Ricci curvature bounds for warped products and cones

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Preface

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Summary

In this thesis we prove generalized lower Ricci curvature bounds in the sense of optimal transport for warped products and cones over metric measure spaces, and we prove a general maximal diameter theorem in this context.

In the first part we focus on the case when the underlying spaces are complete Riemann-Finsler manifolds equipped with a smooth reference measure. The proof is based on calculations for the N-Ricci tensor and on the study of optimal transport of absolutely continuous probability measures in warped products. On the one hand, this result covers a theorem of Bacher and Sturm in [11] concerning Euclidean and spherical N-cones. On the other hand, it can be seen in analogy to a result of Bishop and Alexander in the setting of Alexandrov spaces with curvature bounded from below [2]. Because the warped product metric can degenerate we regard a warped product as a singular metric measure space that is in general neither a Finsler manifold nor an Alexandrov space again but a space satisfying a curvature-dimension condition in the sense of Lott, Sturm and Villani. This result is published in [50].

In the second part we treat the case of general metric measure spaces. The main result states that the (K, N)-cone over any metric measure space satisfies the reduced Riemannian curvature-dimension condition $RCD^*(KN, N+1)$ if and only if the underlying space satisfies $RCD^*(N-1, N)$. The proof uses a characterization of reduced Riemannian curvature-dimension bounds by Bochner's inequality

$$\Delta|\nabla u|^2 \geq \langle \nabla u, \nabla \Delta u \rangle + (N-1)|\nabla u|^2 + \frac{1}{N}(\Delta u)^2$$

that was first established for general metric measure spaces by Erbar, Kuwada and Sturm in [34] and announced independently by Ambrosio, Mondino and Savaré [8]. By application of this result and the Gigli-Cheeger-Gromoll splitting theorem [37] we also prove a maximal diameter theorem for metric measure spaces that satisfy the reduced Riemannian curvature-dimension condition $RCD^*(N-1, N)$. This generalizes the classical maximal diameter theorem for Riemannian manifolds which was proven by Cheng in [28]. These results are contained in the preprint [49].

1 Introduction

Optimal transport has become a powerful tool for the study of metric measure spaces. In particular, synthetic lower Ricci curvature bounds for singular spaces introduced by Lott-Villani [56] and Sturm [78, 79] in terms of optimal transport give a complete new picture on the geometric meaning of Ricci curvature itself. The approach has been extremely successful and a huge number of results and many applications have been obtained in recent years. Still, the field is in rapid progress and pushed further by many mathematicians deepening the understanding of the geometry even for smooth spaces.

Particular objects of interest in connection with these developments are warped products and cones over metric measure spaces. These are construction principles for metric spaces that are known to behave nicely with respect to curvature bounds in the context of Alexandrov spaces.

In this thesis we will show that warped products and cones also preserve Ricci curvature bounds in the sense of the synthetic definition of Lott, Sturm and Villani in precisely the same way as in the case of sectional curvature. In particular, we obtain a result for cones over general metric measure spaces whose consequence is a maximal diameter theorem.

Outline of the chapter. In the introduction we will briefly survey the steps that lead to the current state of the art in the field of metric measure spaces with synthetic Ricci curvature bounds, and we present the main results of the thesis.

We begin in Section 1.1 with the definition of Ricci curvature in a smooth context. We explain how Gromov initiated the problem of defining synthetic Ricci curvature bounds by his pre-compactness theorem for Riemannian manifolds with a uniform lower Ricci curvature bound. After an excursion on metric spaces with curvature bounded from below in the sense of Alexandrov, we recall the theory on Ricci limit spaces that was developed by Cheeger and Colding. In Section 1.2 we describe how optimal transport solves the problem of defining Ricci curvature bounds in a singular framework. In Section 1.3 we explain the relevance of warped products and cones for curvature bounds and we state the main results of the thesis, namely Theorem A, Theorem B and Theorem C. Finally, in Section 1.4 we recall the maximal diameter theorem from smooth Riemannian geometry and its version for Ricci limit spaces. Moreover, we state a general maximal diameter theorem for metric measure spaces that contains the previous ones as special cases.

1.1 Ricci curvature in Riemannian geometry

In Riemannian geometry Ricci curvature means the information that is encoded by the so-called Ricci tensor (named after the Italian mathematician Gregorio Ricci-Curbastro). That is a symmetric 2-tensor field that arises as trace of the Riemannian curvature tensor of a Riemannian manifold M. More precisely, the Ricci curvature in direction of a tangent vector is the mean of the Gauß curvature of all planes in the tangent space, which are perpendicular to each other and intersect at this vector. Lower bounds for the Ricci tensor play a crucial role in numerous classical and modern theorems of Riemannian differential geometry. Let us mention a few results focusing on geometric aspects where M is a d-dimensional Riemannian manifold. They will be motivation and background for the thesis.

- The Bonnet-Myers diameter estimate.

$$\mathrm{ric}_M \geq d-1 \implies \mathrm{diam}_M \leq \pi.$$

- The Cheeger-Gromoll splitting principle [27]. If $\mathrm{ric}_M \geq 0$ and M contains an infinite geodesic line, then it is isometric to a Cartesian product of the form $\mathbb{R} \times M'$ where M' is another Riemannian manifold of non-negative Ricci curvature.

- The Bishop-Gromov volume growth estimate [42, Lemma 5.3.bis].

$$\mathrm{ric}_M \geq K(d-1) \implies \frac{\mathrm{vol}_M(B_R(x))}{\mathrm{vol}_{\mathbb{S}^d_K}(B_R(o))} \leq \frac{\mathrm{vol}_M(B_r(x))}{\mathrm{vol}_{\mathbb{S}^d_K}(B_r(o))} \quad \text{for } 0 < r < R$$

  where $\mathbb{S}^d_K$ is the model space of constant curvature K.

The Bishop-Gromov comparison was actually established by M. Gromov based on a previous volume growth result by R. Bishop (see Section III.3 in [21]). Its consequences are far-reaching. Indeed, Gromov's original intention was not the inequality itself but he wanted to apply a pre-compactness result that was proved earlier by himself for families of metric measure spaces [42, Proposition 5.2]. More precisely, the Bishop-Gromov comparison implies that a family of Riemannian manifolds, which admit a uniform lower bound for the Ricci curvature, a uniform upper bound for the diameter and a uniform upper bound for the dimension, is a uniformly totally bounded class of metric spaces and therefore pre-compact with respect to Gromov-Hausdorff convergence (see Theorem 7.4.15 in [18]).

Now, the following questions seem natural. If we consider a Cauchy sequence in this context, what is the limit space and which properties does it have? The Gromov-Hausdorff distance is a complete metric on the space of compact metric spaces. Hence, within this class we will find a limit. But one will realize immediately that limits of Riemannian manifolds in Gromov-Hausdorff sense will not be manifolds anymore but rather singular length metric spaces.

The program of understanding the *Ricci limit spaces* has been performed by J. Cheeger and T. Colding in a series of articles [23, 24, 25, 26]. They were able to establish many fine and global properties for Ricci limit spaces, e.g. almost everywhere existence of unique tangent cones, a splitting theorem and estimates on the Hausdorff dimension of singularities. The techniques they use are very sophisticated and based on a deep understanding of the Riemannian geometry of the converging sequence. For example, the splitting theorem in this context has to be stated as an *almost splitting theorem*, which is a quantitative version of the original result by Cheeger and Gromoll. In this way, they were able to deduce a complete picture of the geometry of Ricci limit spaces that has been further accomplished by recent contributions of T. Colding, A. Naber and S. Honda [29, 30, 46].

So why bothering about these limits? From the viewpoint of Riemannian geometry the main motivation is the following. Sequences of Riemannian manifolds and their limits arise quite often in proofs. Hence, if we have a priori information on limit spaces, we can exclude certain events and accomplish the proof by a contradiction argument. A good example for this type of technique and the use of the work of Cheeger and Colding is the proof of the generalized Margulis Lemma by V. Kapovitch and B. Wilking [48]. Nevertheless, Cheeger and Colding's approach cannot explain in which sense the limit space itself has bounded Ricci curvature. Since limits are usually singular, there is no chance to establish a curvature tensor in the sense of a smooth Riemannian manifold. This viewpoint is even more interesting in light of the previous motivation. We would like to study limit spaces as objects on their own and not by their approximation.

On the other hand, giving an intrinsic notion of Ricci curvature bounds becomes even more compulsive if we consider the precompactness result as generalization of the following phenomena. Imagine a smooth function $f : [a, b] \to \mathbb{R}$. Convexity of f can be expressed either as non-negativity of its second derivative, or by saying that a line that connects two points on its graph always lies above the graph. It is clear that the former only makes sense for smooth functions while the latter is stable under uniform convergence whose limits are in general not smooth. This type of stability also appears in the context of *lower bounded sectional curvature*. The definition of sectional curvature involves derivatives of the Riemannian metric up to second order. It is more restrictive than Ricci curvature but a generalization to a metric space framework is provided by A. D. Alexandrov's comparison principle. Roughly speaking, the idea goes as follows. By Topogonov's theorem lower (and upper) bounded sectional curvature for a smooth Riemannian manifold is equivalent to the metric statement that triangles are thicker (respectively thinner) than appropriate comparison triangles in a model space of constant curvature. Now, this property makes no use of the differential structure and can be generalized to any length space. Even more, it turns out that this definition is again stable under Gromov-Hausdorff convergence [19, 18] that is a kind of C^0-convergence in the space of metric spaces. Finite dimensional Alexandrov spaces admit very nice local and global properties and in some sense they are *almost* Riemannian.

Hence, the question is wether there is a synthetic notion of generalized lower Ricci curvature that is stable under Gromov-Hausdorff convergence. It was finally answered by optimal transportation theory.

1.2 Synthetic Ricci curvature bounds by optimal transport

The origin of optimal transportation is a problem that was posed by G. Monge in 1781. Imagine a certain good that has to be transported from one location to another one given a cost function that measures the transportation cost per unit from one point in the origin to another point in the destination. What is the optimal strategy that minimizes the total transportation cost? A rigorous mathematical formulation would be as follows. Given two probability measures μ and ν on measurable spaces X and Y respectively and a measurable cost function $c : X \times Y \to \mathbb{R}$ one is looking for a measurable map $T : X \to Y$ such that $T_*\mu = \nu$ and the total cost

$$\int_X c(x, T(x))d\mu(x)$$

is minimized with respect to any map that pushes μ to ν. Although easy to formulate, this problem turned out to be rather tough. In particular, one should not expect to find a map as one can see from the most simple configuration $\mu = \delta_x$ and $\nu = \frac{1}{2}(\delta_y+\delta_z)$ for any triple of points $y \neq z$. The problem was unsolved until L. Kantorovitch gave a reasonable reformulation in terms of couplings in the 40s of the last century [47]. Instead of looking for an optimal map one allows the measure μ to split. Then, the problem's reformulation is: Find a probability measure π on $X \times Y$ with marginals μ and ν such that

$$\int_{X\times Y} c(x, y)d\pi(x, y)$$

is minimized with respect to any coupling between μ and ν. Under lower semi continuity of the cost function c the so-called *Monge-Kantorovitch problem* can be solved by an application of Prohorov's theorem.

Starting from Kantorovitch's work optimal transport turned out to be a very powerful machinery spreading out in many fields of modern mathematics. From a geometer's point of view the most striking contributions were made by Y. Brenier and R. McCann. Let us consider a Riemannian manifold M and the Monge-Kantorovitch problem where we set $X = Y = M$ and $c = \frac{1}{2}\,\mathrm{d}_M^2$. McCann proved in [59] - based on earlier work of Rachev-Rüschendorf [69], and Brenier [17] who treated the Euclidean situation - that for pairs of absolutely continuous probability measures on M one actually can find an optimal map that solves the Monge-Kantorovitch problem. Moreover, any optimal map comes from the gradient of a c-concave function φ and takes the form

$$p \longmapsto T(p) = \exp_p(-\nabla\varphi_p).$$

The notion of c-concavity is a special form of concavity particular adapted to the cost function c (see Section 3.2). From this representation one also gets a dynamical picture of optimal transport on Riemannian manifolds.

$$t \mapsto (T_t)_*\mu = \exp(-t\nabla\varphi)_*\mu,$$

describes a geodesic of probability measures with respect to the *L^2-Wasserstein distance* d_W. For any metric space (X, d_X) this is a metric on the space of probability measures with finite second moment $\mathcal{P}_2(X)$ and was already introduced by Kantorovitch himself. It is defined by the square root of the minimal optimal cost between probability measures with respect to d_X^2. The representation of geodesics of this metric by gradients of c-concave functions opened the door to a fully new view on the geometry of Riemannian manifolds. One began to study its geometry in terms of the *L^2-Wasserstein space* $(\mathcal{P}_2(X), \mathrm{d}_W)$.

McCann's second major contribution was the idea to study convexity properties of functionals on the L^2-Wasserstein space [58]. McCann called this *displacement convexity* to emphasize that this is not convexity in the sense of functionals on the linear vector space of probability measures. Later, Cordero-Erausquin, McCann and Schmuckenschläger [31] proved that a Ricci curvature bound for M directly affects the convexity of the Shannon entropy that is defined by

$$\mathrm{Ent}(\mu) = \int_M \log \rho \; d\mu \quad \text{if} \quad \mu = \rho \; d\,\mathrm{vol}_M$$

and $+\infty$ otherwise. This connection was already suggested by Otto and Villani in [65]. Let us briefly describe the idea. Since a c-concave function φ is also semi-concave in the classical sense, the optimal map between absolutely continuous probability measures μ and ν is differentiable almost everywhere by results of Victor Bangert [13], and a transformation formula holds: $g(x) = \det DT(x) f(T(x))$ where g and f are the densities with respect to vol_M of μ and ν, respectively. Jacobi field calculus and the Ricci curvature bound for M yield an ordinary differential inequality for the functional determinant $\det DT_t(x) = y_x(t)$

$$\log y_x'' \leq -\frac{1}{N}(\log y_x')^2 - \varkappa\theta^2 \leq -\varkappa\theta^2. \tag{1.2.1}$$

where $N \geq \dim M$, $\theta = \mathrm{d}_M(x, T_t(x))$ and $\varkappa$ is the lower Ricci curvature bound for M. From these two ingredients one can prove that the Shannon entropy is $\varkappa$-convex.

This observation was very motivating. Shortly after the work of McCann and his collaborators, Sturm and von Renesse were also able to prove the backward direction in [82]. That is, $\varkappa$-convexity of the Shannon entropy implies that the Ricci tensor is bounded from below by $\varkappa$. This characterization was used by Sturm in [78] and independently by Lott and Villani in [56] to gave a purely synthetic notion of lower Ricci curvature bounds for a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$. They define lower bounded Ricci curvature by $\varkappa$-convexity of the entropy on the L^2-Wasserstein. We

can see that a reference measure is necessary to make sense of absolutely continuous probability measures. Their approach turned out to be very successful. Not only stability under measured Gromov-Hausdorff convergence holds but also a tensorization property, a globalization result and numerous functional inequalities.

Results on Ricci curvature bounds in Riemannian geometry typically involve the dimension of the underlying space. But convexity of the Shannon entropy also allows examples of infinite dimensional type like the real line equipped with the Gauß measure or the abstract Wiener space. For this reason Sturm suggested in [79] to use the full information of inequality (1.2.1) reformulated as

$$(y_x^{\frac{1}{N}})'' \leq -\frac{\varkappa}{N}\theta^2 y_x^{\frac{1}{N}} \tag{1.2.2}$$

for the definition of a *curvature-dimension condition*. One uses the concavity described by (1.2.2) in its integrated form and the Rény entropy $S_N(\mu) = -\int_X \rho^{-\frac{1}{N}}\, d\mu$ (if $d\mu = \rho d\,\mathrm{m}_X + \mu_S$) to include also a dimension parameter N in the definition. The rigorous formulation of the curvature-dimension condition $CD(\varkappa, N)$ and its reduced cousin $CD^*(\varkappa, N)$ is given in Section 2.1 and we omit it at this point. Using the curvature-dimension condition Sturm was able to deduce many geometric properties for the metric measure space in question. For example, a sharp Bonnet-Myers estimate holds for $CD(N-1, N)$-spaces [79].

However, Ricci curvature bounds in the sense of optimal transport cannot distinguish between Riemannian and non-Riemannian type spaces, e.g. any finite dimensional Banach space equipped with the standard Lebesgue measure satisfies a curvature-dimension condition. In this context non-Riemannian means that the induced Laplace-type operator is not linear. Hence, a splitting principle in the sense of Cheeger and Gromoll cannot be true. This was unsatisfactory since the splitting theorem is the model case for rigidity of Riemannian manifolds under lower Ricci curvature bounds. Hence, for the purpose of being able to prove Riemannian-type rigidity results a more restrictive condition was needed. This was done by Ambrosio, Gigli and Savaré in [6]. Their approach makes the linearity of the Laplacian part of the definition where they had to give a rigorous meaning to the objects in question in the context of metric measure spaces. Their approach was inspired by previous work of Cheeger [22]. Surprisingly, the *Riemannian Ricci curvature bound* $RCD(\varkappa, \infty)$ is again equivalent to another unified property, the so-called *evolution variational inequality*. Earlier, Ambrosio-Gigli-Savaré started a program where they identify the evolution of the heat semigroup P_t associated to the Laplacian of a metric measure space as gradient flow w.r.t. the Shannon entropy [4]. The EVI is formulated in terms of this gradient flow

$$\frac{1}{2}\frac{d}{dt}\,\mathrm{d}_W^2(\mu, P_t\psi) + \frac{\varkappa}{2}\,\mathrm{d}_W^2(\mu, P_t\psi) \leq \mathrm{Ent}(\mu) - \mathrm{Ent}(P_t\psi) \tag{1.2.3}$$

for any absolutely continuous probability measure $\psi d\,\mathrm{m}_X \in \mathcal{P}_2(X)$ and any probability measure $\mu \in \mathcal{P}^2(X)$. Equation (1.2.3) gives also an alternative characterization of

gradient flows in a metric space framework and the really astonishing fact is that it rules out any non-Riemannian space. Again, the EVI cannot capture the dimension of the space but Erbar, Kuwada and Sturm gave a modified formulation of EVI in [34] for curvature *and* dimension. One impressive implication of this approach is that a Cheeger-Gromoll splitting principle holds recently proven by N. Gigli [37].

Another aspect of the EVI was that it provides a connection to curvature-dimension bounds in the sense of Bakry and Emery. This is a synthetic formulation of Ricci curvature in the setting of diffusion semigroups and Dirichlet forms. The starting point is the *Bochner formula*

$$\Delta|\nabla u|^2 \geq \langle \nabla u, \nabla \Delta u \rangle + \varkappa |\nabla u|^2 + \frac{1}{N}(\Delta u)^2$$

that can be interpreted as a purely algebraic condition. This approach was introduced by D. Bakry and M. Emery in the 80s of the last century and was also very successful where one of the main contributors was M. Ledoux [53]. However, one depends on a algebraic framework that is usually not available for metric measure spaces. By the work of Erbar, Kuwada and Sturm [34] an equivalence between Ricci bounds in the sense of optimal transport and a modified *Bakry-Emery condition* has been established (the result has been announced independently by Ambrosio, Mondino and Savaré). The former is also known as the *Lagrangian* picture of Ricci curvature that focuses on the quantitative behavior of geodesics. The latter is the *Eulerian* picture where one studies functions and their gradient vector fields. For a more detailed explanation of these notions we refer to Chapter 14 of [80].

1.3 Cones and warped products over metric measure spaces

The concept of warped product between metric spaces B and F is a generalization of the well-known Cartesian product. The second factor F is perturbed by a non-negative Lipschitz function f on the first factor B. This construction is quite standard in Riemannian and metric geometry where warped products play the role of model spaces that show up in numerous situations. The most prominent example is the Euclidean cone with first factor $[0, \infty)$ and warping function $f(r) = r$. In this case the metric is induced by the following semi-metric.

$$\mathrm{d}_{\mathrm{Con}_K}((s,x),(t,y)) = \sqrt{s^2 + t^2 - 2st\cos \mathrm{d}_F(x,y)}.$$

A special feature of warped products and Euclidean cones is that they behave quite nicely under curvature bounds. In the setting of Alexandrov spaces it is standard (e.g. [18]) that the Euclidean cone has curvature bounded from below (CBB) by 0 if and only if the underlying metric space has CBB by 1. For general warped products a similar result was proven by Alexander and Bishop in [2].

In this thesis we will establish analogous results for metric measure spaces with Ricci curvature bounded from below in the sense of Lott, Sturm and Villani. In the first part we will focus on the situation where the underlying spaces are smooth. More precisely, we deduce a curvature-dimension bound for the warped product between B, F and f where B is a Riemannian manifold, F is a weighted Riemann-Finsler manifold and f a smooth function provided suitable conditions for the spaces and the warping function f hold. This result is interesting because examples that mainly inspired the definition of curvature-dimension in the sense of Lott, Sturm and Villani come from Gromov- Hausdorff limits of Riemannian manifolds with a uniform lower bound on the Ricci tensor. Hence, we obtain a new class of examples of metric measure spaces with synthetic Ricci curvature bounds underlining the relevance of the new approach. The main theorem that we will prove in Chapter 3 is

Theorem A. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that* $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. *In any case F satisfies $CD((N-1)K_F, N)$ such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

Then the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$.

The N-warped product is a generalization of the corresponding concept for metric spaces that also involves a suitable reference measure. The precise definition is given in Section 2.2.

In the second part of this thesis, which begins with Chapter 4 we will consider warped products over metric measure spaces. The general framework requires different techniques. The solution is to apply the characterization result of Erbar, Kuwada and Sturm but the prize we pay is that we can prove results only in the framework of Riemannian Ricci curvature bounds. The main results only deal with so-called (K, N)-cones but we conjecture that Theorem A is also true in this context. In the second part we prove the following two theorems.

Theorem B. *Let (F, d_F, m_F) be a metric measure space that satisfies $RCD^*(N-1, N)$ for $N \geq 1$ and* $\operatorname{diam}_F \leq \pi$. *Let $K \geq 0$. Then the (K, N)-cone* $\mathrm{Con}_{N,K}(F)$ *satisfies $RCD^*(KN, N+1)$.*

Theorem C. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space. Suppose the (K, N)-cone* $\mathrm{Con}_{N,K}(F)$ *over F satisfies $RCD^*(KN, N+1)$ for $K \in \mathbb{R}$ and $N \geq 0$. Then*

(1) if $N \geq 1$, F satisfies $RCD^*(N-1, N)$ and $\operatorname{diam} F \leq \pi$,

(2) if $N \in [0, 1)$, F is a point, or $N = 0$ and F consists of exactly two points with distance π.

1.4 The maximal diameter theorem

The most basic geometric property of metric measure spaces satisfying positive Ricci curvature bounds is the Bonnet-Myers estimate. If the space satisfies $RCD^*(N-1,N)$, the diameter is bounded by π. But what happens when the bound is attained? What are the extremal spaces in the Bonnet-Myers estimate?

In the context of Riemannian manifolds such a *maximal diameter theorem* was proven by Cheng in [28] and it provides a rather strong rigidity result. It states that an n-dimensional Riemannian manifold, that has Ricci curvature bounded from below by $n-1$ and attains the maximal diameter π, is the standard sphere $\mathbb{S}^n$. However, a result of Anderson [9] shows that already small perturbations of the diameter destroy this rigidity. Namely, for any even dimension $n \geq 4$ and any $\epsilon > 0$ one can find a Riemannian manifold M_ϵ that satisfies a Ricci bound of $n-1$ and contains points $x, y \in M_\epsilon$ with $\mathrm{d}_{M_\epsilon}(x,y) = \pi - \epsilon = \operatorname{diam} M_\epsilon$ but M_ϵ is not even homeomorph to a sphere for any $\epsilon > 0$. This is in contrast to the situation of Riemannian manifolds with *sectional curvature bounded from below by* 1 where Cheng's rigidity result holds as well but perturbations of the metric do not affect the homeomorphism class of the space as long as $\operatorname{diam}_M > \frac{\pi}{2}$ by a result of Grove and Shiohama [43].

Later, Cheeger and Colding studied this behavior in more detail and gave a refined version of Anderson's result in [23]. They prove that any n-dimensional Riemannian manifold with lower Ricci curvature bound $n-1$ and almost maximal diameter is close in the Gromov-Hausdorff distance to a *spherical suspension* $[0,\pi] \times_{\sin} Y$ over some geodesic metric space Y. Especially, Cheeger and Colding obtain the following maximal diameter theorem for Ricci limit spaces.

Theorem. *Let (X, d_X) be a Ricci limit space of a sequence of n-dimensional Riemannian manifolds M_i with $\mathrm{ric}_{M_i} \geq n-1$ and there are points $x, y \in X$ such that $\mathrm{d}_X(x,y) = \pi$, then there exists a length space (Y, d_Y) with $\operatorname{diam} Y \leq \pi$ such that $[0,\pi] \times_{\sin} Y = X$.*

Hence, in the case of positively curved Ricci limit spaces with maximal diameter one does not get a sphere in general but a spherical suspension. One of the main results of this thesis - obtained as a corollary of Theorem B, Theorem C and the recently established Gigli-Cheeger-Gromoll splitting theorem in the context of $RCD(0,N)$-spaces [37] - is a maximal diameter theorem for RCD^*-spaces.

Theorem D. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space such that $RCD^*(N, N+1)$ holds for $N \geq 0$. If $N = 0$, we assume that $\operatorname{diam} F \leq \pi$. Let x, y be points in F such that $\mathrm{d}_F(x,y) = \pi$. Then, there exists a metric measure space $(F', \mathrm{d}_{F'}, \mathrm{m}_{F'})$ such that $(F, \mathrm{d}_F, \mathrm{m}_F)$ is isomorphic to $[0,\pi] \times_{\sin}^N F'$ and*

(1) if $N \geq 1$, $(F', \mathrm{d}_{F'}, \mathrm{m}_{F'})$ satisfies $RCD^*(N-1,N)$ and $\operatorname{diam} F' \leq \pi$,

(2) if $N \in [0,1)$, F' is a point, or $N = 0$ and F' consists of exactly two points with distance π.

In particular, our theorem includes the result of Cheeger and Colding and it also provides a new proof of the maximal diameter theorem of Cheng since a spherical n-cone, that is a $(n+1)$-dimensional Riemannian manifold, has necessarily to be the standard sphere $\mathbb{S}^{n+1}$.

1.5 Outline of the thesis

This thesis is divided into two parts. Chapter 2 and 3 constitute the first part, and its main results have been published in [50]. In Chapter 2 we provide preliminary material that is used in the thesis. For example, the Wasserstein space, the curvature-dimension condition, the definition of warped products and an introduction to Riemann-Finsler manifolds are presented. In Chapter 3 we prove Theorem A and give some applications.

The second part of the thesis is Chapter 4 and 5. The results of this part are submitted for publication and available in a different form in [49]. In Chapter 4 we provide further preliminary material concerning first order calculus for metric measure spaces. In particular, we present Dirichlet forms, Riemannian Ricci curvature bounds and skew products. Finally, in Chapter 5 we prove Theorem B, Theorem C and the maximal diameter theorem.

2 Preliminaries, part 1

Outline of the chapter. This Chapter provides definitions and results that will be constantly used in this thesis.

In Section 2.1 we introduce elementary notions from the theory of metric measure spaces and Wasserstein geometry, which become the framework for the definition of synthetic Ricci curvature bounds in the sense of Lott, Sturm and Villani. In Section 2.2 we give the definition of warped products between metric measure spaces and as a special case we introduce so-called (K, N)-cones. We give an overview on results established by Bishop and Alexander in the context of Alexandrov spaces. Finally, in Section 2.3 we focus on the smooth situation. We briefly repeat the definition of weighted Riemann-Finsler manifolds, their elementary properties and the warped product construction in this setting. We also give the definition of the N-Ricci tensor. The content of Section 2.3 will only be used in Chapter 3.

2.1 Ricci curvature bounds in the sense of optimal transport

Starting point of the thesis is the following definition of metric measure spaces.

Definition 2.1.1 (Metric measure space)**.** Let (X, d_X) be a complete and separable metric space, and let m_X be a locally finite Borel measure on $(X, \mathcal{O}_X)$ with full support. That is, for all $x \in X$ and all sufficiently small $r > 0$ the volume $\mathrm{m}_X(B_r(x))$ of balls centered at x is positive and finite. A triple $(X, \mathrm{d}_X, \mathrm{m}_X)$ will be called *metric measure space*.

$\mathcal{O}_X$ denotes the topology of open sets with respect to d_X. $\mathcal{O}_X$ generates the corresponding Borel σ-algebra.

Length spaces. The length of a continuous curve $\gamma : [a, b] \subset \mathbb{R} \to X$ is defined as

$$\mathrm{L}(\gamma) := \sup_T \sum_{i=0}^{n-1} d_X(\gamma(t_i), \gamma(t_{i+1})).$$

The supremum is taken with respect to $\{(t_i)_{i=0}^n\} =: T \subset [a, b]$ with $a = t_0 < \cdots < t_n = b$. A curve γ is said to be rectifiable if $\mathrm{L}(\gamma) < \infty$ and the length of a rectifiable curve is independent of reparametrizations. Any rectifiable curve admits a natural parametrization. More precisely, there is a monotone continuous map φ that maps

$[a,b]$ onto $[0,\mathrm{L}(\gamma)]$ such that $\gamma = \widetilde{\gamma}\circ\varphi$ where $\widetilde{\gamma} : [0,\mathrm{L}(\gamma)] \to X$ satisfies $\mathrm{L}(\widetilde{\gamma}|_{[s,t]}) = t-s$ [18, Proposition 2.5.9]. The metric speed of a curve γ is defined as

$$|\dot{\gamma}(t)| = \lim_{h\to 0} \frac{\mathrm{d}_X(\gamma(t),\gamma(t+h))}{h}$$

if the limit exists. If γ is absolutely continuous, then one can prove [18, Theorem 2.7.6] that its metric speed exists almost everywhere and

$$\mathrm{L}(\gamma) = \int_a^b |\dot{\gamma}(t)|dt. \tag{2.1.1}$$

The statement of Theorem 2.7.6 in [18] is for Lipschitz curves, but one can see that the proof also works if the curve is just absolutely continuous. For a complete picture on this subject we refer to [18].

(X,d_X) is called *length space* if $\mathrm{d}_X(x,y) = \inf \mathrm{L}(\gamma)$ for all $x,y \in X$, where the infimum runs over all absolutely continuous curves γ in X connecting x and y. (X,d_X) is called *geodesic space* if every two points $x,y \in X$ are connected by a curve γ such that $\mathrm{d}_X(x,y) = \mathrm{L}(\gamma)$. Distance minimizing curves of constant speed are called *geodesics*. A length space, which is complete and locally compact, is a geodesic space ([18, Theorem 2.5.23]). (X,d_X) is called *non-branching* if for every quadruple (z,x_0,x_1,x_2) of points in X for which z is a midpoint of x_0 and x_1 as well as of x_0 and x_2, it follows that $x_1 = x_2$.

Wasserstein geometry. $\mathcal{P}_2(X,\mathrm{d}_X) = \mathcal{P}_2(X)$ denotes the *L^2-Wasserstein space* of probability measures μ on $(X,\mathcal{O}_X)$ with finite second moments, which means that $\int_X \mathrm{d}_x^2(x_0,x)d\mu(x) < \infty$ for some (hence all) $x_0 \in X$. The *L^2-Wasserstein distance* $\mathrm{d}_W(\mu_0,\mu_1)$ between two probability measures $\mu_0,\mu_1 \in \mathcal{P}_2(X,\mathrm{d}_X)$ is defined as

$$\mathrm{d}_W(\mu_0,\mu_1) = \sqrt{\inf_{\pi} \int_{X\times X} \mathrm{d}_X^2(x,y)\, d\pi(x,y)}. \tag{2.1.2}$$

Here the infimum ranges over all *couplings* of μ_0 and μ_1, i.e. over all probability measures on $X\times X$ with marginals μ_0 and μ_1. $(\mathcal{P}_2(X,\mathrm{d}_X),\mathrm{d}_W)$ is a complete separable metric space. The subspace of m_X-absolutely continuous measures is denoted by $\mathcal{P}_2(X,\mathrm{d}_X,\mathrm{m}_X)$.

A minimizer of (2.1.2) always exists and is called *optimal coupling* between μ_0 and μ_1. A subset $\Gamma \subset X\times X$ is called *d_X^2-cyclically monotone* if and only if for any $k \in \mathbb{N}$ and for any family $(x_1,y_1),\dots,(x_k,y_k)$ of points in Γ the inequality

$$\sum_{i=1}^k \mathrm{d}_X^2(x_i,y_i) \le \sum_{i=1}^k \mathrm{d}_X^2(x_i,y_{i+1})$$

holds with the convention $y_{k+1} = y_1$. Given probability measures μ_0,μ_1 on X, there exists a d_X^2-cyclically monotone subset $\Gamma \subset X\times X$ that contains the support of any

optimal coupling.

A probability measure Π on $\Gamma(X)$ - the set of geodesics in X - is called *dynamical optimal transference plan* if and only if the probability measure $(e_0, e_1)_*\Pi$ on $X \times X$ is an optimal coupling of the probability measures $(e_0)_*\Pi$ and $(e_1)_*\Pi$ on X. Here and in the sequel $e_t : \Gamma(X) \to X$ for $t \in [0,1]$ denotes the evaluation map $\gamma \mapsto \gamma_t$. An absolutely continuous curve μ_t in $\mathcal{P}_2(X, \mathrm{d}_X, \mathrm{m}_X)$ is a geodesic if and only if there is a dynamical optimal transference plan Π such that $(e_t)_*\Pi = \mu_t$.

Definition 2.1.2 (Reduced curvature-dimension condition, [10])**.** Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space. It satisfies the condition $CD^*(\varkappa, N)$ for $\varkappa \in \mathbb{R}$ and $N \in [1, \infty)$ if for each pair $\mu_0, \mu_1 \in \mathcal{P}^2(X, \mathrm{d}_X, \mathrm{m}_X)$ there exists an optimal coupling q of $\mu_0 = \rho_0 \, \mathrm{m}_X$ and $\mu_1 = \rho_1 \, \mathrm{m}_X$ and a geodesic $\mu_t = \rho_t \, \mathrm{m}_X$ in $\mathcal{P}^2(X, \mathrm{d}_X, \mathrm{m}_X)$ connecting them such that

$$\int_X \rho_t^{-1/N'} \rho_t d \, \mathrm{m}_X \geq \int_{X \times X} \left[\sigma^{(1-t)}_{\varkappa, N'}(\mathrm{d}_X) \rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{\varkappa, N'}(\mathrm{d}_X) \rho_1^{-1/N'}(x_1) \right] dq(x_0, x_1) \tag{2.1.3}$$

for all $t \in (0,1)$ and all $N' \geq N$ where $\mathrm{d}_X := \mathrm{d}_X(x_0, x_1)$. In the case $\varkappa > 0$, the *volume distortion coefficients* $\sigma^{(t)}_{\varkappa, N}(\cdot)$ for $t \in (0,1)$ are defined by

$$\sigma^{(t)}_{\varkappa, N}(\theta) = \frac{\sin\left(\sqrt{\varkappa/N}\theta t\right)}{\sin\left(\sqrt{\varkappa/N}\theta\right)}$$

if $0 \leq \theta < \sqrt{\frac{N}{\varkappa}}\pi$ and by $\sigma^{(t)}_{\varkappa, N}(\theta) = \infty$ if $\varkappa\theta^2 \geq N\pi^2$. If $\varkappa\theta^2 = 0$, one sets $\sigma^{(t)}_{0, N}(\theta) = t$, and in the case $\varkappa < 0$ one has to replace $\sin\left(\sqrt{\frac{\varkappa}{N}}\cdot\right)$ by $\sinh\left(\sqrt{\frac{-\varkappa}{N}}\cdot\right)$. In particular, the space is connected.

Definition 2.1.3 (Curvature-dimension condition, [79])**.** Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space. It satisfies the curvature-dimension condition $CD(\varkappa, N)$ for $\varkappa \in \mathbb{R}$ and $N \in [1, \infty)$ if we replace in Definition 2.1.2 the coefficients $\sigma^{(t)}_{\varkappa, N}(\theta)$ by

$$\tau^{(t)}_{\varkappa, N}(\theta) = \begin{cases} \infty & \text{if } \varkappa\theta^2 > (N-1)\pi^2, \\ t^{1/N} \cdot \sigma^{(t)}_{\varkappa, N-1}(\theta)^{1-1/N} & \text{if } \varkappa\theta^2 \leq (N-1)\pi^2 \ \ \& \ \ N > 1, \\ t & \text{if } \varkappa\theta^2 \leq 0 \ \ \& \ \ N = 1. \end{cases}$$

By definition a single point satisfies $CD(\varkappa, N)$ for any $\varkappa > 0$ and $N = 1$. This is the original condition that was introduced by Sturm in [79].

Remark 2.1.4. If the metric measure space is a Riemannian manifold, the reduced and non-reduced condition are equivalent and one conjectures that this should hold also in a more general framework. It is clear that $CD(0, N) = CD^*(0, N)$ and in any case, there are the following implications

$$\begin{aligned} CD(\varkappa, N) &\implies CD^*(\varkappa, N) \quad \text{for any } \varkappa \in \mathbb{R} \text{ and } N \geq 1, \\ CD^*(\varkappa, N) &\iff CD_{loc}(\varkappa, N) \quad \text{for any } \varkappa \in \mathbb{R} \text{ and } N > 1 \end{aligned}$$

(see [10, 32]) where the definition of $CD_{loc}(\varkappa, N)$ can be found for example in [79]. It turns out that the reduced curvature-dimension is more suitable for applications because of the easier form of the coefficients $\sigma^{(t)}_{\varkappa,N}$.

We point out the exceptional character of the case $N = 1$. In particular, there are no metric measure spaces that satisfy $CD(\varkappa, 1)$ for $\varkappa > 0$ except points.

Doubling property. A metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ that satisfies $CD^*(\varkappa, N)$ for some $\varkappa \in \mathbb{R}$ and $N \geq 1$, satisfies a doubling property on each bounded subset $X' \subset \operatorname{supp} \mathrm{m}$ (see [79, 10]). More precisely, there exists a constant $C > 0$ such that for each $r > 0$ and $x \in X'$ with $B_{2r}(x) \subset X'$ we have

$$\mathrm{m}_X(B_{2r}(x)) \leq C\, \mathrm{m}_X(B_r(x)) \tag{2.1.4}$$

In particular each bounded closed subset is compact and $(X, \mathrm{d}_X, \mathrm{m}_X)$ is locally compact. If $\varkappa \geq 0$ or $N \geq 1$ the doubling constant can be chosen uniformily for the whole space and is $\leq 2^N$. Then, we also say the metric measure space satisfies a doubling property. The doubling estimate can be refined. That is, there is $C > 0$ and $N \geq 1$ such that

$$\mathrm{m}_X(B_R(x)) \leq C \left(\frac{R}{r}\right)^N \mathrm{m}_X(B_r(x)) \quad \text{for any } r < R \text{ and } x \in X. \tag{2.1.5}$$

Measure contraction property. If $(X, \mathrm{d}_X, \mathrm{m}_X)$ is non-branching then the reduced curvature-dimension condition $CD^*(\varkappa, N)$ implies the measure contraction property $MCP(\varkappa, N)$ by a result of Cavalletti and Sturm [20] where $\varkappa \in \mathbb{R}$ if $N > 1$ and $\varkappa = 0$ if $N = 1$. There are two different definitions of the measure contraction property by Ohta in [61] and by Sturm in [79]. The latter is more restrictive and implies the former. In a non-branching situation the definitions coincide and it can be stated as follows:

Definition 2.1.5 (Measure contraction property, [60, 79])**.** Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a non-branching metric measure space. Then it satisfies the measure contraction property $MCP(\varkappa, N)$ if for any $x \in X$, for any measurable subset $A \subset X$ with $\mathrm{m}_X(A) < \infty$ (and $A \subset B_{\pi\sqrt{(N-1)/\varkappa}}(x)$ if $\varkappa > 0$) and the unique L^2-Wasserstein geodesic Π such that $\delta_x = (e_0)_*\Pi$ and $\mathrm{m}_X(A)^{-1}\, \mathrm{m}_X = (e_1)_*\Pi$ we have

$$d\, \mathrm{m}_X \geq (e_t)_* \left(\tau^{(t)}_{\varkappa,N}(\mathrm{L}(\gamma))^N\, \mathrm{m}_X(A) d\Pi(\gamma)\right).$$

Again, by definition a single point satisfies $MCP(\varkappa, 1)$ for any $\varkappa > 0$, and $\varkappa > 0$ and $N = 1$ can only appear in this case.

A corollary of the measure contraction property Ohta is the Bonnet-Myers Theorem.

Theorem 2.1.6 (Generalized Bonnet-Myers Theorem, [60]). *Assume that a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $MCP(\varkappa, N)$ for some $\varkappa > 0$ and $N > 1$. Then the diameter of (X, d_X) is bounded by $\pi\sqrt{\frac{N-1}{\varkappa}}$. Especially, a metric measure space that is non-branching and satisfies the reduced curvature-dimension condition $CD^*(\varkappa, N)$ for $\varkappa > 0$ and $N > 1$ has bounded diameter by $\pi\sqrt{\frac{N-1}{\varkappa}}$.*

Remark 2.1.7. One can easily see that the generalized Bonnet-Myers Theorem is an immediate consequence of the condition $CD(\varkappa, N)$ for $\varkappa > 0$ and $N > 1$ even without any non-branching assumption.

2.2 Warped products and cones

Let (B, d_B) and (F, d_F) be length spaces that are complete and locally compact. Let $f : B \to \mathbb{R}_{\geq 0}$ be locally Lipschitz. Let us consider a continuous curve $\gamma = (\alpha, \beta) : [a, b] \to B \times F$. We define the length of γ by

$$\mathrm{L}(\gamma) := \sup_T \sum_{i=0}^{n-1} \left(\mathrm{d}_B(\alpha(t_i), \alpha(t_{i+1}))^2 + f(\alpha(t_{i+1}))^2\, \mathrm{d}_F(\beta(t_i), \beta(t_{i+1}))^2\right)^{\frac{1}{2}}$$

where the supremum is taken with respect to $\{(t_i)_{i=0}^n\} =: T \subset [a, b]$ with $a = t_0 < \cdots < t_n = b$. We call a curve $\gamma = (\alpha, \beta)$ in $B \times F$ admissible if α and β are rectifiable in B and F, respectively, and for admissible curves one can see that $\mathrm{L}(\gamma) < \infty$. If α and β are absolutely continuous, then

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{|\dot{\alpha}(t)|^2 + (f \circ \alpha)^2(t)|\dot{\beta}(t)|^2}dt.$$

L is a length-structure on the class of admissible curves. For details see [18] and [1]. We can define a semi-distance between (p, x) and (q, y) by

$$\inf \mathrm{L}(\gamma) =: \mathrm{d}_C((p, x), (q, y)) \in [0, \infty)$$

where the infimum ranges over all admissible curves γ that connect (p, x) and (q, y).

Definition 2.2.1. The warped product of metric spaces (B, d_B) and (F, d_F) with respect to a locally Lipschitz function $f : B \to \mathbb{R}_{\geq 0}$ is given by

$$(C := B \times F/_\sim, \mathrm{d}_C) =: B \times_f F$$

where the equivalence relation $\sim$ is given by

$$(p, x) \sim (q, y) \iff \mathrm{d}_C\left((p, x), (q, y)\right) = 0$$

and the metric distance is $\mathrm{d}_C\left([(p, x)], [(q, y)]\right) := \mathrm{d}_C\left((p, x), (q, y)\right)$.

Remark 2.2.2. One can see that

$$C = (\mathring{B} \times_{\mathring{f}} F) \dot{\cup} f^{-1}(\{0\}) \text{ where } B \backslash f^{-1}(\{0\}) =: \mathring{B} \text{ and } \mathring{f} = f|_{\mathring{B}}.$$

We will often make use of the notation $\mathring{C} := \mathring{B} \times_{\mathring{f}} F$. $B \times_f F$ is a length space. Completness and local compactness follow from the corresponding properties of B and F. It follows that $B \times_f F$ is geodesic. Especially for every pair of points we find a geodesic between them.

The next two theorems by Alexander and Bishop describe the behavior of geodesics in warped products.

Theorem 2.2.3 ([1, Theorem 3.1])**.** *For a minimizer $\gamma = (\alpha, \beta)$ in $B \times_f F$ with $f > 0$ we have*

1. *β is pregeodesic in F and has speed proportional to $f^{-2} \circ \alpha$. .*
2. *α is independent of F, except for the length of β.*
3. *If β is non-constant, γ has a parametrization proportional to arclength satisfying the energy equation $\frac{1}{2}v^2 + \frac{1}{2f^2} = E$ almost everywhere, where v is the speed of α and E is constant.*

Theorem 2.2.4 ([2, Theorem 7.3])**.** *Let $\gamma = (\alpha, \beta)$ be a minimizer in $B \times_f F$ that intersects $f^{-1}(\{0\}) =: X$.*

1. *If γ has an endpoint in X, then α is a minimizer in B.*
2. *β is constant on each determinate subinterval.*
3. *α is independent of F, except for the distance between the endpoint values of β. The images of the other determinate subintervals are arbitrary.*

Remark 2.2.5. A pregeodesic is a curve, whose length is distance minimizing but not necessarily of constant speed. A determinate subinterval J of definition for β is an interval, where $f \circ \alpha$ does not vanish, e.g. $t \in J$ if $f \circ \alpha(t) > 0$.

Definition 2.2.6. For a metric space (X, d_X), the *K-cone* $\mathrm{Con}_K(X)$ is a metric space defined as follows:

- $(I_K \times X)/_\sim$ where $I_K = \begin{cases} [0, \pi/\sqrt{K}] & \text{if } K > 0 \\ [0, \infty) & \text{if } K \leq 0 \end{cases}$
  and $(s, x) \sim (t, y) \Leftrightarrow (s, x) = (t, y) \vee s = t \in \partial I_K$.
- For $(x, s), (x', t) \in (I_K \times X)/_{(X \times \partial I_K)}$

$$\mathrm{d}_{\mathrm{Con}_K}((x, s), (x', t)) := \begin{cases} \cos_K^{-1}\left(\cos_K(s)\cos_K(t) + K \sin_K(s)\sin_K(t)\cos\left(d(x, x') \wedge \pi\right)\right) & \text{if } K \neq 0 \\ \sqrt{s^2 + t^2 - 2st\cos\left(d(x, x') \wedge \pi\right)} & \text{if } K = 0. \end{cases}$$

where $\sin_K(t) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}t)$ and $\cos_K(t) = \cos(\sqrt{K}t)$ for $K > 0$ and $\sin_K(t) = \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}t)$ and $\cos_K(t) = \cosh(\sqrt{-K}t)$ for $K < 0$.

The K-cone with respect to (X, d_X) is a length (resp. geodesic) metric spaces if and only if (X, d_X) is a length (resp. geodesic) at distances less than π (see [18, Theorem 3.6.17] for $K = 0$).

Remark 2.2.7. If $\mathrm{diam}_X \leq \pi$, the K-cone coincides with the warped product $I_K \times_{\sin_K} X$. This follows easily from Theorem 2.2.3 and from the fact that $\mathring{I}_K \times_{\sin_K} (0, \pi)$ equals the open upper half plane of the 2-dimensional model space of constant curvature K and we can express distances there in polar coordinates.

We have to introduce a reference measure on C, which reflects the warped product construction. In general we define

Definition 2.2.8 (N-warped product)**.** Let $(B, \mathrm{d}_B, \mathrm{m}_B)$ and $(F, \mathrm{d}_F, \mathrm{m}_F)$ be length metric measure spaces. For $N \in [0, \infty)$, the *N-warped product* $(C, \mathrm{d}_C, \mathrm{m}_C) = B \times_f^N F$ of B, F and f is a metric measure space defined as follows:

- $C := B \times_f F = (B \times F/_\sim, \mathrm{d}_C)$
- $d\,\mathrm{m}_C(p, x) := \begin{cases} f^N(p) d\,\mathrm{m}_B(p) \otimes d\,\mathrm{m}_F(x) & \text{on } \mathring{C} \\ 0 & \text{on } C \backslash \mathring{C}. \end{cases}$

In the setting of K-cones we can introduce a measure in the same way. We call the resulting metric measure space a (K, N)-cone.

$\mathcal{F}K$-concavity. $f : B \to \mathbb{R}_{\geq 0}$ is said to be $\mathcal{F}K$-concave if its restriction to every unit-speed geodesic γ satisfies

$$f \circ \gamma(t) \geq \sigma^{(1-t)} f \circ \gamma(0) + \sigma^{(t)} f \circ \gamma(\theta) \text{ for all } t \in [0, \theta]$$

where $\theta = \mathrm{L}(\gamma)$. For the definition of $\sigma^{(t)} = \sigma^{(t)}_{1,1}(\theta)$ see the remark directly after Definition 2.1.3. This is equivalent to that $f \circ \gamma$ is a sub-solution of

$$\begin{aligned} u'' &= -Ku \quad \text{on } (0, \theta) \\ u(0) &= f \circ \gamma(0), \quad u(\theta) = f \circ \gamma(\theta). \end{aligned}$$

Alexandrov spaces. We briefly present well-known results in the setting of Alexandrov spaces with curvature bounded from below (CBB). A nice introduction to Alexandrov spaces can be found in [18]. Let B and F be finite-dimensional Alexandrov spaces with CBB by K and K_F respectively. We assume that f is $\mathcal{F}K$-concave and

$$K_F \geq K f^2(p) \quad \text{and} \quad Df_p \leq \sqrt{K_F - K f^2(p)} \quad \forall p \in B. \tag{2.2.1}$$

Df_p is the modulus of the gradient of f at p in the sense of Alexandrov geometry (see for example [68]). For $\mathcal{F}K$-concave functions on finite dimensional Alexandrov spaces Df_p is always well-defined and, if B is a Riemannian manifold, this notion coincides with the usual one where Df_p can always be replaced by $|\nabla f_p|$.

In [2] Alexander and Bishop prove the following result.

Proposition 2.2.9 ([2, Proposition 3.1])**.** *For an $\mathcal{F}K$-concave function $f : B \to [0, \infty)$ on some Alexandrov space B with CBB by K, the condition (2.2.1) is equivalent to*

1. *If $(f|\partial B)^{-1}(0) = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $(f|\partial B)^{-1}(0) \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.*

Remark 2.2.10. In the proof of the previous proposition Alexander and Bishop especially deduce the following result. Let f be $\mathcal{F}K$-concave and assume it is not identical 0. Let B be as in Proposition 2.2.9. Then f is positive on non-boundary points: $f^{-1}(\{0\}) \subset \partial B$. Especially $(f|\partial B)^{-1}(0) = f^{-1}(0)$.

Proposition 2.2.11 ([2, Proposition 7.2])**.** *Let $f : B \to \mathbb{R}_{\geq 0}$ and B as in the previous proposition. Suppose $X = f^{-1}(\{0\}) \neq \emptyset$ and $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$. Then we have: Any minimizer in $B \times_f F$ joining two points not in X, and intersecting X, consists of two horizontal segments whose projections to F are $\pi/\sqrt{K_F}$ apart, joined by a point in X.*

The main theorem of Alexander and Bishop concerning warped products is:

Theorem 2.2.12 ([2, Theorem 1.2])**.** *Let B and F be complete, finite-dimensional spaces with CBB by K and K_F respectively. Let $f : B \to \mathbb{R}_{\geq 0}$ be an $\mathcal{F}K$-concave, locally Lipschitz function satisfying the boundary condition (†). Set $X = f^{-1}(\{0\}) \subset \partial B$.*

1. *If $X = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $X \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.*

Then the warped product $B \times_f F$ has CBB by K.

(†) If $B^\dagger$ is the result of gluing two copies of B on the closure of the set of boundary points where f is nonvanishing, and $f^\dagger : B^\dagger \to \mathbb{R}_{\geq 0}$ is the tautological extension of f, then $B^\dagger$ has CBB by K and $f^\dagger$ is $\mathcal{F}K$-concave.

If we assume that $\partial B \subset f^{-1}(\{0\})$, then $(f|\partial B)^{-1}(0) = f^{-1}(\{0\}) = \partial B$ and the boundary condition (†) does not play a role.

2.3 Riemann-Finsler Manifolds

In this section we investigate smooth metric measure spaces in more detail. More precisely, this is the class of Riemann-Finsler manifolds equipped with a smooth reference measure. We recall the definition.

Definition 2.3.1 (Riemann-Finsler manifolds)**.** A Finsler structure on a C^∞-manifold M is a function $\mathcal{F}_M : TM \to [0, \infty)$ satisfying the following conditions:

(1) (Regularity) $\mathcal{F}_M$ is C^∞ on $TM\backslash 0_M$, where $0_M : M \to TM$ with $0_M|_p = 0 \in TM_p$ denotes the zero section of TM.

(2) (Positive homogeneity) For any $v \in TM$ and positive number $\lambda > 0$ we have $\mathcal{F}_M(\lambda v) = \lambda \mathcal{F}_M(v)$.

(3) (Strong convexity) Given local coordinates $(x^i, v^i)_{i=1}^n$ on $\pi^{-1}(U) \subset TM$ for $U \subset M$, then

$$(g_{i,j}(v)) := \left(\frac{1}{2} \frac{\partial^2 (\mathcal{F}_M^2)}{\partial v^i \partial v^j}(v) \right) \tag{2.3.1}$$

is positive-definite at every $v \in \pi^{-1}(U)\backslash 0$.

We call $(g_{i,j})_{1 \le i,j \le n}$ fundamental tensor and $(M, \mathcal{F}_M)$ a Riemann-Finsler manifold. $(g_{i,j})_{i,j}$ can be interpreted as Riemannian metric on the vector bundle

$$\bigcup_{v \in TM\backslash 0_M} TM_{\pi(v)} \to TM$$

that associates to every $v_p \in TM_p$ again a copy of TM_p itself. An important property of the fundamental tensor for us is its invariance under vertical rescaling:

$$g_{i,j}(v) = g_{i,j}(\lambda v) \text{ for every } \lambda > 0.$$

The Finsler structure induces a distance that makes the Riemann-Finsler manifold a metric space except for the symmetry of the distance. Because we only consider symmetric metrics, we additionally assume

(4) (Symmetry) $\mathcal{F}_M(v) = \mathcal{F}_M(-v)$.

The definition of Riemann-Finsler manifolds includes the class of Riemannian manifolds. If $(M, \mathcal{F}_M)$ is purely Riemannian, we will write $\mathcal{F}_M = g_M$.

Remark 2.3.2. Although we assume the Finsler structure $\mathcal{F}_M$ to be C^∞-smooth (what we will call just smooth) outside the zero section, the lack of regularity at 0_M is worse than one would expect. Namely $\mathcal{F}_M^2$ is C^2 on TM if and only if $\mathcal{F}_M$ is Riemannian. Otherwise we only get a regularity of order $C^{1+\alpha}$ for some $0 < \alpha < 1$. (For the statement that we have C^2 if and only if we are in a smooth Riemannian setting, see Proposition 11.3.3 in [73].) This fact has important consequences for warped products in the setting of Riemann-Finsler manifolds.

A weighted Riemann-Finsler manifold is a triple $(M, \mathcal{F}_M, \mathrm{m}_M)$ where $(M, \mathcal{F}_M)$ is a Riemann-Finsler manifold and m_M is a positive Radon measure. In this context the measure m_M is assumed to be smooth. That means, if we consider M in local coordinates, the measure m_M is absolutely continuous with respect to $\mathcal{L}^n$ and the density is a smooth and positive function. We remark that there is no canonical volume for Riemann-Finsler manifolds like in the purely Riemannian case. In the Riemannian case we are always able to write $d\,\mathrm{m}_M = e^{-\Psi(x)} d\,\mathrm{vol}_M(x)$ for some smooth function $\Psi : M \to \mathbb{R}$. We also use the notation $(M, \mathrm{m}_M) = (M, \Psi)$.

Definition 2.3.3 (N-Ricci curvature)**.** Given a complete n-dimensional Riemannian manifold M equipped with its Riemannian distance d_M and weighted with a smooth measure $d\,\mathrm{m}_M(x) = e^{-\Psi(x)} d\,\mathrm{vol}_M(x)$ for some smooth function $\Psi : M \to \mathbb{R}$. Then for each real number $N > n$ the N-Ricci tensor is defined as

$$\begin{aligned}\mathrm{ric}^{N,\mathrm{m}_M}(v) := \mathrm{ric}^{N,\Psi}(v) &:= \mathrm{ric}(v) + \nabla^2\Psi(v) - \frac{1}{N-n}\nabla\Psi \otimes \nabla\Psi(v) \\ &= \mathrm{ric}(v) - (N-n)\frac{\nabla^2 e^{-\Psi\frac{1}{N-n}}(v)}{e^{-\Psi\frac{1}{N-n}}(p)}\end{aligned}$$

where $v \in TM_p$. For $N = n$ we define

$$\mathrm{ric}^{N,\Psi}(v) := \begin{cases} \mathrm{ric}(v) + \nabla^2\Psi(v) & \nabla\Psi(v) = 0 \\ -\infty & \text{else.} \end{cases}$$

For $1 \leq N < n$ we define $\mathrm{ric}^{N,\Psi}(v) := -\infty$ for all $v \neq 0$ and 0 otherwise.

We switch again to the setting of weighted Riemann-Finsler manifolds $(M, \mathcal{F}_M, \mathrm{m}_M)$. Ohta introduced in [62] the N-Ricci tensor for a weighted Riemann-Finsler manifold that we define now. For $v \in TM_p$ choose a vector field V on a neighborhood $U \ni p$ such that $v = V_p$ and every integral curve of V is a geodesic. That is always possible and we call such a vector field a geodesic vector field. Because of the strong convexity property of $\mathcal{F}_M$ the vector field V induces a Riemannian structure on U by

$$g_p^V := \sum_{i,j=1}^{n} (g_{i,j})(V_p) dx_p^i \otimes dx_p^j \quad \text{for all } p \in U.$$

and we have the following representation $d\,\mathrm{m}_M = e^{-\Psi_V} d\,\mathrm{vol}_{g^V}$ on U for some smooth function Ψ_V. Then for $N \geq 1$ the N-Ricci tensor at v is defined as

$$\mathrm{ric}^{N,\mathrm{m}_M}(v) := \mathrm{ric}^{N,\Psi_V}(v).$$

The benefit of this definition is the following result.

Theorem 2.3.4 ([79], [62])**.** *A weighted complete Riemann-Finsler manifold without boundary $(M, \mathcal{F}_M, \mathrm{m}_M)$ satisfies the condition $CD(\varkappa, N)$ if and only if*

$$\mathrm{ric}^{N,\mathrm{m}_M}(v, v) \geq \varkappa \mathcal{F}_M^2(v) \quad \textit{for all } v \in TM.$$

Warped products over Riemann-Finsler manifolds. Let B and F metric measure spaces and let f be a locally Lipschitz function. We assume that $\mathring{B}$ and F are Riemannian manifolds with dimension d and n respectively and $\mathring{f} = f|_{\mathring{B}}$ is smooth. The Riemannian warped product with respect to $\mathring{f}$ is defined in the following way:

$$\mathring{C} := \mathring{B} \times_{\mathring{f}} F := (\mathring{B} \times F, g)$$

where the Riemannian metric g is given by

$$g := (\pi_B)^* g_B + (f \circ \pi_B)^2 (\pi_F)^* g_F.$$

g_B and g_F are the Riemannian metrics of $\mathring{B}$ and F respectively. The length of a Lipschitz-continuous curve $\gamma = (\alpha, \beta)$ in $\mathring{C}$ with respect to the metric g is given by

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{g_B(\dot{\alpha}(t), \dot{\alpha}(t)) + f^2 \circ \alpha(t) g_F(\dot{\beta}(t), \dot{\beta}(t))} dt$$

So the Riemannian distance on $\mathring{C}$ is defined by $|(p,x),(q,y)| = \inf \mathrm{L}(\gamma)$ where the infimum is with respect to all Lipschitz curves that are joining (p,x) and (q,y) in $\mathring{C}$. It is easy to see that the Riemannian warped product $\mathring{C}$ as metric space embeds in the metric space warped product C and the metrics coincide on $C \backslash f^{-1}(\{0\})$.

Now we define warped products between Riemann-Finsler manifolds $(B, \mathcal{F}_B)$ and $(F, \mathcal{F}_F)$ with respect to a smooth function $f : B \to [0, \infty)$ and exactly like in the Riemannian case we can define a warped product Finsler structure explicitly on $\mathring{B} \times F$ by

$$\mathcal{F}_{B \times_f F} := \sqrt{\mathcal{F}^2_{\mathring{B}} \circ (\pi_B)_* + (f \circ \pi_B)^2 \mathcal{F}^2_F \circ (\pi_F)_*}.$$

The notion of warped product for Riemann-Finsler manifolds is already known and studied by several authors (for example see [52]). By Remark 2.3.2 it is clear that $\mathcal{F}_{B \times_f F}$ is no Finsler structure on $\mathring{B} \times F$ in the sense of our definition. It cannot be smooth on $T\mathring{B} \times 0_F$ unless F is Riemannian and analogously it cannot be smooth on $0_B \times TF$ unless B is Riemannian. Especially it is only possible to define the fundamental tensor where $\mathcal{F}_{B \times_f F}$ is smooth. We can also consider the warped product between Riemann-Finsler manifolds as metric spaces that we introduced before. Again the two definitions provide the same notion of length and therefore they produce the same complete metric space, that we call again $B \times_f F = C$.

To obtain a metric measure space we introduce again a suitable measure that is precisely the one of Definition 2.2.8. In the purely Riemannian setting this yields the following simplifications. We have $d\,\mathrm{m}_B = e^{-\Psi_1} d\,\mathrm{vol}_B$ and $d\,\mathrm{m}_B = e^{-\Psi_2} d\,\mathrm{vol}_B$. The Riemannian volume of $\mathring{C}$ is

$$d\,\mathrm{vol}_{\mathring{C}} = f^n d\,\mathrm{vol}_{\mathring{B}}\, d\,\mathrm{vol}_F$$

For $N \in [1, \infty)$ we have

$$d\,\mathrm{m}_{\mathring{C}} = f^N e^{-(\Psi_1 + \Psi_2)} d\,\mathrm{vol}_{\mathring{B}}\, d\,\mathrm{vol}_F = f^{N-n} e^{-(\Psi_1 + \Psi_2)} d\,\mathrm{vol}_{\mathring{C}}\,.$$

We define a function Φ on $\mathring{B}$ as follows

$$e^{-\Phi(p)} = f^{N-n}(p) \Longrightarrow \Phi(p) = -(N-n)\log f(p).$$

So the measure $\mathrm{m}_{\mathring{C}}$ has the density $e^{-(\Phi+\Psi_1+\Psi_2)}$ with respect to $d\,\mathrm{vol}_C$.

Remark 2.3.5. It will turn out that curvature bounds for B in the sense of Alexandrov are essential in our proof of Theorem A where we show a curvature-dimension condition for $B \times_f^N F$. But a non-Riemannian Riemann-Finsler manifold will not satisfy such a bound. So it is convenient to assume that at least $(B, \mathcal{F}_B)$ is purely Riemannian with $\mathcal{F}_B^2 = g_B$. In this case the fundamental tensor at $v_{(p,x)}$ where $\mathcal{F}_{B\times_f F}$ is smooth becomes

$$g_{i,j}(v_{(p,x)}) = \begin{cases} (g_B)_{i,j}(p) & \text{if} \quad 1 \le i,j \le d \\ \frac{1}{2} f^2(p) \frac{\partial^2 (\mathcal{F}_F^2)}{\partial v^{i-d} \partial v^{j-d}}((\pi_F)_* v_{(p,x)}) & \text{if} \quad d+1 \le i,j \le d+n \\ 0 & \text{otherwise.} \end{cases}$$

3 Ricci curvature bounds for warped products

Introduction. This chapter is devoted to the proof of the following theorem for warped products over smooth metric measure spaces.

Theorem A. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that* $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. *In any case F satisfies $CD((N-1)K_F, N)$ such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

Then the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$.

Why can we expect such a result? In the case of warped products between Riemannian manifolds it is possible to calculate the Ricci tensor of $\mathring{B} \times_{\hat{f}} F$ explicitly. Consider $\xi + v \in T\mathring{C}_{(p,x)} = TB_p \oplus TF_x$. Then the formula is

$$\begin{aligned}\operatorname{ric}_{\mathring{C}}(\xi + v) = &\operatorname{ric}_B(\xi) - n\frac{\nabla^2 f_p(\xi)}{f(p)} \\ &+ \operatorname{ric}_F(v) - \left(\frac{\Delta f(p)}{f(p)} + (n-1)\frac{|\nabla f|_p^2}{f^2(p)}\right) g_C(v,v)\end{aligned} \tag{3.0.1}$$

and can be found in [64, chapter 7, 43 corollary]. We remark that $g_C(v,v) = f^2(p)g_F(v,v)$. $\nabla^2 f$ denotes the Hessian of f, which can be defined as follows. For $v, w \in TM_p$ we choose vector fields X and Y such that $X_p = v$ and $Y_p = w$. Then

$$\nabla^2 f(v,w) = X_p Y f - (\nabla_X Y)_p f \tag{3.0.2}$$

where ∇ denotes the Levi-Civita-connection of g. We set $\nabla^2 f(v) = \nabla^2 f(v,v)$. Because of (3.0.2) we will always choose vector fields in the way we did above to do calculation. The results will be independent of this choice. In the smooth setting $\mathcal{F}K$-concavity for a smooth f becomes $\nabla^2 f(v) \leq -fK|v|^2$ for any $v \in TB$. Then, if $\operatorname{ric}_B \geq (d-1)K$, f is $\mathcal{F}K$-concave, and $|\nabla f|^2 + Kf^2 \leq K_F$ everywhere in B, we get

$$\operatorname{ric}_F(v) \geq (n-1)K_F|v|^2 \implies \operatorname{ric}_{B\times_f F}(\xi + v) \geq (n+d-1)K|\xi + v|^2 \tag{3.0.3}$$

for every $v \in TF$ and for every $\xi + v \in TB \times_f F = TB \oplus TF$, respectively.

But even in the smooth setting one problem still occurs. When we allow the function f to vanish - that will happen in most of the interesting cases - the metric tensor degenerates and the warped product under consideration is no longer a manifold. Especially, there is no notion for the Ricci tensor at singular points. One strategy to solve this problem could be to cut the singularities and consider only what is left. But in general that space neither will be complete nor strictly intrinsic.

But the warped product together with its distance function and its volume measure is still a complete length metric measure space where the curvature-dimension condition in the sense of Lott, Sturm and Villani can be defined without problem. Hence, we state our result in terms of that condition and use tools from optimal transportation theory to circumvent the problem that comes from the singularities. A first step in this direction was done by Bacher and Sturm in [11] where they show that the Euclidean cone over some Riemannian manifold satisfies $CD(0, N)$ if and only if the underlying space satisfies $CD(N-1, N)$. Our main theorem is an analog of this in the setting of warped products and Finsler manifolds.

One can see that the curvature bound that holds by (3.0.3) on the regular part passes to the whole space provided the given assumptions are fulfilled. In general that will not be true. For example consider $N = 1$ and the Euclidean cone over $F = \mathbb{R}$. Then (3.0.1) is still true where f does not degenerate, but the cone does not satisfy any curvature-dimension condition (see [11]). The main part of the proof is to show that the set of singularity points does not affect the optimal transportation of mass and therefore, does not affect any type of convexity of any entropy functional on the L^2-Wasserstein space.

Outline of the chapter. In Section 3.1 we prove the formula for the N-Ricci tensor of warped products. We generalize formula 3.0.1 to weighted Riemann-Finsler manifolds. In Section 3.2 we give a detailed analysis of optimal transport in warped products where the first part also holds in a non-smooth framework. Finally, in Section 3.3 we prove Theorem A and give some immediate corollaries and applications.

3.1 Ricci tensor of warped products over Riemann-Finsler manifolds

Ricci tensor for Riemannian warped products. Proposition 3.1.1 is a generalization of formula (3.0.1) for N-warped products for all $N \in [1, \infty)$, where $\mathring{B} = B \backslash f^{-1}(0)$ is Riemannian and (F, Ψ) is a weighted Riemannian manifold. We will use the calculus for horizontal and vertical vector fields from [64]. For some vector field X on B there is a unique horizontal lift $\widetilde{V}$ that is a vector field on $\mathring{C}$ such that $(\pi_B)_* \widetilde{V} = V \circ \pi$. Similar, there is unique vertical lift $\widetilde{V}$ for some vector field V on F.

We remind the reader on the following relations that come from straightforward calculations or can be found in [64]. Consider vector fields X, Y on $\mathring{B}$ and V, W on F and their horizontal and vertical lifts $\widetilde{X}, \widetilde{Y}, \widetilde{V}, \widetilde{W}$, respectively. $\widetilde{\nabla}$ denotes the Levi-Civita-connection of $\mathring{B} \times_{\mathring{f}} F$.

(1) $\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla^{\mathring{B}}_X Y}$,

(2) $\widetilde{\nabla}_{\widetilde{X}} \widetilde{V} = \widetilde{\nabla}_{\widetilde{V}} \widetilde{X} = \frac{Xf}{f} \circ \pi_B \widetilde{V}$,

(3) $\widetilde{\nabla}_{\widetilde{V}} \widetilde{W} = -\frac{\langle \widetilde{V}, \widetilde{W} \rangle}{f} \circ \pi_B \widetilde{\nabla f} + \widetilde{\nabla^F_V W}$.

Proposition 3.1.1. *Let $\mathring{B}$ be a Riemannian manifold and (F, Ψ) be a weighted Riemannian manifold. Let $N \geq 1$ and $\mathring{f} : \mathring{B} \to (0, \infty)$ is smooth. $\mathring{B} \times_f F = \mathring{C}$ is the associated N-warped product of $\mathring{B}$, (F, Ψ) and $\mathring{f}$. Consider $\xi + v \in T\mathring{C}_{(p,x)} = TB_p \oplus TF_x$ and vector fields X and V with $X_p = \xi$ and $V_x = v$ and their horizontal and vertical lifts $\widetilde{X}$ and $\widetilde{V}$ on $\mathring{C}$. Then we have*

$$\begin{aligned} \mathrm{ric}^{N+d,\Phi+\Psi}_{\mathring{C}}(\xi + v) = &\mathrm{ric}_{\mathring{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)} \\ &+ \mathrm{ric}^{N,\Psi}_F(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f_p)^2}{f^2(p)} \right) |\widetilde{V}_{(p,x)}|^2. \end{aligned}$$

Φ is given by $e^{-\Phi} = f^{N-n}$. If the reference measure on F is the Riemannian volume and if we identify $\mathrm{ric}^{N,1}_F$ with ric_F, the previous formula reduces to

$$\begin{aligned} \mathrm{ric}^{N+d,\Phi+1}_{\mathring{C}}(\xi + v) = &\mathrm{ric}_{\mathring{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)} \\ &+ \mathrm{ric}_F(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f_p)^2}{f^2(p)} \right) |\widetilde{V}_{(p,x)}|^2. \end{aligned}$$

We recall that $|\widetilde{V}_{(p,x)}|^2 = f(p)|V_x|^2$.

Proof. First we assume $N > n$ where n is the dimension of F. We can calculate the $(N+d)$-Ricci tensor of the warped product explicitly by using formula (3.0.1). Let us first prove the second formula in the proposition. That means the reference measure on F is simply the Riemannian volume. We have to find espressions for the first and second derivative of $\Phi : \mathring{C} \to \mathbb{R}$. We remind the reader of formula (3.0.2). Set $\bar{W} = \widetilde{X} + \widetilde{V}$.

$$\nabla^2\Phi(\bar{W}) = \nabla^2\Phi(\widetilde{X}) + \nabla^2\Phi(\widetilde{V}) + 2\underbrace{\nabla^2\Phi(\widetilde{X}, \widetilde{V})}_{=0},$$

where $\widetilde{X}$ and $\widetilde{V}$ are lifts of vector fields X and V. We have

$$\begin{aligned}\nabla^2\Phi(\widetilde{X}) &= \widetilde{X}\widetilde{X}\Phi - (\nabla_{\widetilde{X}}\widetilde{X})\Phi = XX\Phi - (\nabla_X X)\Phi = \nabla^2\Phi(X)\\ &= -(N-n)\left(X\Big(\frac{Xf}{f}\Big) - \frac{\nabla_X Xf}{f}\right)\\ &= -(N-n)\left(\Big(XXf\cdot f - Xf\cdot Xf\Big)\frac{1}{f^2} - \frac{\nabla_X Xf}{f}\right)\end{aligned}$$

$$\nabla^2\Phi(\widetilde{V}) = \widetilde{V}\widetilde{V}\Phi - (\nabla_{\widetilde{V}}\widetilde{V})\Phi = -(N-n)\frac{|\nabla f|^2}{f^2}|\widetilde{V}|^2$$

$$\frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(\bar{W}) = \frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(X) = (N-n)\frac{1}{f^2}(Xf\cdot Xf)$$

So the $(N+d)$-Ricci curvature becomes

$$\begin{aligned}\mathrm{ric}_{\mathring{C}}^{N+d,\Phi}(\bar{W}) =& \,\mathrm{ric}_{\mathring{C}}(\bar{W}) - (N-n)\frac{1}{f^2}(Xf\cdot Xf)\\ &- (N-n)\left(\frac{1}{f^2}\big(XXf\cdot f - Xf\cdot Xf\big) - \frac{\nabla_X Xf}{f} + \frac{|\nabla f|^2}{f^2}|\widetilde{V}|^2\right)\\ =& \,\mathrm{ric}_{\mathring{C}}(\bar{W}) - (N-n)\left(\frac{1}{f^2}XXf\cdot f - \frac{\nabla_X Xf}{f} + \frac{|\nabla f|^2}{f^2}|\widetilde{V}|^2\right)\\ =& \,\mathrm{ric}_{\mathring{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{|\nabla f|^2}{f^2}\right)|\widetilde{V}|^2\end{aligned}$$

Now we change the measure on F. There is a reference measure $e^{-\Psi}d\,\mathrm{vol}_F$ on F for a function $\Psi : F \to \mathbb{R}$.

$$\nabla^2\Psi(\bar{W}) = \underbrace{\nabla^2\Psi(\widetilde{X})}_{=0} + \nabla^2\Psi(\widetilde{V}) + 2\nabla^2\Psi(\widetilde{X},\widetilde{V})$$

$$\begin{aligned}\nabla^2\Psi(\widetilde{V}) =& \widetilde{V}\widetilde{V}\Psi - (\nabla_{\widetilde{V}}\widetilde{V})\Psi\\ =& VV\Psi - (\nabla_V V)\Psi + \langle\nabla f,\nabla\Psi\rangle\frac{1}{f}|V|^2\\ =& \nabla^2\Psi(V,V) + \langle\nabla f,\nabla\Psi\rangle\frac{1}{f}|V|^2 = \nabla^2\Psi(V,V)\end{aligned}$$

$$\nabla^2\Psi(\widetilde{V},\widetilde{X}) = \widetilde{V}\widetilde{X}\Psi - (\nabla_{\widetilde{V}}\widetilde{X})\Psi = -\frac{1}{f}\widetilde{X}f\cdot\widetilde{V}\Psi$$

$$(\nabla(\Psi+\Phi))\otimes(\nabla(\Psi+\Phi))(\bar{W}) = (\nabla\Psi\otimes\nabla\Psi)(\widetilde{V}) + (\nabla\Phi\otimes\nabla\Phi)(\bar{W}) - 2\frac{N-n}{f}\widetilde{X}f\cdot\widetilde{V}\Psi$$

The $(N+d)-$Ricci curvature becomes

$$\begin{aligned}
\mathrm{ric}_{\mathring{C}}^{N+d,\Phi+\Psi}(\bar{W}) &= \mathrm{ric}_{\mathring{C}}(\bar{W}) + \nabla^2(\Phi+\Psi)(\bar{W}) - \frac{1}{N-n}\left(\nabla(\Phi+\Psi)\otimes\nabla(\Phi+\Psi)(\bar{W})\right)\\
&= \mathrm{ric}_{\mathring{C}}(\bar{W}) + \nabla^2\Phi(\bar{W}) - \frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(\bar{W})\\
&\quad + \nabla^2\Psi(\bar{W}) - \frac{1}{N-n}\left((\nabla\Psi\otimes\nabla\Psi)(\widetilde{V}) - 2\frac{N-n}{f}\widetilde{X}f\cdot\widetilde{V}\Psi\right)\\
&= \mathrm{ric}_{\mathring{C}}^{N+d,\Phi}(\bar{W}) + \nabla^2\Psi(\bar{W}) - \frac{1}{N-n}(\nabla\Psi\otimes\nabla\Psi)(\widetilde{V}) + 2\frac{1}{f}\widetilde{X}f\cdot\widetilde{V}\Psi\\
&= \mathrm{ric}_{\mathring{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{|\nabla f|^2}{f^2}\right)|\widetilde{V}|^2\\
&\quad + \nabla^2\Psi(V,V) - 2\frac{1}{f}\widetilde{X}f\cdot\widetilde{V}\Psi\\
&\quad - \frac{1}{N-n}(\nabla\Psi\otimes\nabla\Psi)(V) + 2\frac{1}{f}\widetilde{X}f\cdot\widetilde{V}\Psi\\
&= \mathrm{ric}_{\mathring{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F^{N,\Psi}(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{|\nabla f|^2}{f^2}\right)|\widetilde{V}|^2
\end{aligned}$$

Now we consider the case $N=n$. $\nabla(\Psi+\Phi)(\xi+v)\neq 0$ for $\xi+v\in TC_{(p,x)}$, where $\xi\in TB_p$ and $v\in TF_x$, is equivalent to $\nabla\Psi(v)\neq 0$ or $\nabla\Phi(\xi)\neq 0$. So by definition the left hand side of our formula evaluated at (ξ,v) is $=-\infty$ if and only if the right hand side is $=-\infty$.

If $\nabla(\Psi+\Phi)(\xi+v)=0$, then choose again vector fields V and X with $V_x=v$ and $X_p=\xi$, repeat the above calculation for $N>n$ and evaluate the formula at (ξ,v). The terms with $\frac{1}{N-n}$ disappear. Let $N\to n$ and get the desired result.

If $N<n$ then $\mathrm{ric}_{\widetilde{C}}^{N+d,\Psi+\Phi}=-\infty$ and $\mathrm{ric}_F^{N,\Psi}=-\infty$ by definition. □

Finsler Case. Now we treat the case of Finsler manifolds.

Proposition 3.1.2. *Let (F,m_F) be a weighted Finsler manifold. $N\geq 1$, $f:\mathring{B}\to(0,\infty)$ and $\mathring{B}$ are as in the previous proposition. $\mathring{B}\times_f F=\mathring{C}$ is the associated N-warped product. Then we have*

$$\begin{aligned}
\mathrm{ric}_{\mathring{C}}^{N+d,\mathrm{m}_C}(\xi+v) &= \mathrm{ric}_{\mathring{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)}\\
&\quad + \mathrm{ric}_F^{N,\mathrm{m}_F}(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f_p)^2}{f^2(p)}\right)\mathcal{F}_{\mathring{C}}(v)^2
\end{aligned}$$

where $\xi+v \in T\mathring{C}_{(p,x)}$ with $v \neq 0$ (Especially the $N+d$-Ricci tensor of $\mathring{C}$ is well-defined at $\xi + v$ because $\mathcal{F}_{\mathring{C}}$ is smooth in this direction).

Proof. Choose $\xi + v \in T\mathring{C}_{(p,x)}$ such that $\mathcal{F}_{\mathring{C}}(\xi + v) = 1$ and with $v \neq 0$ and a unit-speed geodesic $\gamma = (\alpha, \beta) : [-\epsilon, \epsilon] \to \mathring{C}$ with $\dot{\alpha}(0) = \xi$ and $\dot{\beta}(0) = v$. We set $\gamma(-\epsilon) = (p_0, x_0)$ and $\gamma(\epsilon) = (p_1, x_1)$. We choose ϵ small such that $\mathrm{L}(\gamma) = 2\epsilon$ is sufficiently far away from the cut radius of the endpoints. Up to reparametrization β is geodesic in F between x_0 and x_1 by Theorem 2.2.3. Let $\bar{\beta} : [0, \mathrm{L}] \to F$ be the unit-speed reparametrization of β. That means there exists an $s : [-\epsilon, \epsilon] \to [0, L]$ such that $\bar{\beta} \circ s = \beta$. L is the length of β. There exists $t_0 \in [0, \mathrm{L}]$ such that $\bar{\beta}(t_0) = \beta(0) = x$. We have

$$\dot{\beta}(t) = s'(t)\dot{\bar{\beta}}(s(t)) = \mathcal{F}_{F}(\dot{\beta}(t))\dot{\bar{\beta}}(s(t)).$$

We extend this last observation to the flow of geodesic vector fields $\bar{V}$ and $\bar{W}$.

$$\bar{V} = \nabla \mathrm{d}_{F}(x_0, \cdot)$$

is a smooth geodesic vector field on some neighborhood U of x. We choose U small enough such that $x_0, x_1 \notin U$ and it does not intersect with the cut locus of x_0. Then, if we restrict the image of $\bar{\beta}$ to U, it is an integral curve of $\bar{V}$. We can define a Riemannian metric $g^{\bar{V}}$ on U with respect to this geodesic vector field and then we can represent the measure m_{F} with a positive smooth density $\Psi_{\bar{V}}$ with respect to $d\,\mathrm{vol}_{g^{\bar{V}}}$. Additionally, we have

$$\mathcal{F}_{F}(\lambda v) = \sqrt{g^{\bar{V}}(\lambda v, \lambda v)} \text{ and } \mathrm{ric}^{N,\mathrm{m}_{F}}(\lambda v) = \mathrm{ric}^{N,\Psi_{\bar{V}}}(\lambda v) \text{ for all } \lambda > 0 \tag{3.1.1}$$

Consider the vector field

$$\bar{W} = \nabla \mathrm{d}_{\mathring{C}}((p_0, x_0), \cdot)$$

restricted to $(\mathring{B} \times U) \cap \big\{(p, x) : (\pi_{F})_*(\bar{W}_{(p,x)}) \neq 0\big\} = \mathring{U}$, which is open, where π_{F} is the projection from $\mathring{C}$ to F. Every intergral curve of $\bar{W}$ coincides in $\mathring{U}$ with a unit-speed geodesic from (p_0, x_0) to a point $(\bar{p}, \bar{x}) \in \mathring{U}$. And especially, as we mentioned above, the projection to F of each such geodesic is a pre-geodesic that connects x_0 and $\bar{x}$ in F. Thus the vertical projections of integral curves of $\bar{W}$ are integral curves of $\bar{V}$ after reparametrization. For an arbitrary integral curve $\gamma = (\alpha, \beta)$ of $\bar{W}$ we do the following explicit computation

$$\begin{aligned}
(\pi_{F})_*(\bar{W}_{\gamma(t)}) &= (\pi_{F})_*(\dot{\gamma}(t)) \\
&= (\pi_{F})_*(\dot{\alpha}(t) + \dot{\beta}(t)) = \dot{\beta}(t) \\
&= \mathcal{F}_{F}(\dot{\beta}(t))\dot{\bar{\beta}}(s(t)) \\
&= \mathcal{F}_{F}(\dot{\beta}(t))\bar{V}_{\bar{\beta}(s(t))} = \mathcal{F}_{F}(\dot{\beta}(t))\bar{V}_{\beta(t)} \\
\Rightarrow \qquad (\pi_{F})_*(\bar{W}_{(r,p)}) &= \mathcal{F}_{F}((\pi_{F})_*\bar{W}_{(r,p)})\bar{V}_{p} \quad \forall (r, p) \in \mathring{U}
\end{aligned}$$

Thus $\mathcal{F}_F((\pi_F)_*\bar{W})^{-1}\bar{W} =: W$ is π_F-related to $\bar{V}$.

We remember that by definition

$$\mathcal{F}^2_{\mathring{C}} := \mathcal{F}^2_{\mathring{B}} \circ (\pi_B)_* + (f \circ \pi)^2 \mathcal{F}^2_F \circ (\pi_F)_*.$$

The Finslerian N-Ricci tensor with respect to $d\,\mathrm{m}_C = d\mathrm{vol}_{\mathring{B}} \otimes f^N d\,\mathrm{m}_F$ for vectors of $\bar{W}$ is the Riemannian N-Ricci tensor of $g^{\bar{W}}$ with respect to $\Psi_{\bar{W}}$. The components of $g^{\bar{W}}$ are $(g_B)_{i,j}(p)$ if $1 \le i,j \le d$ and

$$\begin{aligned} g^{\bar{W}}_{i,j}|_{(p,x)} &= \frac{1}{2}\frac{\partial^2(\mathcal{F}^2_{\mathring{C}})}{\partial v^i \partial v^j}(\bar{W}_{(p,x)}) \\ &= \frac{1}{2} f^2(p) \frac{\partial^2(\mathcal{F}^2_F)}{\partial v^{i-d}\partial v^{j-d}}((\pi_F)_*\bar{W}_{(p,x)}) \\ &= \frac{1}{2} f^2(p) \frac{\partial^2(\mathcal{F}^2_F)}{\partial v^{i-d}\partial v^{j-d}}(\mathcal{F}_F((\pi_F)_*\bar{W}_{(p,x)})\bar{V}_x) = f^2(p) g^{\bar{V}}_{i-d,j-d}|_x \end{aligned}$$

if $d+1 \le i,j \le n+d$. The last equality holds because the fundamental tensor is homogenous of degree zero. But then $g^{\bar{W}}|_{(p,x)} = (g_B)|_p + f^2(p)(g^{\bar{V}})|_x$ for all $(p,x) \in \mathring{U}$ and we can apply the formula (3.0.1), that especially holds at $\xi + v$. Together with (3.1.1) this yields the desired formula for $\xi + v$. If $\mathcal{F}_{\mathring{C}}(\xi + v) \neq 1$, consider the normalized vector, repeat everything and use (3.1.1) to get the same result. □

3.2 Optimal Transport in warped products

The next theorem is not tied to the context of Finsler manifolds but a purely metric space result.

Theorem 3.2.1. *Let (B, d_B) be a complete Alexandrov space with CBB by K and let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space satisfying the $((N-1)K_F, N)$-MCP for $N \ge 1$ and $K_F > 0$ and diam$(F) \le \pi/\sqrt{K_F}$. Let $f : B \to \mathbb{R}_{\ge 0}$ be some $\mathcal{F}K$-concave function such that $X = \partial B = f^{-1}(\{0\})$, $X \neq \emptyset$ and*

$$Df_p \le \sqrt{K_F} \text{ for all } p \in X.$$

Consider $C = B \times_f F$. Let Π be an optimal dynamical transference plan in C such that $(e_0)_*\Pi$ is absolutely continuous with respect to m_C and $\mathsf{spt}\Pi = \Gamma$. Then the set

$$\Gamma_X := \{\gamma \in \mathsf{spt}\Pi : \exists t \in (0,1) : \gamma(t) \in X\}$$

has Π-measure 0.

Proof. For the proof we set $\mathrm{d}_F(x,y) = |x,y|$ and $\mathrm{d}_C((r,x),(s,y)) = |(r,x),(s,y)|$. We can assume that for all $\gamma \in \Gamma$ there is a $t \in (0,1)$ such that $\gamma(t) \in X$ and without loss of generality $K_F = 1$. We set $\mu_t = (e_t)_*\Pi$ and $\mathsf{spt}\mu_t = \Omega_t$. $\pi = (e_0, e_1)_*\Pi$ is an optimal plan between μ_0 and μ_1. We assume that $\Omega_0 \cap X = \emptyset$. For the proof we use the following results of Ohta

Theorem 3.2.2 ([60]). *If a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the $(\varkappa, N)$-MCP for some $\varkappa > 0$ and $N > 1$, then, for any $x \in M$, there exists at most one point $y \in M$ such that $\mathrm{d}_X(x, y) = \pi\sqrt{(N-1)/\varkappa}$.*

Lemma 3.2.3 ([61]). *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space satisfying the $(\varkappa, N)$-MCP for some $\varkappa > 0$ and $N > 1$. If $\operatorname{diam}_X = \mathrm{d}_X(p, q) = \pi\sqrt{\frac{N-1}{\varkappa}}$, then for every point $z \in X$, we have $\mathrm{d}_X(p, z) + \mathrm{d}_X(z, q) = \mathrm{d}_X(p, q)$. In particular, there exists a geodesic from p to q passing through z.*

We want to show that μ_0 is actually concentrated on the graph of some map $\varphi : p_1(\Omega_0) \subset B \to F$ where $p_1 : B \times_f^N F \to B$ is the projection map. Then since the measure μ_0 is absolutely continuous with respect to the product measure $f^N d\mathcal{H}_B^d \otimes d\,\mathrm{m}_F$, its total mass has to be zero by Fubinis' theorem and the fact that m_F contains no atoms. We define φ as follows. Choose $(p, x) \in \Omega_0$ that is the starting point of some transport geodesic $\gamma = (\alpha, \beta)$. If $(p, \widetilde{x}) \in \Omega_0$, we show that $x = \widetilde{x}$. So φ can be defined by $p \mapsto x$.

Let $\gamma, \widetilde{\gamma} \in \Gamma$ be transport geodesics starting in (p, x) and $(p, \widetilde{x})$, respectively. For the moment we are only concerned with $\gamma = (\alpha, \beta)$. It connects (p, x) and (q, y), and since it passes through X, by Proposition 2.2.11 it decomposes into $\gamma|_{[0,\tau)} = (\alpha_0, x)$, $\gamma(\tau) = s \in X$ and $\gamma|_{(\tau,1]} = (\alpha_1, y)$ where $x, y \in F$ such that $|x, y| = \pi$. We deduce an estimate for $|(p, \widetilde{x}), (q, y)|$. By Lemma 3.2.3 there exists a geodesic from x to y passing through $\widetilde{x}$. So by Theorem 2.2.3 it is enough to consider $B \times_f [0, \pi]$ instead of C. We have $x = 0$ and $y = \pi$. $(\alpha_0, \widetilde{x})$ is a minimizer between $(p, \widetilde{x})$ and s and especially $|s, (p, \widetilde{x})| = |s, (p, x)|$.

We will essentially use a tool introduced in the proof of Proposition 7.1 in [2]. There the authors define a nonexpanding map Ψ from a section of the constant curvature space S_K^3 into $B \times_f [0, \pi]$. For completeness we repeat its construction: B is an Alexandrov space, so the following is well-defined. $C_K(\Sigma_s)$ denotes the K-cone over Σ_s where Σ_s denotes the space of directions of s in B. s is the point where γ intersects ∂B. So we can write down the gradient exponential map in s

$$\exp_s : C_K(\Sigma_s) \to B, \quad (t, \sigma) \mapsto c_\sigma(t)$$

where c_σ denotes the quasi-geodesic that corresponds to $\sigma \in \Sigma_s$. The gradient exponential is a generalisation of the well-known exponential map in Riemannian geometry and it is non-expanding and isometric along cone radii that correspond to minimizers in B. Quasigeodesic were introduced by Alexandrov and studied in detail by Perelman and Petrunin in [67]. $\widetilde{B}$ denotes the doubling of B, that is the gluing of two copies of B along their boundaries. By a theorem of Perelman (see [66]) it is again an Alexandrov space with the same curvature bound. For s the space of direction $\widetilde{\Sigma}_s$ in $\widetilde{B}$ is simply the doubling of Σ_s.

We make the following observations. $\alpha_0 \star s \star \alpha_1$ has to be a geodesic in $\widetilde{B}$ between p and q where p and q lie in different copies of B, respectively. Otherwise there would be a shorter curve $\widetilde{\alpha}_0 \star \widetilde{s} \star \widetilde{\alpha}_1$ that would also give a shorter path between (p, x) and

(q, y) in C. We denote by α_1^+ and α_0^- the right hand side and the left hand side tangent vector at s, respectively. By reflection at ∂B we get another curve that is again a geodesic. This curve results from α_0 and α_1 that were interpreted as curves in the other copy of B, respectively. Two cases occur.

If $\alpha_0^+(t) \neq \alpha_1^-(t)$ in $\widetilde{\Sigma}_s$ then we get two pairs of directions with angle π. The case when $\alpha_0^+(t) = \alpha_1^-(t)$ will be discussed at the end. Now, in an analog way as one step before, we see that $\widetilde{\Sigma}_s$ is a spherical suspension with respect to each of these pairs and that all 4 directions we consider lie on a geodesic loop $c : [0, 2\pi]/_{\{0\sim 2\pi\}} \to \widetilde{\Sigma}_s$ of length 2π. We set $\{v_1, v_2\} = \mathrm{Im}c \cap \partial\Sigma_s$. Because the second curve was obtained by reflection, clearly we have $|\alpha_0^+, v_1| = |\alpha_0^-, v_1|$ and $|\alpha_0^+, v_2| = |\alpha_0^-, v_2|$ and analogously for α_1^+ and α_1^-. So we see that there is an involutive isometry of $\mathrm{Im}c$ fixing $\{v_1, v_2\}$. But then $|v_1, v_2|$ has to be π. We use a parametrization by arclength such that $c(0) = v_1$ and $c(\pi) = v_2$ and consider $c|_{[0,\pi]} = c : [0, \pi] \to \Sigma_s$.

Now consider the space S_K^2 of dimension 2 in $\mathbb{R}^3$ and $S_K^2 \cap (\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}) =: \mathring{S}_K^2$. We introduce polar coordinates

$$\big(\sin_K(\varphi)\cos(\vartheta), \sin_K(\varphi)\sin(\vartheta), \cos_K(\varphi)\big)$$
$$\text{where } \vartheta \in [0, \pi] \text{ and } \varphi \in I_K := \begin{cases} [0, \pi/\sqrt{K}] & \text{if } K > 0 \\ [0, \infty) & \text{if } K \leq 0 \end{cases}$$

and the K-cone map $\widetilde{\Psi} : \mathring{S}_{K,}^2 \to C_K(\Sigma_s)$, $\widetilde{\Psi}(\varphi, \vartheta) = (\varphi, c(\vartheta))$, which is an isometry onto $C_K(\mathrm{Im}c \cap \Sigma_s)$.

We consider $\mathring{S}_K^2 \times_\Phi [0, \pi] =: \mathring{S}_K^3$ where $\Phi(\varphi, \vartheta) = \sin \circ \mathrm{d}_{\partial \mathring{S}_K^2}(\varphi, \vartheta) = \sin_K \varphi \sin \vartheta$ and $\partial \mathring{S}_K^2 = \{(\varphi, \vartheta) : \vartheta = 0 \text{ or } = \pi\} \simeq_{\frac{1}{\sqrt{K}}} S^1$ and define the following map

$$\Psi = \exp_s \circ \widetilde{\Psi} \times \mathrm{id}_{[0,\pi]} : \mathring{S}_K^2 \times_\Phi [0, \pi] = \mathring{S}_K^3 \to B \times_f [0, \pi].$$

From the proof of Proposition 7.1 in [2] we know that Ψ is still nonexpanding and an isometry along cone radii that correspond to minimizers in B. The essential ingredient is $\sin_K(\varphi) \leq f(\alpha(\varphi))$ for any geodesic α in B.

Exactly as in the case of K-cones one can see that the distance of $\mathring{S}_K^2 \times_\Phi [0, \pi]$ is explicetly given by

$$\begin{aligned} \cos_K |(\varphi_0, \vartheta_0, x_0), (\varphi_1, \vartheta_1, x_1)| = {} & \cos_K \varphi_0 \cos_K \varphi_1 \\ & + K \sin_K \varphi_0 \cos \vartheta_0 \sin_K \varphi_1 \cos \vartheta_1 \\ & + K \sin_K \varphi_0 \sin \vartheta_0 \sin_K \varphi_1 \sin \vartheta_1 \cos(x_0 - x_1). \end{aligned}$$

For $K > 0$ we deduce the desired estimate

$$\begin{aligned}
\cos_K |(p,\widetilde{x}),(q,y)| &= \cos_K |\Psi((\varphi_0,\vartheta_0,\widetilde{x})),\Psi((\varphi_1,\vartheta_1,y))| \\
&\geq \cos_K |(\varphi_0,\vartheta_0,\widetilde{x})),(\varphi_1,\vartheta_1,y)| \\
&= \cos_K \varphi_0 \cos_K \varphi_1 + K \sin_K \varphi_0 \cos\vartheta_0 \sin_K \varphi_1 \cos\vartheta_1 \\
&\quad + K \sin_K \varphi_0 \sin\vartheta_0 \sin_K \varphi_1 \sin\vartheta_1 \cos(\widetilde{x}-y) \\
&\geq \cos_K \varphi_0 \cos_K \varphi_1 \\
&\quad + K \sin_K \varphi_0 \sin_K \varphi_1 (\cos\vartheta_0 \cos\vartheta_1 - \sin\vartheta_0 \sin\vartheta_1) \\
&= \cos_K \varphi_0 \cos_K \varphi_1 + K \sin_K \varphi_0 \sin_K \varphi_1 \cos(\vartheta_0+\vartheta_1) \\
&\geq \cos_K \varphi_0 \cos_K \varphi_1 - K \sin_K \varphi_0 \sin_K \varphi_1 \\
&= \cos_K(\varphi_0+\varphi_1) = \cos_K(|s,(p,\widetilde{x})| + |s,(q,y)|) \\
&= \cos_K(|s,(p,x)| + |s,(q,y)|) = \cos_K(|(p,x),(q,y)|) \qquad (3.2.1)
\end{aligned}$$

$$\Longrightarrow |(p,\widetilde{x}),(q,y)| \leq |(p,x),(q,y)| \qquad (3.2.2)$$

with equality in the second inequality if and only if $|y,\widetilde{x}| = \pi$. The case $K \leq 0$ follows in the same way but we have to be aware of reversed inequalities and minus signs that will appear. We get the same estimate for (p,x) and $(\widetilde{q},\widetilde{y})$. By optimality of the plan we have

$$|(p,\widetilde{x}),(q,y)|^2 + |(p,x),(\widetilde{q},\widetilde{y})|^2 \geq |(p,x),(q,y)|^2 + |(p,\widetilde{x}),(\widetilde{q},\widetilde{y})|^2 \qquad (3.2.3)$$

and from that we have equality in (3.2.1). So we get $|x,\widetilde{y}| = \pi$ and $|y,\widetilde{x}| = \pi$. But by Ohta's theorem antipodes are unique and thus we get $y = \widetilde{y}$ and $x = \widetilde{x}$.

The case when $\alpha_0^+(t) = \alpha_1^-(t)$ works as follows. The last identity implies w.l.o.g. $\mathrm{Im}\alpha_1 \subset \mathrm{Im}\alpha_0$. We define a map from the K-cone into the warped product

$$\mathring{\Psi} : I_K \times_{\sin_K} [0,\pi] \to B \times_f [0,\pi] \text{ by } (\varphi,x) \mapsto (\alpha_0(\varphi),x).$$

Again $\mathring{\Psi}$ is nonexpanding. By following the lines of Bacher/Sturm in [11] we get the same estimate as in (3.2.1). □

Existence of optimal maps. We have already mentioned that the Finsler structure on $\mathring{C}$ is not smooth, or more precisely $\mathcal{F}_C^2$ is C^1 but not C^2 at any $v \in T\mathring{B}_p \oplus O_F$. So we cannot apply the classical existence theorem for optimal maps. But the special situation of warped products allows to prove the existence of optimal maps by following the lines given in chapter 10 of [80]. There, the cost function comes from a Lagrangian living on a Riemannian manifold. It is easy to see that the Riemannian structure is not so important. Actually, the Lagrangian viewpoint fits perfectly well

to our setting if we consider $L : T\mathring{C} \to \mathbb{R}$ with $L(v) = \mathcal{F}_C^2(v)$. The associated action functional is

$$\mathcal{A}(\gamma) = \int_0^1 \mathcal{F}_C^2(\dot{\gamma}(t))dt$$

where $\gamma : [0,1] \to \mathring{C}$ is an absolutely continuous curve. Minimizers of this action functional are just the constant speed geodesics of $\mathring{C}$. We have the following theorem.

Theorem 3.2.4. *Given $\mu, \nu \in \mathcal{P}^2(\mathring{C})$ that are compactly supported and such that μ is absolutely continuous with respect to m_C. Take compact sets $Y \supset \operatorname{supp}\nu$ and $X = \bar{U}$ such that $\operatorname{supp}\mu \subset U$. Then there exists a $\frac{1}{2}d^2$-concave function $\varphi : X \to \mathbb{R}_{\geq 0}$ relative to (X,Y) such that the following holds: $\pi = (Id_{\mathring{C}}, T)_*\mu$ is a unique optimal coupling of (μ,ν), where $T : X \to Y$ is a measurable map and defined μ-almost everywhere by $T((p,x)) = \gamma^{(p,x)}(1)$ where $\gamma^{(p,x)}$ is a constant-speed geodesic and uniquely determined by $-d\varphi_{(p,x)}(\dot{\gamma}^{(p,x)}(0)) = \mathcal{F}^2(\dot{\gamma}^{(p,x)}(0))$.*

For completeness we give a self-contained presentation of the proof from [80] where our discussion closely follows [59] and [62].

Proposition 3.2.5. *For any $(p,x) \in \mathring{C}$ and any $\xi + v \in T\mathring{C}_{(p,x)}$ there is a unique geodesic γ starting in (p,x) with initial tangent vector $\dot{\gamma}(0) = \xi + v$.*

Proof. If $v = 0$, $\gamma(t) = (\alpha(t), \beta(0))$ is a geodesic in B and hence uniquely determined by $\dot{\alpha}(0)$. Otherwise we have $\mathcal{F}_F^2(\dot{\beta})f^4(\alpha) = const =: c$ (see Theorem 2.2.3) and α is determined by

$$\nabla_{\dot{\alpha}}\dot{\alpha} = -\nabla \tfrac{c}{2f^2}|_\alpha$$

and $\alpha(0)$ and $\dot{\alpha}(0)$. Together with the uniqueness property of geodesics in F, the statement follows. □

In this section c stands for the cost function $c((p,x),(q,y)) = \frac{1}{2}\,\mathrm{d}_C((p,x),(q,y))^2 = \frac{1}{2}\inf\mathcal{A}(\gamma)$ where the infimum is taken with respect to absolutely continuous curves connecting (p,x) and (q,y). We need some background information on c-concave functions where we also refer to [59].

Definition 3.2.6. Let $X, Y \subset \mathring{C}$ be compact. Given an arbitrary function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$, its c-transform $\varphi^c : Y \to \mathbb{R} \cup \{-\infty\}$ relative to (X,Y) is defined by

$$\varphi^c((q,y)) := \inf_{(p,x)\in X} \{c((p,x),(q,y)) - \varphi((p,x))\}\,.$$

Similar we define the c-transform of a function $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ relative to (Y,X). A function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be c-concave relative to (X,Y) if it is not identical $-\infty$ and if there is a function $\psi : Y \to \mathbb{R} \cup \{\infty\}$ such that $\psi^c = \varphi$.

Lemma 3.2.7. *If φ is c-concave relative to (X,Y), then it is Lipschitz continuous with respect to d_C and the Lipschitz constant is bounded above by some constant depending only on X and Y.*

Remark 3.2.8. Since a c-concave function is Lipschitz continuous, it is differentiable almost everywhere. We also have that $d\varphi : \mathring{C} \to T^*\mathring{C}$ is measurable (see [59, Lemma 4]).

Definition 3.2.9. Let M be a manifold and $f : M \to \mathbb{R}$ a function. A co-vector $\alpha \in T^*M_x$ is called subgradient of f at x if we have

$$f(\sigma(1)) \geq f(\sigma(0)) + \alpha(\dot{\sigma}(0)) + o(\mathcal{F}(\dot{\sigma}(0))$$

for any geodesic $\sigma : [0,1] \to M$ with $\sigma(0) = x$. The set of subgradients at x is denoted by $\partial^*_- f(x)$. Analogously we can define the set $\partial^*_+ f(x)$ of supergradients at x.

Remark 3.2.10. If f admits a sub- and supergradient at x, it is differentiable at x and $\partial^*_- f(x) = \partial^*_+ f(x) = \{df_x\}$ ([80, Proposition 10.7]).

Proposition 3.2.11. *Suppose $\gamma : [0,1] \to \mathring{C}$ is a constant speed geodesic joining (p,x) and (q,y). Then $f(\cdot) = c(\cdot,(q,y))$ has supergradient $-d_v\mathcal{F}^2_{\mathring{C}}|_{\dot{\gamma}(0)} \in T^*\mathring{C}_{\gamma(0)}$ at (p,x) where*

$$d_v\mathcal{F}^2_{\dot{\gamma}(0)}(w) = \frac{d}{dt}\mathcal{F}^2(\dot{\gamma}(0) + tw) \quad \text{for } w \in T\mathring{C}|_{\gamma(0)}$$

Proof. Let $(\widetilde{p},\widetilde{x})$ and $(\widetilde{q},\widetilde{y})$ are points that are very close to (p,x) and (q,y) such that there are unique geodesics $\sigma_0, \sigma_1 : [0,1] \to \mathring{C}$ between (p,x) and $(\widetilde{p},\widetilde{x})$ and between (q,y) and $(\widetilde{q},\widetilde{y})$, respectively. Let $\widetilde{\gamma}$ be an arbitrary curve that connects $(\widetilde{p},\widetilde{x})$ and $(\widetilde{q},\widetilde{y})$. Then we have by the formula of first variation

$$\begin{aligned}\int_0^1 \mathcal{F}^2_C(\dot{\widetilde{\gamma}}(t))dt = \int_0^1 \mathcal{F}^2_C(\dot{\gamma}(t))dt \\ + d_v\mathcal{F}^2_C|_{\dot{\gamma}(1)}(\dot{\sigma}_1(0)) - d_v\mathcal{F}^2_C|_{\dot{\gamma}(0)}(\dot{\sigma}_0(0)) + o\big(\sup_{t\in[0,1]} \mathrm{d}_C(\gamma(t),\widetilde{\gamma}(t))\big).\end{aligned}$$

Hence we can proof for some $\widetilde{\gamma}$ with $(\widetilde{q},\widetilde{y}) = (q,y)$ that

$$\begin{aligned}c((\widetilde{p},\widetilde{x}),(q,y)) &\leq \int_0^1 \mathcal{F}^2_C(\dot{\widetilde{\gamma}}(t))dt \\ &\leq c((p,x),(q,y)) - d_v\mathcal{F}^2_C|_{\dot{\gamma}(0)}(\dot{\sigma}_0(0)) + o\big(\mathrm{d}_C((p,x),(\widetilde{p},\widetilde{x}))\big),\end{aligned}$$

which means that $c(\cdot,(q,y))$ has supergradient $-d_v\mathcal{F}^2_C|_{\dot{\gamma}(0)}$. For more details we refer to [80, Proposition 10.15]. □

Lemma 3.2.12. *Let $X, Y \subset \mathring{C}$ be two compact subsets and $\varphi : X \to \mathbb{R}$ be a c-concave function. If φ is differentiable in $(p,x) \in X$, and*

$$c((p,x),(q,y)) = \varphi((p,x)) + \varphi^c((q,y)), \tag{3.2.4}$$

then there is a geodesic $\gamma = \gamma^{(p,x)}$ between (p,x) and (q,y) satisfying $-d\varphi_{(p,x)}(\dot{\gamma}(0)) = \mathcal{F}^2(\dot{\gamma}(0))$. The point (q,y) and the geodesic γ are uniquely determined by (p,x) and φ.

Proof. By definition of c-concave functions we have $\geq$ in (3.2.4) for any pair of points. Now choose (p,x) and (q,y) such that (3.2.4) holds and φ is differentiable at (p,x). Then we have for any $(\widetilde{p},\widetilde{x})$

$$\varphi((\widetilde{p},\widetilde{x})) - \varphi((p,x)) \leq c((\widetilde{p},\widetilde{x},(q,y)) - c((p,x),(q,y))$$

Instead of the point $(\widetilde{p},\widetilde{x})$ we insert a curve $\sigma : (0,\epsilon) \to X$ (parametrized by arclength). Then we deduce

$$d\varphi_{(p,x)}(\dot{\sigma}) = \frac{d}{d\epsilon}\varphi \circ \sigma|_{\epsilon=0} \leq \liminf_{\epsilon\to 0} \frac{c(\sigma(\epsilon),(q,y)) - c((p,x),(q,y))}{\epsilon}$$

It follows that $d\varphi_{(p,x)}$ is a subgradient of $c(\cdot,(q,y))$ at (p,x). But by the previous proposition $c(\cdot,(q,y))$ has also a supergradient at (p,x). Thus it is differentiable at (p,x) with

$$-d_v\mathcal{F}^2_{\mathring{C}}|_{\dot{\gamma}(0)} = dc(\cdot,(q,y))_{(p,x)} = d\varphi_{(p,x)} \tag{3.2.5}$$

where γ is some geodesic that connects (p,x) and (q,y). Now we know that $\mathcal{F}^2_C$ is strictly convex in v and C^1. Thus the co-vector $d_v\mathcal{F}^2_{\mathring{C}}|_{\dot{\gamma}(0)}$ determines $\dot{\gamma}(0)$ uniquely by $d_v\mathcal{F}^2_{\mathring{C}}|_{\dot{\gamma}(0)}(w) = \mathcal{F}^2(w)$ and therefore γ by Proposition 3.2.5. So (3.2.5) and the strict convexity of $\mathcal{F}^2_C$ with respect to v determines y uniquely. □

Remark 3.2.13. On $T\mathring{B} \oplus TF\backslash 0_F$ we have that $\dot{\gamma}^{(p,x)}$ coincides with the gradient of $-\varphi$ at (p,x), that can be defined via Legendre transformation, and $\gamma^{(p,x)}(t) = \exp(-t\nabla\varphi_{(p,x)})$. On $T\mathring{B} \oplus 0_F$ it coincides with the gradient that comes from the Riemannian structure on B. The map $(p,x) \mapsto \dot{\gamma}^{(p,x)}$ is measurable because φ is Lipschitz (see Remark 3.2.8) and the transformation $\alpha \in T^*C_{(x,p)} \mapsto \alpha^* \in TC_{(p,x)}$ is continuous, where α^* is uniquely determined by $\alpha(\alpha^*) = \mathcal{F}^2(\alpha^*)$. Now one can deduce that also $(p,x) \mapsto \gamma^{(p,x)}(1)$ is measurable by considering the "exponential map" separately on $T\mathring{B} \oplus TF\backslash 0_F$ and $T\mathring{B} \oplus 0_F$. In [80, Theorem 10.28] measurability of T is deduced by applying a measurable selection theorem.

Proof of Theorem 3.2.4. Let π be an optimal transference plan. By Kantorovich duality there exists a c-concave function φ such that $\varphi((p,x))+\varphi^c((q,y)) \leq c((p,x),(q,y))$ everywhere on $\operatorname{supp}\pi \subset X \times Y$, with equality π-almost surely. Since φ is differentiable m_C-almost surely and since μ is absolutely continuous with respect to m_C, we can define T by Lemma 3.2.12 m_C-almost surely by $T((p,x)) = \gamma^{(p,x)}(1)$ where $\gamma^{(p,x)}$ is uniquely given by $-d\varphi_{(p,x)}(\dot{\gamma}(0)) = \mathcal{F}^2(\dot{\gamma}(0))$. Thus π is concentrated on the graph of T, or equivalently $\pi = (Id_{\mathring{C}}, T)_*\mu$. Now from [62, Lemma 4.9] we know that in our setting $-d\varphi$ is unique among all maximizers (φ,φ^c) of Kantorovich duality as long as the initial measure is absolutely continuous. So also T and π are unique. □

3.3 Proof of the main results and applications

Proof of Theorem A. Let $\partial B \neq \emptyset$. For non-constant f we have $K_F > 0$. Otherwise, the warped product is just the ordinary Euclidean product, K has to be nonpositiv and the result is the tensorization property of the CD-condition (see [32]). In the case of $N > 1$ the curvature-dimension condition for $(F, \mathrm{d}_F, \mathrm{m}_F)$ implies $((N-1)K_F, N)$-MCP ([70]). If $N = 1$, then by assumption we have $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. So in any case Theorem 3.2.1 yields that positive mass will never transported through the set of singularity points X. So one could think to apply Theorem 2.3.4 to get the result because on $\mathring{B} \times_f^N F$ the N-Ricci tensor is bounded in the correct way by Proposition 3.1.1 and our assumptions.

Two problems occur. First, the warped product without its singularity points is not geodesically complete. But if we consider some displacement interpolation between bounded and absolutely continuous measures in $\mathring{B} \times_f^N F$ then as we have seen the transport geodesics do not intersect X. So by truncation we can find an ε-environment of the singularity set such that the transport takes place in the complement of this environment and the exceptional mass can be chosen arbitrarily small. Then in the case where F is Riemannian the calculus that was introduced in [31] is available like in the complete setting and one gets convexity of the Jacobian of the optimal map along the transport geodesics, which yields to the curvature-dimension condition (see also [11]). When $\partial B = \emptyset$ this step is redundant because no singularity points appear.

Second, if F is Finslerian, the warped product structure is not smooth on $T\mathring{B} \times O_F$. So we cannot follow the lines of [62] as we did with [31] in the Riemannian case. But we know, if $\gamma = (\alpha, \beta)$ is a geodesic in $\mathring{B} \times_f^N F$ then by Theorem 2.2.3 β is a pre-geodesic. So either β is constant and α is a geodesic in B, or there exists a strictly monotone reparametrization s such that $\bar{\beta} = \beta \circ s$ is a constant speed geodesic in F. We use this fact to circumvent the problem that comes from the non-smoothness. The idea is to split the initial measure of some optimal mass transportation in $\mathring{B} \times_f^N F$ in two disjoint parts that will follow one of these two kinds of geodesics either. To do so we need that a point $(p, x) \in \operatorname{supp} \mu_0$ already determines the transport geodesic that starts in (p, x) uniquely. But this follows from the existence of an optimal map.

So we proceed as follows. Let μ_0 and μ_1 be absolutely continuous probability measures in $\mathring{C}$. We assume w.l.o.g. that μ_0 and μ_1 are compactly supported. Otherwise, we have to choose compact exhaustions of $\mathring{C} \times \mathring{C}$ and to consider the restriction of the plan to these compact sets. For this we also refer to [77, Lemma 3.1]. By Theorem 3.2.4 there is an unique optimal map $T : X \to Y$ between μ_0 and μ_1. So the unique optimal plan is given by $(\mathrm{id}, T)_* \mu_0 = \pi$ and the associated optimal dynamical plan is given by $\gamma_* \mu_0 = \Pi$ where $\gamma : \operatorname{supp} \mu_0 \to \mathcal{G}(\mathring{C})$ with $(p, x) \mapsto \gamma^{(p,x)}$. The geodesic in

$\mathcal{P}^2(\mathring{C})$ with respect to L^2-Wasserstein distance is given by $\mu_t = (\gamma_t^{(p,x)})_*\mu_0$. We have

$$\operatorname{supp}\Pi = \underbrace{\left\{\gamma : \dot\gamma \in T\mathring{B}\times 0_F\right\}}_{=:\Gamma_a} \dot\cup \underbrace{\left\{\gamma : \dot\gamma \in T\mathring{B}\times TF\backslash 0_F\right\}}_{=:\Gamma_b}.$$

We set $\Pi(\Gamma_a)^{-1}\Pi|_{\Gamma_a} =: \Pi_a$ and $\Pi(\Gamma_b)^{-1}\Pi|_{\Gamma_b} = \Pi_b$ that are again optimal dynamical plans. The corresponding L^2-Wasserstein geodesics are $(e_t)_*\Pi_a = \mu_{a,t}$ and $(e_t)_*\Pi_b = \mu_{b,t}$. They are absolutely continuous with densities $\rho_{a,t}$ and $\rho_{b,t}$ and have disjoint support for any $t \in [0,1]$ because of the optimal map and since $\mathring{C}$ is non-branching (see [10, Lemma 2.6]). We have for any $t \in [0,1]$

$$\rho_t d\,\mathrm{m}_C = \mu_t = \Pi(\Gamma_a)\mu_{a,t} + \Pi(\Gamma_b)\mu_{b,t} = \Pi(\Gamma_a)\rho_{a,t}d\,\mathrm{m}_C + \Pi(\Gamma_b)\rho_{b,t}d\,\mathrm{m}_C\,.$$

So the Rényi entropy functional from Definition 2.1.3 splits for any $t \in [0,1]$

$$\int_M \rho_t^{1-1/N'} d\,\mathrm{m}_C = \Pi(\Gamma_a)^{1-1/N'}\int_M \rho_{a,t}^{1-1/N'} d\,\mathrm{m}_C + \Pi(\Gamma_b)^{1-1/N'}\int_M \rho_{b,t}^{1-1/N'} d\,\mathrm{m}_C$$

for any $N' \geq N$. So it suffices to show displacement convexity along Π_a and Π_b separately.

We begin with Π_a. We can approximate Π_a in L^2-Wasserstein distance arbitrarily close by

$$\frac{1}{n}\sum_{i=1}^n \Pi_{a,B}^i \otimes \nu_i$$

where $\Pi_{a,B}^i$ are geometric optimal transference plans in (B, d_B) and ν_i are disjoint absolutely continuous propability measures in F. So it suffices to show displacement convexity along $\Pi_{a,B}^i$. But since B has CBB by K and f is $\mathcal{F}K$-concave, $(B, \mathrm{d}_B, f^N d\,\mathrm{vol}_B)$ satisfies $CD((N+d-1)K, N+d)$ (see [79, Theorem 1.7]) and the desired convexity in $\Pi_{a,B}$ follows at once.

Now consider Π_b. We know a priori that the transport geodesics only follow smooth directions of the Finslerian warped product structure. So we can consider $\mathcal{F}_C^2$ restricted to $T\mathring{B}\times TF\backslash 0_F$. We get the exponential map on $T\mathring{B}\times TF\backslash 0_F$ and we also can define the Legendre transformation, that yields gradient vector fields. Especially, if we consider an optimal transport that follows only smooth direction, the techniques from [62] can be applied. Thus there exists an optimal map T_b of the form $T_b((p,x)) = \exp(-\nabla\varphi_{(p,x)})$ for some c-concave function φ. To make this more precise we can consider the complement of an ϵ-neighborhood $\mathcal{U}_\epsilon$ of $T\mathring{B}\times 0_F$ and restrict the initial measure $\mu_{b,0}$ of Π_b to the set

$$U_\epsilon = \left\{(p,x) \in \operatorname{supp}\mu_{b,0} : \dot\gamma^{(p,x)}(0) \notin \mathcal{U}_\epsilon\right\}.$$

U_ϵ is measurable because the mapping $(p,x) \in\mapsto \dot\gamma^{(p,x)}$ is measurable. Again the exceptional mass can be chosen arbitrarily small. The optimal map T, which has been

derived in Theorem 3.2.4, restricted to U_ϵ has to coincide with T_b because optimality is stable under restriction and because of uniqueness of optimal maps. Especially we can deduce $\mu_{b,t} = (T_t)_*\mu_{b,0}$ where $T_t((p,x)) = \exp(-t\nabla\varphi_{(p,x)})$. Again by results from [62] we know φ is second order differentiable at least on U_ϵ. Hence the Jacobian of T_t exists and satisfies - because of Proposition 3.1.2 and our assumptions - the correct convexity condition. Finally one can follow the lines of section 8 in [62]. □

Corollary 3.3.1. *Let B be a complete, d-dimensional space with CBB by K that is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth. Assume $\emptyset \neq \partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N > 1$. Then the following statements are equivalent*

(i) (F, m_F) satisfies $CD((N-1)K_F, N)$ with $K_F \geq 0$ and
$$|\nabla f|_p \leq \sqrt{K_F} \text{ for all } p \in \partial B.$$

(ii) The N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Proof. Only one direction is left. Assume the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$. Proposition 3.1.2 yields that

$$(N+d-1)K\mathcal{F}^2_{\mathring{B}\times_{\mathring{f}}F}(\widetilde{V}) \leq \mathrm{ric}_F^{N,\mathrm{m}_F}(V) - \left(\tfrac{\Delta f(p)}{f(p)} + (N-1)\tfrac{|\nabla f|_p^2}{f^2(p)}\right)\mathcal{F}^2_{\mathring{B}\times_{\mathring{f}}F}(\widetilde{V}) \tag{3.3.1}$$

where $V \in TF_x$ is arbitrary and $\widetilde{V} \in T\mathring{C}_{(p,x)}$ such that $(\pi_F)_*\widetilde{V} = V$. The last inequality is equivalent to

$$(N+d-1)Kf^2(p)\mathcal{F}_F^2(V) \leq \mathrm{ric}_F^{N,\mathrm{m}_F}(V) - \left(\Delta f(p)f(p) + (N-1)|\nabla f|_p^2\right)\mathcal{F}_F^2(V). \tag{3.3.2}$$

Now we can choose $p \in B$ independent from V and thus, we let p tend to the non-empty boundary of B. Then $\Delta f(p)f(p)$ tends to 0 because Δf is smooth on B (included the boundary) and we get

$$(N-1)|\nabla f|_p^2\mathcal{F}_F^2(V) \leq \mathrm{ric}_F^{N,\mathrm{m}_F}(V) \tag{3.3.3}$$

for all $p \in \partial B$ and all $V \in TF$. This inequality implies that $(N-1)|\nabla f|^2$ is bounded from above on ∂B by $\mathcal{F}_F^2(V)^{-1}\,\mathrm{ric}_F^{N,\mathrm{m}_F}(V)$ for arbitrary $V \in TF$. So we can set $\sup_{p\in\partial B}|\nabla f|_p^2 = K_F < \infty$. For any $\epsilon > 0$ we find $p \in \partial B$ such that $(N-1)|\nabla f|_p^2 > (N-1)K_F - \epsilon$. Then we get from (3.3.3)

$$\left((N-1)K_F - \epsilon\right)\mathcal{F}_F^2(V) \leq \mathrm{ric}_F^{N,\mathrm{m}_F}(V)$$

and since $\epsilon > 0$ is arbitrary we get the desired curvature bound. □

Corollary 3.3.2. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ a function such that it is smooth and satisfies $\nabla^2 f = -Kf$ on $B\backslash\partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N > 1$. Then the following statements are equivalent*

(i) (F, m_F) satisfies $CD((N-1)K_F, N)$ with $K_F \in \mathbb{R}$ such that
   1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
   2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

(ii) The N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Proof. Assume the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$. Like in the proof of previous corollary we can deduce (3.3.1). Now we have $\Delta f = -dKf$ on $B\backslash\partial B$ and we can deduce

$$\mathrm{ric}_F^{N,\mathrm{m}_F}(V) \geq (N-1)\left(Kf^2(p) + |\nabla f|_p^2\right)\mathcal{F}_F^2(V)$$

for all $p \in B\backslash\partial B$ and all $V \in TF$. Like in the proof of the previous corollary this inequality implies that $|\nabla f|^2 + Kf^2$ is bounded on $B\backslash\partial B$. So we can set $\sup_{p\in B\backslash\partial B} |\nabla f|_p^2 + Kf^2(p) =: K_F$. (Since f is $\mathcal{F}K$-concave, $|\nabla f|_p^2 + Kf^2(p)$ is actually constant on B (see for example [2]).) This yields

$$K_F \geq Kf^2(p) \qquad \forall p \in B.$$

Then by Proposition 2.2.9 this is equivalent to the conditions 1. and 2. in the theorem and as in Corollary 3.3.1 the N-Ricci tensor of F is bounded by $K_F(N-1)$. □

Remark 3.3.3. If $B = [0, \pi/\sqrt{K}]$ and $f = \sin_K$ (with appropriate interpretation if $K \leq 0$) and if $\operatorname{diam} F \leq \pi$, the associated warped products are K-cones. If F is a Riemannian manifold in this setting we get the theorem of Bacher and Sturm from [11]. However, if F is Finslerian, the result is new.

Corollary 3.3.4. *For any real number $N > 1$, $CD(N-1, N)$ for a weighted Finsler manifold is equivalent to $CD(K \cdot N, N+1)$ for the associated (K, N)-cone.*

Remark 3.3.5. Like in the theorem of Alexander and Bishop our result can be extended to the case where B satisfies a suitable boundary condition (†).

(†): *If $B^\dagger$ is the result of gluing two copies of B on the closure of the set of boundary points where f does not vanishing, and $f^\dagger : B^\dagger \to \mathbb{R}_{\geq 0}$ is the tautological extension of f, then $B^\dagger$ has CBB by K and $f^\dagger$ is $\mathcal{F}K$-concave.*

The proof of the main theorem in this situation is exactly the same since (†) implies that the warped product $C\backslash\partial C$ is geodesic. We do not go into details and refer to [2].

Remark 3.3.6. Theorem 3.2.1 is true when B is an Alexandrov space and F some general metric measure space. So it is reasonable to assume that our main result also could hold in a non-smooth context and we conjecture the following

Conjecture 3.3.7. *Let (B, d_B) be a complete Alexandrov space with $\dim_B = d$ and let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space. Let $f : B \to \mathbb{R}_{\geq 0}$ be some continuous function such that $\partial B \subset f^{-1}(\{0\})$. Assume that (F, m_F) satisfies $CD((N-1)K_F, N)$ and f is $\mathcal{F}K$-concave such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.*

Then the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Remark 3.3.8. In [11] there is an example where the Euclidean cone over some Riemannian manifold with $CD(N-1, N)$ produces a metric measure space satisfying $CD(0, N+1)$ but that is not an Alexandrov space with curvature bounded from below. They consider $F = \frac{1}{\sqrt{3}}S^2 \times \frac{1}{\sqrt{3}}S^2$, which satisfies $CD(3,4)$ but has sectional curvature 0 for planes spanned by vectors that lie in different spheres. Then the sectional curvature bound for the cone explodes when one gets nearer and nearer to the apex. For general warped products the same phenomenon occurs what can be seen at once from the formula of sectional curvature for warped products. Choose any closed n-dimensional Riemannian manifold with Ricci curvature bounded from below by $(n-1)K_F$ and with sectional curvature $K_F(V_x, W_x) = 0$ for some vectors V_x and W_x in $TF|_x$ (for example choose $\lambda S^m \times \lambda S^m$ where $m + m = n$ and λ is an appropriate scaling factor, that produces the Ricci curvature bound $(n-1)K_F$). Let B be a Riemannian manifold with boundary and sectional curvature bigger than $K \in \mathbb{R}$ and f is $\mathcal{F}K$-concave and satisfies the assumption of the theorem. (for example choose B as the upper hemisphere of S^d and f as the first nontrivial eigenfunction of the Laplacian of this sphere. Especially f vanishes at the boundary of B and $|\nabla f|_{\partial B} = 1$. See [21]). The sectional curvature of the plane $\Pi_{(p,x)}$ spanned by vectors $(X_p, V_x), (Y_p, W_x)$ in $T(B \times_f F)_{(p,x)}$ is

$$\begin{aligned} K(\Pi_{(p,x)}) &= K_B(X_p, Y_p)|X_p|^2|Y_p|^2 - f(p)\left[|W_x|^2\nabla^2 f(X_p, X_p) + |V_x|^2\nabla^2 f(Y_p, Y_p)\right] \\ &\quad + \frac{1}{f^2(p)}\left[K_F(V_x, W_x) - |\nabla f_p|^2\right]|\widetilde{V}_x|^2|\widetilde{W}_x|^2 \\ &= K_B(X_p, Y_p)|X_p|^2|Y_p|^2 - f(p)\left[|W_x|^2\nabla^2 f(X_p, X_p) + |V_x|^2\nabla^2 f(Y_p, Y_p)\right] \\ &\quad - \frac{1}{f^2(p)}|\nabla f_p|^2|\widetilde{V}_x|^2|\widetilde{W}_x|^2. \end{aligned}$$

Hence the sectional curvature of planes $\Pi_{(p_n,x)} \subset T(\overset{\circ}{B} \times_f^N F)_{(p_n,x)}$ as above explodes to $-\infty$ if we choose a sequences (X_{p_n}) and (Y_{p_n}) such that p_n tends to vanishing points of f. On the other hand the Ricci curvature is still bounded by 0 by formula (3.0.1). Especially there is also no upper bound for the sectional curvature.

Another application of Theorem 3.2.1 is the following corollary, which modifies a theorem by Lott that was proven in [55].

Corollary 3.3.9. *Let (B, g) be a compact, n-dimensional Riemannian manifold with distance function d_B and with CBB by K. Let $f : B \to \mathbb{R}_{\geq 0}$ be a smooth and $\mathcal{F}K$-concave function. Let $N \in \mathbb{N}$ such that $N \geq n$ and set $q = N - n$. Assume*

$$|\nabla f|_p^2 \leq K_F \quad \forall p \in \partial \operatorname{supp} f. \tag{3.3.4}$$

Then $(\operatorname{supp} f, \mathrm{d}_B, f^q d \operatorname{vol}_B)$ is the measured Gromov-Hausdorff limit of a sequence of compact geodesic spaces (M_i, d_i) of Hausdorff dimension N satisfying $CD((N-1)K, N)$.

Proof. Consider the q-warped product $M_i = B \times_{g_i} \frac{i}{\sqrt{K_F}} S^q$ where $g_i = \frac{1}{i} f$. The assumption implies

$$|\nabla g_i|_p^2 \leq \frac{K_F}{i^2} \quad \forall p \in \partial \operatorname{supp} f.$$

Then by our main theorem M_i satisfies $CD((N-1)K, N)$ for any i and $(M_i)_i$ converges to $(\operatorname{supp} f, \mathrm{d}_B, f^q d \operatorname{vol}_B)$ in measured Gromov-Hausdorff sense for $i \to 0$. □

We have the Conjecture 3.3.7 but at the moment we are not able to prove it. But one could ask if it is true when F is a warped product itself and satisfies a curvature-dimension bound in the sense of our main theorem. In this situation F would not be a manifold and singularities would occur. However the proof of the following corollary shows that an iterated warped product is essentially again a simple warped product.

Corollary 3.3.10. *Let B_2 be complete, d_2-dimensional space with CBB by K_2 such that $B_2 \backslash \partial B_2$ is a Riemannian manifold and let $f_2 : B_2 \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K_2$-concave and smooth on $B_2 \backslash \partial B_2$. Assume $\emptyset \neq \partial B_2 \subseteq f_2^{-1}(\{0\})$. Let B_1 be complete, d_1-dimensional Riemannian manifold with CBB by K_1 where $K_1 \geq 0$ such that*

$$|\nabla f_2|_p \leq \sqrt{K_1} \text{ for all } p \in \partial B_2.$$

Let $f_1 : B_1 \to \mathbb{R}_{\geq 0}$ be a smooth and $\mathcal{F}K_1$-concave. Assume $\emptyset \neq \partial B_1 \subseteq f_1^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. In any case F satisfies $CD((N-1)K_F, N)$ where $K_F \geq 0$ such that

$$|\nabla f_1|_p \leq \sqrt{K_F} \text{ for all } p \in \partial B_1.$$

Then the $N + d_1$-warped product $B_2 \times_{f_2}^{N+d_1} \left(B_1 \times_{f_1}^{N} F\right)$ satisfies $CD((N + d_1 + d_2 - 1)K_2, N + d_1 + d_2)$.

Proof. First we see that

$$B_2 \times_{f_2}^{N+d_1} \left(B_1 \times_{f_1}^{N} F\right) = \left(B_2 \times_{f_2}^{d_1} B_1\right) \times_{f_2 f_1}^{N} F.$$

as metric measure spaces. This comes from the fact that the warped product measure in both cases is

$$f_2^{N+d_1} d\,\mathrm{vol}_{B_2} \otimes \big(f_1^N d\,\mathrm{vol}_{B_1} \otimes d\,\mathrm{m}_F\big) = (f_1 f_2)^N \big(f_2^{d_1} d\,\mathrm{vol}_{B_2} \otimes d\,\mathrm{vol}_{B_1}\big) \otimes d\,\mathrm{m}_F$$

and the warped product metrics coincide because in both cases the length structure is given by

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{|\dot\alpha_2(t)|^2 + f_2^2 \circ \alpha_2(t) |\dot\alpha(t)_1|^2 + f_2^2 \circ \alpha_2(t) f_1^2 \circ \alpha_1(t) \mathcal{F}_F^2(\dot\beta(t))}\, dt.$$

Hence it is enough to check that $(B_2 \times_{f_2}^{d_1} B_1) \times_{f_2 f_1}^{N} F$ satisfies the required curvature-dimension bound.

We know by Theorem 2.2.12 that $B_2 \times_{f_2} B_1 =: B$ is a space with CBB by K_2. It is easy to see that its boundary is $B_2 \times_{f_2} \partial B_1 = \partial B$ and that the singularity points ∂B_2 are a subset of ∂B. It follows that $B \backslash \partial B$ is a Riemannian manifold. Then we know that if $(p_2, p_1) \in \partial B$, we have $f_2(p_2) f_1(p_1) = 0$ and so $\partial B \subset f^{-1}(\{0\})$ where $f = f_2 f_1$. Then we can calculate that f is $\mathcal{F}K_2$-concave and that it satisfies

$$|\nabla f|_{(p_2,p_1)} \leq \sqrt{K_F} \quad \text{for all } (p_2, p_1) \in \partial B,$$

where the modulus of the gradient is taken with respect to the warped product metric of $B_2 \times_{f_2}^{d_1} B_1$. Thus the assumptions of Theorem A are fulfilled and the result follows. □

4 Preliminaries, part 2

Outline of the chapter. This chapter is the beginning of the second part of the thesis where we leave the smooth framework. It provides necessary definitions and results from metric measure space calculus and Dirichlet form theory.

In Section 4.1 we repeat the definition of upper gradients and related results. Then, we introduce the Cheeger energy and the minimal weak upper gradient as established by Ambrosio, Gigli and Savaré. In Section 4.2 we switch to the field of symmetric Dirichlet forms. After elementary properties have been presented, we give the definition of the so-called Bakry-Emery curvature-dimension condition for symmetric Diffusion operators that was used by Bakry, Emery and Ledoux. We first present the classical approach that assumes the existence of a nice functional algebra, followed by a reformulation by Ambrosio, Gigli and Savaré that is also applicable to more general situations. Then, we also present a smooth example that will be important for the warped product construction. In Section 4.3 we introduce Riemannian Ricci curvature bounds and present the fundamental connections between the Eulerian and the Lagrangian picture of Ricci curvature proved by Erbar, Kuwada and Sturm. At the end of this chapter, in Section 4.4 we give the definition of skew products between Dirichlet forms that was introduced by Fukushima and Oshima.

4.1 Differential calculus for metric measure spaces

Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space. We denote by $L^p(X, \mathrm{m}_X) =: L^p(\mathrm{m}_X)$ for $p \in [0, \infty]$ the Lebesgue spaces with respect m_X.

Poincaré inequalities. A Borel function $g : X \to [0, \infty]$ is an upper gradient of a continuous function $u : X \to \mathbb{R}$ if for any rectifiable curve $\gamma : [0, 1] \to X$

$$|u(\gamma_0) - u(\gamma_1)| \leq \int_0^1 g(\gamma(t))|\dot{\gamma}(t)|dt. \tag{4.1.1}$$

If the metric speed does not exist, the right hand side is infinity. We say that a metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ supports a weak local (q, p)-Poincaré inequality with $1 \leq p \leq q < \infty$ if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all continuous u, any upper gradient g of u and any point $x \in X$ and $r > 0$

$$\left(\int_{B_r(x)} |u - u_{B_r(x)}|^q d\,\mathrm{m}_X \right)^{\frac{1}{q}} \leq Cr \left(\int_{B_{\lambda r}(x)} g^p d\,\mathrm{m}_X \right)^{\frac{1}{p}}. \tag{4.1.2}$$

If $\lambda = 1$, we say X supports a strong (q,p)-Poincaré inequality. Some authors also use the term Poincaré-Sobolev inequality for the case $q > 1$ and $(1,p)$-Poincaré inequalities are just called p-Poincaré inequality (see for example [45]). In the following we say that X supports a (local) Poincaré inequality if it supports a weak local $(1,1)$-Poincaré inequality.

Remark 4.1.1. (i) Under a doubling property for $(X, \mathrm{d}_X, \mathrm{m}_X)$ weak local Poincaré inequalities imply strong ones.

(ii) By Hölder's inequality, a weak local $(1,p)$-Poincaré inequality implies a weak local $(1,p')$-Poincaré inequality for $p' \geq p$.

Definition 4.1.2. Let $u : X \to \mathbb{R}$ be a continuous function. The local slope (or local Lipschitz constant or pointwise Lipschitz constant) is the Borel function Lip given by

$$\operatorname{Lip} u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{\mathrm{d}_X(x,y)}.$$

$\operatorname{Lip} u$ is an upper gradient for u [22, proposition 1.11].

Remark 4.1.3. If a metric measure space satisfies a doubling property, Hajlasz and Koskela proved in [45, Theorem 5.1, 1.] that a weak local $(1,p)$-Poincaré inequality also implies a (q,p)-Poincaré inequality for $q < \frac{pN}{N-p}$ if the doubling constant satisfies $C \leq 2^N$ and $p < N$. This is the case if the space satisfies the condition $CD(0,N)$. In particular, $(X, \mathrm{d}_X, \mathrm{m}_X)$ supports a weak local $(2,2)$-Poincaré inequality.

Von Renesse proved the following result.

Theorem 4.1.4 ([81])**.** *Suppose that the metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $MCP(\varkappa, N)$ in the sense of Sturm and for m_X^2-a.e. pair $(x,y) \in X^2$ there is a unique geodesic. Then $(X, \mathrm{d}_X, \mathrm{m}_X)$ supports a weak local $(1,1)$-Poincaré inequality.*

Remark 4.1.5. If the metric measure space is assumed to be non-branching, the unique-geodesic-property is implied and it actually does not matter if we consider the MCP in the sense of Ohta or in the sense of Sturm. There is also an extension of this result by Rajala, who proved a Poincaré inequality in the setting of $CD(\varkappa, N)$-spaces for any $N \in [1, \infty]$ without non-branching assumptions.

Cheeger energy and Sobolev spaces. We want to define Sobolev spaces and a notion of modulus of gradient on a suitable class of functions. There are several authors that gave different definitions (see [22, 72, 44]). Here, we follow the approach of Ambrosio, Gigli and Savaré. Their main result from [5] (see also [4]) states that for metric measure spaces in the sense of Definition 2.1.1 most of the different approaches coincide and give the same notion of Sobolev space and modulus of a gradient. The key is a non-trivial approximation by Lipschitz functions that we will use as starting

point for our presentation. For any Borel function $u : X \to \mathbb{R}$ in $L^2(\mathrm{m}_X)$ the Cheeger energy $\mathrm{Ch}^X(u)$ is defined by

$$\mathrm{Ch}^X(u) = \frac{1}{2} \inf \left\{ \liminf_{h\to\infty} \int_X (\mathrm{Lip}\, u_h)^2 \, d\,\mathrm{m}_X : u_h \text{ Lipschitz}, \|u_h - u\|_{L^2(\mathrm{m}_X)} \to 0 \right\}. \tag{4.1.3}$$

Then the L^2-Sobolev space is given by $D(\mathrm{Ch}^X) = \left\{ u \in L^2(\mathrm{m}_X) : \mathrm{Ch}^X(u) < \infty \right\}$. The associated norm is $\|u\|^2_{D(\mathrm{Ch}^X)} = \|u\|^2_{L^2} + 2\,\mathrm{Ch}^X(u)$. An important fact is that Ch is not a quadratic form in general.

Definition 4.1.6. Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space. If the Cheeger energy Ch^X is a quadratic form, we call $(X, \mathrm{d}_X, \mathrm{m}_X)$ infinitesimal Hilbertian.

Another result from [5] is that Ch^X can be represented by

$$\mathrm{Ch}^X(u) = \frac{1}{2} \int_X |\nabla u|^2_w d\,\mathrm{m}_X \quad \text{if } u \in D(\mathrm{Ch}) \tag{4.1.4}$$

and $+\infty$ otherwise where $|\nabla u|_w : X \to [0, \infty]$ is Borel measurable and called the minimal weak upper gradient of u. The notion of minimal weak upper gradient is motivated by the next definition that we take from [6].

Let $\gamma : J \to X$ be an absolutely continuous curve, that is, there exists $g \in L^1(J, dt)$ such that

$$\mathrm{d}_X(\gamma(s), \gamma(t)) \le \int_s^t g(\tau) d\tau \quad \text{for } s, t \in J, \ \ s < t. \tag{4.1.5}$$

In particular, γ is rectifiable. Then, γ has a well-defined metric speed $|\dot\gamma(\cdot)| \in L^1(J, dt)$ that satisfies (4.1.5) and the length is given by (2.1.1). We denote with $AC^p(J, X)$ the subset of $AC^1(J, X)$ such that the metric speed is in $L^p(J, dt)$.

We say that $u : X \to \mathbb{R} \cup \{\infty\}$ is "Sobolev along 2-almost every curve" if $u \circ \gamma$ coincides a.e. in $[0, 1]$ and in $\{0, 1\}$ with an absolutely continuous map $u_\gamma : [0, 1] \to \mathbb{R}$ for 2-almost every curve $\gamma \in C([0, 1], X)$. A subset $A \subset C([0, 1], X)$ is 2-negligible in this sense if $\Pi(A) = 0$ for any 2-test plan Π. A probability measure Π on $C([0, 1], X)$ is called 2-test plan if it is concentrated on $AC^2([0, 1], X)$, $\int \int_0^1 |\dot\gamma(t)|^2 dt d\Pi(\gamma) < \infty$ and $(e_t)_*\Pi \le C(\Pi)\,\mathrm{m}_X$ for some constant $C(\Pi) > 0$. This notion was introduced in [6]. Hence, if we want to check a property for 2-almost every curve, it is sufficient to consider $\gamma \in AC^2([0, 1], X)$.

Definition 4.1.7. For u that is Sobolev along almost every curve, an m_X-measurable function $G : X \to [0, \infty]$ is a *weak* upper gradient of u if

$$|u(\gamma_0) - u(\gamma_1)| \le \int_0^1 G(\gamma(t))|\dot\gamma(t)| dt \quad \text{for 2-almost every curve } \gamma.$$

Any function $u \in D(\mathrm{Ch})$ is Sobolev along 2-almost every curve and $|\nabla u|_w$ is a minimal weak upper gradient in the following sense: If G is a weak upper gradient of u, then $|\nabla u|_w \le G$ m_X-a.e. .

Remark 4.1.8. An upper gradient g for some continuous u is also a weak upper gradient in the sense of the previous definition. The converse is in general not true. We have $|\nabla u|_w \leq \operatorname{Lip} u$ a.e. , but no equality in general. If we assume a doubling property and a local Poincaré inequality, there is the following result of Cheeger.

Theorem 4.1.9 ([22]). *If $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a complete and intrinsic metric measure space that provides a doubling property and supports a local Poincaré inequality, then for any function $u : X \to \mathbb{R}$ that is locally Lipschitz, we have $\operatorname{Lip} u = |\nabla u|_w$ m_X-a.e..*

The minimal weak upper gradient admits a stability property.

Theorem 4.1.10 (Stability theorem, [5]). *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a complete and separable metric measure space. Let $u_n \in D(\mathrm{Ch}^X)$ such that $u_n \to u \in L^2(\mathrm{m}_X)$ pointwise m_X-a.e. and assume $|\nabla u_n|_w \in L^2(\mathrm{m}_X)$ converges weakly to $g \in L^2(\mathrm{m}_X)$. Then $u \in D(\mathrm{Ch}^X)$ and $g = |\nabla u|_w$ m_X-a.e. .*

Remark 4.1.11. In the introduction of [5] the authors remark that a complete and separable metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ whose balls have finite measure (hence, it fits in our Definition 2.1.1), supports a weak local $(1,1)$-Poincaré inequality with constants $C > 0$ and $\lambda \geq 1$ if and only if it holds for any Lipschitz function u and upper gradient $\operatorname{Lip} u$. More precisely, supporting a Poincaré inequality is equivalent to that for any Lipschitz function u on X, for any $x \in X$ and $r > 0$ such that $\mathrm{m}_X(B_r(x)) > 0$, it holds

$$\int_{B_r(x)} |u - \bar{u}_{B_r(x)}| d\,\mathrm{m}_X \leq Cr \int_{B_{\lambda r}(x)} \operatorname{Lip} u d\,\mathrm{m}_X \,. \tag{4.1.6}$$

Remark 4.1.12. If we assume that $(X, \mathrm{d}_X, \mathrm{m}_X)$ is locally compact and that the Cheeger energy Ch^X of $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a quadratic form, Ch^X yields also a strongly local Dirichlet form $(\mathrm{Ch}^X, D(\mathrm{Ch}^X))$ on $L^2(\mathrm{m}_X)$. Lipschitz functions are dense in $D(\mathrm{Ch}^X)$ with respect to the Energy norm (see Proposition 4.10 in [6]). Additionally, if we assume that the space X is compact, Lipschitz function are dense in $C_0(X)$ with respect to uniform convergence by application of the Stone-Weierstraß Theorem. Hence, Ch^X is a regular Dirichlet form and Lipschitz functions are a core in the sense of Dirichlet forms.

4.2 Preliminaries on Dirichlet forms

At the end of the last section we saw that the Cheeger energy is a strongly local Dirichlet form provided the underlying space is infinitesimal Hilbertian. Dirichlet forms are well-studied objects in the field of stochastic processes. In the following we will give a brief introduction to it and also to the Bakry-Emery curvature-dimension condition that was also used to define Ricci curvature in an abstract framework.

4.2.1 Dirichlet forms and their Γ-operator

We consider a locally compact and separable Hausdorff space $(X, \mathcal{O}_X)$ and a positive and σ-finite Radon measure m_X on X such that $\mathrm{supp}\,[m_X] = X$. Let $(\mathcal{E}^X, D(\mathcal{E}^X))$ be a symmetric Dirichlet form on $L^2(\mathrm{m}_X)$ where $D(\mathcal{E}^X)$ is a dense subset of $L^2(\mathrm{m}_X)$. A symmetric Dirichlet form is an $L^2(\mathrm{m}_X)$-lower semi-continuous, quadratic form that satisfies the Markov property. Dirichlet forms are closed, that is, the domain $D(\mathcal{E}^X)$ is a Hilbert space with respect to the energy norm that comes from the inner product

$$(u,u)_{D(\mathcal{E}^X)} = (u,u)_{L^2(\mathrm{m}_X)} + \mathcal{E}^X(u,u).$$

There is a self-adjoint, negative-definite operator $(L^X, D_2(L^X))$ on $L^2(\mathrm{m}_X)$. Its domain is

$$D_2(L^X) = \left\{u \in D(\mathcal{E}^X) \colon \exists v \in L^2(\mathrm{m}_X) : -(v,w)_{L^2(\mathrm{m}_X)} = \mathcal{E}^X(u,w)\ \forall w \in D(\mathcal{E}^X)\right\}.$$

We set $v =: L^X u$. $D_2(L^X)$ is dense in $L^2(\mathrm{m}_X)$ and equipped with the topology given by the graph norm. L^X induces a strongly continuous Markov semi-group $(P_t^X)_{t\geq 0}$ on $L^2(X, m_X)$. The relation between form, operator and semi-group is standard (see [36]).

A Dirichlet form is called regular if $\mathcal{E}^X$ possesses a core. A core of $\mathcal{E}^X$ is by definition a subset $\mathcal{C}^X$ of $D(\mathcal{E}^X) \cap C_0(X)$ such that $\mathcal{C}^X$ is dense in $D(\mathcal{E}^X)$ with respect to the energy norm and dense in $C_0(X)$ with respect to uniform convergence where $C_0(X)$ is the set of continuous functions with compact support in X. We say that a symmetric form is strongly local if $\mathcal{E}^X(u,v) = 0$ whenever $u, v \in D(\mathcal{E}^X)$ and $(u+a)v = 0$ m_X-almost surely in X for some $a \in \mathbb{R}$.

Definition 4.2.1 (Γ-operator for Dirichlet forms)**.** Set $D^\infty(\mathcal{E}^X) = D(\mathcal{E}^X) \cap L^\infty(\mathrm{m}_X)$. Then $D^\infty(\mathcal{E}^X)$ is an algebra (see [16]) and for $u, \varphi \in D^\infty(\mathcal{E}^X)$ the following operator is well-defined

$$\Gamma^X(u;\varphi) := \mathcal{E}^X(u, u\varphi) - \frac{1}{2}\mathcal{E}^X(u^2, \varphi).$$

It can be extended by continuity to any $u \in D(\mathcal{E}^X)$. We call $\mathbb{G}$ the set of functions $u \in D(\mathcal{E}^X)$ such that the linear form $\varphi \mapsto \Gamma^X(u;\varphi)$ can be represented by an absolutely continuous measure w.r.t m_X with density $\Gamma^X(u) \in L^1_+(\mathrm{m}_X)$. If $\mathcal{E}^X$ is symmetric, we get the following representation

$$\mathcal{E}^X(u,u) = \int_X \Gamma^X(u)\, d\,\mathrm{m}_X \quad \& \quad \Gamma^X(u;\varphi) = \int_X \Gamma^X(u,u)\varphi\, d\,\mathrm{m}_X \tag{4.2.1}$$

for any $u \in \mathbb{G}$ and $\varphi \in D^\infty(\mathcal{E}^X)$. By polarization we can extend the Γ-operator as trilinear form as follows

$$\Gamma^X(u,v;\varphi) = \frac{1}{2}\left(\Gamma^X(u;\varphi) + \Gamma^X(v;\varphi) - \Gamma^X(u-v;\varphi)\right), \quad u, v \in D(\mathcal{E}^X),\ \varphi \in D^\infty(\mathcal{E}^X)$$

If $\mathbb{G} = D(\mathcal{E}^X)$, we say $\mathcal{E}^X$ admits a "carré du champ" or Γ-operator. Fundamental properties of $\Gamma^X : D(\mathcal{E}^X) \times D(\mathcal{E}^X) \to L^1(\mathrm{m}_X)$ are positivity, symmetry, bilinearity and continuity (see [16, Proposition 4.1.3]).

Leibniz rule. Strong locality of $\mathcal{E}^X$ implies strong locality of Γ^X: $1_U \cdot \Gamma^X(u,v) = 0$ for all $u, v \in \mathbb{G}$ and for all open sets U on which u is constant(see [74, Appendix]) and the Leibniz rule: For all $u, v, w \in \mathbb{G}$ such that $v, w \in L^\infty(\mathrm{m}_X)$ it holds $v \cdot w \in \mathbb{G}$ and

$$\Gamma^X(u, v \cdot w) = \Gamma^X(u,v) \cdot w + v \cdot \Gamma^X(u,w) \tag{4.2.2}$$

(see [74, Appendix]). One can prove the following

Lemma 4.2.2. *Assume $\mathbb{G} = D(\mathcal{E}^X)$ and $\mathrm{m}_X(X) < \infty$. (4.2.2) also holds for $u, v, w \in D(\mathcal{E}^X)$ with $v, \Gamma^X(v) \in L^\infty(\mathrm{m}_X)$.*

Proof. Consider $w_n = (w \wedge -n) \vee n \in D(\mathcal{E}^X) \cap L^\infty(\mathrm{m}_X)$ for $n \in \mathbb{N}$. w_n is bounded and by Theorem 1.4.2 (iii) in [36] $(w_n)_n$ converges to w in $D(\mathcal{E}^X)$. We can apply (4.2.2) for $v, w_n - w_m \in D(\mathcal{E}^X) \cap L^\infty$:

$$\begin{aligned}\mathcal{E}^X(v(w_n - w_m)) = \int_X \big[&\Gamma^X(w_n - w_m)v^2 \\ &+ 2v(w_n - w_m)\Gamma^X(v, w_n - w_m) + (w_n - w_m)^2\Gamma^X(v)\big]\, d\,\mathrm{m}_X\end{aligned}$$

Since $v, \Gamma^X(v) \in L^\infty(\mathrm{m}_X)$, the right hand side converges to 0 if $n, m \to \infty$, hence, $v \cdot w_n$ is a Cauchy sequence in $D(\mathcal{E}^X)$ that converges to $g = v \cdot w$. Thus, $v \cdot w \in D(\mathcal{E}^X)$ and similar one can show the formula (4.2.2). □

Chain rule. We say $\mathcal{E}^X$ is of diffusion type if L^X satisfies the following chain rule. Let η be in $C^2(\mathbb{R})$ with $\eta(0) = 0$. If $u \in D_2(L^X)$ with $\Gamma(u) \in L^2(X, m_X)$ and $\eta(u) \in D(L^X)$, then

$$L^X\eta(u) = \eta'(u)L^X u + \eta''(u)\Gamma^X(u). \tag{4.2.3}$$

This is the case when $\mathbb{G} = D(\mathcal{E}^X)$ (see [16, Corollary 6.1.4]).

Intrinsic distance. If $\mathcal{E}^X$ is strongly local and admits a "carré du champ" operator, we can define $D_{loc}(\mathcal{E}^X)$ as follows. $u \in D_{loc}(\mathcal{E}^X)$ if $u \in L^2_{loc}(\mathrm{m}_X)$ and for any compact set K there exists $v \in D(\mathcal{E}^X)$ such that $v = u$ m_X-a.e. on K. Hence, for any $u \in D_{loc}(\mathcal{E}^X)$ there exists $\Gamma^X(u) \in L^1_{loc}(\mathrm{m}_X)$. The intrinsic distance of $\mathcal{E}^X$ is defined by

$$\mathrm{d}_{\mathcal{E}^X}(x,y) = \sup\left\{u(x) - (y) : u \in D_{loc}(\mathcal{E}^X) \cap C(X),\ \Gamma^X(u) \le 1\ \ \mathrm{m}\text{-a.e.}\right\}.$$

The intrinsic distance is not a metric in general but a pseudo-metric since there can be points $x \neq y$ with $\mathrm{d}_{\mathcal{E}^X}(x,y) = 0, \infty$. For the rest of this article we always assume that $\mathcal{E}^X$ is a strongly local and regular Dirichlet form with $\mathbb{G} = D(\mathcal{E}^X)$. Then we will call $\mathcal{E}^X$ also admissible. We say the Dirichlet form $\mathcal{E}^X$ is strongly regular if $\mathrm{d}_{\mathcal{E}^X}$ is a metric and the topology of $\mathrm{d}_{\mathcal{E}^X}$ coincides with the original one.

We say that $\mathcal{E}^X$ supports a weak local (q,p)-Poincaré inequality with $1 \le p \le q < \infty$ if for any $u \in D(\mathcal{E}^X)$ (4.1.2) holds where we replace g by $\sqrt{\Gamma^X(u)}$.

Remark 4.2.3. Let $\mathcal{E}^X$ a strongly local and strongly regular Dirichlet form and let $\mathrm{d}_{\mathcal{E}^X}$ be its intrinsic distance. Assume that closed balls $\bar{B}_r(x)$ are compact for any $r > 0$ and $x \in X$. Assume a doubling property holds and $\mathcal{E}^X$ supports a weak local $(2,2)$-Poincaré inequality. Then

(1) P_t^X admits an α-Hölder-continuous kernel and is a Feller semi-group.

(2) P_t^X is $L^2 \to L^\infty$-ultracontractiv: $\|P_t^X\|_{L^2 \to L^\infty} \le 1$.

(3) If $\mathrm{m}_X(X) < \infty$, harmonic functions are constant, and X is connected.

$L^2 \to L^\infty$-ultracontractivity actually comes from an upper bound for the heat kernel (see [40, Chapter 14.1] and [76, Theorem 4.1]).

4.2.2 The Bakry-Emery curvature-dimension condition

In this section we introduce the curvature-dimension condition for Dirichlet forms in the sense of Bakry, Emery and Ledoux. The specific feature of this approach is the existence of an algebra $\mathcal{A}^X$ of bounded measurable functions on X that is dense in $D_2(L^X)$ and in all L^p-spaces, stable by L^X and stable by composition with C^∞-functions of several variables that vanish at 0. We call such an algebra admissible. The algebra allows to introduce notions of curvature and dimension on a purely algebraic level and provides a calculus that simplifies proofs significantly.

A consequence of the existence of an admissible algebra is that the "carré du champ"-operator for elements in $\mathcal{A}^X$ is obtained by the following rule

$$\Gamma^X(u) = \frac{1}{2} L^X(u^2) - u L^X u \quad \text{for all } u \in \mathcal{A}^X.$$

Provided $D(\mathcal{E}^X) = \mathbb{G}$, this rule is consistent with Definition 4.2.1 (see [16], section I.4). Replacing L^X by Γ^X in the definition of the carré du champ we can define the so-called iterated carré du champ or Γ_2-operator

$$2\Gamma_2^X(u,v) = L^X \Gamma^X(u,v) - \Gamma^X(u, L^X v) - \Gamma^X(v, L^X u) \quad \text{for all } u, v \in \mathcal{A}^X.$$

We write $\Gamma^X(u)$ for $\Gamma^X(u,u)$ and similarly for Γ_2^X.

Definition 4.2.4 (Classical Bakry-Emery curvature-dimension condition)**.** Assume there is an admissible algebra $\mathcal{A}^X$ for $\mathcal{E}^X$. Then $\mathcal{E}^X$ satisfies the "classical" Bakry-Emery curvature-dimension condition $BE(\varkappa, N)$ of curvature $\varkappa \in \mathbb{R}$ and dimension $1 \le N < \infty$ if

$$\Gamma_2^X(u) \ge \varkappa \Gamma^X(u) + \frac{1}{N}\big(L^X u\big)^2 \quad \text{for all } u \in \mathcal{A}^X. \tag{4.2.4}$$

The inequality is understood to hold m_X-almost everywhere in X. Similar, the condition $BE(\varkappa, \infty)$ holds if $\Gamma_2^X(u) \ge \varkappa \Gamma^X(u)$ m_X-a.e. for all $u \in \mathcal{A}^X$ and $BE(\varkappa, N)$ implies $BE(\varkappa, \infty)$.

In many situations an algebra $\mathcal{A}^X$ is not available. To overcome this problem, in [3] the Definition 4.2.4 was reformulated in an "intrinsic" way that also makes sense without the admissible algebra. For the rest of this section we will briefly present this approach. A more detailed description can be found in [3]. We still consider a regular and strongly local Dirichlet form $\mathcal{E}^X$ on some admissible space X like in Section 4.2.1. The Γ_2-operator can be defined in a weak sense by

$$2\Gamma_2^X(u,v;\varphi) = \Gamma^X(u,v;L^X\varphi) - \Gamma^X(u,L^Xv;\varphi) - \Gamma^X(v,L^Xu;\varphi)$$

for $u,v \in D(\Gamma_2)$ and $\varphi \in D_+^{b,2}(L^X)$ where $D(\Gamma_2^X) := \{u \in D_2(L^X) : L^Xu \in D(\mathcal{E}^X)\}$ and the set of test functions is denoted by

$$D_+^{b,2}(L^X) := \big\{\varphi \in D_2(L^X) : \varphi, L^X\varphi \in L^\infty(X,\mathrm{m}_X), \varphi \geq 0\big\}.$$

We set $\Gamma_2^X(u,u;\varphi) = \Gamma_2^X(u;\varphi)$. Now we can state the curvature-dimension condition in a weak sense.

Definition 4.2.5 (Bakry-Emery curvature-dimension condition)**.** Let $\varkappa \in \mathbb{R}$ and $N \geq 1$. We say that $\mathcal{E}^X$ satisfies the intrinsic Bakry-Emery curvature-dimension condition (or just Bakry-Emery condition) $BE(\varkappa,N)$ if for every $u \in D(\Gamma_2^X)$ and $\varphi \in D_+^{b,2}(L^X)$, we have

$$\Gamma_2^X(u;\varphi) \geq \varkappa\Gamma^X(u;\varphi) + \frac{1}{N}\int_X (L^Xu)^2\varphi d\,\mathrm{m}_X\,. \tag{4.2.5}$$

In this case we have that $\mathbb{G} = D(\mathcal{E}^X)$ (see [3, Corollary 2.3]). Hence, $\mathcal{E}^X$ is of diffusion-type. As before we can also define $BE(\varkappa,\infty)$ and the implications $BE(\varkappa,N) \Rightarrow BE(\varkappa,N') \Rightarrow BE(\varkappa,\infty)$ for $N' \geq N$ hold as well.

Remark 4.2.6. If we assume there is an admissible algebra, then the Bakry-Emery condition $BE(\varkappa,N)$ is always understood in the sene of Definition 4.2.4 without further comment.

Theorem 4.2.7 (Bakry-Ledoux gradient estimate)**.** *Let $\mathcal{E}^X$ be an admissible Dirichlet form. The estimate (4.2.5) for $\varkappa \in \mathbb{R}$, $N \geq 1$ and any $(u,\varphi) \in D(\Gamma_2^X)$ with $\varphi \geq 0$ is equivalent to the following gradient estimate. For any $u \in \mathbb{G}$ and $t > 0$, P_t^Xu belongs to $\mathbb{G}$ and we have*

$$\Gamma^X(P_t^Xu) + \frac{1-e^{-2\varkappa t}}{N\varkappa}(L^XP_t^Xu)^2 \leq e^{-2\varkappa t}P_t^X\Gamma^X(u)\ \ \mathrm{m}_X\text{-a.e. in } X. \tag{4.2.6}$$

Proof. $\rightarrow$ The proof of the theorem in this form can be found in [3] (see also [12, 34]).

Remark 4.2.8. If there is an admissible algebra that is stable with respect to P_t^X, the definitions 4.2.4 and 4.2.5 are consistent. On the one hand, Defintion 4.2.5 and the

existence of an admissible algebra $\mathcal{A}^X$ imply that for any test function $\varphi \in D_+^{b,2}(L^X)$ and any $u \in \mathcal{A}^X$

$$\int_X \Gamma_2^X(u)\varphi d\,\mathrm{m}_X \geq \varkappa\Gamma(u;\varphi) + \frac{1}{N}\int_X (L^X u)^2 \varphi d\,\mathrm{m}_X\,.$$

Then we can replace $\varphi \in D_+^{b,2}(L^X)$ by any bounded and measurable function $\varphi \geq 0$ by using the mollifying property of P_t^X, exactly like in [3] and [34]. This implies the classical Bakry-Emery condition in the sense of Definition 4.2.4 for $u \in \mathcal{A}^X$. On the other hand, if we assume the Bakry-Emery condition in the sense of Definition 4.2.4 for some admissible algebra $\mathcal{A}^X$ that is also stable under P_t^X, we can apply the following lemma.

Lemma 4.2.9. *Assume there is a subset $\Xi \subset D_2(L^X)$ that is dense with respect to the graph norm and stable under the Markovian semi-group P_t^X, and assume we have*

$$\Gamma_2^X(u;\varphi) \geq \varkappa\Gamma^X(u;\varphi) + \frac{1}{N}\int_X (L^X u)^2\varphi d\,\mathrm{m}_X \quad \textit{if } u \in D(\Gamma_2^X)\cap\Xi \textit{ and } \varphi \in D_+^{b,2}(L^X).$$

Then $\mathcal{E}^X$ satisfies $BE(\varkappa, N)$.

Proof. We have to show (4.2.5) for any $u \in D(\Gamma_2^X)$ and any $\varphi \in D_+^{b,2}(L^X)$. We choose a sequence $u_n \in \Xi$ such that $u_n \to u$ in $D_2(L^X)$. Then we have also convergence in $\|\cdot\|_{D(\mathcal{E})}$ since for $v_n = u_n - u$

$$\mathcal{E}(v_n) = -(v_n, L^X(v_n))_{L^2(\mathrm{m}_X)} \leq \frac{1}{2}\|v_n\|^2_{L^2(\mathrm{m}_X)} + \frac{1}{2}\|L^X v_n\|^2_{L^2(\mathrm{m}_X)}\,.$$

$\Gamma^X(\cdot,\cdot)$ is continuous bilinear form form $D(\mathcal{E}^X)\times D(\mathcal{E}^X)$ to $L^1(X,\mathrm{m}_X)$. Hence

$$\begin{aligned}
\int_X \Gamma^X(u_n,u_n)L^X\varphi dm_X &\to \int_X \Gamma^X(u,u)L^X\varphi dm_X\\
\int_X \Gamma^X(u_n,u_n)\varphi d\,\mathrm{m}_X &\to \int_X \Gamma^X(u_n,u_n)\varphi d\,\mathrm{m}_X\\
\int_X (L^X u_n)^2\varphi d\,\mathrm{m}_X &\to \int_X (L^X u)^2\varphi d\,\mathrm{m}_X\,.
\end{aligned} \tag{4.2.7}$$

We also see that $P_t^X u_n, P_t^X u \in D^2(L^X)$ for all $t > 0$ and that $P_t^X u_n \to P_t^X u$ with respect to the graph and the energy norm since $P_t^X : L^2(\mathrm{m}_X) \to L^2(\mathrm{m}_X)$ is a bounded operator and $L^X P_t^X u = P_t^X L^X u$ for any $u \in D^2(L^X)$. Consequently, the convergence statements in (4.2.7) hold also for $P_t^X u_n, P_t^X u$ instead of u_n, u. Now, we use Lemma 1.3.3 from [36] that states that $P_t^X L^2(\mathrm{m}_X) \subset D(\mathcal{E}^X)$ and

$$\mathcal{E}^X(P_t^X u, P_t^X u) \leq \frac{1}{2t}\left((u,u)_{L^2} - (P_t^X u, P_t^X u)\right) \leq \mathcal{E}(u,u) \text{ for } u \in D(\mathcal{E}^X). \tag{4.2.8}$$

Hence, $P_t^X L^X u_n = L^X P_t^X u_n, P_t^X L^X u = L^X P_t^X u \in D(\mathcal{E}^X)$ and $P_t^X L^X u_n \to P_t^X L^X u$ in energy norm. By continuity of $\Gamma^X : D(\mathcal{E}^X)^2 \to L^1(\mathrm{m}_X)$ it follows

$$\int_X \Gamma^X(P_t^X u_n, L^X P_t^X u_n)\varphi d\,\mathrm{m}_X \to \int_X \Gamma^X(P_t^X u, L^X P_t^X u)\varphi d\,\mathrm{m}_X\,.$$

Since we assume that Ξ is stable with respect to P_t^X, the Γ^2-estimate holds for $P_t^X u_n$ for any $t > 0$ and any n, and because of the previous observation it also holds for $P_t^X u$.

Now we can follow the same strategy like in the proof of theorem 4.6 in [38]. Since φ and $L^X\varphi$ belong to $L^\infty(\mathrm{m}_X)$ and since $L^X P_t^X u = P_t^X L^X u$ for any $u \in D^2(L^X)$ it suffices to show that

$$\lim_{t\to 0} \Gamma^X(P_t^X u, P_t^X \widetilde{u}) = \Gamma^X(u, \widetilde{u}) \text{ weakly in } L^1(\mathrm{m}_X)$$

for any $u, \widetilde{u} \in D(\mathcal{E}^X)$. But this follows from (4.2.8) and Schwartz' inequality. □

4.2.3 Some examples of Dirichlet forms

In this section we consider some examples in more detail. They will play an important role later in this thesis.

Assumption 4.2.10. Let (B, g) be a smooth, d-dimensional Riemannian manifold with or without boundary and $\mathrm{ric}_B \geq K(d-1)$. We set $\mathring{B} = B\backslash\partial B$ and assume that $(\mathring{B}, \mathrm{d}_B)$ is geodesically convex. The latter will actually not be used explicitely, but, in particular, it means that B is a geodesic space. In Chapter 2 and 3 we also assumed that $\mathring{B}$ has CBB by K but in the following this will not be needed.

We consider the standard Dirichlet form with Dirichlet boundary conditions. More precisely, this is the form closure of

$$\mathcal{E}^B(u) = \int_B |\nabla u|_g^2 d\,\mathrm{vol}_B \text{ where } u \in C_0^\infty(\mathring{B}).$$

Its domain is $D(\mathcal{E}^B) = W_0^{1,2}(\mathring{B}, d\,\mathrm{vol}_B)$. The associated self-adjoint operator L^B is the Dirichlet Laplace operator Δ^B with domain $D^2(L^B) = W_0^{2,2}(\mathring{B}, \mathrm{vol}_B)$. In this context, we have $\Gamma^B(u) = |\nabla u|_g^2$. Any smooth function $u \in C^\infty(\mathring{B})$ satisfies the Bochner-Weitzenböck inequality

$$\Gamma_2^B(u) = \frac{1}{2}\Delta^B|\nabla u|_g^2 - \langle \nabla u, \nabla\Delta^B u\rangle = \mathrm{ric}_B(\nabla u) + \|\nabla^2 u\|_{HS}^2. \tag{4.2.9}$$

Assumption 4.2.11. We also consider a smooth $f : B \to \mathbb{R}_{\geq 0}$ in $D^2(L^B)$ such that $f|_{\partial B} = 0$, f is positive in $\mathring{B}$ and f is $\mathcal{F}K$-concave in the sense that $\nabla^2 f(v) \leq -Kf|v|_g^2$ for any $v \in TB$, where $\nabla^2 f$ denotes the Hessian of f with respect to g. Since B is geodesic, this equivalent to the definition of $\mathcal{F}K$-concavity from Section 2.2.

Example 4.2.12 (1-dimensional model space). Let B be of the form $I_K = [0, \pi/\sqrt{K}]$ for $K > 0$ and $[0, \infty)$ for $K \leq 0$. The corresponding operator is $L^{I_K} = d^2/dx^2$ and its domain is $W_0^{2,2}(I_K, dx)$. Consider $f : I_K \to \mathbb{R}_{\geq 0}$ in $D_2(L^{I_K})$ that is given by

$$f(t) = \sin_K(t) := \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & \text{for } K > 0 \\ t & \text{for } K = 0 \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}t) & \text{for } K < 0. \end{cases}$$

Consider $(B, g, f^N d\operatorname{vol}_B)$. We can define a symmetric form $\mathcal{E}^{B,f^N}$ on $L^2(B, f^N d\operatorname{vol}_B)$ by

$$\mathcal{E}^{B,f^N}(u) = \int_B |\nabla u|_g^2 f^N d\operatorname{vol}_B \quad \text{for } u \in C_0^\infty(\mathring{B}). \tag{4.2.10}$$

Then $\mathcal{E}^{B,f^N}$ is closable on $C_0^\infty(\mathring{B})$ (see [36, Theorem 6.3.1], [57]), and it becomes a strongly local and regular Dirichlet form. $\Gamma^{B,f^N}(u) = \Gamma^B(u) = |\nabla u|_g^2$ for any $u \in C_0^\infty(\mathring{B})$. L^{B,f^N} denotes the corresponding self adjoint operator.

Remark 4.2.13. If we replace in (4.2.10) $C_0^\infty(\mathring{B})$ by $C_0^\infty(B)$, we obtain the Dirichlet form with Neumann boundary conditions. When the boundary of B is empty, then the form coincides with the form with Dirichlet boundary conditions. In general, this is not the case. But if the boundary ∂B is a polar set in the sense of Grigor'yan and Masamune (see [41]), it is also true. In the weighted situation that we are considering the boundary is a polar set in this sense. In particular, it follows that $C_0^\infty(B) \subset D(\mathcal{E}^{B,\sin_K^N})$. This is actually equivalent to Markov uniquness of the diffusion operator L^{B,f^N} on $C_0^\infty(B)$). We refer to [33, Chapter 3] for a complete presentation of this terminology.

Proposition 4.2.14. *For $u \in C^\infty(\mathring{B}) \cap D^2(L^{B,f^N})$ there is an explicit formula for the generator of $\mathcal{E}^{B,f^N}$ given by*

$$\big(L^{B,f^N}u\big)(p) = \big(\Delta^B u\big)(p) + \frac{N}{f(p)} \langle \nabla f, \nabla u \rangle |_p \quad \textit{for any } p \in \mathring{B}. \tag{4.2.11}$$

Proposition 4.2.15. *Let $(B, g, f^N \operatorname{vol}_B)$ be as above. Then for any $u \in C^\infty(\mathring{B})$ the following Γ_2-estimate holds pointwise everywhere in $\mathring{B}$:*

$$\Gamma_2^{B,f^N}(u) \geq (d + N - 1)K|\nabla u|_g^2 + \tfrac{1}{d+N}\big(L^{B,f^N}u\big)^2. \tag{4.2.12}$$

Proof. Since there is the classical Bochner-Weitzenböck identity and since f is $\mathcal{F}K$-concave, we get pointwise for any $u \in C^\infty(\mathring{B})$

$$\begin{aligned} \Gamma_2^{B,f^N}(u) &= \operatorname{ric}_B(\nabla u) + \|\nabla^2 u\|_{HS}^2 - \frac{N}{f}\nabla^2 f(\nabla u) + \frac{N}{f^2}\langle \nabla f, \nabla u\rangle\langle \nabla u, \nabla f\rangle \\ &\geq (d-1)K|\nabla u|_g^2 + \frac{1}{d}\left(\Delta^B u\right)^2 + NK|\nabla u|_g^2 + \frac{1}{N}\big(\frac{N}{f}\langle \nabla f, \nabla u\rangle\big)^2 \\ &\geq (d + N - 1)K\Gamma^B(u) + \frac{1}{d+N}\big(L^{B,f^N}u\big)^2. \end{aligned} \tag{4.2.13}$$

Details can be found in chapter 14 of [80]. □

4.3 Riemannian Ricci curvature bounds for metric measure spaces

In this section we give the definition of the Riemannian curvature-dimension condition that was first introduced by Ambrosio, Gigli and Savaré in [6] and generalized to finite dimensions by Erbar, Kuwada and Sturm in [34].

Definition 4.3.1 ([34], Riemannian curvature-dimension condition)**.** A metric measure space $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the (reduced) Riemannian curvature-dimension condition $RCD^*(\varkappa, N)$ if

(i) $(X, \mathrm{d}_X, \mathrm{m}_X)$ is infinitesimal Hilbertian and

(ii) $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies the condition $CD^*(\varkappa, N)$.

Similar, we can define the condition $RCD(\varkappa, N)$ and $RCD(0, N) = RCD^*(0, N)$.

Proposition 4.3.2 ([6])**.** *Assume $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $RCD^*(\varkappa, N)$ for $\varkappa \in \mathbb{R}$ and $N \geq 1$. Then Ch^X is an admissible Dirichlet form in the sense of section 4.2 and the intrinsic distance $\mathrm{d}_{\mathrm{Ch}^X}$ coincides with d_X.*

Remark 4.3.3 ([39]). An important property of metric measure measure spaces that satisfy the condition $RCD^*(\varkappa, N)$ is that for any optimal transport between measures $\mu_0, \mu_1 \in \mathcal{P}^2(X)$ where $\mu_0 << \mathrm{m}_X$ there is a unique optimal coupling that is induced by a map and the corresponding dynamical optimal transference plan is concentrated on a set of non-branching geodesics [39, Theorem 1.1]. Let us mention two corollaries of this property.

Corollary 4.3.4 ([39])**.** *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that satisfies $RCD^*(\varkappa, N)$ for $\varkappa \in \mathbb{R}$ and $N \geq 1$. Then, for every $x \in \mathrm{supp}_X$ and for m_X-a.e. $y \in \mathrm{supp}_X$ there is unique geodesic that connects x and y.*

Theorem 4.3.5. *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfy $RCD^*(\varkappa, N)$ where $\varkappa \in \mathbb{R}$ if $N > 1$ and $\varkappa = 0$ if $N = 1$. Then it satisfies the measure contraction property $MCP(\varkappa, N)$.*

Proof. → The theorem is a corollary of several results by Cavalletti, Gigli, Sturm and Rajala and can be found in this form in [39]. The difference between this theorem and the result of Chapter 2 is that the non-branching assumption is not needed anymore, and it does not matter if we consider the MCP in the sense of Ohta or in the sense of Sturm.

Remark 4.3.6. Under the condition $RCD^*(\varkappa, N)$ the Markov semi-group P_t admits important regularity properties [6]. If $u \in D(\mathcal{E}^X)$ and $\Gamma(u) \in L^\infty$, $P_t u$ has a Lipschitz representative, denoted by $\widetilde{P}_t u$ ([6, Theorem 6.1, Theorem 6.2]). Especially, any $u \in D(\mathcal{E}^X)$ with $\Gamma(u) \in L^\infty$ hat a Lipschitz representative $\widetilde{u}$ such that $|\nabla \widetilde{u}| \leq \| |\nabla u|_w \|_{L^\infty}$.

Under stronger conditions, namely $L^2 \to L^\infty$-ultracontractivity, we even have that $\widetilde{P}_t u$ is Lipschitz for any $f \in L^2$ ([6, Remark 6.4]). Especially, this is the case when the space satisfies $RCD^*(\varkappa, N)$.

Assumption 4.3.7. $(X = \operatorname{supp} \mathrm{m}_X, \mathrm{d}_X, \mathrm{m}_X)$ is a geodesic metric measure space. Every $u \in D(\mathrm{Ch}^X)$ with $|\nabla u|_w \leq 1$ a.e. admits a 1-Lipschitz representative.

The main result of Erbar, Kuwada and Sturm in [34] is

Theorem 4.3.8. *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that satisfies the condition $RCD^*(\varkappa, N)$. Then*

(1) $BE(\varkappa, N)$ holds for $(\mathrm{Ch}^X, D(\mathrm{Ch}^X))$.

Moreover, if $(X, \mathrm{d}_X, \mathrm{m}_X)$ is a metric measure space that is infinitesimal Hilbertian, satisfies the Assumption 4.3.7 and $(\mathrm{Ch}^X, D(\mathrm{Ch}^X))$ satisfies the condition $BE(\varkappa, N)$, then

(2) $(X, \mathrm{d}_X, \mathrm{m}_X)$ satisfies $CD^*(\varkappa, N)$, i.e. the condition $RCD^*(\varkappa, N)$.

Proof. $\to$ [34, Theorem 7, Theorem 4.1, Theorem 4.3, Theorem 4.8, Proposition 4.9].

Corollary 4.3.9. *Let $K > 0$ and $N \geq 1$. $\mathcal{E}^{I_K, \sin_K^N}$ satisfies $BE(NK, N+1)$.*

Proof. $\mathcal{E}^{I_K, \sin_K^N}$ coincides with the Cheeger energy of $(I_K, \sin_K^N r dr)$ and $(I_K, \sin_K^N r dr)$ satisfies the condition $CD(KN, N+1)$ (see [79]). Hence, the result follows. □

Theorem 4.3.10. *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that satisfies $RCD^*(\varkappa, N)$ for $\varkappa > 0$ and $N > 1$. Then the spectrum of the associated Laplace operator L^X is discrete and the first non-zero eigenvalue is $\geq \varkappa \frac{N}{N-1}$.*

Proof. First, we verify the following, slightly more general result.

Proposition 4.3.11. *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that is infinitesimal Hilbertian and $\mathrm{m}_X(X) < \infty$. Assume it satisfies a volume doubling property with doubling constant $C \leq 2^N$ for $2 < N$ and admits a $(1,2)$-Poincaré inequality. Then the spectrum of the corresponding self-adjoint operator L^X is discrete.*

Proof of the Proposition. By Remark 4.1.3 the metric measure space X admits a $(2,2)$-Poincaré inequality that implies a $(2^*, 2)$-Sobolev inequality of the form

$$\left(\int_{B_x(R)} u^{2^*} d\,\mathrm{m}_X \right)^{\frac{2}{2^*}} \leq A \int_{B_x(R)} u^2 d\,\mathrm{m}_X + B \int_{B_x(R)} (\operatorname{Lip} u)^2 d\,\mathrm{m}_X, \tag{4.3.1}$$

for any $u \in D(\mathrm{Ch}^X) \cap C_0(X)$ where $2^* = \frac{2N}{N-2}$. This was proven for example in [76, Theorem 2.6] in the context of strongly local and regular Dirichlet forms. Then, a Rellich-Kondrachov compactness Theorem holds by results of Hailasz and Koskela [45, Theorem 8.1]. Finally, we can procede by induction exactly as for Riemannian manifolds (e.g. [14, Chapter III, Theorem 18]). □

For the proof of the theorem we note that the condition $RCD(\varkappa, N)$ implies that the space admits a $(1,2)$-Poincaré inequality, the right volume doubling property holds and $\mathrm{m}_X(X) < \infty$. Hence, we can apply the previous proposition. The second statement directly follows from the Bakry-Emery characterization of the Riemannian curvature-dimension condition. Choose any eigenfunction u with eigenvalue λ. Since X is compact, an admissible test function is $\varphi = 1$. Then the condition $BE(\varkappa, N)$ implies

$$\begin{aligned} 0 &\geq \int_X \Gamma^X(u, L^X u) d\,\mathrm{m}_X + \varkappa \int_X \Gamma^X(u) d\,\mathrm{m}_X + \frac{1}{N} \int_X (L^X u)^2 \, d\,\mathrm{m}_X \\ &= -\lambda \int_X \Gamma^X(u,u) d\,\mathrm{m}_X + \lambda\varkappa \int_X u d\,\mathrm{m}_X + \frac{\lambda^2}{N} \int_X u^2 d\,\mathrm{m}_X \\ &= \int_X u^2 d\,\mathrm{m}_X \left(-\lambda^2 + \lambda\varkappa + \tfrac{\lambda^2}{N}\right) \end{aligned}$$

from which follows $\lambda \geq \varkappa \frac{N}{N-1}$. □

Remark 4.3.12. The conclusion of the previous theorem is also true if $\varkappa = 0$ and $N = 1$. Then $\lambda_1 \geq 1$. It follows since in this case $F \simeq \lambda\mathbb{S}^1$ or $F \simeq \lambda[0,\pi]$ for some $0 < \lambda \leq 1$. The diameter bound implies that F is compact and there are points $x, y \in F$ such that $\mathrm{diam}_F = \mathrm{d}_F(x,y)$ and there is at least one geodesic between x and y. Hence, the Hausdorff dimension has to be 1 and F consists of finitely many geodesic segments that connect x and y since the measure is assumed to be locally finite. But the curvature-dimension condition implies that there can be at most two geodesics.

Another striking consequence of Riemannian Ricci curvature bounds is the splitting theorem. For Riemannian manifolds with non-negative Ricci curvature bounds, this was proven by Cheeger and Gromoll [27]. N. Gigli proved the result for general $RCD(0,N)$ spaces:

Theorem 4.3.13 (N. Gigli, [37])**.** *Let $(X, \mathrm{d}_X, \mathrm{m}_X)$ be a metric measure space that satisfies $RCD(0, N+1)$ for $N \geq 0$ and contains a geodesic line. Then $(X, \mathrm{d}_X, \mathrm{m}_X)$ is isomorphic to the Cartesian product of $(\mathbb{R}, \mathrm{d}_{Eucl}, \mathcal{L}^1)$ and another metric measure space $(X', \mathrm{d}_{X'}, \mathrm{m}_{X'})$ such that*

(1) $(X', \mathrm{d}_{X'}, \mathrm{m}_{X'})$ is $RCD(0,N)$ if $N \geq 1$,

(2) X' is just a point if $N \in [0,1)$.

Here "isomorphic" means that there is a measure preserving isometry.

4.4 Skew products between Dirichlet forms

In this section we define skew and N-skew products for Dirichlet forms. The notion of skew product is well-known and has been introduced by Fukushima and Oshima

in [35]. A N-skew product is a slight modification of that where we also change the topology of the underlying space.

We briefly describe our framework. Let B and $f \in D_2(L^B) \cap C^\infty(B)$ be as in Assumption 4.2.10 and 4.2.11 of Section 4.2.3. Let $\mathcal{E}^F$ be a regular and strongly local Dirichlet form on some admissible space F. Consider $(B \times F, \mathcal{O}_B \otimes \mathcal{O}_F)$ with $\mathrm{m}_C = f^N d\,\mathrm{vol}_B \otimes d\,\mathrm{m}_F$ and the tensor product $C_0^\infty(B) \otimes D(\mathcal{E}^F)$. $\mathcal{O}_B \otimes \mathcal{O}_F$ is the product topology.

The elements of $C_0^\infty(B) \otimes D(\mathcal{E}^F)$ are functions of the form $\sum_{i=1}^k u_1^i u_2^i$ for some finite $k \in \mathbb{N}$ and $u_1^i \in C_0^\infty(B)$ and $u_2^i \in D(\mathcal{E}^F)$. We will follow this convention in the rest of the thesis. In the literature the tensor product between infinite dimensional Hilbert spaces $\mathcal{H}_i$ for $i = 1, 2$ means that one also takes the closure with respect to the induced inner product. Later, this construction will also appear and we use the notation $\overline{\mathcal{H}_1 \otimes \mathcal{H}_2}$.

Definition 4.4.1 (Skew product)**.** Consider the closure of the following densely defined symmetric form on $L^2(B \times F, f^N d\,\mathrm{vol}_B \otimes d\,\mathrm{m}_F)$:

$$\mathcal{E}^C(u) = \int_F \mathcal{E}^{B,f^N}(u^x) d\,\mathrm{m}_F(x) + \int_B \mathcal{E}^F(u^p) f^{N-2}(p) d\,\mathrm{vol}_B(p) < \infty \tag{4.4.1}$$

for $u \in C_0^\infty(\mathring{B}) \otimes D(\mathcal{E}^F)$ where $u^x = u(\cdot, x)$ and $u^p = u(p, \cdot)$ are the horizontal respectively vertical sections of u. $(B \times F, \mathcal{O}_B \otimes \mathcal{O}_F, \mathcal{E}^C)$ is called skew product between B, f and $\mathcal{E}^F$.

Remark 4.4.2. A general skew product is defined in the following way. Consider two regular Dirichlet forms $\mathcal{E}^1$ and $\mathcal{E}^2$ on $L^2(X_1, \mathrm{m}_1)$ and $L^2(X_2, \mathrm{m}_2)$ respectively and a *smooth* Radon measure μ on X_1 (Here, *smooth* is meant in the sense of [36]). Another Dirichlet form $\mathcal{E}^\mu$ is given by

$$\mathcal{E}^\mu(u) = \mathcal{E}^1(u) + \|u\|_\mu^2 \text{ on } D(\mathcal{E}^\mu) = \left\{ u \in D(\mathcal{E}^1) : \|u\|_\mu < \infty \right\}.$$

If $\mathcal{C}^\mu$ and $\mathcal{C}^2$ are cores for $\mathcal{E}^\mu$ and $\mathcal{E}^2$ respectively then the skew product is the well-defined form closure of

$$\mathcal{E}(u) = \int_{X_2} \mathcal{E}^1(u^x) d\,\mathrm{m}_2(x) + \int_{X_1} \mathcal{E}^2(u^p) d\mu(p)$$

where $u \in \mathcal{C}^\mu \otimes \mathcal{C}^2$. The reader can easily convince himself that this coincides with Definition 4.4.1 if we set $\mathcal{E}^1 = \mathcal{E}^{B,f^N}$, $\mathcal{E}^2 = \mathcal{E}^F$ and $\mu = f^{N-2} d\,\mathrm{vol}_B$. By results of Fukushima, Oshima and Okura (see [35, 63]) the skew product is a well-defined, closed, regular and strongly local Dirichlet form and $\mathcal{C}^\mu \otimes \mathcal{C}^2$ is a core. Therefore, in our situation $\mathcal{E}^C$ is also regular and strongly local, and $C_0^\infty(\mathring{B}) \otimes \mathcal{C}_F$ is a core for $\mathcal{E}^C$ if $\mathcal{C}^F$ is a core for $\mathcal{E}^F$.

The next proposition is a Fubini-type result and was proven by Okura.

Proposition 4.4.3 ([63]). *Let $\mathcal{E}^C$ be a skew product like in Definition 4.4.1. Consider $u \in D(\mathcal{E}^C)$. Then $u^x \in D(\mathcal{E}^{B,f^N})$ for m_F-almost every $x \in F$ and $u^p \in D(\mathcal{E}^F)$ for vol_B-almost every $p \in B$ and we have*

$$\mathcal{E}^C(u) = \int_F \mathcal{E}^{B,f^N}(u^x) d\,\mathrm{m}_F(x) + \int_B \mathcal{E}^F(u^p) f^{N-2}(p) d\,\mathrm{vol}_B(p), \tag{4.4.2}$$

Especially $\mathcal{E}^C$ admits a Γ-operator if and only if $\mathcal{E}^F$ does so, and in this case we have for $u \in D(\mathcal{E}^C)$

$$\Gamma^C(u)(p,x) = \Gamma^B(u^x)(p) + \tfrac{1}{f^2(p)}\Gamma^F(u^p)(x) \quad \mathrm{m}_C\text{-a.e. .}$$

Corollary 4.4.4. *Let $\mathcal{E}^C$ be skew product like in Definition 4.4.5. Then $C_0^\infty(\mathring{B}) \otimes D^2(L^F) \subset D^2(L^C)$ and*

$$\left(L^C u\right)(p,x) = \left(L^{B,f^N} u^x\right)(p) + \tfrac{1}{f^2(p)}\left(L^F u^p\right)(x) \quad \text{for } \mathrm{m}_C\text{-a.e. } (p,x) \in B \times F. \tag{4.4.3}$$

Proof. We consider $u \in C_0^\infty(\mathring{B}) \otimes D^2(L^F)$ and $v \in D(\mathcal{E}^C)$. Then $u^x \in C_0^\infty(\mathring{B})$ for every x and $u^p \in D^2(L^F)$ for every p, and $v^x \in D(\mathcal{E}_{B,f^N})$ for m_F-almost every x and $v^p \in D(\mathcal{E}^F)$ for vol_B-almost every p. Hence

$$\mathcal{E}^{B,f^N}(u^x, v^x) = \left(L^{B,f^N} u^x, v^x\right)_{L^2(f^N d\,\mathrm{vol}_B)} \quad \text{and} \quad \mathcal{E}^F(u^p, v^p) = \left(L^F u^p, v^p\right)_{L^2(\mathrm{m}_F)}$$

for m_F-almost every x and for vol_B-almost every p. This and Proposition 4.4.3 implies

$$\begin{aligned}
\mathcal{E}^C(u,v) &= \int_F \mathcal{E}^{B,f^N}(u^x, v^x) d\,\mathrm{m}_F(x) + \int_B \tfrac{1}{f^2(p)} \mathcal{E}^F(u^p, v^p) f^N(p) d\,\mathrm{vol}_B(p) \\
&= -\int_F \left(L^{B,f^N} u^x, v^x\right)_{L^2(f^N\,\mathrm{vol}_B)} d\,\mathrm{m}_F(x) \\
&\qquad - \int_B \tfrac{1}{f^2(p)} \left(L^F u^p, v^p\right)_{L^2(\mathrm{m}_F)} f^N(p) d\,\mathrm{vol}_B(p) \\
&= -\int_C \left[L^{B,f^N} u^x(p) + \tfrac{1}{f^2(p)} L^F u^p(x)\right] v(p,x) d\,\mathrm{m}_C(p,x).
\end{aligned}$$

Then we also see that $L^{B,f^N} u^x(p) + f^{-2}(p) L^F u^p(x)$ is L^2-integrable with respect to m_C. First, we consider $u = u_1 \otimes u_2 \in C_0^\infty(\mathring{B}) \otimes D^2(L^F)$. Then u is L^2-integrable with respect to m_C since

$$\begin{aligned}
2\left\|L^{B,f^N} u + \tfrac{1}{f^2} L^F u\right\|^2_{L^2(\mathrm{m}_C)} \le{}& \left\|L^{B,f^N} u_1\right\|^2_{L^2(f^N d\,\mathrm{vol}_B)} \left\|u_2\right\|^2_{L^2(\mathrm{m}_F)} \\
&+ \left\|\tfrac{u_1}{f}\right\|^2_{L^2(f^N d\,\mathrm{vol}_B)} \left\|L^F u_2\right\|^2_{L^2(\mathrm{m}_F)} < \infty.
\end{aligned}$$

In particular, we used that u_1 is smooth with compact support in $\mathring{B}$. Hence, $u_1 \otimes u_2 \in D^2(L^C)$ and (4.4.3) holds. In general, any $u \in C_0^\infty(\mathring{B}) \otimes D^2(L^F)$ has the form

$$u = \sum_{i=1}^k u_1^i \otimes u_2^i = \sum_{i=1}^k u^i$$

where $L^C u^i$ is L^2-integrable. Then, by linearity of $L^{B,f^N} + \frac{1}{f^2} L^F$ and by triangle inequality also $L^{B,f^N} u^x + \frac{1}{f^2} L^F u^p$ is L^2-integrable and (4.4.3) holds. □

N-skew products. We will introduce a slight modification of Definition 4.4.1. The underlying space of $\mathcal{E}^C$ is $B \times F$ equipped with the product topology but in general the intrinsic distance $\mathrm{d}_{\mathcal{E}^C}$ induces a different topology that we will describe in more detail. Let us define an equivalence relation on $B \times F$ as follows:

$$(p,x) \sim (q,y) \Longleftrightarrow \Big(p = q \in \partial B\Big) \text{ or } \Big(p = q \in B \text{ and } x = y \in F\Big).$$

Then we can consider the quotient space $B \times F/_\sim = C$ and the corresponding projection map $\pi : B \times F \to C$. Obviously, we have the following decomposition

$$C = \partial B \,\dot\cup\, \mathring{B} \times F.$$

A subset $V \subset C$ is open if and only if $\pi^{-1}(V) \subset B \times F$ is open. We denote the corresponding topology by $\mathcal{O}_C$. This is precisely the topology of the metric warped product as in Definition 2.2.1. If u is continuous with respect to $\mathcal{O}_C$ then $u \circ \pi = \widetilde{u}$ is continuous with respect to $\mathcal{O}_B \otimes \mathcal{O}_F$. By abuse of notation we will also write $u = \widetilde{u}$ when the meaning is clear. If $\mathcal{E}^F$ is strongly regular, one can define a family of "open balls" that generates the quotient topology, i.e. any open set is a union of elements from this family. First, we pick $(p,x) = [p,x] \in \mathring{B} \times F$ and we consider $\bar{\epsilon}_{[p,x]} = \inf_{q \in \partial B} \mathrm{d}_{\mathcal{E}^B}(p,q)$. Then, admissible ϵ-balls around $[p,x]$ are

$$B^C_\epsilon([p,x]) := \{[q,y] \in C : \mathrm{d}_{\mathcal{E}^B}(q,p) + \mathrm{d}_{\mathcal{E}^F}(x,y) < \epsilon\} \subset \mathring{B} \times F \subset C$$

for $0 < \epsilon < \bar{\epsilon}_{[p,x]}$. For $p = [p,x] \in \partial B \subset C$ the corresponding ϵ-balls are

$$B^C_\epsilon([p,x]) = \{[q,y] \in C : \mathrm{d}_{\mathcal{E}^B}(q,p) < \epsilon\} \subset C$$

for $0 < \epsilon < \bar{\epsilon}_{[p,x]} = \infty$. The family of all admissible balls is denoted by

$$\mathcal{B} = \left\{B^C_\epsilon([p,x]) : [p,x] \in C,\ 0 < \epsilon < \bar{\epsilon}_{[p,x]}\right\}.$$

It is not hard to check that elements from $\mathcal{B}$ are open with respect to $\mathcal{O}_C$ and that $\mathcal{B}$ is a generator for $\mathcal{O}_C$. We can pushforward the measure m_C to C and denote it also by m_C. $\partial B \subset C$ is a set of measure zero. Hence, C keeps its product structure m_C-almost everywhere.

Definition 4.4.5 (N-skew product)**.** Assume $\mathcal{E}^F$ is a strongly local, regular and strongly regular Dirichlet form and B and f as in Definition 4.4.1. Consider $(C, \mathcal{O}_C) = C$ and the measure m_C. We can define a Dirichlet form $\mathcal{E}^C$ on $L^2(\mathrm{m}_C)$ as in Definition 4.4.1. We call $(C, \mathrm{m}_C, \mathcal{E}^C)$ the N-skew product between B, f and $\mathcal{E}^F$ and we will write $\mathcal{E}^C = \mathcal{E}^B \times^N_f \mathcal{E}^F = B \times^N_f \mathcal{E}^F$. $B \times^N_f \mathcal{E}^F$ is strongly local and regular.

Consider the intrinsic distance $\mathrm{d}_{\mathcal{E}^C}$ of $\mathcal{E}^C = B \times^N_f \mathcal{E}^F$ on C. The topology that is induced by $\mathrm{d}_{\mathcal{E}^C}$ is denoted by $\mathcal{O}_{\mathrm{d}}$.

Lemma 4.4.6. *If $\mathcal{E}^F$ is strongly regular, then $B \times_f^N \mathcal{E}^F$ is strongly regular, i.e. $\mathcal{O}_{\mathrm{d}} = \mathcal{O}_C$. Closed ϵ-balls with respect to $\mathrm{d}_{\mathcal{E}^C}$ are compact if this holds for $\mathrm{d}_{\mathcal{E}^F}$.*

Proof. Consider $u \in D_{loc}(\mathcal{E}^C) \cap C_0(C)$ with $\Gamma^C(u) \leq 1$ m_C-almost everywhere. Since $\mathcal{E}^C$ is strongly local, we assume that $u \in D(\mathcal{E}^C)$. It follows that $\widetilde{u}^x \in C_0(B)$ and $\widetilde{u}^p \in C_0(F)$ for every $x \in F$ and every $p \in B$. From Proposition 4.4.3 we also have that $\widetilde{u}^x \in D(\mathcal{E}^{B,f^N})$ for m_F-a.e. $x \in F$ and $\widetilde{u}^p \in D(\mathcal{E}^F)$ for dr-a.e. $p \in B$ and $\Gamma^F(u^p) \leq 1$ m_F-a.e. and $\Gamma^B(u^x) \leq 1$ dr-a.e. . Hence, for m_C-a.e. $(p,x),(q,y) \in B \times F$

$$\begin{aligned}\widetilde{u}(p,x) - \widetilde{u}(q,y) &= \widetilde{u}(p,x) - \widetilde{u}(q,x) + \widetilde{u}(q,x) - \widetilde{u}(q,y) \\ &\leq \mathrm{d}_{\mathcal{E}^B}(p,q) + \mathrm{d}_{\mathcal{E}^F}(x,y).\end{aligned}$$

Then by continuity of $\widetilde{u}$ and $\mathrm{d}_{\mathcal{E}^B} + \mathrm{d}_{\mathcal{E}^F}$ with respect to $\mathcal{O}_B \otimes \mathcal{O}_F$ the estimate holds for all $p,q \in B$ and $x,y \in F$. Since u was arbitrary, we obtain

$$\mathrm{d}_{\mathcal{E}_C}([p,x],[q,y]) \leq \mathrm{d}_{\mathcal{E}^B}(p,q) + \mathrm{d}_{\mathcal{E}^F}(x,y)$$

for all $[p,x],[q,y] \in C$. This says that balls in $\mathcal{B}$ of admissible radius ϵ are contained in balls of radius ϵ with respect to $\mathrm{d}_{\mathcal{E}^C}$. Since $\mathcal{E}^F$ and $\mathcal{E}^B$ are strongly regular this implies $\mathrm{d}_{\mathcal{E}^C} < \infty$ and $\mathcal{O}_C \subseteq \mathcal{O}_{\mathrm{d}}$.

On the other hand, $D_{loc}(\mathcal{E}^{B,f^N}) \otimes D_{loc}(\mathcal{E}^F) \subset D_{loc}(\mathcal{E}^C)$ and $C_0(\mathring{B}) \otimes C_0(F) \subset C_0(C/_\sim)$ implies

$$\mathrm{d}_{\mathcal{E}^B}(p,q) + \mathrm{d}_{\mathcal{E}^F}(x,y) \leq M\, \mathrm{d}_{\mathcal{E}^C}((p,x),(q,y))$$

for some constant $M > 0$ and for any $p,q \in \mathring{B}$ and and for any $x,y \in F$. The constant M depends on the minimum of f on some closed neighborhood of p,q in $\mathring{B}$. If we consider $u = u_1 \otimes 1$, we also obtain that

$$\mathrm{d}_{\mathcal{E}^B}(p,q) \leq \mathrm{d}_{\mathcal{E}^C}([p,x],[q,y])$$

for $p \in \partial B$, $q \in B$ and $x,y \in F$. It follows that ϵ-balls with respect to $\mathrm{d}_{\mathcal{E}^C}$ are contained in ϵ-balls from $\mathcal{B}$ for ϵ sufficiently small. Hence, $\mathcal{O}_{\mathrm{d}} \subseteq \mathcal{O}_C$ and $\mathrm{d}_{\mathcal{E}^C}([p,x],[q,y]) > 0$ if $[p,x] \neq [q,y]$.

The second statement follows since $\pi^{-1}\left(\overline{B_\epsilon([p,x])}\right)$ is compact for any $[p,x]$. □

5 Riemannian Ricci curvature bounds for cones

Introduction. In this chapter we will prove the following two theorems on synthetic Ricci curvature bounds for cones over metric measure spaces.

Theorem B. *Let (F, d_F, m_F) be a metric measure space that satisfies $RCD^*(N-1, N)$ for $N \geq 1$ and $\operatorname{diam}_F \leq \pi$. Let $K \geq 0$. Then the (K,N)-cone $\operatorname{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$.*

Theorem C. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space. Suppose the (K,N)-cone $\operatorname{Con}_{N,K}(F)$ over F satisfies $RCD^*(KN, N+1)$ for $K \in \mathbb{R}$ and $N \geq 0$. Then*

(1) if $N \geq 1$, F satisfies $RCD^*(N-1, N)$ and $\operatorname{diam} F \leq \pi$,

(2) if $N \in [0,1)$, F' is a point, or $N = 0$ and F consists of exactly two points with distance π.

By application of Theorem B, Theorem C and the Gigli-Cheeger-Gromoll splitting theorem we prove a maximal diameter theorem for RCD^*-spaces.

Theorem D. *Consider a metric measure space $(F, \mathrm{d}_F, \mathrm{m}_F)$ that satisfies $RCD^*(N, N+1)$ for $N \geq 0$. If $N = 0$, we assume that $\operatorname{diam} F \leq \pi$. Let x, y be points in F such that $\mathrm{d}_F(x,y) = \pi$. Then, there exists a metric measure space $(F', \mathrm{d}_{F'}, \mathrm{m}_{F'})$ such that $(F, \mathrm{d}_F, \mathrm{m}_F)$ is isomorphic to $I_K \times^N_{\sin_K} F'$ and*

(1) $(F', \mathrm{d}_{F'}, \mathrm{m}_{F'})$ satisfies $RCD^*(N-1, N)$ and $\operatorname{diam} F' \leq \pi$ if $N \geq 1$,

(2) if $N \in [0,1)$, F' is a point, or $N = 0$ and F' consists of exactly two points with distance π.

We briefly sketch the main ideas for the proof of Theorem B. One would like to adopt the proof of Theorem A in Chapter 3. It follows the Lagrangian interpretation of curvature-dimension bounds that comes from optimal transport. One deduces the convexity of the entropy functional along Wasserstein geodesics directly from bounds for the Ricci tensor. The main difficulty is to deal with singularity points where the underlying space differs from an Euclidean product. It turns out that the curvature-dimension bound for F guarantees that the optimal transport of absolutely continuous measures does not see these singularities and consequently, singularities do not affect the convexity of the entropy. Now, one is tempted to prove the theorem for general metric measure spaces along the same strategy by deducing the convexity

of the entropy directly from the convexity of the entropy of the underlying space F. The statement that singularities can be neglected also holds in the general framework by Theorem 3.2.1. However, as simple the definition of the cone metric might be, the relation of optimal transport in the cone and optimal transport in the underlying space is rather complicated as can be seen from easy examples. Hence, we need to follow a different strategy.

The Bakry-Emery condition captures the Eulerian picture of curvature-dimension bounds. This viewpoint has already been used by Bakry, Emery and Ledoux to deduce many results from Riemannian geometry in the setting of diffusion semigroups and Dirichlet forms only relying on the existence of a nice algebra of functions. Now, we would like to follow their strategy using the result of Erbar, Kuwada and Sturm on the characterization of Riemannian curvature-dimension bounds to prove our theorem. We remind the reader to the situation of smooth Riemannian manifolds. For some warped product $B \times_f F$ between Riemannian manifolds the Ricci tensor can be calculated explicitly at any point and is given by formula (3.0.1).

Now, we switch to the setting of strongly local and regular Dirichlet forms that satisfy a Bakry-Emery curvature-dimension condition. That is, we replace the metric measure space F by a Dirichlet form $\mathcal{E}^F$ that admits an admissible algebra $\mathcal{A}^F$. From Section 4.4 we know that $B \times_f^N \mathcal{E}^F$ is a Dirichlet form. $C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ is an algebra, though it is not necessarily admissible. But it is enough to compute the corresponding Γ_2-operator. Then, one could hope to establish a similar formula as (3.0.1). And indeed, in Section 5.2, we obtain the following inequality that holds pointwise m_C-almost everywhere

$$\begin{aligned}\Gamma_2^{B\times_f^N F}(u)(p,x) \geq \Gamma_2^B(u^x)(p) + \tfrac{1}{f^2(p)}\Delta^F u^p \tfrac{\langle \nabla f, \nabla u^x\rangle_p}{f(p)} \\ + \tfrac{1}{f^4(p)}\Gamma_2^F(u^p)(x) - \left(\tfrac{\Delta^B f(p)}{f(p)} + (N-1)\tfrac{|\nabla f_p|^2}{f^2(p)}\right)\tfrac{1}{f^2(p)}|\nabla u_x^p|^2 \quad (5.0.1)\end{aligned}$$

for any $u \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$. If we use the same curvature and concavity conditions on B and f as in Theorem A and the Bakry-Emery condition $BE(K_F(N-1), N)$ for $\mathcal{E}^F$, we obtain a sharp Γ_2^C-estimate for $u \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$.

At this point, we have the right Γ_2-estimate for N-skew products. But we do not know yet if it yields the full Bakry-Emery curvature-dimension condition. More precisely, we do not know if $C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ is a dense subset of $D_2(L^C)$ with respect to the graph norm, and indeed, it might be false that $\mathcal{E}^C$ satisfies a curvature-dimension condition even when a Γ_2-estimate holds on a large class of functions. For example, consider the metric N-cone over $F = \mathbb{S}^1_{2\pi}$ that is a 1-dimensional sphere of diameter 2π. In [11] Bacher and Sturm prove that it cannot satisfy a curvature-dimension condition in the sense of Lott-Sturm-Villani. This can be seen from the behavior of optimal transport since the cone in the case of a big circle is a kind of covering and if mass is transported from on sheet to another, the cheapest way to do it is to go through the origin and this destroys any convexity of the entropy. This situation

can be avoided if and only if the diameter of the underlying space is smaller than π. On the other hand, consider the N-skew product $[0,\infty) \times_r^N \mathrm{Ch}^{\mathbb{S}^1_{2\pi}}$ where $\mathrm{Ch}^{\mathbb{S}^1_{2\pi}}$ is the Cheeger energy of $\mathbb{S}^1_{2\pi}$. From our result one sees that the Γ_2 estimate holds even when the diameter is bigger than π. Hence, provided $[0,\infty) \times_r^N \mathrm{Ch}^{\mathbb{S}^1_{2\pi}}$ is the Cheeger energy of the metric cone, $C_0^\infty((0,\infty)) \otimes C_0^\infty(\mathbb{S}^1_{2\pi})$ cannot be dense in the domain of the self-adjoint operator.

Another observation is related to this problem. We remind the reader of the following fact. It is known (see [71, Appendix to Section X.I, Example 4]) that the Laplace operator that acts on smooth functions with compact support in $\mathbb{R}^{N+1}\setminus\{0\}$ is essentially self-adjoint if and only if $N \geq 3$ where $N \in \mathbb{N}$. But this situation exactly corresponds to the case of an Euclidean cone over $\mathbb{S}^N$ with admissible algebra $\mathcal{A}^F = C^\infty(\mathbb{S}^N)$. So in this case in general the operator L^C restricted to $C_0^\infty((0,\infty)) \otimes C^\infty(\mathbb{S}^N)$ will provide more than one self-adjoint extension and the Friedrich's extension does not need to coincide with the closure of $C_0^\infty((0,\infty)) \otimes C^\infty(\mathbb{S}^N)$ with respect to the graph norm. So, we cannot hope that $C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ will be dense in the domain of L^C in general.

But we will see that in the Eulerian picture that is described by the Γ_2-estimate, the crucial quantity is not the diameter but the first positive eigenvalue of L^F. For metric measure spaces that satisfy $RCD^*(N-1,N)$ there is a spectral gap $\lambda_1 \geq N$. It allows to prove the density of an admissible class of function in the domain of L^C in the case of cones. Additionally, we obtain a complete picture about how the spectral gap of L^F enters the proof, and this should be seen in comparison to the Lagrangian viewpoint of Bacher and Sturm. Hence, we can establish a Bakry-Emery condition for cones in the sense of Dirichlet forms, and finally, we can use again the equivalence with the RCD^*-condition to prove Theorem 4.3.8. The technical problem that remains is to prove that the intrinsic distance of cones in the sense of Dirichlet forms is the corresponding cone metric over the metric space.

Outline of the chapter. In section 5.1, we establish a fundamental connection between N-warped products over metric measure spaces and N-skew products over Dirichlet forms. More precisely, we focus on the most simple construction, namely $B = I_K$ and $f = \sin_K$ for $K \geq 0$. In Section 5.2 we prove Γ_2-estimates for general N-skew products with respect to the classical approach of Bakry and Emery. In Section 5.3 we prove that the self-adjoint operator that belongs to the cone over some Dirichlet form is essentially self-adjoint if restricted to a nice subset of its domain provided the spectrum of the underlining Dirichlet form is discrete and satisfies a spectral gap estimate. In Section 5.4 we prove the full Bakry-Emery curvature-dimension condition for spherical cones over Dirichlet forms that satisfy the Bakry-Emery condition itself. Finally, in Section 5.5 we combine all the previous results to prove Theorem B and Theorem C by using the equivalence between the Lagrangian and the Eulerian viewpoint of Ricci curvature. As corollary of these results and the Gigli-Cheeger-Gromoll splitting theorem we obtain the maximal diameter theorem.

Assumptions. Throughout this chapter we assume the following.

(i) In Section 5.2 we consider a Riemannian manifold B and a smooth function f like in Assumption 4.2.10 and 4.2.11 of Section 4.2.3. Let $\mathcal{E}^{B,f^N}$ the corresponding Dirichlet form with drift like in Section 4.2.3. In Section 5.1, 5.3, 5.4 and 5.5 we assume $B = I_K$ and $f = \sin_K$ for $K \geq 0$. In Section 5.3 and 5.4 we assume $K > 0$.

(ii) $(F, \mathrm{d}_F, \mathrm{m}_F)$ is a metric measure space as in Definition 2.1.1. We always assume that F is compact. It follows that $\mathrm{m}_F(F) < \infty$. If F is infinitesimal Hilbertian, the Cheeger energy Ch^F is a regular and strongly local Dirichlet form on $L^2(\mathrm{m}_F)$.

(iii) On the other hand, in Section 5.2, 5.3 and 5.4 we consider an arbitrary Dirichlet form $\mathcal{E}^F$ on $L^2(\mathrm{m}_F)$ that is regular, strongly regular and strongly local. We assume that it admits a volume doubling property and supports a $(2,2)$-Poincaré inequality. Since we assume that the space is compact, closed balls are compact and we can apply the results of Remark 4.2.3. If the metric measure space $(F, \mathrm{d}_F, \mathrm{m}_F)$ satisfies a Riemannian curvature-dimension condition, its Cheeger energy Ch^F clearly fits into this framework by (2.1.4), Remark 4.1.3 and Proposition 4.3.2, and Ch^F satisfies the corresponding Bakry-Emery condition by Theorem 4.3.8.

5.1 Warped product versus skew product

The case when F satisfies $RCD^*(N-1,N)$. We want to analyze the intrinsic distance of the N-skew product $\mathcal{E}^C = I_K \times^N_{\sin_K} \mathrm{Ch}^F$ in more detail where F is a metric measure space that is infinitesimal Hilbertian. The main result will be that the intrinsic distance of $I_K \times^N_{\sin_K} \mathrm{Ch}^F$ coincides with $\mathrm{d}_{\mathrm{Con}_K}$ if F satisfies the Riemannian curvature-dimension condition $RCD^*(N-1,N)$. The key is the following proposition. Γ^C denotes the Γ-operator of $\mathcal{E}^C$.

Proposition 5.1.1. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a length metric measure space that satisfies a volume doubling property, supports a local Poincaré inequality and is infinitesimal Hilbertian. Assume $\operatorname{diam} F \leq \pi$ and let $K \geq 0$. Then $D(I_K \times^N_{\sin_K} \mathrm{Ch}^F) \subset D(\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)})$ and for any $u \in D(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$ we have*

$$|\nabla u|^2_w(r,x) \leq \Gamma^{I_K,\sin^N_K}(u^x)(r) + \tfrac{1}{\sin^2_K}|\nabla u^r|_w(p) \quad \mathrm{m}_C\text{-a.e.} \tag{5.1.1}$$

where $u^x(r) = u(r,x)$ and $u^r(x) = u(r,x)$ and $\Gamma^{I_K,\sin^N_K}(u) = \Gamma^{I_K}(u) = u'$.
Especially, the result holds if $(F, \mathrm{d}_F, \mathrm{m}_F)$ satisfies the condition $RCD^*(N-1,N)$.

Proof. We follow the proof of Lemma 6.12 in [6] and use the following elementary lemma from [7]:

Lemma 5.1.2. *Let $d(s,t) : (0,1)^2 \to \mathbb{R}$ be a map that satisfies*

$$|d(s,t) - d(s',t)| \le |v(s) - v(s')|, \;\; |d(s,t) - d(s,t')| \le |v(t) - v(t')|$$

for any $s,t,s',t' \in (0,1)$, for some locally absolutely continuous map $v : (0,1) \to \mathbb{R}$ and let $\delta(t) := d(t,t)$. Then δ is locally absolutely continuous in $(0,1)$ and

$$\frac{d}{dt}\delta|_t \le \limsup_{h\to 0} \frac{d(t,t) - d(t-h,t)}{h} + \limsup_{h\to 0} \frac{d(t,t+h) - d(t,t)}{h} \;\; dt\text{-a.e. in } (0,1).$$

Proof. →[7, Lemma 4.3.4]

1. We recall that $\operatorname{diam}_F \le \pi$ implies $\operatorname{Con}_{N,K}(F) = I_K \times^N_{\sin_K} F$. Consider $u \in C_0^\infty(\mathring{I}_K) \otimes \operatorname{Lip}(F)$. u is Lipschitz with respect to $\mathrm{d}_{\operatorname{Con}_K}$. Let $\gamma = (\alpha,\beta) : [0,1] \to (\partial I_K)^c_\varepsilon \times F$ be a curve in $AC^2(\operatorname{Con}_{K,N}(F))$ where $(\partial I_K)^c_\varepsilon = I_K \backslash \overline{(\partial I_K)}_\varepsilon$. Then, one can check that $\alpha \in AC^2(\mathring{I}_K)$ and $\beta \in AC^2(F,\mathrm{d}_F)$ and there is $g \in L^2((0,1),dt)$ such that

$$\mathrm{d}_F(\beta_s,\beta_t) \le \int_s^t g(\tau)d\tau \quad \text{and} \quad |\alpha_t - \alpha_s| \le \int_s^t g(\tau)d\tau \quad \text{for } s<t \in [0,1].$$

For $K > 0$ we have the following estimates (and similar for $K = 0$).

$$\begin{aligned}\mathrm{d}_{\operatorname{Con}_K}((r,y),(r,x)) &= \cos_K^{-1}(\cos^2 r + \sin^2 r \cos \mathrm{d}_F(x,y)) \\ &= \cos_K^{-1}(1 - \sin^2 r(1 - \cos \mathrm{d}_F(x,y))) \le \cos_K^{-1}(1 - \tfrac{1}{2}\sin^2 r\, \mathrm{d}_F^2(x,y))\end{aligned} \tag{5.1.2}$$

$$\cos_K^{-1}(1 - \tfrac{1}{2}x^2) = x + o(x^2) \;\text{ for }\; x \to 0 \tag{5.1.3}$$

Then we can see that for $s < s'$ and $t < t'$

$$\begin{aligned}|u(\alpha_s,\beta_t) - u(\alpha_s,\beta_{t'})| &\le L\, \mathrm{d}_{\operatorname{Con}_K}((\alpha_s,\beta_t),(\alpha_s,\beta_{t'})) \\ &\le L \cos_K^{-1}(1 - \tfrac{1}{2}\sin^2\alpha_s\, \mathrm{d}_F^2(\beta_t,\beta_{t'})) \\ &\le \widetilde{M}\, \mathrm{d}_F(\beta_t,\beta_{t'}) \le M \int_t^{t'} g(\tau)d\tau\end{aligned}$$

$$|u(\alpha_s,\beta_t) - u(\alpha_{s'},\beta_t)| \le L\, \mathrm{d}_{\operatorname{Con}_K}((\alpha_s,\beta_t),(\alpha_{s'},\beta_t)) \le L|\alpha_s - \alpha_{s'}| \le M \int_s^{s'} g(\tau)d\tau$$

where L is a Lipschitz constant of u and $M, \widetilde{M} > 0$ are constants. Hence, we can apply Lemma 5.1.2 and we obtain

$$\left|\frac{d}{dt}(u\circ\gamma)(t)\right| \le \limsup_{h\to 0} \frac{|u(\alpha_{t-h},\beta_t) - u(\alpha_t,\beta_t)|}{h} + \limsup_{h\to 0} \frac{|u(\alpha_t,\beta_{t+h}) - u(\alpha_t,\beta_t)|}{h}$$

for a.e. $t \in [0,1]$. By definition of the local Lipschitz constant Lip and by the elementary estimate $2ab \le a^2 + b^2$ for any $a, b \in \mathbb{R}$, it follows that

$$\begin{aligned}\left|\frac{d}{dt}(u\circ\gamma)(t)\right| &\le \operatorname{Lip} u^{\beta_t}(\alpha_t)|\dot\alpha(t)| + \operatorname{Lip} u^{\alpha_t}(\beta_t)|\dot\beta(t)| \\ &\le \sqrt{(\operatorname{Lip} u^{\beta_t})^2(\alpha_t) + \tfrac{1}{\sin_K^2(\alpha_t)}(\operatorname{Lip} u^{\alpha_t})^2(\beta_t)}\sqrt{|\dot\alpha(t)|^2 + \sin_K^2(\alpha_t)|\dot\beta_t|^2} \\ &=: G(\gamma(t))|\dot\gamma(t)|\end{aligned}$$

for a.e. $t \in [0,1]$. If we want to check that G is a weak upper gradient of u, we only need consider curves in $(\partial I_K)^c_\epsilon \times F$ since u has compact support in $\mathring{I}_K \times^N_{\sin_K} F$. Hence, integration with respect to t on both sides shows that G is a weak upper gradient of u. It follows

$$|\nabla u|_w(r,x) \le G(r,x) \quad \mathrm{m}_C\text{-a.e. .} \tag{5.1.4}$$

Since $(F, \mathrm{d}_F, \mathrm{m}_F)$ satisfies a volume doubling property and supports a local Poincaré inequality, Cheeger's theorem (Theorem 4.1.9) states that $\operatorname{Lip} u^r = |\nabla u^r|_w$ m_F-a.e. . Then the square of the right hand side of (5.1.4) equals

$$G(r,x)^2 = ((u^x)'(r))^2 + \tfrac{1}{\sin_K^2 r}|\nabla u^r|_w^2(x) \quad \mathrm{m}_C\text{-a.e. .}$$

2. By the definition of skew products $C_0^\infty(\mathring{I}_K)\otimes D(\mathrm{Ch}^F)$ is dense in $D(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$. Hence, for any $u \in D(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$ there is a sequence $u_n \in C_0^\infty(\mathring{I}_K)\otimes D(\mathrm{Ch}^F)$ that converges to u with respect to the energy norm of $I_K \times^N_{\sin_K} \mathrm{Ch}^F$, and we will find a subsequence such that

$$\Gamma^{I_K,\sin_K^N}(u^x_{n_i}) + \tfrac{1}{\sin_K^2}|\nabla u^r_{n_i}|_w^2 \longrightarrow \Gamma^C(u) = \Gamma^{I_K,\sin_K^N}(u^x) + \tfrac{1}{\sin_K^2}|\nabla u^r|_w^2 \quad \mathrm{m}_C\text{-a.e. .}$$

The left hand side of (5.1.4) converges weakly in $L^2(\mathrm{m}_C)$ (after taking another subsequence) and the limit is the minimal weak upper gradient of u. This follows from the stability theorem for minimal weak upper gradients in [5] (see Theorem 4.1.10). More precisely, we can argue as follows. Since $|\nabla u_n|_w \in L^2(\mathrm{m}_C)$ is a bounded sequence, we find a subsequence u_{n_i} such that $|\nabla u_{n_i}|_w$ converges weakly to $g = |\nabla u|_w \in L^2(\mathrm{m}_C)$ by the stability theorem. Especially, we have

$$\int_C |\nabla u_{n_i}|_w \varphi d\,\mathrm{m}_C \to \int_C |\nabla u|_w \varphi d\,\mathrm{m}_C$$

for any non-negative test function $\varphi \in L^2(\mathrm{m}_C)$. Hence, inequality (5.1.4) is preserved in the limit m_C-a.e. and we have

$$|\nabla u|_w^2(r,x) \le \Gamma^{I_K,\sin_K^N}(u^x)(r) + \tfrac{1}{\sin_K^2}|\nabla u^r|_w^2(x) \quad \mathrm{m}_C\text{-a.e.} \tag{5.1.5}$$

and in particular, $u \in D(I_K \times^N_{\sin_K} \mathrm{Ch}_F)$ implies $u \in D(\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)})$. □

Lemma 5.1.3. *Let* $(F, \mathrm{d}_F, \mathrm{m}_F)$ *satisfy* $RCD^*(N-1, N)$ *for* $N \geq 1$ *and* $\operatorname{diam} F \leq \pi$. *Let* $\mathrm{Con}_{N,K}(F)$ *be the corresponding* (K, N)*-cone for* $K \geq 0$. *Then* $\mathrm{Con}_{N,K}(F)$ *satisfies a volume doubling property and supports a local Poincaré inequality.*

Proof. We assume $N > 1$ since the case $N = 1$ is already clear by Remark 4.3.12 and Theorem A. We will use the following theorem of Ohta from [61].

Theorem 5.1.4. *If the metric measure space* $(F, \mathrm{d}_F, \mathrm{m}_F)$ *satisfies* $MCP(N-1, N)$ *in the sense of Ohta and if* $\operatorname{diam}_F \leq \pi$ *then the associated* $(0, N)$*-cone satisfies* $MCP(0, N+1)$ *in the sense of Ohta.*

Hence, in the case $K = 0$ we proceed as follows. When $(F, \mathrm{d}_F, \mathrm{m}_F)$ satisfies $RCD^*(N-1, N)$, Theorem 4.3.5 implies $MCP(N-1, N)$ that implies a measure contraction property $MCP(0, N+1)$ for $\mathrm{Con}_{N,0}(F)$ in the sense of Ohta. In particular, $\mathrm{Con}_{N,K}(F)$ satisfies a volume doubling property by results of Ohta in [60] and supports a local Poincaré inequality by Theorem 4.1.4. The latter follows since the condition $RCD^*(N-1, N)$ implies that for every $x \in F$ and m_X-a.e. $y \in F$ there is a unique geodesic by Corollary 4.3.4. This property is inherited by the cone because of Theorem 2.2.3 and since $\operatorname{diam}_F \leq \pi$. Hence, MCP à la Ohta is the same as MCP à la Sturm and we can apply Theorem 4.1.4 by von Renesse.

The case $K > 0$ can be covered in the same way. Assume without loss of generality that $K = 1$. By following straightforwardly Ohta's proof of Theorem 5.1.4 in [61] we can prove the analogous result for $(1, N)$-cones where one should use the following formula for the projection of a geodesic $\gamma = (\alpha, \beta) : [0, 1] \to \mathrm{Con}_{N,1}(F) \setminus \{\text{singularities}\}$ to $[0, \pi]$.

$$\cos \alpha(t) = \sigma_{1,1}^{(1-t)}(\mathrm{L}(\gamma))\alpha(0) + \sigma_{1,1}^{(t)}(\mathrm{L}(\gamma))\alpha(1).$$

Alternatively, one can use Theorem 5.1.4 directly and compare the metric and the measure of the spherical cone around the origin with the metric of the Euclidean cone around the origin. More precisely, one can find constants $m, M > 0$ such that

$$\frac{1}{M}\,\mathrm{d}_{\mathrm{Con}_K} \leq \mathrm{d}_{\mathrm{Con}_0} \leq \frac{1}{m}\,\mathrm{d}_{\mathrm{Con}_K} \quad \text{and} \quad \frac{1}{M}\sin_K^N r \leq r^N \leq \frac{1}{m}\sin_K^N r.$$

From this estimates one can easily deduce the doubling property and the Poincaré inequality in a neighborhood of the origin from the corresponding results for the 0-cone. Away from the singularities the same argument works by comparison with the direct product $(I_K \times F, \mathrm{d}_{Eukl} \times \mathrm{d}_F, \mathcal{L}^1 \otimes \mathrm{m}_F)$. □

Theorem 5.1.5. *Let* $(F, \mathrm{d}_F, \mathrm{m}_F)$ *be a metric measure space satisfying* $RCD^*(N-1, N)$ *for* $N \geq 1$ *and* $\operatorname{diam} F \leq \pi$. *Then the intrinsic distance* $\mathrm{d}_{\mathcal{E}^C}$ *of* $\mathcal{E}^C = I_K \times_{\sin_K}^N \mathrm{Ch}^F$ *coincides with* $\mathrm{d}_{\mathrm{Con}_K}$.

Proof. By remark 5.1.8 we know that in any case $\operatorname{diam} F \leq \pi$, and $I_K \times_{\sin_K}^N F = \mathrm{Con}_{N,K}(F)$ by Remark 2.2.7. We only check the case $K > 0$.

1. We know from Proposition 5.1.1 that $D(I_K \times^N_{\sin_K} \mathrm{Ch}^F) \subset D(\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)})$ and for any $u \in D(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$

$$|\nabla u|^2_w \leq \Gamma^{I_K,\sin^N_K}(u^x) + \tfrac{1}{\sin^2_K}|\nabla u^r|_w \quad \mathrm{m}_C\text{-a.e.} \tag{5.1.6}$$

where $u^x(r) = u(r,x)$ and $u^r(x) = u(r,x)$. For the intrinsic distance of $\mathcal{E}^C$ we need to consider $u \in \mathcal{L}_{C,loc} = C(I_K \times F/\!\sim, \mathcal{O}_C) \cap \mathcal{L}_{loc}$ where

$$\mathcal{L}_{loc} := \left\{\psi \in D_{loc}(I_K \times^N_{\sin_K} \mathrm{Ch}^F) : \sqrt{\Gamma^C(\psi)} \leq 1\, \mathrm{m}_C\text{-a.e. in } I_K \times F/\!\sim\right\}.$$

One has to prove that u is 1-Lipschitz with respect to $\mathrm{d}_{\mathrm{Con}_K}$. We will follow an argument that was suggested to the author by Tapio Rajala.

First, $\Gamma^C(u) \leq 1$ m_C-a.e. implies $|\nabla u|_w \leq 1$ m_C-a.e. by (5.1.6). $|\nabla u|_w$ is a weak upper gradient and $\mathrm{Con}_{N,K}(F)$ satisfies the measure contraction property $MCP(N, N+1)$ by the proof of the previous lemma. Consider two points $p, q \in \mathrm{Con}_{N,K}(F)$, $B_\epsilon(q) \subset \mathrm{Con}_{N,K}(F)$, $\mu_0 = \mathrm{m}_C(B_\epsilon(q))^{-1}\, \mathrm{m}_C\,|_{B_\epsilon(q)}$ and the unique optimal displacement interpolation μ_t between $\mu_0 = \mu$ and $\mu_1 = \delta_p$. Let Π be the corresponding dynamical transference plan. Because of the measure contraction property $(\mu_t)_{t\in[0,t_0]}$ is a 2-test plan for any $t_0 < 1$. Hence

$$\int |u(\gamma_1) - u(\gamma_0)| d\Pi(\gamma) \leq \int \int_0^1 |\nabla u|_w(\gamma(t))\mathrm{L}(\gamma) dt d\Pi(\gamma) \leq \mathrm{d}_W(\delta_p, \mu_1).$$

where d_W is the L^2-Wasserstein metric of $\mathrm{Con}_{N,K}(F)$. In the last inequality we use that $\mu_t \leq C(t)\, \mathrm{m}_C$ for some $C(t) > 0$ and any $t < 1$ and $|\nabla u|_w \leq 1$ m_C-a.e. . If $\epsilon \to 0$, we obtain

$$|u(p) - u(q)| \leq \mathrm{d}_W(\delta_p, \delta_q) = \mathrm{d}_{\mathrm{Con}_K}(p, q).$$

This yields

$$\mathrm{d}_{\mathcal{E}^C}((s,y),(r,x)) = \sup\{u(s,y) - u(r,x) : u \in \mathcal{L}_{C,loc}\} \leq \mathrm{d}_{\mathrm{Con}_K}((r,x),(s,y)) \tag{5.1.7}$$

for all $(r,x),(s,y) \in \mathrm{Con}_{N,K}(F)$.

2. On the other hand, we define $g((p,x)) = \mathrm{d}_{\mathrm{Con}_K}((p,x),(q,y))$ for some $(q,y) \in I_K \times_{\sin_K} F$ where

$$\mathrm{d}_{\mathrm{Con}_K}((p,x),(q,y)) = \cos^{-1}_K \underbrace{\left(\cos_K(p)\cos_K(q) + K \sin_K(p)\sin_K(q)\cos \mathrm{d}_F(x,y)\right)}_{=:h(p,x)}.$$

$h \in D_{loc}(\mathcal{E}^{I_K,\sin^N_K}) \otimes D(\mathrm{Ch}^F)$ since $\cos_K, \sin_K \in D_{loc}(\mathcal{E}^{I_K,\sin^N_K})$ and $\cos \mathrm{d}_F(\cdot, q), 1 \in D(\mathrm{Ch}^F)$. We can calculate $\Gamma^C(g)$ explicitly. We get

$$\Gamma^C(g) = \left(\left(\cos^{-1}_K\right)'(h(p,x))\right)^2 \Gamma^C(h)(p,x) = \frac{1}{1-h^2(p,x)}\Gamma^C(h)(p,x)$$

Then, a straightforward calculation using the chain rule and $\Gamma^F(\mathrm{d}_F(\cdot,y)) \leq 1$ yields

$$\begin{aligned}\Gamma^C(h)(p,x) =& \Gamma^{I_K}(\cos_K p \cos_K q) + 2\Gamma^{I_K}(\cos_K p \cos_K q, \sin_K p \sin_K q)\cos \mathrm{d}_F(x,y) \\ &+ \Gamma^{I_K}(\sin_K p \sin_K q)\cos^2 \mathrm{d}_F(x,y) + \frac{\sin_K^2 q \sin_K^2 p}{\sin_K^2 p}\Gamma^F(\cos \mathrm{d}_F(x,y)) \\ \leq& 1 - h^2(p,x)\end{aligned}$$

Hence $\Gamma^C(g) \leq 1$, $g \in \mathcal{L}_{C,loc}$ and

$$g((p,x)) - g((q,y)) = g((p,x)) = \mathrm{d}_{\mathrm{Con}_K}((p,x),(q,y)) \leq \mathrm{d}_{\mathcal{E}^C}((p,x),(q,y))$$

by definition of $\mathrm{d}_{\mathcal{E}^C}$. Hence, we obtain that $\mathrm{d}_{\mathcal{E}^C} = \mathrm{d}_{\mathrm{Con}_K}$. □

Corollary 5.1.6. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space satisfying $RCD^*(N-1,N)$ for $N \geq 1$ and $\operatorname{diam} F \leq \pi$. Then $I_K \times^N_{\sin_K} \mathrm{Ch}^F = \mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$.*

Proof. We will use the following theorem of Koskela and Zhou in [51].

Theorem 5.1.7. *Let $\mathcal{E}^X$ be a regular, strongly local and strongly regular Dirichlet form on $L^2(X,\mathrm{m}_X)$. Suppose $(X, \mathrm{d}_{\mathcal{E}^X}, \mathrm{m}_X)$ satisfies a doubling property. Then $\mathrm{Lip}(X) \subset D_{loc}(\mathcal{E})$, $\Gamma^X(u)$ exists for any $u \in \mathrm{Lip}(X)$ and $\Gamma^X(u) \leq \mathrm{Lip}(u)^2$ m_X-a.e. .*

$\mathrm{d}_{\mathrm{Con}_K} = \mathrm{d}_{\mathcal{E}^C}$ by Theorem 5.1.5 and $\mathrm{d}_{\mathrm{Con}_K}$ induces the topology of the underlying space $I_K \times F/\sim$. Theorem 5.1.3 implies the doubling property for $\mathrm{Con}_{N,K}(F)$. Then, by Theorem 5.1.7 and Proposition 5.1.1 we get that any Lipschitz function u with respect to $\mathrm{d}_{\mathrm{Con}_K}$ is in $D_{loc}(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$ and

$$\Gamma^C(u) = \mathrm{Lip}(u) = |\nabla u|^2_w \quad \mathrm{m}_C\text{-a.e. .} \tag{5.1.8}$$

By the definition of the Cheeger energy this implies the result. □

The case when $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$. The main results of this paragraph will be that the metric measure space $(F, \mathrm{d}_F, \mathrm{m}_F)$ is infinitesimal Hilbertian, $\mathrm{d}_{\mathrm{Ch}^F} = \mathrm{d}_F$, $\mathrm{d}_{\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}} = \mathrm{d}_{\mathrm{Con}_K}$ and $\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)} = I_K \times^N_{\sin_K} \mathrm{Ch}^F$ provided that $\mathrm{Con}_{N,K}(F)$ satisfies $RCD(KN, N+1)$. In particular, Ch^F is strongly regular.

Remark 5.1.8. We remind the reader on a result by Bacher and Sturm from [11]. They show the following. If the $(K,1)$-cone over some 1-dimensional space satisfies $CD(0,2)$ then the diameter of the underlying space is bounded by π. It is easy to see that their proof can be extended to any (K,N)-cone of any dimension bound N and any parameter K. Thus, if $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$, then $\operatorname{diam} F \leq \pi$, and consequently, $\mathrm{Con}_{N,K}(F) = I_K \times^N_{\sin_K} F$ by Remark 2.2.7.

Lemma 5.1.9. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space. Assume the (K,N)-cone $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$ for $N \geq 1$ and $K \geq 0$. Then $(F, \mathrm{d}_F, \mathrm{m}_F)$ satisfies a volume doubling property, supports a local Poincaré inequality and $(F, \mathrm{d}_F, \mathrm{m}_F)$ is infinitesimal Hilbertian.*

Proof. We prove the result for $K > 0$. The general case follows in the same way. Consider

$$x \in F \mapsto (1, x) \in \{1\} \times F \subset \mathrm{Con}_{N,K}(F).$$

We can find constants $M > m > 0$ such that

$$\frac{1}{M}\,\mathrm{d}_{\mathrm{Con}_K} \le \max\{|\cdot - \cdot|, \mathrm{d}_F\} \le \frac{1}{m}\,\mathrm{d}_{\mathrm{Con}_K} \;\;\&\;\; \frac{1}{M}\sin_K^N \le 1 \le \frac{1}{m}\sin_K^N$$

in an ϵ-neighborhood of $\{1\} \times F \subset \mathrm{Con}_{N,K}(F)$. On the one hand, from this we can easily deduce the volume doubling property for F. Pick a point $x \in F$ and let $r > 0$. Then

$$\begin{aligned}
4r\,\mathrm{m}_F(B_{2r}(x)) &\le \frac{1}{m}\sin_K^N r dr \otimes \mathrm{m}_F([-2r+1, 2r+1] \times B_{2r}(x)) \\
&\le \frac{1}{m}\,\mathrm{m}_C(B_{2Mr}(1, x)) \\
&\le \frac{1}{m} C \left(\frac{2M}{m}\right)^N \mathrm{m}_C(B_{mr}(1, x)) \\
&\le \frac{1}{m} C \left(\frac{2M}{m}\right)^N \mathrm{m}_C([-r+1, r+1] \times B_r(x)) \\
&\le 2^N C \left(\frac{M}{m}\right)^{N+1} 2r\,\mathrm{m}_F(B_{2r}(x)).
\end{aligned}$$

We used the volume doubling property of $\mathrm{Con}_{N,K}(F)$ in the third inequality. On the other hand, we also obtain that the space F supports a weak local Poincaré inequality because of the bi-Lipschitz invariance of this property. For example, we can follow the method that is provided in Section 4.3 of [15].

Now, we will check that F is infinitesimal Hilbertian. For any Lipschitz function u on $\mathrm{Con}_{N,K}(F)$ we see

$$\begin{aligned}
(\mathrm{Lip}\,u)(r, x) &= \limsup_{(s,y)\to(r,x)} \frac{|u(s, y) - u(r, x)|}{\mathrm{d}_{\mathrm{Con}_K}((s, y), (r, x))} \\
&\ge \limsup_{(s,y)\to(r,x), r=s} \frac{|u(r, y) - u(r, x)|}{\mathrm{d}_{\mathrm{Con}_K}((r, y), (r, x))} \\
&\ge \limsup_{y\to x} \frac{|u^r(y) - u^r(x)|}{\sin_K(r)|x, y|} = \frac{1}{\sin_K(r)}\,\mathrm{Lip}\,u^r(x). \qquad (5.1.9)
\end{aligned}$$

The second last inequality comes from (5.1.2) and (5.1.3). Following the steps in paragraph 1 of the proof of Proposition 5.1.1 we can see that (5.1.4) holds for $C_K^\infty(\mathring{I}_K) \otimes u$ where $u \in \mathrm{Lip}(F)$. There, we did not use that F is infinitesimal Hilbertian. By locality of the minimal weak upper gradient (5.1.4) also holds for $1 \otimes u$. Then (5.1.4) and (5.1.9) imply

$$\mathrm{Lip}(1 \otimes u)(r, x) = \tfrac{1}{\sin_K r}\,\mathrm{Lip}\,u(x) \text{ for a.e. } r \in [0, \pi/\sqrt{K}] \text{ and } \mathrm{m}_F\text{-a.e. } x \in F.$$

Then, $\int_F \operatorname{Lip} u d\,\mathrm{m}_F$ has to be a quadratic form on $\operatorname{Lip}(F)$ and its form closure is by definition the Cheeger energy Ch^F. □

Lemma 5.1.10. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space and assume $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$ for $K \geq 0$ and $N \geq 1$. Then $\mathrm{d}_{\mathrm{Ch}^F} = \mathrm{d}_F$. In particular, Ch^F is strongly regular in the sense of Dirichlet forms.*

Proof. We assume $K > 0$. The case $K = 0$ follows in the same way. Consider $u(\cdot) = \mathrm{d}_F(x, \cdot) \in D(\mathrm{Ch}^F)$. u satisfies $|\nabla u|_w \leq 1$. Hence, $\mathrm{d}_{\mathrm{Ch}^F} \geq \mathrm{d}_F$. The converse inequality is obtained as follows. Consider $u \in D_{loc}(\mathrm{Ch}^F) \cap C(F)$ with $|\nabla u|_w \leq 1$. Let $\epsilon > 0$. We choose $\delta > 0$ such that $\frac{1}{\sin_K r} \leq 1+\epsilon$ if $r \in B_{2\delta}(\pi/2)$. Let $u_1 \in C_0^\infty(\mathring{I}_K)$ such that $u_1 \leq 1$ and $u_1|_{B_\delta(\pi/2)} = 1$. $u_1 \otimes u \in D_{loc}(\mathcal{E}^C)$ and

$$|\nabla(u_1 \otimes u)|^2 = (u_1')^2 u^2 + \tfrac{u_1^2}{\sin_K^2} |\nabla u|_w^2 \in L^\infty(\mathrm{m}_C)$$

In particular, it follows that $|\nabla(u_1 \otimes u)| = \frac{1}{\sin_K} |\nabla u|_w \leq 1 + \epsilon$ on $B_\delta(\pi/2) \times F$. Since $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$, this implies that $u_1 \otimes u$ admits a Lipschitz representative and the Lipschitz constant is locally less than $1 + \epsilon$ on some neighborhood of $\pi/2 \times F$. This can be seen from standard arguments like in paragraph 1 of the proof of Proposition 5.1.5. Hence, for any $x, y \in F$ such that $\mathrm{d}_F(x, y)$ is small, we have

$$|u(x) - u(y)| \leq (1+\epsilon)\, \mathrm{d}_{\mathrm{Con}_K}((\pi/2, x), (\pi/2, y)) \leq (1+\epsilon)\, \mathrm{d}_F(x, y).$$

It follows that $\mathrm{d}_{\mathrm{Ch}^F} \leq (1+\epsilon)\, \mathrm{d}_F$ locally. Now, d_F is geodesic by the remark directly after Definition 2.2.6. We can conclude that $\mathrm{d}_{\mathrm{Ch}^F} \leq (1+\epsilon)\, \mathrm{d}_F$ globally, and since $\epsilon > 0$ was arbitrary, we have $\mathrm{d}_{\mathrm{Ch}^F} \leq \mathrm{d}_F$. □

Lemma 5.1.11. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space that satisfies a volume doubling property and supports a Poincaré inequality. Assume $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$ for $K \geq 0$ and $N \geq 1$. Then the intrinsic distance $\mathrm{d}_{\mathcal{E}^C}$ of $\mathcal{E}^C = I_K \times^N_{\sin_K} \mathrm{Ch}^F$ coincides with $\mathrm{d}_{\mathrm{Con}_K}$.*

Proof. Since F satisfies a volume doubling property, supports a local Poincaré inequality and is infinitesimal Hilbertian, we can apply Proposition 5.1.1. Then, we have for any $u \in D(I_K \times^N_{\sin_K} \mathrm{Ch}^F)$

$$|\nabla u|_w^2 \leq \Gamma^C(u) \;\; \mathrm{m}_C\text{-a.e. .} \tag{5.1.10}$$

$\mathrm{Con}_{N,K}(F)$ satisfies a Riemannian curvature-dimension condition. Hence, $|\nabla u|_w \leq \Gamma^C(u) \leq 1$ implies u is 1-Lipschitz and (5.1.7) holds. On the other hand, we can proceed as in the proof of Theorem 5.1.5 and obtain that $\mathrm{d}_{\mathcal{E}^C} = \mathrm{d}_{\mathrm{Con}_K}$. □

Corollary 5.1.12. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space that satisfies a volume doubling property and supports a Poincaré inequality. Assume $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$ for $K \geq 0$ and $N \geq 1$. Then $I_K \times^N_{\sin_K} \mathrm{Ch}^F = \mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$.*

Proof. We can follow the proof of Corollary 5.1.6. □

We summarize the results of this section in the following corollary.

Corollary 5.1.13. *Let* $(F, \mathrm{d}_F, \mathrm{m}_F)$ *be a metric measure space and* $K \geq 0$. *Assume*

1. $(F, \mathrm{d}_F, \mathrm{m}_F)$ *satisfies* $RCD^*(N-1, N)$ *for* $N \geq 1$ *and* $\mathrm{diam}_F \leq \pi$, *or*
2. $\mathrm{Con}_{N,K}(F)$ *satisfies* $RCD^*(KN, N+1)$ *for* $N \geq 1$.

Then $\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)} = I_K \times^N_{\sin_K} \mathrm{Ch}^F$, $\mathrm{d}_{\mathrm{Ch}^F} = \mathrm{d}_F$ *and* $\mathrm{d}_{\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}} = \mathrm{d}_{\mathrm{Con}_{N,K}(F)}$.

Remark 5.1.14. In particular, F is infinitesimal Hilbertian if $\mathrm{Con}_{N,K}(F)$ satisfies $RCD^*(KN, N+1)$, and $\mathrm{Con}_{N,K}(F)$ is infinitesimal Hilbertian if F satisfies $RCD^*(N-1, N)$.

5.2 Proof of classical estimates for skew products

In this section and in the next two sections we consider a Dirichlet form $\mathcal{E}^F$ on $L^2(\mathrm{m}_F)$ like in the general assumption in the beginning of this chapter. We assume there is an admissible algebra $\mathcal{A}^F$ for $\mathcal{E}^F$. This enables us to do calculations classically. Let B be a Riemannian manifold like in Assumpition 4.2.10 and let $f \in D_2(L^B)$ be smooth and $\mathcal{F}K$-concave like in Assumpition 4.2.11.

Theorem 5.2.1. $\mathcal{E}^F$ *satisfies* $BE((N-1)K_F, N)$ *for* $N \geq 1$ *and* $K_F \in \mathbb{R}$ *such that*

$$\Gamma^B(f) + Kf^2 \leq K_F \text{ on } B.$$

Set $\mathcal{A}^C = C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$. *Then the* N*-skew product* $\mathcal{E}^C = B \times^N_f \mathcal{E}^F$ *satisfies*

$$\Gamma_2^C(u) \geq (N+d-1)K\Gamma^C(u) + \frac{1}{N+d}\left(L^C u\right)^2 \tag{5.2.1}$$

pointwise m_C*-almost everywhere and for any* $u \in \mathcal{A}^C$.

Remark 5.2.2. In particular, we can apply the theorem if $B = I_K$, $f = \sin_K$ and $K_F = 1$. In section 5.4, we will prove analogous results for Dirichlet forms that satisfy the intrinsic Bakry-Emery curvature-dimension condition. The advantages of the classical approach are that calculations really can be done pointwise and that the structure of formulas and inequalities becomes more clear.

Proof. Every element of $\mathcal{A}^C$ is of the form $\sum_{i=1}^k u_1^i u_2^i$ for $k \in \mathbb{N}$, but we will check (5.2.1) only for elements of the form $u + v = u_1 \otimes u_2 + v_1 \otimes v_2$ where $u_1, v_1 \in C_0^\infty(\mathring{B})$ and $u_2, v_2 \in \mathcal{A}^F$. The case of arbitrary finite sum follows in the same way. We compute $\Gamma_2^C(u)$, $\Gamma_2^C(v)$ and $\Gamma_2^C(u, v)$ explicitly m_C-a.e. . A straightforward calculation

yields (see also the first paragraph of the proof of Theorem 5.4.4 where the calculation is peformed in more detail):

$$\begin{aligned}2\Gamma_2^C(u,v) &= 2\Gamma_2^B(u_1,v_1)v_2u_2 + \tfrac{u_1v_1}{f^4}2\Gamma_2^F(u_2,v_2)\\ &\quad + \Gamma^B(u_1,v_2)\tfrac{1}{f^2}L^F(u_2v_2) - \Gamma^B\big(\tfrac{u_1}{f^2},v_1\big)L^F(u_2)v_2 - \Gamma^B\big(u_1,\tfrac{v_1}{f^2}\big)u_2L^F(v_2)\\ &\quad + \big(L^{B,f}(\tfrac{u_1v_1}{f^2}) - L^{B,f}(u_1)v_1\tfrac{1}{f^2} - u_1L^{B,f}(v_1)\tfrac{1}{f^2}\big)\Gamma^F(u_2,v_2) \qquad (5.2.2)\end{aligned}$$

We set

$$2\Gamma_2^C(u,v) - \big(2\Gamma_2^B(u_1,v_1)v_2u_2 + \tfrac{u_1v_1}{f^4}2\Gamma_2^F(u_2,v_2)\big) =: (\mathcal{J})$$

The chain rule for Γ^B and L^B yields:

$$\begin{aligned}\Gamma^B\big(\tfrac{u_1}{f^2},v_1\big) &= \tfrac{1}{f^2}\Gamma^B(u_1,v_1) - \tfrac{2u_1}{f^3}\Gamma^B(f,v_1)\\ L^{B,f}(\tfrac{u_1v_1}{f^2}) &= \tfrac{v_1}{f^2}L^{B,f}(u_1) + \tfrac{u_1}{f^2}L^{B,f}(v_1) - \tfrac{2u_1v_1}{f^3}L^{B,f}(f)\\ &\quad - \tfrac{4v_1}{f^3}\Gamma^B(u_1,f) - \tfrac{4u_1}{f^3}\Gamma^B(v_1,f) + \tfrac{6u_1v_1}{f^4}\Gamma^B(f) + \tfrac{2}{f^2}\Gamma^B(u_1,v_1)\\ \tfrac{N}{f}\Gamma^B(f,\tfrac{u_1v_1}{f^2}) &= \tfrac{Nv_1}{f^3}\Gamma^B(f,u_1) + \tfrac{Nu_1}{f^3}\Gamma^B(f,v_1) - \tfrac{2Nu_1v_1}{f^4}\Gamma^B(f).\end{aligned}$$

We glue this back into $(\mathcal{J})$ and another straightforward calculation yields.

$$\begin{aligned}(\mathcal{J}) &= \tfrac{2u_1}{f^3}\Gamma^B(f,v_1)L^F(u_2)v_2 + \tfrac{2v_1}{f^3}\Gamma^B(f,u_1)L^F(v_2)u_2\\ &\quad - \tfrac{2u_1v_1}{f^3}\big(L^B(f) + \tfrac{N-1}{f}\Gamma^B(f)\big)\Gamma^F(u_2,v_2)\\ &\quad + \underbrace{\big(-\tfrac{4v_1}{f^3}\Gamma^B(u_1,f) - \tfrac{4u_1}{f^3}\Gamma^B(v_1,f) + \tfrac{4u_1v_1}{f^4}\Gamma^B(f) + \tfrac{4}{f^2}\Gamma^B(u_1,v_1)\big)}_{=:(I)(u_1,v_1)}\Gamma^F(u_2,v_2)\end{aligned}$$

In the case $u = v$ we obtain the following simplification

$$(\mathcal{J}) = \tfrac{4u_1}{f^3}\Gamma^B(f,u_1)L^F(u_2)u_2 - \tfrac{2u_1^2}{f^3}\big(L^B(f) + \tfrac{N-1}{f}\Gamma^B(f)\big)\Gamma^F(u_2) + (I)(u_1)\Gamma^F(u_2). \qquad (5.2.3)$$

Now, we can compute the Γ_2-operator for elements of the form $u_1 \otimes u_2 + v_1 \otimes v_2 = u + v \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ for m_C-a.e. point $(p,x) \in \mathring{C}$:

$$\begin{aligned}2\Gamma_2^C(u+v)(p,x) &= 2\Gamma_2^B(u^x)(p) + \tfrac{1}{f^4(p)}2\Gamma_2^F(u^p)(x)\\ &\quad + \tfrac{4}{f^3(p)}\Gamma^B(f,u^x)(p)L^F(u^p)(x) - \tfrac{2}{f^3(p)}\big(L^B(f)(p) + \tfrac{N-1}{f(p)}\Gamma^B(f)(p)\big)\Gamma^F(u^p)(x)\\ &\quad + 2(I)(u_1,v_1)(p)\Gamma^F(u_2,v_2)(x) + (I)(u_1)(p)\Gamma^F(u_2)(x) + (I)(v_1)(p)\Gamma^F(v_2)(x)\end{aligned}$$

We denote the last line in the previous equation with $(II)(p,x)$. First, assume

$u_1(p)v_1(p) \neq 0$. Then, it can be rewritten in the following form

$$\begin{aligned}(II) &= \tfrac{2u_1v_1}{f^2}\big(-4\Gamma^B(\ln|u_1|,\ln f) - 4\Gamma^B(\ln|v_1|,\ln f)\\ &\qquad + 4\Gamma^B(\ln f) + 4\Gamma^B(\ln|u_1|,\ln|v_1|)\big)\Gamma^F(u_2,v_2)\\ &\quad + \tfrac{u_1^2}{f^2}\big(-8\Gamma^B(\ln|u_1|,\ln f) + 4\Gamma^B(\ln f)\\ &\qquad + 4\Gamma^B(\ln|u_1|)\big)\Gamma^F(u_2) + \tfrac{v_1^2}{f^2}\big(-8\Gamma^B(\ln|v_1|,\ln f) + ...\big)\Gamma^F(v_2)\\ &= \tfrac{8}{f^2}\left\langle \nabla\ln(\tfrac{f}{|u_1|}), \nabla\ln(\tfrac{f}{|v_1|})\right\rangle|_p\Gamma^F(u^p,v^p)(x)\\ &\quad + \tfrac{4}{f^2}\left|\nabla\ln(\tfrac{f}{|u_1|})\right|_p^2\Gamma^F(u^p)(x) + \tfrac{4}{f^2}\left|\nabla\ln(\tfrac{f}{|v_1|})\right|_p^2\Gamma^F(v^p)(x)\end{aligned}$$

We choose an orthonormal basis $(e_i)_{1,\dots,d}$ with respect to the Riemannian metric at TB_p and write

$$\nabla\ln(\tfrac{f}{|u_1|})|_p = \textstyle\sum_{i=1}^d a^i e_i \quad\text{and}\quad \nabla\ln(\tfrac{f}{|v_1|})|_p = \sum_{i=1}^d c^i e_i.$$

Then we obtain

$$\begin{aligned}\tfrac{f^2(p)}{4}(II) &= \sum_i^d 2a^ic^i\Gamma^F(u^p,v^p)(x) + \sum_i^d (a^i)^2\Gamma^F(u^p)(x) + \sum_i^d (c^i)^2\Gamma^F(v^p)(x)\\ &= \sum_i^d \Gamma^F(a^iu^p + c^iv^p)(x) \geq 0 \quad \text{for } \mathrm{m}_C\text{-almost every } (p,x) \qquad (5.2.4)\end{aligned}$$

since $\Gamma^F \geq 0$ m_F-a.e. In the case where $v_1(p) = 0$ and $u_1(p) \neq 0$, we get

$$\begin{aligned}(II) &= \tfrac{4}{f^2}\left\langle\nabla\ln(f/|u_1|), \nabla v_1\right\rangle|_p\Gamma^F(u^p,v_2)(x)\\ &\quad + \tfrac{4}{f^2}\left|\nabla\ln(f/|u_1|)\right|_p^2\Gamma^F(u^p)(x) + \tfrac{4}{f^2}\left|\nabla v_1\right|_p^2\Gamma^F(v_2)(x)\end{aligned}$$

and when we set $\nabla v_1|_p = \sum_i^d \alpha^i e_i$, similar as before we obtain that

$$\tfrac{f^2(p)}{4}(II) = \sum_i^d \Gamma^F(a^iu^p + \alpha^iv_2)(x) \geq 0. \qquad (5.2.5)$$

In the same way we can deal with the other cases. If we would consider an arbitrary $u \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ of the form $\sum_j^k u_{1,j} \otimes u_{2,j} = \sum_j^k u_j$, $\frac{f^2}{4}(II)$ would take the form $\sum_i^d \Gamma^F(\sum_j^k a_j^i u_j^p) \geq 0$ and all the other calculations are the same.
It follows in any case that

$$\begin{aligned}2\Gamma_2(u)(p,x) &\geq 2\Gamma_2^B(u^x)(p) + \tfrac{1}{f^4(p)}2\Gamma_2^F(u^p)(x)\\ &\quad + \tfrac{4}{f^3(p)}\Gamma^B(f,u^x)L^F(u^p)\\ &\quad - \tfrac{2}{f^3(p)}\left(L^Bf(p) + \tfrac{N-1}{f(p)}\Gamma^B(f)(p)\right)\Gamma^F(u^p)(x) \quad \mathrm{m}_C\text{-a.e.}\end{aligned}$$

for any $u \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$. This estimate becomes an equality if $u = f \otimes u_2$. From (4.2.13) we have

$$\Gamma_2^{B,f^N}(u) \geq (d+N-1)K\Gamma^B(u) + \frac{1}{d}\left(L^B u\right)^2 + \frac{1}{N}\big(\tfrac{N}{f}\Gamma^B(f,u)\big)^2 \tag{5.2.6}$$

for any function $u \in C_0^\infty(\mathring{B})$. Now we apply the curvature-dimension conditions for $\mathcal{E}^F$ and $\mathcal{E}^{B,f^N}$, inequality (5.2.6), and the assumptions on f. First, we see that

$$L^B f + \frac{N-1}{f}\Gamma^B(f) \leq -dKf + \frac{N-1}{f}\left(K_F - Kf^2\right) \quad \text{everywhere in } B.$$

Then it follows that

$$\begin{aligned}
2\Gamma_2^C(u) &\geq 2(d+N-1)K\Gamma^B(u^x) + 2\frac{1}{d}\left(L^B u^x\right)^2 + 2\frac{1}{N}\big(\tfrac{N}{f}\Gamma^B(f,u^x)\big)^2 \\
&\quad + \tfrac{2}{f^4}\Big((N-1)K_F\Gamma^F(u^p) + \tfrac{1}{N}\big(L^F u^p\big)^2\Big) \\
&\quad + \tfrac{2}{f^3}\Gamma^B(f,u^x)2L^F(u^p) + \tfrac{2}{f^3}\Big(dKf - \tfrac{N-1}{f}\left(K_F - Kf^2\right)\Big)\Gamma^F(u^p) \\
&= 2\Big((d+N-1)K\Gamma^B(u_1) + \frac{1}{d}\left(L^B u^x\right)^2 + \frac{1}{N}\big(\tfrac{N}{f}\Gamma^B(f,u^x)\big)^2\Big) \\
&\quad + \tfrac{2}{f^2}(N+d-1)K\Gamma^F(u^p) \\
&\quad + \tfrac{1}{N}\tfrac{2}{f^4}\big(L^F u^p\big)^2 + \tfrac{1}{N}\tfrac{2N}{f^3}\Gamma^B(f,u^x)2L^F(u^p) \\
&= 2\big((d+N-1)K\Gamma^B(u^x) + \frac{1}{d}\left(L^B u^x\right)^2\big) + \tfrac{2}{f^2}(N+d-1)K\Gamma^F(u^p) \\
&\quad + \frac{1}{N}\Big(2\big(\tfrac{N}{f}\Gamma^B(f,u^x)\big)^2 + \tfrac{2}{f^4}\left(L^F u^p\right)^2 + \tfrac{2N}{f^3}\Gamma^B(f,u^x)2L^F(u^p)\Big) \\
&= 2\Big((N+d-1)K\Gamma^B(u^x) + \frac{1}{d}(L^B u^x)^2\Big) \\
&\quad + \tfrac{2}{f^2}(N+d-1)K\Gamma^F(u^p) + \tfrac{2}{N}\Big(\tfrac{N}{f}\Gamma^B(f,u^x) + \tfrac{1}{f^2}L^F u^p\Big)^2 \ \ \mathrm{m}_C\text{-a.e. .}
\end{aligned}$$

We apply the following elementary equality

$$\tfrac{1}{d}a^2 + \tfrac{1}{N}b^2 = \tfrac{1}{N+d}\left(a+b\right)^2 + \tfrac{d}{(N+d)N}(b - \tfrac{N}{d}a)^2 \tag{5.2.7}$$

for all $d, N \geq 1$ and for all $a, b \in \mathbb{R}$. Hence

$$\begin{aligned}
\Gamma_2^C(u) &\geq (N+d-1)K\Gamma^B(u^x) + \tfrac{1}{f^2}(N+d-1)K\Gamma^F(u^p) \\
&\quad + \frac{1}{d}\left(L^B u^x\right)^2 + \frac{1}{N}\big(\tfrac{N}{f}\Gamma^B(f,u^x) + \tfrac{1}{f^2}L^F u^p\big)^2 \\
&\geq (N+d-1)K\Gamma^C(u) + \tfrac{1}{N+d}\big(L^B u^x + \tfrac{N}{f}\Gamma^B(f,u^x) + \tfrac{1}{f^2}L^F u^p\big)^2 \ \ \mathrm{m}_C\text{-a.e. .}
\end{aligned}$$

This is the right Γ_2-estimate since $L^C u = L^B u^x + \frac{N}{f}\Gamma^B(f,u^x) + \frac{1}{f^2}L^F u^p$ for any $u \in C_0^\infty(\mathring{B}) \otimes \mathcal{A}^F$ because of Corollary 4.4.4. □

Theorem 5.2.3. *Let $\mathcal{A}^F$ be an admissible algebra for $\mathcal{E}^F$. $\mathcal{E}^C = I_K \times^N_{\sin_K} \mathcal{E}^F$ satisfies the Γ_2-estimate (5.2.1) of curvature NK and dimension $N+1$ for $K \geq 0$ and $N \geq 1$ on $C_0^\infty(\mathring{I}_K) \otimes \mathcal{A}^F$ if and only if $\mathcal{E}^F$ satisfies $BE(N-1, N)$.*

Proof. We just need to prove the "only if" direction. We assume a Γ_2-estimate for $(L^C, \mathcal{A}^C)$ and deduce the curvature-dimension condition for $(L^F, \mathcal{A}^F)$. We have to show that the Γ_2-estimate holds pointwise a.e. in F for any $u_2 \in \mathcal{A}^F$. From calculations in the previous proof we have the identity (5.2.3) for Γ_2^C in the following form

$$\begin{aligned}\Gamma_2^C(u_1 \otimes u_2) = \Gamma_2^{I_K, \sin_K^N}(u_1)u_2^2 + \frac{u_1^2}{\sin_K^4}\Gamma_2^F(u_2) \\ + \frac{u_1}{\sin_K^3}\cos_K \ u_1' 2L^F(u_2)u_2 - \frac{u_1^2}{\sin_K^3}\left(-K\sin_K + \tfrac{N-1}{\sin_K}\cos_K^2\right)\Gamma^F(u_2) \\ + \Big(\frac{2}{\sin_K^2}(u_1')^2 - \frac{4u_1}{\sin_K^3}\cos_K \ u_1' + \frac{2u_1^2}{\sin_K^4}\cos_K^2\Big)\Gamma^F(u_2). \qquad (5.2.8)\end{aligned}$$

for any $u_1 \in C_0^\infty(\mathring{B})$ and any $u_2 \in \mathcal{A}^F$ m_C-a.e. in $I_K \times F$. We consider some open set $U \subset \mathring{I}_K$ and we choose $u_1 \in C_0^\infty(\mathring{I}_K)$ such that $u_1 = \sin_K$ on U. By the special choice of u_1 (5.2.8) reduces to

$$\begin{aligned}\Gamma_2^C(u_1 \otimes u_2) = \Gamma_2^{I_K, \sin_K^N}(u_1)u_2^2 + \frac{u_1^2}{\sin_K^4 r}\Gamma_2^F(u_2) \\ + \frac{u_1}{\sin_K^3}\cos_K \cdot u_1' \cdot 2L^F(u_2)u_2 - \frac{u_1^2}{\sin_K^3}\left(-K\sin_K + \tfrac{N-1}{\sin_K}\cos_K^2\right)\Gamma^F(u_2)\end{aligned}$$

for m_C-almost every (r, x) in $U \times F$. In the case of $(I_K, \sin_K^N)$ the identity (4.2.13) in the proof of Proposition 4.2.15 implies

$$\Gamma_2^{I_K, \sin_K^N}(u_1) = (u_1'')^2 + \tfrac{N}{\sin_K}K\sin_K \cdot (u_1')^2 + \tfrac{N}{\sin_K^2}(\cos_K \cdot u_1')^2$$

everywhere in $\mathring{I}_K$ for $u_1 \in C_0^\infty(\mathring{I}_K)$. Hence, we obtain

$$\begin{aligned}\frac{u_1^2}{\sin_K^4}\Gamma_2^F(u_2) &= \Gamma_2^C(u_1 \otimes u_2) - \Gamma_2^{I_K, \sin_K^N}(u_1)u_2^2 \\ &\quad - \frac{u_1}{\sin_K^3}\cos_K \cdot u_1' \cdot 2L^F(u_2)u_2 + \frac{u_1^2}{\sin_K^3}\left(-K\sin_K + \tfrac{N-1}{\sin_K}\cos_K^2\right)\Gamma^F(u_2) \\ &= \Gamma_2^C(u_1 \otimes u_2) - (u_1'')^2u_2^2 - \tfrac{N}{\sin_K}K\sin_K \cdot (u_1')^2u_2^2 - \tfrac{N'}{\sin_K^2}(\cos_K \cdot u_1')^2 u_2^2 \\ &\quad - \frac{u_1}{\sin_K^3}\cos_K \cdot u_1' \cdot 2L^F(u_2)u_2 + \frac{u_1^2}{\sin_K^3}\left(-K\sin_K + \tfrac{N-1}{\sin_K}\cos_K^2\right)\Gamma^F(u_2).\end{aligned}$$

m_C-a.e. in $U \times F$. On the other side the Γ_2-estimate for $\mathcal{E}^C$ gives for any $u_1 \in C_0^\infty(\mathring{B})$ and any $u_2 \in \mathcal{A}^F$

$$\begin{aligned}\Gamma_2^C(u_1u_2) \geq NK\big((u_1')^2u_2^2 + \tfrac{u_1^2}{\sin_K^2}\Gamma^F(u_2)\big)\\ + \frac{1}{N+1}\left(L^{I_K}u_1u_2 + \tfrac{N}{\sin_K}\Gamma^{I_K}(\sin_K, u_1)u_2 + \tfrac{u_1}{\sin_K^2}L^Fu_2\right)^2 \quad \mathrm{m}_C\text{-a.e. .}\end{aligned}$$

Since $u_1(r) = \sin_K r$ and $u_1'(r) = \cos_K r$, we get after some cancelations at (r, x)

$$\begin{aligned}\frac{1}{\sin_K^2}\Gamma_2^F(u_2) \geq (N-1)(\cos_K^2 + K\sin_K^2)\frac{1}{\sin_K^2}\Gamma^F(u_2) - \frac{\cos_K^2}{\sin_K^2}2L^F(u_2)u_2\\ - \left(-K\sin_K\right)^2 u_2^2 - \frac{N}{\sin_K^2}\left(\cos_K\cdot\ \cos_K\right)^2 u_2^2\\ + \frac{1}{N+1}\left(-K\sin_K u_2 + \frac{N}{\sin_K}\cos_K^2 u_2 + \frac{1}{\sin_K}L^Fu_2\right)^2 \qquad (5.2.9)\end{aligned}$$

m_C-a.e. in $U \times F$. We consider the last term on right side in (5.2.9) in more detail. From the identity (5.2.7) we deduce

$$\begin{aligned}&\frac{1}{N+1}\left(-K\sin_K u_2 + \frac{N}{\sin_K}\cos_K^2 u_2 + \frac{1}{\sin_K}L^Fu_2\right)^2\\ &= \left(-K\sin_K u_2\right)^2 + \frac{1}{N}\left(\frac{N}{\sin_K}\cos_K^2 u_2\right)^2 + \frac{1}{N}\left(\frac{1}{\sin_K}L^Fu_2\right)^2 + \frac{2}{\sin_K^2}\cos_K^2 u_2L^Fu_2\\ &\quad - \frac{1}{(N+1)N}\underbrace{\left(\frac{N}{\sin_K}\cos_K^2 u_2 + \frac{1}{\sin_K}L^Fu_2 + NK\sin_K u_2\right)^2}_{=\frac{1}{\sin_K^2}\left(L^Fu_2+Nu_2\right)^2}\end{aligned}$$

It follows

$$\Gamma_2^F(u_2) \geq (N-1)\Gamma^F(u_2) + \frac{1}{N}\left(L^Fu_2\right)^2 - \frac{1}{(N+1)N}\left(L^Fu_2 + Nu_2\right)^2 \qquad (5.2.10)$$

at x for m_C-almost every tuple $(r, x) \in U \times F$. But since (5.2.10) does not depend on $r \in U$ anymore, we can conclude (5.2.10) holds for m_F-a.e. $x \in F$.

We fix x where (5.2.10) holds. $\mathcal{E}^F$ is strongly local. So we can add functions that are locally constant in a neighborhood of x without affecting the value of $L^Fu_2(x)$, $\Gamma^F(u_2)(x)$ and $\Gamma_2^F(u_2)(x)$. Thus we replace u_2 by $\widetilde{u}_2$ such that $\widetilde{u}_2 = u_2 + C$ locally at x where $C = -u_2(x) - \frac{1}{N}L^Fu_2(x)$. Then $(L^Fu_2)(x) = (L^F\widetilde{u}_2)(x)$, $\Gamma(u_2)(x) = \Gamma(\widetilde{u}_2)(x)$ and $\Gamma_2^F(u_2)(x) = \Gamma(\widetilde{u}_2)(x)$ and $\left(L^F\widetilde{u}_2 + N\widetilde{u}_2\right)^2$ vanishes at x. Hence, we obtain the desired estimate for u_2 at m_F-almost every x and we obtain the condition $BE(N\text{–}1, N)$ for F. □

5.3 Essentially selfadjoint operators

We briefly recall some basic notations. Suppose $(\mathcal{H}_i, \|\cdot\|_{\mathcal{H}_i})_{i\in\mathbb{N}}$ is a sequence of Hilbert spaces. Its direct sum is a Hilbert space that is given by

$$\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i = \left\{ v := (v_i)_{i\in\mathbb{N}} : v_i \in \mathcal{H}_i \text{ such that } \|v\|_{\mathcal{H}}^2 := \sum_{i=1}^{\infty} \|v_i\|_{\mathcal{H}_i}^2 < \infty \right\}$$

where the inner product is $\sum_{i=1}^{\infty}(v_i, u_i)_{\mathcal{H}_i} = (v, u)_{\mathcal{H}}$. Additionally, we introduce

$$\sum_{i=1}^{\infty} \mathcal{H}_i = \Big\{ v := (v_i)_{i\in\mathbb{N}} : v_i \in \mathcal{H}_i \text{ and } v_i = 0 \text{ except for finitely many } i \in \mathbb{N} \Big\}.$$

We still consider a Dirichlet form $\mathcal{E}^F$ like in the previous section. We assume the spectrum of $-L^F$ is discrete and there is a spectral gap for the first positive eigenvalue $\lambda_1 \geq N$. Then, there is a spectral decomposition of $L^2(\mathrm{m}_F)$ with respect to the eigenvalues of $-L^F$.

$$L^2(\mathrm{m}_F) = \bigoplus_{i=0}^{\infty} E_i = E_0 \oplus \underbrace{\left(L^2(\mathrm{m}_F)/E_0\right)}_{=:E^{\perp}}$$

where $E_i \subset D^2(L^F)$ is the eigenspace to the ith eigenvalue λ_i. In particular, E_0 is the space of harmonic functions. We recall the properties of Remark 4.2.3.

(1) P_t^F admits an α-Hölder-continuous kernel and is a Feller semi-group.

(2) P_t^F is $L^2 \to L^\infty$-ultracontractiv: $\|P_t^F\|_{L^2\to L^\infty} \leq 1$.

(3) If $\mathrm{m}_F(F) < \infty$, harmonic functions are constant, and F is connected.

Theorem 5.3.1. *Assume $\lambda_1 \geq N \geq 1$. Let L^C be the self-adjoint operator of the N-skew product $\mathcal{E}^C = I_K \times_{\sin_K}^N \mathcal{E}^F$ for $K \geq 0$ and $N \geq 1$. Let $\mathcal{A}$ be dense in the domain of $L^{I_K,\sin_K^N}$. Then*

$$\Xi = [\mathcal{A} \otimes E_0] \oplus \sum_{i=1}^{\infty} C_0^{\infty}(\mathring{I}_K) \otimes E_i$$

is dense in the domain of L^C with respect to the graph norm.

Proof. **1.** We denote the restriction of $\mathcal{E}^F$ and $(\cdot,\cdot)_{L^2(\mathrm{m}_F)}$ to $E_\perp$ by $\mathcal{E}_\perp^F = \mathcal{E}^F|_{E_\perp\times E_\perp}$ and $(\cdot,\cdot)_{E_\perp}$ respectively. There is a self-adjoint, densely defined operator $L_\perp^F$ on $E_\perp$ that corresponds to $\mathcal{E}_\perp^F$. It is easy to see that $D^2(L_\perp^F) = D^2(L^F) \cap E_\perp$, $L_\perp^F u_\perp = L^F u_\perp$ and $\operatorname{spec} L_\perp^F = \operatorname{spec} L^F \setminus \{\lambda_0\}$.

Consequently, also $L^2(\mathrm{m}_C) = \overline{L^2(I_K, \sin_K^N dr) \otimes L^2(\mathrm{m}_F)}$ can be decomposed orthogonally into

$$L^2(\mathrm{m}_C) = U_0 \oplus U_\perp$$

where $U_\perp = \bigoplus_{i=1}^\infty U_i$ and $U_i = L^2(I_K, \sin_K^N dr) \otimes E_i$. For $u = u_1 \otimes u_2 \in U_0$ and $v = v_1 \otimes v_2 \in U_\perp$ with $u_1, v_1 \in L^2(I_K, \sin_K^N dr)$, $u_2 \in E_0$ and $v_2 \in E_\perp$ we have

$$\mathcal{E}^C(u,v) = \mathcal{E}^{I_K, f^N}(u_1, u_2) \underbrace{(u_2, v_2)_{L^2(m_F)}}_{=0} + (u_1, u_2)_{L^2(f^{N-2} d\,\mathrm{m}_F)} \underbrace{\mathcal{E}^F(u_2, v_2)}_{=0} = 0$$

since $\mathcal{E}^F(u_2, v_2) = -(L^F u_2, v_2)_{L^2(\mathrm{m}_F)} = 0$ and $E_0 \perp E_\perp$ in $L^2(\mathrm{m}_F)$. Thus we can decompose $\mathcal{E}^C$ orthogonally as follows

$$\mathcal{E}^C = \mathcal{E}^C|_{U_0 \times U_0} \oplus \mathcal{E}^C|_{U^\perp \times U^\perp} =: \mathcal{E}_0^C \oplus \mathcal{E}_\perp^C.$$

One can easily check that $u = u_0 \oplus u_\perp \in D(\mathcal{E}^C)$ if and only if $u_0 \in D(\mathcal{E}_0^C)$ and $u_\perp \in D(\mathcal{E}_\perp^C)$ and that $u = u_0 \oplus u_\perp \in D^2(L^C)$ if and only if $u_0 \in D^2(L_0^C)$ and $u_\perp \in D^2(L_\perp^C)$ and we have $L^C = L_0^C + L_\perp^C$. In the following we will consider L_0^C and $L_\perp^C$ separately.

2. $L_\perp^C$ is a densely defined operator on $U_\perp$ with

$$D^2(L_\perp^C) = D^2(L^C) \cap U_\perp = D^2(L^C) \cap \overline{L^2(I_K, \sin_K^N dr) \otimes E_\perp}.$$

$C_0^\infty(\mathring{I}_K) \otimes D^2(L^F)$ is a subset of $D^2(L^C)$, hence, $C_0^\infty(\mathring{I}_K) \otimes D^2(L_\perp^F)$ is a subset of $D^2(L_\perp^C)$. For $u_1 \in C_0^\infty(\mathring{I}_K)$ and $u_2 \in D^2(L_\perp^F)$ we have

$$L_\perp^C(u_1 \otimes u_2) = L^{I_K, \sin_K^N} u_1 \otimes u_2 + \frac{u_1}{\sin_K^2} \otimes L_\perp^F u_2.$$

For all $i \in \mathbb{N} \setminus \{0\}$ we set $\widetilde{U}_i = U_i \cap \left[C_0^\infty(\mathring{I}_K) \otimes D^2(L_\perp^F)\right] = C_0^\infty(\mathring{I}_K) \otimes E_i$ and consider the restriction of $L_\perp^C$ to $\widetilde{U}_i$

$$\begin{aligned} L_\perp^C|_{\widetilde{U}_i} = L^C|_{C_0^\infty(\mathring{I}_K) \otimes E_i} &= \left(L^{I_K, \sin_K^N} + \tfrac{-\lambda_i}{\sin_K^2}\right)|_{C_0^\infty(\mathring{I}_K)} \otimes \mathrm{id}_{E_i} \\ &= \left(L^{I_K} + \tfrac{N}{\sin_K} \Gamma^{I_K}(\sin_K, \cdot) - \tfrac{\lambda_i}{\sin_K^2}\right)|_{C_0^\infty(\mathring{I}_K)} \otimes \mathrm{id}_{E_i} =: L_{\perp,i}^C \end{aligned}$$

We define

$$\sum_{i=1}^\infty L_{\perp,i}^C = L_\perp^C|_{\sum_{i=1}^\infty C_0^\infty(\mathring{I}_K) \otimes E_i} =: \widetilde{L}_\perp^C.$$

We will show that $\widetilde{L}_\perp^C$ is essentially self-adjoint. Then, the unique self adjoint extension of $\widetilde{L}_\perp^C$ has to coincide with $L_\perp^C$. In particular, $\sum_{i=1}^\infty C_0^\infty(\mathring{I}_K) \otimes E_i$ is dense in $D^2(L_\perp^C)$ with respect to the graph norm. It is sufficient to show that the operator $(L^{I_K} + \frac{N}{\sin_K} \Gamma(\sin_K, \cdot) - \frac{1}{\sin_K^2} \lambda_i)|_{C_0^\infty(\mathring{I}_K)}$ is essentially self-adjoint for every $\lambda_i \in \operatorname{spec} L^F$

(see [71, ch X, problem 1.a]).
We follow the proof of Theorem X.11 in [71]. Consider the unitary transformation

$$U : L^2(I_K, \sin_K^N dr) \to L^2(I_K, dr), \quad \varphi(r) \mapsto \sqrt{\sin_K^N}\varphi(r).$$

$C_0^\infty(\mathring{I}_K)$ is invariant under U and $L^{I_K,\sin_K^N} - \frac{\lambda_i}{\sin_K^2}$ takes the form

$$U\left(\frac{d^2}{dr^2} + \frac{N}{\sin_K r}\frac{d\sin_K r}{dr}\frac{d}{dr} - \frac{\lambda_i}{\sin_K^2 r}\right)U^{-1} = \frac{d^2}{dr^2} - \underbrace{\left(\frac{N^2}{4}\cos^2 r - \frac{N}{2} - \lambda_i\right)\frac{1}{\sin_K^2 r}}_{V(r)}.$$

We get a Schrödinger-type operator defined on $C_0^\infty(\mathring{I}_K)$. The question, whether such an operator is essentially self-adjoint, is a classical problem from quantum mechanics. It was answered by Herman Weyl, who analyzed the solutions of the following ordinary differential equation $-\varphi'' + V\varphi = \lambda\varphi$. One says that $V(r)$ is in the limit circle case at $r \in \partial I_K$ (we assume that $\partial I_K = \{0, \infty\}$ for $K \leq 0$) if for some λ, all solutions φ are locally square integrable around r. Otherwise we say V is in the limit point case at r.

Theorem 5.3.2 (Weyl's limit point-limit circle criterion)**.** *Let V be a continuous real-valued function on $\mathring{I}_K$. Then, the operator $H = -d^2/dr^2 + V$ is essentially self-adjoint on $C_0^\infty(\mathring{I}_K)$ if and only if V is in the limit point case at any $r \in \partial I_K$.*

For the particular case that we consider, the limit point case at ∞ for $K \leq 0$ is easy to check (see Theorem X.8 in [71] and the next corollary). The case $r = 0$ for $K \leq 0$ and $r = 0$ and $r = \pi/\sqrt{K}$ for $K > 0$ follows from the next theorem.

Theorem 5.3.3. *Let V be continuous and positive near zero. If $V(r) \geq \frac{3}{4}\frac{1}{(r-r_0)^2}$ for $r \to r_0 \in \partial I_K \setminus \{\infty\}$ then $H = -d^2/dr^2 + V(r)$ is in the limit point case at r_0. If for some $\epsilon > 0$, $V(r) \leq (\frac{3}{4} - \epsilon)\frac{1}{(r-r_0)^2}$ near r_0, then H is in the limit circle case.*

Proof. $\to$ Theorem X.10 in [71].
So far we did not use the spectral gap of $-L^F$. We have $\lambda_i \geq N$ for any positive $\lambda_i \in \operatorname{spec} -L^F$. But then, the operator $-\frac{d^2}{dr^2} - \frac{N\cos}{\sin_K}\frac{d}{dr} + \frac{\lambda_i}{\sin_K^2}$ is essentially self-adjoint on $C_0^\infty((0,\infty))$ since for $r \to 0$

$$\begin{aligned}\left(\frac{N^2}{4}\cos^2 r - \frac{N}{2} - \lambda_i\right)\frac{1}{\sin_K^2 r} &\sim \left(\frac{N(N-2)}{4} + \lambda_i\right)\frac{1}{r^2} \\ &\geq \left(\frac{N(N-2)}{4} + N\right)\frac{1}{r^2} \geq \left(\frac{N(N+2)}{4}\right)\frac{1}{r^2} \geq \frac{3}{4r^2}\end{aligned}$$

where the last inequality holds if $N \geq 1$. Analogously for $r \to \pi$.

For $\mathcal{E}_0^C$ we cannot follow this strategy (see the next Remark). But since $\mathcal{A}$ is assumed to be dense in the domain of $L^{I_K,\sin_K^N}$, we obtain that

$$\Xi = [\mathcal{A} \otimes E_0] \oplus \sum_{i=1}^{\infty} C_0^\infty(\mathring{I}_K) \otimes E_i \subset D^2(L_\top^C) \oplus D^2(L_\perp^C) = D^2(L^C)$$

is dense in $D^2(L^C)$. □

Remark 5.3.4. For $u_1 \otimes c \in C_0^\infty(\mathring{I}_K) \otimes \mathbb{R}$ we have that $L_0^C\,(u_1 \otimes c) = L^{I_K,\sin_K^N} u_1 \otimes c$. But in general, $L^{I_K,\sin_K^N}|_{C_0^\infty(\mathring{I}_K)}$ is not essentially self-adjoint. More precisely, under the transformation U it becomes

$$\mathring{L} = L^{I_K,\sin_K^N}|_{C_0^\infty(\mathring{I}_K)} = \frac{d^2}{dr^2} - \left(\frac{N^2}{4}\cos^2 r - \frac{N}{2}\right)\frac{1}{\sin_K^2 r}$$

We see that

$$\left(\frac{N^2}{4}\cos^2 - \frac{N}{2}\right)\frac{1}{\sin_K^2} \sim \frac{N(N-2)}{4}\frac{1}{r^2} \geq \frac{3}{4r^2} \quad \text{if } N \geq 3$$

for $r \to 0$ and

$$\left(\frac{N^2}{4}\cos^2 - \frac{N}{2}\right)\frac{1}{\sin_K^2} \leq \frac{3}{4r^2} - \epsilon \quad \text{for some } \epsilon > 0 \text{ if } N < 3$$

for $r \to 0$. Analogously for $r \to \pi$. Hence, in the case $N \geq 3$ we can choose $C_0^\infty(\mathring{I}_K)$ as dense subset $\mathcal{A}$ in $D^2(L^{I_K,\sin_K^N})$ since the closure of $C_0^\infty(\mathring{I}_K)$ is the only self-adjoint extension. On the other hand, in the case $1 \leq N < 3$ the operator $\mathring{L}$ is not essentially self-adjoint. Then there are more than one self-adjoint extensions A_α of $\mathring{L}$ and

$$D^2(\mathrm{CL}(\mathring{L})) \subsetneq D(A_\alpha) \subset D((\mathring{L})^*)$$

where $\mathrm{CL}(\mathring{L})$ denotes the closure with respect the graph norm and $(\mathring{L})^*$ is the adjoint operator. Hence $C_0^\infty(\mathring{I}_K)$ cannot be dense in the domain $D^2(L^{I_K,\sin_K^N})$ if $1 \leq N < 3$. But we can choose

$$\mathcal{A}_0 := \bigcup_{t>0} P_t^{I_K,\sin_K^N}\left[C^\infty(\mathring{I}_K) \cap L^2(\sin_K^N r dr)\right] \subset D(\Gamma_2^{I_K,\sin_K^N}) \cap C^\infty(\mathring{I}_K).$$

This is a dense subset in the domain of $L^{I_K,\sin_K^N}$ since it is stable with respect to the semigroup, and it consists of functions that are smooth in $\mathring{I}_K$ since they solve a parabolic PDE with smooth coefficients in $\mathring{I}_K$.

Remark 5.3.5. In the following the set Ξ will play an important role. Therefore, we will consider the cases $\lambda_1 \geq 3$ and $\lambda_1 \in [1,3)$, that appear in the previous theorem, separately. In the first case we choose $\mathcal{A} = C_0^\infty(\mathring{I}_K)$, in the second case we choose $\mathcal{A} = \mathcal{A}_0$.

Lemma 5.3.6. *Consider $\mathcal{E}^F$ and $I_K \times^N_{\sin_K} \mathcal{E}^F$ as before. Assume L^C has a discrete spectrum and let E_i be the eigenspace for the i-th eigenvalue. Consider $u = \sum_{i=1}^k u_1^i \otimes u_2^i \in \Xi$. Then*

$$P_t^C u = \sum_{i=0}^{k} P_t^{I_K, \sin_K^N, \lambda_i} u_1^i \otimes u_2^i \quad \text{for any } t > 0. \tag{5.3.1}$$

where $P_t^{I_K, \sin_K^N, \lambda_i}$ is the semi-group that is generated by $L^{I_K, \sin_K^N, \lambda_i}$.

Proof. $\longrightarrow$ [63, proof of Lemma 3.3]

Remark 5.3.7. In any case $\lambda_1 \geq 1$ we also define

$$\Xi' := \bigcup_{t>0} P_t^C \ \Xi = \left[\mathcal{A}_0 \otimes E_0\right] \oplus \left[\sum_{i=1}^{\infty} P_t^{I_K, \sin_K^N, \lambda_i} C_0^\infty(\mathring{I}_K) \otimes E_{\lambda_i}\right].$$

Ξ' is dense in $D^2(L^C)$ and stable with respect to P_t^C. Ξ' will provide a suitable class of test function.

Lemma 5.3.8. *Consider $I_K \times^N_{\sin_K} \mathcal{E}^F$, the corresponding operator L^C and $u \in \Xi'$ of the form*

$$u = u_1 \otimes u_2 = P_t^C(\widetilde{u}_1 \otimes u_2) = P_t^{I_K, \sin_K^N, \lambda} \widetilde{u}_1 \otimes u_2 \in D^2(L^C)$$

where u_2 is an eigenfunction for the eigenvalue $\lambda \in \{0\} \cup [N, \infty)$ of $-L^F$ with $N \geq 1$, and $\widetilde{u}_1 \in C_0^\infty(\mathring{I}_K)$ if $\lambda > 0$ and $\widetilde{u}_1 \in \mathcal{A}_0$ if $\lambda = 0$. Then

$$L^C u = L^{I_K, \sin_K^N, \lambda} u_1 \otimes u_2.$$

Proof. We choose $v = v_1 \otimes v_2$ where $v_1 \in C_0^\infty(\mathring{I}_K) \subset D(\mathcal{E}^{I_K, \sin_K^N, \lambda})$ and $v_2 \in D(\mathcal{E}^F)$. Then

$$\begin{aligned}
-(L^C u, v)_{L^2(\mathrm{m}_C)} &= \int_F \mathcal{E}^{I_K, \sin_K^N}(u^x, v^x) d\,\mathrm{m}_F(x) + \int_{I_K} \tfrac{1}{f^2(p)} \mathcal{E}^F(u^r, v^r) \sin_K^N r dr \\
&= \int_F \left[\mathcal{E}^{I_K, \sin_K^N}(u_1, v_1) + \int_{I_K} \tfrac{\lambda u_1 v_1}{\sin_K^2} \sin_K^N r dr\right] u_2 v_2 d\,\mathrm{m}_F \\
&= \mathcal{E}^{I_K, \sin_K^N, \lambda}(u_1, v_1) \int_F u_2 v_2 d\,\mathrm{m}_F = -(L^{I_K, \sin_K^N, \lambda} u_1 \otimes u_2, v)_{L^2(\mathrm{m}_C)}
\end{aligned}$$

Since $C_0^\infty(\mathring{I}_K) \otimes D(\mathcal{E}^F)$ is dense in $L^2(\mathrm{m}_C)$, the last identity holds for any $v \in L^2(\mathrm{m}_C)$ and the statement follows. □

5.4 Bakry-Emery condition for skew products

5.4.1 Regularity estimates

The last lemma of the previous section provides a representation formula for the semigroup of the N-skew product $I_K \times^N_{\sin_K} \mathcal{E}^F$. Now, we want to understand the regularity of the semigroup

$$P_t^{I_K,\sin_K^N,\lambda} u$$

for $\lambda \geq N \geq 1$, $K > 0$ and $u \in C_0^\infty(\mathring{I}_K)$. This might be done by studying the corresponding Sturm-Liouville operator

$$L^{I_K,\sin_K^N,\lambda_i}|_{C_0^\infty(\mathring{I}_K)} = \frac{d^2}{dr^2} + N\frac{\cos_K}{\sin_K}\frac{d}{dr} - \frac{\lambda_i}{\sin_K}$$

and its eigenfunctions (e.g. [54, Chapter 5, §5]). We will go another way and use the result by Bacher and Sturm from [11] that states that Theorem B already holds if the underlying space is a weighted Riemannian manifold (see Theorem A). Then, we also use Theorem 4.3.8, which connects the Bakry-Emery condition for Dirichlet forms with the Riemannian curvature-dimension condition to deduce L^∞-bounds for the gradient of $P_t^{I_K,\sin_K^N,\lambda} u$.

Proposition 5.4.1. *Let $\lambda \geq N \geq 1$ and $K > 0$. Consider the essentially self-adjoint operator $L^{I_K,\sin_K^N,\lambda}|_{C_0^\infty(\mathring{I}_K)}$ and the corresponding semi-group $P_t^{I_K,\sin_K^N,\lambda}$. Then*

$$\Gamma^{I_K}(P_t^{I_K,\sin_K^N,\lambda} u) = \left((P_t^{I_K,\sin_K^N,\lambda} u)'\right)^2 \in L^\infty(I_K, \sin_K^N r dr).$$

for $u \in C_0^\infty(\mathring{I}_K)$.

Proof. Let us assume that $K = 1$ and we consider the metric measure space

$$F = (I_{\bar{K}}, \sin_{\bar{K}}^{N-1} dr) \quad \text{for } \bar{K} \geq 1 \text{ such that } \bar{K}N = \lambda.$$

F satisfies the condition $RCD^*(\bar{K}(N-1), N)$. We have the Dirichlet form

$$\mathcal{E}^{I_{\bar{K}},\sin_{\bar{K}}^{N-1}} = \mathrm{Ch}^F \quad \text{on} \quad L^2(\sin_{\bar{K}}^{N-1} dr).$$

By Theorem 4.3.9 it satisfies the Bakry-Emery condition $BE(\bar{K}(N-1), N)$.

The first non-negative eigenvalue of the corresponding self-adjoint operator equals $\bar{K}N = \lambda$. An eigenfunction is given by $\cos_{\bar{K}}$ what easily can be checked. Since $1 \leq \bar{K}$, F also satisfies $RCD^*(N-1, N)$ and we can consider the metric $(1, N)$-cone $[0, \pi] \times^N_{\sin} F$. By the result of Bacher and Sturm from [11] it satisfies $CD^*(N, N+1)$ but also $RCD^*(N, N+1)$ because of Corollary 5.1.13. By Theorem 4.3.8 the Cheeger

energy $\mathrm{Ch}^{\mathrm{Con}_{N,K(F)}}$ of $[0,\pi]\times^N_{\sin} F$ satisfies $BE(N,N+1)$. It implies a Bakry-Emery gradient estimate

$$|\nabla P^C_t u|^2_w \leq e^{-2Nt} P^C_t |\nabla u|^2_w \tag{5.4.1}$$

for $u \in D(\mathrm{Ch}^{\mathrm{Con}_{N,K(F)}})$. By the main results from Section 5.1 the Cheeger energy of the metric cone coincides with the N-skew product $I_1 \times^N_{\sin} \mathrm{Ch}^F$ in the sense of Dirichlet forms and we have

$$|\nabla u|^2_w = ((u^x)')^2 + \tfrac{1}{\sin^2}((u^r)')^2 = \Gamma^{[0,\pi]}(u^x) + \tfrac{1}{\sin^2}\Gamma^{I_{\bar{K}}}(u^r).$$

In particular, the curvature-dimension condition implies that the metric $(1,N)$-cone satisfies volume doubling and supports a Poincaré inequality. Hence, P^C_t is $L^2 \to L^\infty$-ultracontractive by Remark 4.2.3.

We choose $u = u_1 \otimes u_2$ where $u_1 \in C^\infty_0((0,\pi))$ and $u_2 \in E_1$. E_1 denotes the eigenspace of λ. Lemma 5.3.6 implies that $P^C u = P^{[0,\pi],\sin^N,\lambda} u_1 \otimes u_2$ and (5.4.1) becomes

$$\Gamma^{[0,\pi]}(P^{[0,\pi],\sin^N,\lambda}_t u_1)u_2^2 + \frac{1}{\sin^2}(P^{[0,\pi],\sin^N,\lambda}_t u_1)^2 \Gamma^{I_{\bar{K}}}(u_2) \leq e^{-2Nt} P^C_t \Gamma^C(u) \in L^\infty(\mathrm{m}_C).$$

This implies that

$$\Gamma^{[0,\pi]}(P^{[0,\pi],\sin^N,\lambda}_t u_1) = \left((P^{[0,\pi],\sin^N,\lambda}_t u_1)'\right)^2 \in L^\infty(\sin^N r dr)$$

that is the statement in the case $K = 1$. □

Remark 5.4.2. At this point we can make an important remark on the regularity of test functions $u \in \Xi'$. Consider a strongly local, regular and strongly regular Dirichlet form $\mathcal{E}^F$ that satisfies $BE(N-1,N)$ and a volume doubling property and supports a local Poincaré inequality. Assume that closed balls are compact. Then remark 4.2.3 implies $L^2 \to L^\infty$-ultracontractivity for P^F_t and it follows that

$$P^F_t \Gamma^F(u) \in L^\infty(\mathrm{m}_F)$$

for any $u \in D(\mathcal{E}^F)$. Hence, if we consider eigenfunctions of L^F, the Bakry-Ledoux gradient estimate implies

$$\Gamma^F(P^F_t u) = e^{-\lambda t}\Gamma^F(u) \leq P^F_t \Gamma^F(u) \in L^\infty(\mathrm{m}_F)$$

and especially $u, \Gamma^F(u) \in L^\infty(\mathrm{m}_F)$. Then, the previous proposition implies for

$$u = u_1 \otimes u_2 \in P^C_t\left[C^\infty_0(\mathring{I}_K) \otimes E_\lambda\right] = P^{I_K,\sin^N,\lambda}_t C^\infty_0(\mathring{I}_K) \otimes E_\lambda \quad \text{and} \quad \lambda \geq 1$$

that $u,\ \Gamma^C(u),\ L^C u \in L^\infty(\mathrm{m}_C)$. The same conclusion holds for $u = u_1 \otimes u_2 \in \mathcal{A}_0 \otimes E_0$ because $(I_K, \sin^N_K r dr)$ satisfies $RCD^*(N,N+1)$. Hence, for any $u \in \Xi'$ we have $u,\ \Gamma^C(u),\ L^C u \in L^\infty(\mathrm{m}_C)$.

Remark 5.4.3. Consider $u \in P_t^{I_K,\sin_K^N,\lambda} C_0^\infty(\mathring{I}_K)$ for $\lambda \geq 1$. We know that $\Gamma^{I_K}(u) \in L^\infty$. Especially, we have

$$\int_{I_K} \Gamma^{I_K}(u,\varphi) \sin_K^N r dr \leq \|u\|_{L^\infty(\sin_K^N r dr)} < \infty$$

for any $\varphi \in C_0^\infty(\mathring{I}_K)$ with $\|\sqrt{\Gamma^{I_K}(\varphi)}\|_{L^\infty} \leq 1$. Hence, there exists a Radon measure $-\mathbf{\Delta} u$ on I_K such that

$$\int_{I_K} \varphi d(-\mathbf{\Delta} u) = \int_{I_K} \Gamma^{I_K}(u,\varphi) \sin_K^N r dr = \int_{I_K} L^{I_K,\sin_K^N} \varphi u \sin_K^N r dr.$$

$L^{I_K,\sin_K^N,\lambda} u \in L^2(\sin_K^N r dr)$ and for $\varphi \in C_0^\infty(\mathring{I}_K)$ we obtain

$$\begin{aligned} -\int_{I_K} L^{I_K,\sin_K^N,\lambda} u\varphi \sin_K^N r dr &= \mathcal{E}^{I_K,\sin_K^N,\lambda}(u,\varphi) \\ &= \int_{I_K} \Gamma^{I_K}(u,\varphi) \sin_K^N r dr + \int_{I_K} \frac{\lambda}{\sin_K^2} u\varphi \sin_K^N r dr \\ &= \int_{I_K} \varphi d(-\mathbf{\Delta} u) + \int_{I_K} \frac{\lambda}{\sin_K^2} u\varphi \sin_K^N r dr. \end{aligned}$$

In the case $u \in \mathcal{A}_0$ and $\lambda = \lambda_0 = 0$ these identities are already true where $d\mathbf{\Delta} u = L^{I_K,\sin_K^N} u \sin_K^N r dr$.

5.4.2 Proof of the Bakry-Emery condition for (K,N)-cones

We recall that $\mathcal{E}^F$ is assumed to be strongly regular, and closed balls are compact since F is compact. This implies that the same properties also hold for $I_K \times_{\sin_K}^N \mathcal{E}^F$ by Lemma 4.4.6. Additionally, we assume that besides $\mathcal{E}^F$ also $I_K \times_{\sin_K}^N \mathcal{E}^F$ satisfies a volume doubling property and supports a local $(2,2)$-Poincaré inequality. Hence, we can use the results of Remark 4.2.3 also on the level of $I_K \times_{\sin_K}^N \mathcal{E}^F$. We assume $K > 0$.

Theorem 5.4.4. *Let $\mathcal{E}^F$ satisfy $BE(N-1,N)$. Assume the spectrum of L^F is discrete and the first positive eigenvalue of $-L^F$ satisfies $\lambda_1 \geq N$. Let $K > 0$. Assume also that $\mathcal{E}^C = I_K \times_{\sin_K}^N \mathcal{E}^F$ satisfies a volume doubling property and supports a local $(2,2)$-Poincaré inequality. Then $I_K \times_{\sin_K}^N \mathcal{E}^F$ satisfies $BE(KN, N+1)$.*

Proof. Consider

$$\Xi = [\mathcal{A} \otimes E_0] \oplus \left[\sum_{i=1}^\infty C_0^\infty(\mathring{I}_K) \otimes E_i\right]$$

from Theorem 5.3.1. In the case $N \geq 3$ we set $\mathcal{A} = C_0^\infty(\mathring{I}_K)$, and in the case $N < 3$ we set $\mathcal{A} := \mathcal{A}_0 \subset D(\Gamma_2^{I_K,\sin_K^N}) \cap C^\infty(\mathring{I}_K)$ like in Remark 5.3.4. Consider $u = \sum_{i=0}^k u_1^i \otimes u_2^i \in \Xi$. We have

$$\begin{aligned} L^C u &= L^{B,\sin_K^N} u^x + \tfrac{1}{\sin_K^2} L^F u^p \\ &= \sum_{i=0}^k \big(L^{B,\sin_K^N} u_1^i u_2^i + \tfrac{u_1^i}{\sin_K^2} L^F u_2^i\big) = \textstyle\sum_{i=0}^k \big(L^{B,\sin_K^N} u_1^i + \tfrac{\lambda_i}{\sin_K^2} u_1^i\big) u_2^i \\ &\in [L^{I_K,\sin_K^N}\mathcal{A} \otimes E_0] \oplus \sum_{i=1}^\infty C_0^\infty(\mathring{I}_K) \otimes E_i \subset D(\mathcal{E}^C). \end{aligned}$$

In particular, $\Xi \subset D(\Gamma_2^C)$. We remind on the regularity properties of eigenfunctions of L^F and of test function $\varphi \in \Xi'$ (see Remark 5.4.2). Hence, $\Gamma_2^C(u,v;\varphi)$ is well-defined for any $u, v \in \Xi$ and any test function $\varphi \in \Xi'$.

1. We compute $\Gamma_2(u,v;\varphi)$ like in the proof of Theorem 5.2.1. First, we consider the case $N \geq 3$. Let $u = u_1 \otimes u_2 \in C_0^\infty(\mathring{I}_K) \otimes E_i$ and $v = v_1 \otimes v_2 \in C_0^\infty(\mathring{I}_K) \otimes E_j$ for $i, j > 0$. We take a test function $\varphi \in \Xi'$ of the form

$$\varphi = \varphi_1 \otimes \varphi_2 \in P_t^{I_K,\sin_K^N,\lambda} C_0^\infty(\mathring{I}_K) \otimes E_\lambda \subset \Xi' \ \text{ if } \lambda > 0$$

or $\varphi \in \mathcal{A}_0 \otimes E_0$ if $\lambda = 0$ (see the definition of Ξ'). In any case φ, $\Gamma^C(\varphi)$, $L^C\varphi \in L^\infty(\mathrm{m}_C)$ by Remark 5.4.2. By definition we have

$$2\Gamma_2^C(u,v;\varphi) = \int_C \Gamma^C(u,v) L^C\varphi d\,\mathrm{m}_C - \int_C \Gamma^C(u, L^C v)\varphi d\,\mathrm{m}_C - \int_C \Gamma^C(v, L^C u)\varphi d\,\mathrm{m}_C$$

We set

$$\begin{aligned} \int_C \Gamma^C(u,v) L^C\varphi d\,\mathrm{m}_C &= \int_C \big(\Gamma^{I_K}(u_1,v_1)u_2v_2 + \tfrac{u_1v_2}{\sin_K^2}\Gamma^F(u_2,v_1)\big) L^C\varphi d\,\mathrm{m}_C =: (I) \\ \int_C \Gamma^C(u, L^C v)\varphi d\,\mathrm{m}_C &= \int_C \Gamma^C(u, L^{I_K,\sin_K^N} v_1 v_2 + \tfrac{v_1}{\sin_K^2} L^F v_2)\varphi d\,\mathrm{m}_C =: (II) \end{aligned}$$

We consider (I):

$$(I) := \underbrace{\int_F \int_{I_K} \Gamma^{I_K}(u_1,v_1)u_2v_2 L^C\varphi \sin_K^N r dr d\,\mathrm{m}_F}_{=:(I)_1} + \underbrace{\int_F \int_{I_K} \tfrac{u_1v_1}{\sin_K^2}\Gamma^F(u_2,v_2) L^C\varphi \sin_K^N r dr d\,\mathrm{m}_F}_{=:(I)_2}$$

One can easily check that all integrals are well-defined. For example, we see that $(I)_1 < \infty$ since $\Gamma^{I_K}(u_1, v_1) L^{I_K, \sin_K^N, \lambda} \lambda_1 \in C_0^\infty(\mathring{I}_K)$ and $u_2 v_2 \in L^1(\mathrm{m}_F)$. We can calculate $(I)_1$ because of Proposition 4.4.3 and Remark 5.4.3:

$$\begin{aligned}
(I)_1 &= \int_{I_K} \underbrace{\Gamma^{I_K}(u_1, v_1)}_{\in C_0^\infty(\mathring{I}_K)} L^{I_K, \sin_K^N, \lambda} \varphi_1 \sin_K^N r dr \int_F u_2 v_2 \varphi_2 d\,\mathrm{m}_F \\
&= \left[\int_{I_K} \Gamma^{I_K}(u_1, v_1) d\boldsymbol{\Delta} \varphi_1 - \int_{I_K} \Gamma^{I_K}(u_1, v_1) \tfrac{\varphi_1 \lambda}{\sin_K^2} \sin_K^N r dr \right] \int_F u_2 v_2 \varphi_2 d\,\mathrm{m}_F \\
&= \Bigg[\int_{I_K} L^{I_K, \sin_K^N} \left(\Gamma^{I_K}(u_1, v_1) \right) \varphi_1 \sin_K^N r dr \\
&\qquad\qquad - \int_{I_K} \Gamma^{I_K}(u_1, v_1) \tfrac{\varphi_1 \lambda}{\sin_K^2} \sin_K^N r dr \Bigg] \int_F u_2 v_2 \varphi_2 d\,\mathrm{m}_F \\
&= \int_C L^{I_K, \sin_K^N} \left(\Gamma^{I_K}(u_1, v_1) \right) u_2 v_2 \varphi d\,\mathrm{m}_C + \int_C \Gamma^{I_K}(u_1, v_1) \tfrac{1}{\sin_K^2} u_2 v_2 \varphi_1 L^F \varphi_2 d\,\mathrm{m}_C \\
&= \int_C L^{I_K, \sin_K^N} \left(\Gamma^{I_K}(u_1, v_1) \right) u_2 v_2 \varphi d\,\mathrm{m}_C + \int_C \Gamma^{I_K}(u_1, v_1) \tfrac{1}{\sin_K^2} L^F(u_2 v_2) \varphi_1 \varphi_2 d\,\mathrm{m}_C
\end{aligned}$$

We remark that $u_2 v_2 \in D(L_1^F)$ for any $u_2, v_2 \in D_2(L^F)$ (for example see [16, Section I.4, Theorem 4.2.2]) and we have

$$L_1^F(u_2 v_2) = L^F u_2 v_2 + u_2 L^F v_2 + \Gamma^F(u_2, v_2) \quad \& \quad \int_F \varphi L_1^F u d\,\mathrm{m}_F = \int_F u L^F \varphi d\,\mathrm{m}_F \tag{5.4.2}$$

if $u \in D(L_1^F)$ and $\varphi \in D_2(L^F) \cap L^\infty(\mathrm{m}_F)$. The operator L_1^F with domain $D(L_1^F) \subset L^1(\mathrm{m}_F)$ is the smallest closed extension of L^F to $L^1(\mathrm{m}_F)$ and there is an associated semi group $P_{t,1}^F : L^1(\mathrm{m}_F) \to L^1(\mathrm{m}_F)$. The second equation in (5.4.2) comes from

$$\int_F P_{t,1}^F u \cdot \varphi d\,\mathrm{m}_F = \int_F u \cdot P_t^F \varphi d\,\mathrm{m}_F \quad \text{for any } u \in L^1(\mathrm{m}_F) \text{ and } \varphi \in L^\infty(\mathrm{m}_F)$$

that follows for instance from the existence of a bounded, continuous heat kernel (see Remark 4.2.3) and Fubini's theorem. Next, we consider $(I)_2$. Similar as before we obtain

$$\begin{aligned}
(I)_2 &= \int_{I_K} \underbrace{\frac{u_1 v_1}{\sin_K^2}}_{\in C_0^\infty(\mathring{I}_K)} L^{I_K, \sin_K^N, \lambda} \varphi_1 \sin_K^N r dr \int_F \Gamma^F(u_2, v_2) \varphi_2 d\,\mathrm{m}_F \\
&= \Bigg[\int_{I_K} L^{I_K, \sin_K^N} \left(\tfrac{u_1 v_1}{\sin_K^2} \right) \varphi_1 \sin_K^N r dr \\
&\qquad\qquad - \int_{I_K} \tfrac{u_1 v_1}{\sin_K^2} \lambda \varphi_1 \sin_K^N r dr \Bigg] \int_F \varphi_2 \Gamma^F(u_2, v_2) d\,\mathrm{m}_F \\
&= \int_C L^{I_K, \sin_K^N} \left(\tfrac{u_1 v_1}{\sin_K^2} \right) \Gamma^F(u_2, v_2) \varphi d\,\mathrm{m}_C + \int_C \tfrac{u_1 v_1}{\sin_K^2} \varphi_1 \Gamma^F(u_2, v_2) L^F \varphi_2 d\,\mathrm{m}_C
\end{aligned}$$

Then, we consider (II)

$$(II) = \underbrace{\int_F \int_{I_K} \Gamma^C(u, L^{I_K,\sin_K^N} v_1 v_2)\varphi \sin_K^N r dr d\,\mathrm{m}_F}_{=:(II)_1} + \underbrace{\int_F \int_{I_K} \Gamma^C(u, \tfrac{v_1}{\sin_K^2} L^F v_2)\varphi \sin_K^N r dr d\,\mathrm{m}_F}_{=:(II)_2}.$$

We can also calculate $(II)_1$ and $(II)_2$:

$$\begin{aligned}(II)_1 &= \int_C \Gamma^{I_K}(u_1, L^{I_K,\sin_K^N} v_1) u_2 v_2 \varphi d\,\mathrm{m}_C \\ &\qquad + \int_F \int_{I_K} \tfrac{u_1 L^{I_K,\sin_K^N} v_1}{\sin_K^2} \Gamma^F(u_2, v_2)\varphi \sin_K^N r dr d\,\mathrm{m}_F \\ (II)_2 &= \int_F \int_{I_K} \Gamma^{I_K}(u_1, \tfrac{v_1}{\sin_K^2}) u_2 L^F v_2 \varphi \sin_K^N r dr d\,\mathrm{m}_F \\ &\qquad + \int_F \int_{I_K} \tfrac{v_1 u_1}{\sin_K^4} \Gamma^F(u_2, L^F v_2)\varphi \sin_K^N r dr d\,\mathrm{m}_F\end{aligned}$$

Finally, we add everything up and obtain

$$\begin{aligned}&2\Gamma_2^C(u, v, \varphi) \\ &= \int_C 2\Gamma_2^{I_K,\sin_K^N}(u_1, v_1) u_2 v_2 \varphi d\,\mathrm{m}_C + \int_{I_K} 2\tfrac{u_1 v_1}{\sin_K^2} \Gamma_2^F(u_2, v_2; \varphi_2)\varphi_1 \sin_K^N r dr \\ &\quad + \int_C \Big[\Gamma^{I_K}(u_1, v_1)\tfrac{1}{\sin_K^2} L^F(u_2 v_2) - \Gamma^{I_K}(u_1, \tfrac{v_1}{\sin_K^2}) u_2 L^F v_2 - \Gamma^{I_K}(v_1, \tfrac{u_1}{\sin_K^2}) v_2 L^F u_2 \\ &\qquad - \Big(\tfrac{u_1 L^{I_K,\sin_K^N} v_1}{\sin_K^2} + \tfrac{v_1 L^{I_K,\sin_K^N} u_1}{\sin_K^2} - L^{I_K,\sin_K^N}\big(\tfrac{u_1 v_1}{\sin_K^2}\big)\Big)\Gamma^F(u_2, v_2)\Big]\varphi d\,\mathrm{m}_C \quad (5.4.3)\end{aligned}$$

We remark that we cannot replace

$$\int_{I_K} \Gamma_2^{I_K,\sin_K^N}(u_1, v_1)\varphi_1 \sin_K^N r dr \quad \text{by} \quad \Gamma_2^{I_K,\sin_K^N}(u_1, v_1; \varphi_1)$$

since φ_1 is not necessarily in $D_2(L^{I_K,\sin_K^N})$ if $\lambda_i > 0$. $\Gamma_2^C(u, v; \varphi)$ and also the right hand side in the last equation is linear in φ. Hence, the last equation also holds for any $\varphi = \sum_{i=1}^k \varphi_1^i \otimes \varphi_2^i \in \Xi'$. Hence $2\Gamma_2^C(u, v; \varphi)$ looks exactly like the weak version of equation (5.2.2) in the proof of Theorem 5.2.1 where we have proven the corresponding classical Γ_2-estimate. Now, if we choose test functions $\varphi \in \Xi'$ with $\varphi \geq 0$, we can proceed exactly like in the proof of Theorem 5.2.1 where one should use (5.4.2) to compute $L^F(u_2 v_2)$. Finally, we obtain the sharp Γ_2-estimate in a weak form for $u \in \Xi$ and $\varphi \in \Xi'$ with $\varphi \geq 0$.

2. Now, we deal with the case $1 \leq N < 3$. We compute $\Gamma_2^C(u,v;\varphi)$ exactly like in the case $N \geq 3$ but we have to consider the case when $u_1 \otimes u_2 \in \mathcal{A}_0 \otimes E_0$ and $v_1 \otimes v_2 \in C_0^\infty(\mathring{I}_K) \otimes E_i$ for $i > 0$ separately. Any other case is already covered by the previous paragraph. We recall that $u_2 = const = m \in \mathbb{R}$ because of Remark 4.2.3. We can compute $\Gamma_2^C(u,v;\varphi)$ for $\varphi \in \Xi'$ exactly like in the previous paragraph since terms of the form $u_1 v_1$, $\Gamma^{I_K}(u_1,v_1)$ and $L^{I_K,\sin_K^N}(u_1 v_1)$ are in $C_0^\infty(\mathring{I}_K)$. We obtain again formula (5.4.3).

The only case that we still have to check is $u = v \in \mathcal{A}_0 \otimes E_0$. It is not covered, yet, since $u_1^2 \notin C_0^\infty(\mathring{I}_K)$. First, let $\varphi = \varphi_1 \otimes \varphi_2 \in P_t^{I_K,\sin_K^N,\lambda} \otimes E_i$. We know that $u = u_1 \otimes m \in D(\mathcal{E}^C)$, $u_1 \in \mathcal{A}_0 \subset D(\Gamma_2^{I_K,\sin_K^N})$ and $\Gamma^C(u) = \Gamma^{I_K}(u_1)m^2$. Hence,

$$\begin{aligned}
\Gamma_2^C(u;\varphi) &= \int_C \tfrac{1}{2} L^C \varphi \Gamma^C(u) d\,\mathrm{m}_C - \int_C \Gamma^C(u, L^C u)\varphi d\,\mathrm{m}_C \\
&= \int_C \tfrac{1}{2} L^{I_K,\sin_K^N,\lambda_i} \varphi_1\varphi_2 \Gamma^{I_K}(u_1) m^2 \sin_K^N r dr d\,\mathrm{m}_F \\
&\qquad - \int_C \Gamma^{I_K}(u_1, L^{I_K,\sin_K^N} u_1) m^2 \varphi_1\varphi_2 \sin_K^N r dr d\,\mathrm{m}_F \\
&= \int_F m^2\varphi_2 d\,\mathrm{m}_F \int_{I_K} \Big[\tfrac{1}{2}\Gamma^{I_K}(u_1) L^{I_K,\sin_K^N,\lambda_i}\varphi_1 - \Gamma^{I_K}(u_1, L^{I_K,\sin_K^N} u_1)\varphi_1\Big] \sin_K^N r dr
\end{aligned}$$

Since φ_2 is an eigenfunction of L^F, the right hand side is 0 unless $\lambda_i = 0$ and $\varphi_2 \neq 0$. We conclude that $\Gamma_2^C(u;\varphi) \neq 0$ for $\varphi \in \Xi'$ only if $\varphi_2 = const \neq 0$. In any case:

$$\Gamma_2^C(u;\varphi) = \int_F m^2\varphi_2 d\,\mathrm{m}_F\, \Gamma_2^{I_K,\sin_K^N}(u_1;\varphi_1). \tag{5.4.4}$$

This is just (5.4.3) where we replace $\Gamma_2^{I_K,\sin_K^N}(u_1)\varphi_1$ by $\Gamma_2^{I_K,\sin_K^N}(u_1;\varphi_1)$. But we can proceed like at the end of the previous paragraph. Because $\mathcal{E}^{I_K,\sin_K^N}$ satisfies $BE(KN, N+1)$ we can bound (5.4.4) by

$$\Gamma_2^C(u;\varphi) \geq m^2 \int_F \int_{I_K} \big[KN\varphi\Gamma^{I_K,\sin_K^N}(u_1) + \frac{1}{N+1}(L^{I_K,\sin_K^N} u_1)^2\varphi\big] \sin_K^N r dr d\,\mathrm{m}_F$$

if $\varphi \geq 0$. Hence, for $u \in \Xi$ and $\varphi \in \Xi'$ with $\varphi \geq 0$ we have the desired Γ_2-estimate.

3. We extend this estimate to any function $u \in D(\Gamma_2^C)$ (and test functions $\varphi \in \Xi'$ with $\varphi \geq 0$). We choose a sequence $u_n \in \Xi$ that converges to $u \in D(\Gamma_2)$ in $D^2(L^C)$. First, we obtain that

$$\begin{aligned}
\int_C \Gamma^C(u_n,u_n) L^C\varphi dm_C &\to \int_C \Gamma^C(u,u) L^C\varphi dm_C, \\
\int_C \Gamma^C(u_n,u_n)\varphi d\,\mathrm{m}_C &\to \int_C \Gamma^C(u,u)\varphi d\,\mathrm{m}_C, \\
\int_C (L^C u_n)^2\varphi d\,\mathrm{m}_C &\to \int_C (L^C u)^2\varphi d\,\mathrm{m}_C\,.
\end{aligned} \tag{5.4.5}$$

We still need to show convergence of $\int_C \Gamma^C(u_n, L^C u_n)\varphi d\,\mathrm{m}_C$. Since $u_n, L^C u_n, \varphi \in D(\mathcal{E}^C)$ and $\varphi, \Gamma(\varphi) \in L^\infty(\mathrm{m}_C)$, we can apply the Leibniz rule (4.2.2) for Γ^C. We obtain

$$\begin{aligned}\int_C \Gamma(u_n, L^C u_n)\varphi d\,\mathrm{m}_C &= \int_C \Gamma(u_n, L^C u_n \varphi) d\,\mathrm{m}_C - \int_C \Gamma^C(u_n, \varphi) L^C u_n d\,\mathrm{m}_C \\ &= -\underbrace{\int_C (L^C u_n)^2 \varphi d\,\mathrm{m}_C}_{\longrightarrow \int_C (L^C u)^2 \varphi d\,\mathrm{m}_C} - \int_C \Gamma^C(u_n, \varphi) L^C u_n d\,\mathrm{m}_C\,.\end{aligned}$$

Consider the second term on the right hand side.

$$\begin{aligned}&\left| \int_C \Gamma^C(u_n, \varphi) L^C u_n d\,\mathrm{m}_C - \int_C \Gamma^C(u, \varphi) L^C u d\,\mathrm{m}_C \right| \\ &\leq \int_C |\Gamma^C(u_n, \varphi) L^C(u_n - u)| + |\Gamma^C(u_n - u, \varphi) L^C u|\, d\,\mathrm{m}_C \\ &\leq \|\Gamma^C(\varphi)\|_{L^\infty} \Bigg[\underbrace{\int_C \Gamma^C(u_n) d\,\mathrm{m}_C}_{\to \int_C \Gamma^C(u) d\,\mathrm{m}_C} \underbrace{\int_C (L^C(u_n - u))^2 d\,\mathrm{m}_C}_{\to 0} \\ &\qquad + \underbrace{\int_C \Gamma^C(u_n - u) d\,\mathrm{m}_C}_{\to 0} \int_C (L^C u)^2 d\,\mathrm{m}_C \Bigg]\end{aligned}$$

Since $\varphi \in \Xi'$, we have that $\|\Gamma^C(\varphi)\|_{L^\infty} < \infty$. It follows that

$$\int_C \Gamma^C(u_n, \varphi) L^C u_n d\,\mathrm{m}_C \to \int_C \Gamma^C(u, \varphi) L^C u d\,\mathrm{m}_C \quad \text{for } u_n \to u \text{ in } D^2(L^C)$$

and consequently

$$\int_C \Gamma^C(u_n, L^C u_n)\varphi d\,\mathrm{m}_C \to \int_C \Gamma^C(u, L^C u)\varphi d\,\mathrm{m}_C \quad \text{for } u_n \to u \text{ in } D^2(L^C)$$

for any $u \in D(\Gamma_2^C)$ and for any test function $\varphi \in \Xi'$ with $\varphi \geq 0$.

4. Finally, we show that the Γ_2-estimate holds for any admissible test function $\varphi \in D_+^{b,2}(L^C)$. Since we assume $K > 0$, the measure m_C is finite and we can assume that $\varphi \geq M > 0$ for some positive constant $M \in D^2(L^C)$. Consider a sequence $\varphi_n \in \Xi$ that converges to φ in $D^2(L^C)$. Then, we also have $P_t^C \varphi \geq M$ and $P_t^C \varphi_n \to P_t^C \varphi$ in $D^2(L^C)$ for all $t > 0$. Since we assume that $\mathcal{E}^C$ satisfies volume doubling and supports a Poincaré inequality, is strongly regular and admits that closed balls are compact (see Lemma 4.4.6) there is an upper bound for the heat kernel (see [76, Corollary 4.2], Remark 4.2.3) that is equivalent to $L^2 \to L^\infty$-ultracontractivity of the semigroup P_t^C (see [40, Chapter 14.1] and Remark 4.2.3). Hence, $P_t^C \varphi_n \to P_t^C \varphi$ and $L^C P_t^C \varphi_n \to$

$L^C P_t^C \varphi$ in $L^\infty(\mathrm{m}_C)$. Since $P_t^C \varphi \geq M > 0$, we deduce that $P_t^C \varphi_n \in D_+^{b,2}(L^C) \cap \Xi'$ for n sufficiently big. Then, the results from the previous paragraphs state that

$$\int_C \left(\tfrac{1}{2}\Gamma^C(u) L^C P_t^C \varphi_n - \Gamma^C(u, L^C u) P_t^C \varphi_n\right) d\,\mathrm{m}_C \geq \int_C \left(KN\Gamma^C(u) + \tfrac{1}{N+1}(L^C u)^2\right) P_t^C \varphi_n \,\mathrm{m}_C\,.$$

Hence, if $n \to \infty$

$$\int_C \left(\tfrac{1}{2}\Gamma^C(u) L^C P_t^C \varphi - \Gamma^C(u, L^C u) P_t^C \varphi\right) d\,\mathrm{m}_C \geq \int_C \left(KN\Gamma^C(u) P_t^C \varphi + \tfrac{1}{N+1}(L^C u)^2 P_t^C \varphi\right) \mathrm{m}_C$$

for $u \in D(\Gamma_2)$ and $\varphi \in D_+^{b,2}$ with $\varphi \geq M > 0$ because of the L^∞-convergence of $P_t^C \varphi_n$ and $P_t^C L^C \varphi_n$. Then we also let $M \to 0$ and the inequality holds for any test function of the form $P_t^C \varphi$ where $\varphi \in D_+^{b,2}$. Finally, by application of Lebesgue's dominated convergence theorem one can check that $P_t^C \varphi$ and $P_t^C L^C \varphi$ converges to φ and $L^C \varphi$, respectively, w.r.t. weak-$*$ convergence if $\varphi \in D_+^{b,2}(L^C)$, and we obtain the Γ_2-estimate for any $u \in D(\Gamma_2^C)$ and for any $\varphi \in D_+^{b,2}(L^C)$. □

5.5 Proof of the main results

Proof of Theorem B. First, let $K > 0$. The Cheeger energy Ch^F of $(F, \mathrm{d}_F, \mathrm{m}_F)$ is a strongly local, regular and strongly regular Dirichlet form that satisfies $BE(N-1, N)$ by Theorem 4.3.8. By Theorem 4.3.10 and the following remark the spectrum of the associated Laplace operator L^F is discrete and the first positive eigenvalue of $-L^F$ satisfies $\lambda_1 \geq N$. By Theorem 5.1.5 and Corollary 5.1.6 we know that $I_K \times_{\sin_K}^N \mathrm{Ch}^F = \mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$ and $\mathrm{d}_{\mathrm{Con}_K} = \mathrm{d}_{\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}}$. Lemma 5.1.3 states that $\mathrm{Con}_{N,K}(F)$ satisfies a volume doubling property and supports a local Poincaré inequality. Especially, $\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$ supports a $(2,2)$-Poincaré inequality by Remark 4.1.3. $\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$ is also strongly regular and $\mathrm{Con}_{N,K}(F)$ is compact since $(F, \mathrm{d}_F, \mathrm{m}_F)$ is compact, $K > 0$ and its Cheeger energy Ch^F is strongly regular (Lemma 4.4.6). Hence, we can apply Theorem 5.4.4 and $I_K \times_{\sin_K}^N \mathrm{Ch}^F$ satisfies $BE(KN, N+1)$.

Finally, we want to apply the backward direction of Theorem 4.3.8. Results of Sturm from [75] (see Remark 4.2.3 and Remark 4.1.3) state a Feller property for the corresponding semigroup P_t^C of $\mathrm{Con}_{K,N}(F)$. Thus, we can apply Theorem 3.15 from [3] that states that in this case any $u \in D(\mathcal{E}^C)$ with $\sqrt{\Gamma^C(u)} \in L^\infty(\mathrm{m}_C)$ has a continuous representative. Consequently, any such $u \in D(\mathcal{E}^C)$ is Lipschitz continuous with respect to the intrinsic distance of $I_K \times_{\sin_K}^N \mathrm{Ch}^F$ that again coincides with $\mathrm{d}_{\mathrm{Con}_K}$ by Corollary 5.1.13. Thus, the regularity Assumption 4.3.7 is satisfied and Theorem 4.3.8 yields the condition $RCD^*(NK, N+1)$ if $K > 0$.

The case $K = 0$ follows from the case $K > 0$. The rescaled space $\mathrm{Con}_{N,K/n^2}(F)$ converges with respect to pointed measured Gromov-Hausdorff convergence to $\mathrm{Con}_{N,0}(F)$ if $n \to \infty$. To see this, for instance, we can adopt the proof of Theorem 10.9.3 in [18] to prove Gromov-Hausdorff convergence. We also obtain a family of ϵ-isometries

between r-balls around 0 in $\mathrm{Con}_{N,K/n^2}(F)$ and r-balls around 0 in $\mathrm{Con}_{N,0}(F)$ for $\frac{1}{n^2} \leq \epsilon$ and n big enough. Then, the pushfowards of $\sin^N_{K/n^2} dr \otimes d\,\mathrm{m}_F$ w.r.t to these ϵ-isometries converge weakly to $r^N dr \otimes d\,\mathrm{m}_F\,|_{B_r(0)}$ as $\epsilon \to 0$. Hence, we obtain pointed measured Gromov-Hausdorff convergence. Especially, this is measured Gromov-Hausdorff convergence of r-balls around 0. Hence, $\mathrm{Con}_{N,0}(F)$ satisfies $RCD^*(0, N+1) = RCD(0, N+1)$ by the usual stability property of the condition CD under measured Gromov-Hausdorff convergence (see for examples [79]). □

Proof of Theorem C. First, let us consider the case $N \geq 1$. Remark 5.1.8 states that $\operatorname{diam} F \leq \pi$ and $\mathrm{Con}_{N,K}(F) = I_K \times^N_{\sin_K} F$ in any case when $N \geq 1$. We need to check the condition $RCD^*(N-1, N)$ for $(F, \mathrm{d}_F, \mathrm{m})$. Corollary 5.1.13 implies that $(F, \mathrm{d}_F, \mathrm{m}_F)$ is infinitesimal Hilbertian. By Proposition 5.1.11 and Corollary 5.1.12 the intrinsic distance of $\mathcal{E}^C = I_K \times^N_{\sin_K} \mathrm{Ch}^F$ is the K-cone distance $\mathrm{d}_{\mathrm{Con}_K}$ and the Cheeger energy of the (K,N)-cone coincides with $\mathcal{E}^C$. Theorem 4.3.8 implies the condition $BE(KN, N+1)$ for $I_K \times^N_{\sin_K} \mathcal{E}^F$.

One can check that $C_0^\infty(\mathring{I}_K) \otimes D(\Gamma_2^F) \subset D(\Gamma_2^C)$ and $1 \otimes D_+^{b,2}(L^F) \subset D_+^{b,2}(L^C)$. Hence, we can again derive formula (5.4.3) in precisely the same way as in the proof of Theorem 5.4.4 for $u_1 \otimes u_2 \in C_0^\infty(\mathring{I}_K) \otimes D(\Gamma_2^F)$ and $1 \otimes \varphi_2 \in 1 \otimes D_+^{b,2}(L^F)$. Now, we can follow the proof of Theorem 5.2.1 and we obtain

$$\begin{aligned}\int_F L^F\varphi\Gamma^F(u) d\,\mathrm{m}_F - \int_F \Gamma^F(u, L^F u)\varphi d\,\mathrm{m}_F \\ \geq (N-1)\int_F \Gamma^F(u)\varphi d\,\mathrm{m}_F + \frac{1}{N}\int_F \left(L^F u_2\right)^2 \varphi d\,\mathrm{m}_F \\ - \frac{1}{(N+1)N}\int_F \left(L^F u_2 + NK_F u_2\right)^2 \varphi d\,\mathrm{m}_F \end{aligned} \tag{5.5.1}$$

for any $u \in D(\Gamma_2^F)$ and any $\varphi \in D_+^{b,2}(L^F)$. We want to deduce $RCD^*(N-1, N)$ for F. However, we cannot apply the argument of Theorem 5.2.3 directly since pointwise estimates for the Bochner inequality do not make sense. But like in the proof of Theorem 4.3.8 (more precisely, see Proposition 4.7 in [34]), we get a gradient estimate of the following type:

$$|\nabla P_t^F u_2|^2 + \frac{c(t)}{N}\left(|L^F P_t^F u_2|^2 - \tfrac{1}{(N+1)} P_t^F \left(L^F u_2 + NK_F u_2\right)^2\right) \leq e^{-2Kt} P_t^F |\nabla u_2|^2 \tag{5.5.2}$$

m_F-a.e. in F for any $u_2 \in D^2(L^F)$. We sketch the argument briefly. Consider

$$h(s) := e^{-2(N-1)s}\int_F P_s^F \varphi |\nabla P_{t-s}^F u_2|^2 d\,\mathrm{m}_F\,.$$

One estimates the derivative of h as:

$$\begin{aligned}
h'(s) &= 2e^{-2(N-1)s}\int_F \big(-(N-1)P_s\varphi|\nabla P^F_{t-s}u_2|^2 + \tfrac{1}{2}L^F P_s\varphi|\nabla P_{t-s}u_2|^2 \\
&\qquad\qquad - P_s\varphi\Gamma^F(P_{t-s}u_2, L^F P_{t-s}u_2)\big)\,d\,\mathrm{m}_F \\
&\geq 2e^{-2(N-1)s}\int_F P_s\varphi\left(\tfrac{1}{N}\left(L^F P_{t-s}u_2\right)^2 - \tfrac{1}{(N+1)N}\left(L^F P_{t-s}u_2 + NK_F P_{t-s}u_2\right)^2\right)d\,\mathrm{m}_F \\
&\geq 2e^{-2(N-1)s}\int_F \varphi\left(\tfrac{1}{N}\left(L^F P_t u_2\right)^2 - \tfrac{1}{(N+1)N}P_t\left(L^F u_2 + NK_F u_2\right)^2\right)d\,\mathrm{m}_F
\end{aligned}$$

where we used (5.5.1) in the first and Jensen's inequality in the second inequality. Finally, we integrate h' from 0 to t and the rest of the proof is exactly the same as in Proposition 4.9 in [34].

We remark that F satisfies a doubling property and supports a local Poincaré inequality and by Lemma 5.1.10 we have that $\mathrm{d}_{\mathrm{Ch}^F} = \mathrm{d}_F$, which implies that Ch^F is strongly local. Thus, by the results of Sturm (Remark 4.2.3) the associated semigroup is Feller and has a continuous kernel. Then we proceed as follows.

For $u_2 \in D^2(L^F)$ we consider $P^F_s\left(L^F u_2 + NK_F u_2\right) = L^F P^F_s u_2 + NK_F P^F_s u_2 =: v_2$ and for $x \in F$ we define $v_{2,x} = v_2 - v_2(x)$. $v_{2,x}$ is continuous on F and $v_{2,x}(x) = 0$. We consider $P^F_t(v^2_{2,x})$ that is jointly continuous in $z \in F$ and $t \geq 0$. For instance, this follows since $v^2_{2,x} \in C(F) \cap L^\infty(\mathrm{m}_F)$ and since we have a nice upper bound for the heat kernel associated to Ch^F because of results of Sturm in [75]. Then, to prove that $P^F_t(v^2_{2,x})$ is jointly continuous, we can copy the proof of the corresponding result in $\mathbb{R}^n$. It holds that $P^F_0(v^2_{2,x})(x) = v_{2,x}(x) = 0$. Hence, for any $\epsilon > 0$ and any $x \in F$ there is $\delta_x > 0$ and $\tau_x > 0$ such that $|P^F_t(v^2_{2,x})(y)| < \epsilon$ for any $y \in B_{\delta_x}(x)$ and $0 < t < \tau_x$. Since F is compact, there is a finite collection $(x_i)_{i=1}^k$ of points such that $B_{\delta_{x_i}}(x_i)_{i=1,\dots,k}$ is a covering of F. We set $\tau = \min_{i=1,\dots,k}\tau_i$.
Now we choose $x_i \in F$ with $B_{\delta_i}(x_i)$ and we set $\delta_i = \delta_{x_i}$. Consider

$$P^F_s u_2 - NK_F P^F_s u_2(x_i) - L^F P^F_s u_2(x_i) =: \bar{v}_{2,x_i} \in D^2(L^F)$$

and insert it in (5.5.2) for $t < \tau$.

$$\begin{aligned}
&|\nabla P^F_t \bar{v}_{2,x_i}|^2 + c(t)\frac{1}{N}\left(L^F P^F_t \bar{v}_{2,x_i}\right)^2 \\
&\qquad - \frac{c(t)}{N(N+1)}\underbrace{P^F_t\left(L^F \bar{v}_{2,x_i} + NK_F \bar{v}_{2,x_i}\right)^2}_{(*)} \leq e^{-2Kt}P^F_t\left|\nabla \bar{v}_{2,x_i}\right|^2.
\end{aligned}$$

We can see that

$$\begin{aligned}
(*) &= P^F_t\left(L^F P^F_s u_2 + NK_F P^F_s u_2 - NK_F P^F_s u_2(x_i) - L^F P^F_s u_2(x_i)\right)^2 \\
&= P^F_t \underbrace{\left(v_2 - v_2(x_i)\right)^2}_{(v_{2,x_i})^2}
\end{aligned}$$

For any $y \in B_{\delta_i}(x_i)$ we get $|(*)(y)| = |P_t^F\left(\widetilde{v}_{2,x_i}^2\right)(y)| < \epsilon$. From that and since $\bar{v}_{2,x_i}$ differs form $P_s^F u_2$ only by a constant, we get for any $0 < t < \tau$ and m_F-a.e. $y \in B_{\delta_i}(x_i)$

$$|\nabla P_t^F P_s^F u_2|^2(y) + \frac{c(t)}{N}\left(|L^F P_t^F P_s^F u_2|^2(y) - \frac{1}{N+1}\epsilon\right) \leq e^{-2Kt} P_t^F |\nabla P_s^F u_2|^2(y).$$

The last inequality does not depend on x_i anymore and since $\epsilon > 0$ is arbitrary, we obtain

$$|\nabla P_t^F P_s^F u_2|^2 + \frac{c(t)}{N}|L^F P_t^F P_s^F u_2|^2 \leq e^{-2Kt} P_t^F |\nabla P_s^F u_2|^2$$

for $0 < t < \tau$ and m_F-a.e. for $u_2 \in D^2(L^F)$. Then we can also let s go to 0

$$|\nabla P_t^F u_2|^2 + \frac{c(t)}{N}|L^F P_t^F u_2|^2 \leq e^{-2Kt} P_t^F |\nabla u_2|^2 \text{ for } 0 < t < \tau$$

and finally, we can follow the proof of Theorem 4.8 in [34] to obtain the condition $BE(N-1,N)$. Now, similar like in the previous theorem, this implies $RCD^*(N-1,N)$ for $(F, \mathrm{d}_F, \mathrm{m}_F)$. We only need to check the Assumption 4.3.7. The condition $RCD^*(KN, N+1)$ for $\mathrm{Con}_{N,K}(F)$ implies that every $u \in D(\mathrm{Ch}^{\mathrm{Con}_{N,K}(F)})$ such that $\Gamma^C(u) \in L^\infty(\mathrm{m}_C)$ admits a Lipschitz representative and Theorem 5.1.6 states that $I_K \times_{\sin_K}^N \mathrm{Ch}^F = \mathrm{Ch}^{\mathrm{Con}_{N,K}(F)}$. This easily implies that also $u \in D(\mathrm{Ch}^F)$ such that $\Gamma^F(u) \in L^\infty(\mathrm{m}_F)$ admits a Lipschitz representative with respect to d_F.

For the case $N \in [0,1)$ we argue by contradiction. First, we see that F has to be discrete. Otherwise, we would find a geodesic γ in F (see the remark directly after Definition 2.2.6), and consequently the cone over $\mathrm{Im}\gamma$ would be a 2-dimensional subset in $\mathrm{Con}_{N,K}(F)$. This contradicts the condition $RCD^*(KN, N+1)$ for $\mathrm{Con}_{N,K}(F)$ that implies that the Hausdorff dimension of $\mathrm{Con}_{N,K}(F)$ cannot be bigger than $N+1 < 2$. Then, assume there are two points x, y in F with $\mathrm{d}_F(x,y) < \pi$. Hence, by the definition of the cone metric there is no continuous curve between $(1,x)$ and $(1,y)$ whose length is ϵ-close to $\mathrm{d}_{\mathrm{Con}_K}((1,x),(1,y))$. The only continuous curve that connects $(1,x)$ and $(1,y)$ consists of the segments that connect each of this points with the nearest origin and its length is $\mathrm{d}_{\mathrm{Con}_K}(o,(1,x)) + \mathrm{d}_{\mathrm{Con}_K}(o,(1,y)) > \mathrm{d}_{\mathrm{Con}_K}((1,x),(1,y))$. But since $\mathrm{Con}_{N,K}(F)$ satisfies a curvature-dimension condition, it has to be an intrinsic metric space what contradicts the previous observation. Thus, there can only be points in F that have distance π. F can only have at most two points since otherwise we will find an optimal transport between absolutely continuous measures in $\mathrm{Con}_{N,K}(F)$ that is essentially branching, and this contradicts the RCD^*-condition. For example, assume there are three points. The geodesics between (s,x), (t,y) and (r,z) for $s,t,r \leq 1$ consist exactly of segments that connect the origin. Hence, one can consider an absolutely continuous measure that is concentrated on one segment and the transport to an absolutely continuous measure that is concentrated equally on the two other segments. In the case where F is just one point we see that

$I_K \times^N_{\sin_K} F = (I_K, \sin^N_K)$. Otherwise, if F has two points with distance π, N has to be 0 and we see that $\mathrm{Con}_{0,K}(F) = \frac{1}{\sqrt{K}}\mathbb{S}^1$. □

Proof of Theorem D. $(F, \mathrm{d}_F, \mathrm{m}_F)$ is compact with $\operatorname{diam} F \leq \pi$. Hence, in any case $\mathrm{Con}_0^{N+1}(F) = [0,\infty) \times_r^{N+1} F$ and by Theorem B $\mathrm{Con}_0^{N+1}(F)$ is a metric measure space that satisfies $RCD^*(0, N+2)$. Since $\mathrm{d}_F(x,y) = \pi$, there is a geodesic line in $\mathrm{Con}_0^{N+1}(F)$. Thus, by the first part of the splitting Theorem 4.3.13, $\mathrm{Con}_{N+2,0}(F) =: X$ splits into $X = \mathbb{R} \times X'$ where $X' = (X', \mathrm{d}_{X'}, \mathrm{m}_{X'})$ denotes a metric measure space that satisfies $RCD^*(0, N+1)$. One can easily see that X' is a metric cone over $F' = F \cap X'$, that F' is a geodesic space and that F' embeds geodesically in F.

Consider $(1,f), (1,g)$ in $\{1\} \times F$. We find $r, s > 0$, $i, j \in [-1,1]$ and $f', g' \in F'$ such that

$$\mathrm{d}_X((1,f),(1,g))^2 = 2 - 2\cos \mathrm{d}_F(f,g) = r^2 + s^2 - 2rs \cos \mathrm{d}_{F'}(f',g') + |i-j|^2.$$

Because the metric on X is precisely given by the metric product of $|\cdot - \cdot|^2$ and $\mathrm{d}_{X'}$, the Pythagorean theorem holds. Hence $i^2 + r^2 = 1$. It follows that

$$\cos \mathrm{d}_F(f,g) = ij + (1-i^2)^{\frac{1}{2}}(1-j^2)^{\frac{1}{2}} \cos \mathrm{d}_{F'}(f',g').$$

There are unique numbers $\theta, \varphi \in [0,\pi]$ such that $i = \cos\theta$ and $j = \cos\varphi$. Thus, there is an isometry between (F, d_F) and the metric 1-cone with respect to F'. In particular, F is a topological suspension in the sense of Ohta's topological splitting result in [61] and the measure has the form $d\,\mathrm{m}_F = \sin^N d\,\mathrm{m}_{F'}$ for some Borel measure $\mathrm{m}_{F'}$ on F'. Hence, F is a (K,N)-cone over $(F', \mathrm{d}_{F'}, \mathrm{m}_{F'})$. Finally, Theorem C yields the result. □

Corollary 5.5.1. *Let $(F, \mathrm{d}_F, \mathrm{m}_F)$ be a metric measure space that satisfies $RCD^*(N-1, N)$ for $N \geq 0$. Assume there are points $x_i, y_i \in F$ for $i = 1, \ldots, n$ with $n > N$ such that $\mathrm{d}_F(x_i, y_i) = \pi$ for any i and $\mathrm{d}_F(x_i, x_j) = \frac{\pi}{2}$ for $i \neq j$. Then $N = n - 1 \in \mathbb{N}$ and $(F, \mathrm{d}_F, \mathrm{m}_F) = \mathbb{S}^N$.*

Proof. First, we consider $x_0, y_0 \in F$ with $\mathrm{d}_F(x_0, y_0) = \pi$. The maximal diameter theorem implies that F is a spherical suspension with respect to some metric measure spaces F_0' that satisfies $RCD(N-2, N-1)$ where the pair (x_0, y_0) corresponds to the two origins of $I_K \times^N_{\sin_K} F'$. If we consider another pair (x_1, y_1), we obtain another suspension structure. Hence, we find a loop $s : [0, 2\pi]/_{\{0 \sim 2\pi\}} \to S$ in F that is geodesic for small distances and intersects with F' at x_1 since $\mathrm{d}_F(x_0, x_1) = \frac{\pi}{2}$. But this also implies that $\mathrm{d}_F(y_0, y_1) = \frac{\pi}{2}$ and $y_1 \in F'$. Since F' embeds geodesically into F, we have $\mathrm{d}_{F'}(x_1, y_1) = \pi$

Then, we also obtain for any other pair $x_i, y_i \in F$ for $i \geq 1$ that $x_i, y_i \in F'$, $\mathrm{d}_{F'}(x_i, y_i) = \pi$ and $\mathrm{d}_{F'}(x_i, x_j) = \frac{\pi}{2}$ for $i \neq j$. Hence, we can proceed by induction and the second part of the maximal diameter theorem tells us that that after finitely many steps no further decomposition is possible and $F = \mathbb{S}^k$ for some $k \in \mathbb{N}$. But then, $n - 1 = N = k$. □

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