjective, and W_0^I is the Weyl group of $G^{I,0}$. We may assume that $n_w \in N^I$ for all $w \in W_0^I$. The uniqueness in (A.3) implies

$$(A.4) \quad G^I = \coprod_{w \in W_0^I} U_w^I n_w B^I.$$

Let $w_0 \in W_0$ be the longest element. The length l on W_0 is I -invariant, and hence $w_0 \in W_0^I$. This implies that $U_{w_0}^I n_{w_0} B^I$ is the unique stratum of maximal dimension in (A.4). Since $U^{\text{op},I} = n_{w_0}^{-1}U_{w_0}^I n_{w_0}$, the density in (A.2) follows. \square

Let $Q_+ \subset X^*(T)$ be the semigroup generated by R_+ , and denote by $(Q_I)_+$ the image of Q_+ under the canonical projection $X^*(T) \rightarrow X^*(T^I)$. The group of characters $X^*(T^I)$ is equipped with the dominance order as follows. For $\mu, \lambda \in X^*(T^I)$, define $\lambda \leq \mu$ if and only if $\mu - \lambda \in (Q_I)_+$.

Denote by $X^*(T)_+$ the semigroup of dominant weights, and let $X^*(T^I)_+$ be the semigroup defined as the image of $X^*(T)_+$ under the canonical projection $X^*(T) \rightarrow X^*(T^I)$. Let $\mu \in X^*(T^I)$. An algebraic representation $\rho: G^I \rightarrow \text{GL}(V)$ is said to be of *highest weight* μ if

- i) μ appears with a non-zero multiplicity in the restriction $\rho|_{T^I}$, and
- ii) if $\lambda \in X^*(T^I)$ appears in $\rho|_{T^I}$ with non-zero multiplicity, then $\lambda \leq \mu$.

Remark A.2. Let w_0 be the longest element in the finite Weyl group $W_0 = W_0(G, T)$. Since I acts by pinned automorphisms, we have $w_0 \in W_0^I$, and it follows that w_0 acts on $X^*(T^I)$. Then property ii) implies that $w_0\mu \leq \lambda \leq \mu$, for all $\lambda \in X^*(T^I)$ appearing in $\rho|_{T^I}$ with non-zero multiplicity.

If G^I is connected reductive, then T^I is a torus by Lemma A.1. In this case, it is well-known that there exists for every $\mu \in X^*(T^I)_+$ a unique up to isomorphism irreducible representation of highest weight μ , and that every irreducible representation is of this form. Moreover, the multiplicity of the μ -weight space is 1, cf. [17, Chapter II.2].

Corollary A.3. *Let G be a connected reductive group over an algebraically closed field C . Let (G, B, T, X) be a pinning of G , and let I be a subgroup of the pinning preserving automorphisms of order prime to the characteristic of C .*

i) For every $\mu \in X^(T^I)_+$ there exists a unique up to isomorphism irreducible representation ρ_μ of G^I of highest weight μ , and every irreducible representation of G^I is of this form.*

ii) The multiplicity of the μ -weight space is 1.

Proof. We follow the argument of Zhu [34, Lemma 4.10]. Let $\bar{\mu}$ be the image of μ under the restriction $X^*(T^I) \rightarrow X^*(T^{I,0})$, and let $\rho_{\bar{\mu}}$ be the unique irreducible representation of highest weight $\bar{\mu}$, cf. [17, Chapter II.2]. Frobenius reciprocity and Lemma A.1 imply

$$(A.5) \quad \text{ind}_{G^{I,0}}^{G^I}(\rho_{\bar{\mu}}) \simeq \bigoplus_{\chi \in X^*(\pi_0(T^I))} \rho \otimes \chi,$$

where ρ is an irreducible representation of G^I which restricts to $\rho_{\bar{\mu}}$. Here, the χ 's are considered as G^I -representations by inflation along $G^I \rightarrow \pi_0(G^I) \simeq \pi_0(T^I)$. This shows that there is a unique $\chi \in X^*(\pi_0(T^I))$ such that $\rho_{\mu} = \rho \otimes \chi$ is of highest weight μ . Conversely, (A.5) implies that every irreducible representation of G^I is a direct summand of some induction, and hence is of the form ρ_{μ} for some $\mu \in X^*(T^I)$. This proves i). Part ii) is easily deduced from (A.5). \square

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