A Potpourri of
PARTITION PROPERTIES

Dissertation zur Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von Thilo Volker Weinert
aus
Hanau

Bonn, February 2014

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät

1. Gutachter: Prof. Dr. Stefan Geschke
2. Gutachter: Prof. Dr. Peter Koepke
Tag der Promotion: Montag, der 2. Juni 2014
Erscheinungsjahr: 2014

Zusammenfassung

Die Dissertation beschäftigt sich mit Fragen, die einen Bezug zum Konzept der Partition bzw. der Färbung besitzen. Sie behandelt diese im Kontext der axiomatischen Mengenlehre nach Zermelo und Fraenkel. Mit Ausnahme von Kapitel 4 wird auch stets die Gültigkeit des Auswahlaxioms angenommen.

Zum einen werden bestimmte Kardinalzahlcharakteristika des Kontinuums, die einen solchen Bezug aufweisen, untersucht. Bisherige Analysen selbiger stammen von Frick, Geschke, Goldstern, Kojman, Kubiś und Schipperus. Im ersten Teil der Arbeit werden Bezüge zu anderen bekannten Kardinalzahlcharakteristika hergestellt. So wird beispielsweise $\mathfrak{r} \leqslant \mathfrak{hm}_3$ bewiesen. Darüber hinaus werden, unter Benutzung der von Zapletal entwickelten Theorie des idealisierten Erzwingens, Eigenschaften der Struktur möglicher Beweise der relativen Widerspruchsfreiheit von $\mathfrak{hm}_{\min} < \mathfrak{r}$ abgeleitet.

Der Rest der Dissertation handelt von Aussagen im Partitionskalkül nach Erdős und Rado. Dieser Rest zerfällt wiederum in einen ersten Teil, in dem lineare Ordnungen untersucht werden, und einen zweiten, in dem das Interesse Ordinalzahlen gilt.

Bezüglich linearer Ordnungen wird in Kapitel 5 gezeigt, dass für jede zerstreute lineare Ordnung τ und jede natürliche Zahl n eine zerstreute lineare Ordnung φ mit der Eigenschaft $\varphi \to (\tau, n)^2$ existiert. Dies ergänzt bekannte Resultate von Erdős, Hajnal, Larson und Milner. Des weiteren werden in Kapitel 4 eine Reihe von Resultaten für Färbungen von Paaren, Tripeln und Quadrupeln lexikographisch geordneter Folgen von Nullen und Einsen ohne Verwendung des Auswahlaxioms bewiesen. Diese Aussagen sind in Modellen, in denen das Axiom der Determiniertheit gilt, besonders interessant. Für diesen Kontext werden drei Vermutungen motiviert und formuliert. Diese Resultate können als Analoga zu ähnlichen Resultaten von Erdős, Milner und Rado unter Annahme des Auswahlaxioms angesehen werden. Der Exponent der Partitionsrelation ist dann jedoch typischerweise um eins niedriger.

Die Analyse von Ordinalzahlen in den Kapiteln 6 und 7 kombiniert bekannte Methoden aus der endlichen und unendlichen Kombinatorik, um eine Reihe neuer

Resultate zu beweisen und bekannte zu verallgemeinern. Gewisse Resultate gelten nur unter der Annahme von Martins Axiom. Kapitel 6 konzentriert sich auf die Zurückführung von Problemen der infinitären Kombinatorik auf solche der finitären Kombinatorik. Kapitel 7 hingegen beleuchtet dann diese endlichen Probleme genauer. Im Stile der endlichen Ramseytheorie werden obere Schranken für die kleinste natürliche Zahl mit der jeweils in Frage stehenden Eigenschaft gefunden. So wird beispielsweise eine kubische obere Schranke in m für das kleinste n so dass $\omega^2 n \to (\omega^2 m, 3)^2$ gefunden. Dies ist interessant, da sich für die analogen Probleme, die sich ergeben, wenn man ω^2 jeweils durch ω und 1 ersetzt, leicht quadratische obere Schranken herleiten lassen.

Kapitel 8 schließlich handelt vom Milner-Prikry-Problem. Es wird nicht gelöst, jedoch wird im Kontext des Problems eine Aussage bewiesen, die nach Meinung des Autors einen Hinweis zur Lösung geben könnte.

Preface

This is a thesis on set theory. The author hopes to attain a doctoral degree submitting it. Despite this he hopes that it contains some value beyond that. Set theory is a branch of modern mathematics and—if one believes the rumours—a somewhat esoteric one. The American Mathematical Society has introduced a system to classify all kinds of mathematics. It has three levels. A two-digit number specifies the top level. The second level is given by a letter and the third, again, by a two-digit number. 83F05, for example, would specify the field of relativistic cosmology. Set theory is classified as 03E and so it belongs to Mathematical Logic, classified as 03. Considering this one might get the impression that every mathematical specialisation can be seen as an esoteric occupation and it surely can. Still, something seems to be different here as the relationship of the average mathematician towards mathematical logic tends to remind the author of the outlook the average person has towards mathematics in general. Clearly, mathematics as a whole can be look upon as an esoteric pursuit. It is esoteric because everything takes so much time. It takes time to learn a subject, it takes time to read an article, it takes even more time to write one. More than in other fields? Yes, much more, at least if one takes a page in a journal as the unit of measurement.

So why should the relationship between logicians and other mathematicians be like the one between mathematicians in general and anyone else? Any mathematician who is not completely incompetent has some understanding of logic. And everyone in our contemporary society who does not want to be tricked on a regular basis at least needs a basic understanding of addition and subtraction of rational numbers. Being constantly exposed to something will often not encourage a person's curiosity, neither is it going to spark imagination. Having spent the day in a dark basement a painter might actually be more creative in the evening than if he had been, say, parachuting on this day. This might be at the heart of the matter.

According to the American Mathematical Society this thesis deals with Partition Relations(03E02), Ordinal and Cardinal Numbers(03E10), Cardinal Characteristics of the Continuum(03E17), the Axiom of Choice and Related Propositions(03E25), Consistency and Independence Results(03E35), the Continuum Hypothesis and Martin's Axiom(03E50) and Determinacy Principles(03E60).

That the author ended up writing this thesis on these topics is the result of both many coincidences and quite a few number of concious decisions not all of which can or should be made transparent in a preface.

This is not the first thesis in set theory which has been written in Bonn-Beuel. The author knows of Adrian Richard David Mathias and of Ioanna Matilde Dimitriou who did the same before.

The author wishes to apologise in advance for anything which might stand in the way of smooth reading. Perfection can never be reached in this world.

Bonn Beuel, 6^{th} February 2014, a Thursday

Contents

1 Introduction

Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Arthur William Russell, 3rd Earl Russell

With the benefit of hindsight one could say that the story of set theory began with Cantor's paper [874Ca] in which he showed that within every interval on the real line there is a transcendental real number. However, at this point he did not emphasise the fact that this argument shows there to be many more real numbers than algebraic ones[1].

The Continuum Hypothesis

Four years later, in 1878, Cantor published the follow-up paper [878Ca]. While the main point of the paper lies in proving the equinumerosity of Euclidean spaces of any finite positive dimension, Cantor states for the first time what is now known as the continuum hypothesis towards its end. The continuum hypothesis claims the following:

(CH) Any infinite set of real numbers is either equinumerous to the set of natural numbers or to the set of all real numbers.

The continuum hypothesis rose to prominence at the latest when it was on the top of David Hilbert's list of 23 open problems he compiled for the second International Congress of Mathematicians in 1900 in Paris. Hilbert presented ten of these problems in his lecture on 8th August at the Sorbonne.

After Cantor achieved partial results by showing that in important point-classes sets of reals cannot assume an intermediate cardinality a major step towards understanding CH was taken by Gödel who showed in [940Gö] that CH is consistent with ZFC. In [963Co] Cohen proved that CH's negation is consistent with ZFC as well.

Ramsey Theory

In [930Ra] Ramsey proved what became known as Ramsey's Theorem. It states that however one distributes the sets of natural numbers of a given fixed finite size into finitely many classes, there is always an infinite set of natural numbers such that all its subsets of this very size are in the same

[1] It seems to be a common belief that this were also the first proof of the existence of transcendental numbers. This belief, however, is actually a misconception. The existence of a transcendental number was first shown by Joseph Liouville in [844Li].

class. As the title of the paper suggests, its primary topic was mathematical logic rather than combinatorics. Ramsey's Theorem was proved with a specific application in mind but Ramsey Theory soon became interesting in its own right. It got off the ground some years later when Pál Erdős developed an interest in it. In [935ES] he and George Szekeres showed that among any five points in the plane no three of which are collinear there are always four forming a convex quadrilateral. Many more theorems of this sort appeared in the following years. The general idea always is that if a structure is rich enough in a specified sense there is a substructure which is—again in a specified sense—ordered. Ramsey Theory entered Set Theory when Pál Erdős and Richard Rado published their seminal paper [956ER] in which they introduced what is now known as the arrow-notation or Hungarian notation. Over the years more and more variants of this notation were introduced.

We are now going to explain the meaning of those variants of the arrow-notation used in this thesis. Suppose τ is an order-type, κ is a cardinal larger than 1, X is some set and $\chi\colon [X]^\tau \longrightarrow \kappa$ is a function. Such functions are often referred to as *colourings*. We call an $H \subset X$ *homogeneous* for χ if χ is constant on $[H]^\tau$. By writing

$$\varphi \to (\psi)^\tau_\kappa$$

we mean that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $H \in [X]^\psi$ which is homogeneous for χ. Speaking about a set's size in this context most often simply refers to its cardinality. Sometimes, however, as in this thesis, we want to speak about its size in a more structural sense. Then "size" refers to something else as well such as a topology or an ordering. In this thesis "size" will refer to the order-type throughout. Increasing the size of any of the parameters on the right side strengthens the relation while increasing φ weakens it.

The second variant of the Hungarian notation we want to mention is the *square-bracket partition relation*. By writing

$$\varphi \to [\psi]^\tau_\kappa$$

we mean that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $N \in [X]^\psi$ such that $\chi \restriction [N]^\tau$ does not map onto κ, i.e. there is an

$\alpha < \kappa$ such that $\chi^{-1}``\{\alpha\} \subset [X]^\tau \setminus [N]^\tau$. Note that if $\kappa > 1$ the round-bracket partition relation implies the square-bracket partition relation. For $\kappa = 2$ they are equivalent. This relation behaves similarly to the round-bracket partition relation under a change of parameters, the sole exception is that increasing the number of colours κ does not result in a stronger but a weaker relation. An important result by Stevo Todorcevic on this subject is the following.

Theorem 1.1 (ZFC, [987To]). $\omega_1 \not\rightarrow [\omega_1]^2_{\aleph_1}$.

Note that this is a very strong result because it claims that for ω_1 even one of the weakest partition relations is false. We mention this since it is a fact one should keep in mind reading Chapter 4.

Now we are going to introduce the *asymmetric partition relation*. The basic idea behind it is that one may search for homogeneous sets of different sizes—depending on the colour. By writing

$$\varphi \rightarrow (\psi_0, \psi_1, \dots)^\tau_\kappa$$

we mean that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $\alpha < \kappa$ and an $H \in [X]^{\psi_\alpha}$ such that $\chi``[H]^\tau = \{\alpha\}$. An alternative notation for κ a natural number consists in writing $\varphi \rightarrow (\psi_0, \dots, \psi_{\kappa-1})^\tau$.

Similarly, by writing

$$\varphi \rightarrow [\psi_0, \psi_1, \dots]^\tau_\kappa$$

we mean that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $\alpha < \kappa$ and an $N \in [X]^{\psi_\alpha}$ such that $\chi``[N]^\tau \subset \kappa \setminus \{\alpha\}$. Again $\varphi \rightarrow [\psi_0, \dots, \psi_{\kappa-1}]^\tau$ is an alternative notation for natural κ. Chapter 8 deals with such a relation.

One can also consider partition relations with alternatives. So

$$\varphi \rightarrow (\psi^0_0 \vee \dots \vee \psi^0_{k_0}, \psi^1_0 \vee \dots \vee \psi^1_{k_1}, \dots)^\tau_\kappa$$

means that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $\alpha < \kappa$, an $\ell \leqslant k_\alpha$ and an $H \in [X]^{\psi^\alpha_\ell}$ such that $\chi``[H]^\tau = \{\alpha\}$. For finite κ another way of expressing this is $\varphi \rightarrow (\psi^0_0 \vee \dots \vee \psi^0_{k_0}, \dots, \psi^{\kappa-1}_0 \vee \dots \vee \psi^{\kappa-1}_{k_{\kappa-1}})^\tau$.

Finally, this relation has a square-bracket version as well.

$$\varphi \to [\psi_0^0 \vee \cdots \vee \psi_{k_0}^0, \psi_0^1 \vee \cdots \vee \psi_{k_1}^1, \ldots]_\kappa^\tau$$

means that for any set X of size φ and any colouring $\chi\colon [X]^\tau \longrightarrow \kappa$ there is an $\alpha < \kappa$, an $\ell \leqslant k_\alpha$ and an $N \in [X]^{\psi_\ell^\alpha}$ such that $\chi``[N]^\tau \subset \kappa \setminus \{\alpha\}$. For finite κ another way of expressing this is $\varphi \to [\psi_0^0 \vee \cdots \vee \psi_{k_0}^0, \ldots, \psi_0^{\kappa-1} \vee \cdots \vee \psi_{k_{\kappa-1}}^{\kappa-1}]^\tau$.

As "size" usually refers to the cardinality or to the order-type and as such an ordering is often well-founded there often is no need to consider a partition relation with alternatives. Chapter 4 is an exception.

Cardinal Characteristics of the Continuum

In mathematics it often happens that the solution of an open problem neither leads to it being canonized in textbooks and forgotten about regarding research activities nor to the problem being plainly forgotten about. In many cases people rather tweak the problem just so much that the solution does not work any more. This is no completely unsystematic way of generating new problems and testing the methods employed in the solution of the original problem.

If we, for the sake of the argument, regard the proof of CH's independence from ZFC as a solution then this is precisely what happened to the continuum problem. It led to the development of the research area of *Cardinal Characteristics of the Continuum*. The idea is to consider families of reals and ask which size they must have in order to satisfy a certain combinatorial property. Here *real* refers, as is common in set theory, to some infinite hereditarily countable object, e.g. an element of the Cantor space ${}^\omega 2$ or the Baire space ${}^\omega\omega$. Most of the time it is a ZFC-result that countable families fail to satisfy the property in question. Thus a cardinal characteristic of the continuum is typically an uncountable cardinal no larger than the continuum. Of course this means that CH completely trivialises the subject. To give a paradigmatic example consider the dominating number.

Definition 1.2. The *dominating number* $\mathfrak{d}$ is the least size of an $F \subset {}^\omega\omega$ such that for all $g \in {}^\omega\omega$ there is an $f \in F$ such that for all $n < \omega$ we have $g(n) < f(n)$.

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathrm{non}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathrm{cof}(\mathcal{M}) \arrow[r, dash] \arrow[d, dash] & \mathrm{cof}(\mathcal{N}) \arrow[r, dash] \arrow[dd, dash] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[r, dash] \arrow[d, dash] & \mathfrak{d} \arrow[d, dash] & & \\
\aleph_1 \arrow[r, dash] & \mathrm{add}(\mathcal{N}) \arrow[r, dash] & \mathrm{add}(\mathcal{M}) \arrow[r, dash] & \mathrm{cov}(\mathcal{M}) \arrow[r, dash] & \mathrm{non}(\mathcal{N}) &
\end{tikzcd}
$$

Figure 1.1: Cichoń's diagram, for details cf. [995BJ].

$\mathfrak{d}$ is also an example for a certain class of cardinal characteristics, the *covering numbers*. For any ideal I on the reals, $\mathrm{cov}(I)$ is defined as the least size of a family of members of the ideal whose union is the set of all reals. $\mathfrak{d}$ is simply $\mathrm{cov}(\mathcal{K}_\sigma)$ where $\mathcal{K}_\sigma$ is the ideal of countable unions of compact sets in the Baire space, cf. [010Bl].

A synonym for "cardinal characteristic" is "cardinal invariant". The latter name, however, might be seen as slightly misleading as the whole point of the enterprise is to prove that their values vary throughout what has been called the generic multiverse, cf. [012Ha].

For any two thus defined cardinal characteristics of the continuum, $\mathfrak{x}$ and $\mathfrak{y}$, the question arises whether one can prove the inequality $\mathfrak{x} \leqslant \mathfrak{y}$ in ZFC or whether there is a model of ZFC + "$\mathfrak{y} < \mathfrak{x}$". Two main strands of cardinal characteristics of the continuum have been analysed. Those derived from the ideals of Lebesgue measure zero sets and meagre sets, respectively, on the one hand and several characteristics associated to the algebra $\mathcal{P}(\omega)/\mathrm{fin}$ on the other. They are often arranged in two diagrams, called Cichoń's diagram and van Douwen's diagram. Whenever a line connects two cardinal characteristics $\mathfrak{x}$ and $\mathfrak{y}$ where $\mathfrak{y}$ is above $\mathfrak{x}$ this says that $\mathfrak{x} \leqslant \mathfrak{y}$ is provable in ZFC. In the case of Cichoń's diagram this also holds when $\mathfrak{y}$ is to the right of $\mathfrak{x}$.

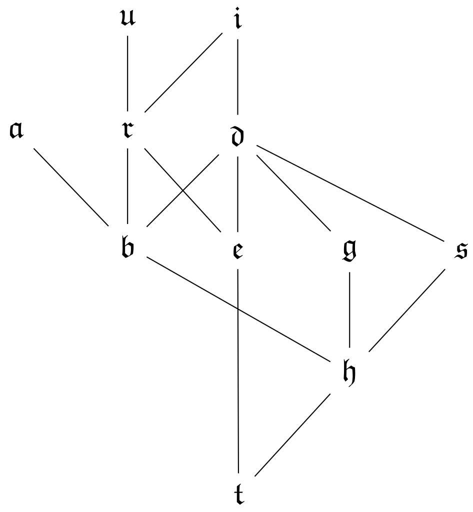

Figure 1.2: van Douwen's diagram, for details cf. [010Bl].

Souslin's Hypothesis

After the independence of CH had been established one of the famous problems in set theory left open was Souslin's hypothesis.

We say that a linear order has the *countable chain condition* if every collection of mutually disjoint non-empty open intervals is countable. A *Souslin line* is a complete dense linear order satisfying the countable chain condition without endpoints which is not isomorphic to the real line.

(SH) There are no Souslin lines.

As it happened again and again in the history of mathematics, the name is somewhat misleading. Souslin never hypothesised SH and it is argued in [011Ka] that if one wished to name something "Souslin's hypothesis", $\neg$SH might have been a better choice.

Souslin's hypothesis had an important influence on the development of set theory as the first proof of its consistency in [971ST] introduced *iterated forcing* into set theory. While iterated forcing is not central to this thesis its development made approaching questions regarding cardinal characteristics of the continuum possible in a systematic way. Moreover it led to the formulation of a new axiom called “Martin’s Axiom”.

Other Hypotheses

Early on, CH was used to prove certain statements which do not derive from the axioms of ZFC alone. Cf. e.g. [954BS]. After the proof of CH’s consistency this was an straightforward method to prove statements to be consistent with ZFC without having to deal with the subtleties of Gödel’s constructible universe L every time. CH’s negation is on average much less suitable than CH to decide statements about the continuum. Although it might be argued that the technique of forcing is more accessible than the theory of L it was clearly desirable to have an axiom describing a somewhat canonical scenario of $\mathsf{ZFC} + \neg\,\mathsf{CH}$. Such an axiom was introduced by Donald Anthony Martin and Robert Solovay in [970MS]. Subsequently it became known as Martin’s Axiom. It is a weakening of the continuum hypothesis which is consistent with the continuum being any regular uncountable cardinal (if ZF is consistent). We are going to abbreviate it by MA. It is defined as follows:

$$(\mathsf{MA}) \qquad \forall\kappa < \mathfrak{c}\colon\ \mathsf{MA}_\kappa$$

We shall not define MA_κ. It turns out that $\mathsf{MA}_{\aleph_1}$ decides many of the statements CH decides and in fact often differently. We provide some examples for this phenomenon.

- $\mathsf{MA}_{\aleph_1} \Rightarrow \forall n < \omega\colon \omega_1\omega \to (\omega_1\omega, n)^2$ while $\mathsf{CH} \Rightarrow \omega_1\omega \not\to (\omega_1\omega, 3)^2$.
- $\mathsf{MA}_{\aleph_1} \Rightarrow \forall n < \omega\colon \omega_1\omega^2 \to (\omega_1\omega^2, n)^2$ while $\mathsf{CH} \Rightarrow \omega_1\omega^2 \not\to (\omega_1\omega^2, 3)^2$.
- $\mathsf{MA}_{\aleph_1} \Rightarrow \mathrm{add}(\mathcal{N}) > \aleph_1$ while $\mathsf{CH} \Rightarrow \mathrm{add}(\mathcal{N}) = \aleph_1$.
- $\mathsf{MA}_{\aleph_1} \Rightarrow \mathfrak{t} > \aleph_1$ while $\mathsf{CH} \Rightarrow \mathfrak{t} = \aleph_1$.

The first two statements were proved by Baumgartner in [989Ba] and Erdős and Hajnal in [971EH], respectively. The third one was already given in [970MS]. The last statement is not the whole truth: Over a prolonged period it was shown that $\mathfrak{t} > \aleph_1$ is equivalent to $\mathsf{MA}_{\aleph_1}$ for σ-centred posets, cf. [981Be] and [XXXMS]. There are, however, also situations where one has to do more than considering $\mathsf{MA}_{\aleph_1}$ and CH:

- $\forall \alpha < \omega_1 \colon \omega_1 \to (\omega_1, \alpha)^2$ is consistent with ZFC while $\mathsf{CH} \Rightarrow \omega_1 \not\to (\omega_1, \omega + 2)^2$.

- $\mathsf{MA}_{\aleph_1} \Rightarrow \mathsf{SH}$ while SH is independent from CH.

Martin's Axiom is going to be employed in Chapter 6 to prove Theorem 6.24.

Later on even more axioms capturing the specifics of models obtained by iterated forcing were formulated. The Proper Forcing Axiom settling questions independent from Martin's Axiom was defined and proved consistent relative to the existence of a supercompact cardinal by Baumgartner, cf. [984Ba]. In [004CP] Ciesielski and Pawlikowski showed how to capture an essential part of the combinatorics of the reals in models attained by the iteration of specific notions of forcing. About the same time Zapletal started to develop the theory of idealised forcing in [004Za]. This theory shows that the empirical fact that there often is a natural choice for a notion of forcing increasing the cardinal characteristic $\operatorname{cov}(I)$ for an ideal I on the reals can be made precise. $\mathsf{CPA}(I)$ is an axiom describing properties of models generated by the forcing notion associated to the ideal I. The theory of idealised forcing can be employed to show that if a cardinal characteristic is consistently smaller than $\operatorname{cov}(I)$ then in most cases it is implied to be so by $\mathsf{CPA}(I)$. This phrasing is, admittedly, not very precise but the theory has its subtleties.

$\mathsf{CPA}(I)$ is going to be used in Chapter 3. For more details regarding it and the theory of idealised forcing cf. [004CP], [004Za], [008Za] and [009Sa].

Determinacy

A thread of research in set theory of the reals is the analysis of *regularity properties*. The most famous ones are the property of Baire, Lebesgue measurability and the Ramsey property. In a certain way it would be nice for all sets to possess these regularity properties.

(BP) All sets of reals have the property of Baire.

(LM) All sets of reals are Lebesgue measurable.

But alas!

Theorem 1.3 (ZFC, [905Vi]). $\neg\,\mathsf{LM}$.

The Vitali set from [905Vi] also shows Theorem 1.4.

Theorem 1.4 (ZFC). $\neg\,\mathsf{BP}$.

Theorem 1.5 (AC, [983Ka]). $\omega \not\to (\omega)^{\omega}_{2}$.

Although by the 1960s the Axiom of Choice was widely accepted Jan Mycielski and Hugo Steinhaus, in [962MS], introduced a statement into set theory contradicting it. Consider an ordinal α and a set S. Any set $G \subset {}^{\alpha}S$ defines a *game* of length α. The idea is that in each step 2β such that $2\beta < \alpha$ Player 1 plays an element of S and in each step $2\beta + 1$ Player 2 plays such an element too.

Player 1 and Player 2 jointly generate a sequence $\vec{s} := \langle s_\gamma \mid \gamma < \alpha \rangle$ of elements of S. Player 1 wins the game if $\vec{s} \in G$, otherwise Player 2 wins. A *strategy* for Player i is a function f with $\operatorname{ran}(f) \subset S$ and $\operatorname{dom}(f) = \bigcup_{2\beta+i-1<\alpha} {}^{\beta}S$. The strategy is *winning* for Player 1 if all $\vec{s}$ satisfying $\forall\beta < \Omega(2\beta{+}1 < \alpha \to s_{2\beta+1} := f(\langle s_{2\gamma} \mid \gamma \leqslant \beta\rangle))$ are elements of G and it is winning for Player 2 if no $\vec{s}$ satisfying $\forall\beta < \Omega(2\beta < \alpha \to s_{2\beta} := f(\langle s_{2\gamma+1} \mid \gamma < \beta\rangle))$ is

Player 1	s_0		s_2		s_4		s_6		s_8		s_{10}		$\dots$
Player 2		s_1		s_3		s_5		s_7		s_9		s_{11}	$\dots$

Figure 1.3: A game between Player 1 and Player 2

an element of G. Note that at most one player can have a winning strategy. If exactly one player has a wining strategy we say that G is *determined*. John von Neumann proved in [928vN] the Minimax Theorem from which it can be easily deduced that G is determined if $\alpha < \omega$. In [953GS] Gale and Stewart, using the Axiom of Choice, showed how to define an undetermined game of length ω. The *Axiom of Determinacy* states the following.

(AD) Every subset of ${}^{\omega}2$ is determined.

A strengthening of AD is the *Axiom of Real Determinacy*, ADR.

(ADR) Every subset of ${}^{\omega}\mathbb{R}$ is determined.

It was known from AD's beginning that it contradicts the Axiom of Choice. Stanisław Sławomir Świerczkowski, Dana Scott and Jan Mycielski([964My2]) noted that it implies that every countable family whose union has at most the size of the continuum has a choice function. Whereas the Axiom of Choice allows one to prove the existence of nonmeasurable sets or sets not having the property of Baire AD forestalls this possibility.

Theorem 1.6 (ZF + AD, [964MŚ]). LM.

Theorem 1.7 (ZF + AD, [964My2]). BP.

It could be argued that the Axiom of Choice, Martin's Axiom and the Axiom of Determinacy should not be called axioms because they are usually on opportunistic grounds quite freely assumed or not assumed. This thesis is no exception in this respect. They could have rather be called hypotheses or postulates like the Continuum Hypothesis or the Parallel Postulate. This being said we are going to use the common terminology.

At this point we would like to remind the reader of a classical result[2] of Donald Anthony Martin from 1973. Its statement is referred to as the *strong partition property*.

Theorem 1.8 (ZF + AD, [981K]). $\omega_1 \to (\omega_1)^{\omega_1}_{\mathbb{R}}$.

[2]He did not publish it, the date stems from [003Ka].

This result is remarkable. In a choice-context it would be plain nonsense since there would be an injection $\iota\colon \omega_1 \hookrightarrow \mathbb{R}$ and one could colour any set just by the image of its least element under ι. Thus in a determinacy-context the nonexistence of such an injection is just a simple corollary of Theorem 1.8. Whereas by the Axiom of Choice there cannot be any cardinal number κ such that $\kappa \to (\kappa)^\omega_2$ and we have $\omega_1 \not\to [\omega_1]^2_{\omega_1}$ both these theorems are radically contradicted by the conclusion of Theorem 1.8. Another result worth mentioning is the following.

Theorem 1.9 ($\mathsf{ZF} + \mathsf{ADR}$, [976Pr]). $\omega \to (\omega)^\omega_2$.

The conclusion of this theorem is sometimes phrased in a different way by saying that all sets of reals have the property of Ramsey.

On the Structure of This Thesis

Originally, this thesis was intended to mainly deal with Combinatorial Cardinal Characteristics of the Continuum, to be more precise with the homogeneity numbers which are going to be defined in Chapter 2.

Chapter 3 contains results from this area of research.

During his doctoral studies and somewhat motivated by a talk by Rene Schipperus in the Bonn Logic Group's Oberseminar the author developed an interest in Ramsey Theory. The author believes in the value of problem-driven research in mathematics which is not to say that he disregards approaches with a different emphasis. The Chapters 6 and 7 grew out of this interest at a time when progress on other questions was unsatisfactory. These chapters deal with partition relations of the form $\alpha \to (\beta, n)^2$ for ordinals α, β of a certain structure and a natural number n. They continue investigations on ordinal partition relations from [974Ba], [989Ba] and [997LM].

In Chapter 5 the author proves a theorem about partitions of pairs of countable scattered linear orders he originally expected to find in the literature.

Chapter 8 contains the result of the author's interest in the Milner-Prikry-Problem. He did not spend much time on it since it was neither the supposed primary topic of his thesis nor were there hopes to believe in the existence of an easily discoverable proof.

Finally, Chapter 4 contains the results of intensively contemplating the splitting-types—which are defined in Chapter 2—while having been working in the area of partition relations. It contains one positive result about the existence of a partition relation and many counterexamples. Within this chapter the author likes most the conjectures 4.16, 4.17 and 4.18 since they connect two hitherto scarcely interacting strings of set-theoretical research, one rooted in mathematical logic and concerned with deriving strong partition properties for ordinal numbers from the Axiom of Determinacy and one in the tradition of infinite combinatorics focusing on the aforementioned problem-driven research. Also, the author considers the interplay between a preformal intuition about a structureless continuum and the almost unavoidable presence of structure in any sharply defined linearly ordered set in a formal framework being an old theme in set theory. It might be argued that the interplays both between arithmetic and geometry as well as between intuition and formalism have inspired work of mathematicians—which is now seen as central—in the past as for example in the cases of the arithmetisation of geometry by Descartes or the reduction of the continuum to set theory by Dedekind, [960De]. In the author's opinion this makes the aforementioned conjectures all the more interesting.

In the above the chapters are listed in the order their results had been proved with the exception of Chapters 2 and 3 which had developed slowly over the whole time. Yet the arrangement of the thesis follows topical considerations. The chapters dealing with Cardinal Characteristics of the Continuum precede the chapters about partition relations for linear orders which fail to be well-orders (or anti-well-orders) which in turn precede the chapters focusing on partition relations for ordinal numbers.

In general, the theorems of this thesis are proved in ZFC, that is axiomatic set theory following Zermelo and Fraenkel together with the Axiom of Choice, AC. The sole exception is Chapter 4 where several theorems are proved which would be simple corollaries of known theorems in a choice-context. The point there is that they are provable in ZF alone, without recurring to the Axiom of Choice. One theorem, numbered 4.12, is proved using BP. So in Chapter 4 just as in this introduction the theory used is mentioned after the number of the statement in question. For lack of necessity this is done in no other chapters.

Chapters 6 and 7 were accepted for publication in the paper [014We] by Combinatorica. Chapters 5 and 8 have been compiled into the paper [XXXWe] and submitted to a mathematical journal. It is planned to do the same with the contents of Chapter 4.

For details regarding the notation not mentioned in this introduction we refer the reader to the Notation section on page 119.

2

The Sequence of Homogeneity Numbers

> Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning.
>
> — Sir Winston Leonard Spencer-Churchill

We shall now introduce the sequence of cardinal characteristics around which the author's research mainly revolved in the past years.

Preliminaries

Consider the ordering of ${}^{<\omega+1}2$ given by end-extension. For $x, y \in {}^{\omega}2$ let $\mathrm{glb}(x, y) \in {}^{<\omega+1}2$ be the greatest lower bound of x and y. For $s \in {}^{<\omega}2$ let $\mathrm{lt}(s)$ be the length of the sequence s. Let $\Delta(x, y) := (\mathrm{lt} \circ \mathrm{glb})(x, y)$. Let n be a natural number, let π be a permutation of n and let $\{x_0, \ldots, x_n\}_< = X \in [{}^{\omega}2]^{n+1}$.

Definition 2.1. We say that X is of *splitting type* π if and only if

$$\langle \Delta(x_{\pi(m)}, x_{\pi(m)+1}) \mid m < n \rangle$$

is strictly descending.

We let $\mathfrak{st}(X)$ denote the splitting type of X. Let Π_n denote the set of all permutations of n. Note that the function $\tau \colon [{}^{\omega}2]^{n+1} \longrightarrow \Pi_n$ assigning its type to every unordered $(n+1)$-tuple is not total. There is no need for concern though since no generality is lost by focusing our attention on sets $X \subset {}^{\omega}2$ with the property that τ is total on $[X]^{n+1}$.

Consider a colouring $\chi \colon [{}^{\omega}2]^n \longrightarrow 2$. We call a set $X \subset {}^{\omega}2$ *weakly homogeneous* with respect to χ if and only if there is a function $f \colon \Pi_n \longrightarrow 2$ such that $\chi = f \circ \tau$, i.e. the colour, assigned by χ, of an $(n+1)$-element subset of X only depends on its type. This is depicted in the following commutative diagram.

$$\begin{tikzcd} {}^{\omega}2 \arrow[r, "\chi"] \arrow[d, "\tau"] & 2 \\ \Pi_n \arrow[ur, dotted, "f"] & \end{tikzcd}$$

Figure 2.1: Blass's Theorem as a commutative diagram

We are now able to quote an intriguing classical result of Blass.

Theorem 2.2 ([981Bl]). Let n be a natural number and let $\chi\colon [{}^{\omega}2]^n \longrightarrow 2$ be continuous. Then there exists a perfect set which is weakly homogeneous with respect to χ.

The Theorem of Mycielski and Taylor

For this thesis it is important to note that Theorem 2.2 actually holds in a much stronger version because the property of continuity requested from the colouring χ can be relaxed. That one can do this is due to a folklore variation of a theorem by Mycielski and Taylor, cf. [964My], [973GP], [978Ta] and [013D].

Theorem 2.3 (Folklore). Let $n \in \omega \setminus 2$ and let $\chi\colon [{}^{\omega}2]^n \longrightarrow 2$ be a colouring with the property of Baire. Then there exists a perfect set $P \subset {}^{\omega}2$ such that χ is continuous on P.

Proof. Let n and χ be as in the theorem. Let $b\colon \omega \longleftrightarrow {}^{<\omega}2$ be nowhere order-reversing, i.e. $\forall m < \omega \forall \ell < m\colon b(m) \not\subset b(\ell)$. Let $U^0_{\langle\rangle} := {}^{\omega}2$. Suppose that in step $m < \omega$ we have defined $U^m_{b(0)}, \ldots, U^m_{b(m)}$. Choose disjoint basic open sets

$$U^{m,0}_{b(m+1)\frown\langle 0\rangle}, U^{m,0}_{b(m+1)\frown\langle 1\rangle} \subset U^m_{b(m+1)} \setminus \bigcup_{\langle j,k,\ell\rangle \in (m+2-n)\times(m+2-n)\times\binom{m-1}{n-2}} N^{j,\ell}_k$$

and let $U^{m,0}_{b(\ell)} := U^m_{b(\ell)}$ for all $\ell \in \{k \mid k \leqslant m \wedge b(k) \not\sqsubset b(m+1) \wedge \nexists j \in m+1 \setminus (k+1)\colon b(k) \sqsubset b(j)\}$. If $m+1 < n$ then set $U^{m+1}_{b(m+1)\frown\langle i\rangle} := U^{m,0}_{b(m+1)\frown\langle i\rangle}$ for both $i < 2$. If, however, $n < m+2$ let $S_m := \{b(\ell) \frown \langle i\rangle \mid \ell \leqslant m \wedge i < 2 \wedge \nexists k \in n+1 \setminus (\ell+1)\colon b(\ell) \sqsubset b(k) \wedge \nexists i < 2\colon b(\ell) \frown \langle i\rangle = b(m+1)\}$ and let $\langle \{s^\ell_0, \ldots, s^\ell_{n-2}\}_{<_{\mathrm{lex}}} \mid \ell < \binom{m-1}{n-2}\rangle$ be an enumeration of all $[S_m]^{n-1}$. Now in substep $\ell < \binom{m-1}{n-2}$, since χ has the property of Baire, there is an $i^{m+2-n}_\ell < 2$ and basic open sets $U^{m,\ell+1}_{s^\ell_k} \subset U^{m,\ell}_{s^\ell_k}$ for $k+1 < n$ as well as $U^{m,\ell+1}_{b(m+1)\frown\langle i\rangle} \subset U^{m,\ell}_{b(m+1)\frown\langle i\rangle}$ for $i < 2$ such that $M^{m+2-n}_\ell := \chi^{-1}``\{i^{m+2-n}_\ell\} \setminus (U^{m,\ell+1}_{b(m+1)\frown\langle 0\rangle} \times U^{m,\ell+1}_{b(m+1)\frown\langle 1\rangle} \times U^{m,\ell+1}_{s^\ell_0} \times \cdots \times U^{m,\ell+1}_{s^\ell_{n-2}})$ is meagre. Let $F^{m+2-n}_\ell := \{N^{m+2-n,\ell}_k \mid k < \omega\}$ be a family of nowhere dense sets such that $\bigcup F^{m+2-n}_\ell = M^{m+2-n}_\ell$. After these finitely many substeps set $U^{m+1}_{b(m+1)\frown\langle i\rangle} := U^{m,\binom{m-1}{n-2}}_{b(m+1)\frown\langle i\rangle}$ for $i < 2$ and $U^{m+1}_s := U^{m,\binom{m-1}{n-2}}_s$ for all $s \in S_m$. After ending the whole construction, let

$$\begin{aligned} \mu\colon {}^\omega 2 &\longrightarrow {}^\omega 2 \\ \xi &\longmapsto \bigcup_{\ell<\omega} \bigcap_{m<\omega} U^m_{\xi\restriction\ell}. \end{aligned}$$

μ is a monomorphism and $P := \mathrm{ran}(\mu)$ is perfect. Note that for each $s \in {}^{<\omega}2$ there is an $\ell < \omega$ such that for all $m \in \omega \setminus \ell$ we have $U^\ell_s = U^m_s$. For any $\Xi := \{\xi_0, \ldots, \xi_{n-1}\}_{<_{\mathrm{lex}}} \in [P]^n$ there is a maximal $m < \omega$ such that $1 < \overline{\overline{\Xi \cap U^m_{b(m+1)}}}$. Let $\ell < \binom{m-1}{n-2}$ be such that $\xi_k \in U^{m,\ell+1}_{s^\ell_k}$ for all $k < n-1$. But then by construction $\chi(\Xi) = i^{m+2-n}_\ell$. Since we threw out all the exceptional nowhere dense sets one after the other, the same colour is given to any tuple in $P \cap (U^{m,\ell+1}_{b(m+1)\frown\langle 0\rangle} \times U^{m,\ell+1}_{b(m+1)\frown\langle 1\rangle} \times U^{m,\ell+1}_{s^\ell_0} \times \cdots \times U^{m,\ell+1}_{s^\ell_{n-2}})$ and thus we know that χ is continuous on P. □

Note that Theorem 2.3 allows us to generalise Theorem 2.2.

Corollary 2.4. Let $n \in \omega \setminus 2$ and let $\chi\colon [{}^\omega 2]^n \longrightarrow 2$ have the property of Baire. Then there exists a perfect set which is weakly homogeneous with respect to χ.

Note that by Theorem 1.7, under AD Theorem 2.2 would hold for all colourings χ.

Now we may define the sequence mentioned in the introduction.

Definition 2.5. Let n be a natural number and let $\chi\colon [{}^{\omega}2]^n \longrightarrow 2$ be a continuous colouring. Let

$$\mathfrak{hm}(\chi) := \min(\{\overline{\overline{F}} \mid \bigcup F = {}^{\omega}2 \wedge \forall X \in F\colon X \text{ is} \\ \text{weakly homogeneous with respect to } \chi\})$$

and let $\mathfrak{hm}_n := \sup\{\mathfrak{hm}(\chi) \mid \chi \text{ is a continuous } n\text{-colouring}\}$.

Twelve years ago, Geschke proved the following.

Theorem 2.6 ([002G]). After a countable support iteration of length ω_2 of notions of Sacks forcing over a model of $\mathsf{ZFC} + \mathsf{CH}$ we have $\mathfrak{hm}_2 = \aleph_1 < \mathfrak{c}$ holding in the final model.

Another interesting theorem regarding $\mathfrak{hm}_2$ was proved in [004G].

Theorem 2.7. There are two continuous pair-colourings, $c_{\min}$ and $c_{\max}$ such that:

1. For every Polish space X and every continuous pair-colouring $c : [X]^2 \longrightarrow 2$ with $\mathfrak{hm}(c) > \aleph_0$,
$$\mathfrak{hm}(c) = \mathfrak{hm}(c_{\min}) \text{ or } \mathfrak{hm}(c) = \mathfrak{hm}(c_{\max})$$
2. There is a model of set theory in which $\mathfrak{hm}(c_{\min}) = \aleph_1$ and $\mathfrak{hm}(c_{\max}) = \aleph_2$.

The colouring $c_{\min}$ is defined as follows:

Definition 2.8.

$$\begin{aligned} c_{\min}\colon {}^{\omega}2 &\longrightarrow 2 \\ \{x, y\}_< &\longmapsto \Delta(x, y)(2) \end{aligned}$$

This means that $\mathfrak{hm}_2 = \mathfrak{hm}(c_{\max})$. From now on we are going to abbreviate $\mathfrak{hm}(c_{\min})$ as $\mathfrak{hm}_{\min}$.

$\mathfrak{hm}_{\min}$ has a funny property. Although it is a nontrivial cardinal characteristic in the sense that it can be uncountable and smaller than the continuum the following theorem by Geschke holds true.

Theorem 2.9 ([`002G`]). $\mathfrak{hm}^+_{\min} \geqslant \mathfrak{c}$.

$\mathfrak{hm}_{\min}$ actually turns out to be quite big in yet another sense.

Theorem 2.10 ([`006Ge`]). $\operatorname{cof}(\mathcal{N}) \leqslant \mathfrak{hm}_{\min}$.

So $\mathfrak{hm}_{\min}$ is at least as large as each nontrivial characteristic in Cichoń's diagram.

3

Relations to Other Cardinal Characteristics

It is a very sad thing that nowadays there is so little useless information.

Oscar Fingel
O'Flahertie Wills Wilde

The last chapter ended with the last word on $\mathfrak{hm}_{\min}$'s position within Cichoń's diagram. So it is natural to take a look at van Douwen's diagram.

$\mathfrak{hm}_{\mathbf{min}}$ and the Reaping Number

For any $x, y \subset \omega$ we say that x *splits* y if both $y \cap x$ and $y \setminus x$ are infinite. A family $F \subset [\omega]^\omega$ is called *splittable* if there is an $s \subset \omega$ which splits all $x \in F$. A family $F \subset [\omega]^\omega$ which is not splittable is called *unsplittable*.

Definition 3.1. The *reaping number* $\mathfrak{r}$ is the minimal size of an unsplittable family.

Problem 3.2. Is $\mathfrak{r} \leqslant \mathfrak{hm}_{\min}$?

A natural approach to get a consistency result is to start with a model of Zermelo-Fraenkel Set Theory where the continuum hypothesis holds true and to add splitting reals with an iteration of length ω_2 with countable support. However, no notion of forcing which the author considered for the single step was mild enough not to also add a real which is not an element of any of the homogeneous sets coded in the ground model. A natural notion of forcing to consider is the one where conditions are splitting trees which are ordered by inclusion. These were defined by Otmar Spinas in [004Sp]. An equivalent notion of forcing was defined before by Saharon Shelah in [992Sh].

Definition 3.3. A perfect tree $T \subset {}^{<\omega}2$ is *splitting* if and only if there is an $f \in {}^\omega\omega$ such that for all $i < 2$, all $m < \omega$, all $s \in T(m)$ and all $n \in \omega \setminus f(m)$ there is a $t \in T(n)$ such that $t \geqslant_T s$ and $t(\mathrm{lt}(t)) = i$.

This notion of forcing is, in a sense, the most natural one since if a new real were a branch of a perfect tree in the ground model which is not splitting, then it could not be a splitting real. Very often, taking as conditions the smallest sets of reals possible with respect to a certain goal is the correct approach. Even if it does not work, it often shows a precise reason why this is so. In this sense, it is sometimes possible to extract a proof in ZFC. Theorem 3.9 below, proved by Zapletal, shows that this can even be made precise. To state it we first have to quote some definitions due to him as well.

Definition 3.4 ([008Za], 6.1.9, also cf. [004Za], B.2.3)**.** Let X be a Polish space. A cardinal characteristic $\mathfrak{x}$ is *very tame* if it is defined as the minimum size of a set A with properties $\psi(A)$ and $\forall x \in \mathbb{R}\, \exists y \in A\colon \vartheta(x,y)$ where $\psi(A) = \forall x_0, x_1, \dots \in A\, \exists y_0, y_1, \dots \in A\colon \tau(\vec{x}, \vec{y})$ for some arithmetic formula τ, ϑ is an analytic formula, and ZFC proves that for every countable set $a \subset \mathbb{R}$ with $\psi(a)$ there is a set $A \supset a$ such that $\overline{\overline{A}} = \mathfrak{x}$ and A is a witness for $\mathfrak{x}$.

Definition 3.5 ([008Za], 2.1.16)**.** A σ-ideal I on a Polish space X is ZFC-*correct* if it is defined by a formula ψ with a possible real parameter r (so that $I = \{A \subset X \mid \psi(A, r)\}$) and every transitive model M of a large fragment of ZFC containing r is correct about I on its analytic sets (so that if $s \in M$ is a code for an analytic set A_s then $\psi(A_s, r) \leftrightarrow M \models \psi(A_s, r)$).

Definition 3.6 ([008Za], 3.9.21)**.** A σ-ideal I on a Polish space X satisfies the *third dichotomy* if every I-positive analytic set contains a Borel I-positive subset.

Definition 3.7 ([008Za], 5.1.3)**.** A σ-ideal I on a Polish space X is *iterable* if

1. the ideal is ZFC-correct;
2. it satisfies the third dichotomy 3.6;
3. for every transitive countable model M of set theory and every condition $B \in P_I \cap M$ the set $\{x \in B \mid x \text{ is } M\text{-generic for } P_I\}$ is I-positive.

Definition 3.8 ([008Za], 2.1.21, also cf. [004Za], C.0.7)**.** A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset {}^\omega 2 \times X$ the set $\{y \in {}^\omega 2 \mid A_y \in I\}$ is coanalytic.

Theorem 3.9 ([004Za], 5.1.7, also cf. [008Za], 6.1.16)**.** Suppose that I is an iterable homogeneous $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal. Suppose that $\mathfrak{x}$ is a very tame cardinal characteristic. If $\mathfrak{x} < \operatorname{cov}(I)$ holds in some inner model of ZFC containing all ordinals or its generic extension, then $\aleph_1 = \mathfrak{x} < \operatorname{cov}(I)$ holds in every generic extension of every inner model of ZFC containing all ordinals whenever this extension satisfies $\mathsf{CPA}(I)$.

Definition 3.10 ([008Za], 3.2.1)**.** Suppose that I, J are σ-ideals on the respective underlying Polish spaces X, Y. I, J are *perpendicular* ($I \perp Y$) if there are a Borel I-positive set $B \subset X$, a Borel J-positive set $C \subset Y$ and a Borel set $D \subset B \times C$ such that the vertical sections of the set D are J-small and the horizontal sections of its complement are I-small.

Proposition 3.11 (LC, [008Za], 3.2.2)**.** Suppose that $\mathbb{P}_I$ is a proper forcing and J is generated by a universally Baire collection of Borel sets. Then $I \perp J$ if and only if some condition in the poset $\mathbb{P}_I$ forces $\dot{C} \cap V \in J$ for some J-positive Borel set C. If the ideal J is ZFC-correct then the large cardinal assumption is not necessary.

Theorem 3.12 ([994So], Theorem 1)**.** If I is an ideal on a Polish space, I is σ-generated by closed sets and $A \in \mathcal{P}(X) \setminus I$ is analytic then there is a a $B \subset A$ such that $B \notin I$ and B is G_δ.

Proposition 3.13 ([008Za], 2.1.22)**.** If a σ-ideal I on a Polish space X is provably $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then it has a ZFC-correct definition.

Theorem 3.14 ([008Za], Theorem 4.1.2)**.** Suppose that I is a σ-ideal on a Polish space X generated by closed sets. The forcing $\mathbb{P}_I$ is proper. The forcing $\mathbb{P}_I$ preserves the Baire category and has the continuous reading of names.

Lemma 3.15 ([004Za], Lemma C.0.9)**.** Suppose that P is a $\mathbf{\Sigma}^1_1$ family of compact subsets of ${}^\omega\omega$ ordered by inclusion, and as a poset it is proper with continuous reading of names. Then the ideal I generated by analytic sets without a subset in P is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

The proof of the following lemma is extracted from the proofs of Proposition 2.1.2 and Theorem 4.1.2 in [008Za].

Lemma 3.16. Suppose I is an ideal σ-generated by a family of closed sets, M is a transitive countable model of set theory and $b \in P_I \cap M$. Then $\{x \in B \mid x \text{ is } M\text{-generic for } P_I\} \notin I$.

Proof. Let $\mathcal{O}$ be a countable basis for the topology of X. Suppose towards a contradiction that $C := \{x \in B \mid x \text{ is } M\text{-generic for } P_I\} \in I$. This means that there is a collection $\{F_n \mid n < \omega\}$ of closed sets in I such that $C = \bigcup_{n<\omega} F_n$. Let $\langle D_i \mid i < \omega\rangle$ be an enumeration of the open dense sets in $\mathbb{P}_I \cap M$. We are inductively defining a sequence $\langle B_i \mid i < \omega\rangle$ such that both $B_{i+1} \in D_i \cap M$ an $B_{i+1} \cap F_i = \emptyset$ for all $i < \omega$. In order to do this suppose that in step $i < \omega$ we have defined B_i. Since we have

$$B_i \subset F_i \cup \bigcup_{O \in \mathcal{O} \wedge F_i \subset X \setminus O} B_i \cap O$$

and $\mathcal{O}$ is countable there must be an $O \in \mathcal{O}$ such that $B_i \cap O \in \mathcal{P}(X \setminus F_n) \setminus I$. Because D_{i+1} is dense and M elementary we may choose a $B_{i+1} \in D_{i+1}$ such that $B_{i+1} \subset B_i \cap O$. Now $\{B \in P_I \mid \exists i < \omega \colon B_i \subset B\}$ defines a generic filter over M. Setting $x := \bigcup \bigcap G$ we have found a generic point x outside of C, a contradiction! $\square$

Now let us consider Theorem 3.9 in the context of Problem 3.2. It is easy to write $\mathfrak{r}$ as a covering number. We have $\mathfrak{r} = \operatorname{cov}(J)$ where J is the ideal σ-generated by the family $\{\{x \in {}^\omega 2 \mid \forall n \in a \colon x(n) = i\} \mid a \in [\omega]^\omega\}$. This is also mentioned in [008Za], section 4.1.7. Note that this family consists of closed sets.

We are going to show that J is both iterable and $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Secondly we are going to demonstrate—which is not difficult—that $\mathfrak{hm}_{\min}$ is very tame. Finally, we are going to use this to argue that $\mathfrak{r} \leqslant \mathfrak{hm}_{\min}$ if J is homogeneous. This falls short of both solving Problem 3.2 as well as solving the following problem.

Problem 3.17 ([008Za], 7.1.3)**.** Prove that some of the forcings presented in [008Za] are not homogeneous.

It should be remarked that there have been signs already that J might be inhomogeneous:

Proposition 3.18 ([008Za], 4.1.29)**.** *$J \perp \mathcal{N}$.*

By Proposition 3.11 this implies that some condition in $\mathbb{P}_I$ collapses the outer Lebesgue measure. In particular, forcing with the countable support iteration of $\mathbb{P}_I$ of length ω_2 increases $\operatorname{non}(\mathcal{N})$ to $\aleph_2$. Zapletal comments on Proposition 3.18 by saying that he is unaware of whether some other condition in $\mathbb{P}_J$, i.e. splitting tree forcing, preserves outer Lebesgue measure.

We are going to need the following theorem by Spinas.

Theorem 3.19 ([004Sp], Theorem 1.2)**.** *Let $A \subset {}^\omega 2$ be analytic and $a \in [\omega]^\omega$. Then A is countably splitting on a if and only if there exists a splitting tree p for a such that $[p] \subset A$.*

Lemma 3.20. *J is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$.*

Proof. It is easy to see that ${}^\omega 2 \setminus J$ consists of the sets which are countably splitting on ω. By Theorem 3.14 $\mathbb{P}_J$ has the continuous reading of names and is proper. Now observe that by Theorem 3.19 $\mathbb{P}_J$ can be viewed as the collection of sets of branches of splitting trees. This is an analytic collection of compact sets. Now by Lemma 3.15 the ideal J' generated by an analytic sets without a subset in $\mathbb{P}_J$ is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$. By Theorem 3.19 we have $J' = J$. □

Lemma 3.21. *J is iterable.*

Proof. We need to show the three conditions defining iterability to apply to J.

1. J is provably $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$ by the proof of Lemma 3.20. Therefore it is ZFC-correct by Proposition 3.13.
2. J satisfies the third dichotomy 3.6 thanks to Theorem 3.12.
3. The third condition applies by Lemma 3.16. □

Remark 3.22.

$$\mathfrak{hm}_{\min} = \min\{\overline{\overline{A}} \mid \forall x \in {}^\omega 2 \exists y \in A\colon \vartheta(x, y)\}$$

Here $\vartheta(x, y)$ shall denote the statement "x is a branch of a $c_{\min}$-homogeneous tree coded by y". Formally this may be achieved by setting

$$\begin{aligned} \vartheta(x, y) :=& \vartheta_0(x, y) \vee \vartheta_1(x, y) \text{ where} \\ \vartheta_i(x, y) :=& \forall n < \omega\colon x(2n + i) = y\big(2^n - 1 + \sum_{k<n+i} 2^k \cdot x(2k + 1 - i)\big). \end{aligned}$$

Note that ϑ is analytic and hence $\mathfrak{hm}_{\min}$ is very tame.

We are now going to consider the splitting tree forcing $\mathbb{P}_{\text{splitting}}$ defined in and before Definition 3.3. Forcingwise $\mathbb{P}_J$ and $\mathbb{P}_{\text{splitting}}$ are isomorphic but combinatorially we are now going to argue with $\mathbb{P}_{\text{splitting}}$.

Lemma 3.23. Let $X_{c_{\min}}$ the set of trees branching only at even or only at odd levels. There is a name for a real $\dot{f}$ such that $\mathbb{1} \Vdash_{\mathbb{P}_{\text{splitting}}}$ "$\nexists T \in X_{c_{\min}} \cap M \colon \dot{f} \in [T]$", i.e. there is a (new) real which is not in any $c_{\min}$-homogeneous set of the ground model.

Proof. Let g denote the canonical generic real, i.e. $\bigcup \bigcap_{p \in G} [p]$ where G is the $\mathbb{P}_{\text{splitting}}$-generic filter. We are not going to define the name explicitly. Instead we are going to show the following which immediately yields the statement above:

There is a maximal antichain $A \subset \mathbb{P}_{\text{splitting}}$ and for every $a \in A$ a strictly increasing sequence $\langle n_i^a \mid i < \omega \rangle$ of natural numbers such that for $\dot{f}_a$ being a name for the function $f \colon \omega \longrightarrow \omega$ defined by $m \mapsto g(n_i^a)$ we have $\forall a \in A \colon a \Vdash_{\mathbb{P}_{\text{splitting}}}$ "$\nexists T \in X_{c_{\min}} \cap M \colon \dot{f}_a \in [T]$".

We are going to construct a dense set D instead of a maximal antichain. The maximal antichain A then can be easily found as a subset of D. So let p be any splitting tree and let us define $d \in D$ such that $d \leqslant p$ and $\langle n_i^d \mid i < \omega \rangle$ inductively.

d will be the fusion of an inductively defined sequence $\langle d_i \mid i < \omega\rangle$ of splitting trees.

(Induction Hypothesis) Let $d_0 := p$ and $m_0 := 0$.

(Induction Step) We are now going to describe how to obtain d_{i+1} from d_i. At first we define a sequence $\langle k^i_j \mid j < 2^{\overline{\overline{T(m_i)}}+2} + 2\rangle$ of levels and then a set of nodes S_i on level $n_{i+1} := k^i_{2^{\overline{\overline{T(n_i)}}+2}+1}$ of the tree d_{i+1}. The tree d_{i+1} will then consist of all nodes in d_i compatible with a node in S_i. The sequence $\langle k^i_j \mid j < 2^{\overline{\overline{T(n_i)}}+2}+2\rangle$ is given inductively by $k^i_0 := n_i$ and $k^i_{j+1} := \max\{K(t) \mid t \in d_i \wedge l(t) = k^i_j\}$. Let $e_i\colon N_i \longleftrightarrow T(n_i)$ be an enumeration of level n_i where N_i is the appropriate natural number. For each node t above or on level k^i_1 we say that t *is of type* $\langle i, 0, j\rangle$ if and only if $e_i(j) \leqslant_T t$ and $t(k^i_1 - 1) = 0$. Correspondingly we define t to be of type $\langle i, 1, j\rangle$ if and only if $e_i(j) \leqslant_T t$ and $t(n_i - 1) = 1$.

Now a node t in d_i on level n_{i+1} shall belong to S_i if and only if the following holds true:

If t is of type $\langle i, b, j_0\rangle$, $f(j_0, j_1) = b$ and $j_1 > 0$ then $t(k^i_{2j_1}) = 0$. Here f is given by

$$\begin{aligned} f\colon \omega \times \omega &\longrightarrow 2, \\ (j_0, j_1) &\longmapsto \text{the } j_0^{\text{th}} \text{ bit in the binary expansion of } j_1, \\ &\qquad \text{that is } \lfloor j_1 \cdot 2^{-j_0}\rfloor - 2\lfloor 2^{-1}\lfloor j_1 \cdot 2^{-j_0}\rfloor\rfloor. \end{aligned}$$

We now can set $d_{i+1} := \{t \in d_i \mid \exists s \in S_i\colon t \leqslant_{d_i} s \vee s \leqslant_{d_i} t\}$ thus concluding the induction step.

d shall now be the fusion of the d_i, i.e. $d := \bigcap_{i<\omega} d_i$. d is a splitting tree, as for example witnessed by

$$\begin{aligned} K\colon d &\longrightarrow \omega \\ t &\longmapsto k_1^{\min\{i<\omega \mid l(t) \leqslant n_i\}}. \end{aligned}$$

This is the case because the construction ensures that for every $i < \omega$, for every node $t \in T(n_i)$ and for each level $\ell \in \omega \setminus k^i_2$ there is a node $t' \geqslant t$ on level ℓ with $t'(\ell-1)=0$ but also a $t'' \geqslant t$ on level ℓ with $t''(\ell-1)=1$. This is because $t'(k_1^i-1)$ is either 0 or 1 and by construction for at least one of the two possibilities one enjoys complete freedom of choice.

Claim 3.24. Now for g defined by $m \mapsto f(k_1^m)$ we have the following:

$$d \Vdash_{\mathbb{P}_{\text{splitting}}} \text{“}\nexists T \in X_{c_{\min}} \cap M \colon \dot{g} \in [T]\text{”}. \tag{3.1}$$

*Proof of Claim 3.24.*Claim The idea here is that we “seal” the splittings between the levels n_i and k_1^i in such a way that every attempt to eliminate all possibilities to choose, for a node t on level n_i, a node t' on level k_1^i above t with prescribed last bit necessarily produces a level on which all remaining nodes in the tree have only zeros.

In order to see this suppose it would not work and let p witness this. I.e. let $p \leqslant \mathbb{P}_{\text{splitting}} d$ be a condition satisfying $p \Vdash_{\mathbb{P}_{\text{splitting}}}$ “$\exists T \in X_{c_{\min}} \cap M : \dot{g} \in [T]$”. By the maximal principle we can find a $q \leqslant p$ and a tree $t \in X_{c_{\min}}$ satisfying $q \Vdash_{\mathbb{P}_{\text{splitting}}}$ “$\dot{g} \in T$”. Let us suppose that T branches on even levels. But since q is still a splitting tree it cannot have infinitely many levels with only zeros. Therefore by construction there have to be levels with both zeros and ones cofinitely often. So choose i odd and large enough such that all levels k_j^i are of this kind. Now by construction this means that for every $j_1 < 2^{\overline{\overline{T(n_i)}}+2}+2$ there is a j_0 and a node t on level n_{i+1} of type $\langle i, 1-f(j_0,j_1), j_1\rangle$. By finite combinatorics there have to exist j_1 and j_1' for which this is witnessed by the same j_0 but different bs, i.e. there are nodes t, t' on level n_{i+1} such that

- t is of type $\langle i, 1-f(j_0,j_1), j_1\rangle$,
- t' is of type $\langle i, 1-f(j_0,j_1'), j_1'\rangle$,
- $f(j_0,j_1)+f(j_0,j_1')=1$.

By cutting back t and t' to level k_1^i we see that there are in fact two nodes s and s' in q ending in different bits lying above the same node on level n_i. Let us call this node u. The condition $q(u)$ already determines f on $k_1^{i-1}+1$, i.e. g on i. Now since $q \Vdash_{\mathbb{P}_{\text{splitting}}}$ “$\dot{g} \in T$” we know that it also determines

g *at* i, i.e. $q \Vdash_{\mathbb{P}_{\text{splitting}}}$ "$\dot{g}(i) = b$" for a certain bit b. On the other hand we can choose one of the nodes s, s' ending in $(1 - b)$, say s, and strengthen q to $q(s)$. Then we should have $q(s) \Vdash_{\mathbb{P}_{\text{splitting}}}$ "$g(i) = 1 - b$" which is a contradiction. ©

□

So Lemma 3.23 tells us a little bit more about the effect of splitting-tree-forcing on $\mathfrak{hm}_{\min}$ as Proposition 3.18 does about its effect on the outer Lebesgue measure. A consistency proof fails not only below some conditions but below all of them. Now we are finally able to state a result about $\mathfrak{r}$ and $\mathfrak{hm}_{\min}$ which, unfortunately, is only partial.

Proposition 3.25. If J is homogeneous then $\mathfrak{r} \leqslant \mathfrak{hm}_{\min}$.

Proof. Suppose towards a contradiction that J is homogeneous and that $\mathfrak{hm}_{\min} < \mathfrak{r}$ is consistent with ZFC. We work within a model M of $\mathsf{ZFC} + \mathfrak{hm}_{\min} < \mathfrak{r}$. By Lemma 3.21 J is iterable. By Lemma 3.20 J is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and by Remark 3.22 $\mathfrak{hm}_{\min}$ is very tame. Let Q be a countable support iteration of splitting-tree-forcing-notions of length ω_2. Let G be Q-generic over M. Clearly, $M[G] \models \mathsf{CPA}(J)$. By Lemma 3.23 $M[G] \models \mathfrak{c} = \mathfrak{hm}_{\min} = \omega_2$. This contradicts Theorem 3.9. □

The author considers it likely that J is not homogeneous. About the question whether $\mathfrak{hm}_{\min} < \mathfrak{r}$ is consistent he wants to hedge his bets. He had the opportunity to discuss Problem 3.2 with Saharon Shelah at the conference "Trends in Set Theory" in Warsaw.

$\mathfrak{hm}_{\min}$ and Doughnuts

Attempting to prove the consistency of $\mathfrak{hm}_{\min} < \mathfrak{r}$, a model worth checking is the one gained by iterating Silver forcing with countable supports in length ω_2. For the definition of Silver forcing , cf. [971Gr]. One would have to check that all reals added by this iteration are branches through some $c_{\min}$-homogeneous tree in the ground model. However by an easy argument of Geschke this already fails for an iteration of length 1. This argument can be employed to prove another inequality between cardinal characteristics.

Definition 3.26 ([OOODH]).

- Let $K \in [\omega]^\omega$ and let $H \subset K$ be such that $K \setminus H \in [K]^\omega$. We call the set $(H, K) \coloneqq \{X \mid X \subset K \wedge H \subset X\}$ a *doughnut*.
- We call a set $X \subset \mathcal{P}(\omega)$ *completely doughnut null* if for every doughnut Y there is a doughnut $Z \subset Y$ with $Z \cap X = \emptyset$.
- The collection of completely doughnut null sets is denoted by v^0.

Theorem 3.27. $\operatorname{cov}(v^0) \leqslant \mathfrak{hm}_{\min}$.

Proof. Consider the following function:

$$\begin{aligned} G\colon \mathcal{P}(\omega) &\longrightarrow {}^\omega 2, \\ x &\longmapsto (k \mapsto \overline{\overline{x \cap \{\min\{\ell < \omega \mid \ell + \overline{\overline{x \cap (\ell+1)}} \geqslant k\}\}}}). \end{aligned}$$

G takes the indicator function of x and successively replaces every zero by a zero and every one by two ones.

Let $\{H_\alpha \mid \alpha < \mathfrak{hm}_{\min}\}$ be a family of $\mathfrak{hm}_{\min}$ $c_{\min}$-homogeneous sets covering the Cantor space. Let $\hat{H}_\alpha \coloneqq G^{-1}``H_\alpha$ for $\alpha < \mathfrak{hm}_{\min}$. Clearly, $\{\hat{H}_\alpha \mid \alpha < \mathfrak{hm}_{\min}\}$ covers $\mathcal{P}(\omega)$ so it remains to be shown that $\hat{H}_\alpha \in v^0$ for every $\alpha < \mathfrak{hm}_{\min}$. So let $\alpha < \mathfrak{hm}_{\min}$ and let (I_0, I_1) be a doughnut. We have to find a doughnut $(K_0, K_1) \subset (I_0, I_1)$ which is disjoint from $\hat{H}_\alpha$.

So let $i < 2, y \in {}^\omega 2$ be such that H_α is coded by y, i.e. $\forall x \in H_\alpha\colon \vartheta_i(x, y)$. Let $k \coloneqq \min(I_1 \setminus I_0)$ and let $\ell \coloneqq \min(I_1 \setminus (I_0 \cup \{k\}))$. Let $j \coloneqq \ell + \overline{\overline{\ell \setminus I_i}}(2)$ and define $J_j \coloneqq I_j \triangle \{k\}$ and $J_{1-j} \coloneqq I_{1-j}$.

Note that $J_0 \cap \ell = J_1 \cap \ell$ and $\overline{\overline{J_0 \cap \ell}} + \ell$ has parity i. Now take any $z \in (J_0, J_1)$ and set

$$j \coloneqq y\Big(2^{\frac{\overline{\overline{J_0 \cap \ell}} + \ell - i}{2}} - 1 + \sum_{m < \frac{\overline{\overline{J_0 \cap \ell}} + \ell + i}{2}} 2^m G(z)(2m + 1 - i)\Big).$$

Note that j does not depend on the choice of z. Now let $K_j := J_j \triangle \{l\}$ and $K_{1-j} := J_{1-j}$. Now assume towards a contradiction that there is an $x \in (K_0, K_1) \cap \hat{H}_\alpha$.

- Suppose $\ell \in x$. Then $j = 0$ by definition of (K_0, K_1). Since $x \in \hat{H}_\alpha$ we have $G(x) \in H_\alpha$ and since y codes H_α we get $G(x)(\overline{\overline{K_0 \cap \ell}} + \ell) = 0$. But on the other hand $G(x)(\overline{\overline{K_0 \cap \ell}} + \ell) = \overline{\overline{x \cap \{\ell\}}} = 1$. Contradiction!
- Suppose $\ell \notin x$. Then $j = 1$ by definition of (K_0, K_1). Since $x \in \hat{H}_\alpha$ we have $G(x) \in H_\alpha$ and since y codes H_α we get $G(x)(\overline{\overline{K_0 \cap \ell}} + \ell) = 1$. But on the other hand $G(x)(\overline{\overline{K_0 \cap \ell}} + \ell) = \overline{\overline{x \cap \{\ell\}}} = 0$. Contradiction! □

$\mathfrak{hm}_3$ and the Reaping Number

While the question about the possible relations of $\mathfrak{r}$ to $\mathfrak{hm}_{\min}$ remains open it turns out that it is possible to give an answer about its relation to $\mathfrak{hm}_3$.

Theorem 3.28. $\mathfrak{r} \leqslant \mathfrak{hm}_3$.

Proof. Suppose towards a contradiction that $\mathfrak{r} > \mathfrak{hm}_3$. Let

$$\begin{aligned} \chi \colon [{}^\omega 2]^3 &\longrightarrow 2, \\ \{x, y, z\}_{\text{lex}} &\longmapsto \begin{cases} x(\Delta(y, z)) \text{ if and only if } \mathfrak{st}(\{x, y, z\}) = \text{id}), \\ 0 \text{ else,} \end{cases} \end{aligned}$$

and let $F = \{T_\alpha \mid \alpha < \mathfrak{hm}_3\}$ be a family of $\mathfrak{hm}_3$ weakly χ-homogeneous perfect trees such that $\bigcup_{T \in F}[T] = {}^\omega 2$. We may assume w.l.o.g. that the covering is done with perfect sets since every set weakly homogeneous with respect to a continuous colouring is contained in a weakly homogeneous perfect set. For any perfect tree $T \subset {}^{<\omega}2$ let $\varepsilon \colon {}^{<\omega}2 \longleftrightarrow T$ be the embedding preserving not only the tree-order but also the lexicographic order and mapping onto the set of splitting nodes of T. Now for any $\alpha < \mathfrak{hm}_3$, let $i_\alpha < 2$ be the colour assigned to triples of branches through T of type id, cf. Definition 2.1. For any $\alpha < \mathfrak{hm}_3$ we define a set of $\aleph_0$ infinite sets of natural numbers $\{x_{\omega\alpha+k} \mid k < \omega\}$ as follows. $x_{\omega\alpha}$ is the set of levels of splitting nodes on the rightmost branch, i.e. it is defined as $(\mathrm{lt} \circ \varepsilon)``s$ where s is the sequence of ones of order-type ω. Now for any $k < \omega$ let s_k be the sequence of k ones and let $X_{\omega\alpha+k+1}$ be the set of levels of splitting nodes in $T_\alpha(\varepsilon(s_k))$.

Consider the family $X := \{x_\alpha \mid \alpha < \mathfrak{hm}_3\}$. Since $\mathfrak{hm}_3 < \mathfrak{r}$ there is a $y \in [\omega]^\omega$ splitting every element of X. Let f_y be the characteristic function of y. By assumption there is an $\alpha < \mathfrak{hm}_3$ such that $f_y \in [T_\alpha]$. Because y splits $x_{\omega\alpha} = \{\ell_0, \ell_1, \dots\}$ there is a $k < \omega$ such that $f_y(\ell_k) = 0$. But y also splits $x_{\omega\alpha+k+1}$ so there is an $m \in x_{\omega\alpha+k+1}$ such that $f_y(m) + i_\alpha = 1$. But then $f_y \notin [T_\alpha]$, contradiction! □

Open Problems

Many more questions may be asked about the relationship between $\mathfrak{hm}_{\min}$ and other cardinal characteristics of the continuum. We shall give a quick overview here to indicate possible directions of future research.

Given that $\mathrm{cof}(\mathcal{N}) \leqslant \mathfrak{hm}_{\min}$ we know that all cardinal characteristics from Cichoń's diagram lie below all the homogeneity numbers. Since the dominating number $\mathfrak{d}$ has a place both in Cichoń's and in van Douwen's diagram, the only cardinal characteristics whose relation to the homogeneity numbers can be of any interest, left in van Douwen's diagram, are $\mathfrak{a}$, $\mathfrak{i}$, $\mathfrak{r}$ and $\mathfrak{u}$. The consistency of $\mathfrak{hm}_{\min} < \mathfrak{a}$, even if true, might be difficult to prove since even the consistency of $\mathfrak{d} < \mathfrak{a}$ had long been an open problem which was positively settled by Shelah in [004Sh] in an entirely non-trivial way. This leaves $\mathfrak{i}$ but there are reasons why one might want to consider other cardinal characteristics.

One reason only applies to possible relations of cardinal characteristics to $\mathfrak{hm}_{\min}$. It is that even the consistency of $\mathfrak{hm}_{\min} < \mathfrak{r}$ is an open problem. Hypothetically, the consistency of $\mathfrak{hm}_{\min} < \mathfrak{i}$ could be easier to prove but $\mathfrak{r}$ is a simpler cardinal characteristic. It is simpler in the following sense. Both $\mathfrak{r}$ and $\mathfrak{i}$ are defined as the minimal size of a family with a certain property. But in the case of $\mathfrak{r}$ this property is preserved under the action of taking superfamilies whereas with $\mathfrak{i}$ this is not the case. Now even for the higher homogeneity numbers one might prefer to analyse their relation to simpler cardinal characteristics first. Hence it might be fruitful to look outside the realm of the canonised cardinal characteristics. First, however, we are going to define $\mathfrak{a}$, $\mathfrak{i}$ and $\mathfrak{u}$.

Definition 3.29. We call two sets x and y of natural numbers *almost disjoint* if and only if their intersection is finite. A family F of sets of natural numbers is an *almost disjoint family* if and only if any two distinct elements of F are almost disjoint. A *maximal almost disjoint family* or *mad family* for short is an almost disjoint family F such that for all $x \in [\omega]^\omega \setminus F$ the family $F \cup \{x\}$ is not almost disjoint.

Definition 3.30. The *almost disjoint number* $\mathfrak{a}$ is the least size of an infinite mad family.

Definition 3.31. A family F of sets of natural numbers is called *independent* if and only if for any $X, Y \in [F]^{<\omega}$ the set

$$\bigcap_{z \in X \vee \omega \setminus z \in Y} z$$

is infinite. F is maximal if for all $x \in [\omega]^\omega \setminus F$ the family $F \cup \{x\}$ fails to be independent.

Definition 3.32. The *independence number* $\mathfrak{i}$ is the least size of a maximal independent family.

Definition 3.33. A *filter* F on a set X is a family $F \subset \mathcal{P}(X)$ which is closed under intersections and supersets. This means that for any $a, b \in F$ we have $a \cap b \in F$ and for any $a \in F$ and $b \subset X$ with $b \supset a$ we have $b \in F$.

Definition 3.34. An *ultrafilter* U on a set X is a filter on X which is maximal. This means that for any filter F on X with $F \supset U$ we have $F = U$. An ultrafilter is *principal* if there is an element $e \in X$ such that for all $Y \subset X$ we have $Y \in U$ if and only if $e \in Y$, otherwise it is called *nonprincipal*.

Definition 3.35. A *base* for a filter F on a set X is a set B such that $F = \{a \mid \exists b \in B \colon b \subset a\}$.

Definition 3.36. The *ultrafilter number* $\mathfrak{u}$ is the minimal size of a base of a nonprincipal ultrafilter on ω.

Question 3.37. Is $\mathfrak{hm}_{\min} < \mathfrak{a}$ consistent with ZFC?

Question 3.38. Is $\mathfrak{hm}_{\min} < \mathfrak{i}$ consistent with ZFC?

Question 3.39. Is $\mathfrak{hm}_{\min} < \mathfrak{u}$ consistent with ZFC?

In [004B], Balcar, Hernández-Hernández and Hrušák considered cardinal characteristics derived from the algebra $\langle \mathrm{Dense}(\mathbb{Q}), \subset \rangle$ instead of $\mathcal{P}(\omega)/\mathrm{fin}$. They defined analogues to $\mathfrak{t}$, $\mathfrak{h}$, $\mathfrak{s}$, $\mathfrak{r}$, and $\mathfrak{i}$. Since they proved $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}$, $\mathfrak{h}_{\mathbb{Q}} \leqslant \mathrm{add}(\mathcal{M})$, $\mathfrak{s}_{\mathbb{Q}} \leqslant \mathfrak{d}$ and $\mathfrak{i}_{\mathbb{Q}} = \mathfrak{i}$ the only cardinal characteristic from this list which is worth looking at in addition is $\mathfrak{r}_{\mathbb{Q}}$.

For any dense $x, y \subset \mathbb{Q}$ we say that x *dense-splits* y if both $y \cap x$ and $y \setminus x$ are dense. A family $F \subset [\mathbb{Q}]^{\mathrm{dense}}$ is called *dense-splittable* if there is an $s \subset \mathbb{Q}$ which dense-splits all $x \in F$. A family $F \subset [\mathbb{Q}]^{\mathrm{dense}}$ which is not dense-splittable is called *dense-unsplittable*.

Definition 3.40. The *dense reaping number* $\mathfrak{r}_{\mathbb{Q}}$ is the minimal cardinality of a dense-unsplittable family.

Question 3.41. Is $\mathfrak{hm}_{\min} < \mathfrak{r}_{\mathbb{Q}}$ consistent with ZFC?

It is an interesting observation that the proof of Theorem 3.28 does not straightforwardly generalise from $\mathfrak{r}$ to $\mathfrak{r}_{\mathbb{Q}}$. So it seems tempting to ask yet another question.

Question 3.42. Is $\mathfrak{hm}_3 < \mathfrak{r}_{\mathbb{Q}}$ consistent with ZFC?

In [000C] Cichoń, Krawczyk, Majcher-Iwanow and Węglorz defined dual versions of the cardinal characteristics in van Douwen's diagram by considering partitions of ω instead of subsets. We are employing the notation introduced by Halbeisen in [998Ha] and referring to them using capital gothic letters. In [000C] it is proved that $\mathfrak{A} = \mathfrak{c}$ and it had already been proved in [986Ma] that $\mathfrak{T} = \aleph_1$. It is shown in [000C] that $\mathfrak{H} \leqslant \mathfrak{S}$ and that $\mathfrak{R} \leqslant \mathfrak{r}$. Keeping $\mathfrak{hm}_{\min}$ in mind this leaves $\mathfrak{S}$ as an interesting open case. In [998Ha] a variant $\mathfrak{S}'$ is defined for which it is obvious that $\mathfrak{S} \leqslant \mathfrak{S}'$.

Definition 3.43. A *partition of* ω is a family F of pairwise disjoint nonempty sets of natural numbers such that $\bigcup F = \omega$. If X and Y are partitions of ω then X is *finer* than Y or, equivalently, Y is *coarser* than X, if and only if for all $b \in Y$ there is a $Z \subset X$ such that $b = \bigcup Z$. We also say that X is a *refinement* of Y or that Y is a *coarsening* of X. We say that partitions X and Y are *compatible* if and only if there is a joint coarsening Z which is infinite. They are *orthogonal* if the only joint coarsening is $\{\omega\}$. A partition X *splits* a partition Y if and only if X and Y are compatible but there is a coarsening Z of Y such that X and Z are orthogonal. A partition X is *almost coarser* than a partition Y if there is a finite set f of natural numbers such that all joint coarsenings of X and f are coarser than Y. If $\{\{n\} \mid n < \omega\}$ is almost coarser than X we call X *trivial*. A family of partitions $\mathcal{S}$ is called *splitting* if and only if for every infinite non-trivial partition X there is an $S \in \mathcal{S}$ such that S splits X.

After this preparatory definition we can finally define the dual splitting number.

Definition 3.44. The *dual splitting number* $\mathfrak{S}'$ is the least size of a splitting family of infinite partitions having an infinite element.

Question 3.45. Is $\mathfrak{hm}_{\min} < \mathfrak{S}'$ consistent with ZFC?

For more details regarding $\mathfrak{a}$, $\mathfrak{i}$ and $\mathfrak{u}$ we refer the reader to [010Bl], for details regarding the dense-subset-variants to [004B] and for details regarding $\mathfrak{S}'$ to [998Ha].

Finally, there are questions regarding solely the homogeneity numbers.

Problem 3.46. Is "$\forall n < \omega\colon \mathfrak{hm}_n < \mathfrak{c}$" consistent with ZFC?

Problem 3.47. Determine $x \subset \omega \setminus 2$ such that $\mathfrak{hm}_n < \mathfrak{hm}_{n+1}$ is consistent with ZFC if and only if $n \in x$.

By a result of Frick we know that $2 \in x$, cf. [008Fr].

The final problem one should mention is one which both the author and his advisor considered to be among the hardest on the topic.

Question 3.48. Is $\aleph_1 < \mathfrak{hm}_{\min} < \mathfrak{c}$ consistent with ZFC?

The only class of models in which $\mathfrak{hm}_2 < \mathfrak{c}$ is presently known to hold is the class of models attained by an iteration of Sacks or Miller-lite forcing notions with countable support of length a cardinal of cofinality at least ω_2. For a definition of Miller-lite forcing cf. [006Ge]. Although this leaves a number of parameters open to variation this does not help much with the questions concerning us because the supports are going to turn any cardinal less than the iteration's length into an ordinal less than ω_2 in the generic extension. Even a product of forcing notions does not work. So it seems that for answering Question 3.48 in the affirmative one would have to iterate notions of forcing but neither using finite nor countable supports. In particular, one could not rely on the well-developed machinery of proper forcing as for example laid out in [998Sh].

Summarising this chapter we provide a diagram encompassing Cichoń's diagram, van Douwen's diagram, the cardinal characteristics $\mathfrak{r}_{\mathbb{Q}}$, $\mathfrak{S}'$, Geschke's result that $\mathrm{cof}(\mathcal{N}) \leqslant \mathfrak{hm}_{\min}$ and ours that $\mathfrak{r} \leqslant \mathfrak{hm}_3$.

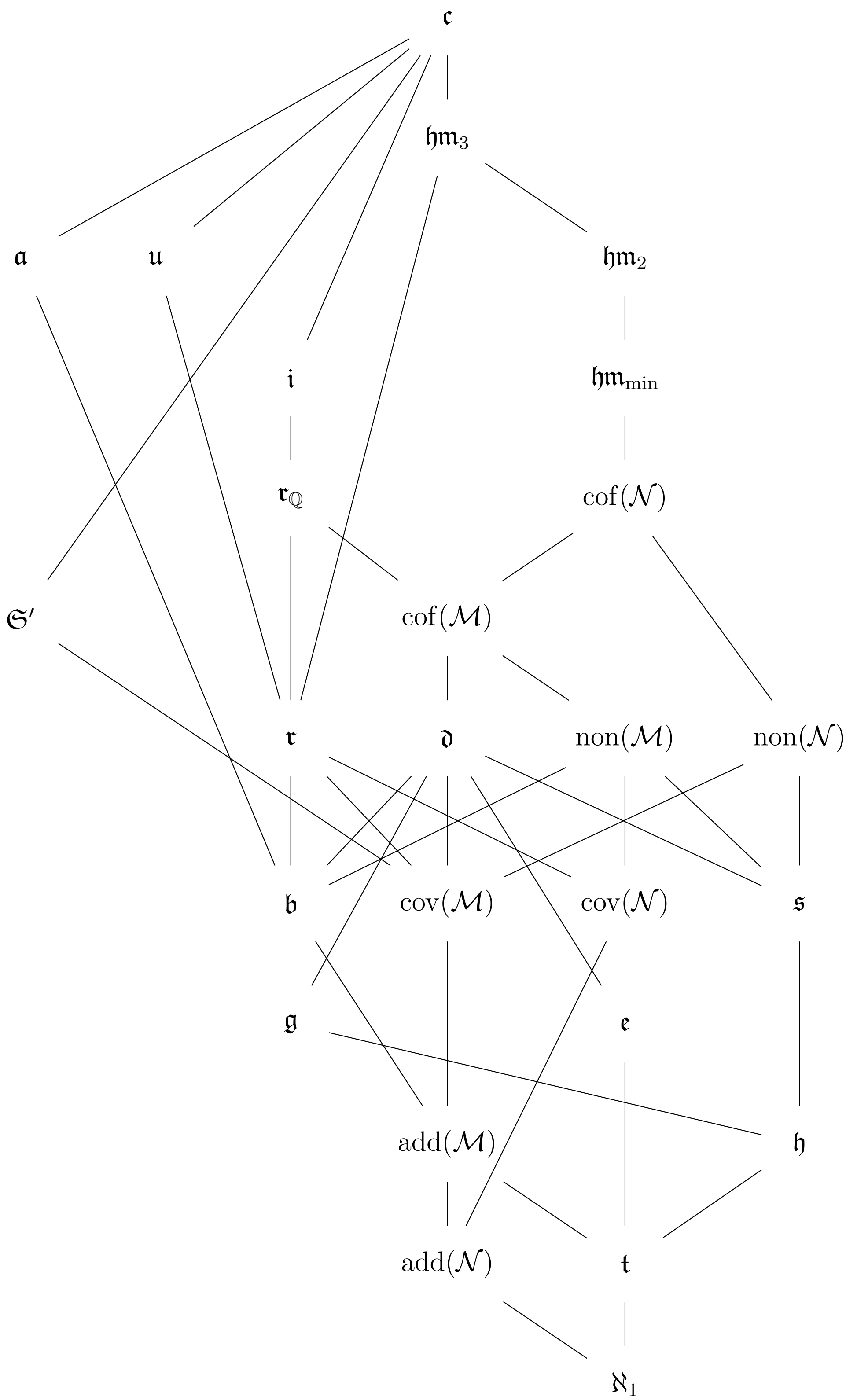

Figure 3.1: An encyclopaedic diagram

4

Partition Relations for Linear Orders in a Non-Choice Context

> In mathematics the art of asking questions is more valuable than solving problems.
>
> Georg Ferdinand Ludwig Philipp Cantor

Studying the splitting types of n-tuples of reals necessary to state Theorem 2.2 one may realise that many arguments about partition relations, especially for general counterexamples in the presence of the Axiom of Choice, can be slightly rephrased to yield quite similar counterexamples in a different context. To be more precise, one can construct, without AC, analogue results—in which the parameters which are natural numbers are increased by one or two—for the linear ordering $\langle {}^{\alpha}2, <_{\text{lex}} \rangle$ where α is an ordinal.

From the Point of View of the Axiom of Choice

There is a fact which is only very scarcely stated because it is very easy to prove. The fact of the matter is the following.

Fact 4.1 (ZFC)**.** There is no order-type φ such that

$$\varphi \to (\omega^*, \omega)^2.$$

When one considers a well-ordering of φ and puts a pair into the first class if the natural order and the well-order agree on it, and into the second if they disagree, a homogeneous set would have to have a descending chain in the well-ordering, which is absurd. One can use similar arguments to show the following theorems.

Theorem 4.2 (ZFC)**.** There is no order-type φ such that

$$\varphi \to (\omega^* + \omega, 4)^3.$$

Theorem 4.3 (ZFC)**.** There is no order-type φ such that

$$\varphi \to (\omega + \omega^*, 4)^3.$$

These theorems were first proved by Arthur H. Kruse in [965Kr] and later reproved by Erdős, Milner and Rado in [971E]. In the latter paper the following theorem was proved as well.

Theorem 4.4 (ZFC)**.** There is no order-type φ such that

$$\varphi \to (\omega + \omega^* \vee \omega^* + \omega, 5)^3.$$

In this paper the authors also mentioned that they were unable to decide the existence of an order-type φ such that $\varphi \to (\omega + \omega^* \vee \omega^* + \omega, 4)^3$. The problem of the existence of such a type remains unsolved to this day.

Plain Vanilla ZF

From this point on any use of the Axiom of Choice is rescinded. The goal of this section is to prove propositions and theorems analogous to Fact 4.1 and Theorems 4.2, 4.3 and 4.4. We consider partitions of subsets of linear orders embeddable into some $\langle {}^{\alpha}2, <_{\text{lex}} \rangle$ into two parts. We are primarily interested in the question for which choices of order-types for sought-after homogeneous sets and which sizes of the subsets partitioned we can have a positive partition relation at all. That this could be an interesting endeavour was somewhat suggested by noticing the following proposition to hold true.

Proposition 4.5 (ZF)**.** There is no ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega^*, \omega)^3.$$

For the proof of the proposition above and for many of the proofs following we are referring to the length of the splitting node of the branches x and y, i.e. the largest node which is an initial segment of both x and y, by $\Delta(x, y)$. Formally,

$$\Delta(b, c) := \min\{\gamma < \alpha \mid b(\gamma) \neq c(\gamma)\} \text{ for } b, c \in {}^{\alpha}2.$$

Proof. Assume towards a contradiction that there was such an α and let $\mathcal{L} := \langle {}^{\alpha}2, <_{\text{lex}} \rangle$. Consider the following colouring:

$$\begin{aligned} \chi\colon [{}^{\alpha}2]^3 &\longrightarrow 2, \\ \{b, c, d\}_{<_{\text{lex}}} &\longmapsto \begin{cases} 0 \text{ if and only if } \Delta(b, c) < \Delta(c, d), \\ 1 \text{ else.} \end{cases} \end{aligned}$$

Now assume there is a set $X \in [\mathcal{L}]^{\omega^*}$ homogeneous for χ in colour 0. Let $\langle x_i \mid i < \omega \rangle$ a strictly descending enumeration of X. Since X is homogeneous in colour 0 we have $\Delta(x_{i+2}, x_{i+1}) < \Delta(x_{i+1}, x_i)$ for all $i < \omega$. But $\operatorname{ran}(\Delta) = \alpha$ and α is well-ordered. Contradiction!

So there is an $X \in [\mathcal{L}]^{\omega}$ homogeneous for χ in colour 1. Note that no $\{x, y, z\}_{<_{\text{lex}}} \in [\mathcal{L}]^3$ can satisfy $\Delta(x, y) = \Delta(y, z)$. So if $\langle x_i \mid i < \omega \rangle$ is a strictly ascending enumeration of X then $\langle \Delta(x_i, x_{i+1}) \mid i < \omega \rangle$ is a strictly descending sequence of ordinals—contradiction! $\square$

Proposition 4.6 (ZF + DC + BP)**.** $\langle {}^{\omega}2, <_{\text{lex}}\rangle \to (\langle {}^{\omega}2, <_{\text{lex}}\rangle)^2_2$.

Proof. Suppose BP holds true and $\chi\colon [{}^{\omega}2]^2 \longrightarrow 2$ is any colouring. By Lemma 2.3 we may assume w.l.o.g. that it is continuous. Then by Theorem 2.2 we find a homogeneous perfect set, that is, a set order-isomorphic to ${}^{\omega}2$. □

Proposition 4.7 (ZF + DC + BP)**.** $\langle {}^{\omega}2, <_{\text{lex}}\rangle \to (\langle {}^{\omega}2, <_{\text{lex}}\rangle, 1 + \omega^* \vee \omega + 1)^3_2$.

Proof. Suppose BP holds true and $\chi\colon [{}^{\omega}2]^3 \longrightarrow 2$ is any colouring. By Lemma 2.3 we may assume w.l.o.g. that it is continuous. Then by Theorem 2.2 we may suppose w.l.o.g. that the colour of a triple only depends on its splitting-type. Now consider the sets

$$X_i := \{x \in {}^{\omega}2 \mid \overline{\overline{\{n < \omega \mid x(n) = i\}}} < 2\}.$$

X_0 has order type $\omega + 1$ and X_1 has order-type $1 + \omega^*$. For both i the sets X_i only contain triples of one splitting type. Now if triples of both splitting types get colour 0 then we have a set of order type $\langle {}^{\omega}2, <_{\text{lex}}\rangle$ which is homogeneous in colour 0. If one of the splitting types gets colour 1 then we have a set homogeneous in colour 1, either of order-type $\omega + 1$ or of order-type $1 + \omega^*$. □

Theorem 4.8 (ZF)**.** There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}}\rangle \to (\langle {}^{\omega+1}2, <_{\text{lex}}\rangle, \aleph_0)^3.$$

Proof. Let $b\colon \alpha \leftrightarrow \omega$ be a bijection. Let $\beta(x, y)$ be an abbreviation for $b(\Delta(x, y))$ for $x, y \in {}^{\alpha}2$. Consider the following hypergraph:

$$E := \{\{x_0, x_1, x_2\}_{<_{\text{lex}}} \mid \Delta(x_0, x_1) < \Delta(x_1, x_2) \wedge \beta(x_1, x_2) < \beta(x_0, x_1)\}.$$

Suppose that there was an $X \in [{}^{\alpha}2]^{\aleph_0}$ such that $[E]^3 \subset E$. If X contains order-type ω^* let $\langle x_n \mid n < \omega\rangle$ be an order-reversing enumeration of a part of X. Then $\langle \Delta(x_{n+1}, x_n) \mid n < \omega\rangle$ is an infinite descending sequence of ordinals.

If X contains order-type ω let $\langle x_n \mid n < \omega\rangle$ be an order-preserving enumeration of a part of X. Then $\langle \beta(x_n, x_{n+1}) \mid n < \omega\rangle$ is an infinite descending enumeration of natural numbers.

Now suppose that there was an $X \in [{}^{\alpha}2]^{{}^{\omega+1}2}$ such that $[X]^3 \subset [{}^{\alpha}2]^3 \setminus E$. Let $\iota\colon {}^{\omega+1}2 \longleftrightarrow X$ be an order-preserving injection and let

$$f_\delta\colon \omega+1 \longleftrightarrow 2$$
$$\gamma \longmapsto \begin{cases} 0 \text{ if and only if } \gamma = \delta, \\ 1 \text{ else.} \end{cases}$$

for $\delta < \omega + 2$. Let $x_\delta := \iota(f_\delta)$ for $\delta < \omega + 1$. We have $\Delta(x_n, x_{n+1}) < \Delta(x_\omega, x_{\omega+1})$ for every $n < \omega$. Pick $n < \omega$ such that $\beta(x_n, x_\omega) > \beta(x_\omega, x_{\omega+1})$. Then $\{x_n, x_\omega, x_{\omega+1}\} \in E$. $\square$

Now we are able to prove choiceless analogues to the theorems in [971E].

Theorem 4.9 (ZF)**.** There is no ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega^* + \omega, 5)^4.$$

Proof. Assume towards a contradiction that there were such an ordinal α. Consider the hypergraph $G := \langle {}^{\kappa}2, E_0 \rangle$ with hyper-edge-relation

$$E_0 := \{\{x_0, x_1, x_2, x_3\}_{<_{\text{lex}}} \mid \Delta(x_1, x_2) < \Delta(x_0, x_1), \Delta(x_2, x_3)\}.$$

First assume that there would be a $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$ independent from G, i.e. $[Z]^4 \cap E_0 = \emptyset$. Let $s \in {}^{<\alpha}2$ be the longest common initial segment of the elements of Z, i.e. $\forall z \in Z, \gamma < \operatorname{lt}(s)\colon z(\gamma) = s(\gamma)$. Then for $Z_0 := \{z \in Z \mid z(\operatorname{lt}(s)) = 0\}$ and $Z_1 := \{z \in Z \mid z(\operatorname{lt}(s)) = 1\}$ we have $\operatorname{otyp}(Z_0) = \omega^*$ and $\operatorname{otyp}(Z_1) = \omega$. Now for any $\{x_0, y_0\}_{<_{\text{lex}}} \in [Z_0]^2$ and any $\{x_1, y_1\}_{<_{\text{lex}}} \in [Z_1]^2$ we obtain $\Delta(y_0, x_1) < \Delta(x_0, y_0), \Delta(x_1, y_1)$. Therefore we have found $\{x_0, y_0, x_1, y_1\}_{<_{\text{lex}}} \in [Z]^4 \cap E_0$. Contradiction!

Now assume that there would be a $Q \in [{}^{\alpha}2]^5$ being a complete sub-hypergraph of G, i.e. $[Q]^4 \subset E_0$. Suppose $Q = \{q_0, q_1, q_2, q_3, q_4\}_{<\text{lex}}$. Choose $i \in 3 \setminus 1$ such that $\Delta(q_i, q_{i+1})$ is minimal. Suppose $i = 1$. Then $\Delta(q_2, q_3) > \Delta(q_1, q_2)$ and hence $Q \setminus \{q_0\} \notin E_0$. Now suppose $i = 2$. Then $\Delta(q_1, q_2) > \Delta(q_2, q_3)$ and hence $Q \setminus \{q_4\} \notin E_0$. Contradiction! $\square$

Theorem 4.10 (ZF)**.** There is no ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega + \omega^*, 5)^4.$$

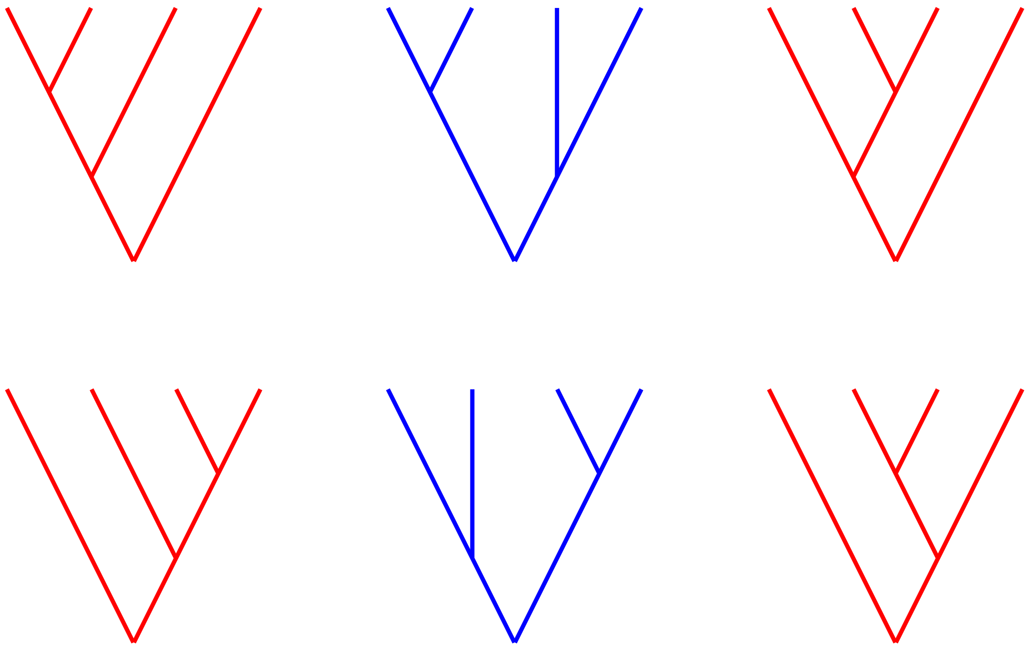

Figure 4.1: Colouring of the splitting types for the proof of Theorem 4.9

Proof. Assume that there were such an α. Consider the hypergraph $G :=$ $\langle {}^{\alpha}2, E_1\rangle$ with hyper-edge-relation

$$E_1 := \{\{x_0, x_1, x_2, x_3\}_{<_{\text{lex}}} \mid \Delta(x_0, x_1), \Delta(x_2, x_3) < \Delta(x_1, x_2)\}.$$

Suppose $Y \in [{}^{\alpha}2]^{\omega+\omega^*}$. Let $s \in {}^{<\alpha}2$ be the longest common initial segment of the elements of Y. Then for $Y_0 := \{y \in Y \mid y(\text{lt}(y)) = 0\}$ and $Y_1 := \{y \in Y \mid y(\text{lt}(y)) = 1\}$ we have $\omega \leqslant \text{otyp}(Y_0)$ or $\omega^* \leqslant \text{otyp}(Y_1)$. W.l.o.g. suppose that $\omega \leqslant \text{otyp}(Y_0)$. So let $\langle y_n \mid n < \omega\rangle$ be an ascending sequence of elements of Y_0 and let x_3 be some element in Y_1. Since κ is well-founded there is an $n < \omega$ such that $\Delta(y_n, y_{n+1}) < \Delta(y_{n+1}, y_{n+2})$. Set $x_i := y_{n+i}$ for $i < 3$. Then $\{x_i \mid i < 4\}_{<_{\text{lex}}} \in [Y]^4 \cap E_1$.

Now let $Q \in [{}^{\alpha}2]^5$. Suppose $Q = \{q_0, q_1, q_2, q_3, q_4\}_{<_{\text{lex}}}$. Choose $i < 2$ such that $\Delta(q_{i+1}, q_{i+2})$ is maximal. Then $\Delta(q_{2-i}, q_{3-i}) < \Delta(q_{i+1}, q_{i+2})$ and hence $Q \setminus \{q_{4i}\} \notin E_1$. $\square$

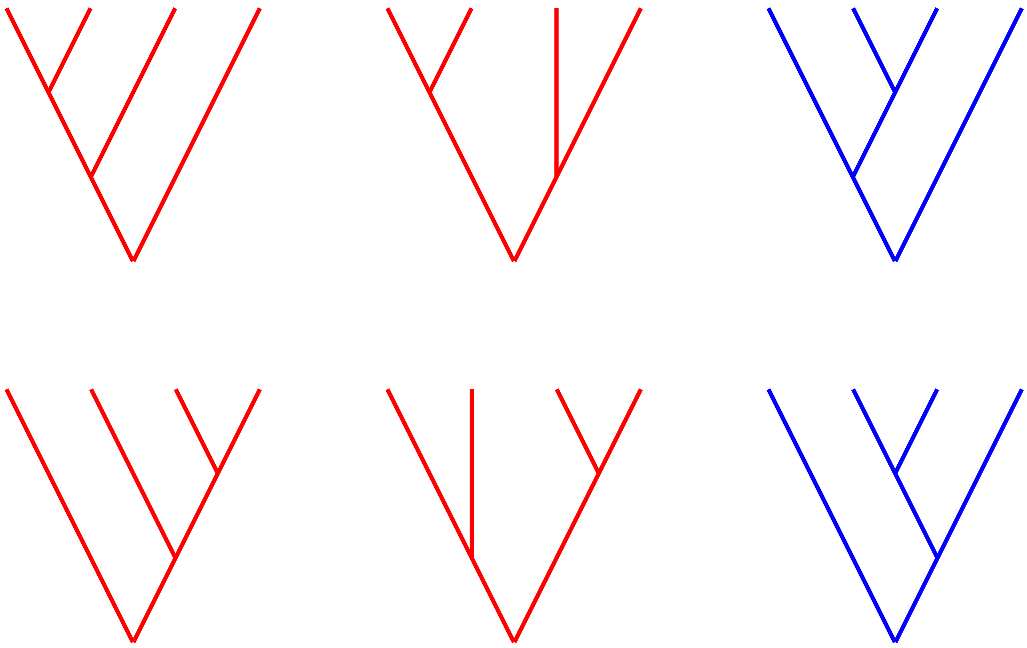

Figure 4.2: Colouring of the splitting types for the proof of Theorem 4.10

Theorem 4.11 (ZF)**.** There is no ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega^* + \omega \vee \omega + \omega^*, 7)^4.$$

Proof. Consider the hypergraph $G := \langle {}^{\alpha}2, E_2 \rangle$ with hyper-edge-relation

$$E_2 := \bigcup_{i<2} E_i.$$

It has already been proved above that $[Z]^4 \not\subseteq [{}^{\alpha}2]^4 \setminus E_0$ for every $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$ and that $[Y]^4 \not\subseteq [{}^{\alpha}2]^4 \setminus E_1$ for every $Y \in [{}^{\alpha}2]^{\omega+\omega^*}$. Thus it only remains to be shown that $[S]^4 \not\subseteq E_2$ for every $S \in [{}^{\alpha}2]^7$. So let $S = \{s_0, \ldots, s_6\}$. Let $n < 6$ be such that $\Delta(s_n, s_{n+1})$ is minimal. Suppose w.l.o.g. that $n < 3$. If $\Delta(s_5, s_6) < \Delta(s_4, s_5) < \Delta(s_3, s_4)$ then $\{s_3, \ldots, s_6\} \in [{}^{\alpha}2]^4 \setminus E_2$. So suppose that there is an $m \in 6 \setminus 3$ such that $\Delta(s_m, s_{m+1}) < \Delta(s_{m+1}, s_{m+2})$. But then $\{s_n, s_m, s_{m+1}, s_{m+2}\} \in [{}^{\alpha}2]^4 \setminus E_2$. $\square$

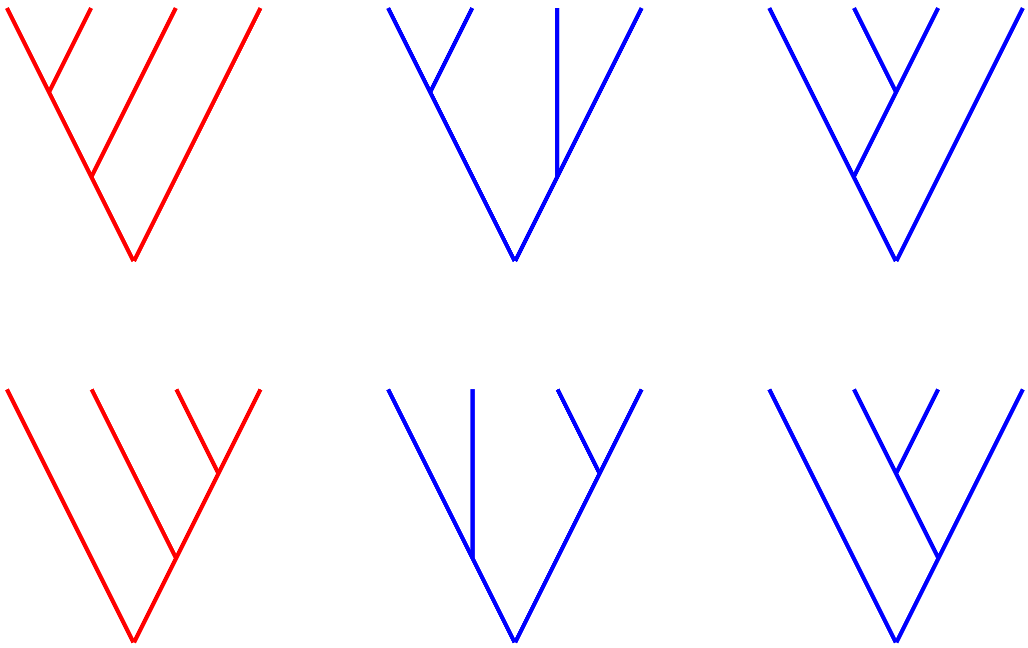

Figure 4.3: Colouring of the splitting types for the proof of Theorem 4.11

This statement is in fact less artificial than it might seem at first sight since $\{\omega^* + \omega, \omega + \omega^*\}$ is a basis for the class of linear orders φ such that neither φ nor φ^* is an ordinal.

Theorem 4.12 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{BP}$)**.**

$$\langle {}^{\omega}2, <_{\text{lex}} \rangle \to (\omega + 1 + \omega^* \vee 1 + \omega^* + \omega + 1, 5)^4.$$

Proof. Suppose that there is an $E \subset [{}^{\omega}2]^4$ such that no $Y \in [{}^{\omega}2]^{\omega+\omega^*}$ satisfies $[Y]^4 \subset [{}^{\omega}2]^4 \setminus E$ and no $Z \in [{}^{\omega}2]^{\omega^*+\omega}$ satisfies $[Z]^4 \subset [{}^{\omega}2]^4 \setminus E$. We will show that there is a $P \in [{}^{\omega}2]^5$ such that $[P]^4 \subset E$.

Recall that by Theorem 2.2 in connection with Lemma 2.3 and Theorem 1.7 we can suppose w.l.o.g. that whether or not a quadruple belongs to E only depends on the splitting-type. So consider quadruples $Q := \{q_0, q_1, q_2, q_3\}_{<_{\text{lex}}}$. If quadruples of type $\Delta(q_0, q_1) < \Delta(q_1, q_2) < \Delta(q_2, q_3)$ are in E then for quintuples $P := \{p_0, p_1, p_2, p_3, p_4\}_{<_{\text{lex}}}$ with $\Delta(p_0, p_1) < \Delta(p_1, p_2) < \Delta(p_2, p_3) < \Delta(p_3, p_4)$ we have $[P]^4 \subset E$. This argument works analogously for quadruples with $\Delta(q_2, q_3) < \Delta(q_1, q_2) < \Delta(q_0, q_1)$.

So suppose w.l.o.g. that if a quadruple Q has one of these types then $Q \notin E$. Consider the following sets.

$$Y := \{y \in {}^{\omega}2 \mid \forall m < \omega (y(m) \equiv m(2) \to \forall n \in \omega \setminus m \colon y(n) \equiv m(2))\},$$
$$Z := \{y \in {}^{\omega}2 \mid \overline{\overline{\{n \in \omega \setminus 1 \mid y(n) \neq y(0)\}}} \leqslant 1\}.$$

Y has order-type $\omega + 1 + \omega^*$ while Z has order-type $1 + \omega^* + \omega + 1$. Consider the following types.

1. $\Delta(q_1, q_2) < \Delta(q_2, q_3) < \Delta(q_0, q_1)$,
2. $\Delta(q_1, q_2) < \Delta(q_0, q_1) < \Delta(q_2, q_3)$,
3. $\Delta(q_0, q_1) < \Delta(q_2, q_3) < \Delta(q_1, q_2)$,
4. $\Delta(q_2, q_3) < \Delta(q_0, q_1) < \Delta(q_1, q_2)$.

Y neither contains the first nor the second type. Z neither contains the third nor the fourth type. So our last w.l.o.g.-assumption together with our non-homogeneity-assumption for sets of order-type $\omega + 1 + \omega^*$ or $\omega^* + \omega$ implies that for one the first two and for one of last two of the following types we have $Q \in E$. Depending on the combination for one of the following four splitting-types of quintuples $P = \{p_0, p_1, p_2, p_3, p_4\}$,

$$\Delta(p_1, p_2) < \Delta(p_3, p_4) < \Delta(p_2, p_3) < \Delta(p_0, p_1),$$
$$\Delta(p_2, p_3) < \Delta(p_0, p_1) < \Delta(p_1, p_2) < \Delta(p_3, p_4),$$
$$\Delta(p_2, p_3) < \Delta(p_3, p_4) < \Delta(p_0, p_1) < \Delta(p_1, p_2),$$
$$\Delta(p_1, p_2) < \Delta(p_0, p_1) < \Delta(p_3, p_4) < \Delta(p_2, p_3),$$

we have $[P]^5 \subset E$. □

Now that we have given several negative partition relations for colourings of quadruples excluding homogeneous quintuples in Theorems 4.9, 4.10, 4.14 and 4.15 and excluding homogeneous septuples in Theorem 4.11 the obvious question is "What about sextuples?". In order to give a partial answer to this we point out that for ordinals $\alpha \in \Omega \setminus \omega 2$ there are two new possible splitting pattern for sets in $[{}^{\alpha}2]^{\omega + \omega^*}$. Considering the proof of Theorem 4.12 these have to be taken into consideration additionally.

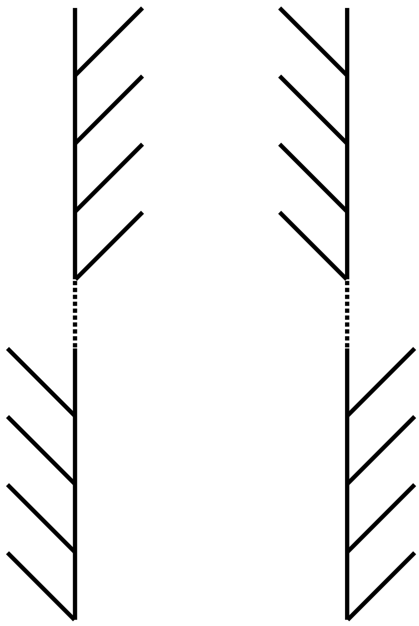

Figure 4.4: Two possible splitting patterns of sets of order-type $\omega + \omega^*$ in ${}^{\alpha}2$

Theorem 4.13 (ZF)**.** There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega^* + \omega \vee \omega + \omega^*, 6)^4.$$

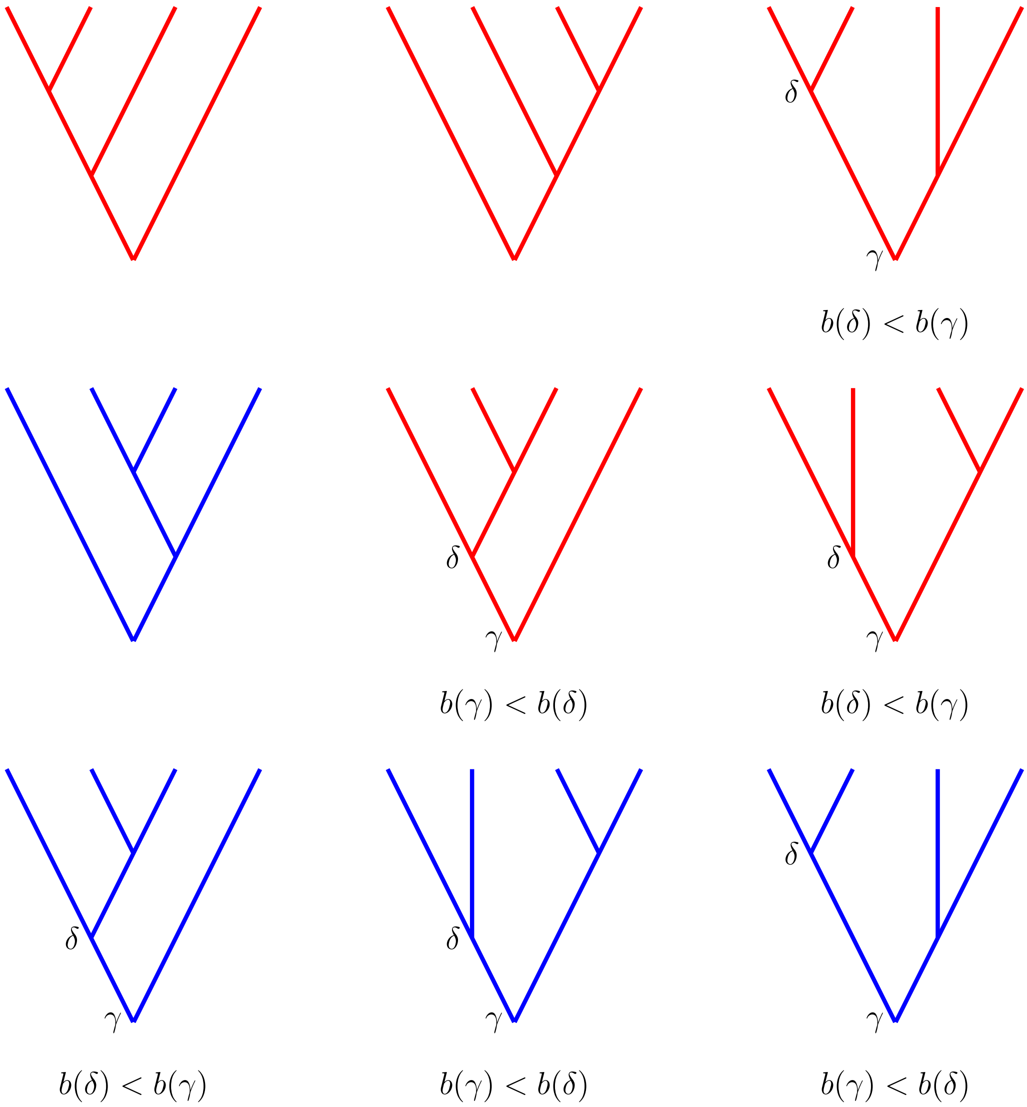

Figure 4.5: Colouring of the splitting types for the proof of Theorem 4.13

Proof. Let $\alpha < \omega_1$. Fix a bijection $b\colon \alpha \longleftrightarrow \omega$. Let $\beta(x,y)$ be an abbreviation for $b(\Delta(x,y))$ for $x,y \in {}^{\alpha}2$. Consider the hypergraph $G := \langle {}^{\alpha}2, E_3\rangle$ with hyper-edge-relation

$$\begin{aligned}
E_3 :=&(E_4 \cap E_7) \cup (E_5 \cap E_8) \cup E_6 \text{ where}\\
E_4 :=&\{\{x_0,x_1,x_2,x_3\}_{<_{\text{lex}}} \mid \Delta(x_1,x_2) < \Delta(x_0,x_1), \Delta(x_2,x_3)\},\\
E_5 :=&\{\{x_0,x_1,x_2,x_3\}_{<_{\text{lex}}} \mid \Delta(x_2,x_3) < \Delta(x_0,x_1) < \Delta(x_1,x_2)\},\\
E_6 :=&\{\{x_0,x_1,x_2,x_3\}_{<_{\text{lex}}} \mid \Delta(x_0,x_1) < \Delta(x_2,x_3) < \Delta(x_1,x_2)\},\\
E_7 :=&\{\{x_0,x_1,x_2,x_3\}_{<_{\text{lex}}} \mid \beta(x_1,x_2) < \beta(x_0,x_1)\},\\
E_8 :=&\{\{x_0,x_1,x_2,x_3\}_{<_{\text{lex}}} \mid \beta(x_0,x_1) < \beta(x_2,x_3)\}.
\end{aligned}$$

Note that the E_i's with $i \in 7 \setminus 4$ are pairwise disjoint.

$\diamondsuit$ Let $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$. Suppose towards a contradiction that $[Z]^4 \subset [{}^{\alpha}2]^4 \setminus E_3$. Let $e\colon \mathbb{Z} \longleftrightarrow Z$ be the order-preserving enumeration of Z. Let $z \in \mathbb{Z}$ be such that $\Delta(e(z), e(z+1)) < \Delta(e(y), e(y+1))$ for every $y \in \mathbb{Z} \setminus \{z\}$. Then $\{e(y) \mid y \in \mathbb{Z} \wedge y > z\}$ has order-type ω and $\{e(y) \mid y \in \mathbb{Z} \wedge y \leqslant z\}$ has order-type ω^*. Let $\langle x_n | n < \omega\rangle$ be an order-preserving enumeration of $\{x \in Z \mid y > e(z)\}$ and let $\langle y_n \mid n < \omega\rangle$ be an order-reversing enumeration of $\{y \in Z \mid y \leqslant e(z)\}$. Note that $\Delta(y_m, x_n) = \Delta(y_0, x_0)$ for all $m, n < \omega$. Since $\beta(y_0, x_0)$ is natural there exists an $n < \omega$ such that $\beta(y_0, x_0) < \beta(y_{n+1}, y_n)$. But then $\{y_{n+1}, y_n, x_0, x_1\} \in E_4 \cap E_7 \subset E_3$, a contradiction!

$\diamondsuit$ Let $H \in [{}^{\alpha}2]^6$. We write $H = \{h_0, \dots, h_5\}_{<_{\text{lex}}}$. Suppose towards a contradiction that $[H]^4 \subset E_3$. First note that for no quadruple $\{q_0, q_1, q_2, q_3\}_{<_{\text{lex}}} = Q \in [H]^4$ we have $\Delta(q_0,q_1) < \Delta(q_1,q_2) < \Delta(q_2,q_3)$ since we would have $Q \notin \bigcup_{i \in 7\setminus 4} E_i \supset E_3$ then. For the same reason we have $\Delta(q_2,q_3) < \Delta(q_1,q_2) < \Delta(q_0,q_1)$ for no such quadruple. But then we know that

$$\begin{aligned}
\Delta(h_2,h_3) &< \Delta(h_0,h_1),\\
\Delta(h_0,h_1) &< \Delta(h_1,h_2),\\
\Delta(h_2,h_3) &< \Delta(h_4,h_5),\\
\Delta(h_4,h_5) &< \Delta(h_3,h_4),\\
\forall m < 3, n \in 6 \setminus 3\colon \Delta(h_m,h_n) &= \Delta(h_2,h_3).
\end{aligned}$$

Consider the quadruple $Q := \{h_0, h_1, h_4, h_5\}$. We have $Q \in E_4$. From $Q \in [H]^4$ we obtain $Q \in E_3$. Together with the aforementioned disjointness, $Q \in E_7$ follows. So $\beta(h_1, h_4) < \beta(h_0, h_1)$. Now consider the quadruple $T := \{h_0, h_1, h_2, h_5\}$. It is easy to see that $T \in E_5$. Now $T \in [H]^4$ implies $T \in E_3$. By disjointness, $T \in E_8$. So $\beta(h_0, h_1) < \beta(h_2, h_5)$. But then

$$\beta(h_0, h_1) < \beta(h_2, h_5) = \beta(h_1, h_4) < \beta(h_0, h_1),$$

a contradiction!

$\diamondsuit$ This is the most difficult case. Let $Y \in [{}^{\alpha}2]^{\omega+\omega^*}$. Suppose towards a contradiction that $[Y]^4 \subset [{}^{\alpha}2]^4 \setminus E_3$. Note that because of $E_6 \subset E_3$ no quadruple $\{q_0, q_1, q_2, q_3\}_{<_{\text{lex}}} = Q \in [Y]^4$ can satisfy $\Delta(q_0, q_1) < \Delta(q_2, q_3) < \Delta(q_1, q_2)$. We distinguish some more cases.

- There is a node $s \in {}^{<\alpha}2$ which is a splitting node of two elements in Y such that $X := \{x \in Y \mid s \frown \langle 1 \rangle \sqsubset x\}$ has order-type $\gamma + \omega^*$ for some $\gamma < \omega + 1$. Let $\langle x_n | n < \omega \rangle$ be the order-reversing enumeration of the right part of X. Then let $n < \omega$ be such that $\Delta(x_{n+1}, x_n)$ is minimal. Let $y \in Y$ be such that $s \frown \langle 0 \rangle \sqsubset y$. Then $\{y, x_{n+2}, x_{n+1}, x_n\} \in E_6 \subset E_3$, a contradiction.
- There is no such node. Let s_0 be the first splitting node of elements of Y. For every $n < \omega$ let s_{n+1} be the first splitting node of elements of Y extending $s_n \frown \langle 0 \rangle$. If $\sup_{n<\omega} \operatorname{lt}(s_n) = \alpha$ then Y would only have had order-type ω^* so we may assume that $s := \bigcup_{n<\omega} s_n$ is a node in ${}^{<\alpha}2$. We also know by the assumption defining this subcase that $X := \{y \in Y \mid s \sqsubset y\}$ has order-type ω. Let $\langle x_n \mid n < \omega \rangle$ be an order-preserving enumeration of X. Let $n < \omega$ be such that $\operatorname{glb}(x_n, x_{n+1})$ is the lowest splitting-node above s. Note that for all $y \not\sqsupset s$ we have $\{x_n, x_{n+1}, x_{n+2}, y\} \in E_5$. Also note that because of the assumption defining this subcase we can choose a $y \in Y$ with $y \not\sqsupset s$ such that $\beta(x_{n+2}, y) > \beta(x_n, x_{n+1})$. But then $\{x_n, x_{n+1}, x_{n+2}, y\} \in E_5 \cap E_8 \subset E_3$, again a contradiction. $\square$

Theorem 4.14 (ZF)**.** There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega + 2 + \omega^* \vee \omega^* + \omega, 5)^4.$$

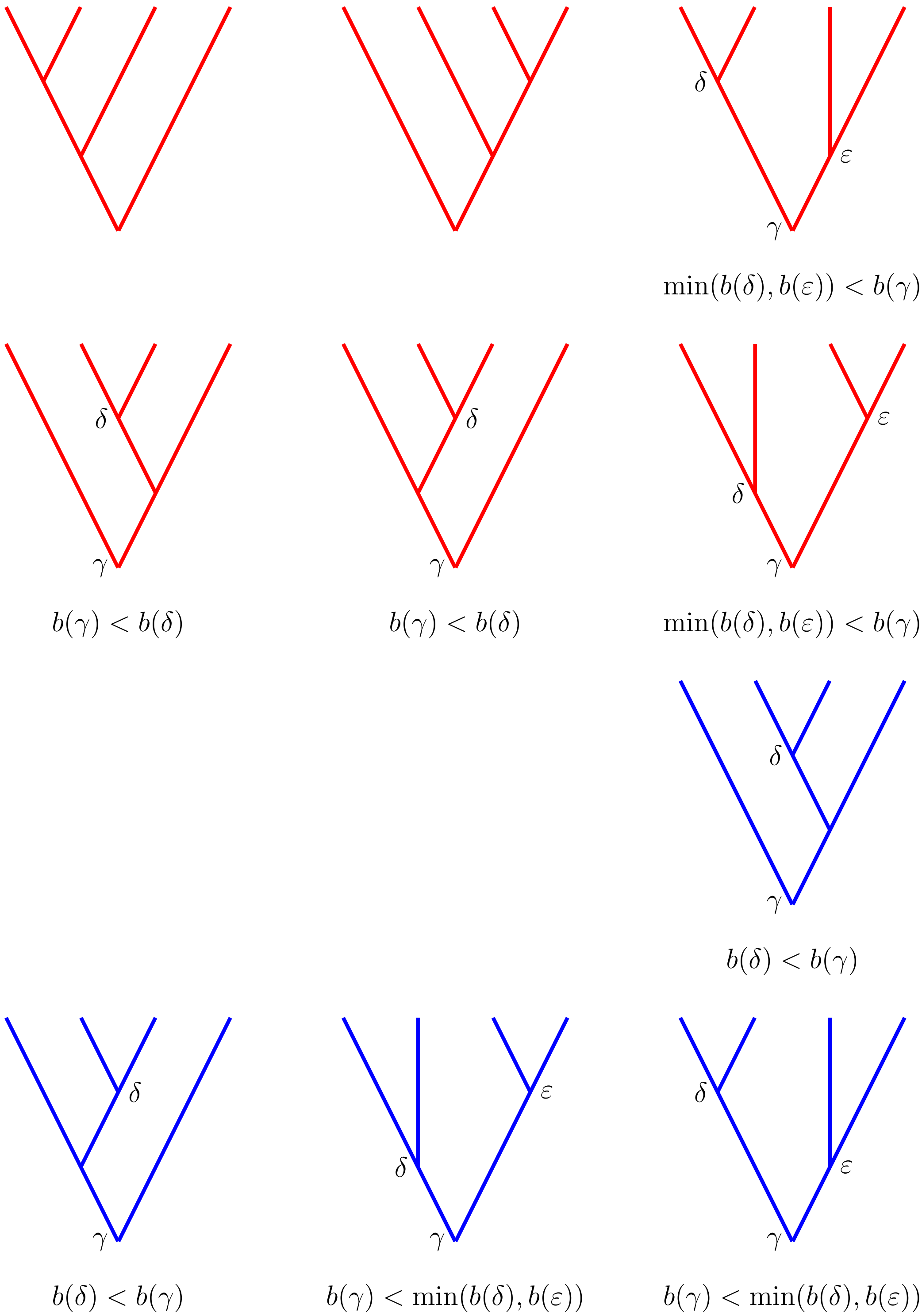

Figure 4.6: Colouring of the splitting types for the proof of Theorem 4.14

The proof is similar to the proof of Theorem 4.13.

Proof. Suppose there were such an α. We can construct a counterexample similarly to the one for Theorem 4.13. For this, consider the hypergraph $G := \langle {}^{\alpha}2, E_9\rangle$ with

$$\begin{aligned} E_9 :=&(E_4 \cap E_7 \cap E_{10}) \cup (E_5 \cap E_{11}) \cup (E_6 \cap E_{11}) \text{ where} \\ E_{10} :=&\{\{x_0, x_1, x_2, x_3\}_{<_{\text{lex}}} \mid \beta(x_1, x_2) < \beta(x_2, x_3)\}, \\ E_{11} :=&\{\{x_0, x_1, x_2, x_3\}_{<_{\text{lex}}} \mid \beta(x_1, x_2) < \beta(x_0, x_3)\}. \end{aligned}$$

$\diamondsuit$ Let $Z \in [{}^{\alpha}2]^{\omega^*+\omega}$. Suppose towards a contradiction that $[Z]^4 \subset [{}^{\alpha}2]^4 \setminus E_9$. Let $e\colon \mathbb{Z} \longrightarrow Z$ be the order-preserving enumeration of Z. Let $z \in \mathbb{Z}$ be such that $\Delta(e(z), e(z+1)) < \Delta(e(y), e(y+1))$ for any $y \in \mathbb{Z} \setminus \{z\}$. Let $\langle x_n \mid n < \omega\rangle$ be an order-preserving enumeration of $\{y \in Z \mid y > e(z)\}$ and let $\langle y_n \mid n < \omega\rangle$ be an order-reversing enumeration of $\{y \in Z \mid <\leqslant e(z)\}$. We have $\Delta(y_m, x_n) = \Delta(y_0, x_0)$ for all $m, n < \omega$. Since $\beta(y_0, x_0)$ is natural there exists an $m < \omega$ such that $\beta(y_0, x_0) < \beta(y_{m+1}, y_m)$. Similarly, there is an $n < \omega$ such that $\beta(y_0, x_0) < \beta(x_n, x_{n+1})$. Then $\{y_{n+1}, y_m, x_n, x_{n+1}) \in E_4 \cap E_7 \cap E_{10}$, a contradiction!

$\diamondsuit$ Now suppose that $P \in [{}^{\alpha}2]^5$. We write $P = \{p_0, \ldots, p_4\}_{<_{\text{lex}}}$. Suppose towards a contradiction that $[P]^4 \subset E_9$. For no quadruple $\{q_0, \ldots, q_3\}_{<_{\text{lex}}} = Q \in [P]^4$ we have $\Delta(q_0, q_1) < \Delta(q_1, q_2) < \Delta(q_2, q_3)$ or $\Delta(q_2, q_3) < \Delta(q_1, q_2) < \Delta(q_0, q_1)$ since this would imply $Q \notin \bigcup_{i \in 7\setminus 4} E_i \supset E_9$. Then one of the following two cases applies.

- One possibility is that $\Delta(p_2, p_3) < \Delta(p_0, p_1) < \Delta(p_1, p_2)$ and $\Delta(p_2, p_3) < \Delta(p_3, p_4)$ and $\Delta(p_m, p_n) = \Delta(p_2, p_3)$ for all $m < 3$ and $n \in 5 \setminus 3$. Now consider the quadruple $Q := \{p_1, p_2, p_3, p_4\}$. We have $Q \in E_4$. From $Q \in [P]^4$ we get $Q \in E_9$. By disjointness we have $Q \in E_7$. So $\beta(p_2, p_3) < \beta(p_1, p_2)$. Now consider the quadruple $T := \{p_0, p_1, p_2, p_3\}$. We have $T \in E_5$. Now $T \in [P]^4$ implies $T \in E_9$. By disjointness, $T \in E_{11}$. So $\beta(p_1, p_2) < \beta(p_0, p_3)$, but then

$$\beta(p_2, p_3) < \beta(p_1, p_2) < \beta(p_0, p_3) = \beta(p_2, p_3),$$

  a contradiction!

- The other possible constellation is to have $\Delta(p_1, p_2) < \Delta(p_0, p_1)$, $\Delta(p_1, p_2) < \Delta(p_3, p_4) < \Delta(p_2, p_3)$ and $\Delta(p_m, p_n) = \Delta(p_1, p_2)$ for all $m < 2$ and $n \in 5 \setminus 2$. Consider the quadruple $Q := \{p_0, p_1, p_2, p_3\}$. We have $Q \in E_4$. From $Q \in [P]^4$ we get $Q \in E_9$. By disjointness, $Q \in E_{10}$ so $\beta(p_1, p_2) < \beta(p_2, p_3)$. Now consider the quadruple $T := \{p_1, p_2, p_3, p_4\}$. We have $T \in E_6$. $T \in E_9$ since $T \in [P]^4$. Now $T \in E_{11}$ by disjointness, so $\beta(p_2, p_3) < \beta(p_1, p_4)$. But then

$$\beta(p_1, p_2) < \beta(p_2, p_3) < \beta(p_1, p_4) = \beta(p_1, p_2),$$

a contradiction!

$\diamondsuit$ Let $Y \in [{}^{\alpha}2]^{\omega+2+\omega^*}$. Suppose towards a contradiction that $[Y]^4 \subset [{}^{\alpha}2]^4 \setminus E_9$. We distinguish two subcases.

- There is a node $s \in {}^{<\alpha}2$ which is a splitting node of two elements in Y such that for $X_i := \{x \in Y \mid s \frown \langle i \rangle\}$ for $i < 2$, we have that $\operatorname{otyp}(X_0) \geqslant \omega$ and $\operatorname{otyp}(X_1) \geqslant \omega^*$. In this case let $\langle x_n \mid n < \omega \rangle$ be an order-preserving enumeration of the left part of X_0 and let $\langle y_n \mid n < \omega \rangle$ be an order-reversing enumeration of the right part of X_1. Clearly for any $m, n < \omega$ we have $\Delta(x_m, y_n) = \Delta(x_0, y_0)$. Since $\beta(x_0, y_0)$ is natural and $m \mapsto \beta(x_m, x_{m+1})$ and $n \mapsto \beta(y_{n+1}, y_n)$ are injections we find $m, n < \omega$ such that $\beta(x_m, x_{m+1}), \beta(y_{n+1}, y_n) \in \omega \setminus (\beta(x_0, y_0) + 1)$ which implies $\{x_m, x_{m+1}, y_{m+1}, y_m\} \in E_4 \cap E_7 \cap E_{10} \subset E_9$, a contradiction.
- There is no such node. Let $y_1, y_2 \in Y$ be such that $\operatorname{otyp}(\{y \in Y \mid y < y_1\}) = \omega$, $\operatorname{otyp}(\{y \in Y \mid y_2 < y\}) = \omega^*$ and $y_1 < y_2$. Let s_0 be the lowest splitting node of elements of Y. For any $n < \omega$ let $i_n < 2$ be the uniquely determined $i < 2$ such that $\{y \in Y \mid y \sqsupset s_n \frown \langle i \rangle\}$ is infinite. Now let s_{n+1} be the first splitting node of elements of Y extending $s_n \frown \langle i_n \rangle$ and let $x_n \in Y$ be such that $x_n \sqsupset s_n \frown \langle 1 - i_n \rangle$. There is an $n < \omega$ such that $\beta(y_1, y_2) < \beta(y_1, x_n)$. Choose $y_0 \in Y \setminus \{y_1, y_2, x_n\}$ such that $y_0 \sqsupset s_n \frown \langle i_n \rangle$ and $\{y_0, y_1, y_2, x_n\} \in E_5 \cup E_6$. This is possible because only finitely many elements of Y were branching off below s_n by our case assumption. But $\beta(y_1, x_n) = \beta(y_0, x_n)$ and hence $\beta(y_1, y_2) < \beta(y_0, x_n)$. So $\{y_0, y_1, y_2, x_n\} \in E_{11}$. But then $\{y_0, y_1, y_2, x_n\} \in E_9$, a contradiction. $\square$

Theorem 4.15 (ZF)**.** There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{\text{lex}} \rangle \to (\omega + \omega^* \vee 2 + \omega^* + \omega, 5)^4.$$

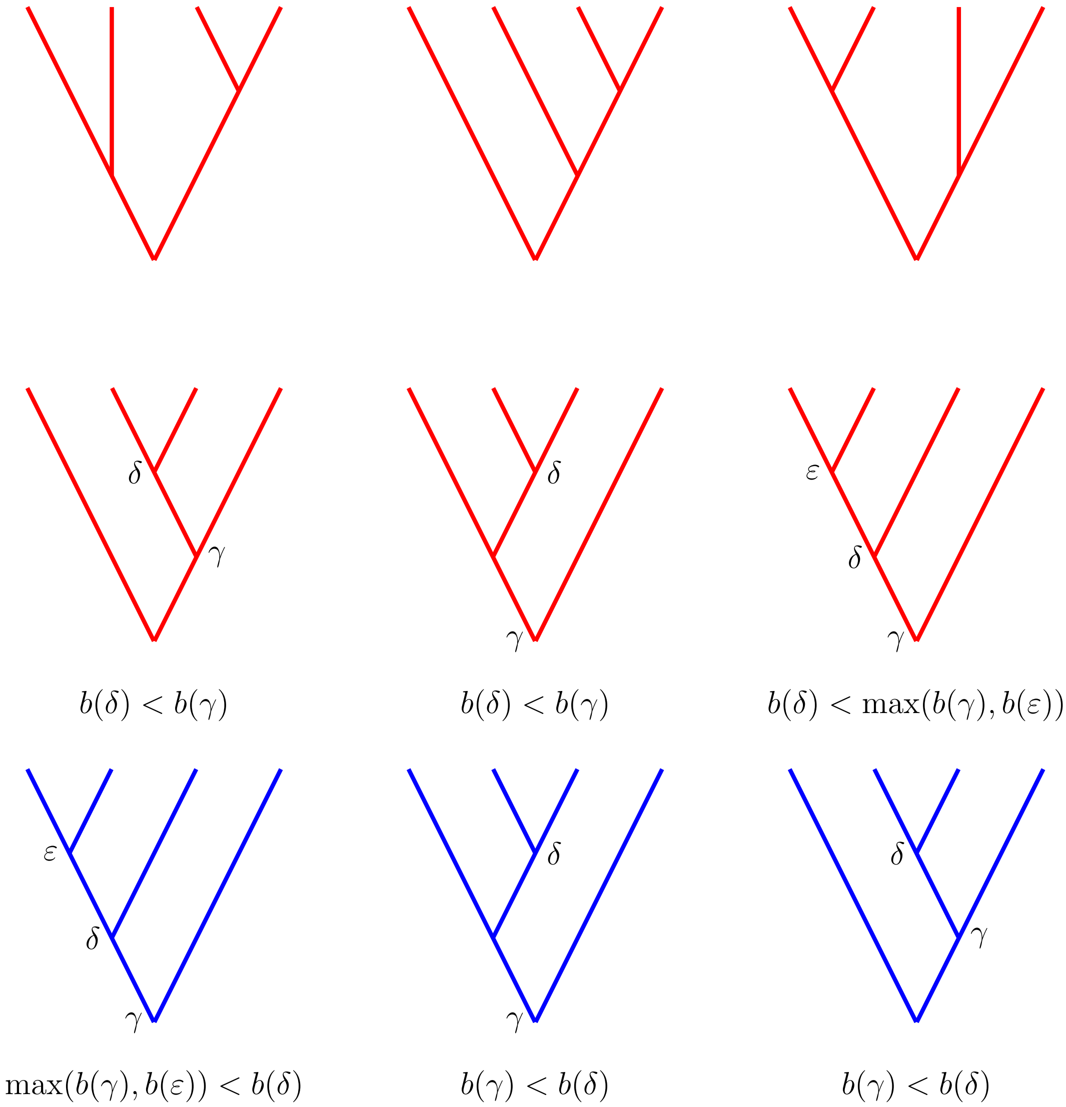

Figure 4.7: Colouring of the splitting types for the proof of Theorem 4.15

Proof. Suppose there were such an α. Again we are using β to construct a counterexample. So consider the hypergraph $G := \langle {}^{\alpha}2, E_{12} \rangle$ with

$$\begin{aligned} E_{12} :=& (E_{13} \setminus (E_7 \cup E_{10})) \cup (E_5 \setminus E_{10}) \cup (E_6 \setminus E_{10}) \text{ where} \\ E_{13} :=& \{\{x_0, x_1, x_2, x_3\}_{<_{\text{lex}}} \mid \Delta(x_2, x_3) < \Delta(x_1, x_2) < \Delta(x_0, x_1)\}. \end{aligned}$$

$\diamondsuit$ Let $Y \in [{}^{\alpha}2]^{\omega+\omega^*}$ and suppose towards a contradiction that $[Y]^4 \subset [{}^{\alpha}2]^4 \setminus E_{12}$. As in the proof of Theorem 4.13 we distinguish two subcases.

- If there is a node $s \in {}^{<\alpha}2$ which is a splitting node of two elements in Y such that $X := \{y \in Y \mid s \frown \langle 1 \rangle \sqsubset y\}$ has order-type $\gamma + \omega^*$ for some $\gamma < \omega + 1$, let $\langle x_n \mid n < \omega \rangle$ be the order-reversing enumeration of the right part of X. Let $m < \omega$ be such that $\Delta(x_{m+1}, x_m)$ is minimal and $y \in Y$ such that $s \frown \langle 0 \rangle \sqsubset y$. Surely now for any $n \in \omega \setminus (m+1)$ we have $Q := \{y, x_{n+1}, x_n, x_m\} \in E_6$. Choose $n \in \omega \setminus (m+1)$ so that $\beta(x_{n+1}, x_n) > \beta(x_n, x_m)$. Then $Q \notin E_{10}$ and hence $Q \in E_{12}$, a contradiction.
- If there is no such node we let s_0 be the first splitting node of elements of Y and fix some $y \in Y$ with $y \sqsupset s_0 \frown \langle 1 \rangle$. For any $n < \omega$ we let s_{n+1} be the first splitting node of elements of Y extending $s_n \frown \langle 0 \rangle$. If $\sup_{n<\omega} \operatorname{lt}(s_n) = \alpha$ then Y would only have had order-type ω^* so we may assume that $s := \bigcup_{n<\omega} s_n$ is a node in ${}^{<\alpha}2$. Of course $X := \{y \in Y \mid s \sqsubset y\}$ has order-type ω. Let $\langle x_n \mid n < \omega \rangle$ be an order-preserving enumeration of X and let $m < \omega$ be such that $\Delta(x_m, x_{m+1})$ is minimal. Clearly for all $n \in \omega \setminus (m+1)$ we have $\{x_m, x_n, x_{n+1}, y\} \in E_5$. Then let $n \in \omega \setminus (m+1)$ be such that $\beta(x_n, x_{n+1}) > \beta(x_m, y)$. But then $\{x_m, x_n, x_{n+1}, y\} \notin E_{10}$. So $\{x_m, x_n, x_{n+1}, y\} \in E_{12}$.

$\diamondsuit$ Now let $P \in [{}^{\alpha}2]^5$ and suppose towards a contradiction that $[P]^4 \subset E_{12}$. Then the definition of E_{12} implies that for no quadruple $\{q_0, q_1, q_2, q_3\}_{<_{\text{lex}}} = Q \in [P]^4$ we have $\Delta(x_0, x_1) < \Delta(x_1, x_2) < \Delta(x_2, x_3)$ or $Q \in E_4$. This means that one of the following four cases applies.

- Suppose first that $\Delta(p_3, p_4) < \Delta(p_2, p_3) < \Delta(p_1, p_2) < \Delta(p_0, p_1)$. Let $Q_i := \{p_0, \ldots, p_{i+3}\}$ for $i < 2$. $Q_0 \notin E_{10}$ since $Q_0 \in E_{12}$. This means that $\beta(p_1, p_2) \geqslant \beta(p_2, p_3)$. On the other hand we have $Q_1 \notin E_7$ since $Q_1 \in E_{12}$.

This gives us $\beta(p_2, p_3) \geqslant \beta(p_1, p_2)$. So we have $\beta(p_1, p_2) = \beta(p_2, p_3)$. But clearly, $\Delta(p_1, p_2) \neq \Delta(p_2, p_3)$, contradicting b being one-to-one.

- Now suppose that $\Delta(p_3, p_4) < \Delta(p_2, p_3) < \Delta(p_0, p_1) < \Delta(p_1, p_2)$. Consider the quadruple $Q_1 := \{p_1, p_2, p_3, p_4\}$. We have $Q_1 \in E_{13}$ and since $E_{13} \subset [{}^{\alpha}2]^4 \setminus (E_5 \cup E_6)$ and $Q_1 \in E_{12}$ we have $Q_1 \notin E_7$. This means that $\beta(p_2, p_3) \geqslant \beta(p_1, p_2)$. Now consider the quadruple $Q_0 := \{p_0, p_1, p_2, p_3\}$. Clearly, $Q_0 \in E_5$. Because of $Q_0 \in E_{12}$ we also have $Q_0 \notin E_{10}$ so $\beta(p_1, p_2) \geqslant \beta(p_2, p_3)$. So together we have $\beta(p_1, p_2) = \beta(p_2, p_3)$ while $\Delta(p_1, p_2) \neq \Delta(p_2, p_3)$, again contradicting b being one-to-one.
- Another variant is the situation $\Delta(p_0, p_1) < \Delta(p_3, p_4) < \Delta(p_2, p_3) < \Delta(p_1, p_2)$. Consider $Q_1 := \{p_1, p_2, p_3, p_4\}$. It is easy to see that $Q_1 \in E_{13}$. Since $Q_1 \in E_{12}$ we have $Q_1 \notin E_7$ and hence $\beta(p_2, p_3) \geqslant \beta(p_1, p_2)$. Also consider $Q_0 := \{p_0, p_1, p_2, p_3\}$. $Q_0 \in E_6$ and from $Q_0 \in E_{12}$ we obtain $Q_0 \notin E_{10}$, so $\beta(p_1, p_2) \geqslant \beta(p_2, p_3)$. Together this yields $\beta(p_1, p_2) = \beta(p_2, p_3)$. Since b is one-to-one this implies $\Delta(p_1, p_2) = \Delta(p_2, p_3)$ which is a contradiction.
- Finally, consider the case where $\Delta(p_3, p_4) < \Delta(p_0, p_1) < \Delta(p_2, p_3) < \Delta(p_1, p_2)$. Look at $Q_1 := \{p_1, p_2, p_3, p_4\}$. Clearly, $Q_1 \in E_{13}$. Because of $Q_1 \in E_{12}$ we have $Q_1 \notin E_7$ so, on the one hand, $\beta(p_2, p_3) \geqslant \beta(p_1, p_2)$. On the other hand we can consider $Q_0 := \{p_0, p_1, p_2, p_3\}$. $Q_0 \in E_6$ and since $Q_0 \in E_{12}$ we get $Q_0 \notin E_{10}$ which means $\beta(p_1, p_2) \geqslant \beta(p_2, p_3)$. So $\beta(p_1, p_2) = \beta(p_2, p_3)$ and b being one-to-one yields $\Delta(p_1, p_2) = \Delta(p_2, p_3)$, a contradiction.

$\diamondsuit$ Finally suppose that there is a $Z \in [{}^{\alpha}2]^{2+\omega^*+\omega}$ such that $[Z]^4 \subset [{}^{\alpha}2]^4 \setminus E_{12}$. We have to distinguish two cases.

- First, suppose that there is a node $s \in {}^{<\alpha}2$ which is a splitting node of elements in Z such that with $X := \{z \in Z \mid z \sqsupset s \frown \langle 1 \rangle\})$ we have $\mathrm{otyp}(X) \geqslant \omega^*$. W.l.o.g. we may suppose that $\mathrm{otyp}(X) = \omega^*$. Now let $z_0 \in Z$ such that $z_0 \sqsupset s \frown \langle 0 \rangle$ and let $z_3 \in X$ be such that for its immediate predecessor z', $\Delta(z', z_3)$ is minimal. Let $\langle x_n \mid n < \omega \rangle$ be a descending sequence of elements of X. There has to be an $n < \omega$ such that $\beta(x_{n+1}, x_n) > \max(\beta(z_0, x_{n+1}), \beta(x_n, z_3))$. Then $\{z_0, x_{n+1}, x_n, z_3\} \in E_6 \setminus E_{10} \subset E_{12}$, a contradiction.

- Now suppose that there is no such node. Let s_0 be the lowest splitting node of elements of Z. For any $n < \omega$ let s_{n+1} be the least splitting node of elements of Z which lies above $s_n \frown \langle 0 \rangle$. Let $s := \sup_{n<\omega} s_n$. Since for no $n < \omega$ we have $\mathrm{otyp}(\{z \in Z \mid z \sqsupset s_n \frown \langle 1 \rangle\}) \geqslant \omega^*$ and $\mathrm{otyp}(Z) = 2 + \omega^* + \omega$ we know that there are two elements $z_0, z_1 \in Z$ such that $z_0, z_1 \sqsupset s$. Let z_3 be the rightmost element of Z. For any $n < \omega$ and any $z' \in Z$ such that $z' \sqsupset s_n \frown \langle 1 \rangle$ we have $\{z_0, z_1, z', z_3\} \in E_{13}$. Now we can choose an $n < \omega$ such that $(b \circ \mathrm{lt})(s_n) > \max(\beta(z_0, z_1), (b \circ \mathrm{lt})(s_0))$ and a $z_2 \in Z$ such that $z_2 \sqsupset s_n \frown \langle 1 \rangle$. Then $\{z_0, z_1, z_2, z_3\} \notin E_7 \cup E_{10}$ and hence $\{z_0, z_1, z_2, z_3\} \in E_{12}$ which is a contradiction. $\square$

It should be noted that all the partition results above involving $\langle {}^{\omega}2, <_{\mathrm{lex}} \rangle$ could be stated again for λ, the order-type of the real number continuum, in place of $\langle {}^{\omega}2, <_{\mathrm{lex}} \rangle$ since both orderings are bi-embeddable.

Three Conjectures

In light of the Theorems 4.13, 4.14, 4.15, 1.8 and 1.9 the author dares to state the following three conjectures.

Conjecture 4.16 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{ADR}$)**.** $\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle \to (\omega^* + \omega \vee \omega + \omega^*, 6)^4$.

Conjecture 4.17 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{ADR}$)**.** $\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle \to (\omega^* + \omega \vee \omega + 2 + \omega^*, 5)^4$.

Conjecture 4.18 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{ADR}$)**.** $\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle \to (2 + \omega^* + \omega \vee \omega + \omega^*, 5)^4$.

The rationale of this is roughly as follows. A partition relation is usually shown to be false by straightforwardly defining a partition witnessing its falsity. This is usually done exploiting some structure. Often but not always the existence of this structure is only known by virtue of the Axiom of Choice.

It very much seems that without the Axiom of Choice there is not enough structure of the appropriate kind left in $\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle$ to falsify the statements of Conjectures 4.16, 4.17 and 4.18. Originally, the author believed that a proof might be possible by using the methods developed in this chapter together with the strong partition properties for ω and ω_1. This idea, however, might have been too optimistic. A combination of the methods of this chapter with those employed to prove the strong partition properties for ω and ω_1 could quite likely be enough to show the three conjectures above to hold.

Rephrasings

Finally it should be noted that Theorems 4.11, 4.12 and 4.13 as well as Conjecture 4.16 allow for alternative formulations. These are equivalent with the sole exception of Theorem 4.20 which is in fact stronger than Theorem 4.12 yet yielded by the proof of Theorem 4.12.

Theorem 4.19 (ZF)**.** There is no ordinal number α such that

$$\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \to (\text{neither well-ordered nor anti-well-ordered}, 7)^4.$$

Theorem 4.20 (ZF + DC + BP)**.**

$$\langle {}^{\omega}2, <_{\mathrm{lex}} \rangle \to (\text{closed but neither well-ordered nor anti-well-ordered}, 5)^4.$$

Theorem 4.21 (ZF)**.** There is no countable ordinal number α such that

$$\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \to (\text{neither well-ordered nor anti-well-ordered}, 6)^4.$$

Conjecture 4.22 (ZF + DC + ADR)**.**

$$\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle \to (\text{neither well-ordered nor anti-well-ordered}, 6)^4.$$

5

A Theorem for Scattered Linear Orders

> If people did not sometimes do silly things, nothing intelligent would ever get done.
>
> — Ludwig Wittgenstein

Recall that an order-type φ is called *scattered* if and only if $\eta \not\leq \varphi$, that is, if it is impossible to embed the rational numbers order-preservingly into it. Scattered orders have been analysed before. In [908Ha] Hausdorff proved that the class of countable scattered order-types is identical to the smallest class of order-types containing the singleton which is closed under taking unions of length ω and ω^*. In [971La] Laver proved Fraïsse's Conjecture by showing the class of countable scattered orderings to be better-quasi-ordered by the embeddability relation.

A quite comprehensive account of results, not all of them combinatorial, about linear orderings was given in [982Ro]. Considering partition relations and excluding the results of well-orders the landscape of theorems looks as follows. Erdős and Rado proved in [956ER] that $\eta \to (\eta, \aleph_0)^2$.Erdős and Hajnal proved in [963EH] that for any countable scattered linear order τ we have that $\tau \to (\varphi, \aleph_0)^2$ implies $\varphi \in \{n, n+\omega^*, \omega+n \mid n < \omega\}$. Larson proved in [974La] that $(\omega^*\omega)^\omega \to ((\omega^*\omega)^\omega, n)^2$ for all natural numbers n. There she also showed that $(\omega^*\omega)^{nk} \to ((\omega^*\omega)^n, k)^2$ for all natural numbers k and n. The following result enriches the picture. It is heavily inspired by the proof of the theorem in [972EM].[1]

Theorem 5.1. For every countable scattered linear order τ and every natural number n there is a countable scattered linear order φ such that $\varphi \to (\tau, n)^2$.

Proof. First note that by induction on $n < \omega$ we get the following partition relation for any linear order τ:

$$\tau^n \to (\tau)^1_n. \tag{5.1}$$

Now suppose that for some $n \in \omega \setminus 2$ and two countable linear orders τ and φ we have $\tau \to (\varphi, n)^2$. We are going to construct a countable linear order ψ such that $\psi \to (\varphi, n+1)^2$ with the property that ψ is scattered if and only if τ is. Note that since $\varphi \to (\varphi, 2)^2$ holds this is going to prove the theorem. Let $b\colon \varphi \longleftrightarrow \omega$. We define

$$\psi := \sum_{\nu<\varphi} \tau^{b(\nu)}.$$

So let $c\colon [\psi]^2 \longrightarrow 2$ be any colouring. Suppose that no $X \in [\psi]^{n+1}$ is homogeneous in colour 1. We are now inductively going to construct a set of

[1]The author was somewhat surprised not to find Theorem 5.1 in the literature.

order type φ which is homogeneous in colour 0. Suppose that in step m of the induction we have constructed a set $X_m = \{x_0, \ldots, x_{m-1}\}$ of order type m which is homogeneous in colour 0. Now consider the following colouring:

$$\begin{aligned} d\colon \tau^m &\longrightarrow m+1, \\ \xi &\longmapsto \min(\{m\} \cup \{k \mid k < m \wedge c(\{x_k, \langle b^{-1}(m), \xi\rangle\}) = 1\}). \end{aligned}$$

We distinguish two cases.

1. $\operatorname{ran}(d) \subset m$. Then by (5.1) there are $X \in [\tau^m]^\tau$ and $k < m$ such that $c(\{x_k, \langle b^{-1}(m), \xi\rangle\}) = 1$ for all $\xi \in X$. If there were an $H \in [X]^n$ homogeneous in colour 1 then $H \cup \{x_k\}$ would be homogeneous in colour 1 and of size $n+1$ which is a contradiction. Since $\tau \to (\varphi, n)^2$ it follows that within X there is a set of order type τ which is homogeneous in colour 0.

2. $\exists x \in \tau^m\colon d(x) = m$. By definition of d this means that $X_m \cup \{x_m\}$ is homogeneous in colour 0. So we set $x_m \coloneqq x$ and $X_{m+1} \coloneqq X_n \cup \{x_m\}$ and continue with our construction.

If this construction never stops with case 1 then after ω steps we let $Z \coloneqq \bigcup_{m<\omega} X_m = \{x_m \mid m < \omega\}$. Z has order-type φ and is homogeneous of colour 0. $\square$

Corollary 5.2. For any countable scattered linear order τ there is a countable scattered linear order φ such that for all natural numbers n we have $\varphi \to (\tau, n)^2$.

To see this one simply has to glue together the orders ψ_n witnessing the theorem for a natural number n. To be more precise, if for all natural numbers n we have $\psi_n \to (\tau, n)^2$ then we have $\varphi \to (\tau, n)^2$ for all natural numbers n where

$$\varphi \coloneqq \sum_{n<\omega} \psi_n.$$

Conjecture 5.3. For any countable scattered linear order τ there exists a countable scattered linear order $\varphi \geqslant \tau$ such that for all natural numbers n we have $\varphi \to (\tau, n)^2$.

Trying to prove the analogue theorem for orders of cardinality $\aleph_1$ in the same fashion one would need a scattered order φ_τ such that $\varphi_\tau \to (\tau)^1_\omega$ for any given scattered order τ. However, lemma 1 in [OO3KS] shows that this is not possible.

We dare to state the following conjecture.

Conjecture 5.4. Whether for any scattered linear order φ of size at most $\aleph_1$ and any natural number n there is a scattered linear order ψ of size $\aleph_1$ such that $\psi \to (\varphi, n)^2$, is independent from ZFC.

Recall that an order-type is called *σ-scattered* if it can be presented as a countable union of scattered order-types.

It is natural to conjecture that a generalisation of Theorem 5.1 is possible by replacing "scattered" by "σ-scattered". So we state another conjecture.

Conjecture 5.5. For any σ-scattered linear order τ of cardinality at most $\aleph_1$ and any natural number n there is a σ-scattered linear order φ of cardinality at most $\aleph_1$ such that $\varphi \to (\tau, n)^2$.

6

Partition Relations for Ordinals of Simple Structure

> If people do not believe that mathematics is simple, it is only because they do not realise how complicated life is.
>
> John von Neumann

The partition calculus as introduced by Erdős and Rado in [956ER] is of interest both to set theorists and combinatorialists. While the former are usually more interested in statements with consistency strength which are hence independent over ZFC the latter are normally interested in finitary problems. In between falls a class of problems the æsthetic appeal of which is in the author's opinion usually underestimated—partition relations between countable ordinals.

People working in this area seem to have been mainly interested in evaluating partition relations of the form $\alpha \to (\alpha, n)$ for countable α and natural n. A good example is [999Sc], the doctoral thesis of Rene Schipperus which won him the Sacks prize. There are three cases in which partition relations of the form $\alpha \to (\beta, n)$ for $\beta < \alpha$ both countable and n natural have been analysed in some generality. This was the case for both α and β being either finite powers of ω, finite multiples of ω or natural numbers—the last case of course being by far the most popular. In this last case people went to great lengths to calculate the $r(m, n)$'s, see for example [995MR]. There are a few results which do not fall into one of these categories, for example in [969Mi], [969HS3], [969HS] and [969HS2].

In the following, we are going to be concerned with arc-coloured digraphs, i.e. directed graphs with coloured arcs.

Definition 6.1. A *coloured digraph* is a triple $\langle V, A, c\rangle$ such that V is a set (of vertices), A is a binary irreflexive asymmetric relation on V—a set of arcs—and $c\colon A \longrightarrow \operatorname{ran}(c)$ is a colouring of A.

We will be especially interested in three-coloured digraphs, i.e. in the case where $\operatorname{ran}(c) = 3$. Remember that a complete digraph is called a *tournament*. In the interest of brevity we will speak of *triples* instead of three-person-trichromatic-tournaments from now on.

In [956ER] Erdős and Rado prove the following theorem[1].

Theorem 6.2. Let $m, n \in \omega \setminus 2$ and denote by $\ell_0 = \ell_0(m, n)$ the least finite number ℓ possessing the following property.

Property P_{mn}. Whenever $\rho(\lambda, \mu) < 2$ for $\{\lambda, \mu\} \in [\ell]^2$, then there is either $\{\lambda_0, \ldots, \lambda_{m-1}\} \in [\ell]^m$ such that $\rho(\lambda_\alpha, \lambda_\beta) = 0$ for $\alpha < \beta < m$ or there is $\{\lambda_0, \ldots, \lambda_{n-1}\} \in [\ell]^n$ such that $\rho(\lambda_\alpha, \lambda_\beta) = 1$ for $\{\alpha, \beta\} \in [n]^2$.

[1]The author changed the notation partially to his own.

Then $\omega\ell_0 \to (m, \omega n)^2$ but $\gamma \not\to (m, \omega n)^2$ for $\gamma < \omega\ell$. Moreover, if $\ell_1 \to (m, m, n)^2$, then $\ell_0 \leqslant \ell_1$.

Note that a possible alternative formulation of all but the last sentence of this theorem would nowadays be the following.

Theorem 6.3. The partition relation $\omega\ell \to (\omega m, n)$ holds true if and only if for every digraph C on ℓ vertices, C contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are transitive.

Theorems 6.2 and 6.3 are equivalent by a well-known fact:

Fact 6.4. If a tournament has a cycle then it has one of length 3.

Theorem 6.2 was later generalised by Baumgartner, cf. [974Ba], in the following way:

Theorem 6.5. Let κ be any infinite cardinal. The partition relation $\kappa\ell \to (\kappa m, n)$ holds true if and only if for every digraph C on ℓ vertices, C contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are transitive.

Following [997LM] we are writing I_m for an independent set of size m, L_n to denote a transitive tournament of size n and $r(I_m, L_n)$ for the least number of vertices such that any digraph with so many of them either has to contain an independent set of size m or an L_n as an induced subgraph. This allows us to phrase Theorem 6.5 by simply writing

$$r(\kappa m, n) = \kappa r(I_m, L_n)$$

for any infinite cardinal κ.

Added in Proof

The author believed for quite some time that he was the first to discover that $r(\omega^2 2, 3) = \omega^2 10$. Only later he learned that this fact already appeared in [969HS2]. Neither a proof was given nor was there a formulation of a finitary problem equivalent to the calculation of the $r(I_m, A_n)$'s or an upper bound for the $r(I_m, A_3)$'s provided.

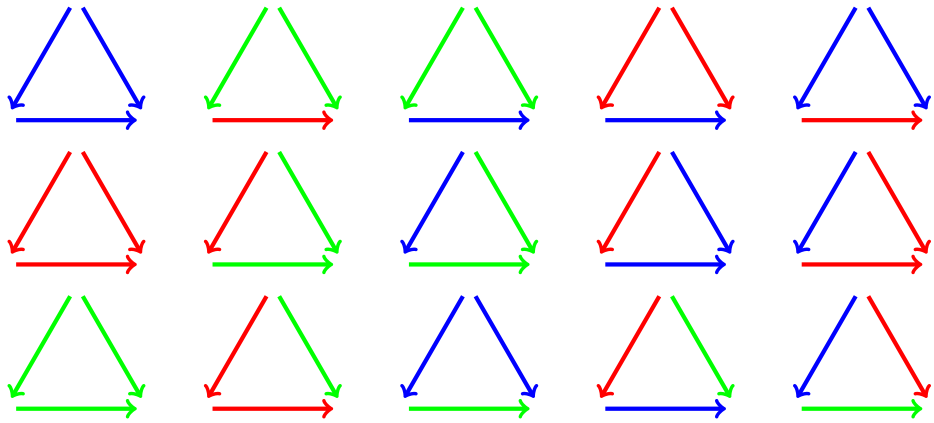

Figure 6.1: The agreeable triples

Determining a Ramsey Number

In this section we mainly build on work of Pál Erdős and Richard Rado in the aforementioned [956ER] and Ernst Specker in [957Sp]. The following eclectic definition is justified by Theorem 6.11 the proof of which explains the background of it.

Definition 6.6. We identify 0 with blue, 1 with red and 2 with green arrows. A triple is called *agreeable* if and only if it is one of those shown in Figure 6.1.

If a triple is not agreeable we call it *disagreeable*. Note that all agreeable triples are *transitive* rather than *cyclic*, i.e. there is both a vertex emitting two arrows and a vertex absorbing two arrows.

Fact 6.7. A triple is disagreeable if and only if it is either cyclic, regardless of the colouring or one of the transitive triples not listed in Figure 6.1. This is depicted in Figure 6.2.

We remind the reader of the definition of weakly compacts.

Definition 6.8. $\kappa \in \Omega \setminus 3$ is called weakly compact if and only if $\kappa \to (\kappa)^2_\omega$.

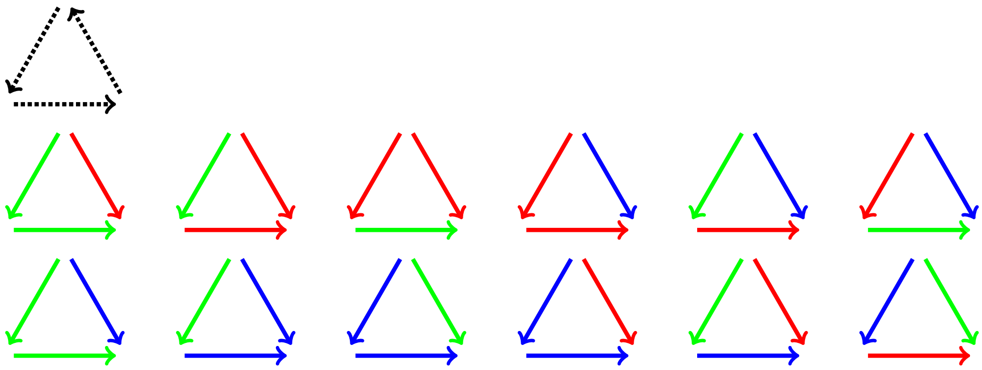

Figure 6.2: The disagreeable triples

Fact 6.9. For all $\kappa \in \Omega$ we have $\kappa \to (\kappa)^2_2$ if and only if $\kappa \in \{0, 1, 2, \omega\}$ or κ is weakly compact.

Fact 6.10. $\kappa \to (\kappa)^2_2$ if and only if $\forall n < \omega \, \forall \alpha < \kappa \colon \kappa \to (\kappa)^n_\alpha$.

Both facts are corollaries of Theorem 7.6 of [003Ka] which can be found on page 76.

Theorem 6.11. Let $\kappa \in \Omega \setminus 3$ satisfy $\kappa \to (\kappa)^2_2$.
The partition relation $\kappa^2\ell \to (\kappa^2 m, n)$ holds true if and only if every coloured digraph $C = \langle \ell, A, c\rangle$ with $\operatorname{ran}(c) = 3$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.

Proof. $\diamondsuit$ Let us first assume that the finite combinatorial characterisation above holds true. Given any colouring $\chi \colon [\kappa^2\ell]^2 \longrightarrow 2$ we define a new colouring χ' as follows:

$$\begin{aligned} \chi' \colon [\kappa]^4 &\longrightarrow 2^{4\ell^2} \\ \{h, j, i, k\}_< &\longmapsto \sum_{f,g<\ell} \big(\chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\})2^{4(\ell f+g)} \\ &\qquad +\chi(\{\kappa^2 f + \kappa h + i, \kappa^2 g + \kappa j + k\})2^{4(\ell f+g)+1} \\ &\qquad +\chi(\{\kappa^2 f + \kappa h + k, \kappa^2 g + \kappa j + i\})2^{4(\ell f+g)+2} \\ &\qquad +\chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa h + i\})2^{4(\ell f+g)+3}\big) \end{aligned}$$

Now we use Fact 6.10 in the form $\kappa \to (\kappa)^4_{2^{4\ell^2}}$ thereby finding an $X \in [\kappa]^\kappa$ homogeneous for χ'.

Now fix a surjection $b\colon X \twoheadrightarrow \{\langle \nu, \gamma\rangle \mid \nu < \ell \wedge \gamma < \kappa + 1\}$ and monotonic enumeration functions $e_\nu\colon \kappa \longleftrightarrow b^{-1}``(\{\langle \nu, \kappa\rangle\})$ such that

- $\forall \nu < \ell \, \forall \gamma < \kappa + 1\colon \overline{\overline{b^{-1}``(\{\langle \nu, \gamma\rangle\})}} = \kappa$,
- $\forall \nu < \ell \, \forall \gamma < \kappa \, \forall \delta \in b^{-1}``(\{\langle \nu, \gamma\rangle\}) : e_\nu(\gamma) < \delta$.

Now let $Y := \{\kappa^2 f + \kappa h + j \mid f < \ell \wedge h \in b^{-1}(\langle f, \kappa\rangle) \wedge j \in b^{-1}(\langle f, e_f^{-1}(h)\rangle)\}$.

In prose what's happening is this: We distribute the elements of X to signify multiples of 1 or κ in the ℓ blocks of size κ^2. No two ordinals may appear in two roles and within one κ^2-block the 1-coordinate is always larger than the κ-coordinate. Note that $Y \in [\kappa^2 \ell]^{\kappa^2 \ell}$.

Observe that the colour of an element $\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2$—where we suppose that $h < i$—is completely determined by f, g and whether $h < j < i < k$, $h < i < j < k$, $h < i < k < j$ or $h = i < j < k$.

Now we define a coloured digraph $C = \langle \ell, A, c\rangle$ as follows. If for $\{f, g\} \in [\ell]^2$ we have

- $\forall \{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2\colon \chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\}) = 0$ then there is no arc between f and g in C.
- $\exists \{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2\colon \chi(\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\}) = 1$ where we again assume that $h < i$ and
  - ◇ $h < i < k < j$ then we let $f \mapsto_b g$.
  - ◇ $h < i < j < k$ then we let $f \mapsto_r g$.
  - ◇ $h < j < i < k$ then we let $f \mapsto_g g$.

  In this case there may very well be more than one way to put an arc between f and g and to colour it—clearly there could be six possible ways to do this yet it suffices to do it in one way only.

Our remark above shows us that this is well-defined. Since ℓ is natural we do not even have to use the Axiom of Choice.

Now we may use our combinatorial statement. If $I \in [\ell]^m$ is independent in C then obviously

$$Z := \{\{\kappa^2 f + \kappa h + j, \kappa^2 g + \kappa i + k\} \in [Y]^2 \mid \{f, g\} \subset I \wedge h, i, j, k < \kappa\}$$

is a homogeneous set of size $\kappa^2 m$ in colour 0. Thus let us assume that there is a subtournament S of C of size n such that all triples in S are agreeable.

Let $\langle m_0, m_2, \ldots, m_{n-1}\rangle$ be the sequence of natural numbers defined by the following backwards recursion:

- $m_n := 0$,
- $m_{i-1} := 3m_i + 2$.

We have $m_i := 3^{n-i} - 1$. Let furthermore $\langle v_0, v_1, \ldots, v_{n-1}\rangle$ be a sequence of vertices such that $S := \{v_i \mid i < n\}$.

Now we are going to inductively construct the 1-homogeneous set $\{\kappa a_i + b_i \mid i < n\}$ where a_i and b_i are ordinals less than κ for $i < n$.

Let us first inductively define a sequence $\langle s_i \mid i < m_0\rangle$ of ordinals less than κ. Let $s_0 := 0$. Given $\langle s_j \mid j \leqslant i\rangle$ choose $\alpha_i < \kappa$ such that $e_{v_j}(\alpha_i) \geqslant s_i$ for all $j < n$ and then choose $s_{i+1} < \omega$ such that $\forall j < n \, \forall k \leqslant i \colon b^{-1}``\{\langle v_j, \alpha_k\rangle\} \cap s_{i+1} \setminus s_i \supsetneq \emptyset$.

In step $i < n$ suppose we have already defined a 1-homogeneous set of size i.

The following will be the induction hypothesis.

1. $\min\{j \in \omega \setminus 1 \mid \exists k < \omega \colon \{a_h, b_h \mid h < i\} \cap s_{j+k+1} \setminus s_{j+k} \supsetneq \emptyset \wedge \{a_h, b_h \mid h < i\} \cap s_{k+1} \setminus s_k \supsetneq \emptyset\} > m_{i+1}$,
2. $\forall j, k < i \colon v_j \mapsto_b v_k \leftrightarrow a_j < a_k < b_k < b_j$,
3. $\forall j, k < i \colon v_j \mapsto_r v_k \leftrightarrow a_j < a_k < b_j < b_k$,
4. $\forall j, k < i \colon v_j \mapsto_g v_k \leftrightarrow a_j < b_j < a_k < b_k$.

This is done as follows, let $L_\kappa := \max(\{a_h \mid h < i \wedge v_h \mapsto v_i\} \cup \{b_h \mid h < i \wedge v_h \mapsto_g v_i\})$ and $U_\kappa := \min(\{a_h \mid h < i \wedge v_i \mapsto v_h\} \cup \{b_h \mid h < i \wedge v_h \mapsto_b v_i\} \cup \{b_h \mid h < i \wedge v_h \mapsto_r v_i\})$.

Claim 6.12. $L_\kappa < U_\kappa$.

Proof of Claim 6.12.

Suppose not. The claim could fail in the following ways:

- $\max\{a_h \mid h < i \wedge v_h \mapsto v_i\} \geqslant \min\{a_j \mid j < i \wedge v_i \mapsto v_j\}$. Fix h and j witnessing this. The induction hypothesis implies that $v_j \mapsto v_h$. But then v_h, v_i and v_j form a cyclic and thus disagreeable triple. Contradiction!
- $\max\{a_h \mid h < i \wedge v_h \mapsto v_i\} \geqslant \min\{b_j \mid j < i \wedge v_j \mapsto_b v_i\}$. As before fix witnesses h and j. By induction hypothesis we have $v_j \mapsto_g v_h$ and Fact 6.7 implies that $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{a_h \mid h < i \wedge v_h \mapsto v_i\} \geqslant \min\{b_j \mid j < i \wedge v_j \mapsto_r v_i\}$. Fix h and j witnessing this. The induction hypothesis implies that $v_j \mapsto_g v_h$. A look at Fact 6.7 shows that $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h \mid h < i \wedge v_h \mapsto_b v_i\} \geqslant \min\{a_j \mid j < i \wedge v_i \mapsto v_j\}$. Fix h, j witnessing this. By induction hypothesis we now either have $v_j \mapsto v_h$, $v_h \mapsto_b v_j$ or $v_h \mapsto_r v_j$. In every case $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h \mid h < i \wedge v_h \mapsto_g v_i\} \geqslant \min\{b_j \mid h < i \wedge v_j \mapsto_b v_i\}$. Let h, j witness this. As before the induction hypothesis implies that either $v_h \mapsto_b v_j$, $v_j \mapsto_r v_h$ or $v_j \mapsto_g v_h$. Again by Fact 6.7 $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- $\max\{b_h \mid h < i \wedge v_h \mapsto_g v_i\} \geqslant \min\{b_j \mid h < i \wedge v_j \mapsto_r v_i\}$. Fix witnesses h and j to this fact. By induction hypothesis we now either have $v_h \mapsto_b v_j$, $v_j \mapsto_r v_h$ or $v_j \mapsto_g v_h$. Fact 6.7, again, proves $\{v_h, v_i, v_j\}$ to be disagreeable. Contradiction! ©

Let $h < m_0$ be minimal such that $L_\kappa < s_h$. Then take

$$a_i \in e_{v_i}``(\kappa) \cap s_{h+m_i+1} \setminus s_{h+m_i}.$$

Now we are going to define b_i. Let $L_1 := \max(\{a_i\} \cup \{a_h \mid h < i \wedge v_i \mapsto_r v_h\} \cup \{b_h \mid h < i \wedge v_i \mapsto_b v_h\} \cup \{b_h \mid h < i \wedge v_h \mapsto_r v_i\})$ and $U_1 := \min(\{a_h \mid h < i \wedge v_i \mapsto_g v_h\} \cup \{b_h \mid h < i \wedge v_h \mapsto_b v_i\} \cup \{b_h \mid h < i \wedge v_i \mapsto_r v_h\})$.

Claim 6.13. $L_1 < U_1$.

Proof of Claim 6.13. As above we distinguish cases and reach contradictions.

- Suppose $a_i \geqslant \min\{a_h \mid h < i \wedge v_i \mapsto_g v_h\}$. Since $\{a_h \mid h < i \wedge v_i \mapsto_g v_h\} \subset \{a_h \mid h < i \wedge v_i \mapsto v_h\}$ we have $\min\{a_h \mid h < i \wedge v_i \mapsto_g v_h\} \geqslant \min\{a_h \mid h < i \wedge v_i \mapsto v_h\}$ and since $\min\{a_h \mid h < i \wedge v_i \mapsto v_h\} \geqslant U_\kappa$ it follows that $a_i \geqslant U_\kappa$. Contradiction!
- If $a_i \geqslant \min\{b_h \mid h < i \wedge v_h \mapsto_b v_i\}$ then analogously we get $\min\{b_i \mid h < i \wedge v_h \mapsto_b v_i\} \geqslant U_\kappa$ implying $a_i \geqslant U_\kappa$. Contradiction!
- Suppose $a_i \geqslant \min\{b_h \mid h < i \wedge v_i \mapsto_r v_h\}$. Similarly, we have $\min\{b_h \mid h < i \wedge v_h \mapsto_r v_i\} \geqslant U_\kappa$ and therefore $a_i \geqslant U_\kappa$. Contradiction!
- Suppose $\max\{a_h \mid h < i \wedge v_i \mapsto_r v_h\} \geqslant \min\{a_j \mid j < i \wedge v_i \mapsto_g v_j\}$. Fix witnesses h and j to this fact. The induction hypothesis implies $v_j \mapsto v_h$. Fact 6.7 shows that in all three cases $\{v_h, v_i, v_j\}$ is disagreeable.
- Suppose $\max\{a_h \mid h < i \wedge v_i \mapsto_r v_h\} \geqslant \min\{b_j \mid j < i \wedge v_j \mapsto_b v_i\}$. Once more fix witnesses h, j. Again the induction hypothesis implies that $v_j \mapsto_g v_h$. Fact 6.7 implies that $\{v_h, v_i, v_j\}$ is a disagreeable triple. Contradiction!
- In case $\max\{a_h \mid h < i \wedge v_i \mapsto_r v_h\} \geqslant \min\{b_j \mid j < i \wedge v_i \mapsto_r v_j\}$ we can again fix h and j witnessing this and use the induction hypothesis to see that $v_j \mapsto_g v_h$. Checking Fact 6.7 assures us of the disagreeableness of $\{v_h, v_i, v_j\}$. Contradiction!
- In case $\max\{b_h \mid h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{a_j \mid j < i \wedge v_i \mapsto_g v_j\}$ let us once more fix witnesses h and j to this fact and use the induction hypothesis. We must either have $v_j \mapsto v_h$ or $v_h \mapsto_b v_j$ or $v_h \mapsto_r v_j$. Fact 6.7 shows that in all five cases $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- Suppose that $\max\{b_h \mid h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{b_j \mid j < i \wedge v_j \mapsto_b v_i\}$. Fix witnesses h and j and use the induction hypothesis in order to see that we either have $v_h \mapsto_b v_j$ and hence a cyclic triple $\{v_h, v_i, v_j\}$ or we have either $v_j \mapsto_r v_h$ or $v_j \mapsto_g v_h$. In both of the last two cases $\{v_h, v_i, v_j\}$ is disagreeable by Fact 6.7. Contradiction!
- If $\max\{b_h \mid h < i \wedge v_i \mapsto_b v_h\} \geqslant \min\{b_j \mid j < i \wedge v_i \mapsto_r v_j\}$ we fix witnesses h and j again. By induction hypothesis we have $v_h \mapsto_b v_j$ or $v_j \mapsto_r v_h$ or

$v_j \mapsto_g v_h$. Fact 6.7 assures us once again of $\{v_h, v_i, v_j\}$'s disagreeability in any case. Contradiction!

- Suppose $\max\{b_h \mid h < i \wedge v_h \mapsto_r v_i\} \geqslant \min\{b_j \mid j < i \wedge v_j \mapsto_b v_i\}$. Let h and j be witnesses to this fact. The induction hypothesis implies that either $v_h \mapsto_b v_j$ or $v_j \mapsto_r v_h$ or $v_j \mapsto_g v_h$. By Fact 6.7 in any case $\{v_h, v_i, v_j\}$ is disagreeable. Contradiction!
- If $\max\{b_h \mid h < i \wedge v_h \mapsto_r v_i\} \geqslant \min\{b_j \mid j < i \wedge v_i \mapsto_r v_j\}$ we fix witnesses h and j. By induction hypothesis we either have $v_h \mapsto_b v_j$ or $v_j \mapsto_r v_h$ or $v_j \mapsto_g v_h$. By Fact 6.7 $\{v_h, v_i, v_j\}$ would be disagreeable in each case. Contradiction!
- If $\max\{b_h \mid h < i \wedge v_h \mapsto_r v_i\} \geqslant \min\{a_j \mid j < i \wedge v_i \mapsto_g v_j\}$ we can again fix witnesses h, j to this fact. By induction hypothesis we can conclude that either $v_j \mapsto v_h$ in which case $\{v_h, v_i, v_j\}$ is cyclic and in particular disagreeable or that either $v_h \mapsto_b v_j$ or $v_h \mapsto_r v_j$. Fact 6.7, however, tells us that $\{v_h, v_i, v_j\}$ is disagreeable in these cases too. Contradiction! ©[2]

Now let $h < m_0$ be minimal such that $L_1 < s_h$. Take

$$b_i \in b^{-1}``(\{\langle v_i, e^{-1}(a_i)\rangle\}) \cap s_{h+2m_i+1} \setminus s_{h+2m_i}.$$

By Claims 6.12 and 6.13 and condition 1. of the inductive hypothesis we have chosen a_i and b_i such that $a_i < U_\kappa$ and $b_i < U_1$. So by definition of the bounds L_κ, U_κ, L_1 and U_1 conditions 2. through 4. of our inductive hypothesis are fulfilled. We also made sure that 1. holds true for the next step.

But then, after n steps, by the definition of our coloured digraph C, we see that $\{\kappa a_i + b_i \mid i < n\}$ is homogeneous of colour 1.

[2] There are basically two ideas behind this abundance of case distinctions. First one can conceive of a completeness theorem for linear orders. Unless one cannot derive a contradiction there has to be a model. The other is the fact that if a tournament has a cycle it has one of length 3. Here it only gets a little bit more involved since one has to simultaneously consider the ω-coordinates and the 1-coordinates.

◇ Now suppose towards a contradiction that there is a a 3-coloured digraph $D = \langle l, A, c \rangle$ of size l such that both all its independent sets have size less than m and every subtournament S of D of size n contains a disagreeable triple.

Now we define a colouring χ as follows:

$$\chi \colon [\omega^2 \ell]^2 \longrightarrow 2$$

$$\{\omega^2 e + \omega g + i, \omega^2 f + \omega h + j\} \longmapsto \begin{cases} 1 \text{ if } g < h < j < i \text{ and } e \mapsto_b f, \\ 1 \text{ if } g < h < i < j \text{ and } e \mapsto_r f, \\ 1 \text{ if } g < i < h < j \text{ and } e \mapsto_g f, \\ 0 \text{ if } e \text{ and } f \text{ are unconnected}, \\ 0 \text{ if } \overline{\overline{\{g, h, i, j\}}} < 4, \\ 0 \text{ if } e = f \text{ or } i < g \text{ or } j < h. \end{cases}$$

In the naming of the pair of ordinals we can of course assume without loss of generality that $g \leqslant h$.

Now if there were a set $H \in [\kappa^2 \ell]^{\kappa^2 m}$ which is 0-homogeneous for χ there would be a set $S \in [\ell]^m$ such that for all $s \in S$ we have $\operatorname{otyp}(\{\kappa^2 s + \alpha \in H \mid \alpha < \kappa^2\}) = \kappa^2$. Since all independent sets in D have size less than m it follows that we have $e \mapsto f$ for $e, f \in S$. Now if $e \mapsto_g f$, choose first $g \in \{k \mid k < \kappa \wedge \overline{\overline{\{k' \mid k' < \kappa \wedge \kappa^2 e + \kappa k + k' \in H\}}} = \kappa\}$, then $i \in \{k \mid k \in \kappa \setminus (g+1) \wedge \kappa^2 e + \kappa g + k \in H\}$, then $h \in \{k \mid k \in \kappa \setminus (i+1) \wedge \overline{\overline{\{k' \mid k' < \kappa \wedge \kappa^2 f + \kappa k + k' \in H\}}} = \kappa\}$ and finally $j \in \{k \mid k \in \kappa \setminus (h+1) \wedge \kappa^2 f + \kappa h + k \in H\}$. But now we have found $\kappa^2 e + \kappa g + i, \kappa^2 f + \kappa h + j \in H$ such that $\chi(\{\kappa^2 e + \kappa g + i, \kappa^2 f + \kappa h + j\}) = 1$ which contradicts the assumption that H was 0-homogeneous. If instead of $e \mapsto_g f$ we have $e \mapsto_r f$ or $e \mapsto_b f$ we can argue similarly, we only have to change the sequence of choices.

Assume that there is a set $H \in [\omega^2 \ell]^n$ which is 1-homogeneous for χ. Considering the last clause in the definition of χ we conclude that no κ^2-block can contain more than one element of H, i.e. we have $\overline{\overline{H \cap \kappa^2(k+1) \setminus \kappa^2 k}} < 2$

for every $k < \ell$. So $S := \{s \mid s < \ell \wedge H \cap \kappa^2(s+1) \setminus \kappa^2 s \supsetneq \emptyset\}$ has size n. Moreover, the fourth clause in the definition of χ implies that every two elements of S are connected by an arrow. So S spans a subtournament of D. Hence S has to contain a disagreeable triple $T = \{t_0, t_1, t_2\} \in [S]^3$. Take $\kappa^2 t_0 + \kappa u_0 + v_0, \kappa^2 t_1 + \kappa u_1 + v_1, \kappa^2 t_2 + \kappa u_2 + v_2 \in H$. Now we distinguish four cases

- T is cyclic. Suppose we have $t_0 \mapsto t_1 \mapsto t_2 \mapsto t_0$. Considering the first three clauses of the definition of χ we may conclude that for the elements chosen above we have $u_0 < u_1 < u_2 < u_0$. Contradiction!
- Suppose $t_0 \mapsto_r t_1$ or $t_0 \mapsto_g t_1$, that $t_2 \mapsto_b t_1$ or $t_1 \mapsto_r t_2$ or $t_1 \mapsto_g t_2$ and that $t_0 \mapsto_b t_2$. Then, using the definition of χ once again, we arrive at $v_0 < v_1 < v_2 < v_0$. Contradiction!
- Suppose that $t_0 \mapsto_g t_1$, that $t_1 \mapsto t_2$ and that $t_0 \mapsto_b t_2$ or $t_0 \mapsto_r t_2$. Then $v_0 < u_1 < u_2 < v_0$ follows. Contradiction!
- Suppose that $t_0 \mapsto_g t_1$, that $t_2 \mapsto_b t_1$ or $t_2 \mapsto_r t_1$ and that either $t_0 \mapsto_b t_2$ or $t_2 \mapsto_r t_0$ or $t_2 \mapsto_g t_0$. Then $v_0 < u_1 < v_2 < v_0$. Contradiction!

Note that modulo a renaming of t_0, t_1 and t_2 the above clauses cover all disagreeable triples. So this concludes the second part of the proof. $\square$

The author wants to point out that if you believe in enough reflection you might believe something like "There are stationarily many weakly compact cardinals in the universe." so Theorem 6.11 really says something about many ordinals.

We can now also state—not in complete earnest—a joint generalisation of Theorems 6.3 and 6.11. In this artificial sense being agreeable is a generalisation of being transitive.

Theorem 6.14. For $i \in 3 \setminus 1$ and $\kappa \in \Omega \setminus 3$ such that $\kappa \to (\kappa)^2_2$ the partition relation $\kappa^i l \to (\kappa^i m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\operatorname{ran}(c) = 2i - 1$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.

The special case the author was actually most interested in, however, is this one:

Corollary 6.15. The partition relation $\omega^2 l \to (\omega^2 m, n)$ holds true if and only if every coloured digraph $C = \langle l, A, c\rangle$ with $\mathrm{ran}(c) = 3$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are agreeable.

Henceforth, A_n is a graph on n vertices such that all its induced subgraphs on three vertices are agreeable. This allows us to define the Ramsey number $r(I_m, A_n)$ to be the smallest natural number such that any digraph on $r(I_m, A_n)$ vertices either contains an independent set of size m or an induced subgraph on n vertices all triples of which are agreeable. Since A_n does not denote a single graph but a class of graphs this might be seen as a slight abuse of notation, or at least a shorthand. Hence the equivalence between the infinitary partition problems and finite digraph combinatorics above can be also condensed into the following formula:

$$r(\kappa^2 m, n) = \kappa^2 r(I_m, A_n) \text{ for } \kappa = \omega \text{ or } \kappa \text{ weakly compact.}$$

Theorem 6.16. $r(I_2, A_3) > 9$.

Proof. The statement above holds because there is the trichromatic nine-person tournament shown in Figure 6.3. Algebraically, this can be seen as the tournament on $\mathbb{Z}_9$ where from every $i \in \mathbb{Z}_9$ there is a green arrow pointing at $i+1$, red arrows pointing both at $i+2$ and $i+3$ and a blue arrow pointing at $i+4$.

A close observation reveals that every triple in this tournament is disagreeable thereby showing $\omega^2 9 \not\to (\omega^2 2, 3)$.

There even is a dichromatic counterexample which is shown in Figure 6.4. This is algebraically interpretable as the tournament on $\mathbb{Z}_3 \times \mathbb{Z}_3$ where from every $\langle i, j\rangle$ there is a green arrow pointing at $\langle i+1, j\rangle$ and red arrows pointing at $\langle i, j+1\rangle$ and $\langle i+1, j+1\rangle$ and $\langle i+2, j+1\rangle$.

Showing this to be a counterexample is easy: Either a triple is one of three completely coloured in green—these are disagreeable—or it shares exactly one point with each of these triples—then it forms a disagreeable

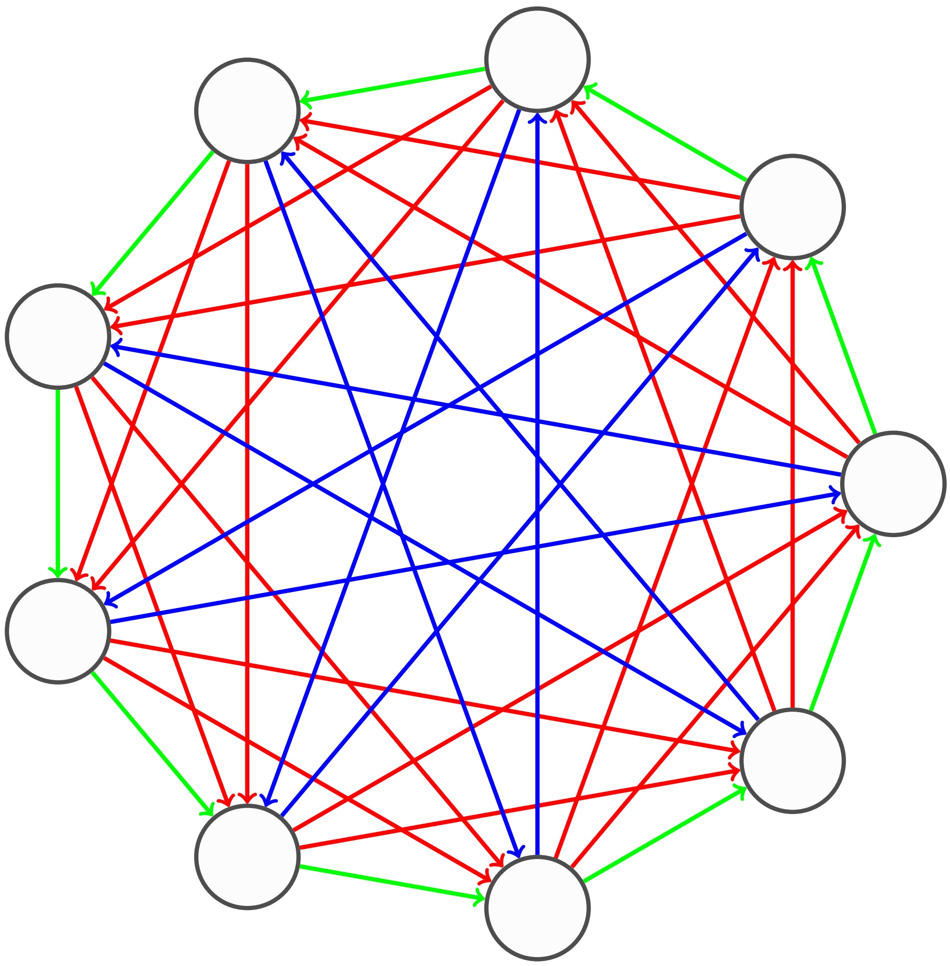

Figure 6.3: An edge-coloured tournament showing $r(I_2, A_3) > 9$

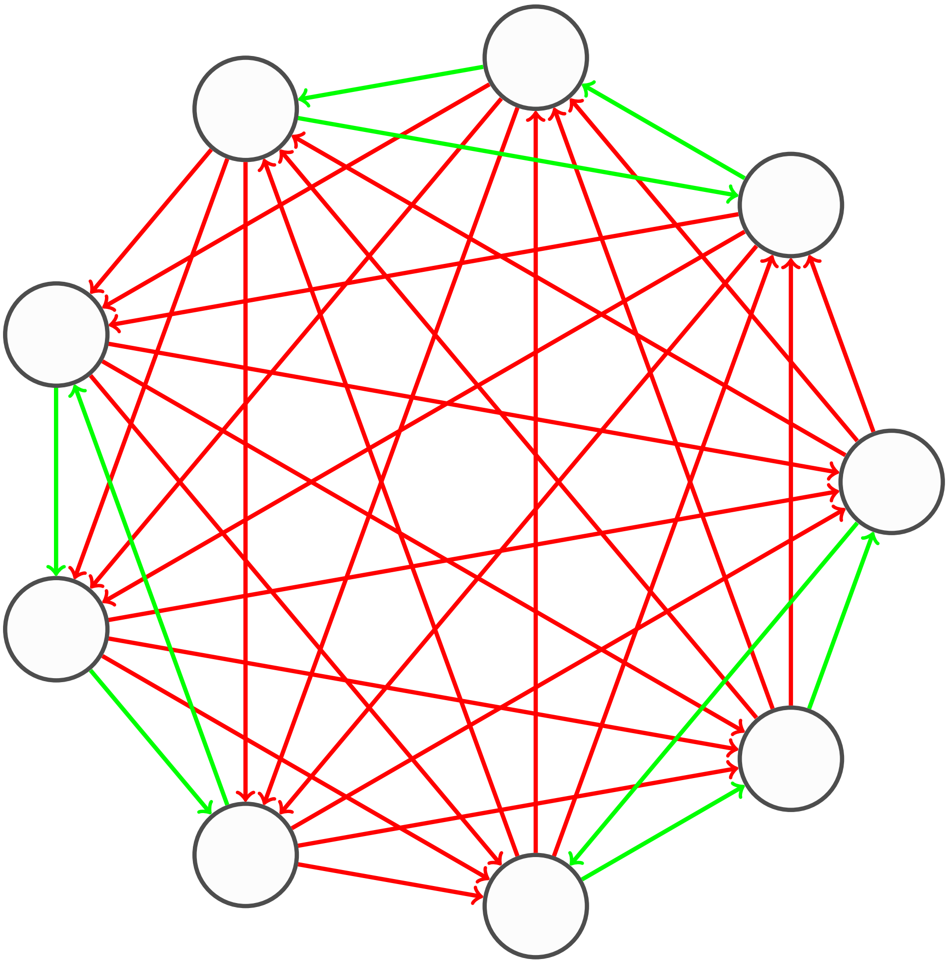

Figure 6.4: Another edge-coloured tournament showing $r(I_2, A_3) > 9$

triple completely coloured in red—or it shares two points with one and one with another—then it is either one of two green/red-coloured disagreeable triples. □

These counterexamples were not found in a very systematic way so there might or might not be others.

Some easy observations quickly yield upper bounds for $r(I_2, A_3)$, for example by ordering the vertices and then colouring a pair of them in one of six colours depending on both whether the direction of the arc agrees with the ordering or not and the original colour of the arc. Since all monochromatic transitive triples are agreeable this yields $r(I_2, A_3) \leqslant r(3,3,3,3,3,3)$. Using $r(3,3,3) = 17$ from [955GG] together with the recursive formula, we obtain $r(I_2, A_3) \leqslant 1898$.

A much better bound is found by observing that every four-person-tournament contains a transitive triple. So clearly, $r(I_2, A_3) \leqslant r(4,4,4)$. The recursive formula together with the result that $r(3,3,4) \leqslant 31$ from [998PR] yields $r(4,4,4) \leqslant 236$ so $r(I_2, A_3) \leqslant 236$ follows. Finally by observing that every six-person-tournament coloured in either blue or red contains an agreeable triple we get $r(I_2, A_3) \leqslant r(4,6)$. In [997MR] it is shown that $r(4,6) \leqslant 41$ so we have $r(I_2, A_3) \leqslant 41$.

In fact far better results are possible.

Fact 6.17. In a tournament on ten vertices containing no agreeable triple there can be at most ten green arrows, thirty red arrows and ten blue arrows. So there are at least five green, twenty-five red and five blue arrows.

Proof. Take any tournament on ten vertices not containing any agreeable triple.

Suppose that there are eleven green arrows. By the pigeonhole principle there has to be a vertex where two green arrows leave. But then this vertex together with the targets of the two green arrows forms an agreeable triple.

Suppose that there are eleven blue arrows. Again by the pigeonhole principle there has to be a vertex where two blue arrows leave. Analogously, this vertex together with the targets of the two blue arrows forms an agreeable triple.

Suppose that there are thirty-one red arrows. By the pigeonhole principle there has to be a vertex where four red arrows leave. Then all of the targets of these arrows have to be connected by green arrows to one another. Fix one of these target-vertices v. Since it is adjacent to at least three green arrows at least two green arrows are leaving from there or two green arrows are arriving there. Suppose two green arrows are leaving. Then v together with the targets of these arrows forms an agreeable triple. If on the other hand two green arrows are arriving then again v, together with the sources of these arrows forms an agreeable triple.

The second statement follows immediately after one observes that a tournament on ten vertices has $\binom{10}{2}$, i.e. 45 arrows. □

Lemma 6.18. Any tournament T on ten vertices containing no agreeable triple contains three green triples, i.e. we have $v_0, \dots, v_9 \in V_T$ with $v_1 \mapsto_g v_2 \mapsto_g v_3 \mapsto_g v_1$ and $v_4 \mapsto_g v_5 \mapsto_g v_6 \mapsto_g v_4$ and $v_7 \mapsto_g v_8 \mapsto_g v_9 \mapsto_g v_7$.

Proof. First observe that any such tournament has to contain two such green triples. This is because by Fact 6.17 there have to be at least twenty-five red arrows and hence five vertices where three red arrows leave. Since the targets of the three red arrows leaving such a vertex have to form a green triple and since any vertex in such a green triple may be the target of at most three red arrows the pigeonhole principle implies that there have to be at least two green triples.

Now suppose towards a contradiction that there are exactly two such green triples $\{v_0, v_1, v_2\}$ and $\{v_3, v_4, v_5\}$ and that there are six vertices where three red arrows leave. Again the targets of these red arrows have to form green triples. Since two green triples cannot overlap without producing an agreeable triple we can apply the pigeonhole principle and conclude that each vertex in each of both triples has to receive three red arrows. Now since three red arrows each are hitting v_0, v_1 and v_2 the sources of them also form a green triple. Since there are only two of them and v_0, v_1 and v_2 are connected to each other by green arrows we know that the sources of them can only be v_3, v_4 and v_5. So we have $v_i \mapsto_r v_j$ for every $i \in 6 \setminus 3$ and $j \in 3$. But since v_3 is also hit by three red arrows and is already connected in the opposite direction with v_0, v_1 and v_2 there has to be a third green triple.

An analogous argument works for the case that there are six vertices where three red arrows *leave*.

So there are at most five vertices where three red arrows leave and at most five vertices where three red arrows arrive. So there are at most twenty-five red arrows. By Fact 6.17 we may restrict our attention to the case of ten green, twenty-five red and ten blue arrows. Since no vertex is either target or source of more than one green arrow the remaining four green arrows have to form a cycle. So without loss of generality we have $v_6 \mapsto_g v_7 \mapsto_g v_8 \mapsto_g v_9 \mapsto_g v_6$.

Now we argue as before. Consider again the five vertices where three red arrows are leaving. The targets of the three red arrows leaving such a vertex must form a green triple. By the pigeonhole principle every vertex in one of the two green triples has to receive three red arrows each. But then again their sources form a green triple. This shows that without loss of generality we have $v_i \mapsto_r v_j$ for every $i \in 6 \setminus 3$ and $j \in 3$.

In total there are five vertices with three red arrows leaving and there are five vertices with three red arrows arriving. Both the targets of the former and the sources of the latter have to form green triples. Since every vertex may emit or absorb at most three red arrows it follows that there are two-element-sets $\{w_0, w_1\}$ and $\{w_2, w_3\}$ with $w_0 \mapsto_g w_1$ and $w_2 \mapsto_g w_3$ such that $v_i \mapsto_r w_j$ for every $i \in 3$ and $j \in 2$ and $w_i \mapsto_r v_j$ for every $i \in 4 \setminus 2$ and $j \in 6 \setminus 3$.

Now we are going to distinguish some—not altogether different—cases. Note that since there are ten blue arrows and a vertex can emit at most one, every vertex has to emit exactly one.

◇ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 2$. Since there are only two possible targets—the elements of the set $\{v_6, \ldots, v_9\} \setminus \{w_0, w_1\}$—for the blue arrows emitted by the vertices $v_0, \ldots, v_5, w_0, w_1$ one of them has to be target of at least four blue arrows which contradicts Fact 6.17.

◇ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 3$. Let $X := \{w_0, w_1\} \cap \{w_2, w_3\}$. Clearly, $\overline{\overline{X}} = 1$. Let $w := \bigcup X$. The only possible target for the blue arrow leaving w is $v := \bigcup(\{v_6, \ldots, v_9\} \setminus \{w_0, w_1, w_2, w_3\})$ since for each $i \in 4$ we either have $w = w_i$ or $w \mapsto_g w_i$ or $w_i \mapsto_g w$ and for all $i \in 6$ we either have $w \mapsto_r v_i$ or

$v_i \mapsto_r w$. Now since the vertices $v_0, \ldots, v_5$ all also have to emit a blue arrow and the sources of blue arrows pointing at the same vertex have to be connected by green arrows, lest there be an agreeable triple, we know that none of these blue arrows can point at v. None can point at w either since w was already shown to be connected differently to every other vertex. So let $x \coloneqq \bigcup(\{w_0, w_1\} \setminus \{v, w\})$ and $y \coloneqq \bigcup(\{w_2, w_3\} \setminus \{v, w\})$. Since $v_i \mapsto_r x$ for any $i \in 3$ it follows that $v_i \mapsto_b y$ for any $i \in 3$ Similarly $y \mapsto_r v_i$ for any $i \in 6 \setminus 3$ implies $v_i \mapsto_b x$ for any $i \in 6 \setminus 3$. But now we see that x and y cannot both emit a blue arrow hence we have a contradiction.

$\diamondsuit$ $\overline{\overline{\{w_0, w_1, w_2, w_3\}}} = 4$. Remember that $v_6, \ldots, v_9$ form a green four-cycle. Since $\{v_6, \ldots, v_9\} = \{w_0, \ldots, w_3\}$ and $w_0 \mapsto_b w_1$ and $w_2 \mapsto_g w_3$ we can conclude $w_0 \mapsto_g w_1 \mapsto_g w_2 \mapsto_g w_3 \mapsto_g w_0$. Now recall that we focused on the case where there are five vertices emitting three red arrows. In total there are twenty-five red arrows so every arrow has to emit at least two red arrows. Now consider the vertex w_0. Because it can only send one red arrow to w_2 and none to v_0, v_1, v_2, w_1 or w_3 it has to send one to v_3, v_4 or v_5. But then w_0 and w_2 would have to be connected by a green arrow making either $\{w_0, w_1, w_2\}$ or $\{w_2, w_3, w_0\}$ an agreeable triple. We have a contradiction again. $\square$

Theorem 6.19. $r(I_2, A_3) \leqslant 10$.

Proof. Suppose towards a contradiction that there is a trichromatic tournament on ten vertices $v_0, \ldots, v_9$ not containing any agreeable triple. By Lemma 6.18 we know that there are three green triples $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$. Since no vertex can emit more than one green arrow and no vertex can absorb more than one green arrow either, we know that v_0 has to be connected to $v_1, \ldots, v_9$ purely by blue and red arrows. In the situation in which no triple is agreeable we know that the targets of three red arrows starting at v_0 have to form a green triple. The same holds for the sources of three red arrows ending at v_0 and the sources of three blue arrows ending at v_0. Moreover we know that v_0 may emit at most one blue arrow. So we may distinguish three cases up to isomorphism:

♢ $v_1, v_2, v_3 \mapsto_b v_0$,
$v_4, v_5, v_6 \mapsto_r v_0$
$v_0 \mapsto_r v_7, v_8$ and $v_0 \mapsto_b v_9$.
In this case $\{v_0, v_7, v_9\}$ is agreeable.

♢ $v_1, v_2, v_3 \mapsto_b v_0$,
$v_0 \mapsto_r v_4, v_5, v_6$
$v_7, v_8 \mapsto_r v_0$ and $v_0 \mapsto_b v_9$.
Here $\{v_0, v_8, v_9\}$ is agreeable.

♢ $v_1, v_2, v_3 \mapsto_r v_0$,
$v_0 \mapsto_r v_4, v_5, v_6$,
$v_7, v_8 \mapsto_b v_0$ (and either $v_9 \mapsto_b v_0$ or $v_0 \mapsto_b v_9$).

We shall now prove two claims:

Claim 6.20. The green triple $\{v_7, v_8, v_9\}$ is connected purely by red arrows to one of the other two green triples.

Proof of Claim 6.20. First note that no red arrow can point from a v_i with $i \in 4 \setminus 1$ to a v_j with $j \in 10 \setminus 7$, otherwise $\{v_0, v_i, v_j\}$ would be agreeable. Similarly, no red arrow may point from a v_i with $i \in 10 \setminus 7$ to a v_j with $j \in 7 \setminus 4$, otherwise $\{v_0, v_i, v_j\}$ would be agreeable. Since at least two blue arrows are leaving the set $\{v_7, v_8, v_9\}$ heading towards v_0 and there can be at most one blue arrow leaving a vertex, we conclude that there can be at most one blue arrow leaving $\{v_7, v_8, v_9\}$ in the direction of any vertex v_i with $i \in 10 \setminus 1$. So we have either that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_4, v_5, v_6\}$ are either incoming red or incoming blue or that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_1, v_2, v_3\}$ are either outgoing red or incoming blue.

Suppose the latter happens. Assume towards a contradiction that $v_i \mapsto_b v_j$ for $i \in 4 \setminus 1$ and $j \in 10 \setminus 7$. Since at most one blue arrow may leave a vertex we know that $v_k \mapsto_r v_i$ for both $k \in 10 \setminus (7 \cup \{j\})$. Now choose $k \in 10 \setminus (7 \cup \{j\})$ such that $v_k \mapsto_g v_j$. Then $\{v_i, v_j, v_k\}$ is agreeable, contradiction!

So assume now that the former happens, i.e. that all arrows between $\{v_7, v_8, v_9\}$ and $\{v_4, v_5, v_6\}$ are either incoming red or incoming blue. Assume towards a contradiction that for some $i \in 7 \setminus 4$ and some $j \in 10 \setminus 7$ we have $v_i \mapsto_b v_j$. Since at most one blue arrow may leave any vertex it follows that $v_i \mapsto_r v_k$ for $k \in 10 \setminus (7 \cup \{j\})$. Choose $k \in 10 \setminus (7 \cup \{j\})$ such that $v_j \mapsto_g v_k$. Then $\{v_i, v_j, v_k\}$ is agreeable, contradiction! ©

Claim 6.21. In both green triples $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ there are at least two vertices connected by a red arrow to a vertex in the other green triple.

Proof of Claim 6.21. Suppose this would fail for $\{v_1, v_2, v_3\}$. Then two vertices, v_i and v_j with $\{i, j\} \in [4 \setminus 1]^2$ were connected to $\{v_4, v_5, v_6\}$ purely by blue arrows. But there are six such connections to be made and since at most one blue arrow may leave a vertex only the usage of five blue arrows is allowed here—contradiction. The proof for the other case works completely analogously. ©

Now we are moving ever closer to a contradiction. By Claim 6.20 we know that the green triple $\{v_7, v_8, v_9\}$ is connected purely by red arrows to one of the other two green triples. Suppose $\{v_1, v_2, v_3\}$ is this triple. Since at most one blue arrow may leave the green triple $\{v_7, v_8, v_9\}$ in the direction of any vertex v_i with $i \in 10 \setminus 1$ and not more than one blue arrow can leave each vertex v_i with $i \in 7 \setminus 4$ we know that five arrows between $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$ have to be red. So in particular at least two vertices in $\{v_4, v_5, v_6\}$ need to have red connections to vertices in the green triple $\{v_7, v_8, v_9\}$. Using Claim 6.21 and a variant of the pigeonhole principle we may choose a vertex v_i with $i \in 7 \setminus 4$ which has a red connection both to a vertex v_j with $j \in 4 \setminus 1$ and to a vertex v_k with $k \in 10 \setminus 7$. Since we supposed the truth of Claim 6.20 to be witnessed by $\{v_1, v_2, v_3\}$ we know that also v_j and v_k are connected by red arrows. If we now recall that all red arrows between $\{v_1, v_2, v_3\}$ and $\{v_7, v_8, v_9\}$ have to point from the latter to the former, we know that in order for $\{v_i, v_j, v_k\}$ to be disagreeable the only possibility left is $v_i \mapsto_r v_k \mapsto_r v_j \mapsto_r v_i$. But then we have $v_j \mapsto_r v_0 \mapsto_r v_i$ and $v_j \mapsto_r v_i$, hence $\{v_0, v_i, v_j\}$ is agreeable—contradiction. If the truth

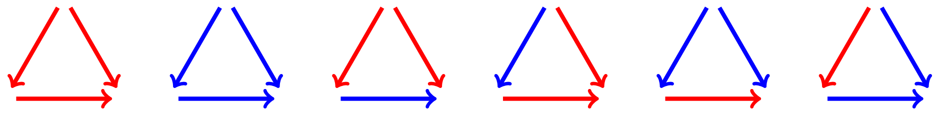

Figure 6.5: The strongly agreeable triples

of Claim 6.20 is witnessed by $\{v_4, v_5, v_6\}$ instead, the proof works totally analogously. This proves the theorem. $\square$

Strong Agreeability

Definition 6.22. A triple is called *strongly agreeable* if and only if it is agreeable and does not contain any green arrow. In other words, it is strongly agreeable precisely if it is among those shown in Figure 6.5.

Theorem 6.23. Let κ be weakly compact and $\lambda \in \kappa \setminus \omega$ be a cardinal. The partition relation $\kappa\lambda\ell \to (\kappa\lambda m, n)$ holds true if and only if every edge-coloured digraph $C = \langle \ell, A, c\rangle$ with $\operatorname{ran}(c) = 2$ contains an independent set of size m or there is a subtournament S of C induced by a set of n vertices such that all triples in S are strongly agreeable.

The following proof of Theorem 6.23 is heavily inspired by [974Ba].

Proof. $\diamondsuit$ Towards a contradiction let us assume that the finite combinatorial characterisation above holds true yet there is neither a 0-homogeneous set of size $\kappa\lambda m$ nor a 1-homogeneous n-tuple. Let furthermore $\chi : [\kappa\lambda\ell]^2 \longrightarrow 2$ be any colouring. W.l.o.g. we may assume that $\chi``(\kappa(\lambda k{+}\alpha{+}1)\setminus\kappa(\lambda k{+}\alpha)) = 1$ for any $\alpha < \lambda$ and $k < \ell$. We define a new colouring ξ as follows:

$$\begin{aligned}\xi\colon [\kappa]^2 &\longrightarrow {}^{[\lambda\ell]^2}4\\ \{\alpha,\beta\}_< &\longmapsto \langle \{\gamma,\delta\}_<, 2\chi(\{\kappa\gamma+\alpha, \kappa\delta+\beta\}) + \chi(\{\kappa\gamma+\beta, \kappa\delta+\alpha\})\rangle\end{aligned}$$

Now we use Fact 6.10 in the form $\kappa \to (\kappa)^2_{2^\lambda}$ thereby finding an $X \in [\kappa]^\kappa$ homogeneous for ξ. Set $\psi := \bigcup \xi``X$. We have $\psi \colon [\lambda\ell]^2 \longrightarrow 4$.

Let $\langle\langle b_j, c_j\rangle \mid j < 2\ell(\ell-1)\rangle$ be an enumeration of $\ell^2 \setminus \Delta$.

We are going to inductively construct for every $i < \ell$ a sequence of sets $\langle Y_i^j \mid j \leqslant 2\ell(\ell-1)\rangle$ by letting $Y_i^0 := \lambda(i+1) \setminus \lambda i$ and for any $j < \ell(\ell-1)$ letting $Y_{b_j}^{2j+1} := B$ and $Y_{c_j}^{2j+1} := C$ if there are sets $B \in [Y_{b_j}^{2j}]^\lambda$, $C \in [Y_{c_j}^{2j}]^\lambda$ with $\overline{\overline{\{y \in C \mid \psi(\{x,y\}) \equiv 1(2)\}}} < \lambda$ for all $x \in B$. If there are no such sets, let $Y_{b_j}^{2j+1} := Y_{b_j}^{2j}$ and $Y_{c_j}^{2j+1} := Y_{c_j}^{2j}$. Similarly, let $Y_{b_j}^{2j+2} := B$ and $Y_{c_j}^{2j+2} := C$ if there are sets $B \in [Y_{b_j}^{2j+1}]^\lambda$, $C \in [Y_{c_j}^{2j+1}]^\lambda$ with $\overline{\overline{\{y \in C \mid \psi(\{x,y\}) \in 4 \setminus 2\}}} < \lambda$ for all $x \in B$. If there are no such sets, let $Y_{b_j}^{2j+2} := Y_{b_j}^{2j+1}$ and $Y_{c_j}^{2j+2} := Y_{c_j}^{2j+1}$. Generally, let $Y_i^{2j+2} := Y_i^{2j}$ for $j < \ell(\ell-1)$ and $i \in l \setminus \{b_j, c_j\}$. Finally, we let $Z_i := Y_i^{2\ell(\ell-1)}$ for all $i < \ell$.

Now we define a digraph $D = \langle \ell, A, c\rangle$ by setting for $i, j < \ell$:

$$i \mapsto_b j \Longrightarrow \exists \alpha \in Z_i : \overline{\overline{\{\beta \in Z_j \mid \psi(\{\alpha,\beta\}) \equiv 1(2)\}}} = \lambda, \tag{6.1}$$

$$\Longleftrightarrow \forall C \in [Z_j]^\lambda : \overline{\overline{\{\eta \in Z_i \mid \overline{\overline{\{\nu \in C \mid \psi(\{\eta,\nu\}) \equiv 1(2)\}}} < \lambda\}}} < \lambda, \tag{6.2}$$

$$i \mapsto_r j \Longrightarrow \exists \alpha \in Z_i : \overline{\overline{\{\beta \in Z_j \mid \psi(\{\alpha,\beta\}) \in 4 \setminus 2\}}} = \lambda, \tag{6.3}$$

$$\Longleftrightarrow \forall C \in [Z_j]^\lambda \overline{\overline{\{\eta \in Z_i \mid \overline{\overline{\{\nu \in C \mid \psi(\{\eta,\nu\}) \in 4 \setminus 2\}}} < \lambda\}}} < \lambda. \tag{6.4}$$

We have $\Rightarrow$ instead of $\Leftrightarrow$ here for a reason. This definition neither necessarily tells us the direction of an arrow—since i and j could change roles—nor does it determine its colour uniquely—as we have $3 \in \{n \in 4 \setminus 2 \mid n \equiv 1(2)\}$. We only need to follow the principle that whenever these conditions allow us to set an arrow we do so. The equivalences hold because of the definition of the Z_i's. We will use the negations of (6.1) and (6.3) in the case where we define the 0-homogeneous set and (6.2) and (6.4) in the case in which we define the 1-homogeneous set.

After having thus defined the digraph we may employ our finitary hypothesis and distinguish two cases:

- There is $N \in [\ell]^n$ such that all elements of $[N]^3$ are strongly agreeable. We are going to inductively construct a 1-homogeneous set H. Since no strongly agreeable triple is cyclic we may choose an enumeration $e : n \longleftrightarrow N$ in such a way that $i < j$ if and only if $e(i) \mapsto e(j)$. We proceed by induction. Using (6.2) and (6.4) we may choose an $\alpha_i \in Z_{e(i)}$ such that

  - ◇ $\forall j < i (e(i) \mapsto_b e(j) \to \psi(\{\alpha_j, \alpha_i) \equiv 1(2))$,
  - ◇ $\forall j < i (e(i) \mapsto_r e(j) \to \psi(\{\alpha_j, \alpha_i) \in 4 \setminus 2)$,
  - ◇ $\forall j \in n \setminus (i+1)(e(i) \mapsto_b e(j) \to \overline{\overline{\{\beta \in Z_{e(j)} \mid \psi(\{\alpha_0, \beta\}) \equiv 1(2)\}}} = \lambda)$,
  - ◇ $\forall j \in n \setminus (i+1)(e(i) \mapsto_r e(j) \to \overline{\overline{\{\beta \in Z_{e(j)} \mid \psi(\{\alpha_0, \beta\}) \in 4 \setminus 2\}}} = \lambda)$.

  Now choose this and let $S \in [\kappa]^{2^n}$ be such that $\forall \eta \in S\, \forall \nu \in \eta \cap S\, \exists \zeta \in X \cap \eta \setminus \nu$. Let $\langle s_\eta \mid \eta < 2^n \rangle$ be the increasing enumeration of S and let $e : n \longleftrightarrow N$. Now let $k_{-2} := 0$, $k_{-1} := 2^n$ and $k_0 := 2^{n-1}$. By induction we choose $\gamma_i \in X \cap s_{k_i} \setminus s_{k_i - 1}$ and k_{i+1} such that the following conditions are met.

  - ◇ If $e(j) \mapsto_b e(i+1)$ then $k_{i+1} < k_j$.
  - ◇ If $e(j) \mapsto_r e(i+1)$ then $k_{i+1} > k_j$.
  - ◇ $|k_{i+1} - k_j| \geqslant 2^{n-i-1}$ for every $j \leqslant i$.

  This we do for every $i < n$. Note that we are always able to choose the parameters in this way because all elements of $[N]^3$ are strongly agreeable. Finally, we may define $H := \{\kappa\alpha_i + \gamma_i \mid i < n\}$. H is 1-homogeneous.

- There is an independent $M \in [\ell]^m$. There are again two cases to distinguish:

  - ◇ λ is regular. This case is easy. Let $Y_0 := \emptyset$ and for limit ordinals ν let $Y_\nu := \bigcup_{\eta < \nu} X_\eta$. For any $\eta < \lambda$, $i \in M$ and an enumeration $f : m \longleftrightarrow M$ let

    (6.5)
    $$Y_{m\eta+i+1} := Y_{m\eta+i} \cup \{\min(\{\nu \in Z_{f(i)} \mid \forall \zeta \in Y_{m\eta+i} : \psi(\{\nu, \zeta\}) = 0\})\}.$$

    At the end, let $Y := Y_\lambda$. M is independent so the formulas on the right sides of (6.1) and (6.3) are jointly negated.

The regularity of λ now ensures that the minimum referred to in (6.5) always exists.

◇ λ is singular. Here we have to be a bit more careful. We have to ensure that we do not spoil our chances of defining a homogeneous set before our inductive construction is finished. So we thin out the Z_i's further as follows:
For each $i \in M$ let $\eta_i \in \lambda cf.\ (\lambda) + 1 \setminus \lambda$ be the least ordinal such that there exists an enumeration $e_i : \eta_i \longleftrightarrow Z_i$ such that

$$\nu \longmapsto \max(\{\overline{\{\zeta \in Z_j \mid \psi(\{e_i(\nu), \zeta\}) > 0\}} \mid j \in M \setminus \{i\}\})$$

is nondecreasing. Fix such η_i and e_i. Similar to the case before we set $Y_0 := \emptyset$, for limit ordinals ν we let $Y_\nu := \bigcup_{\zeta<\nu} X_\zeta$ and for $\nu < \lambda$, $i \in M$ and an enumeration $f : m \longleftrightarrow M$, we define

$$Y_{m\nu+i+1} := Y_{m\nu+i} \cup \{e_{f(i)}(\min(\{\zeta < \lambda \mid \forall \rho \in Y_{m\nu+i} : \psi(\{e_{f(i)}(\zeta), \rho\}) = 0\}))\}. \tag{6.6}$$

Once more, set $Y := Y_\lambda$. Again, the independence of M implies that the right sides of (6.1) and (6.3) are jointly negated. By rearranging the Z_i's using the e_i's we ensured that the minimum in (6.6) always exists.

Let $\langle X_\eta \mid \eta < \lambda\ell\rangle$ be a partition of X into λ sets of size κ. We set $H := \{\kappa\alpha + \gamma \mid \alpha \in Y \wedge \gamma \in X_\alpha\}$. A little contemplation shows that H is 0-homogeneous for χ.

◇ Suppose that the finite combinatorial characterisation above fails, i.e. there is an edge-2-coloured digraph $D = \langle \ell, A, c\rangle$ neither containing an independent set of size m nor a set of size n only inducing strongly agreeable triples. We define a colouring $\chi : [\kappa\lambda\ell]^2 \longrightarrow 2$ as follows.

$$\chi : [\kappa\lambda\ell]^2 \longrightarrow 2,$$

$$\{\kappa(\lambda j+\alpha)+\gamma, \kappa(\lambda k+\beta)+\delta\}_< \longmapsto \begin{cases} 1 \text{ if } j \mapsto_b k \wedge \alpha < \beta \wedge \delta < \gamma, \\ 1 \text{ if } k \mapsto_b j \wedge \beta < \alpha \wedge \gamma < \delta, \\ 1 \text{ if } j \mapsto_r k \wedge \alpha < \beta \wedge \gamma < \delta, \\ 1 \text{ if } k \mapsto_b j \wedge \beta < \alpha \wedge \delta < \gamma, \\ 0 \text{ else.} \end{cases}$$

- Suppose that χ would admit a 0-homogeneous set $X \in [\kappa\lambda\ell]^{\kappa\lambda m}$. Then there would have to be a set $Y \in [\ell]^m$ such that $\operatorname{otyp}(X \cap \kappa\lambda(k+1) \setminus \kappa\lambda k) = \kappa\lambda$ for any $k \in Y$. Since D contains no independent set of size m we may choose $\{j,k\} \in [Y]^2$ with $j \mapsto k$. Let $\alpha := \min\{\eta < \lambda \mid \operatorname{otyp}(X \cap \kappa(\lambda j+\eta+1) \setminus \kappa(\lambda j+\eta)) = \kappa\}$ and let $\beta := \min\{\eta \in \lambda \setminus (\alpha+1) \mid \operatorname{otyp}(X \cap \kappa(\lambda k+\eta+1) \setminus \kappa(\lambda k+\eta) = \kappa\}$.

  We distinguish two cases:

  ◇ $c(\{j,k\}) = 0$. Let $\gamma := \min\{\eta < \kappa \mid \kappa(\lambda k+\beta)+\eta \in X\}$ and let $\delta := \min\{\eta \in \kappa \setminus (\gamma+1) \mid \kappa(\lambda j+\alpha)+\eta \in X\}$. Obviously, $\chi(\{\kappa(\lambda j+\alpha)+\delta, \kappa(\lambda k+\beta)+\gamma\}) = 1$ contradicting the 0-homogeneity of X.

  ◇ $c(\{j,k\}) = 1$. Let $\gamma := \min\{\eta < \kappa \mid \kappa(\lambda j+\alpha)+\eta \in X\}$ and let $\delta := \min\{\eta \in \kappa \setminus (\gamma+1) \mid \kappa(\lambda k+\beta)+\eta \in X\}$. Obviously, $\chi(\{\kappa(\lambda j+\alpha)+\gamma, \kappa(\lambda k+\beta)+\delta\}) = 1$ contradicting the 0-homogeneity of X.

- Now suppose that χ would admit a 1-homogeneous n-tuple $X \in [\kappa\lambda\ell]^n$. By definition of χ we have that $\{\kappa(\lambda j+\alpha)+\gamma, \kappa(\lambda k+\beta)+\delta\} \in [X]^2$ for $j,k < \ell$; $\alpha,\beta < \lambda$ and $\gamma,\delta < \kappa$ implies $j \neq k$; $\alpha \neq \beta$ and $\gamma \neq \delta$. Consider $Y := \{k < \ell \mid \exists \eta < \kappa\lambda : \kappa\lambda + \eta \in X\}$. We have $Y \in [\ell]^n$ and in fact Y induces a subdigraph S of D. W.l.o.g. we may assume that S is a tournament because if $\{j,k\} \in [Y]^2$ is independent we have that $X \cap ((\kappa\lambda(j+1) \setminus \kappa\lambda j) \cup (\kappa\lambda(k+1) \setminus \kappa\lambda k) \subset X$ is 0-homogeneous.

  So Y is a tournament and contains a triple $\{i,j,k\} \in [Y]^3$ which is not strongly agreeable. By definition of Y we may conclude that there are $\alpha,\beta,\gamma < \lambda$ and $\delta,\eta,\zeta < \kappa$ such that $\kappa(\lambda i+\alpha)+\delta, \kappa(\lambda j+\beta)+\eta, \kappa(\lambda k+\gamma)+\zeta \in X$. We may distinguish three cases:

- $\diamond$ It is a 3-cycle. Assume $i \mapsto j \mapsto k \mapsto i$. Since X is 1-homogeneous we may, by definition of χ, conclude that $\alpha < \beta$, $\beta < \gamma$ and $\gamma < \alpha$. Contradiction!
- $\diamond$ We have $i \mapsto_b j$, $j \mapsto_b k$ and $i \mapsto_r k$. By 1-homogeneity of X and per definitionem of χ we may conclude that $\delta > \eta$, $\eta > \zeta$ and $\delta < \zeta$. Contradiction!
- $\diamond$ We have $i \mapsto_r j$, $j \mapsto_r k$ and $i \mapsto_b k$. Again we may use the 1-homogeneity of X and the definition of χ to conclude that $\delta < \eta$, $\eta < \zeta$ and $\delta > \zeta$. Contradiction!

Note that modulo a renaming of i, j and k these three cases exhaust all possibilities.

This concludes the second part of the proof. $\square$

By S_n we denote the class of graphs G on n vertices such that all triples in G are strongly agreeable and by $r(I_m, S_n)$ the smallest natural number ℓ such that all digraphs on ℓ vertices either contain an independent set of size m or an induced subgraph on n vertices all triples of which are strongly agreeable. As before, we may use this notation to condense the equivalences proved above into a single formula:

$$r(\kappa\lambda m, n) = \kappa\lambda r(I_m, S_n) \text{ for } \kappa \text{ weakly compact and any cardinal } \lambda \in \kappa \setminus \omega.$$

Analogously to [989Ba], we can prove another pretty theorem.

Theorem 6.24 ($\mathsf{MA}_{\aleph_1}$)**.** $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$.

Let $\chi\colon [\omega_1\omega\ell]^2 \longrightarrow 2$ be any colouring. We define $E := \{P \in [\omega_1\omega\ell]^2 \mid \chi(P) = 1\}$.

Let $Z := \bigcup_{\alpha<\Omega}[\omega_1(\alpha+1) \setminus \omega_1\alpha]^{\aleph_1}$.

For $X \in Z$ let $\alpha_X < \Omega$ be such that $X \in [\omega_1(\alpha_X+1) \setminus \omega_1\alpha_X]^{\aleph_1}$. Let $A_X := \{\xi < \omega_1 \mid \omega_1\alpha_X + \xi \in X\}$. We say that the pair $\langle X, Y\rangle$ is

- $\diamondsuit$ in constellation *blue* if and only if $X, Y \in Z$ and

$$\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists \gamma \in A_C, \beta \in A_B \setminus (\gamma+1)\colon \{\omega_1\alpha_X+\beta, \omega_1\alpha_Y+\gamma\} \in E.$$

◇ in constellation *red* if and only if $X, Y \in Z$ and

$$\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists \beta \in A_B, \gamma \in A_C \setminus (\beta+1)\colon \{\omega_1\alpha_X+\beta, \omega_1\alpha_Y+\gamma\} \in E.$$

◇ *free* if and only if $\forall B \in [X]^{\aleph_1}, C \in [Y]^{\aleph_1} \exists D \in [B]^{\aleph_1}, F \in [C]^{\aleph_1} \forall \delta \in D, \eta \in F\colon \{\delta, \eta\} \notin E$.

Let us call a set $S \subset \Omega$ weakly independent if for all $\alpha, \beta < \Omega$ with $X = S \cap \omega_1(\alpha+1) \setminus \omega_1\alpha$ and $Y = S \cap \omega_1(\beta+1) \setminus \omega_1\beta$ and $\overline{\overline{X}} = \overline{\overline{Y}} = \aleph_1$ the pair $\langle X, Y \rangle$ is free.

For the proof we are going to use Theorem 3.1 from [989Ba]. By limiting its scope to the case $\alpha = \omega$ we get the following theorem.

Theorem 6.25. Assume $\mathsf{MA}_{\aleph_1}$. Let $E \subset [\omega_1\omega]^2$ be a graph and let $W \in [\omega_1\omega]^{\omega_1\omega}$ be weakly independent. Then there exists $H \in [W]^{\omega_1\omega}$ which is independent such that $\forall n < \omega\colon \overline{\overline{H \cap \omega_1(n+1) \setminus \omega_1 n}} \in \{0, \aleph_1\}$ and $H \cap \omega_1(n+1) \setminus \omega_1 n \supsetneq \emptyset$ implies $\overline{\overline{W \cap \omega_1(n+1) \setminus \omega_1 n}} \leqslant \aleph_0$ for all natural n.

Proof of Theorem 6.24. Even without $\mathsf{MA}_{\aleph_1}$ it is easy to see that $r(\omega_1\omega m, n) \geqslant \omega_1\omega r(I_m, S_n)$. Let D be a 2-coloured digraph showing $\ell < r(I_m, S_n)$. Now consider the following colouring:

$$\chi : [\omega_1\omega\ell]^2 \longrightarrow 2,$$
$$\{\omega_1\omega h + \omega_1 j + \alpha, \omega_1\omega i + \omega_1 k + \beta\}_< \longmapsto \begin{cases} 1 \text{ if } h \mapsto_b i \wedge j < k \wedge \alpha < \beta, \\ 1 \text{ if } i \mapsto_b h \wedge k < j \wedge \beta < \alpha, \\ 1 \text{ if } h \mapsto_r i \wedge j < k \wedge \beta < \alpha, \\ 1 \text{ if } i \mapsto_r h \wedge k < j \wedge \alpha < \beta, \\ 0 \text{ else.} \end{cases}$$

Analogously to the proof of Theorem 6.23 this now shows $\omega_1\omega\ell \not\rightarrow (\omega_1\omega m, n)$.

Now assume $\mathsf{MA}_{\aleph_1}$. Let $\ell := r(I_m, S_n)$. We are going to show $\omega_1\omega\ell \rightarrow (\omega_1\omega m, n)$ by adapting the method of [989Ba] for our purposes.

Note that if a pair $\langle X, Y\rangle$ is not free there are $X' \in [X]^{\aleph_1}$ and $Y' \in [Y]^{\aleph_1}$ such that $\langle X', Y'\rangle$ is in one of the two other constellations. If at least one of these three properties applies to a pair then we call it *decided*.

We are choosing ℓ-tuples $\langle X_0^F, \ldots, X_{\ell-1}^F\rangle$ with $F \in [\omega]^{<\omega}$ by recursion on $\max(F)$.

◇ If $\overline{\overline{F}} = 1$ and $i < \ell$ let $X_i^F \in [\omega_1(\omega i + \bigcup F + 1) \setminus \omega_1(\omega i + \bigcup F)]^{\aleph_1}$ such that for all $G \in [\bigcup F]^{<\omega}$ and $j \in \ell \setminus \{i\}$ either $\langle X_j^G, X_i^F\rangle$ is free or there is an $X \in [X_j^G]^{\aleph_1}$ such that $\langle X, X_i^F\rangle$ is in one of the other two constellations above. This can be achieved by thinning out repeatedly.

◇ If $\overline{\overline{F}} \in \omega \setminus 2$ let $\mu := \max(F)$ and $G := F \setminus \{\mu\}$. Again by thinning out repeatedly for all $i < \ell$ choose $X_i^F \subset X_i^G$ such that for all $j \in \ell \setminus \{i\}$ we have that $\langle X_i^F, X_j^{\{\mu\}}\rangle$ is decided.

Claim 6.26. For every natural number f there is an $F \in [\omega]^f$ and an $H \in [\omega]^\omega$ such that for all $g, h < \ell$ the facts, both whether or not for $x, y \in [H]^i$ with $i \in F$ and $\max(x) < \min(y)$ we have that $\langle X_g^x, X_h^y\rangle$ is free or not free and what the constellation of a pair $\langle X_g^x, X_h^y\rangle$ with $\min(y) \in x$ and $x, y \in [H]^i$ is, only depend on g and h.

Proof of Claim 6.26. Using Ramsey's Theorem, we thin ω to some $H_0 \in [\omega]^\omega$ on which the freeness of some such $\langle X_g^x, Y_h^y\rangle$ for $x, y \in [H_0]^1$ only depends on $\{g, h\}$ and whether $\max(x) < \min(y)$ or $\max(y) < \min(x)$.

Because of the nature of our recursive construction the constellation of a pair $\langle X_g^x, X_h^y\rangle$ with $\min(y) \in x$ is determined by the constellation of the pair $\langle X_g^{x\cap(\min(y)+1)}, X_h^{\{\min(y)\}}\rangle$. The constellation of this pair, however, can be made canonical as well using Ramsey's Theorem. So inductively do this for $x \cap \min(y) = i$ to arrive at some H_i' for $i < \omega$ and after that find an $H_{i+1} \in [H_i']^\omega$ on which the freeness/non-freeness for $x, y \in [H_{i+1}]^{i+1}$ with $\max(x) < \min(y)$ is canonical, i.e. only depends on g and h. This yields a sequence $H_0 \supset H_0' \supset H_1 \supset H_1' \supset \ldots$

Since there are only finitely many possible patterns of both freeness/non-freeness and constellations we may find an $F \in [\omega]^f$ and an H in the sequence which is sufficiently thin such that these patterns are the same for all $i \in F$. ©

Use the claim to find F of a sufficiently (for what follows) large finite size $2L$, say $2^{n+1}-2$.

Now we define a digraph by letting there be arrows as follows:

$g \mapsto_b h$ implies that $\langle X_g^x, X_h^{\{\max(x)\}}\rangle$ generally is in constellation *blue*,
$g \mapsto_r h$ implies that $\langle X_g^x, X_h^{\{\max(x)\}}\rangle$ generally is in constellation *red*.

Moreover, whenever one of these conditions applies an arrow is set.

Now first suppose there were an independent set $I \in [\ell]^m$.

Let $\langle h_i \mid i < \omega\rangle$ be the ascending enumeration of H and pick some $f \in F$. Let $Z := \{\{h_{fi}, \dots, h_{f(i+1)-1}\} \mid i < \omega\}$. Then the union of all the X_g^x's for $g \in I$ and $x \in Z$ is a weakly independent set of size $\omega_1\omega m$. This is because by our recursive construction and by definition of our digraph $\langle X_g^{x\setminus\{\max(x)\}}, X_h^y\rangle$ is free whenever $\max(x) \in y$ and there is no arrow between g and h . We now may use Theorem 6.25 to thin this out to some truly independent set of the same order-type, i.e. some set of order-type $\omega_1\omega m$ which is homogeneous for χ in colour 0.

Now suppose that our digraph contains no independent set of size m. Since we chose ℓ to be $r(I_m, S_n)$ we know that there is a strongly agreeable set $S \in [\ell]^n$.

Let $\langle f_i \mid i < L\rangle$ be an ascending enumeration of F. Define $a_i := \{h_{f_i}, \dots, h_{f_L}\}$ and let $A_g^i := X_g^{a_i}$.

Claim 6.27. Let $g, h \in S$, $\{i,j\} \in [L]^2$ and $i < j$. Now if $g \mapsto_b h$ then $\langle A_g^i, A_h^j\rangle$ is in constellation blue and if $g \mapsto_r h$ then it is in constellation red.

Proof of Claim 6.27. Consider $x := \{h_{f_i}, \dots, h_{f_{j-1}}\}$ and $y := \{h_{f_j}, \dots, h_{f_{2j-i-1}}\}$.

We know that $\langle X_g^x, X_h^y\rangle$ is not free. Because of $X_g^y \subset X_h^{\{h_{f_j}\}}$ the pair $\langle X_g^x, X^{\{h_{f_j}\}}\rangle$ is not free. By our recursive construction we get that $\langle X_g^{x\cup\{h_{f_j}\}}, X_h^{\{h_{f_j}\}}\rangle$ is decided. The colour of the arrow between g and h in S tells us how. Since $A_h^i \subset X_h^{x\cup\{h_{f_j}\}}$ and $A_g^j \subset X_g^{\{h_{f_j}\}}$ the claim follows. ©

Now we are going to construct the complete graph $G \in [\omega_1\omega\ell]^n$ with $[G]^2 \subset E$. To this end let $\langle s_i \mid i < n\rangle$ be an enumeration of S such that

$s_i \mapsto_r s_j$ or $s_j \mapsto_b s_i$ for all $i < j < n$. This is possible since all triples in S are strongly agreeable. To start the construction, i.e. in step 0, choose the middle element j_0 of L. This divides L into two intervals of size, say, $2^{n-1} - 1$. Let us call $\{-1, L\}$ *the border.*

Inductively, in step $i < n$ we haven chosen j_k's with $k < i$ such that the minimal distance between either of them and between one of them and the border is $2^{n-i} - 1$. Now we choose an j_{i+1} such that the minimal distance between two of the j_{k+1}'s for $k \leqslant i$ and between of them and the border is $2^{n-i-1} - 1$ and such that $j_{i+1} > j_k$ if and only if $s_k \mapsto s_i$ for all $k \leqslant i$. Again, strong agreeability allows us to do this.

Finally, we choose an ordinal $\alpha_0 \in A^{j_0}_{s_0}$ in step 0 such that for all $i \in n \setminus 1$ we have $\overline{\overline{\{\alpha \in A^{j_i}_{s_i} \mid \{\alpha_0, \alpha\} \in E\}}} = \aleph_1$. Such an ordinal exists for otherwise there would be an $i < n$ and uncountably many $\alpha \in A^{j_0}_{s_0}$ which are only connected to countably many things in $A^{j_i}_{s_i}$ by E. Because of the way in which we sorted the elements of S this would contradict the constellation of $\langle A^{j_0}_{s_0}, A^{j_i}_{s_i} \rangle$. Now thin out the $A^{j_i}_{s_i}$ for $i \in n \setminus 1$ to those uncountable subsets of ordinals the unordered pairs with α_0 of which are in E. Generally, in step $i < n$ choose an ordinal $\alpha_i \in A^{j_i}_{s_i}$ such that $\{\alpha_k, \alpha_i\} \in E$ for all $k < i$ and thin out the $A^{j_k}_{s_k}$'s for $k \in n \setminus (i + 1)$. In this way we may construct our complete graph of size n, i.e. our set homogeneous for χ in colour 1. $\square$

Now we may take a look at the easiest examples.

Theorem 6.28. $r(I_2, S_3) = 6$.

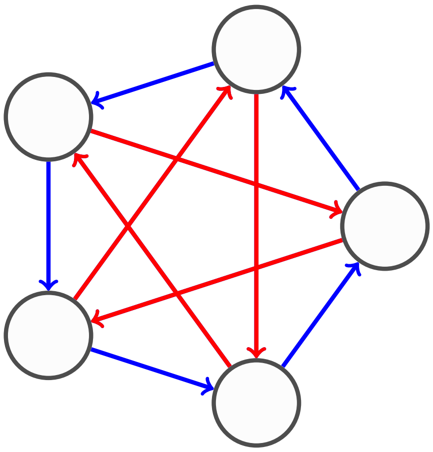

Figure 6.6: An edge-coloured tournament showing $r(I_2, S_3) > 5$

Proof. The lower bound is provided by the counterexample in Figure 6.6, for the upper bound one can argue as follows:

Consider any vertex v in a six-element tournament—if the digraph is not a tournament we are finished immediately. Now $N_B^-(v) + N_B^+(v) + N_R^-(v) + N_R^+(v) \geqslant 5$ so we may conclude, using the pigeonhole principle, that one of these four neighbourhoods contains at least two elements. Then v together with this neighbourhood is strongly agreeable. □

Theorem 6.29. $r(I_3, S_3) = 15$.

Proof.

The lower bound is given by the counterexample in Figure 6.7. It can be viewed as a digraph on $\mathbb{Z}_{14}$ where $v \mapsto_b v + 1$, $v \mapsto_r v + 2$, $v \mapsto_r v - 3$ and $v \mapsto_b v - 4$ for all $v \in \mathbb{Z}_{14}$.

The upper bound is established by Theorem 7.2 below. □

Corollary 6.30.
Let κ be the least weakly compact cardinal. Then $r(\kappa\aleph_{\varepsilon_{\omega 7+9}}3, 3) = \kappa\aleph_{\varepsilon_{\omega 7+9}}15$.

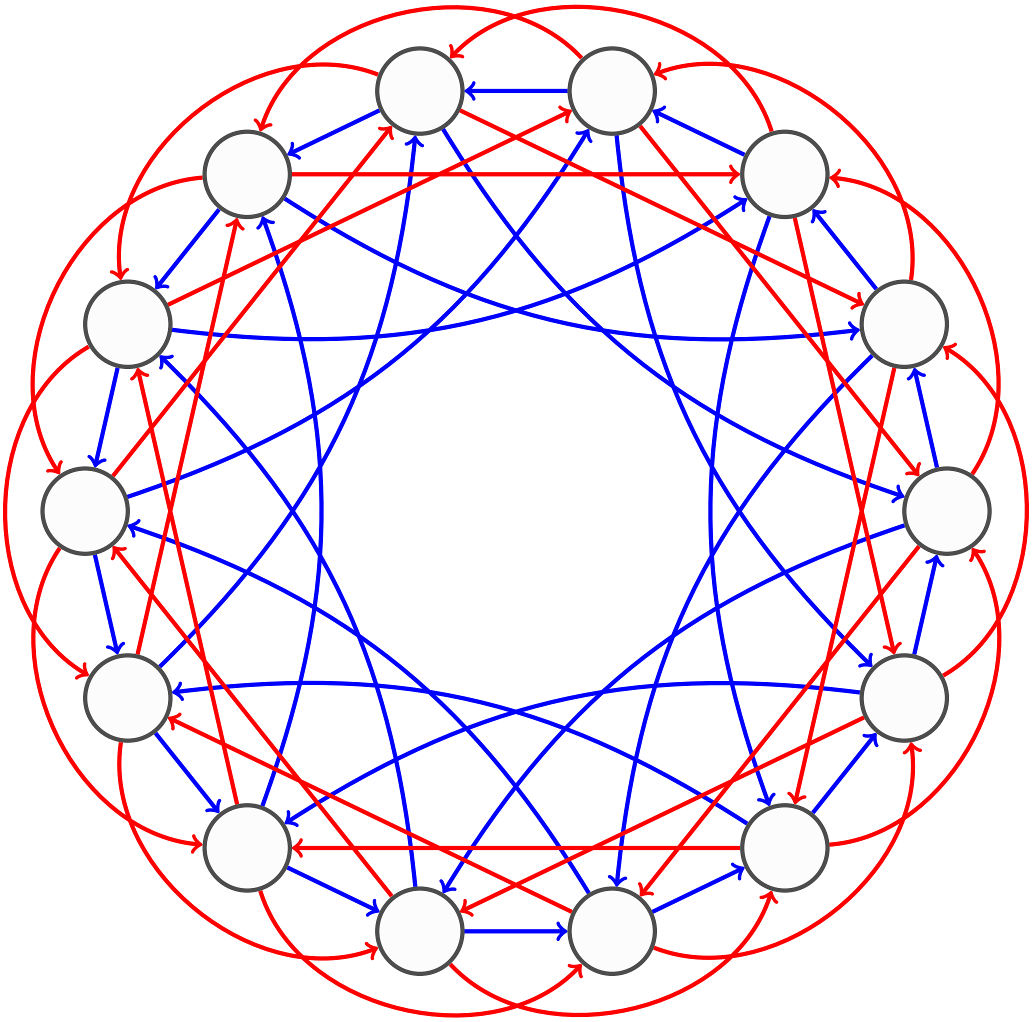

Figure 6.7: An edge-coloured digraph showing $r(I_3, S_3) > 14$

7

General Bounds

> . . . it is true that a mathematician who is not somewhat of a poet, will never be a perfect mathematician.
>
> — Karl Theodor Wilhelm Weierstraß

Upper Bounds for Independent versus Strongly Agreeable Sets

We now are going to provide several general upper bounds for various Ramsey numbers. We start with something easy—an upper bound for the Ramsey numbers $r(I_m, S_3)$.

Lemma 7.1. $r(I_{m+1}, S_3) \leqslant r(I_m, S_3) + 4m + 1$ for all $m \in \omega \setminus 2$.

Proof. Let D be any edge-dichromatic digraph on $r(I_m, S_3)+4m+1$ vertices. Fix any vertex v. Since $\overline{\overline{N_B^+(v)}}+\overline{\overline{N_B^-(v)}}+\overline{\overline{N_R^+(v)}}+\overline{\overline{N_R^-(v)}}+\overline{\overline{I(v)}} = r(I_m, S_3)+4m$ it follows that one of the following cases has to obtain:

◇ $\overline{\overline{N_B^+(v)}} \geqslant m+1$.

◇ $\overline{\overline{N_B^-(v)}} \geqslant m+1$.

◇ $\overline{\overline{N_R^+(v)}} \geqslant m+1$.

◇ $\overline{\overline{N_R^-(v)}} \geqslant m+1$.

◇ $\overline{\overline{I(v)}} \geqslant r(I_m, S_3)$.

In the first four cases either the respective neighbourhood is independent or two vertices in it are connected by an arrow in which case they form an S_3 together with v. In the last case the neighbourhood either contains an S_3 or an independent set of size m which together with v would form an independent set of size $m+1$. □

Theorem 7.2. For all $m \in \omega \setminus 2$ we have $r(I_m, S_3) \leqslant m(2m-1)$.

Proof. The proof proceeds by induction. The base case is established by Theorem 6.28. For the inductive step by Lemma 7.1 we only have to check that

$$(m+1)(2(m+1)-1) = m(2m-1)+4m+1,$$

which is true. □

We continue now by providing an upper bound for $r(I_2, S_n)$.

Lemma 7.3. For all $n \in \omega \setminus 3$ we have $r(I_2, S_{n+1}) \leqslant 4r(I_2, S_n) - 2$.

Proof. Let $T = \langle V, A, c\rangle$ be a bicoloured tournament on $4r(I_2, S_n) - 2$ vertices. Fix any vertex $v \in V$. Since $\overline{\overline{N_B^-(v) + N_B^+(v) + N_R^-(v) + N_R^+(v)}} = 4r(I_2, S_n) - 3$ one of these neighbourhoods N has cardinality at least $r(I_2, S_n)$ by the pigeonhole principle. Hence there is an $S \in [N]^n$ in which all triples are strongly agreeable. So if we set $X := S \cup \{v\}$, then $X \in [V]^{n+1}$ and all triples in X are strongly agreeable. □

Theorem 7.4. For any $n \in \omega \setminus 3$ we have

$$r(I_2, S_n) \leqslant \frac{4^{n-1} + 2}{3}.$$

Proof. By induction. The base case is given by Theorem 6.28. The induction step also works:

$$4 \cdot \frac{4^{n-1} + 2}{3} - 2 = \frac{4^{n-1+1} + 4 \cdot 2 - 2 \cdot 3}{3} = \frac{4^{n+1-1} + 2(4-3)}{3}.$$ □

Again, note that if n is an integer, $\frac{4^{n-1}+2}{3}$ is an whole number as well. It is easy to see by induction that $4^{n-1} \equiv 1(3)$ for any $n \in \omega \setminus 1$. Hence $4^{n-1} + 2 \equiv 0(3)$ for any $n \in \omega \setminus 1$.

Now we want to give a general upper bound for $r(I_m, S_n)$.

Lemma 7.5. For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$:

$$r(I_{m+1}, S_{n+1}) \leqslant r(I_m, S_{n+1}) + 4r(I_{m+1}, S_n) - 3$$

Proof. Suppose that there is a digraph D on $r(I_m, S_{n+1}) + 4r(I_{m+1}, S_n) - 3$ vertices neither containing an I_{m+1} nor an S_{n+1}. Fix any vertex $v \in D$. Then $\overline{\overline{N_B^-(v)}} \geqslant r(I_{m+1}, S_n)$, $\overline{\overline{N_B^+(v)}} \geqslant r(I_{m+1}, S_n)$, $\overline{\overline{N_R^-(v)}} \geqslant r(I_{m+1}, S_n)$, $\overline{\overline{N_R^+(v)}} \geqslant r(I_{m+1}, S_n)$ or $\overline{\overline{I(v)}} \geqslant r(I_m, S_{n+1})$. In the first four cases, the neighbourhood in question contains itself an I_{m+1} or an S_n which forms an S_{n+1} together with v. In the last case, either $I(v)$ contains itself an S_{n+1} or an I_m which forms an I_{m+1} together with v. □

Theorem 7.6. For all $m \in \omega \backslash 2$ and all $n \in \omega \backslash 3$ we have $r(I_m, S_n) \leqslant u(m,n)$ where

$$u(m,n) := \frac{1}{4}\Big(3 + \sum_{i=0}^{n-1} \binom{i+m-2}{i} 4^i\Big). \tag{7.1}$$

Proof. By induction. First, we verify that $u(2,3) = 6$. Recall that we have $r(I_2, S_3) = 6$ by Theorem 6.28. Next we check that $u(2,n)$ agrees with the formula from Theorem 7.4. We have

$$u(2,n) = \frac{1}{4}\big(3 + \sum_{i=0}^{n-1} 4^i\big).$$

Consider $v(n) := u(2,n+1) - u(2,n)$. Clearly, $v(n) = 4^{n-1}$. It is also easy to see that

$$\frac{4^n + 2}{3} - \frac{4^{n-1}+2}{3} = 4^{n-1}.$$

Induction on n provides the desired result.

It is even easier to see, using well-known properties of the binomial coefficients, that $u(m,3) = m(2m-1)$.

Now we can check that in fact

$$u(m+1,n+1) = u(m,n+1) + 4u(m+1,n) - 3. \tag{7.2}$$

Then with Lemma 7.5 the theorem will follow. The following calculation proves (7.2):

$$\begin{aligned}
& u(m, n+1) + 4u(m+1, n) - 3 \\
= \; & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-2}{i} 4^i + 4 \sum_{i=0}^{n-1} \binom{i+m-1}{i} 4^i\Big) \\
= \; & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-2}{i} 4^i + \sum_{i=1}^{n} \binom{i+m-2}{i-1} 4^i\Big) \\
= \; & \frac{1}{4}\Big(4 + \sum_{i=1}^{n} \Big(\binom{i+m-2}{i} + \binom{i+m-2}{i-1}\Big) 4^i\Big) \\
= \; & \frac{1}{4}\Big(4 + \sum_{i=1}^{n} \binom{i+m-1}{i} 4^i\Big) \\
= \; & \frac{1}{4}\Big(3 + \sum_{i=0}^{n} \binom{i+m-1}{i} 4^i\Big) \\
= \; & u(m+1, n+1).
\end{aligned}$$

□

Upper Bounds for Independent versus Agreeable Sets

We are now going to provide an upper bound for the Ramsey numbers $r(I_n, A_3)$.

Lemma 7.7. $r(I_{n+1}, A_3) \leqslant r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1$ for all $n \in \omega \setminus 2$.

Proof. Let there be an edge-trichromatic digraph $\langle V, A\rangle$ with $r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1$ vertices. Pick any vertex v such that $\overline{\overline{N_B^+(v)}}$ is maximal. Note that $\overline{\overline{V \setminus \{v\}}} = r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 2$. Now we may distinguish several cases. Note that necessarily at least one of them must hold!

◊ $\overline{\overline{N_B^-(v)}} \geqslant n + 1$. Since we chose v to have maximum blue out-degree we might immediately proceed with the next case.

◇ $\overline{\overline{N_B^+(v)}} \geqslant n+1$. In this case there are either two vertices $w, x \in N_B^+(v)$ which are connected by an arrow—in which case $\{v, w, x\}$ is an agreeable triple—or $N_B^+(v)$ forms an independent set of size $n+1$. In both cases we have found a set homogeneous in the sought-after-sense.

◇ $\overline{\overline{N_R^-(v)}} \geqslant r(I_{n+1}, L_3)$. If there are $w, x \in N_R^-(v)$ which are connected by either a blue or a red arrow then $\{v, w, x\}$ is an agreeable triple. So let us assume that all pairs of vertices in $N_R^-(v)$ are either connected by green arrows or not connected at all. Our case condition implies that we either have an independent set of size $n+1$ or a transitive green triple—which is agreeable.

◇ $\overline{\overline{N_R^+(v)}} \geqslant r(I_{n+1}, L_3)$. Works exactly like the case before.

◇ $\overline{\overline{N_G^-(v)}} \geqslant n+1$. This and the next case work like the second. There are either two vertices $w, x \in N_G^-(v)$ which are connected by an arrow—then $\{v, w, x\}$ is agreeable—or $N_G^-(v)$ is an independent set of size $n+1$.

◇ $\overline{\overline{N_G^+(v)}} \geqslant n+1$. Again, either there are two vertices which are connected thus forming an agreeable triple together with v or $N_G^+(v)$ is independent.

◇ $\overline{\overline{I(v)}} \geqslant r(I_n, A_3)$. Now, in the last case $I(v)$ either contains an agreeable triple or is itself independent of size n. In the first case we are finished immediately and in the second $I(v) \cup \{v\}$ is an independent set of size $n+1$. □

Corollary 7.8. $r(\omega^2(m+1), 3) \leqslant r(\omega^2 m, 3) + \omega r(\omega(m+1), 3)2 + \omega^2(4m-1)$ for all $m \in \omega \setminus 2$.

This is easy to see using Theorem 6.5, Theorem 6.11 and Lemma 7.7.

Theorem 7.9. For all $m \in \omega \setminus 2$ we have

$$r(I_m, A_3) \leqslant \frac{(2m+1)(m^2+4m-6)}{3}. \tag{7.3}$$

Proof. By induction. The base case is $m = 2$ where Formula (7.3) is verified by Theorem 6.19. Let us assume that the formula holds up to some $m \in \omega \setminus 2$. Using Lemma 7.7 we get

$$r(I_{m+1}, A_3) \leqslant r(I_m, A_3) + 2r(I_{m+1}, L_3) + 4m - 1. \tag{7.4}$$

Larson and Mitchell—see Lemma 4.1 of [997LM]—showed that $r(I_m, L_3) \leqslant n^2$, so the right side of (7.4) is bounded from above as follows:

$$\begin{aligned} &\leqslant \frac{(2m+1)(m^2+4m-6)}{3} + 2(m+1)^2 + 4m - 1 \\ &= \frac{(2(m+1)+1)((m+1)^2+4(m+1)-6)}{3}. \end{aligned}$$

□

Note that the fraction in Formula (7.3) is a whole number if m is an integer. If $m \equiv 1(3)$ then $2m + 1 \equiv 0(3)$ and we are fine. If, however, $m \equiv 0(3)$ then $m^2 + 4m - 6 \equiv 0(3)$ and if $m \equiv 2(3)$ then $m^2 + 4m - 6 \equiv 0(3)$ too.

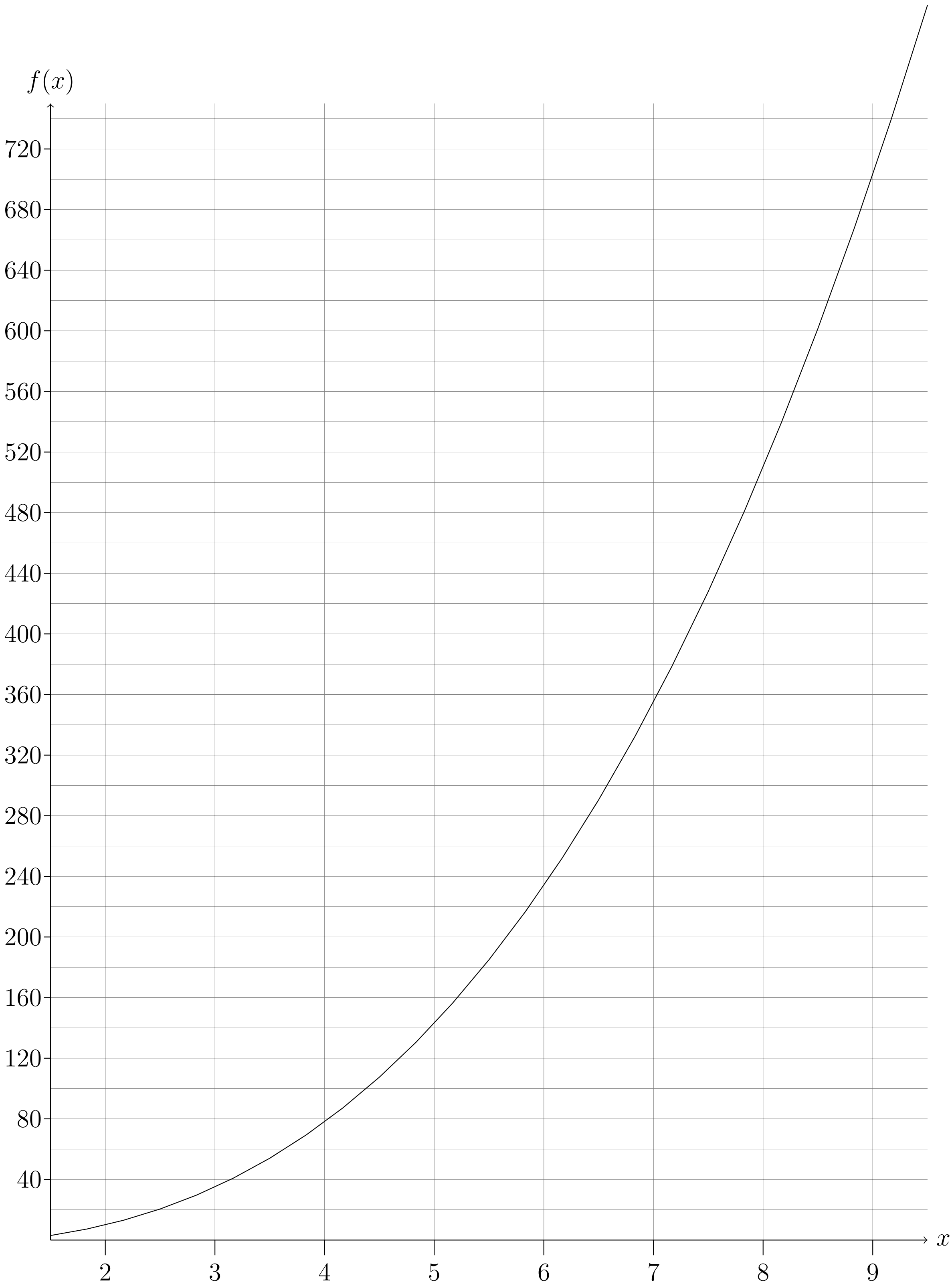

Figure 7.1: The graph of the upper bound (7.3)

A Lower Bound

Theorem 7.10. $r(I_{mn+1}, A_3) > (r(I_{m+1}, L_3) - 1)(r(I_{n+1}, L_3) - 1)$.

Proof. Let $D_0 = \langle V_0, A_0 \rangle$ be a digraph on $r(I_m, L_3) - 1$ vertices neither containing a transitive triple nor an independent set of size $m + 1$ and let $D_1 = \langle V_1, A_1 \rangle$ be a digraph on $r(I_n, L_3) - 1$ vertices neither containing a transitive triple nor an independent set of size $n + 1$. Then consider the coloured digraph $D_0 * D_1 := \langle V_0 \times V_1, A_2, c \rangle$ where

$$\begin{aligned} A_3 =& \{\langle\langle v_0, v_1\rangle, \langle v_2, v_1\rangle\rangle \mid v_0, v_2 \in V_0 \wedge v_1 \in V_1 \wedge \langle v_0, v_2\rangle \in A_0\}, \\ A_4 =& \{\langle\langle v_0, v_1\rangle, \langle v_2, v_3\rangle\rangle \mid v_0, v_2 \in V_0 \wedge v_1, v_3 \in V_1 \wedge \langle v_1, v_3\rangle \in A_1\}, \\ A_2 =& A_3 \cup A_4,\ c``A_3 = \{0\} \text{ and } c``A_4 = \{1\}. \end{aligned}$$

Clearly the independent sets in $D_0 * D_1$ have size at most mn and the only transitive triples in $D_0 * D_1$ contain two red arrows which either originate from the same vertex or point at the same. Both types of triples are disagreeable. □

Some Tables

At the end we take the liberty of supplying a table of some Ramsey numbers $r(\alpha, n)$ known today for countable ordinals α and natural numbers n. Following the example of Radziszowski in [994Ra] we also supply a complementary table containing references. If a reference is given above another, then the upper one gives the upper bound and the lower one gives the lower bound.

	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
ω	ω	ω	ω	ω	ω	ω	ω	ω
$\omega+1$	$\omega 2+1$	$\omega 3+1$	$\omega 4+1$	$\omega 5+1$	$\omega 6+1$	$\omega 7+1$	$\omega 8+1$	$\omega(m-1)+1$
$\omega+2$	$\omega 2+4$	$\omega 3+7$	$\omega 4+11$	$\omega 5+16$	$\omega 6+22$	$\omega 7+29$	$\omega 8+37$	$\omega(m-1)+\frac{m(m-1)}{2}+1$
$\omega+3$	$\omega 2+8$	$\omega 3+16$						
$\omega+n$	$\omega 2+r(n,3)+n-1$							
$\lambda 2$	$\lambda 4$	$\lambda 8$	$\lambda 14$	$\lambda 28$				
$\lambda 3$	$\lambda 9$							
ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
ω^2+1	$\omega^2 2+1$	$\omega^2 3+1$	$\omega^2 4+1$	$\omega^2 5+1$	$\omega^2 6+1$	$\omega^2 7+1$	$\omega^2 8+1$	$\omega^2(m-1)+1$
ω^2+2	$\omega^2 2+4$	$\omega^2 3+7$	$\omega^2 4+11$	$\omega^2 5+16$	$\omega^2 6+22$	$\omega^2 7+29$	$\omega^2 8+37$	$\omega^2(m-1)+\frac{m(m-1)}{2}+1$
ω^2+3	$\omega^2 2+8$	$\omega^2 3+16$						
ω^2+n	$\omega^2 2+r(n,3)+n-1$							
$\omega^2+\omega$	$\omega^2 4+\omega$							
$\omega^2 2$	$\omega^2 10$							
ω^3	ω^4	ω^4	ω^5	ω^5	ω^5	ω^5	ω^6	$\omega^{2+\lceil \mathrm{ld}(m)\rceil}$
ω^3+n	$\omega^4+r(n,3)$	$\omega^4+\omega^3+r(n,4)+n-1$						
ω^4	ω^7	ω^7	ω^{10}	ω^{10}	ω^{10}	ω^{10}		
ω^{5+n}	ω^{9+2n}	ω^{9+2n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{13+3n}	ω^{17+4n}	$\omega^{1+(4+n)\lceil \mathrm{ld}(m)\rceil}$
ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω	ω^ω
ω^{ω^2}	ω^{ω^2}	ω^{ω^2}						
$\kappa\lambda 2$	$\kappa\lambda 6$							
$\kappa\lambda 3$	$\kappa\lambda 15$							

Table 7.1: Finite and transfinite Ramsey numbers

	3	4	5	6	7	8	9	m
3		[955GG]	[955GG]	[964Ké]	[968GY] [966Ka]	[992MM] [982GR]	[982GR] [966Ka]	
4	[955GG]	[955GG]	[995MR] [965Ka]					
ω	[930Ra]	[930Ra]	[930Ra]	[930Ra]	[930Ra]	[930Ra]	[930Ra]	[930Ra]
$\omega+1$	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]
$\omega+2$	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]
$\omega+3$	[969HS3]	[969HS3]						
$\omega+n$	[969HS3]							
$\lambda 2$	[956ER]/[974Ba]	[964EM]/[974Ba]	[970RP]/[974Ba]	[970RP]/[974Ba]				
$\lambda 3$	[974Be]/[974Ba]							
ω^2	[957Sp]	[957Sp]	[957Sp]	[957Sp]	[957Sp]	[957Sp]	[957Sp]	[957Sp]
ω^2+1	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]
ω^2+2	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]	[969HS3]
ω^2+3	[969HS3]	[969HS3]						
ω^2+n	[969HS3]							
$\omega^2+\omega$	[969HS2]							
$\omega^2 2$	[969HS]							
ω^3	[957Sp]	[969HS2]	[974No]	[974No]	[974No]	[974No]	[974No]	[974No]
ω^3+n	[969Mi]	[969HS2]						
ω^4	[975No]	[975No]	[010HL]	[010HL]	[010HL]	[010HL]		
ω^{5+n}	[979No]	[979No]	[979No]	[979No]	[979No]	[979No]	[979No]	[979No]
ω^ω	[974La]	[974La]	[974La]	[974La]	[974La]	[974La]	[974La]	[974La]
ω^{ω^2}	[XXXDL]	[XXXDL]						
$\kappa\lambda 2$	This thesis!							
$\kappa\lambda 3$	This thesis!							

Table 7.2: References to upper and lower bounds for finite and transfinite Ramsey numbers

Combinations of the methods used to find the numbers above would probably generalise some results.

Problems, Questions & Conjectures

To the author the most interesting open question in the topic of this chapter now is the following:

Question 7.11. What is the order of growth of the function $m \mapsto r(I_m, A_3)$?

Only having obtained a cubic upper bound for $m \mapsto r(I_m, A_3)$ so far we state the following modest conjecture:

Conjecture 7.12. There exists an $\varepsilon > 0$ such that $\varepsilon n^{2+\varepsilon} < r(I_n, A_3)$ for all natural n.

The author was considerably surprised by the counterexample in Figure 6.4 providing the lower bound for Theorem 6.19 since there is no blue arrow in it. Hence the following seems to be a natural question:

Question 7.13. Are sharp lower bounds for $r(I_m, A_n)$, for m and n natural numbers, always attainable by counterexamples using only red and green arrows?

The author conjectures that the answer to this question is "No".

The author was not able to come up with a formula for the upper bounds given by the recurrence relations for $r(I_m, A_n)$. Perhaps an expert on recurrence relations can find one.

Problem 7.14. Find a formula similar to the one in Theorem 7.6 for a general upper bound for $r(I_m, A_n)$.

It is fairly obvious that the methods presented in this chapter could be extended to determine Ramsey numbers for even more decomposable ordinals. The related finite combinatorics might turn out to be extremely difficult, though. We mention just one example as an open problem.

Problem 7.15. Determine $r(\kappa\lambda\omega 2, 3)$ for $\kappa > \lambda$ both weakly compact!

This research was mainly motivated by an interest in countable ordinals. Although we have not discussed Ramsey numbers involving ordinals in $\omega_1 \setminus \omega^3$ here at all there are many open problems there. As a teaser we mention the following:

Problem 7.16. Determine $r(\omega^{\omega 2}, 3)$.

Ordinals α satisfying $\alpha \to (\alpha, n)^2$ for all $n < \omega$ are called *partition ordinals*. The author in vain tried to strengthen Theorem 6.24 by replacing $\mathsf{MA}_{\aleph_1}$ by "$\omega_1\omega$ is a partition ordinal". So we leave this as an open question.

Question 7.17. Does $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$ whenever $\omega_1\omega$ is a partition ordinal?

A variant of this problem is the following:

Question 7.18. Does $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$ whenever $\omega_1\omega^2$ is a partition ordinal?

At the end we want to mention that there are problems in Ramsey theory which—presumably—could have been attacked algorithmically but were not. The author conjectures that $r(I_4, L_3)$ and $r(I_3, L_4)$ could easily be algorithmically determined. The same should hold true for $r(I_2, S_4)$. Meanwhile the author is rather sceptical about the possibilities to determine $r(I_3, A_3)$ or $r(I_2, A_4)$ without considerable further mathematical insight.

8

A Thought on the Milner-Prikry-Problem

> We must know, we will know.
>
> David Hilbert

Whereas many results in the Ramsey Theory for pair colourings are known the general situation, and even the one for triple colourings, is less well understood. In the finite case only one nontrivial Ramsey number for triple colourings is known. It was shown in [991MR] that $13 \to (4)^3_2$ while it was known by [969Is] that $12 \not\to (4)^3_2$. The result in [991MR] was a computer proof or, rather, a computation.

Long before that Erdős and Rado had shown in [956ER] that $\omega_1 \not\to (\omega_1, 4)^3$, that $\omega_1 \not\to (\omega+2, \omega)^3$ and that for any uncountable type τ such that neither $\tau \geqslant \omega_1$ nor $\tau \geqslant \omega_1^*$ and any $k < \omega$ one has $\tau \to (\omega+k, 4)^3$.

Galvin showed $\tau \to (\omega+1)^r_s$ for $r, s < \omega$ and $\tau \to (\omega)^1_\omega$ in [970Ga] thus generalising a result of Erdős and Rado from [956ER] who had shown this for $\tau = \omega_1$. In [986MP] Milner and Prikry proved that $\omega_1 \to (\omega+k, 4)^3$ for any $k < \omega$ and later improved their result in [991MP] to $\omega_1 \to (\omega 2 + 1, 4)^3$. In the latter paper they emphasise that their method of proof does not allow for a straightforward generalisation consisting in the replacement of ω_1 by any τ satisfying $\tau \to (\omega)^1_\omega$.

In [986MP] Milner and Prikry wrote that it would be natural to conjecture the following:

$$\tau \to (\omega)^1_\omega \Longrightarrow \forall \alpha < \omega_1 \forall n < \omega \colon \tau \to (\alpha, n)^3.$$

This statement can be seen as a conjecture since they refer back to it as such in [991MP].

In [000Jo] Jones showed that for any infinite cardinal κ and any order-type τ the relation $\tau \not\to (\omega)^1_{2^\kappa}$ implies $\tau \not\to (\kappa+2, \omega)^3$. Later—in [007Jo]—he strengthened Milner's and Prikry's result by showing that $\tau \to (\omega)^1_\omega$ implies $\tau \to (\omega+k, l)^3$ for any $k, l < \omega$. Finally, in [008Jo] he proved $\tau \to (\omega 2 + 1, 4)^3$ for all τ satisfying $\tau \to (\omega)^1_\omega$.

The author considers Milner's and Prikry's conjecture to be a beautiful problem because it connects the finite, the countably infinite and the uncountable. He wants to hedge his bets but is of the opinion that if a counterexample to this conjecture exists, it is not all too surprising that it has not been found yet. Furthermore, the author considers it possible that the natural way to look at this question would be to allow for more than two colours. The following theorem is not very deep but it might hint at two states of affairs. Firstly, that considering more colours might shed more light on the situation and secondly, that in the context of this problem, ordinals like 5 or $\omega 2$ are perhaps less random points below ω_1 than one might guess at first glance.

Theorem 8.1. Suppose τ is an order-type such that $\tau \not\to (\omega)^1_\omega$. Then

$$\tau \not\to [\omega 2, \omega 2, \omega 2, \omega + 1, 5]^3.$$

Proof. Let $f\colon \tau \longrightarrow \omega$ witness that $\tau \not\to (\omega)^1_\omega$. Note that f assumes infinitely many values on every set in $[\tau]^\omega$. We define

$$\begin{aligned} &g\colon [\tau]^3 \longrightarrow 5 \\ &\{\alpha,\beta,\gamma\}_< \longmapsto \begin{cases} 0 \text{ if and only if } f(\alpha) \leqslant f(\gamma) < f(\beta), \\ 1 \text{ if and only if } f(\beta) < f(\alpha) < f(\gamma), \\ 2 \text{ if and only if } f(\beta) < f(\gamma) \leqslant f(\alpha), \\ 3 \text{ if and only if } f(\gamma) < f(\alpha) < f(\beta), \\ 4 \text{ if and only if } f(\alpha) \leqslant f(\beta) \leqslant f(\gamma) \text{ or } f(\gamma) \leqslant f(\beta) \leqslant f(\alpha). \end{cases} \end{aligned}$$

($\omega 2$) Let $X \in [\tau]^{\omega 2}$. Let $\langle \zeta_\nu \mid \nu < \omega 2 \rangle$ be the order-preserving enumeration of X. We have to show that $3 \subset g``[X]^3$.

To show that there is a triple $\{\alpha,\beta,\gamma\}_< \in [X]^3$ such that $g(\{\alpha,\beta,\gamma\}) = 0$ set $\alpha \coloneqq \zeta_0$, let $m < \omega$ be such that $f(\zeta_{\omega+m}) \geqslant f(\alpha)$ and set $\gamma \coloneqq \zeta_{\omega+m}$. Finally let $n < \omega$ be such that $f(\zeta_n) > f(\gamma)$ and set $\beta \coloneqq \zeta_n$.

For $g(\{\alpha,\beta,\gamma\}) = 1$ set $\beta \coloneqq \zeta_\omega$, let $m < \omega$ be such that $f(\zeta_m) > f(\beta)$ and set $\alpha \coloneqq \zeta_m$. Then let $n < \omega$ be such that $f(\zeta_{\omega+n}) > f(\alpha)$ and set $\gamma \coloneqq \zeta_{\omega+n}$.

For $g(\{\alpha,\beta,\gamma\}) = 2$ set $\beta \coloneqq \zeta_\omega$, let $m < \omega$ be such that $f(\zeta_{\omega+m}) > f(\beta)$ and set $\gamma \coloneqq \zeta_{\omega+m}$. Finally, let $n < \omega$ be such that $f(\zeta_n) \geqslant f(\gamma)$ and set $\alpha \coloneqq \zeta_n$.

($\omega + 1$) Let $X \in [\tau]^{\omega+1}$ and let $\langle \zeta_\nu \mid \nu < \omega + 1 \rangle$ be the order-preserving enumeration. Set $\gamma \coloneqq \zeta_\omega$, let $m < \omega$ be such that $f(\zeta_m) > f(\gamma)$ and set $\alpha \coloneqq \zeta_m$. Then let $n < \omega$ be such that $f(\zeta_n) > f(\alpha)$ and set $\beta \coloneqq \zeta_n$.

(5) Let $\{\alpha_0, \ldots, \alpha_4\}_< \in [\tau]^5$ be given. Assume that $\{g(\{\alpha_0, \alpha_i, \alpha_4\}) \mid i \in 4 \setminus 1\} \subset 4$. For every $i \in 4 \setminus 1$ we either have $f(\alpha_i) < f(\alpha_0), f(\alpha_4)$

or $f(\alpha_i) > f(\alpha_0), f(\alpha_4)$. By the pigeonhole principle this implies that there is a pair $\{i, j\}_< \in [4 \setminus 1]^2$ such that either $f(\alpha_i), f(\alpha_j) < f(\alpha_0), f(\alpha_4)$ or $f(\alpha_i), f(\alpha_j) > f(\alpha_0), f(\alpha_4)$.

If $f(\alpha_i), f(\alpha_j) < f(\alpha_0), f(\alpha_4)$ while $f(\alpha_i) \leqslant f(\alpha_j)$ or $f(\alpha_i), f(\alpha_j) > f(\alpha_0), f(\alpha_4)$ while $f(\alpha_i) \geqslant f(\alpha_j)$ then $g(\{\alpha_i, \alpha_j, \alpha_4\}) = 4$.

Otherwise, i.e. if $f(\alpha_i), f(\alpha_j) < f(\alpha_0), f(\alpha_4)$ while $f(\alpha_i) > f(\alpha_j)$ or $f(\alpha_i), f(\alpha_j) > f(\alpha_0), f(\alpha_4)$ while $f(\alpha_i) < f(\alpha_j)$ then $g(\{\alpha_0, \alpha_i, \alpha_j\}) = 4$. $\square$

Epilogue

I think I'll stop here.

Sir Andrew John Wiles

The author decided to speak in the first person at at least one point in this thesis. I want to express my gratitude. The following account is not comprehensive, there are more people whom I am indebted to.

One can achieve so much more if one has help from other people. I received plenty of help. I got funding from the DFG for the project "Continuous Ramsey theory in higher dimension" which would not have happened if it had not been for my advisor, Stefan Geschke, to apply for it. Despite this I am thankful that he always had time and energy to discuss mathematics. Above all I am thankful for being able to expand the topic of my thesis beyond the originally planned one. Speaking of this, I also wish to thank Rene Schipperus for getting me interested in the subject of partition relations by giving a talk in the Oberseminar in Bonn. Furthermore, I would like to thank Jean Larson with whom I enjoyed discussing mathematics and who crucially pointed out to me the papers [974Ba] and [989Ba]. Thanks also belong to Lionel Ngyuen Van Thé who sent copies of the papers [969HS3], [969HS] and [969HS2] and my former office mate Benjamin Seyfferth who provided a crash course in French.

I had the privilege to share offices with many nice people, Merlin Carl, Ioanna Dimitriou, Anne Fernengel, Ingrid Irmer and Benjamin Seyfferth. It was always refreshing and often enlightening to discuss the meta-level of mathematics with Merlin, calling the topic "philosophy of mathematics" might be etymologically correct but, at the same time, runs the risk of giving an impression of too narrow a topic. I wish to thank Ingrid for providing glimpses into another subject of mathematics and, moreover, for inviting me to an unforgettable party in 2010.

Without further mentioning any details I wish to thank Uri Avraham, Dana Bartošová, Andrés Caicedo, Raphaël Carroy, Sam Coskey, Philipp Döbler, Mirna Džamonja, Fred Galvin, Martin Goldstern, Dina Hess, Bernhard Irrgang, Peter Koepke, Menachem Kojman, Peter Komjath, Yurii Khomskii, Robert Kucharczyk, Benedikt Löwe, Philipp Lücke, Menachem Magidor, Adrian Mathias, Gregory McKay, Yann Pequignot, Philipp Schlicht, David Schrittesser, Omar Selim, Saharon Shelah, Katie Thompson, Stevo Todorcevic, Ian Walsh, Marek Wyszkowski and Jindra Zapletal.

Near the end, I would like to thank Wolfgang Wohofsky for his reliable enthusiasm to discuss mathematics, often in a Viennese Kaffeehaus. The Viennese Kaffeehaus is, although possibly past its heyday, still a great place for intellectual endeavours.

I am also grateful to Bianca Bernt for being the best pen-pal one could wish to have.

Finally, I thank my mother, who was always willing to help where help was needed.

Bonn Beuel, 7$^{\text{th}}$ February 2014, a Friday

Notation

In this thesis we use the terminology which is nowadays fairly common. Ordinals are von-Neumann-ordinals, in particular an ordinal is identical to the set of its ordinal predecessors. Therefore $\alpha \setminus \beta$ denotes the half-open interval from β to α containing the former but not the latter. Ω denotes the class of all ordinals and $[\alpha]^\beta$ denotes the set of subsets of α of size β where "size" can refer either to a cardinal or an order type. For a function f we denote its domain by $\mathrm{dom}(f)$ and its range by $\mathrm{ran}(f)$. $f``X$ for $X \subset \mathrm{dom}(f)$ is defined to be $\{y \in \mathrm{ran}(f) \mid \exists x \in X \colon f(x) = y\}$, i.e. the pointwise image of X under the function f. ${}^AB$ denotes the set of all functions from A to B. If A is a well-order and B is a linear order, then we can consider the lexicographic order-relation on ${}^AB$, we denote it by $<_{\mathrm{lex}}$. $\overline{\overline{X}}$ denotes the cardinality of a set X and $\mathrm{otyp}(X)$ denotes its order-type. For two order-types τ and φ, $\tau \leqslant \varphi$ means that every set of order-type τ can be embedded into any set of order-type φ order-preservingly. The reverse of any order-type τ is denoted by τ^*. The order-type of the rational numbers, the unique countable dense linear order without end-points, is, following [897Ca], denoted by η. $\langle s_0, \ldots, s_k\rangle$ denotes an ordered $k+1$-tuple. Correspondingly, $\langle s_\alpha \mid \alpha < \beta\rangle$ denotes a sequence of length β. By writing $\{k_0, \ldots, k_\ell\}_<$ we denote the set $\{k_0, \ldots, k_\ell\}$ and give the additional information that $k_m < k_{m+1}$ for any $m < \ell$.

Moreover, we employ the common notation for finite Ramsey numbers also in the more general case of ordinals. So $r(\beta, n) = \alpha$ means that α is the least ordinal such that $\alpha \to (\beta, n)^2$ holds true.

When v, w are vertices of a directed graph, $v \mapsto_b w$ means that there is a blue arrow from v to w while $v \mapsto_r w$ says that there is a red arrow from v to w; $v \mapsto_g w$ claims the existence of a green arrow from v to w and finally $v \mapsto w$ means that there is an arrow of some kind from v to w. Sometimes we want to talk about there being an arrow of a certain colour between

v and w without claiming anything about its direction. For this we write $v -_b w$—there is a blue arrow between v and w, $v -_r w$—there is a red arrow between v and w, or $v -_g w$—there is a green arrow between v and w. The blue in-neighborhoods of a vertex v will be denoted by $N_B^-(v)$ and the blue out-neighborhoods by $N_B^+(v)$. $N_R^-(v)$, $N_R^+(v)$, $N_G^-(v)$ and $N_G^+(v)$ have their expected denotations. The set of vertices independent from v is denoted by $I(v)$.

In this thesis, a digraph has at most one arrow between any two vertices and never any from a vertex to itself.

For the graph-theoretic terminology we follow [009BG]. Some of the notation was inspired by [009GS].

List of Figures

List of Tables

Bibliography

[XXXDL] Carl Darby and Jean Ann Larson. A positive partition relation for a critical countable ordinal. In preparation.

[XXXMS] Maryanthe Malliaris and Saharon Shelah. Cofinality spectrum theorems in model theory, set theory and general topology. http://math.uchicago.edu/~mem/Malliaris-Shelah-CST.pdf.

[XXXWe] Thilo Weinert. Two small theorems in infinitary structural Ramsey theory. Submitted, 8 pages, http://www.math.uni-bonn.de/people/weinert/twosmalltheorems.pdf.

[014We] Thilo Weinert. Idiosynchromatic poetry. Accepted for Publication by Combinatorica, 36 pages, 2014, http://www.math.uni-bonn.de/people/weinert/Poetry.pdf.

[013D] François Gilbert Dorais, Rafał Filipów and Tomasz Natkaniec. On some properties of Hamel bases and their applications to Marczewski measurable functions. *Cent. Eur. J. Math.*, 11(3):487–508, 2013, doi:10.2478/s11533-012-0144-1.

[012Ha] Joel David Hamkins. The set-theoretic multiverse. *Rev. Symb. Log.*, 5(3):416–449, 2012, doi:10.1017/S1755020311000359, http://dx.doi.org/10.1017/S1755020311000359.

[011Ka] Akihiro Kanamori. Historical remarks on Suslin's problem. In *Set theory, arithmetic, and foundations of mathematics: theorems, philosophies*, volume 36 of *Lect. Notes Log.*, pages 1–12. Assoc. Symbol. Logic, La Jolla, CA, 2011, http://math.bu.edu/people/aki/18.pdf.

[010Bl] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010, doi:10.1007/978-1-4020-5764-9_7, http://dx.doi.org/10.1007/978-1-4020-5764-9_7.

[010HL] Andras Hajnal and Jean Ann Larson. *Handbook of set theory. Vol. 1*, chapter Partition relations, pages Vol. 1: xiv+736 pp. Springer, Dordrecht, 2010, http://www.math.rutgers.edu/%7Eahajnal/newhaj-lar.pdf.

[009BG] Jørgen Bang-Jensen and Gregory Gutin. *Digraphs.* Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, second edition, 2009, doi:10.1007/978-1-84800-998-1, http://dx.doi.org/10.1007/978-1-84800-998-1. Theory, algorithms and applications.

[009Sa] Marcin Sabok. *On Idealized Forcing.* PhD thesis, 2009, http://www.math.uni.wroc.pl/~sabok/PRACE/phd1.pdf.

[009GS] Agelos Georgakopoulos and Philipp Sprüssel. On 3-coloured tournaments. Preprint, April 2009, http://www.arxiv.org/pdf/0904.1967.pdf.

[008Fr] Stefanie Frick. *Continuous Ramsey Theory in Higher Dimensions.* PhD thesis, Freie Universität Berlin, 2008, http://www.diss.fu-berlin.de/diss/servlets/MCRFileNodeServlet/FUDISS_derivate_000000005434/thesisonline16409.pdf.

[008Jo] Albin Lester Jones. Partitioning triples and partially ordered sets. *Proc. Amer. Math. Soc.*, 136(5):1823–1830, 2008, doi:10.1090/S0002-9939-07-09170-8, http://dx.doi.org/10.1090/S0002-9939-07-09170-8.

[008Za] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2008, doi:10.1017/CBO9780511542732, http://dx.doi.org/10.1017/CBO9780511542732.

[007Jo] Albin Lester Jones. More on partitioning triples of countable ordinals. *Proc. Amer. Math. Soc.*, 135(4):1197–1204, 2007, doi:10.1090/S0002-9939-06-08538-8, http://dx.doi.org/10.1090/S0002-9939-06-08538-8.

[006Ge] Stefan Geschke. A dual open coloring axiom. *Ann. Pure Appl. Logic*, 140(1-3):40–51, 2006, doi:10.1016/j.apal.2005.09.003, http://dx.doi.org/10.1016/j.apal.2005.09.003.

[004B] Bohuslav Balcar, Fernando Hernández-Hernández and Michael Hrušák. Combinatorics of dense subsets of the rationals. *Fund. Math.*, 183(1):59–80, 2004, doi:10.4064/fm183-1-4, http://dx.doi.org/10.4064/fm183-1-4.

[004CP] Krzysztof Ciesielski and Janusz Pawlikowski. *The covering property axiom, CPA*, volume 164 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004, doi:10.1017/CBO9780511546457, http://dx.doi.org/10.1017/CBO9780511546457. A combinatorial core of the iterated perfect set model.

[004G] Stefan Geschke, Martin Goldstern and Menachem Kojman. Continuous Ramsey theory on Polish spaces and covering the plane by functions. *J. Math. Log.*, 4(2):109–145, 2004, doi:10.1142/S0219061304000334, http://dx.doi.org/10.1142/S0219061304000334.

[004Sh] Saharon Shelah. Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and fs linearly ordered iterated forcing. *Acta Mathematica*, 192:187–223, 2004, http://arxiv.org/pdf/math/0012170.pdf. Also known under the title "Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea?". math.LO/0012170.

[004Sp] Otmar Spinas. Analytic countably splitting families. *J. Symbolic Logic*, 69(1):101–117, 2004, doi:10.2178/jsl/1080938830, http://dx.doi.org/10.2178/jsl/1080938830.

[004Za] Jindřich Zapletal. Descriptive set theory and definable forcing. *Mem. Amer. Math. Soc.*, 167(793):viii+141, 2004.

[003Ka] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[003KS] Péter Komjáth and Saharon Shelah. A partition theorem for scattered order types. *Combin. Probab. Comput.*, 12(5-6):621–626, 2003, doi:10.1017/S0963548303005686, http://dx.doi.org/10.1017/S0963548303005686. Special issue on Ramsey theory.

[002G] Stefan Geschke, Menachem Kojman, Wieslaw Kubiś and René Schipperus. Convex decompositions in the plane and continuous pair colorings of the irrationals. *Israel J. Math.*, 131:285–317, 2002, doi:10.1007/BF02785863, http://dx.doi.org/10.1007/BF02785863.

[000C] Jacek Cichoń, Adam Krawczyk, Barbara Majcher-Iwanow and Bogdan Węglorz. Dualization of the van Douwen diagram. *J. Symbolic Logic*, 65(2):959–968, 2000, doi:10.2307/2586580, http://dx.doi.org/10.2307/2586580.

[000DH] Carlos Augusto Di Prisco De Venanzi and James Marston Henle. Doughnuts, floating ordinals, square brackets, and ultraflitters. *J. Symbolic Logic*, 65(1):461–473, 2000, doi:10.2307/2586548, http://dx.doi.org/10.2307/2586548.

[000Jo] Albin Lester Jones. A short proof of a partition relation for triples. *Electron. J. Combin.*, 7:Research Paper 24, 9, 2000, http://www.combinatorics.org/Volume_7/Abstracts/v7i1r24.html.

[999Sc] Rene J. Schipperus. *Countable partition ordinals.* ProQuest LLC, Ann Arbor, MI, 1999, http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9938851. Thesis (Ph.D.)–University of Colorado at Boulder.

[998Ha] Lorenz Halbeisen. On shattering, splitting and reaping partitions. *Math. Logic Quart.*, 44(1):123–134, 1998, doi:10.1002/malq.19980440109, http://dx.doi.org/10.1002/malq.19980440109.

[998PR] Konrad Piwakowski and Stanisław P. Radziszowski. $30 \leq R(3,3,4) \leq 31$. *J. Combin. Math. Combin. Comput.*, 27:135–141, 1998, http://www.cs.rit.edu/~spr/PUBL/paper36.pdf.

[998Sh] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[997LM] Jean Ann Larson and William John Mitchell. On a problem of Erdős and Rado. *Ann. Comb.*, 1(3):245–252, 1997, doi:10.1007/BF02558478, http://dx.doi.org/10.1007/BF02558478.

[997MR] Brendan Damien McKay and Stanisław P. Radziszowski. Subgraph counting identities and Ramsey numbers. *J. Combin. Theory Ser. B*, 69(2):193–209, 1997, doi:10.1006/jctb.1996.1741, http://dx.doi.org/10.1006/jctb.1996.1741.

[995BJ] Tomek Bartoszyński and Haim Judah. *Set theory.* A K Peters Ltd., Wellesley, MA, 1995. On the structure of the real line.

[995MR] Brendan Damien McKay and Stanisław P. Radziszowski. $R(4,5) = 25$. *J. Graph Theory*, 19(3):309–322, 1995, doi:10.1002/jgt.3190190304, http://dx.doi.org/10.1002/jgt.3190190304.

[994Ra] Stanisław P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994, http://www.combinatorics.org/Surveys/index.html.

[994So] Sławomir Solecki. Covering analytic sets by families of closed sets. *J. Symbolic Logic*, 59(3):1022–1031, 1994, doi:10.2307/2275926, http://dx.doi.org/10.2307/2275926.

[992MM] Brendan Damien McKay and Zhang Ke Min. The value of the Ramsey number $R(3,8)$. *J. Graph Theory*, 16(1):99–105, 1992, doi:10.1002/jgt.3190160111, http://dx.doi.org/10.1002/jgt.3190160111.

[992Sh] Saharon Shelah. Vive la différence. I. Nonisomorphism of ultrapowers of countable models. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 357–405. Springer, New York, 1992, doi:10.1007/978-1-4613-9754-0_20, http://dx.doi.org/10.1007/978-1-4613-9754-0_20.

[991MP] Eric Charles Milner and Karel Libor Prikry. A partition relation for triples using a model of Todorčević. *Discrete Math.*, 95(1-3):183–191, 1991, doi:10.1016/0012-365X(91)90336-Z, http://dx.doi.org/10.1016/0012-365X(91)90336-Z. Directions in infinite graph theory and combinatorics (Cambridge, 1989).

[991MR] Brendan Damien McKay and Stanisław P. Radziszowski. The first classical Ramsey number for hypergraphs is computed. In *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms (San Francisco, CA, 1991)*, pages 304–308, New York, 1991. ACM, http://www.cs.rit.edu/~spr/PUBL/paper25.pdf.

[989Ba] James Earl Baumgartner. Remarks on partition ordinals. In *Set theory and its applications (Toronto, ON, 1987)*, volume 1401 of *Lecture Notes in Math.*, pages 5–17. Springer, Berlin, 1989, doi:10.1007/BFb0097328, http://dx.doi.org/10.1007/BFb0097328.

[987To] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3-4):261–294, 1987, doi:10.1007/BF02392561, http://dx.doi.org/10.1007/BF02392561.

[986Ma] Pierre Matet. Partitions and filters. *J. Symbolic Logic*, 51(1):12–21, 1986, doi:10.2307/2273937, http://dx.doi.org/10.2307/2273937.

[986MP] Eric Charles Milner and Karel Libor Prikry. A partition theorem for triples. *Proc. Amer. Math. Soc.*, 97(3):488–494, 1986, doi:10.2307/2046243, http://dx.doi.org/10.2307/2046243.

[984Ba] James Earl Baumgartner. Applications of the proper forcing axiom. In *Handbook of set-theoretic topology*, pages 913–959. North-Holland, Amsterdam, 1984.

[983Ka] Ilias George Kastanas. On the Ramsey property for sets of reals. *J. Symbolic Logic*, 48(4):1035–1045 (1984), 1983, doi:10.2307/2273667, http://dx.doi.org/10.2307/2273667.

[982GR] Charles Miller Grinstead and Sam M. Roberts. On the Ramsey numbers $R(3, 8)$ and $R(3, 9)$. *J. Combin. Theory Ser. B*, 33(1):27–51, 1982, doi:10.1016/0095-8956(82)90055-7, http://dx.doi.org/10.1016/0095-8956(82)90055-7.

[982Ro] Joseph Goeffrey Rosenstein. *Linear orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982.

[981Be] Murray G. Bell. On the combinatorial principle $P(\mathfrak{c})$. *Fund. Math.*, 114(2):149–157, 1981, http://matwbn.icm.edu.pl/ksiazki/fm/fm114/fm114114.pdf.

[981Bl] Andreas Blass. A partition theorem for perfect sets. *Proc. Amer. Math. Soc.*, 82(2):271–277, 1981, doi:10.2307/2043323.

[981K] Alexander Sotirios Kechris, Eugene Meyer Kleinberg, Yiannis Nicholas Moschovakis and William Hugh Woodin. The axiom of determinacy, strong partition properties and nonsingular measures. In *Cabal Seminar 77–79 (Proc. Caltech-UCLA Logic Sem., 1977–79)*, volume 839 of *Lecture Notes in Math.*, pages 75–99. Springer, Berlin, 1981, doi:10.1007/BFb0090236.

[979No] Eva Nosal. Partition relations for denumerable ordinals. *J. Combin. Theory Ser. B*, 27(2):190–197, 1979, doi:10.1016/0095-8956(79)90080-7, `http://dx.doi.org/10.1016/0095-8956(79)90080-7`.

[978Ta] Alan D. Taylor. Partitions of pairs of reals. *Fund. Math.*, 99(1):51–59, 1978, `http://matwbn.icm.edu.pl/tresc.php?wyd=1&tom=99`.

[976Pr] Karel Libor Prikry. Determinateness and partitions. *Proc. Amer. Math. Soc.*, 54:303–306, 1976, `http://www.jstor.org/stable/2040805`.

[975No] Eva Nosal. *On arrow relations w(k,w) 2(2) (m,n,k less than w): A study in the partition calculus.* ProQuest LLC, Ann Arbor, MI, 1975, `http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:NK25043`. Thesis (Ph.D.)–University of Calgary (Canada).

[974Ba] James Earl Baumgartner. Improvement of a partition theorem of Erdös and Rado. *J. Combinatorial Theory Ser. A*, 17:134–137, 1974.

[974Be] Jean-Claude Bermond. Some Ramsey numbers for directed graphs. *Discrete Math.*, 9:313–321, 1974.

[974La] Jean Ann Larson. A short proof of a partition theorem for the ordinal ω^ω. *Ann. Math. Logic*, 6:129–145, 1973–1974.

[974No] Eva Nosal. On a partition relation for ordinal numbers. *J. London Math. Soc. (2)*, 8:306–310, 1974, `http://jlms.oxfordjournals.org/content/s2-8/2/306.full.pdf`.

[973GP] Fred Galvin and Karel Prikry. Borel sets and Ramsey's theorem. *J. Symbolic Logic*, 38:193–198, 1973, `http://www.jstor.org/stable/pdfplus/2272055.pdf`.

[972EM] Pál Erdős and Eric Charles Milner. A theorem in the partition calculus. *Canad. Math. Bull.*, 15:501–505, 1972, http://www.renyi.hu/~p_erdos/1972-03.pdf.

[971E] Pál Erdős, Eric Charles Milner and Richard Rado. Partition relations for η_α-sets. *J. London Math. Soc. (2)*, 3:193–204, 1971, http://www.renyi.hu/~p_erdos/1971-16.pdf.

[971EH] Pál Erdős and Andras Hajnal. Ordinary partition relations for ordinal numbers. *Period. Math. Hungar.*, 1(3):171–185, 1971, http://www.renyi.hu/~p_erdos/1971-15.pdf.

[971Gr] Serge Grigorieff. Combinatorics on ideals and forcing. *Ann. Math. Logic*, 3(4):363–394, 1971, http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.215.5089&rep=rep1&type=pdf.

[971La] Richard Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971.

[971ST] Robert Martin Solovay and Stanley Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)*, 94:201–245, 1971.

[970Ga] Fred Galvin. A letter addressed to Paul Erdős dated October 12th. 1970.

[970MS] Donald Anthony Martin and Robert Martin Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2(2):143–178, 1970.

[970RP] Kenneth Brooks Reid, Jr. and Ernest Tilden Parker. Disproof of a conjecture of Erdős and Moser on tournaments. *J. Combinatorial Theory*, 9:225–238, 1970.

[969HS] Labib Haddad and Gabriel Sabbagh. Calcul de certains nombres de Ramsey généralisés. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A1233–A1234, 1969.

[969HS2] Labib Haddad and Gabriel Sabbagh. Nouveaux résultats sur les nombres de Ramsey généralisés. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A1516–A1518, 1969.

[969HS3] Labib Haddad and Gabriel Sabbagh. Sur une extension des nombres de Ramsey aux ordinaux. *C. R. Acad. Sci. Paris Sér. A-B*, 268:A1165–A1167, 1969.

[969Is] John Rolfe Isbell. $N(4, 4; 3) \geq 13$. *J. Combinatorial Theory*, 6:210, 1969.

[969Mi] Eric Charles Milner. Partition relations for ordinal numbers. *Canad. J. Math.*, 21:317–334, 1969.

[968GY] Jack Edward Graver and James Yackel. Some graph theoretic results associated with Ramsey's theorem. *J. Combinatorial Theory*, 4:125–175, 1968.

[966Ka] J. G. Kalbfleisch. *Chromatic Graphs and Ramsey's theorem.* PhD thesis, University of Waterloo, January 1966.

[965Ka] J. G. Kalbfleisch. Construction of special edge-chromatic graphs. *Canad. Math. Bull.*, 8:575–584, 1965, `http://cms.math.ca/cmb/v8/cmb1965v08.0575-0584.pdf`.

[965Kr] Arthur H. Kruse. A note on the partition calculus of P. Erdős and R. Rado. *J. London Math. Soc.*, 40:137–148, 1965, `http://jlms.oxfordjournals.org/content/s1-40/1/137.full.pdf`.

[964EM] Pál Erdős and Leo Moser. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:125–132, 1964, `http://www.renyi.hu/~p_erdos/1964-22.pdf`.

[964Ké] Gerzson Kéry. On a theorem of Ramsey. *Mat. Lapok*, 15:204–224, 1964.

[964MŚ] Jan Mycielski and S. Świerczkowski. On the Lebesgue measurability and the axiom of determinateness. *Fund. Math.*, 54:67–71, 1964, `http://matwbn.icm.edu.pl/ksiazki/fm/fm54/fm5417.pdf`.

[964My] Jan Mycielski. Independent sets in topological algebras. *Fund. Math.*, 55:139–147, 1964, `http://matwbn.icm.edu.pl/tresc.php?wyd=1&tom=55`.

[964My2] Jan Mycielski. On the axiom of determinateness. *Fund. Math.*, 53:205–224, 1963/1964, `http://matwbn.icm.edu.pl/ksiazki/fm/fm53/fm53120.pdf`.

[963Co] Paul Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963, `http://www.ncbi.nlm.nih.gov/pmc/articles/PMC221287/pdf/pnas00240-0135.pdf`.

[963EH] Pál Erdős and Andras Hajnal. On a classification of denumerable order types and an application to the partition calculus. *Fund. Math.*, 51:117–129, 1962/1963, `http://www.renyi.hu/~p_erdos/1962-06.pdf`.

[962MS] Jan Mycielski and Hugo Steinhaus. A mathematical axiom contradicting the axiom of choice. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 10:1–3, 1962.

[960De] Richard Dedekind. *Stetigkeit und irrationale Zahlen.* 6te unveränderte Aufl. Friedr. Vieweg & Sohn. Braunschweig, 1960, `http://www.dima.unige.it/~bartocci/csm12/dedekind_1872.pdf`. The URL does not link to the sixth edition.

[957Sp] Ernst Specker. Teilmengen von Mengen mit Relationen. *Comment. Math. Helv.*, 31:302–314, 1957, `http://www.digizeitschriften.de/dms/img/?PPN=PPN358147735_0031&DMDID=dmdlog15`.

[956ER] Pál Erdős and Richard Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc.*, 62:427–489, 1956, `http://www.ams.org/journals/bull/1956-62-05/S0002-9904-1956-10036-0/S0002-9904-1956-10036-0.pdf`.

[955GG] Robert Ewing Greenwood, Jr. and Andrew Mattei Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, 7:1–7, 1955, `http://cms.math.ca/cjm/v7/cjm1955v07.0001-0007.pdf`.

[954BS] Frederick Bagemihl and H. D. Sprinkle. On a proposition of Sierpiński's which is equivalent to the continuum hypothesis. *Proc. Amer. Math. Soc.*, 5:726–728, 1954.

[953GS] David Gale and F. M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.

[940Gö] Kurt Gödel. *The Consistency of the Continuum Hypothesis*. Annals of Mathematics Studies, no. 3. Princeton University Press, Princeton, N. J., 1940.

[935ES] Pál Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935, http://www.numdam.org/item?id=CM_1935__2__463_0.

[930Ra] Frank Plumpton Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930, http://plms.oxfordjournals.org/content/s2-30/1/264.full.pdf.

[928vN] John von Neumann. Zur Theorie der Gesellschaftsspiele. *Math. Ann.*, 100(1):295–320, 1928, doi:10.1007/BF01448847, http://dx.doi.org/10.1007/BF01448847.

[908Ha] Felix Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Math. Ann.*, 65(4):435–505, 1908, doi:10.1007/BF01451165, http://dx.doi.org/10.1007/BF01451165.

[905Vi] Giuseppe Vitali. Sul problema della misura dei gruppi di punti di una retta. *Bologna, Tip. Gamberini e Parmeggiani*, ?:231–235, 1905, http://www.math.binghamton.edu/paul/505-S08/2.pdf. The URL links to a translation.

[897Ca] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Math. Ann.*, 49(2):207–246, 1897, doi:10.1007/BF01444205, http://dx.doi.org/10.1007/BF01444205.

[878Ca] Georg Ferdinand Ludwig Philipp Cantor. Ein Beitrag zur Mannifaltigkeitslehre. *J. Reine Angew. Math.*, 84:242–258, 1878, http://www.digizeitschriften.de/dms/resolveppn/?PPN=gdzppn002156806.

[874Ca] Georg Ferdinand Ludwig Philipp Cantor. Über eine Eigenschaft des Inbegriffs aller reellen algebraischen Zahlen. *J. Reine Angew. Math.*, 77:258–262, 1874, http://www.cs.elte.hu/~badam/matbsc/11o/cantor1874.pdf.

[844Li] Joseph Liouville. 1844.

Index