

Two Models for the Homotopy Theory of Cocomplete Homotopy Theories

DISSERTATION

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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aus

Lublin

Bonn 2014

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

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 2. Gutachter: Prof. Dr. Peter Teichner
- Tag der Promotion: 21. Juli 2014
Erscheinungsjahr: 2014

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Introduction

The topic of this thesis is abstract homotopy theory, that is the theory of homotopy theories or, more precisely, the *homotopy theory of homotopy theories*. In order to explain what exactly we mean by a “homotopy theory” and the “homotopy theory of homotopy theories” we will give a brief overview of various approaches to abstract homotopy theory. They will be very roughly classified into two types: the classical ones in the spirit of Quillen’s *homotopical algebra*¹ and the modern ones in the spirit of *higher*

¹Usually, the phrase “homotopical algebra” is used to refer to Quillen model categories. Here, we extend its meaning to various related notions such as Brown’s categories of fibrant objects or Thomason model categories.

category theory. The distinction between these types is rather informal, but it reflects practical differences in their actual usage. The main result of this thesis is a precise comparison between certain notions of homotopical algebra and higher category theory.

Homotopical algebra: classical models of homotopy theories

In the past 50 years many different approaches to abstract homotopy theory have been introduced. Perhaps surprisingly, the first such approach, the theory of *model categories*, remains one of the most intricate ones to the present day. Model categories were introduced by Quillen [Qui67]. He defined a model category as a category equipped with three classes of morphisms: weak equivalences, cofibrations and fibrations subject to certain conditions that axiomatized well-known methods of algebraic topology and put them into an abstract framework. This framework proved to be very powerful and widely applicable and today it constitutes one of the main tool-sets of homotopy theory. An important feature of the theory of model categories is that it allows for comparisons between different homotopy theories via the notion of a *Quillen adjoint pair*. A typical example of a problem that can be solved using model categories is that classical colimits are usually not homotopy invariant and hence they have to be replaced by better behaved *homotopy colimits*. If \mathcal{M} is a model category and J is a small category and we can find a model structure on the category of diagrams \mathcal{M}^J such that the colimit functor $\text{colim}_J: \mathcal{M}^J \rightarrow \mathcal{M}$ is a left Quillen functor (i.e. the left part of a Quillen adjoint pair), then we can define the associated homotopy colimit functor as the *left derived functor* of colim_J . Dually, homotopy limit functors can be defined as the right derived functors of classical limit functors. This is achieved by replacing ill-behaved diagrams by better ones, i.e. their (co)fibrant replacements, before applying (co)limit functors. Contributions to the theory of model categories made by various authors are far too numerous to be listed here. Let us just recommend [Hir03], [Hov99] and [Joy08, Appendix E] as general references.

Even though model categories are very versatile it was not long until mathematicians realized that not every theory with homotopical content fits easily into this framework. K. Brown [Bro73] was the first to propose an alternative approach, namely *categories of fibrant objects* (which will be referred to as *fibration categories* in this thesis). Brown observed that the abstract notions of cofibrations and fibrations remain to be useful under a weaker axiomatization than the one used to define model categories.² A fibration category is a category equipped with two classes of morphisms: weak equivalences and fibrations subject to conditions that follow from the axioms of a model category but are, in fact, satisfied by a larger class of examples as discussed in Section 1.4. There is, of course, the dual theory of *cofibration categories* and this is the notion that we will concentrate on throughout most of this thesis. Moreover, so called *exact functors* are a counterpart to Quillen functors and it is still possible to construct homotopy colimit functors as left derived functors in the case of cofibration categories and dually for fibration categories. (The construction is similar to but not quite the same as for model

²Brown's motivating example was the homotopy theory of sheaves of spectra. A model category presenting this homotopy theory was eventually constructed in [Jar87].

categories as explained in Sections 1.3 and 3.3.) Cofibration and fibration categories never became nearly as popular as model categories, but since they were first introduced a number of contributions has been made by, among the others, Anderson [And78], Baues [Bau89, Bau99], Cisinski [Cis10] and Rădulescu-Banu [RB06]. Moreover, Waldhausen [Wal85] introduced a closely related notion of a *category with cofibrations and weak equivalences* (nowadays usually called a *Waldhausen category*) for the purpose of developing a general framework for algebraic K-theory. Subsequently, a close connection to abstract homotopy theory was made by Cisinski [Cis10b].

It is also worth pointing out that more approaches in a similar spirit are possible. For example, in 1995 Thomason [Wei01] introduced a modification of the notion of a model category that addressed certain technical shortcomings³ of Quillen’s original axioms.

While abstract homotopy theory in the spirit of Quillen’s homotopical algebra was being developed throughout the years, an important conceptual progress has been made by realizing that in model categories (and other similar structures) all the homotopical information is contained in the class of weak equivalences and the remaining structure plays only an auxiliary role. A *relative category* is a category equipped with a class of morphisms, called weak equivalences, subject to no special conditions other than being closed under composition and containing all the identities. The first important contribution to the theory of relative categories was made by Gabriel and Zisman [GZ67] who introduced a useful method of constructing the homotopy category of a (nice enough) relative category called the *calculus of fractions*. This method is an important motivation for the central construction of this thesis as explained on p. 51. Later, Dwyer and Kan [DK80, DK80b, DK80c] defined the *simplicial localization* of an arbitrary relative category \mathcal{C} , i.e. certain simplicial category LC that enhances the homotopy category of \mathcal{C} in the sense that $\pi_0 LC \cong \text{Ho}\mathcal{C}$. They also verified that if \mathcal{C} carries a model structure, then the mapping spaces obtained this way are weakly equivalent to the mapping spaces coming from the model structure via so called *framings*. Thus they have indeed demonstrated that all the homotopical content of a model category is contained in its weak equivalences. This statement was made into a sharp result (that will be later stated more precisely) by Barwick and Kan [BK12]. Morphisms of relative categories are *relative functors*, i.e. functors that preserve weak equivalences, but this formalism is not structured enough to yield a reasonable theory of derived functors. However, *homotopical categories* were introduced in [DHKS04] as relative categories satisfying the “2 out of 6 property” where it was observed that they are much better behaved than general relative categories. In fact, it is possible to use homotopical categories as an abstract framework for derived functors, but constructing derived functors still requires using richer structures of homotopical algebra.

³This shortcoming is that it is not known in general how to construct a model structure on the category of diagrams in a model category. (Co)fibration categories also alleviate this problem to some extent as discussed in Section 1.3.

Higher category theory: modern models of homotopy theories

Since Quillen introduced homotopical algebra, a completely new approach to abstract homotopy theory has been invented coming from higher category theory. It would be unrealistic to adequately summarize the history of higher category theory here. We will only briefly mention the aspects most relevant to the topic at hand. A broader historical perspective can be found in [Sim12, Chapter 1] and concise mathematical overviews in [Ber10] and [Por04].

Informally speaking, a *higher category* is a category-like structure that, in addition to objects and morphisms between them, has 2-morphisms between morphisms, 3-morphisms between 2-morphisms etc., possibly ad infinitum. Moreover, these higher morphisms are equipped with composition operations which are associative but only in a weak sense, i.e. up to natural equivalences specified by higher morphisms. Making this casual description into a precise definition is a big challenge which is still not resolved in full generality.

Fortunately, in abstract homotopy theory we are not forced to consider arbitrary higher categories but only so called $(\infty, 1)$ -categories, i.e. the ones where all morphisms above dimension 1 are weakly invertible. Such structures can serve as models of homotopy theories where we think of objects as homotopy types in a given homotopy theory, morphisms as maps of these homotopy types, 2-morphisms as homotopies between maps and higher morphisms as higher homotopies. One of the most important reasons why it should be fruitful to think of homotopy theories in terms of higher category theory is that it should provide a good framework for stating universal properties of various homotopy theoretic constructions (e.g. homotopy colimits) which are difficult to express in the language of homotopical algebra. A result of Barwick and Kan discussed in the next subsection demonstrates that $(\infty, 1)$ -categories indeed capture the classical notion of a homotopy theory. The problem of formalizing the notion of an $(\infty, 1)$ -category has been solved in multiple ways, we will mention a few of the most notable ones.

The best developed notion of an $(\infty, 1)$ -category (and the one used in this thesis) is that of a *quasicategory*. It was introduced by Boardman and Vogt in [BV73] under the name *simplicial set satisfying the restricted Kan condition*. The original purpose of this definition was to provide a good context for the treatment of homotopy coherent diagrams as was done by Vogt [Vog73] and Cordier and Porter [Cor82, CP86, CP97]. However, it took quite a long time before the full potential of quasicategories was realized mostly by Joyal and Lurie in the work culminating in [Joy08] and [Lur09]. In Chapter 2 we give a brief treatment of the basic theory of quasicategories. One of the crucial advantages of quasicategories is that they make it easy to state universal properties of homotopy colimits. Informally, a homotopy colimit of a diagram in an $(\infty, 1)$ -category should be given as a *universal cone*, i.e. a cone such that the *mapping space* into any other cone is contractible. Using quasicategories, this definition can be formalized in a practical way as explained in Section 2.2.

Another early definition of $(\infty, 1)$ -categories was via *simplicially enriched categories* (or *simplicial categories*) although it was not initially presented as such. Simplicial categories were considered by Dwyer and Kan [DK80, DK80b] as a part of their work on

simplicial localization mentioned above, but it was not until much later when Bergner [Ber07] established simplicial categories as models of $(\infty, 1)$ -categories. This may seem rather surprising at the first glance since simplicial categories come with strict composition operations. However, as it turns out, when seen from the correct homotopical perspective these strict composition operations already represent all possible “weak composition operations”. A drawback of this approach is that, unlike quasicategories, simplicial categories make it difficult to express universal properties of homotopy colimits and other homotopy theoretic constructions. In fact, such difficulties could be seen as motivations for the development of the theory of homotopy coherent diagrams using quasicategories cited in the previous paragraph.

As an attempt to rectify the problem of composition operations of simplicial categories being too strict, Dwyer, Kan and Smith [DKS89] introduced *Segal categories* (but they did not give them a name). Roughly speaking, a Segal category is a category “weakly enriched” in simplicial sets. The theory of Segal categories and their generalizations was developed extensively by Hirschowitz and Simpson [HS01]. A comprehensive exposition can be found in [Sim12].

Segal categories are more flexible than simplicial categories. However, they are not quite as flexible as one could hope and the difficulties can be traced to the fact that the underlying ∞ -groupoid of an $(\infty, 1)$ -category is not easily accessible from its presentation as a Segal category. A modified approach has been proposed by Rezk [Rez01] who defined *complete Segal spaces* where the underlying ∞ -groupoid is explicitly built into the structure of an $(\infty, 1)$ -category. The theory of complete Segal spaces has various advantages, e.g. it is presented by a model category (see the next subsection) with unusually good properties compared to other models. It is also suitable for internalizing into homotopy theories other than the homotopy theory of spaces.

The original problem of the lack of a precise mathematical definition of an $(\infty, 1)$ -category has been replaced by the problem of having too many such definitions all of which look equally reasonable. However, the multitude of notions of higher categories is not really a problem since they have different advantages. Simplicial categories and Segal categories serve as sources of examples which may not be easy to construct directly as quasicategories or complete Segal spaces which in turn provide good contexts for carrying out higher categorical arguments.

The homotopy theory of homotopy theories

We have argued that the abundance of notions of $(\infty, 1)$ -categories can be helpful provided that we can properly address the question of comparison between various definitions. As it turns out, abstract homotopy theory itself provides a framework for such comparisons. The homotopy theories of each of the four types of $(\infty, 1)$ -categories discussed above have been described as model categories. (Which typically means that these models have been exhibited as fibrant objects of a model category.) This was done by Joyal for quasicategories [Joy08], by Bergner for simplicial categories [Ber07], by Hirschowitz and Simpson for Segal categories [HS01] and by Rezk for complete Segal spaces [Rez01]. It was subsequently proven that all these model categories are *Quillen*

equivalent, i.e. that they present the same homotopy theory which we call the *homotopy theory of $(\infty, 1)$ -categories*. Quillen equivalences between simplicial categories, Segal categories and complete Segal spaces were established by Bergner [Ber07b]. Moreover, Joyal and Tierney [JT07] constructed a Quillen equivalence (two different ones, in fact) between quasicategories and complete Segal spaces.

Since we introduced $(\infty, 1)$ -categories as models of homotopy theories, this leads us to consider the “homotopy theory of homotopy theories”. However, even though we already know that various definitions of an $(\infty, 1)$ -category encode the same notion of a homotopy theory, the two occurrences of “homotopy theory” in the phrase above still have seemingly different meanings.

In order to address this issue we recall from the preceding discussion that the actual content of the model categories above depends on the notions of their weak equivalences and not on the model structures as such. This means that in order to talk about “homotopy theory of homotopy theories” we have to fix a notion of equivalence of homotopy theories. What Dwyer and Kan [DK80c] actually proved is that a Quillen functor between model categories is a Quillen equivalence if and only if it induces an equivalence of their homotopy categories and weak homotopy equivalences of the mapping spaces in their simplicial localizations (i.e. it is a *Dwyer–Kan equivalence* in the modern language). By combining these observations we arrive at the conclusion that if we want to think of model categories or relative categories as homotopy theories they always have to be accompanied by the notions of Quillen equivalences or Dwyer–Kan equivalences. (Similarly, we will define weak equivalences of cofibration categories in Chapter 1.)

This means that there is a way of giving the same meaning to both occurrences of “homotopy theory” in the phrase “homotopy theory of homotopy theories”, namely, by interpreting it as the “relative category of relative categories” with Dwyer–Kan equivalences as weak equivalences. Moreover, it is now a well posed question whether this notion of homotopy theory is equivalent to the higher categorical ones. Namely, we can ask whether the underlying relative category of any of the four model categories above is Dwyer–Kan equivalent to the relative category of relative categories. This is indeed true by the result of Barwick and Kan [BK12, BK12b]. More precisely, they constructed a model structure on the category of relative categories and proved that it is Quillen equivalent to the Rezk model structure for complete Segal spaces.

All these considerations suggest that we should be able to talk about the “ $(\infty, 1)$ -category of $(\infty, 1)$ -categories” as an alternative to the “homotopy theory of homotopy theories”. This is indeed possible and leads to a very interesting result that the “ $(\infty, 1)$ -category of $(\infty, 1)$ -categories” can be characterized axiomatically. This was first done by Toën [Toë05] in the language of homotopical algebra. Namely, he gave sufficient conditions for a model category to be Quillen equivalent to the Rezk model category for complete Segal spaces. Later, Barwick and Schommer-Pries [BSP13] formulated an alternative axiomatization purely in the language of higher category theory. (In fact, their theory applies to (∞, n) -categories, i.e. the ones where morphisms are only required to be weakly invertible above a fixed finite dimension n .)

New results

Just as different notions of $(\infty, 1)$ -categories have different advantages, higher category theory as such has different advantages than homotopical algebra. A good exemplification of these differences is the way both theories approach homotopy invariant constructions such as homotopy colimits. In higher category theory we define them via universal properties, but such definitions do not address the problem of actually constructing homotopy colimits and it seems that every proof of cocompleteness of an $(\infty, 1)$ -category reduces in one way or another to homotopical algebra. On the other hand, while homotopical algebra provides useful tools for explicit constructions of homotopy colimits, it makes it next to impossible to talk about their universal properties. Thus both approaches play important and complementary roles in abstract homotopy theory.

The state of affairs presented above does not explain how homotopical algebra (which we can now understand as structured theory of relative categories) fits into the context of higher category theory. The purpose of this thesis is to solve this very problem.

It should be apparent that while general relative categories present a wide variety of homotopy theories (in fact all of them), model categories and cofibration categories only present some special homotopy theories, i.e. the ones having some specific properties (or perhaps equipped with some specific structure). One of the main results of this thesis is that the homotopy theories presented by cofibration categories are precisely the cocomplete ones. Similar remarks apply to morphisms of homotopy theories. As mentioned, each of the notions discussed above has associated with it a natural notion of a morphism: Quillen functors for model categories, exact functors for cofibration categories and relative functors for relative categories. Again, relative functors present arbitrary morphism of homotopy theories, but Quillen functors and exact functors are more special. In this thesis we prove that exact functors between cofibration categories correspond to homotopy colimit preserving morphisms of cocomplete homotopy theories.

It is important to realize that the comparison of homotopical algebra to higher category theory is an entire family of problems, one for each notion of homotopical algebra. That is because different notions will present different types of homotopy theories, e.g. in contrast to cofibration categories homotopy theories presented by model categories are both complete and cocomplete. This thesis addresses only the case of cofibration categories (and dually fibration categories) and does not seem to apply to model categories. However, our individual techniques are potentially useful even in the theory of model categories.

The main result is that the homotopy theory of cofibration categories is equivalent to the homotopy theory of cocomplete quasicategories. The examples of equivalences of homotopy theory discussed so far suggest that while model categories and Quillen equivalences do not carry more homotopical information than relative categories and Dwyer–Kan equivalences, it is usually much easier to exploit homotopical algebra to construct Quillen equivalences rather than construct Dwyer–Kan equivalences by hand. Unfortunately, the categories of cofibration categories and cocomplete quasicategories do not carry model structures (e.g. since they have no initial objects). We will circumvent this problem by showing that they are both fibration categories.

In Chapter 1 we introduce cofibration categories and summarize the well known techniques of homotopical algebra that will be use throughout this thesis. We introduce morphisms and weak equivalences of cofibration categories which specifies the homotopy theory of cofibration categories. Then we define fibrations of cofibration categories and prove that they make the category of (small) cofibration categories into a fibration category. Finally, we discuss some basic techniques of constructing fibrations and weak equivalences of cofibration categories and we mention some examples which demonstrate versatility of this approach to homotopical algebra.

Chapter 2 contains the basic theory of quasicategories which is mostly cited from [Joy08] and [DS11]. In particular, we establish fibration categories of quasicategories and of cocomplete quasicategories. This chapter contains no new results, except possibly for the existence of the latter fibration category. (The completeness of the homotopy theory of cocomplete quasicategories is discussed in [Lur09], but it is not stated in terms of fibration categories.)

We start Chapter 3 by constructing a functor from cofibration categories to cocomplete quasicategories. To each cofibration category \mathcal{C} we associate a nerve-like simplicial set denoted by $N_f \mathcal{C}$ and called the *quasicategory of frames in \mathcal{C}* . (The letter f in N_f stands either for *frames* since those are the objects in $N_f \mathcal{C}$ or for *fractions* since the morphisms in $N_f \mathcal{C}$ are certain generalizations of left fractions.) The first step in the proof of the main theorem is to show that N_f is an exact functor between the fibration categories mentioned above. (And in particular that it takes values in cocomplete quasicategories since it is not apparent from the definition.) This proof is somewhat involved and occupies the entire Chapter 3.

The second step, presented in Chapter 4, is to prove that N_f is a weak equivalence of fibration categories. To this end we associate with every cocomplete quasicategory \mathcal{D} a cofibration category $Dg \mathcal{D}$ called the *category of diagrams in \mathcal{D}* . This yields a functor Dg which is not exact but is an inverse to N_f up to weak equivalence. This suffices to conclude that N_f is an equivalence of homotopy theories.

We should explain that parts of the arguments outlined above depend on certain set theoretic assumptions. Most of the results are parametrized by a regular cardinal number κ and concern small κ -cocomplete cofibration categories and small κ -cocomplete quasicategories, i.e. the ones admitting κ -small (homotopy) colimits. We will suppress this parameter as much as possible, but there are situations where referring to it is unavoidable. In the first two and a half chapters we set $\kappa = \aleph_0$, i.e. we consider finitely cocomplete homotopy theories. This is done merely to simplify the exposition, the arguments for $\kappa > \aleph_0$ require only minor modifications which are explained in Section 3.3. However, from this point on the distinction between these two cases starts playing a significant role. As it turns out, the case of $\kappa > \aleph_0$ is much easier for technical reasons discussed in the beginning of Section 3.5. The rest of Chapter 3 is split into Section 3.4 which deals with $\kappa > \aleph_0$ and Section 3.5 which deals with $\kappa = \aleph_0$. Similarly, the main part of Chapter 4 is split into Section 4.2 which deals with $\kappa > \aleph_0$ and Section 4.3 which deals with $\kappa = \aleph_0$. The reader is encouraged to read the arguments for $\kappa > \aleph_0$ first.

We work only with small cofibration categories and quasicategories and do not explicitly mention Grothendieck universes, but it is easy to interpret all the results in any

higher universe of interest. It suffices to fix a Grothendieck universe \mathcal{U} with $\kappa \in \mathcal{U}$ and substitute “ \mathcal{U} -small” for “small”. The only non- \mathcal{U} -small categories under consideration are the categories of \mathcal{U} -small κ -cocomplete cofibration categories, of \mathcal{U} -small quasicategories and of \mathcal{U} -small κ -cocomplete quasicategories. They can be taken to be \mathcal{V} -small for some larger universe \mathcal{V} if desirable.

All the results of this thesis dualize to results about fibration categories and complete quasicategories. Dualizing them formally may lead to a convoluted notation, in Section A.2 we suggest some modifications that should make such a translation easier.

Acknowledgments

This thesis was written while I was a doctoral student in *Bonn International Graduate School in Mathematics* and, more specifically, *Graduiertenkolleg 1150 “Homotopy and Cohomology”* and *International Max Planck Research School for Moduli Spaces*. I want to thank everyone involved for creating an excellent working environment. Moreover, parts of this research were carried out while I was participating in the *Algebraic Topology* program at *Mathematical Sciences Research Institute* in Berkeley.

I want to thank Clark Barwick, Bill Dwyer, André Joyal, Chris Kapulkin, Lennart Meier, Thomas Nikolaus, Chris Schommer-Pries, Peter Teichner and Marek Zawadowski for conversations on various topics which were very beneficial to my research.

I am especially grateful to Viktoriya Ozornova and Irakli Patchkoria for reading an early draft of this thesis. Their feedback helped me make many improvements and avoid numerous errors.

Above all, I want to express my gratitude to my supervisor Stefan Schwede whose expertise was always invaluable and without whose support this thesis could not have been written.

1. Cofibration categories

We start this chapter by introducing cofibration categories. The definition stated here is almost the same as (the dual of) Brown’s original definition [Bro73, p. 420]. (What he called *categories of fibrant objects* we call fibration categories.) There are two small differences that will be explained below. We do not commit much space to the discussion of basic properties of cofibration categories, we refer the reader to [RB06] for these. Instead, the purpose of this chapter is to establish the homotopy theory of cofibration categories in the form of a fibration category. This means that we will consider the category of cofibration categories with exact functors as morphisms and we will define weak equivalences and fibrations in this category and verify that they satisfy the duals of the axioms given below.

The readers might wonder why we choose to work with cofibration categories and not fibration categories and why we organize them into a fibration category and not a cofibration category. The first choice is not crucial and all the results readily dualize to fibration categories. We choose to work with cofibration categories since one of our key constructions (see Section 4.1) looks more natural and is easier to motivate than its formal dual for fibration categories. On the other hand, the category of cofibration categories (or fibration categories) does not carry a structure of a cofibration category itself since it has no initial object. Therefore, we are forced to work either with the fibration category of fibration categories or the fibration category of cofibration categories and we choose the latter for convenience.

1.1. Definitions and basic properties

Definition 1.1. A *cofibration category* is a category \mathcal{C} equipped with two subcategories: the subcategory of *weak equivalences* (denoted by $\xrightarrow{\sim}$) and the subcategory of *cofibrations* (denoted by \twoheadrightarrow) such that the following axioms are satisfied. (Here, an *acyclic cofibration* is a morphism that is both a weak equivalence and a cofibration.)

(C0) Weak equivalences satisfy the “2 out of 6” property, i.e. if

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

are morphisms of \mathcal{C} such that both gf and hg are weak equivalences, then so are f , g and h (and thus also hgf).

(C1) Every isomorphism of \mathcal{C} is an acyclic cofibration.

(C2) An initial object exists in \mathcal{C} .

- (C3) Every object X of \mathcal{C} is cofibrant, i.e. if 0 is the initial object of \mathcal{C} , then the unique morphism $0 \rightarrow X$ is a cofibration.
- (C4) Cofibrations are stable under pushouts along arbitrary morphisms of \mathcal{C} (in particular these pushouts exist in \mathcal{C}). Acyclic cofibrations are stable under pushouts along arbitrary morphisms of \mathcal{C} .
- (C5) Every morphism of \mathcal{C} factors as a composite of a cofibration followed by a weak equivalence.

As mentioned, these axioms differ a little from Brown’s formulation. First, the factorization axiom (C5) is stated in a more general form, but it actually follows from Brown’s version by Lemma 1.7. Second, we use the “2 out of 6” property instead of the “2 out of 3” property, but it is not clear whether it is actually stronger, i.e. whether there exists a category satisfying axioms (C1-5) and the “2 out of 3” property but not the “2 out of 6” property. Regardless, this assumption is not a severe one even if such examples exist. Cisinski proved that in any such example one can replace the subcategory of weak equivalences by its saturation (i.e. by the subcategory consisting of all morphisms that become invertible in the homotopy category, which satisfies the “2 out of 6” property) and the axioms above will be satisfied with the same cofibrations, see [Cis10, Proposition 3.16] or [RB06, Theorem 7.2.8(1)]. Moreover, no such example exists if we extend the axioms to include (C6) introduced in Section 3.3, see [Cis10, Proposition 6.2] or [RB06, Lemma 7.2.2 (1)]. The reason why (C0) is useful is that it already implies that weak equivalences are saturated, see Proposition 1.6.

Definitions of (co)fibration categories found throughout the literature vary in details. Since we use [RB06] as our main source we point out that in the terminology of this paper the definition above corresponds to “precofibration categories with all objects cofibrant and the “2 out of 6” property”. Comparisons to other definitions can be found in [RB06, Chapter 2].

The above axioms describe *finitely cocomplete* cofibration categories. Here, cocompleteness really means “homotopy cocompleteness” since cofibration categories do not necessarily have all finite strict colimits, but they have all finite homotopy colimits. Their construction will be discussed in Section 1.3. If we want to consider cofibration categories with more homotopy colimits we need to assume some extra axioms which will be discussed in Section 3.3.

Cofibration categories can be seen as generalizations of model categories. Namely, if \mathcal{M} is a model category, then its full subcategory of cofibrant objects \mathcal{M}_{cof} with weak equivalences and cofibrations inherited from \mathcal{M} satisfies the above axioms. Many of the standard tools of homotopical algebra (that do not refer to fibrations, e.g. left homotopies, cofiber sequences or homotopy colimits) depend only on these axioms and hence are available for cofibration categories, although they sometimes differ in technical details. These techniques are discussed in great detail in [RB06]. There are examples of (co)fibration categories that do not come from model categories. Some of those are presented in Section 1.4.

Before discussing new results about homotopy theory of cofibration categories, we collect some preliminaries, mostly following [RB06]. We fix a cofibration category \mathcal{C} .

Definition 1.2.

- (1) A *cylinder* of an object X is a factorization of the codiagonal morphism $X \amalg X \rightarrow X$ as $X \amalg X \rightarrow IX \xrightarrow{\sim} X$.
- (2) A *left homotopy* between morphisms $f, g: X \rightarrow Y$ via a cylinder $X \amalg X \rightarrow IX \xrightarrow{\sim} X$ is a commutative square of the form

$$\begin{array}{ccc} X \amalg X & \xrightarrow{[f, g]} & Y \\ \downarrow & & \downarrow \sim \\ IX & \longrightarrow & Z. \end{array}$$

- (3) Morphisms $f, g: X \rightarrow Y$ are *left homotopic* (notation: $f \simeq_l g$) if there exists a left homotopy between them via some cylinder on X .

The definition of left homotopies differs from the standard definition as usually given in the context of model categories. Namely, we allow an extra acyclic cofibration $Y \xrightarrow{\sim} Z$ while in the standard definition we would have $Y = Z$ (and the acyclic cofibration would be required to be the identity morphism). This modification is dictated by the lack of fibrant objects in cofibration categories and makes the definition well-behaved for arbitrary Y while the standard definition in a model category is only well-behaved for a fibrant Y .¹

We denote the homotopy category of \mathcal{C} (i.e. its localization with respect to weak equivalences) by $\text{Ho}\mathcal{C}$ and for a morphism f of \mathcal{C} we write $[f]$ for its image under the localization functor $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$. The homotopy category can be constructed in two steps: first dividing out left homotopies and then applying the calculus of fractions.

Proposition 1.3. *The relation of left homotopy is a congruence on \mathcal{C} . Moreover, every morphism of \mathcal{C} that becomes an isomorphism in \mathcal{C} / \simeq_l is a weak equivalence. Thus left homotopic morphisms become equal in $\text{Ho}\mathcal{C}$ and \mathcal{C} / \simeq_l comes equipped with a canonical functor $\mathcal{C} / \simeq_l \rightarrow \text{Ho}\mathcal{C}$.*

Proof. The first statement is [RB06, Theorem 6.3.3(1)]. The remaining ones follow by straightforward “2 out of 3” arguments. □

The next theorem is a crucial tool in the theory of cofibration categories and can be used to verify many of their fundamental properties. It says that up to left homotopy

¹In fact, it is sometimes possible to introduce fibrant objects to a cofibration category even if general fibrations are not available, see [RB06, Definition 7.3.1]. However, this would not get along with some of the techniques we use later on. Namely, *cofibration categories of diagrams* constructed in Section 4.1 typically don’t have “enough fibrant objects”.

all cofibration categories satisfy the left calculus of fractions in the sense of Gabriel and Zisman [GZ67, Chapter I]. This fact was first proven by Brown [Bro73, Proposition I.2] and can be seen as an abstraction of the classical construction of the derived category of a ring, see e.g. [GM96, Theorem III.4.4]. In general, constructing $\mathrm{Ho}\mathcal{C}$ may involve using arbitrarily long zig-zags of morphisms in $\mathrm{Ho}\mathcal{C}$ and identifying them via arbitrarily long chains of relations. However, the previous proposition implies that $\mathcal{C}/\simeq_l \rightarrow \mathrm{Ho}\mathcal{C}$ is also a localization functor and in that case Theorem 1.4 says that it suffices to consider two-step zig-zags (called *left fractions*) up to a much simplified equivalence relation. Our main construction, i.e. the *quasicategory of frames*, can be seen as an enhancement of the calculus of fractions as discussed on p. 51.

Theorem 1.4. *A cofibration category \mathcal{C} satisfies the left calculus of fractions up to left homotopy, i.e.*

- (1) *Every morphism $\varphi \in \mathrm{Ho}\mathcal{C}(X, Y)$ can be written as a left fraction $[s]^{-1}[f]$ where $f: X \rightarrow \tilde{Y}$ and $s: Y \xrightarrow{\sim} \tilde{Y}$ are morphisms of \mathcal{C} .*
- (2) *Two fractions $[s]^{-1}[f]$ and $[t]^{-1}[g]$ are equal in $\mathrm{Ho}\mathcal{C}(X, Y)$ if and only if there exist weak equivalences u and v such that*

$$us \simeq_l vt \text{ and } uf \simeq_l vg.$$

- (3) *If $\varphi \in \mathrm{Ho}\mathcal{C}(X, Y)$ and $\psi \in \mathrm{Ho}\mathcal{C}(Y, Z)$ can be written as $[s]^{-1}[f]$ and $[t]^{-1}[g]$ respectively and a square*

$$\begin{array}{ccc} Y & \xrightarrow{g} & \tilde{Z} \\ s \downarrow \sim & & u \downarrow \sim \\ \tilde{Y} & \xrightarrow{h} & \tilde{Z} \end{array}$$

commutes up to homotopy, then $\psi\varphi$ can be written as $[ut]^{-1}[hf]$.

Proof. Parts (1) and (2) follow from [RB06, Theorem 6.4.4(1)] and (3) from the proof of [RB06, Theorem 6.4.1]. \square

We will need the following two consequences of the calculus of fractions.

Corollary 1.5.

- (1) *Every two morphisms $\varphi, \psi \in \mathrm{Ho}\mathcal{C}(X, Y)$ can be written as fractions $\varphi = [s]^{-1}[f]$ and $\psi = [s]^{-1}[g]$ where s is an acyclic cofibration.*
- (2) *For any two morphisms $f, g: X \rightarrow Y$ of \mathcal{C} the following are equivalent:*
 - $[f] = [g]$,
 - f and g are left homotopic,

- there is a left homotopy from f to g via a fixed cylinder on X .

Proof. For part (1) write $\varphi = [\tilde{s}]^{-1}[\tilde{f}]$ and $\psi = [\tilde{t}]^{-1}[\tilde{g}]$ with

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}} & V \xleftarrow[\sim]{\tilde{s}} Y \\ X & \xrightarrow{\tilde{g}} & W \xleftarrow[\sim]{\tilde{t}} Y \end{array}$$

(we can assume that \tilde{s} and \tilde{t} are acyclic cofibrations by [RB06, Theorem 6.4.5(1a)]) and form a pushout

$$\begin{array}{ccc} Y & \xrightarrow[\sim]{\tilde{s}} & V \\ \tilde{t} \downarrow \sim & & \sim \downarrow u \\ W & \xrightarrow[\sim]{\tilde{g}} & Z \\ & & \downarrow v \end{array}$$

The conclusion follows if we set $f = u\tilde{f}$, $g = v\tilde{g}$ and $s = u\tilde{s} = v\tilde{t}$.

Part (2) follows by [RB06, Lemma 6.3.2(1) and Theorem 6.4.4(1c)]. \square

Here is an important application of the calculus of fractions (which relies on the “2 out of 6” property).

Proposition 1.6. *Every cofibration category \mathcal{C} is saturated, i.e. a morphism of \mathcal{C} is a weak equivalence if and only if it becomes an isomorphism in $\text{Ho}\mathcal{C}$.*

Proof. [RB06, Theorem 7.2.7] \square

Cylinders can also be used to construct factorizations in cofibration categories. Let $f: A \rightarrow B$ be a morphism and IA any cylinder on A . Form a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ IA & \longrightarrow & Mf \end{array}$$

and call the resulting object Mf a *mapping cylinder* of f . Then standard techniques can be used to construct a factorization of f via its mapping cylinder and even to prove the following lemma.

Lemma 1.7. *Let \mathcal{C} be a category equipped with subcategories of weak equivalences and cofibrations that satisfies all the axioms of a cofibration category except the factorization axiom. If \mathcal{C} has cylinders for all objects, then it satisfies the factorization axiom.*

Proof. [Bro73, Factorization lemma, p. 421] \square

In order to define the homotopy theory of cofibration categories we first need a good notion of a morphism between cofibration categories. We will use *exact* functors which

(according to the definition and the lemma below) are essentially homotopy invariant functors that preserve basic finite homotopy colimits, i.e. initial objects and homotopy pushouts. It will follow from the discussion in Section 1.3 that they actually preserve all finite homotopy colimits.

Definition 1.8. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between cofibration categories is *exact* if it preserves cofibrations, acyclic cofibrations, initial objects and pushouts along cofibrations.

Finally, we recall a standard method of verifying homotopy invariance of functors between cofibration categories.

Lemma 1.9 (K. Brown’s Lemma). *If a functor between cofibration categories sends acyclic cofibrations to weak equivalences, then it preserves all weak equivalences. In particular, exact functors preserve weak equivalences.*

Proof. The proof of [Hov99, Lemma 1.1.12] works for cofibration categories. (See also the proof of [Bro73, Lemma 4.1] where this result first appeared.) \square

1.2. Homotopy theory of cofibration categories

We are now ready to introduce the homotopy theory of cofibration categories. For this it is sufficient to define a class of weak equivalences in the category of cofibration categories which is what we will do next. Later, we will proceed to define fibrations of cofibration categories and prove that they satisfy the axioms of a fibration category which will give us a solid grasp of the homotopy theory of cofibration categories.

Definition 1.10. An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* if it induces an equivalence $\mathrm{Ho}\mathcal{C} \rightarrow \mathrm{Ho}\mathcal{D}$.

This notion is closely related to the *Waldhausen approximation properties* first formulated by Waldhausen as criteria for an exact functor to induce an equivalence of the algebraic K-theory spaces [Wal85, Section 1.6]. Later, Cisinski showed that an exact functor satisfies (slightly reformulated) Waldhausen approximation properties if and only if it is a weak equivalence in the sense of the definition above. This result is an important step in the construction of the fibration category of cofibration categories so we state it as Proposition 1.11 with a full proof. A version of this proposition is given in [RB06, Theorems 5.5.1 and 6.1.5(1)].

It is far from obvious that weak equivalences preserve homotopy types of homotopy mapping spaces. This is indeed true by a theorem of Cisinski [Cis10b, Théorème 3.25] which states that a weak equivalence induces an equivalence of the *hammock localizations* in the sense of Dwyer and Kan [DK80b]. While this result will not be used in this thesis, it justifies our choice of weak equivalences of cofibration categories. In fact, our main result implies that they correspond to categorical equivalences of quasicategories and with some additional effort this could be used to rederive Cisinski’s theorem.

Proposition 1.11 ([Cis10, Théorème 3.19]). *An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a weak equivalence if and only if it satisfies the following properties.*

(App1) F reflects weak equivalences.

(App2) Given a morphism $f: FA \rightarrow Y$ in \mathcal{D} , there exists a morphism $i: A \rightarrow B$ in \mathcal{C} and a commutative diagram

$$\begin{array}{ccc} FA & \xrightarrow{f} & Y \\ Fi \downarrow & & \downarrow \sim \\ FB & \xrightarrow{\sim} & Z \end{array}$$

in \mathcal{D} .

Proof. First, assume that F satisfies the two properties above.

We check that $\text{Ho } F$ is essentially surjective. Let X be an object of \mathcal{D} . Apply (App2) to $0 \rightarrow X$ to get an object B of \mathcal{C} and a diagram

$$\begin{array}{ccc} F0 & \longrightarrow & X \\ \downarrow & & \downarrow \sim \\ FB & \xrightarrow{\sim} & Y. \end{array}$$

It follows that $X \cong Y \cong FB$ in $\text{Ho } \mathcal{D}$.

Next, we verify that $\text{Ho } F$ is full. Let $\varphi: FA \rightarrow FB$ be a morphism of $\text{Ho } \mathcal{D}$. It can be written as a fraction $\varphi = [s]^{-1}[f]$ where $f: FA \rightarrow Z$ and $s: FB \xrightarrow{\sim} Z$ are morphisms of \mathcal{D} by Theorem 1.4(1). Let $p: F(A \amalg B) \rightarrow Z$ be a composite of $[f, s]$ and the canonical isomorphism $F(A \amalg B) \rightarrow FA \amalg FB$. Apply (App2) to p to obtain morphisms $a: A \rightarrow D$, $b: B \rightarrow D$ and a commutative square

$$\begin{array}{ccc} F(A \amalg B) & \xrightarrow{p} & Z \\ F[a, b] \downarrow & & \downarrow q \\ FD & \xrightarrow{u} & W \end{array}$$

with q and u weak equivalences. Then we have $qs = u(Fb)$, so Fb is a weak equivalence and thus so is b by (App1). Therefore

$$F([b]^{-1}[a]) = [Fb]^{-1}[Fa] = ([qs]^{-1}[u])([u]^{-1}[qf]) = [s]^{-1}[f] = \varphi.$$

It remains to verify that $\text{Ho } F$ is faithful. Let φ and ψ be morphisms $A \rightarrow B$ in $\text{Ho } \mathcal{C}$ such that $F\varphi = F\psi$. We can write them as fractions $\varphi = [s]^{-1}[f]$ and $\psi = [s]^{-1}[g]$ where $f, g: A \rightarrow C$ and $s: B \xrightarrow{\sim} C$ are morphisms of \mathcal{C} by Corollary 1.5(1). Then we have

$$[Ff] = [Fs][Fs]^{-1}[Ff] = [Fs]F([s]^{-1}[f]) = [Fs](F\varphi)$$

and symmetrically $[Fg] = [Fs](F\psi)$ and hence $[Ff] = [Fg]$. Let $A \amalg A \rightarrow IA \xrightarrow{\sim} A$ be a cylinder. Then $FA \amalg FA \rightarrow F(IA) \xrightarrow{\sim} FA$ is also a cylinder and it follows by Corollary 1.5(2) that there is a left homotopy from Ff to Fg via $F(IA)$, i.e. a square

$$\begin{array}{ccc}
FA \amalg FA & \xrightarrow{[Ff, Fg]} & FC \\
\downarrow & & \downarrow \sim \\
F(IA) & \longrightarrow & X.
\end{array}$$

We form a pushout

$$\begin{array}{ccc}
A \amalg A & \xrightarrow{[f, g]} & C \\
\downarrow & & \downarrow \\
IA & \longrightarrow & D,
\end{array}$$

which is preserved by F . Thus we get an induced morphism $FD \rightarrow X$, to which we apply (App2), obtaining a morphism $D \rightarrow E$ and a diagram

$$\begin{array}{ccc}
FD & \longrightarrow & X \\
\downarrow & & \downarrow \sim \\
FE & \xrightarrow{\sim} & Y.
\end{array}$$

It follows that the composite $FC \rightarrow FD \rightarrow FE$ is a weak equivalence and thus so is $C \rightarrow D \rightarrow E$ by (App1). So we have a diagram

$$\begin{array}{ccc}
A \amalg A & \xrightarrow{[f, g]} & C \\
\downarrow & & \downarrow \sim \\
IA & \longrightarrow & E
\end{array}$$

which proves that $[f] = [g]$. (Note that strictly speaking this square is not a left homotopy since $C \rightarrow E$ is not a cofibration, but this is not needed to conclude that $[f] = [g]$.)

Now let's assume that $\text{Ho } F$ is an equivalence. Since \mathcal{C} and \mathcal{D} are saturated by Proposition 1.6 and $\text{Ho } F$ reflects isomorphisms, it follows that F reflects weak equivalences.

Given a morphism $f: FA \rightarrow Y$, we can find an isomorphism $\varphi: Y \rightarrow FB$ in $\text{Ho } \mathcal{D}$ since $\text{Ho } F$ is essentially surjective. Since $\text{Ho } F$ is full there is a morphism $\gamma: A \rightarrow B$ in $\text{Ho } \mathcal{C}$ such that $F\gamma = \varphi[f]$. Write $\varphi = [q]^{-1}[p]$ with $p: Y \xrightarrow{\sim} Z$ and $q: FB \xrightarrow{\sim} Z$ and $\gamma = [r]^{-1}[g]$ with $g: A \rightarrow C$ and $r: B \xrightarrow{\sim} C$. Form a pushout

$$\begin{array}{ccc}
FB & \xrightarrow{q} & Z \\
Fr \downarrow & & \downarrow t \\
FC & \xrightarrow{s} & W.
\end{array}$$

All morphisms here are weak equivalences and thus we have

$$\begin{aligned} [tpf] &= [s][s]^{-1}[tpf] = [s][Fr][q]^{-1}[pf] = [s][Fr]\varphi[f] = [s][Fr](F\gamma) \\ &= [s][Fr][Fr]^{-1}[Fg] = [s(Fg)]. \end{aligned}$$

Choose a cylinder $A \amalg A \twoheadrightarrow IA \xrightarrow{\sim} A$. Then $FA \amalg FA \twoheadrightarrow F(IA) \xrightarrow{\sim} FA$ is also a cylinder and it follows by Corollary 1.5(2) that there is a left homotopy from tpf to $s(Fg)$ via $F(IA)$, i.e. a square

$$\begin{array}{ccc} FA \amalg FA & \xrightarrow{[tpf, s(Fg)]} & W \\ \downarrow & & \downarrow h \\ F(IA) & \xrightarrow{H} & V \end{array}$$

with h a weak equivalence. Factor g through its mapping cylinder via $k: A \rightarrow Mg$. Then FMg is a mapping cylinder of Fg and we have a commutative square

$$\begin{array}{ccc} FA & \xrightarrow{f} & Y \\ Fk \downarrow & & \downarrow htp \\ FMg & \xrightarrow{[H, hs]} & V, \end{array}$$

where htp is a weak equivalence since all h , t and p are and $[H, hs]$ is a weak equivalence since both hs and the canonical morphism $FC \rightarrow FMg$ are. \square

We are now ready to define fibrations of cofibration categories, but before doing so we briefly explain the duality between cofibration and fibration categories. A *fibration category* is a category \mathcal{F} equipped with subcategories of weak equivalences and *fibrations* such that \mathcal{F}^{op} is a cofibration category (where the fibrations of \mathcal{F} become the cofibrations of \mathcal{F}^{op}). Similarly, an exact functor of fibration categories is a functor that is exact as a functor of the corresponding cofibration categories. As usual, all the results about cofibration categories readily dualize to results about fibration categories. We do not state them separately, but we point out that all the statements in [RB06] are explicitly given in both versions.

Let CofCat denote the category of small cofibration categories and exact functors and FibCat the category of small fibration categories and exact functors. The duality can be stated as an observation that the functor $(-)^{\text{op}}: \text{CofCat} \rightarrow \text{FibCat}$ is an isomorphism of categories. Our goal is to give a structure of a fibration category of CofCat . Since $(-)^{\text{op}}$ is a covariant isomorphism, it is equivalent to giving a structure of a fibration category to FibCat .

Definition 1.12. Let $P: \mathcal{E} \rightarrow \mathcal{D}$ be an exact functor of cofibration categories.

- (1) P is an *isofibration* if for every object $A \in \mathcal{E}$ and an isomorphism $g: PA \rightarrow Y$ there is an isomorphism $f: A \rightarrow B$ such that $Pf = g$.

- (2) It is said to satisfy the *lifting property for factorizations* if for any morphism $f: A \rightarrow B$ of \mathcal{E} and a factorization

$$\begin{array}{ccc} PA & \xrightarrow{Pf} & PB \\ & \searrow j & \nearrow t \\ & & X \end{array}$$

there exists a factorization

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow i & \nearrow s \\ & & C \end{array}$$

such that $Pi = j$ and $Ps = t$ (in particular, $PC = X$).

- (3) It has the *lifting property for pseudofactorizations* if for any morphism $f: A \rightarrow B$ of \mathcal{E} and a diagram

$$\begin{array}{ccc} PA & \xrightarrow{Pf} & PB \\ \downarrow j & & \downarrow v \\ X & \xrightarrow[t]{} & Y \end{array}$$

there exists a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow i & & \downarrow u \\ C & \xrightarrow[s]{} & D \end{array}$$

such that $Pi = j$, $Ps = t$ and $Pu = v$ (in particular, $PC = X$ and $PD = Y$).

- (4) We say that P is a *fibration* if it is an isofibration and satisfies the lifting properties for factorizations and pseudofactorizations.

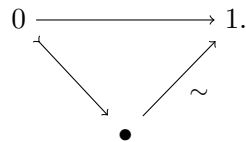
As is usual with constructions of fibration categories, most effort goes into verification of stability of (acyclic) fibrations under pullbacks and construction of factorizations. We begin with the former, in our situation even the existence of pullbacks is not obvious. Each of the above lifting properties is directly motivated by these problems. The first two are needed in Proposition 1.13 where we show that pullbacks along fibrations exist in CofCat . The fact that P is an isofibration implies that the resulting category has all

the required colimits and the lifting property for factorizations implies that it has factorizations. The lifting property for pseudofactorizations is only needed in Proposition 1.14 where we check that acyclic fibrations are stable under pullback.

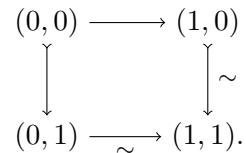
First, we restate the definition in a more technical but convenient way. We define a category $\overline{\text{CofCat}}$ containing the category of cofibration categories CofCat as a non-full subcategory. Objects of $\overline{\text{CofCat}}$ are small categories equipped with two subcategories: the subcategory of weak equivalences and the subcategory of cofibrations such that all identity morphisms are acyclic cofibrations. Morphisms are functors that preserve both weak equivalences and cofibrations.

The definition of fibrations can be rephrased in terms of lifting properties in $\overline{\text{CofCat}}$. Specifically, an exact functor between cofibration categories is a fibration if and only if it has the right lifting property, as a morphism of $\overline{\text{CofCat}}$, with respect to the following functors.

- The inclusion of $[0]$ into $E(1)$ (the groupoid freely generated by an isomorphism $0 \rightarrow 1$).
- The inclusion of $[1]$ (with only identities as weak equivalences or cofibrations) into



- The inclusion of $[1] \times [0]$ (with only identities as weak equivalences or cofibrations) into



A few of the proofs in the remainder of this chapter will use some technical lemmas about limits of cocomplete categories which are delegated to Section A.1 since they are purely categorical and not related directly to homotopy theory.

Let \sqsubset denote the poset of proper subsets of $\{0, 1\}$.

Proposition 1.13. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $P: \mathcal{E} \rightarrow \mathcal{D}$ be exact functors between cofibration categories with P a fibration. Then a pullback of P along F exists in $\overline{\text{CofCat}}$.*

Proof. Form a pullback of P along F in the category of categories.

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{G} & \mathcal{E} \\
Q \downarrow & & \downarrow P \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

Define a morphism f of \mathcal{P} to be a weak equivalence (respectively, a cofibration) if both Gf and Qf are weak equivalences (respectively, cofibrations). Then the above square becomes a pullback in $\overline{\text{CofCat}}$.

Now we check that \mathcal{P} is a cofibration category.

- (C0-1) In \mathcal{P} weak equivalences satisfy “2 out of 6” and all isomorphisms are acyclic cofibrations since this holds in both \mathcal{C} and \mathcal{E} .
- (C2-3) Let $0_{\mathcal{C}}$ be an initial object of \mathcal{C} . By Lemma A.4 there is an initial object $0_{\mathcal{E}}$ of \mathcal{E} such that $P0_{\mathcal{E}} = F0_{\mathcal{C}}$. Then $(0_{\mathcal{C}}, 0_{\mathcal{E}})$ is an initial object of \mathcal{P} by Lemma A.2. Moreover, every object of \mathcal{P} is cofibrant since this holds in both \mathcal{C} and \mathcal{E} .
- (C4) Let $X: \Gamma \rightarrow \mathcal{P}$ be a span with $X_{\emptyset} \rightarrow X_0$ a cofibration. Let S be a colimit of QX in \mathcal{C} , then FS is a colimit of $FQX = PGX$ in \mathcal{D} since F is exact. Lemma A.4 implies that we can choose a colimit T of GX in \mathcal{E} so that $PT = FS$. Then it follows by Lemma A.2 that (S, T) is a colimit of $X = (QX, GX)$ in \mathcal{P} . Thus pushouts along cofibrations exist in \mathcal{P} and both cofibrations and acyclic cofibrations are stable under pushouts since this holds in both \mathcal{C} and \mathcal{E} .
- (C5) Let $f: A \rightarrow B$ be a morphism of \mathcal{P} . Pick a factorization of Qf as

$$QA \twoheadrightarrow C \xrightarrow{\sim} QB$$

in \mathcal{C} . Then $FQf = PGf$ factors as

$$PGA = FQA \twoheadrightarrow FC \xrightarrow{\sim} FQB = PGB$$

and we can lift this factorization to a factorization of Gf as

$$GA \twoheadrightarrow E \xrightarrow{\sim} GB.$$

It follows that

$$A = (QA, GA) \twoheadrightarrow (C, E) \xrightarrow{\sim} (QB, GB) = B$$

is a factorization of f . This completes the verification that \mathcal{P} is a cofibration category.

Next, we need to verify that Q and G are exact. They preserve cofibrations and acyclic cofibrations by the definition of cofibrations and weak equivalences in \mathcal{P} . They also preserve initial objects and pushouts along cofibrations by the construction of these colimits in \mathcal{P} .

It remains to see that the square we constructed is a pullback in the category of cofibration categories, i.e. that given a square

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow P \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

of cofibration categories and exact functors, the induced functor $\mathcal{F} \rightarrow \mathcal{P}$ is also exact. Indeed, it was already observed that it preserves cofibrations and acyclic cofibrations. It also preserves initial objects and pushouts along cofibrations by Lemma A.2. \square

The next proposition will imply the stability of acyclic fibrations under pullbacks. Moreover, in later chapters it will serve as a useful criterion for verifying that an exact functor is a weak equivalence.

Proposition 1.14. *An exact functor $P: \mathcal{C} \rightarrow \mathcal{D}$ is an acyclic fibration if and only if it is a fibration, satisfies (App1) and the right lifting property (in $\overline{\text{CofCat}}$) with respect to the inclusion of $[0]$ into*

$$0 \hookrightarrow 1.$$

Proof. First assume that P satisfies the properties above. We need to check that it satisfies (App2). Let $f: PA \rightarrow Z$ be a morphism of \mathcal{D} . Factor f as a composite of $j: PA \twoheadrightarrow Y$ and $Y \xrightarrow{\sim} Z$ and apply the lifting property above to find a cofibration $i: A \twoheadrightarrow B$ such that $Pi = j$. This yields a diagram

$$\begin{array}{ccc} PA & \xrightarrow{f} & Z \\ Pi \downarrow & & \downarrow \text{id}_Z \\ PB & \xrightarrow{\sim} & Z. \end{array}$$

Conversely, assume that P is an acyclic fibration. We need to check that it satisfies the lifting property above. Consider a cofibration $j: PA \twoheadrightarrow Y$ and apply (App2) to it to get $f: A \rightarrow B$ and a diagram

$$\begin{array}{ccc} PA & \twoheadrightarrow & Y \\ Pf \downarrow & & \downarrow s \\ PB & \xrightarrow{t} & Z \end{array}$$

with both s and t weak equivalences. We factor $[t, s]: PB \amalg_{PA} Y \rightarrow Z$ as a composite of $[t', s']: PB \amalg_{PA} Y \twoheadrightarrow W$ and $W \xrightarrow{\sim} Z$. So we obtain the square on the right

$$\begin{array}{ccc}
PA & \xrightarrow{Pf} & PB \\
\downarrow j & & \downarrow t' \\
Y & \xrightarrow{s'} & W
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow i & & \downarrow v \\
C & \xrightarrow{u} & D
\end{array}$$

with both s' and t' weak equivalences. We can now apply the lifting property for pseudofactorizations to get the square on the left with u and v weak equivalences such that $Pu = s'$, $Pv = t'$ and (most importantly) $Pi = j$. \square

Next, we proceed to the construction of factorizations. This is the first of many situations where we need a way of keeping track of certain homotopical properties of diagrams in cofibration categories. *Homotopical categories* are very convenient for this purpose.

Definition 1.15. A *homotopical category* is a category equipped with a subcategory whose morphisms are called *weak equivalences* such that every identity morphism is a weak equivalence and the “2 out of 6” property holds.

As discussed in the introduction, homotopical categories are models of homotopy theories in their own right, but we will use them merely as a bookkeeping tool. A functor $I \rightarrow J$ between homotopical categories is *homotopical* if it preserves weak equivalences. In particular, for any cofibration category \mathcal{C} and a homotopical category J the homotopical functors $J \rightarrow \mathcal{C}$ will be called *homotopical diagrams*. The notation \mathcal{C}^J will always refer to the category of all homotopical diagrams $J \rightarrow \mathcal{C}$, it is itself a homotopical category with levelwise weak equivalences. If J is a plain category, then it will be considered as a homotopical category with the trivial homotopical structure, i.e. with only isomorphisms as weak equivalences. On the other hand, \widehat{J} will denote J equipped with the largest homotopical structure, i.e. the one where all morphisms are weak equivalences.

Let \mathcal{C} be a cofibration category and let $\text{Sd}[\widehat{1}]$ denote the poset of non-empty subsets of $\{0, 1\}$. Make it into a homotopical poset by declaring all morphisms to be weak equivalences. Call a diagram $X: \text{Sd}[\widehat{1}] \rightarrow \mathcal{C}$ cofibrant if both $X_0 \rightarrow X_{01}$ and $X_1 \rightarrow X_{01}$ are cofibrations in \mathcal{C} . Let PC denote the category of all homotopical cofibrant diagrams $\text{Sd}[\widehat{1}] \rightarrow \mathcal{C}$ (i.e. X such that both $X_0 \rightarrow X_{01}$ and $X_1 \rightarrow X_{01}$ are acyclic cofibrations). Define weak equivalences in PC as levelwise weak equivalences and define a morphism $A \rightarrow X$ to be a cofibration if all

$$\begin{aligned}
&A_0 \rightarrow X_0, \\
&A_1 \rightarrow X_1, \\
&A_{01} \amalg_{A_0} X_0 \rightarrow X_{01} \text{ and} \\
&A_{01} \amalg_{A_1} X_1 \rightarrow X_{01}
\end{aligned}$$

are cofibrations in \mathcal{C} . (Note that this implies that $A_{01} \rightarrow X_{01}$ is a cofibration too.)

The notation $\text{Sd}[\widehat{1}]$ is a special case of the notation that will be introduced later in Chapter 3, but then we will always consider Reedy cofibrant diagrams and not every

cofibrant object in the sense above is Reedy cofibrant. For a Reedy cofibrant object we would require $X_0 \amalg X_1 \rightarrow X_{01}$ to be a cofibration. Similarly, cofibrations above are more general than Reedy cofibrations. (See Definition 1.18 for the definition.) However, this notion reduces easily to the classical one, i.e. a morphism $A \rightarrow X$ is a cofibration in PC if and only if its restrictions along the two non-trivial inclusions $[1] \hookrightarrow \widehat{[1]}$ are Reedy cofibrations. The category PC will serve as a *path object* (i.e. a dual cylinder) in CofCat . The proof of the next proposition is merely an observation that classical arguments about Reedy cofibrations are still valid with this slightly more general definition. Nonetheless, this modification is important since otherwise the diagonal functor in the proof of Theorem 1.17 below wouldn't be exact.

Proposition 1.16. *If \mathcal{C} is a cofibration category, then so is PC with the above weak equivalences and cofibrations.*

Proof.

(C0) Weak equivalences satisfy “2 out of 6” since this holds in \mathcal{C} .

(C1) A morphism $A \rightarrow X$ is an acyclic cofibration if and only if all

$$\begin{aligned} A_0 &\rightarrow X_0, \\ A_1 &\rightarrow X_1, \\ A_{01} \amalg_{A_0} X_0 &\rightarrow X_{01} \text{ and} \\ A_{01} \amalg_{A_1} X_1 &\rightarrow X_{01} \end{aligned}$$

are acyclic cofibrations in \mathcal{C} . Hence every isomorphism is an acyclic cofibration.

(C2-3) The constant diagram of initial objects is cofibrant and initial in PC . Moreover, the definition of a cofibrant object X is equivalent to $0 \rightarrow X$ being a cofibration, thus all objects of PC are cofibrant.

(C4) A cofibration in PC is in particular a levelwise cofibration and thus pushouts along cofibrations in PC exist and are constructed levelwise. Given a pushout square,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in PC we observe that $B_0 \rightarrow Y_0$ and $B_1 \rightarrow Y_1$ are pushouts of $A_0 \rightarrow X_0$ and $A_1 \rightarrow X_1$ so they are cofibrations. The Pushout Lemma says that

$$B_{01} \amalg_{B_0} Y_0 \rightarrow Y_{01} \text{ and } B_{01} \amalg_{B_1} Y_1 \rightarrow Y_{01}$$

are pushouts of

$$A_{01} \amalg_{A_0} X_0 \rightarrow X_{01} \text{ and } A_{01} \amalg_{A_1} X_1 \rightarrow X_{01}$$

so they are cofibrations too. Consequently, $B \rightarrow Y$ is a cofibration in PC . Stability of acyclic cofibrations under pushouts is obtained by combining this argument with the characterization of acyclic cofibrations given in (C1) above.

- (C5) Let $X \rightarrow Y$ be a morphism of PC . For $i \in \{0, 1\}$ factor $X_i \rightarrow Y_i$ as $X_i \twoheadrightarrow Z_i \xrightarrow{\sim} Y_i$ in \mathcal{C} and form pushouts

$$\begin{array}{ccc} X_i & \twoheadrightarrow & Z_i \\ \downarrow & & \downarrow \\ X_{01} & \twoheadrightarrow & W_i. \end{array}$$

Then we have the induced morphisms $W_i \rightarrow Y_{01}$ which make the square

$$\begin{array}{ccc} X_{01} & \twoheadrightarrow & W_0 \\ \downarrow & & \downarrow \\ W_1 & \twoheadrightarrow & Y_{01}. \end{array}$$

commute and thus yield a morphism $W_0 \amalg_{X_{01}} W_1 \rightarrow Y_{01}$. We factor it in \mathcal{C} as

$$W_0 \amalg_{X_{01}} W_1 \twoheadrightarrow Z_{01} \xrightarrow{\sim} Y_{01}.$$

Then Z becomes an object of PC and $X \twoheadrightarrow Z \xrightarrow{\sim} Y$ is a factorization of the original morphism. \square

We are ready to prove the main result of this chapter.

Theorem 1.17. *The category \mathbf{CofCat} with weak equivalences and fibrations as above is a fibration category.*

In fact, \mathbf{CofCat} is a homotopy complete category, i.e. it has all small homotopy limits. This will be explained in Section 3.3.

Proof.

- (C0)^{op} Weak equivalences satisfy “2 out of 6” since they are created from equivalences of categories by $\mathbf{Ho}: \mathbf{CofCat} \rightarrow \mathbf{Cat}$.
- (C1)^{op} Isomorphisms are acyclic fibrations by Proposition 1.14.
- (C2-3)^{op} The category $[0]$ has a unique structure of a cofibration category and it is a terminal cofibration category. Moreover, every cofibration category is fibrant since every category is isofibrant while the lifting properties for factorizations and pseudofactorizations follow from the factorization axiom.

(C4)^{op} Proposition 1.13 says that pullbacks along fibrations exist and by the construction they are also pullbacks in $\widehat{\text{CofCat}}$. Since fibrations are defined by the right lifting property in this category they are stable under pullbacks. This argument also applies to acyclic fibrations by Proposition 1.14 since (App1) is equivalent to the right lifting property with respect to the inclusion $[1] \hookrightarrow \widehat{[1]}$.

(C5)^{op} To verify the factorization axiom it suffices to construct a path object for every cofibration category \mathcal{C} by Lemma 1.7. Let $\text{diag}: \mathcal{C} \rightarrow PC$ be the diagonal functor. It preserves (acyclic) cofibrations since if $X \twoheadrightarrow Y$ is an (acyclic) cofibration in \mathcal{C} , then both $(\text{diag } X)_0 \rightarrow (\text{diag } Y)_0$ and $(\text{diag } X)_1 \rightarrow (\text{diag } Y)_1$ coincide with $X \twoheadrightarrow Y$ while

$$\begin{aligned} & (\text{diag } X)_{01} \amalg_{(\text{diag } X)_0} (\text{diag } Y)_0 \rightarrow (\text{diag } Y)_{01} \\ \text{and } & (\text{diag } X)_{01} \amalg_{(\text{diag } X)_1} (\text{diag } Y)_1 \rightarrow (\text{diag } Y)_{01} \end{aligned}$$

are isomorphisms. It also preserves the pushouts, sequential colimits and coproducts and hence is exact. The evaluation functor $\text{ev}_{0,1} = (\text{ev}_0, \text{ev}_1): PC \rightarrow \mathcal{C} \times \mathcal{C}$ is also exact. Together they form a factorization of the diagonal functor $\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$. We need to show that diag is a weak equivalence and that $\text{ev}_{0,1}$ is a fibration.

Consider the evaluation functor $\text{ev}_{01}: PC \rightarrow \mathcal{C}$. It is a homotopical functor such that $\text{ev}_{01} \text{diag} = \text{id}_{\mathcal{C}}$ and there is a natural weak equivalence $\text{id}_{PC} \rightarrow \text{diag } \text{ev}_{01}$ since all morphisms of $\widehat{\text{Sd}}[1]$ are weak equivalences. It follows that $\text{Ho } \text{diag}$ is an equivalence.

It is easy to see that $\text{ev}_{0,1}$ is an isofibration. The lifting property for factorizations is verified just like the factorization axiom in PC except that now the factorizations $X_i \twoheadrightarrow Z_i \xrightarrow{\sim} Y_i$ are given in advance. The lifting property for pseudofactorizations is handled similarly: let $X \rightarrow Y$ be a morphism in PC and let

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \sim \\ W_i & \xrightarrow{\sim} & Z_i \end{array}$$

be pseudofactorizations of $X_i \rightarrow Y_i$ for $i \in \{0, 1\}$. Form pushouts

$$\begin{array}{ccc} X_i \twoheadrightarrow W_i & & Y_i \xrightarrow{\sim} Z_i \\ \downarrow & & \downarrow \\ X_{01} \twoheadrightarrow U_i & & Y_{01} \xrightarrow{\sim} V_i \end{array}$$

There are induced morphisms $U_i \rightarrow V_i$ which fit into a commutative diagram

$$\begin{array}{ccccc}
U_0 & \longleftarrow & X_{01} & \longrightarrow & U_1 \\
\downarrow & & \downarrow & & \downarrow \\
V_0 & \longleftarrow & Y_{01} & \longrightarrow & V_1
\end{array}$$

and thus induce a morphism $U_0 \amalg_{X_{01}} U_1 \rightarrow V_0 \amalg_{Y_{01}} V_1$ which we pseudofactorize into

$$\begin{array}{ccc}
U_0 \amalg_{X_{01}} U_1 & \longrightarrow & V_0 \amalg_{Y_{01}} V_1 \\
\downarrow & & \downarrow \sim \\
W_{01} & \xrightarrow{\sim} & Z_{01}.
\end{array}$$

Then W and Z form objects of PC which fit into a pseudofactorization

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \sim \\
W & \xrightarrow{\sim} & Z.
\end{array}$$

as required. □

1.3. Cofibration categories of diagrams and homotopy colimits

As already suggested by the two proofs above, Reedy cofibrations play an important role in the theory of cofibration categories. The notion of a Reedy cofibrant diagram (but not really that of a Reedy cofibration) will be essential in the proof of our main theorem. We will not discuss the basic theory of Reedy cofibrations since it is already well covered in the literature. A good general reference is [RV13] which is written from the perspective of Reedy categories and model categories. The theory of diagrams over general Reedy categories requires using both colimits and limits. Thus in the case of cofibration categories we have to restrict attention to a special class of Reedy categories called direct categories where colimits suffice. Specific results concerning Reedy cofibrations in cofibration categories are explained in [RB06] from where we will cite a few most relevant to the purpose of this thesis.

Definition 1.18.

- (1) A category I is *direct* if it admits a functor $\text{deg}: I \rightarrow \mathbb{N}$ that reflects identities (here, we consider \mathbb{N} as a poset with its standard order).
- (2) For a direct category I and $i \in I$, the *latching category* at i is the full subcategory of the slice $I \downarrow i$ on all objects except for id_i . It is denoted by $\partial(I \downarrow i)$.

- (3) Let $X: I \rightarrow \mathcal{C}$ be a diagram in some category and $i \in I$. The *latching object* of X at i is the colimit of the composite diagram

$$\partial(I \downarrow i) \rightarrow I \rightarrow \mathcal{C}$$

where $\partial(I \downarrow i) \rightarrow I$ is the forgetful functor sending a morphism of I (i.e. an object of $\partial(I \downarrow i)$) to its source. The latching object (if it exists) is denoted by $L_i X$ and comes with a canonical *latching morphism* $L_i X \rightarrow X_i$ induced by the inclusion $\partial(I \downarrow i) \rightarrow I \downarrow i$.

- (4) Let \mathcal{C} be a cofibration category. A diagram $X: I \rightarrow \mathcal{C}$ is *Reedy cofibrant* if for all $i \in I$ the latching object of X at i exists and the latching morphism $L_i X \rightarrow X_i$ is a cofibration.
- (5) Let $f: X \rightarrow Y$ be a morphism of Reedy cofibrant diagrams $I \rightarrow \mathcal{C}$. It is called a *Reedy cofibration* if for all $i \in I$ the induced morphism

$$X_i \amalg_{L_i X} L_i Y \rightarrow Y_i$$

is a cofibration (observe that this pushout exists since X is Reedy cofibrant).

The degree functor is not a part of the structure of a direct category, we merely assume it exists. Notions of latching objects, latching morphisms and Reedy cofibrations do not depend on the choice of the degree functor. Even the standard construction of factorizations into Reedy cofibrations and weak equivalences is independent of this choice.

In the definition of a Reedy cofibrant diagram the requirement that the latching objects exist is not a very restrictive one. If a diagram $X: I \rightarrow \mathcal{C}$ is Reedy cofibrant below the degree m , then the latching objects $L_i X$ exist for all $i \in I$ of degree m (provided that I has finite latching categories). Thus it is an acceptable abuse of language to say that a diagram is Reedy cofibrant if all its latching maps are cofibrations.

The main purpose of this section is to construct certain cofibration categories of diagrams and establish some practical criteria for verifying that particular functors between them are weak equivalences or fibrations.

Proposition 1.19. *Let \mathcal{C} be a cofibration category and J a homotopical direct category with finite latching categories.*

- (1) *The category \mathcal{C}_R^J of homotopical Reedy cofibrant diagrams with levelwise weak equivalences and Reedy cofibrations is a cofibration category.*
- (2) *The category \mathcal{C}^J of all homotopical diagrams with levelwise weak equivalences and levelwise cofibrations is a cofibration category.*
- (3) *The inclusion functor $\mathcal{C}_R^J \hookrightarrow \mathcal{C}^J$ is a weak equivalence.*

Proof.

- (1) [RB06, Theorem 9.3.8(1a)]
- (2) [RB06, Theorem 9.3.8(1b)]
- (3) The inclusion functor satisfies the approximation properties of Proposition 1.11 as follows from Lemma 1.22(1) (in fact, from its standard special case of $\mathcal{D} = [0]$ and $I = \emptyset$). \square

The crucial step in the proof of the above proposition is the construction of factorizations. In Lemma 1.22 we revisit that construction in order to prove a more general version which will be a key technical tool in many arguments of this thesis.

A homotopical functor $f: I \rightarrow J$ is a *homotopy equivalence* if there is a homotopical functor $g: J \rightarrow I$ such that gf is weakly equivalent to id_I and fg is weakly equivalent to id_J (where “weakly equivalent” means “connected by a zig-zag of natural weak equivalences”).

Lemma 1.20. *Let \mathcal{C} be a cofibration category and $f: I \rightarrow J$ a homotopical functor where I and J are homotopical direct categories with finite latching categories. Then the induced functor $f^*: \mathcal{C}^J \rightarrow \mathcal{C}^I$ is exact. Moreover, if f is a homotopy equivalence, then f^* is a weak equivalence of cofibration categories. Furthermore, if f induces an exact functor $f^*: \mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$, then it is also a weak equivalence.*

Proof. The functor f^* is clearly exact with respect to the levelwise structures and it is a homotopy equivalence when f is.

For the last statement, consider the commutative square of exact functors

$$\begin{array}{ccc} \mathcal{C}_R^J & \xrightarrow{f^*} & \mathcal{C}_R^I \\ \downarrow & & \downarrow \\ \mathcal{C}^J & \xrightarrow{f^*} & \mathcal{C}^I \end{array}$$

the vertical maps are weak equivalences by Proposition 1.19 so the conclusion follows by “2 out of 3”. \square

The utility of direct categories comes from the fact that it is easy to construct diagrams and morphisms of diagrams inductively. For our purposes it will be most convenient to state this in terms of sieves. A functor $I \rightarrow J$ is called a *sieve* if it is an inclusion of a full downwards closed subcategory, i.e. if it is injective on objects, fully faithful and if $i \rightarrow j$ is a morphism of J such that $j \in I$, then $i \in I$.

Lemma 1.21. *Let $I \hookrightarrow J$ be a sieve between direct categories and $j \in J \setminus I$ an object of a minimal degree. Let $X: I \rightarrow \mathcal{C}$ be a Reedy cofibrant diagram. Then prolongations of X to a Reedy cofibrant diagram $I \cup \{j\} \rightarrow \mathcal{C}$ are naturally bijective with cofibrations $L_j X \twoheadrightarrow X_j$ for varying $X_j \in \mathcal{C}$. ($L_j X$ exists by the minimality of j .)*

Similarly, if X is a Reedy cofibrant diagram over $I \cup \{j\}$ and $f: X|_I \rightarrow Y$ is a Reedy cofibration, then prolongations of f (and Y) to a Reedy cofibration over $I \cup \{j\}$ correspond bijectively to cofibrations $L_j Y \amalg_{L_j X} X_j \twoheadrightarrow Y_j$.

Proof. The only (non-identity) morphisms of $I \cup \{j\}$ missing from I are those going from objects of degree less than $\deg j$ to j and they are encoded by the latching morphism. Similarly, if $f: X \rightarrow Y$ is a morphism (cofibration) of diagrams over I and X is already defined over j , then extensions of f over j correspond to squares

$$\begin{array}{ccc} L_j X & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ L_j Y & \longrightarrow & Y_j \end{array}$$

which in turn correspond to morphisms $L_j Y \amalg_{L_j X} X_j \rightarrow Y_j$ and such an extension is a Reedy cofibration precisely when this morphism is a cofibration. \square

The first part of the next lemma generalizes the standard construction of factorizations into Reedy cofibrations followed by weak equivalences. It says that given a morphism of diagrams $J \rightarrow \mathcal{C}$ and compatible factorizations of its restriction along a sieve $I \hookrightarrow J$ and its image under a fibration $P: \mathcal{C} \rightarrow \mathcal{D}$, there is a factorization of the original morphism compatible with both of them. The other two parts say the same for lifts for pseudofactorizations and for cofibrations (when P is an acyclic fibration as in Proposition 1.14).

Lemma 1.22. *Let $P: \mathcal{C} \rightarrow \mathcal{D}$ be a fibration between cofibration categories. Let J be a homotopical direct category with finite latching categories and $I \hookrightarrow J$ a sieve.*

(1) *Let $f: X \rightarrow Y$ be a morphism in \mathcal{C}^J . If X is Reedy cofibrant,*

$$PX \xrightarrow{k_P} \tilde{Y}_P \xrightarrow[\sim]{s_P} PY \quad \text{and} \quad X|I \xrightarrow{k_I} \tilde{Y}_I \xrightarrow[\sim]{s_I} Y|I$$

are factorizations of Pf and $f|I$ into Reedy cofibrations followed by weak equivalences such that $Pk_I = k_P|I$ and $Ps_I = s_P|I$ (in particular, $P\tilde{Y}_I = \tilde{Y}_P|I$), then there is a factorization

$$X \xrightarrow{k} \tilde{Y} \xrightarrow[\sim]{s} Y$$

of f into a Reedy cofibration followed by a weak equivalence such that $Pk = k_P$, $k|I = k_I$, $Ps = s_P$ and $s|I = s_I$ (in particular, $P\tilde{Y} = \tilde{Y}_P$ and $\tilde{Y}|I = \tilde{Y}_I$).

(2) *Let $f: X \rightarrow Y$ be a morphism in \mathcal{C}^J . If both X and Y are Reedy cofibrant,*

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ \downarrow k_P & & \downarrow l_P \\ \tilde{Y}_P & \xrightarrow[\sim]{s_P} & \tilde{Y}_P \end{array} \quad \text{and} \quad \begin{array}{ccc} X|I & \xrightarrow{f|I} & Y|I \\ \downarrow k_I & & \downarrow l_I \\ \tilde{Y}_I & \xrightarrow[\sim]{s_I} & \tilde{Y}_I \end{array}$$

are pseudofactorizations of Pf and $f|I$ such that $Pk = k_P$, $k|I = k_I$, $Pl = l_P$, $l|I = l_I$, $Ps = s_P$ and $s|I = s_I$ (in particular, $P\tilde{Y}_I = \tilde{Y}_P|I$ and $P\hat{Y}_I = \hat{Y}_P|I$), then there is a pseudofactorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ k \downarrow & & \downarrow l \\ \tilde{Y} & \xrightarrow[s]{} & \hat{Y} \end{array}$$

such that $Pk = k_P$, $k|I = k_I$, $Pl = l_P$, $l|I = l_I$, $Ps = s_P$ and $s|I = s_I$ (in particular, $P\tilde{Y} = \tilde{Y}_P$, $\tilde{Y}|I = \tilde{Y}_I$, $P\hat{Y} = \hat{Y}_P$ and $\hat{Y}|I = \hat{Y}_I$).

(3) If P is acyclic, $X \in \mathcal{C}_R^J$ and

$$PX \xrightarrow{k_P} Z_P \quad X|I \xrightarrow{k_I} Z_I$$

are Reedy cofibrations such that $Pk_I = k_P|I$, then there exists a Reedy cofibration

$$X \xrightarrow{k} Z$$

such that $Pk = k_P$ and $k|I = k_I$ (in particular, $PZ = Z_P$ and $Z|I = Z_I$).

Proof.

(1) It suffices to extend the factorization $f|I = s_I k_I$ over an object $j \in J \setminus I$ of a minimal degree. Then the statement will follow by an induction over the degree.

By the minimality of the degree of j , Reedy cofibrancy of X and since $I \hookrightarrow J$ is a sieve the latching objects $L_j X$ and $L_j \tilde{Y}_I$ exist. Moreover, the induced functor of latching categories $\partial(I \downarrow j) \rightarrow \partial(J \downarrow j)$ is an isomorphism. Thus P sends the morphism $X_j \amalg_{L_j X} L_j \tilde{Y}_I \rightarrow Y_j$ to the analogous morphism $PX_j \amalg_{L_j PX} L_j \tilde{Y}_I \rightarrow PY_j$. The latter factors as

$$PX_j \amalg_{L_j PX} L_j \tilde{Y}_I \hookrightarrow (\tilde{Y}_P)_j \xrightarrow{\sim} PY_j$$

and since P is a fibration we can lift this to a factorization of the former as

$$X_j \amalg_{L_j X} L_j \tilde{Y}_I \hookrightarrow \tilde{Y}_j \xrightarrow{\sim} Y_j.$$

This extends the factorization $f|I = s_I k_I$ over j by Lemma 1.21. The resulting diagram \tilde{Y} is homotopical since it is weakly equivalent to homotopical Y .

(2) This follows by a very similar argument. This time both X and Y are Reedy cofibrant so all the latching objects $L_j X$, $L_j Y$, $L_j \tilde{Y}_I$ and $L_j \hat{Y}_I$ exist. (Where

$j \in J \setminus I$ is an object of a minimal degree as above.) Hence P maps the morphism $X_j \amalg_{L_j X} L_j \tilde{Y}_I \rightarrow Y_j \amalg_{L_j Y} L_j \hat{Y}_I$ to the top morphism of the pseudofactorization

$$\begin{array}{ccc} PX_j \amalg_{L_j PX} L_j P\tilde{Y}_I & \longrightarrow & PY_j \amalg_{L_j PY} L_j P\hat{Y}_I \\ \downarrow & & \downarrow \sim \\ (\tilde{Y}_P)_j & \xrightarrow{\sim} & (\hat{Y}_P)_j \end{array}$$

and since P is a fibration it can be lifted to

$$\begin{array}{ccc} X_j \amalg_{L_j X} L_j \tilde{Y}_I & \longrightarrow & Y_j \amalg_{L_j Y} L_j \hat{Y}_I \\ \downarrow & & \downarrow \sim \\ \tilde{Y}_j & \xrightarrow{\sim} & \hat{Y}_j \end{array}$$

which prolongs the pseudofactorization $l_I f|I = s_I k_I$ over j by Lemma 1.21. The resulting diagrams \tilde{Y} and \hat{Y} are homotopical since they are weakly equivalent to homotopical Y .

- (3) Again, the proof is very similar and it suffices to extend k_I over an object $j \in J \setminus I$ of a minimal degree. Since X is Reedy cofibrant, both $L_j X$ and $L_j Z_I$ exist and we can form a pushout $X_j \amalg_{L_j X} L_j Z_I$ which P maps to the domain of

$$PX_j \amalg_{L_j PX} L_j PZ_I \twoheadrightarrow (Z_P)_j$$

which can be lifted to

$$X_j \amalg_{L_j X} L_j Z_I \twoheadrightarrow Z_j$$

by Proposition 1.14 and since P is acyclic. This prolongs k_I over j by Lemma 1.21. The resulting diagram Z is homotopical since P reflects weak equivalences. \square

The most typical examples of fibrations are restrictions along sieves.

Lemma 1.23. *Let \mathcal{C} be a cofibration category. If I and J are homotopical direct categories with finite latching categories and $f: I \rightarrow J$ a homotopical functor such that for every $i \in I$ the induced functor of the latching categories $\partial(I \downarrow i) \rightarrow \partial(J \downarrow fi)$ is an isomorphism, then the induced functor $f^*: \mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$ is exact.*

Moreover, if f is a sieve, then f^ is a fibration.*

Proof. If f induces isomorphisms of the latching categories, then f^* preserves Reedy cofibrations (and, in particular, Reedy cofibrant diagrams). It also preserves weak equivalences and colimits that exist in \mathcal{C}_R^J so it is exact.

If f is a sieve, then it satisfies the exactness criterion above. Moreover, f^* is a fibration by parts (1) and (2) of Lemma 1.22. \square

The next few lemmas establish some connections between sieves and fibrations which are reminiscent of classical homotopical algebra if we think of sieves as “cofibrations” and sieves $I \hookrightarrow J$ inducing weak equivalences $\mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$ as “acyclic cofibrations”. This does not quite fit into the classical picture since such “cofibrations” do not really belong to the same category as the fibrations do.

Lemma 1.24. *Let $f: I \hookrightarrow J$ be a sieve between homotopical direct categories with finite latching categories and $P: \mathcal{C} \rightarrow \mathcal{D}$ a fibration of cofibration categories. Then the induced exact functor $(f^*, P): \mathcal{C}_R^J \rightarrow \mathcal{C}_R^I \times_{\mathcal{D}_R^I} \mathcal{D}_R^J$*

- (1) *is a fibration,*
- (2) *is an acyclic fibration provided that P is acyclic,*
- (3) *is an acyclic fibration provided that both $f^*: \mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$ and $f^*: \mathcal{D}_R^J \rightarrow \mathcal{D}_R^I$ are weak equivalences.*

Proof. First observe that the pullback in question exists since f^* is a fibration by Lemma 1.23.

- (1) This follows by parts (1) and (2) of Lemma 1.22.
- (2) This follows by (1) above and part (3) of Lemma 1.22.
- (3) This follows by (1) above and a diagram chase. □

Lemma 1.25. *If \mathcal{C} is a fibration category,*

$$\begin{array}{ccc} I & \hookrightarrow & J \\ \downarrow & & \downarrow \\ K & \hookrightarrow & L \end{array}$$

is a pushout square of homotopical direct categories with finite latching categories and both $I \hookrightarrow J$ and $I \hookrightarrow K$ are sieves, then the resulting square

$$\begin{array}{ccc} \mathcal{C}_R^L & \longrightarrow & \mathcal{C}_R^K \\ \downarrow & & \downarrow \\ \mathcal{C}_R^J & \longrightarrow & \mathcal{C}_R^I \end{array}$$

is a pullback of cofibration categories.

Proof. By the construction of pullbacks of cofibration categories it will suffice to verify that a morphism of diagrams over L is a Reedy cofibration if and only if it is one when restricted to both J and K . For this it will be enough to observe that both $J \hookrightarrow L$ and $K \hookrightarrow L$ are sieves and hence for an object $l \in L$ we have either $l \in J$ and then $\partial(J \downarrow l) \rightarrow \partial(L \downarrow l)$ is an isomorphism or $l \in K$ and then $\partial(J \downarrow l) \rightarrow \partial(L \downarrow l)$ is an isomorphism. □

Let $f: I \rightarrow J$ be a homotopical functor of homotopical direct categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ an exact functor of cofibration categories. We say that f has the Reedy left lifting property with respect to F (or F has the Reedy right lifting property with respect to f) if every lifting problem

$$\begin{array}{ccc} I & \xrightarrow{X} & \mathcal{C} \\ f \downarrow & & \downarrow F \\ J & \xrightarrow{Y} & \mathcal{D} \end{array}$$

where X and Y are homotopical Reedy cofibrant diagrams has a solution that is also a homotopical Reedy cofibrant diagram. Such lifting properties will be heavily used in the latter two chapters.

Lemma 1.26. *Let $f: I \hookrightarrow J$ and $g: K \rightarrow L$ be sieves between homotopical direct categories with finite latching categories and $F: \mathcal{C} \rightarrow \mathcal{D}$ an exact functor of cofibration categories. Then there is a natural bijection between Reedy lifting problems (and their solutions) of the forms*

$$\begin{array}{ccccc} I & \longrightarrow & \mathcal{C}_R^L & & (I \times L) \amalg_{I \times K} (J \times K) & \longrightarrow & \mathcal{C} & & K & \longrightarrow & \mathcal{C}_R^J \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ J & \longrightarrow & \mathcal{C}_R^K \times_{\mathcal{D}_R^K} \mathcal{D}_R^L & & J \times L & \longrightarrow & \mathcal{D} & & L & \longrightarrow & \mathcal{C}_R^I \times_{\mathcal{D}_R^I} \mathcal{D}_R^J. \end{array}$$

Proof. This is proven with standard adjointness arguments, e.g. as in [Joy08, Proposition D.1.18], using the fact that a diagram $J \rightarrow \mathcal{C}_R^L$ is Reedy cofibrant if and only if the corresponding diagram $J \times L \rightarrow \mathcal{C}$ is as follows from [RV13, Example 4.6]. \square

Lemma 1.27. *Let $P: \mathcal{C} \rightarrow \mathcal{D}$ be a fibration of cofibration categories. The following are equivalent:*

- (1) P is acyclic,
- (2) P has the Reedy right lifting property with respect to all sieves between direct homotopical categories with finite latching categories,
- (3) P has the Reedy right lifting property with respect to $[0] \hookrightarrow [1]$ and $[1] \hookrightarrow \widehat{[1]}$.

Proof. If P is acyclic, then it has the Reedy right lifting property with respect to all sieves between homotopical direct categories with finite latching categories by Lemma 1.22(3), in particular, with respect to $[0] \hookrightarrow [1]$ and $[1] \hookrightarrow \widehat{[1]}$.

Conversely, by Proposition 1.14 it suffices to see that if P has the Reedy right lifting property with respect to $[0] \hookrightarrow [1]$ and $[1] \hookrightarrow \widehat{[1]}$, then it satisfies (App1) and has the right lifting property in $\overline{\text{CofCat}}$ with respect to the inclusion of $[0]$ into

$$0 \hookrightarrow 1.$$

The latter is equivalent to the Reedy right lifting property with respect to $[0] \hookrightarrow [1]$. To see that the Reedy right lifting property with respect to $[1] \hookrightarrow \widehat{[1]}$ implies (App1) take a morphism $f: X \rightarrow Y$ in \mathcal{C} such that Pf is a weak equivalence. Factor f as

$$X \xrightarrow{j} \widetilde{Y} \xrightarrow{\sim} Y.$$

Then Pj is a weak equivalence by “2 out of 3” and hence so is j by the Reedy right lifting property with respect to $[1] \hookrightarrow \widehat{[1]}$. Thus f is a weak equivalence, too. \square

Lemma 1.28. *If a sieve $f: I \rightarrow J$ between homotopical direct categories has the Reedy left lifting property with respect to all fibrations of cofibration categories, then for every cofibration category \mathcal{C} the induced functor $f^*: \mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$ is an acyclic fibration.*

Proof. Since f is a sieve it will suffice to check that f^* has the Reedy right lifting property with respect to $[0] \hookrightarrow [1]$ and $[1] \hookrightarrow \widehat{[1]}$ by Lemma 1.27. These are equivalent to the Reedy right lifting property of $\mathcal{C}_R^{[1]} \rightarrow \mathcal{C}_R^{[0]}$ and $\mathcal{C}_R^{\widehat{[1]}} \rightarrow \mathcal{C}_R^{[1]}$ with respect to $I \hookrightarrow J$ by Lemma 1.24. \square

The following proposition says that in cofibration categories colimits of Reedy cofibrant diagrams (over finite direct categories) exist and are homotopy invariant. In effect, this yields finite direct homotopy colimits in cofibration categories.

Proposition 1.29. *If I is a finite direct category, then the colimit functor $\mathcal{C}_R^I \rightarrow \mathcal{C}$ exists and is exact.*

Proof. [RB06, Theorem 9.3.5(1)] \square

It is perhaps worth pointing out that this construction does not directly apply to non-cofibrant diagrams, but all direct diagrams can be replaced by Reedy cofibrant ones. This is not directly captured by the definition of a cofibration category as given in the beginning of this chapter since we insisted that all objects are cofibrant. Instead, we can think of the homotopy colimit functor as a zig-zag of exact functors

$$\mathcal{C}^I \xleftarrow{\sim} \mathcal{C}_R^I \xrightarrow{\text{colim}_I} \mathcal{C}.$$

Here, the functor on the left is the one discussed in Proposition 1.19.

Cofibration categories admit all finite homotopy colimits, but finiteness has to be understood in a rather strong sense. Namely, a finite homotopy colimit is a homotopy colimit of a diagram indexed over a category I whose nerve is a finite simplicial set. Such categories coincide with finite direct categories and hence Proposition 1.29 implies existence of finite homotopy colimits in cofibration categories.

Notice that e.g. homotopy colimits of diagrams indexed by non-trivial finite groups are *not* finite homotopy colimits since the nerves of such groups are infinite and hence homotopy colimits over them involve infinite amount of coherence data.

Proposition 1.29 implies that a pushout of two cofibrations in a cofibration category is a homotopy pushout. In fact, a more general and extremely useful statement is true:

a pushout of any morphism along a cofibration is a homotopy pushout. This is known as the Gluing Lemma.

Lemma 1.30 (Gluing Lemma). *Given a commutative cube*

$$\begin{array}{ccccc}
 A_0 & \longrightarrow & B_0 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & A_1 & \longrightarrow & B_1 \\
 & & \downarrow & & \downarrow \\
 X_0 & \longrightarrow & Y_0 & & \\
 \downarrow & \searrow & \downarrow & \dashrightarrow & \downarrow \\
 & & X_1 & \longrightarrow & Y_1
 \end{array}$$

where the indicated morphisms are cofibrations and both front and back squares are pushouts, if the three solid arrows going from the back square to the front square are weak equivalences, then so is the dashed one.

More generally, the conclusion holds provided that both front and back squares are homotopy pushouts, i.e. can be connected by zig-zags of natural weak equivalences to pushouts along cofibrations.

Proof. [RB06, Lemma 1.4.1(1)] □

While the proof of the Gluing Lemma cited above does not state this explicitly, the argument is basically an application of the K. Brown’s Lemma. Recall that Γ is the poset of proper subsets of $\{0, 1\}$. It can be proven (similarly to Proposition 1.16) that there is a cofibration category \mathcal{C}_p^Γ of “partially Reedy cofibrant diagrams” $X: \Gamma \rightarrow \mathcal{C}$, i.e. such that $X_\emptyset \rightarrow X_0$ is a cofibration. The weak equivalences are levelwise and cofibrations are “partial Reedy cofibrations”, i.e. levelwise cofibrations that are Reedy cofibrations when restricted to $\emptyset \rightarrow 0$. One way to motivate the pushout axiom (C4) is that this is what is required for the pushout functor $\text{colim}_\Gamma: \mathcal{C}_p^\Gamma \rightarrow \mathcal{C}$ to be exact. More precisely, stability of (acyclic) cofibrations under pushouts implies that this functor preserves (acyclic) cofibrations.

We will often need to know that certain homotopical functors between homotopical direct categories induce weak equivalences of homotopy colimits. Such functors are called *homotopy cofinal*. For our purposes the following simple criterion is sufficient.

Lemma 1.31. *Let $f: I \rightarrow J$ be a homotopical functor between finite homotopical direct categories and \mathcal{C} a cofibration category. If f induces a weak equivalence $\mathcal{C}_R^J \rightarrow \mathcal{C}_R^I$, then for every homotopical Reedy cofibrant diagram $X: J \rightarrow \mathcal{C}$ the induced morphism $\text{colim}_I f^*X \rightarrow \text{colim}_J X$ is a weak equivalence.*

Proof. The left Kan extension functor $\text{Lan}_f: \mathcal{C}_R^I \rightarrow \mathcal{C}_R^J$ exists, is exact by [RB06, Theorem 9.4.3(1)] and is a left adjoint of f^* . Hence Lan_f is a weak equivalence since f^* is. In particular, the counit $\text{Lan}_f f^*X \rightarrow X$ is a weak equivalence and hence so is the

resulting morphism $\operatorname{colim}_J \operatorname{Lan}_f f^* X \rightarrow \operatorname{colim}_J X$ which coincides with the morphism $\operatorname{colim}_I f^* X \rightarrow \operatorname{colim}_J X$. \square

1.4. Examples

In order to better motivate cofibration and fibration categories we list a number of interesting examples. Neither of them is known to come from a model category and some of them are not known to (and in some cases actually known not to) be equivalent to model categories.

1.4.1. C^* -algebras

Theorem 1.32 ([Sch84, Section 1]). *The category of C^* -algebras carries a structure of a pointed fibration category.*

A streamlined proof of this theorem can be found in [Uuy13, Theorem 2.19] along with a few accompanying results in a similar spirit. Moreover, it is proven [Uuy13, Theorem A.1] that the homotopical category of C^* -algebras does not admit a model structure. This result (originally due to Andersen and Grodal [AG97, Corollary 4.7]) can be phrased in an even stronger way: there is no model category whose underlying fibration category is weakly equivalent to the fibration category of the theorem above. This is because the loop functor fails to have a left adjoint. It follows that not even a cofibration category presenting the homotopy theory of C^* -algebras exists.

1.4.2. Proper homotopy theory

Theorem 1.33 ([BQ01, Theorems 3.6 and 4.5]). *The category of topological spaces with proper maps as morphisms carries a structure of a cofibration category.*

The weak equivalences of this fibration category are *proper homotopy equivalences*. A proper map $f: X \rightarrow Y$ is a proper homotopy equivalence if it admits a proper map $g: Y \rightarrow X$ and homotopies $gf \simeq \operatorname{id}_X$ and $fg \simeq \operatorname{id}_Y$ through proper maps. The cofibrations are *proper (Hurewicz) cofibrations*, i.e. proper maps $A \rightarrow B$ with the proper homotopy extension property. This means that we require that every proper homotopy defined on A whose one end extends over B also extends over B (to a proper homotopy). This category does not carry a structure of a fibration category, e.g. since it has no terminal object.

1.4.3. Homotopy type theory

Theorem 1.34 ([AKL13, Theorem 2.2.5]). *Every categorical model of homotopy type theory carries a canonical structure of a fibration category.*

This category has certain distinguished class of maps that are natural candidates for cofibrations (and would have to be cofibrations if this fibration category was a part of a model category). Unfortunately, it turns out that pushouts along these maps fail to exist in general.

1.4.4. Topological spaces

Most of the remaining examples discuss some well known homotopical categories which admit well know model structures, but in addition they also carry less known structures of (co)fibration categories. They typically have more (co)fibrations than the classical model structures which means that they provide more point-set models of homotopy (co)limits.

We start with the category of topological spaces which has two notable classes of weak equivalences: *homotopy equivalences* and *weak homotopy equivalences*. All of these examples seem to be folklore but we know almost no references.

A map of topological spaces $p: X \rightarrow Y$ is a *Dold fibration* if it has the *weak covering homotopy property*, i.e. for each square on the left

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ i_0 \downarrow & & \downarrow p \\ A \times I & \xrightarrow{H} & Y \end{array} \quad \begin{array}{ccc} A & \xrightarrow{H} & X^I \\ i \downarrow & & \downarrow p_0 \\ B & \xrightarrow{v} & X \end{array}$$

there exists a homotopy $G: A \times I \rightarrow X$ such that $pG = H$ and Gi_0 is homotopic to u fiberwise over Y . Dually, a map $i: A \rightarrow B$ is a *Dold cofibration* if for all squares on the right above there exists a homotopy $G: B \rightarrow X^I$ such that $Gi = H$ and p_0G is homotopic to v relative to A .

Theorem 1.35.

- (1) *The category of topological spaces with homotopy equivalences and Dold fibrations is a fibration category.*
- (2) *The category of topological spaces with homotopy equivalences and Dold cofibrations is a cofibration category.*

Dold fibrations were introduced in [Dol63] and both Dold fibrations and cofibrations are discussed in [tDKP70]. There are more Dold (co)fibrations than classical Hurewicz (co)fibrations.

A *Dold-Serre fibration* is a map satisfying the weak covering homotopy property as above but only for $A = D^m$ for all $m \geq 0$.

Theorem 1.36. *The category of topological spaces with weak homotopy equivalences Dold-Serre fibrations is a fibration category.*

Again, there are more Dold-Serre fibrations than classical Serre fibrations.

One could expect that there is a corresponding notion of a “Dold-Serre cofibration”, but this does not seem to be the case. However, something even better is true.

Theorem 1.37. *The category of topological spaces with weak homotopy equivalences and Hurewicz cofibrations is a cofibration category.*

At the first glance this may seem to come from a mixed model structure in the sense of Cole [Col06], but it does not. This is an attempt to mix in the “wrong direction” which succeeds for delicate point-set reasons. We know from [RB06, Lemma 1.4.3(1)] that it suffices to verify that weak homotopy equivalences and Hurewicz cofibrations satisfy the Gluing Lemma and this holds by [BV73, Appendix, Proposition 4.8(b)]. In fact, by combining this observation with Theorem 1.35(2) one can show that this is even true with Dold cofibrations in the place of Hurewicz cofibrations.

1.4.5. Simplicial and categorical homotopy theory

As we have already illustrated, one can often find classes of (co)fibrations that are larger than ones coming from classical model structures. In fact, it is not difficult to prove that if there is at least one class of (co)fibrations compatible with a given homotopical category, then there is also the largest one. One of the few examples where such class is well understood is the category of simplicial sets.

A simplicial map f is *sharp* if every strict pullback along f is a homotopy pullback. With this definition it is routine to prove the following result.

Theorem 1.38. *The category of all simplicial sets with weak homotopy equivalences and sharp maps is a fibration category.*

Sharp maps were introduced by Rezk [Rez98]. Clearly, a fibration in any fibration category of simplicial sets (with weak homotopy equivalences) is sharp hence this is indeed the largest class of fibrations. Observe that in this fibration category every simplicial set is fibrant.

The next two examples exploit the notion of Dwyer maps to connect category theory to homotopy theory. A *Dwyer map* is a functor f of small categories that is a sieve and factors as $f = gj$ where g is a cosieve and j admits a deformation retraction.

While Thomason [Tho80] doesn’t state this explicitly, a crucial step of his construction of a model structure on small categories is contained in the following theorem.

Theorem 1.39. *The category of small categories with weak homotopy equivalences (i.e. the ones created by the nerve functor from weak homotopy equivalences of simplicial sets) and Dwyer maps is a cofibration category.*

Barwick and Kan [BK12, BK12b] in the construction of their model category of relative categories (which was already discussed in the introduction) used a similar approach. They defined a suitable generalization of Dwyer maps and proved (also implicitly) an analogous result.

Theorem 1.40. *The category of small relative categories with Dwyer–Kan equivalences and Dwyer maps is a cofibration category.*

In both cases there are many more Dwyer maps than cofibrations in their model categories.

2. Quasicategories

This chapter is devoted to a concise summary of the theory of quasicategories. It is well covered in [Joy08] and [Lur09] so we do not go into much detail. Our main goal is to establish a fibration category of finitely cocomplete quasicategories in Theorem 2.17. We follow [Joy08] to demonstrate that the fibration category of all quasicategories can be obtained without constructing the entire Joyal model structure (Theorem 2.3) which makes the proof rather elementary. (A more streamlined exposition of the same results can be found in the appendices to [DS11].) Then we briefly introduce colimits in quasicategories and state their basic properties used in the proof of Theorem 2.17 and later in Chapter 3.

2.1. Homotopy theory of quasicategories

Recall that $E(1)$ is the groupoid freely generated by an isomorphism $0 \rightarrow 1$. Its nerve will be denoted by $E[1]$. Quasicategories are defined as certain special simplicial sets and are to be thought of as models of $(\infty, 1)$ -categories where vertices are objects, edges are morphisms and higher simplices are higher morphisms (or higher homotopies). Functors between quasicategories are just simplicial maps. In particular, maps out of $E[1]$ are equivalences in quasicategories and $E[1]$ -homotopies are natural equivalences between functors. The account of the homotopy theory of quasicategories below closely follows the classical approach to simplicial homotopy theory (see e.g. [GJ99, Chapter I]) with Kan complexes replaced by quasicategories and usual simplicial homotopies replaced by $E[1]$ -homotopies.

Definition 2.1.

- (1) Let $f, g: K \rightarrow L$ be simplicial maps. An $E[1]$ -homotopy from f to g is a simplicial map $K \times E[1] \rightarrow L$ extending $[f, g]: K \times \partial\Delta[1] \rightarrow L$.
- (2) Two simplicial maps $f, g: K \rightarrow L$ are $E[1]$ -homotopic if there exists a zig-zag of $E[1]$ -homotopies connecting f to g . (It suffices to consider sequences instead of zig-zags since $E[1]$ has an automorphism that exchanges the vertices.)
- (3) A simplicial map $f: K \rightarrow L$ is an $E[1]$ -homotopy equivalence if there is a simplicial map $g: L \rightarrow K$ such that fg is $E[1]$ -homotopic to id_L and gf is $E[1]$ -homotopic to id_K .

Definition 2.2.

- (1) A simplicial map is an *inner fibration* if it has the right lifting property with respect to the inner horn inclusions.

- (2) A simplicial map is an *inner isofibration* if it is an inner fibration and has the right lifting property with respect to $\Delta[0] \hookrightarrow E[1]$.
- (3) A simplicial map is an *acyclic Kan fibration* if it has the right lifting property with respect to $\partial\Delta[m] \hookrightarrow \Delta[m]$ for all m .
- (4) A simplicial set \mathcal{C} is a *quasicategory* if the unique map $\mathcal{C} \rightarrow \Delta[0]$ is an inner fibration.

We will refer to $E[1]$ -equivalence between quasicategories as *categorical equivalences* and use them to introduce the homotopy theory of quasicategories. (It is also possible to extend this notion to maps of general simplicial sets, but we have no need to do it.) If K is any simplicial set and \mathcal{C} is a quasicategory, then the relation of “being connected by a single $E[1]$ -homotopy” is already an equivalence relation on the set of simplicial maps $K \rightarrow \mathcal{C}$ by [DS11, Proposition 2.3]. This simplifies the definition of categorical equivalences since it is always sufficient to consider one-step $E[1]$ -homotopies.

Theorem 2.3. *The category of small quasicategories with simplicial maps as morphisms, categorical equivalences as weak equivalences and inner isofibrations as fibrations is a fibration category.*

In fact, this fibration category is homotopy complete, i.e. it admits all small homotopy limits as will be discussed in Section 3.3.

Proof.

- (C0)^{op} Categorical equivalences satisfy “2 out of 6” since by the Yoneda Lemma a functor $\mathcal{C} \rightarrow \mathcal{D}$ is a categorical equivalence if and only if for every quasicategory \mathcal{E} the induced map $[\mathcal{D}, \mathcal{E}]_{E[1]} \rightarrow [\mathcal{C}, \mathcal{E}]_{E[1]}$ is a bijection. (This notation refers to sets of $E[1]$ -homotopy classes of maps between quasicategories.)
- (C1)^{op} Isomorphisms are acyclic fibrations since acyclic fibrations coincide with acyclic Kan fibrations by [Joy08, Theorem 5.15].
- (C2-3)^{op} The simplicial set $\Delta[0]$ is a terminal quasicategory and if \mathcal{C} is a quasicategory, then the unique map $\mathcal{C} \rightarrow \Delta[0]$ is actually an inner isofibration since the map $\Delta[0] \hookrightarrow E[1]$ admits a retraction. Thus all quasicategories are fibrant.
- (C4)^{op} Inner isofibrations are defined via the right lifting property thus they are closed under pullbacks (and quasicategories are closed under pullbacks along inner isofibrations). Moreover, acyclic (inner iso-) fibrations coincide with acyclic Kan fibrations by [Joy08, Theorem 5.15], so they are also closed under pullbacks.
- (C5)^{op} To verify the factorization axiom it suffices (by Lemma 1.7) to construct a path object for every quasicategory \mathcal{C} . Consider the factorization of the codiagonal

$$\partial\Delta[1] \hookrightarrow E[1] \rightarrow \Delta[0].$$

It yields

$$\mathcal{C} \rightarrow \mathcal{C}^{E[1]} \rightarrow \mathcal{C} \times \mathcal{C}$$

which is a factorization into a categorical equivalence followed by an inner isofibration by [Joy08, Proposition 5.20]. \square

This fibration category could also be obtained by restricting weak equivalences and fibrations of the Joyal model structure to the full subcategory of quasicategories (as follows from [Joy08, Theorem 5.22]). However, the proof sketched above does not depend on the existence of the Joyal model structure.

Quasicategories are models for homotopy theories and as such they have homotopy categories. Two morphisms $f, g: x \rightarrow y$ of a quasicategory \mathcal{D} are *homotopic* if there exists a simplex $H: \Delta[2] \rightarrow \mathcal{D}$ such that $H\delta_0 = y\sigma_0$, $H\delta_1 = g$ and $H\delta_2 = f$. The *homotopy category* of \mathcal{D} is the category $\text{Ho } \mathcal{D}$ with the same objects as \mathcal{D} , homotopy classes of morphisms of \mathcal{D} as morphisms and the composition induced by filling horns.

If f is a morphism of a quasicategory \mathcal{C} , then we say that f is an *equivalence* if the simplicial map $f: \Delta[1] \rightarrow \mathcal{C}$ extends to $E[1] \rightarrow \mathcal{C}$. (By [Joy08, Proposition 4.22] a morphism is an equivalence if and only if it becomes an isomorphism in the homotopy category.) Two objects of \mathcal{C} are *equivalent* if they are connected by an equivalence.

We conclude this section by a technical lemma saying that in quasicategories certain outer horns can be filled. Let \mathcal{C} be a quasicategory. A map $X: \Lambda^i[m] \rightarrow \mathcal{C}$ is called a *special outer horn in \mathcal{C}* if $i = 0$ and $X|_{\Delta\{0,1\}}$ is an equivalence or $i = m$ and $X|_{\Delta\{m-1,m\}}$ is an equivalence.

Lemma 2.4. *If $X: \Lambda^i[m] \rightarrow \mathcal{C}$ is a special outer horn and $p: \mathcal{C} \rightarrow \mathcal{D}$ is an inner isofibration between quasicategories, then the diagram*

$$\begin{array}{ccc} \Lambda^i[m] & \xrightarrow{X} & \mathcal{C} \\ \downarrow & & \downarrow \\ \Delta[m] & \longrightarrow & \mathcal{D} \end{array}$$

admits a lift.

Proof. [Joy08, Theorem 4.13] or [DS11, Proposition B.11] \square

2.2. Colimits

We proceed to the discussion of colimits in quasicategories. Such colimits are homotopy invariant by design and they serve as models for homotopy colimits. However, in quasicategories there is no corresponding notion of a “strict” colimit and thus it is customary to refer to “homotopy colimits” in quasicategories simply as colimits. The general theory of colimits is explored in depth in [Lur09, Chapter 4], here we only discuss its most basic aspects.

The quasicategorical notion of colimit is defined using the join construction for simplicial sets. In order to define joins efficiently we briefly introduce augmented simplicial sets. The category Δ_a is defined as the category of finite totally ordered sets of the form $[m]$ for $m \geq -1$ (where $[-1] = \emptyset$). The category of *augmented simplicial sets* is the category of presheaves on Δ_a and is denoted by \mathbf{asSet} . The standard category Δ is a full subcategory of Δ_a , we denote the inclusion functor by $i: \Delta \hookrightarrow \Delta_a$. Precomposition with i is the forgetful functor $i^*: \mathbf{asSet} \rightarrow \mathbf{sSet}$ and it has a right adjoint, the right Kan extension along i denoted by $\mathbf{Ran}_i: \mathbf{sSet} \rightarrow \mathbf{asSet}$. Explicitly, \mathbf{Ran}_i prolongs a simplicial set to an augmented simplicial set by setting the value at $[-1]$ to a singleton.

The category Δ_a carries a (non-symmetric) strict monoidal structure given by concatenation $[m], [n] \mapsto [m] \star [n] \cong [m+1+n]$ with $[-1]$ as the monoidal unit. On morphisms it is also defined by concatenation: $\varphi \star \psi: [k] \star [l] \rightarrow [m] \star [n]$ acts via φ on the first $k+1$ elements and via ψ on the last $l+1$ ones.

Proposition 2.5.

- (1) *The category of augmented simplicial sets carries a closed monoidal structure with the monoidal product, the join $\star: \mathbf{asSet} \times \mathbf{asSet} \rightarrow \mathbf{asSet}$ uniquely characterized by its action on representables $\Delta_a[m], \Delta_a[n] \mapsto \Delta_a([m] \star [n]) \cong \Delta_a[m+1+n]$. The unit is $\Delta_a[-1]$.*
- (2) *The category of simplicial sets carries a monoidal structure with the monoidal product, again called the join, given by $K \star L = i^*(\mathbf{Ran}_i K \star \mathbf{Ran}_i L)$. The unit is the empty simplicial set.*

Proof. The first statement follows from the classical theorem of Day [Day70, Theorem 3.3]. The second one can be proven by observing that \mathbf{Ran}_i embeds \mathbf{sSet} fully and faithfully into \mathbf{asSet} with the essential image consisting of augmented simplicial sets X with X_{-1} a singleton. Under this identification the join of augmented simplicial sets restricts to the join of simplicial sets. □

The category of small categories embeds as a full category of \mathbf{sSet} via the nerve functor and the join product restricts to the category of small categories. Explicitly, given small categories I and J the join $I \star J$ is defined as follows. The set of objects of $I \star J$ is the coproduct of the sets of objects of I and J and

$$(I \star J)(x, y) = \begin{cases} I(x, y) & \text{if } x, y \in I, \\ J(x, y) & \text{if } x, y \in J, \\ * & \text{if } x \in I, y \in J, \\ \emptyset & \text{if } x \in J, y \in I. \end{cases}$$

The composition of $I \star J$ is the unique composition that restricts to the compositions of I and J .

For example $[0] \star J$ is formed by adjoining an initial object to J (a new one if J already had one). If J is discrete, then colimits over $[0] \star J$ are called *wide pushouts*. (They reduce to classical pushouts when J has exactly two objects.)

The join monoidal structure on simplicial set is not closed and the join doesn't preserve all colimits in either of its variables. However, a slightly weaker statement holds. First, we need to observe that for any simplicial set K the functor $K \star -: \mathbf{sSet} \rightarrow \mathbf{sSet}$ lifts to a functor $\mathbf{sSet} \rightarrow K \downarrow \mathbf{sSet}$ (also denoted by $K \star -$.) Such a lift is defined by the following composite

$$\mathbf{sSet} \rightarrow \emptyset \downarrow \mathbf{sSet} \xrightarrow{\text{Ran}_i} \Delta_a[-1] \downarrow \mathbf{asSet} \xrightarrow{\text{Ran}_i K \star -} \text{Ran}_i K \downarrow \mathbf{asSet} \xrightarrow{i^*} K \downarrow \mathbf{sSet}.$$

Proposition 2.6. *For each simplicial set K , the functor $K \star -: \mathbf{sSet} \rightarrow K \downarrow \mathbf{sSet}$ preserves colimits. In particular, the functor $K \star -: \mathbf{sSet} \rightarrow \mathbf{sSet}$ preserves pushouts and sequential colimits and carries coproducts to wide pushouts under K . (The same statement holds for $- \star K$.)*

Proof. For any cocomplete category \mathcal{C} and $X \in \mathcal{C}$ colimits over J in $X \downarrow \mathcal{C}$ are computed as colimits over $[0] \star J$ in \mathcal{C} . Thus a colimit preserving functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a colimit preserving functor $X \downarrow \mathcal{C} \rightarrow FX \downarrow \mathcal{D}$.

It follows that in the composite above all the functors preserve colimits. (Note that Ran_i doesn't preserve all colimits as a functor $\mathbf{sSet} \rightarrow \mathbf{asSet}$ but it does as a functor $\emptyset \downarrow \mathbf{sSet} \rightarrow \Delta_a[-1] \downarrow \mathbf{asSet}$.)

The final statement holds since the inclusion $J \hookrightarrow [0] \star J$ is cofinal whenever J is connected and $[0] \star J$ is the indexing category for wide pushouts if J is discrete. \square

Corollary 2.7. *For each simplicial set K the functor $K \star -: \mathbf{sSet} \rightarrow K \downarrow \mathbf{sSet}$ has a right adjoint denoted by $(X: K \rightarrow M) \mapsto X \setminus M$. ($X \downarrow M$ is called the slice of M under X .)*

Proof. Since $K \star -$ is a colimit preserving functor on a category of presheaves its right adjoint is given by an explicit formula $(X \setminus M)_m = K \downarrow \mathbf{sSet}(K \star \Delta[m], M)$. \square

Lemma 2.8. *Let $P: \mathcal{C} \rightarrow \mathcal{D}$ be an inner isofibration of quasicategories and $X: K \rightarrow \mathcal{C}$ a diagram. Then the induced map $X \setminus \mathcal{C} \rightarrow PX \setminus \mathcal{D}$ is an inner isofibration. In particular, $X \setminus \mathcal{C}$ is a quasicategory.*

Proof. This follows from [Joy08, Theorem 3.19(i) and Proposition 4.10]. \square

For any simplicial set K we define the *under-cone* on K as $K^\triangleright = K \star \Delta[0]$.

Definition 2.9. Let \mathcal{C} be a quasicategory and let $X: K \rightarrow \mathcal{C}$ be any simplicial map (which we consider as a K -indexed diagram in \mathcal{C}).

- (1) A *cone under X* is a diagram $S: K^\triangleright \rightarrow \mathcal{C}$ such that $S|_K = X$.
- (2) A cone S under X is *universal* or a *colimit of X* if for any $m > 0$ and any diagram of solid arrows

$$\begin{array}{ccc} K \star \partial\Delta[m] & \xrightarrow{U} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ K \star \Delta[m] & & \end{array}$$

where $U|_{K^\triangleright} = S$ there exists a dashed arrow making the diagram commute.

- (3) An *initial object* of \mathcal{C} is a colimit of the unique empty diagram in \mathcal{C} .
- (4) A simplicial map $f: K \rightarrow L$ is *cofinal* if for every quasicategory \mathcal{C} and every universal cone $S: L^\triangleright \rightarrow \mathcal{C}$ the induced cone Sf^\triangleright is also universal.
- (5) The quasicategory \mathcal{C} is *finitely cocomplete* if for every finite simplicial set K every diagram $K \rightarrow \mathcal{C}$ has a colimit.
- (6) A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between finitely cocomplete quasicategories is *exact* (or *preserves finite colimits*) if for every finite simplicial set K and every universal cone $S: K^\triangleright \rightarrow \mathcal{C}$ the cone FS is also universal.

For any quasicategory \mathcal{C} and objects $x, y \in \mathcal{C}$ it is possible to construct the *mapping space* $\mathcal{C}(x, y)$, though there is no preferred such construction. A variety of (equivalent) possibilities is discussed in [DS11]. Then an object x is initial if and only if for every y the mapping space $\mathcal{C}(x, y)$ is contractible (see [Lur09, Proposition 1.2.12.4]) and the next lemma allows us to translate this observation to general colimits. However, it turns out that the definition given above is more convenient.

Lemma 2.10. *A cone S under X is universal if and only if it is an initial object of $X \setminus \mathcal{C}$.*

Proof. This follows directly from Corollary 2.7. □

In the remainder of this section we discuss the counterparts of classical statements of category theory saying that colimits are essentially unique and invariant under equivalences. For a quasicategory \mathcal{C} and a diagram $X: K \rightarrow \mathcal{C}$ we let $(X \setminus \mathcal{C})^{\text{univ}}$ denote the simplicial subset of $X \setminus \mathcal{C}$ consisting of these simplices whose all vertices are universal.

Lemma 2.11. *The simplicial set $(X \setminus \mathcal{C})^{\text{univ}}$ is empty or a contractible Kan complex.*

Proof. A simplicial set is empty or a contractible Kan complex if and only if it has the right lifting property with respect to the boundary inclusions $\partial\Delta[m] \hookrightarrow \Delta[m]$ for all $m > 0$. For $(X \setminus \mathcal{C})^{\text{univ}}$ such lifting problems are equivalent to the lifting problems

$$\begin{array}{ccc} K \star \partial\Delta[m] & \xrightarrow{U} & \mathcal{C} \\ \downarrow & \nearrow \text{---} & \\ K \star \Delta[m] & & \end{array}$$

with $U|(K \star \{i\})$ universal for each $i \in [m]$ which have solutions by the definition of universal cones. □

Corollary 2.12. *If $X: K \rightarrow \mathcal{C}$ is a diagram in a quasicategory and S and T are two universal cones under X , then they are equivalent under X , i.e. as objects of $X \setminus \mathcal{C}$.*

Proof. The simplicial set $(X \setminus \mathcal{C})^{\text{univ}}$ is non-empty and thus a contractible Kan complex by the previous lemma. Hence it has the right lifting property with respect to the inclusion $\partial\Delta[1] \hookrightarrow E[1]$ which translates to the lifting property

$$\begin{array}{ccc} K \star \partial\Delta[1] & \xrightarrow{[S, T]} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ K \star E[1] & & \end{array}$$

which yields an equivalence of S and T . \square

Lemma 2.13. *If \mathcal{C} is a quasicategory and X and Y are equivalent objects of \mathcal{C} , then X is initial if and only if Y is.*

Proof. Assume that X is initial and let $U: \partial\Delta[m] \rightarrow \mathcal{C}$ be such that $U|_{\Delta[0]} = Y$. We can consider an equivalence from X to Y as a diagram $f: \Delta[0] \star \Delta[0] \rightarrow \mathcal{C}$. Then by the universal property of X there is a diagram $\Delta[0] \star \partial\Delta[m]$ extending both f and U . (We can iteratively choose extensions over $\Delta[0] \star \Delta[k]$ for all faces $\Delta[k] \hookrightarrow \partial\Delta[m]$.) This diagram is a special outer horn (under the isomorphism $\Delta[0] \star \partial\Delta[m] \cong \Lambda^0[m+1]$) and thus has a filler by Lemma 2.4. Therefore U extends over $\Delta[m]$ and hence Y is initial. \square

2.3. Homotopy theory of cocomplete quasicategories

Our goal is to compare cofibration categories to quasicategories, but we expect cofibration categories to correspond to finitely cocomplete quasicategories, not to arbitrary ones. In this section we will restrict the fibration structure of Theorem 2.3 to the subcategory of finitely cocomplete quasicategories and exact functors.

The next two lemmas and the proposition are quasicategorical analogues of Lemmas A.2 and A.4 and Proposition 1.13. Their proofs are essentially the same, but somewhat more technical.

Lemma 2.14. *Let*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{G} & \mathcal{E} \\ Q \downarrow & & \downarrow P \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

be a pullback square of quasicategories where P is an inner isofibration. Let $S: K^\triangleright \rightarrow \mathcal{P}$ be a cone. If all GS , QS and $PGS = FQS$ are universal, then so is S .

Proof. Under these assumptions the square

$$\begin{array}{ccc}
X \setminus \mathcal{P} & \xrightarrow{G} & GX \setminus \mathcal{E} \\
Q \downarrow & & \downarrow P \\
QX \setminus \mathcal{C} & \xrightarrow{F} & PGX \setminus \mathcal{D}
\end{array}$$

(where $X = S|K$) is also a pullback along an inner isofibration by Lemma 2.8. Hence it suffices to verify the conclusion for initial objects.

Thus assume that $K = \emptyset$ and let $m > 0$ and $U: \partial\Delta[m] \rightarrow \mathcal{P}$ be such that $U|\Delta[0] = S$. Then we have

$$GU|\Delta[0] = GS \text{ and } QU|\Delta[0] = QS$$

and since both GS and QS are initial we can find $V_{\mathcal{E}} \in \mathcal{E}_m$ and $V_{\mathcal{C}} \in \mathcal{C}_m$ such that $V_{\mathcal{E}}|\partial\Delta[m] = GU$ and $V_{\mathcal{C}}|\partial\Delta[m] = QU$. Next, define $\tilde{V}: \partial\Delta[m+1] \rightarrow \mathcal{D}$ by replacing the 1st face of $PV_{\mathcal{E}}\sigma_1|\partial\Delta[m+1]$ with $FV_{\mathcal{C}}$ and $\tilde{W}: \Lambda^1[m+1] \rightarrow \mathcal{E}$ by setting it to $V_{\mathcal{E}}\sigma_1|\Lambda^1[m+1]$.

By the assumption PGS is initial and $\tilde{V}|\Delta[0] = PGS$ so \tilde{V} extends to $V \in \mathcal{D}_{m+1}$. Then we have a commutative square

$$\begin{array}{ccc}
\Lambda^1[m+1] & \xrightarrow{\tilde{W}} & \mathcal{E} \\
\downarrow & & \downarrow P \\
\Delta[m+1] & \xrightarrow{V} & \mathcal{D}
\end{array}$$

which admits a lift W since P is an inner isofibration and $0 < 1 < m+1$. We have $FV_{\mathcal{C}} = PW\delta_1$ and thus $(V_{\mathcal{C}}, W\delta_1)$ is an m -simplex of \mathcal{P} whose boundary is U . Hence S is initial. \square

Lemma 2.15. *Let $P: \mathcal{C} \rightarrow \mathcal{D}$ be an inner isofibration, $X: K \rightarrow \mathcal{C}$ a diagram and $T: K^{\triangleright} \rightarrow \mathcal{D}$ a colimit of PX . If X has a colimit in \mathcal{C} which is preserved by P , then there exists a colimit $S: K^{\triangleright} \rightarrow \mathcal{C}$ of X such that $PS = T$.*

Proof. Let $\tilde{S}: K^{\triangleright} \rightarrow \mathcal{C}$ be some colimit of X . Since both T and $P\tilde{S}$ are universal, we have a simplicial map $U: K \star E[1] \rightarrow \mathcal{D}$ such that $U|(K \star \partial\Delta[1]) = [T, P\tilde{S}]$ by Corollary 2.12. The conclusion now follows from Lemmas 2.8 and 2.13. \square

The homotopical content of the next proposition is the same as that of [Lur09, Lemma 5.4.5.5]. However, we need a stricter point-set level statement.

Proposition 2.16. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $P: \mathcal{E} \rightarrow \mathcal{D}$ be exact functors between finitely cocomplete quasicategories with P an inner isofibration. Then a pullback of P along F exists in the category of finitely cocomplete quasicategories and exact functors.*

Proof. Form a pullback of P along F in the category of quasicategories.

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{G} & \mathcal{E} \\
Q \downarrow & & \downarrow P \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

We will check that this square is also a pullback in the category of finitely cocomplete quasicategories and exact functors.

First, we verify that \mathcal{P} has finite colimits. Let $X: K \rightarrow \mathcal{P}$ be a diagram with K finite. Let $S: K^\triangleright \rightarrow \mathcal{C}$ be a colimit of QX , then FS is a colimit of $FQX = PGX$ in \mathcal{D} . Lemma 2.15 implies that we can choose a colimit T of GX in \mathcal{E} so that $PT = FS$. Then it follows by Lemma 2.14 that (S, T) is a colimit of $X = (QX, GX)$ in \mathcal{P} .

It remains to see that given a square

$$\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow P \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}$$

of finitely cocomplete quasicategories and exact functors, the induced functor $\mathcal{F} \rightarrow \mathcal{P}$ preserves finite colimits. Indeed, this follows directly from Lemma 2.14. \square

Theorem 2.17. *The category of small finitely cocomplete quasicategories with exact functors as morphisms, categorical equivalences as weak equivalences and (exact) inner isofibrations as fibrations is a fibration category.*

In fact, this fibration category is homotopy complete, i.e. it has all small homotopy limits. This will be explained in Section 3.3.

Proof. By Theorem 2.3 it suffices to observe that

- (1) a terminal quasicategory is also a terminal finitely cocomplete quasicategory (which is clear),
- (2) a pullback (in the category of all quasicategories) of finitely cocomplete quasicategories and exact functors one of which is an inner isofibration is also a pullback in the category of finitely cocomplete quasicategories which follows by (the proof of) Proposition 2.16,
- (3) for a finitely cocomplete quasicategory \mathcal{C} the functor $\mathcal{C}^{E[1]} \rightarrow \mathcal{C} \times \mathcal{C}$ is an exact functor between finitely cocomplete quasicategories. Indeed, $\mathcal{C}^{E[1]}$ is finitely cocomplete since it is categorically equivalent to \mathcal{C} (by Lemmas 2.10 and 2.13) and $\mathcal{C} \times \mathcal{C}$ is finitely cocomplete by (2). Finally, $\mathcal{C}^{E[1]} \rightarrow \mathcal{C} \times \mathcal{C}$ preserves finite colimits by (2) since both projections $\mathcal{C}^{E[1]} \rightarrow \mathcal{C}$ do. \square

3. Quasicategories of frames in cofibration categories

In this chapter we will associate to every cofibration category \mathcal{C} a corresponding quasicategory called the *quasicategory of frames in \mathcal{C}* obtaining an exact functor between the fibration categories established in Chapters 1 and 2. We will eventually prove that this functor is a weak equivalence between these fibration categories. However, verification that our construction takes values in finitely cocomplete quasicategories and that it is an exact functor takes some effort and it will occupy this entire chapter.

3.1. Definitions and basic properties

Before introducing quasicategories of frames we need to explain a preliminary construction which will play an essential role in the remainder of this thesis.

Let Δ_{\sharp} denote the subcategory of injective maps in Δ and let J be a homotopical category. We construct a direct homotopical category DJ and a homotopical functor $p_J: DJ \rightarrow J$ as follows. The underlying category of DJ is the slice $\Delta_{\sharp} \downarrow J$, i.e. objects are all functors $[m] \rightarrow J$ for all m and a morphism from $x: [m] \rightarrow J$ to $y: [n] \rightarrow J$ is an injective order preserving map $\varphi: [m] \hookrightarrow [n]$ such that $x = y\varphi$. The functor $p_J: \Delta_{\sharp} \downarrow J \rightarrow J$ is defined by evaluating $[m] \rightarrow J$ at m and the weak equivalences in DJ are created by p_J . Then DJ is homotopical category, p_J is a homotopical functor and DJ is also direct (by setting the degree of $[m] \rightarrow J$ to m). We can think of DJ as a *direct approximation* to J . Observe that D is a functor from homotopical categories to homotopical categories and that DJ has a non-trivial homotopical structure even if J has the trivial one (unless J is empty). This construction has multiple motivations which will be given right after the definition of quasicategories of frames below.

First, we need to verify that Reedy cofibrant diagrams over DJ are well behaved with respect to homotopical functors $I \rightarrow J$. If f is such a functor we will abbreviate the induced functor $(Df)^*: \mathcal{C}_R^{DJ} \rightarrow \mathcal{C}_R^{DI}$ to f^* to simplify the notation. Recall that \mathcal{C}_R^{DJ} refers to the cofibration category of homotopical Reedy cofibrant diagrams $DJ \rightarrow \mathcal{C}$ with levelwise weak equivalences and Reedy cofibrations which exists by Proposition 1.29 since DJ has finite latching categories.

Lemma 3.1. *Let \mathcal{C} be a cofibration category. If $f: I \rightarrow J$ is a homotopical functor of small homotopical categories, then the induced functor $f^*: \mathcal{C}_R^{DJ} \rightarrow \mathcal{C}_R^{DI}$ is exact. If f is injective on objects and faithful, then f^* is a fibration.*

Proof. Both statements follow from Lemma 1.23 since if f is injective on objects and faithful, then Df is a sieve. \square

For a cofibration category \mathcal{C} we define the *quasicategory of frames in \mathcal{C}* as a simplicial set denoted by $N_f \mathcal{C}$ where $(N_f \mathcal{C})_m$ is the set of all homotopical Reedy cofibrant diagrams

$D[m] \rightarrow \mathcal{C}$ ($[m]$ is a homotopical category with only identities as weak equivalences). The simplicial structure is given by functoriality of D (using Lemma 3.1 to see that simplicial operators preserve Reedy cofibrancy). Since exact functors of cofibration categories preserve Reedy cofibrant diagrams, N_f is a functor from the category of cofibration categories to the category of simplicial sets.

Remark 3.2. As a side note, we point out that this construction can be enhanced as follows. If $\widehat{[n]}$ denotes the homotopical poset $[n]$ with all morphisms as weak equivalences, then the bisimplicial set

$$[m], [n] \mapsto \{\text{homotopical Reedy cofibrant diagrams } D([m] \times \widehat{[n]}) \rightarrow \mathcal{C}\}$$

is a complete Segal space with $N_f \mathcal{C}$ as its 0th row.

This definition has a threefold motivation. First, the objects of $N_f \mathcal{C}$ are called *frames* in \mathcal{C} . They are counterparts to frames in a model category \mathcal{M} , i.e. homotopically constant Reedy cofibrant diagrams $\Delta \rightarrow \mathcal{M}$ which can be used to enrich the homotopy category $\text{Ho} \mathcal{M}$ in the homotopy category of simplicial sets as explained in [Hov99, Chapter 5]. In cofibration categories we are forced to replace Δ by $\Delta_{\#}$ and then homotopically constant diagrams over $\Delta_{\#}$ are precisely the homotopical diagrams over $D[0]$. Again, one can prove using such frames that the homotopy category $\text{Ho} \mathcal{C}$ is enriched in the category of homotopy types, see [Sch13, Theorems 3.10 and 3.17].¹ Our construction can be seen as an alternative way of using frames to enrich $\text{Ho} \mathcal{C}$ in homotopy types, namely, by using the mapping spaces of the quasicategory $N_f \mathcal{C}$.

The second motivation is that $N_f \mathcal{C}$ can be seen as an enhancement of the calculus of fractions. Let $\text{Sd}[m]$ denote the poset of non-empty subsets of m . It can be seen as the full subcategory of $D[m]$ spanned by the non-degenerate simplices of $[m]$ as explained in more detail on p. 58. Homotopical Reedy cofibrant diagrams over $D[m]$ can be seen as resolutions of their restrictions to $\text{Sd}[m]$. Therefore an object of $N_f \mathcal{C}$ is a resolution of an object of \mathcal{C} and its morphism is a resolution of a diagram of the form

$$X_0 \longrightarrow X_{01} \xleftarrow{\sim} X_1,$$

i.e. a left fraction from X_0 to X_1 . Similarly, a 2-simplex of $N_f \mathcal{C}$ is a resolution of a diagram of the form

$$\begin{array}{ccccc}
 & & X_1 & & \\
 & \swarrow \sim & & \searrow & \\
 & X_{01} & & X_{12} & \\
 & \searrow & \xrightarrow{\sim} & \swarrow \sim & \\
 & & X_{012} & & \\
 & \swarrow \sim & \uparrow \sim & \searrow \sim & \\
 X_0 & \longrightarrow & X_{02} & \longleftarrow \sim & X_2
 \end{array}$$

¹This result differs from its counterpart for model categories since it uses presimplicial sets (a.k.a. Δ -sets or semisimplicial sets) as models of homotopy types. Presimplicial sets are less well-behaved than simplicial sets, but their homotopy theory is equivalent to that of simplicial sets.

which consists of two fractions going from X_0 to X_1 and from X_1 to X_2 along with a composite fraction going directly from X_0 to X_2 . Such diagrams simultaneously encode the composition of left fractions and the notion of equivalence of fractions which is made precise in the proof of Lemma 4.10. Higher simplices encode the higher homotopy of the mapping spaces of \mathcal{C} in a similar manner.

It might be tempting to simplify the definition of $N_f \mathcal{C}$ by replacing $D[m]$ with $Sd[m]$. This would not work since functors $Sd[m] \rightarrow Sd[n]$ induced by degeneracy operators $[m] \rightarrow [n]$ do not respect Reedy cofibrant diagrams and thus this modification would not even yield a simplicial set.

Finally, the quasicategory of frames can be motivated by the discussion in Section 3.3 which suggests that homotopical Reedy cofibrant diagrams $DJ \rightarrow \mathcal{C}$ contain the information about all homotopy colimits in \mathcal{C} . In fact, this information can be reduced just to homotopical Reedy cofibrant diagrams $D[m] \rightarrow \mathcal{C}$ as implied by Proposition 3.7. These observations will be formalized in the next theorem that says, among other things, that the functor N_f converts homotopy colimits in the sense of homotopical algebra to colimits in quasicategories. (A more precise statement to this effect is Proposition 3.32.)

Theorem 3.3. *The functor N_f takes values in finitely cocomplete quasicategories and is an exact functor from the fibration category of Theorem 1.17 to the fibration category of Theorem 2.17.*

One part of the proof is quite easy.

Proposition 3.4. *The functor N_f preserves a terminal object and pullbacks along fibrations.*

Proof. The preservation of a terminal object is clear. In order to see that pullbacks are also preserved it suffices to verify that given a pullback square

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{G} & \mathcal{E} \\ Q \downarrow & & \downarrow P \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D}. \end{array}$$

of cofibration categories and exact functors a functor $X: D[m] \rightarrow \mathcal{P}$ is a homotopical Reedy cofibrant diagram if and only if both QX and GX are. This follows since latching objects in \mathcal{P} are computed pointwise in \mathcal{C} and \mathcal{E} by Lemma A.2. \square

We will commit the next section to the verification that N_f preserves (acyclic) fibrations. Before that we need to establish some basic properties of this functor.

First, we will give another version of the D construction. For a simplicial set K we define a homotopical direct category DK as follows. The underlying category of DK is the category of elements of K but *only with face operators* as morphisms, i.e. objects of DK are all simplices of K and a morphism from $x \in K_m$ to $y \in K_n$ is an injective order preserving map $\varphi: [m] \hookrightarrow [n]$ such that $x = y\varphi$.

Such a morphism is a *generating* weak equivalence if $y\nu$ is a degenerate edge of K where $\nu: [1] \rightarrow [n]$ is defined by $\nu(0) = \varphi(m)$ and $\nu(1) = n$. The generating weak equivalences do not necessarily satisfy the “2 out of 6” property (they are not even closed under composition in general). Thus we define the subcategory of weak equivalences as the smallest subcategory containing the generating weak equivalences and satisfying the “2 out of 6” property. Of course, in order to verify that a functor from DK to a homotopical category is homotopical it suffices to check that it sends the generating weak equivalences to weak equivalences.

This construction is functorial in K . Moreover, the next lemma says that if K is the nerve of a category J , then DK coincides with DJ in the sense of the previous definition.

Lemma 3.5. *Let J be a category with the trivial homotopical structure. Then the homotopical categories DJ and DNJ coincide.*

Proof. The underlying categories of DJ and DNJ are the same by definition. The generating weak equivalences of DNJ are mapped to identities by $p_J: DJ \rightarrow J$ and hence it suffices to see that every weak equivalence created by p_J can be obtained from the generating ones by applying the “2 out of 6” property. Let $\varphi, \psi \in DJ$ and consider a morphism $\varphi \rightarrow \psi$ mapped by p_J to an isomorphism $f: x \rightarrow y$ of J . Then we have a diagram

$$\begin{array}{ccccccc}
 & & \sim & & & & \\
 & & \curvearrowright & & & & \\
 x & \longrightarrow & xy & \longrightarrow & xyx & \longrightarrow & xyxy \\
 \downarrow \sim & & \downarrow \sim & & \curvearrowleft \sim & & \\
 \varphi & \longrightarrow & \psi & & & &
 \end{array}$$

in DJ where $xyxy$ denotes the sequence

$$x \xrightarrow{f} y \xrightarrow{f^{-1}} x \xrightarrow{f} y$$

and the remaining objects in the first row are its initial segments. The indicated morphisms are generating weak equivalences and hence by “2 out of 6” $\varphi \rightarrow \psi$ is also a weak equivalence of DNJ . \square

Lemma 3.6. *The functor $D: \mathbf{sSet} \rightarrow \mathbf{Cat}$ (i.e. when we disregard the homotopical structures of DK s) preserves colimits.*

Proof. Since $N: \mathbf{Cat} \rightarrow \mathbf{sSet}$ is fully faithful it reflects colimits (see [Bor94, Proposition 2.2.9]). Thus it will suffice to verify that the composite functor $K \mapsto NDK$ preserves colimits. This follows from the fact that

$$(NDK)_m = \coprod_{[j_0] \hookrightarrow [j_1] \hookrightarrow \dots \hookrightarrow [j_m]} K_{j_m}. \quad \square$$

Let $X: DK \rightarrow \mathcal{C}$ be a homotopical Reedy cofibrant diagram. For each simplex $x: \Delta[m] \rightarrow K$ consider the restriction $x^*X: D[m] \rightarrow \mathcal{C}$ which is an m -simplex of $N_f \mathcal{C}$. (Recall that x^* is an abbreviation of $(Dx)^*$.) These simplices fit together to form a simplicial map $K \rightarrow N_f \mathcal{C}$.

Proposition 3.7. *Let \mathcal{C} be a cofibration category and K a simplicial set. The map described above is a natural bijection between*

- *the set of homotopical Reedy cofibrant diagrams $DK \rightarrow \mathcal{C}$*
- *and the set of simplicial maps $K \rightarrow N_f \mathcal{C}$.*

Proof. Denote the former set by $R(DK, \mathcal{C})$ and observe that $R(D-, \mathcal{C})$ is a contravariant functor from simplicial sets to sets. The statement says that this functor is representable and the representing object is $N_f \mathcal{C}$. This will follow by Lemma A.6 if we can verify that if we consider any simplicial set K as a colimit of its simplices, then this colimit is preserved (i.e. carried to a limit) by $R(D-, \mathcal{C})$.

First, note that the functor $\text{Cat}(D-, \mathcal{C})$ carries colimits to limits by Lemma 3.6. Since $R(D-, \mathcal{C})$ is a subfunctor of $\text{Cat}(D-, \mathcal{C})$ it will suffice to see that a diagram $X: DK \rightarrow \mathcal{C}$ is homotopical and Reedy cofibrant if and only if for all $x \in K_m$ the induced diagram x^*X is homotopical and Reedy cofibrant. The cofibrancy statement follows by (the argument of) Lemma 3.1.

It is clear that if X is homotopical then so are all x^*X . In order to prove the converse it suffices to consider the generating weak equivalences of DK . Let $x \in K_m, y \in K_n$ and $\varphi: [m] \hookrightarrow [n]$ be such that $x = y\varphi$ and $y\nu$ is a degenerate edge where $\nu: [1] \rightarrow [n]$ is defined by $\nu(0) = \varphi(m)$ and $\nu(1) = n$. We need to prove that $X\varphi$ is a weak equivalence in \mathcal{C} . First, let's assume that $\varphi(m) = n$, then φ is a weak equivalence when seen as a morphism $\varphi \rightarrow \text{id}_{[n]}$ in $D[n]$. Therefore $X\varphi = (y^*X)\varphi$ is a weak equivalence since y^*X is a homotopical diagram. Next, assume that $\varphi(m) < n$, then ν is injective and can be seen as a morphism $y\nu \rightarrow y$ in DK and we have a commutative diagram on the left in Δ_{\sharp} which can be reinterpreted as a diagram in the middle in DK which in turn yields the diagram on the right in \mathcal{C} (here $\varepsilon_i: [0] \rightarrow [k]$ is the morphism with image i).

$$\begin{array}{ccccc}
[0] & \xrightarrow{\varepsilon_m} & [m] & & y\varepsilon_m & \xrightarrow{\varepsilon_m} & y\varphi & & X(y\varepsilon_m) & \xrightarrow{X\varepsilon_m} & X(y\varphi) \\
\varepsilon_0 \downarrow & & \downarrow \varphi & & \varepsilon_0 \downarrow & & \downarrow \varphi & & X\varepsilon_0 \downarrow & & \downarrow X\varphi \\
[1] & \xrightarrow{\nu} & [n] & & y\nu & \xrightarrow{\nu} & y & & X(y\nu) & \xrightarrow{X\nu} & Xy
\end{array}$$

Now, ε_m and ν are weak equivalences when seen as morphisms of $D[m]$ and $D[n]$ respectively. Thus $X\varepsilon_m$ and $X\nu$ are weak equivalences. The edge $y\nu$ is degenerate, i.e. $y\nu = y\varepsilon_n\sigma_0$, so the diagram $(y\nu)^*X: D[1] \rightarrow \mathcal{C}$ factors through $(y\varepsilon_n)^*X: D[0] \rightarrow \mathcal{C}$. Since all morphisms of $D[0]$ are weak equivalences it follows that $(y\nu)^*X$ sends all morphisms, including ε_0 above, to weak equivalences thus $X\varepsilon_0$ is a weak equivalence and hence so is $X\varphi$. \square

This immediately implies the following.

Corollary 3.8. *Let $i: K \rightarrow L$ be a simplicial map and $F: \mathcal{C} \rightarrow \mathcal{D}$ an exact functor between cofibration categories. Then $N_f F$ has the right lifting property with respect to i if and only if F has the Reedy right lifting property with respect to Di . \square*

Our goal is to find some general procedure of solving such lifting problems.

3.2. Reedy lifting properties

The results of Section 1.3 give criteria for verifying Reedy lifting properties. In this section we verify these criteria for the inner horn inclusions $D\Lambda^i[m] \hookrightarrow D[m]$ and for $D[0] \rightarrow DE[1]$.

The case of inner horn inclusions will be handled by comparing both $D[m]$ and $D\Lambda^i[m]$ to $[m]$ and various “generalized inner horns”.

Lemma 3.9. *For every $m \geq 0$ the functor $p_{[m]}: D[m] \rightarrow [m]$ is a homotopy equivalence of homotopical categories.*

Proof. Let $f: [m] \rightarrow D[m]$ be the functor that sends $i \in [m]$ to the standard inclusion $[i] \hookrightarrow [m]$. This is a homotopical functor and we have $p_{[m]}f = \text{id}_{[m]}$. We will verify that $fp_{[m]}$ is weakly equivalent to $\text{id}_{D[m]}$ which will finish the proof.

To this end define $s: D[m] \rightarrow D[m]$ as follows. Represent an object $x \in D[m]$ as a non-empty finite non-decreasing sequence of elements of $[m]$. Then $s(x)$ is obtained by inserting one extra occurrence of each of the elements $0, 1, \dots, p_{[m]}(x)$ into x . Every such element i is added “at the end” of the (possibly empty) block of i s already present in x . This explains the functoriality of s . Namely, given $\varphi: x \rightarrow y$ and $i \leq p_{[m]}(x)$, the map $s(\varphi)$ acts on the “old” occurrences of i as φ does and sends the “new” occurrences to the “new” occurrences. Thus the functor s is homotopical and admits natural weak equivalences

$$\text{id} \xrightarrow{\sim} s \xleftarrow{\sim} fp_{[m]}$$

where the map on the left inserts x onto the “old” occurrences in $s(x)$ and the right one inserts $fp_{[m]}(x)$ onto the “new” ones. \square

Let $A \subseteq [m]$, we define the *generalized horn* $\Lambda^A[m]$ as the simplicial subset of $\Delta[m]$ generated by its codimension 1 faces lying opposite of vertices not in A . Observe that $\Lambda^{\{i\}}[m] = \Lambda^i[m]$.

Lemma 3.10. *The inclusion functor $D\Lambda^{\{1, \dots, m-1\}}[m] \hookrightarrow D[m]$ induces a weak equivalence $\mathcal{C}_R^{D[m]} \rightarrow \mathcal{C}_R^{D\Lambda^{\{1, \dots, m-1\}}[m]}$ for every cofibration category \mathcal{C} and each $m \geq 2$.*

Proof. It suffices to verify the statement for the levelwise structures by Lemma 1.20 and hence it will be enough to show that the composite $D\Lambda^{\{1, \dots, m-1\}}[m] \hookrightarrow D[m] \rightarrow [m]$ induces a weak equivalence with respect to the levelwise structures.

In the diagram

$$\begin{array}{ccccc}
D[m-2] & \xrightarrow{\delta_{m-1}} & D[m-1] & & \\
\downarrow \delta_0 & \searrow & \downarrow \delta_{m-1} & \searrow & \\
& [m-2] & \xrightarrow{\delta_{m-1}} & [m-1] & \\
& \downarrow & \downarrow & \downarrow & \\
D[m-1] & \xrightarrow{\delta_0} & D\Lambda^{\{1, \dots, m-1\}}[m] & & \\
& \searrow & \downarrow & \searrow & \\
& [m-1] & \xrightarrow{\delta_0} & [m] &
\end{array}$$

the back square is a pushout of two sieves hence it induces a homotopy pullback of the associated categories of Reedy cofibrant diagrams by Lemma 1.25. The front square is a pushout along a sieve, but the vertical map is not a sieve. Nonetheless, the conclusion of Lemma 1.25 holds because of a particularly simple form of the latching categories in totally ordered sets so that a map of diagrams $[m-1] \rightarrow \mathcal{C}$ is a Reedy cofibration if and only if it is one when restricted along both δ_0 and δ_{m-1} . Hence both squares induce homotopy pullbacks on levelwise categories of diagrams by Lemma 1.20 and then the assumptions of the Gluing Lemma are satisfied by Lemma 3.9 which finishes the proof. \square

An *interval* is a subset of $[m]$ of the form $\{x \in [m] \mid i \leq x \leq j\}$ for some $i \leq j \in [m]$. In the next lemma we will consider generalized horns $\Lambda^A[m]$ with $A \subseteq [m]$ such that $[m] \setminus A$ is not an interval (e.g. $A = \{1, \dots, m-1\}$). Such horns are called *generalized inner horns*.

Lemma 3.11. *Let $A \subseteq B$ be subsets of $[m]$ whose complements are not intervals. Then the inclusion $\Lambda^B[m] \hookrightarrow \Lambda^A[m]$ is a composite of pushouts of inner horn inclusions in dimensions at most $m - |A|$. Moreover, all these horns are attached along injective maps.*

Proof. This follows by the proof of [Joy08, Proposition 2.12 (iv)]. (The proposition itself is less specific, but the inductive step in its proof amounts exactly to the statement above.) \square

Proposition 3.12. *The functor N_f carries fibrations of cofibration categories to inner fibrations.*

Proof. By Lemmas 1.26 and 1.24 it suffices to check that $D\Lambda^i[m] \hookrightarrow D[m]$ induces a weak equivalence $\mathcal{C}_R^{D[m]} \rightarrow \mathcal{C}_R^{D\Lambda^i[m]}$ for every cofibration category \mathcal{C} and $0 < i < m$. By Lemma 3.10 it will be enough to check this for $D\Lambda^{\{1, \dots, m-1\}}[m] \hookrightarrow D\Lambda^i[m]$.

That follows by an induction with respect to m since this inclusion is built out of pushouts of horn inclusions in dimensions below m by Lemma 3.11. Since these are pushouts along injective maps Lemma 1.25 says that they induce pullbacks of cofibration categories of Reedy diagrams. \square

Next, we move to $[0] \hookrightarrow DE[1]$ which will be dealt with by constructing an explicit contraction of $DE[1] = DE(1)$.

Lemma 3.13. *The functor $f: [0] \rightarrow DE(1)$ given by the sequence $0 \in DE(1)$ is a homotopy equivalence of homotopical categories.*

Proof. The proof is similar to that of Lemma 3.9. This time objects of $DE(1)$ are represented as arbitrary finite non-empty binary sequences. Let $p: DE(1) \rightarrow [0]$ be the unique functor to $[0]$ and let $s: DE(1) \rightarrow DE(1)$ append a new 0 to every sequence. (As before, $s(\varphi)$ acts on “old” elements as φ and sends the “new” 0 to the “new” 0.) Every morphism of $E(1)$ is an isomorphism so the homotopical structure on $DE(1)$ is the maximal one. Hence the functor s is homotopical and admits natural weak equivalences

$$\text{id} \xrightarrow{\sim} s \xleftarrow{\sim} fp$$

where the map on the left inserts x onto the “old” occurrences in $s(x)$ and the right one inserts $fp(x)$ onto the “new” 0. \square

Before completing the main result of this section we record a corollary which will considerably simplify constructions of $E[1]$ -homotopies in the final chapter.

Corollary 3.14. *For a cofibration category \mathcal{C} a homotopical Reedy cofibrant diagram $X: D[1] \rightarrow \mathcal{C}$ is an equivalence when seen as a morphism of $N_f \mathcal{C}$ if and only if it is homotopical with respect to $D[1]$.*

Proof. If X is an equivalence, then it extends to $DE[1]$. Hence it is homotopical with respect to $D[1]$.

Conversely, consider a diagram

$$\begin{array}{ccccc} & & \cong & & \\ & & \curvearrowright & & \\ D[0] & \longrightarrow & D[\widehat{1}] & \longrightarrow & DE[1] \\ & \cong \downarrow & & \downarrow \cong & \\ [0] & \xrightarrow{\cong} & [1] & & \end{array}$$

where the indicated maps are homotopy equivalences, the vertical ones by (the proof of) Lemma 3.9, the top one by Lemma 3.13 and the bottom one by direct inspection. Hence so is the map $D[\widehat{1}] \rightarrow DE[1]$ which is also a sieve so that the induced restriction functor $\mathcal{C}_R^{DE[1]} \rightarrow \mathcal{C}_R^{D[\widehat{1}]}$ is an acyclic fibration and thus every homotopical Reedy cofibrant diagram on $D[\widehat{1}]$ extends to one on $DE[1]$. \square

Proposition 3.15. *The functor N_f carries fibrations of cofibration categories to isofibrations.*

Proof. By Lemma 1.26 it suffices to check that $D[0] \hookrightarrow E(1)$ induces a weak equivalence $\mathcal{C}_R^{DE(1)} \rightarrow \mathcal{C}_R^{D[0]}$ for every cofibration category \mathcal{C} . Lemma 3.13 asserts that this is the case for the composite

$$[0] \hookrightarrow D[0] \hookrightarrow DE(1)$$

while Lemma 3.9 says the same for the first functor. Thus the conclusion follows by 2 out of 3. \square

Proposition 3.16. *The functor N_f carries acyclic fibrations of cofibration categories to acyclic Kan fibrations.*

Proof. This follows from Lemmas 1.26 and 1.27 and the fact that $D\partial\Delta[m] \hookrightarrow D[m]$ is a sieve for all m . \square

This concludes the verification of all lifting properties necessary for the exactness of N_f . In the remainder of this section we will derive some further lifting properties which will be useful later.

Occasionally, it will be convenient to consider *marked simplicial complexes* instead of simplicial sets. Recall from the classical simplicial homotopy theory that an *ordered simplicial complex* is a poset P equipped with a family of finite, non-empty totally ordered subsets of P (called *simplices*) such that

- a non-empty subset of a simplex is a simplex,
- for each $x \in P$ the singleton $\{x\}$ is a simplex.

Simplicial complexes with an underlying poset P can be identified with simplicial subsets of NP (containing all vertices of NP). This is the point of view that we will adopt to define a marked version of this notion.

Definition 3.17. A *marked simplicial complex* is a simplicial set K equipped with an embedding $K \hookrightarrow NP$ where P is a homotopical poset.

On one hand, a marked simplicial complex K is more special than a generic simplicial set which will make it easier to construct diagrams on DK . On the other, the additional structure will allow more flexible arguments. (Just like *marked simplicial sets* provide some extra flexibility to the theory of quasicategories. In fact, marked simplicial complexes can be seen as certain special marked simplicial sets.)

We extend the definition of DK to a marked simplicial complex K as follows. The underlying category of DK is the same as previously, but the homotopical structure is created by the inclusion $DK \hookrightarrow DP$. This agrees with the old definition when P has the trivial homotopical structure.

Moreover, for a marked simplicial complex K we define a homotopical poset $\text{Sd } K$ as the full subcategory of DK spanned by the non-degenerate simplices of K and with the homotopical structure inherited from DP . The category $\text{Sd } K$ is known as the *barycentric subdivision* of K hence the notation. (By analogy we may think of DK as the

fat barycentric subdivision of K .) It is indeed a poset since its objects can be identified with finite non-empty totally ordered subsets of P that correspond to non-degenerate simplices of K (just as in the classical definition of an ordered simplicial complex above) and morphisms with inclusions of such subsets. With this interpretation an inclusion $A \subseteq B$ is a weak equivalence if and only if $\max A \rightarrow \max B$ is a weak equivalence of P . (Of course, if P has the trivial homotopical structure, then this condition reduces to $\max A = \max B$.) In the case when $K = NP$ we will usually write $\text{Sd } P$ in place of $\text{Sd } K$.

The next two lemmas will allow us to reduce constructions of diagrams over DK to constructions of diagrams over $\text{Sd } K$.

Lemma 3.18. *For any marked simplicial complex K the inclusion $f: \text{Sd } K \rightarrow DK$ is a homotopy equivalence.*

Proof. The construction is a minor modification of the one used in Lemma 3.9. Let P denote the underlying homotopical poset of K . We define $q_K: DK \rightarrow \text{Sd } K$ by sending each simplex of K seen as a map $[k] \rightarrow P$ to its image and $s: DK \rightarrow DK$ by inserting one extra occurrence of each $p \in P$ that is already present in a given $x \in DK$. Just as in Lemma 3.9 a new occurrence is inserted at the end of the block of the old occurrences which yields analogous weak equivalences

$$\text{id} \xrightarrow{\sim} s \xleftarrow{\sim} fq_K.$$

Moreover, $q_K f = \text{id}_{\text{Sd } K}$ which finishes the proof. \square

Lemma 3.19. *Let $K \hookrightarrow L$ be an injective map of finite marked simplicial complexes (which means that it covers an injective homotopical map of the underlying homotopical posets). Then for every cofibration category \mathcal{C} the inclusion $DK \cup \text{Sd } L \hookrightarrow DL$ induces an acyclic fibration $\mathcal{C}_R^{DL} \rightarrow \mathcal{C}_R^{DK \cup \text{Sd } L}$.*

Proof. We have the following pushout square of sieves between homotopical direct categories on the left and hence a pullback square of cofibration categories on the right by Lemma 1.25.

$$\begin{array}{ccc} \text{Sd } K & \hookrightarrow & \text{Sd } L \\ \downarrow & & \downarrow \\ DK & \hookrightarrow & DK \cup \text{Sd } L \end{array} \quad \begin{array}{ccc} \mathcal{C}_R^{DK \cup \text{Sd } L} & \twoheadrightarrow & \mathcal{C}_R^{DK} \\ \downarrow & & \downarrow \\ \mathcal{C}_R^{\text{Sd } L} & \twoheadrightarrow & \mathcal{C}_R^{\text{Sd } K} \end{array}$$

The fibration $\mathcal{C}_R^{DK} \rightarrow \mathcal{C}_R^{\text{Sd } K}$ is acyclic by Lemma 3.18 and hence so is $\mathcal{C}_R^{DK \cup \text{Sd } L} \rightarrow \mathcal{C}_R^{\text{Sd } L}$. Moreover, we have a triangle of fibrations

$$\begin{array}{ccc} \mathcal{C}_R^{DL} & \twoheadrightarrow & \mathcal{C}_R^{\text{Sd } L} \\ & \searrow & \nearrow \\ & \mathcal{C}_R^{DK \cup \text{Sd } L} & \end{array}$$

where $\mathcal{C}_R^{DL} \rightarrow \mathcal{C}_R^{\text{Sd } L}$ is acyclic again by Lemma 3.18 and thus so is $\mathcal{C}_R^{DL} \rightarrow \mathcal{C}_R^{DK \cup \text{Sd } L}$. \square

For future reference we will reinterpret lifting properties for special outer horns in terms of certain homotopical structures on categories $D\Lambda^0[m]$ and $D\Lambda^m[m]$.

For each $m > 1$ let $\langle m \rangle$ denote the homotopical poset with the underlying poset $[m]$ and $0 \xrightarrow{\sim} 1$ as the only non-identity weak equivalence. Similarly, let $[m]$ denote the homotopical poset with the underlying poset $[m]$ and $m-1 \xrightarrow{\sim} m$ as the only non-identity weak equivalence. Let $\Lambda^0\langle m \rangle$ and $\Lambda^m[m]$ denote the outer horns seen as marked simplicial complexes with the underlying homotopical posets $\langle m \rangle$ and $[m]$.

Lemma 3.20. *For every cofibration category \mathcal{C} the inclusion $D\Lambda^0\langle m \rangle \hookrightarrow D\langle m \rangle$ induces a weak equivalence $\mathcal{C}_R^{D\langle m \rangle} \rightarrow \mathcal{C}_R^{D\Lambda^0\langle m \rangle}$.*

The same holds for $D\Lambda^m[m] \hookrightarrow D[m]$.

Proof. By Lemma 1.28 it will suffice to see that the inclusion $D\Lambda^0\langle m \rangle \hookrightarrow D\langle m \rangle$ has the Reedy left lifting property with respect to all fibrations of cofibration categories.

By Proposition 3.7 every Reedy lifting problem of $D\Lambda^0\langle m \rangle \hookrightarrow D\langle m \rangle$ against a fibration of cofibration categories $P: \mathcal{C} \rightarrow \mathcal{D}$ is equivalent to a problem of lifting $\Lambda^0\langle m \rangle \hookrightarrow \langle m \rangle$ against $N_f P$ where the latter is an inner isofibration by Propositions 3.12 and 3.15 and the horn is special by Corollary 3.14. Hence it has a solution by Lemma 2.4.

The same argument works for $D\Lambda^m[m] \hookrightarrow D[m]$ since Lemma 2.4 applies to both types of special horns. \square

Let $[k + \tilde{1} + m]$ denote a homotopical category with underlying category $[k + 1 + m]$ and $k \xrightarrow{\sim} k + 1$ as the only non-identity weak equivalence. Let $\Lambda^{[k]}[k + \tilde{1} + m]$ denote the generalized horn $\Lambda^{[k]}[k + 1 + m]$ seen as a marked simplicial complex with the underlying homotopical poset $[k + \tilde{1} + m]$. The next lemma is a generalization of the previous one.

Lemma 3.21. *The inclusion $D\Lambda^{[k]}[k + \tilde{1} + m] \hookrightarrow D[k + \tilde{1} + m]$ has the Reedy left lifting property with respect to all fibrations of cofibration categories. Hence for any cofibration category \mathcal{C} it induces a weak equivalence $\mathcal{C}_R^{D[k + \tilde{1} + m]} \rightarrow \mathcal{C}_R^{D\Lambda^{[k]}[k + \tilde{1} + m]}$.*

Proof. The case of $k = 0$ is just the previous lemma (with m replaced by $1 + m$). The case of $k > 0$ can be reduced to the case of $k = 0$ as follows. We have $[k + 1 + m] \cong [k] \star [m]$ and $\Lambda^{[k]}[k + 1 + m] \cong \Delta[k] \star \partial\Delta[m]$ and hence it will suffice to solve every lifting problem

$$\begin{array}{ccc} \Delta[k] \star \partial\Delta[m] & \xrightarrow{X} & \mathcal{C} \\ \downarrow & & \downarrow P \\ \Delta[k] \star \Delta[m] & \xrightarrow{Y} & \mathcal{D} \end{array}$$

where X and Y send the edge $k \rightarrow k + 1$ to an equivalence and P is an inner isofibration (by Proposition 3.7). This problem is equivalent to

$$\begin{array}{ccc} \{k\} \star \partial\Delta[m] & \longrightarrow & X' \setminus \mathcal{C} \\ \downarrow & & \downarrow \\ \{k\} \star \Delta[m] & \longrightarrow & Y' \setminus \mathcal{D} \end{array}$$

where X' and Y' are the restrictions of X and Y to $\Delta[k-1]$ so that the resulting horn is special (under identifications $\{k\} \star \Delta[m] \cong \Delta[1+m]$ and $\{k\} \star \partial\Delta[m] \cong \Lambda^0[1+m]$). It has a solution by the case of $k=0$. \square

3.3. Infinite homotopy colimits

The next step is to verify that $N_f \mathcal{C}$ is finitely cocomplete. This proof is rather involved, but it turns out that this is largely due to certain technicalities which disappear if we assume that \mathcal{C} has some infinite homotopy colimits.

In this section we explain how infinite homotopy colimits can be introduced to cofibration categories and how the results discussed so far can be extended to this context.

Infinite homotopy colimits in cofibration categories

We will consider the following axioms in addition to axioms (C0-5) of Chapter 1.

(C6) Cofibrations are stable under sequential colimits, i.e. given a sequence of cofibrations

$$A_0 \twoheadrightarrow A_1 \twoheadrightarrow A_2 \twoheadrightarrow \dots$$

its colimit A_∞ exists and the induced morphism $A_0 \rightarrow A_\infty$ is a cofibration. Acyclic cofibrations are stable under sequential colimits.

(C7- κ) Coproducts of κ -small families of objects exist. Cofibrations and acyclic cofibrations are stable under κ -small coproducts.

Axiom (C7) is parametrized by a regular cardinal number κ . (And if we write (C7) we will take it to refer to all small coproducts.) A set is κ -small if its cardinality is strictly less than κ . In particular, \aleph_0 -small sets are precisely finite sets and \aleph_1 -small sets are precisely countable sets. We say that a cofibration category is

- κ -cocomplete for $\kappa > \aleph_0$ if it satisfies (C6) and (C7- κ),
- cocomplete if it satisfies (C6) and (C7).

Again, the words “ κ -cocomplete” and “cocomplete” are really shorthands for “homotopy κ -cocomplete” and “homotopy cocomplete”. We will justify below that κ -cocomplete cofibration categories indeed have all κ -small homotopy colimits. Axioms (C0-5) imply (C7- \aleph_0) and we will sometimes refer to finitely cocomplete cofibration categories as \aleph_0 -cocomplete cofibration categories. Similarly, the axioms (C0-6) imply (C7- \aleph_1) which is therefore redundant in the definition of a homotopy \aleph_1 -cocomplete cofibration category. This name will be abbreviated to a *countably cocomplete* cofibration category.

Next, we introduce κ -cocontinuous functors which (according to the definition and K. Brown’s Lemma) are essentially homotopy invariant functors that preserve certain basic κ -small homotopy colimits. It will be explained later in this section that they actually preserve all κ -small homotopy colimits.

Definition 3.22.

- (1) For $\kappa > \aleph_0$ a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between κ -cocomplete cofibration categories is κ -*cocontinuous* if it preserves cofibrations, acyclic cofibrations, pushouts along cofibrations, colimits of sequences of cofibrations and κ -small coproducts.
- (2) A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between cocomplete cofibration categories is *cocontinuous* if it preserves cofibrations, acyclic cofibrations, pushouts along cofibrations, colimits of sequences of cofibrations and small coproducts.

Just as in the case of countably cocomplete cofibration categories, preservation of countable coproducts follows from preservation of colimits of sequences of cofibrations and thus it is redundant in the definition of an \aleph_1 -cocontinuous functor. (But then preservation of an initial object has to be assumed explicitly.)

By extension, exact functors in the sense of Chapter 1 will be sometimes referred to as \aleph_0 -*cocontinuous*.

The notions of (κ) -cocomplete cofibration categories and (κ) -cocontinuous functors dualize to the notions of (κ) -complete fibration categories and (κ) -continuous functors.

The category of small κ -cocomplete cofibration categories and κ -cocontinuous functors will be denoted by \mathbf{CofCat}_κ .

All the results about cofibration categories proven or cited in Chapter 1 and Section 3.2 readily generalize to κ -cocomplete cofibration categories. The correct statements can be obtained by replacing phrases

- “cofibration category” with “ κ -cocomplete cofibration category”,
- “exact functor” with “ κ -cocontinuous functor”,
- “finite direct category” with “ κ -small direct category”.

The proofs will occasionally require extra arguments, but they are all routine and completely analogous to the ones already given for finitely cocomplete cofibration categories. For example, an updated version of Proposition 1.13 says that in the category \mathbf{CofCat}_κ pullbacks along fibrations exist. The main modification is that we need to verify that the resulting pullback \mathcal{P} satisfies axioms (C6) and (C7- κ). The proofs are essentially the same as the proof of (C4) except that we use Lemma A.3 instead of Lemma A.2.

We have restricted attention to \mathbf{CofCat}_κ only for convenience. If we want to consider cocomplete cofibration categories, we cannot assume that they are small. However, all the results of Chapter 1 apply to this case in the sense that cocomplete cofibration categories form a fibration category in a higher Grothendieck universe as explained in the introduction.

The updated Proposition 1.29 says that κ -cocomplete cofibration categories have κ -small direct homotopy colimits. This can be used to motivate axioms (C6) and (C7) just like the Gluing Lemma motivated (C4). Namely, (C6) is used to show that the colimit functor $\text{colim}_{\mathbb{N}}: \mathcal{C}_{\mathbb{R}}^{\mathbb{N}} \rightarrow \mathcal{C}$ is exact. More precisely, stability of (acyclic) cofibrations under sequential colimits implies that $\text{colim}_{\mathbb{N}}$ preserves (acyclic) cofibrations, see [RB06,

Lemma 9.3.4(1)]. Similarly, (C7) implies that $\operatorname{colim}_J: \mathcal{C}_R^J \rightarrow \mathcal{C}$ is exact for J discrete. The case of all the other direct categories is reduced to these two and the Gluing Lemma as in the proof of [RB06, Theorem 9.3.5(1)].

This handles the case of *direct* homotopy colimits and, as was pointed out before, for $\kappa = \aleph_0$ restricting to direct categories was essential. However, for $\kappa > \aleph_0$ κ -cocomplete cofibration categories have all κ -small homotopy colimits, i.e. the ones indexed by arbitrary κ -small categories. Their construction is more complicated and uses categories of the form DJ introduced in Section 3.1. In fact, one of the main reasons for introducing this construction is that the problem of computing homotopy colimits over J can be reduced to the problem of computing homotopy colimits over DJ which is direct.

The way this works is that a homotopical diagram $X: J \rightarrow \mathcal{C}$ contains the same homotopical information as $p_J^*X: DJ \rightarrow \mathcal{C}$. In fact, homotopical diagrams over DJ are these that are (weakly equivalent to the ones) pulled back along p_J from homotopical diagrams over J . This is made precise as follows. The category \mathcal{C}^J of all homotopical diagrams $J \rightarrow \mathcal{C}$ has a structure of a cofibration category with levelwise weak equivalences and cofibrations by [RB06, Theorem 9.5.5(1)].² Moreover, $p_J^*: \mathcal{C}^J \rightarrow \mathcal{C}^{DJ}$ is a weak equivalence of cofibration categories by [RB06, Theorem 9.5.8(1)].

As a result, just as in the case of direct homotopy colimits, the homotopy colimit functor can be thought of as a zig-zag of exact functors

$$\mathcal{C}^J \xrightarrow[p_J^*]{\sim} \mathcal{C}^{DJ} \xleftarrow{\sim} \mathcal{C}_R^{DJ} \xrightarrow{\operatorname{colim}_{DJ}} \mathcal{C}.$$

These results were used by Cisinski to prove that every cofibration category has an associated derivator [Cis10, Corollaire 6.21], see also [RB06, Theorem 10.3.2].

Infinite colimits in quasicategories

The results of Chapter 2 also generalize to κ -cocomplete quasicategories, in fact, in an even more straightforward manner since the notion of a colimit of a diagram $K \rightarrow \mathcal{C}$ is completely uniform in K and there is no need to distinguish between cases depending on the cardinality of K .

A quasicategory \mathcal{C} to be κ -cocomplete if it has colimits indexed over all κ -small simplicial sets. Similarly, a functor between κ -cocomplete quasicategories is κ -cocontinuous if it carries universal cones under all κ -small diagrams to universal cones.

All the results of Chapter 2 remain correct when we replace phrases “finitely cocomplete quasicategory” and “exact functor” with “ κ -cocomplete quasicategory” and “ κ -cocontinuous functor” respectively. This time proofs require no modifications.

Completeness of fibration categories of cofibration categories and quasicategories

The discussion in the two previous subsections implies that $\operatorname{CofCat}_\kappa$ and $\operatorname{QCofCat}_\kappa$ are fibration categories for all regular cardinals κ . In fact, they are both complete, i.e.

²Note that this means that \mathcal{C}^J can be made into a cofibration category for an arbitrary (κ -small) J which is not known for model categories.

satisfy axioms (C6)^{op} and (C7)^{op}. We state the upgraded theorems explicitly for future reference.

Theorem 3.23. *The category CofCat_κ of small κ -cocomplete cofibration categories and κ -cocontinuous functors with weak equivalences and fibrations defined as in Chapter 1 is a complete fibration category.*

Theorem 3.24. *The category QCat of small quasicategories with simplicial maps as morphisms, categorical equivalences as weak equivalences and inner isofibrations as fibrations is a complete fibration category.*

Theorem 3.25. *The category QCat_κ of small κ -cocomplete quasicategories with κ -cocontinuous functors as morphisms, categorical equivalences as weak equivalences and inner isofibrations as fibrations is a complete fibration category.*

Proofs of these theorems are routine modifications of the proofs of their counterparts discussed in Chapters 1 and 2. For example in the proof of Theorem 3.23 we need to check that limits of towers of fibrations exist in CofCat_κ . This is very similar to the arguments of Proposition 1.13. Namely, given a sequence

$$\mathcal{C}_0 \longleftarrow \mathcal{C}_1 \longleftarrow \mathcal{C}_2 \longleftarrow \dots$$

of fibrations of κ -cocomplete cofibration categories we form the limit of the underlying categories and define weak equivalences and fibrations levelwise. Then we use the lifting property for factorizations to show that this limit has factorizations. Similarly, we use the isofibration property and Lemmas A.2, A.3 and A.4 to show that it has all the required colimits.

Finally, we state an updated version of Theorem 3.3.

Theorem 3.26. *The functor $N_f: \text{CofCat}_\kappa \rightarrow \text{QCat}_\kappa$ is a continuous functor of complete fibration categories. In particular, it takes values in κ -cocomplete quasicategories and κ -cocontinuous functors.*

This theorem clearly generalizes Theorem 3.3. In the rest of this chapter we will proceed with the proof of the general statement. (Observe that modifications sketched above imply that Proposition 3.4 generalizes to say that N_f preserves limits of towers of fibrations and products.)

3.4. Cocompleteness: the infinite case

In order to complete the proof of Theorem 3.26 it remains to verify that N_f takes values in κ -cocomplete quasicategories and κ -cocontinuous functors. From this point on the cases of finitely cocomplete cofibration categories and κ -cocomplete cofibration categories for $\kappa > \aleph_0$ will diverge. The general approaches to both cases are still analogous, but they differ in technical details and there seems to be no way of presenting them in a completely uniform manner. The presence of infinite homotopy colimits allows us to use simpler

constructions so we will consider the case of $\kappa > \aleph_0$ first. The remaining case of $\kappa = \aleph_0$ will be covered in the next section.

First, we need a few preliminary lemmas. Recall that if I is a discrete category, then colimits over $[0] \star I$ are called wide pushouts. A wide pushout of a diagram $X : [0] \star I \rightarrow \mathcal{C}$ will be denoted by

$$\coprod_{X_0}^{i \in I} X_i.$$

The inclusion of the m th vertex $\Delta[0] \rightarrow K \star \Delta[m]$ is cofinal which suggests that colimits over $D(K \star \Delta[m])$ should be given by evaluating diagrams at any simplex containing that vertex.

Lemma 3.27. *Let \mathcal{C} be a κ -cocomplete cofibration category and K a κ -small simplicial set. If $X : D(K \star \Delta[m]) \rightarrow \mathcal{C}$ is a homotopical Reedy cofibrant diagram, then the induced morphism*

$$X_{[m]} \rightarrow \operatorname{colim}_{D(K \star \Delta[m])} X$$

is a weak equivalence.

Proof. The morphism in question factors as

$$X_{[m]} \rightarrow \operatorname{colim}_{D[m]} X \rightarrow \operatorname{colim}_{D(K \star \Delta[m])} X$$

where the first morphism is a weak equivalence by Lemmas 1.31 and 3.9. Thus it will be enough to check that the second one is.

It will suffice to verify that this statement holds when K is a simplex and that it is preserved under coproducts, pushouts along monomorphisms and colimits of sequences of monomorphisms.

Let $K = \Delta[k]$ and let ι be the composite $[m] \hookrightarrow [k] \star [m] \cong [k+1+m]$. Then we have a commutative square

$$\begin{array}{ccc} X_\iota & \longrightarrow & \operatorname{colim}_{D[m]} X \\ \downarrow & & \downarrow \\ X_{\operatorname{id}_{[k+1+m]}} & \longrightarrow & \operatorname{colim}_{D[k+1+m]} X \end{array}$$

where the left morphism is a weak equivalence since X is homotopical and so are the horizontal ones by the argument above. Thus the right morphism is also a weak equivalence.

Next, consider a pushout square

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ B & \longrightarrow & L \end{array}$$

such that the statement holds for A , B and K . The functor $-\star\Delta[m]$ preserves pushouts by Proposition 2.6 and so does D by Lemma 3.6. Thus in the cube

$$\begin{array}{ccccc}
DA & \xrightarrow{\quad\quad\quad} & DK & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& & D(A\star\Delta[m]) & \xrightarrow{\quad\quad\quad} & D(K\star\Delta[m]) \\
& & \downarrow & & \downarrow \\
DB & \xrightarrow{\quad\quad\quad} & DL & & \\
& \searrow & \downarrow & \searrow & \\
& & D(B\star\Delta[m]) & \xrightarrow{\quad\quad\quad} & D(L\star\Delta[m])
\end{array}$$

both the front and the back faces are pushouts along sieves and the conclusion follows by [RB06, Theorem 9.4.1 (1a)], Lemma A.5 and the Gluing Lemma.

The case of colimits of sequences of monomorphisms is similar and we omit it.

The case of coproducts is also similar, but there is a difference in the fact that $-\star\Delta[m]$ doesn't preserve coproducts. Instead, it sends coproducts to wide pushouts under $\Delta[m]$. Thus if we have a κ -small family $\{K_i \mid i \in I\}$ of κ -small simplicial sets and a diagram $X: D((\coprod_i K_i) \star \Delta[m]) \rightarrow \mathcal{C}$, then there is a canonical isomorphism

$$\coprod_{i \in I}^{\text{colim}_{D[m]} X} (\text{colim}_{D(K_i \star \Delta[m])} X) \cong \text{colim}_{D((\coprod_{i \in I} K_i) \star \Delta[m])} X.$$

The conclusion follows by the fact that in a cofibration category all the structure morphisms of a wide pushout of acyclic cofibrations are again acyclic cofibrations. (By Lemma 1.31 since $\widehat{[0] \star I}$ is contractible to its cone object as a homotopical category.) \square

Note that for any simplicial set K there is a unique functor $p_K: D(K^\triangleright) \rightarrow (DK)^\triangleright$ that restricts to the identity of DK and sends all the objects not in DK to the cone point of $(DK)^\triangleright$. This functor is homotopical. In the next lemma we use it to compare colimits over DK and $D(K^\triangleright)$.

Lemma 3.28. *Let \mathcal{C} be a κ -cocomplete cofibration category, K a κ -small simplicial set and $X: DK \rightarrow \mathcal{C}$ a homotopical Reedy cofibrant diagram. Consider a morphism $f: \text{colim}_{DK} X \rightarrow Y$ and the corresponding cone $\tilde{T}: (DK)^\triangleright \rightarrow \mathcal{C}$. If T is any Reedy cofibrant replacement of $p_K^* \tilde{T}$ relative to DK (which exists by Lemma 1.22), then f factors as*

$$\text{colim}_{DK} X \rightarrow \text{colim}_{D(K^\triangleright)} T \xrightarrow{\sim} Y.$$

Proof. To verify that the above composite agrees with f it suffices to check that it agrees upon precomposition with $X_x \rightarrow \text{colim}_{DK} X$ for all $x \in DK$. That's indeed the case since $T|_{DK} = X$.

It remains to check that the latter morphism is a weak equivalence. In the diagram

$$\begin{array}{ccc}
\operatorname{colim}_{D(K^\triangleright)} T & \longrightarrow & Y \\
\uparrow & \nearrow & \\
T_0 & &
\end{array}$$

the left morphism is a weak equivalence by Lemma 3.27 and so is the diagonal one since T is a cofibrant replacement of $p_K^* \tilde{T}$. Therefore the top morphism is also a weak equivalence. \square

We will need an augmented version of the D construction. In fact, we will only need to apply it to $[m]$ and $\partial\Delta[m]$ so we define it only in these cases.

We will denote by $D_a[m]$ the category of all order preserving maps $[k] \rightarrow [m]$ including the one with $[k] = [-1] = \emptyset$. A morphism from $x: [k] \rightarrow [m]$ to $y: [l] \rightarrow [m]$ is an injective order preserving map $\varphi: [k] \hookrightarrow [l]$ such that $x = y\varphi$. In other words, $D_a[m]$ is obtained from $D[m]$ by adjoining an initial object. The homotopical structure on $D_a[m]$ is an extension of the one on $D[m]$ where $[-1] \rightarrow [m]$ is not weakly equivalent to any other object. We will also consider a slightly richer homotopical structure $\tilde{D}_a[m]$ where $[-1] \rightarrow [m]$ is weakly equivalent to all the constant maps with the value 0.

The homotopical categories $D_a\partial\Delta[m]$ and $\tilde{D}_a\partial\Delta[m]$ are the full homotopical subcategories of $D_a[m]$ and $\tilde{D}_a[m]$ spanned by the non-surjective maps $[k] \rightarrow [m]$ (i.e. by the simplices of $\partial\Delta[m]$ including the “ (-1) -dimensional” one).

Similarly, the homotopical posets $\operatorname{Sd}_a[m]$, $\tilde{\operatorname{Sd}}_a[m]$, $\operatorname{Sd}_a\partial\Delta[m]$ and $\tilde{\operatorname{Sd}}_a\partial\Delta[m]$ are the full homotopical subcategories of $D_a[m]$, $\tilde{D}_a[m]$, $D_a\partial\Delta[m]$ and $\tilde{D}_a\partial\Delta[m]$ respectively spanned by their objects that are injective as maps $[k] \rightarrow [m]$.

Lemma 3.29. *The restriction functors*

$$\begin{array}{ll}
\mathcal{C}_R^{D_a[m]} \rightarrow \mathcal{C}_R^{\operatorname{Sd}_a[m]} & \mathcal{C}_R^{D_a\partial\Delta[m]} \rightarrow \mathcal{C}_R^{\operatorname{Sd}_a\partial\Delta[m]} \\
\mathcal{C}_R^{\tilde{D}_a[m]} \rightarrow \mathcal{C}_R^{\tilde{\operatorname{Sd}}_a[m]} & \mathcal{C}_R^{\tilde{D}_a\partial\Delta[m]} \rightarrow \mathcal{C}_R^{\tilde{\operatorname{Sd}}_a\partial\Delta[m]}
\end{array}$$

are all acyclic fibrations.

Proof. All these functors are induced by sieves so they are fibrations. We will construct a homotopy inverse to $f: \tilde{\operatorname{Sd}}_a[m] \hookrightarrow \tilde{D}_a[m]$ which will restrict to homotopy inverses of all the other sieves in question. The construction is a minor modification of the one used in Lemma 3.9 (and essentially the same as in Lemma 3.18). Namely, we define $q: \tilde{D}_a[m] \rightarrow \tilde{\operatorname{Sd}}_a[m]$ by sending each $[k] \rightarrow [m]$ to its image and $s: \tilde{D}_a[m] \rightarrow \tilde{D}_a[m]$ by inserting one extra occurrence of each $i \in [m]$ that is already present in a given $x \in \tilde{D}_a[m]$. Just as in Lemma 3.9 a new occurrence is inserted at the end of the block of the old occurrences which yields analogous weak equivalences

$$\operatorname{id} \xrightarrow{\sim} s \xleftarrow{\sim} fq.$$

Moreover, $qf = \operatorname{id}_{\tilde{\operatorname{Sd}}_a[m]}$ which finishes the proof. \square

Homotopical Reedy cofibrant diagrams on $D_a[1]$ will be used to encode cones on diagrams in $N_f \mathcal{C}$ and the ones which are homotopical with respect to $\tilde{D}_a[1]$ will correspond to the universal cones. The following lemma (and, more directly, Lemma 3.31 below) will translate between the universality of such cones in $N_f \mathcal{C}$ and strict colimits of the corresponding diagrams in \mathcal{C} .

Lemma 3.30. *The two functors*

- (1) $\mathcal{C}_R^{\tilde{Sd}_a[m]} \rightarrow \mathcal{C}_R^{\tilde{Sd}_a \partial \Delta[m]}$ and
- (2) $\mathcal{C}_R^{\tilde{D}_a[m]} \rightarrow \mathcal{C}_R^{\tilde{D}_a \partial \Delta[m]}$

induced by the inclusion $\partial \Delta[m] \hookrightarrow \Delta[m]$ are acyclic fibrations.

Proof. Both inclusions $\tilde{Sd}_a \partial \Delta[m] \hookrightarrow \tilde{Sd}_a[m]$ and $\tilde{D}_a \partial \Delta[m] \hookrightarrow \tilde{D}_a[m]$ are sieves hence it will be enough to prove that they are homotopy equivalences.

- (1) Consider two homotopical functors $i_0, i_1: Sd_a[m-1] \rightarrow \tilde{Sd}_a[m]$ defined as $i_0 A = A + 1$ and $i_1 A = i_0 A \cup \{0\}$ for any $A \subseteq [m-1]$. We have $i_0 A \subseteq i_1 A$ and the resulting natural transformation induces an isomorphism of homotopical categories $Sd_a[m-1] \times \widehat{[1]} \rightarrow \tilde{Sd}_a[m]$. It follows that i_0 is a homotopy equivalence since $[0] \hookrightarrow \widehat{[1]}$ is. This homotopy equivalence also restricts to a homotopy equivalence $Sd_a[m-1] \hookrightarrow \tilde{Sd}_a \partial \Delta[m]$ and thus the conclusion follows by the triangle

$$\begin{array}{ccc}
 & Sd_a[m-1] & \\
 \swarrow & & \searrow^{i_0} \\
 \tilde{Sd}_a \partial \Delta[m] & \xrightarrow{\quad} & \tilde{Sd}_a[m].
 \end{array}$$

- (2) We have a square

$$\begin{array}{ccc}
 \tilde{Sd}_a \partial \Delta[m] & \xrightarrow{\quad} & \tilde{Sd}_a[m] \\
 \downarrow & & \downarrow \\
 \tilde{D}_a \partial \Delta[m] & \xrightarrow{\quad} & \tilde{D}_a[m]
 \end{array}$$

where the top functor is a homotopy equivalence by the first part of the lemma and so are the horizontal ones by Lemma 3.29. Therefore so is the bottom one. \square

For every $m > 0$ each object of $D(K \star \Delta[m])$ can be uniquely written as $x \star \varphi$ with $x \in D_a K$ and $\varphi \in D_a[m]$. This yields a functor $r_K: D(K \star \Delta[m]) \rightarrow D_a[m]$ sending $x \star \varphi$ to φ to which we associate the left Kan extension

$$\text{Lan}_{r_K}: \mathcal{C}_R^{D(K \star \Delta[m])} \rightarrow \mathcal{C}_R^{D_a[m]}$$

which can be constructed as

$$(\text{Lan}_{r_K} X)_\varphi = \text{colim}_{D[k]} \varphi^* X$$

where $\varphi: [k] \rightarrow [m]$. Analogously, we have a functor $s_K: D(K \star \partial\Delta[m]) \rightarrow D_a\partial\Delta[m]$ and the associated left Kan extension

$$\text{Lan}_{s_K}: \mathcal{C}_R^{D(K \star \partial\Delta[m])} \rightarrow \mathcal{C}_R^{D_a\partial\Delta[m]}.$$

We form pullbacks (the front and back squares of the cube)

$$\begin{array}{ccccc}
\mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} & \xrightarrow{\quad} & \mathcal{C}_R^{\tilde{D}_a[m]} & & \\
\downarrow & \searrow^{P_K} & \downarrow & \searrow & \\
& \mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])} & \xrightarrow{\quad} & \mathcal{C}_R^{\tilde{D}_a\partial\Delta[m]} & \\
& \downarrow & \downarrow & \downarrow & \\
\mathcal{C}_R^{D(K \star \Delta[m])} & \xrightarrow{\quad} & \mathcal{C}_R^{D_a[m]} & & \\
& \searrow & \downarrow^{\text{Lan}_{r_K}} & \searrow & \\
& \mathcal{C}_R^{D(K \star \partial\Delta[m])} & \xrightarrow{\quad} & \mathcal{C}_R^{D_a\partial\Delta[m]} & \\
& & \downarrow^{\text{Lan}_{s_K}} & & \\
& & \mathcal{C}_R^{D_a\partial\Delta[m]} & &
\end{array}$$

Observe that $\mathcal{C}_R^{\tilde{D}(K \star \Delta[m])}$ and $\mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])}$ are just atomic notations for the pullbacks above, i.e. $\tilde{D}(K \star \Delta[m])$ and $\tilde{D}(K \star \partial\Delta[m])$ are *not* homotopical categories for general K , although they will be interpreted as such when K is a simplex.

Lemma 3.31. *The induced functor $P_K: \mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} \rightarrow \mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])}$ is an acyclic fibration for every κ -small simplicial set K .*

Proof. First, we verify that P_K is a fibration. The categories $\mathcal{C}_R^{\tilde{D}(K \star \Delta[m])}$ and $\mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])}$ are full subcategories of $\mathcal{C}_R^{D(K \star \Delta[m])}$ and $\mathcal{C}_R^{D(K \star \partial\Delta[m])}$ respectively. They are both closed under taking weakly equivalent objects. Hence the lifting properties of the fibration $\mathcal{C}_R^{D(K \star \Delta[m])} \rightarrow \mathcal{C}_R^{D(K \star \partial\Delta[m])}$ are inherited by P_K .

For the rest of the argument it will suffice to check that P_K is a weak equivalence when K is empty or a simplex and that this property is preserved under coproducts, pushouts along monomorphisms and colimits of sequences of monomorphisms.

When K is empty then the top square of the cube above happens to be a pullback and hence P_\emptyset is an acyclic fibration by Lemma 3.30.

For $K = \Delta[k]$ we will check that $P_{\Delta[k]}$ coincides with

$$\mathcal{C}_R^{D[k+\tilde{1}+m]} \rightarrow \mathcal{C}_R^{D\Lambda^{[k]}[k+\tilde{1}+m]}$$

and the conclusion will follow from Lemma 3.21. It is enough to verify that a homotopical Reedy cofibrant diagram $X: D[k+1+m] \rightarrow \mathcal{C}$ is homotopical with respect to $D[k+\tilde{1}+m]$ if and only if the induced morphism

$$\text{colim } X|D[k] \rightarrow \text{colim } X|D[k+1]$$

is a weak equivalence. This follows from Lemma 3.27. The same argument works with $\Lambda^{[k]}[k+1+m]$ in place of $[k+1+m]$, since $\Delta[k+1]$ is contained in $\Lambda^{[k]}[k+1+m]$ for $m > 0$.

If

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ B & \longrightarrow & L \end{array}$$

is a pushout square of simplicial sets such that the conclusion holds for A , B and K , then there is a pullback square of cofibration categories

$$\begin{array}{ccc} \mathcal{C}_R^{\tilde{D}(L \star \Delta[m])} & \longrightarrow & \mathcal{C}_R^{\tilde{D}(B \star \Delta[m])} \\ \downarrow & & \downarrow \\ \mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} & \longrightarrow & \mathcal{C}_R^{\tilde{D}(A \star \Delta[m])} \end{array}$$

and a similar one with $\partial\Delta[m]$ in place of $\Delta[m]$. Hence the conclusion for L follows from the Gluing Lemma.

If K is a colimit of a sequence of monomorphisms $K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \dots$, then $\mathcal{C}_R^{\tilde{D}(K \star \Delta[m])}$ is the limit of the tower of fibrations

$$\dots \rightarrow \mathcal{C}_R^{\tilde{D}(K_2 \star \Delta[m])} \rightarrow \mathcal{C}_R^{\tilde{D}(K_1 \star \Delta[m])} \rightarrow \mathcal{C}_R^{\tilde{D}(K_0 \star \Delta[m])}$$

and analogously for $\mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])}$. Therefore, if P_{K_i} is a weak equivalence for all i , then so is P_K .

The case of coproducts is handled similarly except that $- \star \Delta[m]$ doesn't preserve coproducts but carries them to wide pushouts. Hence $\mathcal{C}_R^{\tilde{D}(\coprod_i K_i \star \Delta[m])}$ is the wide pullback

$$\prod_i \mathcal{C}_R^{\tilde{D}(K_i \star \Delta[m])}.$$

The conclusion follows since the wide pullback functor is an exact functor of fibration categories. \square

We are ready to characterize colimits in $N_f \mathcal{C}$ in terms of homotopy colimits in \mathcal{C} .

Proposition 3.32. *Let \mathcal{C} be a κ -cocomplete cofibration category, K a κ -small simplicial set and $S: K^\triangleright \rightarrow N_f \mathcal{C}$. Then S is universal as a cone under $S|_K$ if and only if the induced morphism*

$$\text{colim}_{DK} S \rightarrow \text{colim}_{D(K^\triangleright)} S$$

is a weak equivalence (with S seen as a homotopical Reedy cofibrant diagram $D(K^\triangleright) \rightarrow \mathcal{C}$ by Proposition 3.7). Such a cone exists under every diagram $K \rightarrow N_f \mathcal{C}$.

Proof. If the morphism above is a weak equivalence let $U: K \star \partial\Delta[m] \rightarrow \mathbf{N}_f \mathcal{C}$ extend S . The corresponding homotopical Reedy cofibrant diagram $\tilde{D}(K \star \partial\Delta[m]) \rightarrow \mathcal{C}$ prolongs to $\tilde{D}(K \star \Delta[m]) \rightarrow \mathcal{C}$ since $\mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} \rightarrow \mathcal{C}_R^{\tilde{D}(K \star \partial\Delta[m])}$ is an acyclic fibration by Lemma 3.31. Hence S is universal.

Conversely, let S be universal. Define $T: D(K^\triangleright) \rightarrow \mathcal{C}$ as in Lemma 3.28 where we take f to be the identity of $\text{colim}_{DK} S$. Then the morphism $\text{colim}_{DK} T \rightarrow \text{colim}_{D(K^\triangleright)} T$ is a weak equivalence and so T is universal by the argument above (which proves the existence statement). Therefore by Corollary 2.12 there exists a homotopical Reedy cofibrant diagram $W: D(K \star E[1]) \rightarrow \mathcal{C}$ which restricts to $[S, T]$ on $D(K \star \partial\Delta[1])$. In the diagram

$$\begin{array}{ccccc} \text{colim}_{DK} S & \longrightarrow & \text{colim}_{D(K^\triangleright)} S & \longleftarrow & S_0 \\ \sim \downarrow & & \downarrow & & \downarrow \\ \text{colim}_{D(K^\triangleright)} T & \longrightarrow & \text{colim}_{D(K \star \Delta[1])} W & \longleftarrow & W_{01} \end{array}$$

both bottom morphisms and the top right one are weak equivalences by Lemma 3.27 and so is the right vertical one since the homotopical structure of $DE[1]$ is the maximal one. It follows that $\text{colim}_{DK} S \rightarrow \text{colim}_{D(K^\triangleright)} S$ is also a weak equivalence. \square

Before completing the proof of Theorem 3.26 we will point out that in certain special cases the above criterion for recognizing universal cones can be simplified considerably.

Example 3.33. A homotopical Reedy cofibrant diagram $X: D[0] \rightarrow \mathcal{C}$ is initial as an object of $\mathbf{N}_f \mathcal{C}$ if and only if the canonical morphism $0 \rightarrow X_0$ is a weak equivalence (where 0 is an initial object of \mathcal{C}). This is because the induced morphism $X_0 \rightarrow \text{colim} X$ is a weak equivalence by Lemmas 1.31 and 3.9.

Example 3.34. For a homotopical Reedy cofibrant diagram $X: D([1] \times [1]) \rightarrow \mathcal{C}$ consider its restriction to $\text{Sd}([1] \times [1])$.

$$\begin{array}{ccccc} X_{0,0} & \longrightarrow & X_{01,00} & \xleftarrow{\sim} & X_{1,0} \\ \downarrow & \dashrightarrow & \downarrow & \nearrow & \downarrow \\ X_{00,01} & & X_{01,01} & \xleftarrow{\sim} & X_{11,01} \\ \uparrow & \dashrightarrow & \downarrow & \nearrow & \uparrow \\ X_{0,1} & \longrightarrow & X_{01,11} & \xleftarrow{\sim} & X_{1,1} \end{array}$$

$X_{00,01} \dashrightarrow X_{01,01} \dashrightarrow X_{11,01}$
 $X_{00,01} \dashrightarrow X_{001,011} \dashrightarrow X_{01,01} \dashrightarrow X_{011,001} \dashrightarrow X_{11,01}$
 $X_{0,1} \dashrightarrow X_{01,11} \dashrightarrow X_{1,1}$

The corresponding square $\Delta[1] \times \Delta[1] \rightarrow \mathbf{N}_f \mathcal{C}$ is a pushout (observe that $\Gamma^\triangleright \cong [1] \times [1]$) if and only if the morphism

$$X_{00,01} \amalg_{X_{0,0}} X_{01,00} \rightarrow X_{001,011} \amalg_{X_{01,01}} X_{011,001}$$

induced by the three dashed arrows above is a weak equivalence. This can be justified by observing that in the square

$$\begin{array}{ccc}
X_{00,01} \amalg_{X_{0,0}} X_{01,00} & \xrightarrow{\sim} & \operatorname{colim}_{D\Gamma} X \\
\downarrow & & \downarrow \\
X_{001,011} \amalg_{X_{01,01}} X_{011,001} & \xrightarrow{\sim} & \operatorname{colim} X
\end{array}$$

both horizontal morphism are weak equivalences. Indeed, they are induced by the composite functors

$$\begin{aligned}
\{(00, 01), (0, 0), (01, 00)\} &\hookrightarrow \operatorname{Sd} \Gamma \hookrightarrow D\Gamma \\
\{(001, 011), (01, 01), (011, 001)\} &\hookrightarrow \operatorname{Sd}([1] \times [1]) \hookrightarrow D([1] \times [1])
\end{aligned}$$

where in both cases the latter functor is a homotopy equivalence by Lemma 3.18 while the former functor is a homotopy equivalence in the first case and cofinal in the second one. The conclusion follows by Lemma 1.31.

Proof of Theorem 3.26. Since we have already verified Propositions 3.4, 3.12 and 3.15 (Proposition 3.4 was generalized to infinite limits in the end of Section 3.3) it remains to check that N_f takes values in κ -cocomplete quasicategories and κ -cocontinuous functors.

It takes values in quasicategories by Proposition 3.12 and they are κ -cocomplete by Proposition 3.32.

Similarly, colimits in quasicategories of frames were characterized in Proposition 3.32 by certain morphisms being weak equivalences and weak equivalences are preserved by exact functors by Lemma 1.9. \square

In the next section we will adapt the arguments above to the case of $\kappa = \aleph_0$. The proof of the main theorem continues in Chapter 4.

3.5. Cocompleteness: the finite case

In this section we will prove that $N_f \mathcal{C}$ is finitely cocomplete for any cofibration category. The arguments of the previous section do not directly apply to this case since they heavily use the existence of colimits of Reedy cofibrant diagrams over categories of the form DK . Unfortunately, DK is infinite even when K is a finite (non-empty) simplicial set. In order to address this problem, we will filter the category DK by finite subcategories

$$D^{(0)}K \hookrightarrow D^{(1)}K \hookrightarrow D^{(2)}K \hookrightarrow \dots$$

and instead of using a colimit of a Reedy cofibrant diagram $X : DK \rightarrow \mathcal{C}$ we will consider the resulting sequence of finite colimits

$$\operatorname{colim}_{D^{(0)}K} X \twoheadrightarrow \operatorname{colim}_{D^{(1)}K} X \twoheadrightarrow \operatorname{colim}_{D^{(2)}K} X \twoheadrightarrow \dots$$

If X is homotopical this sequence stabilizes in the sense that from some point on (depending on K) all morphisms are weak equivalences and this stable value is a homotopy colimit of X . However, there is no universal bound on when such a sequence stabilizes when K varies and hence we are forced to think of that entire sequence as a homotopy colimit of X . It turns out that the proofs of the previous section will work if we carefully substitute such sequences for actual colimits over categories DK . The difficult part is constructing such filtrations with all the desired naturality and homotopy invariance which is the main purpose of this section.

Let J be a homotopical category and A a set of objects of DJ , we denote the sieve generated by A in DJ by $D^A J$. Moreover, when $J = [m]$ (possibly with some non-trivial homotopical structure) we will write objects of $D[m]$ as non-decreasing sequences of elements of $[m]$ often using abbreviations like $0^k 1^l$ to denote the sequence of k 0s followed by l 1s.

The category $D[0]$ can be seen as the category of non-degenerate simplices of a simplicial set S with exactly one non-degenerate simplex in each dimension. As it turns out, the skeleton $\text{Sk}^k S$ is contractible for k even but weakly equivalent to the sphere $\Delta[k]/\partial\Delta[k]$ for k odd. This suggests that the filtration of $D[0]$ by sieves generated by even-dimensional simplices of S should be well-behaved homotopically. We verify that this is the case in the next two lemmas and later generalize it to DK for arbitrary finite simplicial sets K .

Lemma 3.35. *For each k the functor $t: D^{0^k 1} \widehat{[1]} \rightarrow [0]$ is a homotopy equivalence of homotopical categories.*

Proof. Represent objects of $D^{0^k 1} \widehat{[1]}$ as binary sequences and let $j: [0] \rightarrow D^{0^k 1} \widehat{[1]}$ classify the object 1. Next, define $s: D^{0^k 1} \widehat{[1]} \rightarrow D^{0^k 1} \widehat{[1]}$ by appending a trailing 1 to each sequence that doesn't have one. Then there are natural weak equivalences

$$\text{id}_{D^{0^k 1} \widehat{[1]}} \xrightarrow{\sim} s \xleftarrow{\sim} jt.$$

Moreover, we have $tj = \text{id}_{[0]}$ which finishes the proof. □

The images of the composite functors

$$\text{Sd}[k] \hookrightarrow D[k] \rightarrow D[0] \text{ and } \text{Sd } \partial\Delta[k+1] \hookrightarrow D\partial\Delta[k+1] \rightarrow D[0]$$

are both $D^{0^{k+1}}[0]$. In the next lemma we consider the resulting functors

$$t: \text{Sd}[k] \rightarrow D^{0^{k+1}}[0] \text{ and } t: \text{Sd } \partial\Delta[k+1] \rightarrow D^{0^{k+1}}[0].$$

Lemma 3.36. *Let $k \geq 0$ and let \mathcal{C} be a cofibration category. If $X: D^{0^{k+1}}[0] \rightarrow \mathcal{C}$ is a homotopical Reedy cofibrant diagram, then*

(1) *the induced morphism*

$$\text{colim}_{\text{Sd } \Delta[k]} t^* X \rightarrow \text{colim}_{D^{0^{k+1}}[0]} X$$

is a weak equivalence when k is even,

(2) the induced morphism

$$\operatorname{colim}_{\operatorname{Sd} \partial \Delta[k+1]} t^* X \rightarrow \operatorname{colim}_{D^{0^{k+1}}[0]} X$$

is a weak equivalence when k is odd.

Proof. We prove both statements by an alternating induction with respect to k .

The functor $\operatorname{Sd}[0] \rightarrow D^0[0]$ is an isomorphism, so condition (1) holds for $k = 0$.

Next, we assume that condition (2) holds for a given odd k and prove that condition (1) holds for $k + 1$. The category $\operatorname{Sd} \partial \Delta[k + 1]$ is nothing but the latching category of $D^{0^{k+2}}[0]$ at 0^{k+2} and hence the inductive construction of the colimit of X yields a pushout square

$$\begin{array}{ccc} \operatorname{colim}_{\operatorname{Sd} \partial \Delta[k+1]} t^* X & \longrightarrow & \operatorname{colim}_{D^{0^{k+1}}[0]} X \\ \downarrow & & \downarrow \\ \operatorname{colim}_{\operatorname{Sd} \Delta[k+1]} t^* X & \longrightarrow & \operatorname{colim}_{D^{0^{k+2}}[0]} X \end{array}$$

where the top morphism is a weak equivalence by the inductive hypothesis. Since the left vertical morphism is a cofibration, it follows that the bottom morphism is also a weak equivalence.

Finally, we assume that condition (1) holds for a given even k and prove that condition (2) holds for $k + 1$. We have the following diagram of homotopical direct categories

$$\begin{array}{ccccc} \operatorname{Sd} \Lambda^{k+2} \widehat{[k+2]} & \hookrightarrow & \operatorname{Sd} \partial \Delta[k+2] & \longrightarrow & D^{0^{k+2}}[0] \\ \downarrow & & \downarrow & \nearrow & \uparrow \text{id} \\ D^{0^{k+1}} \widehat{[1]} & \hookrightarrow & D^{0^{k+1}} \widehat{[1]}_{0^{k+2}} & \longleftarrow & D^{0^{k+2}}[0] \\ & & \uparrow & & \uparrow \\ & & D^{0^{k+1}} \widehat{[1]} & \longleftarrow & D^{0^{k+1}}[0] \end{array}$$

where the indicated maps are sieves, the top left and bottom right squares are pushouts and all functors respect Reedy cofibrant diagrams by Lemma 1.23. (The functor on the very left is induced by $0^{k+2}1: [k+2] \rightarrow [1]$.) Hence there is an induced diagram in \mathcal{C}

$$\begin{array}{ccccc} \operatorname{colim}_{\operatorname{Sd} \Lambda^{k+2} \widehat{[k+2]}} t^* X & \longrightarrow & \operatorname{colim}_{\operatorname{Sd} \partial \Delta[k+2]} t^* X & \longrightarrow & \operatorname{colim}_{D^{0^{k+2}}[0]} X \\ \downarrow & & \downarrow & \nearrow & \uparrow \text{id} \\ \operatorname{colim}_{D^{0^{k+1}} \widehat{[1]}} t^* X & \longrightarrow & \operatorname{colim}_{D^{0^{k+1}} \widehat{[1]}_{0^{k+2}}} t^* X & \longleftarrow & \operatorname{colim}_{D^{0^{k+2}}[0]} X \\ & & \uparrow & & \uparrow \\ & & \operatorname{colim}_{D^{0^{k+1}} \widehat{[1]}} t^* X & \longleftarrow & \operatorname{colim}_{D^{0^{k+1}}[0]} X \end{array}$$

where the indicated maps are cofibrations and the top left and bottom right squares are pushouts by Lemma A.5. Thus the proof will be completed when we verify that both morphisms

$$\begin{aligned} \operatorname{colim}_{\operatorname{Sd} \Lambda^{k+2} \widehat{[k+2]}} t^* X &\rightarrow \operatorname{colim}_{D^{0^{k+1}} \widehat{[1]}} t^* X \\ \operatorname{colim}_{D^{0^{k+1}} [0]} t^* X &\rightarrow \operatorname{colim}_{D^{0^{k+1}} \widehat{[1]}} X \end{aligned}$$

are weak equivalences. For the former we use Lemmas 3.35, 1.31 and 3.20. For the latter we use Lemmas 3.35 and 1.31 and the inductive assumption. \square

In the next two lemmas we generalize the filtration of $D[0]$ to $D[m]$ for all $m \geq 0$.

Lemma 3.37. *Let \mathcal{C} be a cofibration category. Assume that every fiber of $\varphi: [k] \rightarrow [m]$ has an odd number of elements and let $X: D^\varphi[m] \rightarrow \mathcal{C}$ be a homotopical Reedy cofibrant diagram. Then $X_\varphi \rightarrow \operatorname{colim} X$ is a weak equivalence.*

Proof. We proceed by induction with respect to m (simultaneously for all \mathcal{C} and X). For $m = 0$ the conclusion follows by Lemma 3.36.

If $m > 0$, we will prolong X to the augmented sieve $D_a^\varphi[m]$ by setting the missing value to an initial object of \mathcal{C} which does not change the colimit. If the fiber of φ over m has $k+1$ elements for some even k , then $D_a^\varphi[m] \cong D_a^{\varphi'}[m-1] \times D_a^{0^{k+1}}[0]$. (Here, φ' is the restriction of φ to $\varphi^{-1}[m-1]$.) By applying Lemma 3.36 in the category $\mathcal{C}_R^{D_a^{\varphi'}[m-1]}$ to the corresponding diagram $\tilde{X}: D_a^{0^{k+1}}[0] \rightarrow \mathcal{C}_R^{D_a^{\varphi'}[m-1]}$ we obtain a weak equivalence $\tilde{X}_k \rightarrow \operatorname{colim}_{D_a^{0^{k+1}}[0]} \tilde{X}$ and hence by the inductive assumption the composite

$$X_\varphi = \tilde{X}_{k,\varphi'} \rightarrow \operatorname{colim}_{D_a^{\varphi'}[m-1]} \tilde{X}_k \rightarrow \operatorname{colim}_{D_a^{\varphi'}[m-1]} \operatorname{colim}_{D_a^{0^{k+1}}[0]} \tilde{X} \cong \operatorname{colim} X$$

is also a weak equivalence. \square

For each $k, m \geq 0$ we define sets $A_{k,m}$ and $B_{k,m}$ of objects of $D[m]$. We proceed by induction with respect to m . First, we set $A_{k,0} = B_{k,0} = \{[2k] \rightarrow [0]\}$. For $m > 0$ we set

$$\begin{aligned} B_{k,m} &= \{\varphi: [2k-m] \rightarrow [m] \mid \text{each fiber of } \varphi \text{ has an odd number of elements}\} \\ A_{k,m} &= B_{k,m} \cup \bigcup_{i \in [m]} \delta_i A_{k,m-1}. \end{aligned}$$

We set $D^{(k)}[m] = D^{A_{k,m}}[m]$. (In particular, we have $D^{(k)}[0] = D^{[2k]}[0]$.)

Lemma 3.38. *For every simplicial operator $\chi: [m] \rightarrow [n]$ and $k \geq 0$ we have an inclusion $\chi D^{(k)}[m] \subseteq D^{(k)}[n]$.*

Proof. It suffices to verify the statement when χ is an elementary face or degeneracy operator. For the elementary face operators it follows directly from the definition. Hence assume that $\chi = \sigma_j$ for some $j \in [n]$. We will check that $\sigma_j A_{k,n+1} \subseteq D^{(k)}[n]$ by induction with respect to n .

If $\varphi: [2k - n - 1] \rightarrow [n + 1]$ has all fibers of odd cardinality, then the same holds for $\sigma_j \varphi$ except at the fiber over j . Then $\sigma_j \varphi$ is in the sieve generated by $\varphi': [2k - n] \rightarrow [n]$ obtained by adding one extra element to the fiber of $\sigma_j \varphi$ over j (so that $\varphi' \in A_{k,n}$).

If $\psi \in A_{k,n}$, then $\sigma_j \delta_i \psi$ is either equal to ψ or is of the form $\delta_{i'} \sigma_{j'} \psi$. In the first case the conclusion holds trivially, in the second one it follows by the inductive hypothesis. \square

Now, we can generalize the filtration of $D[m]$ to DK for arbitrary finite K . Let $x \in K_m$ and $k \geq 0$. We define a sieve $D^{(k)}K$ in DK as follows. Write $x = x^\# x^b$ with $x^\#$ non-degenerate and x^b a degeneracy operator. Define x to be an element of $D^{(k)}K$ if $x^b \in D^{(k)}[n]$ (where n is the dimension of $x^\#$). It follows from Lemma 3.38 that this definition coincides with the previous one when K is a simplex.

Lemma 3.39. *Every simplicial map $f: K \rightarrow L$ carries $D^{(k)}K$ to $D^{(k)}L$ for all $k \geq 0$.*

Proof. Let $x \in D^{(k)}K$. Then we have a diagram of simplicial sets

$$\begin{array}{ccccc} \Delta[m] & \xrightarrow{x^b} & \Delta[n] & \xrightarrow{x^\#} & K \\ & \searrow (fx)^b & \downarrow (fx^\#)^b & & \downarrow f \\ & & \Delta[n'] & \xrightarrow{(fx^\#)^\#} & L \end{array}$$

and by definition $x^b \in D^{(k)}[n]$. Lemma 3.38 implies that $(fx^\#)^b \in D^{(k)}[n']$ so that $fx \in D^{(k)}L$. \square

Lemma 3.40. *For all $k \geq m$, a cofibration category \mathcal{C} and a homotopical Reedy cofibrant diagram $X: D^{(k)}[m] \rightarrow \mathcal{C}$ the morphism $X_{[m]} \rightarrow \text{colim}_{D^{(k)}[m]} X$ is a weak equivalence.*

Proof. First, we will check that the morphism $X_{[m]} \rightarrow D^{B_{k,m}}[m]$ is a weak equivalence. Indeed, let P be the subposet of \mathbb{N}^{m+1} consisting of tuples $x = (x_0, \dots, x_m)$ such that each x_i is odd and $x_0 + \dots + x_m \leq 2k - m + 1$. Let $\varphi_x \in D[m]$ be the unique object whose fiber over each $i \in [m]$ has cardinality x_i . Then we have $D^{B_{k,m}}[m] = \text{colim}_{x \in P} D^{\varphi_x}[m]$ by Lemma A.5. For each $x \in P$ the morphism $X_{[m]} \rightarrow \text{colim}_{D^{\varphi_x}[m]} X$ is a weak equivalence by Lemma 3.37. The sequence $(1, \dots, 1)$ is the bottom element of P , hence if we consider P as a homotopical poset with all maps as weak equivalences, then $\{(1, \dots, 1)\} \rightarrow P$ is a homotopy equivalence. It follows by Lemma 1.31 that $X_{[m]} \rightarrow D^{B_{k,m}}[m]$ is a weak equivalence.

We are ready to prove the lemma by induction with respect to m . We have

$$D^{B_{k,m}}[m] \cap D^{\delta_i A_{k,m-1}}[m] = D^{\delta_i A_{k-1,m-1}}[m] \cong D^{A_{k-1,m-1}}[m-1]$$

and $D^{(k)}\partial\Delta[m] = \text{colim}_{\varphi \in \text{Sd } \partial\Delta[m]} D^{\varphi A_{k,m-1}}[m]$ (and the same with $k-1$ in place of k). Hence by the inductive assumption the morphism $\text{colim}_{D^{(k-1)}\partial\Delta[m]} X \rightarrow \text{colim}_{D^{(k)}\partial\Delta[m]} X$ is an acyclic cofibration. This along with the first part of the proof and the pushout square

$$\begin{array}{ccc}
\operatorname{colim}_{D^{(k-1)}\partial\Delta[m]} X & \longrightarrow & \operatorname{colim}_{D^{B_{k,m}[m]}} X \\
\downarrow & & \downarrow \\
\operatorname{colim}_{D^{(k)}\partial\Delta[m]} X & \longrightarrow & \operatorname{colim}_{D^{(k)}[m]} X
\end{array}$$

finishes the proof. \square

Lemma 3.41. *For each k the functor $D^{(k)}: \mathbf{sSet} \rightarrow \mathbf{Cat}$ (i.e. when we disregard the homotopical structures of $D^{(k)}K$ s) preserves colimits.*

Proof. If K is any simplicial set, then $D^{(k)}$ preserves the colimit of its simplices by Lemma 3.6 and the definition of $D^{(k)}K$. Hence for every small category J we have the following sequence of isomorphisms natural in both K and J .

$$\begin{aligned}
\operatorname{Cat}(D^{(k)}K, J) &\cong \operatorname{Cat}(D^{(k)}\operatorname{colim}_{\Delta[m]\rightarrow K} \Delta[m], J) \cong \lim_{\Delta[m]\rightarrow K} \operatorname{Cat}(D^{(k)}[m], J) \\
&\cong \lim_{\Delta[m]\rightarrow K} \mathbf{sSet}(\Delta[m], \operatorname{Cat}(D^{(k)}[-], J)) \cong \mathbf{sSet}(K, \operatorname{Cat}(D^{(k)}[-], J))
\end{aligned}$$

It follows that $J \mapsto \operatorname{Cat}(D^{(k)}[-], J)$ is a right adjoint of $D^{(k)}$ and the conclusion follows. \square

Finally, we are ready to start translating the results of Section 3.4 to the case of $\kappa = \aleph_0$. The following is a counterpart to Lemma 3.27.

Lemma 3.42. *Let \mathcal{C} be a cofibration category and K a finite simplicial set. For every homotopical Reedy cofibrant diagram $X: D(K \star \Delta[m]) \rightarrow \mathcal{C}$ and all $k \geq \dim K + 1 + m$, the induced morphism*

$$X_{[m]} \rightarrow \operatorname{colim}_{D^{(k)}(K \star \Delta[m])} X$$

is a weak equivalence.

Proof. The morphism in question factors as

$$X_{[m]} \rightarrow \operatorname{colim}_{D^{(k)}[m]} X \rightarrow \operatorname{colim}_{D^{(k)}(K \star \Delta[m])} X$$

where the first morphism is a weak equivalence by Lemma 3.36. Thus it will be enough to check that the second one is.

It will suffice to verify that this statement holds when K is empty or a simplex and is preserved under pushouts along monomorphisms. For $K = \emptyset$ the morphism in question is an isomorphism.

Let $K = \Delta[n]$ and let ι be the composite $[m] \hookrightarrow [n] \star [m] \cong [n+1+m]$. Then we have a commutative square

$$\begin{array}{ccc}
X_\iota & \longrightarrow & \operatorname{colim}_{D^{(k)}[m]} X \\
\downarrow & & \downarrow \\
X_{\operatorname{id}_{[n+1+m]}} & \longrightarrow & \operatorname{colim}_{D^{(k)}[n+1+m]} X
\end{array}$$

where the left morphism is a weak equivalence since X is homotopical and so are the horizontal ones by Lemma 3.40. Thus the right morphism is also a weak equivalence.

Next, consider a pushout square

$$\begin{array}{ccc} A & \longrightarrow & K \\ \downarrow & & \downarrow \\ B & \longrightarrow & L \end{array}$$

such that the conclusion holds for A , B and K . The functor $- \star \Delta[m]$ preserves pushouts by Proposition 2.6 and so does $D^{(k)}$ by Lemma 3.41. Thus in the cube

$$\begin{array}{ccccc} D^{(k)}A & \xrightarrow{\quad\quad\quad} & D^{(k)}K & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & D^{(k)}(A \star \Delta[m]) & \xrightarrow{\quad\quad\quad} & D^{(k)}(K \star \Delta[m]) & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ D^{(k)}B & \xrightarrow{\quad\quad\quad} & D^{(k)}L & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & D^{(k)}(B \star \Delta[m]) & \xrightarrow{\quad\quad\quad} & D^{(k)}(L \star \Delta[m]) & \end{array}$$

both the front and the back faces are pushouts along sieves and the conclusion follows by [RB06, Theorem 9.4.1(1a)], Lemma A.5 and the Gluing Lemma (since $\dim L = \max\{\dim B, \dim K\}$). \square

For a cofibration category \mathcal{C} we introduce a new cofibration category $\mathcal{C}_{\mathbb{R}}^{\tilde{\mathbb{N}}}$. (Here, $\tilde{\mathbb{N}}$ does not refer to any homotopical structure on \mathbb{N} , $\mathcal{C}_{\mathbb{R}}^{\tilde{\mathbb{N}}}$ should be seen as an atomic notation.) Its objects are Reedy cofibrant diagrams $X: \mathbb{N} \rightarrow \mathcal{C}$ (i.e. sequences of cofibrations in \mathcal{C}) that are *eventually (homotopically) constant*, i.e. such that there is a number k such that for all $l \geq k$ the morphism $X_k \rightarrow X_l$ is a weak equivalence. A morphism $f: X \rightarrow Y$ of such diagrams is called an *eventual weak equivalence* if there is k such that for all $l \geq k$ the morphism f_l is a weak equivalence in \mathcal{C} . This cofibration category is designed as an enlargement of the cofibration category $\mathcal{C}_{\mathbb{R}}^{\mathbb{N}}$ of *(homotopically) constant* sequences. It is necessary since sequences arising as colimits over filtrations $D^{(-)}K$ are only eventually constant.

Lemma 3.43. *If \mathcal{C} is a cofibration category, then the category $\mathcal{C}_{\mathbb{R}}^{\tilde{\mathbb{N}}}$ with Reedy cofibrations and eventual weak equivalences is again a cofibration category. Moreover, the inclusion $\mathcal{C}_{\mathbb{R}}^{\tilde{\mathbb{N}}} \hookrightarrow \mathcal{C}_{\mathbb{R}}^{\mathbb{N}}$ is a weak equivalence.*

Proof. The construction of the cofibration category $\mathcal{C}_{\mathbb{R}}^{\tilde{\mathbb{N}}}$ is a straightforward modification of the construction of $\mathcal{C}_{\mathbb{R}}^{\mathbb{N}}$, see e.g. [RB06, Theorem 9.3.5(1)].

We will verify the approximation properties. By “2 out of 3” a morphism between homotopically constant sequences is a levelwise weak equivalence if and only if it is an eventual weak equivalence. Hence (App1) holds.

Next, let $X \rightarrow Y$ be a morphism with X homotopically constant and Y eventually constant. Assume that Y is homotopically constant from degree k on. Let \tilde{Y} be Y shifted down by k . Then \tilde{Y} is homotopically constant and iterated structure morphisms of Y yield a morphism $Y \rightarrow \tilde{Y}$ which is an eventual weak equivalence (starting from k). This yields a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \sim \\ \tilde{Y} & \xrightarrow{\text{id}} & \tilde{Y} \end{array}$$

which proves (App2). \square

We define a functor $|-|: DK \rightarrow \mathbb{N}$ by sending $x \in DK$ to the smallest $k \in \mathbb{N}$ such that $x \in D^{(k)}K$. We call $|x|$ the *filtration degree* of x . Here, we do not consider any particular homotopical structure on \mathbb{N} so $|-|$ is not a homotopical functor. We will be interested in the left Kan extension of a homotopical Reedy cofibrant diagram $X: DK \rightarrow \mathcal{C}$ along $|-|$. It can be computed as

$$(\text{Lan}_{|-|} X)_k = \text{colim}_{D^{(k)}K} X.$$

We will denote $(\text{Lan}_{|-|} X)_k$ by $\Phi^{(k)}X$ and when k varies $\Phi^{(-)}X$ will stand for the resulting sequence $\mathbb{N} \rightarrow \mathcal{C}$.

Just as colimits can be defined in terms of cones, left Kan extensions can be defined in terms of certain generalized cones. We describe such cones for Kan extensions along $|-|$. Let $DK \star_{|-|} \mathbb{N}$ denote the *cograph* (or *collage*) of $|-|$ defined as the category whose set of objects is the disjoint union of the sets of objects of DK and \mathbb{N} and

$$(DK \star_{|-|} \mathbb{N})(x, y) = \begin{cases} DK(x, y) & \text{when } x, y \in DK, \\ \mathbb{N}(x, y) & \text{when } x, y \in \mathbb{N}, \\ \mathbb{N}(|x|, y) & \text{when } x \in DK \text{ and } y \in \mathbb{N}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The left Kan extension of $X: DK \rightarrow \mathcal{C}$ along $|-|$ is nothing but an initial extension of X to $DK \star_{|-|} \mathbb{N}$ so that morphisms $\Phi^{(-)}X \rightarrow Y$ in $\mathcal{C}^{\mathbb{N}}$ correspond to diagrams on $DK \star_{|-|} \mathbb{N}$ restricting to X and Y on DK and \mathbb{N} respectively. Such an extension of X is a family of cones under the restrictions of X to all $D^{(k)}K$ s. We will compare them to extensions to $D(K^\triangleright)$ using a functor $p_K: D(K^\triangleright) \rightarrow DK \star_{|-|} \mathbb{N}$ defined as follows. Write an object of $D(K^\triangleright)$ as $x \star \varphi$ with $x \in D^a K$ and $\varphi \in D^a[0]$ and set

$$p_K(x \star \varphi) = \begin{cases} |x \star \varphi| & \text{when } \varphi \in D[0], \\ x & \text{otherwise.} \end{cases}$$

This allows us to state and prove a version of Lemma 3.28 for finitely cocomplete cofibration categories.

Lemma 3.44. *Let \mathcal{C} be a cofibration category, K a finite simplicial set and $X: DK \rightarrow \mathcal{C}$ a homotopical Reedy cofibrant diagram. Consider a morphism $f: \Phi^{(-)}X \rightarrow Y$ and the corresponding cone $\tilde{T}: DK \star_{|\cdot|} \mathbb{N} \rightarrow \mathcal{C}$. If T is any Reedy cofibrant replacement of $p_K^* \tilde{T}$ relative to DK (which exists by Lemma 1.22), then f factors as*

$$\Phi^{(-)}X \rightarrow \Phi^{(-)}T \xrightarrow{\sim} Y$$

where the latter morphism is an eventual weak equivalence (starting at $\dim K + 1$).

Proof. To verify that the above composite agrees with f it suffices to check that at each level k it agrees upon precomposition with $X_x \rightarrow \Phi^{(k)}X$ for all $x \in D^{(k)}K$. That's indeed the case since $T|_{DK} = X$.

It remains to check that the latter morphism is an eventual weak equivalence. For $i \geq \dim K + 1$ in the diagram

$$\begin{array}{ccc} \operatorname{colim}_{D^{(i)}(K^{\triangleright})} T & \longrightarrow & Y_i \\ \uparrow & \nearrow & \\ T_{0^{2i+1}} & & \end{array}$$

the left morphism is a weak equivalence by Lemma 3.42 and so is the diagonal one since T is a cofibrant replacement of $p_K^* \tilde{T}$. Therefore the top morphism is also a weak equivalence. \square

For every $m \geq 0$ each object of $D(K \star \Delta[m])$ can be uniquely written as $x \star \varphi$ with $x \in D^a K$ and $\varphi \in D^a[m]$. This yields a functor $r_K: D(K \star \Delta[m]) \rightarrow D^a[m]$ sending $x \star \varphi$ to φ and to which we can associate the “filtered” left Kan extension functor

$$\operatorname{Lan}_{r_K}^{\operatorname{filt}}: \mathcal{C}_R^{D(K \star \Delta[m])} \rightarrow (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D^a[m]}$$

defined as $(\operatorname{Lan}_{r_K}^{\operatorname{filt}} X)_\varphi = \Phi^{(-)}\varphi^* X$ for $\varphi \in D^a[m]$ which is exact by [RB06, Theorem 9.4.3(1)]. Similarly we have

$$\operatorname{Lan}_{s_K}^{\operatorname{filt}}: (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D(K \star \partial \Delta[m])} \rightarrow (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D^a \partial \Delta[m]}.$$

We form pullbacks (the front and back squares of the cube)

$$\begin{array}{ccccc} \mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} & \xrightarrow{\quad} & (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{\tilde{D}^a[m]} & & \\ \downarrow & \searrow^{P_K} & \downarrow & \searrow & \\ \mathcal{C}_R^{\tilde{D}(K \star \partial \Delta[m])} & \xrightarrow{\quad} & (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{\tilde{D}^a \partial \Delta[m]} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathcal{C}_R^{D(K \star \Delta[m])} & \xrightarrow{\quad} & (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D^a[m]} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathcal{C}_R^{D(K \star \partial \Delta[m])} & \xrightarrow{\quad} & (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D^a \partial \Delta[m]} & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{C}_R^{D(K \star \Delta[m])} & \xrightarrow{\quad} & (\mathcal{C}_R^{\tilde{\mathbb{N}}})^{D^a \partial \Delta[m]} \end{array}$$

$\operatorname{Lan}_{r_K}^{\operatorname{filt}}$ $\operatorname{Lan}_{s_K}^{\operatorname{filt}}$

Observe that $\mathcal{C}^{\tilde{D}(K \star \Delta[m])}$ and $\mathcal{C}^{\tilde{D}(K \star \partial \Delta[m])}$ are just atomic notations for the pullbacks above, i.e. $\tilde{D}(K \star \Delta[m])$ and $\tilde{D}(K \star \partial \Delta[m])$ are *not* homotopical categories for general K , although they will be interpreted as such when K is a simplex.

The following is a finite variant of Lemma 3.31.

Lemma 3.45. *The functor $P_K: \mathcal{C}_R^{\tilde{D}(K \star \Delta[m])} \rightarrow \mathcal{C}_R^{\tilde{D}(K \star \partial \Delta[m])}$ is an acyclic fibration for every finite simplicial set K .*

Proof. The proof is virtually identical to the proof of Lemma 3.31 except that now we do not consider the cases of coproducts and colimits of sequences of monomorphisms and we use Lemma 3.42 in the place of Lemma 3.27. \square

Finally, we can characterize colimits in $N_f \mathcal{C}$ in terms of homotopy colimits in \mathcal{C} in a manner similar to Proposition 3.32.

Proposition 3.46. *Let \mathcal{C} be cofibration category, K a finite simplicial set. A cone $S: K^\triangleright \rightarrow N_f \mathcal{C}$ is universal if and only if the induced morphism*

$$\Phi^{(-)}(S|K) \rightarrow \Phi^{(-)}S$$

is an eventual weak equivalence (where S is seen as a homotopical Reedy cofibrant diagram $D(K^\triangleright) \rightarrow \mathcal{C}$ by Proposition 3.7). Such a cone exists under every diagram $K \rightarrow N_f \mathcal{C}$.

Proof. The proof is almost identical to the proof of Proposition 3.32 except that we use Lemmas 3.45, 3.44 and 3.42 in the place of Lemmas 3.31, 3.28 and 3.27 respectively. \square

The more specific criteria for initial objects and pushouts discussed in Examples 3.33 and 3.34 are valid in the finitely cocomplete case in exactly the same form. This can be justified by observing that $\Phi^{(k)}$ stabilizes at $k = 0$ over $D[0]$ and at $k = 2$ over $D([1] \times [1])$ by Lemma 3.42.

Proof of Theorem 3.3. Since we have already verified Propositions 3.4, 3.12 and 3.15 it remains to check that N_f takes values in finitely cocomplete quasicategories and exact functors.

It takes values in quasicategories by Proposition 3.12 and they are finitely cocomplete by Proposition 3.46.

Similarly, colimits in quasicategories of frames were characterized in Proposition 3.46 by certain morphisms being weak equivalences and weak equivalences are preserved by exact functors by Lemma 1.9. \square

4. Cofibration categories of diagrams in quasicategories

In this chapter we will prove our main result, i.e. that N_f is a weak equivalence of fibration categories. This will be achieved by defining a functor Dg_κ from the category of κ -cocomplete quasicategories to the category of κ -cocomplete cofibration categories. The functor Dg_κ fails to be exact (e.g. it doesn't preserve the terminal object), but it will be verified to induce an inverse to N_f on the level of homotopy categories which is sufficient to complete the proof.

4.1. Construction

Let \mathbf{sSet}_κ denote the category of κ -small simplicial sets. If \mathcal{C} is a κ -cocomplete quasicategory we consider the slice category $\mathbf{sSet}_\kappa \downarrow \mathcal{C}$, we denote it by $Dg_\kappa \mathcal{C}$ and call the *category of κ -small diagrams in \mathcal{C}* . Then we define a morphism

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ & \searrow X & \swarrow Y \\ & & \mathcal{C} \end{array}$$

to be

- a *weak equivalence* if the induced morphism $\operatorname{colim}_K X \rightarrow \operatorname{colim}_L Y$ is an equivalence in \mathcal{C} (more precisely, if for any universal cone $S: L^\triangleright \rightarrow \mathcal{C}$ under Y the induced cone Sf^\triangleright is universal under X),
- a *cofibration* if f is injective.

In particular, such a morphism is a weak equivalence whenever f is cofinal, but there are of course many weak equivalences with f not cofinal. We will make use of the class of *right anodyne maps* which is generated by the *right horn inclusions* $\Lambda^i[m] \hookrightarrow \Delta[m]$ (i.e. the ones with $0 < i \leq m$) under coproducts, pushouts along arbitrary maps, sequential colimits and retracts.

Lemma 4.1. *Every right anodyne map is cofinal.*

Proof. [Lur09, Proposition 4.1.1.3(4)] □

Proposition 4.2. *With weak equivalences and cofibrations as defined above $Dg_\kappa \mathcal{C}$ is a κ -cocomplete cofibration category.*

Proof.

(C0) Weak equivalences satisfy “2 out of 6” since equivalences in \mathcal{C} do.

- (C1) Isomorphisms are weak equivalences since isomorphisms of simplicial sets are cofinal.
- (C2-3) The empty diagram is an initial object and hence every object is cofibrant.
- (C4) Pushouts are created by the forgetful functor $\mathrm{Dg}_\kappa \mathcal{C} \rightarrow \mathbf{sSet}_\kappa$ thus pushouts along cofibrations exist and cofibrations are stable under pushouts. By [RB06, Lemma 1.4.3(1)] it suffices to verify that the Gluing Lemma holds which follows by [Lur09, Proposition 4.4.2.2].
- (C5) It will suffice to verify that in the usual mapping cylinder factorization

$$K \rightarrow Mf \rightarrow L$$

the second map is cofinal. Indeed, we have a diagram

$$\begin{array}{ccc}
 K \times \Delta[0] & \xrightarrow{f} & L \\
 K \times \delta_0 \downarrow & & j \downarrow \\
 K \times \Delta[1] & \longrightarrow & Mf
 \end{array}
 \begin{array}{c}
 \searrow \text{id}_L \\
 \downarrow \\
 L
 \end{array}$$

where the square is a pushout. The map $K \times \delta_0$ is right anodyne by [Joy08, Theorem 2.17] and thus so is j . Hence it is cofinal by Lemma 4.1.

- (C6-7- κ) The proof is similar to that of (C4). (But there is no analogue of [Lur09, Proposition 4.4.2.2] for sequential colimits explicitly stated in [Lur09]. Instead, it follows from more general [Lur09, Proposition 4.2.3.10 and Remark 4.2.3.9].) \square

Lemma 4.3. *A κ -cocontinuous functor $F: \mathcal{C} \rightarrow \mathcal{D}$ induces a κ -cocontinuous functor $\mathrm{Dg}_\kappa F = \mathrm{Dg}_\kappa \mathcal{C} \rightarrow \mathrm{Dg}_\kappa \mathcal{D}$ and thus we obtain a functor $\mathrm{Dg}_\kappa: \mathbf{QCat}_\kappa \rightarrow \mathbf{CofCat}_\kappa$.*

Proof. Colimits in both $\mathrm{Dg}_\kappa \mathcal{C}$ and $\mathrm{Dg}_\kappa \mathcal{D}$ are created in \mathbf{sSet}_κ and thus are preserved by $\mathrm{Dg}_\kappa F$. Cofibrations are clearly preserved and so are weak equivalences since F preserves κ -small colimits. \square

4.2. Proof of the main theorem: the infinite case

For a κ -cocomplete cofibration category \mathcal{C} we define a functor $\Phi_{\mathcal{C}}: \mathrm{Dg}_\kappa \mathbf{N}_f \mathcal{C} \rightarrow \mathcal{C}$ by sending a diagram $X: K \rightarrow \mathbf{N}_f \mathcal{C}$ to $\mathrm{colim}_{DK} X$ (observe that DK is κ -small since K is and $\kappa > \aleph_0$, so this colimit exists in \mathcal{C}). It is clear that $\Phi_{\mathcal{C}}$ is a functor. While we may not be able to choose colimits so that $\Phi_{\mathcal{C}}$ is natural in \mathcal{C} , it is 2-natural, i.e. natural up to coherent natural isomorphism.

Lemma 4.4. *The functor $\Phi_{\mathcal{C}}$ is κ -cocontinuous and a weak equivalence.*

Proof. Preservation of cofibrations follows by [RB06, Theorem 9.4.1(1a)] since if $K \hookrightarrow L$ is an injective map of simplicial sets, then the induced functor $DK \hookrightarrow DL$ is a sieve.

Proposition 3.32 and Lemma 3.27 imply that a morphism f in $\mathrm{Dg}_{\kappa} \mathrm{N}_f \mathcal{C}$ is a weak equivalence if and only if $\Phi_{\mathcal{C}} f$ is. Therefore $\Phi_{\mathcal{C}}$ preserves weak equivalences and satisfies (App1).

It follows from Lemma A.5 that $\Phi_{\mathcal{C}}$ preserves colimits and thus it is κ -cocontinuous.

It remains to check (App2), but it follows directly from Lemma 3.28. \square

Next, we need a functor $\mathcal{D} \rightarrow \mathrm{N}_f \mathrm{Dg}_{\kappa} \mathcal{D}$ for every κ -cocomplete quasicategory \mathcal{D} . Let's start with unraveling the definition of $\mathrm{N}_f \mathrm{Dg}_{\kappa} \mathcal{D}$.

An m -simplex of $\mathrm{N}_f \mathrm{Dg}_{\kappa} \mathcal{D}$ consists of a Reedy cofibrant diagram $K: D[m] \rightarrow \mathrm{sSet}_{\kappa}$ and for each $\varphi \in D[m]$ a diagram $X_{\varphi}: K_{\varphi} \rightarrow \mathcal{D}$. These diagrams are compatible with each other in the sense that they form a cone under K with the vertex \mathcal{D} . Moreover, the entire structure is homotopical as a diagram in $\mathrm{Dg}_{\kappa} \mathcal{D}$, i.e. if $\varphi, \psi \in D[m]$ and $\chi: \varphi \rightarrow \psi$ is a weak equivalence, then the induced morphism $\mathrm{colim}_{K_{\varphi}} X_{\varphi} \rightarrow \mathrm{colim}_{K_{\psi}} X_{\psi}$ is an equivalence in \mathcal{D} .

If $\mu: [n] \rightarrow [m]$, then $(K, X)\mu = (K\mu, X\mu)$ is defined simply by $(K\mu)_{\varphi} = K_{\mu\varphi}$ and $(X\mu)_{\varphi} = X_{\mu\varphi}$.

We can now define a functor $\Psi_{\mathcal{D}}: \mathcal{D} \rightarrow \mathrm{N}_f \mathrm{Dg}_{\kappa} \mathcal{D}$ as follows. For $x \in \mathcal{D}_m$ we set the underlying simplicial diagram of $\Psi_{\mathcal{D}} x$ to $\varphi \mapsto \Delta[k]$ where $\varphi: [k] \rightarrow [m]$ and the corresponding diagram in \mathcal{D} to $x_{\varphi}: \Delta[k] \rightarrow \mathcal{D}$. Then $\Psi_{\mathcal{D}} x$ is homotopical as a diagram $D[m] \rightarrow \mathrm{Dg}_{\kappa} \mathcal{D}$ since any weak equivalence in $D[m]$ induces a right anodyne (and hence cofinal by Lemma 4.1) map of simplices. Clearly, $\Psi_{\mathcal{D}}$ is a functor and is natural in \mathcal{D} .

We will check that $\Psi_{\mathcal{D}}$ is a categorical equivalence by using the following criterion. A suitable generalization of this criterion holds in any model category, see [Vog11].

Lemma 4.5. *A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between quasicategories is a categorical equivalence provided that for every commutative square of the form*

$$\begin{array}{ccc} \partial\Delta[m] & \xrightarrow{u} & \mathcal{C} \\ \downarrow & & \downarrow F \\ \Delta[m] & \xrightarrow{v} & \mathcal{D} \end{array}$$

there exists a map $w: \Delta[m] \rightarrow \mathcal{C}$ such that $w|\partial\Delta[m] = u$ and Fw is $E[1]$ -homotopic to v relative to $\partial\Delta[m]$.

Proof. The class of simplicial maps $K \rightarrow L$ with the lifting property with respect to F as in the statement is closed under coproducts, pushouts and sequential colimits and thus contains all monomorphisms. In particular, if we consider the diagram

$$\begin{array}{ccc} & & \mathcal{C} \\ & & \downarrow F \\ \mathcal{D} & \xrightarrow{\text{id}} & \mathcal{D} \end{array}$$

we obtain a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ and an $E[1]$ -homotopy H from FG to $\text{id}_{\mathcal{D}}$ which in turn yields a diagram

$$\begin{array}{ccc} \mathcal{C} \times \partial\Delta[1] & \xrightarrow{[GF, \text{id}]} & \mathcal{C} \\ \downarrow & & \downarrow F \\ \mathcal{C} \times E[1] & \xrightarrow{FH} & \mathcal{D}. \end{array}$$

This time a lift is an $E[1]$ -homotopy from GF to $\text{id}_{\mathcal{C}}$. Thus F is an $E[1]$ -equivalence. \square

To apply this criterion in our situation we need a method of constructing relative $E[1]$ -homotopies in quasicategories of the form $N_f \mathcal{C}$.

Lemma 4.6. *Let $K \hookrightarrow L$ be an inclusion of marked simplicial complexes, X and Y homotopical Reedy cofibrant diagrams $DL \rightarrow \mathcal{C}$ and $f: X|Sd L \rightarrow Y|Sd L$ a natural weak equivalence such that $f|Sd K$ is an identity transformation. Then X and Y are $E[1]$ -homotopic relative to K as diagrams in $N_f \mathcal{C}$.*

Proof. By Corollary 3.14 it suffices to construct a homotopical Reedy cofibrant diagram $D(L \times \widehat{[1]}) \rightarrow \mathcal{C}$ that restricts to $[X, Y]$ on $D(L \times \partial\Delta[1])$ and to the identity on $D(K \times \widehat{[1]})$ (i.e. to a degenerate edge of $(N_f \mathcal{C})^K$).

First, observe that we have a homotopical diagram $[f, \text{id}]: (Sd L \cup DK) \times \widehat{[1]} \rightarrow \mathcal{C}$ which is Reedy cofibrant when seen as a diagram $Sd L \cup DK \rightarrow \mathcal{C}^{\widehat{[1]}}$. Hence Lemma 3.19 implies that it extends to a Reedy cofibrant diagram $DL \rightarrow \mathcal{C}^{\widehat{[1]}}$. We consider it as a diagram $DL \times \widehat{[1]} \rightarrow \mathcal{C}$ and pull it back to $D(L \times \widehat{[1]}) \rightarrow \mathcal{C}$. It restricts to $[X, Y]$ on $D(L \times \partial\Delta[1])$ and to the identity on $D(K \times \widehat{[1]})$. Thus it can be replaced Reedy cofibrantly relative to $D(L \times \partial\Delta[1] \cup K \times \widehat{[1]})$ by Lemma 1.22 which finishes the proof. \square

Proposition 4.7. *For every κ -cocomplete quasicategory \mathcal{D} the functor $\Psi_{\mathcal{D}}$ is a categorical equivalence.*

Proof. Consider a square

$$\begin{array}{ccc} \partial\Delta[m] & \xrightarrow{x} & \mathcal{D} \\ \downarrow & & \downarrow \Psi_{\mathcal{D}} \\ \Delta[m] & \xrightarrow{Y} & N_f Dg_{\kappa} \mathcal{D}. \end{array}$$

By Lemma 4.5 it will be enough to extend x to a simplex $\widehat{x}: \Delta[m] \rightarrow \mathcal{D}$ and construct an $E[1]$ -homotopy from $\Psi_{\mathcal{D}}\widehat{x}$ to Y relative to $\partial\Delta[m]$.

Let's start by finding \hat{x} . Consider $Y_{[m]}: A_{[m]} \rightarrow \mathcal{D}$. Since Y agrees with $\Psi_{\mathcal{D}}x$ over $\partial\Delta[m]$ the $[m]$ th latching object of Y is $x: \partial\Delta[m] \rightarrow \mathcal{D}$, i.e. we have an induced injective map $\partial\Delta[m] \hookrightarrow A_{[m]}$ and $Y_{[m]}|_{\partial\Delta[m]} = x$. Choose a universal cone $\tilde{Y}_{[m]}: A_{[m]}^{\triangleright} \rightarrow \mathcal{D}$ under $Y_{[m]}$ and consider $\tilde{Y}_{[m]}|_{\partial\Delta[m]^{\triangleright}}$. We have $\partial\Delta[m]^{\triangleright} \cong \Lambda^{m+1}[m+1]$ which is an outer horn. However, $\tilde{Y}_{[m]}|_{\partial\Delta[m]^{\triangleright}}$ is special since $\Psi_{\mathcal{D}}x$ is homotopical and thus extends to $z: \Delta[m]^{\triangleright} \rightarrow \mathcal{D}$ by Lemma 2.4. We set $\hat{x} = z|_{\Delta[m]}$.

By Proposition 3.7 finding an $E[1]$ -homotopy from $\Psi_{\mathcal{D}}\hat{x}$ to Y translates into constructing a homotopical Reedy cofibrant diagram $D([m] \times E(1)) \rightarrow \text{Dg}_{\kappa} \mathcal{D}$ restricting to $[\Psi_{\mathcal{D}}\hat{x}, Y]$ on $D(\Delta[m] \times \partial\Delta[1])$. By Corollary 3.14 it will be sufficient to construct such a diagram on $D([m] \times \widehat{[1]})$ and by Lemma 3.19 it will suffice to define it on $\text{Sd}([m] \times \widehat{[1]})$.

We form a pushout on the left

$$\begin{array}{ccc} \tilde{Y}|_{\partial\Delta[m]^{\triangleright}} & \twoheadrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ z & \twoheadrightarrow & Z \end{array} \quad \begin{array}{ccc} \partial\Delta[m] & \hookrightarrow & A_{[m]} \\ \downarrow & & \downarrow \\ \Delta[m] & \hookrightarrow & B \end{array}$$

in $\text{Dg}_{\kappa} \mathcal{D}$. Its underlying square of simplicial sets is $(-)^{\triangleright}$ applied to the square on the right.

This yields the following sequence of morphisms of $\text{Dg}_{\kappa} \mathcal{D}$ (with morphisms of the underlying simplicial sets displayed below).

$$\begin{array}{ccccccc} \hat{x} & \twoheadrightarrow & z & \twoheadrightarrow & Z & \longleftarrow & \tilde{Y}_{[m]} \longleftarrow Y_{[m]} \\ & & & & & & \\ \Delta[m] & \hookrightarrow & \Delta[m]^{\triangleright} & \hookrightarrow & B^{\triangleright} & \longleftarrow & A_{[m]}^{\triangleright} \longleftarrow A_{[m]} \end{array}$$

The first morphism is a weak equivalence since z is a filler of a special horn. So are the middle two since the underlying maps of simplicial sets preserve the cone points. The last one is also a weak equivalence since $\tilde{Y}_{[m]}$ is universal. All these morphisms are maps of cones under $Y|_{\text{Sd}\partial\Delta[m]} = \Psi_{\mathcal{D}}x|_{\text{Sd}\partial\Delta[m]}$ and hence can be seen as transformations of diagrams over $\text{Sd}[m]$ which restrict to identities over $\text{Sd}\partial\Delta[m]$. The conclusion follows by Lemma 4.6. \square

Before we can prove the main theorem we need to know that Dg_{κ} is a homotopical functor. This in turn requires two technical lemmas. The first one is about left homotopies in cofibration categories. Even though cofibrations in a cofibration category do not necessarily satisfy any lifting property, they can still be shown to have a version of the ‘‘homotopy extension property’’ with respect to left homotopies.

Lemma 4.8. *Let $i: A \twoheadrightarrow B$ be a cofibration in \mathcal{C} . Let $f: A \rightarrow X$ and $g: B \rightarrow X$ be morphisms such that gi is left homotopic to f . Then there exist a weak equivalence $s: X \rightarrow \hat{X}$ and a morphism $\tilde{g}: B \rightarrow \hat{X}$ such that \tilde{g} is left homotopic to sg and $\tilde{g}i = sf$.*

Proof. Pick compatible cylinders on A and B , i.e. a diagram

$$\begin{array}{ccccc}
A \amalg A & \xrightarrow{\quad} & IA & \xrightarrow{\sim} & A \\
i \amalg i \downarrow & & \downarrow & & \downarrow i \\
B \amalg B & \xrightarrow{\quad} & IB & \xrightarrow{\sim} & B
\end{array}$$

such that the induced morphism $IA \amalg_{(A \amalg A)} (B \amalg B) \rightarrow IB$ is a cofibration. Let δ_0 and δ_1 denote the two structure morphisms $A \rightarrow IA$.

Pick a left homotopy

$$\begin{array}{ccc}
A \amalg A & \xrightarrow{[f, gi]} & X \\
[\delta_0, \delta_1] \downarrow & & \downarrow j \sim \\
IA & \xrightarrow{H} & \tilde{X}
\end{array}$$

between f and gi . Then we have in particular $jgi = H\delta_1$ and thus there is an induced morphism $[H, jg]: IA \amalg_A B \rightarrow \tilde{X}$ so we can take a pushout

$$\begin{array}{ccc}
IA \amalg_A B & \xrightarrow{[H, jg]} & \tilde{X} \\
\sim \downarrow & & \downarrow \tilde{j} \sim \\
IB & \xrightarrow{\tilde{H}} & \tilde{X}.
\end{array}$$

Set $s = \tilde{j}j$ and $\tilde{g} = \tilde{H}$. We have $sf = \tilde{g}i$ and \tilde{H} and $\text{id}_{\tilde{X}}$ constitute a left homotopy between \tilde{g} and sg . \square

The second lemma says that up to equivalence all frames are Reedy cofibrant replacements of constant diagrams.

Lemma 4.9. *Any object of $X \in \mathbf{N}_f \mathcal{C}$ is equivalent to a Reedy cofibrant replacement of $p_{[0]}^* X_0$.*

Proof. Let $f: [0] \rightarrow D[0]$ and $s: D[0] \rightarrow D[0]$ be as in the proof of Lemma 3.9 so that $p_{[0]} f = \text{id}_{[0]}$ and there are weak equivalences

$$\text{id} \xrightarrow{\sim} s \xleftarrow{\sim} fp_{[0]}.$$

These equivalences evaluated at X form a diagram $D[0] \times \text{Sd} \widehat{[1]} \rightarrow \mathcal{C}$ which we can pull back along $D \widehat{[1]} \rightarrow D[0] \times \text{Sd} \widehat{[1]}$ and then replace Reedy cofibrantly to obtain a homotopical Reedy cofibrant diagram $Y: D \widehat{[1]} \rightarrow \mathcal{C}$ such that $Y\delta_1 = X$ by Lemma 1.22. By Corollary 3.14 Y is an equivalence and by the construction $Y\delta_0$ is a Reedy cofibrant replacement of $p_{[0]}^* X_0$. \square

Lemma 4.10. *The functor Dg_κ is homotopical.*

Proof. We begin by constructing a natural equivalence $\Theta_{\mathcal{C}}: \text{HoN}_f\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ for every cofibration category \mathcal{C} . We send an object $X: D[0] \rightarrow \mathcal{C}$ to X_0 and a morphism $Y: D[1] \rightarrow \mathcal{C}$ to the composite $[v_1]^{-1}[v_0]$ where v_0 and v_1 are the structure morphisms

$$Y_0 \xrightarrow{v_0} Y_{01} \xleftarrow[\sim]{v_1} Y_1.$$

This assignment is well-defined and functorial by Theorem 1.4.

We check that $\Theta_{\mathcal{C}}$ is an equivalence. It is surjective and full since both $\text{Sd}[0] \hookrightarrow D[0]$ and $D\partial\Delta[1] \cup \text{Sd}[1] \hookrightarrow D[1]$ have the Reedy left lifting property with respect to all cofibration categories by Lemma 3.19. For faithfulness, consider $X, \tilde{X}: D[1] \rightarrow \mathcal{C}$ such that $X|_{D\partial\Delta[1]} = \tilde{X}|_{D\partial\Delta[1]}$ and $\Theta_{\mathcal{C}}(X) = \Theta_{\mathcal{C}}(\tilde{X})$. Since we have already verified that $\Theta_{\mathcal{C}}$ is essentially surjective Lemma 4.9 allows us to assume that $X\delta_0$ is a Reedy cofibrant replacement of $p_{[0]}^*X_1$ so that the structure morphisms of X fit into a cylinder

$$X_1 \amalg X_1 \rightarrow X_{11} \xrightarrow{\sim} X_1.$$

By Theorem 1.4(2) we have a diagram

$$\begin{array}{ccccc} & & X_{01} & & \\ & \nearrow & \downarrow \varphi & \nwarrow & \\ X_0 & & Y & & X_1 \\ & \searrow & \uparrow \tilde{\varphi} & \swarrow & \\ & & \tilde{X}_{01} & & \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image, showing the relationships between $X_0, X_1, X_{01}, \tilde{X}_{01}, Y, \tilde{Y}$ and their respective morphisms $\varphi, \tilde{\varphi}, \nu, \tilde{\nu}, \psi, \chi$.)

where both squares commute up to left homotopy. By Lemma 4.8 we can assume that the left square commutes strictly. Let

$$\begin{array}{ccc} X_1 \amalg X_1 & \xrightarrow{[\varphi\nu, \tilde{\varphi}\tilde{\nu}]} & Y \\ \downarrow [\delta_0, \delta_1] & & \downarrow \psi \\ X_{11} & \xrightarrow{\chi} & \tilde{Y} \end{array}$$

be a left homotopy. Then we can form a diagram

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow & \downarrow \nu & \searrow & \\ & & X_{01} & & X_{11} \\ & \nearrow & \downarrow \psi\varphi & \nwarrow & \\ X_0 & & \tilde{Y} & & X_1 \\ & \searrow & \uparrow \psi\tilde{\varphi} & \swarrow & \\ & & \tilde{X}_{01} & & \end{array}$$

(Note: The diagram above is a simplified representation of the large commutative diagram in the image, showing the relationships between $X_0, X_1, X_{01}, X_{11}, \tilde{X}_{01}, \tilde{Y}$ and their respective morphisms $\nu, \tilde{\nu}, \varphi, \tilde{\varphi}, \psi, \chi$.)

which is a homotopical diagram on $\text{Sd}[2]$ and Reedy cofibrant over $\text{Sd } \partial\Delta[2]$. Thus it can be replaced Reedy cofibrantly without modifying it over $\text{Sd } \partial\Delta[2]$ by Lemma 1.22. Then X , \tilde{X} and $X\delta_0\sigma_0$ provide an extension over $D\partial\Delta[2]$. We know that the inclusion $D\partial\Delta[2] \cup \text{Sd}[2] \hookrightarrow D[2]$ has the Reedy left lifting property with respect to all cofibration categories by Lemma 3.19 so we can find an extension to $D[2]$ which is a homotopy between X and \tilde{X} in $\text{N}_f\mathcal{C}$.

Since equivalences of quasicategories induce equivalences of homotopy categories, it follows that N_f reflects equivalences. Thus Dg_κ is homotopical by Proposition 4.7. \square

Finally, we are ready to prove the main theorem.

Theorem 4.11. *The functor $\text{N}_f: \text{CofCat}_\kappa \rightarrow \text{QCat}_\kappa$ is a weak equivalence of fibration categories.*

Proof. By Theorem 3.26 N_f is continuous. The functor Dg_κ is homotopical by Lemma 4.10 and thus induces a functor on the homotopy categories. Since Ψ is a natural categorical equivalence by Proposition 4.7 the induced transformation $\text{Ho } \Psi$ is a natural isomorphism $\text{id} \rightarrow (\text{Ho } \text{N}_f)(\text{Ho } \text{Dg}_\kappa)$. The transformation Φ is merely 2-natural, but natural isomorphisms of exact functors induce right homotopies in CofCat_κ (by the construction of path objects in the proof of Theorem 1.17). Therefore $\text{Ho } \Phi$ is a natural transformation and by Lemma 4.4 it is an isomorphism $(\text{Ho } \text{Dg}_\kappa)(\text{Ho } \text{N}_f) \rightarrow \text{id}$. Hence $\text{Ho } \text{N}_f$ is an equivalence. \square

4.3. Proof of the main theorem: the finite case

The only part of the previous section that does not work for $\kappa = \aleph_0$ is the construction of a natural weak equivalence $\Phi_{\mathcal{C}}: \text{Dg}_\kappa \text{N}_f\mathcal{C} \rightarrow \mathcal{C}$ for every cofibration category \mathcal{C} . Indeed, $\Phi_{\mathcal{C}}$ was defined using colimits over categories DK which are infinite even for finite simplicial sets K . Instead, we will define a zig-zag of (2-natural) weak equivalences connecting $\text{Dg}_{\aleph_0} \text{N}_f\mathcal{C}$ to \mathcal{C} , namely,

$$\text{Dg}_{\aleph_0} \text{N}_f\mathcal{C} \xrightarrow{\Phi_{\mathcal{C}}^{(-)}} \mathcal{C}_{\mathbb{R}}^{\tilde{\aleph}} \longleftarrow \mathcal{C}_{\mathbb{R}}^{\hat{\aleph}} \xrightarrow{\text{ev}_0} \mathcal{C}.$$

We have already verified that $\mathcal{C}_{\mathbb{R}}^{\hat{\aleph}} \hookrightarrow \mathcal{C}_{\mathbb{R}}^{\tilde{\aleph}}$ is a weak equivalence in Lemma 3.43. Moreover, $\text{ev}_0: \mathcal{C}_{\mathbb{R}}^{\tilde{\aleph}} \rightarrow \mathcal{C}$ is induced by a homotopy equivalence $[0] \rightarrow \hat{\aleph}$ hence it is a weak equivalence, too.

It remains to define $\Phi_{\mathcal{C}}^{(-)}$ and prove that it is also a weak equivalence. For each k and an object $X: DK \rightarrow \text{N}_f\mathcal{C}$ we set $\Phi_{\mathcal{C}}^{(k)}X = \text{colim}_{D^{(k)}K} X$. This colimit exists since $D^{(k)}K$ is finite if K is finite.

Lemma 4.12. *For a cofibration category \mathcal{C} the formula above defines an exact functor $\Phi_{\mathcal{C}}^{(-)}: \text{Dg}_{\aleph_0} \text{N}_f\mathcal{C} \rightarrow \mathcal{C}_{\mathbb{R}}^{\tilde{\aleph}}$. Moreover, it is a weak equivalence.*

Proof. First, we need to verify that $\Phi_{\mathcal{C}}^{(-)}X$ is an eventually constant sequence for all $(K, X) \in \text{Dg}_{\aleph_0} \text{N}_f\mathcal{C}$. Consider X as a diagram in $\text{N}_f\mathcal{C}$ and choose a universal cone

$S: K^\triangleright \rightarrow \mathbf{N}_f \mathcal{C}$. Then Lemma 3.42 implies that $\Phi_{\mathcal{C}}^{(-)} S$ is eventually constant and Proposition 3.46 implies that the induced morphism $\Phi_{\mathcal{C}}^{(-)} S \rightarrow \Phi_{\mathcal{C}}^{(-)} S$ is an eventual weak equivalence. Thus $\Phi_{\mathcal{C}}^{(-)} S$ is eventually constant.

Preservation of cofibrations follows by [RB06, Theorem 9.4.1(1a)] since if $K \hookrightarrow L$ is an injective map of simplicial sets, then the induced functors $D^{(k)} K \cup D^{(k-1)} L \rightarrow D^{(k)} L$ are sieves.

Proposition 3.46 and Lemma 3.42 imply that a morphism f in $\mathbf{Dg}_{\aleph_0} \mathbf{N}_f \mathcal{C}$ is a weak equivalence if and only if $\Phi_{\mathcal{C}}^{(-)} f$ is an eventual weak equivalence. Therefore $\Phi_{\mathcal{C}}^{(-)}$ preserves weak equivalences and satisfies (App1).

It follows from Lemma A.5 that $\Phi_{\mathcal{C}}^{(-)}$ preserves colimits and thus it is exact.

It remains to check (App2), but it follows directly from Lemma 3.44. \square

This yields the proof of Theorem 4.11 in the case of $\kappa = \aleph_0$ since the three weak equivalences described above induce a natural isomorphism $(\mathbf{Ho Dg}_{\kappa})(\mathbf{Ho N}_f) \rightarrow \mathbf{id}$ and the rest of the argument applies verbatim.

A. Appendices

A.1. Limits of cocomplete categories

The purpose of this appendix is to prove a few preliminary results in category theory used in Chapter 1. They are all elementary, but we discuss them in some detail since they serve as blueprints for their much more technical quasicategorical counterparts needed in Chapter 2. Roughly speaking, we are interested in criteria for a limit of a diagram of cocomplete categories to be again cocomplete. Such criteria are easily formulated using *isofibrations*.

Definition A.1. A functor $P: \mathcal{E} \rightarrow \mathcal{D}$ is an *isofibration* if for every object $A \in \mathcal{E}$ and an isomorphism $g: PA \rightarrow Y$ there is an isomorphism $f: A \rightarrow B$ such that $Pf = g$.

In other words, isofibrations are functors with the right lifting property with respect to the inclusion $[0] \hookrightarrow E(1)$ where $E(1)$ is the free groupoid on one isomorphism $0 \rightarrow 1$.

If J is a small category then its *under-cone*¹ is the category J^\triangleright (or $J \star [0]$ in the notation of Chapter 2) obtained by adjoining a new terminal object. This notion can be used to define colimits as follows. Given a diagram $X: J \rightarrow \mathcal{C}$, cones under X can be identified with extensions of X to diagrams $J^\triangleright \rightarrow \mathcal{C}$. Such a cone S is universal if for any other cone T under X there is a unique transformation $S \rightarrow T$ that restricts to id_X . As usual, a *colimit* of X is a universal cone under X , but we often abuse the terminology by referring to the cone object of a universal cone under X as a colimit of X .

Lemma A.2. *Let*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{G} & \mathcal{E} \\ Q \downarrow & & \downarrow P \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

be a pullback square in the category of categories. Let $S: J^\triangleright \rightarrow \mathcal{P}$ be a cone. If all GS , QS and $PGS = FQS$ are universal, then so is S .

Proof. Let $T: J^\triangleright \rightarrow \mathcal{P}$ be a cone such that $T|_J = S|_J$. Then we have $GT|_J = GS|_J$ and $QT|_J = QS|_J$ and since both GS and QS are universal, there are unique transformations $u: GS \rightarrow GT$ and $v: QS \rightarrow QT$ such that $u|_J = \text{id}_{GS|_J}$ and $v|_J = \text{id}_{QS|_J}$. Since

¹This non-standard name is introduced as an alternative to the inconsistent usage of names “cone” and “cocone”. The dual notion is called an *over-cone*. This terminology is supposed to be reminiscent of “under-categories” and “over-categories” since objects of these are “cones under” and “cones over” in the case of $J = [0]$.

$PGS = FQS$ is also universal, it follows that $Pu = Fv$ and thus there is a unique transformation $w: S \rightarrow T$ such that $Gw = u$ and $Qw = v$ which is also the unique transformation $S \rightarrow T$ such that $w|_J = \text{id}_{S|_J}$. Hence S is universal. \square

The same approach also proves the following.

Lemma A.3.

(1) Let

$$\mathcal{C}_0 \longleftarrow \mathcal{C}_1 \longleftarrow \mathcal{C}_2 \longleftarrow \dots$$

be a tower of functors with a limit \mathcal{D} and structure maps $P_i: \mathcal{D} \rightarrow \mathcal{C}_i$. If $S: J^\triangleright \rightarrow \mathcal{D}$ is a cone such that $P_i S$ is universal for all i , then S is universal itself.

(2) Let $(\mathcal{C}_i)_{i \in I}$ be a family of categories. If $(S_i): J^\triangleright \rightarrow \prod_i \mathcal{C}_i$ is a cone such that S_i is universal for all $i \in I$, then S is universal itself. \square

Lemma A.4. Let $P: \mathcal{C} \rightarrow \mathcal{D}$ be an isofibration, $X: J \rightarrow \mathcal{C}$ a diagram and $T: J^\triangleright \rightarrow \mathcal{D}$ a colimit of PX . If X has a colimit in \mathcal{C} which is preserved by P , then there exists a colimit $S: J^\triangleright \rightarrow \mathcal{C}$ of X such that $PS = T$.

In other words, if a colimit is preserved up to isomorphism, then it can be replaced by one that is preserved on the nose.

Proof. Let $\tilde{S}: J^\triangleright \rightarrow \mathcal{C}$ be some colimit of X . Then we have a unique transformation $w: T \rightarrow P\tilde{S}$ such that $w|_J = \text{id}_{PX}$. Since P preserves this colimit, w is a natural isomorphism and we can apply the lifting property of isofibrations to replace the cone object of \tilde{S} by an isomorphic one which maps strictly to the cone object of T . Therefore we obtain a cone S which is isomorphic to \tilde{S} , hence universal, and satisfies $PS = T$. \square

We conclude this section with two small lemmas about colimits.

Lemma A.5. Let I be a small category and $J: I \rightarrow \mathbf{Cat}$ a diagram of small categories. Denote by \bar{J} the colimit of J with the structure maps $\kappa_i: J_i \rightarrow \bar{J}$. Let $X: \bar{J} \rightarrow \mathcal{C}$ be a diagram such that all $X\kappa_i$ have colimits in \mathcal{C} . Then the canonical morphism

$$\text{colim}_{i \in I} \text{colim}_{J_i} X\kappa_i \rightarrow \text{colim}_{\bar{J}} X$$

is an isomorphism (in particular, one colimit exists if and only if the other does).

Proof. Let A be an object of \mathcal{C} . There is a sequence of natural bijections between the following sets.

- The set of natural transformations $X \rightarrow \text{const } A$ in $\mathcal{C}^{\bar{J}}$.
- The set of functors $\bar{J} \times [1] \rightarrow \mathcal{C}$ restricting to $[X, \text{const } A]$ on $\bar{J} \times \{0, 1\}$.

- The set of functors $\text{colim}_{i \in I}(J_i \times [1]) \rightarrow \mathcal{C}$ restricting to $[[X\kappa_i, \text{const } A] \mid i \in I]$ on $\text{colim}_{i \in I}(J_i \times \{0, 1\})$.
- The set of families of natural transformations $X\kappa_i \rightarrow \text{const } A$ in \mathcal{C}^{J_i} for each $i \in I$ compatible with respect to morphisms of I .
- The set of natural transformations $\text{colim}_{J_i} X\kappa_i \rightarrow \text{const } A$ in \mathcal{C}^I .

When A varies each of these items forms a functor $\mathcal{C} \rightarrow \text{Set}$. A representation of the first one is $\text{colim}_{\bar{J}} X$ and a representation of the last one is $\text{colim}_{i \in I} \text{colim}_{J_i} X\kappa_i$. Thus if one of these colimits exists, then so does the other one and they are canonically isomorphic. \square

Let Set_J denote the category of presheaves over a small category J .

Lemma A.6. *Let J be a small category. Then every presheaf on J can be canonically written as a colimit of representables over its category of elements. If a functor $F: \text{Set}_J^{\text{op}} \rightarrow \text{Set}$ preserves these colimits (i.e. carries them to limits) then it is representable and represented by $j \mapsto FJ(-, j)$.*

Proof. The first statement is [Bor94, Theorem 2.15.6] and the second one follows by a straightforward computation. Let $\widehat{F}j = FJ(-, j)$ and $X \in \text{Set}_J$, we have

$$FX = F(\text{colim}_{J(-, j) \rightarrow X} J(-, j)) = \lim_{J(-, j) \rightarrow X} \widehat{F}j = \text{Set}_J(X, \widehat{F}). \quad \square$$

A.2. Homotopy theory of complete homotopy theories

All the results of this thesis can be formally dualized to results about fibration categories and complete quasicategories. However, in a few cases such dualizations can be less than obvious at the first glance. Hence we describe some of them directly.

The first potentially confusing situation is the dualization of the definition of DJ and $p_J: DJ \rightarrow J$ for a homotopical category J (which needs to respect the duality between cofibration and fibration categories as discussed before Definition 1.12). We can proceed as follows. First, we consider J^{op} and associate to it $D(J^{\text{op}})$ and $p_{J^{\text{op}}}: D(J^{\text{op}}) \rightarrow J^{\text{op}}$. Second, we take $(D(J^{\text{op}}))^{\text{op}}$ and $p_{J^{\text{op}}}^{\text{op}}: (D(J^{\text{op}}))^{\text{op}} \rightarrow J$. Actually, the first step was not necessary, we carried it out only to end up with a functor to J instead of J^{op} . Unfortunately, this complicated the notation because of the category $(D(J^{\text{op}}))^{\text{op}}$. This can be simplified by observing that there is an isomorphism $DJ \cong D(J^{\text{op}})$ that is the identity on objects and that reindexes the face operators in the reverse order. Then instead of $(p_{J^{\text{op}}})^{\text{op}}$ we can consider the composite

$$(DJ)^{\text{op}} \cong (D(J^{\text{op}}))^{\text{op}} \rightarrow J$$

which we denote by \bar{p}_J and which is given by evaluating an object $[k] \rightarrow J$ at 0. The homotopical structure of $(DJ)^{\text{op}}$ is now created by \bar{p}_J (or, equivalently, by the above isomorphism $(DJ)^{\text{op}} \cong (D(J^{\text{op}}))^{\text{op}}$) and hence it is not dual to the homotopical structure of DJ used in the main body of the thesis. We end up with the choice of carrying a lot

of opposites around or redefining the homotopical structure of $(DJ)^{\text{op}}$. It seems that the latter option results in a more readable notation.

Similarly, given a marked simplicial complex K we redefine the homotopical structure of $(DK)^{\text{op}}$ (and $(\text{Sd } K)^{\text{op}}$) by restricting it from $(DP)^{\text{op}}$ (where P is the underlying homotopical poset of K) which in turn was created by \bar{p}_J .

Finally, in order to keep the notation consistent we need to redefine the homotopical structure of $(DK)^{\text{op}}$ for a simplicial set K . Let $x \in K_m$ and $y \in K_n$ and let $\varphi: [m] \hookrightarrow [n]$ be a morphism $x \rightarrow y$ in DK , i.e. an injective order preserving map such that $x = y\varphi$. Then it (or rather its dual) is a *generating weak equivalence* if $y\nu$ is a degenerate edge of K where $\nu: [1] \rightarrow [n]$ is given by $\nu(0) = 0$ and $\nu(1) = \varphi(0)$. We define the homotopical structure of $(DK)^{\text{op}}$ by closing the generating weak equivalences under “2 out of 6”.

The second construction that requires some explanation is the *opposite of the category of diagrams in a κ -complete quasicategory*. First of all, we point out that the theory of cocomplete quasicategories dualizes easily. Namely, the category of simplicial sets has an involution $(-)^{\text{op}}: \mathbf{sSet} \rightarrow \mathbf{sSet}$ given by reindexing the face and degeneracy operators in the reverse order. This involution interchanges arguments of the join functor and hence sends the “under-cones” to “over-cones” which translates the definition of colimits to the definition of limits. It follows that a quasicategory is κ -cocomplete if and only if its opposite is κ -complete. If \mathcal{D} is a κ -complete quasicategory, then the associated fibration category is $(D_\kappa(\mathcal{D}^{\text{op}}))^{\text{op}}$. When we unravel this definition (and drop some opposites to simplify the notation) we find out that the underlying category of this fibration category is $\mathbf{sSet}_\kappa^{\text{op}} \downarrow \mathcal{D}$, i.e. an object is a diagram $X: K \rightarrow \mathcal{D}$ (with K κ -small) and a morphism from X to $Y: L \rightarrow \mathcal{D}$ is specified by a simplicial map $f: L \rightarrow K$ such that $Xf = Y$. Moreover, such a morphism $X \rightarrow Y$ is a fibration when f is injective and a weak equivalence when it induces an equivalence $\lim_K X \rightarrow \lim_L Y$.

Keeping these modifications in mind, it is routine to translate all the arguments to fibration categories and complete quasicategories. The dual of the main result says that there are fibration categories of κ -complete fibration categories and κ -complete quasicategories and they are weakly equivalent.

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Summary

The main result of this thesis is a comparison between two major approaches to abstract homotopy theory, namely, *homotopical algebra* and *higher category theory*. These two approaches use different techniques and have different advantages and disadvantages. However, according to the theorems presented here they ultimately have the same content.

Homotopical algebra is a classical theory originally introduced by Quillen [Qui67]. There are a few types of structures implementing this theory, e.g. model categories, cofibration categories and fibration categories. Every such structure presents an individual homotopy theory (e.g. the homotopy of spaces, of spectra, of chain complexes etc.) and provides explicit point-set level tools for carrying out homotopy invariant constructions inside it (or even between varying homotopy theories). Similarly, higher category theory offers its own collection of structures that can be used to present homotopy theories: quasicategories, complete Segal spaces and Segal categories are the most notable ones. A big advantage of higher category theory is that it allows for expressing universal properties of homotopy invariant functors which are difficult to access via homotopical algebra. Conversely, it is disconnected from point-set level models of such functors which are often useful and directly described by homotopical algebra.

The comparisons between various notions of higher category are well-known by now, mostly due to the work of Bergner, Joyal and Tierney [Ber07b, JT07]. Moreover, such comparisons fit into the axiomatic framework of Barwick and Schommer-Pries [BSP13]. These results, however, do not address the relationship between homotopical algebra and higher category theory. This is largely because notions of homotopical algebra are highly structured which imposes strong conditions on the homotopy theories they present which are not captured by the mentioned comparison techniques.

The main theorem of this thesis (Theorem 4.11) says that the homotopy theory of cofibration categories is equivalent to the homotopy of cocomplete quasicategories thus bridging the gap between homotopical algebra and higher category theory. Here “cocomplete” means that the associated homotopy theories admit homotopy colimits (of a specified size). All cofibration categories satisfy this condition by design.

There are four major ingredients in the proof of the main theorem.

The first one is an implementation of the homotopy theory of cofibration categories within homotopical algebra. This is achieved by constructing a fibration category of cofibration categories in Theorem 1.17. On one hand, this makes many standard tools available in the theory of cofibration categories which is crucial in all steps of the argument. On the other, it sets the stage for the comparison with quasicategories since the fibration of cocomplete quasicategories (Theorem 2.17) is easily established thanks to the results of Joyal and Lurie.

The second ingredient is the construction of the quasicategory of frames in any given cofibration category in Section 3.1. This construction enhances the classical calculus of fractions and provides higher categorical models of cocomplete homotopy theories that are particularly well suited for comparisons to the original cofibration categories from which they come.

Another important component contained in Propositions 3.32 and 3.46 is a comparison of point-set level constructions of homotopy colimits in cofibration categories to the universal properties of colimits in the associated quasicategories of frames. This explains how the most crucial difference between homotopical algebra and higher category theory can be overcome.

Finally, the last step (Sections 4.2 and 4.3) builds on results of Lurie and reconstructs a cofibration category from any cocomplete quasicategory. All the methods mentioned above are used to demonstrate that cofibration categories obtained this way do indeed capture all the homotopy theoretic content of the original cocomplete quasicategories thus completing the proof of the main theorem.