On GIT Compactified Jacobians via Relatively Complete Models and Logarithmic Geometry

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Abstract

In this thesis we study modular compactifications of Jacobian varieties attached to nodal curves.
Unlike the case of smooth curves, where the Jacobians are canonical, modular compact objects, these compactifications are not unique.
Starting from a nodal curve C, over an algebraically closed field, we show that some relatively complete models, constructed by Mumford, Faltings and Chai, associated with a smooth degeneration of C, can be interpreted as moduli space for particular logarithmic torsors, on the universal formal covering of the formal completion of the special fiber of this degeneration.
We show that these logarithmic torsors can be used to construct torsion free sheaves of rank one on C, which are semistable in the sense of Oda and Seshadri. This provides a "uniformization" for some compactifications of Oda and Seshadri without using methods coming from Geometric Invariant Theory.
Furthermore these torsors have a natural interpretation in terms of the relative logarithmic Picard functor. We give a representability result for this functor and we show that the maximal separated quotient contructed by Raynaud is a subgroup of it.

"... Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza..."

(Dante Alighieri - *Divina Commedia*,
Inferno, Canto XXVI, 118-120)

Contents

Introduction

The theory of Jacobians of curves is a very old topic in algebraic geometry. It is known that for a smooth curve over a field the moduli functor of line bundles of degree zero is representable by a commutative, proper group scheme, i.e. by an abelian variety.
If we consider singular curves then the Jacobians are no more compact and even worst is the situation if we try to consider a moduli functor for a family of degenerating curves. Indeed it turns out that this functor is representable by a scheme only in very special cases.

In this thesis we study compactifications for relative Jacobians of singular curves having at worst nodal singularities. The reason for which we restrict to this special class of curves is that, in order to find a modular compactification of the moduli space of curves, one can add as boundary points, curves which are stable, hence nodal.
It is known that the Jacobian of a nodal curve over a field is an extension of an abelian variety via a torus. Such geometric objects are called semiabelian varieties. Since they are not compact, it is interesting to find compactifications of them which are also modular.
Historically there have been two trends to pursue a meaningful compactification procedure.
On one hand one can look at the generalized Jacobians as abstract semiabelian varieties and try to compactify them as geometric objects. In this context one ignores the functor, corresponding to the sheaves, but pays attention to the modularity of the semiabelian family. This theory is developed in [FC]. This approach produces objects having a good geometric behavior, also in the relative case, but what is missing is an interpretation in terms of sheaves on the curve.
On the other hand one can look at the functor the Jacobian represents and try to enlarge the category of sheaves one is working with.
Since the difference between the smooth curve and the nodal one is only at finitely many points, it is natural to consider sheaves which behave like line bundles at the smooth points and that differ from a line bundle only at the nodes.
This brings to the theory of torsion free, semistable sheaves and GIT quo-

tients.
Unfortunately, in this theory, the objects one obtains are very difficult to study geometrically due to the presence of an action of a reductive group. Furthermore in this setting the functors one is working with tend to be non-separated in the relative setting.

One interesting feature is that the geometric structure of the objects, one obtains in the limit of the first approach, and the geometric structure of the GIT construction tend to be very similar for nodal curves. We found in the literature the name of stable semiabelic varieties ([AL02]) for such geometric objects.

In this thesis we build a bridge between the two theories also in the relative setting. We show that if we start from a nodal curve over a field and we take a regular smoothing of it, over a complete discrete valuation ring, then some compactifications for the Jacobian of the smoothing, obtained via the Mumford's models as described in [FC], can be interpreted in terms of a functor of invariant sheaves, with a certain pole-growing condition, on the formal universal covering of the formal completion of the smoothing and that these sheaves specialize to semistable ones, in the sense of Oda and Seshadri.

In particular we are able to recover and uniformize some coarse moduli spaces of Oda and Seshadri, without using geometric invariant theory, and we have a functor for the uniformizing object.
Here the word "some" means that we can do this only for particular choices of the polarization one uses to construct the compactified Jacobians of Oda and Seshadri.

Since these invariant sheaves naturally correspond to certain logarithmic torsors, we use the formalism of log-geometry to give functoriality to our construction. In particular we show that the sheaves we obtain have a natural interpretation in terms of the logarithmic Picard functor.
We give a representability result for such functor in the relative setting and we show its connection with the maximal separated quotient constructed by Raynaud in [Ra]. It turns out that this quotient is actually a subgroup of the logarithmic Picard functor.

We should also mention that the correspondence we give here has been already investigate by Alexeev in [AL96] and [AL04] and by Andreatta in [And]. Our approach, although influenced by these ones, is different.

We briefly explain how this thesis is structured.

In the first two chapters we recall the basic facts we need both from the

theory of the Mumford models and from the theory of semistable sheaves.
Chapter 3 is dedicated to recall the construction of the formal covering of the curve and to give an interpretation of the Rayanud extension of the Jacobian in terms of equivariant line bundles on this formal covering.
Chapter 4 contains the definition of the functor we are considering and we prove our correspondence.
In chapter 5 we give a representability result, as algebraic space, for the relative logarithmic Picard functor attached to a family of smooth curves degenerating to a nodal one, over a discrete valuation ring. We also show that the maximal separated quotient of the relative Jacobian, constructed by Raynaud, is a subgroup of the logarithmic Picard functor and that the connected component of the identity of this functor is representable by a separated group scheme.

Since we do not want to interrupt the continuity of the main storyline too much, we have provided the extra necessary details in the appendices.

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Chapter 1

Mumford's models

We recall in this section the notion of relatively complete models and we analyze some basic examples coming from toric geometry we need in this thesis.
Essentially these models provide a sort of "compactifications" of the global semiabelian extension (Raynaud extension of the Jacobian), equipped with an action of the periods, in the sense that they provide integral models on which this action can be extended.
After taking the quotient by the periods action one obtains a compactification of the degenerating semiabelian family.
Unfortunately these models do not arise as solution of a moduli problem, but on the other hand they can be constructed explicitly by writing down an algebra and then taking the relative $Proj$. In order to understand how these algebras come out, one needs to translate the condition of having a semiabelian scheme, with action of the periods, in terms of trivialization of certain canonical torsors attached to the deformation situation.
Before of doing this we need to introduce the main characters.

Let us take an affine base scheme $S = Spec(R)$, where R is a noetherian, normal, integral domain and we also assume that it is complete with respect to a radical ideal I. Let η be the generic point of S. By a family of curves $f : C \to S$ we mean a proper and flat morphism over the base scheme S, with C_η connected and smooth and we assume that the other fibers are nodal or even. We also assume that the irreducible components of the fibers are geometrically irreducible.
We can consider the associated family of Jacobians $J_{C/S} \to S$. In this case the Jacobian is a quasi-projective semiabelian scheme with an S-ample line bundle $\mathcal{L}$ rigidified along the zero section ([D]4.3).
The generic fiber $J_{C_\eta} = \mathrm{Pic}^0_{C_\eta}$ is an abelian scheme and $\mathcal{L}_\eta$ induces an ample line bundle rigidified along the origin ([BLR]9.4.Proposition 4). We will recall something more about $\mathcal{L}$ later.

In the case in which the ideal I is maximal, then the special fiber is a semiabelian scheme, i.e. an extension of the abelian variety, corresponding to the Jacobian of the normalization of the special fiber of the curve, via a torus of rank the rank of the first homology group of the dual graph of the curve. In order to see this, let $s \in S$ be the special point and consider the normalization morphism

$$\pi : \tilde{C}_s \to C_s$$

This gives us an exact sequence

$$0 \to \mathcal{O}^*_{C_s} \to \pi_* \mathcal{O}^*_{\tilde{C}_s} \to \pi_* \mathcal{O}^*_{\tilde{C}_s} / \mathcal{O}^*_{C_s} \to 0$$

The sheaf $\pi_* \mathcal{O}^*_{\tilde{C}_s} / \mathcal{O}^*_{C_s}$ is torsion and $H^0(\pi_* \mathcal{O}^*_{\tilde{C}_s} / \mathcal{O}^*_{C_s}) \cong C^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m(s)$ where Γ is the intersection graph of C_s. By taking cohomology one gets

$$0 \to H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m(s) \to H^1(C_s, \mathcal{O}^*_{C_s}) \to H^1(\tilde{C}_s, \mathcal{O}^*_{\tilde{C}_s}) \to 0$$

The curve $\tilde{C}_s$ is smooth and if C_s is not irreducible it is also disconnected. In particular if V is the set of the irreducible components of C_s and C_v denotes the irreducible curve corresponding to the vertex $v \in V$, one has

$$H^1(\tilde{C}_s, \mathcal{O}^*_{\tilde{C}_s}) = \bigoplus_{v \in V} H^1(\tilde{C}_v, \mathcal{O}^*_{\tilde{C}_v})$$

Since $\tilde{C}_v$ is smooth we have a multidegree map which we extend to $H^1(C_s, \mathcal{O}^*_{C_s})$ by sending the torus $H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m(s)$ to zero. The kernel of the multidegree map gives the desired extension

$$0 \to H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m(s) \to J_{C_s} \to \prod J_{\tilde{C}_v} \to 0$$

Given an abelian scheme A over S, we denote with A^t the abelian scheme representing the functor $\mathrm{Pic}^0_{A/S}$. Let us recall briefly why it exists. The fact that the scheme $A \to S$ is proper and cohomologically flat in dimension zero implies that $\mathrm{Pic}_{A/S}$ is an abelian algebraic space over S ([Afm]theorem 7.3) and every abelian algebraic space over S is a scheme ([FC]Ch.I.Thm.1.9). The line bundle $\mathcal{L}_\eta$ is the pull back of the Poincaré bundle $\tilde{P}_\eta$ on $J_{C_\eta} \times J^t_{C_\eta}$ via the homomorphism induced by the polarization morphism

$$J_{C_\eta} \xrightarrow{(1,\lambda)} J_{C_\eta} \times J^t_{C_\eta}$$

One can show in this way that $\mathcal{L}_\eta$ is isomorphic to $\mathcal{O}(\Theta + (-1)^* \Theta)$ where Θ is the theta divisor([BLR]proof of 9.4.4).

In particular the line bundle $\mathcal{L}_\eta$ is *symmetric* meaning that $(-1)^* \mathcal{L}_\eta \cong \mathcal{L}_\eta$.

Given a subset $I \subset \{1,2,3\}$ let m_I be the morphism $m_I : J_\eta^3 \to J_\eta$ given on functorial points by $m_I(x_1,x_2,x_3) = \sum_{i\in I} x_i$ and $m : J_\eta^2 \to J_\eta$ (resp. $p_i : J_\eta^2 \to J_\eta$) be the multiplication map (resp. the projection on the i-th factor).
Define line bundles

$$\Theta(\mathcal{L}_\eta) := \bigotimes_{I\subset\{1,2,3\}} m_I^*\mathcal{L}_\eta^{\otimes(-1)^{|I|}}$$

and

$$\Lambda(\mathcal{L}_\eta) := m^*\mathcal{L}_\eta \otimes p_1^*\mathcal{L}_\eta^{-1} \otimes p_2^*\mathcal{L}_\eta^{-1}$$

Observe that $\Lambda(\mathcal{L}_\eta)$ is nothing else than $(1,\lambda)^*\tilde{P}_\eta$ on $J_{C_\eta} \times J_{C_\eta}$.
The basic properties of these sheaves in terms of biextensions and cubical structures are recalled in Appendix C.

One of the main results in [FC] is that one can attach to the degenerating couple $(J,\mathcal{L})$ as before a 8-ple $(\tilde{J}, Y, c, c^t, \phi, \tau, \tilde{L}, \psi)$ of "non-degenerating" objects which determines $(J,\mathcal{L})$ up to unique isomorphism.

Let us explain the meaning of these data. For more readability we subdivide this description in subsections.

1.1 Semiabelian part

The symbol $\tilde{J}$ in the 8-ple stays for a semiabelian scheme over S, which is a global extension of an abelian variety A by a torus T over S called the *Raynaud extension* associated with $(J_{C/S},\mathcal{L})$. We assume that the torus is split.
This extension is the object of which we would like to take the quotient, but unfortunately the procedure is not so easy and one has to find a good modification of it as we explain later. The Raynaud extension $\tilde{J}$ has the property that if we consider the formal completion at I then

$$\widehat{\tilde{J}} \cong \widehat{J_{C/S}}$$

and it is functorial in $J_{C/S}$.
In chapter 3 we give a description of this extension in terms of a functor corresponding to sheaves on the analytic/formal cover of the special fiber of the curve. For a more algebraic approach the reader can look at [FC]II.1 or [SGA] 7 I.Exp. IX.7.

One can construct the dual $J_{C/S}^t$ also for semiabelian schemes ([FC]Ch.II) and consider its associated Raynaud extension $\widetilde{J_{C/S}}^t$

$$0 \to T^t \to \widetilde{J_{C/S}}^t \to A^t \to 0$$

The notation is due to the fact that one can show that A^t is actually the dual abelian scheme of the abelian part of $\widetilde{J}_{C/S}$.

As we explain in the next section there is a bijection between the set of global semiabelian schemes

$$0 \to T \to G \to A \to 0$$

over S and the group $Hom(X(T), A^t)$ where $X(T)$ denotes the character group of T. In particular the couple

$$\widetilde{J_{C/S}} \quad \text{and} \quad \widetilde{J_{C/S}}^t$$

corresponds to a couple

$$(c, c^t) \in Hom(X(T), A^t) \times Hom(X(T^t), A)$$

The functoriality gives a duality morphism

$$\tilde{\lambda} : \tilde{J}_{C/S} \to \tilde{J}^t_{C/S}$$

which has to be compatible with the extension structure and with the duality on the abelian part, namely it induces a morphism of sequences

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "i"] \arrow[d, "\phi^\vee"'] & \tilde{J}_{C/S} \arrow[r, "\pi"] \arrow[d, "\tilde{\lambda}"] & A \arrow[r] \arrow[d, "\lambda_A"] & 0 \\
0 \arrow[r] & T^t \arrow[r, "i^t"'] & \tilde{J}^t_{C/S} \arrow[r, "\pi^t"'] & A^t \arrow[r] & 0
\end{tikzcd}$$

The morphism $T \to T^t$ is induced by a morphism $\phi : X(T^t) \to X(T)$ and the compatibility with the diagram implies the rule

$$\lambda_A \circ c^t = c \circ \phi$$

To give an idea of why this is true, as we see in the next section, we can write

$$\tilde{J}_{C/S} = Spec_A\big(\bigoplus_{x \in X} c(x)\big)$$

and

$$\tilde{J}^t_{C/S} = Spec_A\big(\bigoplus_{x \in X^t} c^t(x)\big)$$

The morphism $\tilde{\lambda}^*$ on sections is given by taking the sum of the morphisms

$$\tilde{\lambda}^*_x : c^t(x) \xrightarrow{\lambda_A} c(\phi(x))$$

Furthermore $\tilde{\lambda}$ is uniquely determined by ϕ and λ_A. Indeed given two such $\tilde{\lambda}$ and $\tilde{\lambda}_1$ and a functorial point $g \in \tilde{J}_{C/S}(U)$, for some S-scheme U, then

$$(\tilde{\lambda} - \tilde{\lambda}_1)(g) \in T^t(U)$$

By commutativity we have that for a functorial point $t \in T(U)$

$$(\tilde{\lambda} - \tilde{\lambda}_1)(i(t)) = i^t \circ \phi^\vee(t) - i^t \circ \phi^\vee(t) = 0$$

In particular $\tilde{\lambda} - \tilde{\lambda}_1$ defines a morphism $A \to T^t$ which has to be constant, since A is proper and connected, and then zero because it is a group homomorphism.

1.2 The homomorphisms c and c^t

As promised we describe now the bijective correspondence between semiabelian extensions

$$0 \to T \to G \to A \to 0$$

over S and the group $Hom(X(T), A^t)$. Remember that in our case the scheme S is reduced and connected. We may also assume that the torus T is split. The scheme A, corresponding to the abelian part, is proper, flat and its geometric fibers are reduced and connected. In this case we have an isomorphism

$$\mathcal{O}_U \to f_{U,*}\mathcal{O}_{A_U}$$

for every S-scheme U. Indeed the function

$$y \to \dim_{k(y)} H^0(A_y, \mathcal{O}_{A_y})$$

is constant equal to 1, hence $f_*\mathcal{O}_A$ is locally free by [AV]Ch.2.5. Furthermore by [EGA]III.7.7.6 there exists an $\mathcal{O}_S$-module Q, locally of finite presentation, such that for every $\mathcal{O}_S$-module $\mathcal{F}$ we have a functorial in $\mathcal{F}$ isomorphism

$$f_*(\mathcal{O}_A \otimes \mathcal{F}) \cong Hom(Q, \mathcal{F})$$

The fact that $f_*(\mathcal{O}_A)$ is locally free of rank one implies that Q is locally free of rank one and that f is cohomologically flat in dimension zero. This implies that

$$\mathcal{O}_U \cong f_{U,*}\mathcal{O}_{A_U}$$

functorially in $U \to S$. Besides the morphism $A \to S$ has a section corresponding to the identity. It is well known that under these conditions every $\mathbb{G}m$-torsor over A can be trivialized in the Zariski topology.
This tells us that the T-torsor over A corresponding to $\tilde{J}$ can be Zariski-locally trivialized.

The morphism $\pi : \tilde{J} \to A$ is then affine and we can write $\tilde{J} = Spec_{\mathcal{O}_A}(\pi_* \mathcal{O}_{\tilde{J}})$. The torus acts on $\pi_* \mathcal{O}_{\tilde{J}}$. For every character $x \in X$ we consider the subsheaf

$$\pi_* \mathcal{O}_{\tilde{J}} \supset \mathcal{O}_x := \{f \in \pi_* \mathcal{O}_{\tilde{J}} | f(gt) = x(t) f(g),\ t \in T, g \in \tilde{J}\}$$

These give us a decomposition

$$\pi_* \mathcal{O}_{\tilde{J}} = \bigoplus_{x \in X} \mathcal{O}_x$$

Recall that after an étale base change $U \to A$ the scheme $\tilde{J} \times_A U$ is isomorphic to

$$T \times_A U$$

and after this base change $\mathcal{O}_x$ is the invertible sheaf corresponding to the character $x \in X(T)$. The property of being an invertible sheaf is invariant under étale base change and in this way we see that $\mathcal{O}_x$ are invertible sheaves. Furthermore we have isomorphisms

$$\mathcal{O}_x \otimes_{\mathcal{O}_A} \mathcal{O}_y \xrightarrow{\cong} \mathcal{O}_{x+y} \tag{1.1}$$

induced from the canonical isomorphism after base change.
We recall now that the cohomological flatness in dimension zero and the presence of a section $e : S \to A$ let us to construct an isomorphism

$$\mathrm{Pic}_{e,A/S} \xrightarrow{\cong} \mathrm{Pic}_{A/S} \tag{1.2}$$

where $\mathrm{Pic}_{e,A/S}$ is the étale sheaf corresponding to isomorphism classes of couples (L, α) where L is an invertible sheaf on A and α is a rigidificator, i.e. an isomorphism

$$\alpha : \mathcal{O}_S \to e^* L$$

To see this remember that the, functorial in $U \in Sch/S$, Leray spectral sequence $H^p(U, R^q f_{U,*} \mathbb{G}m) \Rightarrow H^{p+q}(A_U, \mathbb{G}m)$ w.r.t. the étale topology gives an exact sequence

$$0 \to H^1(U, \mathbb{G}m) \to H^1(A_U, \mathbb{G}m) \to \mathrm{Pic}_{A/S}(U) \to 0$$

Again we used the fact that the isomorphism $f_*(\mathcal{O}_A) \cong \mathcal{O}_S$ holds functorially for every base change in Sch/S and the fact that under this condition the presence of a section implies that the pull-back f_U^* is injective in cohomology, so that the induced map

$$\mathrm{Pic}_{A/S}(U) \to H^2(U, \mathbb{G}m)$$

is zero. This means that every element $\xi \in \mathrm{Pic}_{A/S}(U)$ can be lifted to an $M \in H^1(A_U, \mathbb{G}m)$. We are free to change M with the pullback of an element in $H^1(U, \mathbb{G}m)$, in particular if we choose $M \otimes f_U^* e_U^* M^{-1}$ this also surjects to ξ. The sheaf $M \otimes f_U^* e_U^* M^{-1}$ is canonically rigidified along the origin:

$$e_U^*(M \otimes f_U^* e_U^* M^{-1}) \cong e_U^* M \otimes e_U^* M^{-1} \cong \mathcal{O}_U$$

We see in this way that the morphism in 1.2 is surjective and it is easy to see the injectivity.

The identity of $\tilde{J}$ induces a trivialization of $\mathcal{O}_x$, namely we can use the identity of A to pullback $e_A^* \pi_* \mathcal{O}_G$. By definition there is an isomorphism

$$\mathcal{O}_T \cong e_A^* \pi_* \mathcal{O}_G \cong e_A^* \bigoplus_{x \in X} \mathcal{O}_x$$

We can now pullback it with the identity of the torus to get a homomorphism

$$e_T^* e_A^* \mathcal{O}_x \to \mathcal{O}_{T,x} \to \mathcal{O}_S$$

which is an isomorphism.
In particular we can interpret the line bundles $\mathcal{O}_x$ as elements in $\mathrm{Pic}_{e,A/S}$.

Define $\mathrm{Pic}^0_{A/S}$ as the subgroup of classes $[L] \in \mathrm{Pic}_{A/S}$ such that on closed point $\bar{a} \to A$ we have an isomorphism $T_{\bar{a}}^* L \cong L$.
The multiplication map μ on $\tilde{J}$ covers the one of A in particular we have a diagram

$$\begin{tikzcd} \tilde{J} \times \tilde{J} \arrow[r, "\mu"] \arrow[d] & \tilde{J} \arrow[d] \\ A \times A \arrow[r, "m_A"'] & A \end{tikzcd}$$

This implies that the $\mathbb{G}m$-torsors $\mathcal{O}_x$ have a group law compatible with the group law on A and by what we see in Appendix C.0.13 this is equivalent to give a trivialization of the $\mathbb{G}m$-torsor $\Lambda(\mathcal{O}_x)$. Hence we have isomorphisms

$$m_A^* \mathcal{O}_x \cong p_1^* \mathcal{O}_x \bigotimes_{\mathcal{O}_{A \times A}} p_2^* \mathcal{O}_x$$

or in other words $\mathcal{O}_x \in \mathrm{Pic}^0_{e,A/S}$.

In this sketched way we see how, given a semiabelian scheme G with abelian part A, we get a group homomorphism

$$c: \qquad X(T) \longrightarrow \mathrm{Pic}^0_{e,A/S} \cong A^t$$

$$x \longrightarrow \mathcal{O}_x$$

where the property of being an homomorphism follows from the isomorphisms in 1.1.

There is another canonical procedure to produce the sheaves $\mathcal{O}_x$ via push-out. We can consider the associated torsors and see that actually we get a negative push-out. Namely denote with $\underline{\mathcal{O}_x}$ the $\mathbb{G}m$-torsor attached to the sheaf $\mathcal{O}_x$. Given a character $x \in X$ we define $\underline{\mathcal{O}_x}$ as the following push-out

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "x"] & \tilde{J} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "Id"] & 0 \\ 0 \arrow[r] & \mathbb{G}m \arrow[r] & \underline{\mathcal{O}_x} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

To avoid confusion about the signs, we recall that since the functor from invertible sheaves to torsors is contravariant, then the sheaf corresponding to the torsor $\underline{\mathcal{O}_x}$ is $\mathcal{O}_{-x}$ and not $\mathcal{O}_x$. This means that we obtain the torsors of the previous sheaves $\mathcal{O}_x$ via negative push-out.

The previous procedure can be inverted. Indeed given a homomorphism

$$c : X \to \mathrm{Pic}^0_{e,A/S}$$

we can consider the scheme $\tilde{J} = Spec_{\mathcal{O}_A}(\bigoplus_{x \in X} \mathcal{O}_{c(x)})$. The fact that the sheaves $\mathcal{O}_{c(x)}$ lie in $\mathrm{Pic}^0_{e,A/S}$ implies that there are isomorphisms

$$m_A^* \mathcal{O}_{c(x)} \cong p_1^* \mathcal{O}_{c(x)} \otimes p_2^* \mathcal{O}_{c(x)}$$

which allow us to define a multiplication morphism $\mu : \tilde{J} \times \tilde{J} \to \tilde{J}$ covering the multiplication on A. One checks that this procedure inverts the previous construction.

1.3 The action i and the trivialization τ

Define $Y := X(T^t)$. We want to define an action $i : Y \to \tilde{J}_\eta$ which is compatible with the diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[dr, "c^t"'] & \tilde{J}_\eta \arrow[d, "\pi"] \\ & A_\eta \end{tikzcd} \tag{1.3}$$

and we want to explain why this is equivalent to find a trivialization of the $\mathbb{G}m$-torsor $(c, c^t)^* \mathcal{P}_\eta^{-1}$ as biextension (definition in appendix C) w.r.t.

$X \times Y$, where $\mathcal{P}_\eta$ is the Poincaré bundle on $A_\eta \times A_\eta^t$.

First of all for every $y \in Y$ we need a point over $c^t(y)$, i.e. a homomorphism

$$c^t(y)^*(\oplus \mathcal{O}_{x,\eta}) \to \mathcal{O}_{S_\eta}$$

Observe that since the sheaves $\mathcal{O}_{x,\eta}$ are rigidified along the origin we need isomorphisms

$$\tau(x,y)^{-1} : c^t(y)^*\mathcal{O}_{x,\eta} \overset{\cong}{\to} e_A^*\mathcal{O}_{x,\eta} \overset{\cong}{\to} \mathcal{O}_{S_\eta}$$

which can be interpreted as multiplication via some section.
The point $i(y)$ can be now defined by taking the one induced by sum of the maps $\tau(x,y)^{-1}$ over $x \in X$.
The minus sign takes again into account the contravariance between sheaves and torsors. The sections $\tau(x,y)$ have to be compatible with the multiplication map. Since the multiplication map on $\tilde{J}_\eta$ is defined by isomorphisms

$$m_A^*\mathcal{O}_x \cong p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{O}_x$$

pulling this back via $(c^t(y_1), c^t(y_2))$, we need the commutativity of the diagram

$$\begin{tikzcd}
(c^t(y_1), c^t(y_2))^* m_A^*\mathcal{O}_{x_\eta} \arrow[r, "\cong"'] \arrow[d, "\tau(x,y_1+y_2)^{-1}"', "\cong"] & (c^t(y_1), c^t(y_2))^*(p_1^*\mathcal{O}_{x,\eta} \otimes p_2^*\mathcal{O}_{x,\eta}) \arrow[d, "\cong"', "\tau(x,y_1)^{-1}\otimes\tau(x,y_2)^{-1}"] \\
\mathcal{O}_{S_\eta} \arrow[r, "="] & \mathcal{O}_{S_\eta}
\end{tikzcd}$$

namely the relation

$$\tau(x, y_1 + y_2)^{-1} = \tau(x, y_1)^{-1}\tau(x, y_2)^{-1} \tag{1.4}$$

Furthermore the isomorphism $\mathcal{O}_{x_1+x_2} \cong \mathcal{O}_{x_1} \otimes \mathcal{O}_{x_2}$ gives us

$$\tau(x_1 + x_2, y)^{-1} = \tau(x_1, y)^{-1}\tau(x_2, y)^{-1} \tag{1.5}$$

Observe that since $\mathcal{O}_x \in \mathrm{Pic}_A^0$ and since we have a Poincaré bundle $\mathcal{P}$ on $A \times A^t$ it follows that $\mathcal{O}_x \cong (1, c(x))^*\mathcal{P}$.
In particular when we pull back under $c^t(y)$, this gives us an isomorphism

$$\mathcal{O}_{S_\eta} \overset{\tau(x,y)^{-1}}{\longleftarrow} c^t(y)^*\mathcal{O}_{x,\eta} = (c^t(y), c(x))^*\mathcal{P}_\eta$$

If we consider the associated torsors and remember that after passing to torsors we have to change sings and arrow direction we get an isomorphism of $\mathbb{G}m$-torsors

$$1_{X\times Y} \overset{\underline{\tau(x,y)}}{\longrightarrow} (c^t, c)^*\underline{\mathcal{P}}_\eta^{-1}$$

where $1_{X\times Y}$ denotes the trivial $\mathbb{G}m$-torsor over $X \times Y$. Since now the possible signs confusion has been clarified we skip the underlined notation and we write τ (resp. $\mathcal{P}$) for $\underline{\tau}$ (resp. $\underline{\mathcal{P}}$) because it will be clear from the context if we are talking about torsors or sheaves.

Observe now that conditions 1.4 and 1.5 imply that τ defines a trivialization of the Poincaré bundle over the generic fiber as biextension and not only as torsor.

Now the problem is to show that such trivialization can be found. This will be explained later.

1.4 The action on $\tilde{\mathcal{L}}_\eta$ and the trivialization ψ

Consider L the ample line bundle on A defining the principal polarization and define $\tilde{\mathcal{L}} := \pi^* L$ on $\tilde{J}$. We want an action of the periods Y on the line bundle $\tilde{\mathcal{L}}_\eta$ compatible with the given action on $\tilde{J}_\eta$ and to show that this is equivalent to a cubical trivialization of the $\mathbb{G}m$-torsor $i^*\tilde{\mathcal{L}}_\eta^{-1}$ on Y. Again definitions in Appendix C.

To give this action we need to exhibit isomorphisms

$$T^*_{i(y)}\tilde{\mathcal{L}}_\eta \cong \tilde{\mathcal{L}}_\eta$$

The direct image has a decomposition

$$\pi_*\tilde{\mathcal{L}} = \pi_*\pi^* L = \bigoplus_{x\in X} L \otimes \mathcal{O}_x$$

For simplicity we denote $L \otimes \mathcal{O}_x$ with L_x. Recall that the group law on $\tilde{J}$ is given by using isomomorphisms

$$m_x : m_A^*\mathcal{O}_x \cong p_1^*\mathcal{O}_x \otimes p_2^*\mathcal{O}_x$$

In particular we have isomorphisms

$$m_{x,y} : T^*_{c^t(y)}\mathcal{O}_x \stackrel{\cong}{\to} \mathcal{O}_x(c^t(y)) \otimes \mathcal{O}_x$$

giving us the action on the sheaf $\mathcal{O}_{\tilde{J}}$

$$T^*_{i(y)}(\mathcal{O}_{\tilde{J}}) = T^*_{i(y)}(\bigoplus_{x\in X} \mathcal{O}_x) \stackrel{\cong}{\to} \bigoplus_{x\in X} c^t(y)^*\mathcal{O}_x \otimes \mathcal{O}_x \stackrel{\cong}{\to} \bigoplus_{x\in X} \mathcal{O}_x = \mathcal{O}_{\tilde{J}}$$

where the last one is induced by $\sum_{x\in X}\tau(x,y)$. We have now

$$T^*_{i(y)}(\oplus_{x\in X} L \otimes \mathcal{O}_x) = \bigoplus_{x\in X} T^*_{c^t(y)} L \otimes T^*_{c^t(y)}\mathcal{O}_x$$

Applying the definitions we obtain

$$\begin{aligned}T^*_{c^t(y)}L =&T^*_{c^t(y)}L\otimes L^{-1}\otimes L\otimes L(c^t(y))\otimes L(c^t(y))^{-1}\\ =&(1,c^t(y))^*(m_A^*L\otimes p_1^*L\otimes p_2^*L^{-1})\otimes L\otimes L(c^t(y)) =\\ =&\lambda_A\circ c^t(y)\otimes L\otimes L(c^t(y))\\ =&\mathcal{O}_{\phi(y)}\otimes L\otimes L(c^t(y)) =\\ =&L_{\phi(y)}\otimes L(c^t(y))\end{aligned}$$

In particular if we had isomorphisms

$$\psi^{-1}(x) : L(c^t(y)) \to \mathcal{O}_{S,\eta}$$

we could cook up, by taking multiplication via $\sum_{x\in X}\psi(x)^{-1}\tau(x,y)^{-1}$, an isomorphism

$$T^*_{i(y)}(\oplus_{x\in X}L_x) = \bigoplus_{x\in X}T^*_{c^t(y)}L\otimes T^*_{c^t(y)}\mathcal{O}_x = \bigoplus_{x\in X}L_{\phi(y)+x}$$

This explain the meaning of ψ. A careful analysis of the compatibility with the group law shows that on the torsors level ψ induces not only a section but a cubic trivialization

$$\psi : 1_Y \to c^{t,*}L^{-1}$$

Furthermore the cocycle condition gives us the commutativity

$$\begin{tikzcd} T^*_{i(y_1)}T^*_{i(y_2)}L_x \arrow[rr, "\psi(y_2)^{-1}\tau(x,\phi(y_2))^{-1}"] \arrow[dr, "\psi(y_1+y_2)^{-1}\tau(x,\phi(y_1+y_2))^{-1}"'] & & T^*_{i(y_1)}L_{x+\phi(y_2)} \arrow[dl, "\psi(y_1)^{-1}\tau(x+\phi(y_2),\phi(y_1))^{-1}"] \\ & L_{x+\phi(y_2)+\phi(y_1)} & \end{tikzcd}$$

Using the fact that τ is bilinear and symmetric one sees that we need the condition

$$\psi(y_1+y_2)\psi(y_1)^{-1}\psi(y_2)^{-1} = \tau(\phi(y_1),y_2)$$

In a more compact we may write $\Lambda(\psi)$ for the section induced on $\Lambda(c^{t,*}L^{-1})$ by ψ, and we get an equality

$$\Lambda(\psi) = \tau\circ(1_Y\times\phi)$$

Again one has to find such trivializations.

1.5 The Positivity Condition

Assume that the base scheme is affine $S = Spec(R)$ normal, integral domain with fraction field K. Let I be the ideal of the degeneration.
The sheaf $\mathcal{O}_{S,\eta}$ has a natural integral structure given by $\mathcal{O}_S$. In particular under these assumptions we can measure the denominators of $\tau(y, \phi(x))$ and $\psi(y)$ for every $y \in Y$ and $x \in X$. To make this more precise recall that at the level of sheaves we have have generic isomorphisms

$$\mathcal{O}_{S,\eta} \xrightarrow[\cong]{\tau(x,y)} \mathcal{O}_x(c^t(y))_\eta^{-1}$$

and

$$\mathcal{O}_{S,\eta} \xrightarrow[\cong]{\psi(y)} L(c^t(y))_\eta^{-1}$$

Intuitively if a quotient "$\tilde{J}_{C/S}/Y$" would exist, then since on the special fiber we have a toric part, we should have that the periods disappear. We translated the periods in terms of the sections τ and ψ which have coefficients in K. If the periods disappear we expect that the coefficient are integral and not unit. This fact gives an intuition for the following condition. The *positivity condition* is the following:

Condition 1.5.1.

- *For every $y \in Y$ the section $\tau(y, \phi(y))$ extends to a section of the sheaves $(c^t(y), c^t(y))^*\Lambda(L)^{-1}$ over S and it is zero modulo the ideal I if $y \neq 0$*
- *For all but finitely many $y \in Y$, the section $\psi(y)$ extends to a section of the sheaf $c^t(y)^*L^{-1}$ over S and it is zero modulo I.*

We add at this point a little more notation which we need in the next section.
The pullback of the integral structures $\mathcal{O}_x(c^t(y))^{-1} \subset \mathcal{O}_x(c^t(y))_\eta^{-1}$ (resp. $L(c^t(y))^{-1} \subset L(c^t(y))_\eta^{-1}$) under the morphism $\tau(x,y)$ (resp. $\psi(y)$) defines fractional ideals in K, induced by the denominators of $\tau(x,y)$ (resp. $\psi(y)$). We denote these fractional ideals with $I_{x,y}$ (resp. I_y).

1.6 Definition of Mumford's models

We introduce in this section the Mumford models and we recall how the "non-degenerating" data interplay with them.
Since we are interested in jacobians we can assume that $X = Y$ and that the abelian part A of a Raynaud extension is isomorphic to its dual under

the duality morphism. We also assume that the morphism $\phi : Y \to X$ is the identity. Furthermore since we are interested in the models obtained by Namikawa, the definition we give here corresponds to the definition of weak relatively complete model in [FC] Ch.VI.

It is still not clear from the previous sections that starting from the couple $(J_{C/S}, \mathcal{L})$, it is possible to extract what is called in [FC] a split object

$$(\widetilde{J}_{C/S}, X, c, \lambda, \tau, i, \tilde{\mathcal{L}}, \psi, \mathcal{M})$$

especially for the relations between τ and ψ.

It turns out that this is true and for the proof we refer to [FC]Ch.II.Theorem 6.2 or [F85]Satz 1.

Given $G \to S$ a semiabelian scheme over $S = Spec(R)$ we denote with $Z(G)$ the Zariski-Riemann space attached to G over S, which is by definition the space of *all* valuations of the function field $K(G)$ which are non negative on R. Observe that since the abelian part A of G is proper over S, every element $v \in Z(G)$ has a center on A and this center can be interpreted as an S-point of A. We have the following definition.

Definition 1.6.1. Let $S = Spec(R)$ where R is a noetherian, normal domain, $I \subset R$ an ideal such that $rad(I) = I$. Let $(G, X, c, \lambda, \tau, i, \tilde{\mathcal{L}}, \psi, \mathcal{M})$ be a split object over S where the symbols have been explained in the previous sections. A *relatively complete model* for such split object is a 5-tuple $(\tilde{\mathcal{P}}, \mathcal{L}_{\tilde{\mathcal{P}}}, T_g, S_x, \tilde{T}_g)$ where

- $\tilde{\mathcal{P}}$ is an integral scheme, locally of finite type over the abelian part $\pi : \tilde{\mathcal{P}} \to A$, whose generic fiber $\tilde{\mathcal{P}}_\eta$ is isomorphic to G_η,
- $\mathcal{L}_{\tilde{\mathcal{P}}}$ is a rigidified invertible sheaf on $\tilde{\mathcal{P}}$ extending $\pi^*\mathcal{M}|_{\pi^{-1}V}$, where V is the maximal open subset of S where $\tilde{\mathcal{P}}$ and G are isomorphic,
- T_g is an action of G on $\tilde{\mathcal{P}}$ extending the translation action of G_V,
- S_x is an action of X on the couple $(\tilde{\mathcal{P}}, \mathcal{L}_{\tilde{\mathcal{P}}})$ extending the action of X on the generic fiber $(G_\eta, p^*\mathcal{M}_\eta)$ defined by i, and ψ where $p : G \to A$ is the structural morphism,
- $\tilde{T}_g$ is an action of G on the sheaf $\pi^*\mathcal{M}^{-1} \otimes \tilde{\mathcal{L}}_{\tilde{\mathcal{P}}}$ extending the action of G_η on its structure sheaf.

Moreover we require the following conditions:

1. There exists a G-invariant open subscheme U of $\tilde{\mathcal{P}}$ of *finite type* such that $\tilde{\mathcal{P}} = \bigcup_{x \in X} S_x(U)$.

2. There exists a positive integer n, such that the global sections of $\mathcal{L}^n_{\tilde{\mathcal{P}}}$ define a basis for the Zariski topology of $\tilde{\mathcal{P}}$.

3. For every $v \in Z(G)$, let $x_v \in A$ be its center over S, then v has center on $\tilde{\mathcal{P}}$ if and only if for every $z \in X$ there exists $x \in X$ such that $v(I_{z,y} \cdot x_v^*(\mathcal{O}_z)) \geq 0$.

Remark 1.6.2. Actually what we presented here is the definition of weak models in [FC]. We decide to use the weak definition because we are going to use models coming from polyhedral decompositions.

Given a split object over a noetherian normal basis $S = Spec(R)$, one can construct many relatively complete models. The proof is quite long and since we only need the polyhedral case, which is described in detail further on, we quote the general result [FC]Chap III, Proposition 3.3.

The reason for which these models are useful is that they allow us to uniformize jacobians in the category of scheme and at the same time to find compactifications of them. The precise statement is the following.

Theorem 1.6.3. *Let $(\tilde{\mathcal{P}}, \mathcal{L}_{\tilde{\mathcal{P}}}, T_g, S_x, \tilde{T}_g)$ be a relatively complete model for a split object $(G, X, c, i, \lambda, \phi, \tau, \tilde{\mathcal{L}}, \psi, \mathcal{M})$ then the following hold*

1. *every irreducible component of the special fiber $\tilde{\mathcal{P}}_0$ is proper over S_0.*

2. *The group X acts freely on $\tilde{\mathcal{P}}_0$.*

3. *$\tilde{\mathcal{P}}_0$ is connected.*

4. *For each $n \geq 1$ there exists a projective scheme $\mathcal{P}_n$ over the n-th thickening of S_n w.r.t. I and an étale morphism $\pi_n : \tilde{\mathcal{P}}_n \to \mathcal{P}_n$ such that*

   - *π_n induces a quotient morphism as fpqc sheaves,*
   - *the invertible sheaf $\mathcal{L}_{\tilde{\mathcal{P}}_n}$ descends to an ample line bundle L_n on $\mathcal{P}_n$,*
   - *the family $\{\mathcal{P}_n, L_n\}_n$ gives rise to a formal scheme with an ample formal sheaf which algebraizes to a projective irreducible scheme $\mathcal{P}$ with an ample line bundle L,*

5. *the open subscheme*
$$\bigcup_{x \in X} S_x(U)$$
induces an open subscheme of $\mathcal{P}$ denoted with G/X.

6. *the scheme G/X is a semiabelian scheme over S, it operates on $\mathcal{P}$ and $(G/X)_\eta = \mathcal{P}_\eta$ is an abelian variety over η, moreover G/X depends only on the couple $(G, i : X \to G(K))$.*

7. *$\mathcal{P}$ has connected geometric fibers and $(G/X)_\eta$ is schematically dense in $\mathcal{P}$.*

Proof. [FC]Chap.III □

We are now going to analyze the situation in which the dimension of the basis is one. In this case explicit examples are easier to write down using toric geometry and polyhedral decompositions as defined in Appendix B. Even more a complete classification is available by results of Mumford [Mum]6.7, [NamI], [AN], [AL02]5.7.1.
Furthermore in this case we have an interpretation of the Raynaud extension in terms of line bundle on a formal covering of the curve using rigid/formal geometry which allows us to define a functor of sheaves in chapter 4.

Let $S = Spec(R)$ be a complete discrete valuation ring used as base scheme. We want to exhibit $\tilde{\mathcal{P}} \to S$ as relative toric scheme over the abelian part. We choose a uniformizer $\pi \in R$ and we consider the toric structure or the log structure generated by positive powers of π.

For the relative setting is more natural to set up everything in the context of log-geometry, as we will do later, but for the moment we restrict our self to the toric aspect. A standard reference for this situation is ([TE].IV§3).

Let Γ be the intersection graph of the special fiber of our curve over S and define $N_\mathbb{R} = H^1(\Gamma, \mathbb{R})$. In order to obtain a morphism of toric schemes $\tilde{\mathcal{P}} \to S$ we need to construct rational polyhedral cones in $N_\mathbb{R} \oplus \mathbb{R}$ compatible with the morphism $\mathbb{R} \to N_\mathbb{R} \oplus \mathbb{R}$ obtained by sending 1 to $(0, 1)$.

As general remark, given a relative polyhedral decomposition Σ of $N_\mathbb{R} \oplus \mathbb{R}$ over $\mathbb{R}$, then the special fiber of the associated relative toric scheme, corresponds to the cones $\omega \in \Sigma$ which are not contained in $N_\mathbb{R} \oplus \{0\}$. Let us see now how to obtain such cones in a way which is compatible with the Mumford construction.

Assume for the moment that the abelian part is trivial.

This will simplify the notations at the beginning and the general case will follow simply by pushing-out. Define $K = Frac(R)$. The semiabelian scheme G is now a torus $T = Spec(R[w^x]_{x \in X})$, where w^x denotes the character corresponding to $x \in X$ and $X = H_1(\Gamma, \mathbb{Z})$.

The homomorphisms c and c^t are trivial.

The trivialization ψ corresponds to a quadratic function $a : X \to K^*$ and the trivialization of the Poincaré biextension gives a bilinear form

$$b : X \times X \to K^*$$

whose compatibility with a gives us

$$b(x, y)a(x)a(y) = a(x + y)$$

The positivity condition says $v_\pi(b(x, x)) > 0$ for all $x \in N_{\mathbb{Z}}$, where v_π is the valuation of R associated to the uniformizer π.

We recall here that in the situation of the Jacobians, the bilinear form can be computed explicitly in terms of the elements in the ideal I giving rise to the nodes and in terms of the intersection graph Γ. More precisely let $f : C \to S$ be the curve. Around every node $c_e \in C$ the local ring of the curve looks, étale locally, like

$$R[[x_e, y_e]]/(x_e y_e - f_e)$$

where x_e and y_e are indeterminate and $f_e \in \mathfrak{m}_{f(c_e)} \subset R$.
Given $x \in H_1(\Gamma, \mathbb{Z})$ and e an element of the canonical basis of $C_1(\Gamma, \mathbb{Z})$ then x has an e coordinate given by the canonical embedding

$$i : H_1(\Gamma, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z}) \tag{1.6}$$

We can now state the following proposition.

Proposition 1.6.4 ([FC]Chap.III,8.3, [F85]Satz 8). *The bilinear form b is, up to units, given by*

$$b(x, y) = \prod_{e \in E} f_e^{x_e y_e}$$

For an anlytic proof of the previous proposition the reader can look at [Nam]Prop.5 and Theorem 2.
Let now v_π denotes the valuation corresponding to the uniformizer, then we get a bilinear form

$$v_\pi(b(x, y))$$

On the other hand we have a canonical canonical pairing

$$B : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}$$

These two pairings do not coincide in general because the valuations of the f_e may not be one. However since we are working with one dimensional basis, we can always take a base change in order that the f_e have valuation one and that the total space of C is regular.

Furthermore if one denotes with e_k^* the rows of the matrix of i in 1.6, we find that the quadratic form q, given by to $v_\pi(b(-,-))$, belongs to the cone

$$\Delta = \sum_{k \in E} \mathbb{R}_{\geq 0}(e_k^* \cdot e_k^{*,t})$$

We describe later that our construction depends on the Voronoi-Delaunay decomposition that such form induces. Besides by B.4.1 we know that if we pick a form q inside the cone Δ then the associated Voronoi-Delaunay decomposition does not change.
This means that from now on without loss of generality we can assume

$$v_\pi(b(-,-)) = 2B(-,-)$$

Observe that the isomorphism classes of the models corresponding to different $b(-,-)$'s having the same valuation may be different, but this does not matter for our construction.

Let us pick an indeterminate θ and consider the monomials

$$\xi_x = a(x)w^x\theta$$

where

$$x \in H_1(\Gamma, \mathbb{Z}) =: M$$

We have introduced the notation of toric geometry to avoid possible confusions coming out from various dualities.

Following [FC] and [Mum] we define an action of M on θ via

$$S_y^*\theta = a(y)w^y\theta$$

and on the characters via

$$S_y^*(w^x) = w^x b(x,y)$$

Using the compatibilities between $a(-)$ and $b(-,-)$ we see that the action on ξ_x is given by

$$S_y^*(\xi_x) = \xi_{x+y}$$

Remark 1.6.5. Note here the analogy with the complex theory. If θ is a theta function over an abelian variety on the complex numbers with matrix Ω, then w.r.t. the multiplicative notation, i.e. we use complex coordinates $q = e^{-2\pi i z}$, the classical relation

$$S_y^*\theta(q) = \theta(qe^{2\pi i \Omega y}) = e^{-i\pi y^t \Omega y}e^{-2\pi i y z}\theta(q) = e^{-i\pi y^t \Omega y}q^y\theta(q)$$

is exactly what we wrote down by taking $a(y)^{-1} = e^{-i\pi y^t \Omega y}$.

Consider the big graded algebra

$$\mathcal{R} := K[w^x, \theta]_{x \in M}/(w^{x+y} - w^x w^y, w^0 - 1)$$

where the grading is given by powers of θ.
We are interested in models which are normal schemes and it is known from toric geometry that normality corresponds to saturated monoids. For this reason we define $\mathcal{R}_1$ as the *saturation* of the R-subalgebra generated by the translated

$$\left\{ S_y^* \theta \right\}_{y \in M}$$

Very roughly speaking the saturation of the subalgebra generated only by the elements of the star corresponds to the normalization of the compactifications we are interested in and this normalization can be expressed as functor. We explain this at the end of chapter 2. Finally define

$$\tilde{\mathcal{P}} = Proj(\mathcal{R}_1)$$

It carries a natural line bundle $\tilde{L}_{\tilde{\mathcal{P}}} = \mathcal{O}(1)$. The couple $(\tilde{\mathcal{P}}, \tilde{L}_{\tilde{\mathcal{P}}})$ is our candidate for a complete relatively model. Actually one needs to impose an extra condition on the star ([FC] p.62), namely that

$$I_y \cdot I_{y,s} \subset R \quad \forall y \in X, s \in \Sigma \tag{1.7}$$

This condition can always been achieved by taking enough higher powers n of the data (ϕ, ψ) and by considering instead of $H_1(\Gamma, \mathbb{Z})$ the couple $(nH_1(\Gamma, \mathbb{Z}), H_1(\Gamma, \mathbb{Z}))$ ([FC]Ch.3, Lemma 3.2).

We want to study more closely the structure of this scheme.
First of all we look at the valuations

$$v_\pi(a(x)) =: A(x)$$

and

$$v_\pi(b(x,y)) =: 2B(x,y)$$

This gives us a quadratic form A and two times the associated bilinear form with values in $\mathbb{Z}$. By considering field extensions, we get forms with real values, but for us it will be enough to consider rational values.

The generators of the algebra $\mathcal{R}_1$ correspond now to some couples

$$(x, d) \in M_{\mathbb{Q}} \oplus \mathbb{Q}$$

The generic fiber is the original torus as in [Mum]3.1.

We consider now the special fiber. It is known that the $Proj$ construction comes equipped with a natural open covering.
In our case if we fix one generator ξ_c of $\mathcal{R}_1$ with $c \in M$, we get the open $U_c = Spec(R[\frac{\xi_x}{\xi_c}])$. We want now to understand these open subsets in terms of a polyhedral decomposition of $M_\mathbb{R}$ induced by A. Write

$$\frac{\xi_x}{\xi_c} = a(x)a(c)^{-1}w^{x-c}$$

Consider a finite field extension $L \supset K$ and take an L-point

$$z_L \in \tilde{\mathcal{P}}(L) \cong \mathbb{G}m(L)^r$$

The point z_L will extend to the special fiber of U_c if and only if

$$v_L(\frac{\xi_x}{\xi_c}(z_L)) \geq 0 \quad \forall\, x \in M$$

where v_L is the induced valuation on L. In other words

$$A(x) - A(c) + v_L(z_L^{x-c}) \geq 0$$

for all $x \in M$.

By taking field extensions we can think about $v_L(z_L^{x-c})$ as real functional $M_\mathbb{R} \to \mathbb{R}$ which we call

$$l_{x-c}(z_L)$$

There is a more convenient way to normalize this function. Namely varying the field and the points we consider elements $\alpha \in M_\mathbb{R}$ such that we have equalities

$$l_{x-c}(z_L) = -2B(\alpha, x-c)$$

where $B(-,-)$ is the bilinear form introduced before. The convenience is that now we get the relation

$$0 \leq A(x) - A(c) - 2B(\alpha, x-c) = |\alpha - x|_A^2 - |\alpha - c|_A^2 \quad \forall\, x \in M$$

where the norm is computed w.r.t. the quadratic form A.

Thinking about B as matrix, the set of $-2B\alpha$ satisfying this condition form a bounded polyhedron in $N_\mathbb{R}$ called the Voronoi cell at c. The main properties of these cells are explained in Appendix B.
We recall here that the cells and their faces at c can be described as follows. For every cell σ and face F_σ, there exists a finite number of integral points $S_{F_\sigma} = \{a_1, \ldots, a_r\} \subset M$ such that F_σ is the set of points α satisfying the set of inequalities

$$\begin{cases} |\alpha - a_i|_A^2 - |\alpha - c|_A^2 = 0 \\ |\alpha - x|_A^2 - |\alpha - c|_A^2 > 0 \quad \forall\, x \in M \setminus S \end{cases}$$

One sees that taking the differences $a_i - c$, when a_i varies among the vertices of the cells σ for any cell at c, gives rise to a star through c.
Since these polyhedra are bounded, the previous description tells us that $\tilde{\mathcal{P}}$ satisfies the following positivity condition:

all valuations $v \in Z(G)$ which are positive on R and such that for any $x \in X$ exists an integer n such that

$$nv(\pi) \geq v(w^x) \geq -nv(\pi) \tag{1.8}$$

have center on $\tilde{\mathcal{P}}$.
Besides one can also do better and functorially describe the objects where the previous condition is also an only if, in terms of log-geometry. We want to spend some words about this fact in the totally degenerate case and we recall a construction in [FC]Ch.IV. Let $B(X)$ be the space of bilinear forms over X and consider the cone $\tilde{C}(X) \subset B(X)_{\mathbb{R}} \times X_{\mathbb{R}}$ consisting of couples (b, l) where b is positive, semidefinite with rational radical and such that l vanishes on the radical of b. We have a cone $C(X) \subset B(X)_{\mathbb{R}}$ consisting of positive, semidefinite bilinear forms with rational radical hence a surjection $\tilde{C}(X) \to C(X)$.
By reduction theory there exists $GL(X)$-admissible polyhedral decomposition of $C(X)$ and a $(GL(X) \ltimes X)$-admissible polyhedral decomposition of $\tilde{C}(X)$ relative to the one of $C(X)$ (see B.4). One can consider a base scheme S with a log-structure corresponding to the cone $C(X)$.
Given two sections $s, t \in M_S^{gp}$, one can declare that $s|t$ if $s^{-1}t \in M_S$.
We have a universal pairing

$$b : X \times X \to M_S$$

and we can define a sheaf

$$Hom(X, \mathbb{G}m^{log})^{(X)}$$

as

$$\{s \in Hom(X, \mathbb{G}m^{log}) |\ \forall\ x \in X\ \exists y_1, y_2\ :\ b(x, y)|s(x)|b(x, y_1)\}$$

For this sheaf one has by [KKN1]Prop.3.5.4 a decomposition

$$Hom(X, \mathbb{G}m^{log})^{(X)} = \bigcup_{\Delta \in C(X)} V(\Delta)$$

where Δ are the cones of the chosen decomposition and $V(\Delta)$ are log-schemes, similar to the one we will define in chapter 4 .

There is also another way to see the Delaunay-Voronoi cones, namely by using admissible homogeneous principal polarization functions.
In order to adapt this concept to our case, let Q be the convex hull of the set of points $\{(x, A(x))\}_{x \in M_{\mathbb{Z}}} \subset M_{\mathbb{R}} \oplus \mathbb{R}$. The lower envelop of Q describes the graph of a piecewise linear function g whose domains of linearity forms a polyhedral decomposition of $M_{\mathbb{R}}$ which is $-2B(Vor_A)$ where Vor_A is the Voronoi decomposition induced by the quadratic form A.
In particular it is invariant under the translation action of $M_{\mathbb{Z}}$ by the results in Appendix B. The function g is a polarization function in the sense of toric geometry.
In general these decompositions are named according to the following definitions we have taken from [AL02] and [Ol].

Definition 1.6.6. Let X be a lattice isomorphic to $\mathbb{Z}^g$ for some positive integer g. A *paving* of $X \otimes \mathbb{R}$ is a set Σ of polytopes in $X \otimes \mathbb{R}$ such that

1. given $\sigma, \rho \in \Sigma$ then $\sigma \cap \rho \in \Sigma$,
2. any face F_σ of a $\sigma \in \Sigma$ is again in Σ,
3. $X \otimes \mathbb{R} = \bigcup_{\sigma \in \Sigma} \sigma$,
4. for any $\sigma, \rho \in \Sigma$ the relative interiors are disjoint,
5. for any bounded set $Y \in X_{\mathbb{R}}$ the intersection $Y \cap \sigma$ is non empty except for finitely many $\sigma \in \Sigma$.

Definition 1.6.7. A paving Σ of $X \otimes \mathbb{R}$ is called *integral* if for any $\sigma \in \Sigma$ the vertices are in X.

Definition 1.6.8. A paving Σ of $X \otimes \mathbb{R}$ is called X-invariant if it is invariant for the translation action of X.

Definition 1.6.9. A paving Σ of $X \otimes \mathbb{R}$ is called *regular* if there exists a non-homogeneous $\mathbb{R}$-valued quadratic form A on X with positive definite homogeneous part such that Σ is the set of domains of linearity of the function defined by the lower envelope of the convex hull of the set $\{(x, A(x)) \mid x \in X\}$.

Regular pavings are always X-invariant ([Ol]4.1.2).

We consider now the Voronoi decomposition

$$(1, -2B(Vor)) \subset \mathbb{R} \oplus N_{\mathbb{R}}$$

and we take the infinite fan Δ consisting of $\{0\}$ and the cones over $(1, -2B(Vor))$ in $\mathbb{R} \oplus N_{\mathbb{R}}$. This gives us a relative toric scheme locally of finite type.

Having introduced the main characters it is not difficult to show the following.

Theorem 1.6.10 ([AN]). *1. $\tilde{\mathcal{P}}$ is covered by the affine toric schemes*

$$U_c = Spec(R(c)), \quad c \in M_{\mathbb{Z}}$$

where $R(c)$ is the semigroup algebra corresponding to the cone at the vertex $(c, A(c))$ of lattice elements

$$\{(x,d) | d \geq \min 2B(\alpha)\}$$

where α runs between the Voronoi vectors of the maximal dimensional Delaunay cells at c. The scheme U_c is the affine torus embedding over S corresponding to the cone over $(1, -2B(c^\star))$ where $c^\star$ is the Voronoi dual of c.

2. *The action of $M_{\mathbb{Z}}$ on the Voronoi decomposition induces an action S_y on the scheme $\tilde{\mathcal{P}}$ and we have isomorphisms*

$$S_y : U_{c+y} \to U_c$$

3. *There exists an n such that the sections of the sheaf $\tilde{L}^{\otimes n} = \mathcal{O}(1)^{\otimes n}$ define a basis for the Zariski topology.*

4. *There are compatible actions of the torus T on $\tilde{\mathcal{P}}$ and of $T \times \mathbb{G}m$ on $\tilde{L}$.*

5. *For every Delaunay cell $c \in \sigma \in Del$ one has a ring $R(\sigma)$ corresponding to the cone over the Voronoi dual $(1, -2B(\sigma^\star))$. The scheme $U(\sigma) = Spec(R(\sigma))$ is open in $\tilde{\mathcal{P}}$ and in U_c and $U(\sigma_1) \cap U(\sigma_2) = U(\sigma_1 \cap \sigma_2)$.*

6. *If $c \in \sigma$ then $U(\sigma)$ is the localization of $U(c)$ at $\frac{\xi_a}{\xi_c}$ where*

$$a \in M_{\mathbb{Z}} \cap \big(\mathbb{R}(\sigma \setminus c)\big)$$

7. *In the special fiber $\tilde{\mathcal{P}}_0$ the T_0 orbits corresponds bijectively to the Delaunay cells and this correspondence is dimension preserving.*

8. *The special fiber $\overline{O(\sigma)}_0$ of the closure of an orbit σ together with the restriction of $\tilde{L}$ is a projective toric variety over $Spec(k)$ with a T_0-linearized ample line bundle.*

9. *For a maximal dimensional cell σ the multiplicity of $\overline{O(\sigma)}_0$ in $\tilde{\mathcal{P}}_0$ is the denominator of $2B(\sigma^\star)$*

Since the compactifications via geometric invariant theory constructed in [OS] are reduced by [OS]11.4.(3), we can hope to get a comparison, in general, only by considering the reduced structures.
We explain now how to find reduced examples and to this aim we recall the following definition.

Definition 1.6.11. A maximal dimensional Delaunai cell σ with vectors $S = \{a_1, \ldots, a_r\}$ is called *generating* if the vectors in S contain a basis of $M_{\mathbb{Z}}$.

Definition 1.6.12. Given maximal dimensional Delaunai cell σ with vectors $S = \{a_1, \ldots, a_r\}$, the minimal integer n such that the lattice generated by $\{\frac{a_1}{n}, \ldots, \frac{a_r}{n}\}$ is called the *nilpotency* of σ.

We have the following proposition.

Proposition 1.6.13. *Let σ be a maximal-dimensional cell with vectors*

$$S = \{a_1, \ldots, a_r\}$$

then the following holds

1. *if σ is generating then the multiplicity of $\overline{O(\sigma)}_0$ in $\tilde{\mathcal{P}}_0$ is 1.*
2. *after a totally ramified base change of the basis, of degree the nilpotency of σ, the multiplicity of $\overline{O(\sigma)}_0$ in $\tilde{\mathcal{P}}_0$ is 1.*

Proof. 1)By translation invariance we can assume that one of the a_i is the origin. Clearly if for every $x \in M_{\mathbb{Z}}$ we have $2B(\sigma^\star)(x) \in \mathbb{Z}$ then $2B(\sigma^\star)$ is integral. Using the hypothesis on the cell σ, given $x \in M$ we can write it as

$$x = \sum x_i a_i \quad x_i \in \mathbb{Z}$$

By definition of Delaunay vectors we have for any $\alpha \in \sigma^\star$

$$0 = |\alpha - a_i|_A^2 - |\alpha|_A^2 = -2B(\alpha)(a_i) + A(a_i)$$

in particular we have

$$2B(\alpha)(x) = \sum x_i 2B(\alpha, a_i) = \sum_i x_i A(a_i) \in \mathbb{Z}$$

We apply now the point 9 in the previous theorem.

2) If the nilpotency is n then using the vectors $\{\frac{a_1}{n}, \ldots, \frac{a_r}{n}\}$ we reduce to case 1) if we knew $\frac{A(a_i)}{n} \in \mathbb{Z}$. The effect of a ramified base change of degree d is that A is multiplied by d and we are done. □

Example 1.6.14. In order to find non-reduced examples one has to take dimension at least 5. Indeed Voronoi showed in [V] that in dimension less or equal then 4 all Delaunai cells are generating and the first non-generating example occurs in dimension 5 ([E-R],p.796).
We give here an easier example we found in [AN] which lives in dimension 8. First of all recall that for every maximal Delaunay cell σ its dual $\sigma^\star$ is never integral. Let a_i be the vertices of the cell σ and assume that they are

generating. Assume we can find B given by a unimodular matrix E with 2 on the entries of the diagonal. This means that the expression $a_i^t E a_i$ is divisible by 2. From this follows that

$$\alpha^t E a_i = -\frac{1}{2} a_i^t E a_i \in \mathbb{Z}$$

Since E is unimodular and the a_i contains a basis this would give a contradiction because by unimodularity we recover α as integral element, which is not possible. To find such matrix we recall that in dimension 8 there exists a unique unimodular lattice, which is self dual and such that the norm of any lattice vector is even, called “the E_8 lattice”. The Cartan matrix of the E_8 system gives the desired matrix.

Once we have understood the situation for the totally degenerate case, we easily obtain the picture in the non trivial abelian part case. One has a semiabelian scheme

$$0 \to T \to G \to A \to 0$$

and a model $\tilde{\mathcal{P}}$ for the toric part , then one simply considers the contracted product

$$\tilde{\mathcal{P}} \times^T G$$

We have a morphism $q : G \to A$ and there are morphisms

$$\begin{tikzcd} & \tilde{\mathcal{P}} \times G \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \tilde{\mathcal{P}} & & G \end{tikzcd}$$

We take M an ample line bundle on A and we consider the line bundle $\tilde{\mathcal{L}} = p^* \tilde{L} \otimes p_2^* q^* M$. This line bundle descends to a line bundle $\tilde{\mathcal{L}}$ on

$$\tilde{\mathcal{P}} \times^T G$$

which we denote with the same symbol. We have now the following lemma.

Lemma 1.6.15 ([AN]3.24)**.** *The couple* $(\tilde{\mathcal{P}} \times^T G, \tilde{\mathcal{L}})$ *is a relatively complete model.*

1.7 Logarithmic version

We briefly discuss Kajiwara approach to the logarithmic uniformization presented in [Kaj]. It is a specialization of the procedure of the previous chapter in terms of sheaves on the curve and it gives an introduction to our approach of chapter 4. Unfortunately it works only for nodal curves over a field whose irreducible components have not self-intersections. In chapter 4 we explain

how to extend this to the relative setting, when the base scheme is the spectrum of a discrete valuation ring and how to use this to produce semistable sheaves.
The reason for using curves without self-intersection is, as usual, that if one wants to work with Zariski logarithmic structures, one has to avoid self-intersection. We say something more about this property in chapter 4, for a more detailed description the reader can look at [Pah]2.4.

Let us fix a nodal curve $f : C \to S = Spec(k)$ without self-intersections over a field k. We want to describe the uniformization of its jacobian in a functorial way. One considers the standard log-structure M_S on $S = Spec(k)$ defined by the morphism of monoids

$$\mathbb{N} \to k$$

sending everything but 0 to $0 \in k$ and 0 to $1 \in k$. One also takes an essentially semistable log-structure (definition in Appendix D.0.18) M_C on C. This means that locally around a point $c \in C$ the log-structure is defined as follows:

- if c is smooth and $c \in U$ is an affine open neighborhood then
$$M_C|_U = f^* M_S$$
- if c is a node then there is an affine open neighborhood U and charts
$$(\alpha : \mathbb{N}^2 \to M_C|_U, \beta : \mathbb{N} \to M_S, \Delta : \mathbb{N} \to \mathbb{N}^2)$$
such that we have a diagram
$$\begin{tikzcd} \mathbb{N}^2 \arrow[r, "\alpha"] & M_C|_U \\ \mathbb{N} \arrow[u, "\Delta"] \arrow[r, "\beta"] & M_S \arrow[u] \end{tikzcd}$$
where Δ is the diagonal morphism, α sends the two generators to the two functions giving rise to the branches, and β sends the generator to the generator.

Let $\nu : \tilde{C} \to C$ denote the normalization of C. We define E (resp. V) the set of edges (resp. vertices) of the graph Γ induced by C. If one allows Zariski base change on the basis $g : U \to S$ and take the log-structure induced by pullback $M_U = g^* M_S$, then one gets a functorial in U diagram ([Kaj]2.8)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}^*_{C_U} \arrow[r] \arrow[d] & \nu_* \mathcal{O}^*_{\tilde{C}_U} \arrow[r] \arrow[d] & \bigoplus_{e \in E} k_U^\times \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f_U^* M_U^{gp} \arrow[r] & \nu_*(f_U \circ \nu_U)^* M^{gp}_{\tilde{C}_U} \arrow[r] & \bigoplus_{e \in E} M^{gp}_{k_U} \arrow[r] & 0 \end{tikzcd}$$

As for the classical Picard functor, we consider the Zariski sheaves arising from the higher direct image of the previous diagram.
First we need some remarks. One can define a Zariski-sheaf

$$(U, M_U) \to \Gamma(U, M_U^{gp}) =: \mathbb{G}m^{log}(U)$$

where $g : (U, M_U) \to (S, M_S)$ is a fine saturated log-scheme over (S, M_S) such that $M_U \cong g^* M_S$.
Clearly the inclusion

$$0 \to \mathcal{O}_U^* \to M_U^{gp}$$

induces an inclusion

$$0 \to \mathbb{G}m \to \mathbb{G}m^{log}$$

In the classic theory, the Picard-functor for $f : C \to S$ is defined via the Zariski-sheaf

$$U \to H^0(U, R^1 f_* \mathcal{O}_C^*)$$

One can show ([Kaj]2.10) that, since $\tilde{C}$ is smooth, then the sheaf given by

$$(U, g^* M_S) \to H^0(U, R^1 f_{U,*} \nu_{U,*} (f_U \circ \nu_U)^* M_{\tilde{C}_U}^{gp})$$

is isomorphic to the sheaf

$$(U, g^* M_S) \to H^0(U, R^1 f_{U,*} \nu_{U,*} \mathcal{O}_{\tilde{C}_U}^*)$$

where g is the morphism $g : U \to S$.
From the previous diagram we see that we need also to consider a new sheaf, which we denote with $\tilde{\mathcal{P}}^{log}$ and it is defined as follows

$$(U, M_U) \to H^0(U, R^1 f_{U,*} f_U^* M_U^{gp})$$

The choice of the symbol $\tilde{\mathcal{P}}$ we already used in 1.6.1 is not casual as we are going to see in a moment. We define the **logarithmic Picard functor** as the Zariski-sheaf

$$(U, M_U) \to H^0(U, R^1 f_{U,*} M_{C_U}^{gp}) =: \underline{\mathrm{Pic}}_{C/S}^{log}(U)$$

After our considerations if we look at the cohomology sequence of the previous diagram, we obtain the following diagram of functors

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m \arrow[r] \arrow[d] & \mathrm{Pic}_{C/S} \arrow[r] \arrow[d] & \mathrm{Pic}_{\tilde{C}/S} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m^{log} \arrow[r] & \tilde{\mathcal{P}}^{log} \arrow[r] & \mathrm{Pic}_{\tilde{C}/S} \arrow[r] & 0
\end{tikzcd}$$

This is also a push-out via the inclusion $\mathbb{G}m \to \mathbb{G}m^{log}$ by [Kaj]2.13.

We observe immediately the analogies with chapter 1. If one could isolate the degree zero part in the Picard, one would have an action of the semi-abelian scheme $\mathrm{Pic}^0_{C/S}$ on $\tilde{\mathcal{P}}^{log}$ and $\tilde{\mathcal{P}}^{log}$ would also have a morphism to the abelian part $\mathrm{Pic}^0_{\tilde{C}/S}$. Furthermore this object has an interpretation as functor, namely it corresponds to torsors with cocyles in $f^*M^{gp}_S$. This characterization plays an important role in chapter 4.

We define a degree map

$$d : \tilde{\mathcal{P}}^{log} \to \bigoplus_{v\in V} \mathbb{Z}$$

to be zero on $H^1(\Gamma,\mathbb{Z}) \otimes \mathbb{G}m^{log}$ and we take the classic degree map on $\mathrm{Pic}_{\tilde{C}/S}$. Define

$$\tilde{\mathcal{P}}^{log,0} := \ker(d)$$

One has finally a diagram ([Kaj]2.15)

$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma,\mathbb{Z}) \otimes \mathbb{G}m \arrow[r] \arrow[d] & \mathrm{Pic}^0_{C/S} \arrow[r] \arrow[d] & \mathrm{Pic}^0_{\tilde{C}/S} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H^1(\Gamma,\mathbb{Z}) \otimes \mathbb{G}m^{log} \arrow[r] & \tilde{\mathcal{P}}^{0,log} \arrow[r] & \mathrm{Pic}^0_{\tilde{C}/S} \arrow[r] & 0
\end{tikzcd}$$

There is still something which is missing from chapter 1:
the action of the periods.
In order to recover this, one considers the exact sequence, functorial for $(U, M_U) \in fs/S$,

$$0 \to f_U^* M_S^{gp} \to M^{gp}_{C_U} \to \bigoplus_{e\in E} \mathbb{Z} \to 0$$

([Kaj]2.13). The associated long exact sequence in cohomology gives an exact sequence

$$\bigoplus_{e\in E} \mathbb{Z} \to \tilde{\mathcal{P}}^{log} \to \underline{\mathrm{Pic}}^{log}_{C/S} \to 0$$

([Kaj]2.17). Again we want to define a degree map on $\underline{\mathrm{Pic}}^{log}_{C/S}$. Using the graph of the curve we have a sequence

$$C_1(\Gamma,\mathbb{Z}) \xrightarrow{\partial} C_0(\Gamma,\mathbb{Z}) \xrightarrow{s} \mathbb{Z}$$

and we define the degree on $\underline{\mathrm{Pic}}^{log}_{C/S}$ by completing the the diagram

$$\begin{tikzcd}
 & C_1(\Gamma,\mathbb{Z}) \arrow[r] \arrow[d] \arrow[dr, "\partial"'] & \tilde{\mathcal{P}}^{log} \arrow[r] \arrow[d, "d"] & \underline{\mathrm{Pic}}^{log}_{C/S} \arrow[r] \arrow[d, dashed, "d_{log}"] & 0 \\
0 \arrow[r] & im(\partial) \arrow[r] & C_0(\Gamma,\mathbb{Z}) \arrow[r, "s"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

At this point we can finally define

$$\mathrm{Pic}^{log,0}_{C/S} := \ker(d_{log})$$

One sees that the previous diagram induces an exact sequence

$$0 \to H_1(\Gamma, \mathbb{Z}) \to \tilde{\mathcal{P}}^{log,0} \to \mathrm{Pic}^{log,0}_{C/S} \to 0 \tag{1.9}$$

which recall the action of the periods in chapter 1. So far we still do not know if these functors are representable by geometric objects.
Kajiwara is able to reconstruct the Mumford models coming from the Delaunay-Voronoi decomposition by imitating the procedure in chapter 1 as subfunctors of $\mathrm{Pic}^{log,0}_{C/S}$. Namely given a polyhedral decomposition Σ of $H^1(\Gamma, \mathbb{R})$ which is $H_1(\Gamma, \mathbb{Z})$-invariants, he constructs intermediate functors

$$H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m \to T_\Sigma \to \mathbb{G}m^{log}$$

which are representable by schemes locally of finite type with an action of $H_1(\Gamma, \mathbb{Z})$. We want to spend more words about the construction of T_Σ because we use it in chapter 4.
Denote with T the torus $H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$ and with X the group $H_1(\Gamma, \mathbb{Z})$ for notational reasons. Looking at the theorem 1.6.10 and given a polyhedral decomposition Σ having the same properties of the Voronoi one, we have to figure out how we could exhibit the closures $\overline{O}(\sigma)$ in loc.cit. as functor, for any $\sigma \in \Sigma$, in a way that they glue along the faces. If we had a curve over one dimensional basis with generic point $Spec(K)$, one could consider sections of $T(K)$ whose reduction behaves like the points z_L in chapter 1. The solution in the logarithmic world is similar. Assume that the decomposition is induced by a quadratic form with bilinear part B. For any cell $\sigma \in \Sigma$ one considers the cone over it

$$\Delta_\sigma := Cone(1, \sigma) \in \mathbb{Q} \oplus X_{\mathbb{Q}}$$

The form B allows us to define an *integral dual*

$$\Delta_\sigma^\vee \tag{1.10}$$

If π denotes the generator of M_S, then one has a pairing

$$\begin{array}{rrcl} \langle\ ,\ \rangle_B : & (\mathbb{Z} \oplus X \otimes \mathbb{G}m^{log}) \times (\mathbb{Z} \oplus X) & \longrightarrow & \mathbb{G}m^{log} \\ & (d, n) \otimes s \times (e, m) & \longrightarrow & \pi^{de} s^{B(m,n)} \end{array}$$

The condition that a section $s \in M_S^{gp}$ belongs to M_S means that that section has no "poles". One considers the functor on the category of (fine-saturated) log-schemes defined on a (U, M_U) over (S, M_S) by

$$T_\sigma(U, M_U) := \left\{ s \in X \otimes \mathbb{G}m^{log} \mid \langle 1 \otimes \pi + s, (e, m)\rangle_B \in M_U\ \forall (e, m) \in \Delta_\sigma^\vee \right\}$$

and it easy to see that it is representable by the log-scheme

$$(Spec(k[\Delta_\sigma^\vee]/(1,0)), \Delta_\sigma^\vee)$$

which is nothing else that the special fiber $\overline{O}(\sigma)_0$ we obtained in theorem 1.6.10. Once this is done one glues the pieces $\{T_\sigma\}_{\sigma\in\Sigma}$ according to the intersections of the polyhedra in Σ and obtains T_Σ. Since the decomposition Σ has a X-action also the scheme T_Σ has one.

Once this is constructed, one forms the push-out in the category of Zariski sheaves

$$T_\Sigma \times^T \mathrm{Pic}^0_{C/S}$$

This push-out is representable by a log-scheme ([Kaj]1.19) and the log-structure is induced by the one on T_Σ.

By the universal property of the push-out one obtains a unique morphism

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & \mathrm{Pic}^0_{C/S} \arrow[d] \arrow[ddr, bend left] & \\
T_\Sigma \arrow[r] \arrow[drr, bend right] & T_\Sigma \times^T \mathrm{Pic}^0_{C/S} \arrow[dr, dotted, "\exists!"] & \\
 & & \tilde{\mathcal{P}}^{0,log}
\end{tikzcd}$$

$$T_\Sigma \times^T \mathrm{Pic}^0_{C/S} \to \tilde{\mathcal{P}}^{0,log}$$

This morphism is compatible with the action of X given in 1.9 ([Kaj]4.3) and it allows us to define representable proper subschemes

$$(T_\Sigma \times^T \mathrm{Pic}^0_{C/S})/X \subset \mathrm{Pic}^{log,0}_{C/S}$$

The properness follows from the fact that for any zero dimensional cell c one has a complete fan given by finitely many cells having c as one of the vertices.
This construction has been generalized in relative picture in the works [KKN1] and [KKN2], but they do not consider the question whether these compactifications have an interpretation in terms of semi-stable sheaves on the curve.

We modify this construction in chapter 4 in order to attack the relative situation over one dimensional basis. Looking at the multidegrees of the sheaves we get, we realize that this construction has still to be modified if one wants to obtain semistable sheaves of a certain fixed degree.

Besides since stable curves are cohomologically flat in dimension zero we can prove in chapter 5 that, even in the relative 1-dimensional situation, the connected component of the identity of $\mathrm{Pic}^{log}_{C/S}$ is representable by a *separated* group scheme over S (Theorem 5.0.5).
Note that a priori the special fiber of the scheme $\mathrm{Pic}^{log,0}_{C/S}$ defined in chapter 5 is different from the one we defined in this section as the kernel of the degree map.

The only interesting geometric properties we know and we use about $\mathrm{Pic}^{log}_{C/S}$ is that it contains, as subgroup, the maximal separated quotient of the Picard functor constructed by Raynaud in [Ra].

Besides since $\mathrm{Pic}^{log,0}_{C/S}$ of this section contains the relatively compete models obtained via polyhedral decompositions, we believe that interesting modular compactfications have to be more investigated in the logarithmic world rather that in geometric invariant theory world.

Chapter 2

Oda-Seshadri semistability

In this chapter we recall the construction given in [OS] in order to compactify the generalized jacobian of a nodal curve over a field through torsion free semi-stable sheaf. In chapter 4 we see how to relate this construction with the theory of chapter 1.

Let us consider C a proper, connected, reduced curve over an algebraically closed field whose singularities are at worst nodal.
In the paper [OS] the authors construct compactifications of the generalized Jacobian of C, using torsion free, generically rank 1 sheaves as described in Appendix A. Denote the set of these sheaves with the symbol

$$LB(C)$$

Let Γ be graph attached to the curve C, and as usual V (resp. E) denotes the set of its vertices (resp. edges). For each subset $E' \subset E$, we denote with $C(E')$ the partial normalization of C at the nodes corresponding to the subset E'.

If F is a sheaf in $LB(C)$ we obtain a line bundle $\tilde{F} = \{\tilde{F}_v\}_{v\in V}$ on the normalization. Namely if $\nu : \tilde{C} \to C$ is the normalization we take $\tilde{F}$ the line bundle $\nu^* F/\{\text{torsion}\}$. This association defines a notion multidegree map by

$$\deg(\tilde{F}) = (\deg(\tilde{F}_v))_{v\in V} \in C_0(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^{|V|}$$

As explained in Appendix A, such sheaves are in bijective correspondence with line bundles on some partial normalization of the curve. In this way there is a decomposition

$$LB(C) = \coprod_{E' \subset E} \mathrm{Pic}_{C(E')}$$

Given a $F \in LB(C)$ we define

$$E \supset E_F := \left\{ \begin{array}{c} \text{edges corresponding to the nodes} \\ \text{where } F \text{ is not free} \end{array} \right\} \tag{2.1}$$

One attaches to a subset $E' \subset E$ a vector of weights, depending only on the graph, in the following way

$$d(E') := \sum_{v \in V} d(E')_v v \in C_0(\Gamma, \mathbb{Z}) \tag{2.2}$$

where

$$d(E')_v := \# \left\{ \begin{array}{c} \text{edges in } E' \text{ with an end point in } v, \\ \text{with loops counted twice} \end{array} \right\}$$

The group $H_1(\Gamma, \mathbb{Z})$ is endowed with a pairing coming from the canonical pairing on $C_1(\Gamma, \mathbb{Z})$ defined via

$$(\, ,) : \qquad C_1(\Gamma, \mathbb{Z}) \times C_1(\Gamma, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\left(\textstyle\sum_{e \in E} n_e e, \sum_{f \in E} m_f f, \right) \longrightarrow \sum n_e m_f \delta_{e,f}$$

Analogously we have a pairing

$$[\, ,] : \qquad C_0(\Gamma, \mathbb{Z}) \times C_0(\Gamma, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\left(\textstyle\sum_{v \in V} n_v v, \sum_{v \in V} m_w w, \right) \longrightarrow \sum n_v m_w \delta_{v,w}$$

Using the fact that for any torsion free sheaf F we have an exact sequence

$$0 \to F \to \bigoplus_{v \in V} \tilde{F}_v \to \bigoplus_{e \in E \setminus E_F} k \to 0$$

and the easy equality

$$[\sum_{v \in V} v, \frac{1}{2} \sum_{v \in V} d(E')] = |E'|$$

true for every subset of edges $E' \subset E$, it is not difficult to show that the Euler characteristic of a sheaf $F \in LB(C)$ can be computed via the formula

$$\chi(F) = [\sum_{v \in V} v, \sum_{v} \deg \tilde{F}_v + \frac{d(E_F)}{2}] + \chi(\mathcal{O}_C)$$

We consider the sheaves of degree zero

$$LB^0(C) \subset LB(C)$$

which are characterized as the sheaves for which the following equality holds

$$[\sum_{v\in V} v, \sum_{v} \deg \tilde{F}_v + \frac{d(E_F)}{2}] = 0 \tag{2.3}$$

We want now to use these pairings to attach to these sheaves polyhedra in $C_1(\Gamma, \mathbb{R})$ and study the stability of these sheaves in terms of the geometry of the associated polyhedra.
Let

$$K(\Gamma) = Del(C_1(\Gamma, \mathbb{R}), C_1(\Gamma, \mathbb{Z}))$$

be the Delaunay decomposition of the graph as defined in Appendix B. Each polyhedron $D \in K(\Gamma)$ is a half-integer translated of a face of a Voronoi hypercube in $\mathbb{Z}^{|E|}$, namely ([OS]5.1) of the form

$$D = b + V_{E'}(0)$$

where for a subset $E' \subset E$ the Voronoi cell through the origin is defined as

$$V_{E'}(0) := \{\sum_{e\in E'} a_e e, a_e \in \mathbb{R}, |a_e| \leq 1/2\}$$

and $b \in C_1(\Gamma, \mathbb{R})$ is the barycenter defined in 2.5.

The subset $E' =: Supp(D)$ is called the *support*.

The group $H_1(\Gamma, \mathbb{Z})$ acts by translation as subgroup of $C_1(\Gamma, \mathbb{R})$ and as explained in Appendix B the decomposition $K(\Gamma)$ is invariant under this action. We define

$$\bar{K}(\Gamma) := K(\Gamma)/H_1(\Gamma, \mathbb{Z})$$

We want to give a map

$$LB^0(C) \to \bar{K}(\Gamma)$$

Take a sheaf $F \in LB^0(C)$. It is not difficult to show ([OS]10.5) that, up to translation via $H_1(\Gamma, \mathbb{Z})$, there is a unique $\xi \in C_1(\Gamma, \mathbb{Z})$ such that

$$\deg(\tilde{F}) = \partial(\xi + \frac{\sum_{e\in E_F} e}{2}) - \frac{d(E_F)}{2} \tag{2.4}$$

Define the Delaunay polyhedron of F as

$$D_F = \xi + \{\sum_{e\in E_F} a_e e, 0 \leq a_e \leq 1\}$$

It is uniquely attached to F up to $H_1(\Gamma, \mathbb{Z})$-translation. The vector

$$b(D_F) = (\xi + \frac{\sum_{e\in E_F} e}{2}) \tag{2.5}$$

is called the *barycenter* of F.

In this way we attach to a sheaf a polyhedron and we want now to understand what stability means in terms of these polyhedra.

First of all we recall now how the semistability works in this context. We need to define a polarization on the curve.
For each $v \in V$ we take M_v to be a line bundle on C having degree one as line bundle on C_v and zero on the other components. We consider the line bundle on C given by

$$L = \bigotimes_{v \in V} M_v$$

We need something more fine that the usual Hilbert polynomial. To this aim for every multivector $\underline{n} = (n_1, \ldots, n_{|V|}) \in C_0(\Gamma, \mathbb{Z})$ we define

$$L^{\underline{n}} = \bigotimes_{v \in V} M_v^{n_i}$$

Definition 2.0.1. Given a coherent sheaf F on C, then the generalized Hilbert polynomial is defined as

$$P_F(\underline{n}) = \chi(F \otimes L^{\underline{n}}) \tag{2.6}$$

One easily verifies that

$$P_F(\underline{n}) = \sum_{v \in V} \mathrm{rk}(F|_{C_v}) n_v + \chi(F)$$

Given $V_1 \subset V$ a subset of vertices, we can consider the associated subcurve C_{V_1} and define $F_{C_{V_1}}$ to be the maximal subsheaf of F supported on C_{V_1}.

Order the set $C_0(\Gamma, \mathbb{Z})$ by declaring $m \geq n$ if and only if $n_v \geq m_v$ for all $v \in V$. We have the following boundedness result due to Ishida.

Proposition 2.0.2 ([I]). *There exists a positive integer θ and element $\tilde{n} \in C_0(\Gamma, \mathbb{Z})$ and such that if an admissible sheaf of rank one F has the property that* $\deg(\tilde{F}_v) \geq -\theta$ *for every $v \in V$, then the sheaf $F(n)$ is generated by global sections and $H^1(C, F(n)) = 0$ for all $n \geq \tilde{n}$.*

Let us fix θ and $\tilde{n}$ as in the proposition and consider the generalized Hilbert polynomial translated by $\tilde{n}$, namely we define

$$q(n) := q_F(n) = \chi(F(n + \tilde{n}))$$

Let $\mathcal{E}$ be a k-vector space of dimension equal to

$$q(0) = \sum_{v \in V} \tilde{n}_v + \chi(F) = \sum_{v \in V} \tilde{n}_v + \deg F + \chi(\mathcal{O}_C)$$

Consider $Quot_{\mathcal{E},q}$ the Grothendieck's Quot scheme parameterizing flat quotients of $\mathcal{E} \otimes \mathcal{O}_C$ with Hilbert polynomial q. Since the torsion freeness condition is open ([EGA] IV,11.2.1) we get an open subscheme

$$R \subset Quot_{\mathcal{E},q}$$

parameterizing admissible sheaves G of rank one with

$$\mathcal{E} \cong H^0(G) \quad \text{and} \quad H^1(G) = 0$$

Now one uses a trick, namely instead of embedding R into the Graßmannian, as usual, and study the stability there, one uses another embedding.

Let $\mathcal{F}$ be the restriction of the universal sheaf on $Quot_{\mathcal{E},q}$ to R.
Fix $x_1, \ldots, x_N$ smooth closed points on C such that each irreducible component contains at least one of them. The integer N has to be determined later.
Geometric points of the projective space $\mathbb{P}(\mathcal{E})$ correspond to one dimensional quotients $\mathcal{E} \to k$. Given $r \in R$ and $x \in C$, one gets a quotient $\mathcal{E} \to \mathcal{F}(x)$ via the evaluation map. In this way one can define a morphism

$$\begin{array}{rccc} \tau : & R & \to & \mathbb{P}(\mathcal{E})^N \\ & r & \to & (\ldots, \mathcal{F}_r(x_i), \ldots) \end{array}$$

Let us assume that we can choose the points x_i in such a way that the previous morphism is a closed immersion, this can be done by [OS]11.5.
One can try to understand the semistability condition in terms of stable points in $\mathbb{P}(\mathcal{E})^N$. To this aim we recall the following fact, from which most of the constructions involving GIT compactifications of jacobians are derived.

Proposition 2.0.3 ([GIT]Prop.3.4)**.** *The locus of stable points in $\mathbb{P}(\mathcal{E})^N$ is the open set of $(z_1, \ldots, z_N) \in \mathbb{P}(\mathcal{E})^N$ such that for every linear subspace $W \subset \mathbb{P}(\mathcal{E})$ the following holds*

$$\frac{|\{z_i \in W\}|}{N} < \frac{\dim W}{\dim \mathcal{E}}$$

In our situation one obtains interesting subspaces of $\mathcal{E}$ as spaces of sections vanishing on subcurves. The procedure goes as follows. Fix $V_1 \subset V$ a subset of vertices and G an admissible sheaf in the family R. Define $W_{V_1} \subset \mathcal{E}$ as the subspace of sections of $G = F(\tilde{n})$ vanishing on the subcurve defined by V_1. To compute this we define

$$E \supset E_{F(V_1)} := \left\{ \begin{array}{c} e \in E \;\text{ s.t } F \text{ is free at } e \\ \text{and both endpoints of } e \\ \text{are contained in } V_1 \end{array} \right\}$$

It is now immediate to verify that the subsheaf $S_{V_1}(F(\tilde{n}))$, whose global sections correspond to the vector space W_{V_1}, fits in an exact sequence

$$0 \to S_{V_1}(F(\tilde{n})) \to F(\tilde{n}) \to \bigoplus_{v\in V_1} F(\tilde{n})_v \to \bigoplus_{e\in E_{F(V_1)}} k \to 0 \tag{2.7}$$

It is not difficult to show that for large $\tilde{n}$ one has

$$W_{V_1} = \chi(S_{V_1}(F(\tilde{n})))$$

In particular we obtain

$$W_{V_1} = \chi(F(\tilde{n})) - \sum_{v\in V_1} \chi(F(\tilde{n})_v) + |E_{F(V_1)}|$$

We use now the exact sequence

$$0 \to F(\tilde{n}) \to \bigoplus_{v\in V} \tilde{F}(\tilde{n})_v \to \bigoplus_{e\in E\setminus E_F} k \to 0$$

to write

$$\chi(F(\tilde{n})) - \sum_{v\in V_1} \chi(F(\tilde{n})_v) = \sum_{v\in V\setminus V_1} \chi(F(\tilde{n})_v) - |E\setminus E_F|$$

Furthermore we have

$$-|E_{F(V_1)}| + |E\setminus E_F| = |(E\setminus E_F)\cap\{e\in E | \text{at least one end point is in } V\setminus V_1\}|$$

Define a_F as the previous quantity. Putting everything together we obtain

$$\begin{aligned} W_{V_1} &= \sum_{v\in V\setminus V_1} \chi(F(\tilde{n})_v) - a_F = \\ &= \sum_{v\in V\setminus V_1} \big(\deg(F(\tilde{n})_v) + \chi(\mathcal{O}_{C_v})\big) - a_F \end{aligned} \tag{2.8}$$

Let now $\mathcal{O}_C(1)$ be the polarization given by $\mathcal{O}_C(\sum_{i=1}^N x_i)$. Proposition 2.0.3 tells us that semistability for these particular choice of vector subspaces can be translated as the inequality

$$\begin{aligned} \frac{\deg \mathcal{O}_C(1)|_{C_{V\setminus V_1}}}{\deg \mathcal{O}_C(1)} \geq & \frac{\chi(S_{V_1}(F(\tilde{n})))}{\sum_v \tilde{n}_v + \deg(F) + \chi(\mathcal{O}_C)} = \\ = & \frac{\sum_{v\in V\setminus V_1} \big(\deg(F(\tilde{n})_v) + \chi(\mathcal{O}_{C_v})\big) - a_F}{\deg F(\tilde{n}) + \chi(\mathcal{O}_C)} \end{aligned} \tag{2.9}$$

The choice of these subspaces is not restrictive because by [OS]11.5 one can choose $\tilde{n}$ and N such that it suffices to check the condition of Proposition 2.0.3 only for subspaces of the form W_{V_1} for any subset $V_1 \subset V$.

The previous inequality and its various incarnations is the starting point of all the GIT construction the author has found in the literature. A small overview with some computations is given in Appendix A.2.

Let us now fix the total degree of F to be zero so that

$$\sum_v \tilde{n}_v = \sum_{v \in V} \deg(\tilde{F}(\tilde{n}_v))$$

We want now to relate the previous condition with a polyhedral decomposition of $H^1(\Gamma, \mathbb{R})$. One introduces the numbers

$$\lambda_v = \frac{\deg \mathcal{O}_C(1)|_{C_v}}{\deg \mathcal{O}_C(1)} \tag{2.10}$$

and

$$d_v := \# \left\{ \begin{array}{c} \text{edges with at least one vertex in } v \\ \text{and loops counted twice} \end{array} \right\} \tag{2.11}$$

We want to take track of the difference between the degree of the sheaves and $\tilde{n}$. For this reason one defines

$$\phi_v = (\sum_v \tilde{n}_v + \chi(\mathcal{O}_C))\lambda_v - \tilde{n}_v - \chi(\mathcal{O}_{C_v}) + \frac{d_v}{2} \tag{2.12}$$

Observe that

$$\sum_{v \in V} \phi_v = 0$$

because $\sum_{v \in V} \lambda_v = 1$ and $\sum_v (\chi(\mathcal{O}_{C_v}) - \frac{d_v}{2}) = \chi(\mathcal{O}_C)$.

The semistability condition is now translated into the fact that for any $V_1 \subset V$ we have inequalities

$$\chi(S_{V_1}(F(\tilde{n}))) \leq \sum_{v \in V \setminus V_1} (\phi_v + \tilde{n}_v + \chi(\mathcal{O}_{C_v}) - \frac{1}{2} d_v) \tag{2.13}$$

For every subset $E_1 \subset E$ we can consider the graph Γ_1 having the same vertex of Γ but the only edges are the one of E_1. This operation induces boundary

$$\partial_{E_1} : C_1(\Gamma_1, \mathbb{Z}) \to C_0(\Gamma_1, \mathbb{Z})$$

and coboundary

$$\delta_{E_1} : C^0(\Gamma_1, \mathbb{Z}) \to C^1(\Gamma_1, \mathbb{Z})$$

This is useful because using the pairings we have the following.

Proposition 2.0.4 ([OS]4.4)**.** *For any subset of vertices $V_1 \subset V$ and of edges $E_1 \subset E$ the cardinality of edges in E_1 having both end points in V_1 is given by*

$$[\sum_{v\in V_1} v, \sum_{v\in V} \frac{1}{2} d(E_1)_v v] - \frac{1}{2}(\delta_{E_1}(\sum_{v\in V_1} v), \delta_{E_1}(\sum_{v\in V_1} v))$$

Using this proposition, the canonical pairing on $C_0(\Gamma, \mathbb{Z})$ and the formulas 2.5 and 2.4 for the relation between the degree and the barycenter we obtain

$$\begin{aligned}
\chi(F(\tilde{n})) - \sum_{v\in V_1} \chi(F(\tilde{n})_v) + |E_{F(V_1)}| = \\
\Big([\sum_{v\in V} v, \deg \tilde{F}(\tilde{n}) - \frac{d(E\setminus E_F)}{2}] + \sum_{v\in V} \chi(\mathcal{O}_{C_v})\Big) + \\
-\Big([\sum_{v\in V_1} v, \partial b(D_F) - \frac{1}{2} d(E_F) + \tilde{n}] + \sum_{v\in V_1} \chi(\mathcal{O}_{C_v})\Big) + \\
+\Big([\sum_{v\in V_1} v, \frac{1}{2} d(E\setminus E_F)] - \frac{1}{2}(\delta_{E\setminus E_F} \sum_{v\in V_1} v, \delta_{E\setminus E_F} \sum_{v\in V_1} v)\Big) = \\
= [\sum_{v\notin V_1} v, \deg \tilde{F}(\tilde{n}) - \frac{1}{2} d(E\setminus E_F)] + \\
+ \sum_{v\notin V_1} \chi(\mathcal{O}_{C_v}) + \\
-\frac{1}{2}(\delta_{E\setminus E_F} \sum_{v\in V_1} v, \delta_{E\setminus E_F} \sum_{v\in V_1} v)
\end{aligned} \tag{2.14}$$

The right hand side of 2.13 can be now written as

$$[\sum_{v\notin V_1} v, \phi + \tilde{n} + \sum_{v\in V} \chi(\mathcal{O}_{C_v}) v - \frac{1}{2} d(E)]$$

Using again the formula for the degree in 2.4 and simplifying we get the important formula

$$\frac{1}{2}(\delta_{E\setminus E_F} \sum_{v\notin V_1} v, \delta_{E\setminus E_F} \sum_{v\notin V_1} v) \geq [\sum_{v\notin V_1} v, \partial b(D_F) - \phi] \tag{2.15}$$

This formula allows us to translate the ϕ-semistability in terms of projection of the Voronoi polyhedra via the following proposition which characterizes them.

Proposition 2.0.5 ([OS]6.3)**.** *Given a subset $E_F \subset E$ then*

$$C_0(\Gamma,\mathbb{R}) \supset \partial V_{E\setminus E_F}(0) = \left\{ \begin{array}{c} x \in \partial C_1(\Gamma,\mathbb{R}) : \\ \\ [x, \sum_{v\in W} v] \leq \frac{1}{2}(\delta_{E\setminus E_F} \sum_{v\in W} v, \delta_{E\setminus E_F} \sum_{v\in W} v) \\ \\ \textit{for all subsets } W \subset V \end{array} \right\}$$

In particular we find the final relation

$$\phi \in \partial(b(D_F) + V_{E\setminus E_F}(0))$$

Recall now that, by definition, given a torsion free sheaf F we have

$$D_F = b(D_F) - \frac{1}{2}\sum_{e\in E_F} e + V_{E_F}(0)$$

Furthermore using [OS]5.1 we have that

$$(D_F)^\star = b(D_F) + V_{E\setminus E_F}(0)$$

In particular

$$\phi \in \partial D_F^\star$$

We recalled the previous messy computations in order to motivate the following definition.

Definition 2.0.6. Fix a vector $\phi \in \partial C_1(\Gamma,\mathbb{R})$. An admissible sheaf of rank one on the curve C is called ϕ-semistable if $\phi \in \partial D_F^\star$. The set of ϕ-semistable sheaves is denoted with $K_\phi(\Gamma)$.

Conversely the ϕ we defined in 2.12 in terms of a polarization gives an element in $\partial C_1(\Gamma,\mathbb{R})$ and we see in this way how stability can be translated in terms of polyhedra. Since the stability requires to take strict inequalities instead of equalities then the next definition is also explained.

Definition 2.0.7.

1. Denote with $K_{\phi-st}(\Gamma)$ the set of ϕ-stable polyhedra, namely the polyhedra such that $\phi \in \partial($ interior of $D^\star)$ and

$$\dim \partial D^\star = \dim \partial C_1(\Gamma,\mathbb{R})$$

2. Define K_ϕ^0 the polyhedra such that $\phi \in \partial($interior of $D^\star)$.

Let

$$\pi : C^1(\Gamma,\mathbb{R}) \to H^1(\Gamma,\mathbb{R})$$

be the canonical projection. The image of the polyhedra in K_ϕ^0 under π gives a polyhedral decomposition of $H^1(\Gamma,\mathbb{R})$ denoted $Del_\phi(H^1(\Gamma,\mathbb{R}))$ which is

invariant under the translation via elements in $B(H^1(\Gamma,\mathbb{Z}))$ ([OS]6.1), where $B : H_1(\Gamma,\mathbb{R}) \to H^1(\Gamma,\mathbb{R})$ is the morphism induced by duality.
As in the case of vector bundles, in order to obtain representable object, one has to take not only isomorphism classes of sheaves but rather one has to quotient out by the so called S-equivalence relation. Namely in appendix A.1 we recall that there is a notion of Harder-Narasimhan filtration, and S-equivalence means that we identify sheaves having the same graded.
The polyhedra in

$$Del_\phi(H^1(\Gamma,\mathbb{R}))/B(H_1(\Gamma,\mathbb{Z}))$$

have the property that they identify S-equivalent sheaves so that one has in general a finite set morphism

$$K_\phi(\Gamma)/H_1(\Gamma,\mathbb{Z}) \to Del_\phi(H^1(\Gamma,\mathbb{R}))/B(H_1(\Gamma,\mathbb{Z}))$$

In general stability and semistability do not coincide, but one can show that there is an open for the Euclidean topology in $\partial C_1(\Gamma,\mathbb{R})$ for which this holds. We recall this briefly.

Definition 2.0.8. An element $\phi \in \partial C_1(\Gamma,\mathbb{R})$ is called *non-degenerate* if

$$K^0_\phi = K_{\phi-st} = K_\phi(\Gamma)$$

There is the following non-emptiness result about non-degeneracy.

Proposition 2.0.9 ([OS]2.1,6.2,7.6)**.** *There exists an open $U \in \partial C_1(\Gamma,\mathbb{R})$ for the Eucledian topology such that ϕ is non degenerate and for such ϕ there is a bijection*

$$K_\phi(\Gamma) \cong Del_\phi(H^1(\Gamma,\mathbb{R}))$$

Non-degeneracy of the polarization has many useful implications in terms of the geometry on Jac^ϕ_C and of the associated functor ([OS]12.15).

Unfortunately the main example we are interested in is the opposite case, namely when

$$\phi = \partial(\xi + e(E)/2)$$

for $\xi \in C_1(\Gamma,\mathbb{Z})$, where we define $e(E) := \sum_{e\in E} e$.
The reason why this is interesting is the following characterization in which one starts to see a relation with the construction we gave in chapter 1.

Proposition 2.0.10 ([OS]6.2)**.** *If ϕ is equal to $\partial(\xi + e(E)/2)$ for some $\xi \in C_1(\Gamma,\mathbb{Z})$ then*

$$Del_{\partial(\xi+e(E)/2)}(H^1(\Gamma,\mathbb{R})) = \pi(\xi + e(J)/2)) + B(Vor(H_1(\Gamma,\mathbb{R}), H_1(\Gamma,\mathbb{Z})))$$

Define now $R_\phi \subset R$ be the open subset parametrizing ϕ-semistable sheaves. Part of the results important for us in [OS] can be summarized in the following theorem.

Theorem 2.0.11. *Fix* $\phi \in \partial C_1(\Gamma, \mathbb{R})$*. Then*

- *a good quotient* $Jac_\phi(C) = R_\phi/GL(E)$ *exists and it is a projective, reduced algebraic scheme ([OS]11.4)*
- *([OS]12.17) the degree map induces a bijection*

$$Jac_\phi(C)/Pic^0_C \to Del_\phi(H^1(\Gamma, \mathbb{R}))/B(H_1(\Gamma, \mathbb{Z}))$$

Given $D \in Del_\phi(H^1(\Gamma, \mathbb{R}))/B(H_1(\Gamma, \mathbb{Z}))$*, let* $O(D)$ *be the corresponding* Pic^0_C*-orbit in* $Jac_\phi(C)$ *then*

1. *O reverses the inclusions:* D *is a face of* D' *iff* $O(D')$ *is in the closure of* $O(D)$
2. $\dim O(D) + \dim D = \dim Jac_\phi(C)$
3. $O(D) \cong Pic_{C(E\setminus SuppD)}^{\partial b(D) - \frac{d(Supp(D))}{2}}$

We see from the last point already the toric structure.

In order to understand how the Mumford's construction and the uniformization procedure comes in we need to analyze the normalization of these compactifications.
To have an idea of why this is needed one can think about the Tate curve. The normalization of the special fiber of the associated compactified jacobian is a projective line and the associated Mumford's model is an infinite chain of projective lines joined along the infinity and the zero section.
In general the special fiber of the Mumford's model we are considering will consists, except for the abelian part, of infinite copies of the normalization of the special fiber of the compactified jacobian, parametrized by the group $H^1(\Gamma, \mathbb{Z})$, and we will glue them according to the intersections relations of the associated Delaunay polyhedra in $Del_\phi(H^1(\Gamma, \mathbb{R}))$.

Since we are looking for functors it would be good if such normalizations also correspond to sheaves with some property.
It turns out that this is the case and we are going only to analyze the corresponding functor.

For every couple $c \in D \in Del_\phi(H^1(\Gamma, \mathbb{R}))$ of integral vertex c and polyhedron D, having set of vertices $\{c, a_1, \ldots, a_r\}$, for some $a_1, \ldots, a_r \in H^1(\Gamma, \mathbb{R})$, consider the cone through c given by

$$\Delta_D = \mathbb{R}_{\geq 0}(a_1 - c) + \ldots \mathbb{R}_{\geq 0}(a_r - c)$$

The set of cones $\Sigma_c := \{\Delta_D, c \in D\}$ forms a complete fan through c and we obtain a complete toric variety denoted with $Temb(c)$, under the action of the torus

$$T := \mathbb{G}m \otimes H^1(\Gamma, \mathbb{Z})$$

Given a vertex $c \in H^1(\Gamma, \mathbb{R})$ we take $\tilde{c}$ a lift to $C^1(\Gamma, \mathbb{R})$. We need the following result.

Proposition 2.0.12 ([OS]13.2)**.** *The normalization of $Jac_\phi(C)$ can be identified with the disjoint union of the schemes*

$$Pic_C^{\partial\tilde{c}} \times^T Temb(c)$$

for c a zero dimensional polyhedron in $Del_\phi(H^1(\Gamma, \mathbb{R}))/B(H_1(\Gamma, \mathbb{Z}))$. Moreover for $\Delta_D \in \Sigma_c$ there is an isomorphism at level of open orbits

$$Pic_C^{\partial c} \times^T orb(\Delta_D) \cong Pic_{C(E \setminus Supp(D))}^{\partial b(D) - \frac{1}{2} d(SuppD)}$$

Let us explain how the last isomorphism comes in because this plays an important role in chapter 4.
Given a line bundle F we get an exact sequence

$$0 \to F \to \bigoplus_{v \in V} \tilde{F}_v \xrightarrow{a} \bigoplus_{e \in E} k \to 0$$

Vice versa given a surjective homomorphism of $\mathcal{O}_C$-modules

$$\bigoplus_{v \in V} \tilde{F}_v \xrightarrow{a} \bigoplus_{e \in E} k \to 0$$

which is also surjective as morphism of $\mathcal{O}_{\tilde{C}}$-modules, meaning that for every node $p \in C$ the associated morphism

$$k_{p_+} \oplus k_{p_-} \to k$$

is surjective on both factors, then the sheaf $\ker(a)$ is a line bundle.
This correspondence is clearly not 1-1 because if we modify the sheaf

$$\bigoplus_{v \in V} \tilde{F}_v$$

by multiplying with an element of

$$Aut(\mathcal{O}_{\tilde{C}}) = C^0(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$$

we obtain the same kernel.
If we consider the jacobian Pic^0_C, its abelian part, corresponding to $\mathrm{Pic}^0_{\tilde{C}}$, acts on $\bigoplus_{v \in V} \tilde{F}_v$ via tensor product and its toric part $H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$ acts on $\bigoplus_{e \in E} k$ via multiplication modulo $Aut(\mathcal{O}_{\tilde{C}})$-action, namely for the last action we need to lift an element of $H^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$ to an element of $C^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$.

We want now to define an analogous for torsion free sheaves in order to see the action of the different orbits in $Temb(c)$ as functor. We need the following definition.

Definition 2.0.13. • Let S be a k-scheme and $q_e \in C$ be the point corresponding to the node of the edge e. A *presentation* over S is a surjective homomorphism of $\mathcal{O}_{C\times S}$-modules

$$a_S : \tilde{F}_a \to \bigoplus_{e\in E_a} \mathcal{O}_{q_e\times S} \to 0$$

where $E_a \subset E$ is a subset of nodes and $\tilde{F}_a \in \mathrm{Pic}_{\tilde{C}\times S}$.

- a morphism $a_S \to b_S$ between two presentations is a couple (s,t) where $s : \tilde{F}_a \to \tilde{F}_b$ is a morphism of $\mathcal{O}_{\tilde{C}\times S}$-modules,

$$t : \bigoplus_{e\in E_a} \mathcal{O}_{q_e\times S} \to \bigoplus_{e\in E_b} \mathcal{O}_{q_e\times S}$$

is a morphism of $\mathcal{O}_{C\times S}$-modules such that the following diagram commutes

$$\begin{tikzcd} \tilde{F}_a \arrow[r, "a_S"] \arrow[d, "s"'] & \bigoplus_{e\in E_a} \mathcal{O}_{q_e\times S} \arrow[d, "t"] \\ \tilde{F}_b \arrow[r, "b_S"] & \bigoplus_{e\in E_b} \mathcal{O}_{q_e\times S} \end{tikzcd}$$

- If for every geometric point $s \in S$ we have $\deg \tilde{F}_s = d \in C_0(\Gamma, \mathbb{Z})$ we say that a_S has *degree* d. The subset E_a is called the *support* of the presentation.

Clearly the presentations form a functor on Sch/k whose sheafification in the Zariski topology is obtained by quotienting out the pullback of line bundles on the basis. For each node e we have two branches q_e^+, q_e^- on $\tilde{C}$. Given a presentation $a_S : \tilde{F}_a \to \bigoplus_{e\in E_a} \mathcal{O}_{q_e\times S}$ of degree d and $e \in E_a$ we can look at the restriction at a node e

$$a_{S,e} : \tilde{F}_e[q_e^+] \oplus \tilde{F}_e[q_e^-] \to \mathcal{O}_{q_e\times S} \tag{2.16}$$

This is surjective as morphism of $\mathcal{O}_{C\times S}$-modules, but it can fail to be surjective as morphism of $\mathcal{O}_{\tilde{C}\times S}$-modules. Since either $\tilde{F}_e(q_e^+)$ or $\tilde{F}_e(q_e^-)$ are mapped surjectively onto $\mathcal{O}_{q_e\times S}$ and since on the normalization $\tilde{C}$ we have a Poincaré bundle $\mathcal{P}_d$ in degree d, in any case we are parametrizing one dimensional quotients of a two dimensional vector space. This consideration makes immediate the proof of the following proposition.

Proposition 2.0.14 ([OS]12.1)**.** *Let $Pres(d, E_1)$ be the functor of presentations of degree d and support $E_1 = \{e_1, \ldots, e_r\}$ for a nodal curve over a field. It is represented by the $\prod_{e\in E_1} \mathbb{P}^1$-bundle over $Pic^d_{\tilde{C}}$ given by the fiber product*

$$\mathbb{P}(\mathcal{P}_d[q_{e_1}^+] \oplus \mathcal{P}_d[q_{e_1}^-]) \times_{Pic^d_{\tilde{C}}} \cdots \times_{Pic^d_{\tilde{C}}} \mathbb{P}(\mathcal{P}_d[q_{e_r}^+] \oplus \mathcal{P}_d[q_{e_r}^-])$$

The previous proposition fails if we let the curve moving and the presentation functor is not even separated.

We need now to understand when the kernel of a presentation gives a semistable admissible sheaf.
Since a torus embedding is a disjoint union of open orbits it is enough to consider presentations for which the morphism in 2.16 is surjective on both factors. Such presentations are called *strict*. We have the following useful lemma.

Lemma 2.0.15 ([OS]12.6). *Let*

$$a_k : \tilde{F}_a \to \bigoplus_{e \in E_a} k \to 0$$

be a strict presentation over a field $Spec(k)$*. Let* F *be the kernel. There exists a subset* $E_a \subset E$ *and a disjoint union decomposition* $E_a^+ \coprod E_a^- = E_a$ *such that*

$$\tilde{F}_a \cong \tilde{F}(\sum_{e \in E_a^+} q_e^+ + \sum_{e \in E_a^-} q_e^-)$$

as $\mathcal{O}_{\tilde{C}}$*-modules and the set* E_F*, the set of nodes where* F *is not free, is equal to* $E \setminus E_a$*.*

Let us now use this to write down all ϕ-semistable sheaves up to S-equivalence in terms of presentations. We know that they correspond to polyhedra D in $Del_\phi(H^1(\Gamma, \mathbb{R}))$, so we fix one of them and we look for the sheaves corresponding to the fixed one. By definition there is a polyhedron $\tilde{D}$ in $C^1(\Gamma, \mathbb{R})$ projecting down to D. Since the decomposition in $C^1(\Gamma, \mathbb{R})$ is induced by translating the faces of the unit cube, there exist a $\tilde{c} \in C^1(\Gamma, \mathbb{Z})$, a subset $E_1 \subset E$ and a decomposition $E_1 = E_1^+ \coprod E_1^-$ such that

$$\tilde{D} = \tilde{c} + \{ \sum_{e \in E_1^+} (e) t_e + \sum_{e \in E_1^-} (-e) t_e \ : \ t_e \in [0, 1] \}$$

according to the position of $\tilde{D} - \tilde{c}$ w.r.t. the coordinate hyperplanes. Write $\tilde{D}$ as the convex hull of $\tilde{c}, z_1, \ldots, z_r \in C^1(\Gamma, \mathbb{R})$. Let $\Delta_{\tilde{D}}$ be the rational polyedral cone in $C^1(\Gamma, \mathbb{R})$ given by

$$\Delta_{\tilde{D}} = \mathbb{R}_{\geq 0}(z_1 - \tilde{c}) + \cdots + \mathbb{R}_{\geq 0}(z_r - \tilde{c})$$

Pick a line bundle $L \in \mathrm{Pic}^{\partial \tilde{c}}_{\tilde{C}}$ and a presentation of the form

$$L(-\sum_{e \in E_1^+} q_e^+ - \sum_{e \in E_1^-} q_e^-) \to \bigoplus_{e \in E \setminus E_1} k \to 0$$

Consider the torus $\mathcal{T} = C^1(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$ and let $O(\Delta_{\tilde{D}})$ be the main open orbit corresponding to the toric variety defined by $\Delta_{\tilde{D}}$.
The torus orbit $O(\Delta_{\tilde{D}})$ acts via multiplication on $\bigoplus_{e \in E \setminus E_1} k$ and $\mathrm{Pic}^0_{\tilde{C}}$ acts on $L(-\sum_{e \in E_1^+} q_e^+ - \sum_{e \in E_1^-} q_e^-)$.

Define F as the kernel of the previous presentation. By construction it is a torsion free sheaf of degree 0, indeed

$$\chi(F) - \chi(\mathcal{O}_C) = \deg L - |E_1| - |E \setminus E_1| + |E| = \deg L = [\sum_{v \in V} v, \partial \tilde{c}] = 0 \quad (2.17)$$

where the last is zero because the curve is connected. Furthermore it is ϕ-semistable because we have started from a polyhedron in $Del_\phi(H^1(\Gamma, \mathbb{R}))$.

As remarked previously to obtain the orbit in the Jacobian we need to divide out by the torus $\delta C_0(\Gamma, \mathbb{Z}) \otimes \mathbb{G}m$ because the isomorphism class of the kernel is not affected by the action of $Aut(\mathcal{O}_{\tilde{C}})$ and we are done.

This last explicit description is the way in which we will think about ϕ-stability.

Once we have understood how to obtain the degree from the polyhedra we only need to find a Mumford model whose open toric orbits of the special fiber acts on the right quotient of a given presentation.
We need then to take care that the tori involved are the right ones and to this aim we will use proposition 2.0.10 in which the link with chapter 1 is clear. Actually since also the periods are acting is better to consider presentations on the formal covering of the special fiber which we are going to introduce in the next chapter.

Chapter 3

Formal covering

We introduce in this chapter the universal covering of the curve and we explain in which sense it is universal.
First we introduce the notion of *trivial covering*.

Definition 3.0.16. 1. Let X be an absolutely connected rigid analytic space over a complete non-archimedean valued field K. A **trivial covering** of X is a morphism $f : Y \to X$ of rigid spaces such that the restriction to each connected component of Y is an isomorphism.

2. A morphism $f : Y \to X$ of rigid spaces is called an **analytic covering** if there exists an admissible covering $\{U_i\}_{i \in I}$ of X such that each of the induced morphisms $f|_{f^{-1}U_i} : f^{-1}U_i \to U_i$ is a trivial covering.

3. an absolutely connected rigid space X is called **simply connected** if every analytic covering of X is trivial.

4. an analytic covering $u : \Omega \to X$ with X absolutely connected is called **universal** if Ω is simply connected.

Assume now that C_K is absolutely irreducible, non singular, 1-dimensional and projective over K and it comes from a curve with semistable reduction C_s over the ring of integers of K which we have fixed.
In this situation we can construct a universal analytic covering of C_K using the intersection graph of the special fiber. Let Γ be such graph. For each vertex v we denote with C_v the irreducible component of C_s corresponding to v. It is easy to show that if Γ is a tree then C_K is simply connected as in the previous definition. If Γ is not a tree then for each edge we take a Zariski open

$$C_s \supset U(e) = C_s \setminus \bigcup_{v \notin \{\text{end-points of } e\}} C_v$$

For each vertex v we take the Zariski open

$$C_s \supset U(v) = C_v \setminus \bigcup_{w \neq v} C_w$$

If we denote with $r : C_K \to C_s$ the analytic reduction then

$$\{r^{-1}U(e)\}_{e\in E} \cup \{r^{-1}U(v)\}_{v\in V}$$

is a covering inducing r. Consider the universal covering of the graph

$$\mathcal{T} \to \Gamma$$

It is an infinite tree. For each edge $\tilde{e}$ (resp. vertex $\tilde{v}$) of $\mathcal{T}$ we take the affinoid $\Omega(\tilde{e})$ (resp. $\Omega(\tilde{v})$) as a copy of $r^{-1}(U(e))$ (resp. $r^{-1}(U(e))$) where e (resp. v) is the edge of Γ corresponding to $\tilde{e}$(resp. $\tilde{v}$) via the projection $\mathcal{T} \to \Gamma$. One glues the affinoids $\Omega(\tilde{e})$ and $\Omega(\tilde{v})$ according to the intersections in the tree $\mathcal{T}$ and get an analytic covering $u : \Omega \to C$, with reduction $R : \Omega \to \Omega_s$. Since $\mathcal{T}$ can been written as increasing union of finite subtrees $\{T_n\}$ and the reduction $\Omega_{s,n}$ corresponding to the tree T_n is simply connected it follows that Ω is simply connected. Note that the covering group is isomorphic to the fundamental group $\mathcal{G}$ of the graph Γ.

In our case we have fixed also an integral model C for C_K and we can perform the same operation using formal schemes. Let $\hat{C}$ be the formal completion of C at the special fiber of the basis. If we consider the formal open subsets $\hat{U}(e) = Spf\big(\mathcal{O}_C(U(e))\big)$ (resp. $\hat{U}(v) = Spf\big(\mathcal{O}_C(U(v))\big)$) we can repeat the previous construction and we obtain a formal covering

$$\hat{\Omega} \to \hat{\mathcal{C}}$$

which is a topological covering in the Zariski topology. Moreover the scheme $\hat{\Omega}$ is *admissible* because for every open affine $Spec(A) \subset C$ the ring $\hat{A}$ is topologically of finite presentation and flat over $\mathcal{O}_K$. Besides the group $\mathcal{G}$ acts freely and discontinuously in the Zariski topology.

3.1 Raynaud Extension of Jacobians

The Mumford models are constructed starting from a semiabelian scheme which is a *global* extension of an abelian variety by a torus. In the case of curves over the spectrum of a discrete valuation ring we have a semiabelian scheme with generic fiber abelian and special fiber semiabelian and we would like to understand how to get such semiabelian global extension. This extension is usually called *Raynaud's extension* and it can be constructed in a more general situation. For example when the base scheme is normal, an algebraic construction of such extensions can been found in [FC]II.1 or [SGA]7.1.Exp.IX.

We would like to give to the Raynaud extension a modular interpretation in the case of degenerations of curves in terms of line bundles on the analytic covering. We can work either with formal geometry or with rigid geometry.

First recall a notation. Consider a triple $(Y, r_Y, \bar{Y})$ where Y is a rigid space and $r_Y : Y \to \bar{Y}$ is an analytic reduction. Let $U \subset \bar{Y}$ be an open subset, then $r_Y^{-1}U$ is called a *formal open subset* of the triple $r_Y : Y \to \bar{Y}$. Such subsets are admissible.

Consider the category of *formal rigid spaces*: objects are triple $(Y, r_Y, \bar{Y})$ where Y is a rigid space and $r_Y : Y \to \bar{Y}$ is an analytic reduction. A morphism $f : (Y, r_Y, \bar{Y}) \to (X, r_X, \bar{X})$ is a morphism of rigid space $Y \to X$ such that the preimage of a formal open subset is a formal open subset.

A *formal line bundle* on a formal rigid space $(Y, r_Y, \bar{Y})$ is a sheaf of $r_{Y,*}\mathcal{O}_Y^{\circ}$-modules on $\bar{Y}$ which is locally isomorphic to $r_{Y,*}\mathcal{O}_Y^{\circ}$.
In our situation we always have a fixed associated formal scheme. We can use indifferently formal line bundle or line bundles on the formal scheme. The reason for this definition is that we want to impose a degree condition on the restriction to the special fiber and for this reason we need integral sheaves.

Given a formal line bundle L on the triple $(Y, r_Y, \bar{Y})$, we obtain a sheaf on $\bar{Y}$ with coefficients in K via the rule

$$(L \otimes K)(U) := L(U) \otimes K$$

where $U \subset \bar{Y}$ is affine open.

Let $u : \Omega \to C$ be the universal analytic cover of the curve C as before. Since we have reductions, we get a covering of formal rigid spaces

$$u : (\Omega, r_\Omega, \bar{\Omega}) \to (C, r_C, \bar{C})$$

with group $\mathcal{G}$ the fundamental group of the graph for the special fiber C_s.

Given a formal rigid space $(Y, r_Y, \bar{Y})$ and a formal line bundle L on $(\Omega \times Y, r_\Omega \times r_Y, \bar{\Omega} \times \bar{Y})$ we say that L satisfy the condition $\star$ if the following are satisfied

1. for any point $t \in \bar{Y}$ the restriction of L to each irreducible component of $\bar{\Omega} \times \{t\}$ has *degree zero.*
2. for any $\gamma \in \mathcal{G}$ there is an isomorphism $(\gamma \times 1)^* L \otimes K \cong L \otimes p_2^* N_\gamma \otimes K$ for some formal line bundle N_γ on $(Y, r_Y, \bar{Y})$

Remark 3.1.1. In the case that $(Y, r_Y, \bar{Y})$ is the base analytic space then condition 2) implies that the bundle L has a $\mathcal{G}$-action. This follows because the failure for the set of isomorphisms $\gamma^* M \cong M$ to be an action is given by a 2-cocycle with values in a commutative group and this 2-cocycle vanishes because $\mathcal{G}$ is free. Moreover the set of such actions is a torsor under the group $Hom(\mathcal{G}, K^*)$.

We consider now couples (L, α) where L satisfies $\star$ and $\alpha(\gamma)$ is a (fixed) family of isomorphisms

$$\alpha(\gamma) : N_\gamma \to N_1$$

compatible with the isomorphisms in the $\star$ condition. We call the previous condition on couples (L, α) the $\star\star$ condition.

Consider now the group functors $\mathcal{A}$ and $\widetilde{G}$ on the category of formal rigid space defined as

$$\mathcal{A}(Y, r_Y, \bar{Y}) := \left\{ \begin{array}{r} \text{fomal line bundles on } (\Omega \times Y, r_\Omega \times r_Y, \bar{\Omega} \times \bar{Y}) \\ \text{satisfying the conditions in } \star \end{array} \right\} / \cong$$

and

$$\widetilde{G}(Y, r_Y, \bar{Y}) := \left\{ \begin{array}{c} \text{couples } (L, \alpha) \\ \\ \text{on } (\Omega \times Y, r_\Omega \times r_Y, \bar{\Omega} \times \bar{Y}) \\ \text{satisfying } \star\star \end{array} \right\} / \cong \tag{3.1}$$

These functors are group functors and there is an obvious forgetful functor

$$\widetilde{G} \to \mathcal{A}$$

which is also a morphism of group functors. The kernel corresponds to elements of the torus $T := Hom(\mathcal{G}, \mathbb{G}m) = Hom(\mathcal{G}^{ab}, \mathbb{G}m)$ so that we have an extension of abelian group functors

$$0 \to \mathcal{T} \to \tilde{G} \to \mathcal{A} \to 0$$

Observe that once we have a formal line bundle L on $(\Omega, r_\Omega, \bar{\Omega})$ with a $\mathcal{G}$-action α, we obtain a formal line bundle M on (C, r_C, C_s) via the rule

$$M(U) = M(u^{-1}U)^\Gamma$$

The point here is that since the action is free in the Zariski topology we have $M(u^{-1}U) = \prod_\gamma M(\gamma V)$ for some open $V \subset \Omega$ mapping to U. Using GAGA theorem to algebraize the line bundle and [FvdP]Thm.1.5.5, we get an exact sequence

$$0 \to \mathcal{G}^{ab} \to \widetilde{G}(K) \to \mathrm{Pic}^{an,0}_{C_K/K} \to 0$$

The diagram of group functors

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \mathcal{G}^{ab} \arrow[d] & & \\ 0 \arrow[r] & \mathcal{T} \arrow[r] & \widetilde{G} \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] & 0 \\ & & \mathrm{Pic}^{an,0}_{C_K} \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

is actually representable by a diagram of formal analytic spaces ([FvdP]4.1) where all the morphisms are analytic.

Moreover the exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \widetilde{G} \longrightarrow \mathcal{A} \longrightarrow 0$$

of analytic groups can be algebraized ([FvdP]6.2) to an exact sequence of group schemes over the base S which is the spectrum of the ring of integers of the field K. At analytic level we have an isomrophism ([FvdP]4.1)

$$\widetilde{G}/\mathcal{G}^{ab} \cong \mathrm{Pic}^{an,0}_{C_K}$$

Define the "special fiber" of $\mathcal{A}$ as the functor $\overline{\mathcal{A}}$, where we copy the definition of $\mathcal{A}$ but where this time instead of considering formal rigid spaces, we take varieties over the residue field.
Denote with $\tilde{C}$ the normalization of C_s, then one can show ([FvdP]) that

$$\overline{\mathcal{A}} \cong \mathrm{Pic}^0_{\tilde{C}}$$

In particular $\overline{\mathcal{A}}$ is representable. Moreover there is a surjective ([FvdP]1.3.4) reduction functor

$$\mathcal{A} \to \overline{\mathcal{A}}$$

Observe that even though $\mathcal{A}$ is an abelian scheme with good reduction $\overline{\mathcal{A}}$, in general the geometric structure of $\mathcal{A}$ does not respect the geometric structure of $\overline{\mathcal{A}}$. Indeed one can find, for genus at least 4, a smooth, projective curve C_K, with *reducible* stable reduction, such that the corresponding $\mathcal{A}_K$ is not a product of Jacobians ([FvdP94]).

From this chapter we have now that the Raynaud's extension for a Jacobian corresponds to a functor in terms of sheaves on the covering $\Omega \to \hat{C}$ and that this functor is representable by a scheme which is *separated*. We have also a universal invariant Poincaré bundle on $\Omega \times \tilde{G}$ by [FvdP]4.2.

Chapter 4

Construction of the quotient

In this section we describe how the constructions of the previous chapters can be combined together in order to construct the uniformization. We consider a family of stable curves

$$C \to S$$

which is generically smooth and S the spectrum of a complete discrete valuation ring R, whose valuation we denote with v_S. We also assume that the residue field is algebraically closed and that the irreducible components of the special fiber of C are reduced. We consider the formal covering

$$\Omega \to \widehat{C}$$

with group $\mathcal{G}$ the fundamental group of the graph Γ of the special fiber C_0 as in the previous chapter, where $\widehat{C}$ is the formal completion of C along the special fiber.

Given an integral, regular paving Σ of $M_{\mathbb{R}} = H_1(\Gamma, \mathbb{R})$ (definition in 1.6.9) we get, for any $\omega \in \Sigma$, a cone $Cone(1, \omega) \in \mathbb{R} \oplus M_{\mathbb{R}}$ and by taking the integral points of this cone we have a monoid which we denote with $\Delta^{\vee}_{\omega^\star}$.

Later the notation $\omega^\star$ denotes the Voronoi dual.

We assume that the maximal dimensional cells $\omega \in \Sigma$ are all generating (definition in 1.6.11).
Given ω we usually fix an $a_0 \in \omega \cap M$ and we denote the differences of the vertices by $a_i - a_0$ for $i = 0, \dots, r$.
Using this notation we denote with

$$X_{\omega^\star} \subset H_1(\Gamma, \mathbb{Z})$$

the sublattice generated by the differences $a_i - a_0$.

Finally we also assume that the cones $\Delta^{\vee}_{\omega^\star}$ can be generated by couples of the form $(a_i - a_0, c_i)$ with $c_i \in \mathbb{N}$, $i = 1, \ldots, r$.

Definition 4.0.2. Let Σ be an integral, regular paving of $H_1(\Gamma, \mathbb{R})$ and $\omega \in \Sigma$. We define $T_{\omega^\star}$ as the sections

$$\alpha \in Hom(\mathcal{G}, \mathbb{G}m(K))$$

such that

$$v_S(\alpha_K(\overline{\gamma})\mu) \geq 0 \quad \forall(\bar{\gamma}, \mu) \in \Delta^{\vee}_{\omega^\star}$$

where $\bar{\gamma} \in \mathcal{G}/[\mathcal{G}, \mathcal{G}] = H_1(\Gamma, \mathbb{Z})$ is the image of γ under the projection

$$\mathcal{G} \to \mathcal{G}/[\mathcal{G}, \mathcal{G}]$$

It is more useful, for functoriality reasons, to introduce the same notion in the context of log-geometry.
First of all the theory of formal schemes can be generalized in the contest of log-geometry. Essentially one has to replace monoids with topological monoids and require that the defining morphism from the topological monoid to the structure sheaf is continuous. For a more detailed treatment we suggest to the reader [Ho].

We need to put a formal log-structure on $\hat{S}$. We take the log-structure induced from the powers of the uniformizer.

We specialize now to the case where Σ is induced by the Delaunay decomposition induced by the standard pairing on $H_1(\Gamma, \mathbb{R})$.

In section B.4.2 we define a monoid H^{sat}_{Σ}, which is the saturation of the dual monoid corresponding to the Delaunay-Voronoi decomposition of semipositive definite bilinear forms with rational radical, whose paving is coarser than Σ.
We also define there a morphism of monoids

$$h_\Sigma : H_\Sigma \otimes \mathbb{Q} \to \mathbb{Q}$$

with bounded denominators. However as we remarked in B.4.11 in the case of curves we can assume that the image is integral and we obtain in this way a morphism, which we denote with the same letter

$$h_\Sigma : H^{sat}_{\Sigma} \to M_{\widehat{S}}$$

Given $d \in M_{\hat{S}}$ and $f : (T, M_T) \to (\hat{S}, M_{\hat{S}})$ a morphism of fine, saturated log-formal schemes, with $M_T \cong f^* M_{\hat{S}}$ we define $d_T \in M_T$ as the pullback.

Let $M = H_1(\Gamma, \mathbb{Z})$, take a Voronoi cell $\sigma \in H^1(\Gamma, \mathbb{R})$ and $\Delta_\sigma^\vee \subset M_\mathbb{Q} \oplus \mathbb{Q}$ be the cone corresponding to the Delaunay dual $\sigma^\star$.
Since there could be confusion, we recall again here that the σ in this case lives in the N-space. We should have written $\Delta^\vee_{(\sigma^\star)^\star}$ according to our notation, but using the duality between Voronoi and Delaunay cells we have $(\sigma^\star)^\star = \sigma$.

We consider the étale topology where the coverings

$$f_\alpha : (U_\alpha, M_{U_\alpha}) \to (U, M_U)$$

for a fs-log-scheme (U, M_U) over $(\hat{S}, M_{\hat{S}})$ are étale morphisms of formal schemes (classical definition in [G-M] 6.1) such that

$$f_\alpha^* M_U \cong M_{U_\alpha}$$

and

$$U = \cup_\alpha f_\alpha(U_\alpha)$$

Definition 4.0.3. Let T_σ be the sheafification in the étale topology on the log-formal scheme $(\widehat{S}, M_{\widehat{S}})$ of the functor that to a strict morphism $(U, M_U) \to (\widehat{S}, M_{\widehat{S}})$, associates

$$T_\sigma(U) := \left\{ \begin{array}{c} s \in Hom(M, \mathbb{G}m^{log}(U)) \mid \quad (s(m) \mod \mathcal{O}^*) \cdot d_U \in \overline{M}_U \\ \\ \forall \ (m, d_U) \in \Delta_\sigma^\vee \end{array} \right\}$$

This functor will be used to define the actions we are interested in.

Remark 4.0.4. We need some comments of this definition. As explained in 1.7, Kajiwara in [Kaj]3 defined an analogous functor for a fixed curve over a field using the Zariski topology.
Unfortunately log-structures in the Zariski topology do not behaves well for families of curves having in the special fiber components with self-intersection and in general étale log-structures do not come from Zariski log structure. For a detailed proof we remand to ([Ol03]A.1).
We report here just an example. Assume we have a family $C \to S$ whose special fiber is an irreducible curve with one node. In this situation we can still define the étale log-structure M_C induced from the branches of the node of the special fiber C_0.
Consider the morphism of sites

$$\epsilon : C_{\acute{e}t} \to C_{Zar}$$

In [Ol03]A.1 is proved that the étale log-structure M_C descends to a Zariski log-structure if and only if the adjunction map

$$\epsilon^{-1}\epsilon_* M_{C,\acute{e}t} \to M_{C_{\acute{e}t}}$$

is an isomorphism. If $\bar{x}$ is a geometric point over the node and $j : C_0 \to C$ is the inclusion of the special fiber then

$$j^*\overline{M}^{gp}_{C_0,\bar{x}} \cong \mathbb{Z}^2$$

On the other hand if we compute it w.r.t. the Zariski topology this is the constant sheaf $\mathbb{Z}$ and the morphism

$$\epsilon^{-1}\epsilon_* j^*\overline{M}^{gp}_{C_0,\bar{x}} \to j^*\overline{M}^{gp}_{C_0,\bar{x}}$$

corresponds to the diagonal morphism $\mathbb{Z} \to \mathbb{Z}^2$ and the adjunction map is not an isomorphism.

As described in Appendix B we have an action of $H_1(\Gamma, \mathbb{Z})$ which permutes the cones, namely $c \in H_1(\Gamma, \mathbb{Z})$ acts by

$$\mathbb{N} \oplus M \ni (d, l) \to (d, l + dc) \in \mathbb{N} \oplus M$$

and it sends $\Delta^\vee_\sigma$ to $\Delta^\vee_{\sigma-c}$.

Before considering the functor we want to define, we can more concretely look at these objects in terms of toric geometry. We imagine our sections s of T_σ coming from element of the "$N \times \mathbb{R}$"-space, which in our case is $H^1(\Gamma, \mathbb{R}) \times \mathbb{R}$.
Unfortunately under the assumption 4.4 we make later for the polarization ϕ, if we put together the definition of Oda and Seshadri there is something confusing because we called $Del_\phi(H^1(\Gamma, \mathbb{R}))$ as Delaunay but we think about it as a translated of $B(Vor(H_1(\Gamma, \mathbb{R}))$. For this reason we want next to emphasize how our sections s corresponds, up to translation by the z in the assumption and to a factor -2, to the elements in $Del_\phi(H^1(\Gamma, \mathbb{R}))$.

Let k be the residue field and take a uniformizer π. In general we get a base change of this situation. Assume for simplicity we can write $s \in T_\sigma$ as a couple

$$(n, d) \otimes t, \quad \text{where } t \in \mathbb{G}m(k), d \in \mathbb{Q},\ n \in H_1(\Gamma, \mathbb{Z})$$

or

$$s = n \otimes t\pi^d$$

Assume that $\sigma^\star$ is centered at c. First we want to see that, up to a rescaling, the cone $\Delta^\vee_\sigma$ is generated by $\bar{M}_S$ and by the elements

$$(a_i - c, A(a_i) - A(c))$$

where a_i are the vertices of the Delaunay cell $\sigma^\star$ centered at c, A is the associated quadratic form. This follows from Proposition B.4.7 3).

Remark 4.0.5. There are cases in which the conclusion of Proposition B.4.7 3) is true without taking positive rationals multiples.
For example if the cell $\sigma^\star$ is generating (definition in 1.6.11) or if the $a_i - c$ generate $\mathbb{Z}^g$ as semigroup then by [Nak]Lemma 1.3 (ii) and Lemma 1.6 we can omit tensoring with $\mathbb{Q}_{\geq 0}$.
In dimension less equal then 4 the last condition is always satisfied, for example by [Nak]Lemma 1.5.
Again in the case of curves the cells are generating by Appendix B.4.1, hence we can assume that everything is defined over $\mathbb{Z}$.

Let us come back to our action s. Using the pairing and ignoring the units, we get

$$s(a_i - c)\pi^{A(a_i)-A(c)} = t^{(n,a_i-c)}\pi^{(n,a_i-c)+d(A(a_i)-A(c))}$$

write now

$$(\frac{n}{d}, a_i - c) = -2B(\alpha, a_i - c), \quad \text{for some } \alpha \in \mathbb{Q}^r.$$

As already explained, the reason for this rewriting comes from the easy relation

$$\begin{aligned}(\frac{n}{d}, a_i - c) + A(a_i) - A(c) = & \quad -2B(\alpha, a_i - c) + A(a_i) - A(c) = \\ = & \quad -2B(\alpha, a_i - c) + A(a_i - c) + 2B(a_i - c, c) = \\ = & \quad |\alpha - a_i|^2_A - |\alpha - c|^2_A\end{aligned}$$

where the norm is computed w.r.t. the matrix A. By definition this quantity is not negative if and only if $-2B\alpha$ lies in σ. Equivalently if the couple (n, d) lies in the cone

$$Cone(\sigma, 1) \subset N_{\mathbb{R}} \times \mathbb{R}$$

The relation $|\alpha - a_i|^2 - |\alpha|^2 = -2B(\alpha, a_i) + A(a_i)$ tells us that after a translation of the vectors to the origin, namely if x_i corresponds to Delaunay vectors through the origin then consider $x_i = a_i - c$ as the corresponding vectors at c, we have

$$0 \leq v_S(s(a_i - c)a(a_i)a(c)^{-1}) = v_S(s(x_i)a(x_i)b(x_i, c)) \qquad (4.1)$$

so that $s(-)b(-, c) \in T_{\sigma_0}$ where $\sigma_0^\star$ has the origin as centering vertex.

In particular this is compatible with the action defined via

$$c_* s(x_i) := s(x_i)b(x_i, c)^{-1} \qquad (4.2)$$

which gives an isomorphism

$$S_c : T_\sigma \cong T_{\sigma+c}$$

Given $\sigma^\star$ through a vertex c we obtain an open toric orbit $O(\Delta_\sigma)$ as follows. Let a_i be the vertices of $\sigma^\star$ and take the corresponding generators in $\Delta_\sigma^\vee$. We obtain

$$O(\Delta_\sigma) = \left\{ \begin{array}{c} s \in T_\sigma \mid s(a_i - c)a(a_i)a(c)^{-1} \in \mathbb{G}m \ \forall i \\ \text{and} \\ v_S(s(m)d) > 0 \text{ for } (m,d) \notin \Delta_\sigma^\vee \end{array} \right\} \tag{4.3}$$

We call this toric orbit because the expression

$$s(a_i - c)a(a_i)a(c)^{-1} = s(a_i - c)a(a_i - c)b(a_i - c, c)$$

gives a character for the subgroup generated by the $a_i - c$.

Definition 4.0.6. Given a cell σ, with $\sigma^\star$ centered at c, let $X_\sigma \subset M$ be the sublattice generated by the differences $a_i - c$.
For each σ consider the subgroup $\mathcal{G}_\sigma \subset \mathcal{G}$ of elements whose projection map to

$$M = H_1(\Gamma, \mathbb{Z}) = \mathcal{G}/[\mathcal{G}, \mathcal{G}]$$

factorizes through X_σ.

Given $x \in X_\sigma$ we can extend it to $(x, \mu) \in \Delta_\sigma^\vee$ by [KKN1]3.4.7 or by Proposition B.4.7 3).

We finally give the definition of the functor of sheaves which has to correspond to a Mumford's model w.r.t. a paving.

Definition 4.0.7. Let Σ be an integral, regular paving of $H_1(\Gamma, \mathbb{R})$ (definition in 1.6.9) and $\mathcal{A}$ be the functor defined in 3.1 of $\mathcal{G}$-invariant formal line bundles on Ω having degree zero on the restriction of each irreducible component of the special fiber. We define $\widetilde{\mathcal{P}}_\Sigma^{log}$ as the étale sheafification, on fine saturated formal log-schemes over $\widehat{S}$, of the functor of **log-Σ-bounded sheaves** defined as follows.
For any strict log-formal scheme $(\mathcal{U}, M_\mathcal{U}) \in (fs)/\hat{S}$ we consider classes of couples $(L_\mathcal{U}, \alpha_\mathcal{U})$ where $L_\mathcal{U} \in \mathcal{A}(\mathcal{U})$ and $\alpha_\mathcal{U} \in Hom(\mathcal{G}, M_\mathcal{U}^{log})(\mathcal{U})$, up to pullback from the basis, such that there exists a $\sigma \in \Sigma$ for which

$$\alpha_\mathcal{U} \in O(\Delta_\sigma)(\mathcal{U})$$

furthermore we require that for any $\gamma \in \mathcal{G}_{\sigma^\star}$, with extension to $(\overline{\gamma}, \mu) \in \Delta_{\sigma^\star}^\vee$, then we choose actions

$$\gamma^* L_\mathcal{U} \to L_\mathcal{U}$$

which are isomorphic to the action given by multiplication via

$$\alpha_\mathcal{U}(\overline{\gamma})\mu$$

Remark 4.0.8. Observe that in this case the scheme of the log-base change coincides with the base change in the category of schemes because the log-structure is of semistable type ([Kaj]1.8).

Remark 4.0.9. Observe that on $\widehat{S}$-sections the definition gives

$$v_S(\alpha_{\widehat{S}}(\bar{\gamma})\mu) = 0$$

for $(\bar{\gamma}, \mu)$ with $\bar{\gamma} \in X_{\sigma^\star}$. Besides we consider the expression $\alpha_{\hat{S}}(\bar{\gamma})\mu$ linear as elements in a cone. This definition imposes a control of the specialization to the residue field of the action w.r.t. to the group $\mathcal{G}_{\sigma^\star}$. In other words we are imitating the specialization property of the Mumford models we explained in the section 1.6.

As consequence of what we explained in chapter 1 we have the following proposition.

Proposition 4.0.10. *The sheaf $\widetilde{\mathcal{P}}_{\Sigma}^{log}$ is represented by a log-formal scheme which is the formal completion of a relatively complete model.*

Proof. Everything now follows from the theory in chapter 1, [Ol], [KKN1]. We recall the argument for completeness.
First it suffices to consider sections having the boundeness condition w.r.t. a given cell σ and then glue them according to the intersection of the faces.

For a given cell σ we consider the log-formal scheme

$$T_{\sigma^\star} = Spf(R \otimes_{\mathbb{Z}[H_{\Sigma}^{sat}]} \mathbb{Z}[H_{\Sigma}^{sat} \ltimes \Delta_{\sigma^\star}^{\vee}])$$

where the log-structure is induced by the monoid $H_{\Sigma}^{sat} \ltimes \Delta_{\sigma^\star}^{\vee}$. The scheme $T_{\sigma^\star}$ is endowed with a torus action. We claim that in the case where $\mathcal{A}$ is trivial then it represents the sections of $\widetilde{\mathcal{P}}_{\Sigma}^{log}$ having the specialization property w.r.t. $\Delta_{\sigma^\star}^{\vee}$. Indeed let $(\mathcal{U}, M_{\mathcal{U}}) \in (fs)/\hat{S}$ and $\alpha_{\mathcal{U}} \in Hom(\mathcal{G}, M_{\mathcal{U}}^{log})$ be such a section. In order to obtain a log-point of $T_{\sigma^\star}(\mathcal{U})$ it is enough to produce a homomorphism of monoids

$$t^{log} : H_{\Sigma}^{sat} \ltimes \Delta_{\sigma^\star}^{\vee} \to M_{\mathcal{U}}$$

over $M_{\hat{S}}$ commuting with the H_{Σ}^{sat}-action induced by h_{Σ}. Given

$$(h, x, \mu) \in H_{\Sigma}^{sat} \ltimes \Delta_{\sigma^\star}^{\vee}$$

define

$$t^{log}(h, x, \mu) := \alpha_U(x) \cdot \mu \cdot h_{\Sigma}(h)$$

As recalled in B.12, the addition rule in $H_{\Sigma}^{sat} \ltimes \Delta_{\sigma^\star}^{\vee}$ is given by

$$(h, x, \mu) \cdot (k, y, \nu) = (h + k + (x, \mu) * (y, \nu), x + y, \mu + \nu)$$

Since (x,μ) and (y,μ) belong to the same cone we know that $(x,\mu)*(y,\nu)=0$ by [Ol]Lemma 4.1.4. Hence t^{log} defines a homomorphism of monoids, hence we get a log-point of $T_{\sigma^\star}(\mathcal{U})$.

Vice versa given a section in $T_{\sigma^\star}(\mathcal{U})$, we get a homomorphism of monoids

$$t^{log}: H_\Sigma^{sat} \ltimes \Delta_{\sigma^\star}^\vee \to M_{\mathcal{U}}$$

We want to use this to define a section $\alpha_{\mathcal{U}}$. As we recalled in Proposition B.4.7 3) we have $\Delta_{\sigma^\star}^{\vee,gp}=\mathbb{Z}\times X$ where we denote $X=H_1(\Gamma,\mathbb{Z})$ for simplicity. Using the natural inclusion $X\to\{0\}\times X\subset\mathbb{Z}\times X$ we define

$$\alpha_{\mathcal{U}}(x) := t^{log,gp}(0,0,x)$$

where $t^{log,gp}$ is the associated group map. Since $H_\Sigma\to H_\Sigma\ltimes\Delta_{\sigma^\star}^\vee$ is integral (by [Ol]4.1.9), we have that the monoids inject into the respective groups and given $(x,\mu)\in\Delta_{\sigma^\star}^\vee$ we find, using the definitions,

$$\begin{aligned}(\alpha_{\mathcal{U}}(x) \mod \mathcal{O}^*)\mu =& (t^{log,gp}(0,0,x) \mod \mathcal{O}^*)\mu = \\ =& (t^{log}(0,\mu,x) \mod \mathcal{O}^*) \in \overline{M}_{\mathcal{U}}\end{aligned}$$

and we are done.

Let T_Σ be the formal scheme obtained by gluing the $T_{\sigma^\star}$ according to the intersections in the paving.

Assume now that the abelian part is non trivial. Let $\tilde{G}$ be the formal scheme (which is also algebraic) corresponding to the Raynaud extension constructed in chapter 3. We form the contracted product

$$\tilde{G}\times^T T_\Sigma$$

which exists as formal scheme on $\hat{S}$.
Since by [FvdP]4.2 we have a universal invariant Poincaré bundle on the product $\Omega\times\tilde{G}$, we can control the poles of the universal action via the sections of T_Σ by the definition of $\widetilde{\mathcal{P}}_\Sigma^{log}$.
It is now clear that the functor $\widetilde{\mathcal{P}}_\Sigma^{log}$ is representable by the formal scheme $\tilde{G}\times^T T_\Sigma$.
Furthermore it has a fine log-structure induced by the monoids $\Delta_{\sigma^\star}^\vee$ or more precisely by the monoid

$$(\mathbb{N}\oplus H_1(\Gamma,\mathbb{Z}))\ltimes H_\Sigma^{sat}$$

by the results in [Ol]4.1. As in chapter 1, one has a line bundle $\mathcal{L}_{T_\Sigma}$ on T_Σ. As explained in [FvdP]5.1 the functor $\mathcal{A}$, corresponding to the abelian part of $\tilde{G}$, is representable by an abelian scheme A with a principal polarization

$\mathcal{M}$, whose reduction induces the canonical principal polarization on $\mathrm{Pic}^0_{\tilde{C}/k}$, where $\tilde{C}$ is the normalization of the special fiber of the curve. If

$$q : \tilde{G} \to A$$

denotes the associated morphism and

$$p_1 : \tilde{G} \times T_\Sigma \to \tilde{G}$$

(resp.

$$p_2 : \tilde{G} \times T_\Sigma \to T_\Sigma$$

) are the projections, then the line bundle

$$p_2^* \mathcal{L}_{T_\Sigma} \otimes p_1^* q^* \mathcal{M}$$

descends to a line bundle $\widetilde{\mathcal{L}}$ on

$$\tilde{G} \times^T T_\Sigma$$

The argument we have just made étale locally on A, i.e. where we have trivialized the various torsors described in chapter 1 can be repeated word by word using those torsors. The couple $(\tilde{G} \times^T T_\Sigma, \widetilde{\mathcal{L}})$ is the formal completion of a completely relative model by the results explained in chapter 1 or from the fact that the model we presented here is the formal completion of the so called "standard family" in [Ol]4.1 (see also B.4.2 for a definition) and for the standard family the properties characterizing the Mumford models are checked in [Ol].

□

Assumption: in order to consider semistable sheaves, we choose the polarization ϕ of the following form

$$\phi = \partial(\frac{e(E)}{2}) \tag{4.4}$$

Remark 4.0.11. As we explain in A.2.4 our assumption on the polarization is satisfied in the interesting case in which ϕ is induced by an integral translated of the canonical polarization on the curve. We can also choose anyother polarization of the form

$$\phi \in \frac{1}{2}\partial(\sum_e e) + \partial C_1(\Gamma, \mathbb{Z})$$

Besides this condition gives us the characterization of $Del_\phi(H^1(\Gamma, \mathbb{R}))$ as given in 2.0.10.

Remark 4.0.12. It is a good place here to remember the following fact which may bring to confusion. According to our choice of ϕ the decomposition $Del_\phi(H^1(\Gamma,\mathbb{R}))$ is a translated of $B(Vor_A(H_1(\Gamma,\mathbb{R}))$, but the Mumford models are constructed by taking cones over $-2B(Vor_A(H_1(\Gamma,\mathbb{R}))$. This means that when we pass from the section to the semistable polyhedra there is always a factor -2 and a translation which have to be remembered.
This plays a role when one considers the degrees of the sheaves, as we will see in the examples.

Under this assumption we consider Σ-bounded sheaves where Σ is the decomposition induced by the Delaunai cells contructed using B, the canonical pairing on $H_1(\Gamma,\mathbb{R})$.

We consider now, up to units, the bilinear form

$$b(x,y) = \pi^{2B(x,y)}$$

The units we are not considering are of the form $\beta(x,y)$ where β is bilinear in x and y and with value in the residue field. Different choices of this reduction may and will produce different isomorphism classes for our models. However for our construction this does not play a relevant role because we are only interested in assuring that the reductions are possible.

This bilinear form induces an action on the functor $\tilde{\mathcal{P}}^{log}_\Sigma$. Indeed we know that the action of the periods covers the morphism

$$c^t : H_1(\Gamma,\mathbb{Z}) \to A$$

defining the dual Raynaud extension (see diagram 1.3).
We want to explicitly describe the special fiber of this morphism. The special fiber of the abelian part A corresponds to

$$Pic^0_{C^{norm}/k}$$

and the morphism

$$c^t : H_1(\Gamma,\mathbb{Z}) \to Pic^0_{C^{norm}/k}$$

is induced by the morphism

$$C_1(\Gamma,\mathbb{Z}) \to Pic_{C^{norm}/k}$$

given as follows. On fixes an orientation on the graph Γ and the previous morphism is the $\mathbb{Z}$-linear extensions of the map which sends an edge e to the line bundle

$$\mathcal{O}_{C^{norm}}(s(e) - t(e))$$

where $s(e)$ (resp. $t(e)$) is the point in C^{norm} corresponding to the source (resp. the target) of e w.r.t. the chosen orientation.

Furthermore, since the sections s live in the torsors space, in order to avoid a difference in the signs, we fix the homomorphism c such that $c(x)$ corresponds to the negative push-out $\underline{\mathcal{O}}_{-x}$ and also $c^t(x)$ corresponds to its negative point because of the relation

$$\lambda_{Pic_{C_0^{norm}/k}} \circ c^t = c$$

This implies that when we translate the cones via a $y \in H_1(\Gamma, \mathbb{Z})$ then we take the action induced by c^t with the minus sign.

Hence if g denotes the covering morphism

$$g : \Omega \to \widehat{C}$$

then given $y \in H_1(\Gamma, \mathbb{Z})$, a couple

$$(L, s)$$

is sent to the couple

$$(L \otimes g^* c^t(y), y_* s)$$

where

$$y_* s(x) := s(x) b(x, y)^{-1}$$

and also $c^t(y)$ was chosen corresponding to $-y$. In this way we make the signs compatible.

Definition 4.0.13. Under the previous assumption on Σ, we define $\widetilde{\mathcal{P}}^{\phi}$ as the sheaf $\widetilde{\mathcal{P}}^{log}_{\Sigma}$ and $\mathcal{P}^{\phi}$ the sheaf $\widetilde{\mathcal{P}}^{\phi}/H_1(\Gamma, \mathbb{Z})$.

Denote with $\mathcal{P}^{\phi}_0$ the special fiber of $\mathcal{P}^{\phi}$. As functor it can be described as follows.
We take for a formal log-scheme $(\mathcal{U}, M_{\mathcal{U}}) \to (\hat{S}, M_{\hat{S}})$ with reduction U_0 the sections which factorize through the composition

$$U_0 \to \mathcal{U} \to \mathcal{P}^{\phi}$$

We can also consider $\mathcal{P}^{\phi}_0$ as functor on the category of schemes over the special fiber S_0 as follows. For every morphism of schemes $u : U \to S_0$ we take the log-structure on U defined by the pullback $u^* M_{S_0}$ and we consider

$$\mathcal{P}^{\phi}_0(U) := \mathcal{P}^{\phi}(U, u^* M_{S_0})$$

the same holds for its covering $\widetilde{\mathcal{P}}^{\phi}_0$. We can now state our theorem.

Theorem 4.0.14. *Given a polarization ϕ satisfying the assumption given by the equality 4.4 then there exists a natural transformation*

$$\beta : \mathcal{P}_0^{\phi} \to Jac_{C_0}^{\phi}$$

of functors over Sch/S_0, which induces an isomorphism of the corresponding schemes.

The rest of the chapter is devoted to the proof of this theorem.

The first reduction consists of finding β such that the induced map, at scheme level, is a birational morphism

$$\beta : (\mathcal{P}_0^{\phi})^{red} \to Jac_{C_0}^{\phi}$$

which is also a universal homeomorphism.

Remark 4.0.15. As remarked at the end of Appendix B.4.1, the cells of the Delaunay decomposition obtained from the matrix of a graph are generating, hence by proposition 1.6.13 we can assume that $\mathcal{P}_0^{\phi}$ is already reduced.

In order to explain this first reduction we recall a definition.

Definition 4.0.16. A reduced scheme X is called *seminormal* if for every reduced scheme Y and finite bijective morphism $f : Y \to X$ such that for every $y \in Y$ the morphism induces an isomorphism on the residue fields $k(y) \cong k(f(y))$, then the morphism f is an isomorphism.

It is already known from [AL04]5.1 that the scheme $Jac_{C_0}^{\phi}$ is seminormal. In order to make our proof working also in positive characteristic, we actually need a stronger condition, which was introduced by Andreotti and Bombieri.

Definition 4.0.17 ([An-B])**.** A reduced scheme X over a field k is called *weakly normal* if any finite and birational morphism $f : Y \to X$, where Y is a reduced k-scheme, which is a universal homoeomorphism is an isomorphism.

The difference between the two definitions is that in the second one we require that the morphism can induce a purely inseparable extension on the residue fields.
However the property of being a universal homeomorphism is easier to check in our case.

In particular if we show that $Jac_{C_0}^{\phi}$ is weakly normal, then our reduction step, once proved, would imply that β is an isomorphism between the corresponding schemes.

Proposition 4.0.18. *The coarse moduli space $Jac_{C_0}^{\phi}$ is weakly normal and Gorenstein.*

Proof. The proof is given in Appendix E. $\square$

We consider now the Raynaud's extension via formal schemes given by the Jacobian

$$0 \to T \to \tilde{G} \to A \to 0$$

From chapter 3 we know that $\tilde{G}$ is represented by an algebraic scheme over S and that the associated formal functor classifies pairs (L, s) where L is an invariant line bundle on the universal covering $\Omega \to \hat{C}$ which has degree zero on each component and $s \in Hom(\mathcal{G}, \mathbb{G}m(K)^0)$ describes the action.

Since $Jac^{\phi}_{C_0}$ is a union of orbits, it suffices to explain our construction for an orbit $O(\Delta_\sigma)$ (definition given in 4.3). We want to consider the schemes

$$\tilde{G} \times^T O(\Delta_\sigma)$$

as functors.
Our aim is to construct from the functorial points of the $H_1(\Gamma, \mathbb{Z})$-orbit

$$\Big(\bigcup_{y \in H_1(\Gamma, \mathbb{Z})} \tilde{G} \times^T O(\Delta_{\sigma+y}) \Big) \to A$$

a presentation on C_0 and show that the associated kernel is ϕ-semistable and that it does not depend on the periods.

Actually the degree of these sheaves has to be corrected in order to prove the ϕ-semistability, but we can do it in a canonical way, which only depends on the cells.

Definition 4.0.19. For every irreducible component C_v of the special fiber C_0 of $\hat{C}$ we fix a line bundle M_v on C_0 such that its pull-back modulo torsion to the normalization has degree one on C_v and it is trivial on the other components.

Definition 4.0.20. For any $\sigma \in Del_\phi(H^1(\Gamma, \mathbb{R}))$ we take the unique cell $\tilde{D}_\sigma \subset C^1(\Gamma, \mathbb{R})$ in K^0_ϕ (definition in 2.0.7) which surjects to σ.
Assume that $\sigma \in Del_\phi(H^1(\Gamma, \mathbb{R}))$ is a translated via a $c \in H_1(\Gamma, \mathbb{Z})$ of a cell through the origin. Write

$$\tilde{D}_\sigma := w_\sigma + \left\{ \sum_{e \in E^+_\sigma} e t_e + \sum_{e \in E^-_\sigma} (-e) t_e \mid t_e \in [0, 1] \right\}$$

with w_σ a zero dimensional cell and define

$$w'_\sigma = w_\sigma - c$$

Let us take a formal scheme U over $\hat{S}$, with special fiber U_0, and a section

$$(L, s) \in \tilde{G} \times^T O(\Delta_\sigma)(U)$$

Again here we imagine U as log-scheme where the log-structure is induced by pulling back the log-structure on $\hat{S}$.
Using the action we can algebraize the formal line bundle L to a line bundle

$$L/\mathcal{G}$$

over $\widehat{C}$ hence we get a line bundle on the special fiber C_0 which we call $I_{(L,s)}$.

The subgroup X_σ, where the glueing action is described by s, determines, up to a global invertible function on $\prod_{v \in V} C_v$, a subset $E' \subset E$.

Furthermore from chapter 2 and by the duality between Delaunay and Voronoi cells, we can choose E' complementary to $E_\sigma^+ \coprod E_\sigma^-$.

Hence if $\tilde{I}_{(L,s)}$ denotes the pull-back modulo torsion of $I_{(L,s)}$ to the normalization of C_0 and we consider $F_{(L,s)}$ be the kernel of the presentation

$$f_s : \tilde{I}_{(L,s)} \to \bigoplus_{e \in E'} k_{U_0} \to 0$$

obtained by composing with the projection

$$\begin{tikzcd} \tilde{I}_{(L,s)} \arrow[r] \arrow[rr, bend right, "f_s"'] & \bigoplus_{e \in E} k_{U_0} \arrow[r] & \bigoplus_{e \in E'} k_{U_0} \arrow[r] & 0 \end{tikzcd}$$

then it depends only on the couple (L, s).

Actually more is true, namely if fix the invariant sheaf L but we start with an action s, corresponding to a cell σ, which is a translated via a $y \in H_1(\Gamma, \mathbb{Z})$ of a section s_0, corresponding to a cell cell σ_0, then the coefficients of $\bar{\gamma}$ in X_σ, i.e. w.r.t. the differences of the vertices of the cell σ, and the coefficient in X_{σ_0} are the same, by translation invariance. Multiplication via $b(-, y)$ sends s_0 to the section s. Furthermore the action given by $b(-, y)$ also extends to the special fiber.
Using the fact that the period action covers the morphism

$$c^t : H_1(\Gamma, \mathbb{Z}) \to Pic^0_{C_0^{norm}/k}$$

one obtains that the periods produce isomorphisms of sheaves with action

$$(L \otimes g^* c^t(y), s_0) \cong (L, s)$$

hence we get isomorphisms

$$I_{(L\otimes c^t(x),s_0)} \cong I_{(L,s)}$$

and also the kernels $F_{(L\otimes c^t(x),s_0)}$ and $F_{(L,s)}$ are isomorphic because we can find an invertible global function in $\prod_v C_v$ such that the subsets corresponding to E_1 are the same.
In other words the sheaf $F_{(L,s)}$ only depends on the corresponding point in

$$\widetilde{\mathcal{P}}_0^{\phi}/H_1(\Gamma,\mathbb{Z}) = \mathcal{P}_0^{\phi}$$

There is still something to do in order that $F_{(L,s)}$ can be mapped to $Jac^{\phi}_{C_0}$. Namely we have to correct the degree in order that this is zero and in order to obtain semistability. We know how to do this using the theory of chapter 2.

For every component C_v of and every node e through C_v, one of the endpoints $s(e)$ or $t(e)$ belongs to C_v. We choose an orientation on the graph Γ which also determines whether $s(e)$ or $t(e)$ belongs to C_v.

We also have a subset $E_\sigma^+ \coprod E_\sigma^- \subset E$, which is complementary to E'.

We define the divisor D_σ, on the pullback to U_0 of the normalization $\tilde{C}_0$, as the base change to U_0 of the divisor which on the component C_v is $-s(e)$ (resp. $-t(e)$), if e lies in E_σ^+ (resp. E_σ^-), and the zero divisor if it lies in E'.

Furthermore we have also an element

$$\partial w_\sigma = \partial w'_\sigma \in C_0(\Gamma,\mathbb{Z})$$

given by the cell $\tilde{D}_\sigma$, and for any $v \in V$ a line bundle M_v, such that its pullback $\tilde{M}_v$ modulo torsion to the normalization is of degree one on C_v and trivial on the other irreducible components.
The presentation

$$\tilde{I}_{(L,s)} \otimes \Big(\prod_{v\in V} \tilde{M}_{v,U_0}^{(\partial w_\sigma)_v}\Big) \otimes \mathcal{O}(D_\sigma) \xrightarrow{f_s} \prod_{e\in E'} k_{U_0} \to 0$$

has kernel which is torsion free. Call this kernel

$$I_{(L,s)}$$

Observe that the sum of the degree of $\mathcal{O}(D_\sigma)$ and of minus the cardinality E' is $-|E|$. Hence by construction we have

$$\deg_{C_{0,u}} I_{(L,s)} = 0$$

for any closed point $u \in U$. Besides $I_{(L,s)}$ is free precisely at the nodes corresponding to the edges which are not in $E_\sigma^+ \cup E_\sigma^-$ and the corresponding polyhedron is ϕ-semistable by the explicit description we gave in chapter 2 .

Up to now we have defined a morphism from the covering $\tilde{\mathcal{P}}_0^\phi$ to the sheafification of the functor of ϕ-semistable sheaves, which is denoted by W_ϕ in [OS]§12.

There is a morphism of functors $w_\phi : W_\phi \to Jac_{C_0}^\phi$ obtained by contracting ϕ-equivalence classes ([OS] 12.14).

The graded equivalence class of $I_{(L,s)}$ gives the desired point in $Jac_{C_0}^\phi(U_0)$. Since the morphism w_ϕ identifies sheaves which are graded equivalent, we have that $Jac_{C_0}^\phi(U_0)$ only sees the cells up to $H_1(\Gamma, \mathbb{Z})$-translation by the theory of chapter 2.
In our construction when we take presentations which are translated by the periods, we change the divisor D_σ by a translation via $c^t(y)$ where $y \in H_1(\Gamma, \mathbb{Z})$. This means that we change

$$\tilde{D}_\sigma \subset C_1(\Gamma, \mathbb{R})$$

by translating it via an element $y \in H_1(\Gamma, \mathbb{Z})$. The line bundles $\tilde{M}_v^{(\partial w_\sigma)_v}$ do not change because

$$\partial(w_\sigma) = \partial(w_\sigma \pm y)$$

Furthermore the morphisms f_s and $f_{y_* s}$ give presentations with isomorphic glueing and divisors which differs by $H_1(\Gamma, \mathbb{Z})$-translation. This gives the same point in $Jac_{C_0}^\phi$.
Hence we finally get that β_{U_0} descends to a morphism

$$\beta_{U_0} : \mathcal{P}_0^\phi(U_0) \to Jac_{C_0}^\phi(U_0)$$

In this way we complete the definition of β_{U_0} functorially in U_0, where the functoriality is given by pulling back the couples (L, s).

Observe that if two presentations $I_{(L_1,s_1)}$ and $I_{(L_2,s_2)}$ over $Spec(\bar{L}) \to S$ with $\bar{L}$ algebraically closed field, we obtain from our construction, are graded-equivalent then this means that the associated polyhedra are translated via the $H_1(\Gamma, \mathbb{Z})$-action, because we have chosen representative of $\tilde{D}_\sigma$ in $K_\phi^0(\Gamma)$. If the associated points in $Jac_{C_0}^\phi(\bar{L})$ coincide then the couples (L_1, s_1) and (L_2, s_2) we started with have to be equivalent in the sense that there exists $y \in H_1(\Gamma, \mathbb{Z})$ such that

$$L_1 \cong L_2 \otimes g^* c^t(y)$$

and s_1 is a translated by y of s_2.
Hence for every morphism $Spec(\bar{L}) \to S_0$, with $\bar{L}$ algebraically closed, the

morphism

$$\beta_{\bar{L}} : \mathcal{P}_0^{\phi}(\bar{L}) \to Jac_{C_0}^{\phi}(\bar{L})$$

is injective.
From the injectivity over algebraically closed fields follows that β is universally injective.
The explicit description of $Jac_{C_0}^{\phi}$ via toric orbits we gave in 2.0.11 and 2.0.12 tells us that this correspondence at the scheme level is also bijective and it preserves the toric orbits. Namely it is clear that it preserves the toric orbits because $\mathcal{P}_0^{\phi}$ and $Jac_{C_0}^{\phi}$ are constructed using the same polyhedral decomposition by our assumption on ϕ.
The characters of the corresponding orbits are given by differences of the Delaunay vectors in the M-space and the Delaunay cells are the same for both source and target space of β.
To check the surjectivity between schemes it is enough to check it over fields. Over a field L with $Spec(L) \to S_0$ the morphism β_L is a surjective map of schemes by the description of the toric orbits.

Hence the morphism between the corresponding schemes is bijective.

Furthermore it is finite, because quasi-finite and proper.

We want to show that this is also an universal homeomorphism and that it is birational.

Since the morphism β respects the orbits, it also respects the generic points, because these are in bijection with the maximal dimensional orbits.
Indeed the maximal dimensional orbits give sheaves which are actually stable and over the stable locus the functors W_{ϕ} and $Jac_{C_0}^{\phi}$ are isomorphic by [OS]12.14(i).
The structure sheaves at the generic points both for the target and for the source space of β are isomorphic to the structure sheaf at the generic point of the semiabelian scheme $\mathrm{Pic}^0_{C_0/k}$.

It follows that β is birational.

In order to show that β induces a universal homeomorphism we need to check that it is universally bijective and universally closed. We already showed that β is universally injective. In particular it is also bijective after any base change, because surjectivity is preserved under base change. Since it is also proper this means that after any base change we have a closed bijective map.

In this way we see that β is a universal homeomorphism.

The weak normality of $Jac^{\phi}_{C_0}$ we proved in Proposition 4.0.18 shows that β induces an isomorphism of the corresponding schemes and the proof of the theorem 4.0.14 is now complete.

This construction exhibits a way to produce sheaves which are generically line bundles and that specialize to semistable ones in a "separated way", since the Mumford models are separated.

Due to the degree correction we made on the presentations, it seems that, to get separated functors, one has not to fix the total degree of the sheaf but consider appropriate combinations on the multidegrees.

Remark 4.0.21. In this construction the couples (L, s) can be interpreted as logarithmic torsors, namely torsors under the group sheaf $g^*M^{gp}_{\widehat{S}}$ where g is the structural morphism $g : \Omega \to \widehat{S}$.
These are also invariant by the $\mathcal{G}$-action, hence we end up with $f^*M^{gp}_{\hat{S}}$-torsors, generalizing the construction of Kajiwara we recalled in section 1.7. This also suggests that a possible separated functor could be searched inside the logarithmic Picard functor.

4.1 Examples

In the following examples we describe the $\widehat{S}$-points and the $Spec(k)$-points in our construction.

4.1.1 The Tate curve

Let us consider $C \to S$ a smooth genus 1 curve degenerating to a rational curve self-intersecting in one node.
In this case $C^1(\Gamma, \mathbb{R}) = H^1(\Gamma, \mathbb{R}) = \mathbb{R}$ and the polarizations are all equivalent. The quadratic form is the standard one

$$A: \qquad \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longrightarrow x^2$$

Let $S = Spec(R)$, $\pi \in R$ be a uniformizer and k be the residue field.

If we fix a valuation on R, induced by π, and rigidifications, then A can be thought as the valuation of the cubical trivialization of the $\mathbb{G}m$-torsor $i^*\tilde{\mathcal{L}}_\eta$ over $\mathbb{Z}$ and b as the trivialization of the biextension attached to the Poincaré bundle.
The morphisms c and c^t are trivial.

There is only one Namikawa decomposition

$$Del(H^1(\Gamma, \mathbb{Z}))$$

It has zero dimensional cells, corresponding to $\mathbb{Z} \subset \mathbb{R}$, one dimensional cells, given by the segments $[n, n+1]$ for $n \in \mathbb{Z}$, and it is the translated of the Voronoi via $\frac{1}{2}$.
We recall here again that there is always -2 factor when one passes from one construction to the other. Namely for the Mumford construction we need two take $-2B(Vor(H_1(\Gamma, \mathbb{R}))$ and for the Namikawa decomposition we have

$$Del(H^1(\Gamma, \mathbb{Z})) = z + B(Vor(H_1(\Gamma, \mathbb{R}))$$

The fundamental group of the graph $\mathcal{G}$ is $\mathbb{Z}$.

Let us consider first the zero dimensional cells. These correspond to one dimensional orbits. We pick for example

$$\{0\} \in Del_\phi(H^1(\Gamma, \mathbb{R}))$$

Take a section

$$s_0 \in O(\Delta_{\{\frac{1}{2}\}})(\hat{S})$$

The point $\frac{1}{2}$ comes from the fact that we are translating via z.
Write $s_0 = t\pi^\delta$, $t \in \mathbb{G}m(S)$ and δ is a power which has to be determined. We want δ of the form $-2B(\alpha)$ for some α with rational coefficient. Since $z = \frac{1}{2}$, in this case we need to take $\alpha = \frac{1}{2}$. We want to find the associated dual Delaunai cell. Let v be the valuation induced by π. We have to solve the equation

$$v((t\pi^\delta)^x a(x)) = -B(\frac{1}{2}, x) + A(x) = -x + x^2 = 0$$

so that we find that $x = 1$.

We compute, up to units,

$$s_0(1)a(1) = t \in \mathbb{G}m(k)$$

In general we have to take into account also the reduction of $a(1)$ modulo π. If we take any other zero dimensional cell given by $\{n\}$ we have

$$\alpha = \frac{1}{2} + n$$

Using our rules, if s denotes the corresponding section, then we have by definition

$$0 = v(s((1+n)-n)a((1+n)-n)b((1+n)-n,n)) = v((s(1)b(1,n))a(1))$$

so that $s = (-n)_* s_0$ for some s_0 is $\mathcal{O}(\Delta_{\{\frac{1}{2}\}})(\hat{S})$.
In particular the previous expression is not zero on the reduction modulo π.
Let us construct the presentations. We take

$$\tilde{D}_\sigma = \{0\}$$

According to our rules, since the support of $\{0\}$ is empty and the barycenter is zero we need a line bundle of degree zero the irreducible component of the normalization of the special fiber. The normalization is isomorphic to $\mathbb{P}^1$, hence we consider $\mathcal{O}$ the trivial line bundle on $\mathbb{P}^1$. We take the following presentation

$$\mathcal{O} \xrightarrow{f_s} k \to 0$$

where f_s is induced by the glueing datum corresponding to $s_0(1)a(1) = t$.
The kernel of this presentation $I_{(\mathcal{O},s)}$ is clearly a line bundle of degree zero and by varying t we have produced a $\mathbb{G}m$-orbit of them.

The translated by the periods of the previous one are given by

$$\mathcal{O}\Big(- y\big(s(e) - t(e)\big)\Big) \xrightarrow{f_{y*s}} k \to 0$$

for $y \in H_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$.
We recall that we have chosen the "negative" c^t in order to obtain compatible signs.
Changing the presentations in the $H_1(\Gamma, \mathbb{Z})$-orbit of s, these sheaves are mapped to the same point in $Jac^{\phi}_{C_0}(k)$.

Let us take a one dimensional cell, for example

$$\sigma = (0, 1) \in Del(H^1(\Gamma, \mathbb{R}))$$

Since again we have to translate by $\frac{1}{2}$, we get the inequality

$$-\frac{1}{2} < \alpha < \frac{1}{2}$$

which implies

$$v(s(x)a(x)) = |\alpha - x|^2 - |\alpha|^2 > 0 \quad \forall x \in \mathbb{Z}$$

The support is given by e, the only edge in Γ. The cell $\tilde{D}_\sigma$ is given for example by

$$\tilde{D}_\sigma = \{et_e \mid t_e \in (0, 1)\}$$

compatibly with the fact that as zero dimensional representative we have chosen $\{0\}$.
Since

$$v(s(1)a(1)) > 0$$

we get a presentation of the form

$$\mathcal{O}(-s(e)) \to 0 \tag{4.5}$$

We have

$$\chi(I_{(\mathcal{O},s)}) - \chi(\mathcal{O}_{C_0}) = \deg(\mathcal{O}) - \deg(-s(e)) + 1 = 0$$

and by construction the $H_1(\Gamma, \mathbb{Z})$-orbit of the sheaf $I_{(\mathcal{O},s)}$ is the stable sheaf corresponding to the one point we have to add to compactify the generalized Jacobian.

We could also have taken the cell

$$\sigma_{-1} = (-1, 0)$$

which is a translated via -1 of σ.
This cell has

$$\tilde{D}_{\sigma_{-1}} = \{-et_e \mid t_e \in (0,1)\}$$

This gives us the presentation

$$\mathcal{O}(-t(e)) \to 0 \tag{4.6}$$

The action of the periods for these zero dimensional cells is given by multiplication via

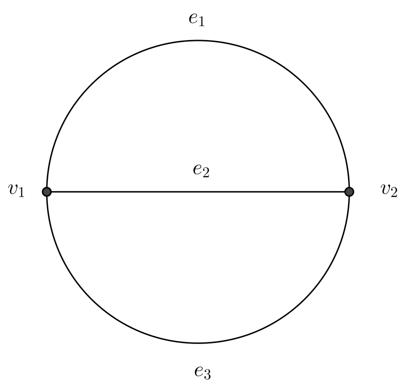
$$c^t(y) = \mathcal{O}\Big(-y\big(s(e) - t(e)\big)\Big)$$

It is a fortunate case that in this example the line bundles $c^t(y)$ are all trivial, in general they provide a non trivial action on the abelian part.

In particular the presentation 4.5 is sent to the presentation 4.6 by multiplication via $c^t(-1)$.
We also know that the image of the presentations 4.5 and 4.6 in $Jac^{\phi}_{C_0}$ are identified to a point.
In this way we see the bijectivity of β.

4.1.2 A two components curve

Let us pick the example where the special fiber of $C \to S = Spec(R)$ is reducible, with 2 rational components meeting in 3 points.
We have two vertices v_1, v_2 and three edges e_1, e_2, e_3. We fix the orientation such that all the edges point to v_2.



Again π denotes a uniformizer in R and k is the residue field of R.
Furthermore we choose a polarization which is a translate of the canonical polarization, namely

$$\phi = \frac{1}{2}(v_1 - v_2) \in C_0(\Gamma, \mathbb{R})$$

The quadratic form on $H_1(\Gamma, \mathbb{R})$ is given by the matrix

$$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The image of the Voronoi decomposition under B is obtained by integral translation of the non regular hexagon, whose set of vertices is the following

$$\{(-1,0),(0,-1),(-1,1),(1,0),(0,1),(1,-1)\}$$

The bounded actions live in -2 times the previous one, i.e. integral translated of the hexagon whose vertices are

$$\{(2,0),(0,2),(2,-2),(-2,0),(0,-2),(-2,2)\}$$

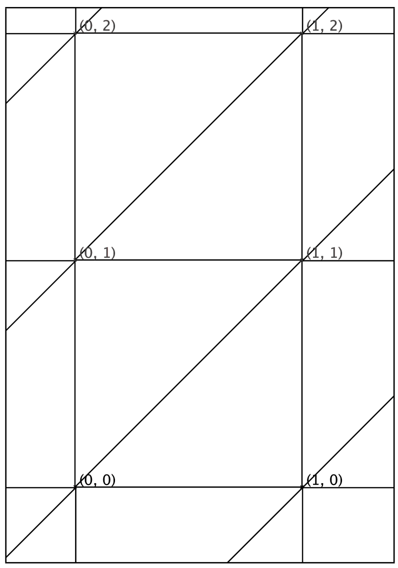
This hexagon gives us the exponents of the uniformizers we have to take to construct the actions.
The Delaunay cells in the M space are given by translating the two triangles T_1 and T_2 whose set of vertices are

$$\{(0,0),(1,1),(0,1)\}$$

and

$$\{(0,0),(1,1),(1,0)\}$$

as in the following picture.



In order to obtain $Del_\phi(H^1(\Gamma, \mathbb{R}))$ we need to do a little bit more. Namely we need to find a vector $x \in C_1(\Gamma, \mathbb{Z})$ such that

$$\phi = \partial(\frac{e(E)}{2} + x)$$

We can take $x = (-1, 0, 0)$. If $p : C^1(\Gamma, \mathbb{R}) \to H^1(\Gamma, \mathbb{R})$ denotes the canonical map, then we know from Proposition B.3.4 that we have

$$p(\frac{e(E)}{2} + x) + B(Vor(H_1(\Gamma, \mathbb{R}))) = (-1, 0) + B(Vor(H_1(\Gamma, \mathbb{R})))$$

This decomposition is $Del_\phi(H^1(\Gamma, \mathbb{R}))$.
Let us call

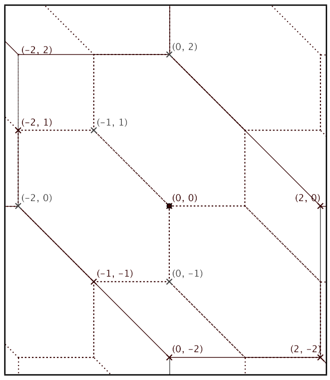
$$c = p(\frac{e(E)}{2} + x)$$

From this description we see that the decomposition $Del_\phi(H^1(\Gamma, \mathbb{R}))$ is obtained by $B(H_1(\Gamma, \mathbb{Z}))$-translation of the non regular hexagon whose set of vertices is the following

$$\{(-2, 0), (-1, -1), (-2, 1), (0, 0), (-1, 1), (0, -1)\}$$

$Del_\phi(H^1(\Gamma, \mathbb{R}))$ -------

$-2B(Vor)$ ———



The special fiber of the corresponding Mumford model $\tilde{\mathcal{P}}^\phi$ is obtained by taking copies of $\mathbb{P}^2$, parametrized by $\mathbb{Z}^2$, gluing them along the coordinates lines and then gluing the vertices.
Let us see how to obtain a $\mathbb{G}m^2$-orbit. We take for example the torus with character group generated by $(0, 1)$ and $(1, 1)$. The associated Vornoi dual is given by the zero dimensional cell $(\frac{1}{3}, \frac{2}{3})$. This gives us the cell

$$\sigma = \{B(\frac{1}{3}, \frac{2}{3}) + c\} = \{(-1, 1)\} \in Del_\phi(H^1(\Gamma, \mathbb{R}))$$

We get $\alpha = (\alpha_1, \alpha_2) = (\frac{1}{3}, \frac{2}{3})$.
The section corresponding to this α is given by $s = (t_1, t_2\pi^{-2})$, where in the exponent we take $-2B(\alpha)$ and $t_1, t_2 \in k^\times$. As already said the corresponding Delaunay cell in the $M = H_1(\Gamma, \mathbb{R})$-space is 2 dimensional and it is given by the integral vectors $(0, 1)$ and $(1, 1)$ and we obtain

$$\begin{aligned} s(0,1)a(0,1) &= t_2\pi^{-2+2} = t_2 \\ s(1,1)a(1,1) &= t_1t_2\pi^{-2+2-2+2} = t_1t_2 \end{aligned} \tag{4.7}$$

These are the points in the torus we have and the corresponding action sends $(k,l) \in \mathbb{Z}^2 \cong X_\sigma$ to

$$(s(0,1)a(0,1))^k(s(1,1)a(1,1))^l = t_2^k(t_1t_2)^l \tag{4.8}$$

For the multidegree we proceed as follows. We need to lift $\{(-1,1)\}$ to $C^1(\Gamma,\mathbb{R})$ for example by using

$$\tilde{D}_\sigma = \{w_\sigma\} = \{(0,1,0)\}$$

Observe that

$$(1,-1) = \partial((0,1,0))$$

Hence we see that we need a line bundle of multidegree $(1,-1)$. Let us fix a line bundle M_1 (resp. M_2) on C_0 such that the pullback $\tilde{M}_1$ (resp. $\tilde{M}_2$) of M_1 (resp. M_2) modulo torsion to the normalization has degree one on the component v_1 (resp. v_2) and it is trivial on v_2 (resp. v_1)

Since the irreducible components are rationals when the have to take the trivial invariant line bundle $\mathcal{O}_\Omega$. A presentation for the couple $(\mathcal{O}_\Omega, s)$ is now given by

$$\tilde{M}_1 \otimes \tilde{M}_2^{-1} \stackrel{f_s}{\to} \bigoplus_{e\in E} k \to 0$$

where f_s is determined by equation 4.8.

A similar computation shows that the other $\mathbb{G}m^2$-orbit, whose characters are generated by the vectors $\{(1,1),(1,0)\}$, corresponds to the zero dimensional cell $(0,0) \in Del_\phi(H^1(\Gamma,\mathbb{R}))$. This lifts for example to

$$(0,0,0)$$

and we take line bundles of multidegree $(0,0)$. The actions are obtained by taking

$$s_0 = (t_1\pi^{-2}, t_2)$$

$t_i \in k^\times$.

In order to see that these are the only distinct orbits we need to see that the sheaves associated with these orbits are not ϕ-equivalent.

Using the general theory of section 2 this follows from the fact that the difference

$$(-1,1) - (0,0)$$

cannot be written as Bv for some vector $v \in H_1(\Gamma,\mathbb{Z})$. Furthermore any other vertex of the hexagon can be translated either into $(-1,1)$ or into $(0,0)$ as follows:

$$(-1,-1) = B(-1,-1)^t + (0,0) \quad (-2,1) = B(-1,0)^t + (0,0)$$

$$(-2,0) = B(-1,-1)^t + (-1,1) \quad (0,-1) = B(0,-1)^t + (-1,1)$$

In this way we see the ϕ-equivalence classes.

Let us see that using the periods we get only one representative for each graded equivalence class, so that the morphism β is injective on S_0 points. Consider for example the cell $(-1,-1)$. Using the previous computation we see that this is a translated via $(-1,-1)$ of the torus corresponding to $(0,0)$. Computing $-2B(\alpha)$ for this cell we get the sections

$$s_1 = (t_1, t_2\pi^2)$$

$t_i \in k^\times$.
Recalling that b is two times the bilinear form of a, we see that $b((-1,-1))$ is up to units (π^{-2}, π^{-2}), hence

$$s_1 = (t_1, t_2\pi^2) = (t_1\pi^{-2+2}, t_2\pi^{0+2}) = s_0 b((-1,-1))^{-1}$$

Let us compute the action we defined on the generating characters. From our theory we know that we need to translate by $b((-1,-1))$ and the same holds for the Delaunay cells where the characters lives. In this way we get

$$s_1(1,0)a(1,0)b((1,0),(-1,-1)) = t_1\pi^{2-2} = t_1$$

and

$$s_1(1,1)a(1,1)b((1,1),(-1,-1)) = t_1t_2\pi^{2-2} = t_1t_2$$

hence we see that it is isomorphic to the action defined by s_0. They are not the same because we only considered b up to units.
We need to find the divisor. We can lift $(-1,-1)$ to the point

$$\tilde{D}_{\{(-1,-1)\}} = \{(-1,0,1)\} \subset C^1(\Gamma, \mathbb{R})$$

observe that if

$$i : H_1(\Gamma, \mathbb{R}) \to C_1(\Gamma, \mathbb{R}) \cong C^1(\Gamma, \mathbb{R})$$

is the inclusion then

$$\tilde{D}_{\{(-1,-1)\}} = \tilde{D}_{\{(0,0)\}} + i((-1,-1))$$

The line bundle

$$\mathcal{O}_{C_0^{norm}}\Big(- \big(s(e_1) - t(e_1) - s(e_3) + t(e_3)\big)\Big)$$

corresponds to

$$c^t(-1,-1)$$

If we define $y = (-1,-1)$ then the kernel of the presentation

$$(\mathcal{O}_{C_0^{norm}}) \otimes c^t(y) \xrightarrow{f_{y*}s_0} \bigoplus_{e\in E} k \to 0$$

is graded equivalent to the kernel of

$$\mathcal{O}_{C_0^{norm}} \xrightarrow{f_{s_1}} \bigoplus_{e\in E} k \to 0$$

Hence the couple

$$(\mathcal{O}_\Omega, s_0)$$

and the couple

$$(\mathcal{O}_\Omega, s_1)$$

are mapped to the same point in $Jac^{\phi}_{C_0}(k)$ if and only if the units t_i's correspond, because the associated polyhedra are translated by an element of $H_1(\Gamma, \mathbb{Z})$.
Furthermore under this condition the couples go to the same point in $\mathcal{P}^{\phi}_0(k)$, because they are in the same $H_1(\Gamma, \mathbb{Z})$-orbit.

We want to consider now a one dimensional orbit. Let us take for example the orbit corresponding to

$$-2B(\alpha) \in (0,-2) + t(-2,2) \quad t \in (0,1)$$

We take the sections of the form $s = (t_1\pi^{\alpha}, t_2\pi^{-2-\alpha})$ with $t_i \in k^{\times}$ and α in the open interval $(-2,0)$.
The associated Delaunay cell in the M space is one dimensional, given by the vector $(1,1)$ and our one dimensional torus is given by

$$s(1,1)a(1,1) = t_1t_2\pi^{\alpha-2-\alpha+2} = t_1t_2 \in k^{\times}$$

Namely this gives us the homomorphism

$$\mathbb{Z} \ni l \to (s(1,1)a(1,1))^l = (t_1t_2)^l$$

The corresponding cell in $Del_\phi(H^1(\Gamma, \mathbb{R}))$ is given by

$$\sigma = (0,0) + \{t(-1,1) \mid t \in [0,1]\}$$

We take a lift of this cell in $C^1(\Gamma, \mathbb{R})$ which is compatible with the representatives we have chosen for $(-1,1)$ and $(0,0)$. Let us consider

$$\tilde{D}_\sigma = (0,0,0) + \{e_2 t \mid t \in [0,1]\}$$

Since the source of e_2 belongs to C_{v_1} then we get presentations

$$\mathcal{O}_{v_1}(-s(e_2)) \times \mathcal{O}_{v_2} \xrightarrow{f_s} k_{e_1} \oplus k_{e_3} \tag{4.9}$$

and by varying the action s and taking the kernel of the presentations we obtain a $\mathbb{G}m$-orbit.
Observe that we could also have taken

$$\tilde{D}_\sigma = (0,1,0) + \{-e_2 t \mid t \in [0,1]\}$$

and with this choice the presentations have form

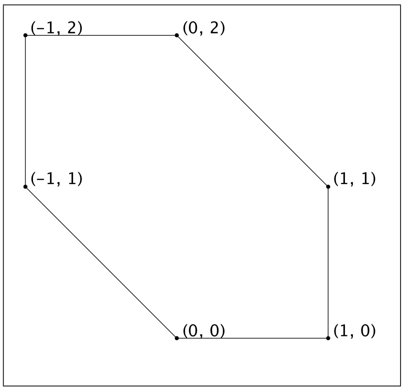
$$\tilde{M}_1 \otimes \tilde{M}_2^{-1}(-t(e_2)) \xrightarrow{f_{s_1}} k_1 \oplus k_3 \to 0 \tag{4.10}$$

The sheaves $(\mathcal{O}_\Omega, s)$ and $(\mathcal{O}_\Omega, s_1)$ give the same point in $\mathcal{P}_0^\phi(k)$ because

$$s(1,1)a(1,1) = t_1 t_2 a(\bar{1},1) = s_1(1,1)a(1,1)$$

and the kernel of the presentation 4.9 and 4.10 are also mapped to the same point in $Jac_{C_0}^\phi(k)$ by section 2 or by [OS] 13.2.

The other orbits are computed analagously.
Let us see how the greded equivalence works on the zero dimensional orbits. In our construction we consider only the orbits corresponding to the translated of the cell σ given as follows



The cell in $C_1(\Gamma, \mathbb{R})$ given by

$$\tilde{D}_\sigma = \{t_1 e_2 + t_2 e_2 - t_3 e_3 \mid t_i \in [0,1]\}$$

surjects onto σ. If we compute the dual we get

$$\tilde{D}_\sigma^\star = \frac{1}{2}(e_1 + e_2 - e_3)$$

and

$$\partial \tilde{D}_\sigma^\star = (\frac{1}{2}, -\frac{1}{2}) = \phi$$

hence we see that this degree is semistable but not stable.
The corresponding presentations are given by the translated of the sheaf

$$L \cong \mathcal{O}_{v_1}(-s(e_1) - s(e_2)) \times \mathcal{O}_{v_2}(-t(e_3))$$

The sheaves which are not free at two nodes also correspond to semistable one and the graded class is equivalent to the previous one.
Indeed let F be a sheaf which is not free at e_1 and e_3. Consider the cell

$$\tilde{D}_{\sigma_{13-}} = \{t_1 e_1 - t_3 e_3 \mid t_i \in [0,1]\}$$

For the dual we obtain

$$\partial \tilde{D}^{\star}_{\sigma_{13-}} = \{(-\frac{1}{2}, \frac{1}{2}) + t(1,-1) \mid t \in [0,1]\} \ni \phi$$

and ϕ is not in the interior. Hence the polyhedron is semistable but not stable and the sheaf F is given by

$$0 \to F \to \mathcal{O}_{v_1}(-s(e_1)) \times \mathcal{O}_{v_2}(-t(e_3)) \to k_2 \to 0$$

As far as the Harder-Narasimhan filtration we consider the subsheaf S_1 given by

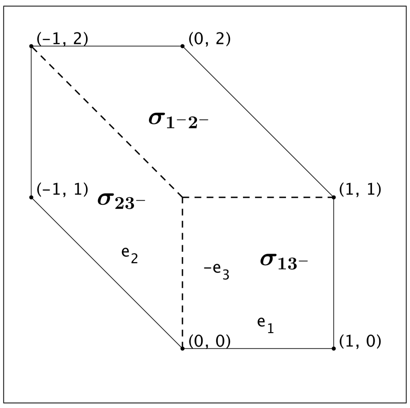
$$0 \to S_1 \to F \to \mathcal{O}_{v_2}(-t(e_3)) \to 0$$

It is immediate to verify that S_1 has degree -2 as sheaf on C_{v_2}.

If $S_1 \cong \mathcal{O}_{v_1}(-s(e_1) - s(e_2))$ we obtain that

$$gr(F) \cong gr(L)$$

Analogously the other polyhedra giving semistable but not stable sheaves whose graded is isomorphic to L can be read from the follwoing picture.



These three cells are not considered in our construction but only the σ. We get in this way a canonical semistable representative on each class.
Observe that if we move the polarization ϕ in the open

$$\{(-\frac{1}{2}, \frac{1}{2}) + t(1,-1) \mid t \in (0,1)\}$$

then σ is no more semistable but the cells σ_{13-}, σ_{1-2-} and σ_{23-} are and they become even stable. The compactifications obtained from these configurations are not obtained from our construction but they can be obtained by giving a functorial interpretation of the models constructed by Alexeev in [AL02].
We would like to investigate this aspect in a future work.

Chapter 5

The Log Picard functor

In this chapter we go through the analysis of the separateness problems related to the constructions we made, in a more general situation. We first recall some well known facts that can be found for example in [BLR], [Ra]. The symbol S will always denote the spectrum of a discrete valuation ring.

Let G be a group scheme over a field k which is locally of finite type and let G' be the connected component of the identity. For any scheme T, let $|T|$ be the corresponding topological space. Consider the subfunctor G^0 defined as

$$G^0(T) = \{g \in G(T) \mid g(|T|) \subset |G'|\}$$

This is representable by a group subscheme of G which is open, connected and of finite type over k (SGA 3,VI_A.2). We recall now some definitions.

Definition 5.0.1. Given a group functor G over Sch/S, such that the fibers are representable and locally of finite type, one defines G^0 as the subfunctor of G whose sections $G^0(T)$ over a scheme $T \in Sch/S$ is the subgroup of elements g in $G(T)$ for which $g|_t \in G_t^0(T)$ for every point $t \in T$.

Definition 5.0.2. A morphism $(X, M_X) \to (S, M_S)$ of log-schemes is called **log-cohomologically flat in dimension zero** if for any nilpotent log-closed immersion $Spec(A_0) \to Spec(A)$ over S defined by square zero ideal, the natural map

$$H^0(X_A, M_{X_A}^{gp}) \to H^0(X_{A_0}, M_{X_{A_0}}^{gp})$$

is surjective.

Definition 5.0.3. 1. Given a morphism of log-schemes

$$f : (C, M_C) \to (S, M_S)$$

the **log Picard stack** on Sch/S is the stack in the étale topology corresponding to the groupoid whose fiber over a scheme $g : T \to S$ is defined by

$$\mathcal{P}ic^{log}(T, g^*M_S) := \{M^{gp}_{C_T}\text{-torsors on } C_{\acute{e}t}\}$$

2. the **log Picard functor**, denoted with $\mathrm{Pic}^{log}_{C/S}(T)$ is the sheafification on $S_{\acute{e}t}$ of the functor of isomorphism classes for log Picard stack. As sheaf it can be written as the sheafification of

$$(T, g^*M_S) \to \frac{\mathcal{P}ic^{log}(T, g^*M_S)}{\cong}$$

The only representablity result the author knows about this functor is the following.

Theorem 5.0.4 ([Ol04]4.4/4.5)**.** *Let $f : (X, M_X) \to (T, M_T)$ be a proper, special (D.0.19) morphism of schemes which is log-cohomologically flat in dimension zero then $Pic^{log}_{C/S}$ is representable by an algebraic space.*

The proof is a careful application of Artin's criterion in [Avd].
Since it is not clear to the author how to check the log-cohomological flatness already in the case of curves, except that for trivial examples where $H^1(X, \mathcal{O}_X) = 0$, we want to use a different procedure in order to show that for semistable curves, over a discrete valuation ring, the classical cohomological flatness allows us to conclude some interesting results on the identity component of the log-Picard functor.
Our result is the following.

Theorem 5.0.5. *Let S be a discrete valuation ring and*

$$f : (X, M_X) \to (S, M_S)$$

be a proper, special, log-semistable curve over S. Assume that X is regular and that the generic fiber is smooth. Consider the following assertions:

1. *the morphism of schemes $f : X \to S$ is cohomologically flat in dimension zero.*

2. *the morphism of log-schemes $f : (X, M_X) \to (S, M_S)$ is log-cohomologically flat in dimension zero.*

3. *$\underline{Pic}^{log}$ is representable by an algebraic space over S and the identity component $(\underline{Pic}^{log})^0$ is representable by a separated group scheme over S.*

We have the following implications 1⇒3, 2⇒3, 1⇒2.

The strategy is very simple. We consider the exact sequence of sheaves in the étale topology

$$0 \to \mathbb{G}m \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

This gives us a long exact sequence

$$f_*\overline{M}_X^{gp} \xrightarrow{\delta} R^1 f_*\mathbb{G}m \to R^1 f_* M_X^{gp} \to R^1 f_*\overline{M}_X^{gp} \to R^2 f_*\mathbb{G}m$$

of group functors. Recall now the following result.

Theorem 5.0.6 ([BrIII]3.2)**.** *Let $f : X \to Y$ be a proper, flat morphism with X and Y locally noetherian and regular, Y of dimension 1, f has one dimensional fibers and the local rings of Y are japanese. Then*

$$R^i f_*\mathbb{G}m = 0 \quad \textit{for } i \geq 2$$

In particular our sequence becomes

$$f_*\overline{M}_X^{gp} \xrightarrow{\delta} R^1 f_*\mathbb{G}m \to R^1 f_* M_X^{gp} \to R^1 f_*\overline{M}_X^{gp} \to 0 \tag{5.1}$$

We have now the following lemma

Lemma 5.0.7. *Assuming the hypotheses of 5.0.5 then the étale sheaf $R^1 f_*\overline{M}_X^{gp}$ is representable by an algebraic space in groups over S.*

Proof. Following [Ol04] we consider the stack $\mathcal{R}^1 f_*\overline{M}_X^{gp}$ whose associated groupoid parametrizes $\overline{M}_X^{gp}$-torsors. The sheaf $R^1 f_*\overline{M}_X^{gp}$ is obtained by taking isomorphism classes.

The sheaf $\overline{M}_X^{gp}$ is a constructible sheaf of $\mathbb{Z}$-modules (by [Ol03]Lemma 3.5) concentrated on the special fiber.

Since f is proper, special and log-semistable, if we take the base change to an affine scheme T, which is the spectrum of a complete local ring with separably closed residue field, we have a decomposition

$$H^0(T, \overline{M}_T) \cong \bigoplus_{c_i \in C(X_0)} \mathbb{N} n_{c_i}$$

and

$$\overline{M}_{X_T}^{gp} \cong \bigoplus_{c \in C(X_0)} \overline{M}_c^{gp}$$

(Appendix D) , where $C(X_0)$ denotes the set of connected components of the singular locus of the special fiber, which we denote with X_0, and $\overline{M}_c$ are "the branches at c" defined as follows

$$\overline{M}_c := \left\{ \begin{array}{c} x \in \overline{M}_{X_0} \text{ such that étale locally} \\ \text{exists } y \in \overline{M}_{X_0} \text{ with } x + y \in (n_c) \end{array} \right\}$$

Write $T = Spec(R)$ and let $t_c \in R$ the function corresponding to the component $c \in C(X)$.
The sheaf $\overline{M}_c^{gp}$ is supported on $t_c = 0$ so we can assume that in R all t_c are zero.
Let Z_c be the corresponding connected component of the locus where f is not smooth (in our case they are just closed points).
If $z \in Z_c$ then, étale locally, around z the scheme X is isomorphic to

$$Spec(R[x_1, x_2]/(x_1x_2))$$

Let J_c be the ideal corresponding to Z_c, i.e. locally given by

$$(x_1, x_2)$$

and we define

$$\nu_c : \tilde{X}_c \to X$$

as the proper transform of the blow-up of X at J_c.
On X_0 the sheaf $\overline{M}_c^{gp}$ is isomorphic to $\nu_{c,*}\mathbb{Z}$.
The same argument repeats whenever we base change with an affine artinian thickening of S

$$Spec(A) \to S$$

In particular, given a surjective morphism $A \to A_0$ of artinian S-algebras, with square zero kernel, we have that the map

$$H^0(X_A, \overline{M}^{gp}_{X_A}) \to H^0(X_{A_0}, \overline{M}^{gp}_{X_{A_0}})$$

is an isomorphism. Indeed by the previous decomposition over complete local rings we have

$$H^0(X_A, \overline{M}^{gp}_{X_A}) \cong \mathbb{Z}^{C(X_0)}$$

Since the set of connected components does not changes under nilpotent thickenings ([Aas] 3.1) we obtain the claim.

This implies that for any object $y : Y \to \mathcal{R}^1 f_* \overline{M}^{gp}_X$ where Y is a scheme, the algebraic group space Aut_y is smooth.
We want now to use the following fact.

Proposition 5.0.8 ([Ol04] Prop. 4.7, [Avd]Appendix)**.** *Let $\mathcal{X}$ be an algebraic stack such that for any object $u : U \to \mathcal{X}$, with U a scheme, the group algebraic space $\underline{Aut}_u$ over U representing the automorphisms of u is smooth. Then the sheaf X corresponding to the presheaf of isomorphism classes of objects in $\mathcal{X}(T)$, for T a scheme, is representable by an algebraic space.*

Using this proposition we need to show that the stack $\mathcal{R}^1 f_* \overline{M}^{gp}_X$ is algebraic.

We apply now Artin's criterion [Avd]5.3 to show that the stack $\mathcal{R}^1 f_* \overline{M}_X^{gp}$ is algebraic.

Let us see that this functor is limit preserving. Take an inductive family $\{A_i\}$ of noetherian rings over S.
We need to see that the functor

$$\varinjlim \mathcal{R}^1 f_* \overline{M}_X^{gp}(A_i) \to \mathcal{R}^1 f_* \overline{M}_X^{gp}(\varinjlim A_i)$$

induces an equivalence of categories. For the full faithfulness we use the same argument in [Ol04]4.10.2. Namely take ξ_1 and ξ_2 objects in $\varinjlim \mathcal{R}^1 f_* \overline{M}_X^{gp}(A_i)$. There is a big index j such that the objects ξ_i give objects in

$$\mathcal{R}^1 f_* \overline{M}_X^{gp}(A_j)$$

By definition there is an affine étale covering

$$Spec(R) \to X_{A_j}$$

where the ξ_i trivializes and for the indices bigger than j we define $R_l := R \otimes A_l$ and $Spec(R'_l) = Spec(R_l) \times_{X_{A_l}} Spec(R_l)$.

Since we are working with groupoids, every morphism is an isomorphism and we have a commutative diagram

$$\begin{tikzcd}
\varinjlim Isom(\xi_1|_{X_{A_i}}, \xi_2|_{X_{A_i}}) \arrow[r] \arrow[d] & Isom(\xi_1|_{\varinjlim X_{A_i}}, \xi_2|_{\varinjlim X_{A_i}}) \arrow[d] \\
\varinjlim Isom(\xi_1|_{R_i}, \xi_2|_{R_i}) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & Isom(\xi_1|_{\varinjlim R_i}, \xi_2|_{\varinjlim R_i}) \arrow[d, shift left] \arrow[d, shift right] \\
\varinjlim Isom(\xi_1|_{R'_i}, \xi_2|_{R'_i}) \arrow[r] & Isom(\xi_1|_{\varinjlim R'_i}, \xi_2|_{\varinjlim R'_i})
\end{tikzcd}$$

with exact columns and the upper horizontal map is already injective.
On $Spec(R_i)$ the objects ξ_j are trivial, hence the isomorphisms correspond to the group $H^0(Spec(R_i), \overline{M}_{R_i}^{gp})$. Using the exactness of the columns in the previous diagram, to show that the upper horizontal arrow is an isomorphism, we have reduced to prove that if $\{Y_i\}$ is an affine filtered inductive limit then we have an isomorphism

$$\varinjlim H^0(Y_i, \overline{M}_{Y_i}^{gp}) \cong H^0(\varinjlim Y_i, \overline{M}_{\varinjlim Y_i}^{gp})$$

The fact that this is an isomorphism follows from [SGA] 4.VII.5.7.

The essential surjectivity is the same formal argument one uses for the classic Picard stack using descent and the fact that for a stack every descent datum

is effective. Let us review this argument.
Take

$$\xi_\infty \in \mathcal{R}^1 f_* \overline{M}_X^{gp}(\varinjlim A_i)$$

and an affine étale cover

$$Spec(R_\infty) \to X_{\varinjlim A_i}$$

where ξ_∞ trivializes. Since X is locally of finite presentation over S then by standard approximation arguments ([EGA]IV.8.14.2), we can find an affine étale scheme $Spec(R) \to X_i$ for some index i such that

$$Spec(R_\infty) = Spec(R) \times_{X_{A_i}} X_{\varinjlim A_i}$$

Let $p : Spec(R_\infty) \to Spec(R)$ be the projection.

We want to use the trivial $\overline{M}_R^{gp}$-torsor 1_R on $Spec(R)$ to find a descend datum on some X_{A_j} for a big index such that the corresponding torsor induces ξ_∞. By construction we have an isomorphism

$$p^* 1_R \cong \xi_\infty|_{R_\infty}$$

Since we are working with trivial torsors this isomorphism corresponds to an element of $H^0(R_\infty, \overline{M}_{R_\infty}^{gp})$. We know now that

$$H^0(R_\infty, \overline{M}_{R_\infty}^{gp}) \cong \varinjlim H^0(Spec(R \otimes A_i), \overline{M}_{Spec(R \otimes A_i)}^{gp})$$

hence we have an isomorphism for some big index

$$1_{R_i} \cong \xi_\infty|_{R_i}$$

The isomorphism

$$p_1^* \xi_\infty|_{R_\infty} \cong p_2^* \xi_\infty|_{R_\infty}$$

gives, by the same reason, an isomorphism

$$\alpha_j : p_1^* 1_{R_j} \cong p_2^* 1_{R_j}$$

on $Spec(R_j) \times_{X_{A_j}} Spec(R_j)$, where perhaps we need to raise the index.
Looking at the cocycle condition we get a cocycle condition for 1_{R_l} where again we could need to raise the index again. The couple $(1_{R_l}, \alpha_l)$ defines an object in

$$\mathcal{R}^1 f_* \overline{M}_X^{gp}(A_l)$$

and by construction when we base change to $\varinjlim A_i$ the corresponding torsor is isomorphic to ξ_∞.

Let now $\nu : \tilde{X}_0 \to X_0$ be the blow up of the singular locus of the special fiber. The special fiber of the proper tranform of X_0 has irreducible components

$$\{\tilde{X}_1, \dots, \tilde{X}_r\}$$

By the Leray spectral sequence applied to ν and the structure of $\overline{M}^{gp}_{X_0}$ we get an inclusion of the form

$$H^1(X_0, \overline{M}^{gp}_{X_0}) \hookrightarrow \oplus H^1(\tilde{X}_i, \mathbb{Z}) \tag{5.2}$$

Since the schemes $\tilde{X}_i$ are irreducible and unibranched and the base is local we have $H^1(\tilde{X}_i, \mathbb{Z}) = 0$ by [SGA] 4.IX.3.6. and this is also true if we take artinian thickenings $A \to A_0$ over S.
In particular for artinian thickenings $A \to A_0$ over S we find, using the decomposition

$$\overline{M}^{gp}_{X_A} \cong \bigoplus_{c \in C(X_{A_0})} \overline{M}^{gp}_c$$

that

$$H^1(X_A, \overline{M}^{gp}_{X_A}) = 0 \tag{5.3}$$

This implies that the obstruction theory is zero and also the deformation theory is zero. In particular the Schlessinger's conditions $(S1)$ and $(S2)$ and the condition (4.1) in [Avd] are also satisfied.

Let us consider the quasi-separation condition. We need a noetherian integral domain A_0 over S. An automorphism of an object is given by an element

$$\alpha \in H^0(X_{A_0}, \overline{M}^{gp}_{X_{A_0}})$$

The set of points $x \in X_{A_0}$ where α induces the identity on $\overline{M}^{gp}_{X_{A_0},x}$ is an open $U \subset X_{A_0}$, because by [Ol03]3.5 it is constructible and stable under generalization.
Let Z be the complement. Since f is proper then $W := f(Z)$ is closed in $Spec(A_0)$ and the complementary $W^c =: V$ is an open containing a dense set of points of finite type by definition of the property [Avd]5.3(4) we want to verify. Given $v \in V$, the fibers X_v do not intersect Z hence they are contained in U. This implies that α induces the identity on the open V.

We need still to check the condition for the completion. Let $R = \varprojlim R/\mathfrak{m}^n$ be a complete local $\mathcal{O}_S$-algebra with residue field of finite type over S.
We need to show that the canonical functor

$$\rho : \mathcal{R}^1 f_* \overline{M}^{gp}_X(R) \to \varprojlim \mathcal{R}^1 f_* \overline{M}^{gp}_X(R/\mathfrak{m}^n)$$

if fully faithful and that for every n the functor induced by projecting to $\mathcal{R}^1 f_* \overline{M}^{gp}_X(R/\mathfrak{m}^n)$ is essentially surjective. Recall that, as already remarked

by Hall and Rydh, there is a typo in [Avd], so that assuming only faithful is not enough.
Define $R_n := R/\mathfrak{m}^n$.
Using the vanishing 5.3 it is enough to deal with the trivial torsors. Let us take a compatible system $\{1_{R_n}, \alpha_n\}$ in

$$\varprojlim \mathcal{R}^1 f_* \overline{M}_X^{gp}(R_n)$$

where α_n are compatible isomorphisms

$$\alpha_n : 1_{X_{R_n}} \to 1_{X_{R_{n+1}}}|_{X_{R_n}}$$

For every n we have an isomorphism

$$\beta_n : 1_{X_R}|_{X_{R_n}} \to 1_{X_{R_n}}$$

and we want to show that we can choose the β_n such that for each step we can lift them compatibly with α_n, namely that we can find

$$\beta_{n+1} : 1_{X_R}|_{X_{R_{n+1}}} \to 1_{X_{R_{n+1}}}$$

such that

$$\beta_{n+1}|_{X_{R_n}} = \alpha_n \circ \beta_n \tag{5.4}$$

Take an arbitrary isomorphism

$$s : 1_{X_R}|_{X_{R_{n+1}}} \to 1_{X_{R_{n+1}}}$$

Let γ_{n+1} be the unique lift of the composition

$$1_{X_R}|_{X_{R_n}} \xrightarrow{\beta_n} 1_{X_{R_n}} \xrightarrow{\alpha_n} 1_{X_{R_{n+1}}}|_{X_{R_n}} \xrightarrow{s^{-1}|_{X_{R_n}}} 1_{X_R}|_{X_{R_n}}$$

Such lift exists because

$$H^0(X_{R_n}, \overline{M}^{gp}_{X_{R_n}}) \cong H^0(X_{R_{n+1}}, \overline{M}^{gp}_{X_{R_{n+1}}})$$

If we now define β_{n+1} as the composition

$$1_{X_R}|_{X_{R_{n+1}}} \xrightarrow{\gamma_{n+1}} 1_{X_R}|_{X_{R_{n+1}}} \xrightarrow{s} 1_{X_{R_{n+1}}}$$

then it satisfies the property 5.4.

In particular ρ is essentially surjective on each step and even in a compatible way.

Let now $\{g_n\}$ be an isomorphism between a couple $\{1_{X_{R_n}}, \alpha_n\}$ and a couple $\{1_{X_{R_n}}, \alpha'_n\}$. Define $X_n := X_{R_n}$. We have compatible isomorphisms β_n and

β'_n from the previous argument. Hence we can consider the commutative diagrams

$$\begin{tikzcd}
1_{X_R}|_{X_n} \arrow[r, "\beta_{n+1}|_{X_n}"] & 1_{X_{n+1}}|_{X_n} \arrow[r, "g_{n+1}|_{X_n}"] & 1_{X_{n+1}}|_{X_n} \arrow[r, "\beta'^{,-1}_{n+1}|_{X_n}"] \arrow[d, "\alpha'^{,-1}_n"] & 1_{X_R}|_{X_n} \arrow[d, "Id"] \\
1_{X_R}|_{X_n} \arrow[u, "Id"] \arrow[r, "\beta_n"'] & 1_{X_n} \arrow[u, "\alpha_n"] \arrow[r, "g_n"'] & 1_{X_n} \arrow[r, "\beta'^{,-1}_n"'] & 1_{X_R}|_{X_n}
\end{tikzcd}$$

From these diagrams we deduce that $\{g_n\}$ gives an element

$$\{\beta'^{,-1}_n \circ g_n \circ \beta_n\} \in \varprojlim H^0(X_n, \overline{M}^{gp}_{X_n})$$

and vice versa every element

$$\{s_n\} \in \varprojlim H^0(X_n, \overline{M}^{gp}_{X_n})$$

gives such $\{g_n\}$ via $\{\beta'_n \circ s_n \circ \beta_n^{-1}\}$.
On the other hand the automorphisms of 1_{X_R} are given by the group

$$H^0(X_R, \overline{M}^{gp}_{X_R})$$

Since

$$H^0(X_R, \overline{M}^{gp}_{X_R}) \cong \varprojlim H^0(X_n, \overline{M}^{gp}_{X_n})$$

we find that ρ is fully faithful.

We checked all conditions of Artin's representability theorem, hence the proof the lemma is now complete. □

Recall now that given a morphism of preschemes $f : X \to S$, with S one dimensional, regular and irreducible, with generic fiber η and a closed subscheme $Z_\eta \subset X_\eta$ of the generic fiber, then there is a unique closed subscheme $Z_1 \subset X$ which is flat over S and such that $Z_{1,\eta} = Z_\eta$ ([EGA] IV.2.8.5).
One can impose the same condition for functors, namely we consider the following definitions we found in [Ra].

Definition 5.0.9. Given a contravariant functor $F : (Sch/S)^o \to Set$ which is a fppf sheaf and G a subsheaf of the generic fiber F_η, the **schematic closure** of G in F is defined as the fppf sheaf generated by morphisms $z : Z \to F$ where Z is a flat scheme over S such that $z_\eta : Z_\eta \to F_\eta$ factorizes through G.

Definition 5.0.10. Define E to be the schematic closure in $\mathrm{Pic}_{X/S}$ of the unity section in $\mathrm{Pic}_{X_\eta/\eta}$.

Definition 5.0.11. The symbol $\mathcal{D}$ denotes the group of divisors with support on the special fiber X_s and $\mathcal{D}_0$ be the subgroup of principal divisors.

Clearly if $A \in \mathcal{D}$ then the line bundle $\mathcal{O}(A)$ is generically trivial. In this way we get a homomorphism

$$\mathcal{D}/\mathcal{D}_0 \to E(S)$$

We recall some properties of E.

Proposition 5.0.12 ([Ra])**.** *Let $X \to S$ be proper, flat and cohomologically flat then*

1. *E is an algebraic space in groups étale over S.*
2. *The morphism $E(S) \to E_s$ is bijective.*
3. *The application $D/D_0 \to E(S)$ is bijective.*
4. *The quotient $Q := Pic_{X/S}/E$ is a separated group scheme over S.*

Assume we can prove that the image of δ in 5.1 is an epimorphism onto E.

We would get an exact sequence

$$0 \to Q := R^1 f_* \mathbb{G}m/E \xrightarrow{i} R^1 f_* M_X^{gp} \to R^1 f_* \overline{M}_X^{gp} \to 0$$

The previous proposition tells us that the left hand side is a separated group scheme over S. Furthermore by lemma 5.0.7 also the right hand side is an algebraic space and we exhibit the functor $R^1 f_* M_X^{gp}$ as an extension of group algebraic spaces.

According to [BrIII]11 and [Ol03]A.1. the previous exact sequence does not change if we consider it in the flat topology.

In particular we exhibit $\underline{\mathrm{Pic}}^{log}$ as an extension of fppf abelian group sheaves which are representable by algebraic spaces. It follows as a consequence of Artin's theorem on representability of flat quotients that $\underline{\mathrm{Pic}}^{log}$ is also an algebraic space in groups ([Aim]7.3).

Let us treat now the separateness question. Since a group algebraic space over a field is always representable by a group scheme ([Afm]4.2) it makes sense to define $(\underline{\mathrm{Pic}}^{log})^0$ as in definition 5.0.1.

Let us also assume for the moment that we can prove that the morphism i in the previous sequence is an *open* morphism.

Since Q is representable by a separated group scheme and the morphism i is an open immersion it follows that $(\underline{\mathrm{Pic}}^{log})^0$ contains an open subgroup which is a separated scheme.

This would imply that $(\underline{\mathrm{Pic}}^{log})^0$ is separated over S by [Ra]3.3.6 and a separated algebraic space in groups over a one dimensional basis is always a scheme by [Anan]Théorème 4.B. In particular $(\underline{\mathrm{Pic}}^{log})^0$ is a separated scheme over S.

First we need now to show that $im\delta$ surjects onto E. Generically there is no difference between $M^{gp}_{X_\eta}$ and $\mathbb{G}m_{X_\eta}$. Indeed M^{gp}_X is étale locally around a node $e \in X_s$ generated by the functions x_e, y_e where

$$\widehat{\mathcal{O}_{X_s,x}} \cong \mathcal{O}_{S,f(x)}[[x,y]]/(x_e y_e - \pi_e)$$

The log structure M_X is defined via pushout

$$\begin{array}{ccc} \alpha^{-1}\mathcal{O}^*_X & \longrightarrow & \overline{M}_X \\ \big\downarrow & & \\ \mathcal{O}^*_X & & \end{array}$$

along $\alpha^{-1}\mathcal{O}^*_X$ where $\alpha : \overline{M}_X \to \mathcal{O}_X$ is a homomorphism of monoid. If we invert π_e then $x_e, y_e \in \mathcal{O}^*_{X_\eta}$, and $\alpha^{-1}(\mathcal{O}^*_{X_\eta}) = \overline{M}^{gp}_{X_\eta}$ and we are done. This implies that it is enough to show that we get a surjection on the special fiber.

Using proposition 5.0.12 it follows that we need to show that the image of δ on the special fiber surjects onto $\mathcal{D}/\mathcal{D}_0$.
For the special fiber we have the following description of δ that can be found in [Ol04] 3.2/3.3.
Let V be the set of connected components of the normalization of X_s. The morphism δ_s is identified with the morphism

$$\mathbb{Z}^{|V|} \longrightarrow H^1(X_s, \mathbb{G}m_{X_s})$$

sending a generator $[X_{s,v}]$, where $X_{s,v}$ is a connected component of $\tilde{X}^{norm}_s$ indexed by $v \in V$, to the line bundle $\mathcal{O}(X_{s,v})|_{X_s}$.
This provides the surjectivity of δ onto E.

We need now to show that i is open. Since a flat morphism locally of finite presentation of algebraic spaces is universally open and a smooth morphism of algebraic spaces is flat and locally of finite presentation it is enough to check that i is smooth.
We know from proposition 5.0.12 that the morphism $\mathrm{Pic} \to \mathrm{Pic}/E$ is étale, hence it is enough to show that $\mathrm{Pic} \to \underline{\mathrm{Pic}}^{log}$ is smooth.
Let Y be a scheme and $Y \to \underline{\mathrm{Pic}}^{log}$ be a morphism. By definition we need to check that the induced morphism

$$Y \times_{\underline{\mathrm{Pic}}^{log}} \mathrm{Pic} \to Y$$

is a smooth morphism of schemes.
As in the classical case, given a nilpotent surjection $A \to A_0$ with kernel I, we have the exponential sequence ([Ol04]4.12.1)

$$0 \to \mathcal{O}_{X_{A_0}} \otimes I \to M^{gp}_{X_A} \to M^{gp}_{X_{A_0}} \to 0$$

Using this sequence and the fact that for every point $t \in Y$ we have (coherent cohomology)

$$H^2(X_t, \mathcal{O}_{X_t}) = 0$$

we see that that Pic and $\underline{\mathrm{Pic}}^{log}$ are smooth over S and locally of finite presentation.
In particular we can assume that Y and $Y \times_{\underline{\mathrm{Pic}}^{log}} \mathrm{Pic}$ are smooth over S. Once we know this on scheme level then it suffices that we check that the map on the tangent spaces of points is an isomorphism. Since E is étale and $\underline{\mathrm{Pic}}^{log}$ is an extension of Pic/E we have for any point x an inclusion $T_x\mathrm{Pic} \hookrightarrow T_{i(x)}\underline{\mathrm{Pic}}^{log}$.
Using the exponential sequence for a thickening we can read the morphism on the tangent spaces from the diagram

$$\begin{tikzcd}
H^0(X_{A_0}, \mathbb{G}m_{X_{A_0}}) \arrow[r] \arrow[d, "a"'] & H^0(X_{A_0}, M^{gp}_{X_{A_0}}) \arrow[d, "b"] \\
H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I) \arrow[r, "t"'] \arrow[d] & H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I) \arrow[d] \\
H^1(X_A, \mathbb{G}m_{X_A}) \arrow[r] \arrow[d] & H^1(X_A, M^{gp}_{X_A}) \arrow[d] \\
H^1(X_{A_0}, \mathbb{G}m_{X_{A_0}}) \arrow[r] & H^1(X_{A_0}, M^{gp}_{X_{A_0}})
\end{tikzcd}$$

We already know, by the inclusion on the tangent spaces, that

$$H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I)/im(a)$$

injects into

$$H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I)/im(b)$$

Remember now that by [EGA] III.7.8.6 or [BLR]8.1.corollary 8 the functor $T \to \Gamma(X_T, \mathcal{O}_{X_T})$ is represented by a vector bundle V over S if and only if $f : X \to S$ is cohomologically flat in dimension zero. In this case the subfunctor

$$T \to \Gamma(X_T, \mathcal{O}^*_{X_T})$$

is represented by an open subgroup scheme ([BLR]8.2,Lemma 10). In particular under our hypothesis on cohomological flatness of the family, this is smooth and as consequence the morphism a has to be zero. In this way we get an injective map

$$t : H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I) \to H^1(X_{A_0}, \mathcal{O}_{X_{A_0}} \otimes I)/im(b)$$

Looking at the lengths it has necessarily to be also surjective and we get smoothness. Observe that as consequence we obtain that this is an isomorphism, that $im(b)$ is zero and we get the log-cohomological flatness.
This completes the proof of the theorem.

Remark 5.0.13. The hypothesis on the regularity of X was only used to show that $R^2 f_* \mathbb{G}m = 0$. One can drop this assumption if one knew that the image of the map

$$R^1 f_* M_X^{gp} \to R^1 f_* \overline{M}_X^{gp}$$

is representable by an algebraic space.

Remark 5.0.14. As the proof shows the maximal separated quotient constructed by Raynaud in [Ra] is a subgroup of the logrithmic Picard functor. In section 1.7 and chapter 4 we have seen that, over DVRs, once one interprets the Mumford's models as spaces parametrizing certain logarithmic torsors, one obtains a map from these models to the functor $\underline{\mathrm{Pic}}^{log}$.
Andreatta in [And] shows that when we take these models over a DVR, corresponding to Jacobians with non-degenerate polarizations, then they provide a compactification for the Néron model of the relative Jacobian.
It would be interesting to know if there exists a proper subscheme of $\underline{\mathrm{Pic}}^{log}$ which is a minimal "Mumford's compactification" of the Néron model. Namely a proper scheme such that all the compactifications of the Néron model, obtained via the Mumford's construction, have a map to it.
For example in dimension one there is only one polarization and a candidate should be the Tate curve.

Appendix A

Stability

A.1 Stability

We recall in this section the properties of stable torsion free sheaves we have used in this work. We fix as setting a nodal curve C over a field k.

Definition A.1.1. A coherent sheaf $\mathcal{F}$ on C is called torsion free if for any $c \in C$ we have

$$depth(\mathcal{F}_c) = 1$$

Outside the nodes such sheaves are locally free. We need to characterize them at a node $c \in C$. Recall the following fact.

Proposition A.1.2 ([Ses82]). *Let $c \in C$ be a node, $\mathfrak{m}_c$ be the maximal ideal in $\mathcal{O}_c$ and L be a torsion free sheaf of rank one on C. Then either*

$$L_c \cong \mathcal{O}_c$$

or

$$L_c \cong \mathfrak{m}_c$$

Looking at the local cohomology it is easy to show that any subsheaf of a torsion free sheaf is torsion free. In particular for rank one such subsheaves arise by looking at torsion free sheaves having generic rank one on a subcurve. We have now the following characterization.

Proposition A.1.3 ([Ses82]). *Let L be a torsion free sheaf of rank one on C and let $\nu : C' \to C$ be the partial normalization of C at the nodes $c \in C$ where $L_c \cong \mathfrak{m}_c$. Let $L' := \nu^* L/torsion$. The sheaf L' is locally free on C' and*

$$L \cong \nu_* L'$$

We want to introduce the notion of stability. For non-smooth curves the notion of stability is not an intrinsic property of the sheaves but rather it depends on the choice of a polarization on the curve.

We define a polarization on the curve as follows. For each $v \in V$ we take M_v to be a line bundle on C having degree d_v as line bundle on the irreducible component C_v and zero on the other components. We consider the polarization on C given by

$$L = \otimes_{v \in V} M_v$$

For every $\underline{n} = (n_1, \ldots, n_{|V|}) \in C_0(\Gamma, \mathbb{Z})$ we define

$$L^{\underline{n}} = \otimes_{v \in V} M_v^{n_i}$$

Definition A.1.4. Given a coherent sheaf F on C, then the generalized Hilbert polynomial is defined as

$$P_F(\underline{n}) = \chi(F \otimes L^{\underline{n}}) \tag{A.1}$$

for $\underline{n}$ large enough.

If F is torsion free generically of rank one and it is free at a subset E_1 of the nodes then it sits in an exact sequence

$$0 \to F \to \bigoplus_{v \in V} \tilde{F}_v \to \bigoplus_{e \in E_1} k \to 0$$

Using this sequence with a mild generalization in the case in which F is only supported on a subcurve we immediately find the following expression

$$P_F(\underline{n}) = \sum_{v \in V} n_v d_v rk(F|_v) + \chi(F)$$

Define the L-rank of F via

$$r_L(F) := \frac{\sum_{v \in V} d_v \mathrm{rk}(F|_{C_v})}{\sum_{v \in V} d_v}$$

and the L-slope via

$$\mu_L(F) := \frac{\chi(F)}{r_L(F)}$$

Declare F to be L-semistable (resp. L-stable) if for any proper subsheaf $G \subset F$ one has the following inequality

$$\mu_L(G) \leq \mu_L(F) \quad \Big(\text{resp.} \quad \mu_L(G) < \mu_L(F) \Big)$$

As in the case of vector bundles, if we fix the L-slope equal to some μ then the category of torsion free sheaves of rank one with $\mu_L(F) = \mu$ is an abelian, noetherian and artinian ([Ses82]). Hence by the Jordan-Hölder theorem we have that for any semistable F we can find a filtration

$$0 = F_{r+1} \subset F_r \subset \cdots \subset F_0 = F$$

with the property that F_i/F_{i-1} is torsion free of rank one, L-stable and with $\mu_L(F_i/F_{i-1}) = \mu_L(F)$. Furthermore the isomorphism class of the sheaf

$$Gr(F) := \bigoplus_{i=0}^{r} F_i/F_{i+1}$$

depends only on the isomorphism class of F.
The concept of semistability given here coincides, in many cases, with the one described by Simpson in [Sim94] (see [AL96] for more details).

Observe that we can find positive integers a, b and choose a vector bundle E on C of the form

$$E = (\oplus^a \mathcal{O}) \oplus (\oplus^b L)$$

in a way that the condition for a fixed slope μ becomes $\chi(F \otimes E) = 0$ and the semi-stability becomes

$$\chi(G \otimes E) \leq 0 \tag{A.2}$$

where G is a subsheaf of F. This point of view has many advantages in the relative setting. If we take a family of curves and a vector bundle E on this family, we can consider sheaves which are semi-stable w.r.t. this vector bundle. This approach combined with the use of generalized theta functions was used in [F96] and [E01] in order to construct compactifications in the relative setting. In the next section we say something more on this approach and on other ones.

A.2 Relation with other constructions

In this sections we recall how the models of Oda and Seshadri have been used by other authors to construct relative modular compactifications. Very roughly speaking they are different incarnations of the inequalities we already saw

$$\frac{\deg \mathcal{O}_C(1)|_{C_{V\setminus V_1}}}{\deg \mathcal{O}_C(1)} \geq \frac{\chi(S_{V_1}(F(\tilde{n})))}{\sum_v \tilde{n_v} + \deg(F) + \chi(\mathcal{O}_C)}$$

These inequalities also appear in [Gi]Prop.1.10.11 and they were used, in a noteworthy construction, from the point of view of the relative case, by Caporaso in [Ca] and then by Pandariphande for vector bundles. It turns out that there is a good behavior of semistability when one considers line bundles with enough big degree ($\geq 10(2g-2)$) and by taking the polarization induced by the canonical sheaf.
As we will describe in a moment these compactifications behave well in families and they also deal with the case of *quasi-stable curves*.

Here by quasi-stable curves we mean semistable curves, having possibly smooth rational components meeting the rest of the curve in two points and these smooth rational components are not permitted to intersect each other. These components are called *exceptional*. The main reason for this definition is that one wants to avoid to work with torsion free sheaves and consider only line bundles. In doing this one has to substitute sheaves, which are not free at a given node, with line bundles on the blow up of the node having degree one on the resulting exceptional component. Caporaso gave in [Ca] modular compactifications in terms of semibalanced bundles whose definition we are going to recall.

Definition A.2.1. Let C be a semistable curve of genus at least 3, d be a positive integer and $\underline{d} \in C_0(\Gamma, \mathbb{Z})$ be a multidegree summing up to d.

1. $\underline{d}$ is called *semibalanced* if for every subcurve $Z \subset C$ we have

$$|\sum_{v \in Z} d_v - d\frac{\deg_Z \omega_C}{\deg \omega_C}| \leq \frac{Z(C \setminus Z)}{2} \tag{A.3}$$

   and for every exceptional component E corresponding to a vertex v_e one has $0 \leq d_{v_e} \leq 1$.

2. $\underline{d}$ is called *balanced* if it is semibalanced and for every exceptional component E corresponding to a vertex v_e one has $d_{v_e} = 1$.

3. $\underline{d}$ is called *stable balanced* if it is balanced and if for every subcurve $Z \subset C$ where $\sum_{v \in Z} d_v$ reaches the equality in A.3 one has that $\overline{C \setminus Z}$ is a union of exceptional components.

In order to understand the relation with the construction in [OS], we give the following lemma.

Lemma A.2.2. *Semibalanced bundles corresponds to ϕ-semistable sheaves if we choose ϕ induced from the canonical polarization.*

Proof. First of all we recall the adjunction formula, which is true also for nodal curves. Denote with ω_C the dualizing sheaf. This is an invertible sheaf because the curve is Gorenstein. Let $Z \subset C$ be a subcurve then the following relation goes true

$$\deg \omega_C|_Z = \deg \omega_Z + Z \cdot (C \setminus Z)$$

Using the previous identity we get

$$\lambda_{C_{V \setminus V_1}}(\deg F(\tilde{n}) + \chi(\mathcal{O}_C)) =$$

$$\frac{\deg \omega_C|_{C_{V \setminus V_1}}}{2g-2} \deg F(\tilde{n}) + \chi(\mathcal{O}_{C_{V \setminus V_1}}) - \frac{C_{V \setminus V_1} \cdot (C \setminus C_{V \setminus V_1})}{2}$$

Recall now that we have

$$\chi(S_{V_1}(F(\tilde{n}))) = \chi(F(\tilde{n})) - \sum_{v\in V_1} \chi(F(\tilde{n})) + |E_{F(V_1)}|$$

Since we are only considering line bundles we have

$$\chi(F(\tilde{n})) - \sum_{v\in V_1} \chi(F(\tilde{n})) = \sum_{v\in V\setminus V_1} \chi(F(\tilde{n})) - |E|$$

and

$$|E_{F(V_1)}| - |E| = -|e \in E|\text{at least one end point is in } V \setminus V_1|$$

Both sides of the inequality are written now in terms of $V \setminus V_1$ so that we replace V_1 with $V \setminus V_1$.
Write now

$$\chi(\mathcal{O}_{C_{V_1}}) = \sum_{v\in V_1} \mathcal{O}(\mathcal{O}_{C_v}) - |e \in E \text{ s.t. both end points are in } V_1|$$

and use

$$\begin{aligned} -|e \in E \text{ s.t. at least one end point is in } V_1|+ \\ +|e \in E \text{ s.t. both end points are in } V_1| = \\ -C_{V_1}(C \setminus C_{V_1}) \end{aligned}$$

to get

$$\frac{\deg \omega_C|_{C_{V_1}}}{2g-2} \deg F(\tilde{n}) - \frac{C_{V_1} \cdot (C \setminus C_{V_1})}{2} \geq \sum_{v\in V\setminus V_1} \deg(F(\tilde{n})_v) - C_{V_1}(C \setminus C_{V_1})$$

this is clearly the condition A.3. □

It turns out that balanced degrees are very nice from the point of view of geometric invariant theory, because one can use them to construct a "universal" compactification of the picard functor over the Deligne-Mumford (stacky) compactification $\overline{\mathcal{M}}_g$.
In order to do this one defines the balanced Picard functor $\overline{\mathcal{P}_{d,g}}$ as the Zariski-sheafification of the functor parametrizing couples (C, L) where C is a quasi-stable curve of genus g and L is a balanced line bundle of degree d on C. We recall Caporaso's results in the next theorem.

Theorem A.2.3 ([Ca]). *Let $g \geq 3$ and $d \geq 10(2g-2)$. There exists a scheme $\overline{P_{g,d}}$ with a morphism $\phi_d : \overline{P_{g,d}} \to \overline{M_g}$ such that*

- *it is projective, reduced, irreducible and Cohen-Macaulay,*
- *the morphism ϕ_d is proper and it is flat over the locus of automorphism free smooth curves,*

- *it coarsely represents the functor $\overline{\mathcal{P}_{d,g}}$ if and only if*

$$(d-g+1, 2g-2) = 1$$

- *for every $[C] \in \overline{\mathcal{M}}_g$ the fiber $\overline{P_{d,C}} := \phi_d^{-1}([C])$ is a projective connected, equidimensional of dimension g scheme and if $[C]$ is smooth and automorphism free then*

$$\phi_d^{-1}([C]) \cong Pic_C^d$$

- *if $[C] \in \overline{M_g}$ is automorphism free then there is an action of the generalized jacobian J_C on $\overline{P_{d,C}}$ and the smooth locus of $\overline{P_{d,C}}$ is isomorphic to a disjoint union of a finite number of copies of J_C*

We explain in the following lemma the case in which the degree is of the form $k(g-1)$ and the polarization is canonical in relation with the parameter ϕ.

Lemma A.2.4. *Assume* $\deg F(\tilde{n}) = k(g-1)$ *for k a positive integer and that the polarization is canonical then*

$$\phi \equiv \partial(\frac{\sum_{e\in E} e}{2}) \mod \partial C_1(\Gamma, \mathbb{Z})$$

Proof. Since in general $\phi \in \partial C_1(\Gamma, \mathbb{R})$, because the curve is connected and $\sum_v \phi_v = 0$, it is enough to show that $\phi - \partial(\frac{\sum_{e\in E} e}{2}) \in C_0(\Gamma, \mathbb{Z})$ or equivalently

$$\phi_v - \partial(\frac{\sum_{e\in E} e}{2})_v \equiv 0 \mod \mathbb{Z}$$

We have the expression

$$\begin{aligned}\phi_v = & -\tilde{n}_v - \frac{1}{2}(\deg \omega_C|_{C_v}) + \frac{\deg \omega_C|_{C_v}}{2g-2}(\deg F(\tilde{n}) + \chi(\mathcal{O}_C)) = \\ & = -\tilde{n}_v + \frac{k-2}{2}(\deg \omega_C|_{C_v})\end{aligned}$$

By adjunction $\deg \omega_C|_{C_v} = \deg \omega_{C_v} + d_v$ where d_v is defined in 2.11. Since $\deg \omega_{C_v}$ is even if k is even we have done. If k is odd we have

$$\phi_v \equiv \frac{d_v(k-2)}{2} \mod \mathbb{Z} \equiv \frac{d_v}{2} \mod \mathbb{Z}$$

Since $d_v - \big(\partial(\sum_{e\in E} e)\big)_v$ is an even number we are done. $\square$

The construction in [OS] has been generalized to the relative picture by Esteves in [E01] using a different technique which allows us to circumvent the GIT quotient. This is a generalization in the singular case of the original construction given by Faltings in [F93] in the case of vector bundles over smooth curves (see also [F96] for the singular case). The idea proceed as follows. We saw in chapter 2 that the compactified Jacobian $Jac_\phi(C)$ can be constructed as quotient

$$R_\phi/PGL(E)$$

Assume instead we can find a line bundle $\mathcal{L}$ on R_ϕ, which is ample and $PGL(E)$-invariant. One could use the sections of this line bundle to produce a rational morphism in some projective space

$$\theta : R_\phi \to \mathbb{P}^r$$

Assume also that one can prove that this rational map is defined everywhere, that the image of θ is closed (the so called semistable reduction theorem) and that θ identifies gr-equivalence classes. Call this image $\mathcal{M}$. In this case one would get a morphism

$$Jac_\phi(C) := R_\phi/PGL(E) \to \mathcal{M}$$

If one could prove that the previous map is also bijective then one would get an isomorphism between $Jac_\phi(C)$ and the seminormalization of $\mathcal{M}$ in the function field of R_ϕ. It turns out that in the case of rank one sheaves we do not need to take the seminormalization [E99]Thm.16.
In order to find such a line bundle one uses a generalization of an old characterization of the theta line bundle on a Jacobian of a smooth curve, due to Mumford, as the inverse of the determinant of cohomology. Let us consider torsion free sheaves of rank one F which are semistable w.r.t. a vector bundle E as at the end of section A.1 with the inequalities A.2. If one considers the line bundle $\mathcal{L}$ as the determinant of the cohomology for the sheaf $E \otimes F$ then Esteves showed in [E01] that it satisfies the properties we required.

An interesting question is to determine, in the relative case, whether the ample line bundle, we obtain from the Mumford construction, corresponds to the inverse of the determinant of the cohomology w.r.t. some vector bundle polarization E or not. For a result in this direction the reader can look at [F].

Another remarkable construction was given in [And]. Andreatta used the geometric description in [OS] of the compactifications to construct relatively complete model for the Jacobian of a nodal curve via deformation theory. His construction works over basis more general than ours but a functorial interpretation of the relatively complete models he obtained, in terms of stable sheaves, is missing.

Alexeev in [AL96] and [AL04] gives a functorial description of the Jacobians obtained by Oda and Seshadri and Simpson in degree $g-1$ as points in the moduli space of stable semiabelic pairs $\overline{AP}_g$ he constructed in [AL02]. He also explains how to produce the data of chapter 1 required to construct the associated point in $\overline{AP}_g$.

Appendix B

Combinatorical aspects

B.1 Delaunay and Voronoi decomposition

We recall the construction of mixed decomposition that can be found in [NamI], [NamII] and [OS].

Fix a non negative integer g and take a positive definite quadratic matrix q on $\mathbb{R}^g$. This form defines a metric $||\cdot||_q$ and a bilinear form $\langle , \rangle_q$ in an obvious way.

Definition B.1.1. Let $\{a_i\}_{i\in I}$ be a finite set of integral vectors in $\mathbb{R}^g$. The convex hull

$$D(\{a_i\}_{i\in I}) = \{\sum t_i a_i \ \sum t_i = 1, t_i \geq 0\}$$

is called a **Delaunay cell** w.r.t. q if there exists a vector $\alpha \in \mathbb{R}^g$ such that

1. for all $i \in I$ we have $||a_i - \alpha||_q = \min_{\{x\in\mathbb{Z}^g\}} ||\alpha - x||_q$
2. for any $\mathbb{Z}^g \ni x \neq a_i$ we have $||\alpha - x||_q > ||a_i - \alpha||_q$

The bilinear form $\langle , \rangle_q$ induces a linear transformation $\mathbb{R}^g \to \mathbb{R}^g$ and we adopt the convention that

$$B : \mathbb{R}^g \to \mathbb{R}^g$$

denotes this linear transformation multiplied by 2.

Definition B.1.2. Given a Delaunay cell $D = D(\{a_i\}_{i\in I})$, the associated **Voronoi cell** is

$$D^\star := \{-B\alpha, \text{ for } \alpha : ||a_i - \alpha||_q = \min_{\{x\in\mathbb{Z}^g\}} ||a_i - x||_q\}$$

We recall now some properties of these cells.

Proposition B.1.3. *1. ([NamI]1.3,1.4) Given a Delaunay cell*

$$D = D(\{a_i\}_{i \in I})$$

then the corresponding Voronoi dual can be expressed as

$$D^\star = \bigcap_{i \in I} \bigcap_{x \in \mathbb{Z}^g} \{y \in \mathbb{R}^g; q(x) + B(x, a_i) + y^\top x \geq 0\}$$

2. *The Delaunay (resp. Voronoi) cells are bounded and they have a finite number of linear faces.*

3. *Every face of a Delaunay (resp. Voronoi) cell is again a Delaunay (resp. Voronoi) cell.*

4. *The intersection of two Delaunay (resp. Voronoi) cells is again a Delaunay (resp. Voronoi) cell.*

5. *The set of* 0*-dimensional Delaunay cells is the set of integral vectors* $\mathbb{Z}^g$.

6. D_1 *is a face of* D_2 *iff* $D_2^\star$ *is a face of* $D_1^\star$.

7. *For any Delaunay cell* D *we have* $\dim D + \dim D^\star = g$.

8. *For any* $y \in \mathbb{Z}^g$ *and Delaunay cell* $D(a_i)$*, the translation* $D(a_i) + y$ *is the Delaunay cell* $D(a_i + y)$ *and*

$$(D + y)^\star = D^\star - B(y)$$

9. *The number of classes of cells modulo* $\mathbb{Z}^g$*-translation is finite.*

10. *For every* $u \in GL(r, \mathbb{Z})$*,* D *is a Delaunay cell (resp.* $D^\star$ *is a Voronoi cell) w.r.t.* q *iff* $D \cdot u^{-1}$ *is a Delaunay cell (resp.* $D^\star u^\top$ *is a Voronoi cell) w.r.t.* $uqu^\top$.

As a consequence we obtain a $\mathbb{Z}^g$-invariant decomposition of $\mathbb{R}^g$.

Definition B.1.4. The polyhedral decomposition of $\mathbb{R}^g$ defined by the Delaunay cells (resp. Voronoi cells) w.r.t. a quadratic form q is called the **Delaunay decomposition** (resp. **Voronoi decomposition**) of $\mathbb{R}^g$ determined by q.

It is also clear from the construction that if q is non negative, then it defines a decomposition on the subspace where q is positive.

B.2 Quotient decompositions

Since we work with quotient decomposition induced from the projection $C^1(\Gamma,\mathbb{R}) \to H^1(\Gamma,\mathbb{R})$ where Γ is some graph we need to understand how the decompositions pass to the quotient. We recall how this can be done.
Assume we have a triple $(E,(,),E_1,L)$ where E is a real vector space of finite dimension, $(,)$ is a non-degenerate bilinear form, E_1 is a vector substpace and $L \subset E$ is a lattice such that $L \cap E_1$ is still a lattice. Using the bilinear form we can find an orthogonal complement E_2 of E_1 in E.
For every $\psi \in E_2$ we can consider the translated $E_{1,\psi} = E_1 + \psi$.
The orthogonal decomposition permits to define a projection

$$\pi : E \to E_{1,\psi}$$

We have Delaunay and Voronoi decomposition on E and one can look at the induced decomposition on E_ψ. We have the following facts

Proposition B.2.1 ([OS]Ch.I)**.** *1. The set of polyhedra $Vor(E_{1,\psi},E,L)$ of the form $V \cap E_{1,\psi}$, where V is a Voronoi polyhedron in E such that the relative interior has non empty intersection with $E_{1,\psi}$ is a polyhedral decomposition of $E_{1,\psi}$ via bounded polyhedra, invariant for the action of $L \cap E_1$ with $Vor(E_{1,\psi},E,L)/L \cap E_1$ a finite set of polyhedra.*

2. *The set of polyhedra $Del_\psi(E_1,E,L)$ of the form $\pi(D(\{a_i\}))$ where $D(\{a_i\})$ defines a Delaunay cell w.r.t. some $\alpha \in E_{1,\psi}$, is a polyhedral decomposition of E_1 by bounded polyhedra, invariant under the translation by the lattice $L \cap E_1$, with $Del_\psi(E_1,E,L)/L \cap E_1$ finite.*

3. *(Duality) Given $D \in Del_\psi(E_1,E,L)$ there exists a unique*

$$D^\star \in Vor(E_{1,\psi},E,L)$$

iff there exists an α defining D contained in $E_{1,\psi}$.

4. *given $D \in Del_\psi(E_1,E,L)$ we have* $\dim D + \dim D^\star = \dim E_1$.

5. *D_1 is a face of D_2 in $Del_\psi(E_1,E,L)$ if and only if $D_2^\star$ is a face of $D_1^\star$.*

6. *The set of zero dimensional polyhedra in $Del_\psi(E_1,E,L)$ is contained in the lattice $\pi(L)$.*

B.3 Decompositions for graphs

In this section see how the specialize the previous construction to the case of graphs.

Let Γ be the graph of a nodal curve. We want to end up with a polyhedral decomposition of the vector space $H^1(\Gamma, \mathbb{R})$. In particular we want to exhibit such decomposition as a quotient of the standard decomposition on $C^1(\Gamma, \mathbb{Z})$, which we are going to define.
The real vector space $C_1(\Gamma, \mathbb{R})$ (resp. $C_0(\Gamma, \mathbb{R})$) is endowed with a standard pairing, namely one has a canonical integral basis given by the edges (resp. the vertices) which gives a decomposition

$$C_1(\Gamma, \mathbb{R}) \cong \bigoplus_{i=1}^{|E|} \mathbb{R}e_i$$

(resp.

$$C_0(\Gamma, \mathbb{R}) \cong \bigoplus_{i=1}^{|V|} \mathbb{R}v_i$$

) and the pairing is given by

$$(e_i, e_j) = \delta_{ij} \tag{B.1}$$

resp.

$$[v_i, v_j] = \delta_{ij} \tag{B.2}$$

where as usual

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

This pairing is clearly positive definite and it induces a Delaunay (resp. a Voronoi) decomposition of $C_1(\Gamma, \mathbb{R})$, denoted with $Del(C_1(\Gamma, \mathbb{R}))$ (resp. $Vor(C_1(\Gamma, \mathbb{R}))$). The same is true for the group with upper index. One usually uses the symbol

$$K(\Gamma) := Del(C_1(\Gamma, \mathbb{R})) \tag{B.3}$$

For every subset $W \subset E$ define the vector

$$C_1(\Gamma, \mathbb{R}) \ni e(|W|) := \sum_{e \in W} e \tag{B.4}$$

This vector plays an important role because for the standard quadratic form Delaunay and Voronoi cells correspond by translating via such vectors.
More precisely we have the following proposition.

Proposition B.3.1 ([OS]5.1)**.** *Each Delaunay cell $D \in Del(C_1(\Gamma, \mathbb{R}))$ is of the form*

$$D = y + \frac{e(|W|)}{2} + \{\sum_{e \in W} t_e e \mid -\frac{1}{2} \leq t_e \leq \frac{1}{2}\}$$

for some $y \in C_1(\Gamma, \mathbb{Z})$ and $W \subset E$. The corresponding Voronoi dual is given by

$$D^{\star} = y + \frac{e(|W|)}{2} + \{ \sum_{e \in E \setminus W} t_e e \mid -\frac{1}{2} \leq t_e \leq \frac{1}{2} \}$$

We can now give a definition.

Definition B.3.2. Given a Voronoi cell

$$V = y + \frac{e(|W|)}{2} + \{ \sum_{e \in W} t_e e \mid \frac{1}{2} \leq t_e \leq \frac{1}{2} \}$$

define its *barycenter* as

$$b(D) = y + \frac{e(|W|)}{2} \tag{B.5}$$

We have the boundary morphism

$$\partial : C_1(\Gamma, \mathbb{R}) \to C_0(\Gamma, \mathbb{R}) \tag{B.6}$$

and its dual map

$$\delta : C^0(\Gamma, \mathbb{R}) \to C^1(\Gamma, \mathbb{R}) \tag{B.7}$$

Using the pairings and boundaries one easily gets orthogonal decompositions as real vector spaces

$$C_0(\Gamma, \mathbb{R}) \cong H^0(\Gamma, \mathbb{R}) \oplus \partial C_1(\Gamma, \mathbb{R}) \quad \text{and} \quad C_1(\Gamma, \mathbb{R}) \cong H_1(\Gamma, \mathbb{R}) \oplus \delta C^0(\Gamma, \mathbb{R})$$

We can now consider the canonical projections

$$\begin{tikzcd} & C_1(\Gamma, \mathbb{R}) \arrow[dl, "\pi''"'] \arrow[dr, "\pi'"] & \\ H_1(\Gamma, \mathbb{R}) & & \delta C^0(\Gamma, \mathbb{R}) \end{tikzcd}$$

induced by the previous orthogonal decompositions. We can restrict the standard paring to this subspaces and this induces a Voronoi decomposition of them. By a result due to Mumford ([OS].5.5) this decomposition is actually induced by the projections of the Delaunay and Voronoi cells of $C_1(\Gamma, \mathbb{R})$ to $H_1(\Gamma, \mathbb{R})$ and to $\delta C_0(\Gamma, \mathbb{R})$, namely we have the following

Proposition B.3.3 ([OS]5.2)**.** *For a graph Γ the following holds*

1. *the Voronoi decomposition of $H_1(\Gamma, \mathbb{R})$ induced by $(,)|_{H_1(\Gamma, \mathbb{R})}$ is formed by the $H_1(\Gamma, \mathbb{Z})$-translated of the faces of*

$$\pi''(\{ \sum_{i \in E} t_j e_j \mid \frac{1}{2} \leq t_j \leq \frac{1}{2} \})$$

2. *the Voronoi decomposition of* $\delta C_0(\Gamma,\mathbb{R})$ *induced by* $(\,,\,)|_{\delta C_0(\Gamma,\mathbb{R})}$ *is formed by the* $\delta C_0(\Gamma,\mathbb{Z})$*-translated of the faces of*

$$\pi'(\{\sum_{i\in E} t_j e_j \mid \frac{1}{2} \leq t_j \leq \frac{1}{2}\})$$

We need no to generalize a little bit more.

Given $\psi \in \delta C^0(\Gamma,\mathbb{R})$ we consider the coset

$$H_1(\Gamma,\mathbb{R})_\psi := H_1(\Gamma,\mathbb{R}) + \psi \subset C_1(\Gamma,\mathbb{R})$$

The set of Voronoi cells V of $C_1(\Gamma,\mathbb{R})$ such that $V^0 \cap H_1(\Gamma,\mathbb{R})_\psi \neq \emptyset$, where $(-)^0$ denotes the relative interior, induces a decomposition on $H_1(\Gamma,\mathbb{R})_\psi$ which is $H_1(\Gamma,\mathbb{Z})$-invariant.
Denote this decomposition with the symbol

$$Vor(H_1(\Gamma,\mathbb{R})_\psi)$$

Given $V \in Vor(H_1(\Gamma,\mathbb{R})_\psi)$ we can consider $V^\star$ and we can take $\pi''(V^\star)$. Varying $V \in Vor(H_1(\Gamma,\mathbb{R})_\psi)$ then the set of $\pi''(V^\star)$ forms a polyhedral decomposition of $H_1(\Gamma,\mathbb{R})$, called **Namikawa decomposition** of $H_1(\Gamma,\mathbb{R})$ via bounded polyhedra, denoted by

$$Del_\psi(H_1(\Gamma,\mathbb{R})) \tag{B.8}$$

This decomposition has the same properties described in proposition B.2.1.

It is useful to consider such decompositions in the dual space $H^1(\Gamma,\mathbb{R})$. Using duality, an element $\psi \in \delta C_0(\Gamma,\mathbb{R})$ corresponds to an element $\psi \in \partial C_1(\Gamma,\mathbb{R})$ and the duality induces a linear morphism

$$B : H_1(\Gamma,\mathbb{R}) \to H^1(\Gamma,\mathbb{R})$$

which in general is not unimodular. Indeed the index

$$[H^1(\Gamma,\mathbb{Z}), B(H_1(\Gamma,\mathbb{Z}))] = [\partial C_1(\Gamma,\mathbb{Z}) : \partial\delta C_0(\Gamma,\mathbb{Z})] \tag{B.9}$$

is equal, by Kirchhoff-Trent theorem, to the number of spanning forests of Γ ([OS]4).

We want now give an example of the previous decomposition which is interesting from the point of view of the Mumford construction and that geometrically is induced from the canonical polarization, as we explain in A.2.4. Define with π the projection $\pi : C^1(\Gamma,\mathbb{R}) \to H^1(\Gamma,\mathbb{R})$.

Proposition B.3.4 ([OS]6.2). *Assume that* $\phi \in \partial C_1(\Gamma,\mathbb{R})$ *is of the form* $\phi = \partial(\frac{1}{2}e(E) + y)$ *for some* $y \in C_1(\Gamma,\mathbb{Z})$ *then*

$$Del_\phi(H^1(\Gamma,\mathbb{Z})) = \pi(\frac{1}{2}e(E) + y) + B(Vor(H_1(\Gamma,\mathbb{R})))$$

B.4 Mixed decomposition

In this section we recall some facts related to a relative version of the Voronoi-Delaunay decomposition. Let $\mathcal{S}_g$ be the vector space of quadratic form of degree g over $\mathbb{R}$.
Denote with $\mathcal{C}_g \subset \mathcal{S}_g$ the cone of positive definite quadratic form and with

$$\mathcal{C}_{\mathbb{Z},r} \subset \mathcal{C}_g \tag{B.10}$$

the set of its integral points.
Define $\overline{\mathcal{S}}_g$ to be the convex hull of the set of non-negative integral quadratic forms.

Definition B.4.1. Two elements q_1 and q_2 in $\overline{\mathcal{S}}_g$ are said equivalent if they induce the same Delaunay decomposition.

Each equivalence class gives a cone inside $\overline{\mathcal{S}}_g$.

Definition B.4.2. The closure Σ of a cone $\Sigma^0 = \Sigma^0(q) = \{q_1 \in \overline{\mathcal{S}}_g\; q_1 \sim q\}$ is called **Delaunay-Voronoi cone** in $\overline{\mathcal{S}}_g$ corresponding to q.

Observe that the group $GL(g, \mathbb{Z})$ acts on $\mathcal{S}_g$ as follows. Let $a \in GL(g, \mathbb{Z})$ and $x \in \mathcal{S}_g$ then one defines

$$g \cdot x := g^t x g$$

The trace map allows us to define a duality on the space of symmetric matrices via

$$\langle x, y\rangle := tr(xy)$$

Definition B.4.3. A cone decomposition $\{\Sigma_i\}$ of $\overline{\mathcal{S}}_g$ is called **admissible** if

1. each Σ_i is a rational convex cone, i.e. it is generated by a finite number of integral forms.
2. every face of Σ_i is again contained in this family for every i and every intersection of two cones in the family is again contained in it.
3. the decomposition is invariant under the natural $GL(g, \mathbb{Z})$-action.
4. there are only a finite number of classes of Σ_i modulo $GL(g, \mathbb{Z})$.

The following is a classical result of Voronoi.

Theorem B.4.4. *The Delaunay-Voronoi decomposition is admissible.*

There are other known admissible decompositions like the perfect and the matroidal. Recently they have been considered by many authors, who showed that the Torelli map can be extended to the toroidal compactifications of $\mathcal{A}_g$ obtained via these decompositions (see [A-B] and [M-V]). We explain this a little bit more in the next section.

Definition B.4.5. Let Σ be a Delaunay-Voronoi cone and $\sigma = D(a_i)$ be a Delaunay cell w.r.t. Σ. Every $q \in \Sigma$ defines a bilinear form $\langle , \rangle_q$. Define the **mixed cone** $V_{\Sigma,\sigma}$ by

$$V_{\Sigma,\sigma} = \{(q, x) \in \Sigma \times \mathbb{R}^g | \ \langle y, y + 2a_i\rangle_q + y \cdot x^\top \geq 0 \ \forall \ y \in \mathbb{Z}^g, a_i\}$$

Definition B.4.6. The union of the $V_{\Sigma,\sigma}$ forms a decomposition of $\overline{\mathcal{S}}_g \times \mathbb{R}^r$ called **mixed decomposition**.

The natural fibration $p : V_{\Sigma,\sigma} \to \Sigma$ has as fiber over an element $q \in \Sigma$ the Voronoi cell corresponding to σ ([NamI]3.2).

We introduce now some dual notions.

Denote with $\widehat{\mathcal{S}}_g$ the dual vector space of $\mathcal{S}_g$ w.r.t. the pairing given by the trace.

Given a cone Σ (resp. a mixed cone $V_{\Sigma,\sigma}$) define the following duals

$$\hat{\Sigma} = \{q \in \widehat{\mathcal{S}}_g | \ \langle q, y\rangle \geq 0 \ \forall \ y \in \Sigma\}$$

$$\hat{V}_{\Sigma,\sigma} = \{(q, x) \in \widehat{\mathcal{S}}_g \times \widehat{\mathbb{R}^g} \mid \langle q, y\rangle + \langle x, z\rangle \geq 0 \ \forall \ (y, z) \in V_{\Sigma,\sigma}\}$$

Furthermore we set

$$\widehat{\mathcal{S}}_{r,\mathbb{Z}} := \{q \in \widehat{\mathcal{S}}_g \mid \langle q, y\rangle \in \mathbb{Z}, \ \forall \ y \in C_{r,\mathbb{Z}}\}$$

$$\widehat{\mathbb{R}^g_{\mathbb{Z}}} := \{x \in \mathbb{R}^g \mid \langle x, z\rangle \in \mathbb{Z}, \ \forall \ y \in \mathbb{Z}^g\}$$

Given $a, x \in \mathbb{Z}^g$, we also introduce the elements $A(x; a) \in \hat{\Sigma}$ by the rule

$$\langle A(x; a), q\rangle = \langle x, x + 2a\rangle_q$$

We recall now some properties of the mixed decompostion

Proposition B.4.7.
1. *Every face V of $V_{\Sigma,\sigma}$ is again a mixed cone ([NamI]3.3).*
2. *The natural action of $GL(g, \mathbb{Z})$ on $\overline{\mathcal{S}}_g \times \mathbb{R}^g$ preserves the mixed decomposition and it is equivariant w.r.t. the projection $\overline{\mathcal{S}}_g \times \mathbb{R}^g \to \overline{\mathcal{S}}_g$ ([NamI]3.4).*
3. *$\hat{V}_{\Sigma,\sigma,\mathbb{Z}} \otimes \mathbb{Q}^+$ is generated by $\hat{\Sigma}_{\mathbb{Z}}$ and $(A(x, a), x)$, $a \in \sigma, x \in \mathbb{Z}^g$ ([NamI].3.6).*
4. *The sets $\hat{\Sigma}$ and $\{(A(x, a), x)$ s.t. $a \in \sigma, x \in \mathbb{Z}^g\}$, generate $\widehat{\mathcal{S}}_{r,\mathbb{Z}} \times \widehat{\mathbb{R}^g}_{\mathbb{Z}}$ as group ([NamI].3.6).*

B.4.1 Matroidal decomposition

In this section we want to recall the construction of a decomposition which is more appropriate for the compactified Torelli map. We also give more details about the reduceness assumption we make in chapter 4. The reference for this section is the paper [M-V].
We want to describe a subcone $\mathcal{S}_g^{mat}$ of $\overline{\mathcal{S}}_g$, which has the following properties:

1. $\mathcal{S}_g^{mat}$ has an admissible decomposition $\{\Sigma_i\}_{i\in I}$;
2. every cone in $\{\Sigma_i\}_{i\in I}$ is also a Delaunay-Voronoi cone;
3. it describe in a better way the image of the Torelli map.

Given a connected graph Γ, denote with c_1 (resp. h_1) the rank of $C_1(\Gamma, \mathbb{Z})$ (resp. $H_1(\Gamma, \mathbb{Z})$). Remember that by the Picard-Lefschetz formula or by Proposition 1.6.4 the monodromy matrix B for a degeneration can be written as linear combinations of the rank-one $(h_1 \times h_1)$-matrices $e_k^* \cdot e_k^{*,t}$ where e_k^* are the rows of the matrix A describing the inclusion $H_1(\Gamma, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z})$. It is known from graph theory that these vectors form a set of totally unimodular vectors.

The $GL_g(\mathbb{Z})$-action on the matrix B is induced by $GL_g(\mathbb{Z})$-right multiplication on A and by left multiplication via permutations matrices.

One declares an integral $(c_1 \times h_1)$-matrix A to be *unimodular* if it can be transformed via $GL_g(\mathbb{Z})$-right multiplication to a totally unimodular matrix and *simple* if every row is not zero and any couple of rows is not proportional. For any simple and unimodular $(c_1 \times h_1)$-matrix A, define the following cone in $\overline{\mathcal{S}}_g$

$$\sigma(A) := \sum_{k=1}^{c_1} \mathbb{R}_{\geq 0} e_k^* \cdot e_k^{*,t}$$

and the matroidal decomposition as

$$\overline{\mathcal{S}}_g \supset \mathcal{S}_g^{mat} := \bigcup_{\{A \text{ unimodular and simple}\}} \sigma(A)$$

There is the following result.

Proposition B.4.8 ([M-V]4.0.6)**.** *The decomposition $\mathcal{S}_g^{mat}$ is admissible.*

The name "matroidal" comes from the fact that the $GL_g(\mathbb{Z})$-orbits are in bijection with the set of simple regular matroids of rank at most g.
These objects have been classified and they can be obtained by applying three formal operations, called 1-sum, 2-sum and 3-sum, to 3 special classes

of matroids (Seymour's theorem). Two of these classes come from graphs.

If we denote with $\mathcal{S}_g^V$ the Delaunay-Voronoi decomposition, then we have the following fact which is a consequence of [E-R].

Proposition B.4.9 ([M-V]4.1.4)**.** $\mathcal{S}_g^{mat} \subset \mathcal{S}_g^V$

Unfortunately this decomposition is much more complicated than the Delaunay-Voronoi one. As explained in [M-V] the cells of the Voronoi-Delaunay decomposition are much more than the cells of the matroidal decomposition already starting from $g = 4$.
Furthermore the maximal dimensional cells have not the same dimension and there is exactly one $\binom{g+1}{2}$-dimensional cell which coincides with the *principal cone.*
It was already known to Namikawa in [Nam] that the image of the compactified Torelli map for curves, whose associated graph is planar, sits inside the principal cone and the authors of [M-V] showed that $GL_g(\mathbb{Z})$-equivalence class of the principal cone coincides with the matroidal class corresponding to the complete graph K_{g+1}. It is interesting that to notice that for $g \geq 4$ this graph is not planar (Kuratowski's theorem), hence this enlarges the previous result of Namikawa.

From this description and by dimensional reasons it is also clear that this decomposition has still to be refined if one want to better describe the image of the Torelli map.
Finally it is interesting for us to notice that the proof of the previous proposition uses the fact that the Voronoi-Delaunay decomposition induced from a matrix in $\mathcal{S}_g^{mat}$ is a *generalized lattice dicing.*

Since it is known that for dicings the maximal cells are generating (definition in 1.6.11) we deduce from proposition 1.6.13 that the special fiber of the models we consider in chapter 4 are reduced.

Another simple way to see this for graphs is by using [OS]Corollary 3.2. Indeed using this corollary and the unimodularity of the vectors $\{e_k^*\}$ one sees that the Delaunay decomposition corresponds to the arrangement of hyperplanes induced by these vectors and that the zero dimensional cells of this arrangement of hyperplane coincide with the lattice $H_1(\Gamma, \mathbb{Z})$, hence the maximal dimensional cells are generating.

B.4.2 Olsson's description

We expose in this section the dual version of the Delaunay-Voronoi cones which allows us to smoothly write the Mumford models in terms of log-geometry.

Essentially we need to take track of the variation of the Delaunay-Voronoi cells.
Let us start with a lattice $X \cong \mathbb{Z}^g$ and with an integral regular paving S of $X_{\mathbb{R}}$ (definition in 1.6.9). For every polytope $\omega \in S$, define $Cone(1,\omega)_{\mathbb{Z}}$ as the set of integral point in the cone over $\{1\} \times \omega \subset \mathbb{R} \times X_{\mathbb{R}}$. Consider the direct limit

$$Q := \varinjlim_{\omega \in S} Cone(1,\omega)_{\mathbb{Z}}^{gp}$$

Since the cones cover $\mathbb{Z} \oplus X$, we have a natural map

$$\rho : \mathbb{Z} \oplus X \to Q$$

Define $\widetilde{H_S} \subset Q$ as the monoid generated by the symbols

$$(d,x) * (e,y) := \rho(d,x) + \rho(e,y) - \rho(d+e, x+y) \quad \forall (d,x),(e,y) \in \mathbb{Z} \oplus X$$

The group X acts $\mathbb{Z} \oplus X$ via

$$y \cdot (d,x) = (d, dy + x)$$

This induces an action on the monoid $\widetilde{H}_S$. This definition is motivated from the fact, we have seen in chapter 1, where the action of the periods on a Mumford algebra is described by isomorphisms

$$S_y^* : \qquad T^*_{c^t(y)}(\mathcal{M}^d \otimes \mathcal{O}_{c(x)}) \xrightarrow{\psi(y)^d \tau(x,y)} \mathcal{M}^d \otimes \mathcal{O}_{x+dy}$$

the factor dy we see corresponds to the section $\psi(y)^d$ we get here.

Definition B.4.10. Given an integral regular paving S then the monoid H_S is defined as follows

$$H_S := \tilde{H}_S / X$$

It is better to think about H_S in a dual way. To explain this remember that given the paving S we have at our disposal a positive definite quadratic function a on $X_{\mathbb{Q}}$. This quadratic function gives us a picewise linear function $\tilde{g} : X_{\mathbb{Q}} \to \mathbb{Q}$ whose domains of linearity are described by the paving S. One can extends this function to a function over $\mathbb{Q} \oplus X_{\mathbb{Q}}$ via the rule

$$g(d,x) = d \cdot \tilde{g}(\frac{x}{d})$$

Using this map we obtain a morphism of *monoids*

$$\tilde{h}_S : \tilde{H}_S^{gp} \otimes \mathbb{Q} \to \mathbb{Q}$$

by defining it on generators via

$$\tilde{h}_S((d,x) * (e,y)) := g(d,x) + g(e,y) - g(d+e, y+x)$$

An easy computation shows that

$$\tilde{h}_S((d,x)*(e,y)) = \tilde{h}_S((d,dz+x)*(e,ez+y)) \quad \forall z \in X$$

so that $\tilde{h}_S$ descends to a morphism

$$h_S : H_S^{gp} \otimes \mathbb{Q} \to \mathbb{Q}$$

Remark B.4.11. Observe that if we move the point $\frac{x}{d}$ inside a maximal cell σ then the function g can be described in terms of $-2B(\alpha, \frac{x}{d})$. As we remarked in 4.0.15 and in the previous section, in the case of curves the Delaunay cells are generating, hence if we forget tensoring with $\mathbb{Q}$ it takes values in $\mathbb{Z}$.

There is a morphism

$$\begin{array}{rccc} \tilde{s}: & X \times X & \longrightarrow & \tilde{H}_S^{gp} \\ & (x,y) & \longrightarrow & (1,x+y)*(1,0)-(1,x)*(1,y) \end{array}$$

such that the composition with the quotient map

$$s : X \times X \to \tilde{H}_S^{gp} \to H_S^{gp}$$

is bilinear and symmetric ([Ol]5.8.2). If b denotes the bilinear part of the quadratic form a, then an easy computation shows the equality

$$\tilde{h}_S \circ \tilde{s} = b$$

In general given $h \in Hom(H_S^{gp}, \mathbb{Q})$ we obtain, by composing with $s \otimes 1_{\mathbb{Q}}$, a bilinear form $h \circ (s \otimes 1_{\mathbb{Q}})$ and we have the following result.

Proposition B.4.12. *Let $a \in \overline{\mathcal{S}}_g$ with paving S and let $\Sigma(a)$ be the cone of positive semidefinite quadratic forms whose associated paving is coarser than S. The following hold.*

1. *The map $s \otimes 1_{\mathbb{Q}}$ induces an isomorphism*

$$Hom(H_S^{gp}, \mathbb{Q}) \cong \Sigma(a)^{gp}$$

which identifies $Hom(H_S, \mathbb{Q}_{\geq 0})$ with the cone $\Sigma(a)$.

2. *Let $F \subset H_S^{sat}$ be a face, then the quotient H_S^{sat}/F is isomorphic to $H_{S'}^{sat}/(torsion)$ for some paving S' such that S is finer than S'*

Proof. [Ol]5.8.16,5.8.18. □

Furthermore this monoid has good geometric properties like being finitely generated ([Ol]4.1.6) and sharp ([Ol]4.1.8).

A similar approach also appears in [KKN1]4.7.

A down to earth way to justify the introduction of this monoid is the following. Assume for simplicity we work over a base scheme V which is the spectrum of a discrete valuation ring with uniformizer π. We have seen in chapter 1 that Mumford models are covered by open U_c, $c \in X$. Assume for the moment that we have generators of the form

$$\xi_{x_i,c} := \pi^{e_i(A(x_i-c)+B(x_i,c))} w^{x_i-c} \quad ,(e_i, x_i) \in \mathbb{Z} \oplus X$$

for the algebra whose $Proj$ gives a Mumford model.
By translation invariance we can reduce to the case $c = 0$. If we consider formal products, we get

$$\xi_{x_i,0}\xi_{x_j,0} = \pi^{e_iA(x_i)+e_jA(x_j)} w^{x_i+y_i}$$

and

$$\xi_{x_i+x_j,0} = \pi^{(e_i+e_j)A(x_i+x_j)} w^{x_i+y_i}$$

In particular we obtain the multiplicative relations

$$\xi_{x_i+x_j,0} = \pi^{e_iA(x_i)+e_jA(x_j)-(e_i+e_j)A(x_i+x_j)} \xi_{x_i,0}\xi_{x_j,0} \tag{B.11}$$

In general to obtain the generators we have to substitute to the expression

$$e_iA(x_i) + e_jA(x_j) - (e_i + e_j)A(x_i + x_j)$$

the one given by the associated picewise linear function as follows

$$g(e_i, x_i) + g(e_j, x_j) - g(e_i + e_j, x_i + x_j)$$

and the element

$$\pi^{g(e_i,x_i)+g(e_j,x_j)-g(e_i+e_j,x_i+x_j)}$$

corresponds to the element

$$(e_i, x_i) * (e_j, x_j) \in H_S^{gp}$$

We use the morphism of monoids

$$H_S \to \mathcal{O}_V$$

induced by the previous description which also defines a log-structure on the scheme V. When we take the reduction modulo π the expression

$$\pi^{g(e_i,x_i)+g(e_j,x_j)-g(e_i+e_j,x_i+x_j)}$$

specializes to a unit precisely if the points (e_i, x_i) and (e_j, x_j) are in the same domain of linearity of $\tilde{g}$ which means that they lie in $Cone(1, \omega)$ for some $\omega \in S$. This is precisely what we described in 1.6.10 and, varying $c \in X$, the equations B.11 describe the multiplication relations for the special fiber of a Mumford model. This computation also motivates the definition in [Ol] for the monoid

$$(\mathbb{N} \oplus X) \rtimes H_S$$

where the addition is defined as follows

$$((d, x), h) + ((e, y), k) := ((d + e, x + y), h + k + (d, x) * (e, y)) \qquad \text{(B.12)}$$

We consider this monoid as graded object where the grade is induced by the $\mathbb{N}$ factor. If A denotes the abelian part of the associated Mumford model, then étale locally the associated Mumford algebra is isomorphic to the graded algebra

$$\mathbb{Z}[(\mathbb{N} \oplus X) \rtimes H_S^{sat}] \otimes_{\mathbb{Z}[H_S^{sat}]} \mathcal{O}_A$$

where the log structure on A is defined by pulling back the log structure on V induced by the monoid $\mathbb{N}$. Again the saturation is needed because we are considering normal models. The Mumford model, (étale locally) obtained by taking the Proj of the previous algebra, is what is called "standard family" in [Ol]. From the properties of these monoids one also obtains that the associated Mumford model is flat (integral morphism of log-schemes) and log-smooth over the basis ([Ol]4.1.11).

Appendix C

Biextensions

Given any line bundle L on a scheme X, we can attach to it a $\mathbb{G}m$-torsor, given by the sheaf of invertible sections, which is the complementary of the zero section in the induced $\mathbb{A}^1$-bundle.

In this appendix we want to explain how to traduce in terms of existence of special sections of some torsors the fact that the line bundle $\mathcal{L}_\eta$ (resp. $\Lambda(\mathcal{L}_\eta)$) has a group law which is compatible with the group law of $\mathbb{G}m$ and the one of J_{C_η} (resp. $J_{C_\eta} \times J_{C_\eta}$).
To formalize this we work in more generality.

Let us fix a topos E and let G and A be abelian groups in E. In our application we have a base scheme S and the topos E is the topos of the S-schemes with the big étale topology.
Consider an extension

$$0 \to G \to L \to A \to 0$$

in E meaning that the morphism $\pi : L \to A$ is an epimorphism and

$$G \cong ker(\pi)$$

By group law, which we denote with $*$, on L compatible with the group laws of G and A, we mean for every pair of functorial points $a_1, a_2 : S \to A$ an isomorphism

$$\varphi_{a_1,a_2} : L_{a_1} \times L_{a_2} \overset{\cong}{\to} L_{a_1+a_2}$$

descending to $L_{a_1} \times^G L_{a_2}$. This means that given $l_i \in L_{a_i}$ and $g \in G$ then

$$g \cdot \varphi_{a_1,a_2}(l_1, l_2) = \varphi_{a_1,a_2}(gl_1, l_2) = \varphi_{a_1,a_2}(l_1, gl_2) \tag{C.1}$$

A candidate for such an isomorphism could be a section s of the torsor

$$\Lambda(L) = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$$

Observe that not every section of the previous torsor is good because if we want a commutative group law this section has to be compatible with the flip isomorphism

$$\xi_{a,b} : \Lambda(L)_{a,b} \cong \Lambda(L)_{b,a}$$

and with the cocycle isomorphism

$$\xi_{a,b,c} : \Lambda(L)_{a+b,c} \otimes \Lambda(L)_{a,b} \cong \Lambda(L)_{a,b+c} \otimes \Lambda(L)_{b,c}$$

We recall this fact with the following proposition.

Proposition C.0.13 ([MB] Ch. I, 2.3.10)**.** *There is a bijective correspondence between group laws on L compatible with the group structure of A and G and sections of $\Lambda(L)$ compatible with the homomorphisms $\xi_{a,b}$ and $\xi_{a,b,c}$.*

It is known that the theorem of the square gives a global section of $\Lambda(\mathcal{L}_\eta)$ satisfying the hypothesis of the previous proposition.

The $\mathbb{G}m$-torsor $\Lambda(\mathcal{L}_\eta)$ is also compatible with the two group laws of $J_{C_\eta} \times J_{C_\eta}$ and to do an analogue of the previous construction but with two base group laws we need to introduce the notion of **biextension**.
Given three groups A, B, G in a topos E and a G torsor P on $A \times B$ we want a group law on P compatible with the group laws on A and B. In particular for sections $a : S \to A$ and $b, b_1 : S \to B$ we want isomorphisms

$$\xi_{a;b,b_1} : P_{a,b} \times^G P_{a,b_1} \to P_{a;b+b_1}$$

and the same for $a, a_1 : S \to A$ and $b : S \to B$

$$\xi_{a,a_1;b} : P_{a,b} \times^G P_{a_1,b} \to P_{a+a_1;b}$$

subject to some compatibilities. We use now the following notation

$$X \wedge Y := X \times^G Y$$

Definition C.0.14. A **biextension** P of $A \times B$ via G is a G-torsor over $A \times B$ such that for every morphism $b : S \to B$ and $a : S \to A$ we have extensions of abelian groups

$$0 \to G \to P_a \to B \to 0$$

and

$$0 \to G \to P_b \to A \to 0$$

such that for any couples $a, a_1 : S \to A$ and $b, b_1 : S \to B$ the following diagram commutes

$$\begin{tikzcd}
P_{a,b} \wedge P_{a,b_1} \wedge P_{a_1,b} \wedge P_{a_1,b_1} \arrow[rr, "\cong"] \arrow[d, "\xi_{a;b,b_1} \times \xi_{a_1;b+b_1}"'] & & P_{a,b} \wedge P_{a_1,b} \wedge P_{a,b_1} \wedge P_{a_1,b_1} \arrow[d, "\xi_{a,a_1;b} \times \xi_{a,a_1;b_1}"] \\
P_{a,b+b_1} \wedge P_{a_1,b+b_1} \arrow[dr, "\xi_{a,a_1;b,b_1}"'] & & P_{a+a_1,b} \wedge P_{a+a_1,b_1} \arrow[dl, "\xi_{a,a_1;b,b_1}"'] \\
& P_{a+a_1,b+b_1} &
\end{tikzcd}$$

In order to understand the structure of biextensions in terms of sections of some torsor one needs the concept of **cubical** structures.
Given as before a topos E, abelian groups A, G in E and a G-torsor L we can consider the G-torsor

$$\Theta(L) = \bigotimes_{I \subset \{1,2,3\}} m_I^* L^{\otimes(-1)^{|I|}}$$

on A^3. The symmetric group in 3 elements S_3 acts on $\Theta(L)$, namely for every $\sigma \in S_3$ and sections $a_1, a_2, a_3 : S \to A$ we have isomorphisms

$$\sigma_{a_1,a_2,a_3} : \Theta(L)_{a_1,a_2,a_3} \overset{\cong}{\to} \Theta(L)_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}}$$

Furthermore we have cocycle isomorphisms defined as follows. Define

$$\mathcal{L} := \Lambda(L)$$

then we need the commitativity of the following diagram

$$\begin{tikzcd}
(\mathcal{L}_{x+y+z,t} \wedge \mathcal{L}^{-1}_{x+y,t} \wedge \mathcal{L}^{-1}_{z,t}) \wedge (\mathcal{L}_{x+y,t} \wedge \mathcal{L}^{-1}_{x,t} \wedge \mathcal{L}^{-1}_{y,t}) \arrow[ddd, bend left, "\cong"] \\
\Theta(L)_{x+y,z,t} \wedge \Theta(L)_{x,y,t} \arrow[u, "\cong"] \arrow[d, "\xi_{x,y,z,t}"] \\
\Theta(L)_{x,y+z,t} \wedge \Theta(L)_{y,z,t} \arrow[d, "\cong"'] \\
(\mathcal{L}_{x+y+z,t} \wedge \mathcal{L}^{-1}_{x,t} \wedge \mathcal{L}^{-1}_{y+z,t}) \wedge ((\mathcal{L}_{y+z,t} \wedge \mathcal{L}^{-1}_{y,t} \wedge \mathcal{L}^{-1}_{z,t}))
\end{tikzcd}$$

Definition C.0.15. A **cubic structure** on a G-torsor L over A is a section τ of $\Theta(L)$ which is S_3-invariant and for any $a, b, c, d : S \to A$ we have

$$\xi_{a,b,c,d}(\tau_{a+b,c,d} \otimes \tau_{a,b,d}) = \tau_{a,b+c,d} \otimes \tau_{b,c,d}$$

We give now the following proposition

Proposition C.0.16 ([MB] Ch.I, 2.5.4). *Given a G-torsor L on A then there is a bijective correspondence between biextentions structures on $\Lambda(L)$ of $A \times A$ via G and cubic structures on L.*

The theorem of the cube gives a cubical structure to $\Theta(\mathcal{L}_\eta)$ and in particular we get a symmetric biextension

$$0 \to \mathbb{G}m \to \Lambda(\mathcal{L}_\eta) \to J_{C_\eta} \times J_{C_\eta} \to 0$$

Appendix D

Log-semistable curves

The standard facts about log-geometry can be found in [K], we recall here some notions that we have used in this work.

Definition D.0.17. Given a separably closed field k then a scheme X over k is called semistable variety if for any closed point $\bar{x} \in X$ there exists an étale neighborhood (U, u) and positive integers $m \leq n$ such that U is étale over

$$Spec(k[X_1, \dots, X_n]/(X_1 \cdot \dots \cdot X_m))$$

and the point u is sent to the point corresponding to the ideal $(X_1, \dots, X_n)$.

Definition D.0.18. A log-smooth morphism $f : (X, M_X) \to (S, M_S)$ is called essentially semistable if for each geometric point $\bar{x} \to X$ the monoids $(f^{-1}\overline{M}_S)_{\bar{x}}$ and $\overline{M}_{X,\bar{x}}$ are free and there exist isomorphisms $(f^{-1}\overline{M}_S)_{\bar{x}} \cong \mathbb{N}^r$ and $\overline{M}_{X,\bar{x}} \cong \mathbb{N}^{r+s}$ such that the induced map

$$(f^{-1}\overline{M}_S)_{\bar{x}} \to \overline{M}_{X,\bar{x}}$$

is "multidiagonal" namely on the generators is given by

$$1_i \to \begin{cases} 1_i & \text{if } i \neq r \\ 1_r + 1_{r+1} + \dots + 1_s & \text{if } i = r \end{cases}$$

Essentially semistable morphism are automatically flat and vertical ([OlU]Lemma2.3). Here vertical means that the cokernel of the map

$$f^* M_S \to M_X$$

is a sheaf of groups.
Let $f : (X, M_X) \to (S, M_S)$ be a morphism of log-schemes. Let $I(\overline{M_S})$ be the set of irreducible elements in $\overline{M}_S$. Define

$$C(X) := \{ \text{ set of connected components of the singular points of } X\}$$

If f is essentially semistable then there is a morphism to the set of irreducible elments

$$s_X : C(X) \to I(\overline{M_S})$$

given by sending a component to the unique irreducible element whose image in $\overline{M}_{X,x}$ is not irreducible.

Definition D.0.19. An essentially semistable morphism of log-schemes $f : (X, M_X) \to (S, M_S)$ is called *special* at a geometric point $\bar{s} \in S$ if the map

$$s_{X_{\bar{s}}} : C(X_{\bar{s}}) \to I(\overline{M}_{S,\bar{s}})$$

induces a bijection between the set of connected components of the singular locus of $X_{\bar{s}}$ and $I(\overline{M}_{S,\bar{s}})$. A morphism is called special if it is special at every closed point.

For general facts about special morphisms we suggest [OlU].
Assume that the base scheme is the spectrum of a field and that

$$f : (X, M_X) \to (S, M_S)$$

is special.

There is an isomorphism, induced by s,

$$\overline{M}_S \cong \mathbb{N}^{C(X)}$$

Given $c \in C(X)$ one defines the subsheaves of "the branches at c"

$$\overline{M}_X \supset \overline{M}_c := \left\{ \begin{array}{c} x \in \overline{M}_X \text{ s.t. étale locally } \exists\, y \in \overline{M}_X \text{ with} \\ x + y \text{ is a multiple of } c \end{array} \right\}$$

whose preimage in M_X gives log-structures M_c. One recovers the log-structure using the previous sheaves by push-out w.r.t. $\mathcal{O}_X^*$, i.e. there is an isomorphism

$$M_X \cong \bigoplus_{c \in C(X), \mathcal{O}_X^*} M_c$$

There is also another way to see these sheaves. A connected component corresponding to a $c \in C(X)$ is defined étale locally around a point x by an ideal of the form

$$J_c = (x_1 \cdots \hat{x}_j \cdots x_r)_{j=1}^r$$

One considers the blowup of X along J_c

$$\nu_c : \tilde{X}_c \to X$$

and shows ([Ol04]2.15) that there is an isomorphism

$$\overline{M}_c \cong \nu_{c,*}\mathbb{N}$$

Locally if $\bar{x}$ is a closed point with branches $x_1, \ldots, x_r$ in $\tilde{X}_c$ then the isomorphism is given by

$$\overline{M}_{c,\bar{x}} \to \bigoplus_{x_i} \mathbb{N}_{x_i}$$

Appendix E

Weak normality

In this section we give a proof of proposition 4.0.18.
Recall that all the schemes we are going to consider are defined over an algebraically closed field k.
We closely follow the proof of the seminormality given in [AL04]5.1. We say something more on the Gorenstein property only in the non-degenerate case. For the general case we make a remark after the proof.

Write $Jac^{\phi}_{C_0}$ as GIT quotient

$$\pi : R(E) \to R(E)/PGL(E) \tag{E.1}$$

where E is a k-vector space of finite dimension and $R(E)$ is an open in some Quot scheme as in chapter 2.

Let us consider weak normality first.

It is enough to show that $R(E)$ is weakly normal. Indeed let

$$X \to R(E)/PGL(E)$$

be the weak normalization of $R(E)/PGL(E)$. By definition it is the maximal finite extension which is birational and a universal homeomorphism. If we pull back to $R(E)$ we get that the morphism

$$x : X \times_{R(E)/PGL(E)} R(E) \to R(E)$$

satisfies the same properties and $R(E)$ is weakly normal, hence x is an isomorphism, which implies that $X \to R(E)/PGL(E)$ is an isomorphism.

In order to show that $R(E)$ is weakly normal one uses a factorization given in the proof of [OS]11.8. Namely there exists a diagram of schemes

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ R(E) & & H \end{tikzcd} \tag{E.2}$$

with Y open in $R(E) \times \mathbb{P}(E^{\vee})$, p_1 and p_2 are formally smooth and surjective and $H \subset Hilb^d_{C_0}$ is the open subset parametrizing *reduced*, 0-dimensional subschemes of C_0 with Euler characteristic d, where d is a fixed integer which can be chosen arbitrarily big, such that these points are all *distinct.*
Hence H corresponds to the open $U \subset S^d(C_0)$ where all the points are distinct. The morphism

$$p : C_0^d \to S^d(C_0)$$

is étale there, i.e. $p|_{p^{-1}U} : p^{-1}U \to U$ is étale.
Let us first consider the case in which the ground field is of characteristic zero. In this situation weak normality coincides with seminormality hence we need only to prove this last property.

Since seminormality can be checked on the local completion ([GT]5.3) and since p_1 and p_2 are formally smooth and surjective, then by [GT]5.5 it is enough to prove that H is seminormal.
Again by smooth descent it is enough to prove that C_0^d is seminormal.
Since C_0 is a nodal curve it is seminormal by [GT]8.1. Once we know this we can apply induction on d and [GT]5.9 to conclude that the product is seminormal.

Observe that this also proves seminormality in positive characteristic.

We need now to consider weak normality when the base field is of positive characteristic.

We prove the weak normality of C_0 as follows.
First recall that it is enough to check weak normality at closed points ([Y]Prop.7). Clearly the smooth points are weakly normal so it is enough to see what happens at the nodes. The curve C_0 and its weak normalization C_0^{wn} are homeomorphic, hence for every node $x \in C_0$ there is only one point $p \in C_0^{wn}$ mapping to x. The map induced on the residue field at those points is a purely inseparable extension, but since the ground field is algebraically closed and the nodes are rational then it is an isomorphism. Since we already know that C_0 is seminormal then $C_0^{wn} \to C_0$ is an isomorphism.

For $d \geq 1$ we use an indirect argument which actually shows that the singularities of C_0^d are better that weakly normal. We make use of the Frobenius splitting. Recall the following definition.

Definition E.0.20. Let A be a Noetherian and excellent ring of positive characteristic p. Let $F : A \to A$ be the Frobenius and assume that this is a finite map. The ring A is called *Frobenius split* if there exists an A-linear splitting of the morphism

$$A \to F_* A$$

As consequence of a theorem of Kunz on the flatness of F_*A for regular rings, it is known that if A is regular then A is Frobenius split.

A trivial observation that we are going to use later is that if the scheme $X = Spec(A)$ is Frobenius split and $d \geq 1$ then also X^d is Frobenius split, by taking products of the split.

Remark E.0.21. Usually by Frobenius split for a scheme X one means that there is a global splitting of the morphism

$$\mathcal{O}_X \to F_* \mathcal{O}_X$$

This is not what we mean here, because the schemes we are considering are in general not globally Frobenius split. We consider only the existence of a local splitting.

We need a lemma.

Lemma E.0.22. *[[BK]Prop.1.2.5] If A is Frobenius split, then it is weakly normal.*

Furthermore it is known that if A is one dimensional, weakly normal over a perfect field and the Frobenius is a finite map then A is Frobenius split.

Since we already remarked that C_0 is weakly normal, it is Frobenius split because one dimensional over a perfect field.

If we consider the product C_0^d, it is also Frobenius split by the product property, hence C_0^d is weakly normal by lemma E.0.22.

In order to show that $R(E)$ is weakly normal it is enough to show that weak normality commutes with smooth morphisms, because of the factorization E.2.

To this aim we give a characterization of the weak normality.
Let A be Mori and noetherian ring and let $\bar{A}$ be the normalization of A. Consider the morphism

$$p_1 : \bar{A} \to (\bar{A} \otimes_A \bar{A})_{red}$$

(resp.

$$p_2 : \bar{A} \to (\bar{A} \otimes_A \bar{A})_{red}$$

) defined via $p_1(\bar{a}) := \bar{a} \otimes 1$ (resp. $p_2(\bar{a}) := 1 \otimes \bar{a}$). Define a subring $C_A \subset \bar{A}$ as the following kernel

$$0 \longrightarrow C_A \longrightarrow \bar{A} \xrightarrow{p_1 - p_2} (\bar{A} \otimes_A \bar{A})_{red} \tag{E.3}$$

First we want to show that the construction of C_A commutes with smooth morphisms.

Lemma E.0.23. *Let $Spec(A)$ be an affine scheme with A a Mori and noetherian ring and*

$$f : Spec(B) \to Spec(A)$$

be a smooth surjective morphism.
We have

$$C_A \otimes_A B \cong C_B$$

Proof. The morphism if faithfully flat with normal fibers, hence if $K(A)$ denotes the field of fractions of A then $K(A) \otimes B$ is normal. The fibers of the morphism $\bar{A} \to \bar{A} \otimes_A B$ are reduced, because these are base change of a morphism with such property ([EGA]IV, 6.8.3.iii)). Under these conditions B is also Mori and we have an isomorphism

$$\bar{B} \cong \bar{A} \otimes_A B$$

by [GS] Theorem 3.2.

Since smooth surjective morphisms commutes the reduceness ([EGA]IV,Prop. 17.5.7) and faithful flatness commutes exactness the claim of the lemma follows. □

We want now to prove the following.

Lemma E.0.24. *The scheme $Spec(C_A)$ corresponds to the weak normalization of $Spec(A)$.*

Proof. It is clear that C_A is reduced. Let us show that $Spec(C_A) \to Spec(A)$ is a universal homeomorphism. Since it is finite and surjective, it is enough to show that this is universally injective. This is equivalent to be radicial or to the fact that the diagonal

$$\Delta_{Spec(C_A)/Spec(A)} : Spec(C_A) \to Spec(C_A \otimes_A C_A)$$

is surjective. We want to prove this last property. It is enough to prove this only for the reduced structures. Observe that since $Spec(\bar{A}) \to Spec(C_A)$ is surjective, we have also that

$$Spec(\bar{A} \otimes_A \bar{A}) \to Spec(C_A \otimes_A C_A)$$

is surjective. Hence on the reduced rings we find that

$$(C_A \otimes_A C_A)_{red} \to (\bar{A} \otimes_A \bar{A})_{red} \tag{E.4}$$

is injective (EGA I, Ch. 1, Cor. 1.2.7). The kernel of the multiplication map

$$(C_A \otimes_A C_A)_{red} \to (C_A)_{red} = C_A$$

is generated by the elements $\{c \otimes 1 - 1 \otimes c$, for $c \in C_A\}$. Since by definition of C_A, these elements go to zero in

$$(\bar{A} \otimes_A \bar{A})_{red}$$

and E.4 is injective we find that the multiplication map on C_A is injective. Hence

$$C_A \cong (C_A \otimes_A C_A)_{red}$$

and the claim follows.

We need now to show that C_A is universal among the reduced Mori rings C such that $Spec(C)$ is birational and universally homeomorphic to $Spec(A)$. Namely that for any such C we can find a morphism

$$Spec(C_A) \to Spec(C)$$

over $Spec(A)$.
Assume we have another $C \to A$ with C reduced and Mori, such that the morphism $Spec(C) \to Spec(A)$ is birational and a universal homeomorphism.
A ring C with these properties is necessarily contained in $\bar{A}$ by [EGA]IV,Cor. 18.12.11.
Furthermore by radiciality the diagonal $Spec(C) \to Spec(C \otimes_A C)$ is surjective hence we have an isomorphism

$$C \cong (C \otimes_A C)_{red}$$

This means that if we take the composition

$$C \to \bar{A} \stackrel{p_1 - p_2}{\longrightarrow} (\bar{A} \otimes_A \bar{A})_{red}$$

this is zero, hence we get a morphism $C \to C_A$ and the universality follows. □

Corollary E.0.25. *Let $Spec(A)$ be an affine scheme with A a Mori ring and*

$$Spec(B) \to Spec(A)$$

be a smooth surjective morphism. The scheme $Spec(B)$ is weakly normal if and only if $Spec(A)$ is weakly normal.

Proof. Clear from the previous two lemmas. □

Applying this corollary to the diagram E.2 we finally obtain the weak normality of $R(E)$.

Let us consider the Gorenstein property for non-degenerate polarizations ϕ. Using [OS]11.3 we know that ϕ is non-degenerate if and only if the ϕ-semistable sheaves are ϕ-stable. Furthermore on the stable locus the morphism π given in E.1 is a principal bundle and by definition this means that π is flat and surjective.
We use now the following lemmas.

Lemma E.0.26 ([WITO]Thm.1$'$.2) **).** *Let $f : X \to Y$ be a flat surjective morphism of preschemes then X is Gorenstein if and only if Y and f are Gorenstein.*

Lemma E.0.27 ([WITO]Thm.PartII)**.** *Let A and B two Gorenstein rings containing a common field K. Assume that $A \otimes_K B$ is noetherian and $A/\mathfrak{m}$ finitely generated over K for each maximal ideal $\mathfrak{m}$ of A. Then $A \otimes_K B$ is also a Gorenstein ring.*

This last lemma implies that $p^{-1}(U)$ is Gorenstein, hence also H is such.
Since the projections from Y in diagram E.2 are formally smooth, lemma E.0.26 tells us that Y is also Gorenstein and that $R(E)$ is.
Using the flatness of π (here is the only point where we use the non-degeneracy of ϕ) and again lemma E.0.26 we conclude that $Jac^{\phi}_{C_0}$ is Gorenstein.

This completes the proof of proposition 4.0.18.

Remark E.0.28. Observe that we know that the model

$$\mathcal{P}^{\phi}_0$$

is Gorenstein and seminormal, even more we know that

$$\omega_{\mathcal{P}^{\phi}_0} \cong \mathcal{O}_{\mathcal{P}^{\phi}_0}$$

by [AN]Lemma 4.2. Since $\mathcal{P}^{\phi}_0$ naturally corresponds to the polarization induced from powers of the canonical bundle of the curve, which is the degenerate case, it is natural to expect that the Gorenstein property also extends to $Jac^{\phi}_{C_0}$ for degeneration polarizations. Unfortunately our proof does not work for general polarizations because there are degenerate cases in which the morphism π in the previous proof is not flat and the reason is that in these examples it contracts positive dimensional fibers to a point. We thank Prof. Viviani for pointing this fact to us. However a complete proof of the fact that $Jac^{\phi}_{C_0}$ is Gorenstein also for degenerate polarizations is given in [CMKV] Theorem B i) using methods different from ours.

Remark E.0.29. Our proof shows that over a perfect field of characteristic p and for a non-degenerate polarization ϕ, the scheme $Jac^{\phi}_{C_0}$ is also Frobenius split.

Indeed as consequence of [HR]Prop.5.4, being Frobenius split descends under faithfully flat morphisms. Hence it is enough to prove that the scheme Y in the previous proof is Frobenius split.
We showed that the product C_0^d is Frobenius split, so it is enough to show that for a smooth surjective morphism, if the base scheme is Frobenius split then also the top space is Frobenius split.
Given

$$f : Spec(B) \to Spec(A)$$

smooth surjective of relative dimension n, we can assume, by working locally, that it decomposes as

$$\begin{tikzcd}
Spec(B) \arrow[r, "g"] \arrow[d, "f"'] & Spec(A[x_1, \dots, x_n]) \arrow[dl] \\
Spec(A) &
\end{tikzcd}$$

where g is étale. It is a well known fact that if

$$h : Spec(S) \to Spec(R)$$

is a surjective and étale morphism, then

$$F_*S \cong F_*R \otimes S$$

hence S splits if R does. In particular it is enough to show that $A[x_1, \dots, x_n]$ is Frobenius split when A is it. If

$$\psi : F_*A \to A$$

is a split for A then the morphism

$$\Psi : F_*(A[x_1, \dots, x_n]) \to A[x_1, \dots, x_n]$$

such that $\Psi|_{F_*A} = \psi$, $\Psi(x_i^{pj}) = x_i^j$ and $\Psi(x_i^j) = 0$ if $p \nmid j$, is a split and we are done.

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