Categorified $\mathcal{U}_q(\mathfrak{sl}_2)$-theory using Bar-Natan's approach

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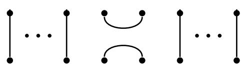
Introduction

The idea of categorification goes back to Crane and Frenkel [CF94] and the method has become more and more popular in recent years. The philosophy of categorification is to replace set-theoretic notions by corresponding category-theoretic notions and so obtain additional structure. Categorification should be viewed as the inverse process of decategorification, where the decategorification procedure that one wants to use has to be specified beforehand. Often, and also in this thesis, the involved decategorification is taking the Grothendieck group of a category or the graded Euler characteristic of a chain complex. Then categorification means that given a module we have to find a category whose Grothendieck group is this module, or given a polynomial we have to find a complex whose graded Euler characteristic is this polynomial.

In the late 1990's, Khovanov [Kho00] categorified the Jones polynomial using Khovanov homology. The Jones polynomial is a classical combinatorial invariant of knots and links. Khovanov homology gives a link invariant strictly stronger than the Jones polynomial in the sense that it can distinguish more links [BN02]. Motivated by this, various authors set out to categorify known structures in the hope of obtaining better invariants or invariants for higher dimensional manifolds.

The Temperley-Lieb algebra TL_n is an algebra over $\mathbb{Z}[q, q^{-1}]$, generated as a module by planar diagrams connecting n upper points to n lower points and a multiplication defined by stacking two elements on top of each other. As an algebra it is generated by elements

$U_i =$  .

The Temperley-Lieb algebra is a quotient of the Hecke algebra and plays an important role in the theory of knot invariants. While giving an alternative description of Khovanov homology, Bar-Natan categorified the Temperley-Lieb algebra in a cobordism language [BN05]. His construction is neither the first nor the only categorification of the Temperley-Lieb algebra [BFK99, Str05, Eli10], but the setting and its variations have been used extensively [MN08, Rus09, CK12, MT07].

The Temperley-Lieb algebra and also the Jones polynomial are closely connected to the representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$. The Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is the quantum enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ of complex trace zero square matrices of size two. The connection is given by the fact that

$$TL_n^{\mathbb{C}} \cong \mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V^{\otimes n}), \tag{1}$$

where $TL_n^{\mathbb{C}}$ is the Temperley-Lieb algebra with coefficients extended to $\mathbb{C}$ and V the quantum version of the natural representation. Using this, one can calculate the Jones polynomial by cutting a knot into certain generating pieces to which one assigns $\mathcal{U}_q(\mathfrak{sl}_2)$-linear maps from $V^{\otimes n}$ to $V^{\otimes m}$. Then the Jones polynomial is obtained as $f(1)$ where $f\colon V^{\otimes 0} \to V^{\otimes 0} = \mathbb{C}(q)$ is a $\mathcal{U}_q(\mathfrak{sl}_2)$-linear map determined by the knot. The isomorphism (1) turns $V^{\otimes n}$ into a $TL_n^{\mathbb{C}}$-module.

Our first goal in this thesis is to categorify the n-fold tensor product $V^{\otimes n}$, not only its endomorphism algebra $\mathrm{End}_{\mathcal{U}_q(\mathfrak{sl}_2)}(V^{\otimes n})$. We want to do this in a way such that we can see the standard basis as well as the action of the Temperley-Lieb algebra. The categorified Temperley-Lieb action should be given by the action of Bar-Natan's categorification. In other categorifications of $V^{\otimes n}$, [FKS06, FSS12], this action is not as direct.

The $\mathcal{U}_q(\mathfrak{sl}_2)$-module $V^{\otimes n}$ splits into weight spaces $(V^{\otimes n})_{2k-n}$ and thus we obtain a categorification of $V^{\otimes n}$ by categorifying each weight space for $k = 0, \dots, n$ and then taking the direct sum. The weight spaces have a special basis, the canonical basis [FK97], which can be described by cup diagrams. Cup diagrams are combinatorial objects given by planar diagrams consisting of half circles.

As a first step of our categorification, we adapt Bar-Natan's construction. The category $\mathrm{Cob}(n)$, which is the foundation of Bar-Natan's categorification of the Temperley-Lieb algebra, consists of objects given by Temperley-Lieb diagrams and morphisms given by cobordisms between them modulo some relations. In our setting, the canonical basis is the easiest to categorify. We categorify the canonical basis analogously to Bar-Natan's categorification of the Temperley-Lieb algebra by defining a category $\mathrm{Cup}(n,k)$ where the objects are given by cup diagrams with two different kinds of boundary points. The morphisms are cobordisms as in Bar-Natan's categorification but we impose additional relations involving the new boundary points. The new relations are motivated by a coloured TQFT that has been used to describe the cohomology of 2-block Spaltenstein varieties in [Scha12].

In this naive categorification, the canonical basis elements are categorified by certain cup diagrams $\mathrm{T}(\lambda)$, where λ is in a finite poset $\Lambda(n,k)$. Summing up the homomorphism spaces $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)$ yields an algebra, called the generalised Khovanov algebra. Generalised Khovanov algebras are a generalisation described by Stroppel in [Str09] of the algebras used in Khovanov's original categorification of the Jones polynomial [Kho00]. They have been extensively studied by Brundan and Stroppel in [BS11a, BS10, BS11b, BS12] and are connected to category $\mathcal{O}$ for $\mathfrak{gl}_n$ [BS11b].

A greater difficulty is the categorification of the standard basis of $V^{\otimes n}$. For that we have to go to the homotopy category $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ of bounded complexes with entries in $\widehat{\mathrm{Cup}}(n,k)$, where $\widehat{\mathrm{Cup}}(n,k)$ is some kind of additive closure of $\mathrm{Cup}(n,k)$ with grading constraints. In $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ we inductively define an exceptional sequence $\mathrm{V}^*(\lambda)$:

Theorem (Definition 6.1.1, Theorem 6.3.2)**.** *There are objects* $\mathrm{V}^*(\lambda)$*,* $\lambda \in \Lambda(n,k)$*, in*

$K^b(\widehat{\mathrm{Cup}}(k,n))$ *that form a graded exceptional sequence, i.e.*

$$\mathrm{Hom}_{K^b(\widehat{\mathrm{Cup}}(n,k))}\big(\mathrm{V}^*(\lambda), \mathrm{V}^*(\mu)\langle l\rangle[j]\big) = \begin{cases} \mathbb{C}, & \text{if } l = 0 = j \text{ and } \lambda = \mu, \\ 0, & \text{if } \lambda \not\geq \mu \text{ or } (l \neq 0 \text{ and } \lambda = \mu) \text{ or } (j \neq 0 \text{ and } \lambda = \mu). \end{cases}$$

Via a certain duality we obtain the complexes $\mathrm{V}(\lambda)$ which finally lead to a categorification of the standard basis:

Theorem (Theorem 7.2.9, Corollary 7.2.10)**.** *The category $K^b(\widehat{\mathrm{Cup}}(k,n))$ categorifies the $(2k-n)$-weight space of $V^{\otimes n}$. More precisely, there is an isomorphism of $\mathbb{C}(q)$-modules*

$$\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\big(K^b(\widehat{\mathrm{Cup}}(k,n))\big) \xrightarrow{\sim} \big(V^{\otimes n}\big)_{2k-n}.$$

Under this isomorphism the $\mathrm{V}(\lambda), \lambda \in \Lambda(n,k)$, are sent to the standard basis v_λ, the $\mathrm{T}(\lambda)$ to the canonical basis and the $\mathrm{V}^*(\lambda)$ to the dual standard basis. Furthermore, $\bigoplus_{k=1}^{n} K^b(\widehat{\mathrm{Cup}}(k,n))$ categorifies $V^{\otimes n}$.

The objects $\mathrm{T}(\lambda)$ lie in what is generated by the exceptional objects. We describe the $\mathrm{T}(\lambda)$ by taking iterated cones of the exceptional objects $\mathrm{V}^*(\mu)$ and giving explicit combinatorial formulas for the exceptional objects appearing in this construction.

Theorem (Theorem 7.1.3)**.** *The object $\mathrm{T}(\lambda)$ is an iterated cone of all the shifted exceptional objects of type $q^{\deg(\mathrm{C}(\lambda)\mu)}\,\mathrm{V}^*(\mu)$ with $\mathrm{C}(\lambda)\mu$ oriented.*

Here, $\mathrm{C}(\lambda)$ is a certain cup diagram and the degree of an orientation $\mathrm{C}(\lambda)\mu$ is also defined in a combinatorial way.

The Temperley-Lieb algebra contains a special idempotent p_n called the Jones-Wenzl projector, which is uniquely determined by

$$p_n^2 = p_n \qquad \text{and} \qquad p_n U_i = 0 = U_i p_n \qquad \text{for } i = 1, \ldots, n-1.$$

The Jones-Wenzl projectors can be used to define the coloured Jones polynomial [MV94]. They are also an important ingredient in the Turaev-Viro invariants of 3-manifolds [TV92].

Via the isomorphism (1) the Jones-Wenzl projector p_n can also be considered as a $\mathcal{U}_q(\mathfrak{sl}_2)$-linear map from $V^{\otimes n}$ to itself. On this side, the Jones-Wenzl projector factorises into a projection operator $\pi_n \colon V^{\otimes n} \to V_n$ composed with an inclusion operator $\iota_n \colon V_n \to V^{\otimes n}$, where V_n is the biggest indecomposable summand of $V^{\otimes n}$. The projection and inclusion satisfy the following properties

$$\iota_n(-).U_i = 0, \qquad \pi_n(-.U_i) = 0, \qquad \pi_n \circ \iota_n = \mathrm{id}, \tag{2}$$

which immediately yield the characterising properties for $p_n = \iota_n \circ \pi_n$.

Using Bar-Natan categorification of the Temperley-Lieb algebra, Cooper and Krushkal [CK12] and Rozansky [Roz10] categorified the Jones-Wenzl projector. Cooper-Krushkal defined a categorification of the Jones-Wenzl projector, called the universal projector, which is a certain chain complex unbounded in one direction and unique up to homotopy. In contrast to the bounded complexes appearing in Khovanov homology, the complex for the universal projector cannot be bounded, since the coefficients in the Jones-Wenzl projector are not polynomials but rational functions. They can be interpreted as infinite power series and so should lead to infinite complexes. The categorified Jones-Wenzl projectors of Cooper-Krushkal and Rozansky have been used to categorify the chromatic polynomial [CHK11] and spin networks [CK12, Hog12].

There is also a categorification of $V^{\otimes n}$ and the Jones-Wenzl projector in a representation theoretic setting by Frenkel, Stroppel and Sussan in [FSS12]. In contrast to [CK12] and [Roz10], there the Jones-Wenzl projector is categorified via a composition, which is missing in the description via universal projectors. A first step of matching the two constructions is to find the factorisation in a setup using the Bar-Natan approach. We also hope that this simplifies the calculations with categorified Jones-Wenzl projectors and so make it easier to calculate categorified spin networks in order to categorify the Turaev-Viro invariants.

Our second goal is to construct a categorification of the Jones-Wenzl projector where one can actually see the factorisation and to compare this with the action of the universal projector. For that we construct a special chain complex $L(\lambda_0)$ in $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ containing all the exceptional objects $\mathrm{V}^*(\lambda)$ in a non-trivial way, which has no analogue in the categorification of Cooper-Krushkal. For this construction we need to construct (up to scalar) unique degree 1 morphisms between the $\mathrm{V}^*(\lambda)$'s and consider how they give rise to degree 2 morphisms. The construction of $L(\lambda_0)$ is motivated by [BS10] seeing the $\mathrm{V}^*(\lambda)$ as a resolution of $L(\lambda_0)$. The actual definition is quite involved and done by showing that there are some (implicit) maps forming a complex which contains the $\mathrm{V}^*(\lambda)$'s. The differential restricted to neighbouring λ's is up to a sign just the explicitly constructed degree 1 map. The complex $L(\lambda_0)$ has the important property that the category $\mathrm{Cob}(n)$ categorifying the Temperley-Lieb algebra acts trivially on it.

Theorem (Definition 10.1.12, Remark 10.1.13, Theorem 10.3.1)**.** *There exists a chain complex $L(\lambda_0)$ in $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ such that $L(\lambda_0).\mathcal{U}_i \simeq 0$ for all i and $[L(\lambda_0)] = \left[\!\left[{n \atop k} \right]\!\right] p_n(v_{\lambda_0})$ in the Grothendieck group.*

Here, $[L(\lambda_0)]$ is the class of $L(\lambda_0)$ in the Grothendieck group $K_0\big(K^b(\widehat{\mathrm{Cup}}(k,n))\big)$ and v_{λ_0} is the standard basis element of $(V^{\otimes n})_{2k-n}$ that is also a canonical basis element. The quantum binomial coefficient $\left[\!\left[{n \atop k} \right]\!\right]$ is an element of $\mathbb{Z}[q,q^{-1}]$.

Using this complex $L(\lambda_0)$ and its endomorphism ring $\mathrm{End}\,\big(L(\lambda_0)\big)$ we can construct two functors satisfying the analogue of (2):

Theorem (Definition 11.1.2, Lemma 11.1.10, Remark 11.1.11, Definition 11.1.13, Lemma 11.1.16, Theorem 11.1.20)**.**
There are functors

$$F\colon K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big) \to D^{co,+}_{C'}\big(\,\mathrm{End}(L(\lambda_0))\text{-}\,\mathrm{gfmod}\,\big)$$

and

$$G\colon D^{co,+}_{C'}\big(\,\mathrm{End}(L(\lambda_0))\text{-}\,\mathrm{gfmod}\,\big) \to K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$$

such that

$$G(-).\mathcal{U}_i \simeq 0, \qquad F(-.\mathcal{U}_i) = 0, \qquad F \circ G_|(-) \cong \mathrm{Id}(-).$$

Here, K_C^- and $D_{C'}^{co,+}$ denote the homotopy category of chain complexes bounded from the right and the derived category of cochain complexes bounded from the left satisfying certain finiteness conditions C and C', respectively, that are needed to obtain finite sums in the construction of F and G. $G_|$ means that we obtain the last equation only when restricting to objects of a subcategory containing the image of F. But this is enough to derive $P \circ P \simeq P$ on objects for $P := G \circ F$ analogously to the uncategorified picture.

The theorem means that F and G "categorify" the projection π_n and the inclusion ι_n, respectively, and hence their composition $G \circ F$ the Jones-Wenzl projector. For a precise statement of this categorification in terms of Grothendieck groups one would need to apply the method of completions of Grothendieck groups from [AS13] to this context.

The main difficulty in the construction of these functors is as follows: In a chain complex of modules over a ring every entry would have elements, but $L(\lambda_0)$ is just a chain complex in an additive category. Therefore, we cannot view $L(\lambda_0)$ as an $\mathrm{End}\big(L(\lambda_0)\big)$-module. The functors F and G should be morally seen as a pair of contravariant functors adjoint to the right.

The universal projectors $\mathbf{P}(n)$ constructed by Cooper-Krushkal and Rozansky are inductively defined and are huge complexes that are impossible to write down explicitly for general n. The construction of Rozansky is in contrast to the one of Cooper-Krushkal inductive in the construction of the complex for a fixed n and does not rely on smaller values for n. Using this, we can explicitly calculate the result of applying the universal projector to $\mathrm{T}(\lambda_0)$ for $k = 0$ and $k = 1$. Since the properties of $\mathbf{P}(n)$ and $G \circ F$ yield automatically that they agree on other $\mathrm{T}(\lambda)$, they agree on all of $\widehat{\mathrm{Cup}}(n,k)$.

Theorem (Proposition 11.3.1, Theorem 11.3.14)**.** *For general n and $k = 0$ or $k = 1$ the functors $G \circ F$ and $(-).\mathbf{P}(n)$ are isomorphic as functors from $\widehat{\mathrm{Cup}}(n,k)$ to $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$.*

Here, $(-).\mathbf{P}(n)$ is the action of the universal projector which is induced by the action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$. Furthermore, $G \circ F$ and $(-).\mathbf{P}(n)$ have the complex $L(\lambda_0)$ as a common fixed point for all n and k.

There are two main difficulties in the programme of this thesis. Firstly, the morphisms in the category $\mathrm{Cup}(n,k)$ are very difficult to understand explicitly. We introduce a combinatorial method to determine the graded dimension of $\mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D)$ by constructing a circle diagram $D\overline{C}$ and applying a certain function $\mathcal{F}_{col}$. The function $\mathcal{F}_{col}$ should be considered as the object part of a certain functor defining a coloured TQFT as described in the appendix. But even with the knowledge of the dimension of the Hom-spaces it is difficult to obtain factorisations. We use the action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$ to better describe the morphisms of $\mathrm{Cup}(n,k)$ and to obtain a factorisation result.

The other main difficulty throughout the thesis is that the categories $\widehat{\mathrm{Cup}}(n,k)$ and $\widehat{\mathrm{Cob}}(n)$ are not abelian, only additive. To get abelian categories we consider the heart of certain t-structures in $K^b\big(\mathrm{Cup}(n,k)\big)$ which contain the $\mathrm{V}^*(\lambda)$'s and $\mathrm{T}(\lambda)$'s. One of the t-structures is motivated by the standard t-structure and the other one is defined by measuring how far a complex is away from being linear, called the linear t-structure. Another fact that shows the importance of the complex $L(\lambda_0)$, is that it is the injective hull of $\mathrm{T}(\lambda_0)$ in the heart of the linear t-structure.

Outline

We define the objects of study first combinatorially, then on the representation theory side and finally categorially. In Chapter 1, we introduce the combinatorial gadgets for our categorification. We define up-down-sequences, cup diagrams and the Temperley-Lieb algebra and investigate their interplay. Chapter 2 defines what we want to categorify. We connect the different bases in $V^{\otimes n}$ to the combinatorial objects from Chapter 1. Furthermore, we consider the special behaviour of the canonical basis under the action of the Jones-Wenzl projector.

In Chapter 3 we start categorifying. We recall Bar-Natan's construction of the category $\mathrm{Cob}(n)$ and the categorification of the Temperley-Lieb algebra. In analogy we define the category $\mathrm{Cup}(n, k)$, give an alternative description for the morphisms and obtain a first categorification result. Analogously to the action of TL_n on cup diagrams described in Chapter 1, in Chapter 4 we define an action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n, k)$ and use this action to describe morphisms in $\mathrm{Cup}(n, k)$ of low degree.

In Chapter 5, we fix notations in homological algebra and gather statements about homotopic complexes that will be used later. The most important concept is the Gaussian elimination, a method to obtain a homotopy equivalent chain complex after deleting certain entries. We recall some spectral sequence arguments to obtain a theorem that is repeatedly used to calculate homomorphism spaces of complexes in $K^b\big(\widehat{\mathrm{Cup}}(n, k)\big)$.

The goal of Chapter 6 is to construct the complexes $\mathrm{V}^*(\lambda)$ and show that they form a graded exceptional sequence. On the way we obtain many results on homomorphisms between $\mathrm{T}(\mu)$ and $\mathrm{V}^*(\lambda)$ in $K^b\big(\widehat{\mathrm{Cup}}(n, k)\big)$ that will be important later on. In Chapter 7, we write the $\mathrm{T}(\lambda)$'s as iterated cones of $\mathrm{V}^*(\mu)$'s, construct $\mathrm{V}(\mu)$ via duality and obtain an iterated cone description of the $\mathrm{T}(\lambda)$'s via the $\mathrm{V}(\mu)$'s from the duality for free. Using this and the naive categorification from Chapter 3, we obtain a categorification of $V^{\otimes n}$ with visible standard basis.

In Chapter 8, we define two t-structures on $K^b\big(\widehat{\mathrm{Cup}}(n, k)\big)$ that both contain the $\mathrm{V}^*(\lambda)$ in the heart. We show that the $\mathrm{T}(\lambda)$ are tilting objects in one of the hearts and simple in the other. Chapter 9 classifies degree 1 morphisms between different $\mathrm{V}^*(\lambda)$'s in $K^b\big(\widehat{\mathrm{Cup}}(n, k)\big)$ and gives an explicit construction. Furthermore, we examine how they give rise to degree 2 morphisms.

We start Chapter 10 by constructing the complex $L(\lambda_0)$. We show that U_i acts trivially on $L(\lambda_0)$, study $\mathrm{End}\big(L(\lambda_0)\big)$ and consider $L(\lambda_0)$ as a linear complex. Using this complex $L(\lambda_0)$ in Chapter 11, we construct the functor F. We define the functor G and consider the composition $F \circ G$. After recalling Cooper-Krushkal's universal projector and Rozansky's construction of it, we describe the action of the universal projector on $\mathrm{T}(\lambda_0)$ for small k to show that it agrees with applying $G \circ F$.

In the Appendix A we recall coloured cobordisms and coloured TQFT. We show that $\widehat{\mathrm{Cup}}(n,k)$ is equivalent to a category defined in analogy to generalised Khovanov algebras.

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Chapter 1

Combinatorics of $\mathbb{S}_n$

This chapter introduces the combinatorics and basic definitions used later on. Although some of the results are interesting on their own, the purpose of some might be unclear until they are used as a crucial ingredient in the proofs of later chapters.

We introduce minimal length coset representatives of $(\mathbb{S}_k \times \mathbb{S}_{n-k})\backslash\mathbb{S}_n$ and their connection to super standard tableaux and up-down-sequences. In particular, we study the natural partial ordering induced by the Bruhat order on the symmetric group. Moreover, we consider the action of the Temperley-Lieb algebra on cup diagrams. In particular, we define the degree of an oriented cup diagram and investigate how it is affected by the action. Lastly, we introduce circle diagrams and a function from them to vector spaces that is used in Chapter 3.

1.1 Minimal length coset representatives

Definition 1.1.1. By s_i we denote the simple transposition $(i, i+1)$ of the symmetric group $\mathbb{S}_n$ and by $l : \mathbb{S}_n \to \mathbb{Z}_+$ the usual length function with respect to simple transpositions. Fix $1 \leq k \leq n$ and denote by $W_{n,k}$ the parabolic subgroup $\mathbb{S}_k \times \mathbb{S}_{n-k}$ of $\mathbb{S}_n$. Let

$$\mathrm{W}^{\min} = \mathrm{W}^{\min}_{n,k} = \{z \in \mathbb{S}_n \mid l(s_j z) > l(z) \; \forall s_j \in W_{n,k}\}.$$

The following lemma explains the name *minimal coset representatives* for $\mathrm{W}^{\min}$, see e.g. [BB05, Corollary 2.4.5(i)] .

Lemma 1.1.2. *In each coset of $W_{n,k}\backslash\mathbb{S}_n$ there exists exactly one element of minimal length given by some $w \in \mathrm{W}^{min}$.*

Recall the following property of $\mathrm{W}^{\min}$, see e.g. [BB05, Lemma 2.3.4]:

Lemma 1.1.3. *An element $w \in \mathbb{S}_n$ belongs to W^{min} if and only if no reduced expression for w starts with $s_j \in W_{n,k}$.*

This means that every reduced expression of $w \in \mathrm{W}^{\min}$ starts with s_k. For example, $s_3 s_2 s_4 s_1 s_5 \in W^{\min}_{6,3}$.

We now consider further properties of elements of $\mathrm{W}^{\min}$ that will be used later.

Lemma 1.1.4. *Let $w \in \mathrm{W}^{min}$ and assume $ws_i \notin \mathrm{W}^{min}$ for some simple transposition s_i. Then $l(ws_i) > l(w)$.*

Proof. Since $w \in \mathrm{W}^{\min}$, we have $l(s_j w) > l(w)$ for all $s_j \in W_{n,k}$, thus

$$l(s_j w) = l(w) + 1 \tag{1.1}$$

for all $s_j \in W_{n,k}$. From $ws_i \notin \mathrm{W}^{\min}$ we obtain that there exists some $s_{j_0} \in W_{n,k}$ with $l(s_{j_0} ws_i) < l(ws_i)$, i.e.

$$l(s_{j_0} ws_i) = l(ws_i) - 1.$$

With $l(ws_i) \leq l(w) + 1$, this yields $l(s_{j_0} ws_i) \leq l(w)$. On the other hand,

$$l(s_{j_0} ws_i) \geq l(s_{j_0} w) - 1 = l(w),$$

where the equality follows from (1.1). Altogether, we have $l(s_{j_0} ws_i) = l(w)$ and thus $l(ws_i) > l(s_{j_0} ws_i) = l(w)$. □

Remark 1.1.5. Let $s_{i_1} \dots s_{i_r}$ and $s_{j_1} \dots s_{j_r}$ be two reduced expressions of some $w \in \mathbb{S}_n$. By [Mat99, Theorem 1.8] we can pass from one reduced expression to another using only the braid relations

$$s_m s_l = s_l s_m \text{ for } |m - l| > 1 \tag{1.2}$$
$$s_l s_m s_l = s_m s_l s_m \text{ for } |m - l| = 1. \tag{1.3}$$

Corollary 1.1.6. *Assume $w \in \mathrm{W}^{min} = \mathrm{W}^{min}_{n,k}$ such that $ws_i \notin \mathrm{W}^{min}$. Then there is a reduced expression of w of one of the following forms*

- $w = s_{i_1} \dots s_{i_r}$ *(and in particular $i \neq k$)* *or*
- $w = s_{l_1} \dots s_{l_t} s_i s_{i\pm1} s_{i_1} \dots s_{i_r}$ *with $l_1, \dots, l_t \in \{1, \dots, n-1\}$*

for some $i_1, \dots, i_r \in \{1, \dots, n-1\}$ with $|i_j - i| > 1$ for all $j = 1, \dots, r$.

Example 1.1.7. Consider $w = s_3 s_2 s_4 s_1 s_5 \in W^{\min}_{6,3}$. Then $ws_2 \notin W^{\min}_{6,3}$, since

$$\begin{aligned} s_3 s_2 s_4 s_1 s_5 s_2 &= s_3 s_2 s_4 s_1 s_2 s_5 = s_3 s_2 s_1 s_4 s_2 s_5 \\ &= s_3 s_2 s_1 s_2 s_4 s_5 = s_3 s_1 s_2 s_1 s_4 s_5 = s_1 s_3 s_2 s_1 s_4 s_5 \end{aligned}$$

and we have the reduced expression $s_3 s_2 s_1 s_4 s_5$ of w which satisfies the second case of the corollary.

Proof(Corollary). Let $s_{j_1} \dots s_{j_p}$ be a reduced expression of w. By the previous lemma we know that $s_{j_1} \dots s_{j_p} s_i$ is a reduced expression of ws_i which can be transformed to a reduced expression starting with s_q, $q \neq k$, using the braid relations by assumption. If this transformation can be achieved using only the relation (1.2), we are in case 1 (and $i = q$). In the other case, where we have to use at least one relation of type (1.3), assume $m_1, \dots, m_a$ is a minimal set of braid relations for the transformation. If $m_1, \dots, m_x$, $x \leq a$, only move the s_i by relations of type (1.2), then there are only s_{j_y}

with $|j_y - i| > 1$ to the right of the s_i, since the braid relations do not change the set of appearing indices. Applying this to $x = a$, we see that there has to be some first m_z of type (1.3) involving the s_i, since otherwise we could permute the s_i to the right again and would gain a reduced expression of w starting with s_q, $q \neq k$, in contradiction to $w \in \mathrm{W}^{\min}$. After having applied $m_1, \ldots, m_{z-1}$, the reduced expression has to be of the form

$$s_{l_1} \ldots s_{l_t} s_i s_{i\pm 1} s_i s_{i_1} \ldots s_{i_r}$$

and by the considerations above we know $|i_j - i| > 1$ for $j = 1, \ldots, r$. Thus

$$ws_i = s_{l_1} \ldots s_{l_t} s_i s_{i\pm 1} s_{i_1} \ldots s_{i_r} s_i$$

and w has a reduced expression of the desired second form.

□

Definition 1.1.8. We define the *super (n,k)-box tableau* as a $k \times (n-k)$-box, where the top-left entry is k and the entries increase by 1 in the rows from left to right and decrease by 1 in the columns from top to bottom.

Example 1.1.9. The super $(9,5)$-box tableau is $\begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline 4&5&6&7\\\hline 3&4&5&6\\\hline 2&3&4&5\\\hline 1&2&3&4\\\hline\end{array}$.

Definition 1.1.10. Let $Y(n,k)$ be the set of tableaux contained in a super (n,k)-box tableau top and left aligned.

Example 1.1.11. $\begin{array}{|c|c|c|}\hline 5&6&7\\\hline 4&5\\\cline{1-2}\end{array} \in Y(9,5)$, since $\begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline 4&5&6&7\\\hline 3&4&5&6\\\hline 2&3&4&5\\\hline 1&2&3&4\\\hline\end{array}$.

Remark 1.1.12. Note that $Y(n,k) \cap Y(n,k') = \varnothing$ for $k \neq k'$, but $Y(n,k)$ and $Y(n',k)$ have nontrivial intersection. For instance

$$\begin{array}{|c|c|c|c|c|}\hline 5&6&7&8&9\\\hline 4&5&6\\\cline{1-3} 3&4&5\\\cline{1-3} 2&3\\\cline{1-2}\end{array} \in Y(10,5) \cap Y(11,5)$$

(but not in $Y(9,5)$).

In other words, given $T \in Y(n,k)$ one can determine the value of k, but n has to be given separately, since T only gives a lower bound for n.

Definition 1.1.13. Given a tableau $T \in Y(n,k)$, the row reading word defines an element in the symmetric group, denoted by $\mathbf{s}(T)$, by sending an entry i to the simple transposition s_i.

Example 1.1.14. $\mathbf{s}\left(\begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline 4&5&6\\\cline{1-3} 3&4&5\\\cline{1-3} 2&3\\\cline{1-2}\end{array}\right) = s_5s_6s_7s_8s_4s_5s_6s_3s_4s_5s_2s_3 \in \mathbb{S}_n$ for $n \geq 9$.

Note that $\mathbf{s}(T)$ is always a reduced expression.

Lemma 1.1.15. *The $w \in \mathrm{W}^{min}$ are in bijection to $Y(n,k)$ via $Y(n,k) \ni T \mapsto \mathbf{s}(T)$.*

Proof. This follows directly from [Str05, Prop A2], since tableaux in $Y(n,k)$ are determined by the rightmost entries of the rows, thus $Y(n,k)$ is in bijection to the $S(n,k)$ defined there. □

Definition 1.1.16. Let

$$\Lambda(n,k) = \left\{\mathbf{a} = a_1a_2\dots a_n \mid a_i \in \{\wedge,\vee\}, \{a_1,\dots,a_n\} = \{\underbrace{\wedge,\dots,\wedge}_{k},\underbrace{\vee,\dots,\vee}_{n-k}\}\right\}.$$

We call a $\lambda \in \Lambda(n,k)$ a $\wedge\vee$-*sequence* or more precisely an (n,k)-$\wedge\vee$-*sequence.* We write $\lambda(i)$ for the ith entry of λ.

We denote by λ_0 the element $\underbrace{\wedge,\dots,\wedge}_{k},\underbrace{\vee,\dots,\vee}_{n-k}$.

Example 1.1.17. $\Lambda(4,2) = \{\wedge\wedge\vee\vee, \wedge\vee\wedge\vee, \vee\wedge\wedge\vee, \wedge\vee\vee\wedge, \vee\wedge\vee\wedge, \vee\vee\wedge\wedge\}$.

Note that the cardinality of $\Lambda(n,k)$ is $\binom{n}{k}$.

Lemma 1.1.18. *There is a canonical bijection $\varphi : \Lambda(n,k) \to \mathrm{W}^{min}$ sending λ_0 to e.*

Proof. $\mathbb{S}_n$ obviously acts on a $\Lambda(n,k)$ from the right by permutation of the a_i's and every $\wedge\vee$-sequence is in the orbit of λ_0. The stabiliser of λ_0 is $\mathbb{S}_k \times \mathbb{S}_{n-k} = W_{n,k}$. So $\Lambda(n,k)$ is in bijection to $W_{n,k}\backslash\mathbb{S}_n$, which in turn is in bijection to $\mathrm{W}^{\min}$ by Lemma 1.1.2. □

Lemma 1.1.19. *Two reduced expressions of the same element in W^{min} are related by a finite sequence of moves $s_is_j = s_js_i$ for $|i-j| > 1$.*

Proof. By Remark 1.1.5 it is enough to show that no reduced expression of an element in $\mathrm{W}^{\min}$ contains the subword $s_is_{i\pm1}s_i$. Assume there is an element $w \in \mathrm{W}^{\min}$ with reduced expression $s_{i_1}\dots s_{i_r}s_is_{i\pm1}s_is_{j_1}\dots s_{j_t}$. By Lemma 1.1.3, also $s_{i_1}\dots s_{i_r}s_is_{i\pm1}s_i$, $s_{i_1}\dots s_{i_r}s_is_{i\pm1}$ and $s_{i_1}\dots s_{i_r}s_i$ are reduced expressions of different elements in $\mathrm{W}^{\min}$. Under the bijection of Lemma 1.1.18, $s \in \mathrm{W}^{\min}$ is send to $\lambda_0 s$. Let $\lambda = \lambda_0 s_{i_1}\dots s_{i_r}$. By considering all the possible $\wedge\vee$-sequence at places $i, i+1, i+2$ resp. $i-1, i, i+1$ in λ we see that two out of λs_i, $\lambda s_is_{i\pm1}$ and $\lambda s_is_{i\pm1}s_i$ have to be equal. But this is a contradiction to the isomorphism. □

Corollary 1.1.20. *There is a bijection $\varphi' : Y(n,k) \to \Lambda(n,k)$ given by $T \mapsto \lambda_0\mathbf{s}(T)$.*

Proof. This is the composition of Lemma 1.1.15 and Lemma 1.1.18. □

Lemma 1.1.21. *Let $T \in Y(n,k)$. We can read off the $\wedge\vee$-sequence $\varphi'(T) \in \Lambda(n,k)$ from the tableaux T embedded in the box super tableau: We start at the lower left corner of the box Young tableau, then go up until we reach the embedded tableau, follow its contours and then go right until we reach the right upper corner of the box Young tableau. This path gives a $\wedge\vee$-sequence by associating a $\wedge$ to going one step up and a $\vee$ to going one step right.*

Example 1.1.22.

7	8	9	10	11	12
6	7	8	9	10	11
5	6	7	8	9	10
4	5	6	7	8	9
3	4	5	6	7	8
2	3	4	5	6	7
1	2	3	4	5	6

$\rightsquigarrow \quad \wedge\wedge\vee\wedge\vee\vee\wedge\vee\vee\wedge\wedge\vee\wedge$

Proof. We prove this by induction on the number of boxes in T. If there is no box, the contour path obviously gives $\lambda_0 = \lambda_0 e$. Assume that the assertion is true when we remove one box. Consider the rightmost box in the lowest row of T and assume it is labelled by i. Since all the boxes with the same label are on a diagonal, the right step given by this box is the ith step in the contour path. Let T' be the tableau without the box. Now, the contour path of T differs by from the one of T' by changing $\vee\wedge$ at places i, $i+1$ to $\wedge\vee$. Since $\mathbf{s}(T) = \mathbf{s}(T')s_i$ and $\lambda_0\mathbf{s}(T')$ is the contour path of T' by induction, the assertion follows. □

Altogether we have the bijections

$$\begin{array}{ccccc} Y(n,k) & \longleftrightarrow & W^{\min}_{n,k} & \longleftrightarrow & \Lambda(n,k) \\ T & \longmapsto & \mathbf{s}(T), \quad w & \xmapsto{\varphi^{-1}} & \lambda_0 w \\ & & & & \text{contour of } T \end{array} \qquad T \xmapsto{\varphi'} \text{contour of } T \tag{1.4}$$

Example 1.1.23. For $n = 8$ and $k = 4$ we have

4	5	6	7
3	4	5	
2	3	4	
1			

$\mapsto \quad s_4s_5s_6s_7s_3s_4s_5s_2s_3s_4s_1 \quad \mapsto \quad \vee\wedge\vee\vee\wedge\wedge\vee\wedge.$

Definition 1.1.24. The *Bruhat order* on $\mathrm{W}^{\min}$ is the partial order defined as follows: For $w, y \in \mathrm{W}^{\min}$ we say $w < y$ if there is a reduced expression $s_{j_1} \ldots s_{j_t}$ of w and some $s_{i_1}, \ldots, s_{i_r}$ such that $s_{j_1} \ldots s_{j_t} s_{i_1} \ldots s_{i_r}$ is a reduced expression of y.

This induces a partial order on $\Lambda(n,k)$: For $\lambda, \mu \in \Lambda(n,k)$ we say that $\lambda < \mu$ if $\varphi(\lambda) < \varphi(\mu)$. With this definition λ_0 is minimal, since $\varphi(\lambda_0) = e$.

Remark 1.1.25. Recall that $l(w)$ for $w \in \mathbb{S}_n$ is equal to the number of inversions of w. Under the bijection

$$\begin{aligned} \mathrm{W}^{\min} &\leftrightarrow \Lambda(n,k) \\ w &\mapsto \lambda_0 w \end{aligned}$$

the number of inversions correlates to the number of transpositions of $\wedge\vee$ to $\vee\wedge$ that are used to go from λ_0 to $\lambda_0 w$. In particular, this means for $ws_i > w$ with $w, ws_i \in \mathrm{W}^{\min}$ that $\lambda_0 w$ has the labels $\wedge\vee$ at places i and $i+1$ and $\lambda_0 ws_i$ has the labels $\vee\wedge$ at places i and $i+1$. Thus, $\lambda < \mu$ if μ arises from λ by a sequence of swapping neighbouring $\wedge\vee$ to $\vee\wedge$.

Note that our definition of $\lambda < \mu$ is reversed to the Bruhat order in [BS10].

Example 1.1.26. In $\Lambda(4,2)$ we have

$$\begin{array}{ccccccccc} & & & & \vee\wedge\wedge\vee & & & & \\ & & & \swarrow & & \nwarrow & & & \\ \wedge\wedge\vee\vee & < & \wedge\vee\wedge\vee & & & & \vee\wedge\vee\wedge & < & \vee\vee\wedge\wedge \\ & & & \nwarrow & & \swarrow & & & \\ & & & & \wedge\vee\vee\wedge & & & & \end{array}$$

and $\vee\wedge\wedge\vee$ and $\wedge\vee\vee\wedge$ are not related.

Definition 1.1.27. We write $\mu \xrightarrow{s_i} \lambda$ (or just $\mu \to \lambda$) if $\lambda > \mu$ and $\lambda = \mu s_i$.

We say that μs_i is *undefined*, if $\varphi(\mu)s_i \notin \mathrm{W}^{\min}$. This is the same as saying that the application of s_i does not change μ.

We say there is a *path from μ to λ*, $\mu \rightsquigarrow \lambda$, if there is a sequence $\lambda_0, \ldots, \lambda_r$ such that $\lambda_0 = \mu$, $\lambda_r = \lambda$ and $\lambda_{i-1} \to \lambda_i$ for $i = 1, \ldots, r$. Hence, $\mu < \lambda$ if and only if there is a path from μ to λ.

Example 1.1.28. In this language, the previous example morphs to

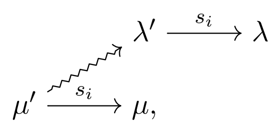
$$\begin{array}{ccccccccc} & & & & \vee\wedge\wedge\vee & & & & \\ & & & \overset{s_1}{\nearrow} & & \overset{s_3}{\searrow} & & & \\ \wedge\wedge\vee\vee & \xrightarrow{s_2} & \wedge\vee\wedge\vee & & & & \vee\wedge\vee\wedge & \xrightarrow{s_2} & \vee\vee\wedge\wedge \\ & & & \underset{s_3}{\searrow} & & \underset{s_1}{\nearrow} & & & \\ & & & & \wedge\vee\vee\wedge & & & & \end{array} .$$

Furthermore, $\vee\wedge\wedge\vee s_2$ is undefined and there is a path for example from $\wedge\vee\wedge\vee$ to $\vee\wedge\vee\wedge$.

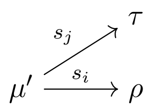
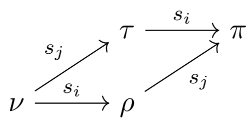
Now we show different facts concerning the previous definitions that will be used in proofs later on.

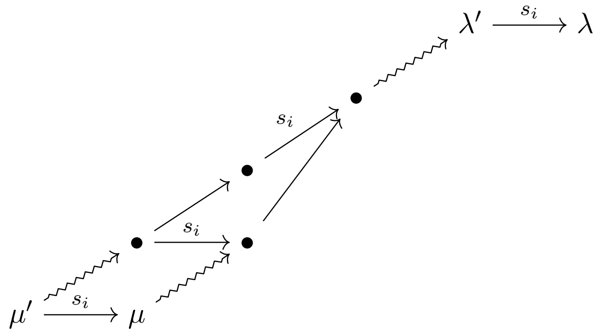
Lemma 1.1.29. *Let $\mu' \leq \lambda'$ and $\lambda' \xrightarrow{s_i} \lambda$, $\mu' \xrightarrow{s_i} \mu$ for some i. Then $\mu \leq \lambda$.*

Proof. The case $\mu' = \lambda'$ is clear. If $\mu' < \lambda'$ then there is a path $\mu' \rightsquigarrow \lambda'$. If this path starts with $\xrightarrow{s_i}$, we have $\mu = \mu' s_i \leq \lambda' < \lambda$ and we are done. So assume that this is not the case, then

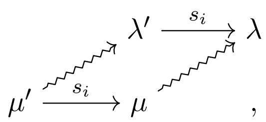


where μ is not on the path $\mu' \rightsquigarrow \lambda'$. We show now that there is also a path from μ' to λ via μ. This will give $\mu \leq \lambda$ as desired.

Consider the following  . Since we can apply s_i and s_j, $i \neq j$, to μ', we have $|i-j| > 1$, because otherwise $\mu'(i) = \wedge$, $\mu'(j) = \wedge$, $\mu'(i+1) = \vee$, $\mu'(j+1) = \vee$ is not possible. Therefore, we can apply s_j to ρ and s_i to τ and arrive at some π:  . We can now apply this to $\mu' \rightsquigarrow \lambda'$ until s_i appears



and get $\mu < \lambda' < \lambda$. If s_i does not appear in $\mu' \rightsquigarrow \lambda'$, we obtain



and again $\mu \leq \lambda$.

□

Corollary 1.1.30. *Let $\lambda \ngeq \mu$ (i.e. $\lambda < \mu$ or they are not comparable) and $\lambda' \xrightarrow{s_i} \lambda$. Then μs_i is undefined or $\lambda' \ngeq \mu$ and $\lambda' \ngeq \mu s_i$.*

Proof. Assume $\lambda' \geq \mu$, then $\lambda \geq \mu$ since $\lambda > \lambda'$ and we have a contradiction. Assume μs_i is defined and $\lambda' \geq \mu s_i$, then $\mu s_i < \mu$ since we already know $\lambda' \ngeq \mu$. Applying the previous lemma for $\mu' = \mu s_i$ we obtain $\mu \leq \lambda$ and thus a contradiction. □

Definition 1.1.31. The *relative length* between $\lambda, \mu \in \Lambda(n,k)$ is defined, following [BS10], as

$$\ell(\lambda,\mu) := \sum_{i=1}^{n} \ell_i(\lambda,\mu),$$

where

$$\ell_i(\lambda,\mu) := \#\{j \mid j \le i \text{ and } \lambda(j) = \vee\} - \#\{j \mid j \le i \text{ and } \mu(j) = \vee\}$$

for $0 \le i \le n$.

So $\ell_i(\lambda,\mu)$ counts the relative inversion, i.e. how many more $\vee$'s there are in λ compared to μ at the positions to the left or equal to the ith place. We note that $\lambda \ge \mu$ if and only if $\ell_i(\lambda,\mu) \ge 0$ for all $1 \le i \le n$.

If $\lambda \ge \mu$, the relative length $\ell(\lambda,\mu)$ is just the minimum number of transpositions of neighbouring $\vee\wedge$ pairs needed to get from λ to μ. Furthermore, if $\lambda \ge \mu$, then $\ell(\lambda,\mu) = \ell(\lambda,\lambda_0) - \ell(\mu,\lambda_0)$. Note that for $w \in \mathrm{W}^{\min}$ and $\lambda = \lambda_0 w$ we have $\ell(\lambda,\lambda_0) = l(w)$.

For example, $\ell(\vee\wedge\wedge\vee, \wedge\vee\vee\wedge) = 1 + 0 - 1 + 0 = 0$ and $\ell(\vee\wedge\vee\wedge, \wedge\vee\wedge\vee) = 2$.

Lemma 1.1.32. *Let $\lambda \to \lambda s_i$, $\mu \to \mu s_i$ and $\mu < \lambda$. Then $\mu s_i \not\geq \lambda$.*

Proof. Since $\mu < \lambda$, we have in particular $\ell(\mu,\lambda_0) < \ell(\lambda,\lambda_0)$. Moreover, $\ell(\mu s_i,\lambda_0) = \ell(\mu,\lambda_0) + 1 \le \ell(\lambda,\lambda_0)$. From $\lambda < \lambda s_i$ we know that $\mu s_i \ne \lambda$. If $\mu s_i > \lambda$, then $0 < \ell(\mu s_i,\lambda) = \ell(\mu s_i,\lambda_0) - \ell(\lambda,\lambda_0)$, but this is a contradiction. □

Lemma 1.1.33. *Let $\lambda \to \lambda s_i$, $\mu \to \mu s_i$, $\lambda \ne \mu$ and $\ell(\lambda,\lambda_0) = \ell(\mu,\lambda_0)$. Then μ and λs_i are not comparable.*

Proof. Assume μ and λs_i are comparable. If $\lambda s_i \le \mu$, then $\lambda < \lambda s_i \le \mu$ and we get a contradiction. Hence, $\lambda s_i > \mu$. From $\ell(\mu,\lambda_0) + 1 = \ell(\lambda,\lambda_0) + 1 = \ell(\lambda s_i,\lambda_0)$ we obtain $\ell(\lambda,\mu) = \ell(\lambda,\lambda_0) - \ell(\mu,\lambda_0) = 1$, hence λs_i has to come from μ by one transposition of a pair $\vee\wedge$, i.e. $\mu \xrightarrow{s_j} \lambda s_i$ for some j. But from $\lambda \to \lambda s_i$, $\mu \to \mu s_i$ we know $\lambda(i) = \wedge$, $\lambda(i+1) = \vee$ and $\mu(i) = \wedge$, $\mu(i+1) = \vee$. So for getting λs_i from μ after applying s_j the only possibility is $j = i$. Thus $\mu s_i = \lambda s_i$, which is a contradiction to $\mu \ne \lambda$. □

Lemma 1.1.34. *If $\lambda \xrightarrow{s_i} \lambda s_i \xrightarrow{s_{i+1}} \lambda s_i s_{i+1}$, then the tableaux $T' = \varphi'^{-1}(\lambda s_i s_{i+1})$ is obtained from $T = \varphi'^{-1}(\lambda)$ by adding two boxes with content i resp. $i+1$ at the end of a common row of T.*

If $\lambda \xrightarrow{s_i} \lambda s_i \xrightarrow{s_{i-1}} \lambda s_i s_{i-1}$, then the tableaux $T' = \varphi'^{-1}(\lambda s_i s_{i-1})$ is obtained from $T = \varphi'^{-1}(\lambda)$ by adding a box with content i at the end of some row and a box with content $i-1$ at the end of the row below and directly below the box with i.

Proof. Since $T = \varphi'^{-1}(\lambda)$, by (1.4) we have $\lambda = \lambda_0 \mathbf{s}(T)$. We want to find T' with $\lambda_0 \mathbf{s}(T') = \lambda s_i s_{i+1} = \lambda_0 \mathbf{s}(T) s_i s_{i+1}$, i.e. with $\mathbf{s}(T') = \mathbf{s}(T) s_i s_{i+1}$. So we have to add two boxes labelled i resp. $i+1$ to T to get T'. The index i determines the label of the box to be added and hence the diagonal where it can be added. So its position is unique and then the box with entry $(i+1)$ must be next to it in the same row.

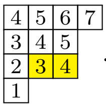
For example $T = \begin{array}{|c|c|c|c|}\hline 4&5&6&7\\\hline 3&4&5\\\cline{1-3} 2\\\cline{1-1} 1\\\cline{1-1}\end{array}$ and $i = 3$, then $T' = \begin{array}{|c|c|c|c|}\hline 4&5&6&7\\\hline 3&4&5\\\cline{1-3} 2&3&4\\\cline{1-3} 1\\\cline{1-1}\end{array}$.

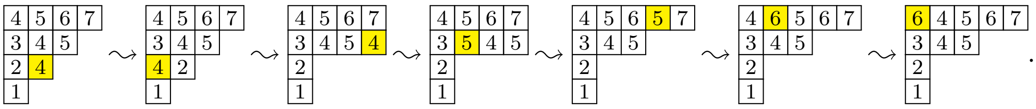
The second case where the boxes have to be added in the same column works similar. □

Lemma 1.1.35. *If* $w \in \mathrm{W}^{min}$ *has a reduced expression* $s_{i_1} \ldots s_{i_r} s_i s_{i\pm 1}$ *it follows that* $s_{i_1} \ldots s_{i_r} s_{i\pm 1} \notin \mathrm{W}^{min}$.

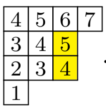
Proof. Consider $T \in Y(n,k)$ such that $w = \mathbf{s}(T)$, i.e. $\mathbf{s}(T)$ and $s_{i_1} \ldots s_{i_r} s_i s_{i\pm 1}$ are reduced expressions of w. Instead of deleting the s_i in $s_{i_1} \ldots s_{i_r} s_i s_{i\pm 1}$ we delete it in $\mathbf{s}(T)$ and then use the relations of $\mathbb{S}_n$ to show that the result has a reduced expression starting with something other than s_k. To better visualise the transformation of the reduced expression, we stay in the language of boxes in tableaux and move these around using the relations of $\mathbb{S}_n$.

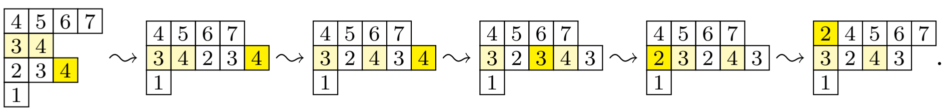
First we consider the case $\pm = +$. By Lemma 1.1.34, in T there is a row r that contains i $(i+1)$ associated to the $s_i s_{i+1}$ at the end of $s_{i_1} \ldots s_{i_r} s_i s_{i+1}$. After we delete i, we can move $i+1$ to the front of the row using (1.2). If r is the first row, then $i+1 \neq k$ since we moved it past the first box. If r is not the first row, we can move $i+1$ to the end of the row above r. Because $T \in Y(n,k)$, we know $(i+1)$ $(i+2)$ has to be in this row. We can change $i+1$ to $i+2$ and move it past them (using (1.3)). Now we move $i+2$ to the front and iterate the argument until we reach the first row.

For example, let $i = 3$ and $T =$  . Then r is the third row from the top and



Now consider the case $\pm = -$. Again by the previous lemma, there is a row r in T that contains $i-1$ (and not i) and in the row r' above there is i and nothing bigger. If we delete i, the row r' is either empty or contains $i-1$ at the end. If the row r' is empty, the $i-1$ is the only entry of the row r and can now move one row up. If r' was the first row, then now the upper left box is not k and we are done. If r' was not the first row, then $i-1$ is by 2 smaller than all the entries above and can be moved by (1.2) until it is the upper left box. Since it still cannot be k, we are again finished. Now assume r' is not empty. Then the rows r' and r have entries $i-t, i-t+1, \ldots, i-2, i-1, i-t-1, i-t, \ldots i-1$. By moving the elements of r' as much as possible to the right by (1.2) and then applying (1.3) repeatedly we get $i-t-1, (i-t, i-t-1), (i-t+1, i-t), (i-t+2, i-t+1), \ldots, (i-2, i-3), (i-1, i-2)$. Now the row r' starts by $i-t-1$ and the entries the rows above r' are all bigger than $i-t$. Thus, $i-t-1$ can be moved by (1.2) to the upper left box and is not equal to k.

For example, let $i = 5$ and $T =$  . Then r' is the second row from the top, r the third and



□

1.2 Combinatorics with cup diagrams and Temperley-Lieb algebras

Definition 1.2.1. The *Temperley-Lieb algebra* TL_n is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $U_1, \ldots, U_{n-1}$ subject to the following relations:

$$U_i^2 = [2]U_i \tag{1.5}$$
$$U_iU_j = U_jU_i, |i-j| > 1 \tag{1.6}$$
$$U_iU_{i\pm1}U_i = U_i, \tag{1.7}$$

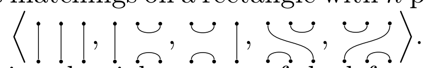
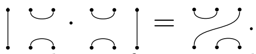
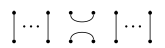
where the quantum integer [2] is defined by $[2] = q + q^{-1}$.

The row reading word defines a Temperley-Lieb element similar to Definition 1.1.13.

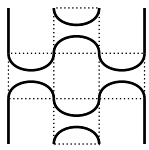
Definition 1.2.2. Let $T \in Y(n, k)$, then the row reading word defines an element in TL_n, denoted by $\mathbf{U}(T)$, by sending an entry i to U_i.

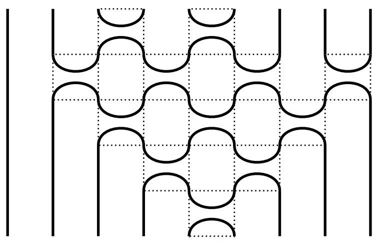
Example 1.2.3. $\mathbf{U}\left(\begin{array}{|c|c|c|c|} \hline 5&6&7&8 \\ \hline 4&5&6 \\ \cline{1-3} 3&4&5 \\ \cline{1-3} 2&3 \\ \cline{1-2} \end{array}\right) = U_5U_6U_7U_8U_4U_5U_6U_3U_4U_5U_2U_3 \in TL_n$ for $n \geq 9$.

Remark 1.2.4. From the definition and the bijections from the previous section, we get: If $s_{i_1} \ldots s_{i_r} \in \mathrm{W}^{\min}$ is reduced and $U = U_{i_1} \ldots U_{i_r}$, then $U = \mathbf{U}(T)$ for $T \in Y(n, k)$.

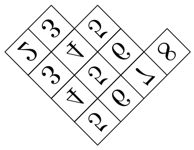
Remark 1.2.5. Recall that TL_n can also be seen diagrammatically as the $\mathbb{Z}[q, q^{-1}]$-module with basis crossingless matchings on a rectangle with n points each on two opposite sides, for example $TL_3 =$  Multiplication of basis elements is performed by putting the right on top of the left and replacing every closed internal loop by the factor [2], regarding a wiggled line as the same as a straightened line. For example,  Obviously, the diagrams  generate TL_n as an algebra and satisfy precisely the relations (1.5)–(1.7) of the U_i's, where i denotes that the cup starts at the ith place.

In this diagrammatic point of view we have a nice description of $\mathbf{U}(T)$ for $T \in Y(n, k)$:

$\mathbf{U}\left(\begin{array}{|c|c|} \hline 2&3 \\ \hline 1&2 \\ \hline \end{array}\right) =$  $\in TL_4$

$\mathbf{U}\left(\begin{array}{|c|c|c|c|} \hline 5&6&7&8 \\ \hline 4&5&6 \\ \cline{1-3} 3&4&5 \\ \cline{1-3} 2&3 \\ \cline{1-2} \end{array}\right) =$  $\in TL_9$

The first row of the tableau corresponds to the lowest diagonal starting at the lowest  and going up diagonally to the right. The second row corresponds to the second diagonal, and so on. So in total the shape of the diagram corresponds to reflecting the tableau and then rotating so that the former upper left box is the lowest:

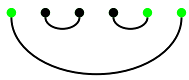
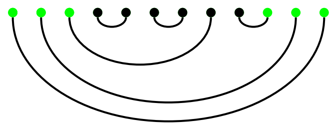


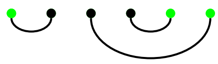
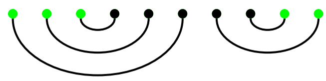
Following [BS11a] and [Scha12] we define cup diagrams:

Definition 1.2.6. Let $\lambda \in \Lambda(n,k)$. The *(extended) cup diagram* $\mathrm{C}(\lambda)$ associated to λ is defined as follows: We enlarge λ by adding k $\vee$'s on the left and $n-k$ $\wedge$'s on the right, i.e. $\underbrace{\vee \dots \vee}_{k} \lambda \underbrace{\wedge \dots \wedge}_{n-k}$,. Then we build the diagram inductively by connecting any adjacent pair $\vee\wedge$ in the region below, and then continuing the process for the sequence with these points excluded. To remember the added $\vee$'s and $\wedge$'s that do not belong to λ, we color the points at the associated position green.

When all $\vee\wedge$'s are connected, no unmatched $\wedge$'s or $\vee$'s can remain. So extending the sequence allows to connect everything with cups.

Example 1.2.7.

$\mathrm{C}(\vee\wedge\vee) =$  , $\mathrm{C}(\vee\wedge\vee\wedge\wedge\vee) =$ 

$\mathrm{C}(\wedge\vee\vee) =$  , $\mathrm{C}(\wedge\wedge\wedge\vee\vee) =$ 

Note that in $\mathrm{C}(\lambda_0)$, the first k black points are connected to left green points and the others to right green points.

Definition 1.2.8. Let $L_{n,k}$ be $2n$ points on a line where the leftmost k and rightmost $n-k$ are coloured green, the rest black.

Example 1.2.9. $L_{7,3} =$

Definition 1.2.10. We define the *set of extended cup diagrams* $eC(n,k)$ as all the crossingless matchings of $L_{n,k}$ with the condition that every arc has at least one black point above it or at the endpoint.

Remark 1.2.11. Note that the condition of having at least one black point below (including endpoints) forbids arcs between two left green points or two right green points.

From the definitions we immediately obtain:

Lemma 1.2.12. *There is a bijection between* $\Lambda(n,k)$ *and* $eC(n,k)$ *given by* $\lambda \mapsto \mathrm{C}(\lambda)$.

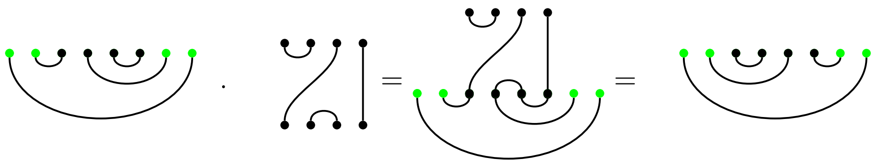
Definition 1.2.13. Let $\widehat{eC}(n,k)$ be the $\mathbb{Z}[q,q^{-1}]$-module with basis $eC(n,k)$.

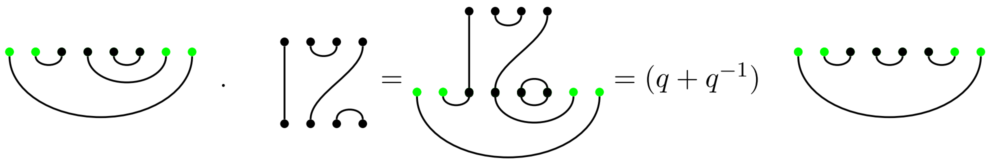
We recall the following well-known crucial fact:

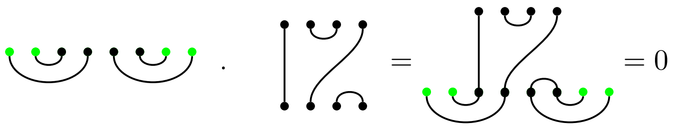
Proposition 1.2.14. *There is a (right) action of TL_n on $\widehat{eC}(n,k)$ given by putting the TL_n diagram on top of an extended cup diagram and smoothing, using $\bigcirc = q+q^{-1}$. If the result is not in $\widehat{eC}(n,k)$, we define the result as zero.*

Proof. Straightforward following the arguments of [LS13]. □

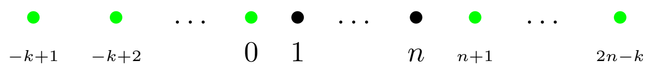
Example 1.2.15.

i) 

ii) 

iii) 

Notation 1.2.16. We number the dots in an extended cup diagram increasing from left to right such that the first black point is labelled 1, i.e.



so that the $\wedge\vee$-sequence λ gives the label $\lambda(i)$ at point i. As a shorthand notation for "the point i in $\mathrm{C}(\lambda)$" we write "$\lfloor i \rfloor_\lambda$". By $t(\lfloor j \rfloor_\lambda)$ we denote the target of the arc starting at $\lfloor j \rfloor_\lambda$ and by $s(\lfloor j \rfloor_\lambda)$ the source of the arc ending at $\lfloor j \rfloor_\lambda$.

We now consider some properties of the action defined above.

Lemma 1.2.17. *Let λ, $\mu \in \Lambda(n,k)$ with $\lambda \xrightarrow{s_i} \mu$. Then $\mathrm{C}(\mu) = \mathrm{C}(\lambda).U_i$.*

Proof. We have $\wedge = \lambda(i)$ and $\vee = \lambda(i+1)$. Here, $\lfloor i \rfloor_\lambda$ is the endpoint of an arc starting in $s(\lfloor i \rfloor_\lambda)$ and $\lfloor i+1 \rfloor_\lambda$ the starting point of an arc ending at $t(\lfloor i+1 \rfloor_\lambda)$. Applying U_i connects $s(\lfloor i \rfloor_\lambda)$ and $t(\lfloor i+1 \rfloor_\lambda)$ as well as $\lfloor i \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$. But this is just $\mathrm{C}(\lambda s_i)$ where $\lambda < \lambda s_i = \mu$ since we changed $\wedge\vee$ to $\vee\wedge$.

□

Corollary 1.2.18. *Let $s = s_{i_1} \dots s_{i_r} \in \mathrm{W}^{min}$ be reduced and define $U := U_{i_1} \dots U_{i_r}$. Then $\mathrm{C}(\lambda_0 s) = \mathrm{C}(\lambda_0).U$.*

Corollary 1.2.19. *As a TL_n-module, $\widehat{eC}(n,k)$ is generated by $\mathrm{C}(\lambda_0)$ and hence so is $\widehat{eC}(n,k)^{\mathbb{C}} := \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \widehat{eC}(n,k)$ as a $TL_n^{\mathbb{C}}$-module.*

Lemma 1.2.20.

$$\mathrm{C}(\lambda).U_i = \begin{cases} (q+q^{-1})\,\mathrm{C}(\lambda) & \text{if } \lambda(i)=\vee, \lambda(i+1)=\wedge, \\ \mathrm{C}(\lambda s_i) \text{ with } \lambda s_i > \lambda & \text{if } \lambda(i)=\wedge, \lambda(i+1)=\vee, \\ \mathrm{C}(\lambda') \text{ with } \lambda' < \lambda & \text{if } \lambda(i)=\lambda(i+1)=\vee \text{ and } t(\lfloor i+1 \rfloor_\lambda) \text{ not green,} \\ & \text{or } \lambda(i)=\lambda(i+1)=\wedge \text{ and } s(\lfloor i \rfloor_\lambda) \text{ not green} \\ 0 & \text{otherwise.} \end{cases}$$

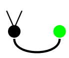
Moreover, the λ' from the third case can be described explicitly: If $\lambda(i)=\lambda(i+1)=\vee$, then $\lambda' = \lambda s_{(i+1,t(\lfloor i+1 \rfloor_\lambda))}$ and if $\lambda(i)=\lambda(i+1)=\wedge$, then $\lambda' = s_{(i,s(\lfloor i \rfloor_\lambda))}$, where by $s_{(a,b)} \in \mathbb{S}_n$ we denote the element $s_a s_{a+1} \dots s_{b-1}$.

Proof. We check all the possibilities for $\lambda(i), \lambda(i+1) \in \{\wedge\vee\}$:

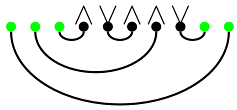
- $\lambda(i)=\vee, \lambda(i+1)=\wedge$: Then $\lfloor i \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$ are connected. Applying U_i creates a circle which we replace by $(q+q^{-1})$, getting $\mathrm{C}(\lambda).U_i = (q+q^{-1})\,\mathrm{C}(\lambda)$.
- $\lambda(i)=\wedge, \lambda(i+1)=\vee$: This case was already proven in Lemma 1.2.17.
- $\lambda(i)=\lambda(i+1)=\vee$: Then $\lfloor i \rfloor_\lambda$ is connected to $t(\lfloor i \rfloor_\lambda)$, $\lfloor i+1 \rfloor_\lambda$ to $t(\lfloor i+1 \rfloor_\lambda)$ and $t(\lfloor i \rfloor_\lambda)$ is to the right of $t(\lfloor i+1 \rfloor_\lambda)$. So if $t(\lfloor i+1 \rfloor_\lambda)$ is green, then so is $t(\lfloor i \rfloor_\lambda)$ and $\mathrm{C}(\lambda).U_i$ is not an element of $\widehat{eC}(n,k)$, so $\mathrm{C}(\lambda).U_i = 0$. If $t(\lfloor i+1 \rfloor_\lambda)$ is not green, then applying U_i connects $\lfloor i \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$ as well as $t(\lfloor i \rfloor_\lambda)$ and $t(\lfloor i+1 \rfloor_\lambda)$. Thus $\mathrm{C}(\lambda).U_i = \mathrm{C}(\lambda s_{(i+1,t(\lfloor i+1 \rfloor_\lambda))})$ and $\lambda s_{(i+1,t(\lfloor i+1 \rfloor_\lambda))} < \lambda$ since it comes from λ by a sequence of $\wedge\vee$ to $\vee\wedge$.
- $\lambda(i)=\lambda(i+1)=\wedge$: Then $\lfloor i \rfloor_\lambda$ is connected to $s(\lfloor i \rfloor_\lambda)$, $\lfloor i+1 \rfloor_\lambda$ to $s(\lfloor i+1 \rfloor_\lambda)$ and $s(\lfloor i+1 \rfloor_\lambda)$ is to the left of $s(\lfloor i \rfloor_\lambda)$. Therefore, if $s(\lfloor i \rfloor_\lambda)$ is green, then so is $s(\lfloor i+1 \rfloor_\lambda)$ and $\mathrm{C}(\lambda).U_i$ is not an element of $\widehat{eC}(n,k)$, so $\mathrm{C}(\lambda).U_i = 0$. If $s(\lfloor i \rfloor_\lambda)$ is not green, then applying U_i connects $\lfloor i \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$ as well as $s(\lfloor i \rfloor_\lambda)$ and $s(\lfloor i+1 \rfloor_\lambda)$. Thus, $\mathrm{C}(\lambda).U_i = \mathrm{C}(\lambda s_{(i,s(\lfloor i \rfloor_\lambda))})$ and $\lambda s_{(i,s(\lfloor i \rfloor_\lambda))} < \lambda$ since it comes from λ by a sequence of $\wedge\vee$ to $\vee\wedge$. □

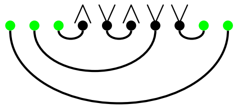
Definition 1.2.21. An *orientation* of a cup with black endpoints is a labelling of the points with $\wedge$ and $\vee$ or $\vee$ and $\wedge$, e.g.  or  but not .

A cup with black endpoints is *clockwise (anticlockwise) oriented*, if its leftmost vertex is labelled $\wedge$ ($\vee$) and its rightmost vertex is labelled $\vee$ ($\wedge$).

An *orientation* of a cup with one black and one green endpoint is a labelling of the black endpoint with $\wedge$ if it is to the right of the green one and with $\vee$ otherwise, e.g.  or . Note that cups with a green point can only be oriented in one way.

Let $\mu \in \Lambda(n,k)$ and $C \in eC(n,k)$. Then we say that $C\mu$ *is oriented*, if, when we label the black points by μ, all the cups are oriented, i.e. only cups of the following form are allowed:

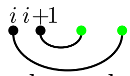
Example 1.2.22. This is oriented: 

whereas the following is not: 

Note that λ always orients $\mathrm{C}(\lambda)$. Furthermore, $\mathrm{C}(\lambda_0)$ can only be oriented by λ_0 since every cup has a green endpoint determining its orientation.

Lemma 1.2.23. *Let $\lambda, \mu \in \Lambda(n,k)$. Assume $\lambda(i) = \lambda(i+1) = \vee$ and $\lfloor i+1 \rfloor_\lambda$ is connected to a green point in $\mathrm{C}(\lambda)$. If $\mathrm{C}(\lambda)\mu$ is oriented, then $\mu(i) = \mu(i+1) = \vee$.*

Assume $\lambda(i) = \lambda(i+1) = \wedge$ and $\lfloor i \rfloor_\lambda$ is connected to a green point in $\mathrm{C}(\lambda)$. If $\mathrm{C}(\lambda)\mu$ is oriented, then $\mu(i) = \mu(i+1) = \wedge$.

Proof. If $\lambda(i) = \lambda(i+1) = \vee$ and $\lfloor i+1 \rfloor_\lambda$ is connected to a green point in $\mathrm{C}(\lambda)$, then $\lfloor i \rfloor_\lambda$ is also connected to a green point:  So only $\mu(i) = \mu(i+1) = \vee$ is possible by definition. The other part follows analogously. □

Lemma 1.2.24. *Assume that $\mathrm{C}(\lambda).U_i = \mathrm{C}(\lambda')$. Then*

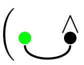
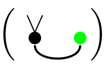
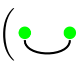
$$\Big\{\nu \mid \mathrm{C}(\lambda')\nu \textit{ is oriented}\Big\} = \Big\{\mu, \mu s_i \mid \mathrm{C}(\lambda)\mu \textit{ is oriented}, \mu s_i \textit{ defined}\Big\}.$$

Proof. In $\mathrm{C}(\lambda)$, $\lfloor i \rfloor_\lambda$ is connected to some $\lfloor a \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$ is connected to some $\lfloor b \rfloor_\lambda$. By assumption, in $\mathrm{C}(\lambda')$, we have an arc between $\lfloor i \rfloor_{\lambda'}$ and $\lfloor i+1 \rfloor_{\lambda'}$ and between $\lfloor a \rfloor_{\lambda'}$ and $\lfloor b \rfloor_{\lambda'}$.

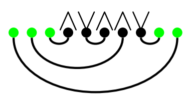
We first show $\supseteq$: Assume μ satisfies that $\mathrm{C}(\lambda)\mu$ is oriented and μs_i is defined. Since μs_i is defined, we know that $\mu(i) \neq \mu(i+1)$. Since μ orients $\mathrm{C}(\lambda)$ and $\mu(i) \neq \mu(i+1)$, we know that $\mu(a) \neq \mu(b)$ (where we see $\mu(c)$ as $\vee$ if c is a left green point and $\mu(c)$ as $\wedge$ if c is a right green point), thus μ orients $\mathrm{C}(\lambda')$. Since $\lfloor i \rfloor_{\lambda'}$ is connected to $\lfloor i+1 \rfloor_{\lambda'}$, we also have that μs_i orients $\mathrm{C}(\lambda')$.

We now show $\subseteq$: Assume $\mathrm{C}(\lambda')\nu$ is oriented. Thus, $\nu(i) \neq \nu(i+1)$ and $\nu(a) \neq \nu(b)$ (where we again see $\nu(c)$ as $\vee$ if c is a left green point and $\nu(c)$ as $\wedge$ if c is a right green point). Since $\nu(i) \neq \nu(i+1)$, νs_i is defined. If $\nu(a) \neq \nu(i)$, then $\mathrm{C}(\lambda)\nu$ is oriented and we are done. If $\nu(a) = \nu(i)$, then $\nu s_i(a) \neq \nu s_i(i)$ and for $\nu' = \nu s_i$ we have that $\mathrm{C}(\lambda)\nu'$ is oriented and $\nu' s_i = \nu$ is defined. □

Definition 1.2.25. Let $\mu \in \Lambda(n,k)$, $C \in eC(n,k)$ such that $C\mu$ is oriented. Then we define the *degree of* $C\mu$, $\deg(C\mu)$, as the number of clockwise oriented cups in $C\mu$, i.e.

deg () = 1, deg () = deg () = deg () = deg () = 0.

For example,

deg () = 0.

Note that we always have $\deg\big(\mathrm{C}(\lambda)\lambda\big) = 0$, since by construction of $\mathrm{C}(\lambda)$ all the cups in $\mathrm{C}(\lambda)\lambda$ are oriented counter-clockwise.

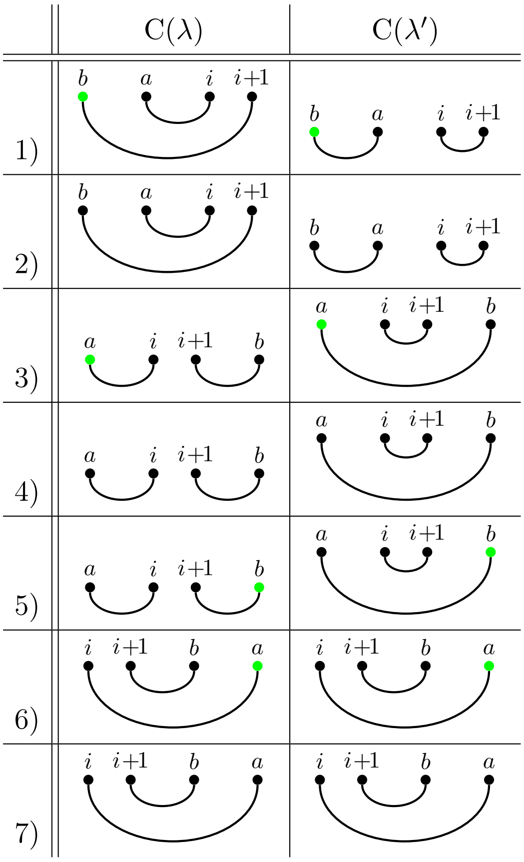
Lemma 1.2.26. *Assume* $\mathrm{C}(\lambda).U_i = \mathrm{C}(\lambda')$ *and pick* μ *such that* $\mathrm{C}(\lambda)\mu$ *is oriented and* μs_i *is defined.*

If $\mu > \mu s_i$*, then* $\deg\big(\mathrm{C}(\lambda')\mu\big) = \deg\big(\mathrm{C}(\lambda)\mu\big) - 1$ *and* $\deg\big(\mathrm{C}(\lambda')\mu s_i\big) = \deg\big(\mathrm{C}(\lambda)\mu\big)$.

If $\mu < \mu s_i$*, then* $\deg\big(\mathrm{C}(\lambda')\mu s_i\big) = \deg\big(\mathrm{C}(\lambda)\mu\big)$ *and* $\deg\big(\mathrm{C}(\lambda')\mu\big) = \deg\big(\mathrm{C}(\lambda)\mu\big) + 1$.

Note that $\mathrm{C}(\lambda')\mu$ *and* $\mathrm{C}(\lambda')\mu s_i$ *are oriented by the previous lemma.*

Proof. Again, in $\mathrm{C}(\lambda)$, $\lfloor i \rfloor_\lambda$ is connected to some $\lfloor a \rfloor_\lambda$ and $\lfloor i+1 \rfloor_\lambda$ is connected to some $\lfloor b \rfloor_\lambda$, whereas in $\mathrm{C}(\lambda')$, we have an arc between $\lfloor i \rfloor_{\lambda'}$ and $\lfloor i+1 \rfloor_{\lambda'}$ and between $\lfloor a \rfloor_{\lambda'}$ and $\lfloor b \rfloor_{\lambda'}$, and the rest is the same. We consider all the possibilities for the arcs involving the endpoints a, b, i, $i+1$:

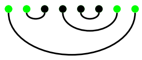


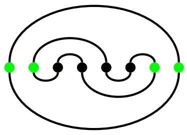
If $\mu > \mu s_i$ then $\mu s_i(i) = \wedge = \mu(i{+}1)$ and $\mu s_i(i{+}1) = \vee = \mu(i)$. Since $\mathrm{C}(\lambda)\mu$ is oriented, we are in one of the cases 1), 2), 4), 6), 7). Furthermore, we have $\mu s_i(a) = \mu(a) = \wedge$ or a is a green point and $\mu s_i(b) = \mu(b) = \vee$ or b is green. Now we check the orientations in the possible cases: For case 4) in $\mathrm{C}(\lambda)\mu$ both arcs are oriented clockwise, as well as in $\mathrm{C}(\lambda')\mu s_i$, whereas in $\mathrm{C}(\lambda')\mu$ only one is oriented clockwise. If we are in cases 1), 2), 6), 7) then in $\mathrm{C}(\lambda)\mu$ one arc is oriented clockwise, as well as in $\mathrm{C}(\lambda')\mu s_i$, where in $\mathrm{C}(\lambda')\mu$ none is oriented clockwise.

If $\mu < \mu s_i$ then $\mu s_i(i) = \vee = \mu(i{+}1)$ and $\mu s_i(i{+}1) = \wedge = \mu(i)$. Since $\mathrm{C}(\lambda)\mu$ is oriented, we are in one of the cases 2), 3), 4), 5), 7). Moreover, we have $\mu s_i(a) = \mu(a) = \vee$ or a is a green point and $\mu s_i(b) = \mu(b) = \wedge$ or b is a green point. Again we check the orientations in the different cases: In cases 3), 4), 5) the arcs in $\mathrm{C}(\lambda)\mu$ are both not oriented clockwise, as well as in $\mathrm{C}(\lambda')\mu s_i$, where in $\mathrm{C}(\lambda')\mu$ there is one cup oriented clockwise. If we are in cases 2) or 7), then in $\mathrm{C}(\lambda)\mu$ only one arc is oriented clockwise, as well as in $\mathrm{C}(\lambda')\mu s_i$, where in $\mathrm{C}(\lambda')\mu$ both are oriented clockwise.

□

Definition 1.2.27. Let $C, D \in eC(n,k)$ be extended cup diagrams. Then $C\overline{D}$ is defined as reflecting D at the horizontal axis and putting it on top of C. We call $C\overline{D}$ a *circle diagram*.

Example 1.2.28. Let $C =$  and $D =$ . Then $C\overline{D} =$

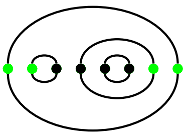
 .

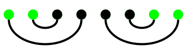
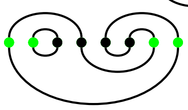
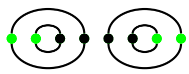
Lemma 1.2.29. *There are exactly n circles in $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$ iff $\mathrm{C}(\lambda) = \mathrm{C}(\mu)$, i.e. $\lambda = \mu$.*

Proof. This is true because every circle contains an even number of black or green points (where it intersects the x-axis) and at least 2. If $\lambda \neq \mu$, i.e. $\mathrm{C}(\lambda) \neq \mathrm{C}(\mu)$, then there are points a, b that are connected in $\mathrm{C}(\lambda)$, but in $\mathrm{C}(\mu)$ the point a is connected to some $c \neq b$. So in $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$ the circle containing a and b contains also c and thus ≥ 3 points. Since there are $2n$ points (black or green) altogether, there are now $< n$ circles. □

Definition 1.2.30. A circle inside a circle diagram is called *red* if it contains more then one right green point or more then one left green point. Other circles that contain a green point are called *green*. Circles without green points are called *black*.

This colouring is motivated by [Str09] and also allows to consider circle diagrams as objects of the category of coloured cobordisms (cf. Section A.1).

Example 1.2.31. With C and D from Example 1.2.28, $C\overline{D}$ consists of two green circles. Moreover, $C\overline{C} =$  consists of one black and three green circles.

Furthermore, for C as before and $D =$  $= \mathrm{C}(\lambda_0)$ we have one red and one green circle in $C\overline{D} =$  . $D\overline{D} =$  has only green circles.

Note that $\mathrm{C}(\lambda)\overline{\mathrm{C}(\lambda)}$ does not contain red circles. It consists solely of green circles if and only if $\lambda = \lambda_0$.

Chapter 2

The background story: The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$

This chapter introduces the objects we want to categorify: The weight spaces of the $\mathcal{U}_q(\mathfrak{sl}_2)$-module $V^{\otimes n}$ and the different bases of $V^{\otimes n}$. We connect the bases to the cup diagrams of the last chapter and to the action of the Temperley-Lieb algebra. After that, we conclude with the definition of the Jones-Wenzl projector and recall some of its properties.

2.1 Finite dimensional representations

For the definitions, we mostly follow [FSS12]. Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q.

Definition 2.1.1. Let $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ be the associative algebra over $\mathbb{C}(q)$ generated by E, F, K, K^{-1} subject to the relations:

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KE &= q^2 EK \\ KF &= q^{-2} FK \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

$\mathcal{U}_q$ is a Hopf algebra with the following comultiplication:

$$\triangle(E) = 1 \otimes E + E \otimes K^{-1}, \qquad \triangle(F) = K \otimes F + F \otimes 1, \qquad \triangle(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}.$$

For a variable t, the *t-quantum integers* are defined as $[k]_t = \sum_{j=0}^{k-1} t^{k-2j-1}$ and the *t-quantum binomial coefficients* as $\begin{bmatrix} n \\ k \end{bmatrix}_t = \frac{[n]_t!}{[k]_t![n-k]_t!}$, where $[n]_t! = [1]_t \cdot [2]_t \cdots [n]_t$. For example, $[1]_t = 1$, $[2]_t = t + t^{-1}$, $[3]_t = t^2 + 1 + t^{-2}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_t = t^4 + t^2 + 2 + t^{-2} + t^{-4}$. Note that $[k]_{-q} = (-1)^{k-1}[k]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{-q} = (-1)^{n+k} \begin{bmatrix} n \\ k \end{bmatrix}_q$. We leave out the index and denote $[-] = [-]_q$ in case $t = q$, the indeterminate from above. Note that for $t = q$ or $t = -q$, the quantum integers and binomial coefficients live in $\mathbb{Q}(q)$.

Let V_n be the irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module with basis $\{v_0, v_1, \dots, v_n\}$ such that

$$K^{\pm 1} v_i = (-q)^{\pm(2i-n)} v_i \qquad E v_i = [i+1]_{-q} v_{i+1} \qquad F v_i = [n-i+1]_{-q} v_{i-1},$$

where we set $v_{-1} = 0 = v_{n+1}$. A direct sum of modules of the form V_n is called a *type I-module*. Note that here we differ from the conventions of [FSS12] by the change of variable $q \mapsto -q$. This is necessary to harmonise with the combinatorial picture.

In the following, we mostly consider the irreducible $\mathcal{U}_q$-module $V = V_1$ and its tensor powers. Recall that $V^{\otimes n}$ is an $\mathcal{U}_q$-module via $G.(v_{i_1} \otimes \ldots \otimes v_{i_n}) = \triangle^n(G)(v_{i_1} \otimes \ldots \otimes v_{i_n})$ for $G \in \{K^{\pm 1}, E, F\}$, where $\triangle^n = (1 \otimes \ldots \otimes 1 \otimes \triangle) \circ \cdots \circ (1 \otimes \triangle) \circ \triangle$.

Definition 2.1.2. For M a finite dimensional $\mathcal{U}_q$-module of type I the *weight space decomposition* is defined as $M = \bigoplus_\beta M_\beta$, where $M_\beta = \{m \in M \mid Km = (-q)^\beta m\}$ are eigenspaces of K.

Example 2.1.3. Since $K.(v_{i_1} \otimes \ldots \otimes v_{i_n}) = K v_{i_1} \otimes \ldots \otimes K v_{i_n} = (-q)^{\#1-\#0} v_{i_1} \otimes \ldots \otimes v_{i_n}$ (where we use the shorthand notation $\#1 - \#0 = \#\{j \mid i_j = 1\} - \#\{j \mid i_j = 0\}$), the space

$$\left(V^{\otimes n}\right)_\beta = \left\langle v_{i_1} \otimes \ldots \otimes v_{i_n} \mid \{i_1, \dots, i_n\} = \{\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k}\}, \beta = 2k - n \right\rangle$$

consists of eigenvectors with eigenvalue β. Since these vectors $v_{i_1} \otimes \ldots \otimes v_{i_n}$, $i_1, \dots, i_n \in \{0, 1\}$ form a basis we have $V^{\otimes n} = \bigoplus (V^{\otimes n})_\beta$.

Definition 2.1.4. Let $\mathbf{a} = (a_{i_1}, \dots, a_{i_n})$ with $a_{i_j} \in \{0, 1\}$. As a shorthand notation we denote $v_{\mathbf{a}} := v_{a_{i_j}} \otimes \ldots \otimes v_{a_{i_n}}$.

Note that we can identify $\mathbf{a}$ with an $\wedge\vee$-sequence $\lambda \in \Lambda(n,k)$ for some k by identifying $\wedge$ with 1 and $\vee$ with 0. Thus, we also write v_λ for standard basis elements of $V^{\otimes n}$ instead of $v_{\mathbf{a}}$. In particular, standard basis elements of $(V^{\otimes n})_{2k-n}$ are of the form v_λ with $\lambda \in \Lambda(n,k)$.

There is a $\mathbb{C}(q)$-bilinear form $(-,-)$ on $V^{\otimes n}$, such that

$$(v_{\mathbf{a}}, v_{\mathbf{b}}) = \begin{cases} 1 & \text{if } \mathbf{a} = \mathbf{b}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

(cf. [FSS12, (5)]).

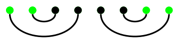
Definition 2.1.5. The *canonical basis* in $(V^{\otimes n})_{2k-n}$ is defined via

$$v_{\heartsuit\lambda} = \sum_{\mu : \mathrm{C}(\lambda)\mu \text{ is or.}} q^{-\deg(\mathrm{C}(\lambda)\mu)} v_\mu,$$

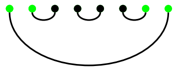
where $\lambda, \mu \in \Lambda(n,k)$. Up to an obvious renormalisation this is the twisted canonical basis from [BS10]. Note that the base change matrix to the standard basis of $(V^{\otimes n})_{2k-n}$ is a triangular matrix with 1's on the diagonal.

Example 2.1.6. For $n = 4$, $k = 2$ we have the following canonical basis of $\left(V^{\otimes 4}\right)_0$:

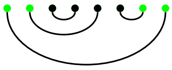
- $v_{\heartsuit\wedge\wedge\vee\vee} = v_{\wedge\wedge\vee\vee} = v_1 \otimes v_1 \otimes v_0 \otimes v_0$

  since $\wedge\wedge\vee\vee$ is the only orientation of 

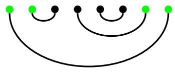
- $v_{\heartsuit\wedge\vee\wedge\vee} = v_{\wedge\vee\wedge\vee} + q^{-1} v_{\wedge\wedge\vee\vee}$

  since $\mathrm{C}(\wedge\vee\wedge\vee) =$  has only the two orientations $\wedge\vee\wedge\vee$ and $\wedge\wedge\vee\vee$ with the right degrees

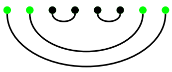
- $v_{\heartsuit\vee\wedge\wedge\vee} = v_{\vee\wedge\wedge\vee} + q^{-1} v_{\wedge\vee\wedge\vee}$

  since $\mathrm{C}(\vee\wedge\wedge\vee) =$ 

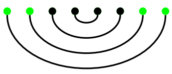
- $v_{\heartsuit\wedge\vee\vee\wedge} = v_{\wedge\vee\vee\wedge} + q^{-1} v_{\wedge\vee\wedge\vee}$

  since $\mathrm{C}(\wedge\vee\vee\wedge) =$ 

- $v_{\heartsuit\vee\wedge\vee\wedge} = v_{\vee\wedge\vee\wedge} + q^{-1} v_{\wedge\vee\vee\wedge} + q^{-1} v_{\vee\wedge\wedge\vee} + q^{-2} v_{\wedge\wedge\vee\vee}$

  since $\mathrm{C}(\vee\wedge\vee\wedge) =$ 

- $v_{\heartsuit\vee\vee\wedge\wedge} = v_{\vee\vee\wedge\wedge} + q^{-1} v_{\vee\wedge\vee\wedge} + q^{-1} v_{\wedge\vee\wedge\vee} + q^{-2} v_{\wedge\wedge\vee\vee}$

  since $\mathrm{C}(\vee\vee\wedge\wedge) =$ 

Note that $v_{\heartsuit\lambda_0} = v_{\lambda_0}$ holds for every n, k, since λ_0 is the only possible orientation of $\mathrm{C}(\lambda_0)$.

It is a well-known fact that $TL_n^{\mathbb{C}} \cong \mathrm{End}_{\mathcal{U}_q}(V^{\otimes n})$, see e.g. [Str05, Proposition 4.2]. With our conventions this isomorphism sends $U_i \in TL_n^{\mathbb{C}}$ to $C_{i,n} = \mathrm{id}^{\otimes(i-1)} \otimes u \otimes \mathrm{id}^{\otimes(n-i-1)}$, where $u : V^{\otimes 2} \to V^{\otimes 2}$ is defined via

$$u(v_r \otimes v_s) = \begin{cases} 0 & \text{if } r = s = 0 \text{ or } r = s = 1, \\ v_1 \otimes v_0 + q v_0 \otimes v_1 & \text{if } r = 0, s = 1, \\ q^{-1} v_1 \otimes v_0 + v_0 \otimes v_1 & \text{if } r = 1, s = 0. \end{cases}$$

In particular, using this isomorphism, we can view $V^{\otimes n}$ as a right $TL_n^{\mathbb{C}}$-module. Also, since u does not change the number of 0's and 1's, we have that $(V^{\otimes n})_\beta$ is an $TL_n^{\mathbb{C}}$-module, too.

Explicitly, when we label again by $\wedge\vee$-sequences, we have

$$v_\mu . U_i = \begin{cases} v_{\mu s_i} + q v_\mu & \text{if } \mu s_i < \mu, \\ v_{\mu s_i} + q^{-1} v_\mu & \text{if } \mu s_i > \mu, \\ 0 & \text{if } \mu s_i = \mu. \end{cases} \tag{2.2}$$

The following lemma gives us the main idea for the categorification in the following chapters.

Proposition 2.1.7. *The assignment* $\mathrm{C}(\lambda) \mapsto v_{\heartsuit\lambda}$ *defines an isomorphism* $\widehat{eC}(n,k)^{\mathbb{C}} \cong (V^{\otimes n})_{2k-n}$ *of* $TL_n^{\mathbb{C}}$*-modules.*

Proof. Since basis elements are sent to basis elements bijectively, we clearly have an isomorphism. It remains to check the compatibility with the $TL_n^{\mathbb{C}}$-action, i.e. to show that $\mathrm{C}(\lambda).U_i \mapsto v_{\heartsuit\lambda}.U_i$ holds:
$\mathrm{C}(\lambda).U_i$ is determined in Lemma 1.2.20. By the formula for $v_\mu.U_i$ we have

$$v_{\heartsuit\lambda}.U_i = \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.} \\ \mu<\mu s_i}} q^{-\deg(\mathrm{C}(\lambda)\mu)}(v_{\mu s_i} + q^{-1}v_\mu) + \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.} \\ \mu>\mu s_i}} q^{-\deg(\mathrm{C}(\lambda)\mu)}(v_{\mu s_i} + qv_\mu).$$

We now consider different cases of entries of λ at places i and $i+1$. If $\lambda(i) = \vee$ and $\lambda(i+1) = \wedge$, then

$$\{\mu \mid \mathrm{C}(\lambda)\mu \text{ is oriented }\} = \{\nu, \nu s_i \mid \nu < \nu s_i, \mathrm{C}(\lambda)\nu \text{ is oriented }\},$$

since the cup belonging to $\lambda(i)$ and $\lambda(i+1)$ can be oriented in two ways. Thus,

$$\begin{aligned} v_{\heartsuit\lambda}.U_i &= \sum_{\substack{\nu:\mathrm{C}(\lambda)\nu \text{ is or.} \\ \nu<\nu s_i}} \left(q^{-\deg(\mathrm{C}(\lambda)\nu)}(v_{\nu s_i} + q^{-1}v_\nu) + q^{-\deg(\mathrm{C}(\lambda)\nu s_i)}(v_\nu + qv_{\nu s_i})\right) \\ &= \sum_{\substack{\nu:\mathrm{C}(\lambda)\nu \text{ is or.} \\ \nu<\nu s_i}} q^{-\deg(\mathrm{C}(\lambda)\nu)}(v_{\nu s_i} + q^{-1}v_\nu + qv_\nu + q^2 v_{\nu s_i}) \\ &= \sum_{\substack{\nu:\mathrm{C}(\lambda)\nu \text{ is or.} \\ \nu<\nu s_i}} q^{-\deg(\mathrm{C}(\lambda)\nu)}(q + q^{-1})(v_\nu + qv_{\nu s_i}) \\ &= (q+q^{-1}) \sum_{\substack{\nu:\mathrm{C}(\lambda)\nu \text{ is or.} \\ \nu<\nu s_i}} (q^{-\deg(\mathrm{C}(\lambda)\nu)}v_\nu + q^{-\deg(\mathrm{C}(\lambda)\nu s_i)}v_{\nu s_i}) = (q+q^{-1})v_{\heartsuit\lambda}. \end{aligned}$$

since $\deg\big(\mathrm{C}(\lambda)\nu s_i\big) = \deg\big(\mathrm{C}(\lambda)\nu\big) - 1$ for $\nu < \nu s_i$ follows from $\lambda(i) = \vee$, $\lambda(i+1) = \wedge$. From Lemma 1.2.20 we know $\mathrm{C}(\lambda).U_i = (q+q^{-1})\,\mathrm{C}(\lambda)$, so we are finished in this case.

If $\lambda(i) = \lambda(i+1) = \vee$ and in $\mathrm{C}(\lambda)$ the point $i+1$ is connected to a green point, then using Lemma 1.2.23 we get that $v_{\heartsuit\lambda}.U_i = 0$. Also, $v_{\heartsuit\lambda}.U_i = 0$ if $\lambda(i) = \lambda(i+1) = \wedge$ and in $\mathrm{C}(\lambda)$ the point i is connected to a green point.

In all the remaining cases we have $\mathrm{C}(\lambda).U_i = \mathrm{C}(\lambda')$ for some λ'. Thus, using Lemma 1.2.24 at the second equality sign and Lemma 1.2.26 at the forth, we get

$$\begin{aligned} v_{\heartsuit\lambda'} &= \sum_{\nu:\mathrm{C}(\lambda')\nu \text{ is or.}} q^{-\deg(\mathrm{C}(\lambda')\nu)}v_\nu \\ &= \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.} \\ \mu s_i \text{ is def.}}} \left(q^{-\deg(\mathrm{C}(\lambda')\mu)}v_\mu + q^{-\deg(\mathrm{C}(\lambda')\mu s_i)}v_{\mu s_i}\right) \end{aligned}$$

$$
\begin{aligned}
=& \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i<\mu}} \left(q^{-\deg(\mathrm{C}(\lambda')\mu)}v_\mu + q^{-\deg(\mathrm{C}(\lambda')\mu s_i)}v_{\mu s_i}\right)\\
&\quad + \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i>\mu}} \left(q^{-\deg(\mathrm{C}(\lambda')\mu)}v_\mu + q^{-\deg(\mathrm{C}(\lambda')\mu s_i)}v_{\mu s_i}\right)\\
=& \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i<\mu}} \left(q^{-\deg(\mathrm{C}(\lambda)\mu)+1}v_\mu + q^{-\deg(\mathrm{C}(\lambda)\mu)}v_{\mu s_i}\right)\\
&\quad + \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i>\mu}} \left(q^{-\deg(\mathrm{C}(\lambda)\mu)-1}v_\mu + q^{-\deg(\mathrm{C}(\lambda)\mu)}v_{\mu s_i}\right)\\
=& \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i<\mu}} q^{-\deg(\mathrm{C}(\lambda)\mu)}(qv_\mu + v_{\mu s_i}) + \sum_{\substack{\mu:\mathrm{C}(\lambda)\mu \text{ is or.}\\ \mu s_i>\mu}} q^{-\deg(\mathrm{C}(\lambda)\mu)}(q^{-1}v_\mu + v_{\mu s_i})\\
=& v_{\heartsuit\lambda}.U_i.
\end{aligned}
$$

□

The previous proposition yields the following well-known fact directly from Corollary 1.2.19.

Corollary 2.1.8. *As TL_n-module, $(V^{\otimes n})_{2k-n}$ is generated by v_{λ_0}.*

Note that equation (2.2) defines the well-known parabolic Hecke module $\mathcal{N}^{\mathfrak{p}}$ [Str05, Lemma 1.4]; in particular, from this observation Corollary 2.1.8 follows immediately.

2.2 Jones-Wenzl projectors

Adapting the definition of [FSS12] to our sign convention, we define:

Definition 2.2.1. Let $|\mathbf{a}|$ is the number of ones in $\mathbf{a}$ and $v^m := \frac{1}{\left[{n \atop m}\right]_{-q}} v_m$.

Define the projection $\pi_n\colon V^{\otimes n} \to V_n$ by the formula

$$\pi_n(v_{\mathbf{a}}) = (-q)^{-l(\mathbf{a})}v^{|\mathbf{a}|} = (-q)^{-l(\mathbf{a})}\frac{1}{\left[{n \atop |\mathbf{a}|}\right]_{-q}}v_{|\mathbf{a}|} \tag{2.3}$$

where $l(\mathbf{a})$ is equal to the number of pairs (i,j) with $i<j$ and $a_i<a_j$.

The inclusion $\iota_n\colon V_n \to V^{\otimes n}$ is the intertwining map

$$v_k \mapsto \sum_{|\mathbf{a}|=k} (-q)^{b(\mathbf{a})} v_{\mathbf{a}} \tag{2.4}$$

where $b(\mathbf{a})$ is the number of pairs (i,j) with $i<j$ and $a_i>a_j$, i.e. $b(\mathbf{a}) = |\mathbf{a}|(n-|\mathbf{a}|) - l(\mathbf{a})$. Note that these morphisms are $\mathcal{U}_q$-equivariant. The composite $p_n = \iota_n \circ \pi_n$ is the *Jones-Wenzl projector*.

Note that when we consider for an 01-sequence $\mathbf{a}$ the corresponding $\wedge\vee$-sequence λ, then $l(\mathbf{a}) = \ell(\lambda, \lambda_0)$. This is true since $l(\mathbf{a})$ counts the number of pairs $(\vee, \wedge)$ where the $\vee$ is to the left of the $\wedge$. But this is exactly the number of swappings of neighbouring $\wedge\vee$ to get from $\wedge \dots \wedge\vee \dots \vee = \lambda_0$ to λ, which is $\ell(\lambda, \lambda_0)$.

Example 2.2.2. In the case $n = 2$ we obtain

$$p_2(v_{10}) = \iota_2\left(\frac{1}{\begin{bmatrix}2\\1\end{bmatrix}_{-q}} v_1\right) = -[2]^{-1}(v_{01} - qv_{10})$$

and $p_2(v_{\heartsuit 01}) = 0$, since

$$\pi_2(v_{\heartsuit 01}) = \pi_2(v_{01} + q^{-1}v_{10}) = (-q)^{-1}v^1 + q^{-1}v^1 = 0.$$

The Jones-Wenzl projector has the following important property (see e.g. [FSS12]).

Proposition 2.2.3. *The endomorphism p_n of $V^{\otimes n}$ is the unique $\mathcal{U}_q$-morphism which satisfies for $1 \leq i \leq n-1$:*

i) $p_n \circ p_n = p_n$

ii) $C_{i,n} \circ p_n = 0$

iii) $p_n \circ C_{i,n} = 0$

Even a stronger version of i)-iii) holds:

Lemma 2.2.4. *a)* $\pi_n \circ \iota_n = \mathrm{id}$

b) $C_{i,n} \circ \iota_n = 0$

c) $\pi_n \circ C_{i,n} = 0$

Proof. a) We compute

$$\begin{aligned}\pi_n \circ \iota_n(v_k) &= \sum_{|\mathbf{a}|=k} (-q)^{b(\mathbf{a})}\pi_n(v_{\mathbf{a}}) = \sum_{|\mathbf{a}|=k} (-q)^{b(\mathbf{a})}(-q)^{-l(\mathbf{a})}\frac{1}{\begin{bmatrix}n\\k\end{bmatrix}_{-q}} v_k \\ &= \frac{1}{\begin{bmatrix}n\\k\end{bmatrix}_{-q}}(-q)^{k(n-k)} \sum_{|\mathbf{a}|=k} (-q)^{-2l(\mathbf{a})} v_k = \frac{1}{\begin{bmatrix}n\\k\end{bmatrix}_{-q}} \begin{bmatrix}n\\k\end{bmatrix}_{-q} v_k = v_k,\end{aligned}$$

where the second last equality holds by substituting q with $-q$ in [Sch12, Proposition 2.2.5].

b) We calculate

$$\begin{aligned}
\iota_n(v_k).U_i &= \sum_{|\lambda|=k} (-q)^{k(n-k)-\ell(\lambda,\lambda_0)} v_\lambda.U_i \\
&= \sum_{\substack{|\lambda|=k \\ \lambda<\lambda s_i}} (-q)^{k(n-k)-\ell(\lambda,\lambda_0)}(v_{\lambda s_i} + q^{-1}v_\lambda) + \sum_{\substack{|\lambda|=k \\ \lambda>\lambda s_i}} (-q)^{k(n-k)-\ell(\lambda,\lambda_0)}(v_{\lambda s_i} + qv_\lambda) \\
&= \sum_{\substack{|\lambda|=k \\ \lambda<\lambda s_i}} (-q)^{k(n-k)-\ell(\lambda,\lambda_0)}(v_{\lambda s_i} + q^{-1}v_\lambda) + \sum_{\substack{|\lambda|=k \\ \lambda<\lambda s_i}} (-q)^{k(n-k)-\ell(\lambda s_i,\lambda_0)}(v_\lambda + qv_{\lambda s_i}) \\
&= \sum_{\substack{|\lambda|=k \\ \lambda<\lambda s_i}} (-q)^{k(n-k)-\ell(\lambda,\lambda_0)}\Big((v_{\lambda s_i} + q^{-1}v_\lambda) - q^{-1}(v_\lambda + qv_{\lambda s_i})\Big) = 0
\end{aligned}$$

using $\ell(\lambda s_i, \lambda_0) = \ell(\lambda, \lambda_0) + 1$ for $\lambda < \lambda s_i$.

c) For $\lambda < \lambda s_i$ with $|\lambda| = k$, we have

$$\begin{aligned}
\pi_n(v_\lambda.U_i) &= \pi_n(v_{\lambda s_i} + q^{-1}v_\lambda) \\
&= \Big((-q)^{-\ell(\lambda s_i,\lambda_0)} + q^{-1}(-q)^{-\ell(\lambda,\lambda_0)}\Big)v^k \\
&= (-q)^{-\ell(\lambda s_i,\lambda_0)}\Big(1 + q^{-1}(-q)\Big) = 0.
\end{aligned}$$

The case $\lambda > \lambda s_i$ follows analogously and the case $\lambda = \lambda s_i$ holds trivially. □

It is especially easy to consider the behaviour of π_n and p_n when applied to most of the canonical basis:

Corollary 2.2.5. *For $\lambda \neq \lambda_0$ we have*

$$\pi_n(v_{\heartsuit\lambda}) = 0 = p_n(v_{\heartsuit\lambda}).$$

On the other hand, for $\lambda = \lambda_0$ we have $\pi_n(v_{\heartsuit\lambda_0}) = v^k$ and

$$p_n(v_{\heartsuit\lambda_0}) = \frac{(-1)^{k+n}}{\begin{bmatrix} n \\ k \end{bmatrix}} \sum_{|\mathbf{a}|=k} q^{-b(\mathbf{a})} v_{\mathbf{a}}.$$

Proof. For $\lambda \neq \lambda_0$ we have $v_{\heartsuit\lambda} = v_{\heartsuit\lambda'}.U_i$ for some λ' and some i by Proposition 2.1.7 and Lemma 1.2.17. Thus, $\pi_n(v_{\heartsuit\lambda}) = 0$ follows from the previous lemma. Since $v_{\heartsuit\lambda_0} = v_{\lambda_0}$ we obtain $\pi_n(v_{\heartsuit\lambda_0}) = v^k$ by definition. Plugging this in yields

$$p_n(v_{\heartsuit\lambda_0}) = \iota_n(v^k) = \frac{1}{\begin{bmatrix} n \\ k \end{bmatrix}_{-q}} \iota_n(v_k) = \frac{(-1)^{k+n}}{\begin{bmatrix} n \\ k \end{bmatrix}} \sum_{|\mathbf{a}|=k} q^{-b(\mathbf{a})} v_{\mathbf{a}}.$$

□

2.3 Jones-Wenzl projectors in the Temperley-Lieb algebra

Recall the following well-known statement (see e.g. [KL94, 3.1.1]):

Proposition 2.3.1. *There is a unique non-zero element* $P_n \in TL_n^{\mathbb{C}}$ *such that*

i) $P_n^2 = P_n$

ii) $P_n U_i = 0 = U_i P_n$ *for all* $i = 1, \dots, n-1$.

This assertion yields together with Proposition 2.2.3 that the Jones-Wenzl projector P_n is sent to p_n under the isomorphism $TL_n^{\mathbb{C}} \cong \mathrm{End}_{\mathcal{U}_q}(V^{\otimes n})$.

Kauffman and Lins [KL94, 3.2] even give an explicit formula which under our conventions transfers to the following:

Lemma 2.3.2. *In the Temperley-Lieb algebra let* $H_i = U_i - q\,\mathrm{Id}$. *For* $s = s_{i_1} \dots s_{i_r}$ *a reduced expression in* S_n *let* $H(s) := H_{i_1} \dots H_{i_n}$ *and* $H(e) := \mathrm{Id}$. *Then*

$$P_n = \frac{1}{[\![n]\!]!} \sum_{s \in S_n} (-q)^{l(s)} H(s),$$

where $[\![n]\!] = (q^{2n} - 1)/(q^2 - 1)$ *and* $[\![n]\!]! = [\![1]\!] \cdot [\![2]\!] \cdots [\![n]\!]$.

Example 2.3.3. For $n = 2$ we obtain using the formula above

$$\begin{aligned} P_2 &= \frac{1}{[\![2]\!]!}\Big((-q)^0 \,\mathrm{Id} + (-q)^1 H(s_1)\Big) = \frac{1}{[\![2]\!]}\Big(\mathrm{Id} + (-q)(U_1 - q\,\mathrm{Id})\Big) \\ &= \frac{1}{1+q^2}\Big((1+q^2)\,\mathrm{Id} - qU_1\Big) = \mathrm{Id} - \frac{q}{1+q^2} U_1 = \mathrm{Id} - \frac{1}{[2]} U_1. \end{aligned}$$

When fixing n and k and applying P_n to the standard basis vector v_{λ_0} in $(V^{\otimes n})_{2k-n}$ using the formula above, then we can restrict the number of s for which we use $H(s)$ in the formula:

Lemma 2.3.4. *Let* $\lambda_0 \in \Lambda(n,k)$. *Then*

$$\sum_{s \in \mathbb{S}_n} (-q)^{l(s)} v_{\lambda_0}.H(s) = [\![k]\!]![\![n-k]\!]! \sum_{s \in \mathrm{W}^{min}_{n,k}} (-q)^{l(s)} v_{\lambda_0}.H(s). \tag{2.5}$$

Proof. By (2.2) $v_{\lambda_0}.U_i = 0$ for all $i \neq k$. Hence, $v_{\lambda_0}.H(s_i) = -q v_{\lambda_0}$ for $i \neq k$. By [BB05, Proposition 2.4.4] every $s \in \mathbb{S}_n$ has a reduced expression of the form $s_{i_1} \dots s_{i_r} s_{j_1} \dots s_{j_l}$ with $i_t \neq k$ for $t = 1, \dots, r$ and $s_{j_1} \dots s_{j_l} \in \mathrm{W}^{\min}$. For s with such a reduced expression we have

$$v_{\lambda_0}.H(s) = (-q)^r v_{\lambda_0}.H(s_{j_1} \dots s_{j_l}).$$

By counting elements we see that for fixed $t = s_{j_1} \dots s_{j_l}$ all possible $s_{i_1} \dots s_{i_r} \in \mathbb{S}_k \times \mathbb{S}_{n-k}$ appear. Thus, in the sum on the left hand side of (2.5) a fixed $v_{\lambda_0}.H(s)$ with $s \in \mathrm{W}^{\min}$ appears in total with factor

$$\sum_{t \in \mathbb{S}_k \times \mathbb{S}_{n-k}} (-q)^{2l(t)} = [\![k]\!]![\![n-k]\!]!,$$

where the equality holds because of [Hag08, Theorem 1.1]. □

Chapter 3

The Bar-Natan approach to categorification

In this chapter, we recall Bar-Natan's categorification [BN05] of the Temperley-Lieb algebra and use a similar construction to define a category which is the foundation for the categorification of $V^{\otimes n}$. But before doing this, we need to recollect some categorical constructions.

3.1 Categorification tools

Definition 3.1.1. A *pre-additive category* is a category $\mathcal{C}$ in which the morphism sets between two given objects are abelian groups and the composition maps are bilinear.

A *($\mathbb{Z}$-)graded pre-additive category* is a category $\mathcal{C}$ where every morphism set is a graded abelian group and the composition is bilinear and respects the grading: For objects A, B, C we have $\mathcal{C}(A,B) = \bigoplus_{i\in\mathbb{Z}} \mathcal{C}(A,B)_i$ with $\mathcal{C}(B,C)_j \circ \mathcal{C}(A,B)_i \subset \mathcal{C}(A,C)_{i+j}$ and $\mathrm{id}_A \in \mathcal{C}(A,A)_0$.

Let $\mathcal{C}$ be a graded pre-additive category. Following [MOS09] we define $\mathcal{C}^{\mathbb{Z}}$ to be the category such that $\mathrm{ob}(\mathcal{C}^{\mathbb{Z}}) = \mathrm{ob}(\mathcal{C}) \times \mathbb{Z}$ and for $A, B \in \mathrm{ob}(\mathcal{C})$ and $i, j \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}((A,i),(B,j)) = \mathcal{C}(A,B)_{i-j}$. Composition is just composition in $\mathcal{C}$.

Note that $\mathcal{C}^{\mathbb{Z}}$ is pre-additive, too.

A *graded pre-additive monoidal category* is a graded pre-additive category that is also monoidal and has the property that the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfies $\mathcal{C}(A,B)_i \otimes \mathcal{C}(A',B')_j \subset \mathcal{C}(A\otimes A', B\otimes B')_{i+j}$ and $\otimes$ is bilinear.

Note that if $\mathcal{C}$ is graded pre-additive monoidal, then $\mathcal{C}^{\mathbb{Z}}$ is pre-additive monoidal, indeed we can define a functor

$$\begin{aligned} \otimes^{\mathbb{Z}} : \mathcal{C}^{\mathbb{Z}} \times \mathcal{C}^{\mathbb{Z}} &\to \mathcal{C}^{\mathbb{Z}} \\ \big((A,i),(B,j)\big) &\mapsto (A\otimes B, i+j) \\ \big(f\colon (A,i) \to (A',i'), g\colon (B,j) \to (B',j')\big) &\mapsto \big(f\otimes g\colon (A\otimes B, i+j) \to (A'\otimes B', i'+j')\big). \end{aligned}$$

If I is the unit in $\mathcal{C}$ and a, l, r are the associativity, left unit and right unit isomorphisms, we define $I^{\mathbb{Z}} = (I, 0)$. Then associativity and unity in $\mathbb{Z}$ allows us to define $o^{\mathbb{Z}} = o \times \mathrm{id}_{\mathbb{Z}}$ for $o \in \{a, l, r\}$ such that the required coherence diagrams obviously commute.

On $\mathcal{C}^{\mathbb{Z}}$ we have an autoequivalence

$$\begin{aligned} \langle 1 \rangle : \mathcal{C}^{\mathbb{Z}} &\to \mathcal{C}^{\mathbb{Z}} \\ (A, i) &\mapsto (A, i+1) \\ \big(f\colon (A,i) \to (B,j)\big) &\mapsto \big(f\colon (A, i+1) \to (B, j+1)\big). \end{aligned}$$

Note that

$$\begin{aligned} f \in \mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}\big((A,i),(B,j)\big) &= \mathcal{C}(A,B)_{i-j} = \mathcal{C}(A,B)_{i+1-(j+1)} \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}\big((A,i+1),(B,j+1)\big). \end{aligned}$$

The autoequivalence has an inverse $\langle -1 \rangle$, such that $\langle 1 \rangle \circ \langle -1 \rangle = \mathrm{id} = \langle -1 \rangle \circ \langle 1 \rangle$. We denote $\langle i \rangle := \underbrace{\langle 1 \rangle \circ \cdots \circ \langle 1 \rangle}_{i}$ and $\langle -i \rangle := \underbrace{\langle -1 \rangle \circ \cdots \circ \langle -1 \rangle}_{i}$

Having this in mind, for objects in $\mathcal{C}^{\mathbb{Z}}$ we also use the notation $A\langle i \rangle := (A, i)$.

By the degree of a morphism in $\mathcal{C}^{\mathbb{Z}}$ we mean its degree in $\mathcal{C}$, that is for a morphism $f \in \mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}\big((A,i),(B,j)\big)$ we have $\deg(f) = i - j$.

Following [BN05] we define the additive closure of a pre-additive category:

Definition 3.1.2. For $\mathcal{C}$ a pre-additive category $\mathrm{Mat}(\mathcal{C})$ is the category with:
Objects: formal finite direct sums (possibly empty) $\bigoplus_{i=1}^{n} O_i$ of objects O_i in $\mathcal{C}$.
Morphisms: For objects $O = \bigoplus_{i=1}^{m} O_i$ and $O' = \bigoplus_{i=1}^{n} O'_i$, a morphism $F\colon O' \to O$ is an $m \times n$ matrix $F = (F_{ij})$ of morphisms $F_{ij}\colon O'_j \to O_i$ in $\mathcal{C}$. Morphisms in $\mathrm{Mat}(\mathcal{C})$ are added using matrix addition and composition is defined by a rule modelled on matrix multiplication:

$$\big((F_{ij}) \circ (G_{jk})\big)_{ik} := \sum_j F_{ij} \circ G_{jk}.$$

If $\mathcal{C}$ is equipped with an autoequivalence $\langle 1 \rangle$, then this induces one on $\mathrm{Mat}(\mathcal{C})$, too. If $\mathcal{C}$ is monoidal, then there is an induced monoidal structure on $\mathrm{Mat}(\mathcal{C})$. Moreover, if there is an autoequivalence $\langle 1 \rangle$ compatible with the monoidal structure on $\mathcal{C}$, then it is still compatible with the monoidal structure on $\mathrm{Mat}(\mathcal{C})$.

For $\mathcal{C}$ a graded pre-additive (or graded pre-additive monoidal) category we denote $\widehat{\mathcal{C}} = \mathrm{Mat}(\mathcal{C}^{\mathbb{Z}})$.

Remark 3.1.3. For $\mathcal{C}$ a graded pre-additive category, $\widehat{\mathcal{C}}$ has as objects direct sums of some $A\langle i \rangle$ and as morphisms matrices of morphisms $f\colon A\langle i \rangle \to B\langle j \rangle$ such that seen as a morphism in $\mathcal{C}$ the morphism f satisfies $\deg(f) + j - i = 0$.

Definition 3.1.4. The *split Grothendieck group* $K_0(\mathcal{A})$ for an additive category $\mathcal{A}$ is defined as

$$K_0(\mathcal{A}) = \mathbb{Z}\big\langle Iso(\mathcal{A})\big\rangle \big/ \big([A \oplus B] = [A] + [B]\big),$$

i.e. it is the free abelian group generated by isomorphism classes of objects of $\mathcal{A}$ modulo the relation specified above.

If $\mathcal{A}$ has an autoequivalence $\langle 1 \rangle$ we set $q^i[A] = [A\langle i \rangle]$ and this makes $K_0(\mathcal{A})$ into a $\mathbb{Z}[q, q^{-1}]$-module.

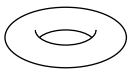
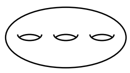
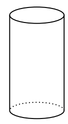
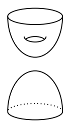
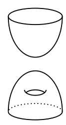
If moreover $\mathcal{A}$ is monoidal, then $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ induces $K_0(\mathcal{A}) \otimes K_0(\mathcal{A}) \to K_0(\mathcal{A})$, endowing $K_0(\mathcal{A})$ with an algebra structure.

In the spirit of this and also since it is better readable, we will often use $A\langle i \rangle$ and $q^i A$ synonymously.

3.2 Categorification of the Temperley-Lieb algebra

We start with defining the category that is used to categorify TL_n.

Definition 3.2.1. Let R be a integral domain in which 2 is invertible. Let $\mathrm{Cob}_R(n)$ be the category with
Objects: Compact 1-manifolds with boundary the n upper and n lower marked points in a rectangle, where the upper points are on the upper side of the rectangle and the lower ones on the lower side. We call the union of the upper and lower marked points P.
Morphisms: A morphism $f\colon A \to B$ is a formal R-linear combination of cobordisms with boundary $A \cup P \times [0,1] \cup B$ (nicely embedded in the (rectangle $\times [0,1]$)) regarded up to boundary-preserving isotopies modulo the following local relations:

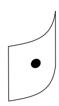
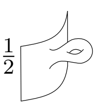
 $= 2,$  $= 0$  $= \frac{1}{2}$  $+ \frac{1}{2}$  . (3.1)

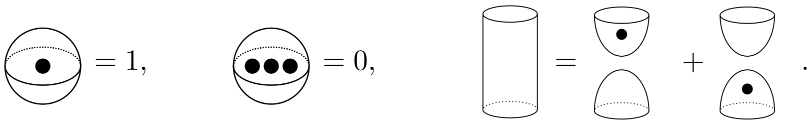
Here, the first relation means that whenever a cobordism contains a torus as a connected component, it may be deleted and replaced by the factor 2. Analogously, the second relation says that every cobordism containing a genus 3 surface is 0. The last relations means that whenever we find a tube inside a cobordism, then we can replace it by $\frac{1}{2}$ times the sum of two cobordisms that arise when we replace the tube inside the cobordism by the first summand of the relation resp. the second summand.

The composition is given by the bilinear extension of composing cobordisms and rescaling, i.e. when we have $f\colon A \to B$ and $g\colon B \to C$, we glue the two copies of B and rescale the cobordism to length 1. For a cobordism f we define a degree via $\deg(f) = n - \chi(f)$, where $\chi(f)$ is the Euler characteristic of the cobordism.

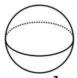
We call the relations (3.1) the *Bar-Natan relations*; the last of the relations is called *neckcutting*. Note that the neckcutting relation is homogeneous of degree 0.

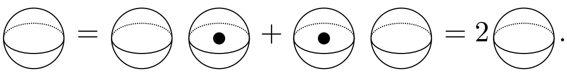
Furthermore, the degree is compatible with composition, so $\mathrm{Cob}_R(n)$ is a graded pre-additive category.

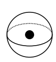
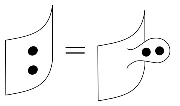
Remark 3.2.2. When we use the shorthand notation  $= \frac{1}{2}$  the Bar-Natan relations become

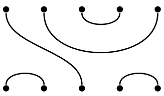
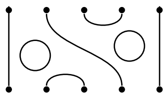
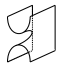
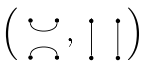
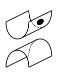
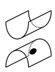
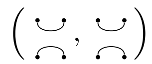


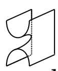
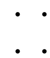
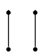
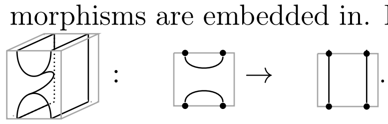
When we refer to the Bar-Natan relations or neckcutting in the future, we will mean this version.

First note that  $= 0$ follows from the relations: When we apply neckcutting to the middle of the sphere we obtain

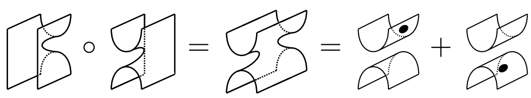


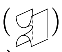
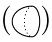
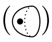
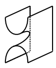
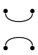
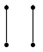
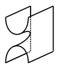
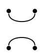
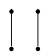
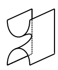
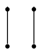
Furthermore, we get  $= 0$ analogously by using neckcutting to cut in the middle between the two •'s and then use  $= 1$. Moreover, we can show  $= 0$, by using neckcutting to cut between a part containing the two •'s and the rest, and then applying that a sphere with two or three •'s is zero.

Example 3.2.3. For example,  and  are objects in $\mathrm{Cob}_R(5)$. In $\mathrm{Cob}_R(2)$ with, for example, $R = \mathbb{Z}[\frac{1}{2}]$, we have  $\in \mathrm{Hom}_{\mathrm{Cob}_R(2)}$  and 5  $- 3$  $\in \mathrm{Hom}_{\mathrm{Cob}_R(2)}$ .

The boundary of  is  $\cup$  $\times [0,1] \cup$ . Note that we usually do not draw the rectangle resp. the cuboid where the objects resp. morphisms are embedded in. Here is an example with these habitats drawn explicitly: .

Here is an example for the composition and an application of neck-cutting:

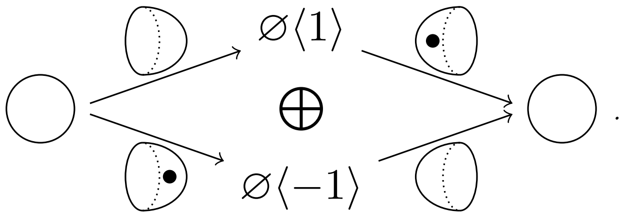


Moreover we know, $\deg($$) = 1$, $\deg($$) = -1$ and $\deg($$) = 1$. Since $\deg\Big($ $:$  $\langle 1\rangle \to$ $\Big) = 0$, we have that  $:$  $\langle 1\rangle \to$  is a morphism in $\mathrm{Cob}_R(n)^{\mathbb{Z}}$, but  $:$  $\langle 2\rangle \to$  is not.

$\mathrm{Cob}(n)$ is even a monoidal category by putting objects or morphisms on top of each other followed by rescaling. Furthermore, the degree defined above is compatible with the monoidal structure, turning $\mathrm{Cob}_R(n)$ into a graded pre-additive monoidal category.

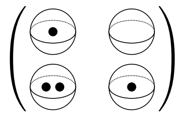
The relations in $\mathrm{Cob}_R(n)$ allow the following delooping argument from [BN07]:

Lemma 3.2.4 (Delooping)**.**
If an object S in $\mathrm{Cob}_R(n)^{\mathbb{Z}}$ *contains a* ○, *then it is isomorphic in* $\mathrm{Mat}\left(\mathrm{Cob}_R(n)^{\mathbb{Z}}\right)$ *to* $S'\langle 1\rangle \oplus S'\langle -1\rangle$, *where S' is S with the* ○ *removed. The isomorphism and its inverse are given by*

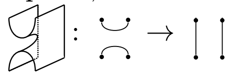


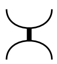
We give the proof to help the reader to get a feeling for the relations.

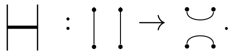
Proof. We check that the compositions of the morphisms given above are the identity: Going from ○ to itself, we get the right hand side of neck-cutting, which is equal to the identity. Now going from $\varnothing\langle 1\rangle \oplus \varnothing\langle -1\rangle$ to itself we obtain

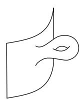
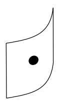
 $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

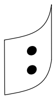
□

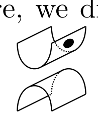
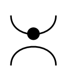
We have a shorthand notation for the saddle cobordism: Between the parts, where the saddle cobordism is applied, we draw a thick line. For example, for 

we draw  and for the saddle cobordism going in the opposite direction we draw



Remark 3.2.5. With  $= 2$  we can write morphisms of $\mathrm{Cob}_R(n)$ as linear combinations of cobordisms without genus and and decorated by several •'s.

Using  $= 0$ we see that morphisms in $\mathrm{Cob}_R(n)$ are linear combinations of cobordisms without genus and up to one • on each connected component.

Also, we can use the Bar-Natan relations to split the cobordism into parts where each connected component has exactly one boundary component, see also [Nao06]. In particular, cobordisms from an object T to itself are linear combinations of $T \times [0,1]$ with at most one dot on each component. Therefore, we draw these summands as T with dots on the components of T. For example, for  we draw  .

Notation 3.2.6. We can consider basis elements of TL_n as objects of the category $\mathrm{Cob}_R(n)$; this holds in particular for $U_i \in TL_n$. To distinguish them, we write $\mathcal{U}_i$ for U_i seen as an object of $\mathrm{Cob}_R(n)$.

Also, most of the time we denote the tensor product $A \otimes B$ in $\mathrm{Cob}_R(n)$ simply by juxtaposition AB.

Note that in this notation, the equalities (1.6) and (1.7) from the definition of the Temperley-Lieb algebra hold now up to isomorphism: $\mathcal{U}_i\mathcal{U}_j \cong \mathcal{U}_j\mathcal{U}_i$ for $|i-j| > 1$ and $\mathcal{U}_i\mathcal{U}_{i\pm1}\mathcal{U}_i \cong \mathcal{U}_i$.

Furthermore, using neckcutting, (1.5) has the counterpart $\mathcal{U}_i\mathcal{U}_i \cong \mathcal{U}_i\langle 1\rangle \oplus \mathcal{U}_i\langle -1\rangle$.

The following categorification result for the Temperley-Lieb algebra can be found in [MN08, Theorem 5.2] and [CK12, Lemma 2.6]:

Theorem 3.2.7. *The category $\widehat{\mathrm{Cob}}_R(n)$ categorifies TL_n in the sense that we have an isomorphism of $\mathbb{Z}[q,q^{-1}]$-algebras*

$$K_0\Big(\widehat{\mathrm{Cob}}_R(n)\Big) \cong TL_n$$

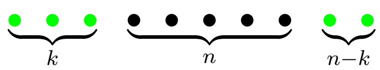
between the split Grothendieck ring and the Temperley-Lieb algebra.

The proof relies heavily on the fact that by using neck-cutting, every object in $\widehat{\mathrm{Cob}}_R(n)$ is isomorphic to a direct sum of objects without circles, and of course, the isomorphisms corresponding to the Temperley-Lieb relations from the previous remark.

3.3 The category $\mathrm{Cup}(n,k)$

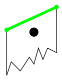
Our goal is to categorify $V^{\otimes n}$ as a TL_n-module such that we can see the standard basis. Having Proposition 2.1.7 in mind, it is conceivable that a first step should be a categorification of $\widehat{eC}(n,k)$ as a $\mathbb{Z}[q,q^{-1}]$-module. For this, analogously to the categorification of TL_n, we now define a new category, where objects are modelled on extended cup diagrams (cf. Definition 1.2.10) with additional circles.

We fix $n \geq k \geq 0$. Let $L_{n,k}$ be the sequence of coloured points

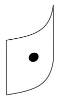
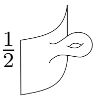


from Definition 1.2.8.

Definition 3.3.1. Let R be a integral domain in which 2 is invertible. Let $\mathrm{Cup}_R(n,k)$ be the category with
Objects: Compact 1-manifolds embedded in a rectangle with boundary $L_{n,k}$ on the upper side of the rectangle, with the additional condition that no component contains two right or two left green points.
Morphisms: A morphism $f\colon A \to B$ is a formal R-linear combinations of (nicely embedded) cobordisms with boundary $A \cup L_{n,k} \times [0,1] \cup B$ regarded up to boundary-preserving isotopies modulo the local Bar-Natan relations and the following additional relations:

1) A connected component containing a dot and one green boundary line is equal to 0:
 $= 0$

2) A connected component containing two left green or two right green boundary lines is equal to 0,

where again we use the notation  $= \frac{1}{2}$  . The composition is given by the bilinear extension of composing cobordisms and rescaling analogously to the composition in $\mathrm{Cob}_R(n)$. Again, we define the degree of a cobordism by $\deg(f) = n - \chi(f)$, which turns $\mathrm{Cup}_R(n, k)$ into a graded pre-additive category.

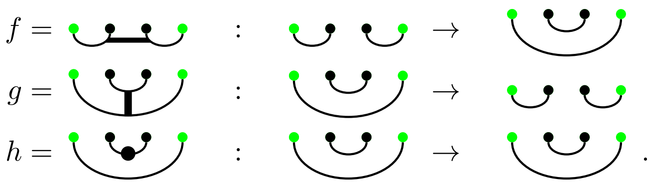
Remark 3.3.2. Note that, since the Bar-Natan relations also hold in $\mathrm{Cup}_R(n, k)$, a version of Remark 3.2.5 is true for $\mathrm{Cup}_R(n, k)$: Cobordisms in $\mathrm{Cup}_R(n, k)$ are linear combinations of cobordisms without genus and and decorated by $\bullet$'s such that every connected component has exactly one boundary component. Furthermore, each connected component is decorated by at most one $\bullet$, and no $\bullet$'s if it also contains green boundary lines.

We also use the shorthand notations for cobordisms from the previous section.

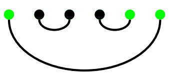
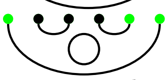
Example 3.3.3. In $\mathrm{Cup}_R(2, 1)$ there are only two objects without circles up to isomorphisms:

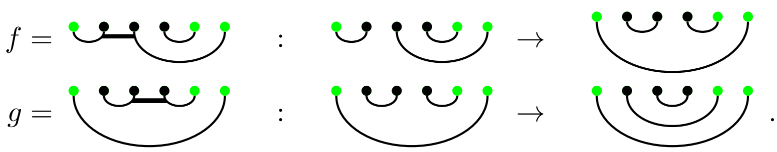
 and  .

Possible morphisms between those are the identity morphisms, the two saddle morphisms and the degree 2 morphism resp.

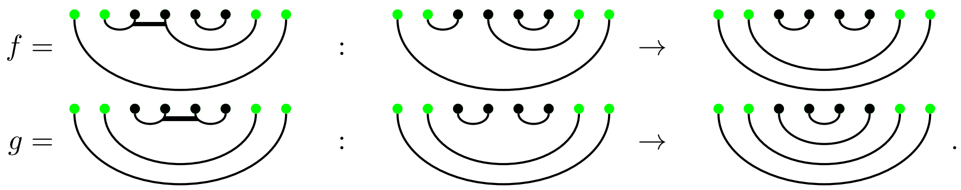


Note that $f \circ g = h$ holds because of neck-cutting and Relation 1), whereas $g \circ h = 0$ and $h \circ f = 0$ also because of Relation 1). Furthermore, $\deg(f) = \deg(g) = 1$ and $\deg(h) = 2$.

In $\mathrm{Cup}_R(3, 1)$ there are 3 objects without circle up to isomorphism:  ,  and  . An example for an object with a circle is given by  . Moreover, the following morphisms f and g in $\mathrm{Cup}_R(3, 1)$ are interesting, since the composition is zero because of Relation 2):

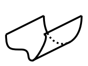
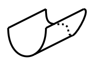
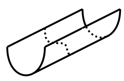


Note that Relation 2) does not imply that the following composition of f and g in $\mathrm{Cup}_R(4,2)$ is zero, which look similar locally:



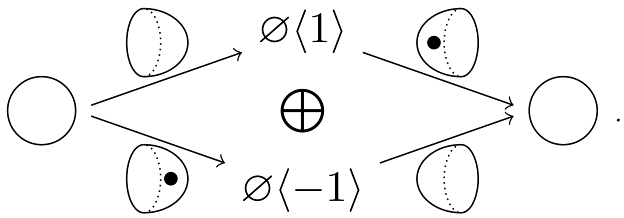
Here relation 2) cannot be applied since some of the boundary lines that were green in the previous example are now black.

Notation 3.3.4. We can consider the extended cup diagrams $\mathrm{C}(\lambda)$ as objects of $\mathrm{Cup}_R(n,k)$. When doing so we denote them by $\mathrm{T}(\lambda)$. Here T stands for tilting, see Proposition 8.3.10. Note that every object of $\mathrm{Cup}_R(n,k)$ without circles is isomorphic to some $\mathrm{T}(\lambda)$. To see this, we observe that differently wiggled cups are isomorphic, since there are obvious degree 0 morphisms between them whose composition is the identity because of isotopy: We have

and now  $\circ$  $= \mathrm{id}_{\cup}$ since  is isotopic to . Analogously, the other composition is the identity, too.

Lemma 3.2.4 also holds for $\mathrm{Cup}_R(n,k)$ instead of $\mathrm{Cob}_R(n)$, since only the Bar-Natan-relations are used in the proof:

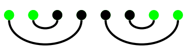
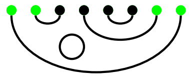
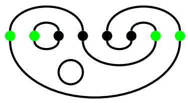
Lemma 3.3.5 (Delooping).
If an object S in $\mathrm{Cup}_R(n,k)^{\mathbb{Z}}$ contains a $\bigcirc$, then it is isomorphic in $\mathrm{Mat}\left(\mathrm{Cup}_R(n,k)^{\mathbb{Z}}\right)$ to $S'\langle 1\rangle \oplus S'\langle -1\rangle$, where S' is S with the $\bigcirc$ removed. The isomorphism is given by



3.4 Alternative description for the morphisms using circle diagrams

Before coming to our first categorification result, we need to know more about the morphisms of our category $\mathrm{Cup}_R(n,k)$. For this, we want to give an alternative description using the function $\mathcal{F}_{col}$ defined below.

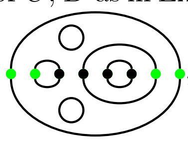
Definition 3.4.1. Analogously to Definition 1.2.27 for C, D objects in $\mathrm{Cup}_R(n,k)$ we define the *circle diagram* $C\overline{D}$ as the result of reflecting D at the horizontal axis and putting it on top of C. Again, a circle inside a circle diagram is called *red* if it contains more then one right green point or more then one left green point. Other circles that contain a green point are called *green*. Circles without green points are called *black*. Note that now $C\overline{D}$ can also contain circles that arise from circles in C or D. These are always black.

Example 3.4.2. For $C =$  and $D =$  the circle diagram $C\overline{D} =$  has exactly one circle of every colour.

Definition 3.4.3. Every circle diagram is the disjoint union of coloured circles. Let O_r be a red, O_g a green and O_b a black circle. We define $\mathcal{F}_{col}$: {circle diagrams} $\to$ $\mathbb{C}$-vector spaces via setting $\mathcal{F}_{col}(O_b) = \mathbb{C}[x]/(x^2)$, $\mathcal{F}_{col}(O_g) = \mathbb{C}$, $\mathcal{F}_{col}(O_r) = 0$ and then tensoring together (over $\mathbb{C}$), i.e. for a circle diagram O we have

$$\mathcal{F}_{col}(O) = \begin{cases} \left(\mathbb{C}[x]/(x^2)\right)^{\otimes b} \otimes_{\mathbb{C}} \mathbb{C}^{\otimes g} & \text{if } r = 0, \\ 0 & \text{if } r > 0, \end{cases}$$

where r is the number of red circles in O, g the number of green and b the number of black ones. Here, we keep the $\mathbb{C}^{\otimes g}$ on purpose to be better able to describe maps between tensor factors later on.

Example 3.4.4. For C, D as in Example 3.4.2 we get $\mathcal{F}_{col}(C\overline{D}) = 0 \otimes \mathbb{C} \otimes \mathbb{C}[x]/(x^2) = 0$. Moreover, $C\overline{C} =$ , so $\mathcal{F}_{col}(C\overline{C}) = \left(\mathbb{C}[x]/(x^2)\right)^{\otimes 3} \otimes \mathbb{C}^{\otimes 3}$.

Our goal now is to show $\mathrm{Hom}_{\mathrm{Cup}_{\mathbb{C}}(n,k)}(C, D) \cong \mathcal{F}_{col}(D\overline{C})$ in Theorem 3.4.12. For an intermediate step, we define categories $\mathrm{Cob}_R(n, m)$ analogously to $\mathrm{Cob}_R(n)$ (cf. Definition 3.2.1). The only difference is that in $\mathrm{Cob}_R(n, m)$ the number of upper and lower points is allowed to be different. We fix n upper and m lower points and call the union of points P.

Definition 3.4.5. Let R be a integral domain in which 2 is invertible and let $m + n$ be even. Then $\mathrm{Cob}_R(n, m)$ is the category with
Objects: Compact 1-manifolds with boundary the n upper and m lower marked points P in a rectangle, where the upper points are on the upper side of the rectangle and the lower ones on the lower side.
Morphisms: A morphism $f \colon A \to B$ is a formal R-linear combination of (nicely embedded) cobordisms with boundary $A \cup P \times [0, 1] \cup B$ regarded up to boundary-preserving isotopies modulo the Bar-Natan relations (3.1). We make this category into a graded one via

$$\deg(f) = \frac{m+n}{2} - \chi(f). \tag{3.2}$$

Note that in this case also a version of Remark 3.2.5 holds: Every morphism in $\mathrm{Cob}_R(n, m)$ is a linear combination of cobordisms in which every connected component has boundary, the boundary of the component is connected itself and every component contains at most one $\bullet$.

Definition 3.4.6. The objects of $\mathrm{Cup}_R(n,k)$ reduce to objects in $\mathrm{Cob}_R(2n,0)$ by forgetting the colouring. Let $I\colon \mathrm{ob}\left(\mathrm{Cup}_R(n,k)\right) \to \mathrm{ob}\left(\mathrm{Cob}_R(2n,0)\right)$ be the colouring forgetting map. Note that this does not extend to a functor in any obvious way.

However, the relations on morphisms in $\mathrm{Cup}_R(n,k)$ are the same as in $\mathrm{Cob}_R(2n,0)$ except for the two additional relations concerning green points. So we have a canonical surjection

$$\Pi\colon \mathrm{Hom}_{\mathrm{Cob}_R(2n,0)}\left(I(C), I(D)\right) \twoheadrightarrow \mathrm{Hom}_{\mathrm{Cup}_R(n,k)}(C,D)$$

given by quotiening out the extra relations.

In the following, we omit the base ring R in case $R=\mathbb{C}$ and simply denote $\mathrm{Cob}(n) := \mathrm{Cob}_{\mathbb{C}}(n)$, $\mathrm{Cup}(n,k) := \mathrm{Cup}_{\mathbb{C}}(n,k)$ and $\mathrm{Cob}(n,m) := \mathrm{Cob}_{\mathbb{C}}(n,m)$. From now on, we restrict ourselves to this case, even though most of the results hold more generally for any R.

Definition 3.4.7. Of course we can define a circle diagram $C\overline{D}$ for C,D objects in $\mathrm{Cob}(2n,0)$ as before and also apply the function $\mathcal{F}_{col}$. Since $C\overline{D}$ contains only black circles we just write $\mathcal{F}(C\overline{D})$ in this setting. In particular, $\mathcal{F}(C\overline{D}) = \left(\mathbb{C}[x]/(x^2)\right)^{\otimes b}$, where b is the number of circles in $C\overline{D}$.

For circle diagrams $C\overline{D}$ with C,D objects of $\mathrm{Cup}(n,k)$ we define $\pi_{C\overline{D}} : \mathcal{F}\big(I(C)\overline{I(D)}\big) \to \mathcal{F}_{Col}(C\overline{D})$ by defining it on every tensor factor and then tensoring together:

$$\begin{aligned}
\pi_{O_b} &: \mathbb{C}[x]/(x^2) \xrightarrow{\mathrm{id}} \mathbb{C}[x]/(x^2)\\
\pi_{O_g} &: \mathbb{C}[x]/(x^2) \to \mathbb{C}\\
&\qquad\qquad 1 \mapsto 1\\
&\qquad\qquad x \mapsto 0\\
\pi_{O_r} &: \mathbb{C}[x]/(x^2) \xrightarrow{0} \{0\}\\
\text{and } \pi_{X\sqcup Y} &= \pi_X \otimes \pi_Y
\end{aligned}$$

for O_b a black, O_g a green, O_r a red circle and X and Y arbitrary disjoint unions of circles.

For objects we obviously have $\mathcal{F}_{Col}(C\overline{D}) = \pi_{C\overline{D}}\Big(\mathcal{F}\big(I(C)\overline{I(D)}\big)\Big)$.

For an alternative approach to this theory using coloured TQFT see Section A.1. The forgetting colour map and the similarities of Π and π_A will allow us to proof our desired isomorphism first for the uncoloured case and then go over to the coloured setting.

The following lemma collects some easy topological sliding properties expressed in terms of morphism spaces in $\mathrm{Cob}(2n,0)$.

Lemma 3.4.8. *a) We have isomorphisms of graded $\mathbb{C}$-vector spaces*

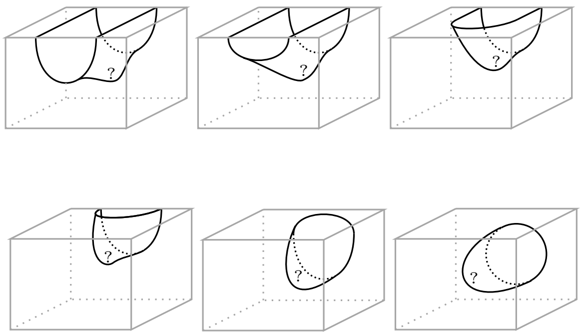
$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Cob}(2,0)}(\cup,\cup) \cong \mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing,\bigcirc)\,\langle 1\rangle \quad \textit{and}\\
&\mathrm{Hom}_{\mathrm{Cob}(2,0)}(\cup,\cup) \cong \mathrm{Hom}_{\mathrm{Cob}(0,0)}(\bigcirc,\varnothing)\,\langle 1\rangle\,.
\end{aligned}$$

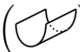
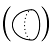
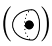
b) More generally, let C, D be objects of $\mathrm{Cob}(2n, 0)$*. Then there are isomorphisms of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C, D) \cong \mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing, D\overline{C})\,\langle n\rangle \;\; \textit{and}$$
$$\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C, D) \cong \mathrm{Hom}_{\mathrm{Cob}(0,0)}(C\overline{D}, \varnothing)\,\langle n\rangle\,,$$

where by $D\overline{C}$ we mean that we reflect C at the horizontal axis, put it on top of D and identify the boundary points and analogously for $C\overline{D}$.

Proof. a) The morphism space $\mathrm{Hom}_{\mathrm{Cob}(2,0)}(\cup, \cup)$ is 2-dimensional with basis given by the identity morphism  and the identity morphism with dot . On the other hand, $\mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing, \bigcirc)$ is also 2-dimensional with basis  and . The assignment  $\mapsto$  and  $\mapsto$  defines an isomorphism of vector spaces. Pictorially, the cap gets slided to the other side:



Since deg () = 0 and deg () = 2 whereas deg () = −1 and deg () = 1, the isomorphism is homogeneous of degree −1 and so the first isomorphism follows. The second one is analogous.

b) Repeatedly applying sliding cups as in a) to basis elements in $\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C, D)$ defines an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C, D) \cong \mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing, D\overline{C}).$$

Each sliding move does not change the Euler characteristic of the cobordism representing the basis vector, but the number of boundary points decreases by 2. Hence by (3.2) the total change in degree is 1 per slide, thus n in total, since C has n cups. If there is a circle, we can write it as $\cap \circ \cup$ and slide the cup and cap separately. The degree does not change by sliding a circle this way. This gives the first isomorphism, the second is analogous.

□

Recall that $\mathrm{Cob}(n)$ is a monoidal category $(\mathrm{Cob}(n), \otimes, \mathrm{Id})$. We have the following basic properties for homomorphism spaces:

Lemma 3.4.9. *Let $S, T \in \mathrm{Cob}(n)$, then there is an isomorphism of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cob}(n)}(S \otimes \mathcal{U}_i, T) \cong \mathrm{Hom}_{\mathrm{Cob}(n)}(S, T \otimes \mathcal{U}_i).$$

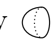
In particular, if $\mathcal{U} := \mathcal{U}_{i_1} \otimes \ldots \otimes \mathcal{U}_{i_r}$*, then* $\mathrm{Hom}_{\mathrm{Cob}(n)}(U, T) \cong \mathrm{Hom}_{\mathrm{Cob}(n)}(\mathrm{Id}, T \otimes \overline{\mathcal{U}})$*, where* $\overline{\mathcal{U}}$ *is obtained from* $\mathcal{U}$ *by reflection at a horizontal axis, i.e.* $\overline{\mathcal{U}} \cong \mathcal{U}_{i_r} \otimes \ldots \otimes \mathcal{U}_{i_1}$.

Proof. Since $\mathcal{U}_i = \cup \circ \cap$ we can apply the sliding move twice to obtain the isomorphism. The Euler characteristic stays the same as well as the number of boundary points, thus so does the degree. □

Lemma 3.4.10. *Let* C, D *be objects of* $\mathrm{Cob}(2n, 0)$ *such that* $D\overline{C}$ *has* r *circles. Then*

$$\mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing, D\overline{C}) \cong \mathcal{F}(D\overline{C}) \langle -n \rangle$$

is an isomorphism of graded vector spaces, when we define the degree on $\mathcal{F}(D\overline{C})$ *as follows: Let* $f = x_1 \otimes \ldots \otimes x_r$ *be a basis element of* $\mathcal{F}(D\overline{C})$*, i.e.* $x_i \in \{1, x\}$*. Define* $\deg(1) = 0$*,* $\deg(x) = 2$ *and* $\deg(f) = n - r + \sum_{i=1}^{r} \deg(x_i)$*.*

Proof. Let $A = \mathbb{C}[x]/(x^2)$, i.e. $\mathcal{F}(D\overline{C}) = A^{\otimes r}$. By Remark 3.2.5, every morphism $f : \varnothing \to D\overline{C}$ is a linear combination of nested  and . We identify  (with deg() $= -1$) with $1 \in A$ and  (with deg() $= 1$) with $x \in A$. Comparing the total degrees gives the shift by $-n$. □

Corollary 3.4.11. *Let* C, D *be objects of* $\mathrm{Cob}(2n, 0)$*. Then there is an isomorphism of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C, D) \cong \mathcal{F}(D\overline{C}).$$

Proof. This follows directly from the previous lemma and Lemma 3.4.8 b). □

Theorem 3.4.12. *Let* C, D *be objects of* $\mathrm{Cup}(n, k)$ *such that* $D\overline{C}$ *has* r *circles. Then there is an isomorphism of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}(C, D) \cong \mathcal{F}_{col}(D\overline{C})$$

with the degree in $\mathcal{F}_{col}(D\overline{C})$ *defined as follows: Let* $f = x_1 \otimes \ldots \otimes x_r$ *be a basis element of* $\mathcal{F}_{col}(D\overline{C})$*, i.e.* $x_i \in \{1, x\}$*. Define* $\deg(1) = 0$*,* $\deg(x) = 2$ *and* $\deg(f) = n - r + \sum_{i=1}^{r} \deg(x_i)$*.*

Proof. Let

$$\Phi\colon \mathrm{Hom}_{\mathrm{Cob}(2n,0)}\big(I(C), I(D)\big) \to \mathcal{F}\big(I(D)\overline{I(C)}\big)$$

be the isomorphism from Corollary 3.4.11. From Remark 3.4.6 we have the surjection

$$\Pi\colon \mathrm{Hom}_{\mathrm{Cob}(2n,0)}\big(I(C), I(D)\big) \twoheadrightarrow \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C, D).$$

Furthermore, from Definition 3.4.7 we have

$$\pi_{D\overline{C}} : \mathcal{F}\big(I(D)\overline{I(C)}\big) \twoheadrightarrow \mathcal{F}_{Col}(D\overline{C})$$

with

$$\mathcal{F}_{Col}(C\overline{D}) = \pi_{C\overline{D}}\Big(\mathcal{F}\big(I(C)\overline{I(D)}\big)\Big).$$

Let $f \in \mathrm{Hom}_{\mathrm{Cob}_R(2n,0)}\big(I(C), I(D)\big)$ be a basis element and let $y = \Phi(f)$. $\Pi(f)$ containing a connected component with two left green lines or two right green lines is equivalent to y having a tensor factor corresponding to a red circle, i.e. $\pi_{D\overline{C}}(y) = 0$. $\Pi(f)$ containing a connected component with a dot and one green boundary line is equivalent to y having a tensor factor coming from an $x \in \mathbb{C}[x]/(x^2)$ corresponding to a green circle, which is send to zero by $\pi_{D\overline{C}}$. Thus, Φ induces an isomorphism $\mathrm{Hom}_{\mathrm{Cup}(n,k)}(C, D) \cong \mathcal{F}_{col}(D\overline{C})$. □

Corollary 3.4.13. *Let* $\mathrm{T}(\lambda), \mathrm{T}(\mu)$ *be objects of* $\mathrm{Cup}(n, k)$ *such that* $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$ *has* r *circles. If there is no red circle in* $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$*, then there is a (up to scalar unique) non-zero degree* $n - r$ *cobordism in* $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)$*. All other cobordisms in* $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)$ *are of higher degree. If there is a red circle in* $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$*, then* $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big) = 0$.

Proof. The degree of the cobordism without •'s is $\deg(1 \otimes \ldots \otimes 1) = n + r \cdot (-1)$. The rest follows from Theorem 3.4.12 above. □

We also obtain the following result, which will be an important ingredient for our categorification (Theorem 3.5.4):

Lemma 3.4.14. *Let* $\lambda, \mu \in \Lambda(n, k)$*. Then*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)_0 = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)_r = 0$ *for all negative* r.

Proof. Let r be the number of circles in $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$. By Lemma 1.2.29 we obtain $r \leq n$ with equality only for $\lambda = \mu$. When $\lambda = \mu$ there cannot be a red circle since every circle runs through exactly two points and there are no arcs connecting two left green or two right green points. Thus, by Corollary 3.4.13 the first assertion follows. We always have $n - r \geq 0$, hence the second assertion follows. □

The morphisms from $\mathrm{T}(\lambda_0)$ to itself are up to scalar only the identity:

Lemma 3.4.15.

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0), \mathrm{T}(\lambda_0)\big)_r \cong \begin{cases} \mathbb{C} & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first case is clear by Lemma 3.4.14. Since there are only green circles in $\mathrm{C}(\lambda_0)\overline{\mathrm{C}(\lambda_0)}$, we have $\mathcal{F}_{col}\big(\mathrm{C}(\lambda_0)\overline{\mathrm{C}(\lambda_0)}\big) \cong \mathbb{C}$. Thus, by Theorem 3.4.12 we obtain $\dim \mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0), \mathrm{T}(\lambda_0)\big) = 1$ and there cannot be more morphisms. □

3.5 First categorification

In this section we finally obtain our first categorification result. For that, we need to consider the following subcategory.

Definition 3.5.1. Let $\mathrm{CupT}(n,k)$ be the full subcategory of $\mathrm{Cup}(n,k)$ with objects $\{\mathrm{T}(\lambda), \lambda \in \Lambda(n,k)\}$.

Remark 3.5.2. The category $\mathrm{Mat}\left(\mathrm{CupT}(n,k)^{\mathbb{Z}}\right)$ is the skeleton of the category $\mathrm{Mat}\left(\mathrm{Cup}(n,k)^{\mathbb{Z}}\right)$: By delooping (Lemma 3.3.5), every object of $\mathrm{Mat}\left(\mathrm{Cup}(n,k)^{\mathbb{Z}}\right)$ is isomorphic to some object of $\mathrm{Mat}\left(\mathrm{Cup}(n,k)^{\mathbb{Z}}\right)$ without any circles, i.e. to some object where no direct summand contains circles. But every object of $\mathrm{Cup}(n,k)$ without a circle is isomorphic to some $\mathrm{T}(\lambda)$ (cf. Notation 3.3.4) and there is no isomorphism $\mathrm{T}(\lambda) \cong \mathrm{T}(\mu)$ for $\lambda \neq \mu$ by Lemma 3.4.14.

The following lemma is a slight generalisation of the main argument in the proof of [MN08, Theorem 5.2]. For our categorification we want to apply it to $\mathrm{Cup\,T}(n,k)$.

Recall that we use the notations qX and $X\langle 1\rangle$ instantaneously for X an object of some $\mathcal{B}^{\mathbb{Z}}$, $\mathcal{B}$ a graded pre-additive category. For $r \in \mathbb{N}[q,q^{-1}]$, $r = \sum_j a_j q^j$ with $a_j \in \mathbb{N}$, we write rX for $\bigoplus_j \underbrace{q^j X \oplus \cdots \oplus q^j X}_{a_j}$.

Lemma 3.5.3. *Let $\mathcal{B}$ be a graded pre-additive category. Let X_i, $i = 1, \ldots, r$ be representatives of the isomorphism classes of objects. Assume that the grading has the following additional properties:*

- $\deg(f) \geq 0$ *for all* $f\colon X_i \to X_j$, *and*
- *if* $\deg(f) = 0$ *for* $f\colon X_i \to X_j$, *then* $i = j$ *and* $f = c \cdot \mathrm{id}$ *for* $c \in \mathbb{C}$.

If we now have an isomorphism $\varphi\colon \bigoplus_i r_i X_i \cong \bigoplus_i r'_i X_i$ in $\mathrm{Mat}(\mathcal{B}^{\mathbb{Z}})$, where $r_\lambda, r'_\lambda \in \mathbb{N}[q,q^{-1}]$ are multiplicities including degree shifts, then $r_i = r'_i$ for all i.

Proof. To restrict to the case where we have only multiplicities in $\mathbb{N}$ we do some preliminary considerations. Recall for $f\colon q^{s_i}X_i \to q^{s_j}X_j$ in $\mathrm{Mat}\left(\mathcal{B}^{\mathbb{Z}}\right)$ we have $\deg(f)+s_j-s_i = 0$. Thus, from the assumptions we obtain $\deg(f) > 0$ if $s_i \neq s_j$. Furthermore, if $s_i = s_j$ we only have $\deg(f) = 0$ if $i = j$ and in this case $f = c\cdot\mathrm{id}$. Consider $q^s X_i \xrightarrow{f} Y \xrightarrow{g} q^s X_i$, where $Y = q^t X_j$ with $t \neq s$ or $i \neq j$. Then we have $\deg(f) > 0$, $\deg(g) > 0$ and thus $\deg(g \circ f) > 0$. But this means that $g \circ f = 0$, since $\deg(g\circ f) - s + s = 0$.

Now assume $\varphi\colon \bigoplus_\alpha r_\alpha Y_\alpha \cong \bigoplus_\alpha r'_\alpha Y_\alpha$ where $Y_\alpha = q^{s_\alpha} X_{t_\alpha}$ and $Y_\alpha \neq Y_\beta$ for $\alpha \neq \beta$ and $r_\alpha, r'_\alpha \in \mathbb{N}$ are multiplicities.

We now follow the proof of [MN08, Thm 5.2] even more closely: We fix any β and set

$$J = \bigoplus_{\alpha\neq\beta} r_\alpha Y_\alpha, \qquad J' = \bigoplus_{\alpha\neq\beta} r'_\alpha Y_\alpha.$$

Then φ and φ^{-1} can be written as 2×2-matrices

$$\varphi = \begin{pmatrix} \varphi_{00} \colon r_\beta Y_\beta \to r'_\beta Y_\beta & \varphi_{01} \colon J \to r'_\beta Y_\beta \\ \varphi_{10} \colon r_\beta Y_\beta \to J' & \varphi_{11} \colon J \to J' \end{pmatrix},$$

$$\varphi^{-1} = \begin{pmatrix} \overline{\varphi}_{00} \colon r'_\beta Y_\beta \to r_\beta Y_\beta & \overline{\varphi}_{01} \colon J' \to r_\beta Y_\beta \\ \overline{\varphi}_{10} \colon r'_\beta Y_\beta \to J & \overline{\varphi}_{11} \colon J' \to J \end{pmatrix}.$$

Calculating the composition $\varphi^{-1}\varphi$ we see that $\mathrm{id}_{r_\beta Y_\beta} = \overline{\varphi}_{00}\varphi_{00} + \overline{\varphi}_{01}\varphi_{10}$. By the considerations above, we know that $\overline{\varphi}_{01}\varphi_{10} = 0$ and φ_{00} and $\overline{\varphi}_{00}$ are matrices with entries the identity up to scalars. Thus, we can write $\varphi_{00} = M\,\mathrm{id}$ and $\overline{\varphi}_{00} = N\,\mathrm{id}$ where M is a $r'_\beta \times r_\beta$-matrix and N a $r_\beta \times r'_\beta$-matrix with coefficients in $\mathbb{C}$. We obtain $NM = 1$ and by repeating the arguments for $\varphi\varphi^{-1}$ also $MN = 1$. So we get $r_\beta = r'_\beta$. □

The lemma gives us our first categorification, a categorification of $\widehat{eC}(n,k)$ (cf. Definition 1.2.13).

Theorem 3.5.4. $\widehat{\mathrm{Cup}}(k,n)$ *categorifies* $\widehat{eC}(n,k)$ *as* $\mathbb{Z}[q,q^{-1}]$*-module, i.e.*

$$K_0\Big(\widehat{\mathrm{Cup}}(k,n)\Big) \cong \widehat{eC}(n,k)$$
$$[\mathrm{T}(\lambda)] \mapsto \mathrm{C}(\lambda).$$

Proof. By Remark 3.5.2 we know that every object of $\widehat{\mathrm{Cup}}(k,n)$ is isomorphic to some sum of (shifted) diagrams without circles, i.e. to some sum of (shifted) $\mathrm{T}(\lambda)$'s. Recall that $[\mathrm{T}(\lambda)\langle l\rangle] = q^l[\mathrm{T}(\lambda)]$. Therefore, every element in the Grothendieck group can be written as some $\sum_j a_j[\mathrm{T}(\lambda)]$ with $a_j \in \mathbb{Z}[q,q^{-1}]$.

To show that the $[\mathrm{T}(\lambda)]$ are a basis, we need to see that nothing else is identified: Assume there is an isomorphism $\varphi\colon \bigoplus_\lambda r_\lambda \mathrm{T}(\lambda) \cong \bigoplus_\lambda r'_\lambda \mathrm{T}(\lambda)$, where r_λ and r'_λ are multiplicities including degree shifts. Then, using Lemma 3.4.14, we can apply the lemma above to $\mathrm{CupT}(n,k)$ and get $r_\lambda = r'_\lambda$.

Thus, the $[\mathrm{T}(\lambda)]$, $\lambda \in \Lambda(n,k)$, are a $\mathbb{Z}[q,q^{-1}]$ basis of the Grothendieck group and by sending $[\mathrm{T}(\lambda)]$ to $\mathrm{C}(\lambda)$ we get the desired isomorphism. □

Remark 3.5.5. Note that the lemma above can also be used to prove the categorification of TL_n as in Theorem 3.2.7, see [MN08, Theorem 5.2].

Furthermore, in view of Proposition 2.1.7 we also have a vector space isomorphism

$$\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\Big(\widehat{\mathrm{Cup}}(k,n)\Big) \cong \left(V^{\otimes n}\right)_{2k-n},$$

but since we have neither the TL_n-action nor the standard basis categorified yet, we do not call it a categorification of $(V^{\otimes n})_{2k-n}$. Our next task is to categorify the TL_n-action.

Chapter 4

Interplay of $\mathrm{Cob}(n)$ and $\mathrm{Cup}(n,k)$

The aim of this chapter is to categorify $\widehat{eC}(n,k)$ as TL_n-module. To achieve this aim, we introduce an action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$. This action is then used to find out more about morphisms in $\mathrm{Cup}(n,k)$. Finally, we consider functors from $\mathrm{Cup}(n,k)$ to $\mathrm{Cup}(n-2,k-1)$ associated to elements in $\mathrm{Cob}(n-2,n)$.

4.1 Categorified Temperley-Lieb algebra actions

Before defining the action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$ we consider the general setting of a monoidal category acting on another category.

Following [JK02] and [Hov99] we define:

Definition 4.1.1. Let $\mathcal{A}$ be a category and $\mathcal{D} = (\mathcal{D}, \otimes, I, a, l, r)$ a monoidal category. A *(right) action of* $\mathcal{D}$ *on* $\mathcal{A}$ is a triple $(.,\alpha,\lambda)$, where $. : \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ is a functor, $\alpha_{A,C,D} : A.(C \otimes D) \to (A.C).D$ a natural isomorphism and $\lambda_A : A.I \to A$ a natural isomorphism, such that the following diagrams commute:

$$\begin{array}{ccccc}
A.\big((C\otimes D)\otimes E\big) & \xrightarrow{\mathrm{id}_A.a_{C,D,E}} & A.\big(C\otimes(D\otimes E)\big) & \xrightarrow{\alpha_{A,C,D\otimes E}} & (A.C).(D\otimes E) \\
\downarrow{\scriptstyle \alpha_{A,C\otimes D,E}} & & & & \downarrow{\scriptstyle \alpha_{A.C,D,E}} \\
\big(A.(C\otimes D)\big).E & & \xrightarrow{\alpha_{A,C,D}.\,\mathrm{id}_E} & & \big((A.C).\big).E
\end{array}$$

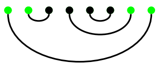
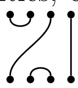
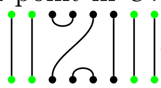
$$\begin{array}{ccc}
A.(I\otimes D) & \xrightarrow{\alpha_{A,I,D}} & (A.I).D \\
& {\scriptstyle \mathrm{id}_A.l_D}\searrow \quad \swarrow{\scriptstyle \lambda_A.\,\mathrm{id}_D} & \\
& A.D &
\end{array}
\qquad
\begin{array}{ccc}
A.(D\otimes I) & \xrightarrow{\alpha_{A,D,I}} & (A.D).I \\
& {\scriptstyle \mathrm{id}_A.r_D}\searrow \quad \swarrow{\scriptstyle \lambda_{A.D}} & \\
& A.D &
\end{array}$$

Remark 4.1.2. Let $\mathcal{A}$ be an additive category and let $\mathcal{D}$ be an additive monoidal category. Assume $\mathcal{D}$ acts on $\mathcal{A}$ such that the action is compatible with the additive structures. Then an action of $\mathcal{D}$ on $\mathcal{A}$ induces an action of $K_0(\mathcal{D})$ on $K_0(\mathcal{A})$.

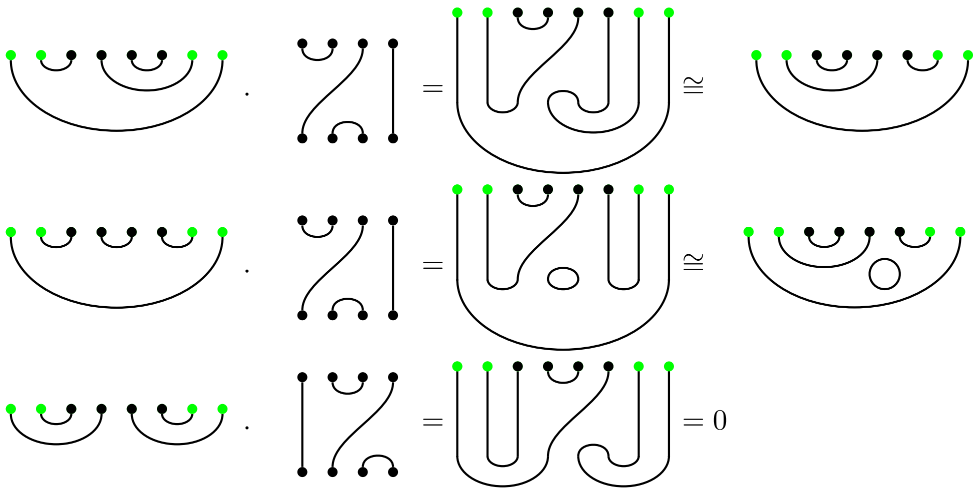
We want to get an action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$ analogously to Proposition 1.2.14. But there is no 0 in $\mathrm{Cup}(n,k)$, so we need to add it artificially:

Definition 4.1.3. $\mathrm{Cup}(n,k)^0$ is the category $\mathrm{Cup}(n,k)$ with an additional object 0. More precisely $\mathrm{ob}\big(\mathrm{Cup}(n,k)^0\big) = \mathrm{ob}\big(\mathrm{Cup}(n,k)\big) \cup \{0\}$ and

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)^0}(C,D) = \begin{cases} \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D) & \text{if } C \neq 0 \neq D, \\ \{0\} & \text{otherwise.} \end{cases}$$

For f an object in $\mathrm{ob}\big(\mathrm{Cob}(n)\big)$ and $C \in \mathrm{ob}\big(\mathrm{Cup}(n,k)\big)$ let $\tilde{f}$ be obtained from f by adding additional identities, one for each green point in C. For example, for $C =$  and $f =$  we obtain $\tilde{f} =$ .
Let $C.f = 0$ if glueing $\tilde{f}$ on top of C is not (even after rescaling) an object in $\mathrm{Cup}(n,k)$. Otherwise, let $C.f$ be the object in $\mathrm{Cup}(n,k)$ obtained from this after appropriate rescaling. Note that $C.f = 0$ if and only if the glueing of $\tilde{f}$ on top of C creates a connected component containing two left green or two right green boundary points.

Example 4.1.4.



Similarly, for cobordisms $\alpha \in \mathrm{Hom}_{\mathrm{Cob}(n)}(f,g)$ and $F \in \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D)$ define $F.\alpha$ by considering the cobordism M obtained by glueing α on top of F with additional identities and rescaling, setting $F.\alpha = 0$ if M is not a morphism in $\mathrm{Cup}(n,k)$.

Moreover, we set $0.f = 0$ and $0.\alpha = 0$ for $f \in \mathrm{ob}\big(\mathrm{Cob}(n)\big)$ and $\alpha \in \mathrm{mor}\big(\mathrm{Cob}(n)\big)$.

Proposition 4.1.5. *By extending the assignment $(F,\alpha) \mapsto F.\alpha$ from above linearly we obtain an action of* $\mathrm{Cob}(n)$ *on* $\mathrm{Cup}(n,k)^0$.

Proof. This follows directly from the definitions and the isotopy relations, since the isotopy relations make morphisms, that compress parts of an object and elongate other parts, into isomorphisms. □

Remark 4.1.6. We have $\widehat{\mathrm{Cup}}(n,k) = \widehat{\mathrm{Cup}(n,k)^0} := \mathrm{Mat}\Big(\big(\mathrm{Cup}(n,k)^0\big)^{\mathbb{Z}}\Big)$ since 0 gets identified with the empty sum when applying $\mathrm{Mat}(-)$.

Corollary 4.1.7. *The action from Proposition 4.1.5 extends to an action of* $\widehat{\mathrm{Cob}}(n)$ *on* $\widehat{\mathrm{Cup}}(n,k)$ *compatible with the additive structures.*

Proof. The action of $\mathrm{Cob}(n)$ on $\mathrm{Cup}(n,k)$ is compatible with degrees since the Euler characteristic is additive under glueing. Thus, the assertion follows directly from the definitions and the remark above. $\square$

The equalities from Lemma 1.2.20 turn now into isomorphisms of objects in $\mathrm{Cup}(n,k)^0$ and $\widehat{\mathrm{Cup}}(n,k)$ respectively in the setup of Proposition 4.1.5 and Corollary 4.1.7:

Lemma 4.1.8. *In $\widehat{\mathrm{Cup}}(n,k)$ we have*

$$\mathrm{T}(\lambda).\mathcal{U}_i \cong \begin{cases} q\,\mathrm{T}(\lambda) \oplus q^{-1}\,\mathrm{T}(\lambda) & \text{if } \lambda(i) = \vee, \lambda(i+1) = \wedge, \\ \mathrm{T}(\lambda.s_i) \text{ with } \lambda.s_i > \lambda & \text{if } \lambda(i) = \wedge, \lambda(i+1) = \vee, \\ \mathrm{T}(\lambda') \text{ with } \lambda' < \lambda & \text{if } \lambda(i) = \lambda(i+1) = \vee \text{ and } t(\lambda(i+1)) \text{ not green} \\ & \text{or } \lambda(i) = \lambda(i+1) = \wedge \text{ and } s(\lambda(i)) \text{ not green,} \\ 0 & \text{otherwise,} \end{cases}$$

where by $t(\lambda(j))$ we denote the target of the arc starting at $\lambda(j)$ and by $s(\lambda(j))$ the source of the arc ending at $\lambda(j)$.

In particular, $\mathrm{T}(\lambda_0).\mathcal{U}_i = 0$ for $i \neq k$ and $\mathrm{T}(\lambda_0).\mathcal{U}_k \cong \mathrm{T}(\lambda_0 s_k)$.

Proof. The proof of the different cases follows step by step Lemma 1.2.20 replacing equalities by isomorphisms.

The assertion for λ_0 follows from the fact that in $\mathrm{C}(\lambda_0)$ the black points $1,\ldots,k$ are connected to left green points and the black points $k+1,\ldots,n$ are connected to right green points. Thus, when $\lambda(i) = \lambda(i+1) = \vee$, then $t(\lambda(i+1))$ is green and also when $\lambda(i) = \lambda(i+1) = \wedge$, then $s(\lambda(i))$ is green. $\square$

Our next goal is to proof the analog of Corollary 1.2.19 in our situation.

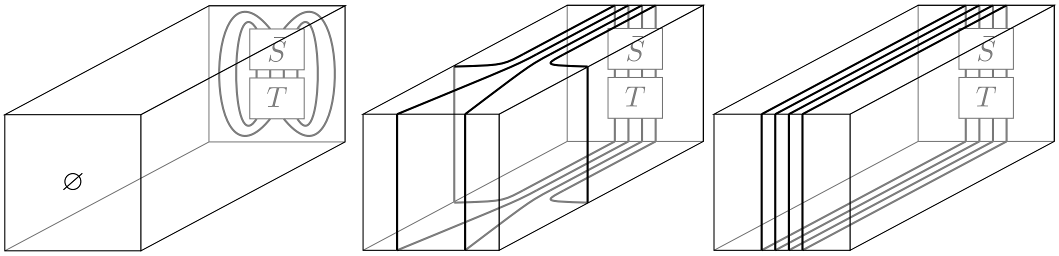
Definition 4.1.9. Let $\mathcal{A}$ be a category with an action of a monoidal category $\mathcal{D}$. Then $\mathcal{A}$ is *generated* by $x \in \mathrm{ob}(\mathcal{A})$ if for every $y \in \mathrm{ob}(\mathcal{A})$ there exists $f \in \mathrm{ob}(\mathcal{D})$ such that $y \cong x.f$ and for every $F : a \to b \in \mathrm{mor}(\mathcal{A})$ there exists $\alpha : f \to g \in \mathrm{mor}(\mathcal{D})$ such that $F' = \mathrm{id}_x .\alpha$ where $F' : x.f \cong a \to b \cong x.g$.

Theorem 4.1.10. $\mathrm{Cup}(n,k)^0$ *is generated by* $\mathrm{T}(\lambda_0)$ *under the action of* $\mathrm{Cob}(n)$.

Before proving this, we have to consider how the morphism $\mathrm{T}(\lambda) \to \mathrm{T}(\mu)$ of minimal degree arises from a morphism in $\mathrm{Cob}(n)$.

Proposition 4.1.11. *Let $\mathrm{T}(\lambda)$ and $\mathrm{T}(\mu)$ be elements in $\mathrm{Cup}(n,k)$ and assume $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda_0).S$ and $\mathrm{T}(\mu) \cong \mathrm{T}(\lambda_0).T$ in $\mathrm{Cup}(n,k)$ with $S,T \in \mathrm{Cob}(n)$. Then there exists a cobordism $\varphi_{S,T} : S \to T$ with $deg(\varphi_{S,T}) = n - l$ where l is the number of circles in $\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)}$. Moreover, if $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda),\mathrm{T}(\mu)\big) \neq 0$, then $(\mathrm{T}(\lambda_0) \times [0,1]).\varphi_{S,T} \neq 0$.*

Proof. Put $\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)} = \mathrm{C}(\lambda_0)T\overline{S}\overline{\mathrm{C}(\lambda_0)}$ on one side of a cuboid and consider the cobordism from $\varnothing$ to $\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)}$ given by nested $\bigcirc$'s. This cobordism, seen as a morphism of $\mathrm{Cob}(0)$ has degree $-l$. Now we pull the boundary strands along the cuboid as follows:

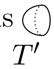


The result describes a cobordism $\mathrm{Id} \to T \otimes \overline{S}$ which has now degree $n-l$, since the Euler characteristic is the same as before moving the strands and we have now boundary points in the degree formula. By Lemma 3.4.9 this corresponds to a cobordism $\varphi_{S,T} : S \to T$ of same degree.

The neckcutting relation cannot be applied to the cobordism $\varphi_{S,T}$ since it cannot be applied to the cobordism from $\varnothing$ to $\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)}$ it came from. Also, $\varphi_{S,T}$ contains no $\bullet$'s. Thus, $(\mathrm{T}(\lambda_0) \times [0,1]).\varphi_{S,T}$ can only be zero by the additional relation 2). But under the isomorphism from Theorem 3.4.12 being 0 because of the additional relation 2) corresponds to a red circle in $\mathrm{C}(\lambda_0).T\overline{\mathrm{C}(\lambda_0).S} = \mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$, which means $\mathrm{Hom}_{\mathrm{Cup}(n,k)}(\mathrm{T}(\lambda), \mathrm{T}(\mu)) = 0$ by Corollary 3.4.13. □

Proof of Theorem 4.1.10. On the level of non-zero objects without circles this follows from Lemma 4.1.8 by the arguments for Lemma 1.2.18. Moreover, since $\mathrm{T}(\lambda_0).\mathcal{U}_i = 0$ for $i \neq k$ by definition, the object 0 can be generated from $\mathrm{T}(\lambda_0)$. In case we have a non-zero object C with circles, we first remove the circles to obtain C', find the element T in Cob(n) such that $C' = \mathrm{T}(\lambda_0).T$ by the start of the proof and then replace T by T' where T' agrees with T except of circles (the difference of C and C') added.

On the level of morphisms we first consider the case where the source and target do not contain circles, i.e. they are isomorphic to some $\mathrm{T}(\lambda)$ and $\mathrm{T}(\mu)$, respectively. Assume $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big) \neq 0$ for $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda_0).S$, $\mathrm{T}(\mu) \cong \mathrm{T}(\lambda_0).T$. Let $\varphi_{S,T}$ be the morphism from Proposition 4.1.11 and let $\psi_{S,T} = (\mathrm{T}(\lambda_0) \times [0,1]).\varphi_{S,T}$. Then by Proposition 4.1.11 we obtain $\psi_{S,T} \neq 0$ and $\deg(\psi_{S,T}) = n - r$ for r the number of circles in $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$. Thus, by Corollary 3.4.13, $\psi_{S,T}$ is the cobordism of minimal degree in $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0).S, \mathrm{T}(\lambda_0).T\big)$. By Remark 3.3.2 basis vectors in $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0).S, \mathrm{T}(\lambda_0).T\big)$ not of minimal degree can be obtained by adding dots to $\psi_{S,T}$ which is the same as adding first dots to $\varphi_{S,T}$ and then applying to $\mathrm{T}(\lambda_0) \times [0,1]$.

If $\mathrm{T}(\lambda_0).S$ and $\mathrm{T}(\lambda_0).T$ contain circles let S' be S without the circles and T' be T without the circles. By Remark 3.3.2, a basis element ψ of $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0).S, \mathrm{T}(\lambda_0).T\big)$ is of the form $\psi' \sqcup d \sqcup b$, where $\psi' \in \mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda_0).S', \mathrm{T}(\lambda_0).T'\big)$, d is  or  on the circles of S and b is  or  on the circles of T. Hence, $\psi = \big(\mathrm{T}(\lambda_0) \times [0,1]\big).(\varphi' \sqcup d \sqcup b)$, where $\varphi' : S' \to T'$ is the cobordism constructed above satisfying $\psi' = \big(\mathrm{T}(\lambda_0) \times [0,1]\big).\varphi'$.

□

Corollary 4.1.12. $\widehat{\mathrm{Cup}}(n,k)$ *is generated by* $\mathrm{T}(\lambda_0)$ *as a* $\widehat{\mathrm{Cob}}(n)$*-module.*

Remark 4.1.13. On the level of Grothendieck groups, the action of $\widehat{\mathrm{Cob}}(n)$ on $\widehat{\mathrm{Cup}}(n,k)$ matches the action of TL_n on $\widehat{eC}(n,k)$, so we have categorified $\widehat{eC}(n,k)$ as an TL_n-module. Corollary 4.1.12 is in fact a categorified version of Corollary 1.2.19.

4.2 More results on morphisms between tiltings

With the action of the $\mathcal{U}_i$ and the tools from Section 3.4 we can now describe the morphisms in $\mathrm{Cup}(n,k)$ more concretely. In particular, we can decide when there are morphisms of low degrees from $\mathrm{T}(\lambda)$ to $\mathrm{T}(\mu)$ when we know how λ and μ are related.

Proposition 4.2.1. *For $\lambda, \mu \in \Lambda(n,k)$, there is an isomorphism of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)^0}\big(\mathrm{T}(\lambda).\mathcal{U}_i, \mathrm{T}(\mu)\big) \cong \mathrm{Hom}_{\mathrm{Cup}(n,k)^0}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu).\mathcal{U}_i\big).$$

Proof. First assume $\mathrm{T}(\lambda).\mathcal{U}_i \neq 0 \neq \mathrm{T}(\mu).\mathcal{U}_i$. By Theorem 3.4.12, there are isomorphisms of graded vector spaces

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda).\mathcal{U}_i, \mathrm{T}(\mu)\big) \cong \mathcal{F}_{Col}\big(\mathrm{C}(\lambda).U_i\overline{\mathrm{C}(\mu)}\big)$$
$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu).\mathcal{U}_i\big) \cong \mathcal{F}_{Col}\big(\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu).U_i}\big).$$

But $\mathrm{C}(\lambda).U_i\overline{\mathrm{C}(\mu)} = \mathrm{C}(\lambda)\overline{\mathrm{C}(\mu).U_i}$ so we get the desired isomorphism. In case $\mathrm{T}(\lambda).\mathcal{U}_i = 0 = \mathrm{T}(\mu).\mathcal{U}_i$, the assertion is clear. Now assume $\mathrm{T}(\lambda).\mathcal{U}_i = 0$ and $\mathrm{T}(\mu).\mathcal{U}_i \neq 0$. In this case, U_i connects two right green points or two left green points of $\mathrm{C}(\lambda)$. Thus, there is a red circle in $\mathrm{C}(\lambda).U_i\overline{\mathrm{C}(\mu)} = \mathrm{C}(\lambda)\overline{\mathrm{C}(\mu).U_i}$ and the assertion follows from $\mathcal{F}_{col}\big(\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu).U_i}\big) = 0$. The remaining case follows analogously.

□

Remark 4.2.2. For $\mathrm{T}(\lambda).\mathcal{U}_i \neq 0 \neq \mathrm{T}(\mu).\mathcal{U}_i$, the isomorphism from the proposition above is given by sliding the $\mathcal{U}_i$ to the other side, analogously to Lemma 3.4.9. Even more, $\psi : \mathrm{T}(\lambda).\mathcal{U}_i \to \mathrm{T}(\mu)$ is isomorphic to some $(\mathrm{T}(\lambda_0) \times [0,1]).\varphi$ for $\varphi : S\mathcal{U}_i \to T$ and $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda_0).S$, $\mathrm{T}(\mu) \cong \mathrm{T}(\lambda_0).T$ by Theorem 4.1.10. When φ is send to φ' under the isomorphism of Lemma 3.4.9, then ψ is send to $\psi' \cong (\mathrm{T}(\lambda_0) \times [0,1]).\varphi'$ under the isomorphism of Proposition 4.2.1.

Lemma 4.2.3. *Let $\lambda \xrightarrow{s_i} \mu$ or $\mu \xrightarrow{s_i} \lambda$. Then*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)_p = \begin{cases} 0 & \text{if } p = 0, \\ \mathbb{C} & \text{if } p = 1. \end{cases}$$

Proof. First assume $\lambda \xrightarrow{s_i} \mu$. Then the first case follows from Lemma 3.4.14. Since $\mathrm{T}(\mu) \cong \mathrm{T}(\lambda).\mathcal{U}_i$, there is a degree 1 morphism $\mathrm{id}\,.\,\mathrm{H}_i : \mathrm{T}(\lambda).\,\mathrm{Id} \to \mathrm{T}(\lambda).\mathcal{U}_i \cong \mathrm{T}(\mu)$ given by the saddle. This is non-zero, since none of the relations can be applied: There is no neck to cut, no $\bullet$ and a single saddle cannot connect two left green or two right green boundary lines when source and target are non-zero. By Corollary 3.4.13 it is unique up to scalar.

For the case $\mu \xrightarrow{s_i} \lambda$ note that $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big) \cong \mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\mu), \mathrm{T}(\lambda)\big)$, since by Theorem 3.4.12 they are isomorphic to $\mathcal{F}_{col}\big(\mathrm{T}(\mu)\overline{\mathrm{T}(\lambda)}\big)$ or $\mathcal{F}_{col}\big(\mathrm{T}(\lambda)\overline{\mathrm{T}(\mu)}\big)$, respectively. But $\mathcal{F}_{col}\big(\mathrm{T}(\mu)\overline{\mathrm{T}(\lambda)}\big) \cong \mathcal{F}_{col}\big(\mathrm{T}(\lambda)\overline{\mathrm{T}(\mu)}\big)$ since $\mathrm{T}(\mu)\overline{\mathrm{T}(\lambda)}$ and $\mathrm{T}(\lambda)\overline{\mathrm{T}(\mu)}$ have the same number of black, green and red circles. □

Notation 4.2.4. We denote the saddle cobordism from $\mathrm{Id} \to \mathcal{U}_i$ by H_i. This is motivated by the pictorial shorthand notation  introduced in Chapter 3. The saddle going in the other direction is denoted by $\overline{\mathrm{H}}_i : \mathcal{U}_i \to \mathrm{Id}$. If, as in the lemma above, $\lambda \xrightarrow{s_i} \mu$, we also denote the saddle cobordism $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda).\mathrm{Id} \xrightarrow{\mathrm{id}.\mathrm{H}_i} \mathrm{T}(\lambda).\mathcal{U}_i \cong \mathrm{T}(\mu)$ by H_i. Analogously, we write $\overline{H}_i : \mathrm{T}(\mu) \to \mathrm{T}(\lambda)$ for the saddle cobordism induced by $\overline{H}_i : \mathcal{U}_i \to \mathrm{Id}$.

Lemma 4.2.5. *Let $\lambda \xrightarrow{s_i} \nu \xrightarrow{s_j} \mu$, then $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\mu), \mathrm{T}(\lambda)\big)_p = 0$ for $p = 0, 1$ and*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\mu), \mathrm{T}(\lambda)\big)_2 = \begin{cases} \mathbb{C} & \text{if there is no red circle in } \mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if there is no red circle, then the basis element of degree 2 is given by $\overline{\mathrm{H}}_i \circ \overline{\mathrm{H}}_j$ and if $i \neq j \pm 1$, then there is no red circle.

Proof. Again, the assertion for $p = 0$ is true because of Lemma 3.4.14. By the Lemma 4.2.3 and Corollary 3.4.13 we know that $\mathrm{C}(\nu)\overline{\mathrm{C}(\lambda)}$ contains $n-1$ circles. Since $\mathrm{C}(\mu) = \mathrm{C}(\nu).U_j$ we go from $\mathrm{C}(\nu)\overline{\mathrm{C}(\lambda)} = \big(\mathrm{C}(\nu).\mathrm{Id}\big)\overline{\mathrm{C}(\lambda)}$ to $\big(\mathrm{C}(\nu).U_j\big)\overline{\mathrm{C}(\lambda)} = \mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$ by changing some $\mid\ \mid$ to . By doing this, either two circles are connected to one or one circle is split in two. So $\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}$ contains either n or $n-2$ circles. But it cannot contain n circles since we already know that there is no degree 0 map from $\mathrm{T}(\mu)$ to $\mathrm{T}(\lambda)$. Thus, by Corollary 3.4.13 there is no degree 1 map and there is (up to scalar) a unique degree 2 map if and only if there are no red circles. Neckcutting and the additional relation 1) cannot be applied to the degree 2 morphism $\overline{\mathrm{H}}_i \circ \overline{\mathrm{H}}_j$, thus it can only be zero by the additional relation 2). But this is equivalent to having a red circle.

It remains to check that there is no red circle if $i \neq j \pm 1$. Since $i \neq j \pm 1$, there is no interaction between U_i and U_j. So red circles can only arise, when U_i or U_j connects two arcs with left green points or two arcs with right green points in $\mathrm{C}(\lambda)$. But $\mathrm{C}(\lambda).U_i = \mathrm{C}(\nu) \neq 0$ and since $i \neq j \pm 1$ we have $\lambda s_i s_j = \lambda s_j s_i$, thus $\lambda \to \lambda s_j$ and $\mathrm{C}(\lambda).U_j = \mathrm{C}(\lambda s_j) \neq 0$. So neither U_i nor U_j can create a red circle. □

Lemma 4.2.6. *Let $\lambda, \mu \in \Lambda(n,k)$. Then there is a non-zero degree 1 map $\mathrm{H} : \mathrm{T}(\lambda) \to \mathrm{T}(\mu)$ if and only if λ and μ differ only by changing not necessarily neighbouring $\wedge\vee$ to $\vee\wedge$ or vice versa.*

Proof. First assume that in λ there is a (not necessarily neighbouring) $\vee\wedge$ which we change to $\wedge\vee$ in μ. Because of the green dots in $\mathrm{C}(\lambda)$, there is a cup directly below the one given by our $\vee\wedge$-pair (with one endpoint to the right and one endpoint to the left of our given cup) and going to $\mathrm{C}(\mu)$ just changes these two nested cups to two neighbouring ones. In between, there is a saddle cobordism which is obviously non-zero. Reading this cobordism the other way gives a cobordism for the other case.

Now assume the existence of the degree 1 map. Since by adding •'s we can only raise the degree by 2, there is no degree 0 map. Thus, by Corollary 3.4.13, there are $n-1$ circles in $\mathrm{C}(\lambda)\overline{\mathrm{C}(\mu)}$. So there are $n-2$ circles containing 2 points and one with 4 points. Let i_1, i_2, i_3, i_4 be the points on the 4-point circle from left to right. In one of $\mathrm{C}(\lambda)$ and

$\mathrm{C}(\mu)$ we have that i_1 is connected to i_2 and i_3 to i_4 and in the other i_1 to i_4 and i_2 to i_3. But by the definition of extended cup diagrams this just means that $\lambda = \mu.(i_2, i_3)$, where $(i_2, i_3) \in \mathbb{S}_n$ is the transposition swapping i_2 with i_3. □

The following lemma tells us conditions for factorising the degree 2 map given by two saddles from $\mathrm{T}(\mu)$ to $\mathrm{T}(\lambda)$ over a special degree 1 map.

Proposition 4.2.7. *Let $\lambda \xrightarrow{s_i} \nu \xrightarrow{s_j} \mu$ and let $\mathrm{H} : \mathrm{T}(\tau) \to \mathrm{T}(\lambda)$ be a non-zero degree 1 map, where $\tau < \lambda$. Assume there is a non-zero degree 1 map H' from $\mathrm{T}(\mu)$ to $\mathrm{T}(\tau)$. Then $j = i \pm 1$.*

Proof. By Lemma 4.1.8 we have $\mathrm{T}(\mu) \cong \mathrm{T}(\lambda).\mathcal{U}_i\mathcal{U}_j$, hence $\mathrm{C}(\mu)\overline{\mathrm{C}(\tau)} = \mathrm{C}(\lambda)U_iU_j\overline{\mathrm{C}(\tau)}$. The existence of H' implies that $\mathrm{C}(\lambda)U_iU_j\overline{\mathrm{C}(\tau)}$ must contain $(n-1)$ circles of which one passes through four points and the others through 2 points each. Analogously, the existence of H implies that $\mathrm{C}(\lambda)\overline{\mathrm{C}(\tau)} = \mathrm{C}(\lambda)\,\mathrm{Id}\,\mathrm{Id}\,\overline{\mathrm{C}(\tau)}$ has $(n-1)$ circles. Hence passing from $\mathrm{C}(\lambda)U_iU_j\overline{\mathrm{C}(\tau)}$ to $\mathrm{C}(\lambda)\,\mathrm{Id}\,\mathrm{Id}\,\overline{\mathrm{C}(\tau)}$ does not change the number of circles.

Now assume $i \neq j \pm 1$. Then, $\lambda(i) = \wedge = \lambda(j)$ and $\lambda(i+1) = \vee = \lambda(j+1)$. Since $\tau < \lambda$ and there is a degree 1 map between $\mathrm{T}(\tau)$ and $\mathrm{T}(\lambda)$, by Lemma 4.2.6 we are in the situation that there are indices $l_1 < l_2$ such that $\lambda = \tau$ except for $\lambda(l_1) = \vee = \tau(l_2)$, $\lambda(l_2) = \wedge = \tau(l_1)$ and l_1 and l_2 are connected in $\mathrm{C}(\lambda)$. By the same argument as in the proof of Lemma 4.2.6 there is a 4 point circle in $\mathrm{C}(\lambda)\overline{\mathrm{C}(\tau)}$ containing the points l_1, l_2. Let $l_1' < l_2'$ be the other two points on this circle. Note that $l_1' < l_1 < l_2 < l_2'$, $\lambda(l_1') = \tau(l_1') = \vee$ and $\lambda(l_2') = \tau(l_2') = \wedge$ holds by the definition of extended cup diagrams.

Because of this and since $\lambda(i) = \wedge$, $\lambda(i+1) = \vee$, only one of $i, i+1$ can be in $\{l_1', l_2', l_1, l_2\}$; the same holds for $j, j+1$. Consequently, the other one is part of a 2-point circle in $\mathrm{C}(\lambda)\overline{\mathrm{C}(\tau)}$. Moreover, $i, i+1$ cannot lie on the same 2-point-circle; the same is true for $j, j+1$.

Now we show that when passing from $\mathrm{C}(\lambda)\,\mathrm{Id}\,\mathrm{Id}\,\overline{\mathrm{C}(\tau)}$ to $\mathrm{C}(\lambda)U_iU_j\overline{\mathrm{C}(\tau)}$ the number of circles does not stay the same. Changing Id to U_l (for $l = i$ or $l = j$) either connects two circles to one or splits one circle into two. The change Id to U_l can only turn one circle into two if the points $l, l+1$ lie on the same circle. We know that at the start, $i, i+1$ cannot lie on the same circle and the same holds for $j, j+1$. We show that $i, i+1, j, j+1$ lie on at least 3 different circles. Then, even after connecting two of them, the remaining two still lie on different circles. Obviously, there cannot be a single circle containing all $i, i+1, j, j+1$. Also, they cannot all lie on two 2-point circles, since considering the up-down-sequence λ those cross the horizontal line in $\vee\wedge\vee\wedge$ or $\vee\vee\wedge\wedge$ which does not contain two different subsequences of $\wedge\vee$. We now consider a 2-point circle and the 4-point circle. We consider merges of the sequences $\vee\wedge$ of the 2-point circle and $\vee\vee\wedge\wedge$ of the 4-point circle with the condition that the $\vee\wedge$ of the 2-point circle are next to each other since otherwise the lines cross. All the possibilities are

$$\vee\wedge\vee\vee\wedge\wedge,\ \vee\vee\wedge\vee\wedge\wedge,\ \vee\vee\vee\wedge\wedge\wedge,\ \vee\vee\wedge\vee\wedge\wedge,\ \vee\vee\wedge\wedge\vee\wedge$$

and we see that there are never two distinct subsequences of $\wedge\vee$.

Therefore, when applying first U_i and then U_j, the number of circles decreases both times, which is a contradiction. So $i \neq j \pm 1$ is not possible. □

Lemma 4.2.8. *For* X, Y, Z *arbitrary objects of* $\mathrm{Cob}(n)$*, the following diagrams commute in* $\mathrm{Cob}(n)$

a)

$$\begin{array}{ccc} X\,\mathcal{U}_i\,Y\,\mathrm{Id}\,Z & \xrightarrow{\mathrm{id}_X\,\mathrm{id}_{\mathcal{U}_i}\,\mathrm{id}_Y\,\mathrm{H}_j\,\mathrm{id}_Z} & X\,\mathcal{U}_i\,Y\,\mathcal{U}_j\,Z \\ \downarrow{\scriptstyle \mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\mathrm{id}_{Id}\,\mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\mathrm{id}_{\mathcal{U}_j}\,\mathrm{id}_Z} \\ X\,\mathrm{Id}\,Y\,\mathrm{Id}\,Z & \xrightarrow[\mathrm{id}_X\,\mathrm{id}_{Id}\,\mathrm{id}_Y\,\mathrm{H}_j\,\mathrm{id}_Z]{} & X\,\mathrm{Id}\,Y\,\mathcal{U}_j\,Z \end{array}$$

b)

$$\begin{array}{ccc} X\,\mathcal{U}_i\,Y\,\mathcal{U}_j\,Z & \xrightarrow{\mathrm{id}_X\,\mathrm{id}_{\mathcal{U}_i}\,\mathrm{id}_Y\,\overline{\mathrm{H}}_j\,\mathrm{id}_Z} & X\,\mathcal{U}_i\,YIdZ \\ \downarrow{\scriptstyle \mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\mathrm{id}_{\mathcal{U}_j}\,\mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\mathrm{id}_{Id}\,\mathrm{id}_Z} \\ XIdY\,\mathcal{U}_j\,Z & \xrightarrow[\mathrm{id}_X\,\mathrm{id}_{Id}\,\mathrm{id}_Y\,\overline{\mathrm{H}}_j\,\mathrm{id}_Z]{} & XIdYIdZ \end{array}$$

where $\mathrm{H}_l : \mathrm{Id} \to \mathcal{U}_l$ *and* $\overline{\mathrm{H}}_l : \mathcal{U}_l \to \mathrm{Id}$ *are the saddles.*

Proof. In a) both compositions are equal to $\mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\mathrm{H}_j\,\mathrm{id}_Z$ and in b) both compositions are equal to $\mathrm{id}_X\,\overline{\mathrm{H}}_i\,\mathrm{id}_Y\,\overline{\mathrm{H}}_j\,\mathrm{id}_Z$. □

Remark 4.2.9. From the lemma above we obtain that when $\mathrm{T}(\lambda).\mathcal{U}_j = 0$, then $\overline{\mathrm{H}}_j \circ \overline{\mathrm{H}}_i \colon \mathrm{T}(\lambda).\mathcal{U}_i\mathcal{U}_j \to \mathrm{T}(\lambda)$ is zero. This is true since it factorises over 0. For example, for $n = 3$ and $k = 1$ we obtain that $\overline{\mathrm{H}}_2 \circ \overline{\mathrm{H}}_1 \colon \mathrm{T}(\lambda_0).\mathcal{U}_1\mathcal{U}_2 \to \mathrm{T}(\lambda_0).\mathcal{U}_1 \to \mathrm{T}(\lambda_0)$ is zero, since $\mathrm{T}(\lambda_0).\mathcal{U}_2 = 0$ by Lemma 4.1.8.

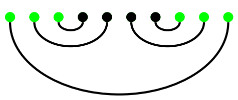
More general, if $\mathrm{T}(\lambda_0).XY\,\mathcal{U}_j\,Z = 0$ for some objects X, Y, Z of $\mathrm{Cob}(n)$, then the cobordism $\overline{\mathrm{H}}_i \circ \overline{\mathrm{H}}_j \colon \mathrm{T}(\lambda_0).X\,\mathcal{U}_i\,Y\,\mathcal{U}_j\,Z \to \mathrm{T}(\lambda_0).XYZ$ equals zero, even if $\mathrm{T}(\lambda_0).X\,\mathcal{U}_i\,YZ \neq 0$.

4.3 Reducing the number of boundary points

The reduction of the number of boundary points introduced in this section will be helpful later when we want to use induction on the number of boundary points.

Let $L^+_{n,k}$ be $L_{n,k}$ with an additional green point on the right and an additional green point on the left. For example $L_{4,2} =$ whereas $L^+_{4,2} =$.

Let $\mathrm{Cup}(n, k)_+$ be the category defined as $\mathrm{Cup}(n, k)$ but with boundary points $L^+_{n,k}$ instead of $L_{n,k}$ and the additional condition that in all objects the outermost left and the outermost right green point are connected.

For example,  is an object of $\mathrm{Cup}_+(4, 2)$.

Note that because of the neckcutting relation and the additional relation 1), every morphism in $\mathrm{Cup}(n, k)_+$ consists of linear combinations of morphisms $f \sqcup d \times [0, 1]$ for d the arc connecting the two outermost green points and f a morphism of $\mathrm{Cup}(n, k)$. It follows that the functor from $\mathrm{Cup}(n, k) \to \mathrm{Cup}(n, k)_+$ given by adding the arc d on objects and $d \times [0,1]$ on morphisms yields an equivalence of categories. On objects, the inverse is given by just deleting the outer arc.

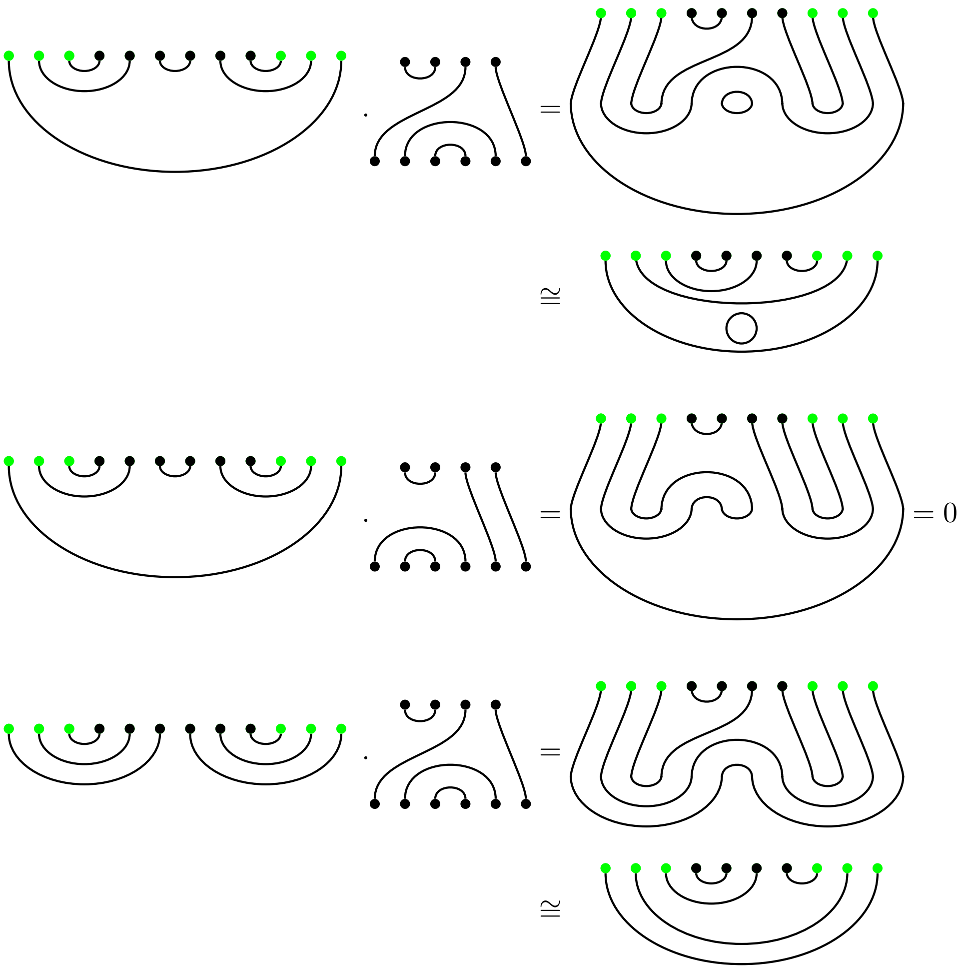
Let $\mathrm{Cup}(n,k)_+^0$ the category $\mathrm{Cup}(n,k)_+$ with an extra object 0 as in Definition 4.1.3.

An object $A \in \mathrm{Cob}(n-2,n)$ induces a functor

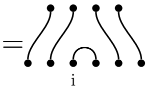
$$\begin{aligned} P_A : \mathrm{Cup}(n,k) &\to \mathrm{Cup}(n-2,k-1)_+^0 \\ C &\mapsto C.A \\ f &\mapsto f.(A \times [0,1]), \end{aligned}$$

where $C.A$ is putting A together with identities on the green points on top of C and rescaling if this gives an element of $\mathrm{Cup}(n-2,k-1)_+$ and 0 otherwise and $f.(A \times [0,1])$ is putting $(A \times [0,1])$ on top of f and rescaling if this is a morphism of $\mathrm{Cup}(n-2,k-1)$ and 0 otherwise. Of course we can extend this functor to $\mathrm{Cup}(n,k)^0$ in the obvious way. Note that every $\mathrm{T}(\lambda) \in \mathrm{Cup}(n,k)$ with $\lambda \neq \lambda_0$ has an arc connecting the two outermost green points. Considering $\mathrm{T}(\lambda_0).A$ we see that either an arc connecting the outermost green points or an arc connecting two right green or two left green points is created. Thus, when $C.A = 0$, then the addition of C connects two left green points or two right green points, since elements of $\mathrm{Cup}(n-2,k-1)_+$ can have circles everywhere,

Example 4.3.1.



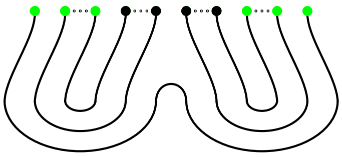
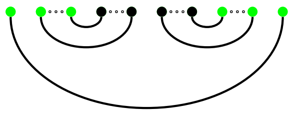
The equivalence $\mathrm{Cup}(n-2,k-1)_+ \cong \mathrm{Cup}(n-2,k-1)$ extends to an equivalence $\mathrm{Cup}(n-2,k-1)_+^0 \cong \mathrm{Cup}(n-2,k-1)^0$. Using this we consider P_A as a functor from $\mathrm{Cup}(n,k)^0 \to \mathrm{Cup}(n-2,k-1)^0$.

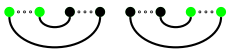
Mostly, we will use a special P_A for $A = \cap_i :=$  and write $-.\cap_i$ instead of $P_{\cap_i}(-)$.

Lemma 4.3.2. *For $\lambda_0^{n,k} \in \Lambda(n,k)$ we have*

$$\mathrm{T}(\lambda_0^{n,k}).\cap_k \cong \mathrm{T}(\lambda_0^{n-2,k-1})$$
$$\mathrm{T}(\lambda_0^{n,k}).\cap_i = 0 \textit{ for } i \neq k.$$

Proof. If $i \neq k$ we connect either two left green points or two right green points. For $i = k$ we have

$\mathrm{T}(\lambda_0^{n,k}).\cap_k =$  $\cong$ 

$\mapsto$  $= \mathrm{T}(\lambda_0^{n-2,k-1})$.

□

Of course, we can find similar statements for $\lambda \neq \lambda_0$ along the lines of Lemma 4.1.8 but we omit this, since we will not need it later on.

Lemma 4.3.3. *For $C, D \in \mathrm{Cup}(n,k)$, there exists an isomorphism of graded vector spaces*

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)^0}(C, D.\mathcal{U}_i) \cong \mathrm{Hom}_{\mathrm{Cup}(n-2,k-1)^0}(C.\cap_i, D.\cap_i). \tag{4.1}$$

Proof. Assume first that $D.\mathcal{U}_i$, $C.\cap_i$ and $D.\cap_i$ are all not zero. Then by Theorem 3.4.12

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)^0}(C, D.\mathcal{U}_i) = \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C, D.\mathcal{U}_i) \cong \mathcal{F}_{Col}(D.\mathcal{U}_i\overline{C})$$
$$\mathrm{Hom}_{\mathrm{Cup}(n-2,k-1)^0}(C.\cap_i, D.\cap_i) = \mathrm{Hom}_{\mathrm{Cup}(n-2,k-1)}(C.\cap_i, D.\cap_i) \cong \mathcal{F}_{Col}(D.\cap_i\overline{C\cap_i}).$$

But $D.\mathcal{U}_i\overline{C}$ and $D.\cap_i\overline{C\cap_i}$ agree except for the green outer circle that $C.\cap_i\overline{D\cap_i}$ lacks. Thus,

$$\mathcal{F}_{Col}\big(C\overline{D.\mathcal{U}_i}\big) \cong \mathcal{F}_{Col}\big(C.\cap_i\overline{D\cap_i}\big) \otimes \mathbb{C} \cong \mathcal{F}_{Col}\big(C.\cap_i\overline{D\cap_i}\big).$$

The degrees also agree, since we go from $n-r$ to $(n-1)-(r-1)$ for r the number of circles in $C\overline{D.\mathcal{U}_i}$ and $\deg(1) = 0$. Hence, the claim holds in this case.

We have $D.\cap_i = 0$ if and only if $D.\mathcal{U}_i = 0$, since this only depends on which arcs are connected by the lower cap of $\mathcal{U}_i$. Thus, we are also done in this case. If $C.\cap_i = 0$ then we are done by the analogous argument since we know $\mathrm{Hom}_{\mathrm{Cup}(n,k)^0}(C, D.\mathcal{U}_i) \cong \mathrm{Hom}_{\mathrm{Cup}(n,k)^0}(C.\mathcal{U}_i, D)$ by Proposition 4.2.1. □

Remark 4.3.4. For $C.\cap_i \neq 0 \neq D.\cap_i$, there is in fact a natural and canonical isomorphism realising (the inverse of) (4.1) given by sliding the $\cup$ to the other side or alternatively by the composition

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Cup}(n-2,k-1)}(C.\cap_i, D.\cap_i) \to \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C.\mathcal{U}_i,\ & D.\mathcal{U}_i) \to \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C, D.\mathcal{U}_i) \\ g \mapsto \tilde{g} \qquad\qquad\qquad & \qquad f \mapsto f \circ s,\end{aligned}$$

where $\tilde{g}$ is obtained from g by adding the identity cobordism of the added cup and s is the saddle cobordism.

Chapter 5

Homological algebra

In this chapter, we summarise the homological algebra that will be needed in the following chapters. As in the first chapter, some statements are exactly what is used later and their purpose might not be clear now. We start by recalling some standard definitions and statements about homotopy equivalences between complexes. After that, we apply the theory of spectral sequences to get Corollary 5.2.5 which will be used extensively later on.

5.1 Homotopy equivalences

We start by recalling some standard constructions to fix the notations and sign conventions.

Let $\mathcal{A}$ be an additive category.

Let $(A, d) = \cdots \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \to \ldots$ be a chain complex with entries in $\mathcal{A}$. The shifted complex $A[p]$ is defined via $A[p]_i = A_{i-p}$ with differential $(-1)^p d$.

Let $Ch(\mathcal{A})$ be the category of chain complexes and $Ch^b(\mathcal{A})$ the category of bounded chain complexes. We use $\simeq$ to denote homotopy equivalences of chain complexes. Let $K(\mathcal{A})$ be the homotopy category of chain complexes and $K^b(\mathcal{A})$ the homotopy category of bounded chain complexes.

Let (A, d_A), (B, d_B) be chain complexes and $f\colon A \to B$ a chain morphism. Then $\mathrm{Cone}(f) = A[1] \oplus B$ is the complex with $\mathrm{Cone}(f)_i = A_{i-1} \oplus B_i$ and differential $d_i\big((a_{i-1}, b_i)\big) = \big(-d_A(a_{i-1}), f_{i-1}(a_{i-1}) + d_B(b_i)\big)$. If X is another chain complex and $g : \mathrm{Cone}(f) \to X$ is a chain map, then $g|_B : B \to X$ is also a chain map and $g|_A : A[1] \to X$ is what we call a *multi-map*, i.e. it does not commute with the differentials.

Recall (see e.g. [Wei94]) that $K(\mathcal{A})$ is a triangulated category with distinguished triangles given by those isomorphic to triangles of the form

$$A \xrightarrow{f} B \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{\pi} A[1].$$

If $\mathcal{D}$ is an additive monoidal category, then for $(X, d^X), (Y, d^Y) \in Ch(\mathcal{D})$ the tensor product $X \otimes Y$ is the complex defined by $(X \otimes Y)_l = \bigoplus_{i+j=l} X_i \otimes Y_j$ with differential

d given by

$$d|_{X_i \otimes Y_j} = \left(d_i^X \otimes \mathrm{id}_{Y_j}, (-1)^i \mathrm{id}_{X_i} \otimes d_j^Y\right) : X_i \otimes Y_j \to X_{i-1} \otimes Y_j \oplus X_i \otimes Y_{j-1}.$$

When $\mathcal{D}$ acts on $\mathcal{A}$, i.e. the functor $. : \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ from Definition 4.1.1 is compatible with the additive structures, for $(A, d^A) \in Ch(\mathcal{A})$ and $(X, d^X) \in Ch(\mathcal{D})$, we define the action $A.X$ as the complex with $(A.X)_n = \bigoplus_{i+j=n} A_i.X_j$ and differential d given by

$$d|_{A_i.X_j} = \left(d_i^A.\mathrm{id}_{X_j}, (-1)^i \mathrm{id}_{A_i}.d_j^X\right) : A_i.X_j \to A_{i-1}.X_j \oplus A_i.X_{j-1}.$$

Recall that the tensor product of homotopic complexes results in homotopic complexes. Of course, the same is true for the action.

In the following we need a notion of some complex being part of another that is weaker than being a subcomplex. We say that A is a *partcomplex* of $B \in Ch(\mathcal{A})$ if every A_i is a summand of B_i and every differential in A appears also in B. For example, $A_2 \xrightarrow{d} A_1 \to 0$ is a partcomplex of $A_2 \xrightarrow{(d,d')} A_1 \oplus A_1' \xrightarrow{c} A_0$. Also, if $A \to B$ is a chain map, then $A[1]$ is a partcomplex of $\mathrm{Cone}(A \to B)$. In particular, every subcomplex is also a partcomplex.

Lemma 5.1.1. *Let (C, d^C) and (D, d^D) be chain complexes in $K(\mathcal{A})$.*

a) Let A be a partcomplex of C with $Hom_{K(\mathcal{A})}(A, D) = 0$. Assume we have $C_i = A_i \oplus B_i$ for all i. Furthermore, assume there is some $f \in Hom_{K(\mathcal{A})}(C, D)$ satisfying $f_{i-1} \circ d^C|_{A_i \to B_{i-1}} = 0$. Then there is some f' with $f \simeq f'$ and $f'|_A = 0$. If moreover $d^C|_{B_i \to A_{i-1}} = 0$, then $f_i'|_B = f_i|_B$.

b) Let A be a partcomplex of C with $Hom_{K(\mathcal{A})}(D, A) = 0$. Assume we have $C_i = A_i \oplus B_i$ for all i. Furthermore, assume there is $f \in Hom_{K(\mathcal{A})}(D, C)$ satisfying $d^C|_{B_i \to A_{i-1}} \circ f_i = 0$. Then there is some f' with $f \simeq f'$ and $f'|_A = 0$. If moreover $d^C|_{A_{i+1} \to B_i} = 0$, then $f_i'|_B = f_i|_B$.

Proof. a) Having $f_{i-1} \circ d^C|_{A_i \to B_{i-1}} = 0$ yields that $f|_A$ is a morphism of chain complexes, so we have a homotopy $h\colon A[1] \to A$ with $f|_A = d^D h + h d^A$. Now $h' = h \oplus 0\colon C[1] \to C$ gives a homotopy between f and $f' = f - (d^D h' + h' d^C)$ since $f - f' = d^D h' + h' d^C$. But

$$f'|_A = f|_A - (d^D h + h d^A + 0) = 0$$

as desired. Now assume in addition $d^C|_{B_i \to A_{i-1}} = 0$. Then

$$f_i'|_B = f_i|_B - (d^D 0 + h d^C|_{B_i \to A_{i-1}} + 0 d^C|_{B_i \to B_{i-1}}) = f_i|_B.$$

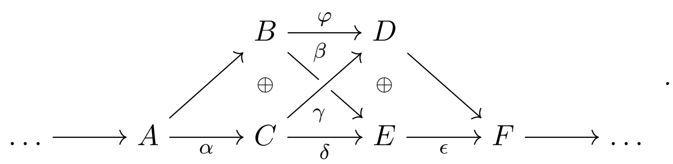
b) This follows analogously.

□

Lemma 5.1.2 (Gaussian Elimination)**.** *Assume we are given a complex X in $Ch(\mathcal{A})$ which looks locally like*

$$\cdots \to A \to B \oplus C \to D \oplus E \to F \to \ldots \tag{5.1}$$

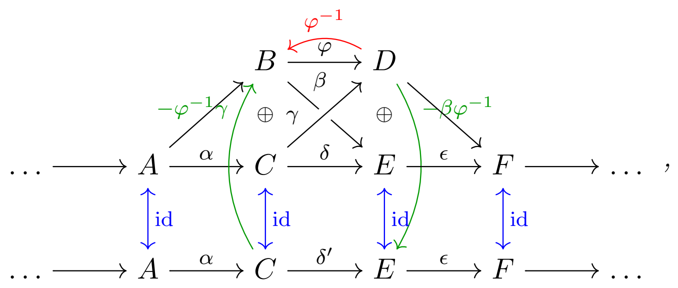
with differentials



Assume $\varphi\colon B \to D$ *is an isomorphism. Then* X *is homotopy equivalent to a complex* X' *which agrees with* X *except that the part* (5.1) *is changed to*

$$\dots \longrightarrow A \xrightarrow{\alpha} C \xrightarrow{\delta - \beta\circ\varphi^{-1}\circ\gamma} E \xrightarrow{\epsilon} F \longrightarrow \dots .$$

A homotopy equivalence is given as follows



where $\delta' := \delta - \beta\circ\varphi^{-1}\circ\gamma$ *and the vertical maps define the morphisms of complexes and the red map going from right to left defines the homotopy. All maps that are not drawn are zero.*

The passage from X *to* X' *is called Gaussian elimination with respect to* φ.

Proof. Explicit calculation, see also [BN07, Lemma 4.2]. □

Corollary 5.1.3. *Let* X, Y, Z, U *in* $Ch^b(\mathcal{A})$ *be complexes. Let*

$$F = \begin{pmatrix} \varphi & \gamma \\ \beta & \delta \end{pmatrix} : X \oplus Y \to Z \oplus U$$

(i.e. $\varphi\colon X \to Z$, $\gamma\colon Y \to Z$, $\beta\colon X \to U$ *and* $\delta\colon Y \to U$*)*

be a chain morphism with $\varphi_i\colon X_i \to Z_i$ *an isomorphism for all* i *and set* $\delta' = \delta - \beta\circ\varphi^{-1}\circ\gamma\colon Y \to U$*. Then*

$$\mathrm{Cone}(X \oplus Y \xrightarrow{F} Z \oplus U) \simeq \mathrm{Cone}(Y \xrightarrow{\delta'} U).$$

The morphisms in either direction are given by

$$\begin{pmatrix} 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & -\beta_{i+1}\circ\varphi_{i+1}^{-1} & \mathrm{id} \end{pmatrix} : X_i \oplus Y_i \oplus Z_{i+1} \oplus U_{i+1} \underset{g}{\overset{f}{\rightleftarrows}} Y_i \oplus U_{i+1} : \begin{pmatrix} \varphi_i^{-1}\circ\gamma_i & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix},$$

the homotopy for the composition $f \circ g$ is trivial and for $g \circ f$ given by

$$\begin{pmatrix} 0 & 0 & \varphi_i^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : X_{i-1} \oplus Y_{i-1} \oplus Z_i \oplus U_i \to X_i \oplus Y_i \oplus Z_{i+1} \oplus U_{i+1}.$$

Proof. We repeatedly apply Gaussian elimination to φ_i, starting with the biggest i which exists, since the complexes are bounded. ☐

Corollary 5.1.4. *Assume we have chain morphisms $f\colon A \to B$ and $\beta\colon C \to A$ of bounded complexes. Then*

$$\begin{pmatrix} \mathrm{id}_C & 0 & 0 & \\ 0 & 0 & \mathrm{id}_A & 0 \end{pmatrix} : \mathrm{Cone}\left(\mathrm{Cone}(C \xrightarrow{f\circ\beta} B) \xrightarrow{\begin{pmatrix} \beta & 0 \\ 0 & \mathrm{id} \end{pmatrix}} \mathrm{Cone}(A \xrightarrow{f} B) \right) \to \mathrm{Cone}(\beta)[1]$$

is a homotopy equivalence with inverse $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & f \\ 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$ *and homotopy*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : C_{i-1} \oplus B_i \oplus A_i \oplus B_{i+1} \quad \to \quad C_i \oplus B_{i+1} \oplus A_{i+1} \oplus B_{i+2}.$$

Proof. We repeatedly apply Gaussian Elimination with respect to $\mathrm{id}\colon B_r \to B_r$ starting with the smallest r. ☐

Lemma 5.1.5. *Let $f\colon A \to B$ be a morphism of complexes and assume $\beta\colon C \xrightarrow{\simeq} A$ and $\alpha\colon B \xrightarrow{\simeq} D$ are homotopy equivalences. Then $\mathrm{Cone}(f) \simeq \mathrm{Cone}(f')$ where $f' = \alpha f \beta$.*

Proof. See e.g. [Ros11a, Prop.3.3]. ☐

Later on, we will need this statement more explicitly for the special case where $\alpha = \mathrm{id}$:

Lemma 5.1.6. *Let $f\colon A \to B$ be a morphism of complexes and $\beta\colon C \to A$ a homotopy equivalence with inverse $\beta'\colon A \to C$ via homotopy maps $H_1\colon C[1] \to C$ and $H_2\colon A[1] \to A$. Then the following maps are homotopy inverses*

$$\begin{pmatrix} \beta & 0 \\ 0 & \mathrm{id} \end{pmatrix} : \mathrm{Cone}(C \xrightarrow{f\circ\beta} B) \quad \rightleftarrows \quad \mathrm{Cone}(A \xrightarrow{f} B) : \begin{pmatrix} \beta' & 0 \\ -f \circ H_2 & \mathrm{id} \end{pmatrix}.$$

Proof. By [Ros11b, Lemma 3.1] and the explicit formulas from its proof (with slightly other signs due to our different sign convention for cones) we know that $\mathrm{Cone}(\beta)$ is homotopic to zero via the homotopy

$$\widetilde{H} = \begin{pmatrix} H_1 + \beta' H_2 \beta - \beta' \beta H_2 & \beta' \\ -(H_2 H_2 \beta - H_2 \beta H_1) & -H_2 \end{pmatrix} : \mathrm{Cone}(\beta) \to \mathrm{Cone}(\beta)[1].$$

Thus, Cone(β)[1] is also homotopic to zero via $\widetilde{H}$. Composing with the the explicit maps from Corollary 5.1.4 we obtain that Cone$\left(\begin{pmatrix}\beta & 0\\ 0 & \mathrm{id}\end{pmatrix}\right)$ is homotopic to zero via

$$H = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{id}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathrm{id} & 0\\ 0 & f\\ 0 & \mathrm{id}\\ 0 & 0 \end{pmatrix} \widetilde{H} \begin{pmatrix} \mathrm{id}_C & 0 & 0 & 0\\ 0 & 0 & \mathrm{id}_A & 0 \end{pmatrix}.$$

The explicit formulas from the proof of [Ros11b, Lemma 3.1] now say that our desired homotopy inverse of $\begin{pmatrix}\beta & 0\\ 0 & \mathrm{id}\end{pmatrix}$ is $H_{12}\colon \mathrm{Cone}(A \to B) \to \mathrm{Cone}(C \to B)[1]$. We calculate

$$H = \begin{pmatrix} H_1 + \beta' H_2 \beta - \beta' \beta H_1 & 0 & \beta' & 0\\ -f(H_2 H_2 \beta - H_2 \beta H_1) & 0 & -f H_2 & \mathrm{id}\\ -(H_2 H_2 \beta - H_2 \beta H_1) & 0 & -H_2 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and obtain $H_{12} = \begin{pmatrix}\beta' & 0\\ -fH_2 & \mathrm{id}\end{pmatrix}$.

□

5.2 Spectral sequences

We start by recalling some facts about total complexes that for example can be found in [HS97]. We then use these together with the general theory of spectral sequences. For an introduction to spectral sequences we refer to [Wei94].

Let $B = (B_{p,q})$ be a double complex with the two differentials $d'\colon B_{p,q} \to B_{p-1,q}$ and $d''\colon B_{p,q} \to B_{p,q-1}$. Instead of the usual total complex which is defined using direct sums we consider here the *(second) total chain complex* $\mathrm{Tot}^{\Pi} B$ defined by

$$\left(\mathrm{Tot}^{\Pi} B\right)_i = \prod_{p+q=i} B_{p,q}$$

with differential $d = d' + d''$.

Definition 5.2.1. Let (C, d_C) and (D, d_D) be chain complexes in $Ch(\mathcal{A})$. The *double hom complex* B is given by $B_{p,q} = \mathrm{Hom}_{\mathcal{A}}(C_{-p}, D_q)$ with differentials

$$d'(f) = (-1)^{p+q+1} f \circ d_C \colon C_{-p+1} \to D_q \text{ and}$$
$$d''(f) = d_D \circ f \colon C_{-p} \to D_{q-1}$$

for $f \in B_{p,q}$.

Then the total complex $\mathbf{Hom}(C, D) := \mathrm{Tot}^{\Pi} B$ is the *chain complex of homomorphism from C to D*. For $f = \{f_{p,q}\colon C_{-p} \to D_q\}$ the differential is given by $(df)_{p,q} = (-1)^{p+q} f_{p+1,q} \circ d_C + d_D \circ f_{p,q+1}$.

Lemma 5.2.2. *Let $C = (\cdots \to C_1 \to C_0 \to 0)$ and $D = (\cdots \to D_1 \to D_0 \to 0)$ be two chain complexes. Then there is a spectral sequence with $E^1_{p,q} = H_q\big(\mathbf{Hom}(C_{-p}, D), d''\big)$ which weakly converges to $H_{p+q}\big(\mathbf{Hom}(C, D)\big)$. If the spectral sequence collapses, then it converges.*

Proof. Consider the double hom complex $B_{p,q} = \mathrm{Hom}(C_{-p}, D_q)$. By assumption, we have $B_{p,q} = 0$ except for $p \leq 0$ and $q \geq 0$. So by [Wei94, 5.6.1] the conditions for [Wei94, 5.5.10] are satisfied and the spectral sequence associated to the double complex has $E^1_{p,q} = H_q(B_{p,*}, d'')$ and weakly converges to $H_{p+q}(\mathrm{Tot}^{\Pi} B) = H_{p+q}\big(\mathbf{Hom}(C, D)\big)$. If the sequence collapses, it is in particular bounded above, so we get convergence by [Wei94, 5.5.10]. □

Corollary 5.2.3. *Let $C = (\cdots \to C_1 \to C_0)$ and $D = (\cdots \to D_1 \to D_0)$ be two chain complexes. Assume $H_l\big(\mathbf{Hom}(C_i, D)\big) = 0$ for all i and all $l \neq 0$. Then the following holds:*

a) The spectral sequence from Lemma 5.2.2 with $E^1_{p,q} = H_q\big(\mathbf{Hom}(C_{-p}, D), d''\big)$ collapses at E^2 at the latest and converges to $H_{p+q}\big(\mathbf{Hom}(C, D)\big)$.

b) $H_0\big(\mathbf{Hom}(C, D)\big) \cong \ker\Big(H_0\big(\mathbf{Hom}(C_0, D)\big) \xrightarrow{\bar{d}} H_0\big(\mathbf{Hom}(C_1, D)\big)\Big)$, where $\bar{d}$ is the map induced by precomposition with d_C.

c) If $H_0\big(\mathbf{Hom}(C_1, D)\big) = 0$, then $H_0\big(\mathbf{Hom}(C, D)\big) \cong H_0\big(\mathbf{Hom}(C_0, D)\big)$.

d) If $H_0\big(\mathbf{Hom}(C_0, D)\big) = 0$, then $H_0\big(\mathbf{Hom}(C, D)\big) = 0$.

e) If $H_0\big(\mathbf{Hom}(C_i, D)\big) = 0$ for all $i \neq j$, then $H_{-j}\big(\mathbf{Hom}(C, D)\big) \cong H_0\big(\mathbf{Hom}(C_j, D)\big)$ and $H_l\big(\mathbf{Hom}(C, D)\big) = 0$ for all $l \neq -j$.

f) Assume that the maps induced by precomposition with d_C

$$H_0\big(\mathbf{Hom}(C_i, D)\big) \to H_0\big(\mathbf{Hom}(C_{i+1}, D)\big)$$

are zero for all i. Then $H_{-j}\big(\mathbf{Hom}(C, D)\big) \cong H_0\big(\mathbf{Hom}(C_j, D)\big)$ for all j.

Proof. a) By assumption, $E^1_{p,q} = 0$ for $q \neq 0$, so E^1 is concentrated in a single row.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ \cdots & \longleftarrow & H_0\big(\mathbf{Hom}(C_2, D)\big) & \longleftarrow & H_0\big(\mathbf{Hom}(C_1, D)\big) & \longleftarrow & H_0\big(\mathbf{Hom}(C_0, D)\big) & \longleftarrow & 0 \\ & & 0 & & 0 & & 0 & & \end{array}$$

(axes p, q)

The differentials of E^1 go horizontally, so E^2 is also concentrated in a single row and therefore the spectral sequences collapses at E^r for $r \leq 2$. Thus, the assertion follows from Lemma 5.2.2.

b) The differentials at the E^1-page are induced by the differential of the double complex $d' = (-1)^{p+q+1}\_ \circ d_C$. Thus,

$$\begin{aligned} H_0\big(\mathbf{Hom}(C,D)\big) \cong E^2_{0,0} \cong H\Big(0 \to H_0\big(\mathbf{Hom}(C_0,D)\big) \xrightarrow{d'} H_0\big(\mathbf{Hom}(C_1,D)\big)\Big) \\ = \ker\Big(H_0\big(\mathbf{Hom}(C_0,D)\big) \xrightarrow{d'} H_0\big(\mathbf{Hom}(C_1,D)\big)\Big). \end{aligned}$$

We can ignore the sign, since it does not change the kernel.

c) This follows from b) since in this case the kernel is just $H_0(\mathbf{Hom}(C_0, D))$.

d) Again, this follows from b) since the kernel is 0.

e) With the additional assumptions, the spectral sequence even collapses at E^1 and only $E^1_{-j,0} = H_0\big(\mathbf{Hom}(C_j, D)\big)$ is non-zero. The assertion follows.

f) If $- \circ d_C$ induces the 0-map, then the spectral sequence collapses at E^1 and $H_{-j}\big(\mathbf{Hom}(C,D)\big) \cong E^1_{-j,0} \cong H_0\big(\mathbf{Hom}(C_j,D)\big)$.

□

Remark 5.2.4. For complexes $C, D \in Ch(\mathcal{A})$, we have

$$H_i\big(\mathbf{Hom}(C,D)\big) = \mathrm{Hom}_{K(\mathcal{A})}\big(C, D[-i]\big)$$

Indeed, we defined $\mathbf{Hom}(C,D)$ via $\mathbf{Hom}(C,D)_i = \prod_p \mathrm{Hom}_{\mathcal{A}}(C_{-p}, D_{i-p})$ with differential given by $(df)_{p,q} = (-1)^{p+q} f_{p+1,q} \circ d_C + d_D \circ f_{p,q+1}$ for $f = \{f_{p,q} \colon C_{-p} \to D_q\}$. So if we label the maps in $\mathrm{Hom}(C_{-p}, D_{i-p})$ by $f_{p,i}$ instead of $f_{p,i-p}$ we get the following formula for the differential:

$$(df)_{p,i-1} = (-1)^{i-1} f_{p+1,i} \circ d_C + d_D \circ f_{p,i}.$$

Thus, for $d = d_i \colon \mathbf{Hom}(C,D)_i \to \mathbf{Hom}(C,D)_{i-1}$ we have

$$\ker d_i = \big\{\{f_{p,i}\}_p \,\big|\, d_D \circ f_{p,i} = (-1)^i f_{p+1,i} \circ d_C\big\} = \mathrm{Hom}_{Ch(\mathcal{A})}(C, D[-i])$$

and

$$\begin{aligned} \operatorname{im} d_{i+1} &= \Big\{\{f_{p,i}\}_p \,\Big|\, \exists h_{i+1} = \{h_{p,i+1}\}_p \colon f_{p,i} = (dh_{i+1})_{p,i} = (-1)^i h_{p+1,i+1} \circ d_C + d_D \circ h_{p,i+1}\Big\} \\ &= \big\{f = \{f_{p,i}\}_p \in \mathrm{Hom}_{Ch(\mathcal{A})}(C, D[-i]) \,\big|\, f \simeq 0\big\}. \end{aligned}$$

A direct consequence of this remark and the previous corollary is:

Corollary 5.2.5. *Let $C = (\cdots \to C_1 \to C_0 \to 0)$ and $D = (\cdots \to D_1 \to D_0 \to 0)$ be two chain complexes. Assume $\mathrm{Hom}_{K(\mathcal{A})}(C_i, D[l]) = 0$ for all i and all $l \neq 0$, where C_i is in homological degree 0. Then the following holds:*

a) If $\mathrm{Hom}_{K(\mathcal{A})}(C_1, D) = 0$, then $\mathrm{Hom}_{K(\mathcal{A})}(C, D) \cong \mathrm{Hom}_{K(\mathcal{A})}(C_0, D)$.

b) If $\mathrm{Hom}_{K(\mathcal{A})}(C_0, D) = 0$, then $\mathrm{Hom}_{K(\mathcal{A})}(C, D) = 0$.

c) *If* $\mathrm{Hom}_{K(\mathcal{A})}(C_i, D) = 0$ *for all* $i \neq j$, *then* $\mathrm{Hom}_{K(\mathcal{A})}(C, D[j]) \cong \mathrm{Hom}_{K(\mathcal{A})}(C_j, D)$ *and* $\mathrm{Hom}_{K(\mathcal{A})}(C, D[l]) = 0$ *for all* $l \neq j$.

d) *In general,* $\mathrm{Hom}_{K(\mathcal{A})}(C, D) \cong \ker\Big(\mathrm{Hom}_{K(\mathcal{A})}(C_0, D) \xrightarrow{-\circ d_C} \mathrm{Hom}_{K(\mathcal{A})}(C_1, D)\Big)$.

e) *Assume that the maps* $\mathrm{Hom}_{K(\mathcal{A})}(C_i, D) \xrightarrow{-\circ d_C} \mathrm{Hom}_{K(\mathcal{A})}(C_{i+1}, D)$ *are zero for all* i. *Then* $\mathrm{Hom}_{K(\mathcal{A})}(C, D[j]) \cong \mathrm{Hom}_{K(\mathcal{A})}(C_j, D)$ *for all* j.

Chapter 6

Exceptional Objects

In this chapter we inductively construct chain complexes $\mathrm{V}^*(\lambda)$ which will turn out to form an exceptional sequence in $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$. In the first section, after the definition, we give an alternative explicit description of the $\mathrm{V}^*(\lambda)$ by cube-complexes and consider some immediate properties. After that we study $\mathrm{V}^*(\lambda).\cap_i$ as a means to show that there are no maps from $\mathrm{T}(\lambda_0)$ to (shifted) $\mathrm{V}^*(\lambda)$'s. In the last section, we consider maps from shifted $\mathrm{T}(\lambda)$'s to $\mathrm{V}^*(\mu)$ in order to investigate maps between the $\mathrm{V}^*(\mu)$ and to finally show that they form a graded exceptional sequence. The $\mathrm{V}^*(\mu)$ will be an important ingredient for categorifying the standard basis of $V^{\otimes n}$ in the next chapter.

6.1 Construction of the exceptional objects $\mathrm{V}^*(\lambda)$

Let $K = K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ be the homotopy category of bounded complexes with entries in $\widehat{\mathrm{Cup}}(n,k)$. Recall that we use $A\langle i\rangle$ and q^iA synonymously.

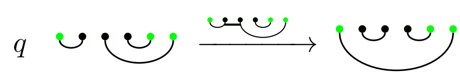
Definition 6.1.1 (Construction of $\mathrm{V}^*(\lambda)$)**.**
We inductively construct objects $\mathrm{V}^*(\lambda)$ in K. Let $\mathrm{V}^*(\lambda_0) = \mathrm{T}(\lambda_0)$ in homological degree 0. Now assume $\mathrm{V}^*(\lambda)$ is already defined. For $\lambda \xrightarrow{s_i} \mu$ we define

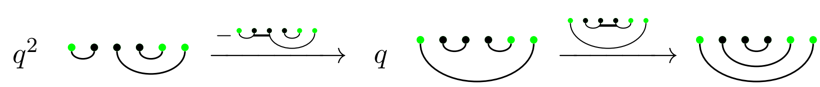
$$\mathrm{V}^*(\mu) = \mathrm{Cone}\Big(q\,\mathrm{V}^*(\lambda) \xrightarrow{\mathrm{H}_i} \mathrm{V}^*(\lambda).\mathcal{U}_i\Big),$$

where $\mathrm{H}_i = \mathrm{id}.\mathrm{H}_i\colon q\,\mathrm{V}^*(\lambda) \cong q\,\mathrm{V}^*(\lambda).\mathrm{Id} \to \mathrm{V}^*(\lambda).\mathcal{U}_i$ given by the saddle cobordism in every homological degree.

Example 6.1.2. For $n=3, k=1$ we have:

$\mathrm{V}^*(\wedge\vee\vee) = \mathrm{T}(\lambda_0) =$ 

$\mathrm{V}^*(\vee\wedge\vee) \cong$ 

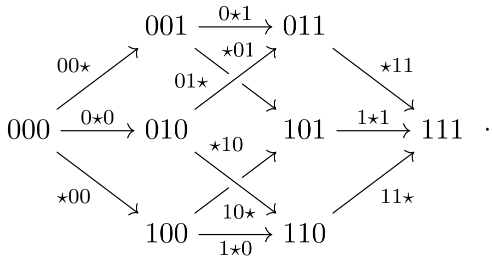
$\mathrm{V}^*(\vee\vee\wedge) \cong$ 

For $\mathrm{V}^*(\vee\vee\wedge)$ note that $\mathrm{V}^*(\vee\wedge\vee).\mathcal{U}_2 \cong 0 \to$  , thus

$$\mathrm{V}^*(\vee\vee\wedge) = \mathrm{Cone}\Big(q\,\mathrm{V}^*(\vee\wedge\vee) \to \mathrm{V}^*(\vee\wedge\vee).\mathcal{U}_2\Big) \cong \mathrm{Cone}\Big(q\,\mathrm{V}^*(\vee\wedge\vee) \to$$  $$\Big)$$

is the complex depicted above.

Definition 6.1.3. Consider the *r-dimensional cube* R_r (cf. [BN05, 2.3]) whose vertices are all the r-letter strings of 0's and 1's. The edges of the cube are marked by r-letter strings of 0's, 1's and precisely one $\star$, where the $\star$ marks the coordinate which changes from 0 to 1 along a given edge. The cube is skewered along its main diagonal from $00\ldots0$ to $11\ldots1$ such that vertices with the same sum of coordinates are on a vertical line.

For example, $R_3 =$ 

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Let $1 \le i_1, \ldots, i_r \le n$. We define a complex $R(i_1, \ldots, i_r)$ in $Ch\big(\widehat{\mathrm{Cob}}(n)\big)$ as follows: For $x \in \{0,1\}$ we define $b(x)$ to be the other value, i.e. $b(x) = 1 - x$.

For $\boldsymbol{w} = (w_1 \ldots w_r)$ with $w_j \in \{0,1\}$ let

$$\boldsymbol{w}(i_1, \ldots, i_r) = q^{\sum_i b(w_i)} B_{i_1} \ldots B_{i_r},$$

where $B_{i_l} = \begin{cases} \mathcal{U}_{i_l} & \text{if } w_l = 1, \\ \mathrm{Id} & \text{if } w_l = 0. \end{cases}$

For $\xi = (\xi_1 \ldots \xi_r)$ an edge-label with $\xi_j = \star$ let

$$\xi(i_1, \ldots, i_r) = (-1)^{\sum_{i>j} b(\xi_i)}\,\mathrm{id} \ldots \mathrm{id}\,\mathrm{H}_{i_j}\,\mathrm{id} \ldots \mathrm{id}, \tag{6.1}$$

where $\mathrm{H}_{i_j} \colon \mathrm{Id} \to \mathcal{U}_{i_j}$ is the saddle cobordism.

Finally, let $R(i_1, \ldots, i_r) = R_r(i_1, \ldots, i_r)$, i.e.

$$R(i_1, \ldots, i_r)_l = \bigoplus_{\boldsymbol{w}: \sum w_i = l} \boldsymbol{w}(i_1, \ldots, i_r)$$

with the differential given by the outgoing edges. Note that the choice of signs for the differential ensures that the square of the differential is zero.

Example 6.1.4. We have, for example, $(001)(3,2,4) = q^{1+1+0}\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_4 = q^2\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_4$ and $(101)(3,2,4) = q\,\mathcal{U}_3\,\mathrm{Id}\,\mathcal{U}_4$. The differential in between is given by $(\star01)(3,2,4) = (-1)^{1+0}\,\mathrm{H}_3\,\mathrm{id}\,\mathrm{id} = -\,\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}$.

Doing this for the whole cube R_3 we obtain

$$R(3,2,4) = \begin{array}{ccccccc} & & q^2\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_4 & \xrightarrow{\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & q\,\mathrm{Id}\,\mathcal{U}_2\mathcal{U}_4 & & \\ & \nearrow^{\mathrm{id}\,\mathrm{id}\,\mathrm{H}_4} & \oplus & {}^{-\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}}\ \ {}_{\mathrm{id}\,\mathrm{id}\,\mathrm{H}_4} & \oplus & \searrow^{\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}} & \\ q^3\,\mathrm{Id}\,\mathrm{Id}\,\mathrm{Id} & \xrightarrow{-\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & q^2\,\mathrm{Id}\,\mathcal{U}_2\,\mathrm{Id} & & q\,\mathcal{U}_3\,\mathrm{Id}\,\mathcal{U}_4 & \xrightarrow{\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & \mathcal{U}_3\mathcal{U}_2\mathcal{U}_4 \\ & \searrow_{\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}} & \oplus & {}^{-\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}}\ \ {}_{\mathrm{id}\,\mathrm{id}\,\mathrm{H}_4} & \oplus & \nearrow_{\mathrm{id}\,\mathrm{id}\,\mathrm{H}_4} & \\ & & q^2\,\mathcal{U}_3\,\mathrm{Id}\,\mathrm{Id} & \xrightarrow[-\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}]{} & q\,\mathcal{U}_3\mathcal{U}_2\,\mathrm{Id} & & \end{array},$$

where $\mathcal{U}_3\mathcal{U}_2\mathcal{U}_4$ is in homological degree 0.

Proposition 6.1.5. *Let $s_{i_1}\dots s_{i_r} \in \mathrm{W}^{min}$ be reduced, where W^{min} is the set of minimal coset representatives as in Definition 1.1.1. Then*

$$\mathrm{V}^*(\lambda_0 s_{i_1}\dots s_{i_r}) \cong \mathrm{T}(\lambda_0).R(i_1,\dots,i_r).$$

Proof. We show this by induction on r. It is obviously true for $r=0$. Now assume it is true for $r-1$, i.e. $\mathrm{V}^*(\lambda_0 s_{i_1}\dots s_{i_{r-1}}) \cong \mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1})$. Then $\mathrm{V}^*(\lambda_0 s_{i_1}\dots s_{i_r})$ is defined as

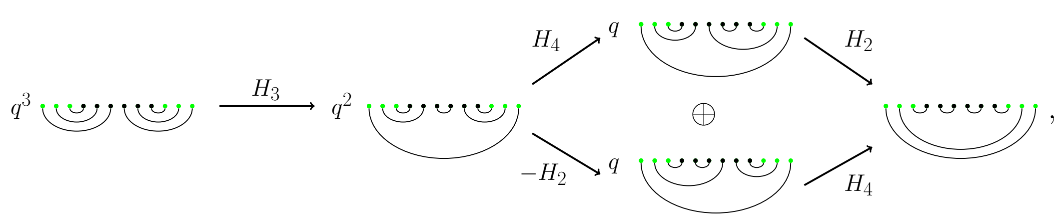
$$\begin{aligned}&\mathrm{Cone}\left(q\,\mathrm{V}^*(\lambda_0 s_{i_1}\dots s_{i_{r-1}}) \xrightarrow{\mathrm{H}_{i_r}} \mathrm{V}^*(\lambda_0 s_{i_1}\dots s_{i_{r-1}}).\mathcal{U}_{i_r}\right)\\ &\qquad\cong \mathrm{Cone}\left(q\,\mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1}) \xrightarrow{\mathrm{H}_{i_r}} \mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1}).\mathcal{U}_{i_r}\right).\end{aligned}$$

Now

$$\mathrm{Cone}\left(q\,\mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1}) \xrightarrow{\mathrm{H}_{i_r}} \mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1}).\mathcal{U}_{i_r} = \mathrm{T}(\lambda_0).R(i_1,\dots,i_r)\right)$$

follows directly from the construction of the cube complex $R(i_1,\dots,i_r)$: The q-shift of $q\,\mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1})$ is taken into account by $q^{\sum_i b(w_i)}$ since $b(w_{i_r}) = 1$ there. Because of the cone, the signs of $q\,\mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1})$ inside the cone are changed by -1 and this is reflected in the formula, since $b(\xi_{i_r}) = 1$. The maps inside $\mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1}).\mathcal{U}_{i_r}$ stay with the same sign since there $b(\xi_{i_r}) = 0$, as prescribed by the cone. The remaining maps all get positive sign, since i_r is the last index. □

Example 6.1.6. For $n=6$ and $k=3$ we obtain $\mathrm{T}(\lambda_0).R(3,2,4) \cong$



since $\mathrm{T}(\lambda_0).X = 0$ for $X \in \left\{q\,\mathrm{Id}\,\mathcal{U}_3\mathcal{U}_4,\ q^2\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_4,\ q^2\,\mathrm{Id}\,\mathcal{U}_2\,\mathrm{Id}\right\}$ and the remaining entries of $R(3,2,4)$ applied to $\mathrm{T}(\lambda_0)$ are isomorphic to what is depicted above.

Lemma 6.1.7. *Let $s_{i_1}\dots s_{i_r} = s_{j_1}\dots s_{j_r}$ be reduced expressions in W^{min}. Then*

$$R(i_1,\dots,i_r) \cong R(j_1,\dots,j_r).$$

Proof. By Lemma 1.1.19 the reduced expressions $s_{i_1} \dots s_{i_r}$ and $s_{j_1} \dots s_{j_r}$ in $\mathrm{W}^{\min}$ differ only by a finite number of moves $s_i s_j = s_j s_i$ for $|i-j| > 1$. Hence, it is enough to consider the case where $(i_1, \dots, i_r)$ and $(j_1, \dots, j_r)$ differ by swapping neighbouring indices at places m and $m+1$. Say $(i_m, i_{m+1}) = (\alpha, \beta)$, $(j_m, j_{m+1}) = (\beta, \alpha)$ and $i_s = j_s$ for $s \notin \{m, m+1\}$. We have the height-isomorphisms

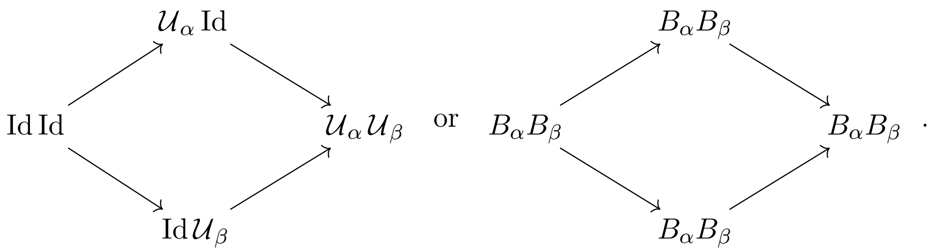
$$\mathcal{U}_\alpha \mathcal{U}_\beta \cong \mathcal{U}_\beta \mathcal{U}_\alpha, \qquad \mathcal{U}_\alpha \mathrm{Id} \cong \mathrm{Id}\, \mathcal{U}_\alpha, \qquad \mathrm{Id}\, \mathcal{U}_\beta \cong \mathcal{U}_\beta \mathrm{Id}, \tag{6.2}$$

where the first holds since $|\alpha - \beta| > 1$. We define $\iota\colon R(i_1, \dots, i_r) \to R(j_1, \dots, j_r)$ by defining it on the direct summands at every homological degree via

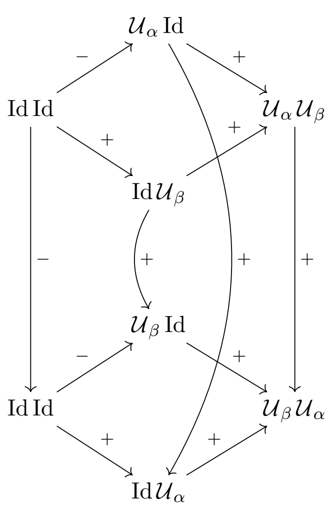
$$\begin{aligned} \boldsymbol{w}(i_1, \dots, i_r) = B_{i_1} \dots B_\alpha B_\beta \dots B_{i_r} \xrightarrow{(-1)^{b(w_m)b(w_{m+1})}\varphi} \; & B_{i_1} \dots B_\beta B_\alpha \dots B_{i_r} \\ &= B_{j_1} \dots B_\beta B_\alpha \dots B_{j_r} \\ &= s_m(\boldsymbol{w})(j_1, \dots, j_r), \end{aligned} \tag{6.3}$$

where φ is given by one of the height-isomorphisms from (6.2) or $\mathrm{Id}\,\mathrm{Id} \cong \mathrm{Id}\,\mathrm{Id}$.

If ι is a chain map, it is clearly an isomorphism. We check that ι is a chain map on every square in the cubes. It suffices to check this locally, since the global situation only differs by an overall sign. Locally at places $(m, m+1)$, a square in $R(i_1, \dots, i_r)$ looks like



In the second case where we have a square in which $B_\alpha B_\beta$ do not change, the commutativity with the differential is obvious. In the other case, we have



The vertical maps are the isomorphisms part of i with sign calculated from (6.3) and the horizontal maps are the differentials with local sign as in (6.1). Since the squares containing vertical maps commute, we are done.

For $s_{i_1} \dots s_{i_r}$ and $s_{j_1} \dots s_{j_r}$ general reduced expressions the isomorphism $R(i_1, \dots, i_r) \cong R(j_1, \dots, j_r)$ is given by a finite composition of (6.3). □

Thus, we get together with Proposition 6.1.5:

Corollary 6.1.8. *The complex representing* $V^*(\mu)$ *does not depend (up to isomorphism of complexes, not just homotopy equivalence) on the reduced expression* $s_{i_1} \dots s_{i_r}$ *for* $\mu = \lambda_0.s_{i_1} \dots s_{i_r}$.

For later reference we calculate the overall signs in a special case of Lemma 6.1.7 using (6.3):

Lemma 6.1.9. *Let* $s_{i_1} \dots s_{i_r} = s_{i_1} \dots \widehat{s_{i_j}} \dots s_{i_r} s_{i_j}$ *be reduced expressions in* W^{min}. *Then* $R(i_1, \dots, i_r)$ *is isomorphic to* $R(i_1, \dots, \widehat{i_j} \dots, i_r, i_j)$ *via*

$$\boldsymbol{w}.(i_1, \dots, i_r) \xrightarrow{(-1)^{b(w_j)(b(w_{j+1})+\dots+b(w_r))}\varphi} \big(s_{(j,r)}(\boldsymbol{w})\big)(i_1, \dots, \widehat{i_j} \dots, i_r, i_j),$$

where $s_{(j,r)} = s_j \dots s_{r-1}$ *moves place* j *to the end and* φ *is a composition of height-isomorphisms as in* (6.2).

Now we study the entries of $V^*(\mu)$. By delooping we know that every summand with a circle appearing in $V^*(\mu)$ is isomorphic to a sum of (shifted) $T(\lambda)$'s. The following lemma specifies which $T(\lambda)$'s can appear.

Lemma 6.1.10. $V^*(\mu)$ *is isomorphic (as a complex) to a complex* X *with* $X_0 = T(\mu)$ *and for* $l \neq 0$, X_l *has entries (shifted)* $T(\lambda)$ *with* $\lambda < \mu$. *In particular, all entries of* X *are (shifted)* $T(\lambda)$ *with* $\lambda \leq \mu$.

Proof. We proceed by induction on $\ell(\mu, \lambda_0)$. For $V^*(\lambda_0) = T(\lambda_0)$ it is obviously true. Now assume it is true for μ' with $\mu' \xrightarrow{s_i} \mu$, thus $V^*(\mu') \cong X'$ with X' satisfying the conditions. Hence, $V^*(\mu) \cong \mathrm{Cone}\big(q\, V^*(\mu') \to V^*(\mu').\mathcal{U}_i\big) \cong \mathrm{Cone}\big(qX' \to X'.\mathcal{U}_i\big)$. By assumption there are no circles in the summands of X' and all summands of X'_l are some $T(\lambda)$ with $\lambda < \mu' < \mu$ for $l \neq 0$ and $X'_0 = T(\mu')$. Therefore, the part qX' of the cone has the asserted form and $V^*(\mu)_0 \cong T(\mu').\mathcal{U}_i \cong T(\mu)$ by Lemma 4.1.8 since $\mu' s_i = \mu$.

We use delooping (Lemma 3.3.5) to resolve all circles in $X'.\mathcal{U}_i$ and consider all possible cases of $T(\lambda).\mathcal{U}_i$ given by Lemma 4.1.8. If $T(\lambda).\mathcal{U}_i \cong q\, T(\lambda) \oplus q^{-1}\, T(\lambda)$ or $T(\lambda).\mathcal{U}_i = 0$ or $T(\lambda).\mathcal{U}_i \cong T(\lambda')$ with $\lambda' < \lambda$, then the assertions are obviously satisfied. In the remaining case we have $T(\lambda).\mathcal{U}_i \cong T(\lambda s_i)$. Since $\lambda < \mu'$, $\mu' \xrightarrow{s_i} \mu$ and $\lambda \xrightarrow{s_i} \lambda s_i$ we obtain $\lambda s_i \leq \mu$ by Lemma 1.1.29. Furthermore, $\lambda s_i \neq \mu' s_i = \mu$, thus $\lambda s_i < \mu$.

□

Lemma 6.1.11. *Let* X *be a complex in* K. *Then*

$$\mathrm{Cone}(qX \xrightarrow{H_i} X.\mathcal{U}_i).\mathcal{U}_i \simeq q^{-1} X.\mathcal{U}_i$$

and the maps for the homotopy equivalence are

$$\begin{pmatrix}0 & \alpha\end{pmatrix} : qX_{n-1}.\mathcal{U}_i \oplus X_n.\mathcal{U}_i\mathcal{U}_i \rightleftarrows q^{-1}X_n.\mathcal{U}_i : \begin{pmatrix}0\\ \beta\end{pmatrix},$$

where $\alpha = \mathrm{id}_X .\alpha'$ *with* $\alpha' = \mathrm{id} \odot \mathrm{id} - \rho \bigcirc \mathrm{id} : \mathcal{U}_i\mathcal{U}_i \to \mathcal{U}_i$, *where* ρ *is identity with a dot, and* $\beta = \mathrm{id}_X .\beta'$ *with* $\beta' = \mathrm{id} \bigcirc \mathrm{id} : \mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_i$.

Proof. We have

$$\mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_i \cong \mathrm{Cone}\left(qX.\mathcal{U}_i \xrightarrow{\mathrm{H}_i\,.\,\mathrm{id}_{\mathcal{U}_i}} X.(\mathcal{U}_i\mathcal{U}_i)\right).$$

Now we apply Lemma 3.3.5 to the second part and get an isomorphism to

$$\mathrm{Cone}\left(qX.\mathcal{U}_i \xrightarrow{\mathrm{id}\oplus\rho'} qX.\mathcal{U}_i \oplus q^{-1}X.\mathcal{U}_i\right),$$

where ρ' is identity with a dot on the lower part of $\mathcal{U}_i$.

We apply Corollary 5.1.3 for $Y = 0$ and get the first assertion. The corollary gives homotopy inverse maps

$$\begin{pmatrix}0 & -\rho'_n & \mathrm{id}\end{pmatrix} : qX_{n-1}.\mathcal{U}_i \oplus qX_n.\mathcal{U}_i \oplus q^{-1}X_n.\mathcal{U}_i \quad \rightleftarrows \quad q^{-1}X_n.\mathcal{U}_i : \begin{pmatrix}0\\0\\ \mathrm{id}\end{pmatrix}.$$

We compose with the isomorphism from Lemma 3.3.5 and get the second assertion.

□

Corollary 6.1.12. *Let* μs_i *be defined. Then*

$$\mathrm{V}^*(\mu).\mathcal{U}_i \simeq \begin{cases} q^{-1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i & \text{if } \mu > \mu s_i, \\ q\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i & \text{if } \mu < \mu s_i. \end{cases}$$

If $\mu < \mu s_i$, *then the homotopy equivalence is given by*

$$\begin{pmatrix}0\\ \beta\end{pmatrix} : \left(\mathrm{V}^*(\mu)\mathcal{U}_i\right)_n \to \left(q^2\,\mathrm{V}^*(\mu).\mathcal{U}_i\right)_{n-1} \oplus \left(q\,\mathrm{V}^*(\mu).\mathcal{U}_i\mathcal{U}_i\right)_n \cong \left(q\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i\right)_n,$$

where $\beta = \mathrm{id}_{\mathrm{V}^*(\mu)} .\beta'$ *and* $\beta' = \mathrm{id} \bigcirc \mathrm{id} : \mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_i$.

Proof. If $\mu > \mu s_i$, then $\mathrm{V}^*(\mu) \cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu s_i) \to \mathrm{V}^*(\mu s_i).\mathcal{U}_i\right)$ and the assertion follows from Lemma 6.1.11 for $X = \mathrm{V}^*(\mu s_i)$. If $\mu < \mu s_i$, then $\mathrm{V}^*(\mu s_i) \cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i\right)$ and the assertion follows from the lemma for $X = \mathrm{V}^*(\mu)$. □

Lemma 6.1.13. *Let* X *be a complex in* K *and* $j \in \{i+1, i-1\}$. *Then*

$$\mathrm{Cone}\left(q\,\mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i) \xrightarrow{\mathrm{H}_j} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_j\right).\mathcal{U}_i \simeq qX.\mathcal{U}_j\mathcal{U}_i[1].$$

Proof. Since $\mathcal{U}_i$ commutes with cones in the weak sense that $\mathrm{Cone}(f).\mathcal{U}_i \cong \mathrm{Cone}(f.\mathcal{U}_i)$, we have

$$\begin{aligned}\mathrm{Cone}\Big(q\,\mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i) \xrightarrow{\mathrm{H}_j} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_j\Big).\mathcal{U}_i \\ \cong \mathrm{Cone}\Big(q\,\mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_i \xrightarrow{\mathrm{H}_j} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_j\mathcal{U}_i\Big).\end{aligned}$$

By Lemma 6.1.11,

$$\begin{pmatrix}0\\ \beta\end{pmatrix} : q^{-1}X.\mathcal{U}_i \xrightarrow{\simeq} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_i$$

is a homotopy equivalence, where $\beta = \mathrm{id}_X.\beta'$ with $\beta' = \mathrm{id} \bigcirc \mathrm{id} : \mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_i$. Thus,

$$\begin{aligned}&\mathrm{Cone}\Big(q\,\mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_i \xrightarrow{\mathrm{H}_j} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_j\mathcal{U}_i\Big) \\ &\simeq \mathrm{Cone}\Big(X.\mathcal{U}_i \xrightarrow{\mathrm{H}_j \circ \binom{0}{\beta}} \mathrm{Cone}(qX \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i).\mathcal{U}_j\mathcal{U}_i\Big) \\ &\cong \mathrm{Cone}\Big(X.\mathcal{U}_i \xrightarrow{\binom{0}{\gamma}} \mathrm{Cone}(qX.\mathcal{U}_j\mathcal{U}_i \xrightarrow{\mathrm{H}_i} X.\mathcal{U}_i\mathcal{U}_j\mathcal{U}_i)\Big) \\ &\cong \mathrm{Cone}\Big(X.\mathcal{U}_i \xrightarrow{\binom{0}{\mathrm{id}}} \mathrm{Cone}(qX.\mathcal{U}_j\mathcal{U}_i \xrightarrow{\iota\circ\mathrm{H}_i} X.\mathcal{U}_i)\Big),\end{aligned}$$

where $\gamma = \mathrm{id}_X.\gamma'$ with $\gamma' = \mathrm{H}_j \circ \beta' \colon \mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_j\mathcal{U}_i$ and in the last step we use the isomorphism $\iota\colon \mathcal{U}_i\mathcal{U}_j\mathcal{U}_i \xrightarrow{\cong} \mathcal{U}_i$ with $\mathrm{id} = \iota \circ \gamma'$. Applying Corollary 5.1.4 with $C = 0$, we obtain

$$\mathrm{Cone}\Big(X.\mathcal{U}_i \xrightarrow{\binom{0}{\mathrm{id}}} \mathrm{Cone}\big(qX.\mathcal{U}_j\mathcal{U}_i \xrightarrow{\iota\circ\mathrm{H}_i} X.\mathcal{U}_i\big)\Big) \simeq qX.\mathcal{U}_i\mathcal{U}_j[1].$$

□

Proposition 6.1.14. *If* $\mu = \mu s_i$ *then* $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$.

Proof. From the assertion we get $\mu = \lambda_0 w$ with $ws_i \notin \mathrm{W}^{\min}$. We prove $\mathrm{V}^*(\lambda_0 w).\mathcal{U}_i \simeq 0$ by induction on $l(w)$. If $l(w) = 0$ then $\mathrm{V}^*(\mu) = \mathrm{T}(\lambda_0)$ and $\mathrm{V}^*(\mu).\mathcal{U}_i = \mathrm{T}(\lambda_0).\mathcal{U}_i = 0$ by Lemma 4.1.8, since from $es_i = s_i \notin \mathrm{W}^{\min}$ we know $i \neq k$. If $l(w) > 0n$ then by Corollary 1.1.6 either there is a reduced expression $w = s_{i_1} \dots s_{i_r}$ and $i \neq k$ or a reduced expression $w = \tau s_i s_{i\pm 1} s_{i_1} \dots s_{i_r}$ for some $\tau = s_{l_1} \dots s_{l_t} \in \mathrm{W}^{\min}$ where in both cases $|i_j - i| > 1$.

In the first case

$$\begin{aligned}\mathrm{V}^*(\mu).\mathcal{U}_i &\cong \mathrm{Cone}\Big(q\,\mathrm{V}^*(\lambda_0 s_{i_1} \dots s_{i_{r-1}}) \to \mathrm{V}^*(\lambda_0 s_{i_1} \dots s_{i_{r-1}}).\mathcal{U}_{i_r}\Big).\mathcal{U}_i \\ &\cong \mathrm{Cone}\Big(q\,\mathrm{V}^*(\lambda_0 s_{i_1} \dots s_{i_{r-1}}).\mathcal{U}_i \to \mathrm{V}^*(\lambda_0 s_{i_1} \dots s_{i_{r-1}}).\mathcal{U}_i\mathcal{U}_{i_r}\Big)\end{aligned}$$

since $\mathcal{U}_i$ and $\mathcal{U}_{i_r}$ commute. But $(s_{i_1} \dots s_{i_{r-1}}) \in \mathrm{W}^{\min}$, $(s_{i_1} \dots s_{i_{r-1}})s_i = s_i(s_{i_1} \dots s_{i_{r-1}}) \notin \mathrm{W}^{\min}$ and $l(s_{i_1} \dots s_{i_{r-1}}) < l(w)$, thus $\mathrm{V}^*(\lambda_0 s_{i_1} \dots s_{i_{r-1}}).\mathcal{U}_i \simeq 0$ and thus $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$ by Lemma 5.1.5.

In the second case assume first $r = 0$. Then for $X = \mathrm{V}^*(\lambda_0\tau)$ in Lemma 6.1.13, we get $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq q\,\mathrm{V}^*(\lambda_0\tau).\mathcal{U}_{i\pm1}\mathcal{U}_i[1]$. Now $l(\tau) < l(w)$ and $\tau s_i s_{i\pm1} \in \mathrm{W}^{\min}$ reduced, hence by Lemma 1.1.35 $\tau s_{i\pm1} \notin \mathrm{W}^{\min}$, thus $\mathrm{V}^*(\lambda_0\tau).\mathcal{U}_{i\pm1} \simeq 0$ and therefore $\mathrm{V}^*(\mu) \simeq 0$.

If $r > 0$ then

$$\begin{aligned}\mathrm{V}^*(\mu).\mathcal{U}_i &\cong \mathrm{Cone}\Big(q\,\mathrm{V}^*(\lambda_0\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}) \to \mathrm{V}^*(\lambda_0\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}).\mathcal{U}_{i_r}\Big).\mathcal{U}_i\\ &\cong \mathrm{Cone}\Big(q\,\mathrm{V}^*(\lambda_0\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}).\mathcal{U}_i \to \mathrm{V}^*(\lambda_0\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}).\mathcal{U}_i\mathcal{U}_{i_r}\Big)\end{aligned}$$

since $\mathcal{U}_i$ and $\mathcal{U}_{i_r}$ commute. But $(\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}) \in \mathrm{W}^{\min}$, $(\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}})s_i \notin \mathrm{W}^{\min}$ and $l(\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}) < l(w)$, thus $\mathrm{V}^*(\lambda_0\tau s_i s_{i\pm1}s_{i_1}\dots s_{i_{r-1}}).\mathcal{U}_i \simeq 0$ and thus $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$. □

Now we know that $\mathrm{V}^*(\mu).\mathcal{U}_i$ is either homotopic to zero or to some other $\mathrm{V}^*(\mu').\mathcal{U}_i$ (up to internal shift).

6.2 Behaviour under reducing the number of boundary points

The first goal of this section is to describe $\mathrm{V}^*(\mu).\cap_i$ explicitly. $\cap_i$ has been defined in Section 4.3 and can be considered as the lower half of $\mathcal{U}_i$. Thus, $\cap_i$ acts on $\mathrm{V}^*(\mu)$ analogously to what we have shown for the action of $\mathcal{U}_i$ above.

Proposition 6.2.1. *a) Let $\mu > \mu s_i$. Then $\mathrm{V}^*(\mu).\cap_i \simeq q^{-1}\,\mathrm{V}^*(\mu s_i).\cap_i$.*

b) Let $\mu = \mu s_i$. Then $\mathrm{V}^*(\mu).\cap_i \simeq 0$.

Proof. a) By Corollary 6.1.12 we know $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq q^{-1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i$ and by Lemma 6.1.11, the maps for the homotopy equivalence are either zero or the identity on the upper cup of $\mathcal{U}_i$. Applying the functor $\cap_i$ yields $\mathrm{V}^*(\mu).\mathcal{U}_i\,.\cap_i \simeq q^{-1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i\,.\cap_i$ where restricted to the circle given by $\mathcal{U}_i\,.\cap_i \cong \cap_i\bigcirc$ the maps for the homotopy equivalence are either zero or the identity. Using delooping (Lemma 3.3.5) we obtain

$$q\,\mathrm{V}^*(\mu).\cap_i \oplus q^{-1}\,\mathrm{V}^*(\mu).\cap_i \simeq \mathrm{V}^*(\mu s_i).\cap_i \oplus q^{-2}\,\mathrm{V}^*(\mu s_i).\cap_i$$

and the maps for this homotopy equivalence are only between $q\,\mathrm{V}^*(\mu).\cap_i$ and $\mathrm{V}^*(\mu s_i).\cap_i$ resp. $q^{-1}\,\mathrm{V}^*(\mu).\cap_i$ and $q^{-2}\,\mathrm{V}^*(\mu s_i).\cap_i$. Hence, $q\,\mathrm{V}^*(\mu).\cap_i \simeq \mathrm{V}^*(\mu s_i).\cap_i$ and $q^{-1}\,\mathrm{V}^*(\mu).\cap_i \simeq q^{-2}\,\mathrm{V}^*(\mu s_i).\cap_i$, thus we obtain the assertion by shifting.

b) By Proposition 6.1.14 we have $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$, thus applying the functor $\cap_i$ we obtain $\mathrm{V}^*(\mu).\mathcal{U}_i\,.\cap_i \simeq 0$. Using delooping (Lemma 3.3.5) to resolve the circle of $\mathcal{U}_i\,.\cap_i \cong \cap_i\bigcirc$ we obtain $q\,\mathrm{V}^*(\mu).\cap_i \oplus q^{-1}\,\mathrm{V}^*(\mu).\cap_i \simeq 0$. Therefore, in particular $q\,\mathrm{V}^*(\mu).\cap_i \simeq 0$ and shifting this yields the assertion.

□

For $\lambda \in \Lambda(n,k)$ let $\lambda^{\dagger_i}$ be λ with places $i, i+1$ deleted. For example, for $\lambda = \vee\vee\wedge\wedge$ we obtain $\lambda^{\dagger_2} = \vee\wedge$.

Proposition 6.2.2. *For $\mu \in \Lambda(n,k)$ we have*

$$\mathrm{V}^*(\mu).\cap_i \simeq \begin{cases} 0 & \text{if } \mu = \mu s_i, \\ \mathrm{V}^*(\mu^{\dagger_i}) & \text{if } \mu < \mu s_i, \\ \mathrm{V}^*(\mu^{\dagger_i}) \langle -1 \rangle & \text{if } \mu > \mu s_i. \end{cases}$$

Proof. We prove this by induction on the relative length $\ell(\mu, \lambda_0)$. For $\ell(\mu, \lambda_0) = 0$, i.e. $\mu = \lambda_0$, we have $\mu = \mu s_i$ for all $i \neq k$ and $\lambda_0^{n-2,k-1} = (\lambda_0^{n,k})^{\dagger_k}$, so this is just Lemma 4.3.2. Now assume the assumption holds for all i and all μ with $\ell(\mu, \lambda_0) < \ell(\lambda, \lambda_0)$. We choose j such that $\lambda s_j < \lambda$, i.e. the assertion holds for λs_j in particular. We distinguish different cases of the distance of i and j.

- $i = j$: By Proposition 6.2.1 a) and induction hypothesis we know

$$\mathrm{V}^*(\lambda).\cap_i \simeq q^{-1}\,\mathrm{V}^*(\lambda s_i).\cap_i \simeq q^{-1}\,\mathrm{V}^*\left((\lambda s_i)^{\dagger_i}\right).$$

  Since $\lambda^{\dagger_i} = (\lambda s_i)^{\dagger_i}$ and $\lambda s_i = \lambda s_j < \lambda$ we are done.

- $|i - j| > 1$: In this case we can slide $\mathcal{U}_j$ and $\cap_i$ past each other and we have $\mathcal{U}_j \cap_i \cong \cap_i \mathcal{U}_{j'}$ where $j' = j$ if $i > j$ and $j' = j - 2$ if $i < j$. Hence,

$$\begin{aligned} \mathrm{V}^*(\lambda).\cap_i &\cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\lambda s_j) \xrightarrow{\mathrm{H}_j} \mathrm{V}^*(\lambda s_j).\mathcal{U}_j\,\right).\cap_i \\ &\cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\lambda s_j).\cap_i \xrightarrow{\mathrm{H}_j} \mathrm{V}^*(\lambda s_j).\cap_i.\mathcal{U}_{j'}\,\right) \\ &\simeq \begin{cases} \mathrm{Cone}(0 \to 0) & \text{if } \lambda s_j = \lambda s_j s_i, \\ \mathrm{Cone}\left(q\,\mathrm{V}^*\left((\lambda s_j)^{\dagger_i}\right) \xrightarrow{\mathrm{H}_j} \mathrm{V}^*\left((\lambda s_j)^{\dagger_i}\right).\mathcal{U}_{j'}\right) & \text{if } \lambda s_j < \lambda s_j s_i, \\ \mathrm{Cone}\left(\mathrm{V}^*\left((\lambda s_j)^{\dagger_i}\right) \xrightarrow{\mathrm{H}_j} q^{-1}\,\mathrm{V}^*\left((\lambda s_j)^{\dagger_i}\right).\mathcal{U}_{j'}\right) & \text{if } \lambda s_j > \lambda s_j s_i, \end{cases} \\ &\cong \begin{cases} 0 & \text{if } \lambda s_j = \lambda s_j s_i, \\ \mathrm{V}^*\left((\lambda s_j)^{\dagger_i} s_{j'}\right) & \text{if } \lambda s_j < \lambda s_j s_i, \\ \mathrm{V}^*\left((\lambda s_j)^{\dagger_i} s_{j'}\right) \langle -1 \rangle & \text{if } \lambda s_j > \lambda s_j s_i, \end{cases} \\ &\cong \begin{cases} 0 & \text{if } \lambda s_j = \lambda s_j s_i, \\ \mathrm{V}^*(\lambda^{\dagger_i}) & \text{if } \lambda s_j < \lambda s_j s_i, , \\ \mathrm{V}^*(\lambda^{\dagger_i}) \langle -1 \rangle & \text{if } \lambda s_j > \lambda s_j s_i, \end{cases} \end{aligned}$$

  where the homotopy equivalence holds by induction hypothesis. For the last two isomorphisms we use that since $|i - j| > 1$ we have $(\lambda s_j)^{\dagger_i} s_{j'} = (\lambda s_j s_j)^{\dagger_i} = \lambda^{\dagger_i}$ and $\lambda^{\dagger_i} > (\lambda s_j)^{\dagger_i}$ since $\lambda > \lambda s_j$. Now we are done since $\lambda = \lambda s_i \Leftrightarrow \lambda s_j = \lambda s_j s_i$ and the same for $<$ or $>$ instead of $=$ because $|i - j| > 1$.

- $i = j+1$: By construction we know $\lambda(j+1) = \wedge$ and $\lambda s_j(j+1) = \vee$. We consider two cases for $\lambda(i+1)$:
  - $\lambda(i+1) = \wedge$: In this case $\lambda(i) = \lambda(i+1) = \wedge$ and $\mathrm{V}^*(\lambda).\cap_i \simeq 0$ by Proposition 6.2.1 b), thus we are done.
  - $\lambda(i+1) = \vee$: In this case $\lambda s_j(i) = \lambda s_j(i+1) = \vee$, thus $\mathrm{V}^*(\lambda s_j).\cap_i \simeq 0$ by Proposition 6.2.1 b). Furthermore, $\mathrm{T}(\alpha).U_{i-1}.\cap_i \cong \mathrm{T}(\alpha).\cap_{i-1}$ for all α, therefore

$$\begin{aligned}\mathrm{V}^*(\lambda).\cap_i &\cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\lambda s_j).\cap_i \to \mathrm{V}^*(\lambda s_j).U_j.\cap_i\right)\\ &\simeq \mathrm{Cone}\left(0 \to \mathrm{V}^*(\lambda s_j).\cap_{i-1}\right) \cong \mathrm{V}^*(\lambda s_j).\cap_{i-1}\,.\end{aligned}$$

    Altogether we have $\lambda(i-1,i,i+1) = \vee\wedge\vee$ and $\lambda s_j(i-1,i,i+1) = \wedge\vee\vee$. Therefore, $\lambda < \lambda s_i$, $\lambda s_j < \lambda s_j s_{i-1}$ and $\lambda^{\dagger_i} = (\lambda s_j)^{\dagger_{i-1}}$. Using the induction hypothesis we obtain

$$\mathrm{V}^*(\lambda).\cap_i \simeq \mathrm{V}^*(\lambda s_j).\cap_{i-1} \simeq \mathrm{V}^*\left((\lambda s_j)^{\dagger_{i-1}}\right) = \mathrm{V}^*(\lambda^{\dagger_i}).$$

- $i = j-1$: This works analogously to the case $i = j+1$.

□

Lemma 6.2.3. *Let B be a complex in $K(\widehat{\mathrm{Cup}}(n,k))$ and A an object of $\widehat{\mathrm{Cup}}(n,k)$ viewed as a complex concentrated in degree* 0*. Then*

$$\begin{aligned}\mathrm{Hom}_{K(\widehat{\mathrm{Cup}}(n,k))}(A.\mathcal{U}_i[j],B) &\cong \mathrm{Hom}_{K(\widehat{\mathrm{Cup}}(n,k))}(A[j],B.\mathcal{U}_i)\\ &\cong \mathrm{Hom}_{K(\widehat{\mathrm{Cup}}(n-2,k-1))}(A.\cap_i[j],B.\cap_i).\end{aligned}$$

Proof. Consider the complexes

$$\cdots \to \mathrm{Hom}(A.\mathcal{U}_i,B_{j+1}) \xrightarrow{d_B\circ-} \mathrm{Hom}(A.\mathcal{U}_i,B_j) \xrightarrow{d_B\circ-} \mathrm{Hom}(A.\mathcal{U}_i,B_{j-1}) \to \dots$$

and

$$\cdots \to \mathrm{Hom}(A,B_{j+1}.\mathcal{U}_i) \xrightarrow{(d_B\,\mathrm{id}_{\mathcal{U}_i})\circ-} \mathrm{Hom}(A,B_j.\mathcal{U}_i) \xrightarrow{(d_B\,\mathrm{id}_{\mathcal{U}_i})\circ-} \mathrm{Hom}(A,B_{j-1}.\mathcal{U}_i) \to \dots,$$

where all Hom are $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}$. The entries of the complexes are isomorphic by Proposition 4.2.1 and these isomorphisms are compatible with the differentials since the differential only acts on the part that is not moved. Thus, the complexes are isomorphic and the first isomorphism of the assertion follows from Remark 5.2.4. The second isomorphism follows analogously using Lemma 4.3.3. □

Lemma 6.2.4. $\mathrm{Hom}_K\left(\mathrm{T}(\lambda_0)\langle l\rangle[j],\mathrm{V}^*(\lambda_0 s_k)\right) = 0$ *for all j,l.*

Proof. By definition, $\mathrm{V}^*(\lambda_0 s_k) \cong q\,\mathrm{T}(\lambda_0) \xrightarrow{\mathrm{H}_k} \mathrm{T}(\lambda_0 s_k)$. There is only a map $\mathrm{T}(\lambda_0)\langle l\rangle \to q\,\mathrm{T}(\lambda_0)$ if $l = 1$ and in this case it is $c\cdot\mathrm{id}$ for some $c \in \mathbb{C}$ by Lemma 3.4.15. Thus, if the map

$$\begin{array}{ccccc}
\mathrm{T}(\lambda_0)\,\langle l\rangle\,[1] & \cong & \mathrm{T}(\lambda_0)\,\langle l\rangle & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle c\cdot \mathrm{id}} & & \downarrow \\
\mathrm{V}^*(\lambda_0 s_k) & \cong & q\,\mathrm{T}(\lambda_0) & \xrightarrow{\mathrm{H}_k} & \mathrm{T}(\lambda_0 s_k)
\end{array}$$

is non-zero, the composition of $c \cdot \mathrm{id}$ with H_k is non-zero, too, i.e. it is not a chain map. Thus, there is no non-zero chain map from $\mathrm{T}(\lambda_0)\,\langle l\rangle\,[1]$ to $\mathrm{V}^*(\lambda_0 s_k)$. By Lemma 4.2.3 and Remark 3.1.3 there is only a non-zero map $\mathrm{T}(\lambda_0)\,\langle l\rangle \to \mathrm{T}(\lambda_0 s_k)$ if $l = 1$ and in this case it is given by $c \cdot \mathrm{H}_k$ for some $c \in \mathbb{C}$. So every non-zero chain map $\mathrm{T}(\lambda_0)\,\langle l\rangle\,[0] \to \mathrm{V}^*(\lambda_0 s_k)$ is homotopic to zero by $c \cdot \mathrm{id} : \mathrm{T}(\lambda_0)\,\langle l\rangle\,[0] \to \mathrm{V}^*(\lambda_0)_1 \cong q\,\mathrm{T}(\lambda_0)$:

$$\begin{array}{ccccc}
\mathrm{T}(\lambda_0)\,\langle l\rangle\,[0] & \cong & 0 & \longrightarrow & \mathrm{T}(\lambda_0)\,\langle l\rangle \\
\downarrow & & \downarrow & \overset{c\cdot \mathrm{id}}{\swarrow} & \downarrow{\scriptstyle c\cdot \mathrm{H}_k} \\
\mathrm{V}^*(\lambda_0 s_k) & \cong & q\,\mathrm{T}(\lambda_0) & \xrightarrow[\mathrm{H}_k]{} & \mathrm{T}(\lambda_0 s_k)
\end{array}$$

□

Theorem 6.2.5. $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], \mathrm{V}^*(\mu)\big) = 0$ *for all* j, l *and* $\mu \neq \lambda_0 \in \Lambda(n,k)$.

Proof. For $k = 0$ or $k = n$ the assertion is clear since there is no $\wedge\vee$-sequence μ different from λ_0 in these cases. Now fix n, k with $n > k > 0$ and assume the assertion is true for all smaller values of n and k. We do induction on $\ell(\mu, \lambda_0)$. If $\ell(\mu, \lambda_0) = 1$, i.e. $\mu = \lambda_0 s_k$, then the assertion is true by the lemma above. Now let $\mu \to \mu s_i$ and assume the assertion is true for μ. By definition $\mathrm{V}^*(\mu s_i) \cong \mathrm{Cone}\big(q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i\big)$, thus we have an exact triangle $q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i \to \mathrm{V}^*(\mu s_i) \to q\,\mathrm{V}^*(\mu)\,[1]$. Therefore, we get a long exact sequence

$$\begin{aligned}
\cdots \to \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], \mathrm{V}^*(\mu).\mathcal{U}_i\big) &\to \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], \mathrm{V}^*(\mu s_i)\big) \\
&\to \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], q\,\mathrm{V}^*(\mu)[1]\big) \to \dots .
\end{aligned}$$

By induction we have

$$\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], q\,\mathrm{V}^*(\mu)[1]\big) \cong \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l-1\rangle\,[j-1], \mathrm{V}^*(\mu)\big) = 0.$$

Thus, after showing $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], \mathrm{V}^*(\mu).\mathcal{U}_i\big) = 0$ we are done. Lemma 6.2.3 yields

$$\begin{aligned}
&\mathrm{Hom}_{K^b(\widehat{\mathrm{Cup}}(n,k))}\big(\mathrm{T}(\lambda_0)\,\langle l\rangle\,[j], \mathrm{V}^*(\mu).\mathcal{U}_i\big) \\
&\qquad \cong \mathrm{Hom}_{K^b(\widehat{\mathrm{Cup}}(n-2,k-1))}\big(\mathrm{T}(\lambda_0).\cap_i\,\langle l\rangle\,[j], \mathrm{V}^*(\mu).\cap_i\big).
\end{aligned}$$

By Lemma 4.3.2 we are done in the cases $i \neq k$ since then $\mathrm{T}(\lambda_0).\cap_i = 0$. In the remaining case $i = k$ we have $\mathrm{T}(\lambda_0).\cap_i \cong \mathrm{T}(\lambda_0')$ for $\lambda_0' = \lambda_0^{n-2,k-1} = \lambda_0^{\dagger_k}$. By Proposition 6.2.2 we know

$$\mathrm{V}^*(\mu).\cap_k \simeq \begin{cases} 0 \\ \mathrm{V}^*(\mu^{\dagger_k})\,\langle m\rangle \end{cases} \qquad \text{for } m \in \{0, -1\}.$$

In case $\mathrm{V}^*(\mu).\cap_k = 0$ we are obviously done. In the other cases we are done since the assertion is true for all smaller n and k, except for the case that $\mu^{\dagger_k} = \lambda_0$. But $\mu^{\dagger_k} = \lambda_0$

is only possible if $\mu = \lambda_0$ or $\mu = \lambda_0 s_k$ and the former one is not a possible value of μ while the latter case is already shown. □

This theorem will be the basic ingredient for the next section.

6.3 Exceptional sequences

Before we can show that our $V^*(\lambda)$ fit into the general setting of exceptional sequences we have to define the exceptional sequences in a graded setting.

Definition 6.3.1. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category with internal grading-shift $\langle - \rangle$. Elements $\{X_\alpha\}_{\alpha\in\Omega}$, for $X_\alpha \in \mathcal{D}$ and Ω a finite poset, form a *graded exceptional sequence* if the following two conditions hold

(E1) $\quad \mathrm{Hom}_{\mathcal{D}}\left(X_\alpha, X_\beta\langle i\rangle[j]\right) = 0$ for $\alpha \ngeq \beta$ and all i, j

(E2) $\quad \mathrm{Hom}_{\mathcal{D}}\left(X_\alpha, X_\alpha\langle i\rangle[j]\right) = \begin{cases} \mathbb{C} & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$

The goal for this section is the following:

Theorem 6.3.2. *The* $V^*(\lambda)$ *form a graded exceptional sequence.*

As a first step, we want to consider maps from shifted $T(\lambda)$ to $V^*(\mu)$. The initial case $\lambda = \lambda_0$ has been already covered in the last section. Now we deduce insight about maps from shifted $T(\lambda)$'s to $V^*(\mu)$ from the initial case $\lambda = \lambda_0$ using the sliding properties from Proposition 4.2.1.

Lemma 6.3.3. *Let* $\lambda \to \lambda s_i$. *Then*

$$\mathrm{Hom}_K\left(T(\lambda s_i)\langle l\rangle[j], V^*(\mu)\right) \cong \mathrm{Hom}_K\left(T(\lambda)\langle l\rangle[j], V^*(\mu).\mathcal{U}_i\right)$$

for all l, j.

Proof. Since $\lambda s_i > \lambda$ we have $T(\lambda s_i) \cong T(\lambda).\mathcal{U}_i$ by Lemma 4.1.8. Thus, the assertion follows directly from Lemma 6.2.3. □

Lemma 6.3.4. *Let* $\mu \in \Lambda(n,k)$ *such that* μs_i *is not defined. Let* $\lambda' \xrightarrow{s_i} \lambda$. *Then for all* j, l *we have*

$$\mathrm{Hom}_K\left(T(\lambda)\langle l\rangle[j], V^*(\mu)\right) = 0.$$

Proof. By Lemma 6.3.3 and Proposition 6.1.14 we obtain

$$\begin{aligned} \mathrm{Hom}_K\left(T(\lambda)\langle l\rangle[j], V^*(\mu)\right) &\cong \mathrm{Hom}_K\left(T(\lambda')\langle l\rangle[j], V^*(\mu).\mathcal{U}_i\right) \\ &\cong \mathrm{Hom}_K\left(T(\lambda')\langle l\rangle[j], 0\right) = 0. \end{aligned}$$

□

Lemma 6.3.5. *Let $\lambda \to \lambda s_i$ and j fixed. If $\mu s_i \neq \mu$ and*

$$\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0 = \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu s_i)\right)$$

for all l, then

$$\mathrm{Hom}_K\left(\mathrm{T}(\lambda s_i)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0$$

for all l.

Proof. By Lemma 6.3.3 we only have to show $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu).\mathcal{U}_i\right) = 0$.

Assume first $\mu s_i > \mu$. Then by definition $\mathrm{V}^*(\mu s_i) = \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i\right)$, thus we have an exact triangle $q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i \to \mathrm{V}^*(\mu s_i) \to q\,\mathrm{V}^*(\mu)[1]$. Therefore, we get a long exact sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], q\,\mathrm{V}^*(\mu)\right) &\to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu).\mathcal{U}_i\right)\\ &\to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu s_i)\right) \to \dots\end{aligned}$$

Since we know the outer parts to be zero, the same follows for the middle part.

Assume now $\mu > \mu s_i$. Then $\mathrm{V}^*(\mu) \cong \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu s_i) \to \mathrm{V}^*(\mu s_i).\mathcal{U}_i\right)$ and by Lemma 6.1.12 $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq q^{-1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i$. As above, we have an exact triangle $q\,\mathrm{V}^*(\mu s_i) \to q\,\mathrm{V}^*(\mu).\mathcal{U}_i \to \mathrm{V}^*(\mu) \to q\,\mathrm{V}^*(\mu s_i)[1]$ which gives us an exact sequence

$$\begin{aligned}\cdots \to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], q\,\mathrm{V}^*(\mu s_i)\right) &\to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], q\,\mathrm{V}^*(\mu).\mathcal{U}_i\right)\\ &\to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) \to \dots\end{aligned}$$

Since we know the outer parts to be zero, we get $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], q\,\mathrm{V}^*(\mu).\mathcal{U}_i\right) = 0$ for all l and thus $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu).\mathcal{U}_i\right) = 0$, too. □

Proposition 6.3.6. *For $\lambda \not\geq \mu$ we have $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ for all j, l.*

Proof. This is clear for $\lambda = \lambda_0$ by Theorem 6.2.5. Now we proceed by induction on $\ell(\lambda, \lambda_0)$. Consider $\lambda' \xrightarrow{s_i} \lambda$ where the claim is already proven for λ'.

Let $\lambda \not\geq \mu$. By Corollary 1.1.30 we have that μs_i is not defined or $\lambda' \not\geq \mu, \mu s_i$.

If μs_i is defined, we get $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ by Lemma 6.3.5 and the induction hypothesis, since we have $\mu, \mu s_i \not\leq \lambda'$.

Now assume μs_i is not defined. Then we get $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ by Lemma 6.3.4. □

Lemma 6.3.7. *Let D be a complex and A be an object in $\widehat{\mathrm{Cup}}(n,k)$. Assume $\mathrm{Hom}_K(A[i], D) = 0$ and consider a complex C with $C_i = A \oplus B$. Let f be a chain map in $\mathrm{Hom}_K(C, D)$ with $f_j = 0$ for $j < i$. Then $f \simeq g$ where $g_i|_A = 0 = g_j$ for $j < i$ and $g_i|_B = f_i|_B$.*

Proof. This is a special case of Lemma 5.1.1 for $B_j = C_j$ for all $j \neq i$ and $B_i = B$. Since $f_{i-1} = 0$ we have $f_{j-1} d_C|_{A_j \to B_{j-1}} = 0$ for $i = j$ and for $i \neq j$ it holds anyway since then $A_j = 0$, thus $g_i|_A = 0$. Also, since $A_{j-1} = 0$ for $j \leq i$, we have $d_C|_{A_{j-1} \to B_j} = 0$, so $g_j|_B = f_j|_B$ for $j \leq i$.

□

Proposition 6.3.8. *Let C, D be complexes in K and $f\colon C \to D$ a chain map. Let $\Gamma \subset \Lambda(n,k)$ and $r \in \mathbb{Z}$. Assume*

a) $\operatorname{Hom}_K\left(\mathrm{T}(\lambda)\langle j\rangle[l], D\right) = 0$ *for all j, l and $\lambda \in \Gamma$,*

b) $f_i = 0$ *for $i < r$,*

c) for all $i \geq r$ we have $C_i \cong \bigoplus_{j=1}^{m_i} \mathrm{T}(\lambda_j^i)\langle i_j\rangle$ *with $\lambda_j^i \in \Gamma$.*

Then $f \simeq 0$.

Proof. This follows inductively using Lemma 6.3.7. We start by applying the lemma to $i = r$ and $A = \mathrm{T}(\lambda_j^r)\langle r_j\rangle$ for $j = 1$ and obtain a homotopy equivalent chain map which is also zero on A. Since $f_{i|_B}$ is not changed, we can go on by applying the lemma for $j = 2$ and so on, until the new homotopy equivalent chain map is zero on all of C_r. Then we go on with C_{r+1} and iterate until we obtain a homotopy equivalent chain map that is 0 on all of C. □

We can now show the condition (E1) of the exceptional sequence where K is the triangulated category and $\Lambda(n,k)$ the poset:

Theorem 6.3.9. $\operatorname{Hom}_K\left(\mathrm{V}^*(\lambda)[j]\langle l\rangle, \mathrm{V}^*(\mu)\right) = 0$ *for all j, l and all $\lambda \not\geq \mu$.*

Proof. For a fixed μ we set $\Gamma = \{\lambda \in \Lambda(n,k) \mid \lambda \not\geq \mu\}$ and let $r = 0$. Let $f \in \operatorname{Hom}_K\left(\mathrm{V}^*(\lambda)[j]\langle l\rangle, \mathrm{V}^*(\mu)\right)$, then condition b) of Proposition 6.3.8 is trivially satisfied. By Proposition 6.3.6, condition a) holds. By Lemma 6.1.10 every $\mathrm{V}^*(\lambda)_i$ is isomorphic to a sum of shifted $\mathrm{T}(\lambda')$ with $\lambda' \leq \lambda$. Assume $\lambda' \geq \mu$. Then we get $\lambda \geq \mu$ and a contradiction. Thus, the last remaining condition for Proposition 6.3.8 is satisfied and it yields $f \simeq 0$. □

The next goal is to show that the $(j \neq 0)$-part of condition (E2) of the exceptional sequence holds.

The $(j < 0)$-case follows analogously to the previous theorem:

Lemma 6.3.10. $\operatorname{Hom}_K\left(\mathrm{V}^*(\lambda)\langle l\rangle[j], \mathrm{V}^*(\lambda)\right) = 0$ *for $j < 0$ and all l.*

Proof. Let $r = 0$ and $\Gamma = \{\mu \in \Lambda(n,k) \mid \mu < \lambda\} \subset \{\mu \in \Lambda(n,k) \mid \mu \not\geq \lambda\}$. By Lemma 6.1.10, condition *c*) of Proposition 6.3.8 is satisfied, by Proposition 6.3.6 condition a) is satisfied, too and condition b) holds trivially, thus Proposition 6.3.8 yields the assertion. □

Proposition 6.3.11. *Let $j \neq 0$. Then* $\operatorname{Hom}_K\left(\mathrm{T}(\lambda)\langle m\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ *for all λ, μ and all m.*

Proof. For $\lambda \not\geq \mu$ this is true by Proposition 6.3.6. So it remains to show the claim for $\lambda \geq \mu$. Fix $j \neq 0$.

We now use double induction. The outer induction is induction on $\ell(\lambda, \mu) = l$: The assertion is already shown for $l < 0$ since $l < 0$ is only possible for $\lambda \not\geq \mu$. Moreover, note that $\ell(\lambda, \mu) = \ell(\lambda, \lambda_0) - \ell(\mu, \lambda_0)$, since we only consider $\lambda \geq \mu$. Assume that the assertion is true for $l - 1 \geq -1$. Now we show it for l and to do this we proceed by inner induction on $\ell(\mu, \lambda_0)$.

For $\ell(\mu, \lambda_0) = 0$, i.e. $\mu = \lambda_0$, the assertion is clear since $\mathrm{V}^*(\lambda_0)$ has no non-zero entries outside of homological degree 0. Now consider μ and assume the assertion is shown for all ν with $\ell(\nu, \lambda_0) < \ell(\mu, \lambda_0)$. Since $\ell(\mu, \lambda_0) > 0$ and $0 \leq l = \ell(\lambda, \mu) = \ell(\lambda, \lambda - 0) - \ell(\mu, \lambda_0)$, we know $\lambda \neq \lambda_0$ and we can choose λ' such that $\lambda' \xrightarrow{s_i} \lambda$. We have either $\lambda' \not\geq \mu$ or $\lambda' \geq \mu$. In the first case we have $\mathrm{Hom}_K\left(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ for all m by Proposition 6.3.6. In the second case consider

$$\ell(\lambda', \mu) = \ell(\lambda', \lambda_0) - \ell(\mu, \lambda_0) = \ell(\lambda, \lambda_0) - 1 - \ell(\mu, \lambda_0) = l - 1.$$

Therefore, also in this case, we already know $\mathrm{Hom}_K\left(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ for all m by induction hypothesis of the outer induction.

Now consider μs_i. We have to distinguish several cases:

- If μs_i is undefined, then $\mathrm{Hom}_K(\mathrm{T}(\lambda)\langle m\rangle[j], \mathrm{V}^*(\mu)) = 0$ holds for all m by Lemma 6.3.4.
- Assume $\mu s_i > \mu$: Now we have either $\lambda' \not\geq \mu s_i$ or $\lambda' \geq \mu s_i$. In the first case, $\mathrm{Hom}_K\left(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu s_i)\right) = 0$ for all m by Proposition 6.3.6. In the second case, we have

  $$\ell(\lambda', \mu s_i) = \ell(\lambda', \lambda_0) - \ell(\mu s_i, \lambda_0) = \ell(\lambda, \lambda_0) - 1 - \ell(\mu, \lambda_0) - 1 = l - 2.$$

  Therefore, $\mathrm{Hom}_K(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu s_i)) = 0$ for all m also in this case by induction hypothesis of the outer induction. With Lemma 6.3.5, we now get $\mathrm{Hom}_K(\mathrm{T}(\lambda)\langle m\rangle[j], \mathrm{V}^*(\mu)) = 0$ for all m.
- Assume $\mu s_i < \mu$: Again, we have either $\lambda' \not\geq \mu s_i$ or $\lambda' \geq \mu s_i$. In the first case, $\mathrm{Hom}_K\left(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu s_i)\right) = 0$ for all m by Proposition 6.3.6. In the second case, we have

  $$\ell(\lambda', \mu s_i) = \ell(\lambda', \lambda_0) - \ell(\mu s_i, \lambda_0) = \ell(\lambda, \lambda_0) - 1 - \ell(\mu, \lambda_0) + 1 = l.$$

  But since $\ell(\mu s_i, \lambda_0) = \ell(\mu, \lambda_0) - 1$, we know $\mathrm{Hom}_K(\mathrm{T}(\lambda')\langle m\rangle[j], \mathrm{V}^*(\mu s_i)) = 0$ for all m by induction hypothesis of the inner induction. Again, we apply Lemma 6.3.5 to get $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle m\rangle[j], \mathrm{V}^*(\mu)\right) = 0$ for all m.

□

Corollary 6.3.12. *Let $X \in K$ be a complex such that $X_i = 0$ for $i \leq 0$. Then for all $\mu \in \Lambda(n,k)$ we have*

$$\mathrm{Hom}_K\left(X, \mathrm{V}^*(\mu)\right) = 0.$$

Proof. We apply Proposition 6.3.8 for $\Gamma = \Lambda(n,k)$ and $r = 1$. The Proposition 6.3.11 yields the only non-trivial condition. □

In particular, we get our needed $(j > 0)$-case and even more:

Corollary 6.3.13. *For all $\lambda, \mu \in \Lambda(n,k)$, $l \in \mathbb{Z}$ and $j > 0$ we have*

$$\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)\langle l\rangle[j], \mathrm{V}^*(\mu)\right) = 0.$$

Now we consider the $(j = 0)$-case:

Proposition 6.3.14. *For all $\lambda \in \Lambda(n,k)$ we have*

$$\mathrm{Hom}\left(\mathrm{T}(\lambda)\langle l\rangle, \mathrm{V}^*(\lambda)\right) \cong \begin{cases} 0 & \text{if } l \neq 0, \\ \mathbb{C} & \text{if } l = 0. \end{cases}$$

Proof. We first consider the $(l = 0)$-case: By Lemma 6.1.10 we know $\mathrm{V}^*(\lambda)_0 \cong \mathrm{T}(\lambda)$, so we can consider the chain map given by the identity from $\mathrm{T}(\lambda)$ to $\mathrm{V}^*(\lambda)$. By Lemma 3.4.14, the identity morphism does not factorise and is up to scalar the only degree 0 morphism, thus it cannot be homotopic to zero and we obtain $\mathbb{C}$ in case $l = 0$.

The case for $l \neq 0$ follows by induction on $\ell(\lambda, \lambda_0)$. By Lemma 3.4.15 it is clear for $\lambda = \lambda_0$. Now let $\lambda < \lambda s_i$. By Lemma 6.3.3 we have $\mathrm{Hom}_K\left(\mathrm{T}(\lambda s_i)\langle l\rangle, \mathrm{V}^*(\lambda s_i)\right) \cong \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle, \mathrm{V}^*(\lambda s_i).\mathcal{U}_i\right)$. As before the distinguished triangle $q\,\mathrm{V}^*(\lambda) \to q\,\mathrm{V}^*(\lambda s_i) \to \mathrm{V}^*(\lambda s_i) \to q\,\mathrm{V}^*(\lambda)[1]$ yields the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle, q\,\mathrm{V}^*(\lambda)\right) \to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle, q\,\mathrm{V}^*(\lambda s_i).\mathcal{U}_i\right) \\ \to \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle, \mathrm{V}^*(\lambda s_i)\right) \to \dots \end{aligned}$$

For $l \neq 1$ the first term is equal to zero by induction and the last term is always zero by Proposition 6.3.6 since $\lambda \not\geq \lambda s_i$. So the middle term is zero for $l \neq 1$, i.e. $0 = \mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l-1\rangle, \mathrm{V}^*(\lambda s_i).\mathcal{U}_i\right) \cong \mathrm{Hom}_K\left(\mathrm{T}(\lambda s_i)\langle l-1\rangle, \mathrm{V}^*(\lambda s_i)\right)$ for $l \neq 1$. □

Corollary 6.3.15. *For all $\lambda \in \Lambda(n,k)$ we have*

$$\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)\langle l\rangle, \mathrm{V}^*(\lambda)\right) = 0 \qquad \text{for } l \neq 0.$$

Proof. Let $f\colon \mathrm{V}^*(\lambda)\langle l\rangle \to \mathrm{V}^*(\lambda)$ be a chain map. By Lemma 6.1.10 $\mathrm{V}^*(\lambda)_0 \cong \mathrm{T}(\lambda)$ and by Proposition 6.3.14 we have $\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle, \mathrm{V}^*(\lambda)\right) = 0$, hence $f \simeq g$ with $g_0 = 0$ by Lemma 6.3.7. Applying Proposition 6.3.8 for $r = 1$, Γ maximal and using Proposition 6.3.11 to have condition a), we obtain $g \simeq 0$. □

Corollary 6.3.16.

$$\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda), \mathrm{V}^*(\lambda)\right) \cong \mathbb{C}$$

Proof. By Proposition 6.3.11 we have $\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)_i, \mathrm{V}^*(\lambda)[l]\right) = 0$ for all i and all $l \neq 0$. Using Lemma 6.1.10, Proposition 6.3.6 yields $\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)_1, \mathrm{V}^*(\lambda)\right) = 0$. Thus, by Corollary 5.2.5 a) we obtain

$$\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda), \mathrm{V}^*(\lambda)\right) \cong \mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)_0, \mathrm{V}^*(\lambda)\right)$$

which gives the assertion by Proposition 6.3.14 since $\mathrm{V}^*(\lambda)_0 \cong \mathrm{T}(\lambda)$ by Lemma 6.1.10. □

In the proof above we used the spectral sequence argument Corollary 5.2.5 a), since we could not apply Lemma 6.3.7 because this needs all maps to the right to be zero. We could also have used the spectral sequence argument (more precisely Corollary 5.2.5 c)) from the previous proof to show Theorem 6.3.9, Corollary 6.3.12 and Corollary 6.3.15.

Altogether (Theorem 6.3.9, Lemma 6.3.10, Corollary 6.3.13, Corollary 6.3.15, Corollary 6.3.16), we finally have proven Theorem 6.3.2.

For later reference we want to collect what we have proven about maps from $\mathrm{T}(\lambda)$ to $\mathrm{V}^*(\mu)$.

Remark 6.3.17. All in all (Prop. 6.3.11, Prop. 6.3.6, Prop. 6.3.14) we have shown in this section:

$$\mathrm{Hom}_K\left(\mathrm{T}(\lambda)\langle l\rangle[k], \mathrm{V}^*(\mu)\right) \cong \begin{cases} 0 & \text{if } k \neq 0, \\ 0 & \text{if } \lambda \not\geq \mu, \\ 0 & \text{if } \lambda = \mu, l \neq 0, \\ \mathbb{C} & \text{if } \lambda = \mu, l = 0 = k. \end{cases}$$

Chapter 7

Categorification

In this chapter we prove that the exceptional objects from Theorem 6.3.2 generate the triangulated category K by showing that every $\mathrm{T}(\lambda)$ can be constructed as an iterated cone from some shifted $\mathrm{V}^*(\mu)$'s. This can be seen as some sort of base change between the $\mathrm{T}(\lambda)$ and the $\mathrm{V}^*(\mu)$. We explicitly determine which $\mathrm{V}^*(\mu)$'s occur in this construction. Then, we define a notion of duality and dual complexes and show that the duals of the $\mathrm{V}^*(\mu)$ categorify the standard basis in $V^{\otimes n}$. In the last section, we categorify the bilinear form on $V^{\otimes n}$.

7.1 Iterated cones

To describe the $\mathrm{T}(\lambda)$ via cones of $\mathrm{V}^*(\mu)$ we first investigate how to write $\mathrm{V}^*(\mu).\mathcal{U}_i$ as a cone of other $\mathrm{V}^*(\mu')$. In Corollary 6.1.12 and Proposition 6.1.14 we already considered $\mathrm{V}^*(\mu).\mathcal{U}_i$, but now we want to describe it only by other $\mathrm{V}^*(\nu)$ without $\mathcal{U}_i$ appearing.

Proposition 7.1.1. *Let $\mu \in \Lambda(n,k)$ and $1 \leq i \leq n$. Then*

$$\mathrm{V}^*(\mu).\mathcal{U}_i \simeq \begin{cases} 0 & \text{if } \mu s_i \text{ is undefined,} \\ \mathrm{Cone}\left(\mathrm{V}^*(\mu s_i)[-1] \to q\,\mathrm{V}^*(\mu)\right) & \text{if } \mu < \mu s_i, \\ \mathrm{Cone}\left(q^{-1}\,\mathrm{V}^*(\mu)[-1] \to \mathrm{V}^*(\mu s_i)\right) & \text{if } \mu > \mu s_i. \end{cases}$$

Proof. The first case is just Proposition 6.1.14. For $\mu s_i > \mu$ we have $\mathrm{V}^*(\mu s_i) = \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu) \to \mathrm{V}^*(\mu).\mathcal{U}_i\right)$ and the assertion follows. In the case $\mu s_i < \mu$ we have $\mathrm{V}^*(\mu) = \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu s_i) \to \mathrm{V}^*(\mu s_i).\mathcal{U}_i\right)$ and since $q\,\mathrm{V}^*(\mu).\mathcal{U}_i \simeq \mathrm{V}^*(\mu s_i).\mathcal{U}_i$ by Corollary 6.1.12 the assertion follows again after a q-shift. □

Lemma 7.1.2. *If $X \in K = K^b(\widehat{\mathrm{Cup}}(n,k))$ is an iterated cone of shifted $\mathrm{V}^*(\mu)$'s, then so is $X.\mathcal{U}_i$.*

Proof. By Proposition 7.1.1 we know that each $\mathrm{V}^*(\mu).\mathcal{U}_i$ is a cone of some $\mathrm{V}^*(\nu)$'s. Since $\mathrm{Cone}(A \to B).\mathcal{U}_i \cong \mathrm{Cone}(A.\mathcal{U}_i \to B.\mathcal{U}_i)$, the assertion follows. □

Theorem 7.1.3. *Every $\mathrm{T}(\lambda)$ is an iterated cone of $\mathrm{V}^*(\mu)$'s. More precisely,* *i)* $\mathrm{T}(\lambda)$ *is an iterated cone of*

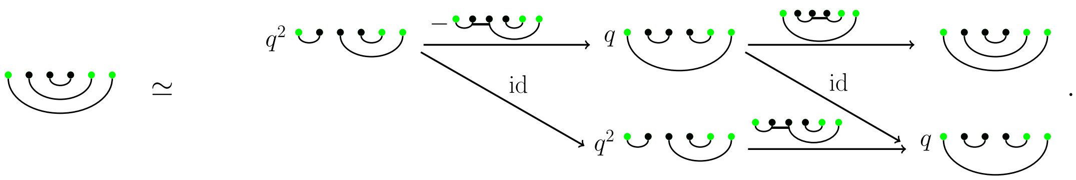
$$\left\{q^{\deg(\mathrm{C}(\lambda)\mu)}\,\mathrm{V}^*(\mu) \mid \mathrm{C}(\lambda)\mu \text{ is oriented}\right\}$$

ii) in the final complex, all $q^{\deg(\mathrm{C}(\lambda)\mu)}\,\mathrm{V}^*(\mu)$ *are right-aligned, i.e. the rightmost non-zero entry is in homological degree* 0*.*

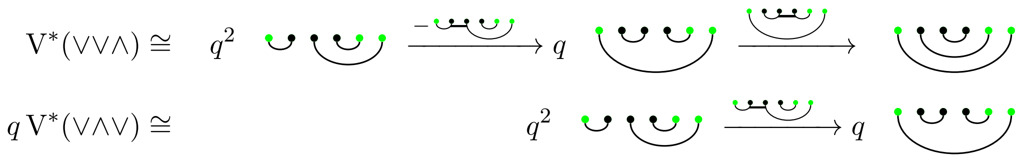
For i) recall that for an extended cup diagram $\mathrm{C}(\lambda)$ and a $\wedge\vee$-sequence μ we defined that μ orients $\mathrm{C}(\lambda)$ when all the cups get oriented (Definition 1.2.21) and the degree of this orientation is the number of clockwise oriented cups (Definition 1.2.25).

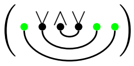
Before the proof we look at an example.

Example 7.1.4. In case $n = 3$ and $k = 1$ we have



by Gaussian elimination (Lemma 5.1.2) with respect to the two identities. Note that the complexes



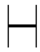
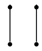
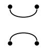
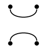
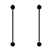
are right-aligned in the right complex, i.e. start on the right in homological degree 0. Furthermore, deg  = 0, deg  = 1 and $\wedge\vee\vee$ does not orient  .

Proof. We first show that $\mathrm{T}(\lambda)$ is an iterated cone of $\mathrm{V}^*(\mu)$'s by induction on $\ell(\lambda, \lambda_0)$. Since $\mathrm{T}(\lambda_0) = \mathrm{V}^*(\lambda_0)$, we clearly have that $\mathrm{T}(\lambda_0)$ is an iterated cone. Assume the assertion is shown for all λ' with $\ell(\lambda', \lambda_0) < \ell(\lambda, \lambda_0)$ and choose $\lambda' \xrightarrow{s_i} \lambda$. Then $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda').\mathcal{U}_i$ by Lemma 4.1.8 and by induction $\mathrm{T}(\lambda')$ is already an iterated cone, hence $\mathrm{T}(\lambda)$ is an iterated cone by Lemma 7.1.2.

We check now that properties i) and ii) hold. For λ_0 they are obviously true. We assume that $\lambda' \xrightarrow{s_i} \lambda$ and $\mathrm{T}(\lambda')$ is an iterated cone of the required form. Applying $\mathcal{U}_i$ changes the appearing $\mathrm{V}^*(\mu)$'s according to Proposition 7.1.1. By Lemma 1.2.24 and Lemma 1.2.26 (for λ' playing the role of λ in the lemmas) we obtain property i). The right alignment of property ii) follows then from Proposition 7.1.1 again, since if $\mathrm{V}^*(\mu)$ is right aligned $\mathrm{V}^*(\mu).\mathcal{U}_i$ is of the form $\mathrm{Cone}(A[-1] \to B)$ with A and B right aligned. □

7.2 Reflected complexes and categorification of $V^{\otimes n}$

In this section we will finally obtain our categorification of $V^{\otimes n}$. But beforehand we need to describe how we obtain the complexes $\mathrm{V}(\lambda)$ by reflection.

Definition 7.2.1. In $\mathrm{Cup}(n,k)$ for a cobordism $f\colon A \to B$ we denote by $f^*\colon B \to A$ the reflected cobordism. For example,  :  →  and  =  :  → .
This extends to $\mathrm{Cup}(n,k)^{\mathbb{Z}}$ via sending $f\colon A\langle i\rangle \to B\langle j\rangle$ to $f^*\colon B\langle -j\rangle \to A\langle -i\rangle$ since $\deg(f) = \deg(f^*)$. By taking $*$ on every component of the matrix of morphisms we can extend $*$ to $\mathrm{Mat}\big(\mathrm{Cup}(n,k)^{\mathbb{Z}}\big) = \widehat{\mathrm{Cup}}(n,k)$.

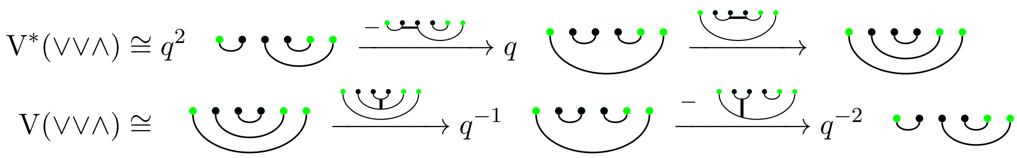
For a complex (A,d) we define the *reflected complex* A^* by setting $A^*_{-k} = \bigoplus_i a_i\langle -m_i\rangle$ if $A_k = \bigoplus_i a_i\langle m_i\rangle$ with differentials $d^*_k\colon A^*_{-k+1} \to A^*_{-k}$.
Schematically:

$$\text{complex}$$
$$\cdots \to \bigoplus_i b_i\langle m_i\rangle \xrightarrow{d_1} \bigoplus_i a_i\langle s_i\rangle \xrightarrow{d_0} \bigoplus_i c_i\langle l_i\rangle \to \dots$$

$$\text{reflected complex}$$
$$\dots \to \bigoplus_i c_i\langle -l_i\rangle \xrightarrow{d_0^*} \bigoplus_i a_i\langle -s_i\rangle \xrightarrow{d_1^*} \bigoplus_i b_i\langle -m_i\rangle \to \dots$$

We denote $\mathrm{V}^*(\lambda)^*$ by $\mathrm{V}(\lambda)$ for $\lambda \in \Lambda(n,k)$.

Example 7.2.2. For $n = 3, k = 1$ we have



where  is in homological degree 0 in both complexes.

Remark 7.2.3. For complexes A, B and a chain map $f\colon A \to B$ we obtain a chain map $f^*\colon B^* \to A^*$ by setting $(f^*)_i = (f_{-i})^*$. Then we have

$$\mathrm{Cone}(A \xrightarrow{f} B)^* = \mathrm{Cone}(B^* \xrightarrow{-f^*} A^*)[-1].$$

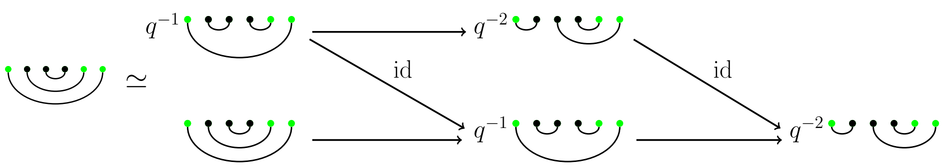
From reflecting the assertion of Theorem 7.1.3, we get that every $\mathrm{T}(\lambda)$ is an iterated cone of the same $\mathrm{V}(\mu)$'s, only shifted in the other direction:

Corollary 7.2.4. *For any $\lambda \in \Lambda(n,k)$, the object $\mathrm{T}(\lambda)$ is an iterated cone of*

$$\big\{q^{-deg(\mathrm{C}(\lambda)\mu)}\,\mathrm{V}(\mu) \mid \mathrm{C}(\lambda)\mu \text{ is oriented}\big\}$$

and in the final complex, all $q^{-\deg(\mathrm{C}(\lambda)\mu)}\,\mathrm{V}(\mu)$ are now left-aligned.

For example, for $n = 3$ and $k = 1$ we have



,

where  is in homological degree 0.

Almost directly from the definition of $K = K^b(\widehat{\text{Cup}}(n,k))$ we obtain the following statement:

Lemma 7.2.5. *Every element of K can be constructed by iteratively taking cones and homological shifts of $\{q^i\,\mathrm{T}(\lambda) \mid i \in \mathbb{Z}, \lambda \in \Lambda(n,k)\} =: \Upsilon$.*

Proof. Every entry of a complex in K is isomorphic to a direct sum of elements in Υ, which we can construct with $A \oplus B = \mathrm{Cone}(A[-1] \xrightarrow{0} B)$. From that we can construct the complexes inductively. □

Using Corollary 7.2.4, we immediately get:

Corollary 7.2.6. *Every element of K can be constructed by iteratively taking cones and homological shifts of $\{q^i\,\mathrm{V}(\lambda) \mid i \in \mathbb{Z}, \lambda \in \Lambda(n,k)\}$.*

Definition 7.2.7. Let $\mathcal{A}$ be an additive category and $\mathcal{K}(\mathcal{A})$ its homotopy category of bounded complexes. Let $K_0(\mathcal{K}(\mathcal{A}))$ be the *triangulated Grothendieck group*, i.e.

$$K_0\big(\mathcal{K}(\mathcal{A})\big) = \mathbb{Z}\left\langle Iso\big(\mathcal{K}(\mathcal{A})\big)\right\rangle \big/ \big([B] = [A] + [C] \text{ for distinguished triangles } A \to B \to C\big)$$

By $Iso\big(\mathcal{K}(\mathcal{A})\big)$ we mean isomorphism classes in $\mathcal{K}(\mathcal{A})$, i.e. classes of homotopy equivalent complexes.

As for $K_0(\mathcal{A})$, if $\mathcal{A}$ has an internal grading shift $\langle - \rangle$ we set $q^i[A] = [A\langle i\rangle]$ and this makes $K_0\big(\mathcal{K}(\mathcal{A})\big)$ into a $\mathbb{Z}[q,q^{-1}]$-module.

By definition, in $K_0\big(\mathcal{K}(\mathcal{A})\big)$ we have $[C \oplus D] = [C] + [D]$ for two complexes C,D and $[C] = \sum_{i=-\infty}^{\infty}(-1)^i[C_i]$, where C_i is seen as a complex in degree 0, and the sum is finite because the complexes are bounded. By e.g. [Ros11b] the triangulated Grothendieck group of $\mathcal{K}(\mathcal{A})$ is canonically isomorphic to the split Grothendieck of $\mathcal{A}$, i.e.

$$K_0\big(\mathcal{K}(\mathcal{A})\big) \cong K_0(\mathcal{A}). \tag{7.1}$$

Remark 7.2.8. For some $\lambda^i \in \Lambda(n,k)$, $i = 1,\ldots,t$, assume that X is an iterated cone of shifted (homologically and internally) $\mathrm{V}(\lambda^i)$ such that in the final complex $V(\lambda^i)\langle s_i\rangle[r_i]$ appears. Then from the formulas above we get the following equality in $K_0(K)$:

$$[X] = \sum (-1)^{r_i} q^{s_i}[V(\lambda^i)].$$

For our main categorification theorem recall the weight space decomposition $V^{\otimes n} = \bigoplus \left(V^{\otimes n}\right)_{2k-n}$ from Section 2.1.

Theorem 7.2.9.

a) $K^b\big(\widehat{\text{Cup}}(k,n)\big)$ *categorifies the $(2k-n)$-weight space of $V^{\otimes n}$. More precisely, there is an isomorphism of $\mathbb{C}(q)$-modules*

$$\Phi\colon \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\Big(K^b\big(\widehat{\text{Cup}}(k,n)\big)\Big) \xrightarrow{\sim} \left(V^{\otimes n}\right)_{2k-n}.$$

b) *Under this isomorphism the* $\mathrm{V}(\lambda), \lambda \in \Lambda(n,k)$*, are sent to the standard-basis* v_λ *whereas the* $\mathrm{T}(\lambda)$ *are sent to the canonical basis from Definition 2.1.5.*

Corollary 7.2.10. $\bigoplus_{k=1}^{n} K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ *categorifies the* $\mathbb{C}(q)$*-vector space* $V^{\otimes n}$ *with weight space decomposition* $\bigoplus_{k=1}^{n} (V^{\otimes n})_{2k-n}$.

Proof of Theorem 7.2.9. We already know $K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big) \cong K_0\big(\widehat{\mathrm{Cup}}(k,n)\big)$ by (7.1). Furthermore, by Theorem 3.5.4, we have an isomorphism of $\mathbb{Z}[q,q^{-1}]$-modules $K_0\big(\widehat{\mathrm{Cup}}(k,n)\big) \cong \widehat{eC}(n,k)$ which sends $[\mathrm{T}(\lambda)]$ to $\mathrm{C}(\lambda)$. After complexification and identification of $\widehat{eC}(n,k)^{\mathbb{C}}$ with $(V^{\otimes n})_{2k-n}$ we get an isomorphism $\Phi'\colon \mathbb{C}(q) \otimes_{Z[q,q^{-1}]} K_0\big(\widehat{\mathrm{Cup}}(k,n)\big) \cong (V^{\otimes n})_{2k-n}$ sending $1 \otimes [\mathrm{T}(\lambda)]$ to $v_{\heartsuit\lambda}$ by Proposition 2.1.7.
From Corollary 7.2.4 and Remark 7.2.8 we obtain

$$[\mathrm{T}(\lambda)] = \sum_{\mu : \mathrm{C}(\lambda)\mu \text{ is oriented}} q^{-\deg(\mathrm{C}(\lambda)\mu)}[\mathrm{V}(\mu)].$$

By Definition 2.1.5 we conclude that

$$\Phi\colon \mathbb{C}(q) \otimes_{Z[q,q^{-1}]} K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big) \xrightarrow{\sim} \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\big(\widehat{\mathrm{Cup}}(k,n)\big) \xrightarrow{\Phi'} \big(V^{\otimes n}\big)_{2k-n}$$

sends $1 \otimes [\mathrm{V}(\mu)]$ to the standard basis vector v_μ. □

Remark 7.2.11. Note that we still have an action of $\widehat{\mathrm{Cob}}(n)$ on $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$, which induces the action of TL_n on $V^{\otimes n}$ on the level of Grothendieck groups. Hence we have categorified the standard basis as well as the action of the Temperley Lieb algebra.

7.3 Categorified bilinear form

We now want to investigate how to obtain the bilinear form on $V^{\otimes n}$ from its categorification $K^b(\widehat{\mathrm{Cup}}(n,k))$.
Using the isomorphism $\Phi\colon \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big) \xrightarrow{\sim} (V^{\otimes n})_{2k-n}$ from Theorem 7.2.9 we define a $\mathbb{Z}[q,q^{-1}]$-bilinear form $(-,-)$ on $K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big)$ via

$$\big([M],[N]\big) := \big(\Phi(1 \otimes [M]), \Phi(1 \otimes [N])\big),$$

where the bilinear form on $V^{\otimes n}$ is defined by (2.1) from Definition 2.1.4.
The goal of this section is to prove the following theorem:

Theorem 7.3.1. *For two complexes* M, N *in* $K := K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$*, we have*

$$\big([M],[N]\big) = \sum_{i,j} (-1)^i \dim \mathrm{Hom}_K\left(M, N^* \langle j \rangle [i]\right) q^j.$$

Before the proof, we need some more properties of $\mathrm{V}(\lambda)$ which follow almost directly from the definition via reflection.

Lemma 7.3.2. *a) There is a representative of* $\mathrm{V}(\lambda)$ *in* K *which is* $\mathrm{T}(\lambda)$ *in homological degree* 0 *and consists of sums of shifted* $\mathrm{T}(\lambda')$ *with* $\lambda' < \lambda$ *in negative homological degrees.*

b) For $\lambda, \mu \in \Lambda(n,k)$ *we have*

$$\mathrm{Hom}_K\left(\mathrm{V}(\mu), \mathrm{T}(\lambda)\langle l\rangle[k]\right) \cong \begin{cases} 0 & \text{if } k \neq 0, \\ 0 & \text{if } \lambda \not\geq \mu, \\ 0 & \text{if } \lambda = \mu, l \neq 0, \\ \mathbb{C} & \text{if } \lambda = \mu, l = 0 = k. \end{cases}$$

Proof. a) For complexes A, B in K we have that $A \cong B$ yields $A^* \cong B^*$ directly by the definitions. Thus, the assertion follows from Lemma 6.1.10.

b) We have $\mathrm{Hom}_K\left(\mathrm{V}(\lambda), \mathrm{T}(\mu)\langle j\rangle[l]\right) \cong \mathrm{Hom}_K\left(\mathrm{T}(\mu)\langle -j\rangle[-l], \mathrm{V}^*(\lambda)\right)$, because every map in $\mathrm{Hom}_K\left(\mathrm{T}(\mu)\langle -j\rangle[-l], \mathrm{V}^*(\lambda)\right)$ gives one in $\mathrm{Hom}_K\left(\mathrm{V}(\lambda), \mathrm{T}(\mu)\langle j\rangle[l]\right)$ by reading the cobordism from the other direction (and vice versa). The same holds of course for homotopies. Thus, the assertion follows from Remark 6.3.17.

□

The following theorem will be used as a basic ingredient of the proof of Theorem 7.3.1.

Theorem 7.3.3. *The* $\mathrm{V}(\lambda)$ *are dual to the* $\mathrm{V}^*(\lambda)$ *in the following sense:*

$$\mathrm{Hom}_K\left(\mathrm{V}(\lambda), \mathrm{V}^*(\mu)\langle t\rangle[j]\right) \cong \begin{cases} \mathbb{C} & \text{if } j = 0 = t \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We distinguish three cases: $\lambda \not\geq \mu$, $\mu \not\geq \lambda$ and $\lambda = \mu$. Here, the case that λ and μ are unrelated is treated twice, but nevertheless all possible relations appear.

First assume $\lambda \not\geq \mu$. By Lemma 7.3.2 a), $\mathrm{V}(\lambda)$ has entries (shifted) $\mathrm{T}(\lambda')$ with $\lambda' \leq \lambda$, in particular $\lambda' \not\geq \mu$. Thus, the assertion follows from Proposition 6.3.6 using Lemma 6.3.7 inductively.

Now assume $\mu \not\geq \lambda$. By Lemma 7.3.2 b) we have $\mathrm{Hom}_K\left(\mathrm{V}(\lambda), \mathrm{T}(\mu')\langle t\rangle[j]\right) = 0$ for $\mu' \not\geq \lambda$ and $\mathrm{V}^*(\mu)$ contains only such $\mathrm{T}(\mu')$ by Lemma 6.1.10. The reflected version of Lemma 6.3.7 holds and we can use this to get the assertion inductively as before.

So only the case $\lambda = \mu$ is left to show. Let $l = \ell(\lambda, \lambda_0)$, then the rightmost non-zero entry of $\mathrm{V}(\lambda)[l]$ is in homological degree 0. The assertion is equivalent to

$$\mathrm{Hom}_K\left(\mathrm{V}(\lambda)[l], \mathrm{V}^*(\lambda)\langle t\rangle[j]\right) \cong \begin{cases} \mathbb{C} & \text{if } j - l = 0 = t, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 6.3.11 we obtain $\mathrm{Hom}_K\left((\mathrm{V}(\lambda)[l])_i, \mathrm{V}^*(\lambda)\langle t\rangle[j]\right) = 0$ for all i and all $j \neq 0$. Since by Lemma 7.3.2a) we have $\mathrm{V}(\lambda)_0 = \mathrm{T}(\lambda)$ and other $\mathrm{V}(\lambda)_i$ contain (shifted) $\mathrm{T}(\lambda')$ with $\lambda' < \lambda$, we obtain $\mathrm{Hom}_K\left((\mathrm{V}(\lambda)[l])_i, \mathrm{V}^*(\lambda)\langle t\rangle\right) = 0$ for $i \neq l$ by Proposition 6.3.6.

Thus, by Corollary 5.2.5 c), we have

$$\operatorname{Hom}_K\big(\mathrm{V}(\lambda)[l], \mathrm{V}^*(\lambda)\langle t\rangle[i]\big) \cong \begin{cases} \operatorname{Hom}_K\big(\mathrm{T}(\lambda), \mathrm{V}^*(\lambda)\langle t\rangle\big) & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$$

But by Proposition 6.3.14, we have

$$\operatorname{Hom}_K\big(\mathrm{T}(\lambda), \mathrm{V}^*(\lambda)\langle t\rangle\big) \cong \begin{cases} \mathbb{C} & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

Lemma 7.3.4. *Let A, B, X be complexes in K, $f\colon A \to B$ and fix $j \in \mathbb{Z}$. Then*

a)
$$\sum_i (-1)^i \dim \operatorname{Hom}_K\big(X, \operatorname{Cone}(f)\langle j\rangle[i]\big) = \sum_i (-1)^i \Big(\dim \operatorname{Hom}_K\big(X, B\langle j\rangle[i]\big) - \dim \operatorname{Hom}_K\big(X, A\langle j\rangle[i]\big)\Big)$$

b)
$$\sum_i (-1)^i \dim \operatorname{Hom}_K\big(\operatorname{Cone}(f), X\langle j\rangle[i]\big) = \sum_i (-1)^i \Big(\dim \operatorname{Hom}_K\big(B, X\langle j\rangle[i]\big) - \dim \operatorname{Hom}_K\big(A, X\langle j\rangle[i]\big)\Big)$$

Proof. We only prove a), since b) is analogous. From the exact triangle $A\langle j\rangle \to B\langle j\rangle \to \operatorname{Cone}(f)\langle j\rangle \to A\langle j\rangle[1]$, we obtain the the long exact sequence

$$\cdots \to \operatorname{Hom}_K\big(X, A\langle j\rangle\big) \to \operatorname{Hom}_K\big(X, B\langle j\rangle\big) \to \operatorname{Hom}_K\big(X, \operatorname{Cone}(f)\langle j\rangle\big) \to \operatorname{Hom}_K\big(X, A\langle j\rangle[1]\big) \to \operatorname{Hom}_K\big(X, B\langle j\rangle[1]\big) \to \dots.$$

Thus, by calculating dimensions using that the complexes are bounded, we get

$$0 = \sum_i (-1)^i \Big(\dim \operatorname{Hom}_K\big(X, A\langle j\rangle[i]\big) - \dim \operatorname{Hom}_K\big(X, B\langle j\rangle[i]\big) + \dim \operatorname{Hom}_K\big(X, \operatorname{Cone}(f)\langle j\rangle[i]\big)\Big).$$

□

Proof of Theorem 7.3.1. Let $\vartheta(M, N) := \sum_{i,j} (-1)^i \dim \operatorname{Hom}_K\big(M, N^*\langle j\rangle[i]\big) q^j$.

We first show $\big([M], [N]\big) = \vartheta(M, N)$ for $M = \mathrm{V}(\lambda)$ and $N = \mathrm{V}(\mu)$. By Theorem 7.2.9 we have

$$\big([\mathrm{V}(\lambda)], [\mathrm{V}(\mu)]\big) = (v_\lambda, v_\mu) = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, by Theorem 7.3.3 we know

$$\vartheta\big(\,\mathrm{V}(\lambda), \mathrm{V}(\mu)\big) = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we get equality in this basic case.

Now we want to show the equality for general complexes in K using Corollary 7.2.6. For that, we consider how $([-],[-])$ and $\vartheta(-,-)$ change under homological and internal shifts and cones.

We obviously have $\vartheta\big(M, N\langle t\rangle\,\big) = q^t\vartheta(M,N) = \vartheta\big(M\langle t\rangle, N\big)$ and $\vartheta\big(M, N[t]\big) = (-1)^t\vartheta(M,N) = \vartheta\big(M[t], N\big)$. Also, $\big([M],[N\langle t\rangle]\big) = q^t\big([M],[N]\big) = \big([M\langle t\rangle],[N]\big)$ and $\big([M],[N[t]]\big) = (-1)^t\big([M],[N]\big) = \big([M[t]],[N]\big)$ holds directly by the definition and the bilinearity.

Furthermore, using Lemma 7.3.4 and Remark 7.2.3, for complexes X, A, B and $f\colon A \to B$ we get

$$\begin{aligned}
\vartheta\big(X, \mathrm{Cone}(A \xrightarrow{f} B)\big) &= (-1)\sum_{i,j}(-1)^i \dim\mathrm{Hom}\Big(X, \mathrm{Cone}\left(B^* \xrightarrow{-f^*} A^*\right)\langle j\rangle\,[i]\Big)q^j \\
&= (-1)\sum_{i,j}(-1)^i\Big(\dim\mathrm{Hom}_K\left(X, A^*\langle j\rangle\,[i]\right) - \dim\mathrm{Hom}_K\left(X, B^*\langle j\rangle\,[i]\right)\Big)q^j \\
&= \vartheta(X,B) - \vartheta(X,A)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta\big(\,\mathrm{Cone}(A \xrightarrow{f} B), X\big) &= \sum_{i,j}(-1)^i \dim\mathrm{Hom}_K\Big(\,\mathrm{Cone}\left(A \xrightarrow{f} B\right), X^*\langle j\rangle\,[i]\Big)q^j \\
&= \sum_{i,j}(-1)^i\Big(\dim\mathrm{Hom}_K\left(B, X^*\langle j\rangle\,[i]\right) - \dim\mathrm{Hom}_K\left(A, X^*\langle j\rangle\,[i]\right)\Big)q^j \\
&= \vartheta(B,X) - \vartheta(A,X).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\big([X], [\mathrm{Cone}(A \xrightarrow{f} B)]\big) = \big([X],[B]-[A]\big) = \big([X],[B]\big) - \big([X],[A]\big) \quad \text{and} \\
&\big([\mathrm{Cone}(A \xrightarrow{f} B)], [X]\big) = \big([B]-[A],[X]\big) = \big([B],[X]\big) - \big([A],[X]\big).
\end{aligned}$$

Therefore, the equality for general complexes follows inductively. □

Remark 7.3.5. Theorem 7.3.1 together with Theorem 7.3.3 shows that under the categorification isomorphism $\Phi\colon \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big) \xrightarrow{\sim} (V^{\otimes n})_{2k-n}$ the class of $\mathrm{V}^*(\lambda)$ gets sent to the dual of the standard basis element v_λ. Hence, the $\mathrm{V}^*(\lambda)$ categorify the dual standard basis.

Chapter 8

Two t-structures on $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$

In this chapter, we show that $\mathrm{V}^*(\lambda)$ is a linear complex, almost exact in some precise sense and contained in the heart of two different t-structures. Up to now we have worked with additive categories, but these t-structures allow us to consider the abelian subcategory given by the heart. For the first t-structure, considering a functor F we define the notion of F-exactness and F-homology to construct an analog of the standard t-structure using F-homology. The second t-structure is built by measuring how far a complex is away from being linear. We study the properties of $\mathrm{T}(\lambda)$ in the two hearts.

8.1 $\mathrm{V}^*(\lambda)$ as a linear complex

The linear complexes which we define now will turn out to form an abelian subcategory of $K = K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$. Linear complexes can be defined in a general setting [MOS09], but we restrict ourselves to the context we need.

Definition 8.1.1. We say that $X \in Ch^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ is a *linear complex* if for all i we have $X_i = \bigoplus_j \mathrm{T}(\lambda_j^i)\langle i\rangle$ for some $\lambda_j^i \in \Lambda(n,k)$.

Before we can show that the $\mathrm{V}^*(\lambda)$'s are homotopy equivalent to linear complexes, we need some preparations.

Definition 8.1.2. Let $F : \widehat{\mathrm{Cup}}(n,k) \to \mathrm{Vsp}$ be a functor, where Vsp stands for the category of finite dimensional vector spaces. A complex C is called *F-exact*, if $F(C)$ is an exact chain complex of vector spaces. The complex C is *F-exact at C_j* if $F(C)$ is exact at $F(C)_j$.

Theorem 8.1.3. *For all $\lambda,\mu \in \Lambda(n,k)$ and all $l \in \mathbb{Z}$ the complex $\mathrm{V}^*(\lambda)$ is $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}\left(\mathrm{T}(\mu)\langle l\rangle, -\right)$-exact at all $\mathrm{V}^*(\lambda)_j$ except at $\mathrm{V}^*(\lambda)_0$.*

Proof. We show that for $i \neq 0$ we have $H_i\Big(\mathrm{Hom}_{\widehat{\mathrm{Cup}}}\left(\mathrm{T}(\mu)\langle l\rangle, \mathrm{V}^*(\lambda)\right)\Big) = 0$. By Remark 5.2.4 we have $H_i\Big(\mathrm{Hom}_{\widehat{\mathrm{Cup}}}\left(\mathrm{T}(\mu)\langle l\rangle, \mathrm{V}^*(\lambda)\right)\Big) \cong \mathrm{Hom}_K\left(\mathrm{T}(\mu)\langle l\rangle[i], \mathrm{V}^*(\lambda)\right)$, which in turn is zero for $i \neq 0$ by Proposition 6.3.11. □

Lemma 8.1.4. *Let $C = (C_*, d_*)$ be a complex in $K^b(\widehat{\mathrm{Cup}}(n,k))$ such that all differentials are degree 1 maps and all cup diagrams contained in C have at most one circle. Let A be a (shifted) cup diagram containing a circle and assume A is a summand of C_j. Assume C is $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}(A,-)$-exact at C_j. Then there is a (non-zero) part of the differential d leaving A, i.e. there is a summand X of C_{j-1} such that the restriction of d_j to $A \to X$ is nonzero.*

Proof. Assume there is no map leaving A and write $C_j = A \oplus B$. Then

$$d_j = (0, g) : A \oplus B = C_j \to C_{j-1}$$

for some $g : B \to C_{j-1}$. By assumption

$$\mathrm{Hom}(A, C_{j+1}) \xrightarrow{d^{j+1}} \mathrm{Hom}(A, C_j) \xrightarrow{d^j} \mathrm{Hom}(A, C_{j-1})$$

is exact in the middle, where $d^m = (d_m \circ -)$. At $\mathrm{Hom}(A, C_j)$ the kernel of d^j equals

$$\begin{aligned} &\big\{(a,b) \in \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(A,B) \cong \mathrm{Hom}(A,C_j) \mid (0 \circ a, g \circ b) = 0\big\} \\ =&\big\{(a,b) \in \mathrm{Hom}(A,A) \oplus \mathrm{Hom}(A,B) \mid g \circ b = 0\big\} \\ =&\,\mathrm{Hom}(A,A) \oplus \ker d^j|_{\mathrm{Hom}(A,B)}. \end{aligned}$$

So in particular $(\mathrm{id}, 0)$ has to be in the image of d^{j+1}. We choose $f \in \mathrm{Hom}(A, C_{j+1})$ such that $d^{j+1}(f) = (\mathrm{id}, 0)$.

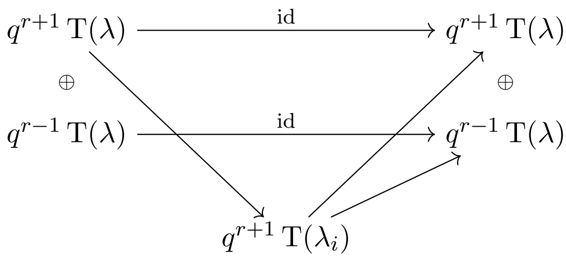
By assumption, $A \cong q^r\,\mathrm{T}(\lambda) \sqcup \bigcirc$ for some r and λ and $C_{j+1} = D_1 \oplus \cdots \oplus D_p$ with $D_i \cong q^{r_i}\,\mathrm{T}(\lambda_i)$ or $D_i \cong q^{r_i}\,\mathrm{T}(\lambda_i) \sqcup \bigcirc$ for some r_i and λ_i. We choose the labelling such that $d_{j+1}|_{D_i} = 0$ precisely for $l \le i \le p$. Since by assumption the differentials are degree 1 maps, by Remark 3.1.3 we have $r_i = r+1$ for all $1 \le i \le l$.

Write $f = (f_1, \ldots, f_p) : A \to D_1 \oplus \cdots \oplus D_p$. Then

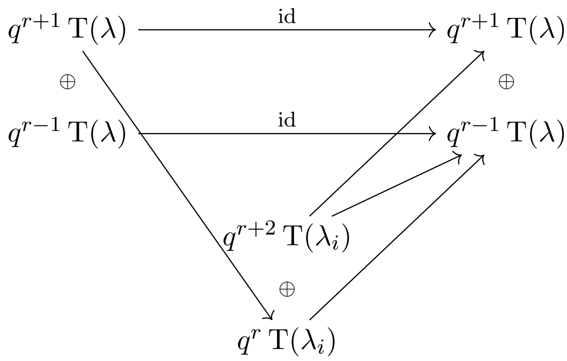
$$\begin{array}{ccc} A & \xrightarrow{\quad\mathrm{id}\quad} & A \\ & {\scriptstyle (f_1,\ldots,f_l)}\searrow \quad \nearrow{\scriptstyle d_{j+1}} & \\ & \bigoplus_{i=1}^{l} D_i & \end{array} \tag{8.1}$$

commutes since $d^{j+1}(f) = (\mathrm{id}, 0)$.

Using delooping (Lemma 3.3.5) we identify $A \cong q^{r+1}\,\mathrm{T}(\lambda) \oplus q^{r-1}\,\mathrm{T}(\lambda)$ and consider the maps induced by the maps in the previous diagram for each $1 \le i \le l$. There are two cases: Either $D_i \cong q^{r+1}\,\mathrm{T}(\lambda_i)$ or $D_i \cong q^{r+1}\,\mathrm{T}(\lambda_i) \sqcup \bigcirc \cong q^{r+2}\,\mathrm{T}(\lambda_i) \oplus q^r\,\mathrm{T}(\lambda_i)$. By Lemma 3.4.14 and Remark 3.1.3 there is only a map $q^x\,\mathrm{T}(\mu) \to q^y\,\mathrm{T}(\nu)$ if $x \ge y$. Thus using delooping, the diagram $\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ & {\scriptstyle f_i}\searrow \ \nearrow{\scriptstyle d_{j+1}} & \\ & D_i & \end{array}$ is either isomorphic to



or to



,

where no arrow means the map is zero. Hence, we have $d_{j+1} \circ f_i|_{q^{r-1}\,\mathrm{T}(\lambda)} = 0$ for all i and also

$$\sum_{i=1}^{l} d_{j+1} \circ f_i|_{q^{r-1}\,\mathrm{T}(\lambda)} = 0.$$

But this is a contradiction to the commutativity of (8.1).

□

Theorem 8.1.5. $\mathrm{V}^*(\mu)$ *is homotopy equivalent to a linear complex.*

Before the proof we want to look at an example.

Example 8.1.6. A complex of the form

$$q^2\,\mathrm{T}(\mu) \to q\,\mathrm{T}(\mu) \sqcup \bigcirc \to \mathrm{T}(\mu) \oplus \mathrm{T}(\nu)$$

with $\mathrm{T}(\mu)\oplus\mathrm{T}(\nu)$ in homological degree 0 is not a linear complex, since $\mathrm{T}(\mu)\sqcup\bigcirc$ appears. Using delooping (Lemma 3.3.5) we see that the complex is isomorphic to

$$q^2\,\mathrm{T}(\mu) \to q^2\,\mathrm{T}(\mu) \oplus \mathrm{T}(\mu) \to \mathrm{T}(\mu) \oplus \mathrm{T}(\nu).$$

Now, all the entries are sums of shifted $\mathrm{T}(\mu')$, but the shifts do not match the homological degrees, so the new complex is still not linear. If we assume that the differentials in the original complex were saddles such that delooping creates identities in the differentials between the two copies of $q^2\,\mathrm{T}(\mu)$ and between the two copies of $\mathrm{T}(\mu)$, then we can apply Gaussian elimination (Lemma 5.1.2) with respect to those identities to obtain a homotopy equivalent complex of the form $0 \to 0 \to \mathrm{T}(\nu)$ which now is clearly linear.

The important point of the following proof is to ensure that when we resolve circles, there are enough identity maps to eliminate the two factors that come from delooping.

Proof of Theorem 8.1.5. We show this by induction on $\ell(\mu, \lambda_0)$. For $\mu = \lambda_0$ it is obviously true. For the following note that since $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}(C, -)$-exactness is homotopy invariant, it is preserved under Gaussian elimination. Furthermore, if we talk about a "map", we always mean some $f : q^i C \to q^j D$ for $C, D \in \mathrm{Cup}(n,k)$, which is part of the differential of the complex.

Consider μ such that the assertion is true for all μ' with $\ell(\mu', \lambda_0) < \ell(\mu, \lambda_0)$. Choose a μ' such that $\mu' \xrightarrow{s_i} \mu$. Before we start, we assign to each summand of $\mathrm{V}^*(\mu) = \mathrm{Cone}\left(q\,\mathrm{V}^*(\mu') \to \mathrm{V}^*(\mu).\mathcal{U}_i\right)$ the label "upper" or "lower" depending on whether it is a summand of the homological shift of $q\,\mathrm{V}^*(\mu')$ or of $\mathrm{V}^*(\mu').\mathcal{U}_i$. By induction, $\mathrm{V}^*(\mu')$ is linear and thus the partcomplex $q\,\mathrm{V}^*(\mu')[1]$ of $\mathrm{V}^*(\mu)$ is linear. In particular, upper summands do not contain circles. Lower summands arise from the application of $\mathcal{U}_i$ to $\mathrm{V}^*(\mu')$, thus they are either a $q^m\,\mathrm{T}(\nu)$ or contain one circle. If the lower summands contain a circle, they are of the form $q^m X = q^m\,\mathrm{T}(\nu) \sqcup \bigcirc = q^m\,\mathrm{T}(\nu).\mathcal{U}_i$. Here $\mathrm{T}(\nu).\mathcal{U}_i = \mathrm{T}(\nu) \sqcup \bigcirc$ hold by Lemma 4.1.8.

We inductively construct a homotopic complex that is linear by reducing step by step the number of circle-summands of $\mathrm{V}^*(\mu)_j$ for minimal j.

After each reduction step, the following conditions $(*)$ hold:

- every summand that still exists has already been a summand at the start (i.e. summands are only deleted, not changed or added)
- for each lower summand of $\mathrm{V}^*(\mu)_l$, $l \geq j$, there is exactly one upper summand with a map from the upper summand to the lower summand and it is of the form $q^{r+1}\,\mathrm{T}(\tau) \xrightarrow{\pm \mathrm{H}_i} q^r\,\mathrm{T}(\tau).\mathcal{U}_i$ for some τ
- all maps are saddles up to scalars

By induction and construction of $\mathrm{V}^*(\mu)$ as a cone this is obviously true in the beginning.

Now let j be minimal such that $\mathrm{V}^*(\mu)_j$ has a summand containing a circle. By construction of $\mathrm{V}^*(\mu)$ this summand is some $q^j X$ with $X = \mathrm{T}(\nu).\mathcal{U}_i$, i.e. $q^j X$ is a lower summand. By Lemma 8.1.4 we know that there is a map leaving $q^j X$.

By condition (*) there is exactly one map α entering $q^j X$ that is coming from an upper summand and it is of the form $q^{r+1}\,\mathrm{T}(\nu) \xrightarrow{\pm \mathrm{H}_i} q^r\,\mathrm{T}(\nu).\mathcal{U}_i$. When we resolve the circle in $q^j X$ using delooping (Lemma 3.3.5) we get $q^{j-1}\,\mathrm{T}(\nu) \oplus q^{j+1}\,\mathrm{T}(\nu)$ and α composed with the delooping isomorphism is an isomorphism onto the summand $q^{j+1}\,\mathrm{T}(\nu)$. Using Gaussian elimination (Lemma 5.1.2) we can delete $q^{j+1}\,\mathrm{T}(\nu) \xrightarrow{\cong} q^{j+1}\,\mathrm{T}(\nu)$. Since $q^j X$ has only this one entering map coming from an upper summand, by Gaussian elimination only maps that go from lower to upper summands are changed, where the lower summand is a summand of $\mathrm{V}^*(\mu)_{j+1}$. These new maps are still saddles (up to scalar), since in the Gaussian elimination process we compose two degree 0 maps with a saddle.

In particular, the map leaving $q^j X$ composed with the resolving-isomorphism is still there. Let Z be its target. By minimality assumption $Z = q^{j-1}\,\mathrm{T}(\nu')$ and the map

$q^jX \to q^{j-1}\,\mathrm{T}(\nu')$ was a saddle. Since X contains a circle and $\mathrm{T}(\nu')$ does not, the saddle s has to connect the circle to another component. Composing this with $\bigcirc$: $q^{j-1}\,\mathrm{T}(\nu) \to q^jX$, which is part of the resolving-isomorphism, we get a non-zero degree 0 map, i.e. an isomorphism $q^{j-1}\,\mathrm{T}(\nu) \to q^{j-1}\,\mathrm{T}(\nu')$. (Note that by Lemma 3.4.14 this means $\nu = \nu'$.)

So we can delete $q^{j-1}\,\mathrm{T}(\nu) \to q^{j-1}\,\mathrm{T}(\nu')$ by Gaussian elimination. This only changes maps going from $\mathrm{V}^*(\mu)_j$ to $\mathrm{V}^*(\mu)_{j-1}$ and does this by composing two degree 0 morphisms with a saddle. Therefore, all morphisms are still saddles (up to scalar). Hence, after each step the conditions $(*)$ are still satisfied.

We iterate until no summand with a circle is left. Since all the surviving summands are summands already present at the start, they have the right degree. Note that the elimination process never eliminates $\mathrm{V}^*(\mu)_0 \cong \mathrm{T}(\mu)$: From the process above that elimination would mean that after delooping $\mathrm{V}^*(\lambda)_1$ contains a summand $\mathrm{T}(\lambda)$. But we know from Lemma 6.1.10 that this cannot be true. □

Since $\mathrm{V}(\lambda)$ is just the reflected complex (cf. Definition 7.2.1) and by definition the reflected complex of a linear complex is again linear, we immediately get

Corollary 8.1.7. $\mathrm{V}(\lambda)$ *is homotopy equivalent to a linear complex.*

8.2 Generalities on t-structures

We start by recalling the definition of a t-structure in a triangulated category, for a more detailed treatment see e.g. [KS94].

Definition 8.2.1. Let $\mathcal{D}$ be a triangulated category and let $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ be full subcategories. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a *t-structure on* $\mathcal{D}$ if the following conditions are satisfied with $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$:

(T1) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$

(T2) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathrm{ob}(\mathcal{D}^{\leq 0})$ and $Y \in \mathrm{ob}(\mathcal{D}^{\geq 1})$

(T3) For any $X \in \mathrm{ob}(\mathcal{D})$, there exists a distinguished triangle $X^0 \to X \to X^1 \to X^0[1]$ in $\mathcal{D}$ with $X^0 \in \mathrm{ob}(\mathcal{D}^{\leq 0})$ and $X^1 \in \mathrm{ob}(\mathcal{D}^{\geq 1})$.

The full subcategory $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure.

The heart has the following important properties:

Proposition 8.2.2. *The heart is an abelian category. A sequence* $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ *in the heart is exact if and only if there exists a distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ *in* $\mathcal{D}$*.*

Proof. See e.g. [KS94, Prop 10.1.11] resp. [KW01, Theorem II.3.1]. □

For later use we need the following property of Ext^1 in the heart, see e.g. [KW01, Lemma II.3.2]:

Lemma 8.2.3. *Let $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, then for $X, Y \in \mathcal{C}$ we have*

$$\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, Y[1]).$$

Remark 8.2.4. The isomorphism φ of Lemma 8.2.3 can be defined as follows: An element a of $\mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ is given by an exact sequence $0 \to Y \to Z \to X \to 0$. By [KS94, Proposition 10.1.11 (iii)] there is a unique $h : X \to Y[1]$ such that $Y \to Z \to X \xrightarrow{h} Y[1]$ is a distinguished triangle in $\mathcal{D}$. We define the map φ by sending a to h.

We will now construct two different t-structures in the triangulated category $K = K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$. The first one should be viewed as an analog of the standard t-structure in the bounded derived category $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$. Since we neither start with an abelian category nor work with the derived category of some module category this construction requires some work.

8.3 The homological t-structure

The first of the two t-structures is build using homology functors (measuring F-exactness).

Definition 8.3.1. Let $F : \widehat{\mathrm{Cup}}(n,k) \to \mathrm{Vsp}$ be a functor. The *F-homology* of a complex $X \in Ch\big(\widehat{\mathrm{Cup}}(n,k)\big)$ is the family of vector spaces $H^F_*(X)$ defined by

$$H^F_i(X) = H_i\big(F(X)\big),$$

which is the ith homology of the complex $F(X)$.

We are mainly interested in the case where F is the functor

$$F(-) = \bigoplus_{\substack{\lambda \in \Lambda(n,k)\\ i \in \mathbb{Z}}} \mathrm{Hom}_{\widehat{\mathrm{Cup}}}(\mathrm{T}(\lambda)\langle i\rangle, -). \tag{8.2}$$

Lemma 8.3.2. *For $X \in K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ and $\lambda \in \Lambda(n,k)$ there are only finitely many $i \in \mathbb{Z}$ such that $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}\big(\mathrm{T}(\lambda)\langle i\rangle, X\big) \neq 0$. Therefore, $F(X)$ is a finite sum for F as in* (8.2).

Proof. By Theorem 3.4.12 all $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\big(\mathrm{T}(\lambda), \mathrm{T}(\mu)\big)$ are finite dimensional, and since there are only finitely many λ and μ, the maximal degree of maps is bounded by some b. By delooping (Lemma 3.3.5) X is isomorphic to a complex containing only direct sums of shifted $q^r\,\mathrm{T}(\lambda)$'s for some $r \in \mathbb{Z}$ and some $\lambda \in \Lambda(n,k)$. Since X is bounded, the appearing q-shifts in X are bounded above by m_a and below by m_b. Hence, if we take i larger than $b + m_a$, then $\mathrm{Hom}_{\widehat{\mathrm{Cup}}}(\mathrm{T}(\lambda)\langle i\rangle, X) = 0$ for all i. Since degrees of maps are non-negative by Lemma 3.4.14, we obtain the same result when taking $i < m_b$. The second assertion follows from the finiteness of $\Lambda(n,k)$. □

From now on we consider the triangulated category $\mathcal{D} = K = K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ together with the functor F from (8.2). We define subcategories for the t-structure in analogous fashion as for the standard t-structure, where we have to note that the inequalities look slightly different since we denote our complexes homologically and not cohomologically. Let $\mathcal{D}^{\geq 0}$ be the full subcategory of $\mathcal{D}$ with objects complexes X with $H_i^F(X) = 0$ for $i > 0$; $\mathcal{D}^{\leq 0}$ is defined analogously with < 0.

Our goal is to show that this gives a t-structure. But before doing this, we need to understand $\mathcal{D}^{\leq 0}$ better.

Lemma 8.3.3. *Let X be an object in $\mathcal{D}^{\leq 0}$. Then $X \simeq Z$ for some complex Z with $Z_j = 0$ for all $j < 0$.*

Proof. By Gaussian elimination (Lemma 5.1.2), the complex X is homotopic to a complex Z that contains no isomorphisms as a part of the differential, i.e. written in the matrix form of Definition 3.1.2 the entries of the differential matrices are never isomorphisms. Since X is an object of $\mathcal{D}^{\leq 0}$, the same holds for Z. Let m be minimal such that $Z_m \neq 0$ and assume that $m < 0$. Since $Z_m \neq 0$ there are some $\lambda^i \in \Lambda(n,k)$ and $l_i \in \mathbb{Z}$ such that $Z_m \cong \bigoplus_{i=1}^r \mathrm{T}(\lambda^i)\langle l_i\rangle$. Now $\mathrm{Hom}_K\big(\mathrm{T}(\lambda^1)\langle l_1\rangle[m], Z\big)$ contains the map given by $\mathrm{T}(\lambda^1)\langle l_1\rangle \hookrightarrow Z_m$ since by minimality of m this is a chain map:

$$\begin{array}{ccccccc}
\mathrm{T}(\lambda_1)\langle l_1\rangle[m] & = & & 0 & \longrightarrow & \mathrm{T}(\lambda_1)\langle l_1\rangle & \\
\downarrow & & & \downarrow & & \downarrow & \\
Z & = & \ldots \longrightarrow & Z_{m+1} & \longrightarrow & Z_m & \longrightarrow 0
\end{array}$$

It is not homotopic to zero, since there are no isomorphisms in the differential of Z. Thus $H_m(\mathrm{Hom}_{\mathrm{Cup}}(\mathrm{T}(\lambda^1)\langle l_1\rangle, Z)) \cong \mathrm{Hom}_K(\mathrm{T}(\lambda^1)\langle l_1\rangle[m], Z) \neq 0$. But this contradicts $H_m^F(Z) = 0$ for $m < 0$. □

Corollary 8.3.4. *(T2) for t-structures holds.*

Proof. Let $X \in \mathrm{ob}(\mathcal{D}^{\leq 0})$ and $Y \in \mathrm{ob}(\mathcal{D}^{\geq 1}) = \mathrm{ob}(\mathcal{D}^{\geq 0})$. Then $\mathrm{Hom}_K(X,Y) \cong \mathrm{Hom}_K(Z,Y)$ for Z as in the lemma above. By assumption, $H_i^F(Y) = 0$ for all $i > -1$, thus $\mathrm{Hom}_K\big(\mathrm{T}(\lambda)\langle j\rangle[i], Y\big) \cong H_i\big(\mathrm{Hom}_{\widehat{\mathrm{Cup}}}(\mathrm{T}(\lambda)\langle j\rangle, Y)\big) = 0$ for all $i \geq 0$, all $j \in \mathbb{Z}$ and all $\lambda \in \Lambda(n,k)$. Since $Z_i = 0$ for $i < 0$ we can use Lemma 6.3.7 analogously to the proof of Proposition 6.3.8 to obtain $\mathrm{Hom}_K(Z,Y) = 0$ and we are done. □

Proposition 8.3.5. *(T3) for t-structures holds.*

Proof. Let X be a complex in K. If $H_0^F(X) \neq 0$ we have $\mathrm{Hom}_K\big(\mathrm{T}(\lambda^i)\langle l_i\rangle[0], X\big) \neq 0$ for finitely many choices $(\lambda^i, l_i) \in \Lambda(n,k) \times \mathbb{Z}$, say $1 \leq i \leq m$. For each such pair (λ^i, l_i) let $f_i^1, \ldots, f_i^{r_i}$ be a basis of $\mathrm{Hom}_K\big(\mathrm{T}(\lambda^i)\langle l_i\rangle[0], X\big)$. We now enlarge X to a complex B constructed inductively, such that $B \in \mathrm{ob}(\mathcal{D}^{\geq 1})$. Let $B_1 = X_1 \oplus A_0$ for

$$A_0 = \bigoplus_i \bigoplus_{j=1}^{r_i} \mathrm{T}(\lambda^i)\langle l_i\rangle.$$

Then,

$$B^0 = \cdots \to X_3 \to X_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} X_0 \to X_{-1} \to \ldots$$

with differentials as in X except for $d_2|_{X_2 \to A_0} = 0$, $d_2|_{X_2 \to X_1} = d_2^X$, $d_1|_{X_1 \to X_0} = d_1^X$ and

$$d_1|_{\bigoplus_{j=1}^{r_i} \mathrm{T}(\lambda^i)\langle l_i \rangle \to X_0} = (f_i^1, \dots, f_i^{r_i})$$

for all i is a complex, since the f_i^j are chain maps. Now $\mathrm{Hom}_K(\mathrm{T}(\lambda^i)\langle l_i \rangle [0], B^0) = 0$ for all i since all the possible maps into X_0 factorise over B_1 via maps of the form $c \cdot \mathrm{id}$, thus $H_0^F(B^0) = 0$. We continue to do the same procedure for 1 instead of 0 and B^0 instead of X and obtain a complex B^1 with $H_j^F(B^1) = 0$ for $j = 0, 1$. Iterating until we reach $m = \max\{l \mid X_l \neq 0\}$ we obtain a complex $B = B^m$ with $H_j^F(B) = 0$ for all $j \geq 0$. Therefore, we have $B \in \mathrm{ob}(\mathcal{D}^{\geq 1})$. Let A be the partcomplex given by the A_j, $j = 0, \dots, m$ but now starting in homological degree 0. By construction, A is concentrated in non-negative homological degrees, thus $A \in \mathrm{ob}(\mathcal{D}^{\leq 0})$. Let $g : B[-1] \to A$ be the chain map given by $g_j|_{X_{j+1} \to A_j} = 0$, $g_j|_{A_j \to A_j} = \mathrm{id}$. Then, for example by iterated Gaussian elimination (Lemma 5.1.2) starting from the left of the complex $\mathrm{Cone}(g)$ we obtain finally $X \simeq \mathrm{Cone}(g)$. In particular, X fits into a distinguished triangle $B[-1] \to A \to X \to B$. By rotation we get the desired triangle $A \to X \to B \to A[1]$. □

Since condition (T1) of the t-structure is clearly satisfied we obtain in total:

Theorem 8.3.6. $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *is a t-structure on* $\mathcal{D}$.

We call this t-structure the *homological t-structure on* K and denote by $\mathcal{C}^h$ its heart.

Proposition 8.3.7. *The* $q^r\,\mathrm{V}^*(\lambda)$*'s are contained in the heart* $\mathcal{C}^h$ *for every* r.

Proof. This follows directly from Theorem 8.1.3. □

Our next goal is to justify our notation $\mathrm{T}(\lambda)$ by showing that these objects are tilting objects in $\mathcal{C}^h$ in the sense of [Soe99]. We start by verifying the assumptions from there.

Proposition 8.3.8. *In the heart* $\mathcal{C}^h$ *of the homological t-structure the* $\mathrm{V}^*(\lambda)\langle j \rangle$*,* $\lambda \in \Lambda(n,k)$*,* $j \in \mathbb{Z}_{\geq 0}$*, are indecomposable objects such that*

1. $\mathrm{Hom}_{\mathcal{C}^h}\big(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle\big) = 0$ *unless* $\lambda \geq \mu$*,*
2. $\mathrm{Ext}^1_{\mathcal{C}^h}\big(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle\big) = 0$ *unless* $\lambda > \mu$*,*
3. $\dim \mathrm{Hom}_{\mathcal{C}^h}\big(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle\big) < \infty$*, and*
   $\dim \mathrm{Ext}^1_{\mathcal{C}^h}\big(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle\big) < \infty$ *for all* λ, μ, m, j.

Proof. By Theorem 6.3.2 we know

$$\mathrm{End}\big(\mathrm{V}^*(\lambda)\langle j \rangle\big) = \mathrm{End}\big(\mathrm{V}^*(\lambda)\big) = \mathbb{C}$$

which is local, thus $\mathrm{V}^*(\lambda)\langle j \rangle$ is indecomposable. Since

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^h}(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle) &= \mathrm{Hom}_K(\mathrm{V}^*(\lambda)\langle m \rangle, \mathrm{V}^*(\mu)\langle j \rangle)\\ &= \mathrm{Hom}_K(\mathrm{V}^*(\lambda)\langle m - j \rangle, \mathrm{V}^*(\mu)),\end{aligned}$$

the first condition holds by Theorem 6.3.9. For the second condition note that by Lemma 8.2.3 we know

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{C}^h}(\mathrm{V}^*(\lambda)\langle m\rangle, \mathrm{V}^*(\mu)\langle j\rangle) &= \mathrm{Hom}_K(\mathrm{V}^*(\lambda)\langle m\rangle, \mathrm{V}^*(\mu)\langle j\rangle[1]) \\ &\cong \mathrm{Hom}_K(\mathrm{V}^*(\lambda)\langle m-j\rangle[-1], \mathrm{V}^*(\mu))\end{aligned}$$

which is zero unless $\lambda \geq \mu$ by Theorem 6.3.9. By Lemma 6.3.10 we also see that it is zero for $\lambda = \mu$, so we have the second condition.

By Theorem 3.4.12 $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\left(\mathrm{T}(\nu), \mathrm{T}(\nu')\right)$ is finite dimensional. Since $\mathrm{V}^*(\lambda)$ can be represented by a bounded complex of shifted $\mathrm{T}(\nu)$'s by Lemma 6.1.10, $\mathrm{Hom}_K\left(\mathrm{V}^*(\lambda)\langle j\rangle, \mathrm{V}^*(\mu)\right)$ is contained in a finite product of $\mathrm{Hom}_{\mathrm{Cup}(n,k)}\left(\mathrm{T}(\nu), \mathrm{T}(\nu')\right)$'s and hence also finite. By the same argument using Lemma 8.2.3 the other part of the third condition follows, too. □

Definition 8.3.9. Analogously to [Soe99] we say that an object M in the heart $\mathcal{C}^h$ *admits a finite* V^**-flag* if $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that $M_i/M_{i-1} \cong q^s\,\mathrm{V}^*(\lambda^i)$ for some $\lambda^i \in \Lambda(n,k)$ and some $s \in \mathbb{Z}$. Moreover, the V^*-flag ends with $\mathrm{V}^*(\lambda)$ if $\mathrm{V}^*(\lambda) \cong M_r/M_{r-1}$.

Proposition 8.3.10. *The* $\mathrm{T}(\lambda)$*'s are indecomposable tilting objects in the sense that they are the unique indecomposable objects in* $\mathcal{C}^h$ *satisfying*

(a) $\mathrm{Ext}^1_{\mathcal{C}^h}\left(\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\langle j\rangle\right) = 0$ *for all* $\nu \in \Lambda(n,k)$ *and all* $j \in \mathbb{Z}$

(b) $\mathrm{T}(\lambda)$ *admits a finite* V^**-flag ending with* $\mathrm{V}^*(\lambda)$.

Proof. Here we want to apply the dual version of [Soe99, Proposition 3.1] under the condition of Proposition 8.3.8. The proof works analogously to the proof of [Soe99, Proposition 3.1]: As induction start one takes $T(\lambda) = \mathrm{V}^*(\lambda)$ for λ minimal. In the induction step, instead of reducing the set $\{\Delta(\nu) \mid \nu \in \Lambda\}$ to $\{\Delta(\nu) \mid \nu \in \lambda, \nu \neq \mu\}$ for μ a smallest element below λ, we have to reduce the set $\{\mathrm{V}^*(\nu)\langle j\rangle \mid \nu \in \Lambda(n,k), j \in \mathbb{Z}\}$ to $\{\mathrm{V}^*(\nu)\langle j\rangle \mid \nu \in \Lambda(n,k), \nu \neq \mu, j \in \mathbb{Z}\}$, where ν is a smallest element below λ which exists since $\Lambda(n,k)$ is finite. The rest works in the same way. Thus, there exist indecomposable objects $T = \mathrm{T}(\lambda)$, $\lambda \in \Lambda(n,k)$, called the indecomposable tilting objects, which satisfy conditions a) and b) and they are unique up to isomorphism.

We check the defining conditions for our $\mathrm{T}(\lambda)$: We have

$$\mathrm{End}_{\mathcal{C}^h}\left(\mathrm{T}(\lambda)\right) = \mathrm{Hom}_K\left(\mathrm{T}(\lambda), \mathrm{T}(\lambda)\right) = \mathrm{Hom}_{\mathrm{Cup}(n,k)}\left(\mathrm{T}(\lambda), \mathrm{T}(\lambda)\right)_0 \cong \mathbb{C}$$

by Lemma 3.4.15, thus $\mathrm{End}_{\mathcal{C}^h}(\mathrm{T}(\lambda))$ is local and hence $\mathrm{T}(\lambda)$ is indecomposable. Since

$$\mathrm{Ext}^1_{\mathcal{C}^h}\left(\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\langle j\rangle\right) \cong \mathrm{Hom}_K\left(\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\langle j\rangle[1]\right)$$

by Lemma 8.2.3, Propositon 6.3.11 yields

$$\mathrm{Ext}^1_{\mathcal{C}^h}\left(\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\langle j\rangle\right) = 0$$

for all ν and all j. We prove the existence of the flags by induction on $\ell(\lambda, \lambda_0)$. It is clear for $\mathrm{T}(\lambda_0) = \mathrm{V}^*(\lambda_0)$. If there is such a flag for some $\mathrm{T}(\lambda)$ and $\lambda \to \lambda s_i$, then there is one for $\mathrm{T}(\lambda s_i) \cong \mathrm{T}(\lambda).\mathcal{U}_i$ by the next lemma (Lemma 8.3.12). □

Remark 8.3.11. Applying the grading shift by j there exists a unique $\mathrm{T}(\lambda)\langle j\rangle$ which satisfies a) and admits a finite V^*-flag ending with $\mathrm{V}^*(\lambda)\langle j\rangle$. An object of the form $T=\bigoplus_{\lambda,j}\mathrm{T}(\lambda)^{\oplus a_{\lambda,j}}\langle j\rangle$ is called tilting.

The above result makes a connection to standard results from representation theory. It is known [BS11a] that the algebra A from Remark A.2.3 is quasi-hereditary (or its category $\mathcal{C}$ of finitely generated A-modules is highest weight) in the sense of Cline, Parshall and Scott. In particular, $\mathcal{C}$ has a class of standard objects $\Delta(\lambda)$ and dual standard objects $\nabla(\lambda)$, $\lambda\in\Lambda(n,k)$. A module T in $\mathcal{C}$ is then tilting if and only if it has both a Δ-filtration and a ∇-filtration. In particular, it is tilting in the above sense by [Don98, Appendix].

Under the identification of $\bigoplus_{\lambda,\mu\in\Lambda(n,k)}\mathrm{Hom}\left(\mathrm{T}(\lambda),\mathrm{T}(\mu)\right)\cong A$ our $\mathrm{V}^*(\lambda)$ and $\mathrm{V}(\lambda)$ correspond to the standard and dual standard objects, resp.

To finish the proof of Proposition 8.3.10 we have to prove the following lemma:

Lemma 8.3.12. *Let $M\in\mathcal{C}^h$ and assume M has a finite V^*-flag. Then $M.\mathcal{U}_i\in\mathcal{C}^h$ and $M.\mathcal{U}_i$ has a finite V^*-flag. If the V^*-flag of M ends with $\mathrm{V}^*(\lambda)$ and $\lambda\to\lambda s_i$, then the V^*-flag of $M.\mathcal{U}_i$ ends with $\mathrm{V}^*(\lambda s_i)$.*

Proof. We do induction on the length r of the filtration. If $r=1$, then $M=q^s\,\mathrm{V}^*(\lambda)$ for some λ. By Proposition 7.1.1 we either have $M.\mathcal{U}_i\simeq 0$, and $M.\mathcal{U}_i$ is in the heart with a filtration of length 0, or there is a distinguished triangle

$$q^t\,\mathrm{V}^*(\nu)[-1]\to q^p\,\mathrm{V}^*(\mu)\to M.\mathcal{U}_i\to q^t\,\mathrm{V}^*(\nu)$$

for some t,p and some ν, μ. If $\lambda\to\lambda s_i$, then $\nu=\lambda s_i$ and $t=s$. By rotation we get the distinguished triangle $q^p\,\mathrm{V}^*(\mu)\to M.\mathcal{U}_i\to q^t\,\mathrm{V}^*(\nu)\to q^p\,\mathrm{V}^*(\mu)[1]$. Since $q^p\,\mathrm{V}^*(\mu)$ and $q^r\,\mathrm{V}^*(\nu)$ are in the heart, so is $M.\mathcal{U}_i$ by [KS94, Proposition 10.1.11]. By Proposition 8.2.2 the sequence

$$0\to q^p\,\mathrm{V}^*(\mu)\to M.\mathcal{U}_i\to q^t\,\mathrm{V}^*(\nu)\to 0$$

is exact in $\mathcal{C}^h$ and thus we obtain a filtration $0\subset q^p\,\mathrm{V}^*(\mu)\subset M.\mathcal{U}_i$ with $M.\mathcal{U}_i\,/q^p\,\mathrm{V}^*(\mu)\cong q^t\,\mathrm{V}^*(\nu)$. In particular, if $\lambda\to\lambda s_i$, then the filtration ends with $\mathrm{V}^*(\lambda s_i)$.

Now assume $r>1$. Then there is an exact sequence $0\to M_{r-1}\to M\to q^s\,\mathrm{V}^*(\lambda)\to 0$ in $\mathcal{C}^h$, where $q^s\,\mathrm{V}^*(\lambda)\cong M/M_{r-1}$, giving a distinguished triangle $M_{r-1}\to M\to q^s\,\mathrm{V}^*(\lambda)\to M_{r-1}[1]$. Since $\mathrm{Cone}(A\to B).\mathcal{U}_i\cong\mathrm{Cone}(A.\mathcal{U}_i\to B.\mathcal{U}_i)$, the triangle $M_{r-1}.\mathcal{U}_i\to M.\mathcal{U}_i\to q^s\,\mathrm{V}^*(\lambda).\mathcal{U}_i\to M_{r-1}.\mathcal{U}_i[1]$ is also distinguished. By [KS94, Proposition 10.1.11] and Proposition 8.2.2 $M.\mathcal{U}_i$ is in $\mathcal{C}^h$ and

$$0\to M_{r-1}.\mathcal{U}_i\xrightarrow{\iota}M.\mathcal{U}_i\xrightarrow{\pi}q^s\,\mathrm{V}^*(\lambda).\mathcal{U}_i\to 0$$

is an exact sequence in $\mathcal{C}^h$. By induction there is a finite V^*-flag $0=M'_0\subset M'_1\subset\cdots\subset M'_l=M_{r-1}.\mathcal{U}_i$ and by the induction start we have an exact sequence

$$0\to q^p\,\mathrm{V}^*(\mu)\to q^s\,\mathrm{V}^*(\lambda).\mathcal{U}_i\xrightarrow{\pi'}q^t\,\mathrm{V}^*(\nu)\to 0$$

for some t, p and μ,ν with $\nu = \lambda s_i$ and $t = s$ if $\lambda \to \lambda s_i$. Let $p = \pi' \circ \pi : M.\mathcal{U}_i \to q^s \mathrm{V}^*(\lambda).\mathcal{U}_i \to q^t \mathrm{V}^*(\nu)$ and let $M'_{l+1} := \ker(p)$. Since $\iota \circ \pi = 0$ we have $M'_l \subset M'_{l+1}$. Furthermore $0 \to M'_{l+1} \to M.\mathcal{U}_i \overset{p}{\to} q^t \mathrm{V}^*(\nu) \to 0$ is exact, thus $M'_{l+1} \subset M'_{l+2} := M.\mathcal{U}_i$ with $M'_{l+2}/M_{l+1} \cong q^t \mathrm{V}^*(\nu)$. It remains to show $M'_{l+1}/M'_l = \ker(p)/M_{r-1}.\mathcal{U}_i \cong q^p \mathrm{V}^*(\mu)$. But this follows from Lemma 8.3.13 below with $A = M_{r-1}.\mathcal{U}_i$, $B = M.\mathcal{U}_i$, $C = q^s \mathrm{V}^*(\lambda).\mathcal{U}_i$, $E = q^t \mathrm{V}^*(\nu)$, $F = \ker(p)$, $X = q^p \mathrm{V}^*(\mu)$ and $Y = \ker(p)/M_{r-1}.\mathcal{U}_i$. □

Lemma 8.3.13. *Assume in an abelian category all sequences in the following diagram are exact*

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & & Y & & & & 0 & & \\
 & \searrow & \uparrow & & & & \downarrow & & \\
 & & F & & & & X & & \\
 & & \uparrow & \overset{f}{\searrow} & & & \downarrow & & \\
0 & \longrightarrow & A & \underset{a}{\longrightarrow} & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow & & & \searrow & \downarrow & & \\
 & & 0 & & & & E & & \\
 & & & & & & \downarrow & \searrow & \\
 & & & & & & 0 & & 0
\end{array}
$$

Then $X \cong Y$.

Proof. We apply the snake-lemma to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & B & \xrightarrow{\mathrm{id}} & B & \longrightarrow & 0 & &
\end{array}
$$

and since $\ker(f) = 0$, $\ker(0) = Y$, $\operatorname{coker}(a) = C$, $\operatorname{coker}(f) = E$, $\operatorname{coker}(0) = 0$ we get an exact sequence

$$0 \to Y \to C \to E \to 0.$$

Thus, by uniqueness of the kernel we obtain $X \cong Y$. □

8.4 The linear t-structure

The second t-structure is constructed such that the heart is given by the linear complexes from Definition 8.1.1. All the important complexes appearing in this thesis will turn out to be (homotopy equivalent to) linear complexes.

Theorem 8.4.1. *Let $\mathcal{D} = K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ and let $\mathcal{D}^{\geq 0}$ be the full subcategory with objects complexes X with $X_i = \bigoplus q^{k_\mu} \mathrm{T}(\mu)$ where $k_\mu - i \geq 0$; $\mathcal{D}^{\leq 0}$ is defined analogously with ≤ 0. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.*

We call this t-structure the *linear t-structure* and denote by $\mathbb{C}^l$ its heart.

Proof. We have that $\mathcal{D}^{\geq n}$ is the full subcategory of complexes X with $X_i = \bigoplus q^{k_\mu}\,\mathrm{T}(\mu)$ where $k_\mu - i \geq n$; analogously for $\leq$. Therefore, $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ is clear.

For (T2) note that because of Corollary 3.4.13 all morphisms in $\mathrm{Cup}\,\mathrm{T}(n,k)$ are of non-negative degree and every complex is isomorphic to one with entries direct sums of shifted $\mathrm{T}(\lambda)$'s. Assume there is $A \in \mathrm{ob}(\mathcal{D}^{\leq 0})$ and $B \in \mathrm{ob}(\mathcal{D}^{\geq 1})$ with $f_j : A_j \to B_j$ not zero for some j. Then there is some summand $q^r\,\mathrm{T}(\mu)$ of A_j and $q^s\,\mathrm{T}(\lambda)$ of B_j such that f_j restricted to those is not zero. Let $g : q^r\,\mathrm{T}(\mu) \to q^s\,\mathrm{T}(\lambda)$ be the restriction. We have $r - j \leq 0$ and $s - j \geq 1$, so $\deg(g) = r - s = r - j - (s - j) < 0$, which is a contradiction to Lemma 3.4.14. Thus, there is no non-zero map from A to B.

Now let $X \in \mathcal{D}$. We write each X_i as $X_i = X_i^- \oplus X_i^+$ where $X_i^- = \bigoplus q^{k_\mu}\,\mathrm{T}(\mu)$ where $k_\mu - i \leq 0$ and $X_i^+ = \bigoplus q^{k_\mu}\,\mathrm{T}(\mu)$ where $k_\mu - i \geq 1$. We have $d_i|_{X_i^-} : X_i^- \to X_{i-1}^- \oplus X_{i-1}^+ = (d_i^-, 0)$ by the proof of (T2). Thus (X^-, d^-) defined by $(X^-)_i = X_i^-$ is a complex and the inclusion $\iota : X^- \to X$ is a morphism of complexes. Using Gaussian elimination (Lemma 5.1.2) with respect to all ι_i to eliminate all summands of X^- we see that $\mathrm{Cone}(\iota)$ is homotopy equivalent to some $Y \in \mathcal{D}^{\geq 1}$. Thus, the desired distinguished triangle for (T3) is $X^- \to X \to Y \to X^-[1]$. □

Corollary 8.4.2. *The linear complexes form an abelian subcategory of* $K^b(\widehat{\mathrm{Cup}}(n,k))$.

Proof. Clearly, the linear complexes are the heart of the linear t-structure which is abelian by Proposition 8.2.2. □

Thus, by Theorem 8.1.5 and Corollary 8.1.7, we have:

Corollary 8.4.3. *The* $\mathrm{V}^*(\lambda)$*'s and* $\mathrm{V}(\lambda)$*'s are contained in the heart* $\mathcal{C}^l$ *of the linear t-structure.*

Proposition 8.4.4. *The* $\mathrm{T}(\lambda)$*'s are simple objects in the heart* $\mathcal{C}^l$.

Proof. Let Z be a linear complex and $f : \mathrm{T}(\lambda) \to Z$ a chain map. We have to show that f is either a monomorphism or 0. Assume $f \neq 0$. By [KS94, (10.1.17)], $\ker f \simeq \tau^{\leq 0}(\mathrm{Cone}(f)[-1])$ and for $X^0 \to X \to X^1 \to X^0[1]$ a triangle as in (T3) we have $\tau^{\leq 0} X \simeq X_0$ by [KS94, Proposition 10.1.4].

Since $f \neq 0$ because of Lemma 3.4.14 we have $Z_0 = Y \oplus \mathrm{T}(\lambda)$ and $f' := f|_{\mathrm{T}(\lambda) \to \mathrm{T}(\lambda)} = c \cdot \mathrm{id}$. Thus, inside $\mathrm{Cone}(f)$ we can apply Gaussian elimination (Lemma 5.1.2) with respect to f' and obtain $\mathrm{Cone}(f) \simeq Z'$ with $Z'_i = Z_i$ for $i \neq 0$ and $Z'_0 = Y$. Now Z' is still a linear complex, thus $(Z'[-1])_i$ contains only summands $q^{r_\mu}\,\mathrm{T}(\mu)$ with $r_\mu - i = 1$, thus $\tau^{\leq 0}(Z'[-1]) \simeq (Z'[-1])^- \simeq 0$, i.e. $\ker f \simeq 0$ and f is a monomorphism. □

Corollary 8.4.5. *The simple objects in* $\mathcal{C}^l$ *are the* $\mathrm{T}(\lambda)\,\langle i \rangle\,[i]$*'s.*

Proof. Let $i \neq 0$ and assume Z is a linear complex such that $0 \neq f : \mathrm{T}(\lambda)\,\langle i \rangle\,[i] \to Z$ is not a monomorphism. Then $Z\,\langle -i \rangle\,[-i]$ is also a linear complex and $f\,\langle -i \rangle\,[-i] : \mathrm{T}(\lambda) \to Z\,\langle -i \rangle\,[-i]$ is not a monomorphism which contradicts Proposition 8.4.4. Thus, the $\mathrm{T}(\lambda)\,\langle i \rangle\,[i]$'s are simple.

Assume some linear complex Y with $Y_i \neq 0$ is simple and Y is not isomorphic to some $\mathrm{T}(\mu)\langle i\rangle[i]$. Let $\mathrm{T}(\lambda)\langle i\rangle$ be a summand of Y_i and consider $f : Y \to \mathrm{T}(\lambda)\langle i\rangle[i]$ given by id on $\mathrm{T}(\lambda)\langle i\rangle$ and 0 elsewhere. To $\mathrm{Cone}(f)$ we can apply Gaussian elimination (Lemma 5.1.2) with respect to id and obtain $\mathrm{Cone}(f) \simeq Z[1]$ where Z is the linear complex with $Z_j = Y_j$ for $j \neq i$ and $Y_i = Z_i \oplus \mathrm{T}(\lambda)\langle i\rangle$. By the formulas from the proof of Proposition 8.4.4 we know

$$\ker f \simeq \tau^{\leq 0}\,\mathrm{Cone}(f)[-1] \simeq \tau^{\leq 0} Z \simeq Z \neq 0,$$

where $\tau^{\leq 0} Z \simeq Z$ since Z is in $\mathcal{C}^l$ and $Z \neq 0$ since Y is not isomorphic to $\mathrm{T}(\lambda)\langle i\rangle[i]$. Thus, f is not a monomorphism and Y cannot be simple. □

We will revisit linear complexes in Chapter 10 when we consider the role of the complex $L(\lambda_0)$ constructed there in the heart of the linear t-structure.

Chapter 9

Morphisms between exceptional objects

In this chapter, we investigate morphisms of degree 1 and 2 between the $\mathrm{V}^*(\lambda)$'s. Understanding these morphisms will be crucial for constructing the complex $L(\lambda_0)$ in the next chapter, which will lead to a categorification of the projection π_n. We give an explicit construction of all degree 1 morphisms and examine how they give rise to degree 2 morphisms.

9.1 Degree 1 morphisms

As a first step to understand morphisms from $q\,\mathrm{V}^*(\lambda s_i)$ to $\mathrm{V}^*(\lambda)$ for $\lambda \to \lambda s_i$, we consider maps from $q\,\mathrm{V}^*(\lambda s_i)_0 \cong q\,\mathrm{T}(\lambda s_i)$ and $q\,\mathrm{V}^*(\lambda s_i)_1$ to $\mathrm{V}^*(\lambda)$.

Lemma 9.1.1. *Let $\lambda \to \lambda s_i$. Then*

$$\mathrm{Hom}_K\left(q\,\mathrm{T}(\lambda s_i), \mathrm{V}^*(\lambda)\right) \cong \mathbb{C}.$$

Proof. By Lemma 4.2.3 there is (up to scalar) only one degree 1 map from $\mathrm{T}(\lambda s_i)$ to $\mathrm{T}(\lambda)$ which gives a chain map $q\,\mathrm{T}(\lambda s_i) \to \mathrm{V}^*(\lambda)$, since $\mathrm{V}^*(\lambda)_0 \cong \mathrm{T}(\lambda)$ by Lemma 6.1.10. Assume that it is homotopic to zero:

$$\begin{array}{ccccccc} & & & & & & q\,\mathrm{T}(\lambda s_i) \\ & & & & & \swarrow & \downarrow \\ \mathrm{V}^*(\lambda) \;\cong & \ldots \longrightarrow & \mathrm{V}^*(\lambda)_2 & \xrightarrow{d_2} & \mathrm{V}^*(\lambda)_1 & \xrightarrow{d_1} & \mathrm{T}(\lambda) \end{array}$$

This means there is a factorisation of the map $f : q\,\mathrm{T}(\lambda s_i) \to \mathrm{T}(\lambda)$ over $\mathrm{V}^*(\lambda)_1$. We have $\deg(f) = 1 = \deg(d_1)$, thus the factorisation map $\mathrm{T}(\lambda s_i) \to \mathrm{V}^*(\lambda)_1$ has to be of degree 0. By Lemma 6.1.10, $V(\lambda)_1$ has entries (shifted) $\mathrm{T}(\mu)$ with $\mu < \lambda < \lambda s_i$. In particular, $\mu \neq \lambda s_i$ and therefore, by Lemma 3.4.14, there is no degree 0 morphism $\mathrm{T}(\lambda s_i) \to \mathrm{V}^*(\lambda)_1$. Thus, there is no such nullhomotopy. □

Proposition 9.1.2. *Let $\lambda \to \lambda s_i$. Then*

$$\operatorname{Hom}_K\big(q\,\mathrm{V}^*(\lambda s_i)_1, \mathrm{V}^*(\lambda)\big) = 0.$$

Proof. We have $\mathrm{V}^*(\lambda s_i) \cong \operatorname{Cone}\big(q\,\mathrm{V}^*(\lambda) \to \mathrm{V}^*(\lambda).\mathcal{U}_i\big)$, thus

$$q\,\mathrm{V}^*(\lambda s_i)_1 \cong q^2\,\mathrm{V}^*(\lambda)_0 \oplus q\,\mathrm{V}^*(\lambda)_1.\mathcal{U}_i \cong q^2\,\mathrm{T}(\lambda) \oplus q\,\mathrm{V}^*(\lambda)_1.\mathcal{U}_i\,.$$

By Lemma 6.1.10, any summand $q^r\,\mathrm{T}(\mu)$ of $\mathrm{V}^*(\lambda)_1$ satisfies $\mu < \lambda$. Using Lemma 4.1.8 gives

$$\mathrm{T}(\mu).\mathcal{U}_i \cong \begin{cases} q\,\mathrm{T}(\mu) \oplus q^{-1}\,\mathrm{T}(\mu), \\ \mathrm{T}(\nu) \text{ for } \nu < \mu < \lambda, \\ \mathrm{T}(\mu s_i) \text{ with } \mu s_i > \mu, \\ 0. \end{cases}$$

But Lemma 1.1.32 provides $\mu s_i \not\geq \lambda$ if $\mu s_i > \mu$. Thus, $q\,\mathrm{V}^*(\lambda)_1.\mathcal{U}_i \cong \bigoplus_{\text{some } \nu:\nu \not\geq \lambda} q^{r_\nu}\,\mathrm{T}(\nu)$ for some r_ν and

$$q\,\mathrm{V}^*(\lambda s_i)_1 \cong q^2\,\mathrm{T}(\lambda) \oplus \bigoplus_{\text{some } \nu:\nu \not\geq \lambda} q^{r_\nu}\,\mathrm{T}(\nu).$$

Therefore, we can apply Remark 6.3.17 to obtain $\operatorname{Hom}_K(q\,\mathrm{V}^*(\lambda s_i)_1, \mathrm{V}^*(\lambda)) = 0$. □

Theorem 9.1.3. *Let $\lambda \to \lambda s_i$. Then*

$$\operatorname{Hom}_K\big(q\,\mathrm{V}^*(\lambda s_i), \mathrm{V}^*(\lambda)\big) \cong \mathbb{C}.$$

Proof. By Remark 6.3.17 we have $\operatorname{Hom}_K\big((q\,\mathrm{V}^*(\lambda s_i))_j, \mathrm{V}^*(\lambda)[l]\big) = 0$ for $l \neq 0$ and all j. By Proposition 9.1.2, we know $\operatorname{Hom}_K\big(q\,\mathrm{V}^*(\lambda s_i)_1, \mathrm{V}^*(\lambda)\big) = 0$, thus the assertion follows from Corollary 5.2.5 a) and Lemma 9.1.1. □

Our next task is to construct a non-trivial element of the homomorphism space $\operatorname{Hom}_K\big(q\,\mathrm{V}^*(\lambda s_i), \mathrm{V}^*(\lambda)\big)$.

Remember that for $\boldsymbol{w} = (w_1, \dots, w_r)$ a vector with entries 0 and 1, $\boldsymbol{w}.(i_1, \dots, i_r)$ was defined as $B_{i_1} \dots B_{i_r}$ (up to shift), where

$$B_{i_l} = \begin{cases} \mathcal{U}_{i_l} & \text{if } w_l = 1, \\ Id & \text{if } w_l = 0. \end{cases}$$

Definition 9.1.4. Let $\lambda \to \lambda s_i = \nu$. We define a collection of maps $f_{\lambda,\nu} : q\,\mathrm{V}^*(\nu) \to \mathrm{V}^*(\lambda)$ as follows:
We know $q\,\mathrm{V}^*(\nu) = \operatorname{Cone}\big(q^2\,\mathrm{V}^*(\lambda) \to q\,\mathrm{V}^*(\lambda).\mathcal{U}_i\big)$. Let $f_{\lambda,\nu} = \alpha \oplus \beta$ where $\beta : q\,\mathrm{V}^*(\lambda).\mathcal{U}_i \xrightarrow{\mathrm{id}.\overline{\mathrm{H}}_i} \mathrm{V}^*(\lambda)$ is given by the saddle from $\mathcal{U}_i$ to Id.

To define the map α we switch to the cube description. Let $\nu = \lambda_0.s_{i_1} \dots s_{i_r}$ where $s_{i_1} \dots s_{i_r} \in \mathrm{W}^{\min}$ is a reduced expression and $s_{i_r} = s_i$. Recall that $\mathrm{V}^*(\nu) \cong \mathrm{T}(\lambda_0).R(i_1, \dots, i_r)$ and the entries of the complex $\mathrm{V}^*(\nu)$ are (up to degree-shift) of the form $\mathrm{T}(\lambda_0).\big(\boldsymbol{w}.(i_1,\dots,i_r)\big)$, where $\boldsymbol{w}$ is a $0,1$-vector as before. In particular, every entry of $\mathrm{V}^*(\nu)$ is determined by its $\boldsymbol{w}$ (cf. Definition 6.1.3, Example 6.1.4).

For $\boldsymbol{w} = (w_1,\dots,w_r)$ let $\overline{\boldsymbol{w}} = (w_1,\dots,w_{r-1})$. Assume $w_j = 1$, then we denote by $\overline{\boldsymbol{w}}{\downarrow}j$ the tuple where w_j is changed to 0. We know that $\overline{\boldsymbol{w}}{\downarrow}i$ describes an entry of $\mathrm{V}^*(\lambda)$ for $i = 1,\dots,r-1$.

Now let $\boldsymbol{w}$ describe an entry of $q\,\mathrm{V}^*(\lambda)$ inside $\mathrm{V}^*(\nu)$, i.e. $w_r = 0$. For each j such that $w_j = 1$ we define a map $\alpha_{\boldsymbol{w},j} : \mathrm{T}(\lambda_0).\big(\boldsymbol{w}.(i_1,\dots,i_r)\big) \to \mathrm{T}(\lambda_0).\big((\overline{\boldsymbol{w}}{\downarrow}j).(i_1,\dots,i_{r-1})\big)$ via $\alpha_{\boldsymbol{w},j} = \eta_{\boldsymbol{w},j}\cdot \mathrm{id}\,\overline{\mathrm{H}}_j\,\mathrm{id}$ where $\eta_{\boldsymbol{w},j} \in \{\pm 1, 0\}$ will be specified later. Finally we define α via $\alpha|_{\mathrm{T}(\lambda_0).(\boldsymbol{w}.(i_1,\dots,i_r))} = \{\alpha_{\boldsymbol{w},j}\}_{j:w_j\neq 0}$. All the other maps are zero.

One crucial step will be to choose the $\eta_{\boldsymbol{w},j}$ such that $f_{\lambda,\nu}$ becomes a chain map (cf. Proposition 9.1.10).

Note that we always have an easy description of $f_{\lambda,\nu}$ in homological degree 0: $(f_{\lambda,\nu})_0 = \beta_0 = \mathrm{id}\,\overline{H}_i$.

Example 9.1.5. Recall

$$R(3,2,4) = \quad \begin{array}{ccccccc} & & q^2\,\mathrm{Id}\,\mathcal{U}_2\,\mathrm{Id} & \xrightarrow{-\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & q\,\mathcal{U}_3\mathcal{U}_2\,\mathrm{Id} & & \\ & {\scriptstyle -\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}}\nearrow & \oplus & {\scriptstyle \mathrm{id}\,\mathrm{id}\,\mathrm{H}_4,\ -\mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & \oplus & \searrow{\scriptstyle \mathrm{id}\,\mathrm{id}\,\mathrm{H}_4} & \\ q^3\,\mathrm{Id}\,\mathrm{Id}\,\mathrm{Id} & \xrightarrow{\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}} & q^2\,\mathcal{U}_3\,\mathrm{Id}\,\mathrm{Id} & & q\,\mathrm{Id}\,\mathcal{U}_2\mathcal{U}_4 & \xrightarrow{\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}} & \mathcal{U}_3\mathcal{U}_2\mathcal{U}_4 \\ & {\scriptstyle \mathrm{id}\,\mathrm{id}\,\mathrm{H}_4}\searrow & \oplus & {\scriptstyle \mathrm{id}\,\mathrm{id}\,\mathrm{H}_4,\ \mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & \oplus & \nearrow{\scriptstyle \mathrm{id}\,\mathrm{H}_2\,\mathrm{id}} & \\ & & q^2\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_4 & \xrightarrow[-\mathrm{H}_3\,\mathrm{id}\,\mathrm{id}]{} & q\,\mathcal{U}_3\,\mathrm{Id}\,\mathcal{U}_4 & & \end{array},$$

from Example 6.1.4. We have

$$R(3,2,4) = \mathrm{Cone}\left(qR(3,2)\,\mathrm{Id} \xrightarrow{H_4} R(3,2).\mathcal{U}_4\right)$$

for

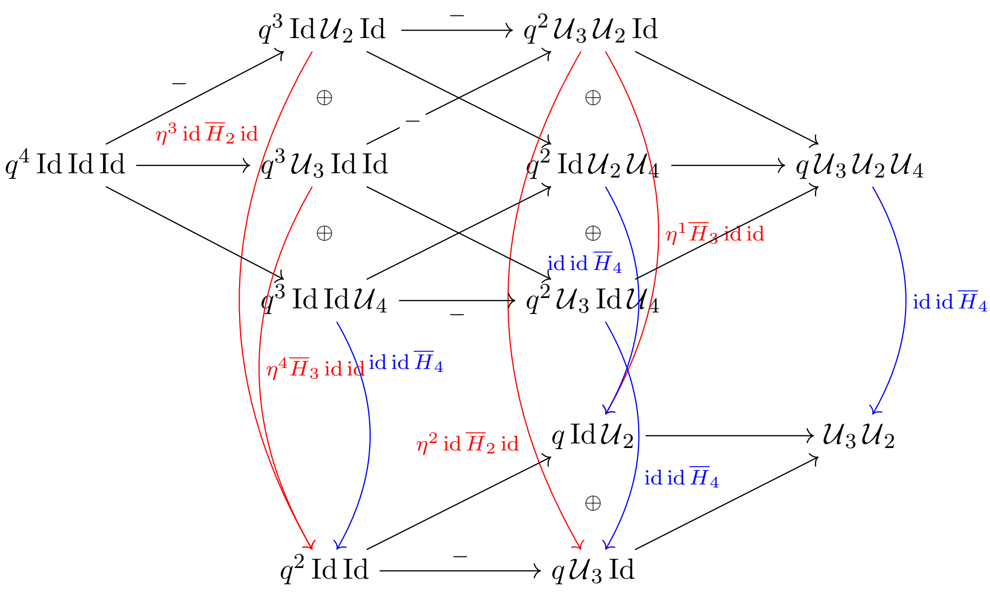
$$R(3,2) = \quad \begin{array}{ccccc} & & q\,\mathrm{Id}\,\mathcal{U}_2 & \xrightarrow{\mathrm{H}_3\,\mathrm{id}} & \mathcal{U}_3\mathcal{U}_2 \\ & {\scriptstyle \mathrm{id}\,\mathrm{H}_2}\nearrow & \oplus & {\scriptstyle \mathrm{id}\,\mathrm{H}_2}\nearrow & \\ q^2\,\mathrm{Id}\,\mathrm{Id} & \xrightarrow{-\mathrm{H}_3\,\mathrm{id}} & q\,\mathcal{U}_3\,\mathrm{Id} & & \end{array}.$$

Now we want to describe a chain map

$$q\,\mathrm{V}^*(\lambda_0 s_3 s_2 s_4) \cong q\,\mathrm{T}(\lambda_0).R(3,2,4) \to \mathrm{T}(\lambda_0).R(3,2) \cong \mathrm{V}^*(\lambda_0 s_3 s_2)$$

as in the definition above.

In a first step we get a collection of maps $qR(3,2,4) \to R(3,2)$:



Here, the blue maps are β and the red maps are α with the $\eta^i \in \{0,1,-1\}$, where we denote $\eta^1 = \eta_{(110),1}$, $\eta^2 = \eta_{(110),2}$, $\eta^3 = \eta_{(010),2}$ and $\eta^4 = \eta_{(100),1}$. Consider for example $\boldsymbol{w} = (110)$, then $\boldsymbol{w}$ describes the entry $\boldsymbol{w}(3,2,4) = q\,\mathcal{U}_3\mathcal{U}_2\,\mathrm{Id}$ inside $R(3,2,4)$. For $j = 1$ we have $w_j = 1$ and $\overline{\boldsymbol{w}}{\downarrow}j = (01)$ describes the entry $(01)(3,2) = q\,\mathrm{Id}\,\mathcal{U}_2$ inside $R(3,2)$. Thus we have some $\alpha_{\boldsymbol{w},j} = \eta^1\overline{H}_3\,\mathrm{id} : q^2\,\mathcal{U}_3\mathcal{U}_2\,\mathrm{Id} \to q\,\mathrm{Id}\,\mathcal{U}_2$. Analogously the other red maps are constructed.

The trick is now to choose the η^i such that we obtain a chain map after applying everything to $\mathrm{T}(\lambda_0)$. In this example one might be able to guess them, in particular since $\mathrm{T}(\lambda_0).\,\mathrm{Id}\,\mathcal{U}_2 = 0 = \mathrm{T}(\lambda_0).\,\mathrm{Id}\,\mathcal{U}_2\,\mathrm{Id}$, but there is still some calculation involved.

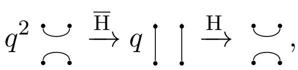
The first part of showing that $f_{\lambda,\nu}$ is a chain map is easy:

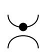
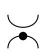
Lemma 9.1.6. *The map $\beta : q\,\mathrm{V}^*(\lambda).\mathcal{U}_i \to \mathrm{V}^*(\lambda)$ from the previous definition is a chain map.*

Proof. This follows directly from Lemma 4.2.8 a), since the signs in $\mathrm{V}^*(\lambda).\mathcal{U}_i$ and $\mathrm{V}^*(\lambda)$ are the same. □

Now we need to work out the coefficients $\eta_{\boldsymbol{w},j}$ from the definition.

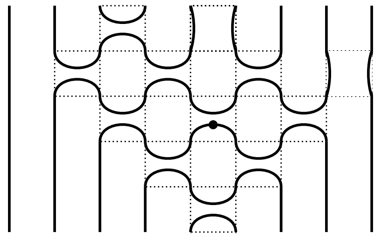
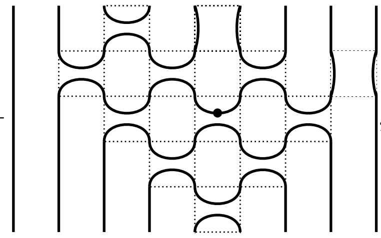
The next notation is motivated by the following: When we consider the composition of the two saddles

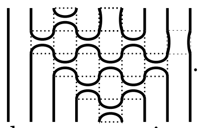


then using neckcutting this composition is equal to  +  in the shorthand notation from Remark 3.2.5.

By $(\bullet_l)\big(\boldsymbol{w}.(i_1,\dots,i_r)\big)$ we denote the sum of the following two cobordisms from $\big(\boldsymbol{w}.(i_1,\dots,i_r)\big)$ to itself: They both consist of $\big(\boldsymbol{w}.(i_1,\dots,i_r)\big) \times [0,1]$ with one dot each. For the first summand, there is a dot on one of the components of $B_{i_l} \times [0,1]$ and for the other one on the other component.

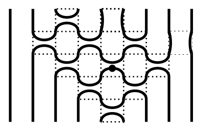
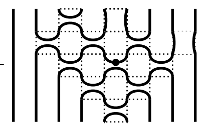
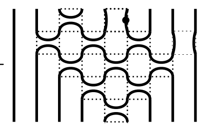
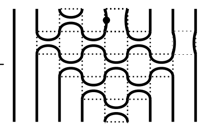
Example 9.1.7. Still using the shorthand notation for cobordisms with $\bullet$'s (Remark 3.2.5), in the simple case of our motivation above we have for $\boldsymbol{w} = (1)$ and $(i_1) = 1$ that $(\bullet_1)\big(\boldsymbol{w}.(i_1)\big) =$  $+$  . In a bigger case, for $(i_1, \ldots, i_r) = (5,6,7,8,4,5,6,3,4,5,2,3)$ and $\boldsymbol{w} = (1,1,1,0,1,0,1,1,1,1,1,1)$ we have

$(\bullet_6)\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big) =$  $+$  ,

since $\boldsymbol{w}.(i_1, \ldots, i_r) =$ . As in Remark 1.2.5, the $\mathcal{U}_{i_j}$ or Id_{i_j} are pictured diagonally inside the boxes starting from the lowest box and counting following the diagonals, and the $\bullet$'s have to be on the two components of the 6th box.

Now let $\mathbf{z} = (z_1, \ldots, z_r)$ be a vector with entries $0, 1, -1$. We define $\mathbf{z}\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big) = \sum_l z_l \cdot (\bullet_l)\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big)$. It is a morphism from $\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big)$ to itself in $\mathrm{Cup}(n,k)$ of degree 2.

Example 9.1.8. In the previous example take $\mathbf{z} = (0,0,0,0,0,1,0,0,0,-1,0,0)$. Then

$\mathbf{z}\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big) =$  $+$  $-$  $-$ 

Finally, by $\mathrm{T}(\lambda_0).\mathbf{z}\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big)$ we denote the previous cobordism applied to $\mathrm{T}(\lambda_0) \times [0,1]$.

Proposition 9.1.9. *Let $\boldsymbol{w}$ be a vector with entries 0 and 1 and fix $(i_1, \ldots, i_r)$ such that $s_{i_1} \ldots s_{i_r} \in \mathrm{W}^{min}$ is reduced. Then there is a choice of $\mathbf{z}$ such that $z_r = -1$ and $\mathrm{T}(\lambda_0).\mathbf{z}\big(\boldsymbol{w}.(i_1, \ldots, i_r)\big) = 0$ for all $\boldsymbol{w}$ with $w_r = 0$.*

Proof. Since the $- \times [0,1]$ plays no role when considering connected components, we reformulate the calculus using the diagrams from Example 9.1.8 above. Applying a diagram to $\mathrm{T}(\lambda_0)$ gives 0 if the diagram has a dot on a line connected to the bottom, so we just use this formulation. Altogether we are considering a $\mathbb{Z}$-linear combination X of diagrams each of which is obtained from a fixed Temperley-Lieb diagram T by adding exactly one dot. The coefficients will turn out to be in $\{\pm 1\}$. We therefore can abbreviate X a the single Temperley-Lieb diagram T equipped with signed dots indicating the coefficients.

First consider $\boldsymbol{w} = (1, \ldots, 1, 1)$. We have $\boldsymbol{w}.(i_1, \ldots, i_r) = \mathcal{U}_{i_1} \ldots \mathcal{U}_{i_r}$. By Remark 1.2.4 we know that $\boldsymbol{w}.(i_1, \ldots, i_r)$ is isomorphic to some $\boldsymbol{\mathcal{U}}(T)$ for $T \in Y(n,k)$ and $\mathcal{U}_{i_r}$ is associated with a box B which is rightmost in a row of T. Let $\{j_1, \ldots, j_d\} \subset \{i_1, \ldots, i_{r-1}\}$

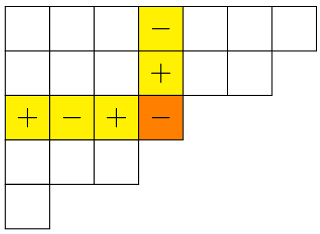
be the entries of the row of B in T, read from right to left, beginning one left to B. We set

$$z_{j_l} = \begin{cases} 1 & \text{if } l \equiv 1 \mod 2, \\ -1 & \text{if } l \equiv 0 \mod 2. \end{cases}$$

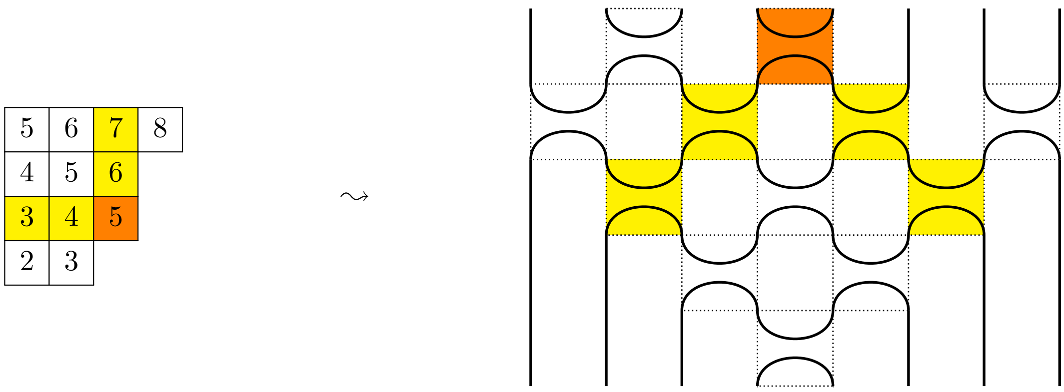
$\{j'_1, \dots, j'_{d'}\} \subset \{i_1, \dots, i_{r-1}\}$ be the entries of the column of B, read from bottom to top, beginning one above B. We set

$$z_{j'_l} = \begin{cases} 1 & \text{if } l \equiv 1 \mod 2, \\ -1 & \text{if } l \equiv 0 \mod 2. \end{cases}$$

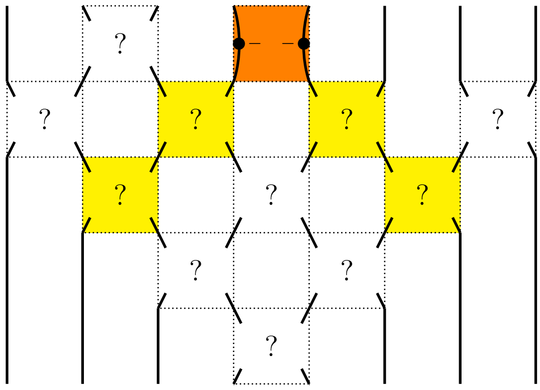
We set $z_r = -1$ and all other $z_i = 0$.



This satisfies the conditions for all $\boldsymbol{w}$ with $w_r = 0$:
Let $\{k_1, \dots, k_r\}$ be a permutation π of $\{i_1, \dots, i_r\}$ such that $\mathcal{U}_{j_1} \dots \mathcal{U}_{j_r} \cong \mathcal{U}_{k_1} \dots \mathcal{U}_{k_r} = \boldsymbol{\mathcal{U}}(T)$, the Temperley-Lieb diagram constructed from the row reading word of T. We know that $\boldsymbol{w}.(i_1, \dots, i_r) \cong \widehat{\boldsymbol{w}}.(k_1, \dots, k_r)$ where $\widehat{\boldsymbol{w}} = \pi(\boldsymbol{w})$. Also $\mathbf{z}\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) = \widehat{\mathbf{z}}\big(\widehat{\boldsymbol{w}}.(k_1, \dots, k_r)\big)$ where $\widehat{\mathbf{z}} = \pi(\mathbf{z})$. Thus, it is enough to show the claim for $\widehat{\mathbf{z}}\big(\widehat{\boldsymbol{w}}.(k_1, \dots, k_r)\big)$. We denote the index associated to the box B in T by a, so $i_k = a$; and let $k_s = i_k$ in the permutation, so $i_k = k_s = a$.

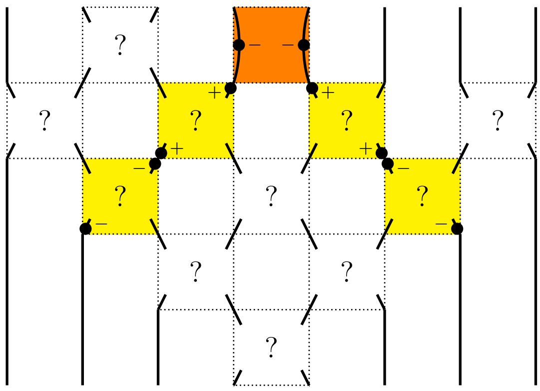


Then $j_1, \dots, j_d = a-1, \dots, a-d$ and $(\bullet_s)\big(\widehat{w}.(k_1, \dots, k_r)\big)$ is the sum of the diagram which has dots on top of the strands a and $a+1$.



Let $j_l = k_{l'}$. Independent of whether $B_{j_l} = \mathcal{U}_{j_l}$ or $= Id$, we know that $(\bullet_{l'})(\widehat{t}.(k_1,\dots,k_r))$ is the sum of the diagram with one dot above B_{j_l} on strand $a-l+1$ and one below B_{j_l} on strand $a-l$.

With the choice of signs for $\mathbf{z}$ as described above, in $\widehat{\mathbf{z}}(\widehat{\boldsymbol{w}}(k_1,\dots,k_r))$ these kill each other and furthermore the one dot on strand a of B_{k_s} except for the lower dot of B_{j_d}. But since we know that the diagram is of $Y(n,k)$-form, this dot is on a line connected to the bottom and the diagram is zero anyway.



Since $k_s = a$ we know that $j'_1,\dots,j'_{d'} = a+1,\dots,a+d'$. Again, for $j'_l = k_{l''}$ independent of whether $B_{j'_l} = \mathcal{U}_{j'_l}$ or $= Id$, $(\bullet_{l''})(\widehat{\boldsymbol{w}}.(k_1,\dots,k_r))$ is the sum of the diagram with one dot above $B_{j'_l}$ on strand $a+l$ and one below $B_{j'_l}$ on strand $a+l+1$. With the choice of signs as before, in $\widehat{\mathbf{z}}(\widehat{\boldsymbol{w}}(k_1,\dots,k_r))$ these kill the dot on strand $a+1$ of B_{k_s} and each other except for the dot below $B_{j'_{d'}}$. But since $j'_{d'}$ is in the first row, this dot is always on a line connected to the bottom. □

Proposition 9.1.10. *With the notation from before and* $\mathbf{z}$ *as in Proposition 9.1.9 let*

$$\eta_{\boldsymbol{w},j} := (-1)^{\sum_{r>l>j} b(w_l)} z_j.$$

Then $(\alpha \oplus \beta)|_{q\mathrm{V}^*(\lambda)}$ *from Definition 9.1.4 is a chain map.*

Before proving this we consider what the map $f_{\lambda,\nu}$ now looks like:

Remark 9.1.11. Let $T \in Y(n,k)$ such that $\boldsymbol{\mathcal{U}}(T) = \mathcal{U}_{i_1} \dots \mathcal{U}_{i_r}$. Note that up to sign the chain map from Definition 9.1.4 $\mathrm{T}(\lambda_0).R(i_1,\dots,i_r) \to \mathrm{T}(\lambda_0).R(i_1,\dots,i_{r-1})$ is given by $\mathrm{id}\,\overline{\mathrm{H}}_{i_j}\,\mathrm{id} : \boldsymbol{w}.(i_1,\dots,i_r) \to \boldsymbol{w}'.(i_1,\dots,i_r)$ where $\boldsymbol{w}' = \overline{\boldsymbol{w}{\downarrow}j}$ for some j such that i_j is in the same row or column of T as i_r and $w_j = 1$. Here, $\boldsymbol{w}{\downarrow}j$ is again setting the jth entry to zero and $\overline{\mathbf{v}}$ means deleting the last entry of $\mathbf{v}$. In particular, the maps α and β from $f_{\lambda,\nu} = \alpha \oplus \beta$ can be described in the same fashion since setting the last entry from 1 to 0 and then deleting it agrees with the other definition of β.

Example 9.1.12. For $k = 1$ and arbitrary n we only have one $\wedge$ in the sequences in $\Lambda(n,k)$ and every $\lambda \in \Lambda(n,k)$ can be described as $\lambda = \lambda_0.s_1 \dots s_t$ for some $t < n$. The tableau associated to λ is a single row of the form $\boxed{1}\boxed{2}\boxed{3}\dots\boxed{r}$. Therefore, for $r+1 < n$ and $\nu = \lambda_0.s_1 \dots s_{t+1}$ the vector $\mathbf{z}$ constructed in Proposition 9.1.9 is of the form $\big((-1)^{t-1}, (-1)^{t-2}, \dots, -1, +1, -1\big)$. Thus, the chain map $f_{\lambda,\nu}$ consists up to sign of all possible saddle cobordisms (i.e. no $\eta_{\boldsymbol{w},j}$ is zero) and the sign is $(-1)^{\sum_{t+1>l>j} b(w_l)}(-1)^{t-j}$ for the map starting at the entry determined by $\boldsymbol{w}$ when $j < t+1$ is deleted and $+1$ if $j = t+1$ is deleted.

Proof of Proposition 9.1.10. We show that all possible squares commute. All squares are given by an starting point determined by $\boldsymbol{w}$ with $w_r = 0$ and an endpoint $\boldsymbol{w}'$. First we consider squares with $\overline{\boldsymbol{w}} \neq \boldsymbol{w}'$. Because of this assumption, the map β is not involved in such squares and only the differentials inside $q\,\mathrm{V}^*(\lambda)$ play a role. Also $\overline{\boldsymbol{w}}$ and $\boldsymbol{w}'$ have to contain the same number of 1's, so $\overline{\boldsymbol{w}} \neq \boldsymbol{w}'$ can only happen if there are $1 \leq i \neq j < r$ such that $\overline{w}_k = w'_k$ for all $k \neq i,j$ and $\overline{w}_i = w_i = 1$, $w'_i = 0$, $\overline{w}_j = w_j = 0$, $w'_j = 1$. Let $\overline{\boldsymbol{w}}^i$ be the same as $\overline{\boldsymbol{w}}$ but for $w_i = 0$ (i.e. $\overline{\boldsymbol{w}}^i = \overline{\boldsymbol{w}{\downarrow}i}$) and let $\boldsymbol{w}^j$ be the same as $\boldsymbol{w}$ except $w_j = 1$. So (up to signs) we consider the square

$$\begin{array}{ccc} \boldsymbol{w} & \xrightarrow{\mathrm{id}\,\mathrm{H}_j\,\mathrm{id}} & \boldsymbol{w}^j \\ {\scriptstyle id\overline{\mathrm{H}}_i id}\downarrow & & \downarrow{\scriptstyle id\overline{\mathrm{H}}_i id} \\ \overline{\boldsymbol{w}}^i & \xrightarrow{\mathrm{id}\,\mathrm{H}_j\,\mathrm{id}} & \boldsymbol{w}'. \end{array}$$

Up to signs, this square commutes by Lemma 4.2.8 a). The map $\boldsymbol{w} \to \boldsymbol{w}^j$ has sign $(-1)^{\sum_{q>j} b(w_q)}$. Since $w_j = 0$ but $(w^j)_j = 1$, the map $\boldsymbol{w}^j \to \boldsymbol{w}'$ has sign

$$(-1)^{\sum_{r>q>i} b(w^j_q)} z_i = (-1)^{\sum_{r>q>i} b(w_q)} z_i (-1)^{\gamma(i,j)},$$

where

$$\gamma(i,j) = \begin{cases} 1 & \text{if } j > i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\boldsymbol{w} \to \boldsymbol{w}^j \to \boldsymbol{w}'$ has in total the sign

$$(-1)^{\gamma(i,j)}(-1)^{\sum_{q>j} b(w_q)} \cdot (-1)^{\sum_{r>q>i} b(w_q)} z_i = -(-1)^{\gamma(i,j)} z_i,$$

since $w_r = 0$. The map $\boldsymbol{w} \to \overline{\boldsymbol{w}}^i$ has sign $(-1)^{\sum_{r>q>i} b(w_q)} z_i$ and $\overline{\boldsymbol{w}}^i \to \boldsymbol{w}'$ has sign

$$(-1)^{\sum_{q>j} b(\overline{w}^i_q)} = (-1)^{\sum_{r>q>j} b(w_q)} (-1)^{\gamma(i,j)+1}.$$

Thus for $\boldsymbol{w} \to \overline{\boldsymbol{w}}^i \to \boldsymbol{w}'$ we have total sign

$$(-1)^{\sum_{r>q>i} b(w_q)} z_i (-1)^{\sum_{r>q>j} b(w_q)} (-1)^{\gamma(i,j)+1} = -(-1)^{\gamma(i,j)} z_i$$

and the signs agree.

If $\boldsymbol{w}' = \overline{\boldsymbol{w}}$, then (up to signs) we have to consider

$$\begin{array}{ccc} \boldsymbol{w} & \xrightarrow{\operatorname{id} \mathrm{H}_j \operatorname{id}} & \{\boldsymbol{w}^j\}_{w_j=0} \\ {\scriptstyle id\overline{\mathrm{H}}_i id}\downarrow & & \downarrow{\scriptstyle id\overline{\mathrm{H}}_j id} \\ \{\overline{\boldsymbol{w}}^i\}_{w_i \neq 0} & \xrightarrow{\operatorname{id} \mathrm{H}_i \operatorname{id}} & \boldsymbol{w}', \end{array}$$

where the vertices other than $\boldsymbol{w}$ and $\boldsymbol{w}'$ are the sum of all possible $\overline{\boldsymbol{w}}^i = \overline{\boldsymbol{w}}{\downarrow}i$ and $\boldsymbol{w}^j$. First going α and then $d_{\mathrm{V}^*(\lambda)}$ gives

$$\begin{aligned} \sum_{i:w_i \neq 0} d_{\overline{\boldsymbol{w}}^i \to \overline{\boldsymbol{w}}} \circ \alpha_{\boldsymbol{w},i} &= \sum_{i:w_i \neq 0} (-1)^{\sum_{j>i} b(\overline{w}_j^i)} \operatorname{id} \mathrm{H}_i \operatorname{id} \circ (-1)^{\sum_{r>j>i} b(w_j)} z_i \operatorname{id} \overline{\mathrm{H}}_i \operatorname{id} \\ &= \sum_{i:w_i \neq 0} (-1)^{\sum_{r>j>i} b(w_j)} (-1)^{\sum_{r>j>i} b(w_j)} z_i \operatorname{id} \mathrm{H}_i \operatorname{id} \circ \operatorname{id} \overline{\mathrm{H}}_i \operatorname{id} \\ &= \sum_{i:w_i \neq 0} z_i \operatorname{id}(\mathrm{H}_i \circ \overline{\mathrm{H}}_i) \operatorname{id}. \end{aligned}$$

Going first d and then β gives $\operatorname{id} \overline{\mathrm{H}}_{i_r} \circ \operatorname{id} \mathrm{H}_{i_r} = \operatorname{id}(\overline{\mathrm{H}}_{i_r} \circ \mathrm{H}_{i_r})$. And going first d and then α gives

$$\begin{aligned} \sum_{i<r:w_i=0} \alpha_{\boldsymbol{w}^i,i} \circ d_{\boldsymbol{w} \to \boldsymbol{w}^i} &= \sum_{i<r:w_i=0} (-1)^{\sum_{r>j>i} b(w_j^i)} z_i \operatorname{id} \overline{\mathrm{H}}_i \operatorname{id} \circ (-1)^{\sum_{j>i} b(w_j)} \operatorname{id} \mathrm{H}_i \operatorname{id} \\ &= \sum_{i<r:w_i=0} (-1)^{\sum_{r>j>i} b(w_j)} z_i (-1)^{\sum_{j>i} b(w_j)} \operatorname{id} \overline{\mathrm{H}}_i \operatorname{id} \circ \operatorname{id} \mathrm{H}_i \operatorname{id} \\ &= \sum_{i<r:w_i=0} -z_i \operatorname{id}(\overline{\mathrm{H}}_i \circ \mathrm{H}_i) \operatorname{id}. \end{aligned}$$

Since for $w_i = 0$ we have $\operatorname{id}(\mathrm{H}_i \circ \overline{\mathrm{H}}_i) \operatorname{id} = (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big)$ and for $w_i \neq 0$ instead $\operatorname{id}(\overline{\mathrm{H}}_i \circ \mathrm{H}_i) \operatorname{id} = (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big)$ by neckcutting, we have to show that the following holds when applied to $\mathrm{T}(\lambda_0) \times [0,1]$

$$\begin{aligned} \sum_{i:w_i \neq 0} z_i (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) &= \sum_{i<r:w_i=0} -z_i (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) + (\bullet_r)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) \\ &\Leftrightarrow \sum_i z_i (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) = 0 \end{aligned}$$

where $z_r = -1$. But this is just Proposition 9.1.9, since $\sum_i z_i (\bullet_i)\big(\boldsymbol{w}.(i_1, \dots, i_r)\big) = \mathbf{z}\big(\boldsymbol{w}.(i_1, \dots, i_r)\big)$ by definition.

□

Remark 9.1.13. For the construction of $f_{\lambda,\nu}$ we assumed that $\nu = \lambda_0.s_{i_1} \dots s_{i_t} s_i$ and $\lambda = \lambda_0.s_{i_1} \dots s_{i_t}$. Now we want to consider the signs (the rest does not change), if

$\nu = \lambda_0.s_{j_1} \dots s_{j_r}$ for $s_{j_1} \dots s_{j_r} = \mathbf{s}(T)$ for $T \in Y(n,k)$ and we delete an entry associated to a lower-right box in T, but possibly not s_{j_r}. Let j_s be the entry we delete and $\lambda = \lambda_0.s_{j_1} \dots \widehat{s_{j_r}} \dots s_{j_r}$. Then by Lemma 6.1.9, we have an isomorphism $\mathrm{T}(\lambda_0).R(j_1,\dots,j_r) \to \mathrm{T}(\lambda_0).R(j_1,\dots,\widehat{j_s},\dots,j_r,j_s)$ given by height isomorphisms from each entry determined by $\boldsymbol{w}$ to the isomorphic entry determined by $\boldsymbol{w'}$ with sign $(-1)^{b(w_{j_s})(b(w_{j_{s+1}})+\dots+b(w_{j_r}))}$. Here $\boldsymbol{w'}$ is defined by

$$w'_i = \begin{cases} w_i & \text{if } i < s, \\ w_{i+1} & \text{if} r > i \geq s, \\ w_s & \text{if } i = r. \end{cases}$$

When considering the map

$$f_{\lambda,\nu} : \mathrm{T}(\lambda_0).R(j_1,\dots,\widehat{j_s},\dots,j_r,j_s) \to \mathrm{T}(\lambda_0).R(j_1,\dots,\widehat{j_s},\dots,j_r)$$

restricted to the entry determined by $\boldsymbol{w'}$, then the signs are as follows: The sign is $(-1)^{\sum_{r>j>i} b(w'_j)} z_i$ if the entry at the ith position is deleted for $i \neq r$ and the sign is $+1$ if $r = s$. By definition of $\boldsymbol{w'}$ we obtain

$$(-1)^{\sum_{r-1>j>i} b(w'_j)} = (-1)^{\sum_{s>j>i} b(w_j)} (-1)^{\sum_{r>j\geq s} b(w_{j+1})}.$$

Now we put this together to a map from $\mathrm{T}(\lambda_0).R(j_1,\dots,j_r)$ to $\mathrm{T}(\lambda_0).R(j_1,\dots\widehat{j_s}\dots,j_r)$: Since we can only delete j_s if $b(w_{j_s}) = 0$, the sign for this is $+1$. And if we delete something else that is in the same row or column as j_s then the sign is

$$\begin{aligned} &(-1)^{b(w_{j_s})(b(w_{j_{s+1}})+\dots+b(w_{j_r}))} (-1)^{\sum_{s>j>i} b(w_j)} (-1)^{\sum_{r>j\geq s} b(w_{j+1})} \\ &= (-1)^{(b(w_{j_s})+1)(b(w_{j_{s+1}})+\dots+b(w_{j_r}))} (-1)^{\sum_{s>j>i} b(w_j)} \\ &= (-1)^{w_{j_s}(b(w_{j_{s+1}})+\dots+b(w_{j_r}))} (-1)^{\sum_{s>j>i} b(w_j)} \end{aligned}$$

multiplied with the appropriate sign given by where in the tableau the entry stands. This appropriate sign for j_i is $(-1)^{l(j_i,j_s)}$ where by $l(j_i,j_s)$ we mean the number of lines one has to cross while going from j_s to j_i in the tableau associated to $j_1,\dots,j_r$.

Finally, we have constructed our desired non-zero homomorphism.

Theorem 9.1.14. *Let $\lambda \to \lambda s_i$, then we know that $f_{\lambda,\lambda s_i}$ is a non-zero element of* $\mathrm{Hom}_K(q\,\mathrm{V}^*(\lambda s_i), \mathrm{V}^*(\lambda))$.

Proof. We just showed that $f_{\lambda,\lambda s_i}$ is a chain map. It remains to show that it is not homotopic to 0: If there was a homotopy, then there would be a factorization of the non-zero map $q\,\mathrm{V}^*(\lambda s_i)_0 \to \mathrm{V}^*(\lambda)_0$ over $\mathrm{V}^*(\lambda)_1$. But as shown in the proof of Lemma 9.1.1 this is not possible. □

The non-zero representatives $f_{\lambda,\nu}$ will be used in the next chapter to construct a complex containing all the $\mathrm{V}^*(\lambda)$ as partcomplexes in a non-trivial way.

9.2 Degree 2 morphisms

The obvious way to get a degree 2 morphism is to compose two degree 1 morphisms. To describe the relations of these compositions we use the notion of diamonds.

Definition 9.2.1. Following [Bra02] and [ES13], we call a quadruple $(\lambda, \nu, \nu', \lambda')$ of distinct elements in $\Lambda(n, k)$ an *(oriented) diamond* if we have

$$\begin{array}{ccccc} & & \nu & & \\ & \swarrow & & \nwarrow & \\ \lambda & & & & \lambda' . \\ & \nwarrow & & \swarrow & \\ & & \nu' & & \end{array}$$

We call the triple (λ, ν, λ') *straight* if $\lambda' \to \nu \to \lambda$ and there is no ν' such that $(\lambda, \nu, \nu', \lambda')$ is a diamond.

We start our investigation of degree 2 morphisms with the case where we have a diamond.

Theorem 9.2.2. *If $(\lambda, \nu, \nu', \lambda')$ is a diamond, then $f_{\lambda',\nu} \circ f_{\nu,\lambda} = f_{\lambda',\nu'} \circ f_{\nu',\lambda}$.*

Proof. Let T be the tableau associated to σ, where $\lambda = \lambda_0.\sigma$. Recall that a right-lower box in the tableau is a box at the bottom of its column and at the right of its row. We get from λ to ν or to ν' by deleting (different) right-lower boxes and from there to λ' by deleting the other right-lower box. Since we have a diamond, these two boxes are neither in the same row nor in the same column. Let $\square_1$ be the box that is deleted while going from λ to ν and $\square_2$ the box for going from λ to ν'. Let L_1 be the set of entries of T in the same row or column as $\square_1$ and define L_2 analogously. Then, by Remark 9.1.11, up to sign, the chain map $f_{\nu,\lambda}$ is given by some $\mathrm{id}\,\overline{\mathrm{H}}_i\,\mathrm{id}$, where $i \in L_1$, and the same is true for $f_{\lambda',\nu'}$. Analogously, chain maps $f_{\nu',\lambda}$ or $f_{\lambda',\nu}$ are up to sign given by some $\mathrm{id}\,\overline{\mathrm{H}}_j\,\mathrm{id}$ with $j \in L_2$. Restricted to any homological degree $f_{\lambda',\nu} \circ f_{\nu,\lambda} = f_{\lambda',\nu'} \circ f_{\nu',\lambda}$ holds up to signs because of the involved height moves from Lemma 4.2.8 b). Thus, it only remains to check the signs.

For the signs, we assume that for $\mu \in \{\lambda, \nu, \nu', \lambda'\}$ we have $\mathrm{V}^*(\mu) = \mathrm{T}(\lambda_0).R(i_1, \ldots, i_r)$ with $i_1, \ldots, i_r$ of tableau-form. So for the signs we can use Remark 9.1.13. Let l_1 be the entry of $\square_1$ and l_2 the one of $\square_2$.

If we delete first the entry l_1 and then l_2, the sign is $+1$, the same for the other way around. Now assume without loss of generality that $\square_1$ is above $\square_2$ in T.

Let $l_1 = i_{s_1}$ and $l_2 = i_{s_2}$, so we know $s_1 < s_2$. We now consider different cases:

If we delete first the entry $k = i_j \in L_1 \setminus \{l_1\}$ and then l_2, then the sign is $(-1)^{w_{s_1}(b(w_{s_1+1})+\cdots+b(w_r))}(-1)^{\sum_{s_1>i>j} b(w_i)} z_k$. For the other way around, we get the sign $(-1)^{w_{s_1}(b(w_{s_1+1})+\ldots\widehat{b(w_{s_2})}\cdots+b(w_r))}(-1)^{\sum_{s_1>i>j} b(w_i)} z_k$. But since we can delete l_2, we know that $b(w_{s_2}) = 0$ and the signs are the same.

If we delete first $k = i_j \in L_2 \setminus \{l_2\}$ and then l_1, then the sign is $(-1)^{w_{s_2}(b(w_{s_2+1})+\cdots+b(w_r))}(-1)^{\sum_{s_2>i>j} b(w_i)} z_k$. For the other way around, we get the sign $(-1)^{w_{s_2}(b(w_{s_2+1})+\cdots+b(w_r))}(-1)^{\sum_{s_1>i>j, i\neq s_1} b(w_i)} z_k$. But since we can delete l_1, we know that $b(w_{s_1}) = 0$ and the signs are the same.

By the same considerations, if we first delete something in $L_2 \setminus \{l_2\}$ and then in $L_1 \setminus \{l_1\}$, then the signs agree, since the $b(w_i)$ that appears in one direction and not in the other is $= 0$ anyway.

□

Now we consider the case where there is no diamond. First, we go back to $\wedge\vee$-sequences and investigate how a straight triple can occur.

Lemma 9.2.3. *If (λ, ν, λ') is straight, then $\nu = \lambda' s_i$ and $\lambda = \nu s_{i\pm1}$.*

Proof. By definition of straight, $\nu = \lambda' s_i$ and $\lambda = \nu s_j$ with $i \neq j$. If $j \neq i \pm 1$, then $\lambda = \lambda' s_i s_j = \lambda' s_j s_i$ and $(\lambda, \nu, \lambda' s_j, \lambda')$ is a diamond which is a contradiction. □

Proposition 9.2.4. *Let (λ, ν, λ') be straight. Then*

$$\mathrm{Hom}_K\left(q^2\,\mathrm{T}(\lambda), \mathrm{V}^*(\lambda')\right) = 0.$$

Proof. By Lemma 9.2.3, we know that $\lambda = \lambda' s_i s_{i\pm1}$.

By Lemma 4.2.5 there is up to scalar at most one degree 2 map from $\mathrm{T}(\lambda' s_i s_{i\pm1})$ to $\mathrm{T}(\lambda')$ and it is given by

$$q^2\,\mathrm{T}(\lambda' s_i s_{i\pm1}) \cong q^2\,\mathrm{T}(\lambda').\mathcal{U}_i\mathcal{U}_{i\pm1} \xrightarrow{\mathrm{id}\,\overline{\mathrm{H}}_{i\pm1}} q\,\mathrm{T}(\lambda').\mathcal{U}_i \xrightarrow{\mathrm{id}\,\overline{\mathrm{H}}_i} \mathrm{T}(\lambda'). \tag{9.1}$$

If it is zero, then we are finished, so assume the opposite. The goal then is to construct a null-homotopy for (9.1).

By Lemma 4.2.8 b), the diagram

$$\begin{array}{ccc} q^2\,\mathrm{T}(\lambda').\mathcal{U}_i\mathcal{U}_{i\pm1} & \xrightarrow{\mathrm{id}\,\overline{\mathrm{H}}_{i\pm1}} & q\,\mathrm{T}(\lambda').\mathcal{U}_i\,\mathrm{Id} \\ \downarrow{\scriptstyle \mathrm{id}\,\overline{\mathrm{H}}_i} & & \downarrow{\scriptstyle \mathrm{id}\,\overline{\mathrm{H}}_i} \\ q\,\mathrm{T}(\lambda').\,\mathrm{Id}\,\mathcal{U}_{i\pm1} & \xrightarrow{\mathrm{id}\,\overline{\mathrm{H}}_{i\pm1}} & \mathrm{T}(\lambda') \end{array} \tag{9.2}$$

commutes.

First assume $\lambda = \lambda' s_i s_{i+1}$.

We know that $\mathrm{T}(\lambda') \cong \mathrm{T}(\lambda'_0).\boldsymbol{\mathcal{U}}(T)$ for some $T \in Y(n,k)$. By Lemma 1.1.34, there is a tableau $T' \in Y(n,k)$ such that $\boldsymbol{\mathcal{U}}(T)\mathcal{U}_i\mathcal{U}_{i+1} \cong \boldsymbol{\mathcal{U}}(T')$ and T and T' differ in the way that T' contains two more boxes, O_i and O_{i+1}, labelled i and $i+1$ at the end of some row.

Assume the boxes O_i, O_{i+1} are in the first row. We have $\mathrm{T}(\lambda').\mathcal{U}_i\mathcal{U}_{i+1} \cong \mathrm{T}(\lambda'_0).\boldsymbol{\mathcal{U}}(T') \cong \mathrm{T}(\lambda_0).\mathcal{U}_k\mathcal{U}_{k+1}\ldots\mathcal{U}_{i-1}\mathcal{U}_i\mathcal{U}_{i+1}?\ldots?$, where $?\ldots?$ stands for some $\mathcal{U}_{j_1}\ldots\mathcal{U}_{j_l}$ that are unimportant for the following calculations. Thus,

$$\begin{aligned} \mathrm{T}(\lambda').\,\mathrm{Id}\,\mathcal{U}_{i+1} &\cong \mathrm{T}(\lambda_0).\mathcal{U}_k\mathcal{U}_{k+1}\ldots\mathcal{U}_{i-1}\,\mathrm{Id}\,\mathcal{U}_{i+1}?\ldots? \\ &\cong \mathrm{T}(\lambda_0).\mathcal{U}_{i+1}\mathcal{U}_k\mathcal{U}_{k+1}\ldots\mathcal{U}_{i-1}\,\mathrm{Id}?\ldots? = 0, \end{aligned}$$

since $i+1 \neq k$ because $i+1 > i$ was in the first row. So using the diagram (9.2), we get that the only map $q^2\,\mathrm{T}(\lambda) \to \mathrm{T}(\lambda')$ is zero.

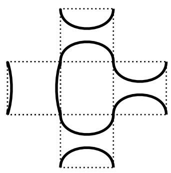
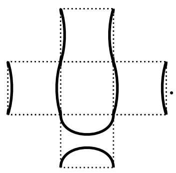
Now assume that the boxes O_i, O_{i+1} are not in the first row of T'. Let O_{i+2} be the box (labelled $i+2$) directly on top of O_{i+1} in T'. We call the associated box in T also O_{i+1}. Let $\boldsymbol{\mathcal{U}}(T) = \mathcal{U}_{i_1}\dots\mathcal{U}_{i_r}$, then $\mathrm{V}^*(\lambda') \cong \mathrm{T}(\lambda_0).R(i_1,\dots,i_r)$. Let $\boldsymbol{\mathcal{U}}(T') = \mathcal{U}_{j_1}\dots\mathcal{U}_{j_{r+2}}$. Let $l, l', s, s' \in \mathbb{N}$ such that $\mathcal{U}_{i_l}$ corresponds to O_{i+2} in T, $\mathcal{U}_{j_{l'}}$ corresponds to O_{i+2} in T', $\mathcal{U}_{j_s}$ corresponds to O_{i+1} in T' and $\mathcal{U}_{j_{s'}}$ to O_i.

Let $\boldsymbol{w} = (1,\dots,1,0,1,\dots,1)$, where there are r entries and the 0 is at place l. Let $\boldsymbol{w}'$ be a $0,1$-vector of length $r+2$, containing 0 only at places l', s and s' and let $\boldsymbol{w}''$ be a $0,1$-vector containing 0 only at place s'.

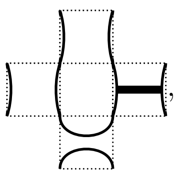
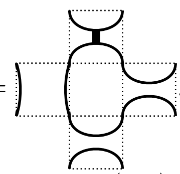
We claim that the following diagram commutes.

$$\begin{array}{ccc}
q\,\mathrm{T}(\lambda').Id\mathcal{U}_{i+1} & \xrightarrow{\mathrm{id}\,\overline{\mathrm{H}}_{i+1}} & \mathrm{T}(\lambda') \\
\downarrow{\scriptstyle\alpha_1} & & \uparrow{\scriptstyle\beta_2} \\
q\boldsymbol{w}''.(j_1,\dots,j_{r+2}) & & (1,\dots,1).(i_1,\dots,i_r) \\
\downarrow{\scriptstyle\alpha_2} & & \uparrow{\scriptstyle\beta_1^{-1}} \\
q\boldsymbol{w}'.(j_1,\dots,j_{r+2}) & & \overline{\boldsymbol{w}}.(j_1,\dots,j_{r+2}) \\
\downarrow{\scriptstyle\alpha_3} & & \uparrow{\scriptstyle\beta_1} \\
q\boldsymbol{w}.(i_1,\dots,i_r) & \xrightarrow{\mathrm{id}\,\mathrm{H}_{i_l}\,\mathrm{id}} & (1,\dots,1).(i_1,\dots,i_r)
\end{array} \tag{9.3}$$

Here, the top and bottom vertical maps $\alpha_1, \alpha_3, \beta_1, \beta_2$ are the obvious isomorphisms of the form $\mathrm{Id}\,\mathcal{U}_i \cong \mathcal{U}_i\,\mathrm{Id} \cong \mathcal{U}_i$ and $\mathcal{U}_i\mathcal{U}_j \cong \mathcal{U}_j\mathcal{U}_i$ only moving the involved $\mathcal{U}_j$ vertically or adding or forgetting Id's. The vertical map α_2 is the isomorphism of the form $\mathcal{U}_{i+1}\mathcal{U}_{i+2}\mathcal{U}_{i+1} \cong \mathcal{U}_{i+1}$, pictorially

 $\cong$ . (9.4)

On the other hand,

$\gamma := \beta_1 \circ (\mathrm{id}\,\mathrm{H}_{i_l}\,\mathrm{id}) \circ \alpha_3 =$ , $\quad \delta := \beta_1 \circ \beta_2^{-1} \circ (\mathrm{id}\,\overline{\mathrm{H}}_{i+1}) \circ \alpha_1^{-1} =$ , (9.5)

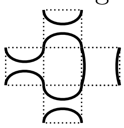
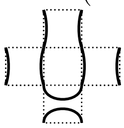
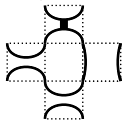
which implies $\gamma \circ \alpha_2 = \delta$. Thus, the diagram commutes and therefore

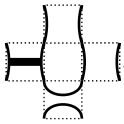
$$\mathrm{id}\,\overline{\mathrm{H}}_{i+1} = \beta_2 \circ (\mathrm{id}\,\mathrm{H}_{i_l}\,\mathrm{id}) \circ \alpha_3 \circ \alpha_2 \circ \alpha_1.$$

Hence, we constructed a homotopy h for the map (9.1), namely $h = \sigma h'$, where

$$h' : q^2\,\mathrm{T}(\lambda' s_i s_{i+1}) \cong q^2\,\mathrm{T}(\lambda').\mathcal{U}_i\mathcal{U}_{i+1} \xrightarrow{\overline{\mathrm{H}}_i} q\,\mathrm{T}(\lambda').\mathcal{U}_{i+1} \xrightarrow{\alpha_3\circ\alpha_2\circ\alpha_1} q\boldsymbol{w}.(i_1,\dots,i_r)$$

and σ is the sign attached to $\operatorname{id} \mathrm{H}_{i_l} \operatorname{id}$ as part of the differential $\mathrm{V}^*(\lambda')_1 \to \mathrm{V}^*(\lambda')_0$ in the complex realising $\mathrm{V}^*(\lambda')$ from Proposition 6.1.5. (Recall that the isomorphism β_2 is used in the interpretation of $\mathrm{V}^*(\lambda')$ as cube complex.) Thus, we are done in the case $\lambda = \lambda' s_i s_{i+1}$.

The case $\lambda = \lambda' s_i s_{i-1}$ works analogously. Namely, if the boxes with entries i, $i-1$, which are now next to each other by Lemma 1.1.34, have no box to the left, then the map (9.1) is zero anyway. If not, we take the box to the left of the $(i-1)$-box. We get again a commutative diagram as in (9.3), which allows us to define the homotopy. The analog of (9.4) is  $\cong$  and the two compositions in (9.5) are 

and . ☐

Note that the straightness assumption is really necessary, otherwise we have a totally different situation:

Lemma 9.2.5. *Let $j \neq i \pm 1$ and $\lambda' \xrightarrow{s_i} \lambda' s_i \xrightarrow{s_j} \lambda$. (In particular, $(\lambda, \lambda' s_i, \lambda')$ is not straight by Lemma 9.2.3.) Then*

$$\operatorname{Hom}_K \left(q^2\, \mathrm{T}(\lambda), \mathrm{V}^*(\lambda')\right) \cong \mathbb{C}.$$

Proof. By Lemma 4.2.5 there exists a non-zero chain map f from $q^2\, \mathrm{T}(\lambda)$ to $\mathrm{V}^*(\lambda')$. By Theorem 8.1.5, $\mathrm{V}^*(\lambda')$ is a linear complex, thus $\mathrm{V}^*(\lambda')_1$ contains only summands $q\, \mathrm{T}(\mu)$ which by Lemma 6.1.10 satisfy $\mu < \lambda'$. Thus, we have $\mathrm{V}^*(\lambda')_1 = \bigoplus q\, \mathrm{T}(\mu)$ for some $\mu < \lambda'$ and $d_{\mathrm{V}^*(\lambda')}|_{q\,\mathrm{T}(\mu)} : q\, \mathrm{T}(\mu) \to \mathrm{V}^*(\lambda')_0 \cong \mathrm{T}(\lambda')$ is a degree 1 map. But by Proposition 4.2.7, there is no degree 1 map from $q^2\, \mathrm{T}(\lambda)$ to $q\, \mathrm{T}(\mu)$, so f cannot factorise, i.e. it cannot be nullhomotopic. ☐

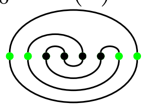
We see that under the straightness assumption there are no degree 2 morphisms:

Theorem 9.2.6. *Let (λ, ν, λ') be straight, then*

$$\operatorname{Hom}_K \left(q^2\, \mathrm{V}^*(\lambda), \mathrm{V}^*(\lambda')\right) = 0.$$

Proof. Since entries of $q^2\, \mathrm{V}^*(\lambda)$ are some (shifted) $\mathrm{T}(\mu)$, we get $\operatorname{Hom}_K(q^2\, \mathrm{V}^*(\lambda)_i, \mathrm{V}^*(\lambda')[l]) = 0$ for $l \neq 0$ and all i by Proposition 6.3.11. Using Proposition 9.2.4, the assertion follows from Corollary 5.2.5 b). ☐

From the previous theorem we obtain in particular that for (λ, ν, λ') straight there is a homotopy $h_{\lambda,\nu,\lambda'}$ such that $f_{\lambda',\nu} \circ f_{\nu,\lambda} \simeq 0$.

Example 9.2.7. Here is an Example where the composition is not zero but only homotopic to zero: Let $n = 4$, $k = 2$. Then for $\lambda = \vee\vee\wedge\wedge$, $\nu = \vee\wedge\vee\wedge$ and $\lambda' = \vee\wedge\wedge\vee$ we have that (λ, ν, λ') is straight (cf. Example 1.1.28). Now consider $\left(f_{\lambda',\nu} \circ f_{\nu,\lambda}\right)_0 = \left(f_{\lambda',\nu}\right)_0 \circ \left(f_{\nu,\lambda}\right)_0 : q^2\, \mathrm{V}^*(\lambda)_0 \to q\, \mathrm{V}^*(\nu)_0 \to \mathrm{V}^*(\lambda')_0$. This map is given by the composition of two saddles from $V^*(\lambda)_0 \cong T(\lambda)$ to $V^*(\lambda')_0 \cong T(\lambda')$. Thus, it is non-zero by Lemma 4.2.5 since $C(\lambda)\overline{C(\lambda')} =$  contains no red circle.

Chapter 10

The main player $L(\lambda_0)$

We have already seen that λ_0 plays an important role in all the constructions. In the next chapter we will construct two functors F and G categorifying the projection and inclusion of the factorisation of the Jones-Wenzl projector. For this, we construct a certain complex $L(\lambda_0)$ that contains all the $\mathrm{V}^*(\lambda)$ in a non-trivial way. After that, we study properties of this complex: We show that $\mathrm{Cob}(n)$ acts trivially on $L(\lambda_0)$. We investigate maps from $L(\lambda_0)$ to itself and construct them for $k = 0$ and $k = 1$. Finally, we show that $L(\lambda_0)$ is the injective hull of $\mathrm{T}(\lambda_0)$ in the heart of the linear t-structure.

10.1 The construction of $L(\lambda_0)$

In this section we want to construct a complex $L(\lambda_0)$ which satisfies $L(\lambda_0).\mathcal{U}_i \simeq 0$ and whose class $[L(\lambda_0)]$ in the Grothendieck group is up to a factor equal to $v_{\lambda_0}.P_n$. Here, v_{λ_0} is the standard basis element in $V^{\otimes n}$ which is also an element of the canonical basis and P_n is the Jones-Wenzl projector (cf. Chapter 2). Having Lemma 2.3.4 in mind, our first attempt is to define

$$L' := \bigoplus_{\lambda \in \Lambda(n,k)} q^{\ell(\lambda,\lambda_0)}\, \mathrm{V}^*(\lambda)[\ell(\lambda,\lambda_0)].$$

Indeed, this L' satisfies the desired condition about its class in the Grothendieck group $K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big)$:

Proposition 10.1.1. *We have*

$$[L'] = \begin{bmatrix}\!\begin{bmatrix} n \\ k \end{bmatrix}\!\end{bmatrix} v_{\lambda_0}.P_n,$$

where $\left[\!\left[{n \atop k} \right]\!\right] = \frac{[\![n]\!]!}{[\![k]\!]![\![n-k]\!]!}$ *and* $[\![n]\!] = (q^{2n}-1)/(q^2-1)$ *as in Lemma 2.3.2.*

Proof. In $K^b\big(\widehat{\mathrm{Cob}}(n)\big)$, let $\mathcal{H}_i = q\,\mathrm{Id} \xrightarrow{\mathrm{H}_i} \mathcal{U}_i$ and $\mathcal{H}(s) = \mathcal{H}_{i_1} \dots \mathcal{H}_{i_r}$ for $s = s_{i_1} \dots s_{i_r} \in \mathbb{S}_n$ a reduced expression. Here, we write the tensor product in $K^b\big(\widehat{\mathrm{Cob}}(n)\big)$ simply by juxtaposition, analogously to the tensor product in $\widehat{\mathrm{Cob}}(n)$.

For $\lambda \to \lambda s_i$ we have $\mathrm{V}^*(\lambda).\mathcal{H}_i \cong \mathrm{V}^*(\lambda.s_i)$, since $\mathrm{Cone}\big(q\,\mathrm{V}^*(\lambda) \xrightarrow{H_i} \mathrm{V}^*(\lambda).\mathcal{U}_i\big)$ and $\mathrm{V}^*(\lambda).\mathcal{H}_i$ agree up to signs and the obvious sign-changing morphism is an isomorphism. Therefore, $\mathrm{V}^*(\lambda_0).\mathcal{H}(s) \cong \mathrm{V}^*(\lambda_0 s)$ for $s \in \mathrm{W}^{\min}$. Thus,

$$\begin{aligned}[L'] &= \sum_{\lambda \in \Lambda(n,k)} (-q)^{\ell(\lambda,\lambda_0)}[\mathrm{V}^*(\lambda)] \\ &= \sum_{s \in \mathrm{W}^{\min}} (-q)^{l(s)}[\mathrm{V}^*(\lambda_0).\mathcal{H}(s)] \\ &= \sum_{s \in \mathrm{W}^{\min}} (-q)^{l(s)} v_{\lambda_0}.H(s)\end{aligned}$$

and the assertion follows from Lemma 2.3.2 and Lemma 2.3.4. □

But since not every μ satisfies $\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$ we cannot have $L'.\mathcal{U}_i \simeq 0$.

Thus, we want to construct a complex having the $\mathrm{V}^*(\lambda)$ as the same places as in L' but differentials in between:

Definition 10.1.2. Let $r = \max\big\{\ell(\lambda,\lambda_0) \mid \lambda \in \Lambda(n,k)\big\} = k(n-k)$. For $0 \le j \le r$ consider the complex

$$M(j) = \bigoplus_{\ell(\lambda,\lambda_0)=j} \mathrm{V}^*(\lambda).$$

We define inductively (depending on certain morphisms g_j, $0 \le j \le r-1$) complexes $L(j)$ as follows: Set $L(r) = M(r)$. Assume $L(s)$ for $r \ge s > j$ is already constructed and we are given a chain map $g_j : qL(j+1) \to M(j)$. Set $L(j) = \mathrm{Cone}(g_j)$.

Note that by construction $[L(0)] = [L']$ is independent of the choices for g_j.

In a perfect world we could construct $L(0)$ by using only maps g_j given by the $f_{\lambda,\nu}$ constructed in Chapter 9. But as we have seen in Theorem 9.2.6 and Example 9.2.7 in general we only know $f_{\lambda',\nu} \circ f_{\nu,\lambda} \simeq 0$ for (λ,ν,λ') straight and not $f_{\lambda',\nu} \circ f_{\nu,\lambda} = 0$, so we cannot obtain a chain complex using this naive approach.

We also need to introduce some signs such that using Theorem 9.2.2 for diamonds we obtain 0 when doing the differential twice.

Definition 10.1.3. Let $\vec{P} := \{(\nu,\lambda) \in \Lambda(n,k) \times \Lambda(n,k) \mid \nu \to \lambda\}$. Let $\sigma : \vec{P} \to \{\pm 1\}$ be a sign assignment such that for every diamond $(\lambda,\nu,\nu',\lambda')$, the signs satisfy

$$\sigma(\nu,\lambda)\sigma(\lambda',\nu)\sigma(\nu',\lambda)\sigma(\lambda',\nu') = -1.$$

In particular we have $\sigma(\nu,\lambda)\sigma(\lambda',\nu) + \sigma(\nu',\lambda)\sigma(\lambda',\nu') = 0$. Such sign assignments exist by [BS10, Section 7]. From now on we fix such a choice.

The goal now is to show the following theorem which tells us that it is possible to choose the g_j such that they contain the $f_{\lambda,\nu}$ together with the new signs:

Theorem 10.1.4. *In each step of the construction from Definition 10.1.2, there is a unique choice of* $g_j : qL(j+1) \to M(j)$ *such that*

$$g_j|_{qM(j+1)} = \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+1} \{\sigma(\nu,\lambda) f_{\nu,\lambda}\}_{\lambda\leftarrow\nu}. \tag{10.1}$$

We will prove this by downward induction. For the base case note that there is a unique $\lambda_r \in \Lambda(n,k)$ of maximal length $\ell(\lambda_r,\lambda_0)$, namely $\lambda_r = \underbrace{\vee\ldots\vee}_{n-k}\underbrace{\wedge\ldots\wedge}_{k}$, and a unique $\lambda_{r-1} \in \Lambda(n,k)$ of length one smaller, namely $\lambda_{r-1} = \lambda_r s_{n-k}$. Thus, the base case $j = r-1$ is true, since $L(r) = \mathrm{V}^*(\lambda_r)$, $M(r-1) = \mathrm{V}^*(\lambda_{r-1})$ and $\lambda_{r-1} \to \lambda_r$, thus by Theorem 9.1.14 the unique map exists.

For the other induction steps, we need some preparation:

Definition 10.1.5. We call $\nu \in \Lambda(n,k)$ *single*, if there is only one μ such that $\nu \to \mu$.

For example, in $\Lambda(4,2)$, the element $\wedge\vee\vee\wedge$ is single since there is only $\wedge\vee\vee\wedge \to \vee\wedge\vee\wedge$, cf. Example 1.1.28.

For ν single, the construction of g_j is in some sense local, i.e. it does not involve the already constructed g_i's.

Proposition 10.1.6. *Assume* $L(j+1)$ *is already constructed. Let* ν *be single and* $\ell(\nu,\lambda_0) = j \leq r-2$. *Then*

a)
$$\mathrm{Hom}_K\big(qL(j+1), \mathrm{V}^*(\nu)\big) \cong \mathbb{C}$$

b) there exist some (by a) unique) $f_j \in \mathrm{Hom}_K\big(qL(j+1), \mathrm{V}^*(\nu)\big)$ *such that*

$$f_j|_{qM(j+1)} = \bigoplus_{\substack{\lambda:\ell(\lambda,\lambda_0)=j+1\\ \lambda\leftarrow\nu}} \sigma(\nu,\lambda) f_{\nu,\lambda}.$$

Proof. By Proposition 6.3.11 we have $\mathrm{Hom}_K\big((qL(j+1))_i[m], \mathrm{V}^*(\nu)\big) = 0$ for all $m \neq 0$. Furthermore, by Lemma 6.1.10

$$\begin{aligned}\big(qL(j+1)\big)_0 = \big(qM(j+1)\big)_0 = q \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j-1} \mathrm{V}^*(\lambda)_0 &\cong \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j-1} q\,\mathrm{T}(\lambda)\\ &= q\,\mathrm{T}(\mu) \oplus \bigoplus_{\substack{\lambda:\ell(\lambda,\lambda_0)=j+1\\ \text{not } \nu\to\lambda}} q\,\mathrm{T}(\lambda),\end{aligned}$$

where μ is given by ν single, i.e. $\nu \to \mu$. Thus, by Proposition 6.3.6 and Lemma 9.1.1 we obtain

$$\mathrm{Hom}_K\big((qL(j+1))_0, \mathrm{V}^*(\lambda)\big) \cong \mathrm{Hom}_K\big(q\,\mathrm{T}(\nu), \mathrm{V}^*(\lambda)\big) \cong \mathbb{C}.$$

Moreover,

$$\big(qL(j+1)\big)_1 = q^2 M(j+2)_0 \oplus qM(j+1)_1 \cong \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+2} q^2\,\mathrm{T}(\lambda) \oplus \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+1} q\,\mathrm{V}^*(\lambda)_1.$$

Since ν is single, for λ with $\ell(\lambda,\lambda_0) = j+2$ we either have that ν and λ are not comparable or (λ,μ,ν) is straight. Thus, by Proposition 6.3.6 and Proposition 9.2.4

$$\mathrm{Hom}_K\left(\bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j-2} q^2\,\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\right) = 0.$$

On the other hand, the λ with $\ell(\lambda,\lambda_0) = j+1$ are either equal to μ or not comparable to ν. If λ is not comparable to ν and $\mathrm{T}(\lambda')$ a shifted summand of $\mathrm{V}^*(\lambda)_1$, then by Lemma 6.1.10 $\lambda' < \lambda$ and therefore $\lambda' < \nu$ or λ' not comparable to ν. So using Proposition 6.3.6 and Proposition 9.1.2, we get

$$\mathrm{Hom}_K\Big(\big(qL(j+1)\big)_1, \mathrm{V}^*(\nu)\Big) \cong \mathrm{Hom}_K\left(\bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j-1} q\,\mathrm{V}^*(\lambda)_1, \mathrm{V}^*(\nu)\right) = 0.$$

Finally, we can apply Corollary 5.2.5 a) to obtain assertion a).

Now let $0 \neq f \in \mathrm{Hom}_K\big(qL(j+1), \mathrm{V}^*(\nu)\big)$. Then $f = g' \oplus g$, where $g' : q^2L(j+2)[1] \to \mathrm{V}^*(\nu)$ is a multi-map and $g : qM(j+1) \to \mathrm{V}^*(\nu)$ is a chain map. By the next lemma (Lemma 10.1.7) and Theorem 6.3.9, we can assume that

$$g|_{\bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j-1,\lambda\neq\mu} q\,\mathrm{V}^*(\lambda)} = 0.$$

Assume that also $g|_{q\,\mathrm{V}^*(\mu)} = 0$. Then g' is a chain map and by Corollary 6.3.12 homotopic to zero. But this is a contradiction to $f \neq 0$, so $g|_{q\,\mathrm{V}^*(\mu)} \neq 0$. By Theorem 9.1.3, we know $\mathrm{Hom}_K\big(q\,\mathrm{V}^*(\mu), \mathrm{V}^*(\nu)\big) = \mathbb{C}$, so there is a scalar τ such that $\tau g|_{q\,\mathrm{V}^*(\mu)} = \sigma(\nu,\mu) f_{\nu,\mu}$. We choose $f_j := \tau f$ which yields assertion b). □

For the last proof we need the following rather general lemma.

Lemma 10.1.7. *Let* $\varphi \in \mathrm{Hom}_K\Big(\mathrm{Cone}\big(A \to (B \oplus C)\big), D\Big)$ *and* $\mathrm{Hom}_K(C,D) = 0$. *Then* φ *is homotopic to some* φ' *with* $\varphi'|_C = 0$.

Proof. This is a special case of Lemma 5.1.1 a). Since we have a cone and a direct sum, the differential of $\mathrm{Cone}\big(A \to (B \oplus C)\big)$ goes from C_i to nothing else except C_{i-1}. □

For considering the harder case where ν is not single, we need the following easy statement.

Lemma 10.1.8. *Assume* $\nu \to \mu_1$, $\nu \to \mu_2$ *and* $\mu_1 \neq \mu_2$. *Then there is a* λ *such that* $(\lambda,\mu_1,\mu_2,\nu)$ *is diamond. Furthermore, if* $\nu \to \delta \to \lambda$, *then* $\delta \in \{\mu_1,\mu_2\}$.

Proof. By definition $\mu_1 = \nu s_i$ and $\mu_2 = \nu s_j$ with $i \neq j$. Also $i \neq j \pm 1$, since otherwise not both s_i and s_j can be applied. We set $\lambda = \nu s_i s_j$ which makes of course $(\lambda,\mu_1,\mu_2,\nu)$ into a diamond. The second part follows from the description of $\lambda = \nu s_i s_j = \nu s_j s_i$. □

Lemma 10.1.9. *Fix* ν *not single and assume that for all diamonds* (λ,μ,μ',ν) *we have*

$$\sigma(\mu,\lambda)\tau_{\nu,\mu} + \sigma(\mu',\lambda)\tau_{\nu,\mu'} = 0$$

for some $\tau_{\nu,\mu}, \tau_{\nu,\mu'} \in \mathbb{C}$. Then there is a constant $c \in \mathbb{C}$ such that

$$\tau_{\nu,\mu} = c \cdot \sigma(\nu, \mu)$$

for all $\nu \to \mu$.

Proof. Since ν is not single, for every $\nu \to \mu_1$ there is some $\nu \to \mu_2$ with $\mu_1 \neq \mu_2$. Thus, by the lemma above there is some λ such that $(\lambda, \mu_1, \mu_2, \nu)$ is a diamond. From $\sigma(\mu_1, \lambda)\tau_{\nu,\mu_1} + \sigma(\mu_2, \lambda)\tau_{\nu,\mu_2} = 0$ we obtain by the definition of $\sigma(-,-)$ that

$$\begin{aligned} \tau_{\nu,\mu_1} &= \tau(\mu_1, \mu_2)\sigma(\nu, \mu_1) \\ \tau_{\nu,\mu_2} &= \tau(\mu_1, \mu_2)\sigma(\nu, \mu_2), \end{aligned}$$

for $\tau(-,-)$ a scalar.

For every μ_0 with $\nu \to \mu_0$, by Lemma 10.1.8, for all $\nu \to \mu \neq \mu_0$, we have $\tau_{\nu,\mu_0} = \tau(\mu_0, \mu)\sigma(\nu, \mu_0)$, so $\tau(\mu_0, \mu) = \tau(\mu_0, \mu')$ for all $\nu \to \mu, \mu'$. Using $\tau(\mu, \mu') = \tau(\mu', \mu)$ for all $\nu \to \mu, \mu'$, we obtain $\tau(-,-) \equiv c$ for some constant c.

□

Proposition 10.1.10. *Assume $L(j+1)$ is constructed using the g_{j+1} satisfying* (10.1)*, i.e.*

$$d_{qL(j+1)}|_{q^2M(j+2)[1] \to qM(j+1)} = \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+2} \left\{\sigma(\mu, \lambda) f_{\mu,\lambda}\right\}_{\lambda \leftarrow \mu}.$$

Then

$$\mathrm{Hom}_K\left(qL(j+1), \mathrm{V}^*(\nu)\right) \cong \mathbb{C}.$$

Proof. As before, by Proposition 6.3.11 we have

$$\mathrm{Hom}_K\left((qL(j+1))_i[m], \mathrm{V}^*(\nu)\right) = 0$$

for all $m \neq 0$. Furthermore,

$$\big(qL(j+1)\big)_0 \cong \bigoplus_{\mu:\ell(\mu,\lambda_0)=j+1} q\,\mathrm{T}(\mu) = \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1 \\ \text{not}\,\nu \to \mu}} q\,\mathrm{T}(\mu) \ \oplus \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1 \\ \nu \to \mu}} q\,\mathrm{T}(\mu).$$

Thus, by Proposition 6.3.6 and Lemma 9.1.1 we obtain

$$\mathrm{Hom}_K\left((qL(j+1))_0, \mathrm{V}^*(\nu)\right) \cong \mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1 \\ \nu \to \mu}} q\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\right) \cong \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1 \\ \nu \to \mu}} \mathbb{C}.$$

Moreover,

$$\big(qL(j+1)\big)_1 \cong \bigoplus_{\mu:\ell(\mu,\lambda_0)=j+2} q^2\,\mathrm{T}(\mu) \oplus \bigoplus_{\mu:\ell(\mu,\lambda_0)=j+1} q\,\mathrm{V}^*(\mu)_1$$

$$= \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\mu}} q^2\,\mathrm{T}(\mu) \oplus \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu,\mu \text{ unrel.}}} q^2\,\mathrm{T}(\mu)$$
$$\oplus \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1\\ \nu\to\mu}} q\,\mathrm{V}^*(\mu)_1 \oplus \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1\\ \text{not } \nu\to\mu}} q\,\mathrm{V}^*(\mu)_1.$$

By Proposition 6.3.6 we know

$$\mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu,\mu \text{ unrel.}}} q^2\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\right) = 0.$$

We have that $\nu \to \delta \to \mu$ can either be straight or part of a diamond. So by Proposition 9.2.4 and Lemma 9.2.5, we get

$$\mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\mu}} q^2\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\right) \cong \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\mu \text{ in diamond}}} \mathbb{C}$$

Furthermore, by Proposition 9.1.2

$$\mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j-1\\ \nu\to\mu}} q\,\mathrm{V}^*(\mu)_1, \mathrm{V}^*(\nu)\right) = 0.$$

If μ is not comparable to ν and $\mathrm{T}(\mu')$ a shifted summand of $\mathrm{V}^*(\mu)_1$, then by Lemma 6.1.10 $\mu' < \mu$ and therefore $\mu' < \nu$ or μ' not comparable to ν; so using Proposition 6.3.6 we get

$$\mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j-1\\ \text{not } \nu\to\mu}} q\,\mathrm{V}^*(\mu)_1, \mathrm{V}^*(\nu)\right) = 0.$$

Thus, altogether, we have

$$\mathrm{Hom}_K\left((qL(j+1))_1, \mathrm{V}^*(\nu)\right) \cong \mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\mu \text{ in diamond}}} q^2\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\right) \cong \bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\mu \text{ in diamond}}} \mathbb{C}$$

Now, by Corollary 5.2.5d), we have to consider

$$\ker\Big(\mathrm{Hom}_K\left((qL(j+1))_0, \mathrm{V}^*(\nu)\right) \xrightarrow{-\circ d_L} \mathrm{Hom}_K\left((qL(j+1))_1, \mathrm{V}^*(\nu)\right)\Big)$$

$$\cong \ker\left(\mathrm{Hom}_K\Big(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1\\ \nu\to\mu}} q\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\Big) \xrightarrow{-\circ d_L} \mathrm{Hom}_K\Big(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+2\\ \nu\to\to\lambda \text{ in diamond}}} q^2\,\mathrm{T}(\lambda), \mathrm{V}^*(\nu)\Big)\right).$$

By assumption $d_L = \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+2} \{\sigma(\mu,\lambda)f_{\mu,\lambda}\}_{\lambda\leftarrow\mu}$, thus we have to precompose with

$$\bigoplus_{\nu\to\to\lambda} \{\sigma(\mu,\lambda)(f_{\mu,\lambda})_0\}_{\lambda\leftarrow\mu} = \bigoplus_{\nu\to\to\lambda} \{\sigma(\mu,\lambda)\overline{\mathrm{H}}_{\mu,\lambda}\}_{\lambda\leftarrow\mu},$$

where $\overline{\mathrm{H}}_{\mu,\lambda} = \overline{\mathrm{H}}_i$ for $\mu \xrightarrow{s_i} \lambda$.

By Lemma 4.2.3, every $f \in \mathrm{Hom}_K\left(\bigoplus_{\substack{\mu:\ell(\mu,\lambda_0)=j+1\\ \nu\to\mu}} q\,\mathrm{T}(\mu), \mathrm{V}^*(\nu)\right)$ is of the form $f = \bigoplus_{\nu\to\mu} c_\mu \overline{\mathrm{H}}_{\nu,\mu}$. Thus,

$$\begin{aligned} f\circ d_L &= \bigoplus_{\substack{\nu\to\to\lambda\\ \text{in diamond}}} \sum_{\nu\to\mu\to\lambda} c_\mu \overline{\mathrm{H}}_{\nu,\mu}\circ\sigma(\mu,\lambda)\overline{\mathrm{H}}_{\mu,\lambda} \\ &= \bigoplus_{\substack{\nu\to\to\lambda\\ (\lambda,\mu,\mu',\nu)\text{ diamond}}} \left(c_\mu \overline{\mathrm{H}}_{\nu,\mu}\circ\sigma(\mu,\lambda)\overline{\mathrm{H}}_{\mu,\lambda} + c_{\mu'}\overline{\mathrm{H}}_{\nu,\mu'}\circ\sigma(\mu',\lambda)\overline{\mathrm{H}}_{\mu',\lambda}\right) \\ &= \bigoplus_{\substack{\nu\to\to\lambda\\ (\lambda,\mu,\mu',\nu)\text{ diamond}}} \left(c_\mu\sigma(\mu,\lambda) + c_{\mu'}\sigma(\mu',\lambda)\right)\overline{\mathrm{H}}_{\nu,\mu}\circ\overline{\mathrm{H}}_{\mu,\lambda}, \end{aligned}$$

since for every $\nu\to\to\lambda$ in a diamond there are unique $\mu_1\neq\mu_2$ such that $(\nu,\mu_1,\mu_2,\lambda)$ is a diamond and they satisfy $\overline{\mathrm{H}}_{\nu,\mu}\circ\overline{\mathrm{H}}_{\mu,\lambda} = \overline{\mathrm{H}}_{\nu,\mu'}\circ\overline{\mathrm{H}}_{\mu',\lambda}$ because the saddles are on different strands since we have a diamond.

Therefore, for f to be in the kernel we need

$$c_\mu\sigma(\mu,\lambda) + c_{\mu'}\sigma(\mu',\lambda) = 0.$$

By Lemma 10.1.9 we get $c_\mu = c\cdot\sigma(\nu,\mu)$ for all $\nu\to\mu$ for a fixed $c\in\mathbb{C}$. Thus, the kernel is isomorphic to $\mathbb{C}$.

□

Proposition 10.1.11. *Let ν be not single and $\ell(\nu,\lambda_0) = j \le r-2$. Assume $L(j+1)$ is constructed using the g_{j+1} satisfying* (10.1), *i.e.*

$$d_{qL(j+1)}|_{q^2M(j+2)[1]\to qM(j+1)} = \bigoplus_{\lambda:\ell(\lambda,\lambda_0)=j+2} \{\sigma(\mu,\lambda)f_{\mu,\lambda}\}_{\lambda\leftarrow\mu}.$$

Then there is a unique $f_j \in \mathrm{Hom}_K\left(qL(j+1),\mathrm{V}^*(\nu)\right)$ such that

$$f_j|_{qM(j+1)} = \bigoplus_{\substack{\lambda:\ell(\lambda,\lambda_0)=j+1\\ \lambda\leftarrow\nu}} \sigma(\nu,\lambda)f_{\nu,\lambda}.$$

Proof. Let $0\neq g\in\mathrm{Hom}_K\left(qL(j+1),\mathrm{V}^*(\nu)\right)$. For μ with $\ell(\mu,\lambda_0) = j+1$ we have $g|_{q\,\mathrm{V}^*(\mu)}$ is a chain map. We can assume that $g|_{q\,\mathrm{V}^*(\mu)} = 0$ if ν and μ are not comparable, since otherwise we can homotope it to 0 using Theorem 6.3.9 and Lemma 10.1.7. Furthermore, if $\nu\to\mu$, then $g|_{q\,\mathrm{V}^*(\mu)} = \tau_\mu f_{\nu,\mu}$ for some $\tau_\mu\in\mathbb{C}$ by Theorem 9.1.3. Now consider λ with $\ell(\lambda,\lambda_0) = j+2$. For every such λ we have a multi-map

$g|_{q^2\,\mathrm{V}^*(\lambda)[1]} : q^2\,\mathrm{V}^*(\lambda)[1] \to \mathrm{V}^*(\nu)$. Moreover, for all $\mu \to \lambda$ with $\ell(\mu, \lambda_0) = j+1$ we have that

$$g|_{q^2\,\mathrm{V}^*(\lambda)[1]} \oplus \bigoplus_{\mu\to\lambda} g|_{q\,\mathrm{V}^*(\mu)} : \mathrm{Cone}\left(q^2\,\mathrm{V}^*(\lambda) \xrightarrow{\bigoplus_{\mu\to\lambda}\sigma(\mu,\lambda)f_{\mu,\lambda}} \bigoplus_{\mu\to\lambda} q\,\mathrm{V}^*(\mu)\right) \to \mathrm{V}^*(\nu)$$

is a chain map.

By Lemma 10.1.8, for $\nu \to \mu_1$, $\nu \to \mu_2$, $\mu_1 \neq \mu_2$, there is some λ such that $(\lambda, \mu_1, \mu_2, \nu)$ is diamond. We apply the chain map-condition when starting at $q^2\,\mathrm{T}(\lambda) \cong q^2\,\mathrm{V}^*(\lambda)_0$.

By Proposition 4.2.7, there is no map $q^2\,\mathrm{T}(\lambda) \to \mathrm{V}^*(\nu)_1$: By Theorem 8.1.5, $\mathrm{V}^*(\nu)$ is a linear complex, thus $\mathrm{V}^*(\nu)_1$ contains only summands $q\,\mathrm{T}(\tau)$ which by Lemma 6.1.10 satisfy $\tau < \nu$. Thus, every map from $q^2\,\mathrm{T}(\lambda)$ to a $q\,\mathrm{T}(\tau)$ is of degree 1 and the same is true for the differential $q\,\mathrm{T}(\tau) \to \mathrm{T}(\nu)$. Thus, Lemma 4.2.7 gives $\lambda = \nu s_i s_{i\pm 1}$, i.e. $\nu \to\to \lambda$ is not part of a diamond, which is a contradiction.

Since $0 = (g|_{q^2\,\mathrm{V}^*(\lambda)[1]})_0 : 0 \to \mathrm{V}^*(\nu)_0$, this forces

$$\begin{aligned}
0 &= \left(\bigoplus_{\nu\to\mu}(g|_{q\,\mathrm{V}^*(\mu)})_0\right) \circ \left(\bigoplus_{\mu\to\lambda}\sigma(\mu,\lambda)(f_{\mu,\lambda})_0\right) \\
&= \left(\bigoplus_{\substack{\nu\to\mu\\ \mu\to\lambda}} \tau_\mu \cdot (f_{\nu,\mu})_0\right) \circ \left(\bigoplus_{\mu\to\lambda}\sigma(\mu,\lambda)(f_{\mu,\lambda})_0\right) \\
&= \sum_{\substack{\nu\to\mu\\ \mu\to\lambda}} \sigma(\mu,\lambda)\tau_\mu \cdot (f_{\nu,\mu})_0 \circ (f_{\mu,\lambda})_0 \\
&= \sigma(\mu_1,\lambda)\tau_{\mu_1} \cdot \left(f_{\nu,\mu_1} \circ f_{\mu_1,\lambda}\right)_0 + \sigma(\mu_2,\lambda)\tau_{\mu_2} \cdot \left(f_{\nu,\mu_2} \circ f_{\mu_2,\lambda}\right)_0,
\end{aligned}$$

where the last equality holds since by Lemma 10.1.8 there is no δ with $\nu \to \delta \to \lambda$ except for μ_1, μ_2. By Theorem 9.2.2, we have $f_{\nu,\mu_1} \circ f_{\mu_1,\lambda} = f_{\nu,\mu_2} \circ f_{\mu_2,\lambda}$, hence

$$\sigma(\mu_1,\lambda)\tau_{\mu_1} + \sigma(\mu_2,\lambda)\tau_{\mu_2} = 0.$$

By Lemma 10.1.9 we obtain $\tau_\mu = c \cdot \sigma(\mu,\lambda)$, which yields $g|_{q\,\mathrm{V}^*(\mu)} = c\sigma(\mu,\lambda)f_{\mu,\lambda}$. Now choosing $f_j := \frac{1}{c} g$ gives the desired result. Uniqueness follows from Proposition 10.1.10. □

Now we have inductively proven Theorem 10.1.4 and can use it to define our desired complex.

Definition 10.1.12. Let the $g_j : qL(j+1) \to M(j)$ be as in Theorem 10.1.4. Then we define $L(\lambda_0) := L(0)$.

We will show that $L(\lambda_0)$ satisfies $L(\lambda_0).\mathcal{U}_i$ in Section 10.3. The next section is dedicated to a special family of examples of $L(\lambda_0)$. Before passing to them we want to make some general observations.

Remark 10.1.13. We can describe the entries of $L(\lambda_0)$ via the entries of the $\mathrm{V}^*(\lambda)$'s: In the ith homological degree of $L(\lambda_0)$ we have

$$L(\lambda_0)_i = \bigoplus_{j+l=i,\ell(\lambda,\lambda_0)=j} q^j\,\mathrm{V}^*(\lambda)_l.$$

Since λ_0 is the only λ that satisfies $\ell(\lambda, \lambda_0) = 0$ we have $L(\lambda_0)_0 = \mathrm{V}^*(\lambda_0)_0 = \mathrm{T}(\lambda_0)$. Furthermore, there is only one $\lambda \in \Lambda(n, k)$ with $\ell(\lambda, \lambda_0) = 1$ and it is $\lambda = \lambda_0 s_k$ since $\lambda_0 s_i = \lambda_0$ for all $i \neq k$. Hence, $L(\lambda_0)_1 = \mathrm{V}^*(\lambda_0 s_k)_0 \cong \mathrm{T}(\lambda_0 s_k)$.

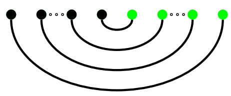
Note that $L(\lambda_0)$ is concentrated in homological degrees 0 to $2k(n-k)$.

Furthermore, we have $[L(\lambda_0)] = \left[\!\left[{n \atop k}\right]\!\right] v_{\lambda_0}.P_n$ as explained at the beginning of this section.

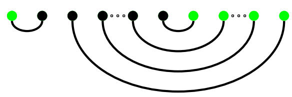
10.2 $L(\lambda_0)$ for special cases

We consider the complex $L(\lambda_0)$ for $k = 0$ and $k = 1$.

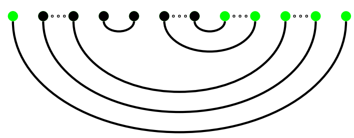
For $k = 0$, we have $\Lambda(n, k) = \{\lambda_0\} = \{\underbrace{\vee \dots \vee}_{n}\}$, thus

$L(\lambda_0) = \mathrm{V}^*(\lambda_0) = \mathrm{T}(\lambda_0) =$  .

For $k = 1$, the $\mathrm{T}(\lambda)$ with $\lambda \in \Lambda(n, k)$ contain exactly one cup that has two black endpoints if $\lambda \neq \lambda_0$. For this section, we introduce a new notation:

$\mathrm{T}(0) := \mathrm{T}(\lambda_0) =$ 

and $\mathrm{T}(i) = \mathrm{T}(\lambda_i)$ where λ_i has the $\wedge$ at place $i + 1$, i.e.

$\mathrm{T}(i) =$  .

The same notation is used for the $\mathrm{V}^*(\lambda)$'s.

Now the morphisms between the $\mathrm{T}(i)$ in $\mathrm{Cup}(n, 1)$ are easy to determine and they are described by the following quiver:

$$\mathrm{T}(0) \underset{\overline{\mathrm{H}}_1}{\overset{\mathrm{H}_1}{\rightleftarrows}} \mathrm{T}(1) \underset{\overline{\mathrm{H}}_2}{\overset{\mathrm{H}_2}{\rightleftarrows}} \mathrm{T}(2) \underset{\overline{\mathrm{H}}_3}{\overset{\mathrm{H}_3}{\rightleftarrows}} \cdots \underset{\overline{\mathrm{H}}_{n-1}}{\overset{\mathrm{H}_{n-1}}{\rightleftarrows}} \mathrm{T}(n-1)$$

modulo the relations

$$\mathrm{H}_{i+1}\,\mathrm{H}_i = 0 = \overline{\mathrm{H}}_i\overline{\mathrm{H}}_{i+1} \tag{10.2}$$

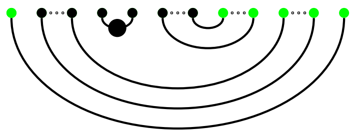
$$\overline{\mathrm{H}}_{i+1}\,\mathrm{H}_{i+1} = \mathrm{H}_i\,\overline{\mathrm{H}}_i \tag{10.3}$$

$$\overline{\mathrm{H}}_0\,\mathrm{H}_0 = 0. \tag{10.4}$$

We have already considered some of the relations for $n = 2, 3$ in Example 3.3.3. The relations can be seen directly from the relations of $\mathrm{Cup}(n, 1)$, but some also follow easier from the theory of Section 3.4: Relation (10.4) follows from Lemma 3.4.15 or alternatively from neckcutting and the additional relation 1) of $\mathrm{Cup}(n, 1)$. Relation (10.2) follows either directly from the additional relation 2) of $\mathrm{Cup}(n, 1)$ or from the fact that

$$\mathrm{Hom}_{\mathrm{Cup}(n,1)}\big(\mathrm{T}(i), \mathrm{T}(i+2)\big) = 0 = \mathrm{Hom}_{\mathrm{Cup}(n,1)}\big(\mathrm{T}(i+2), \mathrm{T}(i)\big)$$

by Corollary 3.4.13, since $\mathrm{T}(i)\overline{\mathrm{T}(i+2)}$ has a red circle. From neckcutting and the additional relation 1) and of $\mathrm{Cup}(n, 1)$ we see that

$\overline{\mathrm{H}}_{i+1}\,\mathrm{H}_{i+1} =$  $= \mathrm{H}_i\,\overline{\mathrm{H}}_i,$

so relation (10.3) holds. Also, for $i \neq 0$, $\mathrm{T}(i)\overline{\mathrm{T}(i)}$ has one black and several green circles. Since we know

$$\dim \mathrm{Hom}_{\mathrm{Cup}(n,1)}\big(\mathrm{T}(i), \mathrm{T}(j)\big)_r = \begin{cases} 1 & \text{if } r = 0, i = j, \\ 1 & \text{if } r = 1, i = j \pm 1, \\ 1 & \text{if } r = 2, i = j \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

from the considerations above and Lemma 4.2.3 the dimensions are as required those are all relations needed.

Furthermore, from the description via hypercubes $\mathrm{V}^*(i) = \mathrm{V}^*(\lambda_0.s_1 \ldots s_i) \cong \mathrm{T}(0).R(1, \ldots, i)$ (cf. Proposition 6.1.5) we obtain

$$\mathrm{V}^*(i) \cong \Big(q^i\,\mathrm{T}(0) \xrightarrow{(-1)^{i-1}\,\mathrm{H}_1} q^{i-1}\,\mathrm{T}(1) \xrightarrow{(-1)^{i-2}\,\mathrm{H}_2} \ldots \xrightarrow{-\,\mathrm{H}_{i-1}} q\,\mathrm{T}(i-1) \xrightarrow{\mathrm{H}_i} \mathrm{T}(i)\Big),$$

since $\mathrm{T}(0).\mathcal{U}_j = 0$ for $j \neq 1$ and $\mathrm{T}(0).U_1 \ldots \mathcal{U}_j \cong \mathrm{T}(j)$. Hence, $\mathrm{V}^*(i)$ is isomorphic to

$$q^i\,\mathrm{T}(0) \xrightarrow{\mathrm{H}_1} q^{i-1}\,\mathrm{T}(1) \xrightarrow{\mathrm{H}_2} \ldots \xrightarrow{\mathrm{H}_{i-1}} q\,\mathrm{T}(i-1) \xrightarrow{\mathrm{H}_i} \mathrm{T}(i)$$

(see also Remark 10.2.1 below) giving us the representative for the isomorphism class of $\mathrm{V}^*(i)$ we will use from now on. The chain maps $f_{\nu,\lambda}$ for $\nu \to \lambda$ from the construction of $L(\lambda_0)$ for this representative are now $f_{i-1,i} : q\,\mathrm{V}^*(i) \to \mathrm{V}^*(i-1)$ given by the saddle $\overline{\mathrm{H}}_j : q\,\mathrm{T}(j) \to \mathrm{T}(j-1)$ in every degree, see also Example 9.1.12.

The following remark allows us to ignore the signs:

Remark 10.2.1. Let W and W' be double complexes with $W = W'$ when forgetting the differentials and assume the differentials differ only by signs. Let $R = \mathrm{Tot}(W)$ and $R' = \mathrm{Tot}(W')$ be the associated total complexes. Then there is an isomorphism ι consisting of maps $\pm\,\mathrm{id}$ only in each degree between R and R', i.e. every $\iota_j : R_j = A_1 \oplus \cdots \oplus A_l \to R'_j = A_1 \oplus \cdots \oplus A_l$ is a diagonal matrix with the entries on the diagonal either id or $-\,\mathrm{id}$. We call such an isomorphism a *$\pm$-isomorphism.*

Indeed, since the squares in the double complex have to anti-commute, when considering the same square in W as in W', then the signs of 0, 2 or 4 maps differ. Thus, it is possible to make the map $\mathrm{id}\colon W \to W'$ into a morphism of double complexes by adding signs going square by square. This gives a $\pm$-isomorphism between the double complexes which induces one between the total complexes.

In particular, viewing the complex as a one row double complex, all the possible signs for the $\mathrm{V}^*(j)$ give rise to $\pm$-isomorphisms. Moreover, if we have a complex R that can be written as the total complex of a double complex W where every $W_{i,j}$ is just a single $q^{r_i}\,\mathrm{T}(t_j)$, then we can ignore the signs of the differentials since every possible choice leads to a complex isomorphic to R.

Lemma 10.2.2. *For $k = 1$, $L(\lambda_0)$ is of the form $\mathrm{Tot}(W)$, where W is the double complex having the $q^i\,\mathrm{V}^*(i)$ as rows with vertical maps given by the $f_{i-1,i}$ (up to signs).*

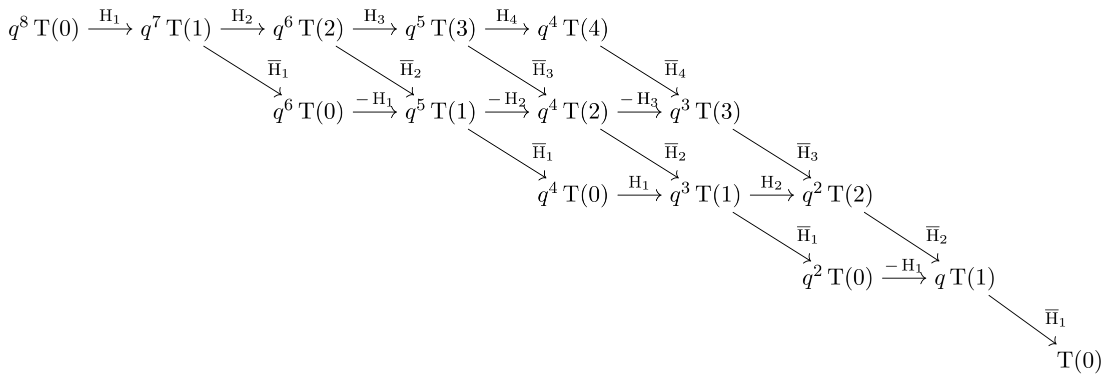
Proof. The only non-zero maps in the definition of $L(\lambda_0)$ (Definition 10.1.2 and 10.1.12) are the $f_{i-1,i}$, since the other maps in the definition would result in maps $q^{r_i}\,\mathrm{T}(i) \to q^{r_j}\,\mathrm{T}(j)$ with $i > j+1$. But there are no $\mathrm{T}(i) \to \mathrm{T}(j)$ for $i \neq j \pm 1$ in $\mathrm{Cup}(n,1)$ due to relation (10.2). Taking repeatedly the cone as in the definition of $L(\lambda_0)$ is (up to signs) the same as taking the total complex of all the maps. $\square$

Remark 10.2.3. Let $L := L(\lambda_0)$, then Remark 10.1.13 reduces in our special case to

$$L_i \cong q^i \bigoplus_{j\leq i,\, j\equiv i \bmod 2} \mathrm{T}(j)$$

for $i \leq n-1$ and $L_i = q^{2i-2(n-1)} L_{2(n-1)-i}$ for $i > n-1$. Up to sign, the differentials are the saddles $\mathrm{H}_{i+1} : q\,\mathrm{T}(i) \to \mathrm{T}(i+1)$ and $\overline{\mathrm{H}}_i : q\,\mathrm{T}(i) \to \mathrm{T}(i-1)$ if $i > 0$. One possible choice of signs is to take the saddles without signs except for $-\,\mathrm{H}_{i+1} : q\,\mathrm{T}(i) \to \mathrm{T}(i+1)$ if and only if this map is part of $q^{2l+1}\,\mathrm{V}^*(2l+1)$.

Example 10.2.4. For $n = 5$ we have $L(\lambda_0) \cong$



10.3 Categorification of a trivial TL_n-module

Now we prove the important property of $L(\lambda_0)$ that made us use the maps $f_{\lambda,\nu}$ in the construction of $L(\lambda_0)$.

Theorem 10.3.1. *$L(\lambda_0).\mathcal{U}_i \simeq 0$ for all i.*

Proof. We show that $L(\lambda_0).\mathcal{U}_i$ satisfies the conditions of the next proposition (Proposition 10.3.2). As in Definition 10.1.2 let $r = k(n-k)$. Our goal is to change the numbering such that $L(\lambda_0).\mathcal{U}_i := L(0).\mathcal{U}_i = Y(r+1) = Y$. For this we choose

$$\left\{X^{i,j} \mid 1 \leq j \leq r_i\right\} = \left\{q^{l(\mu,\lambda_0)}\,\mathrm{V}^*(\mu).\mathcal{U}_i \mid l(\mu,\lambda_0) = r+1-i\right\}$$

for every $1 \leq i \leq r+1$. Thus, we have $X(1) = q^r M(r).\mathcal{U}_i$, $X(2) = q^{r-1}M(r-1).\mathcal{U}_i$ and so on until finally $Y(r+1) = M(0).\mathcal{U}_i$. When we now set $g^2 = g_{r-1}.\,\mathrm{id}_{\mathcal{U}_i}$, $g^3 = g_{r-2}.\,\mathrm{id}_{\mathcal{U}_i}$, ..., $g^{r+1} = g_0.\,\mathrm{id}_{\mathcal{U}_i}$ with g_i as in Definition 10.1.12 we obtain $Y = L(\lambda_0).\mathcal{U}_i$. By Proposition 6.1.14 we know that $q^{l(\mu,\lambda_0)}\,\mathrm{V}^*(\mu).\mathcal{U}_i \simeq 0$ if $\mu = \mu s_i$. All the other μ satisfy $\mu \neq \mu s_i$ and therefore, by Corollary 6.1.12 they split into pairs $\left(q^{l(\mu s_i,\lambda_0)}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i, q^{l(\mu,\lambda_0)}\,\mathrm{V}^*(\mu).\mathcal{U}_i\right)$ for $\mu < \mu s_i$ which are homotopy equivalent as required. Let $l = l(\mu,\lambda_0)$ and $l' = r+1-l$. The homotopy equivalence is

$$f^{l',\mu,\mu s_i} = \begin{pmatrix} 0 \\ \beta \end{pmatrix} : q^l\,\mathrm{V}^*(\mu).\mathcal{U}_i \to q^{l+1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i$$

where $\beta = \mathrm{id}_{\mathrm{V}^*(\mu)}.\beta'$ and $\beta' = \mathrm{id} \bigcirc \mathrm{id} : \mathcal{U}_i \to \mathcal{U}_i\mathcal{U}_i$. By construction

$$g^{l'}|_{q^{l+1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i \to q^l\,\mathrm{V}^*(\mu).\mathcal{U}_i} = \sigma(\mu,\mu s_i) f_{\mu,\mu s_i}.\,\mathrm{id}_{\mathcal{U}_i}\,.$$

But $f_{\mu,\mu s_i}|_{\mathrm{V}^*(\mu).\mathcal{U}_i \to \mathrm{V}^*(\mu)} = \mathrm{id}_{\mathrm{V}^*(\mu)}\,\overline{\mathrm{H}_i}$, so when we compose we get

$$\begin{aligned} g^{l'}|_{q^{l+1}\,\mathrm{V}^*(\mu s_i).\mathcal{U}_i \to q^l\,\mathrm{V}^*(\mu).\mathcal{U}_i} \circ f^{l',\mu,\mu s_i} &= \sigma(\mu,\mu s_i)\,\mathrm{id}_{\mathrm{V}^*(\mu)}\left((\overline{\mathrm{H}}_i\,\mathrm{id}_{\mathcal{U}_i}) \circ (\mathrm{id} \bigcirc \mathrm{id})\right) \\ &= \sigma(\mu,\mu s_i)\,\mathrm{id}_{\mathrm{V}^*(\mu).\mathcal{U}_i}\,. \end{aligned}$$

Finally, for the last condition, we need that for $\mu < \mu s_i$, $\lambda < \lambda s_i$, $\ell(\lambda,\lambda_0) = l = \ell(\mu,\lambda_0)$, we get that μs_i and λ are not comparable, so that there is no map between $q^{l+1}\,\mathrm{V}^*(\mu s_i)$ and $q^l\,\mathrm{V}^*(\lambda)$. But this is just Lemma 1.1.33. $\square$

To show the theorem above we reduce it to the following abstract framework.

Proposition 10.3.2. *Assume we are in the following situation:*

- *We have chain complexes $X^{i,j}$, $i = 1, \dots, s, j = 1, \dots, r_i$ with entries in an additive category*
- *there are complexes $X(i) := \bigoplus_{j=1}^{r_i} X^{i,j}$*
- *a complex Y is constructed inductively via setting $Y(1) = X(1)$, $Y(t) = \operatorname{Cone}\big(Y(t-1) \xrightarrow{g^t} X(t)\big)$ for $t \geq 2$ and some chain maps g^t and finally $Y = Y(s)$*
- *for some i, j we have $X^{i,j} \simeq 0$*
- *all other $X^{i,j}$ split into pairs $(X^{i,t}, X^{i+1,t'})$ such that*
  - *$X^{i,t} \simeq X^{i+1,t'}$ via $f^{i,t,t'} : X^{i+1,t'} \to X^{i,t}$ with $g^{i+1}|_{X^{i,t} \to X^{i+1,t'}} \circ f^{i,t,t'} = \pm \operatorname{id}$*
  - *for $(X^{i,t}, X^{i+1,t'}) \neq (X^{i,\bar{t}}, X^{i+1,\bar{t}'})$ different pairs we have $g^{i+1}|_{X^{i,t} \to X^{i+1,\bar{t}'}} = 0$.*

Then $X \simeq 0$.

Proof. We show this by induction on s. If $s = 1$, then $X = \bigoplus_{j=1}^{r_1} X^{1,j}$ with $X^{1,j} \simeq 0$. Of course, then $X \simeq 0$.

Now for bigger s consider $X(s) = X^{s,1} \oplus \cdots \oplus X^{s,r_s}$. For some j we have $X^{s,j} \simeq 0$, assume wlog. $X^{s,j} \simeq 0$ for $j = m+1, \dots, r_s$. Thus, via $f^s := (\operatorname{id}, \dots, \operatorname{id}, 0, \dots, 0) : X(s) \to X^{s,1} \oplus \cdots \oplus X^{s,m} =: X'(s)$ we have $X(s) \simeq X'(s)$. Therefore, by Lemma 5.1.5,

$$Y = Y(s) = \operatorname{Cone}\left(Y(s-1) \xrightarrow{g^s} X(s)\right) \simeq \operatorname{Cone}\left(Y(s-1) \xrightarrow{f^s \circ g^s} X'(s)\right).$$

Since $f^s \circ g^s$ is just the restriction of g^s, we denote it by $g^s_|$. We know

$$Y(s-1) = \operatorname{Cone}\left(Y(s-2) \xrightarrow{g^{s-1}} X(s-1)\right)$$

(for $s = 2$ set $Y(s-2) = 0$). Hence,

$$\begin{aligned} Y &\simeq \operatorname{Cone}\left(\operatorname{Cone}\left(Y(s-2) \xrightarrow{g^{s-1}} X(s-1)\right) \xrightarrow{g^s_|} X'(s)\right) \\ &= \operatorname{Cone}\left(Y(s-2)[1] \xrightarrow{g^{s-1} \oplus g^{s,2}} \operatorname{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X'(s)\right)\right), \end{aligned}$$

where $g^s_| = g^{s,1} \oplus g^{s,2}$ for $g^{s,1} : X(s-1) \to X'(s)$ a chain map and $g^{s,2} : Y(s-2)[1] \to X'(s)$ a multi-map.

By the assumptions, we know that for every $X^{s,j}$, $j = 1, \dots, m$, there is a partner $X^{s-1,j'}$ satisfying the conditions above. Wlog. we can assume that $X^{s-1,j}$ is the partner of $X^{s,j}$ for $j = 1, \dots, m$. Thus, we have

$$X(s-1) \simeq X^{s,1} \oplus \cdots \oplus X^{s,m} \oplus X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}} =: X'(s-1)$$

via

$$f^{s-1} := (f^{s-1,1,1} \oplus \cdots \oplus f^{s-1,m,m} \oplus \mathrm{id} \oplus \cdots \oplus \mathrm{id}) : X'(s-1) \to X(s-1)$$

and homotopy inverse

$$\overline{f}^{s-1} := (\overline{f}^{s-1,1,1} \oplus \cdots \oplus \overline{f}^{s-1,m,m} \oplus \mathrm{id} \oplus \cdots \oplus \mathrm{id}) : X(s-1) \to X'(s-1)$$

by assumption, where $\overline{f}^{s-1,i,i}$ is the homotopy inverse of $f^{s-1,i,i}$ for $i = 1, \ldots, m$. Thus, by Lemma 5.1.5 we have

$$\mathrm{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X'(s)\right) \simeq \mathrm{Cone}\left(X'(s-1) \xrightarrow{g^{s,1} \circ f^{s-1}} X'(s)\right).$$

and

$$(g^{s,1} \circ f^{s-1})|_{X^{s,i} \to X^{s,j}} = \begin{cases} \pm \mathrm{id} & \text{if } 1 \leq i = j \leq m, \\ 0 & \text{if } 1 \leq i \neq j \leq m, \end{cases}$$

by assumption, since $g^{s,1}|_{X^{s-1,j}} = g^s|_{X^{s-1,j}}$ by construction.

But now, by Lemma 10.3.3 below, we obtain

$$\mathrm{Cone}\left(X'(s-1) \xrightarrow{g^{s,1} \circ f^{s-1}} X'(s)\right) \simeq (X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}})[1]$$

via $\alpha : \mathrm{Cone}(g^{s,1} \circ f^{s-1}) \to X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}$ with $\alpha|_{X^{s,j}} = 0$ for all $j = 1, \ldots, m$ and $\alpha|_{X^{s-1,j} \to X^{s-1,l}} = \delta_{j,l}\, \mathrm{id}$ for all $m+1 \leq j, l \leq r_{s-1}$.

Altogether, $\mathrm{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X'(s)\right) \simeq X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}$ via some

$$h : \mathrm{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X'(s)\right) \to X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}.$$

By Lemma 5.1.6 we know $h = \alpha \circ \begin{pmatrix} \overline{f}^{s-1} & 0 \\ -g^{s,1} \circ H & \mathrm{id} \end{pmatrix}$, where

$$\begin{pmatrix} \overline{f}^{s-1} & 0 \\ -g^{s,1} \circ H & \mathrm{id} \end{pmatrix} : \mathrm{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X(s)\right) \to \mathrm{Cone}\left(X'(s-1) \xrightarrow{f^{s-1} \circ g^{s,1}} X(s)\right)$$

and $H : X(s-1)[1] \to X(s-1)$.

Finally,

$$\begin{aligned} Y &\simeq \mathrm{Cone}\left(Y(s-2)[1] \xrightarrow{g^{s-1} \oplus g^{s,2}} \mathrm{Cone}\left(X(s-1) \xrightarrow{g^{s,1}} X'(s)\right)\right) \\ &\simeq \mathrm{Cone}\left(Y(s-2)[1] \xrightarrow{h \circ (g^{s-1} \oplus g^{s,2})} X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}\right). \end{aligned}$$

The explicit description of $\overline{f}^{s-1}$ and α yields that $h \circ (g^{s-1} \oplus g^{s,2}) : Y(s-2)[1] \to$

$X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}$ satisfies

$$h \circ (g^{s-1} \oplus g^{s,2})|_{X^{s-2,j} \to X^{s-1,i}} = g^{s-1}|_{X^{s-2,j} \to X^{s-1,i}}$$

for all i, j. Therefore, $Y \simeq Y'$ for some Y' that satisfies the conditions for $s-1$ when we choose $X^{s-1,m+1} \oplus \cdots \oplus X^{s-1,r_{s-1}}$ as the new $X(s-1)$. Thus, we are done by induction. □

Lemma 10.3.3. *Let $X^1, \ldots, X^r$ be chain complexes in an additive category. Let $m \leq r$ and $f : X^1 \oplus \cdots \oplus X^r \to X^1 \oplus \cdots \oplus X^m$ be a chain map such that*

$$f|_{X^i \to X^j} = \begin{cases} \pm \mathrm{id} & \text{if } 1 \leq i = j \leq m, \\ 0 & \text{if } 1 \leq i \neq j \leq m. \end{cases}$$

Then $\mathrm{Cone}(f) \simeq (X^{m+1} \oplus \cdots \oplus X^r)[1]$ *via* $\alpha : \mathrm{Cone}(f) \to (X^{m+1} \oplus \cdots \oplus X^r)[1]$ *with* $\alpha_l : (X^1 \oplus \cdots \oplus X^r)_{l-1} \oplus (X^1 \oplus \cdots \oplus X^m)_l \to (X^{m+1} \oplus \cdots \oplus X^r)_{l-1}$ *satisfying*

$$\alpha|_{X^i_l \to X^j_{l-1}} = 0 \text{ for all } i, j \quad \text{and}$$

$$\alpha|_{X^i_{l-1} \to X^j_{l-1}} = \begin{cases} \mathrm{id} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We can apply Corollary 5.1.3 for $X = X^1 \oplus \cdots \oplus X^m = Z$, $Y = X^{m+1} \oplus \cdots \oplus X^r$ and $U = 0$ since by the assumptions $\varphi : X \to Z$ is a diagonal matrix with $\pm\,\mathrm{id}$ on the diagonal, hence an isomorphism. Therefore, $\mathrm{Cone}(f) \simeq \mathrm{Cone}(Y \to 0) = X^{m+1} \oplus \cdots \oplus X^r[1]$. Since $U = 0$, the homotopy equivalence is given by $\alpha : \mathrm{Cone}(f) \to Y[1]$ with $\alpha_i = \begin{pmatrix} 0 \, \mathrm{id} \, 0 \end{pmatrix} : X_{i-1} \oplus Y_{i-1} \oplus Z_i \to Y_{i-1}$. □

10.4 Endomorphisms of $L(\lambda_0)$

The endomorphisms of $L(\lambda_0)$ form a ring which is interesting on its own but will also be used for the functor G in the next chapter.

For this section we denote $L := L(\lambda_0)$ and let $\mathrm{End}(L) := \bigoplus_{i,j \in \mathbb{Z}} \mathrm{Hom}_K(L, L\langle i \rangle [j])$.

Before we can consider $\mathrm{End}(L)$, we need to know more about maps from shifted $\mathrm{T}(\mu)$ into L. This uses the following consequence of Theorem 10.3.1.

Corollary 10.4.1. $\mathrm{Hom}_K\big(\mathrm{T}(\mu)[j]\langle l \rangle, L(\lambda_0)\big) = 0$ *for all $\mu \neq \lambda_0$ and all j, l.*

Proof. For every $\mu \neq \lambda_0$ there is some ν and some i such that $\mu = \nu s_i$. Thus, by Lemma 4.1.8 we know $0 \to \mathrm{T}(\mu) \to 0 \cong (0 \to \mathrm{T}(\nu) \to 0).\mathcal{U}_i$. Therefore, by Lemma 6.2.3, we have $\mathrm{Hom}_K(\mathrm{T}(\mu)[j]\langle l \rangle, L(\lambda_0)) \cong \mathrm{Hom}_K\big(\mathrm{T}(\nu)[j]\langle l \rangle, L(\lambda_0).\mathcal{U}_i\big)$. But by Theorem 10.3.1 we know $L(\lambda_0).\mathcal{U}_i \simeq 0$, so we are finished. □

Proposition 10.4.2. $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)[j]\langle l \rangle, L(\lambda_0)\big) \cong \begin{cases} \mathbb{C} & \text{if } j = l = 0, \\ 0 & \text{otherwise.} \end{cases}$

Proof. By Remark 10.1.13 we know that $L(\lambda_0)_0 = \mathrm{T}(\lambda_0)$ and $L(\lambda_0)_1 \cong q\,\mathrm{T}(\lambda_0.s_k)$. Furthermore, $\mathrm{Hom}_K(\mathrm{T}(\lambda_0), L(\lambda_0)_0) \cong \mathbb{C}$ and it does not factorise through $L(\lambda_0)_1$ by Lemma 3.4.15, i.e. there is no nullhomotopy. Therefore, $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0), L(\lambda_0)\big) \cong \mathbb{C}$. For $l \neq 0$, Lemma 3.4.15 yields

$$\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle, L(\lambda_0)\big) = \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle, L(\lambda_0)_0\big) = 0.$$

For $j < 0$ we trivially have

$$\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle[j], L(\lambda_0)\big) = \mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle, 0\big) = 0.$$

Now assume $j > 0$. We know that $L(\lambda_0)$ contains all $\mathrm{V}^*(\mu)$ as (shifted) partcomplexes with additional maps going to $\mathrm{V}^*(\nu)$'s with $\nu < \mu$. And by Proposition 6.2.5 we have $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle[j], \mathrm{V}^*(\mu)\big) = 0$ for all l, j and $\mu \neq \lambda_0$. Consider now $\alpha : \mathrm{T}(\lambda_0) \to L(\lambda_0)_j$, $j > 0$. Again by Remark 10.1.13 we have

$$L(\lambda_0)_j = \bigoplus_{i+l=j, \ell(\lambda,\lambda_0)=i} q^i\,\mathrm{V}^*(\lambda)_l.$$

Now choose i maximal such that there is a λ with $\ell(\lambda,\lambda_0) = i$ and $\alpha|_{\mathrm{T}(\lambda_0)\to\mathrm{V}^*(\lambda)} \neq 0$. In the construction of $L(\lambda_0)$ we first take $M(a) = \bigoplus_{\ell(\lambda',\lambda_0)=a}$ and then find maps between $M(a)$ and shifted $M(b)$ for different a and b. Thus inside $L(\lambda_0)$ there are no maps between $\mathrm{V}^*(\lambda)$ and other $\mathrm{V}^*(\lambda')$ with $\ell(\lambda',\lambda_0) = \ell(\lambda,\lambda_0)$. Therefore, we can use the next lemma (Lemma 10.4.3) by which α is homotopic to some β with $\beta|_{\mathrm{T}(\lambda_0)\to\mathrm{V}^*(\lambda)} = 0$. Because there are no maps between the $\mathrm{V}^*(\lambda')$ inside $L(\lambda_0)$, we can do the same for the other $\mathrm{V}^*(\lambda')$ with $\ell(\lambda',\lambda_0) = i$ without destroying that the map restricted to $\mathrm{V}^*(\lambda)$ is zero. Then we continue with $i-1$ until all maps are zero. Thus, $\mathrm{Hom}_K\big(\mathrm{T}(\lambda_0)\langle l\rangle[j], L(\lambda_0)\big) = 0$ for $j \neq 0$. □

Lemma 10.4.3. *Consider $C := \mathrm{Cone}(A \xrightarrow{f} B)$ and an object P considered as a chain complex concentrated in homological degree 0. Assume that $\mathrm{Hom}_K(P[j], A) = 0$ for all j. Then every $\alpha \in \mathrm{Hom}_K(P[j], C)$ is homotopic to some β with $\beta|_{P\to A} = 0$.*

Proof. This is a special case of Lemma 5.1.1. Since C is a cone, we have $d_C|_{B\to A} = 0$. □

Armed with the facts above we can calculate the dimensions of $\mathrm{End}(L)$.

Theorem 10.4.4. *We have $\dim \mathrm{End}(L) = \binom{n}{k}$. More precisely,*

$$\dim \mathrm{Hom}_K\big(L, L\langle i\rangle[j]\big) = \begin{cases} r_j & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $r_i := \#\{\lambda \in \Lambda(n,k) \mid 2\ell(\lambda,\lambda_0) = i\}$.

Proof. We first compute $\dim \mathrm{Hom}_K(L, L\langle i\rangle[j])$. By Corollary 10.4.1 and Proposition 10.4.2 we have $\mathrm{Hom}_K\big(\mathrm{T}(\lambda)\langle l\rangle, L[t]\big) = 0$ for all l and all $t \neq 0$. Thus, for fixed s we obtain $\mathrm{Hom}_K(L_i\langle s\rangle, L[t]) = 0$ for all i and $t \neq 0$. By construction the only shifted $\mathrm{T}(\lambda_0)$-summand inside $\mathrm{V}^*(\lambda)$ is $q^{\ell(\lambda,\lambda_0)}\,\mathrm{T}(\lambda_0) = \mathrm{V}^*(\lambda)_{\ell(\lambda,\lambda_0)}$. By Remark 10.1.13 we know that L_i contains r_i summands $q^i\,\mathrm{T}(\lambda_0)$. Thus, by Corollary 10.4.1 and Proposition 10.4.2 when we consider L_i in homological degree 0 then $\mathrm{Hom}_K(L_i\langle s\rangle, L) = 0$ for all $i \neq -s$ and $\dim \mathrm{Hom}_K(L_i\langle -i\rangle, L) = r_i$. Therefore, by Corollary 5.2.5 c) we obtain $\mathrm{Hom}_K(L\langle s\rangle, L[j]) = 0$ for all $j \neq -s$ and $\mathrm{Hom}_K(L\langle s\rangle, L[-s]) \cong \mathrm{Hom}_K(L_{-s}\langle s\rangle, L)$. Thus,

$$\dim \mathrm{Hom}_K(L, L\langle -s\rangle[-s]) = \dim \mathrm{Hom}_K(L\langle s\rangle, L[-s]) = \dim \mathrm{Hom}_K(L_{-s}\langle s\rangle, L) = r_{-s},$$

i.e. $\dim \mathrm{Hom}_K(L, L\langle s\rangle[s]) = r_s$. Now $\dim \mathrm{End}(L) = \sum_i r_i = \binom{n}{k}$, since every $\mathrm{V}^*(\lambda)$ contains exactly one $\mathrm{T}(\lambda_0)$-summand and the number of all $\mathrm{V}^*(\lambda)$ contained in L is the whole $\Lambda(n,k)$, i.e. we have $\sum_i r_i = |\Lambda(n,k)| = \binom{n}{k}$.

□

Definition 10.4.5. $\mathrm{End}(L)$ is a ring with the following multiplication: Let $f \in \mathrm{Hom}_K(L, L\langle i\rangle[j])$ and $g \in \mathrm{Hom}_K(L, L\langle i'\rangle[j'])$. Then

$$g.f = g \circ f \in \mathrm{Hom}_K\left(L, L\left\langle i+i'\right\rangle[j+j']\right),$$

where we use

$$\mathrm{Hom}_K\left(L\langle i\rangle[j], L\left\langle i+i'\right\rangle[j+j']\right) = \mathrm{Hom}_K\left(L, L\left\langle i'\right\rangle[j']\right).$$

Lemma 10.4.6. $\mathrm{End}(L)$ *is a local ring.*

Proof. Let $I = \bigoplus_{i>0} \mathrm{Hom}_K(L, L\langle i\rangle[i])$. Since by Theorem 10.4.4 all non-zero summands of $\mathrm{End}(L)$ are of the form $\mathrm{Hom}_K(L, L\langle i\rangle[i])$ for $i \geq 0$, I is an ideal. Furthermore, $r_0 = \#\{\lambda \in \Lambda(n,k) \mid 2\ell(\lambda,\lambda_0) = 0\} = \#\{\lambda_0\} = 1$, thus I is maximal. □

We now compute $\mathrm{End}(L)$ in the cases $k = 0$ and $k = 1$.

The case $k = 0$ is trivial: $\mathrm{End}(L) = \mathrm{End}(\mathrm{T}(\lambda_0)) \cong \mathbb{C}$ by Lemma 3.4.15.

For the rest of this section we consider $k = 1$ and we use again the notations of Section 10.2.

Proposition 10.4.7. *We have* $L \cong L^*\langle 2(n-1)\rangle[2(n-1)]$ *and the isomorphism is a* $\pm$*-isomorphism, where* * *denotes again the reflection from Definition 7.2.1.*

Proof. By Remark 10.2.3 the two complexes have obviously the same entries. Up to sign, in the reflected complex we still have all the possible saddles as differential. So by Remark 10.2.1, they are $\pm$-isomorphic. □

Definition 10.4.8. For $r \leq n$ let L^r be the complex $\mathrm{Tot}(W^r)$ where W^r is the double complex having the $q^i\,\mathrm{V}^*(i)$ with $i < r$ as rows with vertical maps given by the $f_{i-1,i}$ (up to signs).

Example 10.4.9. In general, we have $L(\lambda_0) = L^n$ (cf. Lemma 10.2.2). For $n = 5$ we have $L^5 = L(\lambda_0)$ (cf. Example 10.2.4) and

$$L^4 \cong \begin{array}{ccccccccccccc}
q^6\,\mathrm{T}(0) & \xrightarrow{-\mathrm{H}_1} & q^5\,\mathrm{T}(1) & \xrightarrow{-\mathrm{H}_2} & q^4\,\mathrm{T}(2) & \xrightarrow{-\mathrm{H}_3} & q^3\,\mathrm{T}(3) & & & & & & \\
& & & \searrow^{\overline{\mathrm{H}}_1} & & \searrow^{\overline{\mathrm{H}}_2} & & \searrow^{\overline{\mathrm{H}}_3} & & & & & \\
& & & & q^4\,\mathrm{T}(0) & \xrightarrow{\mathrm{H}_1} & q^3\,\mathrm{T}(1) & \xrightarrow{\mathrm{H}_2} & q^2\,\mathrm{T}(2) & & & & \\
& & & & & & & \searrow^{\overline{\mathrm{H}}_1} & & \searrow^{\overline{\mathrm{H}}_2} & & & \\
& & & & & & & & q^2\,\mathrm{T}(0) & \xrightarrow{-\mathrm{H}_1} & q\,\mathrm{T}(1) & & \\
& & & & & & & & & & & \searrow^{\overline{\mathrm{H}}_1} & \\
& & & & & & & & & & & & \mathrm{T}(0),
\end{array}$$

$$L^3 \cong \begin{array}{ccccccccc}
q^4\,\mathrm{T}(0) & \xrightarrow{\mathrm{H}_1} & q^3\,\mathrm{T}(1) & \xrightarrow{\mathrm{H}_2} & q^2\,\mathrm{T}(2) & & & & \\
& & & \searrow^{\overline{\mathrm{H}}_1} & & \searrow^{\overline{\mathrm{H}}_2} & & & \\
& & & & q^2\,\mathrm{T}(0) & \xrightarrow{-\mathrm{H}_1} & q\,\mathrm{T}(1) & & \\
& & & & & & & \searrow^{\overline{\mathrm{H}}_1} & \\
& & & & & & & & \mathrm{T}(0),
\end{array}$$

$$L^2 \cong \begin{array}{ccccc}
q^2\,\mathrm{T}(0) & \xrightarrow{-\mathrm{H}_1} & q\,\mathrm{T}(1) & & \\
& & & \searrow^{\overline{\mathrm{H}}_1} & \\
& & & & \mathrm{T}(0)
\end{array} \qquad \text{and } L^1 = \mathrm{T}(0).$$

From the definition we immediately obtain:

Lemma 10.4.10. *L^r contains the L^s for $s < r$ as a subcomplex starting at homological degree 0 on the right.*

Corollary 10.4.11. *For $r < n$, L^n contains a partcomplex called $\overline{L^r}$, which is left-aligned inside L^n, such that $\overline{L^r}\,\langle -2(n-r)\rangle\,[-2(n-r)]$ is $\pm$-isomorphic to L^r.*

Proof. We have the partcomplex L^r in L^n right-aligned, which by going via the $\pm$-isomorphism to $(L^n)^*\,\langle 2(n-1)\rangle\,[2(n-1)]$ is sent to $(L^r)^*\,\langle 2(n-r)\rangle\,[2(n-r)] =: \overline{L^r}$. Thus, the desired $\pm$-isomorphism comes from composing this $\pm$-isomorphisms with the one from Proposition 10.4.7. □

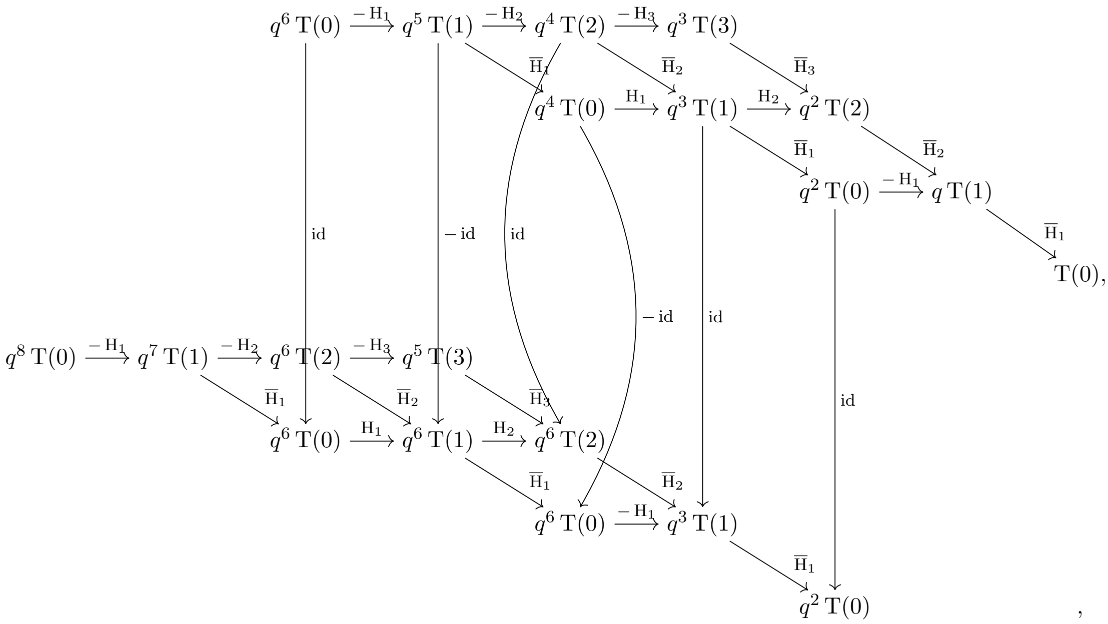
Now we see that the $\mathrm{End}(L^n)$ are well-known rings.

Proposition 10.4.12. $\mathrm{End}(L^n) \cong \mathbb{C}[x]/(x^n)$.

Proof. By Theorem 10.4.4, we know the dimensions and only have to find non-null-homotopic maps corresponding to $x^r \in \mathbb{C}[x]/(x^n)$. We define $f_x : L^n \to L^n\,\langle 2\rangle\,[2]$ as the $\pm$-isomorphism of the partcomplexes $\overline{L^{n-1}}$ of L^n and $L^{n-1}\,\langle 2\rangle\,[2]$ of $L^n\,\langle 2\rangle\,[2]$. It is not null-homotopic, since it contains $\pm\,\mathrm{id} : \mathrm{T}(0)\,\langle 2\rangle \to \mathrm{T}(0)\,\langle 2\rangle$ which does not factorise. Now $(f_x)^r : L^n \to L^n\,\langle 2r\rangle\,[2r]$ is 0 except for a $\pm$-isomorphism between the partcomplex $\overline{L^{n-r}}$ of L^n and $L^{(n-r)}\,\langle 2r\rangle\,[2r]$ of $L^n\,\langle 2r\rangle\,[2r]$ which by the same reasoning is not null-homotopic. □

Note that $\mathrm{End}(L^n)$ is hence isomorphic to the cohomology ring of $\mathbb{CP}^{n-1}$ with complex coefficients.

Example 10.4.13. For $n = 4$ the chain map f_x looks like this:



10.5 Linear complexes revisited

We come back to the heart of the linear t-structure (cf. Section 8.4) to find out what kind of object $L(\lambda_0)$ is in there. But before we can use the abelian structure of the heart, we have to show that $L(\lambda_0)$ is contained in it. Also, we need some knowledge about maps from linear complexes to a homologically shifted $L(\lambda_0)$ in order to calculate Ext-groups.

Lemma 10.5.1. *For $\lambda \in \Lambda(n,k)$ and $m \neq 0$ we have*

$$\mathrm{Hom}_K\big(\mathrm{T}(\lambda)\,\langle j\rangle\,[j], L(\lambda_0)[m]\big) = 0.$$

Proof. We have $\mathrm{Hom}_K\big(\mathrm{T}(\lambda)\,\langle j\rangle\,[j], L(\lambda_0)[m]\big) = \mathrm{Hom}_K\big(\mathrm{T}(\lambda)\,\langle j\rangle\,[j-m], L(\lambda_0)\big)$ and $j - m \neq j$. Thus, the assertion follows directly form Corollary 10.4.1 and Proposition 10.4.2. $\square$

Corollary 10.5.2. *Let X be a linear complex and $m \neq 0$. Then*

$$\mathrm{Hom}_K\big(X, L(\lambda_0)[m]\big) = 0$$

Proof. This follows inductively from Lemma 10.5.1 using Lemma 6.3.7 analogously as for Proposition 6.3.8. $\square$

In the heart of the linear t-structure, $L(\lambda_0)$ is a special object:

Proposition 10.5.3. *$L(\lambda_0)$ is contained in the heart $\mathcal{C}^l$ of the linear t-structure and there it is the injective hull of $\mathrm{T}(\lambda_0)$ via the canonical inclusion $\mathrm{T}(\lambda_0) \to L(\lambda_0)$ given by* $\mathrm{id} : \mathrm{T}(\lambda_0) \to L(\lambda_0)_0$.

Proof. Since the $\mathrm{V}^*(\lambda)$'s are homotopic to linear complexes by Theorem 8.1.5, by the construction of $L(\lambda_0)$ and Lemma 5.1.5 the same holds for $L(\lambda_0)$.

By Proposition 8.4.4, $\mathrm{T}(\lambda_0)$ is simple and $\mathrm{id} : \mathrm{T}(\lambda_0) \to L(\lambda_0)$ is a monomorphism.

We have $\mathrm{Ext}^1_{\mathcal{C}l}\big(X, L(\lambda_0)\big) \cong \mathrm{Hom}_K\big(X, L(\lambda_0)[1]\big) = 0$ by Lemma 8.2.3 and Corollary 10.5.2 for all linear complexes X, thus $L(\lambda_0)$ is injective.

Since $\mathrm{End}(L(\lambda_0))$ is local by Lemma 10.4.6, $\mathrm{id} : \mathrm{T}(\lambda_0) \to L(\lambda_0)$ is an injective hull by the dual statement of [Kra12, Lemma 2.2.3]. □

For the next chapter we need some more facts about maps into L.

Lemma 10.5.4. *Let M be a complex in $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ where no summand of an entry contains a circle. Assume $f : M \to L$ is a chain map such that f_0 contains no degree 0 maps, i.e. when we write $f_0 : M_0 \to L_0$ as matrix then no entry is a degree 0 map. Then $f \simeq 0$.*

Proof. Consider $f_0 : M_0 \to L_0 = \mathrm{T}(\lambda_0)$. If $\mathrm{T}(\lambda_0)$ is a summand of M_0, then by assumption $f_0|_{\mathrm{T}(\lambda_0)} = 0$, since by Lemma 3.4.15 the only map from $\mathrm{T}(\lambda_0)$ to itself is of degree 0. For summands X of M_0 which are not equal to $\mathrm{T}(\lambda_0)$ we know $\mathrm{Hom}_K(X, L) = 0$ by Corollary 10.4.1 and Proposition 10.4.2. Thus by Lemma 6.3.7 $f \simeq f'$ with $f'_0 = 0$. Now we can apply Proposition 6.3.8 for $r = 1$ and Γ maximal and obtain $f' \simeq 0$. □

Definition 10.5.5. We call a complex M a shifted linear complex, if there is some m such that $M[m]$ is a linear complex.

Lemma 10.5.6. *Let M be a shifted linear complex and assume there is a non-nullhomotopic chain map $f : M \to L\langle a\rangle[b]$. Then f consists of degree 0 maps.*

Proof. Since f is non-nullhomotopic if and only if $f' := f\langle -a\rangle[-b] : M\langle -a\rangle[-b] \to L$ is nullhomotopic, we can apply Lemma 10.5.4 to f' and obtain that f'_0 contains a non-zero degree 0 map. But this means that $M\langle -a\rangle[-b]$ is linear and f' consists of degree 0 maps. Thus, the same holds for f. □

Chapter 11

Categorified Jones-Wenzl projectors as a composition

In this chapter, we define two functors that satisfy the properties of the projection and inclusion factorising the Jones-Wenzl projector on a higher level. Therefore, their composition categorifies the Jones-Wenzl projector. Then, we recall Cooper-Krushkal's definition of the universal projector and start to relate the action of the universal projector to our composition of functors. We recall Rozansky's construction of the universal projector and use this to calculate the action of the universal projector for $k = 1$ and general n.

11.1 The categorification of projection and inclusion

The following theorem will motivate the definition of a functor F below which leads to the categorification of the projection π_n that is part of the factorisation of the Jones-Wenzl projector p_n.

Define π_n on $K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)\Big)$ via the isomorphism Φ of Theorem 7.2.9, that means $\pi_n([M]) := \pi_n\big(\Phi(1 \otimes [M])\big)$, where π_n is the projection from Definition 2.2.1.

Theorem 11.1.1. *We have*

$$\pi_n\big([M]\big) = \sum_{i,j} (-1)^i \dim \mathrm{Hom}_K\left(M, L(\lambda_0)\left\langle j\right\rangle[i]\right) q^j v^k,$$

where v^k is as in Definition 2.2.1.

Proof. By Corollary 10.4.1 and Proposition 10.4.2 we have

$$\mathrm{Hom}_K\left(\mathrm{T}(\lambda), L(\lambda_0)\left\langle j\right\rangle[l]\right) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \lambda_0 \text{ and } j = l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $M = \mathrm{T}(\lambda)$ the right hand side reduces to v^k if $\lambda = \lambda_0$ and 0 otherwise. By Theorem 7.2.9 we have $\pi_n([\mathrm{T}(\lambda)]) = \pi_n(v_{\heartsuit\lambda})$, thus, comparing with Corollary 2.2.5, we see that the assertion is true for all $\mathrm{T}(\lambda)$. Of course, both sides are compatible with internal and homological shifts. When the assertion is true for complexes A and B, then putting $\mathrm{Cone}(A \to B)$ into the right hand side, by Lemma 7.3.4 we get $\pi_n([B])-\pi_n([A])$ which is equal to $\pi_n\big([\mathrm{Cone}(A \to B)]\big)$. Therefore, the assertion follows inductively by Lemma 7.2.5. □

Definition 11.1.2. Given a collection of vector spaces $V_{(a,b)}$, $a, b \in \mathbb{Z}$, we denote by

$$V = \bigoplus_{(a,b)} V_{(a,b)} \langle a \rangle [b]$$

the bigraded vector space with graded components $V_{(a,b)}$.

Denote again $L := L(\lambda_0)$.

In particular, for $X \in K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ we have the bigraded $\mathbb{C}$-vector space

$$F(X) = \bigoplus_{i,j} \mathrm{Hom}_K\left(X, L\langle i \rangle [j]\,\right) \langle i \rangle [j].$$

We can view the differential of X as a chain map $d : X \to X[1]$. This map induces a map $\widehat{d}$ via

$$\begin{aligned} F(X)[1] &= \bigoplus_{i,j} \mathrm{Hom}_K\big(X, L\langle i \rangle [j]\big) \langle i \rangle [j+1] \\ &= \bigoplus_{i,j} \mathrm{Hom}_K\big(X[1], L\langle i \rangle [j]\big) \langle i \rangle [j] \to \bigoplus_{i,j} \mathrm{Hom}_K\big(X, L\langle i \rangle [j]\big) \langle i \rangle [j] = F(X) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \varphi \mapsto \varphi \circ d \end{aligned}$$

Therefore, we obtain a degree $(0,-1)$ map $\widehat{d}$ on the bigraded vector space $F(X)$ with $\widehat{d}^2 = 0$. Thus, $F(X)$ can be viewed as a cocomplex of graded vector spaces. F sends homotopy equivalent chain complexes to isomorphic cocomplexes since maps between chain complexes are mapped to precomposition with them, and if $f \circ g \simeq \mathrm{id}$, then $\alpha \circ f \circ g = \alpha$ inside Hom_K.

For an additive category $\mathcal{A}$ let $K^{co}(\mathcal{A})$ and $D^{co}(\mathcal{A})$ be the homotopy or derived category of cocomplexes, resp; similar for other variants of complexes.

We now have a functor

$$F : K^b\big(\widehat{\mathrm{Cup}}(n,k)\big) \to D^{co,b}(\mathbb{C}\text{-gfmod}),$$

where $\mathbb{C}$-gfmod denotes finite dimensional graded $\mathbb{C}$-vector spaces.

Lemma 11.1.3. *The differential $\widehat{d}$ is always zero.*

Proof. Let $Y = L\langle i \rangle [j]$. We know that the differential $\widehat{d}$ sends $f \in \mathrm{Hom}_K(X[1], Y)$ to $f \circ d \in \mathrm{Hom}_K(X, Y)$. We define a homotopy $h : X[1] \to Y$ via $h_t = (-t)f_t$, then

$$d_Y h_i + h_{i-1} d_X = (-i) f_i d_{X[1]} - (i-1) f_i d_X = i f_i d_X - (i-1) f_i d_X = f_i d_X.$$

Thus, $f d_X$ is 0 in $\mathrm{Hom}_K(X, Y)$. □

We want to show that when applying F only the shifted $\mathrm{T}(\lambda_0)$-summands are important. As a first step we see that it is enough to consider complexes of a certain form.

Remark 11.1.4. As before, every complex in $K\big(\widehat{\mathrm{Cup}}(n,k)\big)$ is isomorphic to a complex of sums of shifted $\mathrm{T}(\lambda)$ by delooping (Lemma 3.3.5). Furthermore, it is homotopy equivalent to a complex where additionally there are no degree 0 maps part of the differential, since those can be homotoped away using Gaussian elimination (Lemma 5.1.2). In particular, in these representatives, there are no maps between different (shifted) $\mathrm{T}(\lambda_0)$.

We now fix a notation for the $\mathrm{T}(\lambda_0)$ inside a given complex.

Definition 11.1.5. Let $K^- = K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ the subcategory of bounded to the right complexes of $K\big(\widehat{\mathrm{Cup}}(n,k)\big)$. Given a complex M in K^- we say $\bigoplus_{s=1}^{m_j} \mathrm{T}(\lambda_0)\langle a_j^s\rangle$ is the $\mathrm{T}(\lambda_0)$*-part of* M_j if $M_j \cong \bigoplus_{s=1}^{m_j} \mathrm{T}(\lambda_0)\langle a_j^s\rangle \oplus R$, where no summand of R is isomorphic to some $\mathrm{T}(\lambda_0)\langle l\rangle$ with $l\in\mathbb{Z}$.

We call $\bigoplus_{j=-\infty}^{\infty}\bigoplus_{s=1}^{m_j} \mathrm{T}(\lambda_0)\langle a_j^s\rangle[b_j]$ the $\mathrm{T}(\lambda_0)$*-part of* M. Note that by definition of K^- the sum is finite in every homological degree. Also, we use $[b_j]$ and not $[j]$ since there may be homological degrees without any $\mathrm{T}(\lambda_0)$.

Proposition 11.1.6. *Let M be a complex in K without circles and not containing degree 0 maps in the differential. Let $\bigoplus_{j=-\infty}^{\infty}\bigoplus_{s=1}^{m_j} \mathrm{T}(\lambda_0)\langle a_j^s\rangle[b_j]$ be the $\mathrm{T}(\lambda_0)$-part of M. Then*

$$F(M)\cong\bigoplus_{j,s}\mathbb{C}\langle a_j^s\rangle[b_j].$$

as a bigraded vector space.

Proof. Again, we work with complexes bounded to the right and can assume that $M = (\cdots\to M_2\to M_1\to M_0\to 0)$, because otherwise the same argument works with shifts. We want to apply Corollary 5.2.5 e) for $C = M$ and $D = L\langle l\rangle$. By Corollary 10.4.1 and Proposition 10.4.2 we have $\mathrm{Hom}_K(M_i, L\langle l\rangle[j]) = 0$ for $j\neq 0$. Furthermore, by the same results we know

$$\mathrm{Hom}_K(M_i, L\langle l\rangle) = \mathrm{Hom}_K\left(\bigoplus_{s:a_i^s=l}\mathrm{T}(\lambda_0)\langle l\rangle, L\langle l\rangle\right)\cong\bigoplus_{s:a_i^s=l}\mathbb{C}.$$

But in d_M there are no maps of degree 0 and in particular no maps between different copies of $\mathrm{T}(\lambda_0)\langle l\rangle$, thus the map between $\mathrm{Hom}_K(M_i, L\langle l\rangle)$ and $\mathrm{Hom}_K(M_{i+1}, L\langle l\rangle)$ induced by precomposition with d_M is zero. Therefore, Corollary 5.2.5 e) yields

$$\mathrm{Hom}_K\big(M, L\langle l\rangle[j]\big)\langle l\rangle[j]\cong\mathrm{Hom}_K\big(M_j, L\langle l\rangle\big)\langle l\rangle[j]\cong\bigoplus_{s:a_j^s=l}\mathbb{C}\langle l\rangle[j]$$

and altogether we get the desired result. □

Recall from Definition 10.4.5 the ring $\operatorname{End}(L) = \bigoplus_{i,j} \operatorname{Hom}_K(L, L\langle i\rangle[j])$.

We want to see $F(X)$ is a $\operatorname{End}(L)$-module via postcomposing with elements from $\operatorname{End}(L)$. But the $\operatorname{End}(L)$-action does not preserve the internal grading, forcing us to change it.

Definition 11.1.7. By Theorem 10.4.4 we know $\operatorname{Hom}_K(L, L\langle i\rangle[j]) = 0$ if $i \neq j$ or $i < 0$. Thus we can define a non-negative grading on $\operatorname{End}(L)$ by setting $\operatorname{End}(L) = \bigoplus_{m\geq 0} \operatorname{End}(L)_m = \bigoplus_{m\geq 0} \operatorname{Hom}_K(L, L\langle m\rangle[m])$. To distinguish from $F(L)$, we denote this ring with its grading by $R = \bigoplus_{m\geq 0} R_m$.

We define a new bigrading on $F(X)$ by setting $F(X)_{(a,b)} = \operatorname{Hom}_K(X, L\langle a\rangle[a+b])$, i.e. $\operatorname{Hom}_K(X, L\langle a\rangle[b]) = F(X)_{(a,b-a)}$.

With this new grading $\bigoplus_a F(X)_{(a,b)}$ for fixed b is a graded R-module: For $\varphi \in R_m = \operatorname{Hom}_K(L, L\langle m\rangle[m])$ and $f \in F(X)_{(a,b)} = \operatorname{Hom}_K(X, L\langle a\rangle[a+b])$ we have $\varphi \circ f \in \operatorname{Hom}_K(X, L\langle a+m\rangle[a+b+m]) = F_{(a+m,b)}$.

$F(X)$ is even a complex of R-modules, since the differential is zero and thus compatible with the R-module structure. Furthermore, the morphisms are R-module homomorphisms in every homological degree, since for $\alpha : X \to Y$ and $f \in F(Y)_{(a,b)}$ we have $f \circ \alpha \in F(X)_{(a,b)}$. Moreover, if $\varphi \in R_m$, then this obviously commutes with the maps, since $\varphi \circ (f \circ \alpha) = (\varphi \circ f) \circ \alpha$.

Therefore, F is a functor

$$F : K^b\big(\widehat{\mathrm{Cup}}(n,k)\big) \to D^{co,b}(R\text{-gfmod}),$$

where the category R-gfmod consists of graded R-modules that are finite dimensional over $\mathbb{C}$.

Before we can define G we need to extend the functor F to some unbounded complexes. We cannot enlarge to all of $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ since the result of our functor might not be a cocomplex in R-gfmod otherwise. The following condition is build to guarantee that when applying F we obtain cocomplexes bounded in the right direction with finite dimensional entries in every homological degree.

Definition 11.1.8. Let $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$ be the full subcategory of $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ given by objects X satisfying the *condition C*: There is an $s = s(X) \in \mathbb{Z}$ so that

$$X_i \cong \bigoplus_{\lambda\in\Lambda(n,k),\, j_i\in\mathbb{Z}} (\mathrm{T}(\lambda)\langle j_i\rangle)^{\oplus a_{\lambda,j_i}},$$

where the multiplicity a_{λ,j_i} is almost always zero and $a_{\lambda,j_i} \neq 0$ implies $i - j_i \leq s$. Furthermore, the equation $i - j_i = t$ is only allowed to have finitely many solutions for any fixed t.

Since X is in $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ there is also an $u = u(X)$ such that $X_i \neq 0$ only if $i \geq u$.

Note that $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ is a full subcategory of $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$ since for X in $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ there are only finitely many $X_i \neq 0$ and every X_i is a finite sum of shifted $\mathrm{T}(\lambda)$'s. Thus, we can take $s = i_{\max} - j_{\min}$, the difference of the the maximal

i with $X_i \neq 0$ and the minimal internal shift. Of course, the finiteness condition for solutions of $i - j = t$ is satisfied.

Similarly we enlarge the category $D^{co,b}(R\text{-gfmod})$. Here the condition will guarantee that we obtain finite sums and complexes bounded in one direction when we later apply the functor G defined below.

Definition 11.1.9. Let $D^{co,+}_{C'}(R\text{-gfmod})$ be the full subcategory of $D^{co,+}(R\text{-gfmod})$ of complexes X satisfying the *condition* C': There is a $u = u(X) \in \mathbb{Z}$ such that X_i is supported only in degrees j_i with $i + j_i \geq u$ and the equation $i + j_i = t$ has only finitely many solutions for every t.

Since X is in $D^{co,+}(R\text{-gfmod})$ there is also an $s = s(X) \in \mathbb{Z}$ such that $X_i \neq 0$ only for $i \leq s$.

Lemma 11.1.10. *F extends to functor $F : K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big) \to D^{co,+}_{C'}(R\text{-gfmod})$.*

Proof. We extend F by setting $F(M)_{(a,b)} = \mathrm{Hom}_{K^-}(M, L\langle a\rangle[a+b])$, where $K^- = K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$. We have to check that for M in $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$ this yields a complex in $D^{co,+}_{C'}(R - \mathrm{gfmod})$. By Remark 11.1.4 we can assume that there are no circles and degree 0 maps in M. Let $X := \bigoplus_{j=-\infty}^{\infty} \bigoplus_{r=1}^{m_j} \mathrm{T}(\lambda_0)\langle a^r_j\rangle[b_j]$ be the $\mathrm{T}(\lambda_0)$-part of M. Since M is in $K^-_C(\widehat{\mathrm{Cup}}(n,k))$, X also is in $K^-_C(\widehat{\mathrm{Cup}}(n,k))$, so there are $s, u \in \mathbb{Z}$ such that $b_j \geq u$, $b_j - a^r_j \leq s$ and the equation $b_j - a^r_j = t$ has only finitely many solutions for every t. Since Proposition 11.1.6 holds completely analogously also for K^- we know

$$F(M) \cong \bigoplus_{j,r} \mathbb{C}\langle a^r_j\rangle_{old}[b_j]_{old} = \bigoplus_{j,r} \mathbb{C}\langle a^r_j\rangle[b_j - a^s_j].$$

Since there are only finitely many solutions for $b_j - a^r_j = t$, the sum is finite in every homological degree. The inequality $b_j \geq u$ yields $b_j - a^r_j + a^r_j \geq u$ and $b_j - a^r_j \leq s$ bounds the cocomplex. Hence, $F(M)$ is an object of $D^{co,+}_{C'}(R\text{-gfmod})$. □

Remark 11.1.11. Having Theorem 11.1.1 in mind we want to view the functor F as a categorification of π_n. For every complex X we have $F(X.\mathcal{U}_i) = 0$ since by Lemma 6.2.3 and Theorem 10.3.1

$$\mathrm{Hom}_{K^-}\big(X.\mathcal{U}_i, L\langle l\rangle[j]\big) \cong \mathrm{Hom}_{K^-}\big(X, L.\mathcal{U}_i\langle l\rangle[j]\big) \cong \mathrm{Hom}_{K^-}(X, 0) = 0.$$

Note that $F(X.\mathcal{U}_i) = 0$ is a categorified version of Lemma 2.2.4 c) when we recall that $C_{i,n}$ is just the action of U_i. To make a precise statement on the level of Grothendieck groups one has to carefully deal with the Grothendieck group of categories of unbounded complexes (namely $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$ and $D^{co,+}_{C'}(R\text{-gfmod})$). This extra difficulty was addressed in the context of derived categories of graded abelian categories in [AS13].

Lemma 11.1.12. *We have $F(L\langle a\rangle[b]) \cong R\langle a\rangle[b-a]$ as a graded R-module.*

Proof. Using Theorem 10.4.4 we have

$$\begin{aligned}F(L\langle a\rangle[b]) &= \bigoplus_{i,j}\mathrm{Hom}_{K^-}\big(L\langle a\rangle[b], L\langle i\rangle[j]\big)\langle i\rangle[j-i]\\ &= \bigoplus_{i,j}\mathrm{Hom}_{K^-}\big(L, L\langle i-a\rangle[j-b]\big)\langle i\rangle[j-i]\\ &= \bigoplus_{i}\mathrm{Hom}_{K^-}\big(L, L\langle i-a\rangle[i-a]\big)\langle i\rangle[b-a]\\ &= \Big(\bigoplus_{m}\mathrm{Hom}_{K^-}\big(L, L\langle m\rangle[m]\big)\langle m\rangle\Big)\langle a\rangle[b-a],\end{aligned}$$

where in the last step we substitute $i-a$ by m. As R-module we get that the term $\bigoplus_m \mathrm{Hom}_{K^-}(L, L\langle m\rangle[m])\langle m\rangle$ is R itself. □

The definition of the functor G below is motivated by the fact that in the Grothendieck group of $K^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ we have $[L(\lambda_0)] = \left[\!\left[{n \atop k}\right]\!\right] v_{\lambda_0}.P_n$, cf. Remark 10.1.13. Therefore, if $G\circ F$ wants to categorify the action of P_n, then G should contain copies of L.

Definition 11.1.13 (Definition of G)**.** We define the map $G : D^{co,+}_{C'}(R\text{-gfmod}) \subset D^{co,+}(R\text{-gfmod}) \to K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ as the following composition:

$$\begin{aligned}D^{co,+}(R\text{-gfmod}) &\overset{(1)}{\cong} K^{co,+}\big(\mathcal{P}(R\text{-gfmod})\big) \overset{(2)}{=} K^-\big(\mathcal{F}(R\text{-gfmod})\big)\\ &\overset{(3)}{\to} K^+\big(Ch^b(\widehat{\mathrm{Cup}}(n,k))\big)\\ &\xrightarrow{\mathrm{Tot}} K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)\end{aligned}$$

Here, (1) is just the usual equivalence between the derived category of bounded to the right cocomplexes to the homotopy category of bounded to the right projectives for cocomplexes.

By Lemma 10.4.6 R is local, thus the projective modules $\mathcal{P}(R\text{-gfmod})$ are just the free modules $\mathcal{F}(R\text{-gfmod})$ which yields (2).

The functor (3) is induced by the additive contravariant functor $\mathcal{F}(R\text{-gfmod}) \to Ch^b\big(\widehat{\mathrm{Cup}}(n,k)\big)$ given by sending the object $R\langle j\rangle$ to $L\langle j\rangle[j]$ and a morphism $\varphi : R\langle j\rangle \to R\langle m\rangle$ to $\hat{\varphi} : L\langle m\rangle[m] \to L\langle j\rangle[j]$ constructed as follows: Note that φ is given by multiplication with some element $\tilde{\varphi}\in R_{j-m}$. Under the identification $R = \bigoplus_{i,j}\mathrm{Hom}_K(L, L\langle i\rangle[j])$ this gives a map in $R_{j-m} = \mathrm{Hom}_K(L\langle m\rangle[m], L\langle j\rangle[j])$.

Finally, $\mathrm{Tot} : K^+\big(Ch^b(\widehat{\mathrm{Cup}}(n,k))\big) \to K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ makes a complex of complexes into a double complex by adding signs and then sends it to the total complex.

Apriori it is not clear that taking the total complex in the last step results in a complex in $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$. The next proposition shows that when starting with an element in $D^{co,+}_{C'}(R\text{-gfmod})$ we not only land in $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ but in $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$. Before we state the proposition we want to consider an example.

Example 11.1.14. Let $n=2$ and $k=1$. Consider $\mathbb{C}$ in $D^{co,+}_{C'}(R\text{-gfmod})$, then by (1) and (2) it is send to its free resolution

$$\begin{array}{ccccccccccc} 1 & 0 & & -1 & & -2 & & -3 & & -4 \\ 0 & R & \xleftarrow{\cdot x} & R\langle 2\rangle & \xleftarrow{\cdot x} & R\langle 4\rangle & \xleftarrow{\cdot x} & R\langle 6\rangle & \xleftarrow{\cdot x} & \ldots, \end{array}$$

after that to

$$\begin{array}{ccccccccccc} 1 & 0 & & -1 & & -2 & & -3 & & -4 \\ 0 & L & \xrightarrow{f_x} & L\langle 2\rangle[2] & \xrightarrow{f_x} & L\langle 4\rangle[4] & \xrightarrow{f_x} & L\langle 6\rangle[6] & \xrightarrow{f_x} & \ldots, \end{array}$$

which as double complex looks like

$$\begin{array}{c|cccccc} & 1 & 0 & -1 & -2 & -3 & -4 \\ \hline 0 & & L_0 & & & & \\ & & \uparrow & & & & \\ 1 & & L_1 & & & & \\ & & \uparrow & & & & \\ 2 & & L_2 \longrightarrow & q^2L_0 & & & \\ & & & \uparrow & & & \\ 3 & & & q^2L_1 & & & \\ & & & \uparrow & & & \\ 4 & & & q^2L_2 \longrightarrow & q^4L_0 & & \\ & & & & \uparrow & & \\ 5 & & & & q^4L_1 & & \\ & & & & \uparrow & & \\ 6 & & & & q^4L_2 \longrightarrow & q^6L_0 & \\ & & & & & \uparrow & \\ 7 & & & & & q^6L_1 & \\ & & & & & \uparrow & \\ 8 & & & & & q^6L_2 \longrightarrow & \ldots \end{array}$$

and finally by Tot to

$$\begin{array}{cccccccccccc} 5 & & 4 & & 3 & & 2 & & 1 & & 0 \\ & & & & & & L_2 & \longrightarrow & L_1 & \longrightarrow & L_0 \\ & & & & & & \oplus & \searrow & \oplus & & \\ & & & & q^2L_2 & \longrightarrow & q^2L_1 & \longrightarrow & q^2L_0 & & \\ & & & & \oplus & \searrow & \oplus & & & & \\ & & q^4L_2 & \longrightarrow & q^4L_1 & \longrightarrow & q^4L_0 & & & & \\ & & \oplus & \searrow & \oplus & & & & & & \\ q^6L_2 & \longrightarrow & q^6L_1 & \longrightarrow & q^6L_0 & & & & & & \\ & \searrow & \oplus & & & & & & & & \\ & & \ldots & & & & & & & & \end{array}$$

Proposition 11.1.15. *The functor G from Definition 11.1.13 is well-defined and has image in* $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$.

Proof. Let X be a complex in $D^{co,+}_{C'}(R\text{-gfmod})$. The equivalence (1) can be defined explicitly by sending X to the total complex of the Cartan-Eilenberg resolution. Recall that in the Cartan-Eilenberg resolution $P_{*,*}$ every column $P_{p,*}$ is a projective resolution of X_p [Wei94, Exercise 5.7.1]. We can resolve every indecomposable summand M separately. Assume M is in homological and internal degree 0 and consider its projective resolution. Since the differentials play no role in whether condition C' is satisfied, we denote the projective resolution as if they were all zero. By Theorem 10.4.4, R is not only positively graded but is only non-zero in even degrees. Hence, the projective resolution of M is of the form

$$\bigoplus_{b\leq 0}\bigoplus_{\substack{t_b\geq s\geq 1\\ a_b^s+2b\geq 0}} R\langle a_b^s\rangle[b].$$

Because of $a_b^s+2b\geq 0$, the equation $a_b^s+b=t$ can only have finitely many solutions for every t, since $t+b=a_b^s+2b\geq 0$ can only hold for finitely many $b\leq 0$. Also, $a_b^s+b\geq 0$ holds, since $a_b^s+b=a_b^s+2b-b\geq 0$. When we now resolve the complex

$$X=\bigoplus_{i\leq s}\bigoplus_{\substack{z_i\geq v\geq 1\\ j_v^i+i\geq u\\ j_v^i+i=t \text{ fin.}}} M_{j,v,i}\langle j_v^i\rangle[i],$$

where we again disregard the differentials, and then take the total complex, we obtain

$$\bigoplus_{i\leq s}\bigoplus_{\substack{z_i\geq v\geq 1\\ j_v^i+i\geq u}}\bigoplus_{b\leq 0}\bigoplus_{\substack{t_b\geq s\geq 1\\ a_b^s+2b\geq 0}} R\langle a_b^s+j_v^i\rangle[b+i].$$

Now $b+i\leq s+0\leq s$ and $a_b^s+j_v^i+b+i\geq 0+u=u$. Furthermore, the equation $a_b^s+j_v^i+b+i=t$ implies $u\leq j_v^i+i\leq t-(a_b^s+b)\leq t$, thus they are finitely many solutions for i and j_v^i. Also, for every such solution, the equation $a_b^s+b=t-j_v^i-i$ has finitely many solutions, thus we obtain finitely many solutions in total. Therefore, condition C' is preserved under (1) and (2).

Thus, we need to consider a complex Y in $K^-\big(\mathcal{F}(R\text{-gfmod})\big)$ with

$$Y_i=\bigoplus_{a=1}^{r_i} R\left\langle j_a^i\right\rangle$$

such that $i+j_a^i\geq u$ for some fixed $u\in\mathbb{Z}$, $Y_i\neq 0$ only for $i\leq s$ for some $s\in\mathbb{Z}$ and the equation $i+j_a^i=t$ has only finitely many solutions for every t. Again, for satisfying condition C' it is not important what the differentials of Y look like, so we will assume they are all zero. Then

$$Y=\bigoplus_{i\leq s,a} R\left\langle j_a^i\right\rangle[i]$$

with $i+j_a^i\geq u$. Now applying (3) sends $R\langle a\rangle[b]$ to $L\langle a\rangle[a,b]$, where seen as complex of complexes b denotes the outer and a the inner homological degree. By Proposition 10.5.2 L is a bounded linear complex and thus $L=\bigoplus_{l,\text{some }\lambda\in\Lambda(n,k)}\mathrm{T}(\lambda)\langle l\rangle[l]$ with $r\geq l\geq 0$

when we again disregard the differentials. Thus,

$$\begin{aligned} Y &\overset{(4)}{\mapsto} \bigoplus_{i\leq s,a} L\left\langle j_a^i\right\rangle [j_a^i, i] \\ &= \bigoplus_{\substack{i\leq s\\ a}} \bigoplus_{\substack{\text{some } \lambda\in\Lambda(n,k)\\ l\geq 0}} \mathrm{T}(\lambda)\left\langle l+j_a^i\right\rangle [l+j_a^i, i]. \end{aligned}$$

Now taking the total complex yields a complex Z with

$$Z_m = \bigoplus_{l+j_a^i+i=m} \bigoplus_{\substack{i\leq s\\ a}} \bigoplus_{\substack{\text{some } \lambda\in\Lambda(n,k)\\ l\geq 0}} \mathrm{T}(\lambda)\left\langle l+j_a^i\right\rangle [l+j_a^i+i].$$

Now $m = l + j_a^i + i \geq 0 + u = u$, thus Z is bounded to the right. Furthermore, the equation $l+j_a^i+i = m$ is equivalent to $j_a^i+i = m-l$. We have $m \geq m-l \geq m-r$, i.e. the right hand side can only take finitely many values. Since $a+b = t$ has only finitely many solutions, the same is true for $l+j_a^i+i = m$. Finally, $(l+j_a^i+i)-(l+j_a^i) = i \leq s$, so all the conditions are satisfied. □

Lemma 11.1.16. $G(-).\mathcal{U}_i \simeq 0$

Proof. By construction $G(M)$ for some $M \in D^{co,+}_{C'}(R\text{-gfmod})$ is a total complex of a double complex containing direct sums of shifted L's in every row. By Theorem 10.3.1 we know $L.\mathcal{U}_i \simeq 0$, thus $G(M).\mathcal{U}_i$ is the total complex of a double complex with collapsing rows. Therefore, by [CK12, Lemma 2.12] or alternatively [Hog12, Proposition 7.5] we obtain $G(M).\mathcal{U}_i \simeq 0$. □

Remark 11.1.17. Lemma 11.1.16 is the categorified version of $C_{i,n} \circ \iota_n = 0$ from Lemma 2.2.4. To make a precise statement on the level of Grothendieck groups one has the same problem with unbounded complexes as addressed in Remark 11.1.11.

Lemma 11.1.18. *We have* $G\big(R\langle a\rangle[b]\big) = L\langle a\rangle[a+b]$ *and* $G\circ F\big(L\langle a\rangle[b]\big) = L\langle a\rangle[b]$.

Proof. The functor G sends $R\langle a\rangle[b]$ to $\mathrm{Tot}(L\langle a\rangle[a][b]) = L\langle a\rangle[a+b]$. Together with Lemma 11.1.12 this yields the assertion. □

Remark 11.1.19. Note that we have $G\big(M\langle a\rangle[b]\big) = (GM)\langle a\rangle[a+b]$. Indeed, let $P_\bullet = P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots$ be a projective resolution of M. Then $P_\bullet\langle a\rangle[b] = (P_0\langle a\rangle \leftarrow P_1\langle a\rangle \leftarrow P_2\langle a\rangle \leftarrow \dots)[b]$ is a projective resolution of $M\langle a\rangle[b]$. When $P_\bullet$ is send to $\mathrm{Tot}(Y^0 \to Y^1 \to Y^2 \to \dots)$, then $P_\bullet\langle a\rangle[b]$ is sent to $\mathrm{Tot}(Y^0\langle a\rangle[a] \to Y^1\langle a\rangle[a] \to Y^2\langle a\rangle[a] \to \dots)[b] = \mathrm{Tot}(Y^0 \to Y^1 \to Y^2 \to \dots)\langle a\rangle[a+b]$. In particular, $G\circ F$ is compatible with shifts since $F(X\langle a\rangle[b]) = F(X)\langle a\rangle[b-a]$ by definition. Also, $F\circ G$ is compatible with shifts.

In analogy to Lemma 2.2.4 a) we now want to show the following:

Theorem 11.1.20. *For every* $M \in D^{co,+}_{C',0}(R\text{-gfmod})$ *we have* $F\circ G(M) \cong M$, *where here we denote by* $\cong$ *the isomorphism in the derived category and* $D^{co,+}_{C',0}(R\text{-gfmod})$ *is the full subcategory of complexes in* $D^{co,+}_{C'}(R\text{-gfmod})$ *with zero differential.*

Note that we do not show this for all of the source of G but only for the subcategory that contains the image of F. But this is still enough to get $P \circ P(M) \simeq P(M)$ for $P := G \circ F$ and $M \in K_C^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$ which is the analogy to the property $p_n^2 = p_n$ of the Jones-Wenzl projector.

We start the proof by constructing a map as follows:

Construction 11.1.21. *Let* $(P_\bullet, d_\bullet)$ *be in* $K_{C'}^{co,+}\big(\mathcal{F}(R\text{-}gfmod)\big)$. *Then every* $f \in \ker d_i \subset P_i$ *defines an element* $\widetilde{f} \in F\big(G(P)\big)$.

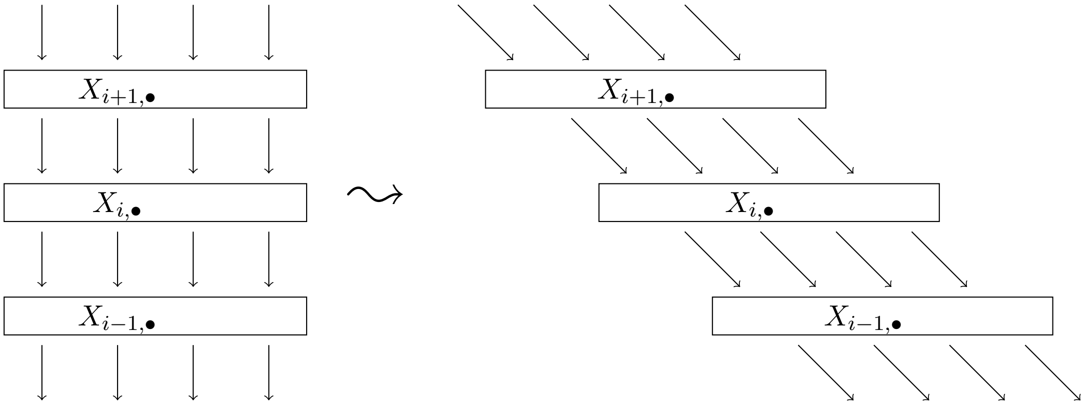
Proof. By definition $G(P_\bullet)$ is the total complex $Z_\bullet$ of a double complex $X_{\bullet,\bullet}$ with entries direct sums of $L\langle a\rangle[a]$ in every row. (Note that we switch rows and columns compared to Example 11.1.14 to better see how the copies of L appear in the total complex.)

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{0,3} & \longrightarrow & X_{0,2} & \longrightarrow & X_{0,1} & \longrightarrow & X_{0,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{-1,3} & \longrightarrow & X_{-1,2} & \longrightarrow & X_{-1,1} & \longrightarrow & X_{-1,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{-2,3} & \longrightarrow & X_{-2,2} & \longrightarrow & X_{-2,1} & \longrightarrow & X_{-2,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{-3,3} & \longrightarrow & X_{-3,2} & \longrightarrow & X_{-3,1} & \longrightarrow & X_{-3,0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

This double complex is bounded to the top since we start in $K^{co,+}$. Moreover, condition C' ensures that there are only finitely many nonzero entries on each diagonal, so that $Z_r = \bigoplus_{i+j=r} X_{i,j}$ is finite for every r. The horizontal differentials, which come from the differentials in each copy of L, are denoted by ∂.

The vertical maps δ are induced from the differentials in $P_\bullet$. More precisely if $P_i = \bigoplus_i R\langle t_i\rangle$, $P_{i+1} = \bigoplus_i R\langle t_i'\rangle$ and $d_i : P_i \to P_{i+1}$, then $X_{i+1,\bullet} = \bigoplus_i L\langle t_i'\rangle[t_i']$, $X_{i,\bullet} = L\bigoplus_i \langle t_i'\rangle[t_i']$ and $\delta : X_{i+1,\bullet} \to X_{i,\bullet}$ is given by the elements of the matrix d_i. Now assume $f \in \ker d_i$. Then $f = \bigoplus_i f_i$ with $f_i \in R\langle t_i\rangle$. Since $R = \mathrm{Hom}_K\left(L, \bigoplus_a L\langle a\rangle[a]\right)$, every f_i defines a chain map $\hat{f}_i : L\langle t_i\rangle[t_i] \to \bigoplus_a L\langle a\rangle[a]$ and $\hat{f} = \bigoplus_i \hat{f}_i : X_{i,\bullet} = \bigoplus_i L\langle t_i\rangle[t_i] \to \bigoplus_a L\langle a\rangle[a]$. Furthermore, $f \in \ker d_i$ means that $\hat{f} \circ \delta = 0$.

The signs in the double complex are chosen such that the differentials in $X_{i,\bullet}$ are given the sign $(-1)^i$. (Note that by Remark 10.2.1 different sign choices lead to isomorphic complexes.) This means, when going to the total complex, $X_{i,\bullet}[i]$ is a partcomplex of $\mathrm{Tot}(X) = G(P_\bullet)$.



Hence, $\hat{f}$ yields a map $\tilde{f} : \mathrm{Tot}(X) \to \bigoplus_a L\langle a\rangle [a][i]$ by defining it as $\hat{f}$ on the part-complex $X_{i,\bullet}[i]$ and 0 elsewhere. Now $\tilde{f}$ is a chain map since $\tilde{f} \circ \delta = 0$. Thus, $\tilde{f} \in F\big(G(P)\big) = \bigoplus_{i,j} \mathrm{Hom}_{K^-}\big(\mathrm{Tot}(X), L\langle i\rangle [j]\big)\langle i\rangle [j]$. □

Lemma 11.1.22. *In the situation of the construction above, the map*

$$\Psi : \bigoplus_i \ker d_i \to F\big(G(P)\big)$$
$$\bigoplus_i f_i \mapsto \bigoplus \tilde{f}_i$$

is surjective.

Proof. We use again the notations from the previous lemma. We fix a, b and set $Y_\bullet := L\langle a\rangle [a+b]$. Let $f : Z_\bullet \to Y_\bullet$ be a chain map. Let $f_{i,j}$ be the restriction of f to the summand $X_{i,j}$ in Z_{i+j}. By condition C' we know that for smaller i the internal degree of elements in P_i gets bigger. This transfers to $X_{\bullet,\bullet}$, where also the internal degree rises when i gets smaller. From Theorem 3.4.12 we know that the maximal degree of maps in $\mathrm{Cup}(n,k)$ and thus in $\widehat{\mathrm{Cup}}(n,k)$ is bounded. Since $Y_\bullet$ is bounded and thus has a bounded maximal internal degree, for $i \ll 0$ we have $f_{i,j} = 0$. We choose i_0 minimal such that $f_{i_0,j} \neq 0$ for some j.

We can also assume $f_{i_0,\bullet}$ is not nullhomotopic, since if it were nullhomotopic we could define a homotopy $f \simeq g$ by Lemma 5.1.1, that might change the $f_{j,\bullet}$ for $j > 0$, but satisfies $g_{j,\bullet} = f_{j,\bullet}$ for $j < i_0$ and $g_{i_0,\bullet} = 0$. By construction, the source of $f_{i_0,\bullet}$ is some $\bigoplus_m L\langle t_m\rangle [t_m][i_0]$, i.e. a shifted linear complex. Thus, by Lemma 10.5.6 $f_{i_0,\bullet}$ is a degree 0 chain map.

By the definition of chain maps, f has to satisfy

$$d' f = f(\partial + \delta),$$

where d' is the differential in $Y_\bullet$ and $\partial + \delta$ the differential of the total complex. In particular, $d' f_{i_0,\bullet} = f_{i_0,\bullet}\partial$ by minimality of i_0. Therefore, f restricts to an ordinary chain map $f_{i_0,\bullet} : X_{i_0,\bullet}[i_0] \to Y_\bullet$.

We now have for all j that

$$d' f_{i_0+1,j} = f_{i_0+1,j-1}\partial + f_{i_0,j}\delta.$$

which is equivalent to $f_{i_0,j}\delta = d' f_{i_0+1,j} - f_{i_0+1,j-1}\partial$. Here δ is a degree 0 map since it comes from a map between linear complexes. On the other hand, d' and ∂ are differentials in linear complexes, i.e. degree 1 maps. Hence, $f_{0,j}\delta = 0$ is the only possibility, i.e. $f_{0,\bullet} \circ \delta = 0$. Now we iterate and obtain that every $f_{i,\bullet} : X_{i,\bullet}[i] \to Y_\bullet$ is a chain maps with $f_{i,\bullet} \circ \delta = 0$. Therefore, it comes as an $\Psi(f')$ for $f' \in \bigoplus_i \ker d_i$.

□

Lemma 11.1.23. *In the situation of the previous lemma, the map $\Psi : \bigoplus_i \ker d_i \to F(G(P))$ satisfies $\Psi(\bigoplus_i \operatorname{im} d_{i-1}) = 0$.*

Proof. Again we use the notations from the previous lemmas. Assume $f \in \ker d_i$ is of the form $f = d_{i-1}\varphi$. Then $\tilde{f} = \Psi(f)$ is of the form $\tilde{f} = \tilde{\varphi}\delta_{i-1} : X_{i,\bullet}[i] \to \bigoplus_{a,b} L\langle a\rangle [b] =: \widetilde{L}$ with $\tilde{\varphi} : X_{i-1,\bullet}[i-1] \to \widetilde{L}[-1]$. Now the homotopy $h : \operatorname{Tot}(X)[1] \to \widetilde{L}$ defined by

$$h|_{X_{l,j}} = \begin{cases} \tilde{\varphi}_j & \text{if } l = i-1, \\ 0 & \text{otherwise,} \end{cases}$$

is a nullhomotopy for $\tilde{f}$: For maps starting at $X_{i,j}$ we have $d'h + h(\partial + \delta) = 0 + 0 + \tilde{\varphi}\delta_{i-1} = \tilde{f}$. For maps starting at $X_{i-1,j}$ we have $d'h + h(\partial + \delta) = d'\tilde{\varphi} + \tilde{\varphi}\partial + 0 = d'\tilde{\varphi} - d'\tilde{\varphi} = 0$, since $\tilde{\varphi}\partial = d'_{[-1]}\tilde{\varphi} = -d'\tilde{\varphi}$. For all other $X_{l,j}$ we have $d'h + h(\partial + \delta) = 0$. □

Putting all of this together we can prove our desired theorem.

Proof of Theorem 11.1.20. Since the projective resolution works separately for every direct summand and every R-module splits into a sum of indecomposable ones [Lan02, Theorem 7.5, Example 7.1], we can assume M is indecomposable. Furthermore, we know that F and G commute with homological shifts, so we can assume that M is in homological degree 0.

First, we consider the case where $M = \mathbb{C}\langle a\rangle$ for some a and $(P_\bullet, d_\bullet)$ is the free resolution of M. Since we know how F and G commute with shifts by Remark 11.1.19 we can assume $a = 0$. By Lemma 11.1.22 and Lemma 11.1.23 we have that

$$\Psi : \bigoplus_i H^i(P_\bullet) = \bigoplus_i \ker d_i / \operatorname{im} d_{i-1} \to F\big(G(P_\bullet)\big).$$

is surjective. Since $P_\bullet$ is a resolution all H^i except for $i = 0$ vanish and $H^\bullet(P_\bullet)$ is quasi-isomorphic to $\mathbb{C}$. Thus, to show that Ψ is injective it suffices to show it on the degree 0 part. The only cohomology is 1-dimensional spanned by $1 \in R = P_0$. But we have $X_{0,\bullet} = L$ and $\tilde{1}_{|X_{0,\bullet}} = \mathrm{id}$. id is not homotopic to zero, since for degree reasons there can be no homotopy.

For the other indecomposable modules M we do induction over the dimension $d = \dim_{\mathbb{C}}(M)$. The case $d = 1$ has already been treated above. Now assume $d > 1$ and let a be the minimal non-zero degree of M. Then there is a surjective R-linear map $\tau \colon M \to \mathbb{C}\langle a\rangle$. Let $M' := \ker(\tau)$, then there is a short exact sequence $0 \to M' \to M \to \mathbb{C}\langle a\rangle \to 0$ with $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} M' + 1$ and M' is isomorphic to a sum of indecomposables of smaller dimension. In particular, we know $F\big(G(M')\big) \cong M'$ in the derived category. Setting $N = M\langle -a\rangle$, $N' = M'\langle -a\rangle$, we still have $F\big(G(N')\big) \cong N'$

by Remark 11.1.19, and the sequence $0 \to N' \to N \to \mathbb{C} \to 0$ is also exact. It is sent to a distinguished triangle $G(N') \to G(N) \to G(\mathbb{C}) \to G(N')[1]$ by G. Thus, for every $b \in \mathbb{Z}$ we obtain a long exact sequence

$$\begin{array}{lll} \ldots \longrightarrow \mathrm{Hom}_{K^-}\big(G(\mathbb{C}),L\langle b\rangle[-1]\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N),L\langle b\rangle[-1]\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N'),L\langle b\rangle[-1]\big) \\ \longrightarrow \mathrm{Hom}_{K^-}\big(G(\mathbb{C}),L\langle b\rangle\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N),L\langle b\rangle\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N'),L\langle b\rangle\big) \\ \longrightarrow \mathrm{Hom}_{K^-}\big(G(\mathbb{C}),L\langle b\rangle[1]\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N),L\langle b\rangle[1]\big) & \longrightarrow \mathrm{Hom}_{K^-}\big(G(N'),L\langle b\rangle[1]\big) \longrightarrow \ldots \end{array}$$

From the calculations for $\mathbb{C}$ we already know $\mathrm{Hom}_{K^-}(G(\mathbb{C}), L\langle b\rangle [j]) = 0$ for $b \neq 0$ or $j \neq 0$. Thus, for $b \neq 0$ or $j \neq -1, 0$ we obtain $\dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N), L\langle b\rangle [j]\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N'), L\langle b\rangle [j]\big)$.

By induction we know that $F\big(G(N')\big)$ is isomorphic to N'. In addition to this, $F\big(G(N')\big)$ is a cocomplex with differential 0 and thus, its cohomology is the same as $F\big(G(N')\big)$ itself. By assumption, N' and also its cohomology is concentrated in homological degree 0. If $\mathrm{Hom}_{K^-}\big(G(N'), L[-1]\big)$ is non-zero, then $F\big(G(N')\big)_{(0,-1)}$ is non-zero by definition, i.e. it has a non-zero entry in homological degree -1. Thus, $\mathrm{Hom}_{K^-}\big(G(N'), L[-1]\big)$ has to be zero. Therefore, we obtain

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N), L\langle b\rangle [j]\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N'), L\langle b\rangle [j]\big) = 0$$

for $b = 0$ and $j = -1$ and for $j = b = 0$ we obtain

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N), L\big) = \dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(\mathbb{C}), L\big) + \dim_{\mathbb{C}} \mathrm{Hom}_{K^-}\big(G(N'), L\big).$$

In total, we have

$$\begin{aligned} \dim_{\mathbb{C}} F\big(G(N)\big) &= \dim_{\mathbb{C}} F\big(G(N')\big) + \dim_{\mathbb{C}} F\big(G(\mathbb{C})\big) \\ &= \dim_{\mathbb{C}} N' + \dim_{\mathbb{C}} \mathbb{C} = \dim_{\mathbb{C}} N. \end{aligned}$$

Of course, we also have $\dim_{\mathbb{C}} F\big(G(M)\big) = \dim_{\mathbb{C}} F\big(G(N)\big) = \dim_{\mathbb{C}} M$. Thus, the map Ψ is also injective in this case and this yields our desired isomorphism $F\big(G(M)\big) \cong M$ in the derived category.

□

Altogether we have shown that for $P := G \circ F$, analogously to the way Lemma 2.2.4 yields the equalities of Proposition 2.2.3, we have

$$P \circ P(X) \simeq P(X) \quad \text{and} \quad P(X).\mathcal{U}_i \simeq 0 \simeq P(X.U_i).$$

for every object X in $K^-_C\big(\widehat{\mathrm{Cup}}(n,k)\big)$. These equalities are the categorified version of the properties of the Jones-Wenzel projector. They are also satisfied by the universal projector defined by Cooper and Krushkal [CK12] which we recall in the next section.

11.2 Universal projectors

We recall Cooper-Krushkal's definition of universal projectors and start to relate it to the composition of the functors G and F defined in the last section. Before continuing this comparison in the next section, we recall Rozansky's construction of the universal projectors.

11.2.1 Cooper-Krushkal's definition

Recall that for $\mathcal{A}$ an additive category we denote by $Ch^-(\mathcal{A})$ and $K^-(\mathcal{A})$ bounded to the right chain complexes in $Ch(\mathcal{A})$ and $K(\mathcal{A})$, resp.

Following [CK12] in the normalisation of [Hog12] we define:

Definition 11.2.1. A chain complex $P \in Ch^-\big(\widehat{\mathrm{Cob}}(n)\big)$ is a *universal projector* if:

(1) The complex is concentrated in non-negative homological degrees, is isomorphic to the identity in degree 0 and Id_n only appears at homological degree 0, i.e.

- $P_0 \cong \mathrm{Id}_n$
- $P_j \ncong \mathrm{Id}_n \oplus D$ for any $D \in \widehat{\mathrm{Cob}}(n)$ for $k > 0$
- $P_j = 0$ for $j < 0$.

(2) P is contractible under turnbacks, i.e. $P \otimes \mathcal{U}_i \simeq 0 \simeq \mathcal{U}_i \otimes P$ for all i.

Here, the tensor product is the usual tensor product of complexes where $\mathcal{U}_i$ is in homological degree 0.

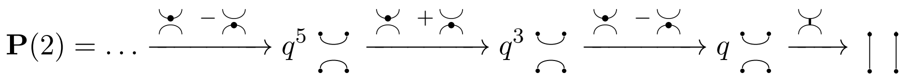
Theorem 11.2.2 (Cooper-Krushkal)**.** *Universal projectors exist and are unique up to homotopy equivalence for every n. They satisfy $P \otimes P \simeq P$.*

A proof can be found in [CK12, Section 3].

Definition 11.2.3. We denote by $\mathbf{P}(n)$ the unique universal projector in $K^-\big(\widehat{\mathrm{Cob}}(n)\big)$.

The universal projector $\mathbf{P}(n)$ satisfies the properties $P_n^2 = P_n$ and $P_nU_i = 0 = U_iP_n$ of the Jones-Wenzl projector on a higher level. For a description on how to impose technical conditions on the Grothendieck groups to make the statement that the universal projector categorifies the Jones-Wenzl projector more precise see [CK12, Sections 2 and 3].

Example 11.2.4. The smallest universal projector, i.e. the one for $n = 2$, looks as follows:



More precisely, $\mathbf{P}(2)_i = q^{2i-1}$ for $i > 0$ and for $i \geq 2$ the differential $d_i\colon \mathbf{P}(2)_i \to \mathbf{P}(2)_{i-1}$ is defined as + for i odd and as - for i even. Using the relations of $\mathrm{Cob}(n)$, one can easily calculate that the differential squares to zero, cf. [CK12, Proposition 4.1]. The calculation for $\mathbf{P}(2) \otimes \mathcal{U}_1 \simeq 0$ is more involved and uses delooping (Lemma 3.2.4) and iterated Gaussian elimination (Lemma 5.1.2), cf. [CK12, Theorem 4.1].

Now we want to compare $\mathbf{P}(n)$ with $G \circ F$.

Lemma 11.2.5. *We have* $\mathrm{T}(\lambda).\mathbf{P}(n) \simeq 0 = G \circ F\big(\mathrm{T}(\lambda)\big)$ *for all* $\lambda \neq \lambda_0$ *in* $\Lambda(n,k)$.

Proof. Since $\lambda \neq \lambda_0$, by Lemma 4.1.8 there is some i and some λ' such that $\mathrm{T}(\lambda) \cong \mathrm{T}(\lambda').\mathcal{U}_i$. Therefore, by definition we have $\mathrm{T}(\lambda).\mathbf{P}(n) \cong \mathrm{T}(\lambda').\big(\mathcal{U}_i \otimes \mathbf{P}(n)\big) \simeq 0$ and $P\big(\mathrm{T}(\lambda)\big) = P\big(\mathrm{T}(\lambda').\mathcal{U}_i\big) = 0$ holds by Remark 11.1.11. □

Proposition 11.2.6. *We have* $L.\mathbf{P}(n) \simeq L = G\big(F(L)\big)$.

Proof. We already know $G\big(F(L)\big) = L$ from Lemma 11.1.18. Furthermore, $L.\mathbf{P}(n)$ is the total complex of the double complex having $L.\mathbf{P}(n)_i$ in the rows. By definition of a universal projector we know that $\mathbf{P}(n)_0 \cong \mathrm{Id}_n$ and $\mathbf{P}(n)_j \cong \bigoplus_{r=1}^{s} \mathcal{U}_{i_r} \otimes D_r$ for some $D_r \in \widehat{\mathrm{Cob}}(n)$, hence by Theorem 10.3.1 we have $L.\mathbf{P}(n)_i \simeq 0$ for $i \neq 0$ and $L.\mathbf{P}(n)_0 \cong L.\mathrm{Id} \cong L$. Thus, by [Hog12, Proposition 7.5], we obtain $L.\mathbf{P}(n) \simeq L$. □

We now consider the functor $.\mathbf{P}(n)$ given by application of $\mathbf{P}(n)$.

Corollary 11.2.7. *If* $\mathrm{T}(\lambda_0).\mathbf{P}(n) \simeq G \circ F\big(\mathrm{T}(\lambda_0)\big)$, *then the functors* $.\mathbf{P}(n)$ *and* $G \circ F$ *are isomorphic as functors from* $\widehat{\mathrm{Cup}}(n,k)$ *to* $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$.

Proof. By Lemma 11.2.5 and the assumption, for every object X in $\widehat{\mathrm{Cup}}(n,k)$ we have a morphism $\eta_X \colon X.\mathbf{P}(n) \to G \circ F(X)$ which is an isomorphism, since both functors are additive and compatible with shifts. Now we consider $f \colon X \to Y$ first for $X \cong q^r\,\mathrm{T}(\lambda)$ and $Y \cong q^s\,\mathrm{T}(\mu)$. If $\mu \neq \lambda_0$ we have $G \circ F(Y) = 0$ and thus $\eta_Y \circ G \circ F(f) = f.\mathbf{P}(n) \circ \eta_X$. If $\lambda \neq \lambda_0$, then $X.\mathbf{P}(n)$ is isomorphic to zero and we also have $\eta_Y \circ G \circ F(f) = f.\mathbf{P}(n) \circ \eta_X$. For the case $\lambda = \mu = \lambda_0$ by Lemma 3.4.15 the only possible morphisms f are $c \cdot \mathrm{id}$ for $c \in \mathbb{C}$ and $r = s$. Since $\mathrm{id}\,.\mathbf{P}(n) = \mathrm{id}_{\mathrm{T}(\lambda_0).\mathbf{P}(n)}$ and $G \circ F(\mathrm{id}) = \mathrm{id}_{G \circ F(\mathrm{T}(\lambda_0))}$ we have $\eta_Y \circ G \circ F(\mathrm{id}) = \mathrm{id}\,.\mathbf{P}(n) \circ \eta_X$. The same holds for $c \cdot \mathrm{id}$, since everything is $\mathbb{C}$-linear. Again by additivity and compatibility with shifts we obtain $\eta_Y \circ G \circ F(f) = f.\mathbf{P}(n) \circ \eta_X$ for general objects X, Y of $\widehat{\mathrm{Cup}}(n,k)$. □

To obtain the homotopy equivalence when applying to $\mathrm{T}(\lambda_0)$ that is a condition in the corollary above, we need a construction of the universal projectors.

11.2.2 Rozansky's construction

Before going on, we need to recall some terminology from [Roz10, Section 2.2.2], adapted to the fact that we denote complexes homologically instead of cohomologically.

We are now back in the general setting that $\mathcal{A}$ is an additive category and we consider complexes in $K(\mathcal{A})$.

Definition 11.2.8. The *homological order* $|C|_\hbar \in \mathbb{Z} \cup \{\infty\}$ of a complex C is defined as

$$|C|_\hbar = \sup\big\{m \mid \exists\, B = (\cdots \to B_{m+1} \to B_m \to 0) \text{ s.t. } C \simeq B\big\}.$$

For example, for $C = 0 \to M \to N \xrightarrow{\mathrm{id}} N \to 0$ with the right 0 in homological degree 0 we have $|C|_\hbar = 3$ since $C \simeq 0 \to M \to 0 \to 0 \to 0$.

A *direct system* $\mathbf{A}$ in $K(\mathcal{A})$ is a sequence of complexes $A^i \in K(\mathcal{A})$, $i \in \mathbb{N}$, connected by chain morphisms:

$$\mathbf{A} = (A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} \dots).$$

A direct system $\mathbf{A}$ is *Cauchy*, if $\lim_{i\to\infty} \big|\operatorname{Cone}(f^i)\big|_\hbar = \infty$.

A direct system $\mathbf{A}$ *has a limit* A, where A is a chain complex, if there exist chain morphisms $A^i \xrightarrow{\tilde{f}^i} A$ such that $\tilde{f}^i \simeq \tilde{f}^{i+1} f^i$ and $\lim_{i\to\infty} \big|\operatorname{Cone}(\tilde{f}^i)\big|_\hbar = \infty$.

Example 11.2.9. Let $A = (\dots \to C_2 \to C_1 \to C_0 \to 0)$ a complex, unbounded to the left. Let $A^i = (C_i \to C_{i-1} \to \dots \to C_1 \to C_0)$ and define $f^i : A^i \to A^{i+1}$ by $f^i_j = \mathrm{id}_{C_j}$ for $0 \le j \le i$. Then the direct system $\mathbf{A} = (A^0 \xrightarrow{f^0} A^1 \xrightarrow{f^1} \dots)$ is a Cauchy since (for example by Gaussian elimination (Lemma 5.1.2)) we have $\operatorname{Cone}(f^i) \simeq (C_{i+1} \to 0 \to \dots \to 0)$, i.e. $|\operatorname{Cone}(f^i)|_\hbar = i+1$.

We even have that A is a limit of $\mathbf{A}$. Indeed, if we define $\tilde{f}^i_j = \mathrm{id}_{C_j}$ for $0 \le j \le i$ we have $\tilde{f}^i = \tilde{f}^{i+1} f^i$. Furthermore, $\operatorname{Cone}(\tilde{f}^i) \simeq (\dots \to C_{i+2} \to C_{i+1} \to 0 \to \dots \to 0)$ by Gaussian elimination, so $|\operatorname{Cone}(f^i)|_\hbar = i+1$.

Proposition 11.2.10 ([Roz10, Theorems 2.5, 2.6])**.** *A direct system* $\mathbf{A}$ *has a limit if and only if it is Cauchy. The limit is unique up to homotopy equivalence. We denote it by* $\varinjlim \mathbf{A}$.

For the application in Corollary 11.3.13 below, we need the following.

Proposition 11.2.11. *Let* $\mathcal{A}$ *be an additive category with an action of an additive monoidal category* $\mathcal{D}$, *i.e. the functor* $. : \mathcal{A} \times \mathcal{D} \to \mathcal{A}$ *from Definition 4.1.1 is compatible with the additive structures. Let* $\mathbf{D} = (D^0 \xrightarrow{f^0} D^1 \xrightarrow{f^1} \dots)$ *be a Cauchy system in* $K(\mathcal{D})$, $\varinjlim \mathbf{D}$ *its limit and* $A \in K(\mathcal{A})$. *Then*

$$A.\mathbf{D} = \left(A.D^0 \xrightarrow{\mathrm{id}.f^0} A.D^1 \xrightarrow{\mathrm{id}.f^1} \dots \right)$$

defines a Cauchy system in $K(\mathcal{A})$ *and* $\varinjlim(A.\mathbf{D}) \simeq A.\varinjlim \mathbf{D}$. *(Here the action of* $K(\mathcal{D})$ *on* $K(\mathcal{A})$ *is as defined in Section 5.1.)*

Proof. By Proposition 11.2.10 it is enough to show that $A.\mathbf{D}$ has limit $A.\varinjlim \mathbf{D}$. Since by assumption $\mathbf{D}$ is Cauchy, it has by Proposition 11.2.10 a unique limit $\varinjlim \mathbf{D}$ in $K(\mathcal{D})$ with associated maps

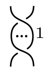
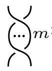
$$D^i \xrightarrow{\tilde{f}_i} \varinjlim \mathbf{D} \tag{11.1}$$

satisfying $\tilde{f}_i \simeq \tilde{f}_{i+1} f_i$ and $\lim_{i\to\infty} |\operatorname{Cone}(\tilde{f}_i)|_\hbar = \infty$. The action of (11.1) on A gives chain maps $A.D^i \xrightarrow{\mathrm{id}\,\tilde{f}_i} A.\varinjlim \mathbf{D}$ satisfying $\mathrm{id}.\tilde{f}_i \simeq \mathrm{id}.\tilde{f}_{i+1}\,\mathrm{id}.f_i$. Since

$$\operatorname{Cone}(\mathrm{id}\,.\tilde{f}_i) = A.\operatorname{Cone}(\tilde{f}_i) \qquad \text{and} \qquad \lim_{i\to\infty} |\operatorname{Cone}(\tilde{f}_i)|_\hbar = \infty$$

we obtain $\lim\limits_{i\to\infty} |\operatorname{Cone}(\mathrm{id}\,.\tilde{f}_i)|_\hbar = \infty$. Thus, $\lim\limits_{\to}(A.\mathbf{D}) \simeq A.\lim\limits_{\to}\mathbf{D}$. □

Now we can recall Rozansky's definition of the categorified Jones-Wenzl projector.

Definition 11.2.12. Following [Roz10] we consider the following braid diagrams with n strands. Let $\beta_{cyl,n} =$  and $\beta_{rot,n} := (\beta_{cyl,n})^n =$  $=:$  the 1-fold full twist (by 360°) and $\omega_m =$  $= (\beta_{rot,n})^m$ the m-fold full twist. For Id the trivial braid let $\mathscr{C}(\mathrm{Id}) = \mathrm{Id}$ as a complex in $K(\widehat{\mathrm{Cob}}(n))$ concentrated in degree 0. For $\sigma_i :=$  we define the complex

$$\mathscr{C}(\sigma_i) = (q\mathcal{U}_i \xrightarrow{\mathrm{H}_i} \mathrm{Id})$$

in $K\big(\widehat{\mathrm{Cob}}(n)\big)$ and for $1 \le i_1, \dots, i_r \le n-1$ let

$$\mathscr{C}(\sigma_{i_1} \dots \sigma_{i_r}) = \mathscr{C}(\sigma_{i_1}) \otimes \dots \otimes \mathscr{C}(\sigma_{i_r}),$$

i.e. vertical composition of braids corresponds to tensor product of complexes.
For example, for $n = 2$

$$\mathscr{C}(\omega_1) = \mathscr{C}(\sigma_1^2) = \mathscr{C}(\sigma_1) \otimes \mathscr{C}(\sigma_1) = \begin{array}{ccc} q^2\mathcal{U}_1\mathcal{U}_1 & \xrightarrow{-\mathrm{id}\,\mathrm{H}_1} & q\mathcal{U}_1\,\mathrm{Id} \\ & \searrow^{\mathrm{H}_1\,\mathrm{id}} \quad \oplus & \searrow^{\mathrm{H}_1\,\mathrm{id}} \\ & q\,\mathrm{Id}\,\mathcal{U}_1 & \xrightarrow{\mathrm{id}\,\mathrm{H}_1} \mathrm{Id}\,\mathrm{Id} \end{array}$$

More generally, $\mathscr{C}(\beta_{cyl,n}) = \mathscr{C}(\sigma_1) \otimes \dots \otimes \mathscr{C}(\sigma_{n-1})$ and $\mathscr{C}(\omega_{m+1}) \cong \mathscr{C}(\omega_m) \otimes \mathscr{C}(\omega_1)$.
Let $\mathbf{B}(n)$ be the direct system

$$\mathscr{C}(\mathrm{Id}) \xrightarrow{f^0} \mathscr{C}(w_1) \xrightarrow{f^1} \mathscr{C}(w_2) \xrightarrow{f^2} \dots,$$

where f^0 is given by $\mathrm{id} : \mathscr{C}(\mathrm{Id}) \to \mathscr{C}(w_1)_0$ and for higher m we have

$$f^m = \mathrm{id}_{\mathscr{C}(\omega_m)} \otimes f^0 : \mathscr{C}(\omega_m) \cong \mathscr{C}(\omega_m) \otimes \mathscr{C}(\mathrm{Id}) \to \mathscr{C}(\omega_m) \otimes \mathscr{C}(\omega_1) \cong \mathscr{C}(\omega_{m+1}).$$

By [Roz10, Theorem 4.4] we have $|\operatorname{Cone}(f^m)|_\hbar \ge 2m(n-1)+1$. In particular, $\mathbf{B}(n)$ is Cauchy and hence has a limit which we denote by $\mathbf{P}(n) = \lim\limits_{\to}\mathbf{B}(n)$.

Theorem 11.2.13 ([Roz10, Theorem 2.7 and (2.27)]). $\mathbf{P}(n)$ *is a universal projector.*[1]

Recall the cube-complexes R_k from Definition 6.1.3. We now define a complex $R'(i_1, \dots, i_r)$ in $Ch\big(\widehat{\mathrm{Cob}}(n)\big)$ similar to the definition there but with $\mathcal{U}_i$ and Id switched and the signs adjusted.

[1]To translate into Rozansky's setting we have to rotate the braids by 90 degrees and collapse the triple grading to a double grading.

Definition 11.2.14. Let $1 \leq i_1, \ldots, i_r \leq n-1$. For $\mathbf{w} = (w_1, \ldots, w_r)$ with $w_j \in \{0,1\}$ let

$$\mathbf{w}(i_1, \ldots, i_r) = q^{\sum_i w_i} B_{i_1} \ldots B_{i_r},$$

where $B_{i_l} = \begin{cases} \mathcal{U}_{i_l}, & \text{if } w_l = 0 \\ Id, & \text{if } w_l = 1. \end{cases}$
For $\xi = (\xi_1, \ldots, \xi_r)$ an edge-label with $\xi_j = \star$ let

$$\xi(i_1, \ldots, i_r) = (-1)^{\sum_{i<j} \xi_i} \operatorname{id} \ldots \operatorname{id} \overline{\mathrm{H}}_{i_j} \operatorname{id} \ldots \operatorname{id},$$

where $\overline{\mathrm{H}}_{i_j} : \mathcal{U}_{i_j} \to \mathrm{Id}$ is the saddle cobordism.
Finally, let $R'(i_1, \ldots, i_r) = R_r(i_1, \ldots, i_r)$, i.e.

$$R'(i_1, \ldots, i_r)_l = \bigoplus_{\mathbf{w}: \sum w_i = l} \mathbf{w}(i_1, \ldots, i_r)$$

with the differential given by the outgoing edges.

Proposition 11.2.15. $\mathscr{C}(\beta_{cyl,n}) = R'(1, \ldots, n-1)$.

Proof. We show inductively that $\mathscr{C}(\sigma_1) \otimes \ldots \otimes \mathscr{C}(\sigma_i) = R'(1, \ldots, i)$. It is obviously true for $i = 1$. Furthermore $R'(1, \ldots, i) \otimes \mathscr{C}(\sigma_{i+1}) = R'(1, \ldots, i+1)$ is clear on objects and the morphisms up to sign. By the definition of tensor product we need to add a sign $(-1)^j$ to maps $R'(1, \ldots, i)_j \otimes \mathscr{C}(\sigma_{i+1})_1 \to R'(1, \ldots, i)_j \otimes \mathscr{C}(\sigma_{i+1})_0$. But for a summand $\mathbf{w}(1, \ldots, i)$ of $R'(1, \ldots, i)_j$ this new edge is $(w_1, \ldots, w_i, \star).(1, \ldots, i+1)$ and thus has sign $(-1)^{\sum_{l=1}^{i} w_l} = (-1)^j$. The signs of other maps are not changed. $\square$

Example 11.2.16. For $n = 4$ we have

$$\mathscr{C}(\beta_{cyl,4}) = \begin{array}{ccccccc}
 & & q^2\mathcal{U}_1\mathcal{U}_2\mathrm{Id} & \xrightarrow{\operatorname{id}\overline{\mathrm{H}}_2\operatorname{id}} & q\mathcal{U}_1\mathrm{Id}\,\mathrm{Id} & & \\
 & \nearrow^{\operatorname{id}\operatorname{id}\overline{\mathrm{H}}_4} & \oplus & {}^{\overline{\mathrm{H}}_3\operatorname{id}\operatorname{id}}\nearrow \searrow_{-\operatorname{id}\operatorname{id}\overline{\mathrm{H}}_4} & \oplus & \searrow^{\overline{\mathrm{H}}_3\operatorname{id}\operatorname{id}} & \\
q^3\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3 & \xrightarrow{\operatorname{id}\overline{\mathrm{H}}_2\operatorname{id}} & q^2\mathcal{U}_1\mathrm{Id}\,\mathcal{U}_3 & & q\,\mathrm{Id}\,\mathcal{U}_2\mathrm{Id} & \xrightarrow{-\operatorname{id}\overline{\mathrm{H}}_2\operatorname{id}} & \mathrm{Id}\,\mathrm{Id}\,\mathrm{Id} \\
 & \searrow_{\overline{\mathrm{H}}_3\operatorname{id}\operatorname{id}} & \oplus & {}^{\overline{\mathrm{H}}_3\operatorname{id}\operatorname{id}}\nearrow \searrow_{-\operatorname{id}\operatorname{id}\overline{\mathrm{H}}_4} & \oplus & \nearrow_{\operatorname{id}\operatorname{id}\overline{\mathrm{H}}_4} & \\
 & & q^2\mathrm{Id}\,\mathcal{U}_2\mathcal{U}_3 & \xrightarrow[-\operatorname{id}\overline{\mathrm{H}}_2\operatorname{id}]{} & q\,\mathrm{Id}\,\mathrm{Id}\,\mathcal{U}_3 & &
\end{array}.$$

Directly from the Proposition 11.2.15 and the definitions we obtain

Corollary 11.2.17. $\mathscr{C}(\omega_m) \cong R'(1, \ldots, n-1)^{\otimes nm}$.

11.3 The action of the universal projector in special cases

The goal of this section is to show $P\big(\mathrm{T}(\lambda_0)\big) \simeq \mathrm{T}(\lambda_0).\mathbf{P}(n)$ for $P := G \circ F$ and F and G as defined in Section 11.1 in the special cases $k = 0$ and $k = 1$. We conjecture that this also holds for general k, but we only have enough explicit information about $\mathrm{End}(L)$ and L for $k = 0, 1$.

We start with $k=0$. First note that this case is very simple: $\mathrm{T}(\lambda_0).\mathbf{P}(n) \cong \mathrm{T}(\lambda_0)$ since $\mathrm{T}(\lambda_0).\mathcal{U}_i = 0$ for all i and $\mathbf{P}(n)_j$ does not contain Id as a summand for $j \neq 0$. Even more, $\mathbf{P}(n)$ acts as the identity on all the complexes in $K^-(\widehat{\mathrm{Cup}}(n,k))$ since all objects are isomorphic to complexes with entries shifted $\mathrm{T}(\lambda_0)$'s.

Proposition 11.3.1. *Let $k=0$, then P and $.\mathbf{P}(n)$ are isomorphic as functors from $\widehat{\mathrm{Cup}}(n,k)$ to $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$.*

Proof. Since $\mathrm{T}(\lambda_0).\mathbf{P}(n) \cong \mathrm{T}(\lambda_0)$, by Corollary 11.2.7 and Proposition 10.4.2 we only have to show $G(\mathbb{C}) \simeq \mathrm{T}(\lambda_0)$. But since $R \cong \mathbb{C}$ the first part of the functors that G is composed of just send $\mathbb{C}$ to $\mathbb{C}$ and the last two send it to $L = \mathrm{T}(\lambda_0)$. □

Now we investigate the special case $k=1$ and again use the notation of Section 10.2. As a first step we compute $P\big(\mathrm{T}(0)\big)$.

Lemma 11.3.2. *Let $f_x : L \to L\langle 2\rangle[2]$ be the element of* $\mathrm{End}(L)$ *associated to $x \in \mathbb{C}[x]/(x^n)$ under the isomorphism of Proposition 10.4.12. Then*

$$\mathrm{Cone}(f_x) \simeq$$
$$\big(q^{2n}\,\mathrm{T}(0) \to q^{2n-1}\,\mathrm{T}(1) \to \cdots \to q^{n+1}\,\mathrm{T}(n-1) \xrightarrow{f} q^{n-1}\,\mathrm{T}(n-1) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)\big),$$

where all the maps are saddles except of $f = \mathrm{H}_{n-1} \circ \overline{\mathrm{H}}_{n-1}$ which is the identity with a dot on the only black cup.

Proof. Recall the subcomplexes L^r of $L = L^n$ from Definition 10.4.8. We have

$$L^n = \mathrm{Cone}(\mathrm{V}^*(n-1)\langle n-1\rangle[n-2] \xrightarrow{f_{n-2,n-1}} L^{n-1}).$$

Thus, by Proposition 10.4.7 and Corollary 10.4.11, we obtain

$$L^n \cong \mathrm{Cone}(\overline{L^{n-1}}[-1] \to \mathrm{V}^*(n-1)^*\langle n-1\rangle[n-1]).$$

Recall from the proof of Proposition 10.4.12 that the map f_x is given by the $\pm$-isomorphism of the partcomplexes $\overline{L^{n-1}}$ of L^n and $L^{n-1}\langle 2\rangle[2]$ of $L^n\langle 2\rangle[2]$. When we apply iterated Gaussian elimination (Lemma 5.1.2) to the $\pm$-isomorphisms in $\mathrm{Cone}(f_x)$, we see that there are no new maps except the composition of $\mathrm{T}(n-1) \to \mathrm{T}(n-2) \xrightarrow{\mathrm{id}} \mathrm{T}(n-2) \to \mathrm{T}(n-1)$. This is true because this is the only possible map between what remains. But the composition is just f. □

Proposition 11.3.3. *We have $P(\mathrm{T}(0)) \simeq \widetilde{Q}$ where $\widetilde{Q} :=$*

$$\begin{aligned}
\ldots \xrightarrow{\bullet} q^{4n-1}\,\mathrm{T}(1) \to q^{4n-2}\,\mathrm{T}(2) \to \cdots \to q^{3n+1}\,\mathrm{T}(n-1)\\
\xrightarrow{\bullet} q^{3n-1}\,\mathrm{T}(n-1) \to \cdots \to q^{2n+2}\,\mathrm{T}(2) \to q^{2n+1}\,\mathrm{T}(1)\\
\xrightarrow{\bullet} q^{2n-1}\,\mathrm{T}(1) \to q^{2n-1}\,\mathrm{T}(2) \to \cdots \to q^{n+1}\,\mathrm{T}(n-1)\\
\xrightarrow{\bullet} q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0),
\end{aligned}$$

where the unlabelled $\to$ are saddles and the $\xrightarrow{\bullet}$ are $\mathrm{H}_i \circ \overline{\mathrm{H}}_i$ for $i=0$ or $i=n-1$.

Before the proof we look at an example of $\widetilde{Q}$.

Example 11.3.4. For $n = 4$ we have $\widetilde{Q} =$

$$\cdots \to q^{18}\,\mathrm{T}(2) \to q^{17}\,\mathrm{T}(1) \xrightarrow{\bullet} q^{15}\,\mathrm{T}(1) \to q^{14}\,\mathrm{T}(2) \to q^{13}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{11}\,\mathrm{T}(3) \to q^{10}\,\mathrm{T}(2) \to$$
$$q^{9}\,\mathrm{T}(1) \xrightarrow{\bullet} q^{7}\,\mathrm{T}(1) \to q^{6}\,\mathrm{T}(2) \to q^{5}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{3}\,\mathrm{T}(3) \to q^{2}\,\mathrm{T}(2) \to q\,\mathrm{T}(1) \to \mathrm{T}(0).$$

Proof. By Proposition 10.4.2 we already know $F(\mathrm{T}(\lambda_0)) = \mathbb{C}$. By Proposition 10.4.12 we have $\mathrm{End}(L) \cong \mathbb{C}[x]/(x^n)$. A projective (and free) resolution of $\mathbb{C}$ in comodules of $\mathbb{C}[x]/(x^n)$-gfmod is given by

$$\mathbb{C} \xleftarrow{\mathrm{pr}} \mathbb{C}[x]/(x^n) \xleftarrow{\cdot x} \mathbb{C}[x]/(x^n)\langle 2\rangle \xleftarrow{\cdot x^{n-1}} \mathbb{C}[x]/(x^n)\langle 2n\rangle \xleftarrow{\cdot x}$$
$$\mathbb{C}[x]/(x^n)\langle 2n+2\rangle \xleftarrow{\cdot x^{n-1}} \mathbb{C}[x]/(x^n)\langle 4n\rangle \xleftarrow{\cdot x} \mathbb{C}[x]/(x^n)\langle 4n+2\rangle \xleftarrow{\cdot x^{n-1}} \dots,$$

where pr is the projection. Thus, G first sends $\mathbb{C}$ to

$$\mathbb{C}[x]/(x^n) \xleftarrow{\cdot x} \mathbb{C}[x]/(x^n)\langle 2\rangle \xleftarrow{\cdot x^{n-1}} \mathbb{C}[x]/(x^n)\langle 2n\rangle \xleftarrow{\cdot x}$$
$$\mathbb{C}[x]/(x^n)\langle 2n+2\rangle \xleftarrow{\cdot x^{n-1}} \mathbb{C}[x]/(x^n)\langle 4n\rangle \xleftarrow{\cdot x} \mathbb{C}[x]/(x^n)\langle 4n+2\rangle \xleftarrow{\cdot x^{n-1}} \dots,$$

then to $W_L :=$

$$L \xrightarrow{f_x} L\langle 2\rangle[2] \xrightarrow{f_{x^{n-1}}} L\langle 2n\rangle[2n] \xrightarrow{f_x} L\langle 2n+2\rangle[2n+2] \xrightarrow{f_{x^{n-1}}} \dots,$$

where $f_x : L \to L\langle 2\rangle[2]$ is the element of $\mathrm{End}(L)$ associated to $x \in \mathbb{C}[x]/(x^n)$ under the isomorphism of Proposition 10.4.12 and $f_{x^{n-1}}$ the one corresponding to x^{n-1}. Finally, $G(\mathbb{C}) = \mathrm{Tot}(W_L)$, so we have to see that this is homotopic to $\widetilde{Q}$.

First note that $f_{x^{n-1}}$ is just the map id between the leftmost $\mathrm{T}(0)$ of L and the rightmost $\mathrm{T}(0)$ of $L\langle 2n-2\rangle[2n-2]$. Furthermore, the total complex consists of several copies of shifted $\mathrm{Cone}(f_x)$. Thus, using Lemma 11.3.2, it is homotopic to the total complex of the double complex associated to

$$\mathrm{Cone}(f_x) \to \mathrm{Cone}(f_x)\langle 2n\rangle[2n] \to \mathrm{Cone}(f_x)\langle 4n\rangle[4n] \to \dots,$$

where the maps are id between the outermost $\mathrm{T}(0)$. Applying Gaussian elimination (Lemma 5.1.2) to these id's, we get the desired result. □

Remark 11.3.5. Let $\mathbf{P}(2)$ be the complex from Example 11.2.4. Then for $n = 2$

$$\mathrm{T}(0).\mathbf{P}(2) = \dots \xrightarrow{\bullet} q^5\,\mathrm{T}(1) \xrightarrow{\bullet} q^3\,\mathrm{T}(1) \xrightarrow{\bullet} q\,\mathrm{T}(1) \to \mathrm{T}(0).$$

This is just $\widetilde{Q}$ in the case $n = 2$, hence by Proposition 11.3.3 this is homotopic to $G(\mathbb{C})$. Thus we know $P(\mathrm{T}(0)) \simeq \mathrm{T}(0).\mathbf{P}(n)$ in the case $n = 2$.

Our next goal is to describe $\mathrm{T}(0).\mathscr{C}(\omega_m)$ as an intermediate step to understand $\mathrm{T}(0).\mathbf{P}(n)$, since $\mathbf{P}(n)$ is defined via the $\mathscr{C}(\omega_m)$. For this we study how certain parts of the cube complex $R'(1, \dots, n-1)$ act on $\mathrm{T}(0)$.

Lemma 11.3.6. *Let $\lambda_0 \in \Lambda(n, 1)$ and $1 \le i_1 < i_2 < \dots < i_r < n$.*

a) We have $\mathrm{T}(0).\mathcal{U}_{i_1}\ldots\mathcal{U}_{i_r} \neq 0$ *if and only if* $i_1 = 1, i_2 = 2, \ldots, i_r = r$. *Furthermore,* $\mathrm{T}(0).\mathcal{U}_1\mathcal{U}_2\ldots\mathcal{U}_r \cong \mathrm{T}(r)$.

b) We have $\mathrm{T}(j).\mathcal{U}_{i_1}\ldots\mathcal{U}_{i_r} \neq 0$ *if and only if* $i_1, \ldots, i_r$ *are consecutive numbers starting with* j *or* $j \pm 1$.

We write $\mathcal{U}_{[i,j]}$ *for* $\mathcal{U}_i\mathcal{U}_{i+1}\ldots\mathcal{U}_j$, $1 \leq i \leq j \leq n-1$.

c) For $i \leq j$, $(i,j) \neq (n-1, n-1)$ *we have*

$$\begin{aligned}&(q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)).\mathcal{U}_{[i,j]}\\ &\cong (0 \to \cdots \to 0 \to q^{i+1}\,\mathrm{T}(j) \to q^{i}\,\mathrm{T}(j) \sqcup \bigcirc \to q^{i-1}\,\mathrm{T}(j) \to 0 \to \cdots \to 0)\\ &\simeq 0\end{aligned}$$

d)

$$\begin{aligned}&(q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)).\mathcal{U}_{n-1}\\ &\cong (q^{n-1}\,\mathrm{T}(n-1) \sqcup \bigcirc \to q^{n-2}\,\mathrm{T}(n-1) \to 0 \to \cdots \to 0)\end{aligned}$$

Proof. a) is clear and b) follows from a) keeping in mind that $\mathcal{U}_i\mathcal{U}_{i\pm1}\mathcal{U}_i \cong \mathcal{U}_i$ and $\mathcal{U}_i\mathcal{U}_i \cong \mathcal{U}_i \sqcup \bigcirc$. For c) and d) note that by b) $\mathrm{T}(l).\mathcal{U}_{[i,j]} \neq 0$ only if $l = i-1, i, i+1$. The homotopy equivalence of c) follows directly from Gaussian elimination (Lemma 5.1.2), since the differentials are saddles adjacent to the $\bigcirc$. ☐

Lemma 11.3.7. *Let* $R = R'(1, \ldots, n-1)$ *be the cube complex from Definition 11.2.14. Then*

a) $\mathrm{T}(0).R \cong \Big(q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)\Big)$

b) $\mathrm{T}(0).(R \otimes R) \simeq$
$$\Big(q^{n+1}\,\mathrm{T}(n-1) \xrightarrow{\bullet} q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)\Big)$$

Proof. a) This follows directly from Lemma 11.3.6 a).

b) Let

$$Q = \mathrm{T}(0).R = \big(q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0)\big).$$

We have $\mathrm{T}(0).(R \otimes R) = \big(\mathrm{T}(0).R\big).R = Q.R$. But $Q.R$ is the total complex of the double complex with rows $W_k = \bigoplus_{|\mathbf{w}|=k} Q.\big(\mathbf{w}(1, \ldots, n-1)\big)$. As such it can be written as iterated cone

$$Q.R = \mathrm{Cone}\Big(\ldots \mathrm{Cone}\,\big(\,\mathrm{Cone}(W_{n-1} \to W_{n-2}) \to W_{n-3}\big) \to \cdots \to W_0\Big)$$

By Lemma 11.3.6 we know $Q.(\mathbf{w}(1, \ldots, n-1))$ is 0 or homotopic to 0 except for $\mathbf{w} = (0, \ldots, 0)$ or $(0, \ldots, 0, 1)$. Thus, by Lemma 5.1.5 and 11.3.6 we obtain

$$Q.R \simeq \mathrm{Cone}(W_1 \to W_0) \simeq \mathrm{Cone}(qQ.\mathcal{U}_{n-1} \to Q)$$

$$\begin{array}{llllll} \simeq q^n\,\mathrm{T}(n-1) \sqcup \bigcirc \longrightarrow q^{n-1}\,\mathrm{T}(n-1) & & & & \\ & \searrow \quad\quad \searrow & & & \\ & q^{n-1}\,\mathrm{T}(n-1) \longrightarrow q^{n-2}\,\mathrm{T}(n-2) \longrightarrow \ldots \longrightarrow \mathrm{T}(0) \end{array},$$

which in turn is homotopic to

$$q^{n+1}\,\mathrm{T}(n-1) \xrightarrow{\bullet} q^{n-1}\,\mathrm{T}(n-1) \to q^{n-2}\,\mathrm{T}(n-2) \to \cdots \to q\,\mathrm{T}(1) \to \mathrm{T}(0).$$

(Here and in the following by writing $q^{r_i}\,\mathrm{T}(i)$ and $q^{r_j}\,\mathrm{T}(j)$ in the same column of a complex we mean that the complex has the direct sum of those as entry at that homological degree. So we will leave out all $\oplus$ when writing down a complex.) To see this consider the left part of the complex and observe

$$\begin{array}{ccc} q^n\,\mathrm{T}(n-1) \sqcup \bigcirc \xrightarrow{h_1} q^{n-1}\,\mathrm{T}(n-1) & & q^{n-1}\,\mathrm{T}(n-1) \xrightarrow{\mathrm{id}} q^{n-1}\,\mathrm{T}(n-1) \\ \searrow^{h_2} & \cong & \text{(crossed arrows: id, 0)} \\ q^{n-1}\,\mathrm{T}(n-1) & & q^{n+1}\,\mathrm{T}(n-1) \xrightarrow{\bullet} q^{n-1}\,\mathrm{T}(n-1). \end{array}$$

(For this isomorphism note that h_1 connects the $\bigcirc$ to a green cup while h_2 connects it to a black one, hence we get 0 and $\bullet$ when we resolve.) Then we apply Gaussian elimination (Lemma 5.1.2) with respect to the upper identity in the last diagram to obtain the desired result.

□

In the previous lemma we saw that applying R and $R \otimes R$ to $\mathrm{T}(0)$ results in subcomplexes of $\widetilde{Q}$. More general, we will see that by applying $R := R'(1, \ldots, n)$ to $\mathrm{T}(0)$ repeatedly, we obtain complexes of the following form.

Definition 11.3.8. For $m \in \mathbb{N}_0$ and $0 \leq r \leq 2(n-1)-1$, let $Q_{m,r}$ be the subcomplex of the complex $\widetilde{Q}$ from Proposition 11.3.3 given by the first $1+m\cdot(2(n-1))+r$ entries, i.e. for $r \leq n-1$ we have (where we leave out the q-shifts for better readability)

$$Q_{m,r} = \mathrm{T}(r) \to \cdots \to \mathrm{T}(1) \to \underbrace{\underbrace{\mathrm{T}(1) \to \cdots \to \mathrm{T}(1)} \to \cdots \to \underbrace{\mathrm{T}(1) \to \cdots \to \mathrm{T}(1)}}_{m} \to \mathrm{T}(0)$$

and for $n \leq r = n-1+r' \leq 2(n-1)-1$ we have

$$\begin{aligned} Q_{m,r} = & \mathrm{T}(n-r') \to \cdots \to \mathrm{T}(n-1) \to \mathrm{T}(n-1) \to \cdots \to \mathrm{T}(1) \to \\ & \underbrace{\underbrace{\mathrm{T}(1) \to \cdots \to \mathrm{T}(1)} \to \cdots \to \underbrace{\mathrm{T}(1) \to \cdots \to \mathrm{T}(1)}}_{m} \to \mathrm{T}(0). \end{aligned}$$

Example 11.3.9. For $n = 4$ we have $Q_{2,2} =$

$$\begin{aligned} & q^{18}\,\mathrm{T}(2) \to q^{17}\,\mathrm{T}(1) \xrightarrow{\bullet} q^{15}\,\mathrm{T}(1) \to q^{14}\,\mathrm{T}(2) \to q^{13}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{11}\,\mathrm{T}(3) \to q^{10}\,\mathrm{T}(2) \to \\ & q^{9}\,\mathrm{T}(1) \xrightarrow{\bullet} q^{7}\,\mathrm{T}(1) \to q^{6}\,\mathrm{T}(2) \to q^{5}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{3}\,\mathrm{T}(3) \to q^{2}\,\mathrm{T}(2) \to q\,\mathrm{T}(1) \to \mathrm{T}(0) \end{aligned}$$

and $Q_{1,5} =$

$$q^{14}\,\mathrm{T}(2) \to q^{13}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{11}\,\mathrm{T}(3) \to q^{10}\,\mathrm{T}(2) \to$$
$$q^{9}\,\mathrm{T}(1) \xrightarrow{\bullet} q^{7}\,\mathrm{T}(1) \to q^{6}\,\mathrm{T}(2) \to q^{5}\,\mathrm{T}(3) \xrightarrow{\bullet} q^{3}\,\mathrm{T}(3) \to q^{2}\,\mathrm{T}(2) \to q\,\mathrm{T}(1) \to \mathrm{T}(0).$$

Lemma 11.3.10.

a) *Let $n \leq r = n-1+r' < 2(n-1)$, then for $i \leq j \leq n-1$ we have $Q_{m,r}.\mathcal{U}_{[i,j]} \simeq 0$ if $i \neq n-r', n-r'-1$. Let $r = 0$, then $Q_{m,r}.\mathcal{U}_{[i,j]} \simeq 0$ if $i \neq 1$.*

b) *For $n \leq r < 2(n-1)$ let $t = 2n-r-1 \geq 2$ and $t \leq j \leq n-1$*

$$Q_{m,r}.(q\mathcal{U}_{[t-1,j]} \xrightarrow{\overline{H}_r} \mathcal{U}_{[t,j]}) \simeq 0.$$

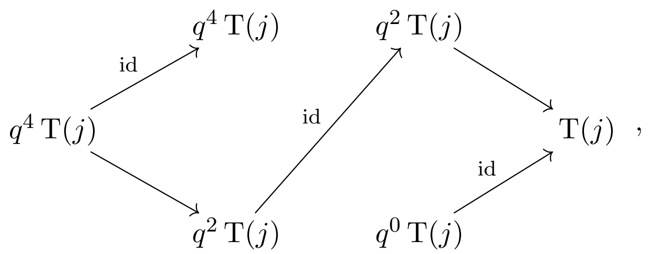
Proof. a) If $i \neq n-1, 1$, this follows analogously to Lemma 11.3.6 c), only that we might have multiple copies of the form

$$q^{s+2}\,\mathrm{T}(j) \to q^{s+1}\,\mathrm{T}(j) \sqcup \bigcirc \to q^{s}\,\mathrm{T}(j), \tag{11.2}$$

which are homotopic to zero. For $i = 1$ or $n-1$ we get summands of the form

$$q^{s+4}\,\mathrm{T}(j) \to q^{s+3}\,\mathrm{T}(j) \sqcup \bigcirc \xrightarrow{\bullet} q^{s+1}\,\mathrm{T}(j) \sqcup \bigcirc \to q^{s}\,\mathrm{T}(j). \tag{11.3}$$

Since the saddles and the $\bullet$ are adjacent to the circle, by delooping (Lemma 3.3.5) we get the isomorphic complex



which is homotopic to zero by applying Gaussian elimination (Lemma 5.1.2) to the id's. The condition $i \neq n-r', n-r'-1$ guarantees that we have no truncated copies of (11.2) and (11.3). If $r = 0$, then the only truncated copy can appear for $i = 1$ since $i = 0$ is not possible.

b) As in part a) we can eliminate (using homotopies) all copies of (11.2) and (11.3) inside $Q_{m,r}.q\mathcal{U}_{[r-1,j]}$ and $Q_{m,r}.\mathcal{U}_{[r,j]}$. Thus we only have to consider the truncated parts of (11.2) and (11.3) that remain. On the left, the complex $Q_{m,r}$ looks like

$$q^{s}\,\mathrm{T}(t) \to q^{s-1}\,\mathrm{T}(t+1) \to \cdots \to q^{s-t+1}\,\mathrm{T}(n-1) \xrightarrow{\bullet} q^{s-t-1}\,\mathrm{T}(n-1) \to$$

for some s. If $t \neq n-2, n-1$, then $Q_{m,r}.(q\mathcal{U}_{[t-1,j]} \xrightarrow{\overline{H}_r} \mathcal{U}_{[t,j]}) \simeq$

$$\begin{array}{llllll} q^{s+1}\,\mathrm{T}(j) \longrightarrow 0 & & \dots & & & \\ \quad \searrow & & & & & \\ & q^{s}\,\mathrm{T}(j) \sqcup \bigcirc \longrightarrow q^{s-1}\,\mathrm{T}(j) \longrightarrow 0 & & & \dots & \end{array}$$

which is a summand of type (11.2) and thus homotopic to zero. For $t = n-2$ the left side of $Q_{m,r}$ looks like

$$\begin{aligned} q^{s}\,\mathrm{T}(n-2) \to q^{s-1}\,\mathrm{T}(n-1) \to q^{s-3}\,\mathrm{T}(n-1) \to q^{s-4}\,\mathrm{T}(n-2) \\ \to q^{s-5}\,\mathrm{T}(n-3) \to q^{s-6}\,\mathrm{T}(n-4)\dots \end{aligned}$$

and we obtain $Q_{m,r}.(q\mathcal{U}_{[t-1,j]} \xrightarrow{\overline{H}_r} \mathcal{U}_{[t,j]}) \simeq$

$$\begin{array}{ll} q^{s+1}\,\mathrm{T}(j) \longrightarrow 0 \qquad \dots & \\ \quad \searrow & \\ & q^{s}\,\mathrm{T}(j) \sqcup \bigcirc \to q^{s-1}\,\mathrm{T}(j) \to q^{s-3}\,\mathrm{T}(j) \to q^{s-4}\,\mathrm{T}(j) \sqcup \bigcirc \to q^{s-5}\,\mathrm{T}(j) \to 0 \end{array}$$

The first three non-zero components and the last three non-zero components form a configuration as in (11.2) and thus are homotopic to zero. For $t = n-1$ the left side of $Q_{m,r}$ looks like

$$q^{s}\,\mathrm{T}(n-1) \to q^{s-2}\,\mathrm{T}(n-1) \to q^{s-3}\,\mathrm{T}(n-2) \to q^{s-3}\,\mathrm{T}(n-3) \to q^{s-4}\,\mathrm{T}(n-4) \to \dots$$

and we have $Q_{m,r}.(q\mathcal{U}_{[t-1,j]} \xrightarrow{\overline{H}_r} \mathcal{U}_{[t,j]}) \simeq$

$$\begin{array}{ccccccccc} q^{s+1}\,\mathrm{T}(j) & \longrightarrow & q^{s-1}\,\mathrm{T}(j) & \longrightarrow & q^{s-2}\,\mathrm{T}(j) \sqcup \bigcirc & \longrightarrow & q^{s-3}\,\mathrm{T}(j) \longrightarrow 0 & & \dots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ & & q^{s}\,\mathrm{T}(j) \sqcup \bigcirc & \longrightarrow & q^{s-2}\,\mathrm{T}(j) \sqcup \bigcirc & \longrightarrow & q^{s-3}\,\mathrm{T}(j) & \longrightarrow 0 & \dots \end{array}.$$

After deleting the type (11.2) part of the first row, a summand of the type (11.3) remains which again is homotopic to zero.

□

Corollary 11.3.11. *a)* $Q_{m,n-1}.R \simeq Q_{m,n}$.

b) For $n \le r < 2(n-1)$ *we have* $Q_{m,r}.R \simeq Q_{m,r+1}$, *where we set* $Q_{m,2(n-1)} := Q_{m+1,0}$.

c) $Q_{m,0}.R \simeq Q_{m,n-1}$.

Proof. a) This follows analogously to Lemma 11.3.7 b) which is the special case $m = 0$: By Lemma 11.3.10, we have $Q_{m,n-1}.\big(\mathbf{w}(1,\dots,n-1)\big) \simeq 0$ for $\mathbf{w} \neq (0,\dots,0)$ or $(0,\dots,0,1)$ by Gaussian elimination (Lemma 5.1.2). Therefore,

$$Q_{m,n-1}.R \simeq \mathrm{Cone}(qQ_{m,n-1}.\mathcal{U}_{n-1} \to Q_{m,n-1}).$$

Analogously to Lemma 11.3.7 b), this is now homotopic to $Q_{m,n}$ after possibly eliminating copies of shifted (11.3) by Gaussian elimination.

b) By Lemma 11.3.10 the only $\mathbf{w}$ for which we need to consider $Q_{m,r}.\mathbf{w}(1,\dots,n-1)$ are $\mathbf{w} = (0,\dots,0)$ and $\mathbf{w} = (0, \dots, 0, \underset{r-1}{1}, 0, \dots, 0)$. Thus, by applying repeated Gaussian elimination to the complex $Q_{m,r}.R$ we get $\mathrm{Cone}(qQ_{m,r}.\mathcal{U}_{r-1} \to Q_{m,r})$. After deleting the shifted summands of the form (11.2) inside $qQ_{m,r}.\mathcal{U}_{r-1}$ only a shifted $\mathrm{T}(r).\mathcal{U}_{r-1} \cong \mathrm{T}(r-1)$ remains at the far left. Together with the cone-map to $Q_{m,r}$ we obtain $Q_{m,r+1}$.

c) By Lemma 11.3.10 a), we have that $Q_{m,0}.\big(\mathbf{w}(1,\dots,n-1)\big) \simeq 0$ for $\mathbf{w} \neq (1,\dots,1,0,\dots0)$ by Gaussian elimination. For $\mathbf{w} = (\underbrace{1,\dots,1}_{j},0,\dots,0)$, we can eliminate all pieces of type (11.3), until only $q^s\,\mathrm{T}(j) \sqcup \bigcirc \to q^{s-1}\,\mathrm{T}(j)$ remain on the left. Thus, we have

$$\begin{array}{llll}
q^{s+n-1}\,\mathrm{T}(n-1) \sqcup \bigcirc \to q^{s+n-2}\,\mathrm{T}(n-1) & & & \\
\quad\searrow \ \ddots \quad \searrow \ \ddots & & & \\
 & q^{s+1}\,\mathrm{T}(1) \sqcup \bigcirc \to q^{s}\,\mathrm{T}(1) & & \\
 & \qquad\quad \searrow \qquad\qquad \searrow & & \\
 & \qquad\qquad q^{s}\,\mathrm{T}(1) \to q^{s-1}\,\mathrm{T}(2) \to & &
\end{array}$$

Now resolving the circles and using Gaussian elimination to eliminate the higher diagonal using half of the lower diagonal gives the desired result.

□

Theorem 11.3.12. *Let $m \in \mathbb{N}_0$, $r < n$ and $Q_{m,r}$ as in Definition 11.3.8. Then we have*

$$\mathrm{T}(0).R^{\otimes mn+r} \simeq Q_{m,r}$$

and in particular $\mathrm{T}(0).\mathscr{C}(\omega_m) \simeq Q_{m,0}$.

Proof. The first part follows inductively from Lemma 11.3.7 a) and Corollary 11.3.11. Using Corollary 11.2.17, we obtain the second assertion. □

We know that $\mathbf{P}(n)$ is the limit of the direct system of the $\mathscr{C}(\omega_m)$'s and the $Q_{m,0}$ are parts of the complex $\widetilde{Q}$. Now we want to put this together to obtain:

Corollary 11.3.13. $\mathrm{T}(0).\mathbf{P}(n) \simeq \widetilde{Q}$ *with the notation from Proposition 11.3.3.*

Proof. Since $\mathbf{P}(n) = \lim\limits_{\to} \mathbf{B}(n)$ (cf. Definition 11.2.12), by Proposition 11.2.11, we have to compute that $\widetilde{Q}$ is a limit of

$$\big(\mathrm{T}(0) \xrightarrow{\mathrm{id}.f^0} \mathrm{T}(0).\mathscr{C}(\omega_1) \xrightarrow{\mathrm{id}.f^1} \mathrm{T}(0).\mathscr{C}(\omega_2) \xrightarrow{\mathrm{id}.f^2} \dots\big).$$

For $i \geq 0$ we define $\tilde{f}^i : \mathrm{T}(0).\mathscr{C}(\omega_i) \to \widetilde{Q}$ via $\mathrm{T}(0).\mathscr{C}(\omega_i) \xrightarrow{g^i} Q_{i,0} \overset{\iota_i}{\hookrightarrow} \widetilde{Q}$, where g^i is given by the homotopy equivalence $\mathrm{T}(0).\mathscr{C}(\omega_i) \simeq Q_{i,0}$ from the previous theorem. Now

$$\begin{aligned}\mathrm{Cone}(\widetilde{f}_i) &\simeq \mathrm{Cone}(Q_{i,0} \xrightarrow{\iota_i} \widetilde{Q}) \\ &\simeq \cdots \to \widetilde{Q}_{2i(n-1)+2} \to \widetilde{Q}_{2i(n-1)+1} \to 0 \to \cdots \to 0,\end{aligned}$$

where the last homotopy follows for example by Gaussian elimination. Therefore, $\left|\mathrm{Cone}(\widetilde{f}_i)\right|_\hbar \geq i \cdot 2(n-1)$, hence $\lim\limits_{i\to\infty} \left|\mathrm{Cone}(\widetilde{f}_i)\right|_\hbar = \infty$. It remains to show that

$$\iota_i \circ g^i \simeq \iota_{i+1} \circ g^{i+1} \circ \mathrm{id}\,.f^i.$$

But this holds since f^i and g^i are both inductively defined via tensoring with R^n:

$$\begin{array}{ccccccc}(\mathrm{T}(0).\mathscr{C}(w_i)).\,\mathrm{Id} & \xrightarrow{\cong} & \mathrm{T}(0).\mathscr{C}(w_i) & \xrightarrow{g^i} & Q_{i,0} & \hookrightarrow\searrow & \\ \big\downarrow{\scriptstyle \mathrm{id}\,.f^0} & & & & \big\downarrow{\scriptstyle \mathrm{id}\,.f^0} & & \\ (\mathrm{T}(0).\mathscr{C}(w_i)).\mathscr{C}(w_1) & \xrightarrow{\cong} & \mathrm{T}(0).\mathscr{C}(w_{i+1}) & \xrightarrow{\bar{g}^{i+1}} & Q_{i,0}.\mathscr{C}(w_1) & \xrightarrow[s]{\simeq} & Q_{i+1,0}\end{array}$$

The square commutes obviously, the triangle commutes by construction and $g^{i+1} = s \circ \bar{g}^{i+1}$. □

Finally, we have altogether:

Theorem 11.3.14. *Let $k = 1$, then P and $.\mathbf{P}(n)$ are isomorphic as functors from $\widehat{\mathrm{Cup}}(n,k)$ to $K^-\big(\widehat{\mathrm{Cup}}(n,k)\big)$.*

Proof. By Corollary 11.2.7 we only have to check $P(\mathrm{T}(0)) \simeq \mathrm{T}(0).\mathbf{P}(n)$. But this follows directly from Corollary 11.3.13 and Proposition 11.3.3. □

Thus, we see that at least for small k our construction agrees with the action of the universal projector.

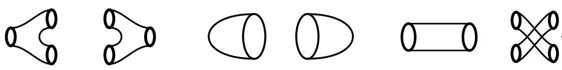
Appendix A

Coloured cobordisms and diagram categories

A.1 Coloured 2-dimensional TQFT

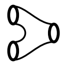
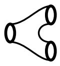
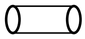
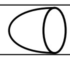
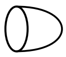
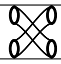
In this section we recall a generalisation of a construction of Khovanov [Kho00] in a way motivated by Stroppel in [Str09]. For more details see [Scha10] and [Scha12].

Let *Cob* be the category of two-dimensional cobordisms up to boundary-preserving diffeomorphisms. By [Koc04] this monoidal category is generated under composition and disjoint union by the cobordisms



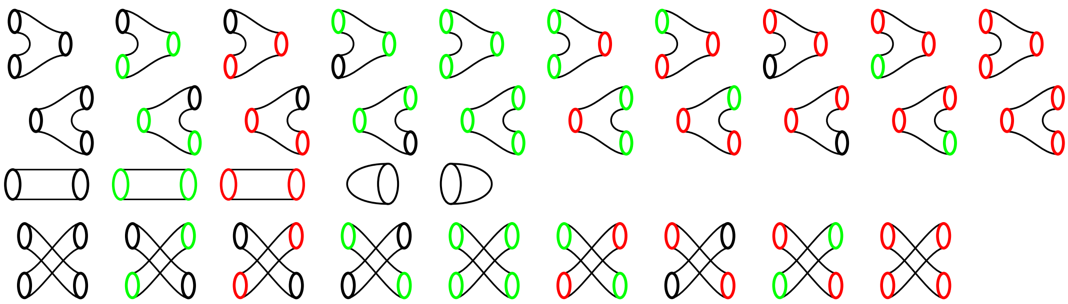
Furthermore, these generators are subject to an explicit list of relations (see. e.g. [Koc04]) saying that the image of the circle under a symmetric monoidal functor to the monoidal category of finite dimensional vector spaces is a commutative Frobenius algebra and every such commutative Frobenius algebra defines such a functor.

When we fix the commutative Frobenius algebra $\mathbb{C}[x]/(x^2)$, then this gives us a 2-dimensional topological quantum field theory $\mathcal{F}$, i.e. a symmetric monoidal functor from *Cob* to the category of vector spaces. $\mathcal{F}$ sends n circles to $\left(\mathbb{C}[x]/(x^2)\right)^{\otimes n}$ and the generators of the morphisms to linear maps as described in the following table:

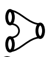
	$1 \otimes 1 \mapsto 1$ $x \otimes 1 \mapsto x$ $1 \otimes x \mapsto x$ $x \otimes x \mapsto 0$
	$1 \mapsto x \otimes 1 + 1 \otimes x$ $x \mapsto x \otimes x$
	$1 \mapsto 1$ $x \mapsto x$
	$1 \mapsto 1$
	$1 \mapsto 0$ $x \mapsto 1$
	$a \otimes b \mapsto b \otimes a$

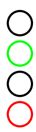
Now we consider coloured cobordisms, i.e. the objects of the monoidal category are circles coloured black, green or red and the morphisms are cobordisms with boundaries coloured accordingly.

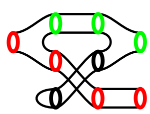
Definition A.1.1. Let *ColCob* be the monoidal category generated under composition and disjoint union by



subject to the relations for *ColCob*. The relations of *ColCob* are exactly the lifts of the relations from *Cob*, i.e. we replace all generating morphisms by coloured generating morphisms in all possible compatible ways. For an explicit list of the relations see [Scha10, A.2].

Note that the cobordisms  and  do not appear in all possible colourings of the boundaries as generators. Our restriction of possibilities is motivated by the application of *CobCob* in [Scha12], where other colourings cannot appear.

Example A.1.2. An example for an object in *ColCob*: 

An example for a morphism in *ColCob*: 

Let Vect be the monoidal category of vector spaces with ordinary tensor product.

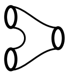
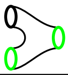
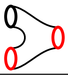
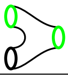
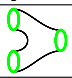
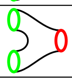
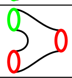
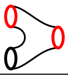
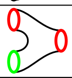
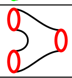
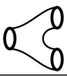
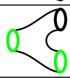
Theorem A.1.3 ([Scha12, Lemma 10.3, Theorem 10.4]). *ColCob is a symmetric monoidal category and a symmetric monoidal functor*

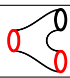
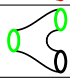
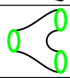
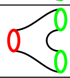
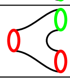
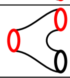
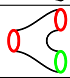
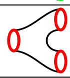
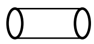
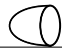
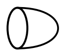
$$\mathcal{F}_{Col} : ColCob \to \mathrm{Vect}$$

can be defined as follows:
Let B be the black circle in ColCob, R the red one and G the green one.

$$\mathcal{F}_{Col}(B) = \mathbb{C}[x]/(x^2), \quad \mathcal{F}_{Col}(G) \ = \mathbb{C}, \quad \mathcal{F}_{Col}(R) = 0.$$

The values of $\mathcal{F}_{Col}$ on generating morphisms can be found in the table below.

	$1 \otimes 1 \mapsto 1$ $x \otimes 1 \mapsto x$ $1 \otimes x \mapsto x$ $x \otimes x \mapsto 0$
	$1 \otimes 1 \mapsto 1$ $x \otimes 1 \mapsto 0$
	$1 \otimes 0 \mapsto 0$ $x \otimes 0 \mapsto 0$
	$1 \otimes 1 \mapsto 1$ $1 \otimes x \mapsto 0$
	$1 \otimes 1 \mapsto 1$
	$1 \otimes 1 \mapsto 0$
	$1 \otimes 0 \mapsto 0$
	$0 \otimes 1 \mapsto 0$ $0 \otimes x \mapsto 0$
	$0 \otimes 1 \mapsto 0$
	$0 \otimes 0 \mapsto 0$
	$1 \mapsto x \otimes 1 + 1 \otimes x$ $x \mapsto x \otimes x$
	$1 \mapsto x \otimes 1$

	$0 \mapsto 0 \otimes 0$
	$1 \mapsto 1 \otimes x$
	$1 \mapsto 0 \otimes 0$
	$0 \mapsto 0 \otimes 0$
	$0 \mapsto 0 \otimes 0$
	$0 \mapsto 0 \otimes 0$
	$0 \mapsto 0 \otimes 0$
	$0 \mapsto 0 \otimes 0$
	$1 \mapsto 1$ $x \mapsto x$
	$1 \mapsto 1$
	$0 \mapsto 0$
	$1 \mapsto 1$
	$1 \mapsto 0$ $x \mapsto 1$
Twists	$a \otimes b \mapsto b \otimes a$

Forgetting the colours defines a monoidal functor $I : ColCob \to Cob$.
For A an object in $ColCob$ we define $i_A : \mathcal{F}_{Col}(A) \to \mathcal{F}\big(I(A)\big)$ by

$$\begin{aligned} i_B : \mathbb{C}[x]/(x^2) &\xrightarrow{\mathrm{id}} \mathbb{C}[x]/(x^2) \\ i_G : \mathbb{C} &\hookrightarrow \mathbb{C}[x]/(x^2) \\ 1 &\mapsto 1 \\ i_R : \{0\} &\hookrightarrow \mathbb{C}[x]/(x^2) \end{aligned}$$

and $i_{X \otimes Y} = i_X \otimes i_Y$

for B, G, R black, green, red circles, respectively, and X, Y arbitrary objects in $ColCob$. For A an object in $ColCob$ we define $\pi_A : \mathcal{F}\big(I(A)\big) \to \mathcal{F}_{Col}(A)$ by

$$\begin{aligned} \pi_B &: \mathbb{C}[x]/(x^2) \stackrel{\text{id}}{\to} \mathbb{C}[x]/(x^2) \\ \pi_G &: \mathbb{C}[x]/(x^2) \to \mathbb{C} \\ &\qquad 1 \mapsto 1 \\ &\qquad x \mapsto 0 \\ \pi_R &: \mathbb{C}[x]/(x^2) \stackrel{0}{\to} \{0\} \\ \text{and } \pi_{X\otimes Y} &= \pi_X \otimes \pi_Y \end{aligned}$$

for B, G, R, X, Y as above.

For objects we obviously have $\mathcal{F}_{Col}(A) = \pi_A\big(\mathcal{F}(I(A))\big)$.

Lemma A.1.4. *For $f : A \to A'$ in $ColCob$ the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}_{Col}(A) & \xrightarrow{i_A} & \mathcal{F}\big(I(A)\big) \\ \Big\downarrow{\scriptstyle \mathcal{F}_{Col}(f)} & & \Big\downarrow{\scriptstyle \mathcal{F}(I(f))} \\ \mathcal{F}_{Col}(A') & \xleftarrow[\pi_{A'}]{} & \mathcal{F}\big(I(A')\big) \end{array}$$

Proof. The equation $\mathcal{F}_{Col}(f) = i_A \circ \mathcal{F}\big(I(f)\big) \circ \pi_{A'}$ is true for the generating cobordisms, see tables above. Thus, it extends to all cobordisms and the lemma follows. □

A.2 Equivalence to a category of diagrams

To obtain a connection to representation theory and diagram algebras, we want to compare the category $\text{Cup}(n, k)$ to the following, more algebraically defined category.

Definition A.2.1. Let $\mathcal{M}_{n,k}$ be the category with:
Objects: $\lambda \in \Lambda(n, k)$
Morphisms: $\text{Hom}(\lambda, \mu) = \mathcal{F}_{col}\big(\text{C}(\mu)\overline{\text{C}(\lambda)}\big)$, where $\mathcal{F}_{col}$ is the functor defined in Section A.1
The composition of morphisms is defined as follows: Consider $f \in \text{Hom}(\mu, \lambda)$ and $g \in \text{Hom}(\nu, \mu)$. There is a cobordism from $\text{C}(\lambda)\overline{\text{C}(\mu)}\,\text{C}(\mu)\overline{\text{C}(\nu)}$ to $\text{C}(\lambda)\overline{\text{C}(\nu)}$ contracting $\overline{\text{C}(\mu)}\,\text{C}(\mu)$ given by (possibly nested) saddle cobordisms. This induces a homomorphism of vector spaces

$$\mathcal{F}_{col}\big(\text{C}(\lambda)\overline{\text{C}(\mu)}\big) \otimes \mathcal{F}_{col}\big(\text{C}(\mu)\overline{\text{C}(\nu)}\big) \to \mathcal{F}_{col}\big(\text{C}(\lambda)\overline{\text{C}(\nu)}\big) \tag{A.1}$$

and thus a composition

$$\text{Hom}(\mu, \lambda) \otimes \text{Hom}(\nu, \mu) \to \text{Hom}(\nu, \lambda).$$

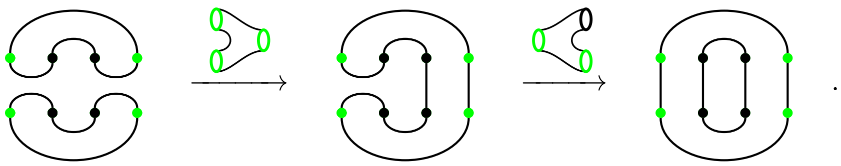
Here, the homomorphism spaces are $\mathbb{C}$-vector spaces and the composition is $\mathbb{C}$-linear.

The grading is defined as follows: Let $f = x_1 \otimes \ldots \otimes x_r$ be a basis element of $\text{Hom}(\lambda, \mu)$, i.e. $x_i \in \{1, x\}$. Define $\deg(1) = 0$, $\deg(x) = 2$ and $\deg(f) = n - r + \sum_{i=1}^{r} \deg(x_i)$.

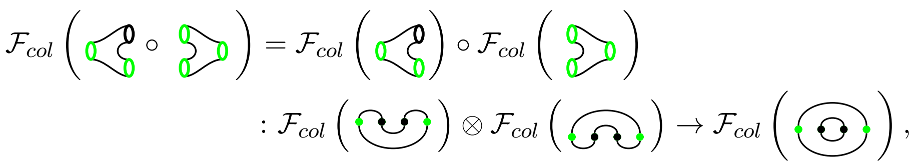
Example A.2.2. Let $n = 2$ and $k = 1$. Then $\mathcal{M}_{n,k}$ has the two objects $\wedge\vee$ and $\vee\wedge$. The composition

$$\mathrm{Hom}\big((\wedge\vee),(\vee\wedge)\big) \otimes \mathrm{Hom}\big((\vee\wedge),(\wedge\vee)\big) \to \mathrm{Hom}\big((\vee\wedge),(\vee\wedge)\big)$$

is defined via the contracting cobordism



Thus, it equals



which is explicitly

$$\begin{aligned} \mathbb{C} \otimes \mathbb{C} &\to \mathbb{C}[x]/(x^2) \otimes \mathbb{C} \\ 1 \otimes 1 &\mapsto x \otimes 1 \end{aligned}$$

with both $1 \in \mathbb{C}$ of degree 1 and $x \otimes 1$ of degree 2.

Remark A.2.3. The vector space $\oplus_{\lambda,\mu\in\Lambda(n,k)} \mathrm{Hom}(\lambda,\mu)$ together with the multiplication given by (A.1) defines an algebra. It agrees with the so-called generalised Khovanov algebra ([Str09, 5.4], [BS11a]) which plays an important role in Lie theory and representation theory.

We want to connect now the topological picture with the more algebraic picture. In Theorem A.2.7 we will establish an equivalence of categories between $\widehat{\mathcal{M}_{n,k}} = \mathrm{Mat}\left(\mathcal{M}_{n,k}^{\mathbb{Z}}\right)$ and $\widehat{\mathrm{Cup}}(n,k) = \mathrm{Mat}\left(\mathrm{Cup}(n,k)^{\mathbb{Z}}\right)$.

An important step is to show that $\mathcal{M}_{n,k}$ and $\mathrm{Cup\,T}(n,k)$ are equivalent. By Corollary 3.4.12 we already know

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cup\,T}(n,k)}\left(\mathrm{T}(\lambda),\mathrm{T}(\mu)\right) &= \mathrm{Hom}_{\mathrm{Cup}(n,k)}\left(\mathrm{T}(\lambda),\mathrm{T}(\mu)\right) \\ &\overset{\Phi'}{\cong} \mathcal{F}_{col}\left(\mathrm{C}(\mu)\overline{\mathrm{C}(\lambda)}\right) = \mathrm{Hom}_{\mathcal{M}_{n,k}}(\lambda,\mu). \end{aligned}$$

For the equivalence, we want to define a functor from $\mathrm{Cup\,T}(n,k)$ to $\mathcal{M}_{n,k}$ on morphisms via this isomorphism. For that we have to check, that the isomorphism is compatible with composition.

Proposition A.2.4. *Let C, D, E objects of* $\mathrm{Cob}(2n,0)$ *without circles. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(D,E)\otimes \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C,D) & \xrightarrow{\ \circ\ } & \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C,E) \\
\big\downarrow{\scriptstyle \Phi\otimes\Phi} & & \big\downarrow{\scriptstyle \Phi} \\
\mathcal{F}(E\overline{D})\otimes\mathcal{F}(D\overline{C}) & \xrightarrow{\ \bar{\circ}\ } & \mathcal{F}(E\overline{C}),
\end{array}$$

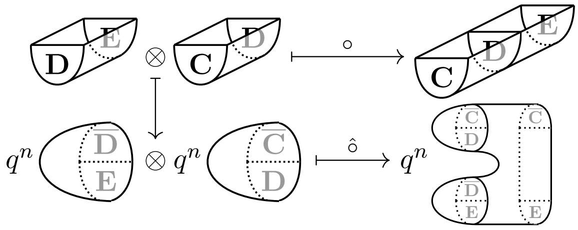
where Φ is the isomorphism from Theorem 3.4.11 and $\bar{\circ}$ is defined similar to the composition (A.1) *in $\mathcal{M}_{n,k}$, only without colours, i.e. it is the functor $\mathcal{F}$ applied to the cobordism $\alpha_{C,D,E}$ that comes from putting $D\overline{C}$ on top of $E\overline{D}$ and contracting $\overline{D}D$.*

Proof. We show that the two squares of the following diagram commute:

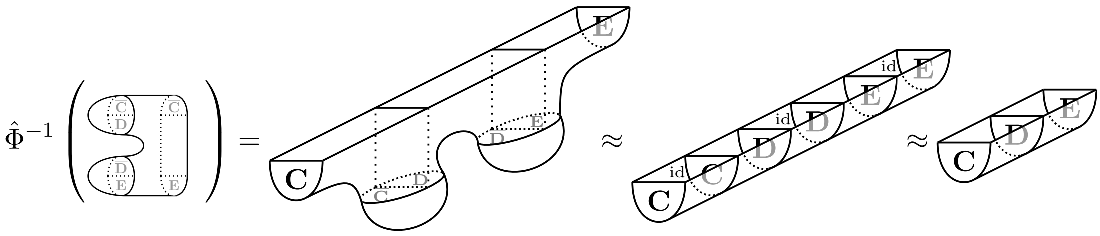
$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Cob}(2n,0)}(D,E)\otimes \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C,D) & \xrightarrow{\ \circ\ } & \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(C,E) \\
\big\downarrow{\scriptstyle \hat{\Phi}\otimes\hat{\Phi}} & & \big\downarrow{\scriptstyle \hat{\Phi}} \\
q^n\,\mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing,E\overline{D})\otimes q^n\,\mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing,D\overline{C}) & \xrightarrow{\ \hat{\circ}\ } & q^n\,\mathrm{Hom}_{\mathrm{Cob}(0,0)}(\varnothing,E\overline{C}) \\
\big\downarrow{\scriptstyle \bar{\Phi}\otimes\bar{\Phi}} & & \big\downarrow{\scriptstyle \bar{\Phi}} \\
\mathcal{F}(E\overline{D})\otimes\mathcal{F}(D\overline{C}) & \xrightarrow{\ \bar{\circ}\ } & \mathcal{F}(E\overline{C}),
\end{array}$$

where $\hat{\circ}$ is defined as $f\otimes g \overset{\hat{\circ}}{\mapsto} \alpha_{C,D,E}\circ(f\otimes g)$.

From the definition of $\hat{\circ}$ and $\bar{\circ}$, the lower square commutes by Lemma 3.4.10 and [BN05, Section 9.1]. The commutativity of the upper square becomes clear when we consider it on schematically pictured elements:



Furthermore,



□

Corollary A.2.5. *Let C, D, E objects of* $\mathrm{Cup}(n,k)$ *without circles. Then the following diagram commutes*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{Cup}(n,k)}(D,E)\otimes \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D) & \xrightarrow{\ \circ\ } & \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,E) \\
\big\downarrow{\scriptstyle \Phi'\otimes\Phi'} & & \big\downarrow{\scriptstyle \Phi'} \\
\mathcal{F}_{col}(E\overline{D})\otimes\mathcal{F}_{col}(D\overline{C}) & \xrightarrow{\ \bar{\circ}\ } & \mathcal{F}_{col}(E\overline{C}),
\end{array}$$

where Φ' is the isomorphism from Theorem 3.4.12 and $\bar{\circ}$ is the composition from Definition A.2.1.

Proof. For I the colouring forgetting map from Definition 3.4.6, the isomorphism Φ' is given by

$$\mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D) \hookrightarrow \mathrm{Hom}_{\mathrm{Cob}(2n,0)}\big(I(C),I(D)\big) \xrightarrow{\Phi} \mathcal{F}\big(I(D)\overline{I(C)}\big) \xrightarrow{\pi} \mathcal{F}_{col}(D\overline{C})$$

with inverse

$$\mathcal{F}_{col}(D\overline{C}) \overset{i}{\hookrightarrow} \mathcal{F}(I(D)\overline{I(C)}) \xrightarrow{\Phi^{-1}} \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(I(C),I(D)) \xrightarrow{\Pi} \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D)$$

Let α_{CDE} be again the contracting cobordism. Consider the following diagram:

$$\begin{array}{ccc}
\mathcal{F}_{col}(E\overline{D}) \otimes \mathcal{F}_{col}(D\overline{C}) & \xrightarrow{\mathcal{F}_{col}(\alpha_{CDE})} & \mathcal{F}_{col}(E\overline{C}) \\
\big\downarrow{\scriptstyle i\otimes i} & & \big\uparrow{\scriptstyle \pi} \\
\mathcal{F}\big(I(E)\overline{I(D)}\big) \otimes \mathcal{F}\big(I(D)\overline{I(C)}\big) & \xrightarrow{\mathcal{F}(\alpha_{CDE})} & \mathcal{F}\big(I(E)\overline{I(C)}\big) \\
\big\downarrow{\scriptstyle \Phi\otimes\Phi} & & \big\uparrow{\scriptstyle \Phi^{-1}} \\
\mathrm{Hom}_{\mathrm{Cob}(2n,0)}\big(I(D),I(E)\big) \otimes \mathrm{Hom}_{\mathrm{Cob}(2n,0)}\big(I(C),I(D)\big) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathrm{Cob}(2n,0)}(I(C),I(E)) \\
\big\downarrow{\scriptstyle \Pi\otimes\Pi} & & \big\uparrow \\
\mathrm{Hom}_{\mathrm{Cup}(n,k)}(D,E) \otimes \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,D) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathrm{Cup}(n,k)}(C,E)
\end{array}$$

The upper square commutes by Lemma A.1.4 and the fact that $\mathcal{F}_{col}(E\overline{D}) \otimes \mathcal{F}_{col}(D\overline{C}) = \mathcal{F}_{col}(E\overline{D} \otimes D\overline{C})$ and $\mathcal{F}\big(I(E)\overline{I(D)}\big) \otimes \mathcal{F}\big(I(D)\overline{I(C)}\big) = \mathcal{F}\big(I(E\overline{D} \otimes D\overline{C})\big)$. The middle square commutes by the proposition above. The lower square commutes except if the additional relations of $\mathrm{Cup}(n,k)$ are used. Thus, if they are not used, the outer square commutes, too.

It remains to check the commutativity of the outer square when an additional relation is used. Assume first, the additional relations of $\mathrm{Cup}(n,k)$ are used when projecting by $\Pi \otimes \Pi$. But then, because $\Pi \circ \Phi \circ i$ is an isomorphism, this corresponds to the zero element in $\mathcal{F}_{col}(E\overline{D}) \otimes \mathcal{F}_{col}(D\overline{C})$ and the outer square commutes for this element since everything is zero. Now assume for basis elements $x \otimes x' \in \mathcal{F}_{col}(E\overline{D}) \otimes \mathcal{F}_{col}(D\overline{C})$ that $\Phi'(x) \otimes \Phi'(x') \neq 0$ but $\Phi'(x) \circ \Phi'(x') = 0$ because of the additional relations. This means $\Pi(\Phi(i(x)) \circ \Phi(i(x'))) = 0$ but $\Phi(i(x)) \circ \Phi(i(x')) \neq 0$. But since $\Pi(y) = 0$ is equivalent to $\pi(\Phi^{-1}(y)) = 0$, we are done in this case, too. □

Corollary A.2.6. *The categories* $\mathrm{CupT}(n,k)$ *and* $\mathcal{M}_{n,k}$ *are equivalent as graded pre-additive categories.*

Proof. By Corollary A.2.5 we know that Φ' is compatible with composition. By construction $\Phi'(\mathrm{id}) = \mathrm{id} = 1 \otimes \ldots \otimes 1$. Thus, we can define a functor $G : \mathrm{CupT}(n,k) \to \mathcal{M}_{n,k}$ on objects via $\mathrm{C}(\lambda) \mapsto \lambda$ and on morphisms via $f \mapsto \Phi'(f)$. G is obviously fully faithful and essentially surjective, hence yields an equivalence of categories. Furthermore, G is compatible with the pre-additive structure and the grading.

□

From Remark 3.5.2 and Corollary A.2.6 the main result of this section follows:

Theorem A.2.7. *For any $n, k \in \mathbb{Z}_{\geq 0}$ with $k \leq n$ there is an equivalence of categories*

$$\widehat{\mathcal{M}_{n,k}} \cong \widehat{\mathrm{Cup}}(n, k).$$

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Zusammenfassung

Diese Dissertation beschäftigt sich mit der Kategorifizierung von darstellungstheoretischen Objekten mit Hilfe von Methoden aus der Topologie und der homologischen Algebra.

In seiner Arbeit über eine alternative Beschreibung der Khovanov-Homologie kategorifizierte Bar-Natan die Temperley-Lieb-Algebra in einer Kobordismus-Sprache. Wir benutzen Bar-Natans Herangehensweise um den $\mathcal{U}_q(\mathfrak{sl}_2)$-Modul $V^{\otimes n}$ zu kategorifizieren. Hierbei ist die Algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ die Quanten-Version der universellen einhüllenden Algebra der komplexen Lie-Algebra $\mathfrak{sl}_2$ und der Modul V ist die Quanten-Version der natürlichen Darstellung.

Der $\mathcal{U}_q(\mathfrak{sl}_2)$-Modul $V^{\otimes n}$ zerfällt in Gewichtsräume $(V^{\otimes n})_{2k-n}$. Indem wir jeden der Gewichtsräume kategorifizieren, erhalten wir eine Kategorifizierung von $V^{\otimes n}$. Die Gewichtsräume haben eine besondere Basis, die kanonische Basis, welche durch sogenannte Cup-Diagramme beschrieben werden kann. Cup-Diagramme sind kombinatorische Objekte, genauer gesagt planare Diagramme welche aus Halbkreisen bestehen. In unserem Aufbau ist die kanonische Basis am einfachsten zu kategorifizieren. Wir tun dies analog zu der Bar-Natan Kategorifizierung der Temperley-Lieb-Algebra, indem wir eine Kategorie $\mathrm{Cup}(n,k)$ definieren, deren Objekten durch Cup-Diagramme gegeben sind. In der Kategorifizierung entsprechen Objekte $\mathrm{T}(\lambda)$, für λ ein Element der partiell geordneten Menge $\Lambda(n,k)$, den Elementen der kanonischen Basis.

Die Standard-Basis von $V^{\otimes n}$ ist schwieriger zu kategorifizieren. Um dies zu tun, müssen wir zur Homotopiekategorie $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ von beschränkten Kettenkomplexen mit Einträgen in $\widehat{\mathrm{Cup}}(k,n)$ übergehen, wobei $\widehat{\mathrm{Cup}}(k,n)$ eine Art additive Vervollständigung von $\mathrm{Cup}(n,k)$ mit Grad-Einschränkungen ist. Für λ in $\Lambda(n,k)$ definieren wir induktiv eine graduierte exzeptionelle Folge $\mathrm{V}^*(\lambda)$ in $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$. Mittels Dualität erhalten wir die Kettenkomplexe $\mathrm{V}(\lambda)$, welche schließlich zu einer Kategorifizierung der Standardbasis führen. Dafür beschreiben wir die Objekte $\mathrm{T}(\lambda)$ als iterierte Kegel von $\mathrm{V}^*(\mu)$s, wobei wir die vorkommenden $\mathrm{V}^*(\mu)$ kombinatorisch bestimmen. Insgesamt erhalten wir, dass $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$ den $(2k-n)$-Gewichtsraum von $V^{\otimes n}$ kategorifiziert. Genauer gesagt gibt es einen Isomorphismus von $\mathbb{C}(q)$-Moduln

$$\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K_0\Big(K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)\Big) \xrightarrow{\sim} \big(V^{\otimes n}\big)_{2k-n}$$

unter dem die Klassen der $\mathrm{V}(\lambda)$ auf die Standardbasis, die der $\mathrm{T}(\lambda)$ auf die kanonische Basis und die der $\mathrm{V}^*(\lambda)$ auf die duale Standardbasis geschickt werden.

Wir betrachten zwei T-Strukturen, welche die Objekte $\mathrm{V}^*(\lambda)$ und $\mathrm{T}(\lambda)$ im Herz enthalten und zeigen, dass die $\mathrm{T}(\lambda)$ Tilting-Objekte in dem Herz der einen und einfache Objekte in dem der anderen sind.

Der Jones-Wenzl-Projektor ist eine spezielle $\mathcal{U}_q(\mathfrak{sl}_2)$-lineare Abbildung $p_n : V^{\otimes n} \to V^{\otimes n}$, die faktorisiert werden kann als $p_n = \iota_n \circ \pi_n$. Hierbei ist $\pi_n \colon V^{\otimes n} \to V_n$ die Projektion auf den größten unzerlegbaren Summanden und $\iota_n : V_n \to V^{\otimes n}$ die Inklusion von diesem. Um den Jones-Wenzl Projektor mitsamt der Faktorisierung zu kategorifizieren, betrachten wir einen speziellen Kettenkomplex $L(\lambda_0)$ in $K^b\big(\widehat{\mathrm{Cup}}(k,n)\big)$, welcher alle

exzeptionellen Objekte $V^*(\lambda)$ auf nicht-triviale Weise enthält. Für die Konstruktion von $L(\lambda_0)$ betrachten wir die (bis auf ein Skalar eindeutigen) Grad-1-Morphismen zwischen den $V^*(\lambda)$s und untersuchen die daraus entstehenden Grad-2-Morphismen. Eine wichtige Eigenschaft des Komplexes $L(\lambda_0)$ ist, dass die Kategorie, welche die Temperley-Lieb-Algebra kategorifiziert, trivial auf ihm wirkt. Des Weiteren ist er die injektive Hülle von $T(\lambda_0)$ im Herz der zweiten T-Struktur.

Mit Hilfe dieses Komplexes $L(\lambda_0)$ und seines Endomorphismenrings $\mathrm{End}\big(L(\lambda_0)\big)$ konstruieren wir zwei Funktoren, welche die Eigenschaften der Projektions- und Inklusionsabbildung auf einer höheren Ebene erfüllen. Zuletzt vergleichen wir die Wirkung des universellen Projektors, einem von Cooper und Krushkal definiertem Komplex, der auch den Jones-Wenzl-Projektor kategorifiziert, mit der Komposition der Funktoren für kleine Werte von k.