Factorization of holomorphic eta quotients

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To the fond memories of Dadai and Boromasi

&

To Baba, Ma, Tatai, Ishita and Shaurjo

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Introduction

The *Dedekind eta function* is defined by an infinite product:

$$\eta(z) := e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) \tag{1}$$

for all $z \in \mathfrak{H}$, where $\mathfrak{H}$ is the *complex upper half plane*, i. e. the set of complex numbers with positive imaginary parts. The function η has its significance in Elementary Number Theory, because $1/\eta$ is the generating function for the ordinary partition function $p : \mathbb{N} \to \mathbb{N}$ (see [1] or [2]) and because η leads to Dedekind sums (via its modular transformation property, see [17] or [15]). The eta function is also relevant in Algebraic Number Theory, viz. $\eta(z_1)/\eta(z_2)$ is an algebraic number if $z_1, z_2 \in \mathfrak{H}$ belong to an imaginary quadratic field (see [10]), and in Analytic Number Theory, e. g., the Fourier coefficients of η^{24} define the Ramanujan τ function (see [7]) and the value of $\eta^{8h(\mathbb{Q}(\sqrt{-D}))}$ at a quadratic irrationality in $\mathfrak{H}$ of discriminant $-D$ is related via the Lerch/Chowla-Selberg formula to the values of the Gamma function with arguments in $D^{-1}\mathbb{Z}$ (see [7], [10], [26], [44]). Also, the eta function has connections with Representation Theory, viz. holomorphic eta quotients appear in Moonshine of finite groups (see [16]) and explicit Fourier expansions of infinitely many eta quotients follow from the Macdonald Identities (see [27]). The study of the modular properties of eta was started 137 years ago by Dedekind (see [12]). But even in the recent past, a lot of research has been done to unearth many interesting features of eta quotients (for example, see [5], [9], [11], [19], [25], [28] and [36]).

The function η is a modular form of weight $1/2$ with a multiplier system (see [20]) on $\mathrm{SL}_2(\mathbb{Z})$. Throughout this thesis, by *modular forms*, we shall mean *modular forms with multiplier systems*. The eta function endow us with a natural supply of explicit examples of modular forms: The usual ways to construct more modular forms from the eta function are to rescale it, to take products of its rescalings or more generally, to consider holomorphic quotients of products of its rescalings and to make linear combination of eta quotients of equal weights. For $d \in \mathbb{N}$, by η_d we denote the rescaling of η by d, defined by $\eta_d(z) := \eta(dz)$. In general, by an *eta quotient* we mean a finite product over $d \in \mathbb{N}$ of the functions $\eta_d^{X_d}$, where $X_d \in \mathbb{Z}$. The lcm of the scaling factors d corresponding to nonzero exponents X_d is called the *level* of an eta quotient. Eta quotients naturally inherit the modular transformation property from η: the level of an eta quotient f is the smallest positive integer N for which f is a weakly holomorphic modular form on $\Gamma_0(N)$. Since η is non-zero on the upper half plane, the eta quotient f is holomorphic if and only if it does not have any pole at the cusps of $\Gamma_0(N)$.

Though it is away from the focus of this thesis, nevertheless it is worth mentioning here that eta quotients satisfy a plethora of linear identities (see Somos's list [40]). In particular, Somos singled out the following remarkable three term identity (see [41]) from his list:

$$\eta\eta_{12}\eta_{15}\eta_{20} + \eta_3\eta_4\eta_5\eta_{60} = \eta_2\eta_6\eta_{10}\eta_{30} \tag{2}$$

which Rogers and Yuttanan proved in [35] (Of course the proof of any such identity is trivial if one uses the modularity properties, by checking a finite number of terms in the Fourier expansions at ∞. The authors here only wished to demonstrate a classical technique with which it could also be proved). Here we point out a distinctive attribute of this identity:

We shall see (Corollary 1.42) that an eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders of vanishing at the cusps $\{1/t\}_{t|N}$ of $\Gamma_0(N)$. Again invertiblity of the valuation matrix (see Proposition 1.41 (b)) over $\mathbb{Q}$ implies that given a tuple of arbitrary rational orders at the cusps $\{1/t\}_{t|N}$ of $\Gamma_0(N)$, there exists a unique eta quotient with rational exponents whose orders at the cusps $\{1/t\}_{t|N}$ of $\Gamma_0(N)$ matches with the given tuple (rational powers of eta are well-defined by (1.64) (see also [18])). So, in particular, given a set of eta quotients on $\Gamma_0(N)$, we may define their gcd to be the eta quotient on $\Gamma_0(N)$ whose order at each cusp $1/t$, $t|N$ of $\Gamma_0(N)$ is the minimum of the orders of the given eta quotients at that cusp. The gcd of the eta quotients in (2) is $f := a^{1/3}b^{1/3}c^{1/6}$, where $a := \eta\eta_{12}\eta_{15}\eta_{20}$ $b := \eta_3\eta_4\eta_5\eta_{60}$ and $c := \eta_2\eta_6\eta_{10}\eta_{30}$. Now, dividing (2) by f, we get a new "ABC"-identity:

$$\frac{a}{f} + \frac{b}{f} = \frac{c}{f}, \tag{3}$$

in which all three terms are of weight 1/3. The reduction of the original weight of an eta quotient identity to such a small weight is very rare, which distinguishes this case from the rest. Moreover, numerical calculations by Zagier suggest that—

as one could perhaps expect from the low weight ($< 1/2$) of the terms—the n-th Fourier coefficients of a/f, b/f and c/f go to zero as $n \to \infty$. This is a surprising feature that of course cannot occur for ordinary modular forms or for eta quotients with integral coefficients, since the Fourier coefficients of such forms are always integral.

I studied several other properties of additive identities of eta quotients of this type, but have not included them in this thesis, in which the focus is about their multiplicative properties instead. In the remainder of this introduction I will explain the type of questions that will be studied here and try to highlight some of the main results.

We say that a holomorphic eta quotient f is *divisible* by a holomorphic eta quotient g if f/g is holomorphic. If f is divisible by g, we call g a *factor* of f. A holomorphic eta quotient f is *irreducible* if it has only the trivial factors, viz. 1 and f. There is a lot of numerical evidence for the following conjecture:

Conjecture (Irreducibility Conjecture)**.** *A rescaling of an irreducible holomorphic eta quotient is irreducible.*

In particular, rescaling of a holomorphic eta quotient is a holomorphic eta quotient of the same weight and also rescaling is compatible with multiplication of eta quotients. Another example of a map which sends eta quotients to eta quotients and also possesses these two properties which we mentioned above is an *Atkin-Lehner involution*:

We define $\odot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$d_1 \odot d_2 := \frac{d_1 d_2}{(d_1, d_2)^2}. \tag{4}$$

For $N \in \mathbb{N}$, by $\mathcal{D}_N$, we denote the set of divisors of N. For $n \in \mathbb{N}$ and $d \in \mathcal{D}_N$, we say that d *exactly divides* N and write $d \| N$ if $(d, N/d) = 1$. We denote the set of such divisors of N by $\mathcal{E}_N$. It follows trivially that $(\mathcal{E}_N, \odot)$ is a boolean group (i. e., each element of $\mathcal{E}_N$ is the inverse of itself) and that $\mathcal{E}_N$ acts on $\mathcal{D}_N$ by $\odot$.

For $N, k \in \mathbb{Z}$, let $\mathbb{E}^!_{N,k}$ (resp. $\mathbb{E}_{N,k}$) be the set of eta quotients (resp. holomorphic eta quotients) of weight $k/2$ on $\Gamma_0(N)$. For $n \in \mathcal{E}_N$, we define the Atkin-Lehner map $\mathrm{al}_{n,N} : \mathbb{E}^!_{N,k} \to \mathbb{E}^!_{N,k}$ by

$$\mathrm{al}_{n,N}\Big(\prod_{d\in\mathcal{D}_N} \eta_d^{X_d}\Big) := \prod_{d\in\mathcal{D}_N} \eta_{n\odot d}^{X_d}. \tag{5}$$

Since $\mathcal{E}_N$ is a boolean group and since it acts on $\mathcal{D}_N$ by $\odot$, it follows trivially that the map $\mathrm{al}_{n,N} : \mathbb{E}^!_{N,k} \to \mathbb{E}^!_{N,k}$ is an involution. In the Section 1.6, we shall see that the definition of Atkin-Lehner involutions of eta quotients which we gave above matches with the usual definition of Atkin-Lehner involutions of modular forms on $\Gamma_0(N)$ up to multiplication by a constant. In particular, that implies:

Proposition. *Let f be an eta quotient on $\Gamma_0(N)$ and let $n \in \mathcal{E}_N$. Then f is holomorphic if and only if so is $\mathrm{al}_{n,N}(f)$.*

Just like the case of rescaling which we saw above, we also have a lot of numerical evidence which suggest the truth of the following assertion:

Conjecture (Irreducibility Conjecture, alternative form)**.** *The image of an irreducible holomorphic eta quotient under an Atkin-Lehner involution is irreducible.*

In Section 2.2, we show that the last two conjectures are equivalent.

Next, we note that the definition of reducibility of an eta quotient allows factors of arbitrary levels. For example, we have

$$\frac{\eta\eta_2\eta_6}{\eta_3} = \frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}} \times \frac{\eta_2^2\eta_{12}}{\eta_4\eta_6}, \tag{6}$$

where a reducible holomorphic eta quotient of level 6 is factored into two holomorphic eta quotients of level 12.

We call a holomorphic eta quotient f of level N *strongly reducible* if it has a nontrivial factor g of some level $N_g \,|\, N$. Certainly, every strongly reducible holomorphic eta quotient is reducible. But it was quite a surprise to find much numerical evidence for the converse:

Conjecture (Reducibility Conjecture)**.** *A holomorphic eta quotient is reducible only if it is strongly reducible.*

For example, the eta quotient of level 6 in (6) has also the following factorization into holomorphic eta quotients of level 6 and level 2:

$$\frac{\eta\eta_2\eta_6}{\eta_3} = \frac{\eta^2\eta_6}{\eta_2\eta_3} \times \frac{\eta_2^2}{\eta} \tag{7}$$

So, the reducible holomorphic eta quotient $\dfrac{\eta\eta_2\eta_6}{\eta_3}$ is indeed strongly reducible.

We shall see in Section 2.2 that the Irreducibility Conjecture above follows from the Reducibility Conjecture. If the Reducibility Conjecture holds for level N, then in particular, it gives an algorithm to check the irreducibility of a holomorphic eta quotient of level N, since strong reducibility is algorithmically verifiable.

Now we give some results towards the Reducibility Conjecture. We shall see their proofs in Section 2.3. For $M, N \in \mathbb{N}$, by $M|N^\infty$ we mean that there exists some $n \in \mathbb{N}$ such that $M|N^n$.

Theorem. *A holomorphic eta quotient of level N is reducible only if it is reducible in some level M with $M|N^\infty$.*

Corollary. *For $N \in \mathbb{N}$ and $M\|N$, if a holomorphic eta quotient of level M is reducible on $\Gamma_0(N)$, then it is strongly reducible.*

Corollary. *If a holomorphic eta quotient f has a factor of a squarefree level, then f is strongly reducible.*

In Section 2.4, we establish the Reducibility Conjecture for prime power levels:

Theorem. *A holomorphic eta quotient of a prime power level is reducible if and only if it is strongly reducible.*

Corollary. *The image of an irreducible holomorphic eta quotient of a prime power level under an Atkin-Lehner involution is irreducible.*

Corollary. *A rescaling of an irreducible holomorphic eta quotient of a prime power level is irreducible.*

In Section 2.5, we prove the existence of an algorithm to check the irreducibility of a holomorphic eta quotient in a finite time without assuming the Reducibility Conjecture, but instead by showing that the level of any factor of a holomorphic eta quotient f of weight $k/2$ and level N is bounded w.r.t. k and N:

Theorem. *For $N, k \in \mathbb{N}$, there exists an effectively computable $M = M(N, k) \in \mathbb{N}$ such that M is divisible by the level of any factor of a holomorphic eta quotient of weight $k/2$ and level N.*

To prove the above theorem, we use a finiteness result of Mersmann:

An eta quotient is called *primitive* if it is not a rescaling of another eta quotient of a smaller level. A holomorphic eta quotient which is primitive and not strongly reducible is called a *simple* holomorphic eta quotient . Such eta quotients were first considered by Zagier, who relying on extensive numerical calculations made two conjectures, one saying that there are only finitely many simple holomorphic eta quotients of a given weight and the other giving a complete list for weight 1/2. In a brilliant piece of work, his student Mersmann established both of these conjectures in 1991 (see [7], [22], [29]). In Chapters 3 and 4 of this thesis, we shall see respectively simplified and much shorter proofs of Mersmann's theorems:

Theorem (Mersmann's First Theorem). *There are only finitely many simple holomorphic eta quotients of any fixed weight.*

Theorem (Mersmann's Second Theorem). *The following fourteen are the only simple holomorphic eta quotients of weight* $\frac{1}{2}$:

$$\eta,\ \frac{\eta^2}{\eta_2},\ \frac{\eta_2^2}{\eta},\ \frac{\eta_2^3}{\eta\eta_4},\ \frac{\eta_2^5}{\eta^2\eta_4^2},\ \frac{\eta\eta_4}{\eta_2},\ \frac{\eta\eta_6^2}{\eta_2\eta_3},\ \frac{\eta^2\eta_6}{\eta_2\eta_3},\ \frac{\eta_2^2\eta_3}{\eta\eta_6},\ \frac{\eta_2\eta_3^2}{\eta\eta_6}$$

$$\frac{\eta_2^2\eta_3\eta_{12}}{\eta\eta_4\eta_6},\ \frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2},\ \frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}},\ \frac{\eta\eta_4\eta_6^5}{\eta_2^2\eta_3^2\eta_{12}^2}.$$

The inclusion of these two chapters in this thesis is inspired by a paragraph on page 117 of [22], where Köhler discusses the formidabilily of the task of simplifying Mersmann's original proofs.

Though Mersmann's First Theorem is effective and though the algorithm of its proof produces quite moderate bounds for prime power levels, for a general level N, the bound gets extremely large as there seems to be no way to combine the bounds for the prime power divisors of N except taking the product over all of them.

We also did extensive computations to formulate an analog of Mersmann's Second Theorem for weight 1. But the list of simple holomorphic eta quotients of weight 1 is unexpectedly larger than the similar list given in Mersmann's Second Theorem (see Appendix A). For example, for weight 1, there are 496 simple holomorphic eta quotients of level 48, 478 simple holomorphic eta quotients of level 60 and 736 simple holomorphic eta quotients of level 144. Also, unlike in the case of weight 1/2, for weight 1 a simple holomorphic eta quotient could be found even at much higher levels. For instance,

$$\frac{\eta\,\eta_4\,\eta_6^2\,\eta_{48}^3\,\eta_{128}^2\,\eta_{768}}{\eta_2\,\eta_3\,\eta_{12}\,\eta_{16}\,\eta_{24}\,\eta_{96}\,\eta_{256}\,\eta_{384}}$$

is a simple holomorphic eta quotient of weight 1 and level 768 .

In Chapter 5, we consider the dual perspective of Mersmann's First Theorem, i. e., instead of considering eta quotients of a particular weight and arbitrary levels, we consider holomorphic eta quotients of a particular level and arbitrary weights. Among other results, here we proved that

Theorem. *The weight of any simple holomorphic eta quotient of level N is less than*

$$\frac{1}{2}\varphi\left(\mathrm{rad}(N)\right)\prod_{\substack{p|N\\ p\ prime}}\big((v_p(N)-1)(p-1)+2\big),$$

where $\mathrm{rad}(N)$ *is the product of the primes dividing N, φ denotes the Euler totient function and $v_p(N)$ denotes the p-adic valuation of N.*

It is easy to show that there exist only finitely many holomorphic eta quotients of a given level and weight (Corollary 1.50). Thus, from the above theorem, we conclude that

Corollary. *There are only finitely many simple holomorphic eta quotients of a given level.*

In particular, that implies:

Corollary. *There are only finitely many irreducible holomorphic eta quotients of a given level.*

In the last chapter, we construct examples of simple holomorphic eta quotients of various levels. For example, we show that

Theorem. *If $N \in \mathbb{N}$ is cubefree, then there exists an explicitly constructible simple holomorphic eta quotient of level N. Moreover, if N is a square, then we may choose this simple holomorphic eta quotient to be irreducible.*

In particular, in Chapter 6, we study simple holomorphic eta quotients of prime power levels intensively. For a prime level p, it is easy to show that the only simple holomorphic eta quotients of level p are η^p/η_p and η_p^p/η. For level p^2, we have:

Theorem. *Let p be a prime. The only simple holomorphic eta quotients of level p^2 are*

$$\frac{\eta^r\eta_{p^2}^{p-r}}{\eta_p},\ 1\le r\le p-1\quad \textit{and}\quad \frac{\eta_p^{sp+1}}{\eta^s\eta_{p^2}^s},\ 1\le s\le p.$$

This implies via the Irreducibility Conjecture for prime power levels (which we prove in Section 2.4):

Corollary. *For any prime p and $m \in \mathbb{N}$, the holomorphic eta quotients*

$$\frac{\eta_m^r \eta_{mp^2}^{p-r}}{\eta_{mp}} \quad \text{and} \quad \frac{\eta_{mp}^{sp+1}}{\eta_m^s \eta_{mp^2}^s}$$

are irreducible for all $r, s \in \mathbb{N}$ with $1 \leq r \leq p-1$ and $1 \leq s \leq p$.

We checked that for each prime $p \in \{2, \ldots, 23\}$, there does not exist any simple holomorphic eta quotient of level p^3. So, we conjecture that

Conjecture. *For a prime p, there does not exist any simple holomorphic eta quotient of level p^3*

For $n > 3$, we have proved that

Theorem. *For any integer $n > 3$ and p prime, the holomorphic eta quotient*

$$f_{p,n} := \begin{cases} \dfrac{\eta_p^p \eta_{p^{n-1}}^{(p-1)^2} \displaystyle\prod_{s=1}^{n/2-1} \eta_{p^{2s-1}}^{p^2-3p+1} \eta_{p^{2s}}^{p^2-2p+2}}{(\eta \eta_{p^n})^{p-1}} & \text{if } n \text{ is even.} \\[2ex] \dfrac{(\eta_p \eta_{p^{n-1}})^p \displaystyle\prod_{s=1}^{n-1} \eta_{p^s}^{p^2-3p+2}}{(\eta \eta_{p^n})^{p-1}} & \text{if } n \text{ is odd and } p \neq 2, \end{cases} \tag{8}$$

is simple of level p^n.

In particular, the above theorem together with the Reducibility Conjecture for prime power levels (which we prove in Section 2.4) implies:

Corollary. *For any integer $n > 3$, the eta quotient $f_{p,n}$ is irreducible.*

The significance of the special simple holomorphic eta quotients $f_{p,n}$ lies in the following conjecture (supported by a good amount of numerical evidence), with which we close this introduction:

Conjecture. *For any integer $n > 3$ and for any odd prime p, there are no simple holomorphic eta quotients of level p^n and of weight greater than that of $f_{p,n}$.*

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Part I

Eta Quotients and their factorization

Chapter 1

Preliminaries

The central characters of this thesis are eta quotients. They are weakly holomorphic modular forms which are nonzero on complex the upper half plane and which has uniform orders at certain sets of cusps of a suitable Hecke subgroup of the full modular group. We give the precise definitions of the relevant terms later in this chapter. We begin the first section by defining cusps.

In classical texts on automorphic forms (see [30], [38]), cusps of a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ are defined as the parabolic points of the subgroup. But usually all we care about are the equivalence classes of the parabolic points of the pertinent group while studying the singularities of a modular curve. Since the set of parabolic points of two mutually commensurable discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ are the same, naturally it became a common practice in modern literature instead (see [8], [13], [42]) to start directly with the set $\wp$ of the parabolic points of the full modular group $\mathrm{SL}_2(\mathbb{Z})$ and to define the cusps of a discrete subgroup $G \subset \mathrm{SL}_2(\mathbb{R})$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$ by the G-equivalence classes in $\wp$. We shall follow the later convention.

1.1 Cusps

In the following, by Γ_1 we denote the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The group Γ_1 acts on $\mathbb{P}^1(\mathbb{Q})$ transitively by *Möbius transformations*:

$$\gamma s = [a\alpha + b\lambda : c\alpha + d\lambda] \tag{1.1}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ and $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$. The restriction of this group action to a finite index subgroup $\Gamma \subset \Gamma_1$ defines an action of Γ on $\mathbb{P}^1(\mathbb{Q})$. The *cusps* of Γ

are the Γ-orbits in $\mathbb{P}^1(\mathbb{Q})$.

More generally, the group $\mathcal{G} := \mathbb{R}^+ \cdot \mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformations. Here by $\mathbb{R}^+$ we denote the set of positive real numbers and by $\mathrm{GL}_2^+(\mathbb{Q})$ we denote the group of 2×2 rational matrices with positive determinant. Let Γ and Γ' be two finite index subgroups of Γ_1 such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$ for some $\mathfrak{g} \in \mathcal{G}$. Then there is a natural map from the set of the cusps of Γ' onto the set of the cusps of Γ:

Lemma 1.1. *Let Γ and Γ' be two finite index subgroups of Γ_1 and let $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$. The endomorphism of $\mathbb{P}^1(\mathbb{Q})$ that maps s to $\mathfrak{g}s$ induces a surjection from the set of the cusps of Γ' to the set of the cusps of Γ.*

Proof. Let us consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}^1(\mathbb{Q}) & \xrightarrow{s \mapsto \mathfrak{g}s} & \mathbb{P}^1(\mathbb{Q}) \\ \downarrow & & \downarrow \\ \Gamma'\backslash\mathbb{P}^1(\mathbb{Q}) & \longrightarrow & \Gamma\backslash\mathbb{P}^1(\mathbb{Q}) \end{array}$$

Let $s_1, s_2 \in \mathbb{P}^1(\mathbb{Q})$ with $s_2 = \gamma' s_1$ for some $\gamma' \in \Gamma'$. Then we have $\gamma\mathfrak{g}s_1 = \mathfrak{g}s_2$, where $\gamma = \mathfrak{g}\gamma'\mathfrak{g}^{-1} \in \Gamma$. In other words, the above diagram commutes. Now a counterclockwise diagram-chasing shows that each element in $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ has a preimage in $\Gamma'\backslash\mathbb{P}^1(\mathbb{Q})$. So, the map $s \mapsto \mathfrak{g}s$ on $\mathbb{P}^1(\mathbb{Q})$ indeed induces a surjection from the set of the cusps Γ' to the set of the cusps of Γ. □

Corollary 1.2. *For two finite index subgroups of Γ_1 which are conjugate to each other by some element of $\mathcal{G}$, there is a natural bijection between their sets of cusps.*

Proof. Let Γ and Γ' be two finite index subgroups of Γ_1 and let $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} = \Gamma$. Then from Lemma 1.1, we know that the map $s \mapsto \mathfrak{g}s$ on $\mathbb{P}^1(\mathbb{Q})$ induces a map from the set of the cusps Γ' onto the set of the cusps of Γ. Similarly, the map $s \mapsto \mathfrak{g}^{-1}s$ on $\mathbb{P}^1(\mathbb{Q})$ induces a map from the set of the cusps of Γ onto the set of the cusps of Γ'. Clearly, the later map is the inverse of the former. Thus, we get a bijection between the sets of the cusps of Γ and Γ'. □

We identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$ via the canonical bijection that maps $[\alpha : \lambda]$ to α/λ if $\lambda \neq 0$ and to ∞ if $\lambda = 0$. For a subgroup $\Gamma \subset \Gamma_1$ and for $s \in \mathbb{P}^1(\mathbb{Q})$, by

$\operatorname{Stab}_s(\Gamma)$ we denote the stabilizer of s in Γ. For example, it follows immediately from (1.1) that

$$\operatorname{Stab}_\infty(\Gamma_1) = \{\pm T^n \mid n \in \mathbb{Z}\}, \tag{1.2}$$

where $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Lemma 1.3. *For $s \in \mathbb{P}^1(\mathbb{Q})$ and for any subgroup $\Gamma \subset \Gamma_1$, the map*

$$\Gamma \backslash \Gamma_1 / \operatorname{Stab}_s(\Gamma_1) \to \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$$

given by $\Gamma \cdot \gamma_1 \cdot \operatorname{Stab}_s(\Gamma_1) \mapsto \Gamma \cdot \gamma_1 s$ *is a bijection.*

Proof. Since Γ_1 acts transitively on $\mathbb{P}^1(\mathbb{Q})$, we can identify $\Gamma_1 / \operatorname{Stab}_s(\Gamma_1)$ and $\mathbb{P}^1(\mathbb{Q})$ via the natural bijection given by $\gamma_1 \cdot \operatorname{Stab}_s(\Gamma_1) \mapsto \gamma_1 s$. □

Lemma 1.4. *A finite index subgroup of Γ_1 has only finitely many cusps.*

Proof. Let Γ be a finite index subgroup of Γ_1. We consider the following diagram:

$$\begin{array}{ccc} \Gamma_1 & \longrightarrow & \Gamma_1 / \operatorname{Stab}_s(\Gamma_1) \\ \downarrow & & \downarrow \\ \Gamma \backslash \Gamma_1 & \longrightarrow & \Gamma \backslash \Gamma_1 / \operatorname{Stab}_s(\Gamma_1) \end{array}$$

Since group operations are associative, this diagram commutes. Since the set $\Gamma \backslash \Gamma_1$ of left cosets of Γ in Γ_1 is finite, the set of double cosets $\Gamma \backslash \Gamma_1 / \operatorname{Stab}_s(\Gamma_1)$ is also finite. Hence by Lemma 1.3, the group Γ has only finitely many cusps. □

We define the *Hecke subgroup* $\Gamma_0(N)$ and the *principal congruence subgroup* $\Gamma(N)$ of level N by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \;\middle|\; c \equiv 0 \pmod{N} \right\}$$

and

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \pmod{N},\, b \equiv 0 \pmod{N} \right\}.$$

A subgroup of Γ_1 that contains $\Gamma(N)$ is called a *congruence subgroup* of level N. The set of such subgroups of Γ_1 is closed under conjugation:

Lemma 1.5. *All the conjugates of a congruence subgroup of level N by the elements of Γ_1 are congruence subgroups.*

Proof. Since $\Gamma(N)$ is the kernel of the canonical homomorphism from Γ_1 to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, $\Gamma(N)$ is a normal subgroup of Γ_1. So, given any congruence subgroup of level N, $\Gamma(N)$ sits inside all its conjugates. □

Since $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is a finite group, it follows from the proof above that $\Gamma(N)$ (and hence any congruence subgroup) has finite index in Γ_1. For example, we have (see [34]):

$$[\Gamma_1 : \Gamma_0(N)] = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right). \tag{1.3}$$

So, we obtain the following corollary of Lemma 1.4:

Corollary 1.6. *A congruence subgroup of Γ_1 has only finitely many cusps.*

In particular, the group $\Gamma_0(N)$ has only finitely many cusps. Below we describe a system of representatives of the cusps of $\Gamma_0(N)$. We denote by $\mathcal{D}_N$ the set of divisors of N. For $x, y \in \mathbb{N}$, by (x, y) we denote the greatest common divisor of x and y.

Proposition 1.7. *The following set contains a unique representative for each cusp of $\Gamma_0(N)$:*

$$\Big\{[\alpha : \lambda] \;\Big|\; \lambda \in \mathcal{D}_N,\ \alpha \in \mathbb{Z},\ (\alpha, \lambda) = 1\Big\}/\sim,$$

where $[\alpha : \lambda] \sim [\beta : \lambda]$ if $\alpha \equiv \beta \pmod{(\lambda, N/\lambda)}$.

Proof. In the following, we shall use the notation '$\sim$' to denote a more general equivalence relation, viz., if $s, s' \in \mathbb{P}^1(\mathbb{Q})$ represent the same cusp of $\Gamma_0(N)$, we shall write $s \sim s'$. The proposition in particular claims that the former equivalence relation mentioned in the statement of the proposition is only a restriction of the later equivalence relation to the set $\{[\alpha : \lambda] \mid \lambda \in \mathcal{D}_N, \alpha \in \mathbb{Z}, (\alpha, \lambda) = 1\}$. Let $[a : b] \in \mathbb{P}^1(\mathbb{Q})$ with $(a, b) = 1$. First we show that $[a : b] \sim [a' : (N, b)]$ for some $a' \in \mathbb{Z}$. For this, it suffices to find integers x, y which satisfy

$$axN + by = (N, b). \tag{1.4}$$

such that $(xN, y) = 1$. Because, then we have a matrix of the form

$$\begin{pmatrix} * & * \\ xN & y \end{pmatrix} \in \Gamma_0(N)$$

which sends $[a:b]$ to some $[a':(N,b)] \in \mathbb{P}^1(\mathbb{Q})$.

If $x, y \in \mathbb{Z}$ satisfy (1.4), then we have $(x,y)=1$. So, it suffices to find integers x, y which satisfy (1.4) such that $(N,y)=1$. Since $(a,b)=1$, there indeed exist $x_0, y_0 \in \mathbb{Z}$ such that

$$ax_0N + by_0 = (aN,b) = (N,b). \tag{1.5}$$

So, all integer solutions of (1.4) are given by the pairs $x_0 + bt/(N,b)$, $y_0 - Nt/(N,b)$ for $t \in \mathbb{Z}$. If $n \in \mathbb{N}$ is coprime to $N/(N,b)$, then the arithmetic progression $\{y_0 - Nt/(N,b)\}_{t\in\mathbb{Z}}$ covers all the residue classes modulo n. In particular, if we choose n to be the product of all the primes which divide N but do not divide $N/(N,b)$, then there exists a $t_0 \in \mathbb{Z}$, such that

$$y_0 - Nt_0/(N,b) \equiv 1 \pmod{n}. \tag{1.6}$$

From (1.5) we have $(y_0, N/(N,b)) = 1$. Hence, $y_0 - Nt_0/(N,b)$ is also not divisible by any prime dividing $N/(N,b)$. Let $y = y_0 - Nt_0/(N,b)$ and $x = x_0 + bt_0/(N,b)$. Then $(N,y)=1$ and x,y satisfy (1.4). (In fact, Dirichlet's theorem on primes in arithmetic progression provides a shortcut to the existence of such a solution: since $(y_0, N/(N,b)) = 1$, one can choose y to be some prime in the arithmetic progression $\{y_0 - Nt/(N,b)\}_{t\in\mathbb{Z}}$ which does not divide N). Thus, we have proved that $[a:b] \sim [a':(N,b)]$ for some $a' \in \mathbb{Z}$.

Next we show that if $[\alpha:\lambda]$ and $[\beta:\lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $\lambda | N$ and $(\alpha,\lambda) = (\beta,\lambda) = 1$, then $[\alpha:\lambda] \sim [\beta:\lambda]$ if and only if $\alpha \equiv \beta \pmod{(\lambda, N/\lambda)}$.

First let us suppose, $[\alpha:\lambda] \sim [\beta:\lambda]$. Then there exists $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ such that $\gamma[\alpha:\lambda] = [\beta:\lambda]$ and hence $\gamma^{-1}[\beta:\lambda] = [\alpha:\lambda]$. In other words,

$$[a\alpha + b\lambda : c\alpha + d\lambda] = [\beta:\lambda] \quad \text{and} \quad [d\beta - b\lambda : -c\beta + a\lambda] = [\alpha:\lambda]. \tag{1.7}$$

Since

$$\gamma^{-1} \begin{pmatrix} a\alpha + b\lambda \\ c\alpha + d\lambda \end{pmatrix} = \begin{pmatrix} \alpha \\ \lambda \end{pmatrix}$$

and since $(\alpha,\lambda) = 1$, we have $(a\alpha + b\lambda, c\alpha + d\lambda) = 1$. Similarly, we have $(d\beta - b\lambda, -c\beta + a\lambda) = 1$. Hence, from (1.7) we get

$$a\alpha + b\lambda = \beta \tag{1.8}$$

$$c\alpha + d\lambda = \lambda \tag{1.9}$$

$$d\beta - b\lambda = \alpha \tag{1.10}$$

$$-c\beta + a\lambda = \lambda \tag{1.11}$$

From (1.11) we get $a = (c/\lambda)\beta + 1$. Since $\lambda|N$ and since $N|c$, indeed we have $\lambda|c$. From (1.8) we get $a\alpha \equiv \beta \pmod{\lambda}$ and therefore,

$$\big((c/\lambda)\beta + 1\big)a \equiv \beta \pmod{\lambda}. \tag{1.12}$$

Since N/λ divides c/λ, from (1.12) we get $\alpha \equiv \beta \pmod{(\lambda, N/\lambda)}$.

Now let us suppose $\alpha \equiv \beta \pmod{(\lambda, N/\lambda)}$. To show that $[\alpha : \lambda] \sim [\beta : \lambda]$, we shall construct a $\gamma \in \Gamma_0(N)$ such that $\gamma\,[\alpha : \lambda] = [\beta : \lambda]$.

Since $\alpha \equiv \beta \pmod{(\lambda, N/\lambda)}$, there exist $n_0, n_1 \in \mathbb{Z}$ such that

$$n_0\lambda + n_1(N/\lambda) = \beta - \alpha.$$

Hence, $\beta - \alpha \equiv n_1(N/\lambda) \pmod{\lambda}$. Since $(\alpha, \lambda) = (\beta, \lambda) = 1$, $\alpha\beta$ is invertible modulo λ. Let $(\alpha\beta)^{-1} \equiv n_2 \pmod{\lambda}$ and let $m \in \mathbb{Z}$ such that $m \equiv n_1 n_2 \pmod{\lambda}$. Then we have

$$\beta - \alpha \equiv m(N/\lambda)\alpha\beta \pmod{\lambda}. \tag{1.13}$$

Hence,

$$\frac{\beta - \alpha - m(N/\lambda)\alpha\beta}{\lambda}$$

is an integer. Let

$$\gamma := \begin{pmatrix} m\beta(N/\lambda) + 1 & \dfrac{\beta - \alpha - m(N/\lambda)\alpha\beta}{\lambda} \\ mN & -m\alpha(N/\lambda) + 1 \end{pmatrix}.$$

Then we have $\gamma \in \Gamma_0(N)$ and $\gamma\,[\alpha : \lambda] = [\beta : \lambda]$. □

Corollary 1.8. *Let $\Gamma \subset \Gamma_1$ be a finite index subgroup that is conjugate to $\Gamma_0(N)$ by some element of $\mathcal{G}$. The total number of cusps of Γ is*

$$\prod_{\substack{p|N \\ p\ prime}} \left(p^{\lfloor v_p(N)/2 \rfloor} + p^{\lceil v_p(N)/2 \rceil - 1}\right),$$

where for each prime p, the integer $v_p(N)$ denotes the p-adic valuation of N.

Proof. From Corollary 1.2 and Proposition 1.7, we know that there are exactly $\sum_{d|N} \varphi((d, N/d))$ cusps of Γ, where φ denotes the Euler totient function. Since for a prime p and a positive integer m, we have

$$\sum_{j=0}^{m} \varphi\big(p^{\min\{j,m-j\}}\big) = p^{\lfloor m/2 \rfloor} + p^{\lceil m/2 \rceil - 1} \tag{1.14}$$

and since the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) := \sum_{d|n} \varphi((d, n/d)) \tag{1.15}$$

is multiplicative, we get

$$f(N) = \prod_{\substack{p|N \\ p \text{ prime}}} f\big(p^{v_p(N)}\big) = \prod_{\substack{p|N \\ p \text{ prime}}} \big(p^{\lfloor v_p(N)/2 \rfloor} + p^{\lceil v_p(N)/2 \rceil - 1}\big).$$

□

Convention 1.9. Let Γ be a finite index subgroup fo Γ_1. Then we would refer to the cusp of Γ represented by $s \in \mathbb{P}^1(\mathbb{Q})$ as the cusp $s \pmod{\Gamma}$. If the group Γ is clear from the context, then we would simply write *the cusp* s instead of *the cusp* $s \pmod{\Gamma}$.

1.2 Modular forms of integral weight

Since the protagonists of our story (viz. holomorphic eta quotients) are modular forms with multiplier systems, in our definition of modular forms, multiplier systems would be innate:

A modular form is a complex-valued holomorphic function on $\mathfrak{H}$ which is invariant up to multiplication by some complex number of modulus 1 when slashed with any element of a suitable matrix group and moreover, the function is also holomorphic at the cusps of the group. We give the necessary definitions below.

The group $\mathrm{GL}_2^+(\mathbb{R})$ of 2×2 real matrices with positive determinants acts on the *complex upper half plane*

$$\mathfrak{H} := \{z \in \mathbb{C} \,\big|\, \mathrm{Im}(z) > 0\}$$

transitively by *Möbius transformations*:

$$\gamma z := \frac{\alpha z+\beta}{\lambda z+\delta} \quad \text{for } \gamma = \begin{pmatrix} \alpha & \beta \\ \lambda & \delta \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}) \text{ and } z \in \mathfrak{H}. \tag{1.16}$$

For $k \in \mathbb{Z}$, we define the *slash operator* $|_k$ on the space of complex-valued functions on $\mathfrak{H}$ as follows:

Let f be a complex-valued function on $\mathfrak{H}$ and let $\gamma = \begin{pmatrix} \alpha & \beta \\ \lambda & \delta \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. We define $f|_k\gamma : \mathfrak{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(z) := \det(\gamma)^{k/2} J(\gamma, z)^{-k}\; f(\gamma z), \tag{1.17}$$

where $J(\gamma, z) := cz+d$. In particular, the *automorphic factor* $J : \mathrm{GL}_2^+(\mathbb{R}) \times \mathfrak{H} \to \mathbb{C}$ satisfies the *cocycle condition*:

$$J(\gamma\gamma', z) = J(\gamma, \gamma' z)\, J(\gamma', z) \tag{1.18}$$

for all $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathfrak{H}$. For each $k \in \mathbb{Z}$, the slash operator $|_k$ defines an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of complex-valued functions on $\mathfrak{H}$:

Lemma 1.10. *Let f be a function on $\mathfrak{H}$, $k \in \mathbb{Z}$ and $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbb{R})$. Then*

$$f|_k(\gamma\gamma') = (f|_k\gamma)|_k\gamma'.$$

Proof. Since $\det(\gamma)$ and $\det(\gamma')$ are positive real numbers, we have

$$\det(\gamma\gamma')^\tau = \det(\gamma)^\tau \cdot \det(\gamma')^\tau \text{ for all } \tau \in \mathbb{C}. \tag{1.19}$$

In particular, (1.19) holds when τ is a half integer. Now, the claim follows immediately from the cocycle condition (1.18). □

Let Γ be a finite index subgroup of Γ_1. Let $k \in \mathbb{Z}$, let f be a meromorphic function on $\mathfrak{H}$ and let $v = v_f$ be a map from Γ to the circle group $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ such that

$$f|_k\gamma = v(\gamma) f \quad \text{for all} \quad \gamma \in \Gamma. \tag{1.20}$$

Then we say that *f transforms like a modular form of weight k with a multiplier system v on* Γ. From (1.20) and from Lemma 1.10, we have

$$v(\gamma\gamma')f = f|_k(\gamma\gamma') = (f|_k\gamma)|_k\gamma' = v(\gamma)v(\gamma')f \tag{1.21}$$

for all $\gamma, \gamma' \in \Gamma$. So, if we assume that f is not identically zero, then the map $v : \Gamma \to \mathbb{T}$ is necessarily a homomorphism.

Let $\Gamma \subset \Gamma_1$ be as before, let $\gamma_1 \in \Gamma_1$, let $s = \gamma_1\infty \in \mathbb{P}^1(\mathbb{Q})$ and let $s \pmod{\Gamma}$ be a cusp of Γ. We have

$$\begin{aligned} \gamma_1^{-1} \operatorname{Stab}_s(\Gamma)\gamma_1 &= \operatorname{Stab}_{\gamma_1^{-1}s}(\gamma_1^{-1}\Gamma\gamma) = \operatorname{Stab}_\infty(\gamma_1^{-1}\Gamma\gamma_1) \\ &\subset \operatorname{Stab}_\infty(\Gamma_1) = \{\pm T^n \,\big|\, n \in \mathbb{Z}\}, \end{aligned} \tag{1.22}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In particular there is a unique $w_s \in \mathbb{N}$ such that either T^{w_s} or $-T^{w_s}$ belongs to the group $\gamma_1^{-1} \operatorname{Stab}_s(\Gamma)\gamma_1$. The integer w_s is called the *width* of the cusp s.

For any subgroup $\Gamma \subset \Gamma_1$, we define the group $\overline{\Gamma}$ as the quotient of the join of the subgroups Γ and $\{\pm I\}$ by the later group:

$$\overline{\Gamma} := (\Gamma \vee \{\pm I\})/\{\pm I\}. \tag{1.23}$$

Here I is the identity matrix in Γ_1. Since both the points on $\mathfrak{H}$ and the elements in $\mathbb{P}^1(\mathbb{Q})$ are invariant under the Möbius transformation by $-I$, for each subgroup $\Gamma \subset \Gamma_1$, there is an action of $\overline{\Gamma}$ on $\mathfrak{H}$ (resp. $\mathbb{P}^1(\mathbb{Q})$) induced by the action of Γ on $\mathfrak{H}$ (resp. $\mathbb{P}^1(\mathbb{Q})$) by Möbius transformations.

Lemma 1.11. *Let Γ be a finite index subgroup of Γ_1 and let $s \in \mathbb{P}^1(\mathbb{Q})$. The width of the cusp s of Γ is well-defined.*

Proof. Let w_s denote the width of the cusp s of Γ. From the definition of width, it easily follows that

$$w_s = [\operatorname{Stab}_s(\overline{\Gamma}_1) : \operatorname{Stab}_s(\overline{\Gamma})]. \tag{1.24}$$

In particular, from (1.24), we get that w_s neither depends on the choice of the matrix $\gamma_1 \in \Gamma_1$ such that $s = \gamma_1\infty$ nor does it change if we replace s with some $s' \in \mathbb{P}^1(\mathbb{Q})$ which represent the same cusp of Γ. So, the width of a cusp is indeed well-defined. □

Under the natural bijections between the sets of cusps (see Lemma 1.1 and Corollary 1.2) of a congruence subgroup $\Gamma \subset \Gamma_1$ and its conjugates by the elements of Γ_1 (see Lemma 1.5), the width of each cusp remains invariant:

Lemma 1.12. *Let Γ and Γ' be two congruence subgroups of Γ_1 such that there exists $\gamma_1 \in \Gamma_1$ with $\gamma_1^{-1}\Gamma\gamma_1 = \Gamma'$. For $s \in \mathbb{P}^1(\mathbb{Q})$ and $s' := \gamma_1^{-1}s$, the widths of the cusps s of Γ and s' of Γ' are the same.*

Proof. Let $\gamma \in \Gamma_1$ be such that $s = \gamma\infty$. Let $\gamma' := \gamma_1^{-1}\gamma$, so $s' = \gamma'\infty$. Then we have

$$\begin{aligned}\gamma^{-1}\operatorname{Stab}_s(\Gamma)\gamma &= \gamma^{-1}\operatorname{Stab}_s(\gamma_1\Gamma'\gamma_1^{-1})\gamma = \gamma^{-1}\gamma_1\operatorname{Stab}_{\gamma_1^{-1}s}(\Gamma')\gamma_1^{-1}\gamma \\ &= \gamma'^{-1}\operatorname{Stab}_{s'}(\Gamma')\gamma'.\end{aligned}$$

Hence, the width of the cusp s of Γ is equal to the width of the cusp s' of Γ'. □

Lemma 1.13. *Let $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = 1$. The width of the cusp s of $\Gamma_0(N)$ is*

$$\frac{N}{(\lambda^2, N)}.$$

Proof. Since $(\alpha, \lambda) = 1$, there exists $\beta, \delta \in \mathbb{Z}$ such that $\gamma_1 = \begin{pmatrix}\alpha & \beta \\ \lambda & \delta\end{pmatrix} \in \Gamma_1$. Let w_s be the width of s. Since, $-I \in \Gamma_0(N)$, we have $T^{w_s} = \begin{pmatrix}1 & w_s \\ 0 & 1\end{pmatrix} \in \gamma_1^{-1}\operatorname{Stab}_s(\Gamma_0(N))\gamma_1$, i.e.

$$\gamma_1 \begin{pmatrix}1 & w_s \\ 0 & 1\end{pmatrix} \gamma_1^{-1} \in \operatorname{Stab}_s(\Gamma_0(N)) \subset \Gamma_0(N). \tag{1.25}$$

Multiplying the matrices, we see that (1.25) holds if and only if

$$\lambda^2 w_s \equiv 0 \pmod{N}. \tag{1.26}$$

Since w_s is the smallest positive integer satisfying (1.26), it follows that $w_s = \dfrac{N}{(\lambda^2, N)}$. □

Let Γ be a finite index subgroup of Γ_1, let $\gamma_1 \in \Gamma_1$, let $s = \gamma_1\infty$ and let w_s be the width of the cusp s of Γ. If $T^{w_s} \in \gamma_1^{-1}\operatorname{Stab}_s(\Gamma)\gamma_1$, we call s a *regular cusp* of Γ. Otherwise, we call it an *irregular cusp*. We define the *irregularity indicator* χ_s by

$$\chi_s := \begin{cases}1 & \text{if } s \text{ is an irregular cusp of } \Gamma. \\ 0 & \text{otherwise.}\end{cases} \tag{1.27}$$

In particular, for the cusp $s = \gamma_1\infty$ of Γ, we have $e^{\pi i \chi_s} T^{w_s} \in \gamma_1^{-1}\operatorname{Stab}_s(\Gamma)\gamma_1$. Let f be a meromorphic function on $\mathfrak{H}$ that transforms like a modular form of weight $k \in \mathbb{Z}$ with a multiplier system v on Γ. We define the *cusp-parameter* $\kappa_s = \kappa_s(v) \in [0,1)$ by

$$\kappa_s := \frac{1}{2\pi i} \operatorname{Log} v(\gamma_1 e^{\pi i \chi_s} T^{w_s} \gamma_1^{-1}), \tag{1.28}$$

where Log denotes the principal branch of complex logarithm. Define $f_{\gamma_1} : \mathfrak{H} \to \mathbb{C}$ by

$$f_{\gamma_1}(z) := e^{-\pi i (k\chi_s + 2\kappa_s) z / w_s} (f|_k \gamma_1)(z) \tag{1.29}$$

Then we have

$$f_{\gamma_1}(z + w_s) = e^{-\pi i (k\chi_s + 2\kappa_s) z / w_s} e^{-2\pi i \kappa_s} \left((f|_k(\gamma_1 e^{\pi i \chi_s} T^{w_s} \gamma_1^{-1}))|_k \gamma_1 \right)(z) = f_{\gamma_1}(z). \tag{1.30}$$

So, f_{γ_1} has a Fourier expansion: $f_{\gamma_1}(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z / w_s}$, from which we get

$$(f|_k \gamma_1)(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i (n + k\chi_s/2 + \kappa_s) z / w_s}. \tag{1.31}$$

Convention 1.14. For all $r \in \mathbb{C}$ we define the function $q^r : \mathbb{C} \to \mathbb{C}$ by

$$q^r(z) := e^{2\pi i r z}. \tag{1.32}$$

Henceforth, we adopt a common abuse of notation by writing q^r instead of $q^r(z)$.

Definition 1.15. Let Γ be a finite index subgroup of Γ_1, let f be a holomorphic function on $\mathfrak{H}$, let $k \in \mathbb{Z}$ and let $v : \Gamma \to \mathbb{T}$ be a homomorphism such that

$$f|_k \gamma = v(\gamma) f \quad \textit{for all} \quad \gamma \in \Gamma. \tag{1.33}$$

Moreover, we presume that f is *meromorphic at the cusps*, i. e, for each $\gamma_1 \in \Gamma_1$, we assume that there exists an $m_s \in \mathbb{Z}$, where $s = \gamma_1 \infty$ such that $f|_k \gamma_1$ has a series expansion of the form

$$(f|_k \gamma_1)(z) = \sum_{n = m_s}^{\infty} a_n q^{(n + k\chi_s/2 + \kappa_s)/w_s}, \tag{1.34}$$

where χ_s is the irregularity indicator, w_s is the width and κ_s is the cusp-parameter of the cusp s of Γ. Then we call f a *weakly holomorphic modular form* of weight k with a multiplier system v on Γ. We define the *order* of f at the cusp s of Γ by

$$\operatorname{ord}_s(f; \Gamma) := m_s + k\chi_s/2 + \kappa_s. \tag{1.35}$$

We call f *holomorphic at the cusp* s if $\mathrm{ord}_s(f;\Gamma) \geq 0$. If f is holomorphic at all the cusps of Γ, we call f a *modular form* of weight k with a multiplier system v on Γ.

Lemma 1.16. *Let Γ be a finite index subgroup of Γ_1, let $s \in \mathbb{P}^1(\mathbb{Q})$ and let f be a weakly holomorphic modular form of integral weight k with a multiplier system v on Γ. The cusp-parameter w.r.t. v at the cusp s of Γ and the order of f at the cusp s are well-defined.*

Proof. Let χ_s be the irregularity indicator and let w_s be the width of the cusp s of Γ. Let $\gamma_1, \gamma_1' \in \Gamma_1$ with $s = \gamma_1\infty = \gamma_1'\infty$. Then $\gamma_1^{-1}\gamma'_1 \in \mathrm{Stab}_\infty(\Gamma_1) = \{\pm T^n \,|\, n \in \mathbb{Z}\}$. Let $n_0 \in \mathbb{Z}$ be such that $\gamma_1' = \pm\gamma_1 T^{n_0}$. Then we have $\gamma_1' e^{\pi i \chi_s} T^{w_s} {\gamma_1'}^{-1} = \gamma_1 e^{\pi i \chi_s} T^{w_s} \gamma_1^{-1}$. So, from (1.28) it follows that the cusp-parameter κ_s is indeed well-defined.

Let $m \in \mathbb{Z}$ be such that $f|_k\gamma_1$ has a series expansion of the form

$$(f|_k\gamma_1)(z) = \sum_{n=m}^{\infty} a_n q^{(n + k\chi_s/2 + \kappa_s)/w_s}, \tag{1.36}$$

Then we have

$$(f|_k{\gamma'}_1)(z) = (\pm 1)^k (f|_k\gamma_1)(z + n_0) = \sum_{n=m}^{\infty} b_n q^{(n + k\chi_s/2 + \kappa_s)/w_s},$$

where $b_n = (\pm 1)^k a_n e^{2\pi i n_0 (n + k\chi_s/2 + \kappa_s)/w_s}$. So, $\mathrm{ord}_s(f;\Gamma)$ is indeed well-defined. □

Convention 1.17. In order to lighten the notation, we shall write $\mathrm{ord}_s(f)$ instead of $\mathrm{ord}_s(f;\Gamma)$ whenever the group Γ is clear from the context.

Lemma 1.18. *Let f_1 and f_2 be weakly holomorphic modular forms on a finite index subgroup Γ of Γ_1. Then at each cusp s of Γ, we have*

$$\mathrm{ord}_s(f_1 f_2) = \mathrm{ord}_s(f_1) + \mathrm{ord}_s(f_2).$$

Proof. Let $f = f_1 f_2$ and let $k = k_1 k_2$, where k_1 and k_2 are respectively the weights of f_1 and f_2. Let $\gamma_1 \in \Gamma_1$ such that $s = \gamma_1\infty$. Now, the claim follows trivially from (1.34) and (1.35), since we have

$$f|_k\gamma_1 = (f_1|_{k_1}\gamma_1) \cdot (f_2|_{k_2}\gamma_1).$$

□

Now we prove a theorem that will be quite useful later. To state the theorem, we recall that by the notation $\mathcal{G}$, we denote the group $\mathbb{R}^+\cdot \mathrm{GL}_2^+(\mathbb{Q})$.

Theorem 1.19. *Let Γ and Γ' be two finite index subgroups of Γ_1 and let $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$. Let f be a weakly holomorphic modular form of integral weight k with a multiplier system v on Γ. Then*

(*a*) *$f|_k\,\mathfrak{g}$ is a weakly holomorphic modular form of weight k on Γ' with the multiplier system $v_{\mathfrak{g}}$ defined by*

$$v_{\mathfrak{g}}(\gamma') := v(\mathfrak{g}\gamma'\mathfrak{g}^{-1}) \text{ for all } \gamma' \in \Gamma'. \tag{1.37}$$

(*b*) *The orders of f and $f|_k\,\mathfrak{g}$ at the cusps of Γ and Γ' are related by*

$$\mathrm{ord}_s(f|_k\,\mathfrak{g}\,;\Gamma') = \frac{\delta^2 w_s}{\det(\mathfrak{g}^*)\, w_{\mathfrak{g}s}}\,\mathrm{ord}_{\mathfrak{g}s}(f;\Gamma) \text{ for } s \in \mathbb{P}^1(\mathbb{Q}), \tag{1.38}$$

where w_s is the width of the cusp s of Γ', $w_{\mathfrak{g}s}$ is the width of the cusp $\mathfrak{g}s$ of Γ, $\mathfrak{g}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary integer matrix that is equal to $\mathfrak{g}$ up to multiplication by a positive real number and $\delta = (a\alpha + b\lambda, c\alpha + d\lambda)$, where $\alpha, \lambda \in \mathbb{Z}$ with $(\alpha, \lambda) = 1$ such that $s = [\alpha : \lambda]$.

Proof. (*a*) For $\gamma' \in \Gamma'$, we have

$$(f|_k\,\mathfrak{g})|_k\gamma' = (f|_k\,\mathfrak{g}\gamma'\mathfrak{g}^{-1})|_k\,\mathfrak{g} = v_{\mathfrak{g}}(\gamma')f|_k\,\mathfrak{g}. \tag{1.39}$$

So, $f|_k\,\mathfrak{g}$ transforms like a modular form of weight k with the multiplier system $v_{\mathfrak{g}}$ on Γ'. Clearly, the orders of vanishing of f and $f|_k\,\mathfrak{g}$ at the points on $\mathfrak{H}$ are related by

$$\mathrm{ord}_P(f|_k\,\mathfrak{g}) = \mathrm{ord}_{\mathfrak{g}P}(f) \text{ for } P \in \mathfrak{H}. \tag{1.40}$$

Therefore, $f|_k\,\mathfrak{g}$ is holomorphic on $\mathfrak{H}$ if and only if so is f. Now from part (*b*) of the theorem and from Lemma 1.1, it follows that $f|_k\,\mathfrak{g}$ is meromorphic at the cusps of Γ' if and only if f is meromorphic at the cusps of Γ. Since f is a weakly holomorphic modular form on Γ, we conclude that $f|_k\,\mathfrak{g}$ is a weakly holomorphic modular form on Γ'.

(*b*) Let r be a positive real number and $\mathfrak{g}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix such that $\mathfrak{g} = r\mathfrak{g}^*$. Let $\gamma = \begin{pmatrix} \alpha & \beta \\ \lambda & \nu \end{pmatrix} \in \Gamma_1$ and let $s = \gamma\infty$ be a cusp of Γ'. Let $a_1, b_1, c_1, d_1 \in \mathbb{Z}$ be such that $\mathfrak{g}^*\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$. Let $\delta = (a_1, c_1)$, let $a_1' = a_1/\delta$ and let $c_1' = c_1/\delta$. Then there exist $x, y \in \mathbb{Z}$ such that $a_1'x + c_1'y = 1$. In other words, $\gamma_1 := \begin{pmatrix} a_1' & -y \\ c_1' & x \end{pmatrix} \in \Gamma_1$. Since $a_1d_1 - b_1c_1 = \det(\mathfrak{g}^*\gamma) = \det(\mathfrak{g}^*)$ and since $\delta = (a_1, c_1)$, we have $u := \dfrac{\det(\mathfrak{g}^*)}{\delta} \in \mathbb{Z}$. Let $\gamma_2 := r\begin{pmatrix} \delta & b_1x + d_1y \\ 0 & u \end{pmatrix}$. Since $\mathfrak{g}\gamma = \gamma_1\gamma_2$, we have

$$((f|_k\mathfrak{g})|_k\gamma)(z) = ((f|_k\gamma_1)|_k\gamma_2)(z) = \frac{\delta^k}{\det(\mathfrak{g}^*)^{k/2}}(f|_k\gamma_1)(\gamma_2(z)). \tag{1.41}$$

Since $\mathfrak{g}s = \gamma_1\infty$, from (1.34) we get

$$(f|_k\gamma_1)(z) = \sum_{n=m_{\mathfrak{g}s}}^{\infty} a_n q^{(n+k\chi_{\mathfrak{g}s}/2+\kappa_{\mathfrak{g}s})/w_{\mathfrak{g}s}} \tag{1.42}$$

where $\chi_{\mathfrak{g}s}$ is the irregularity indicator, $w_{\mathfrak{g}s}$ is the width and $\kappa_{\mathfrak{g}s}$ is the cusp-parameter of the cusp $\mathfrak{g}s$ of Γ. We have

$$\gamma_2(z) = \frac{\delta^2 z}{\det(\mathfrak{g}^*)} + \rho_{\mathfrak{g}s}, \tag{1.43}$$

where $\rho_{\mathfrak{g}s} = \rho_{\mathfrak{g}s}(b_1, d_1) \in \mathbb{C}$ can be chosen from a certain arithmetic progression. So, from (1.41), (1.42) and (1.43), we get that

$$\begin{aligned} ((f|_k\mathfrak{g})|_k\gamma)(z) &= \frac{\delta^k}{\det(\mathfrak{g}^*)^{k/2}} \sum_{n=m_{\mathfrak{g}s}}^{\infty} a_n e^{\frac{2\pi i\gamma_2(z)\,(n+k\chi_{\mathfrak{g}s}/2+\kappa_{\mathfrak{g}s})}{w_{\mathfrak{g}s}}} \\ &= \sum_{n=m_{\mathfrak{g}s}}^{\infty} b_n q^{\frac{\delta^2\,(n+k\chi_{\mathfrak{g}s}/2+\kappa_{\mathfrak{g}s})}{\det(\mathfrak{g}^*)\,w_{\mathfrak{g}s}}}, \end{aligned} \tag{1.44}$$

where $b_n := \dfrac{a_n\delta^k e^{2\pi i\rho_{\mathfrak{g}s}(n+k\chi_{\mathfrak{g}s}/2+\kappa_{\mathfrak{g}s})/w_{\mathfrak{g}s}}}{\det(\mathfrak{g}^*)^{k/2}}$. It follows from (1.35) and (1.44) that

$$\mathrm{ord}_s(f|_k\,\mathfrak{g}\,;\Gamma') = \frac{\delta^2 w_s}{\det(\mathfrak{g}^*)\,w_{\mathfrak{g}s}}\,\mathrm{ord}_{\mathfrak{g}s}(f;\Gamma), \tag{1.45}$$

where w_s is the width of the cusp s of Γ' and $w_{\mathfrak{g}s}$ is the width of the cusp $\mathfrak{g}s$ of Γ. □

From (1.40) and (1.38), it follows that the map $f \mapsto f|_k\mathfrak{g}$ sends a holomorphic modular form to a holomorphic modular form. We call such a map a *holomorphy-preserving map.* In particular, we have:

Corollary 1.20. *Let Γ and Γ' be two finite index subgroups of Γ_1 and let $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$. Let f be a modular form of integral weight k with a multiplier system v on Γ. Then $f|_k\mathfrak{g}$ is a modular form of weight k on Γ' with the multiplier system $v_{\mathfrak{g}}$ defined by* (1.37)*.*

Corollary 1.21. *For two finite index subgroups of Γ_1 which are conjugate to each other by some element of $\mathcal{G}$, there is a natural holomorphy-preserving isomorphism between the spaces of weakly holomorphic modular forms of integral weight k on them.*

Corollary 1.22. *Let Γ be a finite index subgroup of Γ and let $k \in \mathbb{Z}$. For each $\mathfrak{g}$ belonging to the normalizer of Γ in $\mathcal{G}$, the map $f \mapsto f|_k\mathfrak{g}$ is a holomorphy-preserving automorphism of the space of weakly holomorphic modular forms of weight k on Γ.*

Convention 1.23. Henceforth, by a *modular form* we shall mean a *modular form with a multiplier system.*

1.3 Modular forms of rational weight

We have seen in (1.21) that the multiplier system of modular forms of integral weight is necessarily a homomorphism from the relevant matrix group to the circle group. But if we define the automorphic factor for a non-integral weight modular form via the choice of a particular branch of the complex logarithm, then the multiplier system remains no longer a homomorphism in general (see [18], [22], [20]). However, we can get around this problem by extending the relevant matrix group in a suitable way.

We shall define rational weight modular forms in such a way that the well-developed notion of half-integral weight modular forms fits into it. Usually, the automorphic factor of a half-integral weight modular form is defined via the automorphic factor of the Jacobi theta series $\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$ (see [39]). But it could have been defined via the automorphic factor of some other modular form of weight half as well. So, rather than defining the automorphic factor as the one associated with a particular modular form, we replace the relevant matrix group with its central extension by a group of roots of unity and work with the canonical automorphic factor. In particular, for the weight half case, instead of taking a suitable embedding of the relevant matrix group into its central extension by μ_4, we replace our matrix group with its central extension by μ_2, where μ_n denotes the group of complex n-th roots of unity. All the apparent discrepancies with the usual notion of half-integral weight modular forms those may seem to arise due to these changes of convention, would be taken care of by the introduction of the multiplier system.

Let $G := \mathrm{GL}_2^+(\mathbb{R})$ and for $n \in \mathbb{N}$, let $\widetilde{G}_n$ be the central extension of G by the group μ_n given by

$$\widetilde{G}_n := \left\{ (\gamma, \alpha) \;\middle|\; \begin{array}{c} \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ and } \alpha : \mathfrak{H} \to \mathbb{C} \text{ holomorphic with} \\ \alpha(z)^n = \dfrac{cz+d}{\sqrt{\det(\gamma)}} \end{array} \right\}. \tag{1.46}$$

The law of composition in $\widetilde{G}_n$ is defined by

$$(\gamma, \alpha) \cdot (\gamma', \beta) = (\gamma\gamma', \lambda), \text{ where } \lambda(z) = \alpha(\gamma' z)\beta(z) \text{ for all } z \in \mathfrak{H}. \tag{1.47}$$

For any subgroup $H \subset G$, we define the group $\widetilde{H}_n \subset \widetilde{G}_n$ by replacing G with H in (1.46).

For $r \in \mathbb{Q}$, we define the *slash operator* $|_r$ on the space of complex-valued functions on $\mathfrak{H}$ as follows:

Let ℓ be the smallest positive integer such that $\ell r \in \mathbb{Z}$ and let $k = \ell r$. Given a complex-valued function f on $\mathfrak{H}$ and $\widetilde{\gamma} = (\gamma, \alpha) \in \widetilde{G}_\ell$, we define $f|_r \widetilde{\gamma} : \mathfrak{H} \to \mathbb{C}$ by

$$(f|_r \widetilde{\gamma})(z) := \det(\gamma)^{k/2} J(\widetilde{\gamma}, z)^{-k} \; f(\gamma z), \tag{1.48}$$

where $J(\widetilde{\gamma}, z) := \alpha(z)$. From the law of composition in $\widetilde{G}_\ell$, it follows that the *automorphic factor* $J : \widetilde{G}_\ell \times \mathfrak{H} \to \mathbb{C}$ again satisfies the *cocycle condition*:

$$J(\widetilde{\gamma}\widetilde{\gamma}', z) = J(\widetilde{\gamma}, \widetilde{\gamma}' z)\, J(\widetilde{\gamma}', z) \tag{1.49}$$

for all $\widetilde{\gamma}, \widetilde{\gamma}' \in \widetilde{G}_\ell$ and $z \in \mathfrak{H}$. For r and ℓ as above, the slash operator $|_r$ defines an action of $\widetilde{G}_\ell$ on the set of complex-valued functions on $\mathfrak{H}$:

Lemma 1.24. *Let f be a function on $\mathfrak{H}$, let $r \in \mathbb{Q}$ and let ℓ be the smallest positive integer such that $\ell r \in \mathbb{Z}$. Then for $\widetilde{\gamma}, \widetilde{\gamma}' \in \widetilde{G}_\ell$, we have*

$$f|_r(\widetilde{\gamma}\widetilde{\gamma}') = (f|_r\widetilde{\gamma})|_r\widetilde{\gamma}'.$$

Proof. Similar to the integral weight case, follows immediately from the cocycle condition (1.49). □

Let Γ be a finite index subgroup of Γ_1, let $r \in \mathbb{Q}$ and let ℓ be the smallest positive integer such that $\ell r \in \mathbb{Z}$. Let f be a meromorphic function on $\mathfrak{H}$ and let $v = v_f$ be a map from $\widetilde{\Gamma}_\ell$ to the circle group $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ such that

$$f|_r\widetilde{\gamma} = v(\widetilde{\gamma})f \quad \text{for all} \quad \widetilde{\gamma} \in \widetilde{\Gamma}_\ell. \tag{1.50}$$

Then we say that f *transforms like a modular form of weight r with a multiplier system v on* Γ. From (1.50) and from Lemma 1.24, we have

$$v(\widetilde{\gamma}\widetilde{\gamma}')f = f|_r(\widetilde{\gamma}\widetilde{\gamma}') = (f|_r\widetilde{\gamma})|_r\widetilde{\gamma}' = v(\widetilde{\gamma})v(\widetilde{\gamma}')f \tag{1.51}$$

for all $\widetilde{\gamma}, \widetilde{\gamma}' \in \widetilde{\Gamma}_\ell$. So, if we assume that f is not identically zero, then the map $v : \widetilde{\Gamma}_\ell \to \mathbb{T}$ is necessarily a homomorphism.

Definition 1.25. Let Γ be a finite index subgroup of Γ_1. Let f be a meromorphic function on $\mathfrak{H}$ that transforms like a modular form of weight $r \in \mathbb{Q}$ with a multiplier system v on Γ and let $\ell \in \mathbb{N}$ be such that $\ell r \in \mathbb{Z}$. Let $k = \ell r$. Then we call f a *weakly holomorphic modular form* (resp. a *modular form*) of weight r on Γ if f^ℓ is a *weakly holomorphic modular form* (resp. a *modular form*) of weight k on Γ. The *order* of f at a cusp s of Γ is naturally defined by

$$\operatorname{ord}_s(f) := \frac{1}{\ell}\operatorname{ord}_s(f^\ell).$$

In particular, f is meromorphic (resp. holomorphic) at the cusp s if and only if f^ℓ is meromorphic (resp. holomorphic) at s.

A generalization of Theorem 1.19 follows readily:

Theorem 1.19′. *Let Γ and Γ' be two finite index subgroups of Γ_1 and let $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$. Let f be a weakly holomorphic modular form of weight $r \in \mathbb{Q}$ with a multiplier system v on Γ and let ℓ be the smallest positive number such that $\ell r \in \mathbb{Z}$. Then for $\widetilde{\mathfrak{g}} = (\mathfrak{g}, \alpha) \in \widetilde{\mathcal{G}}_\ell$, we have* (*a*) $f|_r\,\widetilde{\mathfrak{g}}$ *is a weakly holomorphic modular form of weight* r *on* Γ' *with the multiplier system* $v_{\widetilde{\mathfrak{g}}}$ *defined by*

$$v_{\widetilde{\mathfrak{g}}}(\widetilde{\gamma}') := v(\widetilde{\mathfrak{g}}\widetilde{\gamma}'\widetilde{\mathfrak{g}}^{-1}) \text{ for all } \widetilde{\gamma}' \in \widetilde{\Gamma}'_\ell. \tag{1.52}$$

(*b*) *The orders of* f *and* $f|_r\,\widetilde{\mathfrak{g}}$ *at the cusps of* Γ *and* Γ' *are related by*

$$\mathrm{ord}_s(f|_k\,\mathfrak{g}\,;\Gamma') = \frac{\delta^2 w_s}{\det(\mathfrak{g}^*)\, w_{\mathfrak{g}s}}\,\mathrm{ord}_{\mathfrak{g}s}(f;\Gamma) \text{ for } s \in \mathbb{P}^1(\mathbb{Q}), \tag{1.53}$$

where w_s *is the width of the cusp* s *of* Γ', $w_{\mathfrak{g}s}$ *is the width of the cusp* $\mathfrak{g}s$ *of* Γ, $\mathfrak{g}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is an arbitrary integer matrix that is equal to* $\mathfrak{g}$ *up to multiplication by a positive real number and* $\delta = (a\alpha + b\lambda, c\alpha + d\lambda)$, *where* $\alpha, \lambda \in \mathbb{Z}$ *with* $(\alpha, \lambda) = 1$ *such that* $s = [\alpha : \lambda]$.

Proof. (*a*) Similar to the integral weight case, follows from Lemma 1.24, part (*b*) and the fact that the orders of f and $f|_r\,\widetilde{\mathfrak{g}}$ at the points of $\mathfrak{H}$ are related by

$$\mathrm{ord}_P(f|_r\,\widetilde{\mathfrak{g}}) = \mathrm{ord}_{\mathfrak{g}P}(f) \text{ for } P \in \mathfrak{H}. \tag{1.54}$$

(*b*) Follows from the fact that $(f|_r\,\widetilde{\mathfrak{g}})^\ell = f^\ell|_k\,\mathfrak{g}$ and from Theorem 1.19(*b*). □

Corollary 1.20′. *Let* Γ *and* Γ' *be two finite index subgroups of* Γ_1 *and let* $\mathfrak{g} \in \mathcal{G}$ *such that* $\mathfrak{g}\Gamma'\mathfrak{g}^{-1} \subset \Gamma$. *Let* f *be a modular form of rational weight* r *with a multiplier system* v *on* Γ *and let* ℓ *be the smallest positive number such that* $\ell r \in \mathbb{Z}$. *Then for* $\widetilde{\mathfrak{g}} = (\mathfrak{g}, \alpha) \in \widetilde{\mathcal{G}}_\ell$, $f|_r\,\widetilde{\mathfrak{g}}$ *is a modular form of weight* r *on* Γ' *with the multiplier system* $v_{\widetilde{\mathfrak{g}}}$ *defined by* (1.52).

Corollary 1.21′. *For two finite index subgroups of* Γ_1 *which are conjugate to each other by some element of* $\mathcal{G}$, *there is a natural holomorphy-preserving isomorphism between the spaces of weakly holomorphic modular forms of rational weight* r *on them.*

Corollary 1.22′. *Let* Γ *be a finite index subgroup of* Γ, *let* $r \in \mathbb{Q}$ *and let* ℓ *be the smallest positive integer such that* $\ell r \in \mathbb{N}$. *For each* $\widetilde{\mathfrak{g}} = (\mathfrak{g}, \alpha) \in \widetilde{\mathcal{G}}_\ell$ *with* $\mathfrak{g}$ *in the normalizer of* Γ *in* $\mathcal{G}$, *the map* $f \mapsto f|_r\,\widetilde{\mathfrak{g}}$ *is a holomorphy-preserving automorphism of the space of weakly holomorphic modular forms of weight* r *on* Γ.

We can extend our definition of modular forms to *generalized modular forms of rational weight* by defining *generalized multiplier systems* which take arbitrary nonzero complex values instead of taking values only in $\mathbb{T}$. The Valence Formula for integral weight modular forms with trivial multiplier systems on Γ_1 also holds for generalized modular forms of arbitrary rational weights:

Theorem 1.26 (Valence Formula)**.** *Let $r \in \mathbb{Q}$ and let $f : \mathfrak{H} \to \mathbb{C}$ be a meromorphic function that transforms like a generalized modular form of weight r on Γ_1 and which is meromorphic at ∞. Then we have*

$$\operatorname{ord}_\infty(f) + \sum_{P \in \Gamma_1 \backslash \mathfrak{H}} \frac{1}{n_P} \operatorname{ord}_P(f) = \frac{r}{12},$$

where n_P is the number of elements in the stabilizer of P in the group $\Gamma_1/\{\pm I\}$. Here I is the identity matrix in Γ_1.

The above theorem will follow from the usual Valence Formula for modular forms with integral weights. But before proceeding further, we make a general observation:

Let Γ be a finite index subgroup of Γ_1, and let f be a meromorphic function on $\mathfrak{H}$ that transforms like a generalized modular form of rational weight r with a generalized multiplier system v on Γ. If f is not identically zero, then similarly as in (1.51), we get that the map $v : \widetilde{\Gamma}_\ell \to \mathbb{C}^*$ is necessarily a homomorphism, where ℓ is the smallest positive integer such that $\ell r \in \mathbb{Z}$. In particular, if $\Gamma = \Gamma_1$, then we have:

Lemma 1.27. *Let $r \in \mathbb{Q}$ and let $f : \mathfrak{H} \to \mathbb{C}$ be a meromorphic function that transforms like a generalized modular form of weight r on Γ_1. Then the multiplier system v of f is a homomorphism from $(\widetilde{\Gamma}_1)_\ell$ to $\mu_{12\ell}$, where ℓ is the least positive integer such that $\ell r \in \mathbb{Z}$ and $\mu_{12\ell}$ is the group of complex 12ℓ-th roots of unity.*

Proof. Since the map $v : (\widetilde{\Gamma}_1)_\ell \to \mathbb{C}^*$ is a homomorphism, so is the map $v^\ell : \Gamma_1 \to \mathbb{C}^*$. Also, we have (see [43])

$$\Gamma_1 \simeq \mu_4 *_{\mu_2} \mu_6, \tag{1.55}$$

i. e. the group Γ_1 is isomorphic to the amalgameted free product of $\mu_4 \simeq \langle S \rangle$ and $\mu_6 \simeq \langle U \rangle$, amalgameted along $\mu_2 \simeq \langle -I \rangle$, where

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; U := TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } I \text{ is the identity matrix in } \Gamma_1.$$

Hence in particular, we have

$$\Gamma_1^{ab} \simeq \mu_4 \vee_{\mu_2} \mu_6 = \mu_{12}, \tag{1.56}$$

i. e. the abelianization of Γ_1 is isomorphic to μ_{12} since the later group is the join of its subgroups μ_4 and μ_6 along μ_2. Since $\mathbb{C}^*$ is abelian, the homomorphism $v^\ell : \Gamma_1 \to \mathbb{C}^*$ factors through Γ_1^{ab}:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\quad v^\ell \quad} & \mathbb{C}^* \\ {\scriptstyle \mathscr{ab}} \downarrow & & \uparrow {\scriptstyle \mathscr{i}} \\ \Gamma_1^{ab} & \xrightarrow{\quad \widetilde{v^\ell} \quad} & \mu_{12} \end{array}$$

where $\mathscr{ab} : \Gamma_1 \to \Gamma_1^{ab}$ is the abelianization map and $\mathscr{i} : \mu_{12} \to \mathbb{C}^*$ is the inclusion. So, we have $v^\ell(\Gamma_1) = \mathscr{i} \circ \widetilde{v^\ell}\,(\Gamma_1^{ab}) \subseteq \mu_{12}$ and therefore, $v^{12\ell} \equiv 1$. Thus, we conclude that v is in fact, a homomorphism from $(\widetilde{\Gamma}_1)_\ell$ to $\mu_{12\ell}$. □

Example 1.28. *The homomorphism from Γ_1 to μ_{12} that maps S to $-i$ and U to $\frac{1-i\sqrt{3}}{2}$ is the multiplier system of the modular form η^2 of weight 1, where η is the Dedekind eta function which we shall see in the next section.*

Corollary 1.29. *Let f be a generalized modular form of rational weight r on Γ_1 and let ℓ be the least positive integer such that $\ell r \in \mathbb{Z}$. Then $f^{12\ell}$ is a modular form of integral weight with the trivial multiplier system on Γ_1.*

From the usual Valence Formula for the integral weight modular forms with trivial multiplier systems (see [21], [46]) and from Corollary 1.29, now Theorem 1.26 follows immediately. □

1.4 The Dedekind eta function and eta quotients

The *Dedekind eta function* is defined by an infinite product:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.57}$$

where $q^r = q^r(z)$ is defined by (1.32). Since the double series

$$S(z) := \sum_{n\in\mathbb{N}} \log(1 - q^n) = -\sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \frac{q^{mn}}{m} \tag{1.58}$$

converges absolutely and uniformly on compact subsets of $\mathfrak{H}$, the function $\eta = e^L$ is holomorphic on $\mathfrak{H}$, where $L : \mathfrak{H} \to \mathbb{C}^*$ is defined by

$$L(z) := \frac{\pi i z}{12} + S(z). \tag{1.59}$$

The function L transforms as follows (see [15] and [33]) under the action of Γ_1 on its argument by Möbius transformation:

$$L(\gamma z) = L(z) + \begin{cases} \dfrac{\pi i b d}{12} & \text{if } c = 0, \\[2ex] \dfrac{1}{2}\log(cz+d) + \dfrac{\pi i}{12} \cdot \dfrac{a + d - 2D(d, |\, c \,|)}{c} - \dfrac{\pi i \operatorname{sgn}(c)}{4} & \text{otherwise,} \end{cases} \tag{1.60}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, where $\log(cz+d)$ is defined by the principal branch of complex logarithm. Here, for $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{N}$ with $(\alpha, \beta) = 1$, the Dedekind symbol $D(\alpha, \beta)$ is defined (See [17]) by $D(\alpha, \beta) := 6\beta s(\alpha, \beta)$, where $s(\alpha, \beta)$ is the Dedekind sum

$$s(\alpha, \beta) := \sum_{m=1}^{\beta} \Big(\Big(\frac{m}{\beta}\Big)\Big)\Big(\Big(\frac{m\alpha}{\beta}\Big)\Big), \tag{1.61}$$

and $((\,)) : \mathbb{Q} \to \mathbb{Q}$ is defined by

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \dfrac{1}{2} & \text{if } x \text{ is not an integer,} \\[2ex] 0 & \text{otherwise.} \end{cases} \tag{1.62}$$

From (1.60), it follows that $\eta = e^L$ is a modular form of weight $1/2$ on Γ_1 and the multiplier system of η is given by

$$v_\eta(\gamma, \pm\sqrt{cz+d}) = \begin{cases} \pm e^{\pi i b d/12} & \text{if } c = 0, \\ \pm e^{\pi i (a + d - 2D(d,|c|) - 3|c|)/(12c)} & \text{otherwise,} \end{cases} \tag{1.63}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, where $\sqrt{cz+d}$ is defined via the principal branch of complex logarithm. For $r \in \mathbb{C}$, we define

$$\eta^r := e^{rL}. \tag{1.64}$$

In particular, for $r \in \mathbb{Q}$, (1.64) provides us with examples of modular forms of arbitrary rational weights.

Since η is a modular form of weight $1/2$ on Γ_1, from Lemma 1.27, it follows that the multiplier system of η is a homomorphism from $(\widetilde{\Gamma}_1)_2$ to μ_{24}. So in particular, from (1.63) we get that $(a+d-2D(d,|\,c\,|))/c$ is an integer. Another proof of the integrality of $(a+d-2D(d,|\,c\,|))/c$ could also be given by using the reciprocity law of Dedekind sums along with the theorem that says that the denominator of the Dedekind sum $s(\alpha,\beta)$ is a divisor of $2\beta(3,\beta)$ (so, in particular the Dedekind symbol $D(\alpha,\beta)$ is always an integer) (See [17], [15] or [33]).

Since the multiplier system of η is given by a 24-th root of unity, using congruences modulo 24, (1.63) could be simplified to the following (See [17], [33], [20] or [32]):

$$v_\eta(\gamma, \pm\sqrt{cz+d}) := \begin{cases} \pm\left(\frac{d}{|c|}\right)\zeta_\gamma & \text{if } c \text{ is odd,} \\ \pm(-1)^{(\mathrm{sgn}(c)-1)(\mathrm{sgn}(d)-1)/4-(c-1)(d-1)/4}\left(\frac{c}{|d|}\right)\zeta_\gamma & \text{if } d \text{ is odd} \end{cases} \tag{1.65}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$, where $\left(\cdot\right)$ denotes the Jacobi symbol and ζ_γ is the 24-th root of unity defined by

$$\zeta_\gamma := e^{\pi i((a+d)c+bd(1-c^2)-3c)/12}. \tag{1.66}$$

Before defining eta quotients, we recall some general facts about rescalings of modular forms. Let f be a weakly holomorphic modular form on a finite index subgroup of Γ_1. For $d \in \mathbb{N}$, we define f_d, the *rescaling of f with d* by $f_d(z) := f(dz)$. For $N \in \mathbb{N}$, by $\mathcal{D}_N$ we denote the set of divisors of N.

Proposition 1.30. *Let $r \in \mathbb{Q}$, $M \in \mathbb{N}$ and let f be a weakly holomorphic modular form of weight r on $\Gamma_0(M)$. Then for $N \in \mathbb{N}$ and $d \in \mathcal{D}_N$,*

(*a*) *f_d is a weakly holomorphic modular form of weight r on $\Gamma_0(MN)$.*

(*b*) *Let $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = 1$ and let $ds := [d\alpha : \lambda]$. Then we have*

$$\mathrm{ord}_s(f_d; \Gamma_0(MN)) = \frac{N(d,\lambda)^2(\lambda^2, M)}{d(\lambda^2, MN)}\,\mathrm{ord}_{ds}(f; \Gamma_0(M)). \tag{1.67}$$

Proof. (*a*) Let $\mathfrak{g}_d := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{G}$. Let ℓ be the smallest positive integer such that $\ell r \in \mathbb{Z}$ and let $\widetilde{\mathfrak{g}_d} := (\mathfrak{g}_d, 1) \in \widetilde{\mathcal{G}}_\ell$. Since $\mathfrak{g}_d\Gamma_0(MN)\mathfrak{g}_d^{-1} \subset \Gamma_0(N)$ and since $f|_r\,\widetilde{\mathfrak{g}_d} = f_d$, the claim follows from Theorem 1.15′(*a*).

(*b*) Since $\mathfrak{g}_d \Gamma_0(MN)\, \mathfrak{g}_d^{-1} \subset \Gamma_0(N)$, from Lemma 1.1, it follows that the endomorphism of $\mathbb{P}^1(\mathbb{Q})$ that maps s to $\mathfrak{g}_d s = ds$ induces a surjection from the set of cusps of $\Gamma_0(MN)$ to the set of cusps of $\Gamma_0(N)$. Now, the claim follows from Theorem 1.15′(*b*) and Lemma 1.13. □

Corollary 1.31. *Let f, M, N, d, s and λ be as above. If $d|\lambda$ and if $(MN)|\lambda^2$, then we have*

$$\mathrm{ord}_s(f_d; \Gamma_0(MN)) = d \cdot \mathrm{ord}_{ds}(f; \Gamma_0(M)).$$

Corollary 1.32. *Let f, M, N, d, s and λ be as in Proposition 1.30. Then*

$$\mathrm{ord}_s(f_d; \Gamma_0(MN)) \in \mathrm{ord}_{ds}(f; \Gamma_0(M)) \cdot \mathbb{N}.$$

Proof. By (1.67), we only require to show that

$$h(M, N, \lambda, d) := \frac{N(d, \lambda)^2(\lambda^2, M)}{d(\lambda^2, MN)} \tag{1.68}$$

is an integer for $d \in \mathcal{D}_N$. Since $h : \mathbb{N}^4 \to \mathbb{N}$ is multiplicative in all its arguments, it is enough to show the integrality of $h(M, N, \lambda, d)$ by assuming that M, N and λ are powers of the same prime and $d \in \mathcal{D}_N$, in which case the proof is quite straightforward. □

Corollary 1.33. *Let f be a weakly holomorphic modular form on Γ_1, let $N \in \mathbb{N}$, let $d \in \mathcal{D}_N$ and let $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = 1$ be a cusp of $\Gamma_0(N)$. Then we have*

$$\mathrm{ord}_s(f_d) = \frac{N(d, \lambda)^2}{d(\lambda^2, N)} \mathrm{ord}_\infty(f) \in \mathrm{ord}_\infty(f) \cdot \mathbb{N}. \tag{1.69}$$

Apparently, it may seem that in (1.69), there is some room for confusion since f is a modular form on both Γ_1 and $\Gamma_0(N)$ and ∞ represents a cusp of either of the groups. However, since the the cusps $\infty\,(\mathrm{mod}\ \Gamma_1)$ and $\infty\,(\mathrm{mod}\ \Gamma_0(N))$ have the same width, we have $\mathrm{ord}_\infty(f; \Gamma_1) = \mathrm{ord}_\infty(f; \Gamma_0(N))$. In other words, (1.69) makes perfect sense either way.

In particular, for N, d, s and λ as above, from Corollary 1.33, we have

$$\mathrm{ord}_s(\eta_d) = \frac{N(d, \lambda)^2}{24d(\lambda^2, N)} \in \frac{1}{24}\mathbb{N}, \tag{1.70}$$

since $\mathrm{ord}_\infty(\eta) = 1/24$.

Corollary 1.34. *Let f be a weakly holomorphic modular form on Γ_1 and let $N \in \mathbb{N}$. If $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ and $s' = [\beta : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = (\beta, \lambda) = 1$ are two cusps of $\Gamma_0(N)$, then for all $d \in \mathcal{D}_N$, we have*

$$\mathrm{ord}_s(f_d) = \mathrm{ord}_{s'}(f_d). \tag{1.71}$$

Corollary 1.35. *Let f be a weakly holomorphic modular form on Γ_1 and let $N \in \mathbb{N}$. For each cusp s of $\Gamma_0(N)$, there exists a divisor d of N such that*

$$\mathrm{ord}_s(f_d) = \mathrm{ord}_\infty(f).$$

Proof. By Corollary 1.33, the claim is equivalent to show that given $\lambda \in \mathbb{N}$, there exists $d \in \mathcal{D}_N$ such that $h(1, N, \lambda, d) = 1$ holds, where $h : \mathbb{N}^4 \to \mathbb{N}$ is defined in (1.68). Since h is multiplicative in all its arguments, we may assume that both N and λ are powers of the same prime p. Then for $\lambda^2 \geq N$, let $d_\lambda = 1$ and for $\lambda^2 < N$, let $d_\lambda = N$. Now it is easy to check that $h(1, N, \lambda, d_\lambda) = 1$. □

The converse of Corollary 1.35 does not hold in general, i. e. unless either N is squarefree or $\mathrm{ord}_\infty(f) = 0$, there are always $d \in \mathcal{D}_N$ for which there are no cusps of $\Gamma_0(N)$ such that $\mathrm{ord}_s(f_d) = \mathrm{ord}_\infty(f)$ holds. For example, for a modular form f on Γ_1 with $\mathrm{ord}_\infty(f) \neq 0$, we have $\mathrm{ord}_s(f_2) \neq \mathrm{ord}_\infty(f)$ for all cusps s of $\Gamma_0(4)$.

Also, unless $M = 1$, for a given λ the quantity $h(M, N, \lambda, d)$ may never be 1 for $d \in \mathcal{D}_N$. For example $h(2, 2, 2, d) \neq 1$ for all $d \in \mathcal{D}_2$.

Corollary 1.36. *Let f be a weakly holomorphic modular form on Γ_1. Let $d \in \mathbb{N}$ and let $\lambda, N, N' \in d\mathbb{N}$ be such that both N and N' divide λ^2. Then for $\alpha, \beta \in \mathbb{Z}$ with $(\alpha, \lambda) = (\beta, \lambda) = 1$ and for $s := [\alpha : \lambda]$, $s' := [\beta : \lambda]$, we have*

$$\mathrm{ord}_s(f_d; \Gamma_0(N)) = \mathrm{ord}_{s'}(f_d; \Gamma_0(N')).$$

Proof. By Corollary 1.31, both sides of the above equation are equal to $d \cdot \mathrm{ord}_\infty(f)$. □

Now we introduce a law of composition on $\odot$ on the set of positive integers, which we shall also encounter later in this chapter. We define $\odot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$d_1 \odot d_2 := \frac{d_1 d_2}{(d_1, d_2)^2}. \tag{1.72}$$

Under this law of composition, $\mathbb{N}$ assumes an unital* commutative algebraic structure of exponent† 2. But it is not a group, because the operation $\odot$ is not associative in general. For example, $2 \odot (2 \odot 4) \neq (2 \odot 2) \odot 4$. Since for every $a, b \in \mathbb{N}$ we have $a \odot ab = b$, $\mathbb{N}$ nearly satisfies the axiom of a quasi-group‡ (see [45]). However, $\mathbb{N}$ is not even a quasi-group as there may be more than one solution to the equation $a \odot x = b$ for $a, b \in \mathbb{N}$. For example, we have $2 \odot 1 = 2 \odot 4$.

For $N \in \mathbb{N}$, though the restriction of $\odot$ from $\mathbb{N}$ to $\mathcal{D}_N$ endows $\mathcal{D}_N$ also with an unital commutative algebraic structure of exponent 2, but in general, $\mathcal{D}_N$ is even further away from being a quasi-group. For example, there is no $d \in \mathcal{D}_4$ with $2 \odot d = 4$. However, for all $N \in \mathbb{N}$, there is a nontrivial subset of $\mathcal{D}_N$ which is an abelian group under $\odot$. We shall see such groups in the last section of this chapter. For now, we return to the rescalings of modular forms and draw yet another conclusion out of Corollary 1.33:

Corollary 1.37. *Let f be a weakly holomorphic modular form on Γ_1 and let $N \in \mathbb{N}$. If $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ and $s' = [\beta : \lambda'] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = (\beta, \lambda') = 1$ are two cusps of $\Gamma_0(N)$ such that $\lambda' = N/\lambda$, then for all $d \in \mathcal{D}_N$, we have*

$$\operatorname{ord}_s(f_d) = \operatorname{ord}_{s'}(f_{\frac{N}{d}}). \tag{1.73}$$

Proof. From (1.72), we get that

$$\frac{N}{d_1} \odot \frac{N}{d_2} = d_1 \odot d_2 \tag{1.74}$$

for all $d_1, d_2 \in \mathcal{D}_N$. So, from Corollary 1.33, we obtain

$$\operatorname{ord}_s(f_d) = \frac{N}{d \odot \lambda} \cdot \frac{1}{(\lambda, N/\lambda)} \cdot \operatorname{ord}_\infty(f) = \frac{N}{\frac{N}{d} \odot \lambda'} \cdot \frac{1}{(\lambda', N/\lambda')} \cdot \operatorname{ord}_\infty(f) = \operatorname{ord}_{s'}(f_{\frac{N}{d}}).$$

□

Next, we define eta quotients. For $N \in \mathbb{N}$, and for an additive subgroup G of $\mathbb{C}$, by $G^{\mathcal{D}_N}$ we denote the set of all maps from the set $\mathcal{D}_N$ to G and we define $\sigma : G^{\mathcal{D}_N} \to G$ by

$$\sigma(X) := \sum_{d \in \mathcal{D}_N} X_d, \tag{1.75}$$

*An *unital algebraic structure* is an algebraic structure that has an identity element.

†We say an unital algebraic structure $\mathscr{A}$ has *exponent* n if the orders of each of its elements divide n, i. e. if for each element $\alpha \in \mathscr{A}$, n compositions of α with itself yields the identity.

‡A *quasi-group* is a set with a binary operation in which each of the equations $ax = b$ and $ya = b$ has a unique solution, for any two elements a and b in the set.

where $X_d \in G$ denotes the image of $d \in \mathcal{D}_N$ under the map $X \in G^{\mathcal{D}_N}$.

Definition 1.38. For $N \in \mathbb{N}$ and for $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define the *eta quotient* η^X by

$$\eta^X := \prod_{d|N} \eta_d^{X_d}. \tag{1.76}$$

We call $\gcd\{d \in \mathcal{D}_N \,|\, X_d \neq 0\}$ the *level* of η^X. In general, for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, we call η^X an eta quotient on $\Gamma_0(N)$.

It follows that if f is an eta quotient on $\Gamma_0(N)$, then the level of f is a divisor of N. The following proposition gives a clarification to the above terminology:

Proposition 1.39. *For $X \in \mathbb{Z}^{\mathcal{D}_N}$, an eta quotient η^X is a weakly holomorphic modular form of weight $\sigma(X)/2$ on $\Gamma_0(N)$.*

Proof. Since η is a modular form of weight $1/2$ on Γ_1, from Proposition 1.30, we know that for each divisor d of N, η_d is a modular form of weight $1/2$ on $\Gamma_0(N)$. So, the claim follows by summing up the weights of $\eta_d^{X_d}$ for all $d|N$. □

Corollary 1.34 implies that for an eta quotient f on $\Gamma_0(N)$ and for two cusps s and s' of $\Gamma_0(N)$, we have

$$\mathrm{ord}_s(f) = \mathrm{ord}_{s'}(f) \tag{1.77}$$

whenever there exist $[\alpha : \lambda], [\beta : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = (\beta, \lambda) = 1$ such that $s = [\alpha : \lambda]$ and $s' = [\beta : \lambda]$.

Conversely, under a pair of necessary assumptions, Condition (1.77) determines whether a generalized modular form is some power of an eta quotient or not:

In [23], Kohnen and Mason showed that if the poles and zeros of a generalized modular form f are supported at the cusps of $\Gamma_0(N)$ and if it has a Fourier expansion at ∞ with rational Fourier coefficients which are p-integral for all but a finite number of primes p, then f satisfies (1.77) if and only if some integral power of f is an eta quotient up to multiplication by some constant.

In particular, it is easy to show that if f belongs to a subspace of generalized modular forms which has an *integral basis*, i. e. each element in the basis has a Fourier expansion at ∞ with integral Fourier coefficients (for example, the space of modular forms of a certain weight with the trivial multiplier system on $\Gamma_0(N)$, (see [38])) and if moreover, every element in this subspace is determined up to a constant by the set of its zeros and poles, then only the assumption that the poles and zeros of f are supported at the cusps of $\Gamma_0(N)$ is enough to ensure the fact that f satisfies (1.77) if and only if f is an eta quotient up to multiplication by some constant.

Corollary 1.36 implies that for an eta quotient f of level M, for $N, \lambda \in M\mathbb{N}$ with $N|\lambda^2$, for $\alpha, \beta \in \mathbb{Z}$ with $(\alpha, \lambda) = (\beta, \lambda) = 1$ and for $s := [\alpha : \lambda]$, $s' := [\beta : \lambda]$, we have

$$\mathrm{ord}_s(f\,; \Gamma_0(N)) = \mathrm{ord}_{s'}(f\,; \Gamma_0(M)). \tag{1.78}$$

1.5 The valuation map

The valuation map is a linear map from the lattice of exponents of the eta quotients on $\Gamma_0(N)$ to the lattice of their orders at different cusps. We elaborate this in the following:

The *lattice of exponents* of the eta quotients on $\Gamma_0(N)$ is $\mathbb{Z}^{\mathcal{D}_N}$. From Proposition 1.7, from (1.70) and from (1.77), it follows that the *lattice of orders* at different cusps of $\Gamma_0(N)$ of the eta quotients on $\Gamma_0(N)$ is a sublattice O_N of $\mathcal{L}_N := \frac{1}{24}\mathbb{Z}^{\mathcal{D}_N}$. More precisely, (1.70) implies that O_N is the image of the *valuation map* $\mathcal{V}_N : \mathbb{Z}^{\mathcal{D}_N} \to \mathcal{L}_N$ given by

$$(\mathcal{V}_N(X))_t := \frac{1}{24}\sum_{d\in\mathcal{D}_N}\frac{N(d,t)^2}{d(t^2,N)}X_d \quad \text{for } X \in \mathbb{Z}^{\mathcal{D}_N} \text{ and } t \in \mathcal{D}_N. \tag{1.79}$$

In particular, for $X \in \mathbb{Z}^{\mathcal{D}_N}$ and for any cusp s of $\Gamma_0(N)$, we have

$$\mathrm{ord}_s(\eta^X) = (\mathcal{V}_N(X))_{(N,\lambda)}, \tag{1.80}$$

where $s = [\alpha : \lambda] \in \mathbb{P}^1(\mathbb{Q})$ with $(\alpha, \lambda) = 1$.

Clearly, $\mathbb{Z}^{\mathcal{D}_N}$ is a free $\mathbb{Z}$-module with basis $\mathbb{B} := \{\delta^{(d)}\}_{d\in\mathcal{D}_N}$, where $\delta^{(d)} \in \mathbb{Z}^{\mathcal{D}_N}$ are defined by

$$\delta_t^{(d)} := \begin{cases} 1 & \text{if } t = d, \\ 0 & \text{otherwise.} \end{cases}$$

The *valuation matrix* $A_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ of level N is the matrix of the valuation map $\mathcal{V}_N$ w.r.t the basis $\mathbb{B}$ of $\mathbb{Z}^{\mathcal{D}_N}$ and the basis $\mathbb{B}' := \{\,\delta_d/24 \;\big|\; \delta_d \in \mathbb{B}\}$ of $\mathcal{L}_N$. So, for $t, d \in \mathcal{D}_N$, the (t, d)-th entry of A_N is given by

$$\frac{N(d,t)^2}{d(t^2,N)}. \tag{1.81}$$

In particular, we have

$$\mathrm{O}_N = \frac{1}{24} A_N \, \mathbb{Z}^{\mathcal{D}_N}. \tag{1.82}$$

Corollary 1.35 says that each row of A_N has some entry equal to 1 and Corollary 1.37 says that for all $t, d \in \mathcal{D}_N$, the (t, d)-th entry and the $(N/t, N/d)$-th entry of A_N are equal. For example, for a prime power p^n, we have

$$A_{p^n} = \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-2} & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-4} & p^{n-3} & p^{n-2} & \cdots & p & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & p & p^2 & \cdots & p^{n-1} & p^{n-2} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}. \tag{1.83}$$

Since for $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have

$$(A_N X)_t = 24 (\mathcal{V}_N(X))_t \ \text{ for all } t \in \mathcal{D}_N, \tag{1.84}$$

from (1.80), it follows that

Lemma 1.40. *For $X \in \mathbb{Z}^{\mathcal{D}_N}$, the eta quotient η^X is holomorphic if and only if $A_N X \geq 0$, i. e. the entries of $A_N X$ are nonnegative.*

In the next proposition, we derive some properties of A_N. For $A \in \mathbb{R}^{\mathcal{D}_N \times \mathcal{D}_N}$ and $d \in \mathcal{D}_N$, we denote by $A(d, \_)$ (resp. $A(\_, d)$) the row (resp. column) of A which is indexed by d and by $|A| \in \mathbb{R}^{\mathcal{D}_N \times \mathcal{D}_N}$ we denote the matrix whose entries are the absolute values of the corresponding entries of A. As in the previous lemma, for $Y \in \mathbb{R}^{\mathcal{D}_N}$, if all the entries of Y are positive, we write $Y > 0$.

Proposition 1.41. *Let $A_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be the valuation matrix of level N.*

(a) We have

$$A_N = \bigotimes_{\substack{p^n \| N \\ p \text{ prime}}} A_{p^n}, \tag{1.85}$$

*where by $\otimes$, we denote the Kronecker product of matrices.**

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing N induces a lexicographic ordering on $\mathcal{D}_N$ with which the entries of A_N are indexed, Equation (1.85) makes sense for all possible orderings of the primes dividing N.

(*b*) *A_N is invertible over $\mathbb{Q}$.*

(*c*) *Let φ and $\psi : \mathbb{N} \to \mathbb{N}$ denote the Euler φ function and the Dedekind ψ function defined by*

$$\varphi(N) := N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad \psi(N) := N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right). \tag{1.86}$$

We define $\mathbb{1}_N, \alpha_N$ and $\beta_N \in \mathbb{Q}^{\mathcal{D}_N}$ by

$$\mathbb{1}_N(t) := 1 \quad \text{for all } t \in \mathcal{D}_N, \tag{1.87}$$

$$\alpha_N := (A_N^{-1})^{\mathrm{T}} \mathbb{1}_N \quad \text{and} \quad \beta_N := |A_N^{-1}|^{\mathrm{T}} \mathbb{1}_N. \tag{1.88}$$

Then we have

$$\alpha_N(t) = \frac{\varphi(t')}{\psi(N)} \quad \text{and} \quad \beta_N(t) = \frac{\psi(t')}{\varphi(N)} \quad \text{for all } t \in \mathcal{D}_N, \tag{1.89}$$

where $t' = (t, N/t)$.

Proof. (*a*) Follows by induction on the number of prime divisors of N.
(*b*) It is easy to verify that for a prime power p^n, the matrix A_{p^n} is invertible with the tridiagonal inverse:

$$A_{p^n}^{-1} = \frac{1}{p^{n-1}(p^2-1)} \begin{pmatrix} p & -p & & & & \\ -1 & p^2+1 & -p^2 & & \mathbf{0} & \\ & -p & p(p^2+1) & -p^3 & & \\ & & \ddots & \ddots & \ddots & \\ & \mathbf{0} & & -p^2 & p^2+1 & -1 \\ & & & & -p & p \end{pmatrix}, \tag{1.90}$$

where for each positive integer $j < n$, the column $A_{p^n}^{-1}(\_\,, p^j)$ is just a cyclic permutation of $A_{p^n}^{-1}(\_\,, p)$ multiplied with $p^{\min\{j-1,n-j-1\}}$. More precisely,

$$p^{n-1}(p^2-1) \cdot A_{p^n}^{-1}(p^i, p^j) = \begin{cases} p & \text{if } i = j = 0 \text{ or } i = j = n, \\ -p^{\min\{j,n-j\}} & \text{if } |i-j| = 1, \\ p^{\min\{j-1,n-j-1\}}(p^2+1) & \text{if } 0 < i = j < n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.91}$$

The general case now follows by (a).
(c) If N is a prime power, the claim follows from (1.90). The general case again follows by multiplicativity. □

Corollary 1.42. *Any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t\in\mathcal{D}_N}$ of $\Gamma_0(N)$ as follows: if $f=\eta^X$ for some $X\in\mathbb{Z}^{\mathcal{D}_N}$, then*

$$X_d = 24\sum_{t\in\mathcal{D}_N} A_N^{-1}(d,t)\,\mathrm{ord}_{1/t}(f) \quad \text{for } d\in\mathcal{D}_N,$$

where $A_N^{-1}(d,t)$ denotes the (d,t)-th entry of A_N^{-1}.

Proof. Follows immediately from (1.80), (1.84) and Proposition 1.41. □

Corollary 1.43. *If $X,X'\in\mathbb{Z}^{\mathcal{D}_N}$ with $X\neq X'$, then $\eta^X\neq\eta^{X'}$.*

Corollary 1.44 (Valence Formula for eta quotients)**.** *Let $k\in\mathbb{Z}$ and let $f=\eta^X$ be an eta quotient of weight $k/2$ on $\Gamma_0(N)$. Then we have*

$$\sum_{t\in\mathcal{D}_N} \alpha_N(t)\cdot\mathrm{ord}_{1/t}(f) = \frac{k}{24}, \tag{1.92}$$

where $\alpha_N:\mathcal{D}_N\to\mathbb{Q}$ is as in (1.89).

Proof. From Proposition 1.39, we know that f is of weight $\sigma(X)/2$. So, we get

$$k=\sigma(X)=\sum_{d\in\mathcal{D}_N}X_d = 24\sum_{d\in\mathcal{D}_N}\sum_{t\in\mathcal{D}_N}A_N^{-1}(d,t)\,\mathrm{ord}_{1/t}(f) = 24\sum_{t\in\mathcal{D}_N}\alpha_N(t)\cdot\mathrm{ord}_{1/t}(f),$$

where the third equality follows from Corollary 1.42 and the fourth equality follows from Proposition 1.41(c). □

Conversely, in [36] Rouse and Webb showed that under a certain condition on N, each nonnegative integral solution in $\{x_t\}_{t\in\mathcal{D}_N}$ of the linear equation

$$\sum_{t\in\mathcal{D}_N}\alpha_N(t)\cdot x_t = \frac{k}{24}, \tag{1.93}$$

implies that there is a holomorphic eta quotient f of weight $k/2$ on $\Gamma_0(N)$ whose order at the cusp $1/t$ of $\Gamma_0(N)$ is equal to x_t.

Corollary 1.45. *Let $k \in \mathbb{Z}$ and let $f = \eta^X$ be an eta quotient of weight $k/2$ on $\Gamma_0(N)$ and let $\mathcal{S}_N$ denote the set of cusps of $\Gamma_0(N)$. Then we have*

$$\sum_{s \in \mathcal{S}_N} \operatorname{ord}_s(f) = \frac{k\,[\Gamma_1 : \Gamma_0(N)]}{24}. \tag{1.94}$$

Proof. From Proposition 1.7, we know that for $t \in \mathcal{D}_N$, there are exactly $\varphi(t, N/t)$ inequivalent cusps with representatives of the form $\alpha/t \in \mathbb{Q}$ with $(\alpha, t) = 1$. Now, (1.77) implies that the order of f at all these cusps are equal to $\operatorname{ord}_{1/t}(f)$. So the sum of the orders of f at all such cusps is equal to $\varphi(t, N/t) \cdot \operatorname{ord}_{1/t}(f)$. Since we have $[\Gamma_1 : \Gamma_0(N)] = \psi(N)$ (see (1.3)), the claim now readily follows from Corollary 1.44 and (1.89). □

We could have also obtained the last three corollaries from the Valence Formula for modular forms on $\Gamma_0(N)$:

Let $r \in \mathbb{Q}$ and let $f : \mathfrak{H} \to \mathbb{C}$ be a meromorphic function that transforms like a modular form of weight r on $\Gamma_0(N)$ and which is also meromorphic at each element of $\mathcal{S}_N$, where $\mathcal{S}_N$ denotes the set of the cusps of $\Gamma_0(N)$. Then the Valence Formula for $\Gamma_0(N)$ (see [6] or [34]) states*:

$$\sum_{s \in \mathcal{S}_N} \operatorname{ord}_s(f) + \sum_{P \in \Gamma_0(N)\backslash\mathfrak{H}} \frac{1}{n_P} \operatorname{ord}_P(f) = \frac{r\,[\Gamma_1 : \Gamma_0(N)]}{12}, \tag{1.95}$$

where n_P is the number of elements in the stabilizer of P in the group $\overline{\Gamma_0(N)}$ which is defined by (1.23).

Convention 1.46. Henceforth by *Valence Formula*, we mean the *Valence Formula for eta quotients* (see Corollary 1.44).

Below we draw a few corollaries to the Valence formula:

Corollary 1.47. *Let f be an eta quotient of weight $k/2$. If f is holomorphic and if $f \neq 1$, then $k \in \mathbb{N}$.*

Proof. Follows from Corollary 1.45, Proposition 1.39 and (1.70). □

*In both of the books cited, the statement of the Valence Formula also encompasses the real weight case. However, we restrict (1.95) only to the case of rational weights, since we have only defined modular forms with such weights and since that is all we require here.

Corollary 1.48. *For a set* $\{u_t \in \mathbb{R} | t \in \mathcal{D}_N\}$*, there are only finitely many weakly holomorphic eta quotients* f *of a fixed weight on* $\Gamma_0(N)$ *such that* $\operatorname{ord}_{1/t}(f) \leq u_t$ *for all* $t \in \mathcal{D}_N$*.*

Proof. By Corollary 1.44, it is enough to show that the equation (1.93) has only a finite number of integral solutions in $\{x_t\}_{t\in\mathcal{D}_N}$ if $x_t \leq u_t$ for all $t \in \mathcal{D}_N$. This is equivalent to say that the equation

$$\sum_{t\in\mathcal{D}_N} \alpha_N(t) \cdot x'_t = K \tag{1.96}$$

has only finitely many solutions in nonnegative integers $\{x'_t\}_{t\in\mathcal{D}_N}$, where

$$K := \sum_{t\in\mathcal{D}_N} \alpha_N(t) \cdot u_t - \frac{k}{24}.$$

Since $\alpha_N(t) > 0$ for all $t \in \mathcal{D}_N$, equation (1.96) has no solutions if $K < 0$. If $K \geq 0$, then for a solution $\{x_t\}_{t\in\mathcal{D}_N}$ of equation (1.96) in nonnegative integers, we have $x'_t \in \mathbb{Z} \cap [0, K/\alpha_N(t)]$ for all $t \in \mathcal{D}_N$. So, the number of solutions of equation (1.96) in nonnegative integers is less than

$$\prod_{t\in\mathcal{D}_N} \Big(\frac{K}{\alpha_N(t)} + 1\Big). \tag{1.97}$$

Hence, the number of the eta quotients f of weight k on $\Gamma_0(N)$ with $\operatorname{ord}_{1/t}(f) \leq u_t$ is also less than the naïve bound provided in (1.97). □

Corollary 1.49. *For a set* $\{l_t \in \mathbb{R} | t \in \mathcal{D}_N\}$*, there are only finitely many weakly holomorphic eta quotients* f *of a fixed weight on* $\Gamma_0(N)$ *such that* $\operatorname{ord}_{1/t}(f) \geq l_t$ *for all* $t \in \mathcal{D}_N$*.*

Proof. Proceeding similarly as in the proof of the previous corollary, we get that the number of the eta quotients f of weight k on $\Gamma_0(N)$ with $\operatorname{ord}_{1/t}(f) \geq l_t$ is less than

$$\prod_{t\in\mathcal{D}_N} \Big(\frac{K'}{\alpha_N(t)} + 1\Big), \tag{1.98}$$

where $K' = \dfrac{k}{24} - \sum_{t\in\mathcal{D}_N} \alpha_N(t) \cdot l_t$. □

Corollary 1.50. *There are only finitely many holomorphic eta quotients of a fixed weight and level.*

More precisely, the number of holomorphic eta quotients of weight k and level N is less than

$$\Big(\frac{k\,\psi(N)}{24}+1\Big)^n, \tag{1.99}$$

where n is the number of divisors of N.

Proof. From the proof of Corollary 1.49. it follows that the number of holomorphic eta quotients of weight k and level N is less than

$$\prod_{t\in\mathcal{D}_N}\Big(\frac{k\,\psi(N)}{24\,\varphi(t')}+1\Big)<\Big(\frac{k\,\psi(N)}{24}+1\Big)^n,$$

where φ and $\psi:\mathbb{N}\to\mathbb{N}$ are as defined in (1.86), $t'=(t,N/t)$ and n is the number of divisors of N. □

A radical improvement to the naïve bound given in (1.99) is implied by Mersmann's Lemma which we shall see in Chapter 3.

1.6 Atkin-Lehner involutions

We have defined a non-associative law of composition on $\mathbb{N}$ in (1.72). Since $\odot$ is multiplicative in both of its arguments, the roots of the obstacle to associativity of this law of composition on $\mathbb{N}$ could certainly be traced back to the restriction of $\odot$ to the set of powers of any prime p. For two nonnegative integers a and b, from (1.72) we have $p^a\odot p^b=p^{|a-b|}$. Hence, for any suitable choice of three nonnegative integers a, b and c (for example, $a=1$, $b=2$, $c=3$) we have

$$p^{|a-|b-c||}=p^a\odot(p^b\odot p^c)\neq(p^a\odot p^b)\odot p^c=p^{||a-b|-c|}. \tag{1.100}$$

To surmount the obstacle to the associativity of $\odot$ posed by (1.100), we must limit our choice of prime powers in a sensible way. One way of doing so is to impose the condition that the nonnegative integers a, b and c be chosen so that either their maximum is attained by at least two of them or one of them is zero. It is easy to check that for such choices of a, b and c we always have:

$$p^a\odot(p^b\odot p^c)=(p^a\odot p^b)\odot p^c. \tag{1.101}$$

Thus, we obtain some subsets of $\mathbb{N}$ which are abelian groups under the law of composition $\odot$:

For $n \in \mathbb{N}$ and $d \in \mathcal{D}_N$, we say that d *exactly divides* N and write $d \| N$ if $(d, N/d) = 1$. We denote the set of such divisors of N by $\mathcal{E}_N$. Now, from (1.101) and from the multiplicativity of $\odot$ in both of its arguments, it follows that:

Lemma 1.51. *For $N \in \mathbb{N}$, $(\mathcal{E}_N, \odot)$ is a boolean group (i. e. each element of $\mathcal{E}_N$ is the inverse of itself). The group $\mathcal{E}_N$ acts on $\mathcal{D}_N$ by $\odot$.*

Since the action of any element of a group acting on a set induces an injection of the set to itself and since $\mathcal{D}_N$ is finite, we get:

Corollary 1.52. *For $N \in \mathbb{N}$, the action of each element of $\mathcal{E}_N$ on $\mathcal{D}_N$ induces an automorphism of $\mathcal{D}_N$.*

For $n \in \mathcal{E}_N$, the set

$$\mathscr{W}_n := \frac{1}{\sqrt{n}} \begin{pmatrix} n\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & n\mathbb{Z} \end{pmatrix} \cap SL_2(\mathbb{R})$$

is a coset of $\Gamma_0(N)$ in its normalizer in $SL_2(\mathbb{R})$. It is also easy to show that the product of two arbitrary matrices from $\mathscr{W}_m$ and $\mathscr{W}_n$ is a matrix in $\mathscr{W}_{m\odot n}$. In particular, since $\mathcal{E}_N$ is a group of exponent 2, the product of any two matrices of $\mathscr{W}_n$ is a matrix in $\mathscr{W}_1 = \Gamma_0(N)$. Hence, each matrix $W \in \mathscr{W}_n$ induces an involution on $\Gamma_0(N)\backslash\mathfrak{H}$ via Möbius transformation on $\mathfrak{H}$. These are called the *Atkin-Lehner involutions*. Any such involution on $\Gamma_0(N)\backslash\mathfrak{H}$ in turn induces an involution on the space of weakly holomorphic modular forms of any fixed weight on $\Gamma_0(N)$ (see Lemma 1.53 and Corollary 1.54 below). These later involutions are also called Atkin-Lehner involutions.

Lemma 1.53. *Let γ be a 2×2 integer matrix with positive determinant such that $\gamma' := \frac{1}{\sqrt{\det(\gamma)}}\gamma$ is an element in the normalizer of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{R})$ and $\gamma'^2 \in \Gamma_0(N)$. Let $r \in \mathbb{Q}$ and let ℓ be the smallest positive number such that $\ell r \in \mathbb{Z}$. Let $\widetilde{\gamma}' = (\gamma', \alpha) \in \widetilde{\mathcal{G}}_\ell$. Then the map*

$$f \mapsto v_f(\widetilde{\gamma}'^2)^{-1/2} f|_r \widetilde{\gamma}' \tag{1.102}$$

*is a holomorphy-preserving involution on the space of weakly holomorphic modular forms of weight r on $\Gamma_0(N)$, where we define $v_f^{-1/2}$ via the principal branch of complex logarithm.**

*The same proof also works if we define $v_f^{-1/2}$ via some other branch of complex logarithm.

Proof. From Corollary 1.22′, we know that $f|_r\widetilde{\gamma}'$ is a weakly holomorphic modular form of weight r on $\Gamma_0(N)$. Also, we have

$$v(\widetilde{\gamma}'^2)^{-1/2}\big(v(\widetilde{\gamma}'^2)^{-1/2}f|_r\widetilde{\gamma}'\big)|_r\widetilde{\gamma}' = v(\widetilde{\gamma}'^2)^{-1}f|_k\widetilde{\gamma}'^2 = f. \tag{1.103}$$

Hence, the map (1.102) is indeed an involution. Furthermore, Corollary 1.22′ implies that this involution preserves holomorphy both at the points on $\mathfrak{H}$ and at the cusps of $\Gamma_0(N)$. □

Corollary 1.54. *For $N \in \mathbb{N}$ and $n \in \mathcal{E}_N$, each $W \in \mathscr{W}_n$ induces a holomorphy-preserving involution on the space of weakly holomorphic modular forms of any fixed weight on $\Gamma_0(N)$.*

The following proposition shows that under Atkin-Lehner involutions, the image of a rescaling of a modular form f on Γ_1 is just another rescaling of f, up to multiplication by some constant:

Proposition 1.55. *Let $r \in \mathbb{Q}$ and let ℓ be the smallest positive number such that $\ell r \in \mathbb{Z}$. Let $N \in \mathbb{N}$, $n \in \mathcal{E}_N$, $W \in \mathscr{W}_n$ and let $\widetilde{W} = (W, \alpha) \in \widetilde{\mathcal{G}}_\ell$. Let f be a weakly holomorphic modular form of weight r on Γ_1 and let v be the multiplier system of f. Then for $d \in \mathcal{D}_N$ we have*

$$f_d|_r\widetilde{W} = \frac{\xi n^{r/2}}{(n,d)^r} f_{n\odot d}\,,$$

where $\xi = \xi(n, d, v, \widetilde{W})$ is a 12ℓ-th root of unity.

Proof. Let $W = \frac{1}{\sqrt{n}}\begin{pmatrix} n\mu & \nu \\ N\lambda & n\delta \end{pmatrix} \in \mathscr{W}_n$. Then $n\mu\delta - N\lambda\nu/n = 1$. So,

$$\gamma_1 := \begin{pmatrix} \mu(n,d) & \nu d/(n,d) \\ N\lambda(n,d)/(nd) & n\delta/(n,d) \end{pmatrix} \in \Gamma_1.$$

Let $\widetilde{\gamma}_1' = (\gamma_1, \beta) \in (\widetilde{\Gamma_1})_\ell$ and let $k = \ell r$. Let

$$\gamma_2 := \frac{1}{\sqrt{n}}\begin{pmatrix} nd/(n,d) & 0 \\ 0 & (n,d) \end{pmatrix} \quad \text{and} \quad \mathfrak{g}_d := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\mathfrak{g}_d W = \gamma_1\gamma_2$, we have

$$\begin{aligned}(f_d|_r\widetilde{W})(z) &= \frac{1}{\alpha(z)^k} f(\mathfrak{g}_d W z) = \frac{1}{\alpha(z)^k} f(\gamma_1\gamma_2 z) = v(\widetilde{\gamma}_1')^{-1}\rho(z)^k f(\gamma_2 z) \\ &= v(\widetilde{\gamma}_1')^{-1}\rho(z)^k f_{n\odot\delta}(z),\end{aligned} \tag{1.104}$$

where v is the multiplier system of f and $\rho(z) := \beta(\gamma_2 z)/\alpha(z)$. We have

$$\rho(z)^\ell = \frac{\beta(\gamma_2 z)^\ell}{\alpha(z)^\ell} = \Big(\frac{N\lambda(n,d)}{nd}\gamma_2 z + \frac{n\delta}{(n,d)}\Big)\Big/\Big(\frac{N\lambda}{\sqrt{n}}z + \sqrt{n}\delta\Big) = \frac{\sqrt{n}}{(n,d)}. \tag{1.105}$$

Since α and β are holomorphic functions on $\mathfrak{H}$ which do not have any zeros on $\mathfrak{H}$ (see (1.46)), the function $\rho : \mathfrak{H} \to \mathbb{C}$ is holomorphic. So, (1.105) implies that ρ is a constant function:

$$\rho \equiv \frac{\zeta n^{1/(2\ell)}}{(n,d)^{1/\ell}}\,,$$

where ζ is an ℓ-th root of unity. Also, from Lemma 1.27, it follows that $v(\widetilde{\gamma}_1')$ is a 12ℓ-th root of unity. Let $\xi := v(\widetilde{\gamma}_1')^{-1}\zeta^k$. Then the claim follows from (1.104). □

Definition 1.56. *For $N, k \in \mathbb{Z}$, let $\mathbb{E}^!_{N,k}$ (resp. $\mathbb{E}_{N,k}$) be the set of eta quotients (resp. holomorphic eta quotients) of weight $k/2$ on $\Gamma_0(N)$. For $n \in \mathcal{E}_N$, we define the endomorphism $\mathrm{al}_{n,N} : \mathbb{E}^!_{N,k} \to \mathbb{E}^!_{N,k}$ by*

$$\mathrm{al}_{n,N}\Big(\prod_{d\in\mathcal{D}_N}\eta_d^{X_d}\Big) := \prod_{d\in\mathcal{D}_N}\eta_{n\odot d}^{X_d}. \tag{1.106}$$

Now, from Lemma 1.53 and Proposition 1.55, we get

Corollary 1.57. *For $N, k \in \mathbb{N}$ and $n \in \mathcal{E}_N$ the map $\mathrm{al}_{n,N} : \mathbb{E}^!_{N,k} \to \mathbb{E}^!_{N,k}$ is a holomorphy-preserving involution.*

Convention 1.58. We adopt the abuse of terminology of calling the involution (1.106) an Atkin-Lehner involution (of level N).

The condition $n \in \mathcal{E}_N$ is essential in Corollary 1.57. In other words, for $n \in \mathcal{D}_N \smallsetminus \mathcal{E}_N$, the map

$$\prod_{d\in\mathcal{D}_N}\eta_d^{X_d} \mapsto \prod_{d\in\mathcal{D}_N}\eta_{n\odot d}^{X_d} \tag{1.107}$$

in general, is neither an involution nor does it preserve holomorphy. For example, $h := \dfrac{\eta_2^3}{\eta\eta_4}$ is a modular form of weight $1/2$ on $\Gamma_0(4)$. Now, for $N = 4$ and for $n = 2 \in \mathcal{D}_4 \smallsetminus \mathcal{E}_4$, the map (1.107) sends h to $h_1 := \dfrac{\eta^3}{\eta_2^2}$ which is not holomorphic at ∞. Also, the map (1.107) sends h_1 to $\dfrac{\eta_2^3}{\eta^2} \neq h$.

We end this chapter with the following lemmas on Atkin-Lehner involutions of eta quotients:

Lemma 1.59. *Let $N \in \mathbb{N}$ and let $n \in \mathcal{E}_N$. Let f, g and h be eta quotients on $\Gamma_0(N)$ such that $f = gh$. Then we have*

$$\mathrm{al}_{n,N}(f) = \mathrm{al}_{n,N}(g) \cdot \mathrm{al}_{n,N}(h). \tag{1.108}$$

Proof. Follows trivially from (1.106). □

Lemma 1.60. *Let f be an eta quotient on $\Gamma_0(N)$ and let $M \in \mathbb{N}$ be a multiple of N. Let $m \in \mathcal{E}_M$. Then we have*

$$\mathrm{al}_{m,M}(f) = (\mathrm{al}_{n,N}(f))_\nu, \tag{1.109}$$

where $n = (m, N)$, $\nu = m/n$ and $(\mathrm{al}_{n,N}(f))_\nu$ denotes the rescaling of the eta quotient $\mathrm{al}_{n,N}(f)$ by ν.

Proof. Since $m \| M$, and $N|M$, we have $n = (m, N)\|N$. So $\mathrm{al}_{n,N}$ is indeed defined by (1.106). Now, the claim follows from the fact that $m \odot d = \nu(n \odot d)$ for all $d \in \mathcal{D}_N$. □

Lemma 1.61. *Let f be an eta quotient on $\Gamma_0(N)$ and let $M \in \mathbb{N}$ be a multiple of N. Let $n \in \mathcal{E}_N$, let $m \in \mathcal{E}_M \cap n\mathbb{Z}$ and let $\nu := m/n$. Then we have*

$$\mathrm{al}_{m,M}(f_\nu) = \mathrm{al}_{n,N}(f). \tag{1.110}$$

Proof. Follows from the fact that $(\nu n) \odot (\nu d) = n \odot d$ for all $d \in \mathcal{D}_N$.. □

Corollary 1.62. *Let f be an eta quotient on $\Gamma_0(N)$ and let f_ν be the rescaling of f by $\nu \in \mathbb{N}$. Then we have*

$$\mathrm{al}_{\nu N,\nu N}(f_\nu) = \mathrm{al}_{N,N}(f). \tag{1.111}$$

Chapter 2

Factorization of holomorphic eta quotients

We say that a holomorphic eta quotient f is *divisible* by a holomorphic eta quotient g if f/g is holomorphic. If f is divisible by g, we call g a *factor* of f. A holomorphic eta quotient f is *irreducible* if it has only the trivial factors, viz. 1 and f. By a descent argument on weights of holomorphic eta quotient (see Corollary 1.47), it follows that:

Lemma 2.1. *Each holomorphic eta quotient is a product of irreducible holomorphic eta quotients, though such a factorization may not be unique.*

In particular, the definition of reducibility of an eta quotient allows factors of arbitrary levels. For example, we have

$$\frac{\eta\eta_2\eta_6}{\eta_3} = \frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}} \times \frac{\eta_2^2\eta_{12}}{\eta_4\eta_6}, \tag{2.1}$$

where a reducible holomorphic eta quotient of level 6 is factored into two holomorphic eta quotients of level 12.

2.1 A conjecture on reducibility

Let f be a holomorphic eta quotient on $\Gamma_0(M)$ (i. e. the level of f divides M). If there is a holomorphic eta quotient g on $\Gamma_0(M)$ which is a nontrivial factor of f, we say that f is *reducible on* $\Gamma_0(M)$. In particular, if f is reducible on $\Gamma_0(N)$, where N is the level of f, we say that f is *strongly reducible.* Certainly, every strongly reducible holomorphic eta quotient is reducible. But it was quite a surprise to find much numerical evidence for the converse:

Conjecture 2.2 (Reducibility Conjecture)**.** *A holomorphic eta quotient is reducible only if it is strongly reducible.*

For example, the eta quotient of level 6 in (2.1) has also the following factorisation into holomorphic eta quotients of level 6 and level 2:

$$\frac{\eta\eta_2\eta_6}{\eta_3} = \frac{\eta^2\eta_6}{\eta_2\eta_3} \times \frac{\eta_2^2}{\eta}. \tag{2.2}$$

So, the eta quotient $\dfrac{\eta\eta_2\eta_6}{\eta_3}$ is indeed strongly reducible. Below we give an example of an eta quotient, which is not strongly reducible:

Lemma 2.3. *For any prime p, the eta quotient $\dfrac{\eta^p}{\eta_p}$ is not strongly reducible.*

Proof. From (1.57), we see that the Fourier expansions of η^p and η_p are

$$\eta^p(z) = q^{\frac{p}{24}} + \mathcal{O}(q^{\frac{p}{24}+1}) \text{ and } \eta_p(z) = q^{\frac{p}{24}} + \mathcal{O}(q^{\frac{25p}{24}}).$$

So, $\operatorname{ord}_\infty(\eta^p/\eta_p) = 0$. Let $f := \eta^p/\eta_p$ and suppose, f is reducible on $\Gamma_0(p)$ by a factor g. Since $\operatorname{ord}_\infty(f) = 0$, we have

$$\operatorname{ord}_\infty(g) = -\operatorname{ord}_\infty(f/g).$$

But as both g and f/g are holomorphic, from the above equality, we get that

$$\operatorname{ord}_\infty(g) = \operatorname{ord}_\infty(f/g) = 0. \tag{2.3}$$

Also, since $\operatorname{ord}_\infty(f) = 0$, from (1.77) and from the Valence formula, we get that $\operatorname{ord}_0(f) = (p^2-1)/24$. Again, from (1.77) and from Corollary 1.42, it follows that if the orders of two eta quotients on $\Gamma_0(p)$ match at the cusps 0 and ∞ ,then the eta quotients are identical. Since $1 \neq g \neq f$, we get

$$0 < \operatorname{ord}_0(g) < \frac{p^2-1}{24}. \tag{2.4}$$

Let $X \in \mathbb{Z}^{\mathcal{D}_p}$ be such that $g = \eta^X$. Then from Corollary 1.42, (2.3), (2.4) and (1.91), we get that $-1 < X_p < 0$. Thus, we get a contradiction! So, the eta quotient η^p/η_p is not strongly reducible. □

From Corollary 1.57 and Lemma 1.59, it follows that

Lemma 2.4. *Let f be a holomorphic eta quotient on $\Gamma_0(N)$ and let $n\|N$. Then f is reducible on $\Gamma_0(N)$ if and only if so is $\mathrm{al}_{n,N}(f)$.*

Corollary 2.5. *For any prime p, the eta quotient $\dfrac{\eta_p^p}{\eta}$ is not strongly reducible.*

Proof. Follows from Lemma 2.3 and Lemma 2.4, since $\mathrm{al}_{p,p}(\eta^p/\eta_p) = \eta_p^p/\eta$. □

2.2 Some consequences of the conjecture

In this section, we shall see some implications of the Reducibility Conjecture: For example, a good lot of numerical evidence suggest the truth of the following assertions:

Conjecture 2.6 (Irreducibility Conjecture)**.** *The image of an irreducible holomorphic eta quotient under an Atkin-Lehner involution is irreducible.*

Conjecture 2.7 (Irreducibility Conjecture, alternative form)**.** *A rescaling of an irreducible holomorphic eta quotient is irreducible.*

Lemma 2.8. *Conjecture* 2.6 *and Conjecture* 2.7 *are equivalent.*

Proof. From Corollary 1.62, we see that Conjecture 2.7 is implied by Conjecture 2.6. Conversely, from Lemma 1.61 and Lemma 2.4, it follows that Conjecture 2.7 implies Corollary 1.62. □

The lemma below shows that Conjecture 2.6 holds assuming the truth of the Reducibility Conjecture:

Lemma 2.9. *The image of an irreducible holomorphic eta quotient under an Atkin-Lehner involution is not strongly reducible.*

Proof. Let f be an irreducible holomorphic eta quotient of level M and let

$$g := \mathrm{al}_{n,N}(f)$$

for some $N \in M\mathbb{N}$ and for some $n\|N$. Suppose, g is strongly reducible. Then, since the level of g divides N, g is reducible on $\Gamma_0(N)$. So, by Lemma 2.4, f must be reducible on $\Gamma_0(N)$. Thus, we get a contradiction! So, g is not strongly reducible. □

In particular, Corollary 1.62 and Lemma 2.9 together imply:

Corollary 2.10. *A rescaling of an irreducible holomorphic eta quotient is not strongly reducible.*

In the next section, we shall show that if for some $n \in \mathbb{N}$, the Reducibility Conjecture holds for the holomorphic eta quotients whose levels have at most n distinct prime divisors, then Conjecture 2.6 and hence, Conjecture 2.7 also hold if the initial eta quotients under consideration in these later conjectures are of such levels. Furthermore, in a later section, we shall show that the Reducibility conjecture holds for eta quotients of prime power levels.

If the Reducibility Conjecture holds for level N, then in particular, it gives an algorithm to check the irreducibility of a holomorphic eta quotient of level N:

Let f be a holomorphic eta quotient of level N. The Reducibility Conjecture implies that if f is reducible, then there exists a holomorphic eta quotient $g \notin \{1, f\}$ on $\Gamma_0(N)$, such that

$$\operatorname{ord}_s(g) \leq \operatorname{ord}_s(f) \tag{2.5}$$

at all cusps s of $\Gamma_0(N)$. Now, (1.70) and (2.5) together implies that $\operatorname{ord}_s(g)$ belongs to the finite set $[0, \operatorname{ord}_s(f)] \cap \frac{1}{24}\mathbb{Z}$ for each cusp s of $\Gamma_0(N)$. Since $\Gamma_0(N)$ has only finitely many cusps, it follows that the search for such a g halts after a finite amount of time.

Some improvements of the above algorithm are immediate:
(a) The Reducibility Conjecture implies Conjecture 2.7 and the contrapositive of the later is:

If a rescaling of a holomorphic eta quotient g is reducible, then so is g.

On the other hand, it is trivial that if an eta quotient is reducible, then so is any rescaling of it. Therefore, as the above algorithm assumes the Reducibility conjecture, to check the irreducibility of a holomorphic eta quotient $f = \eta^X$ of level N by the above algorithm is the same as to check whether the primitive holomorphic eta quotient

$$f_0 := \prod_{d \in \mathcal{D}_N} \eta_{d/\alpha}^{X_d}$$

of level N/α (of which, f is a rescaling by α), is strongly reducible or not, where α is the greatest common divisor of the elements of the set $\{d \in \mathcal{D}_N \,|\, X_d \neq 0\}$. Here, by a *primitive* eta quotient, we mean an eta quotient that is not the rescaling of some other eta quotient.

(b) Let f, f_0 and α be as above. Then from (1.77), it follows that f_0 is strongly reducible if and only if there exists a holomorphic eta quotient g on $\Gamma_0(N/\alpha)$ such that Condition (2.5) with f replaced by f_0 is satisfied at each cusp in the subset $\{1/t\}_{t|(N/\alpha)}$ of cusps of $\Gamma_0(N/\alpha)$. Now, from Corollary 1.42, it follows that to see whether f_0 is strongly reducible we only require to check if all the entries of

$$v' := 24A_{N/\alpha}^{-1}v$$

are integers, where $v \in \frac{1}{24}\mathbb{Z}^{\mathcal{D}_{N/\alpha}}$ (see (1.70)) with $v_t \in [0, \mathrm{ord}_{1/t}(f_0)]$ for all $t \in \mathcal{D}_{N/\alpha}$ and $A_{N/\alpha}$ is the valuation matrix (see 1.81). The algorithm halts either if we find an integral v' as above, in which case f_0 is strongly reducible by $\eta^{v'}$ or if none of the finitely may possibilities for v leads to any v' with integer entries as above, in which case f_0 is not strongly reducible.

(c) In (b), in order to search for a v' with integer entries, we do not need to go through the complete set of possibilities for v. Rather by unimodular Gaussian elimination, we may reduce $A_{N/\alpha}^{-1}$ to a triangular matrix which would allow us to search for an integral v' through a tree, determining one entry of v' at a time and consequently reducing the time complexity of the algorithm (see Chapter 4 in [22]).

2.3 Level reduction of the factors

In this section, we prove a partial result towards the Reducibility Conjecture. For $N \in \mathbb{N}$ and $M \in \mathcal{D}_N$, we define the linear map $\mathcal{P}_{M,N} : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}^{\mathcal{D}_M}$ by

$$(\mathcal{P}_{M,N}(X))_d := \sum_{t \in \mathcal{D}_{N/M}} X_{dt} \quad \text{for } X \in \mathbb{Z}^{\mathcal{D}_N} \text{ and } d \in \mathcal{D}_M. \tag{2.6}$$

Let $\mathbb{E}^!_N := \bigcup_{k \in \mathbb{Z}} \mathbb{E}^!_{N,k}$ (resp. $\mathbb{E}_N := \bigcup_{k \in \mathbb{Z}} \mathbb{E}_{N,k}$) be the group (resp. monoid) of eta quotients (resp. holomorphic eta quotients) on $\Gamma_0(N)$. Then for $M \in \mathcal{D}_N$, the map $\mathcal{P}_{M,N}$ induces the homomorphism $\mathrm{p}_{M,N}\colon \mathbb{E}^!_N \to \mathbb{E}^!_M$ given by

$$\mathrm{p}_{M,N}(\eta^X) := \eta^{\mathcal{P}_{M,N}(X)}. \tag{2.7}$$

It follows trivially from the above definition that for $m \in \mathbb{N}$, with $(m, N) = 1$, rescaling by m is a section of the map $\mathrm{p}_{N,mN} : \mathbb{E}^!{}_{mN} \to \mathbb{E}^!{}_N$. In other words, the following holds:

Lemma 2.11. *Let f be an eta quotient on $\Gamma_0(N)$ and let $m \in \mathbb{N}$ with $(m, N) = 1$. Then we have*

$$\mathrm{p}_{N,mN}(f_m) = f.$$

This leads to the following:

Lemma 2.12. *Let f be an eta quotient on $\Gamma_0(N)$ and let $M \in \mathbb{N}$ be a multiple of N. Let $m \in \mathcal{E}_M$ be such that $(m/(m,N), N) = 1$. Then we have*

$$\mathrm{p}_{N,\nu N} \circ \mathrm{al}_{m,M}(f) = \mathrm{al}_{n,N}(f), \tag{2.8}$$

where $n = (m, N)$ and $\nu = m/n$.

Proof. Let $g := \mathrm{al}_{n,N}(f)$ and let $\nu = m/n$. Then from Lemma 1.60, we have $\mathrm{al}_{m,M}(f) = g_\nu$. Since g is of level N and since $(\nu, N) = 1$, from Lemma 2.11, we get $\mathrm{p}_{N,\nu N}(g_\nu) = g$. □

We could have also obtained the last lemma as an implication of Lemma 2.13(c), Lemma 2.15 and Corollary 2.17 below.

In (2.7), if M is an exact divisor of N, then the homomorphism $\mathrm{p}_{M,N}$ possess some useful properties which we describe below. If $X \in \mathbb{Z}^{\mathcal{D}_N}$ has no negative entries, then we write $X \geq 0$.

Lemma 2.13. *For $N \in \mathbb{N}$ and $M \in \mathcal{D}_N$, let $\mathrm{p}_{M,N} : \mathbb{E}^!_N \to \mathbb{E}^!_M$ be the homomorphism defined above. If $M \| N$, then the following statements hold:*

(*a*) *The homomorphism $\mathrm{p}_{M,N}$ preserves weight.*

(*b*) *Also, $\mathrm{p}_{M,N}$ is a holomorphy-preserving homomorphism.*

(*c*) *If $n \in \mathcal{D}_M$ and if f is an eta quotient of level n, then $\mathrm{p}_{M,N}(f) = f$.*

Proof. (*a*) By Proposition 1.39, it suffices to show that $\sigma(\mathcal{P}_{M,N}(X)) = \sigma(X)$. This follows from the following:

$$\sum_{d \in \mathcal{D}_M} \sum_{t \in \mathcal{D}_{N/M}} X_{dt} = \sum_{d \in \mathcal{D}_N} X_d,$$

where the equality holds because $(M, N/M) = 1$.
(*b*) We require to show that $\mathcal{V}_M(\mathcal{P}_{M,N}(X)) \geq 0$ if $\mathcal{V}_N(X) \geq 0$ (see (1.80)).

By $P_{M,N} \in \mathbb{Z}^{\mathcal{D}_M} \times \mathbb{Z}^{\mathcal{D}_N}$, we denote the matrix of the linear map $\mathcal{P}_{M,N}$ with respect to the canonical bases of $\mathbb{Z}^{\mathcal{D}_M}$ and $\mathbb{Z}^{\mathcal{D}_N}$:

$$P_{M,N}(d,t) = \begin{cases} 1 & \text{if } t/d \in \mathcal{D}_{N/M}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

where $P_{M,N}(d,t)$ denotes the (d,t)-th entry of $P_{M,N}$. From (1.84) and from Proposition 1.41, we get

$$\mathcal{V}_M(\mathcal{P}_{M,N}(X)) = A_M P_{M,N} A_N^{-1} \mathcal{V}_N(X).$$

Hence, it suffices to show that none of the entries of the matrix $A_M P_{M,N} A_N^{-1}$ are negative. Since $(M, N/M) = 1$, from Proposition 1.41 we have

$$A_N^{-1} = A_M^{-1} \otimes A_{N/M}^{-1}. \tag{2.10}$$

Also, from (2.9), it is easy to check that

$$P_{M,N} = I_M \otimes \mathbb{1}_{N/M}^{\mathrm{T}}, \tag{2.11}$$

where $I_M \in \mathbb{Z}^{\mathcal{D}_M \times \mathcal{D}_M}$ is the identity matrix and $\mathbb{1}_{N/M} \in \mathbb{Z}^{\mathcal{D}_{N/M}}$ is defined similarly as in Proposition 1.41(c). So, from (2.10) and (2.11), we get

$$A_M P_{M,N} A_N^{-1} = I_M \otimes (\mathbb{1}_{N/M}^{\mathrm{T}} A_{N/M}^{-1}) = I_M \otimes \mathbb{a}_{N/M}^{\mathrm{T}} \geq 0, \tag{2.12}$$

where the second equality and the last inequality follows from Proposition 1.41(c).

(c) Since $n|M$ and since $(M, N/M) = 1$, we have $(n, M/N) = 1$. Let $X \in \mathcal{D}_N$ be such that $f = \eta^X$. Since f is of level n, we have

$$X_d = 0 \text{ for all } d \nmid n. \tag{2.13}$$

So, for $d \in \mathcal{D}_N$, we have

$$(\mathcal{P}_{M,N}(X))_d = \sum_{t \in \mathcal{D}_{N/M}} X_{dt} = \begin{cases} X_d & \text{1f } d|n, \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where the last equality follows from (2.13) and the fact that $(n, N/M) = 1$. Thus, we obtain that $\mathrm{p}_{M,N}(f) = f$. □

For $M \in \mathcal{D}_N$ which is not an exact divisor of N, the map $\mathrm{p}_{M,N}\colon \mathbb{E}^!_N \to \mathbb{E}^!_M$ in general, does not have any of the three properties described in Lemma 2.13. For example, we have $2 \in \mathcal{D}_4 \smallsetminus \mathcal{E}_4$. Since $\mathrm{p}_{2,4}(\eta_2) = \eta\eta_2$, this map has neither property (a) nor property (c) of Lemma 2.13. In fact, the map $\mathrm{p}_{2,4}$ does not have property (b) either, since we have $\mathrm{p}_{2,4}(\eta^2/\eta_2) = \eta/\eta_2$, where η^2/η_2 is a holomorphic modular form on $\Gamma_0(4)$ but η/η_2 is not holomorphic at ∞.

Corollary 2.14. *Let f be a holomorphic eta quotient of level N. If there exists an $M\|N$ such that $\mathrm{p}_{M,N}(f)$ is irreducible, then f is also irreducible.*

For example, $h := \dfrac{\eta_2^2\eta_3^2\eta_5^5\eta_{12}^2\eta_{20}^5\eta_{30}^{25}}{\eta\eta_4\eta_6^5\eta_{10}^{10}\eta_{15}^{10}\eta_{60}^{10}}$ is a holomorphic eta quotient of level 60 and we have

$$\mathrm{p}_{5,60}(h) = \frac{\eta_5^5}{\eta}.$$

From Corollary 2.5, we know that η_5^5/η is not strongly reducible. Hence, by Theorem 2.36 below, it follows that η_5^5/η is irreducible. So, by Corollary 2.14, we get that h is irreducible.

For $M\|N$, the following lemma lets us interchange the order of the operation of $\mathrm{p}_{M,N}$ with that of the Atkin-Lehner involutions on $\mathbb{E}^!_N$:

Lemma 2.15. *For $N \in \mathbb{N}$ and $n, M \in \mathcal{E}_N$, we have*

$$\mathrm{p}_{M,N} \circ \mathrm{al}_{n,N} = \mathrm{al}_{(n,M),M} \circ \mathrm{p}_{M,N}.$$

Proof. Follows easily from Lemma 1.51 and Corollary 1.52. □

Lemma 2.16. *Let $N \in \mathbb{N}$, $N' \in \mathcal{D}_N$, $M \in \mathcal{E}_N$, $M' := (M, N')$, $\nu := N/N'$ and let $\mu := (M, \nu)$. Let f be an eta quotient on $\Gamma_0(N')$. Then we have*

$$\mathrm{p}_{M,N}(f_\nu) = (\mathrm{p}_{M',N'}(f))_\mu.$$

Proof. Let $f_\nu = \eta^X$ for some $X \in \mathbb{Z}^{\mathcal{D}_N}$. Since $M\|N$, each element of $\mathcal{D}_N$ is uniquely represented as a product dt, where $d \in \mathcal{D}_M$ and $t \in \mathcal{D}_{N/M}$. Since f is of level N', for $d \in \mathcal{D}_M$ and $t \in \mathcal{D}_{N/M}$, we have $X_{dt} = 0$ unless $d = \mu d'$ and $t = \nu' t'$ for some $d'|M'$ and $t'|(N'/M')$, where $\nu' := \nu/\mu$. Now, the claim follows trivially from (2.6) and (2.7). □

Corollary 2.17. *Let $N \in \mathbb{N}$, $N' \in \mathcal{D}_N$, $M \in \mathcal{E}_N$ and let $M' := (M, N')$. Let f be an eta quotient on $\Gamma_0(N')$. Then we have*

$$\mathrm{p}_{M,N}(f) = \mathrm{p}_{M',N'}(f).$$

Corollary 2.18. *Let $N \in \mathbb{N}$, $N' \in \mathcal{D}_N$, $M \in \mathcal{E}_N$, $M' := (M, N')$, $\nu := N/N'$ and let $\mu := (M, \nu)$. Let f be an eta quotient on $\Gamma_0(N')$. Then we have*

$$\mathrm{p}_{M,N}(f_\nu) = (\mathrm{p}_{M,N}(f))_\mu.$$

Proposition 2.19. *For $n \in \mathbb{N}$, if the Reducibility Conjecture holds for the holomorphic eta quotients whose levels have at most n distinct prime divisors, then the image of an irreducible holomorphic eta quotient of such a level under an Atkin-Lehner involution is irreducible.*

Proof. Let f be an irreducible holomorphic eta quotient of some level $\mathcal{N}$ that has at most n distinct prime divisors and suppose that there exists a multiple M of $\mathcal{N}$ in $\mathbb{N}$ and $m \in \mathcal{E}_M$ such that $\mathrm{al}_{m,M}(f)$ is reducible. Then there exists an $N \in M\mathbb{N}$ such that $\mathrm{al}_{m,M}(f)$ is reducible on $\Gamma_0(N)$. Let $N' \| N$ be such that N' is divisible only by all the primes which divide $\mathcal{N}$ and let $M' := (N', M)$. It follows that $\mathcal{N}$ divides M'. Let $m' := (m, M')$. Then we have

$$\mathrm{p}_{N',N} \circ \mathrm{al}_{m,M}(f) = \mathrm{p}_{M',M} \circ \mathrm{al}_{m,M}(f) = \mathrm{al}_{m',M'} \circ \mathrm{p}_{M',M}(f) = \mathrm{al}_{m',M'}(f), \tag{2.15}$$

where the first equality holds by Corollary 2.17, the second by Lemma 2.15 and the third by Lemma 2.13(c). Since $\mathrm{al}_{m,M}(f)$ is reducible on $\Gamma_0(N)$, from (2.15) and from Lemma 2.13, it follows that $\mathrm{al}_{m',M'}(f)$ is reducible on $\Gamma_0(N)$. Again, since M' has at most n distinct prime divisors, according to our assumption, the Reducibility Conjecture holds for eta quotients of level M'. Therefore, $\mathrm{al}_{m',M'}(f)$ is reducible on $\Gamma_0(M')$. Hence, Lemma 2.4 implies that f is reducible. Thus, we get a contradiction! □

Similarly as we saw in the last section, the last proposition and Corollary 1.62 together imply:

Corollary 2.20. *For $n \in \mathbb{N}$, if the Reducibility Conjecture holds for the holomorphic eta quotients whose levels have at most n distinct prime divisors, then a rescaling of an irreducible holomorphic eta quotient of such a level is irreducible.*

For $m \in \mathbb{N}$, by $\mathrm{rad}(m)$, we denote the *radical* of m, i. e. the product of distinct prime divisors of m. The following lemma helps us to trim the levels of the factors of a reducible holomorphic eta quotient:

Lemma 2.21 (Level Lowering Lemma)**.** *Let f be a holomorphic eta quotient on of level M which is reducible on $\Gamma_0(N)$ for some $N \in M\mathbb{N}$. Let $N = mn$ with* $\mathrm{rad}(m)|M$ *and $(n, M) = 1$. Let g be a factor of f on $\Gamma_0(N)$. Then we have*

$$f = \mathrm{p}_{m,N}(g) \times \mathrm{p}_{m,N}(f/g).$$

Proof. Since $M\,|\,N = mn$ and since $(n, M) = 1$, we have $M|m$. Again Since $(n, M) = 1$ and since $\mathrm{rad}(m)|M$, we get that $m\|N$. So, Lemma 2.13 implies that the map $\mathrm{p}_{m,N}\colon \mathbb{E}^!_{N'} \to \mathbb{E}^!_m$ preserves weight and is also a holomorphy-preserving homomorphism. Since the level of f divides m, from Lemma 2.13(c), we have

$$f = \mathrm{p}_{m,N'}(f) = \mathrm{p}_{m,N'}(g) \times \mathrm{p}_{m,N'}(h).$$

□

For example, the eta quotient $\dfrac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2}$ of level 12 is a factor of the eta quotient $\dfrac{\eta_2^8}{\eta^2\eta_4^2}$ of level 4. So by Lemma 2.21, the later is strongly reducible:

$$\frac{\eta_2^8}{\eta^2\eta_4^2} = \frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2} \times \frac{\eta_2^3\eta_6^2}{\eta_3\eta_{12}} = \mathrm{p}_{4,12}\Big(\frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2}\Big) \times \mathrm{p}_{4,12}\Big(\frac{\eta_2^3\eta_6^2}{\eta_3\eta_{12}}\Big) = \frac{\eta_2^3}{\eta\eta_4} \times \frac{\eta_2^5}{\eta\eta_4}.$$

In particular, from Lemma 2.21, we get:

Theorem 2.22. *A holomorphic eta quotient of level N is reducible only if it is reducible on some $\Gamma_0(m)$ with $\mathrm{rad}(m)|N$.*

Corollary 2.23. *For $N \in \mathbb{N}$ and $M\|N$, if a holomorphic eta quotient of level M is reducible on $\Gamma_0(N)$, then it is strongly reducible.*

Corollary 2.24. *If a holomorphic eta quotient f has a factor of a squarefree level, then f is strongly reducible.*

2.4 Two results on reducibility

In this section, we shall prove the Reducibility Conjecture for eta quotients of prime power levels. By Theorem 2.22, it is enough to show that if an eta quotient of a prime power level p^n is reducible on $\Gamma_0(p^m)$ for some $m > n$, then it is strongly reducible. This will follow from a corollary of Theorem 2.31 which we prove below.

For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define $|X| \in \mathbb{Z}^{\mathcal{D}_N}$ by $|X|_d := |X_d|$. For two eta quotients $f = \eta^X$ and $g = \eta^Y$ respectively on $\Gamma_0(M)$ and $\Gamma_0(N)$ with mutually coprime M and N, we define the eta quotient $f \otimes g$ on $\Gamma_0(MN)$ by

$$f \otimes g := \eta^{X\otimes Y} = \prod_{d\in\mathcal{D}_N}\prod_{\delta\in\mathcal{D}_M} \eta_{d\delta}^{X_dY_\delta}. \tag{2.16}$$

Then by definition, we have:

$$f \otimes g = g \otimes f. \tag{2.17}$$

Contrary to the fact that Kronecker product of matrices is not commutative, (2.17) holds since in (2.16), with changing the order of the eta quotients f and g, we also change the order of the indexing set, viz. $\mathcal{D}_{MN} \simeq \mathcal{D}_M \times \mathcal{D}_N$.

The lemmas below also follow trivially from the above definition:

Lemma 2.25. *Let f_1, f_2 and g be eta quotients such that each of the levels of f_1 and f_2 is coprime to the level of g. Let $f := f_1 f_2$. Then we have*

$$f \otimes g = (f_1 \otimes g) \cdot (f_2 \otimes g).$$

Corollary 2.26. *Let f and g be holomorphic eta quotients with mutually coprime levels. If either f or g is reducible (resp. strongly reducible), then so is $f \otimes g$.*

Lemma 2.27. *Let f and g be holomorphic eta quotients with mutually coprime levels and let $n \in \mathbb{N}$. Then*

$$f^n \otimes g = f \otimes g^n.$$

Lemma 2.28. *Let f be an eta quotient of level N. Then for $m \in \mathbb{N}$ with $(m, N) = 1$, we have*

$$f_m = f \otimes \eta_m,$$

where f_m (resp. η_m) denotes the rescaling of f (resp. η) by m.

Lemma 2.29. *Let f and g be two eta quotients respectively on $\Gamma_0(M)$ and $\Gamma_0(N)$ with mutually coprime M and N. Let the respective weights of f and g be $k/2$ and $\ell/2$. Then we have*

$$\mathrm{p}_{M,MN}(f \otimes g) = f^\ell \text{ and } \mathrm{p}_{N,MN}(f \otimes g) = g^k.$$

To prove the next theorem, we require a relation between the orders of the eta quotients related by $\otimes$, at different cusps of the relevant groups:

Lemma 2.30. *Let f_1 and f_2 be two eta quotients respectively on $\Gamma_0(N)$ and $\Gamma_0(N')$ with $(N, N') = 1$. Let $f = f_1 \otimes f_2$. Let $t_1 \in \mathcal{D}_N$ and $t_2 \in \mathcal{D}_{N'}$ Let $1/t_1$, $1/t_2$ and $1/(t_1 t_2)$ be cusps of $\Gamma_0(N)$, $\Gamma_0(N')$ and $\Gamma_0(NN')$ respectively. Then we have*

$$\mathrm{ord}_{\frac{1}{t_1 t_2}}(f) = 24 \cdot \mathrm{ord}_{\frac{1}{t_1}}(f_1) \cdot \mathrm{ord}_{\frac{1}{t_2}}(f_2). \tag{2.18}$$

Proof. By (1.80), it suffices to show that

$$\mathcal{V}_{NN'}(X \otimes X') = 24\,\mathcal{V}_N(X) \otimes \mathcal{V}_{N'}(X'). \tag{2.19}$$

Since $(N, N') = 1$, from Proposition 1.41 (a) we have

$$A_{NN'} = A_N \otimes A_{N'}.$$

So, we get

$$A_{NN'}(X \otimes X') = (A_N X) \otimes (A_{N'} X').$$

Hence, the claim follows by (1.84). □

Theorem 2.31. *Let p be a prime and let $n, N \in \mathbb{N}$ be such that $N = p^n M$, where $(M, p) = 1$. Let f be a holomorphic eta quotient on $\Gamma_0(N)$ which has a factor $f_1 = \eta^X$ of level Np^ℓ for some $\ell \in \mathbb{N}$. Let $X' \in \mathbb{Z}^{\mathcal{D}_M}$ be such that $X'_d = X_{dp^{n+\ell}}$ for all $d \in \mathcal{D}_M$ and let $f'_1 := \eta^{X'}$. Let g be a holomorphic eta quotient on $\Gamma_0(M)$ such that*

$$\operatorname{ord}_{1/t}(g) \leq |\operatorname{ord}_{1/t}(f'_1)| \tag{2.20}$$

for all $t \in \mathcal{D}_M$. Then

$$g \otimes \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{\ell-1}}.$$

is a factor of f.

Proof. We shall proceed by induction on ℓ. So, first we consider the case where $\ell = 1$:

Since $f_1 \in \mathbb{E}_{Np}$ a factor of f, so is also $f_2 := f/f_1 \in \mathbb{E}_{Np}$. For all $r \in \mathcal{D}_{Np}$, we define $a, b \in \dfrac{1}{24}\mathbb{Z}^{\mathcal{D}_{Np}}$ by $a_r := \operatorname{ord}_{1/r}(f_1)$ and $b_r := \operatorname{ord}_{1/r}(f_2)$. Since f_1 and f_2 are holomorphic, we have

$$a_r, b_r \geq 0 \text{ for all } r \in \mathcal{D}_{Np}. \tag{2.21}$$

Since $f_1 = \eta^X$, from Corollary 1.42 we get

$$X = 24 A_{Np}^{-1} a = 24\,(A_M^{-1} \otimes A_{p^{n+1}}^{-1})\,a, \tag{2.22}$$

where the second equality follows from Proposition 1.41 (a). Since $X'_d = X_{dp^{n+1}}$ for all $d \in \mathcal{D}_M$, from (2.22) we obtain

$$\begin{aligned} X' &= 24\,(A_M^{-1} \otimes A_{p^{n+1}}^{-1}(p^{n+1}, \_\,))\,a \\ &= \frac{24}{p^{n-1}(p^2-1)}\,A_M^{-1}\,(a' - a''), \end{aligned} \tag{2.23}$$

where $a', a'' \in \mathbb{Z}^{\mathcal{D}_M}$ are defined by $a'_t := a_{tp^{n+1}}$ and $a''_t := a_{tp^n}$ for all $t \in \mathcal{D}_M$. Above, (2.23) holds since we have

$$A^{-1}_{p^{n+1}}(p^{n+1}, p^n) = -\frac{1}{p^{n-1}(p^2-1)}, \quad A^{-1}_{p^{n+1}}(p^{n+1}, p^{n+1}) = \frac{1}{p^{n-1}(p^2-1)}$$

and all other entries of $A^{-1}_{p^{n+1}}(p^{n+1}, \_)$ are zeros (see (1.90) or (1.91)).

Now, from (2.23) and (1.84), we get:

$$a'_t - a''_t = p^{n-1}(p^2-1)\,\mathcal{V}_M(X')_t.$$

Since both a'_t and a''_t are nonnegative (see (2.21)), it follows that

$$\max\{a'_t, a''_t\} \geq p^{n-1}(p^2-1)\,|\mathcal{V}_M(X')_t| \text{ for all } t \in \mathcal{D}_M. \tag{2.24}$$

Let $Y \in \mathbb{Z}^{\mathcal{D}_{Np}}$ such that $f_2 = \eta^Y$. We define $Y' \in \mathbb{Z}^{\mathcal{D}_M}$ by $Y'_d := Y_{dp^{n+1}}$ for all $d \in \mathcal{D}_M$. Let $(X+Y)' := X' + Y'$. Then similarly as we deduced (2.23), we get

$$Y' = \frac{24}{p^{n-1}(p^2-1)}\,A_M^{-1}(b' - b''), \tag{2.25}$$

where $b', b'' \in \mathbb{Z}^{\mathcal{D}_M}$ are defined by $b'_t := b_{tp^{n+1}}$ and $b''_t := b_{tp^n}$ for all $t \in \mathcal{D}_M$. Since f_1 and f_2 are of level Np, both X' and Y' are nonzero. Again, since $f = f_1 f_2 = \eta^{X+Y}$ is of level N, we have

$$X' + Y' = (X+Y)' = 0. \tag{2.26}$$

Thus, we obtain

$$a' + b' = a'' + b'' \geq p^{n-1}(p^2-1)\cdot|\mathcal{V}_M(X')| \tag{2.27}$$

where the equality follows from (2.23), (2.25) and (2.26), whereas the inequality follows from (2.24).

From (1.70), we get that the orders of $\eta^p_{p^n}/\eta_{p^{n-1}}$ at the cusps $\{1/p^\ell\}_{0\leq\ell\leq n+1}$ of $\Gamma_0(p^{n+1})$ are as follows:

$$\operatorname{ord}_{\frac{1}{p^\ell}}\left(\frac{\eta^p_{p^n}}{\eta_{p^{n-1}}}\right) = \begin{cases} 0 & \text{if } 0 \leq \ell < n, \\ p^{n-1}(p^2-1)/24 & \text{otherwise.} \end{cases} \tag{2.28}$$

Let $h := g \otimes (\eta^p_{p^n}/\eta_{p^{n-1}})$. Then from Lemma 2.30 and (2.28), we get that

$$\operatorname{ord}_{\frac{1}{tp^\ell}}(h) = \begin{cases} 0 & \text{if } 0 \leq \ell < n, \\ p^{n-1}(p^2-1)\cdot\operatorname{ord}_{\frac{1}{t}}(g) & \text{otherwise,} \end{cases} \tag{2.29}$$

for all $t \in \mathcal{D}_M$. Also, for all $t \in \mathcal{D}_M$, we have

$$\operatorname{ord}_{\frac{1}{tp^{n+1}}}(f) = a'_t + b'_t, \ \ \operatorname{ord}_{\frac{1}{tp^n}}(f) = a''_t + b''_t \ \text{ and } \ \mathcal{V}_N(X')_t = \operatorname{ord}_{\frac{1}{t}}(f'_1) \tag{2.30}$$

where the last equality follows from (1.80).

Since f is holomorphic and since $\operatorname{ord}_{1/t}(g) \leq |\operatorname{ord}_{1/t}(f'_1)|$ for all $t \in \mathcal{D}_M$, from (2.27), (2.29), (2.30) and from (1.77), we get that

$$\operatorname{ord}_s(f) \geq \operatorname{ord}_s(h)$$

at each cusp s of $\Gamma_0(Np)$. Hence, h is indeed a factor of f.

Now, let us assume that the claim holds for ℓ_0 for some $\ell_0 \in \mathbb{N}$. Below, we show that then it holds also for $\ell = \ell_0 + 1$:

Let $N' := Mp^m$, where $m = n + \ell_0$. For the sake of clarity, we redefine the notations which we have also used in the previous case, as follows:

Let f be a holomorphic eta quotient on $\Gamma_0(N)$ which has a factor $f_1 = \eta^X$ of level $Np^\ell = N'p$. Let $X' \in \mathbb{Z}^{\mathcal{D}_M}$ be such that $X'_d = X_{dp^{n+\ell}} = X_{dp^{m+1}}$ for all $d \in \mathcal{D}_M$ and let $f'_1 := \eta^{X'}$. Let $Y \in \mathbb{Z}^{\mathcal{D}_M}$ be such that the eta quotient $g := \eta^Y$ on $\Gamma_0(M)$ satisfies

$$\operatorname{ord}_{1/t}(g) \leq |\operatorname{ord}_{1/t}(f'_1)|$$

for all $t \in \mathcal{D}_M$. Then from the previous case, it follows that the holomorphic eta quotient $h_1 := g \otimes (\eta_{p^m}^p/\eta_{p^{m-1}})$ of level $N' = Np^{\ell_0}$ is a factor of f. Let $Z \in \mathbb{Z}^{\mathcal{D}_{N'}}$ be such that $h_1 = \eta^Z$ and let $Z' \in \mathbb{Z}^{\mathcal{D}_M}$ be such that $Z'_d = Z_{dp^{n+\ell_0}}$ for all $d \in \mathcal{D}_M$. Since

$$\eta^Z = h_1 = g \otimes (\eta_{p^m}^p/\eta_{p^{m-1}}) = \eta^Y \otimes (\eta_{p^m}^p/\eta_{p^{m-1}}),$$

we have $Z'_d = Z_{dp^m} = pY_d$ for all $d \in \mathcal{D}_M$. So, $h'_1 := \eta^{Z'} = g^p$ is a holomorphic eta quotient on $\Gamma_0(M)$. Now, by the induction hypothesis, it follows that

$$h_1 \otimes \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{\ell_0 - 1}} = g^p \otimes \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{\ell_0 - 1}} = g \otimes \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{\ell - 1}}$$

is a factor of f, where the last equality holds by Lemma 2.27. □

Corollary 2.32. *Let p be a prime and let $M \in \mathbb{N}$ be such that $(M, p) = 1$. Let a holomorphic eta quotient f of level $N = Mp^n$ have a factor f_1 of level Np^ℓ for some integer $\ell > 1$. If there exists a holomorphic eta quotient $g \neq 1$ on $\Gamma_0(M)$ which satisfies Condition* (2.20)*, where f'_1 is the same as defined in Theorem* 2.31*, then f is strongly reducible.*

Proof. Since $g \neq 1$, from Theorem 2.31 and Lemma 2.25, it follows that $g \otimes \dfrac{\eta_{p^n}^p}{\eta_{p^{n-1}}}$ is a nontrivial factor of f. □

The above corollary does not hold for $\ell = 1$, since in that case f itself could be equal to $g \otimes (\eta_{p^n}^p/\eta_{p^{n-1}})$. But certainly, if g is strongly reducible, then so is f (see Lemma 2.25).

Corollary 2.33. *For a prime p, if a holomorphic eta quotient f of level p^n has a factor of level p^m for some $m > n$, then*

$$\left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}^p}\right)^{p^{m-n-1}}$$

is a factor of f.

Proof. Let f_1 be a factor of level p^m of f and let $X \in \mathcal{D}_{p^{n+1}}$ be such that $f_1 = \eta^X$. In particular, since f_1 is of level p^m, we have $X_{p^m} \neq 0$. Let $f_1' = \eta^{X_{p^m}}$. Then, $g = \eta$ satisfies Condition (2.20). So, Theorem 2.31 implies that

$$\eta \otimes \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{m-n-1}} = \left(\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}\right)^{p^{m-n-1}}$$

is a factor of f. □

Corollary 2.34. *Let p be a prime. If a holomorphic eta quotient f of level p^n has a factor of level p^m for some $m > n+1$, then f is strongly reducible by the factor*

$$\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}.$$

To extend the above corollary to the case $m = n+1$, the following lemma would be very useful. For proving this lemma, we would require a special case (i. e. the case where $M = 1$) of some statements which we have seen during the proof of Theorem 2.31. Instead of repeating the same arguments, we shall recall them from the proof of the theorem as needed.

Lemma 2.35. *For a prime p, if a holomorphic eta quotient f of level p^n has a factor of level p^{n+1}, then*

$$f \neq \frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}.$$

Proof. Let f_1 be a factor of f of level p^{n+1}. Then also $f_2 := f/f_1$ is such a factor of f. Let $X, Y \in \mathbb{Z}^{\mathcal{D}_{p^{n+1}}}$ be such that $f_1 = \eta^X$ and $f_2 = \eta^Y$. Then for $M = 1$, from (2.23) and (2.25), we get

$$X_{p^{n+1}} = \frac{24(a' - a'')}{p^{n-1}(p^2-1)} \quad \text{and} \quad Y_{p^{n+1}} = \frac{24(b' - b'')}{p^{n-1}(p^2-1)}, \tag{2.31}$$

where $a' := \mathrm{ord}_{1/p^{n+1}}(f_1)$, $a'' := \mathrm{ord}_{1/p^n}(f_1)$, $b' := \mathrm{ord}_{1/p^{n+1}}(f_2)$ and $b'' := \mathrm{ord}_{1/p^n}(f_2)$. In particular, since f_1 and f_2 are holomorphic, we have

$$a', a'', b', b'' \geq 0. \tag{2.32}$$

Since f_1 and f_2 are of level p^{n+1}, neither $X_{p^{n+1}}$ nor $Y_{p^{n+1}}$ is zero but we have

$$X_{p^{n+1}} + Y_{p^{n+1}} = 0, \tag{2.33}$$

because $f = f_1 f_2$ is of level p^n. So, without loss of generality, we may assume that $X_{p^{n+1}} > 0$. Then (2.31), (2.32) and (2.33) together imply that

$$a' + b' = a'' + b'' \geq b'' = b' + \frac{p^{n-1}(p^2-1)}{24} X_{p^{n+1}} \geq \frac{p^{n-1}(p^2-1)}{24}. \tag{2.34}$$

Suppose, if possible $f = \eta_{p^n}^p / \eta_{p^{n-1}}$. Since $f = f_1 f_2$, the sum of the orders of f_1 and f_2 at the cusps $\{1/p^j\}_{0 \leq j \leq n+1}$ of $\Gamma_0(p^{n+1})$ is then given by (2.28). Since both f_1 and f_2 are holomorphic, they have nonnegative orders at all the cusps. So, from (2.28), it follows that for $0 \leq j < n$, we have

$$\mathrm{ord}_{1/p^j}(f_1) = \mathrm{ord}_{1/p^j}(f_2) = \mathrm{ord}_{1/p^j}(\eta_{p^n}^p / \eta_{p^{n-1}}) = 0. \tag{2.35}$$

Since $f = f_1 f_2$, we also have

$$\mathrm{ord}_{1/p^n}(f) = \mathrm{ord}_{1/p^n}(f_1) + \mathrm{ord}_{1/p^n}(f_2) = a'' + b''$$

and

$$\mathrm{ord}_{1/p^{n+1}}(f) = \mathrm{ord}_{1/p^{n+1}}(f_1) + \mathrm{ord}_{1/p^{n+1}}(f_2) = a' + b'.$$

Also, since $f = \eta_{p^n}^p / \eta_{p^{n-1}}$, (2.28) implies that all the equalities in (2.34) hold. In particular, we have $a'' = b' = 0$ and $a' = b'' = p^{n-1}(p^2-1)/24$, i. e.

$$\mathrm{ord}_{1/p^n}(f_1) = \mathrm{ord}_{1/p^{n+1}}(f_2) = 0 \tag{2.36}$$

and

$$\mathrm{ord}_{1/p^{n+1}}(f_1) = \mathrm{ord}_{1/p^n}(f_2) = \frac{p^{n-1}(p^2-1)}{24}. \tag{2.37}$$

Now, from Corollary 1.42, (2.35), (2.36), (2.37) and (1.91) it follows that

$$X_{p^n} = 24 \cdot A_{p^{n+1}}(p^n, p^{n+1}) \cdot \mathrm{ord}_{1/p^{n+1}}(f_1) = -1/p \notin \mathbb{Z}.$$

Thus, we get a contradiction! Therefore, we conclude that $f \neq \eta_{p^n}^p/\eta_{p^{n-1}}$. □

So, we can extend Corollary 2.34 to the following:

Corollary 2.34′. *Let p be a prime. If a holomorphic eta quotient f of level p^n has a factor of level p^m for some $m > n$, then f is strongly reducible by the factor*

$$\frac{\eta_{p^n}^p}{\eta_{p^{n-1}}}.$$

Now, Corollary 2.22 and Corollary 2.34′ together imply that:

Theorem 2.36. *A holomorphic eta quotient of a prime power level is reducible if and only if it is strongly reducible.*

The last theorem and Proposition 2.19 together imply:

Corollary 2.37. *The image of an irreducible holomorphic eta quotient of a prime power level under an Atkin-Lehner involution is irreducible.*

Also, from Theorem 2.36 and Corollary 2.20, we obtain:

Corollary 2.38. *A rescaling of an irreducible holomorphic eta quotient of a prime power level is irreducible.*

In particular, from Lemma 2.3, Corollary 2.5, Theorem 2.36 and Corollary 2.38, we get:

Corollary 2.39. *For a prime p and $m \in \mathbb{N}$, the holomorphic eta quotients*

$$\frac{\eta_{mp}^p}{\eta_m} \quad \text{and} \quad \frac{\eta_m^p}{\eta_{mp}}$$

are irreducible.

We shall also see examples of some special families of irreducible holomorphic eta quotients in a later chapter.

2.5 Checking irreducibility

In Section 2.2, we have seen that the Reducibility Conjecture implies an algorithm to check the irreducibility of a holomorphic eta quotient in a finite time. We prove the existence of such an algorithm without assuming the conjecture, by showing that the level of any factor of a holomorphic eta quotient f of level N and weight k is bounded w.r.t. N and k (this suffices, since Corollary 1.50 implies that there are only finitely many holomorphic eta quotients if a pair of bounds on their weights and levels are given and the proof of Corollary 1.49 gives a naïve algorithm to list all such eta quotients (for a more efficient algorithm for producing a complete list of holomorphic eta quotients for any given pair of weight and level, see Chapter 4 in [22] and also consult [36] in this regard)). We shall use a finiteness result of Mersmann on simple holomorphic eta quotients:

A holomorphic eta quotient is called *simple* if it is primitive and not strongly reducible. In 1991, in his excellent thesis, Mersmann proved the following conjecture of Zagier (see [7], [29]):

Theorem 2.40 (Mersmann's First Theorem)**.** *There are only finitely many simple holomorphic eta quotients of any particular weight.*

The above theorem could be made effective (see Chapter 3), i. e. we could restate it as follows:

Theorem 2.40′. *For each $k \in \mathbb{N}$, there exists an effectively computable $M_k \in \mathbb{N}$ such that if there exists any simple holomorphic eta quotient of weight less than or equal to $k/2$ and level N, then $N \,|\, M_k$.*

For $k \in \mathbb{N}$, we call the smallest possible value for M_k (as defined above) the *Mersmann bound for weight $k/2$.*

Convention 2.41. *Henceforth, by M_k, we shall denote the Mersmann bound for weight $k/2$.*

For example, we have $M_1 = 12$ (see Theorem 4.1).

Theorem 2.42. *The level of any factor of a holomorphic eta quotient of level M and weight k is a divisor of $\mathcal{N}(M,k)^2$, where $\mathcal{N}(M,k)$ is the least common multiple of M_k and the following number:*

$$\prod_{\substack{p \text{ prime} \\ p<2k \text{ or } p\,|\,M \\ p^b \| M,\ p^a \leq 2k\psi(p^b) < p^{a+1}}} p^{a+1}, \tag{2.38}$$

where $\psi : \mathbb{N} \to \mathbb{N}$ *is defined by* (1.86).

Proof. Let f be a holomorphic eta quotient of level M and weight k and let g be a factor of f. Since the claim holds trivially for the trivial factors of f, we may assume that $g \notin \{1, f\}$. Also, without loss of generality, we may assume that g is irreducible of weight less than or equal to $k/2$. Let N be the level of g and suppose, $N \not| M'^2$, where $M' := \mathcal{N}(M, k)$. By Theorem 2.40$'$, there exists $\nu \in \mathbb{N}$ such that $g = h_\nu$ for a holomorphic eta quotient h of some level $N' \,|\, M_k \,|\, M'$. Since $\nu N' = N \not| M'^2$, it follows that $\nu \not| M'$. So, there exists a prime $p|\nu$ such that

$$v_p(\nu) \geq v_p(M'), \tag{2.39}$$

where v_p denotes the p-adic valuation. Since both M and N' divide M' and since $N = \nu N'$, the above inequality implies:

$$v_p(\nu) \geq v_p(M) \text{ and } 2v_p(\nu) \geq v_p(N). \tag{2.40}$$

Let $m := p^{v_p(M)}$, $n := p^{v_p(N)}$, $n' := p^{v_p(N')}$, $\mu := p^{v_p(\nu)}$ and let f', g' and h' be the respective images of f, g and h under the homomorphism $\mathrm{p}_{n,N} : \mathbb{E}^!_N \to \mathbb{E}^!_n$. From (2.6) and (2.7), it follows that the levels of f', g' and h' are m, n and n', respectively. Since $n\|N$, from Lemma 2.13, it follows that f', g' and h' are holomorphic eta quotients . Since g is a factor of f and since $\mathrm{p}_{n,N}$ is a homomorphism, g' is a factor of f'. Since $g = h_\nu$, from Corollary 2.18, we get $g' = h'_\mu$. Since $g \neq 1$, from Lemma 2.13, it follows that $g' \neq 1$. Since $h'_\mu = g'$, we get $h' \neq 1$. Since h' is holomorphic and $h' \neq 1$, there exists $s \in \mathbb{P}^1(\mathbb{Q})$ such that $\mathrm{ord}_s(h'; \Gamma_0(n')) > 0$. So, from (1.70), we get that

$$\mathrm{ord}_s(h'; \Gamma_0(n')) \geq \frac{1}{24}. \tag{2.41}$$

In particular, by (1.77), we may choose s to be of the form $[1 : \lambda]$ for some $\lambda \in \mathbb{N} \cup \{0\}$. Let $s' := [1 : \mu\lambda] \in \mathbb{P}^1(\mathbb{Q})$. Since $g' = h'_\mu$ and since by (2.40), we have $n \,|\, \mu^2$, from (2.41) and from Corollary 1.31, we get

$$\mathrm{ord}_{s'}(g'; \Gamma_0(n)) \geq \frac{\mu}{24}. \tag{2.42}$$

Again, by (2.40), we have $m \,|\, \mu$. So, from the proof of Proposition 1.7, we get that s' and $[1 : m]$ represent the same cusp of $\Gamma_0(m)$. Now, from (1.78), it follows that

$$\mathrm{ord}_{s'}(f'; \Gamma_0(n)) = \mathrm{ord}_{1/m}(f'; \Gamma_0(m)) \leq \frac{k\psi(m)}{12}, \tag{2.43}$$

where the last inequality holds by the Valence formula (see Corollary 1.44), since from Lemma 2.13(a), it follows that the weight f' is k. Since $m = p^{v_p(M)} \| M$, from (2.39) and from the definition of $M' = \mathcal{N}(M,k)$ (see in particular, (2.38)), a strengthening of the first inequality in (2.40) follows:

$$\mu > 2k\psi(m). \tag{2.44}$$

Now, the above inequality together with (2.42) and (2.43) implies that

$$\mathrm{ord}_{s'}(g';\Gamma_0(n)) > \mathrm{ord}_{s'}(f';\Gamma_0(n)). \tag{2.45}$$

So, g' can not be a factor of f'. Thus we get a contradiction! Hence, we must have

$$N < \mathcal{N}(M,k)^2.$$

□

We recall from Theorem 2.22 that an eta quotient of level N is reducible only if it is reducible on some $\Gamma_0(M)$ with $\mathrm{rad}(M)|N$. Now, we define the *restricted Mersmann bound* for level N and weight $k/2$ to be the least positive integer $M_{N,k}$ such that if there exists any simple holomorphic eta quotient of weight less than or equal to $k/2$ and level M with $\mathrm{rad}(M)|N$, then $M|M_{N,k}$. In particular, Mersmann's First Theorem implies that the set $\{M_{N,k} \mid N \in \mathbb{N}\}$ is finite. So, M_k is the least common multiple of the elements in this set. Certainly, we can also write:

$$M_k = \sup_{N\in\mathbb{N}} M_{N,k}. \tag{2.46}$$

On the other hand, from Level Lowering Lemma (see Lemma 2.21), it follows that $M_{N,k}$ is the largest divisor of M_k whose radical divides N. In particular, the effective computability of M_k guarantees the effective computability of $M_{N,k}$.

Corollary 2.43. *Any reducible holomorphic eta quotient of level M and weight k has a factor of some level dividing $\mathcal{N}_0(M,k)^2$ where $\mathcal{N}_0(M,k) \in \mathbb{N}$ is the least common multiple of $M_{N,k}$ and the following number:*

$$\prod_{\substack{p \text{ prime} \\ p^b \| M,\ ,b>0 \\ p^a \leq 2k\psi(p^b) < p^{a+1}}} p^{a+1}. \tag{2.47}$$

Proof. Follows from Theorem 2.22 and Theorem 2.42. □

Theorem 2.42 or Corollary 2.43 shows that checking irreducibility of a holomorphic eta quotient is possible without resort to the Reducibility Conjecture. However, the Mersmann bounds M_k are not explicitly known for any $k > 1$ due to the immense computational complexity of the presently known algorithm for determining them (see Chapter 3) and also, the present upper bounds for them are too large for all practical purposes.

Part II

Holomorphic eta quotients of a particular weight

Chapter 3

Simple holomorphic eta quotients

About 25 years ago, based on extensive numerical calculations, Zagier formulated two conjectures on simple holomorphic eta quotients , one asserting the finiteness of the set of such eta quotients of any particular weight and the other giving a complete list of such eta quotients of weight 1/2. Both of these conjectures were subsequently established by his student Mersmann in a brilliant thesis.

In this chapter and the next, we give respectively simplified and shorter proofs of Mersmann's theorems (see Theorem 2.40 and Theorem 4.1). The inclusion of these two chapters in this thesis is inspired by a paragraph on page 117 of [22], where Köhler discusses the formidabilily of the task of simplifying Mersmann's original proofs. For the convenience of the reader, here we restate Mersmann's First Theorem:

Theorem (Mersmann's First Theorem)**.** *There are only finitely many simple holomorphic eta quotients of any particular weight.*

A basic step towards establishing the above theorem is to prove the following lemma:

Lemma (Mersmann's Lemma)**.** *Let $f = \eta^X$ is a holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$. Then*

$$\|X\| \leq k \cdot \prod_{\substack{p \text{ prime} \\ p|N}} \left(\frac{p+1}{p-1}\right)^{\min\{2,\, v_p(N)\}}, \tag{3.1}$$

where $v_p(N)$ *denotes the* p*-adic valuation of* N *and* $\|X\| := \sigma(|X|)$*. Here* $|X| \in \mathbb{Z}^{\mathcal{D}_N}$ *(resp.* $\sigma : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}$*) is as defined in the beginning of Section* 2.4 *(resp. in* (1.75)*).*

The last inequality gives a bound on the number of holomorphic eta quotients of weight k and level N. Comparing it with the dimension (see [13] or [42]) of the space of modular forms of weight k and level N, we see that for any $k \in \mathbb{N}$, there are infinitely many $N \in \mathbb{N}$ such that the space of modular forms of weight k and level N does not have enough eta quotients to make up for a basis. This gives a partial answer to a question asked by Ono in [31] about classification of the spaces of modular forms which are spanned by eta quotients.

Actually, Mersmann proved a variant of the above lemma in [29]. In the next section, we prove a more general result for weakly holomorphic eta quotients which would be quite useful later and in particular, from which Mersmann's Lemma would follow (see Corollary 3.2). Independently of this work, in [36], Rouse and Webb have also established the inequality (3.1) and have drawn a similar conclusion as in the above paragraph about the spaces of modular forms which are not generated by eta quotients.

3.1 A generalization of Mersmann's Lemma

In order to formulate and prove our next result, we require the folowing notations:

$$v_+ \in \mathbb{R}^{\mathcal{D}_N} : \text{ the positive component of } v \in \mathbb{R}^{\mathcal{D}_N},\ v_+ = \tfrac{1}{2}(v + |v|).$$

$$v_- \in \mathbb{R}^{\mathcal{D}_N} : \text{ the negative component of } v \in \mathbb{R}^{\mathcal{D}_N},\ v_- = \tfrac{1}{2}(v - |v|).$$

$$\|v\|_\pm \in \mathbb{R} : \text{ the sum of the entries of } |v_\pm|,\ \|v\|_\pm = \|v_\pm\| = \pm\sigma(v_\pm).$$

Lemma 3.1 (Generalized Mersmann's Lemma)**.** *Let* $f = \eta^X$ *be an eta quotient of weight* $k/2$ *on* $\Gamma_0(N)$*. Then we have*

$$\|X\| \leq k \cdot F(N) + G(N) \cdot \|\widehat{A}_N X\|_-, \tag{3.2}$$

where

$$F(N) := \frac{\psi(N)}{\varphi(N)} \cdot \prod_{\substack{p \text{ prime} \\ p^2|N}} \frac{p+1}{p-1}\,, \quad G(N) := \left(\frac{1}{\psi(N)} + \frac{1}{\varphi(N)}\right) \cdot \prod_{\substack{p \text{ prime} \\ p^2|N}} \left(1 + \frac{1}{p}\right), \tag{3.3}$$

ψ, $\varphi : \mathbb{N} \to \mathbb{N}$ are as defined in (1.86) *and $\widehat{A}_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ is the symmetrized valuation matrix of level N, defined by*

$$\widehat{A}_N(t, \_) = (t, N/t) \cdot A_N(t, \_) \text{ for all } t \in \mathcal{D}_N, \tag{3.4}$$

where A_N is the valuation matrix of level N.

Corollary 3.2 (Mersmann's Lemma)**.** *Let $f = \eta^X$ be a holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$. Then we have*

$$\|X\| \le kF(N). \tag{3.5}$$

Corollary 3.3. *Let $f = \eta^X$ be a weakly holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$ with $\|X\| \ge kF(N) + \varepsilon$. Then we have*

$$\|\widehat{A}_N X\|_- \ge \varepsilon/G(N). \tag{3.6}$$

Corollary 3.4. *Let $f = \eta^X \neq 1$ be a weakly holomorphic eta quotient of weight $k/2$ on $\Gamma_0(N)$, where $k \le 0$. Then we have*

$$\|\widehat{A}_N X\|_- \ge \begin{cases} 2/G(N) & \text{if } k = 0, \\ |k| \cdot (F(N)+1)/G(N) & \text{otherwise.} \end{cases} \tag{3.7}$$

Proof. If $\sigma(X) = k = 0$ (see Proposition 1.39) but $X \neq 0$, then $X \in \mathbb{Z}^{\mathcal{D}_N}$ has at least two nonzero entries. Hence, $\|X\| \ge 2$ and so, the claim for the case $k = 0$ follows from (3.2).

For the case $k < 0$, from (3.2), we have $G(N) \cdot \|\widehat{A}_N X\|_- \ge \|X\| + |k| \cdot F(N)$. Since we have $\|X\| \ge |k|$, the claim follows. □

Proof of Lemma 3.1. Let $\widehat{A}_N$ be the symmetrized valuation matrix (see (3.4) of level N (see Section 1.5) and let $Y := \widehat{A}_N X$. Then we have

$$\|X\| = \|\widehat{A}_N^{-1} Y\| \le \|\widehat{A}_N^{-1} Y_+\| + \|\widehat{A}_N^{-1} Y_-\|. \tag{3.8}$$

To not leave room for any possible confusion, we mention here that by $\widehat{A}_N^{-1}$, we denote $(\widehat{A}_N)^{-1}$. Let $\mathcal{Q}_N \subset \mathcal{D}_N$ be such that $d \in \mathcal{Q}_N$ only if $d^2 | N$. Then for all $t \in \mathcal{D}_N$, we have $(t, N/t) \in \mathcal{Q}_N$ and for all $d \in \mathcal{Q}_N$, we have $(d, N/d) = d$. For $d \in \mathcal{Q}_N$, let

$$y_d := \sum_{\substack{t \in \mathcal{D}_N \\ Y_t < 0 \\ (t, N/t) = d}} |Y_t|. \tag{3.9}$$

Then from Proposition 1.41(c) and from (3.4), we get

$$\mathbb{1}_N^{\mathrm{T}} \widehat{A}_N^{-1} Y_- = -\sum_{d\in\mathcal{Q}_N} y_d \frac{\varphi(d)}{d\psi(N)} \quad \text{and} \quad \|\widehat{A}_N^{-1} Y_-\| \le \mathbb{1}_N^{\mathrm{T}} |\widehat{A}_N^{-1}|\,|Y_-| = \sum_{d\in\mathcal{Q}_N} y_d \frac{\psi(d)}{d\varphi(N)}\,. \tag{3.10}$$

Again by Proposition 1.41 (c), by (3.4) and by (3.10), it follows that

$$\frac{1}{\psi(N)}\Big(\min_{d\in\mathcal{Q}_N} \frac{\varphi(d)}{d}\Big)\mathbb{1}_N^{\mathrm{T}} Y_+ \le \mathbb{1}_N^{\mathrm{T}} \widehat{A}_N^{-1} Y_+ = k - \mathbb{1}_N^{\mathrm{T}} \widehat{A}_N^{-1} Y_- = k + \sum_{d\in\mathcal{Q}_N} y_d \frac{\varphi(d)}{d\psi(N)}\,. \tag{3.11}$$

So, we obtain

$$\begin{aligned} \|\widehat{A}_N^{-1} Y_+\| &\le \mathbb{1}_N^{\mathrm{T}} |\widehat{A}_N^{-1}| Y_+ \le \frac{1}{\varphi(N)}\Big(\max_{d\in\mathcal{Q}_N} \frac{\psi(d)}{d}\Big)\mathbb{1}_N^{\mathrm{T}} Y_+ \\ &\le \frac{\psi(N)}{\varphi(N)}\Big(\max_{d\in\mathcal{Q}_N} \frac{\psi(d)}{\varphi(d)}\Big)\bigg(k + \sum_{d\in\mathcal{Q}_N} y_d \frac{\varphi(d)}{d\psi(N)}\bigg) \end{aligned} \tag{3.12}$$

where the first inequality is trivial, the second holds by Proposition 1.41 (c) and by (3.4), whereas and the third inequality holds by (3.11) and by the fact that $\psi(d)/d$ and $\varphi(d)/d$ attain respectively the maximum and minimum for the same values of d in $\mathcal{Q}_N$.

Now, from (3.8), (3.10) and (3.12), we get

$$\begin{aligned} \|X\| &\le k \cdot \prod_{\substack{p \text{ prime} \\ p|N}} \Big(\frac{p+1}{p-1}\Big)^{\min\{2,\, v_p(N)\}} + \Big(\max_{d\in\mathcal{Q}_N} \frac{\psi(d)}{d}\Big)\bigg(\frac{1}{\varphi(N)} + \frac{1}{\psi(N)}\bigg) \cdot \|\widehat{A}_N X\|_- \\ &\le k \cdot F(N) + G(N) \cdot \|\widehat{A}_N X\|_-\,. \end{aligned} \tag{3.13}$$

□

3.2 A simplified proof of the finiteness

To briefly describe the strategy of proving Mersmann's First Theorem (see Theorem 2.40), first we need some notations and background:

We denote by $\mathbb{N}$ the set of positive integers. For $N \in \mathbb{N}$, by $\mathcal{D}_N$ we denote the set of its divisors. For any two real matrices B and C of equal size, $B \ge C$ means that $B_{i,j} \geq C_{i,j}$ for all i, j. For $X \in \mathbb{Z}^{\mathcal{D}_N}$, the valuation* of the eta quotient

$$\eta^X := \prod_{d|N} \eta(dz)^{X_d}$$

at the cusps of the relevant congruence subgroup is given by a linear map $L_N : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}^{\mathcal{D}_N}$. Thus, holomorphy of η^X is equivalent to the condition $L_N(X) \geq 0$ and irreducibility of η^X is equivalent to saying that in each decomposition of the form $X = X' + X''$, either $L_N(X')$ or $L_N(X'')$ has a negative entry.

Now, Mersmann's strategy was to find a decompostion $X = X' + X''$ using the simplicity of η^X such that if its level N is sufficiently big w.r.t. its weight, then all the negative entries in $L_N(X')$ can not be subdued by the corresponding positive entries of $L_N(X'')$, which contradicts with the holomorphy of η^X.

With the same basic strategy as above, we provide a short proof of Mersmann's First Theorem Let us fix the following notations :
$N = P_1^{e_1} P_2^{e_2} \cdots P_m^{e_m}$, where $P_1 < P_2 < \cdots < P_m$ are primes.
For any $d \| N$, there exists a canonical bijection between $\mathbb{Z}^{\mathcal{D}_N}$ and $\mathbb{Z}^{D(\frac{N}{d}) \times D(d)}$. If $d = P_i^{e_i}$, then we denote the image of $X \in \mathbb{Z}^{\mathcal{D}_N}$ by $X^{(i)}$ under this bijection. *i. e.*, if we set $N_i := \frac{N}{P_i^{e_i}}$ then

$$X^{(i)}_{\nu, P_i^j} = X_{\nu P_i^j}\,, \quad \nu | N_i \text{ and } 0 \leq j \leq e_i\,. \tag{3.14}$$

For any nonnegative integer $j \leq e_i$, we call $X_j^{(i)} := \{X_{\nu, P_i^j}\}_{\nu \in D(N_i)}$ the j-th column of $X^{(i)}$.

Let η^X be a simple holomorphic eta quotient of weight $\frac{k}{2}$ and level N. From (3.1), we get

$$\|X\| \leq k\, F(N). \tag{3.15}$$

Mersmann's First Theorem will follow from (3.15) if we can show that N is bounded above by some function of k. Let
$\delta_i := 1 +$ the highest number of consecutive zero columns in $X^{(i)}$.
$F_m := F(p_1 p_2 \cdots p_m)$, where $p_1 = 2$, $p_2 = 3, \ldots p_m$ are the first m primes.
It is easy to note that $F(N) \leq F_m$ and that $F_m = O(\log^2 m)$ as $m \to \infty$. Later in this section, we shall show that there exists a constant C_k such that for all P_i we have

$$P_i^{\delta_i} < C_k F(N)^5. \tag{3.16}$$

* By valuation, here we mean a suitably normalized valuation.

Hence in particular, for $i = m$ we get $P_m = O_k(\log^{10} m)$. Since $P_m \geq p_m \sim m \log m$, this bounds m and all primes $P_i | N$.*

For each i, $X^{(i)}$ has $e_i + 1$ columns. Since η^X is primitive, the first column of $X^{(i)}$ is nonzero (see Proposition 6.6.(iii)). Again, from the definition of level (see Introduction), it follows that the last column of $X^{(i)}$ is nonzero. Therefore, the number of nonzero columns in $X^{(i)}$ is at most $\frac{e_i}{\delta_i} + 1$. Hence, from (3.15) we get

$$\frac{e_i}{\delta_i} + 1 < k\, F(N). \tag{3.17}$$

Since (3.16) and (3.17) together bound e_i, we have only finitely many possibilities for N if k is given. □

Now, we construct a decreasing function $g : \mathbb{Z} \to (0, 2]$ such that if η^X is a simple holomorphic eta quotient of weight $\frac{k}{2}$ and level N, then for any prime $P_i | N$, (3.16) is satisfied if we put

$$C_k = \frac{k}{2\, g(k-1)}\,. \tag{3.18}$$

For all $n \leq 0$, we set $g(n) = 2$. For $n > 0$, we define $g(n)$ inductively as follows :

Let

$$g_0(n) := G(M_n), \tag{3.19}$$

where M_n is the least positive integer M such that

$$G(M) < g(n-1)\,. \tag{3.20}$$

Let M_n' be the least upper bound of the set of integers $M \in \mathbb{N}$ such that for each prime power $p^{e_p} \| M$,

$$\frac{F(M)^5}{2p^{r_{p,n}(M)}\,(1+\frac{1}{p})}\left(n + \frac{g_0(n)}{F(M)^2+1}\right) \geq g(n-1) - g_0(n)\,, \tag{3.21}$$

where $r_{p,n}(M) \in \mathbb{Z}$ such that

$$r_{p,n}(M) - 1 < \frac{e_p + 1}{n\,F(M) + g_0(n)} \leq r_{p,n}(M) \tag{3.22}$$

* In fact, this naïve bound may be very large. For example, we have $C_1 = \frac{1}{4}$. The order of magnitude of the largest prime p_m for which the inequation $p_m < F_m/4$ holds is 10^{14}. Whereas actually, the greatest prime divisor of the level of a simple holomorphic eta quotient of weight $\frac{1}{2}$ can be at most 3, as we shall see in the 5th section.

Then

$$g(n) := \begin{cases} g_0(n) & \text{if } M_n = M_n', \\ \min\limits_{M_n \le M \le M_n'} G(M) & \text{otherwise.} \end{cases} \tag{3.23}$$

In order to prove (3.16), we need the following lemmas. Given a matrix T and a real interval I, we denote by T_I the submatrix of T consisting of its successive columns with indices in I.

For any $N'|N$, let $s_{N'} : \mathbb{Z}^{D(N')} \to \mathbb{Z}^{\mathcal{D}_N}$ be a section of the projection $\pi_{N'} : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}^{D(N')}$ such that for all $Z \in \mathbb{Z}^{D(N')}$ and $d|N$, we have

$$s_{N'}(Z)_d = \begin{cases} Z_d & \text{if } d|N' \\ 0 & \text{otherwise.} \end{cases}$$

We define $\widetilde{s}_{N'} := s_{N'} \circ \pi_{N'}$.

Lemma 3.5. *For any $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have $(\widehat{A}_N X)^{(i)} = \widehat{A}_{N_i} X^{(i)} \widehat{A}_{P_i^{e_i}}$.* *

Lemma 3.6. *Let $a, b \in \mathbb{Z}$ with $0 \le a < b \le e_i$ such that $X^{(i)}_{(a,b)} = 0$ and let $N_a = P_i^a N_i$. Then*

$$\|\widehat{A}_N X\|_- > P_i^{e_i - a} \left\| \widehat{A}_{N_a} \pi_{N_a}(X) \right\|_- - \frac{N\psi(N)}{P_i^{b-a}(1 + \frac{1}{P_i})\varphi(N)} \|X - \widetilde{s}_{N_a}(X)\| .$$

Lemma 3.7. *If $\widehat{A}_N X \not\geq 0$, then*

$$\|\widehat{A}_N X\|_- \geq g(\sigma(X)) \frac{\varphi(N)^2}{\psi(N) + \varphi(N)} .$$

Considering eta quotients of the form $\frac{\prod_{i=1}^n \eta(b_i z)}{\eta(z)}$ with $b_1 > 1$ and $b_i := \prod_{j=1}^{i-1} b_j + 1 \ \forall\ i \in \mathbb{Z}_{>1}$ and using Lemma 3.7, it can be shown that $g(n)$ decays at least double exponentially as n grows.

We shall prove these lemmas in the next section.

Proof of (3.16). Let $a, b \in \mathbb{Z}$ with $0 \le a < b \le e_i$ such that $X^{(i)}_{(a,b)} = 0$ and

* In fact, for any $d \| N$, if we denote by $X^{[d]}$ the image of $X \in \mathbb{Z}^{\mathcal{D}_N}$ in $\mathbb{Z}^{D(N_d) \times D(d)}$, where N_d denotes the greatest divisor of N that is coprime to d, then we have $(\widehat{A}_N X)^{[d]} = \widehat{A}_{N_d} X^{[d]} \widehat{A}_d$.

$a-b=\delta_i$. For ease of notation, we write $p=P_i$, $N_a=p^aN_i$ and $e=e_i$. From the primitivity of η^X and from the definition of level it follows that both $\widetilde{s}_{N_a}(X)$ and $(X-\widetilde{s}_{N_a}(X))$ are nonzero. Let $k_1:=\sigma(\widetilde{s}_{N_a}(X))$ and
$k_2:=\sigma(X-\widetilde{s}_{N_a}(X))$. So, $k_1+k_2=k$. Now, if k_1 or $k_2\leq 0$, then by (3.7), we have respectively $\widehat{A}_N\widetilde{s}_{N_a}(X)\ngeq 0$ or $\widehat{A}_N(X-\widetilde{s}_{N_a}(X))\ngeq 0$. Otherwise, $0<k_1,k_2<k$. Since η^X is irreducible, we still have either $\widehat{A}_N\widetilde{s}_{N_a}(X)\ngeq 0$ or $\widehat{A}_N(X-\widetilde{s}_{N_a}(X))\ngeq 0$. Therefore if necessary, replacing X by $\widetilde{X}$ where $\eta^{\widetilde{X}}=\mathrm{al}_{n,N}(\eta^X)$ for some $n\in\mathcal{E}_N$ with $p|n$ (hence, replacing a by $e-b$ and b by $e-a$), we may assume that $k_1<k$ and

$$\widehat{A}_N\widetilde{s}_{N_a}(X)\ngeq 0\,. \tag{3.24}$$

We have

$$\widetilde{s}_{N_a}(X)^{(i)}\widehat{A}_{p^e}=\big(\pi_{N_a}(X)^{(i)}\,\big|\,0\big)\left(\begin{array}{c|c} p^{e-a}\widehat{A}_{p^a} & B\\ \hline 0 & 0\end{array}\right)$$

where the j-th column of $B=p^{e-a-j}$(the last column of $\widehat{A}_{p^a}$), for all $j\leq e-a$. Hence from (3.24), via Lemma 3.5 we get

$$\widehat{A}_{N_a}\pi_{N_a}(X)\ngeq 0\,. \tag{3.25}$$

Since

$$\|X-\widetilde{s}_{N_a}(\widehat{A}_NX)\|\leq\|X_+\|=\frac{1}{2}(\|X\|+k)\,, \tag{3.26}$$

from Lemma 3.6, we have

$$\|\widehat{A}_NX\|_->p^{e-a}\|A_{N_a}\pi_{N_a}(X)\|_--\frac{N\psi(N)}{2p^{b-a}(1+\frac{1}{p})\varphi(N)}(\|X\|+k)\,. \tag{3.27}$$

Since $\|\widehat{A}_NX\|_-=0$, from (3.27), (3.25), Lemma 3.7 and (3.15) , we get

$$\frac{kN\psi(N)}{2p^{b-a}(1+\frac{1}{p})\varphi(N)}\Big(\frac{\psi(N)^2}{\varphi(N)^2}+1\Big)>p^{e-a}g(k_1)\frac{\varphi(N_a)^2}{\psi(N_a)+\varphi(N_a)}\,. \tag{3.28}$$

By definition, g is a decreasing function and it can be verified easily by cross multiplication that

$$p^{e-a}\frac{\varphi(N_a)^2}{\psi(N_a)+\varphi(N_a)}\geq\frac{\varphi(N)^2}{\psi(N)+\varphi(N)}\,. \tag{3.29}$$

Hence, we can replace the right hand side of the inequality (3.28) by

$$g(k-1)\frac{\varphi(N)^2}{\psi(N)+\varphi(N)}$$

to obtain (3.16) by (20). □

3.3 Proof of the lemmas

Proof of Lemma 3.5. Follows from the facts that $\widehat{A}_N = \widehat{A}_{P_i^{e_i}} \otimes \widehat{A}_{N_i}$ and these matrices are symmetric.* □

Proof of Lemma 3.6. To lighten the notation, we write $p = P_i$ and $e = e_i$. From Lemma 3.5, we have

$$(\widehat{A}_N X)^{(i)} = \widehat{A}_{N_i} \widetilde{s}_{N_a}(X)^{(i)} \widehat{A}_{p^e} + \widehat{A}_{N_i} \left(X^{(i)} - \widetilde{s}_{N_a}(X)^{(i)} \right) \widehat{A}_{p^e} . \tag{3.30}$$

Therefore, $\widetilde{s}_{N_a}(\widehat{A}_N X)^{(i)} = \left(\pi_{N_a}(\widehat{A}_N X)^{(i)} \middle| 0 \right) = \left((\widehat{A}_N X)^{(i)}_{[0,a]} \middle| 0 \right) =$

$$\widehat{A}_{N_i} \left(\pi_{N_a}(X)^{(i)} \middle| 0 \right) \left(\begin{array}{c|c} p^{e-a} \widehat{A}_{p^a} & 0 \\ \hline 0 & 0 \end{array} \right) + \widehat{A}_{N_i} \left(X^{(i)} - \widetilde{s}_{N_a}(X)^{(i)} \right) \left(\begin{array}{c|c} 0 & 0 \\ \hline p^{e-|j-k|} \underset{\substack{b \le j \le e \\ 0 \le k \le a}}{} & 0 \end{array} \right), \tag{3.31}$$

where the first two equalities are trivial and the third holds by (3.30). By Lemma 3.5, the absolute value of the sum of the negative entries in the 1st term of (3.31) is $p^{e-a} \| \widehat{A}_{N_a} \pi_{N_a}(X) \|_-$. The sum of positive entries in the 2nd term of (3.31) is less than or equal to

$$\mathbb{1}^t_{N_i} \widehat{A}_{N_i} \left| X^{(i)} - \widetilde{s}_{N_a}(X)^{(i)} \right| \left(\begin{array}{c|c} 0 & 0 \\ \hline p^{e-|j-k|} \underset{\substack{b \le j \le e \\ 0 \le k \le a}}{} & 0 \end{array} \right) \mathbb{1}_{p^e} .$$

Now, replacing N by N_i in (1.89), after left-multiplication with $\widehat{A}_{N_i}$ and using (3.4), we get†

$$\mathbb{1}^t_{N_i} \widehat{A}_{N_i} \le \frac{N_i \psi(N_i)}{\varphi(N_i)} \mathbb{1}^t_{N_i} . \tag{3.32}$$

Since

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline p^{e-|j-k|} \underset{\substack{b \le j \le e \\ 0 \le k \le a}}{} & 0 \end{array} \right) \mathbb{1}_{p^e} \le \frac{p^{e-(b-a)}}{1 - \frac{1}{p}} \mathbb{1}_{p^e} , \tag{3.33}$$

* See Lemma 4.3.1 in [3].

† Since $\widehat{A}_{N_i} > 0$, left-multiplication by $\widehat{A}_{N_i}$ does not alter the direction of the inequality.

the sum of the positive entries in the second term of (3.31) is less than or equal to

$$\frac{p^{e-(b-a)}N_i\psi(N_i)}{(1-\frac{1}{p})\varphi(N_i)}\mathbb{1}^t_{N_i}\big|X^{(i)}-\widetilde{s}_{N_a}(X)^{(i)}\big|\mathbb{1}_{p^e}=\frac{N\psi(N)}{p^{b-a}(1+\frac{1}{p})\varphi(N)}\|X-\widetilde{s}_{N_a}(X)\|\,.$$

Thus, we have

$$\|\widehat{A}_N X\|_- \geq \|\widetilde{s}_{N_a}(\widehat{A}_N X)\|_- \geq p^{e-a}\|\widehat{A}_{N_a}\pi_{N_a}(X)\|_- - \frac{N\psi(N)}{p^{b-a}(1+\frac{1}{p})\varphi(N)}\|X-\widetilde{s}_{N_a}(\widehat{A}_N X)\|\,.$$

□

Proof of Lemma 3.7. The lemma holds trivially for $N = 1$. We proceed by induction on N. Let $M > 1$ be an integer and let us assume that the lemma holds for all $N < M$. Let $X \in \mathbb{Z}^{D(M)}$ such that $A_M X \ngeq 0$. Let $n := \mathbb{1}^t_M X$. If $n \leq 0$, then the lemma holds by (3.7). So, let us assume $n > 0$. By the definition of g, the lemma holds trivially for M if $M \leq N_n$. So, we may assume that

$$M > N_n\,. \tag{3.34}$$

Since $g_0(n) \geq g(n)$,* if $\|X\| \geq n\frac{\psi(M)^2}{\varphi(M)^2} + g_0(n)$, the claim holds by (3.6). So, we may also assume that

$$\|X\| < n\frac{\psi(M)^2}{\varphi(M)^2} + g_0(n)\,. \tag{3.35}$$

Since $M > M'_n$, there exists a prime $p = P_i$ dividing M such that †

$$\frac{F(M)}{2p^{r_{p,n}(M)}(1+\frac{1}{p})}\left(n + g_0(n)\frac{\varphi(M)^2}{\psi(M)^2+\varphi(M)^2}\right) < g(n-1) - g_0(n)\,, \tag{3.36}$$

Let $p^e \| M$. Since $A_M X \ngeq 0$, X is nonzero. Hence, $X^{(i)}$ has at least one nonzero column.

If $X^{(i)}_a$ is the only nonzero column of $X^{(i)}$, then if necessary, replacing X by $\widetilde{X}$, where $\eta^{\widetilde{X}} = \mathrm{al}_{m,M}(\eta^X)$ for some $m \in \mathcal{E}_M$ with $p|m$ (thereby replacing a by $e-a$), we may assume that $a < e$. Let $M_a := \frac{M}{p^{e-a}}$. By the same argument as in the proof of (3.16),‡ we see that $A_{M_a}\pi_{M_a}(X) \ngeq 0$. So, we have

$$\|A_M X\|_- \geq \|\widetilde{s}_{M_a}(A_M X)\|_- = p^{e-a}\|A_{M_a}\pi_{M_a}(X)\|_-\,, \tag{3.37}$$

* See (3.19) and (3.23).
† See (3.21).
‡ See (3.25).

where the inequality is trivial and the last equality holds as we have $X^{(i)} = \widetilde{s}_{M_a}(X)^{(i)}$, since $X_a^{(i)}$ is the only nonzero column of $X^{(i)}$.* Again,

$$p^{e-a}\|A_{M_a}\pi_{M_a}(X)\|_- \geq g(\sigma(\pi_{M_a}(X)))\frac{p^{e-a}\varphi(M_a)^2}{\psi(M_a)+\varphi(M_a)} \geq g(n)\frac{\varphi(M)^2}{\psi(M)+\varphi(M)}\,, \tag{3.38}$$

where the first inequality holds by the induction hypothesis and the second by the fact that $\sigma(\pi_{M_a}(X)) = \sigma(X) = n$. † Thus, from (3.37) and (3.38) the claim follows in this case.

Otherwise, $X^{(i)}$ has at least two nonzero columns $X_a^{(i)}$ and $X_b^{(i)}$ such that $X_{(a,b)}^{(i)} = 0$. We choose a and b such that $b - a = 1+$ the highest number of consecutive zero columns in $X^{(i)}$. Since $X^{(i)}$ has $e+1$ columns, the number of nonzero columns of $X^{(i)}$ is greater than or equal to $\frac{e+1}{b-a}$. Hence, from (3.35) we get

$$\frac{e+1}{b-a} < n\frac{\psi(M)^2}{\varphi(M)^2} + g_0(n)\,. \tag{3.39}$$

Let $M_a := \frac{M}{p^{e-a}}$ and $m_a := \sigma(\pi_{M_a}(X))$. Since $A_M\widetilde{s}_{M_a}(X) + A_M(X - \widetilde{s}_{M_a}(X)) \not\geq 0$, either $A_M\widetilde{s}_{M_a}(X) \not\geq 0$ or $A_M(X - \widetilde{s}_{M_a}(X)) \not\geq 0$. Hence by the same argument as in the proof of (3.16),‡ we may assume that $m_a < n$, $A_{M_a}\pi_{M_a}(X) \not\geq 0$ and

$$\|A_M X\|_- > p^{e-a}\|A_{M_a}\pi_{M_a}(X)\|_- - \frac{M\psi(M)}{2p^{b-a}(1+\frac{1}{p})\varphi(M)}(\|X\| + n)\,. \tag{3.40}$$

From (3.35), (3.40) and the induction hypothesis, we get

$$\|A_M X\|_- > p^{e-a}g(m_a)\frac{\varphi(M_a)^2}{\psi(M_a)+\varphi(M_a)} - \frac{M\psi(M)}{2p^{b-a}(1+\frac{1}{p})\varphi(M)}\left(n\Big(\frac{\psi(M)^2}{\varphi(M)^2}+1\Big) + g_0(n)\right). \tag{3.41}$$

Again, by the same reasoning as in (3.29), we can replace the first term on the right hand side of the inequality (3.41) by $g(n-1)\frac{\varphi(M)^2}{\psi(M)+\varphi(M)}$ to obtain

$$\|A_M X\|_- > \frac{\varphi(M)^2}{\psi(M)+\varphi(M)}\left(g(n-1) - \frac{F(M)}{2p^{b-a}(1+\frac{1}{p})}\left(n + g_0(n)\frac{\varphi(M)^2}{\psi(M)^2+\varphi(M)^2}\right)\right) \tag{3.42}$$

* See (3.31).
† See (3.29).
‡ See (3.25) and (3.27).

Since (3.39) implies that $b - a \geq r_{p,n}(M)$, * from (3.36) we get

$$g(n-1) - \frac{F(M)}{2p^{b-a}(1+\frac{1}{p})}\left(n + g_0(n)\frac{\varphi(M)^2}{\psi(M)^2+\varphi(M)^2}\right) > g_0(n) \geq g(n)\,. \qquad (3.43)$$

From (3.42) and (3.43), the claim follows. □

* See (3.22).

Chapter 4

Holomorphic eta quotients of weight 1/2

4.1 Zagier's list

Around 1988, from his extensive numerical calculations Zagier obtained a list of 14 primitive* holomorphic eta quotients of weight 1/2 and he conjectured that these are the only primitive holomorphic eta quotients weight 1/2. Within the next few years, his student Mersmann also established this conjecture. In this chapter we shall see a short proof of it. Precisely, Zagier's conjecture, i. e. Mersmann's Second Theorem states:

Theorem 4.1 (Mersmann's Second Theorem)**.** *The following fourteen are the only simple holomorphic eta quotients of weight* $\frac{1}{2}$:

$$\eta,\ \frac{\eta^2}{\eta_2},\ \frac{\eta_2^2}{\eta},\ \frac{\eta_2^3}{\eta\eta_4},\ \frac{\eta_2^5}{\eta^2\eta_4^2},\ \frac{\eta\eta_4}{\eta_2},\ \frac{\eta\eta_6^2}{\eta_2\eta_3},\ \frac{\eta^2\eta_6}{\eta_2\eta_3},\ \frac{\eta_2^2\eta_3}{\eta\eta_6},\ \frac{\eta_2\eta_3^2}{\eta\eta_6}$$

$$\frac{\eta_2^2\eta_3\eta_{12}}{\eta\eta_4\eta_6},\ \frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2},\ \frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}},\ \frac{\eta\eta_4\eta_6^5}{\eta_2^2\eta_3^2\eta_{12}^2}.$$

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* Since any holomorphic eta quotient of weight 1/2 is irreducible, here primitivity and simplicity are synonymous.

4.2 A short proof of exhaustiveness of the list

The naïve upper bounds those are obtained from the results given in the previous sections for the highest levels of the primitive and irreducible holomorphic eta quotients of even very small weights are extremely large. Nevertheless, in the following we will see that the highest level for primitive holomorphic eta quotients of weight $\frac{1}{2}$ is only 12. Note that any eta quotient of weight $\frac{1}{2}$ is irreducible.

Lemma 4.2. *Let $N \in \mathbb{Z}_{>0}$, $X \in \mathbb{Z}^{\mathcal{D}_N} \setminus \{0\}$ and p be a prime dividing N. Write $N = p^e N'$ with $p \nmid N'$ and let $Z_p := X^{(p)} \mathbb{1}_{p^e}$. If $\widehat{A}_N X \geq 0$, then*

$$\widehat{A}_{N'}(Z_p + \ell X_j^{(p)}) \geq 0 \text{ and } Z_p + \ell X_j^{(p)} \neq 0.$$

for all $\ell \in \mathbb{Z}_{[0,p-2]}$ and $j \in \mathbb{Z}_{[0,e]}$.

Proof. Let η_j denote a vector of size $e+1$ whose $(j+1)$-th entry is 1 and all other entries are 0. It can be easily checked that $A_{p^e}^{-1}(u_{p^e} + \ell\eta_j) > 0$ for all $\ell \in \mathbb{Z}_{[0,p-2]}$ and for all $j \in \mathbb{Z}_{[0,e]}$ By Lemma 3.5, we get

$$\widehat{A}_{N'}(Z_p + \ell X_j^{(p)}) = \widehat{A}_{N'} X^{(p)}(u_{p^e} + \ell\eta_j) = Y^{(p)} \widehat{A}_{p^e}^{-1}(u_{p^e} + \ell\eta_j) \geq 0$$

for all $\ell \in \mathbb{Z}_{[0,p-2]}$ and for all $j \in \mathbb{Z}_{[0,e]}$. Since $Y \neq 0$ and $\widehat{A}_{p^e}^{-1}(u_{p^e} + \ell\eta_j) > 0$, we have $\widehat{A}_{N'}(Z_p + \ell X_j^{(p)}) \neq 0$. □

Corollary 4.3. *With the same assumptions as in Lemma 4.2, if $p \geq 3$ and $\mathbb{1}_N^t X = 1$, then*

$$\mathbb{1}_{N'}^t X_j^{(p)} \geq 0 \text{ for all } j \in \mathbb{Z}_{[0,e]}.$$

Lemma 4.4. *Let $N = 2^e$ for some $e \in \mathbb{Z}_{>2}$. If $Y := \widehat{A}_N X \geq 0$, then $|X_1| + |X_N| \leq 2 \cdot \mathbb{1}_N^t X$ and if the equality holds, then both of X_1 and X_N are even.*

Proof. Let η_j denote a vector of size $e+1$ whose $(j+1)$-th entry is 1 and all other entries are 0. Let $w := (\operatorname{sgn}(X_1)\eta_0^t + \operatorname{sgn}(X_N)\eta_e^t - 2 \cdot \mathbb{1}_N^t)\widehat{A}_N^{-1}$. Then $|X_1| + |X_N| - 2 \cdot \mathbb{1}_N^t X = wY$. It can be easily checked that exactly two entries of w (say, w_{i_1} and w_{i_2}) are 0 and the rest of its entries are negative (see Remark 1.90). Hence,

$$|X_1| + |X_N| \leq 2 \cdot \mathbb{1}_N^t X. \tag{4.1}$$

If the equality holds in (4.1), then all the entries of Y except Y_{i_1} and Y_{i_2} must be 0. So,

$$X = Y_{i_1}(\text{The } i_1\text{-th column of } \widehat{A}_N^{-1}) + Y_{i_2}(\text{The } i_2\text{-th column of } \widehat{A}_N^{-1}).$$

We have, $X \in \mathbb{Z}^{(e+1)}$, $e \geq 3$ and $\widehat{A}_N^{-1}$ tridiagonal. Hence by Remark 1.90, Y_i(The i-th column of $\widehat{A}_N^{-1}) \in \mathbb{Z}^{(e+1)}$ for all $i \in \{i_1, i_2\}$. And since $N = 2^e$, both of X_1 and X_N must be even. □

Remark 4.5. By a straightforward computation using Lemma 3.1, it can be checked that if $N|144$, then all possible solutions for X such that η^X is a primitive eta quotient of weight $\frac{1}{2}$ and level N, are as follows:

$$N = 1 \quad : \quad X = 1$$

$$N = 2 \quad : \quad X = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$N = 4 \quad : \quad X = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$$

$$N = 6 \quad : \quad X = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$$N = 12 \quad : \quad X = \begin{pmatrix} -2 \\ 5 \\ -2 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -2 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Lemma 4.6. *Let $a, b \in \mathbb{Z}_{>0}$ and let $N = 2^a 3^b$. If there exists a primitive eta quotient of weight $\frac{1}{2}$ and level N, then $a \leq 4$ and $b \leq 2$.*

Proof. Let η^X denote a primitive eta quotient of weight $\frac{1}{2}$ and level N and let for all primes $p|N$, Z_p be as in Lemma 4.2. We have, $\widehat{A}_{3^b}Z_2 \geq 0$, $\widehat{A}_{2^a}Z_3 \geq 0$ (see Lemma 4.2) and $\mathbb{1}_{3^b}^t Z_2 = \mathbb{1}_{2^a}^t Z_3 = \mathbb{1}_N X = 1$ (see Remark 1).

If $b \geq 3$, then there exists $j_0 \in \{0, b\}$ and $i_0 \in \{j_0 - 1, j_0 + 1\}$ such that $\mathbb{1}_{2^a}^t X_{i_0}^{(3)} =$ the $(i_0{+}1)$-th entry of $Z_2 = \mathbb{1}_{2^a}^t X_{j_0}^{(3)} =$ the $(j_0{+}1)$-th entry of $Z_2 = 0$ (see Corollary 4.3). Therefore, $\mathbb{1}_{2^a}^t(Z_3{+}X_{i_0}^{(3)}) = \mathbb{1}_{2^a}^t(Z_3{+}X_{i_0}^{(3)}{+}4X_{j_0}^{(3)}) = 1$ and $\widehat{A}_{2^a}(Z_3{+}X_{i_0}^{(3)}) \geq 0$ (see Lemma 4.2). Also, proceeding similarly as in the proof of Lemma 4.2, we get $\widehat{A}_{2^a}(Z_3 + X_{i_0}^{(3)} + 4X_{j_0}^{(3)}) \geq 0$. Since η^X is a primitive eta quotient of level N, $X_{j_0}^{(3)} \neq 0$. So, the pair $\{Z_3{+}X_{i_0}^{(3)},\, Z_3{+}X_{i_0}^{(3)}{+}4X_{j_0}^{(3)}\}$ gives two distinct holomorphic eta quotients of weight $\frac{1}{2}$ such that their corresponding exponents are congruent modulo 4 and their levels divide 2^a. But such a pair does not exist (see Lemma 4.4 and Remark 4.5). Thus we get a contradiction!

If $a \geq 5$, since $\widehat{A}_{2^a}Z_3 \geq 0$, $\widehat{A}_{2^a}(Z_3 + X_0^{(3)}) \geq 0$ and $\widehat{A}_{2^a}(Z_3 + X_b^{(3)}) \geq 0$ (see Lemma 4.2), we see that each of Z_3, $Z_3 + X_0^{(3)}$ and $Z_3 + X_b^{(3)}$ has 0 as either its first or its last entry (see Lemma 4.4).

Therefore, if both of the first and the last entries of Z_3 are 0, then each of $X_0^{(3)}$ and $X_b^{(3)}$ has 0 at its one end. If both of them has 0 at the same end (which it have to be if $b = 1$), then the column at that end of $X^{(2)}$ is entirely 0, which contradicts with either the primitivity or the level of η^X.
Otherwise, $X_0^{(3)}$ and $X_b^{(3)}$ has their respective nonzero entries at different ends. Now, proceeding as before, we see that $\widehat{A}_{2^a}(Z_3 + X_1^{(3)} + 4X_0^{(3)}) \geq 0$. But both of the first and the last entries of $Z_3 + X_1^{(3)} + 4X_0^{(3)}$ must be odd, sum of their absolute values being $2 \cdot \mathbb{1}_{2^a}^t(Z_3 + X_1^{(3)} + 4X_0^{(3)})$ (see Lemma 4.4). Thus we get a contradiction! Hence, there exists a unique $j_0 \in \{0, a\}$ such that $\mathbb{1}_{3^b}X_{j_0}^{(2)} =$ the $(j_0 + 1)$-th entry of $Z_3 = 0$. Let $\imath$ and $\jmath$ denote the indices of the columns of $X^{(2)}$, which are respectively adjacent to $X_{j_0}^{(2)}$ and $X_{a-j_0}^{(2)}$. Since η^X is a primitive eta quotient of level N, $X_{j_0}^{(2)} \neq 0$. Now, considering all twelve possibilities for Z_3 and hence all possibilities for $\mathbb{1}_{3^b}^t X_j^{(2)}\ \forall\, j \in \mathbb{Z}_{[0,a]}$ (see Lemma 4.4 and Remark 4.5) and proceeding as before, we see that the pair

$$\left\{\alpha Z_2 - \operatorname{sgn}\left(\mathbb{1}_{3^b}^t X_\beta^{(2)}\right) X_\beta^{(2)} + X_\imath^{(2)} + X_{j_0}^{(2)}\ ,\ \alpha Z_2 - \operatorname{sgn}\left(\mathbb{1}_{3^b}^t X_\beta^{(2)}\right) X_\beta^{(2)} + X_\imath^{(2)} + 3X_{j_0}^{(2)}\right\},$$

where $\alpha = \begin{cases} 3 & \text{if } \mathbb{1}_{3^b}^t X_{a-j_0}^{(2)} = -2 \\ 2 & \text{otherwise} \end{cases}$ and $\beta = \begin{cases} a - j_0 & \text{if } \mathbb{1}_{3^b}^t X_{a-j_0}^{(2)} \neq 2 \\ \jmath & \text{if } \mathbb{1}_{3^b}^t X_{\jmath}^{(2)} = -1 \end{cases}$

gives two distinct eta quotients of weight $\frac{1}{2}$ such that their corresponding exponents are congruent modulo 2 and their levels divide 3^b. But such a pair does not exist (see Corollary 4.3). Thus we get a contradiction! □

Proposition 4.7. *Any holomorphic eta quotient of weight* $\frac{1}{2}$ *is the rescaling of some eta quotient whose level divides* 12.

Proof. Let us consider holomorphic eta quotients of weight $\frac{1}{2}$ and level $N \in \mathbb{Z}_{>1}$. Let $p_{(N)}$ denote the largest prime divisor of N. For $p_{(N)} \leq 3$, the proposition holds (see Corollary 4.3, Lemma 4.4, Lemma 4.6 and Remark 4.5). We proceed by induction on $p_{(N)}$. Let us assume that the proposition holds for all N with $p_{(N)} < p$ for some prime $p \in \mathbb{Z}_{>3}$.

Now, let $p_{(N)} = p$, $e := v_p(N)$ and $N' := \frac{N}{p^e}$. Let η^X be a primitive eta quotient of weight $\frac{1}{2}$ and level N and let Z_p be as in Lemma 4.2. Then $\widehat{A}_{N'} Z_p \geq 0$ (see Lemma 4.2). Since $p > 3$ and since η^X is a primitive eta quotient of level N, there exists $j_0 \in \{0, e\}$ such that $X_{j_0}^{(p)} \neq 0$ and $\mathbb{1}_{N'}^t X_{j_0}^{(p)} = 0$ (see Corollary 4.3). Therefore, $\mathbb{1}_{N'}^t (Z_p + 3X_{j_0}^{(p)}) = \mathbb{1}_{N'}^t Z_p = 1$ and $\widehat{A}_{N'}(Z_p + 3X_{j_0}^{(p)}) \geq 0$ (see Lemma 4.2). From the induction hypothesis and Remark 4.5, we see that for the pair $\{Z_p, Z_p + 3X_{j_0}^{(p)}\}$, there are only the follwoing two possibilities:

$$(i) \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad (ii) \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -2 \\ 5 \\ -2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Considering either possibility, we see that $\widehat{A}_{N'}(Z_p + X_{j_1}^{(p)}) \not\geq 0$, where $j_1 \in \mathbb{Z}_{[0,e]}$ is the unique index (see Corollary 4.3) such that $\mathbb{1}_{N'}^t X_{j_1}^{(p)} = 1$. Thus we get a contradiction! (see Lemma 4.2) □

Proposition 3 along with Remark 6 completes the proof of Theorem 2. □

Remark 4.8. The translation $z \mapsto z + \frac{1}{2}$ or equivalently the sign transformation $q \mapsto -q$ gives an involution of Zagier's list of eta quotients (see Theorem 2) by interchanging respectively the first seven eta quotients with the last seven, up to multiplication by a 48-th root of unity. Hence, one can restate the Mersmann's Second Theorem as:

There exist only three holomorphic eta quotients of weight $\frac{1}{2}$ up to the sign transformation, Atkin-Lehner involutions and rescaling. e.g.,

$$\eta(z)\,, \quad \frac{\eta(z)^2}{\eta(2z)}\,, \quad \frac{\eta(z)\eta(6z)^2}{\eta(2z)\eta(3z)}\,.$$

Remark 4.9. From [47] and [48], one notes that all the eta quotients in Zagier's list (see Theorem 2) are obtained from specializations of the Jacobi triple product identity:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}y)(1+x^{2n-1}y^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2}y^n, \text{ where } |x|<1 \text{ and } y \neq 0. \tag{4.2}$$

i. e. every eta quotient in the list can be obtained by substituting $x = \xi e^{\pi i a z}$ and $y = \zeta\xi e^{\pi i b z}$ in the left hand side of (2) and multiplying it with $Ce^{\frac{\pi i b^2 z}{4a}}$ for some $\xi \in U_4$, $\zeta \in U_6$, $C \in U_6 \cup \{\frac{1}{2}\}$ and $a, b \in \mathbb{Z}_{[1,6]}$, where U_n denotes the group of complex n-th roots of unity. From the corresponding changes on the right hand side of (4.2), we see that every eta quotient in the list has a series expansion like

$$C\sum_{n=-\infty}^{\infty}(\pm 1)^{\frac{n(n+1)}{2}}\zeta^n e^{\frac{\pi i(2an+b)^2 z}{4a}}\,. \tag{4.3}$$

Expressing all eta quotients in Zagier's list in the above form and simplifying, we see that each of them is a linear combination (with coefficients in $\mathbb{Z}_{[-2,2]} \cup \{\frac{1}{2}\}$) of theta series

$$\theta_{n_0,\frac{1}{t}} := \sum_{\substack{n \equiv n_0 \,(\mathrm{mod}\ 24) \\ n\in\mathbb{Z}}} e^{\frac{2\pi i n^2 z}{t}} \tag{4.4}$$

for some $n_0 \in \mathbb{Z}$ and $t|24$ (see Section 2.1 in [37]). Simplifying further, we see that each eta quotient in the list can also be expressed as

$$\sum_{n=0}^{\infty} \varepsilon(n) e^{\frac{2\pi i n^2 z}{t}} \tag{4.5}$$

for some $t|24$, where depending on the eta quotient under consideration, $\varepsilon : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{[-2,2]}$ may or may not be a character (see Table 1 in [48] or Chapter 8 in [22]). From Theorem 2, it follows that all holomorphic eta quotients of weight $\frac{1}{2}$ are obtained from specializations of the Jacobi triple product and up to rescaling, they can be expressed in the forms (8), (10) and as linear combinations of theta series of the form (9).

Part III

Holomorphic eta quotients of a particular level

Chapter 5

Irreducible holomorphic eta quotients

We recall that a holomorphic eta quotient is called *irreducible* if it is not a product of holomorphic eta quotients of smaller weights and that any eta quotient of weight 1/2 is irreducible, since 1/2 is the smallest possible weight for an eta quotient. So, in particular, η_p is irreducible for any prime p Also, from Corollary 2.39, we know that the eta quotients η^p/η_p and η_p^p/η are irreducible. It is easy to show that the above three are the only irreducible holomorphic eta quotients of the prime level p. In this chapter, we shall show that finiteness of irreducible holomorphic eta quotients of a given level also holds in general.

5.1 The finiteness

For $N \in \mathbb{N}$, we call an eta quotient f *irreducible on* $\Gamma_0(N)$ if the level of f divides N and if f is not reducible on $\Gamma_0(N)$. We shall show that

Theorem 5.1. *For $N \in \mathbb{N}$, there are only finitely many holomorphic eta quotients which are irreducible on $\Gamma_0(N)$.*

In particular, the above theorem implies that

Corollary 5.2. *There are only finitely many irreducible holomorphic eta quotients of any particular level.*

We recall (see Section 1.5) that holomorphic eta quotients on $\Gamma_0(N)$ has a natural bijection with the lattice points in a cone $\mathcal{K}_N$ of dimension equal to the number of divisors of N. Under this bijection, the holomorphic eta quotients which are irreducible on $\Gamma_0(N)$, correspond to the lattice points in $\mathcal{K}_N$ which can not be written as the sum of any two nonzero lattice points in $\mathcal{K}_N$. We show that all such lattice points should be contained in a bounded polytope in $\mathcal{K}_N$, which gives off the finiteness, thus proving the theorem that we stated above:

Proof of Theorem 5.1. We show that there are only finitely many irreducible holomorphic eta quotients on $\Gamma_0(N)$. Let

$$\eta^X := \prod_{d|N} \eta(dz)^{X_d}$$

be an eta quotient on $\Gamma_0(N)$. Then η^X is holomorphic if and only if $A_N X \geq 0$, where A_N is the valuation matrix of level N (see Section 1.5). Since A_N is invertible, the cone

$$\mathcal{K}_N := \{X \in \mathbb{R}^{\mathcal{D}_N} | A_N X \geq 0\} \tag{5.1}$$

is generated by nonnegative linear combinations of the columns of A_N^{-1}, i. e.

$$\mathcal{K}_N = \Big\{ \sum_{d|N} C_d u_d \,\Big|\, C_d \in \mathbb{R}_{\geq 0} \text{ for all } d|N \Big\},$$

where u_d is the column of A_N^{-1} indexed by $d \in \mathcal{D}_N$. As u_d has rational entries, for each $d|N$, there exists a smallest positive integer m_d such that $v_d := m_d u_d$ has integer entries. Also, we have

$$\mathcal{K}_N = \Big\{ \sum_{d|N} C_d v_d \,\Big|\, C_d \in \mathbb{R}_{\geq 0} \text{ for all } d|N \Big\}.$$

Let

$$\mathcal{Z}_N := \Big\{ \sum_{d|N} C_d v_d \,\Big|\, C_d \in [0,1] \text{ for all } d|N \Big\}. \tag{5.2}$$

Since $\mathcal{Z}_N$ is compact, there are only finitely many lattice points in it. So, Theorem 5.1 follows from the lemma below:

Lemma 5.3. *If η^X is a holomorphic eta quotient which is irreducible on $\Gamma_0(N)$, then $X \in \mathcal{Z}_N \cap \mathbb{Z}^{\mathcal{D}_N}$.*

Proof. We show that each lattice point in $\mathcal{K}_N$ can be written as a nonnegative integral linear combination of lattice points in $\mathcal{Z}_N$. Let $a = \sum_{d|N} a_d v_d$ be a lattice point in $\mathcal{K}_N$, where $a_d \in \mathbb{R}_{\geq 0}$ for all $d|N$. Let $a' = \sum_{d|N} \lfloor a_d \rfloor v_d$. As v_d for each $d \in \mathcal{D}_N$ and a have integer entries, both a' and $a - a'$ are lattice points. Moreover, we have $a - a' \in \mathcal{Z}_N$. So, $a = \sum_{d|N} \lfloor a_d \rfloor v_d + (a - a')$ is a nonnegative integral linear combination of lattice points in $\mathcal{Z}_N$.

As holomorphic eta quotients on $\Gamma_0(N)$ correspond to lattice points in the cone $\mathcal{K}_N$, each eta quotient of on $\Gamma_0(N)$ can be written as a product of holomorphic eta quotients corresponding to lattice points in $\mathcal{Z}_N$. In other words, the holomorphic eta quotients which are irreducible on $\Gamma_0(N)$, can only correspond to the lattice points in $\mathcal{Z}_N$.

□

5.2 An upper bound on weight

In particular, the proof of Theorem 5.1 leads to a way for the estimation of an upper bound of the maximum possible weight of an irreducible holomorphic eta quotient of level N. Let $\kappa : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\kappa(N) = \varphi\left(\mathrm{rad}(N)\right) \prod_{\substack{p \text{ prime} \\ p^b \| N,\, b>0}} \big((b-1)(p-1)+2\big), \tag{5.3}$$

where $\mathrm{rad}(N)$ denotes the product of the distinct prime divisors of N. We shall show that

Theorem 5.4. *The weight of any holomorphic eta quotient which is irreducible on $\Gamma_0(N)$ is less than $\frac{1}{2}\kappa(N)$.*

From the above theorem, we get

Corollary 5.5. *The weight of any irreducible holomorphic eta quotient of level N is less than $\frac{1}{2}\kappa(N)$.*

Proof of Theorem 5.4. Let p be a prime. We know that each holomorphic eta quotient which is irreducible on $\Gamma_0(N)$ corresponds to some lattice point in the polytope $\mathcal{Z}_{p^n}$. From (1.90), we recall that

$$A_{p^n}^{-1}=\frac{1}{p^{n-1}(p^2-1)}\begin{pmatrix} p & -p & & & & \\ -1 & p^2+1 & -p^2 & & \text{\huge 0} & \\ & -p & p(p^2+1) & -p^3 & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\huge 0} & & -p^2 & p^2+1 & -1 \\ & & & & -p & p \end{pmatrix}, \tag{5.4}$$

So, from the proof of Theorem 5.1 and (5.4), we get

$$\mathcal{Z}_{p^n}=\Big\{\sum_{i=0}^{n}C_i v_{p,i}\,\Big|\,C_0,\dots,C_n\in[0,1]\Big\},$$

where $v_{p,i}\in\mathbb{Z}^{\mathcal{D}_{p^n}}$ is the column of $A_{p^n}^{-1}$ indexed by p^i, multiplied with the least positive integer such that the resulting vector has integer entries, i. e.

$$v_{p,0}=\begin{pmatrix} p\\ -1\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},\ v_{p,1}=\begin{pmatrix} -p\\ p^2+1\\ -p\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},\ v_{p,2}=\begin{pmatrix} 0\\ -p\\ p^2+1\\ -p\\ 0\\ \vdots\\ 0\end{pmatrix},\dots,\ v_{p,n}=\begin{pmatrix} 0\\ 0\\ 0\\ \vdots\\ 0\\ -1\\ p\end{pmatrix}. \tag{5.5}$$

Summing up the entries of each of the above vectors, we get

$$\sigma(v_{p,i})=\begin{cases} p-1 & \text{if } i=0 \text{ or } i=n,\\ (p-1)^2 & \text{if } 1\le i\le n-1.\end{cases} \tag{5.6}$$

For $N\in\mathbb{N}$, we define the *normalized valuation matrix* $\widetilde{A}_N\in\mathbb{Z}^{\mathcal{D}_N\times\mathcal{D}_N}$ by

$$\widetilde{A}_N(t,\_):=\frac{N(t,N/t)}{\varphi(N)\psi(N)\operatorname{rad}(N)}A_N(t,\_)\quad\text{for all } t\in\mathcal{D}_N. \tag{5.7}$$

Now, let $N=\prod_{j=1}^{m}p_j^{n_j}$, where p_j is prime for all $j\in\{1,\dots,m\}$. Then by multiplicativity of the multiplier of the columns of A_N in (5.7) and from Proposition 1.41, we have

$$\widetilde{A}_N=\widetilde{A}_{p_1^{n_1}}\otimes\cdots\otimes\widetilde{A}_{p_m^{n_m}}. \tag{5.8}$$

Also, (5.4) and (5.7) together imply that for a prime p and $n \in \mathbb{N}$, we have

$$\widetilde{A}_{p^n}^{-1}(\_\,, p^j) = v_{p,j}, \tag{5.9}$$

where $v_{p,j}$ for $0 \le j \le n$ are as in (5.5). Since each column of $\widetilde{A}_N^{-1}$ is the Kronecker product of some columns of $\widetilde{A}_{p_1^{n_1}}^{-1}, \ldots, \widetilde{A}_{p_m^{n_m}}^{-1}$, from the definition of $\mathcal{Z}_N$, it follows immediately that

$$\mathcal{Z}_N = \left\{ \sum_{\underline{i} \in I} C(\underline{i})\, v_{p_1,i_1} \otimes \cdots \otimes v_{p_m,i_m} \,\middle|\, C(\underline{i}) \in [0,1] \text{ for all } \underline{i} \in \mathcal{I} \right\}, \tag{5.10}$$

where $\underline{i} = (i_1, \ldots, i_m)$ and $\mathcal{I} = \{0,1,\ldots,n_1\} \times \cdots \times \{0,1,\ldots,n_m\}$.* For $\mathfrak{z} = \sum_{\underline{i}} C(\underline{i})\, v_{p_1,i_1} \otimes \cdots \otimes v_{p_m,i_m} \in \mathcal{Z}_N$, we have

$$\sigma(\mathfrak{z}) = \sum_{\underline{i} \in \mathcal{I}} C(\underline{i}) \cdot \sigma(v_{p_1,i_1}) \cdots \sigma(v_{p_m,i_m}).$$

From (5.6), we know $\sigma(v_{p_j,i_j}) > 0$ for all prime p_j and for all $i_j \in \{0,1,\ldots,n_j\}$. Hence,

$$\max_{\mathfrak{z} \in \mathcal{Z}_N} \sigma(\mathfrak{z}) \;=\; \max \sum_{\substack{\underline{i} \in \mathcal{I} \\ C(\underline{i}) \in [0,1]}} C(\underline{i}) \prod_{j=1}^{m} \sigma(v_{p_j,i_j}) \;=\; \sum_{\underline{i} \in \mathcal{I}} \prod_{j=1}^{m} \sigma(v_{p_j,i_j}) \;=\; \prod_{j=1}^{m} \sum_{i_j=0}^{n_j} \sigma(v_{p_j,i_j}). \tag{5.11}$$

Therefore, it follows from (5.6) that

$$\max_{\mathfrak{z} \in \mathcal{Z}_N} \sigma(\mathfrak{z}) \;=\; \prod_{j=1}^{m} \big((n_j - 1)(p_j - 1)^2 + 2(p_j - 1) \big) \;=\; \kappa(N). \tag{5.12}$$

In particular, for $w \in \mathcal{Z}_N$, we have

$$\sigma(w) = \kappa(N) \text{ if and only if } w = \sum_{\underline{i} \in \mathcal{I}} v_{p_1,i_1} \otimes \cdots \otimes v_{p_m,i_m}, \tag{5.13}$$

i. e. $\sigma(w) = \kappa(N)$ if and only if w is the sum of $|\mathcal{I}|$ lattice points, viz. $v_{p_1,i_1} \otimes \cdots \otimes v_{p_m,i_m}$ in $\mathcal{Z}_N$, which in particular implies that the eta quotient corresponding to w is reducible on $\Gamma_0(N)$. □

*The bijection between $\mathcal{I}$ and $\mathcal{D}_N$ is given by $\underline{i} \mapsto p_1^{i_1} \cdots p_m^{i_m}$.

Corollary 5.6. *For $N \in \mathbb{N}$, there are at most $\varphi(N)\psi(N)\,\mathrm{rad}\,(N)/N$ holomorphic eta quotient which are irreducible on $\Gamma_0(N)$.*

Proof. In the proof of Theorem 5.4, we saw that the holomorphic eta quotients which are irreducible on $\Gamma_0(N)$, correspond to a set of integer points in the fundamental parallelepiped of the lattice generated by the columns of $\widetilde{A}_N^{-1}$, where $\widetilde{A}_N$ is the normalized valuation matrix (see (5.7)). So, the volume of this fundamental parallelepiped i. e. the determinant of $\widetilde{A}_N^{-1}$ is an upper bound for the number of holomorphic eta quotients which are irreducible on $\Gamma_0(N)$. By multiplicativity (see (5.8)), it suffices to show that the determinant of $\widetilde{A}_{p^n}^{-1}$ is $p^{n-1}(p^2-1)$, which follows easily by induction on n. □

5.3 A consequence of the Irreducibility Conjecture

Let A_N denote the valuation matrix of level N. We conjecture that:

Conjecture 5.7. *For $d|N$, let $v_d = m_d u_d$, where $u_d = A_N^{-1}(\_\,, d)$ and m_d is the smallest positive integer such that $m_d u_d \in \mathbb{Z}^{\mathcal{D}_N}$. Then η^{v_d} is irreducible for all $d \in \mathcal{D}_N$.*

We show that unless N is squarefree, there exists at least one $d \in \mathcal{D}_N$ such that η^{v_d} is an irreducible holomorphic eta quotient:

Theorem 5.8. *For $d|N$, let $v_d = m_d u_d$, where $u_d = A_N^{-1}(\_\,, d)$ and m_d is the smallest positive integer such that $m_d u_d \in \mathbb{Z}^{\mathcal{D}_N}$.*

(*a*) *The holomorphic eta quotient η^{v_d} is not reducible on $\Gamma_0(N)$.*

(*b*) *If $d \in \mathcal{D}_{N/\mathrm{rad}(N)}$, then η^{v_d} is irreducible.*

Proof. (*a*) For $t \in \mathcal{D}_N$, we have

$$(A_N v_d)_t = \begin{cases} m_d & \text{if } t = d, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose, η^{v_d} is reducible on $\Gamma_0(N)$. Then there exists $v', v'' \in \mathbb{Z}^{\mathcal{D}_N} \smallsetminus \{0\}$ with $v_d = v' + v''$ such that $A_N v' \geq 0$ and $A_N v'' \geq 0$. Hence, there exist $m'_d, m''_d > 0$ with $m_d = m'_d + m''_d$ such that

$$(A_N v')_t = \begin{cases} m'_d & \text{if } t = d, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (A_N v'')_t = \begin{cases} m''_d & \text{if } t = d, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have $v' = m'_d \cdot u_d$ and $v'' = m''_d \cdot u_d$. Since $m'_d, m''_d < m_d$ and since m_d is the smallest positive integer such that $m_d u_d \in \mathbb{Z}^{\mathcal{D}_N}$, we conclude that $v' \notin \mathbb{Z}^{\mathcal{D}_N}$ and $v'' \notin \mathbb{Z}^{\mathcal{D}_N}$. Thus we get a contradiction! Hence, η^{v_d} not reducible on $\Gamma_0(N)$.

(b) Suppose, for some $d \in \mathcal{D}_{N/\mathrm{rad}(N)}$ and some multiple M of N, the holomorphic eta quotient η^{v_d} is reducible on $\Gamma_0(M)$. Then by Theorem 2.22, we may assume that $\mathrm{rad}(M) = \mathrm{rad}(N)$. We have, $v_d = \widetilde{A}_N^{-1}(\_\,, d)$, where $\widetilde{A}_N$ is as defined in (5.7). Since $N|M$, since $\mathrm{rad}(M) = \mathrm{rad}(N)$ and since $d \in \mathcal{D}_{N/\mathrm{rad}(N)}$, we have $d \in \mathcal{D}_{M/\mathrm{rad}(M)}$. Now, it follows from (5.8), (5.9) and (5.5) that $\widetilde{A}_M^{-1}(\_\,, d)$ is the same as $\widetilde{A}_N^{-1}(\_\,, d)$ augmented with a suitable number of zeros. In other words, we have $\eta^{v'_d} = \eta^{v_d}$, where $v'_d := \widetilde{A}_M^{-1}(\_\,, d)$. Now, from part ($a$), it follows that $\eta^{v'_d}$ is not reducible on $\Gamma_0(M)$. Thus we get a contradiction! Hence, for $d \in \mathcal{D}_{N/\mathrm{rad}(N)}$, η^{v_d} is irreducible. □

Next, we show that the Irreducibility Conjecture implies the conjecture above:

Lemma 5.9. *Conjecture* 2.6 *implies Conjecture* 5.7*.*

Proof. For $N \in \mathbb{N}$, by Theorem 5.8, the claim for irreducibility in Conjecture 5.7 holds for all of $d \in \mathcal{D}_{N/\mathrm{rad}(N)}$. So, we may assume that $d \in \mathcal{D}_N \smallsetminus \mathcal{D}_{N/\mathrm{rad}(N)}$. Let n be the largest divisor of d such that $n \| N$ and let $d' := n \odot d$, where $\odot$ is as defined in (1.72). Then we have $d' \in \mathcal{D}_{N/\mathrm{rad}(N)}$ and it is easy to show that $\eta^{v_d} = \mathrm{al}_{n,N}(\eta^{v'_d})$. Since, by Theorem 5.8, $\eta^{v'_d}$ is irreducible, Conjecture 2.6 implies that so is η^{v_d}. □

5.4 The common multiple with the least weight

In Section 5.2, we saw that no holomorphic eta quotient which is irreducible on $\Gamma_0(N)$ can have an weight greater than or equal to $\kappa(N)/2$. In this section, we show that $\kappa(N)/2$ is the smallest possible weight for an eta quotient f such that for each holomorphic eta quotient g which is irreducible on $\Gamma_0(N)$, f/g is holomorphic:

Theorem 5.10. *There exists a holomorphic eta quotient F_N of weight $\frac{1}{2}\kappa(N)$ on $\Gamma_0(N)$ such that F_N is a factor of a holomorphic eta quotient h on $\Gamma_0(N)$ if and only if h is divisible by each holomorphic eta quotient which is irreducible on $\Gamma_0(N)$.*

Proof. From Lemma 5.3, we know that if η^X is a holomorphic eta quotient which is irreducible on $\Gamma_0(N)$, then $X \in \mathcal{Z}_N \cap \mathbb{Z}_N$, where

$$\mathcal{Z}_N = \Big\{ \sum_{d|N} C_d v_d \,\Big|\, C_d \in [0,1] \text{ for all } d|N \Big\}$$

and for $d \in \mathcal{D}_N$, $v_d = m_d u_d$, where $u_d = A_N^{-1}(\_\,, d)$ and m_d is the smallest positive integer such that $m_d u_d \in \mathbb{Z}^{\mathcal{D}_N}$.

Let $Z = \sum_{d|N} C_d v_d \in \mathcal{Z}_N$ and let $\frac{\alpha}{\lambda} \in \mathbb{Q}$ with $\lambda|N$ and $(\alpha, \lambda) = 1$ be an arbitrary cusp of $\Gamma_0(N)$. Then from (1.80) and (1.84), we get

$$\operatorname{ord}_{\frac{\alpha}{\lambda}}(\eta^Z) = \frac{1}{24}(A_N Z)_\lambda = \frac{m_\lambda}{24} C_\lambda. \tag{5.14}$$

Let $Y = \sum_{d|N} v_d \in \mathcal{Z}_N$ and let $F_N = \eta^Y$. Then F_N is a holomorphic eta quotient on $\Gamma_0(N)$ and it follows from (5.13) that F_N has weight $\frac{1}{2}\kappa(N)$. Let $\frac{\alpha}{\lambda} \in \mathbb{Q}$ with $\lambda|N$ and $(\alpha, \lambda) = 1$ be an arbitrary cusp of $\Gamma_0(N)$ (See Proposition 1.7). From (5.14), we get

$$\max_{Z \in \mathcal{Z}_N} \operatorname{ord}_{\frac{\alpha}{\lambda}}(\eta^Z) = \max_{C_\lambda \in [0,1]} \frac{m_\lambda}{24} C_\lambda = \frac{m_\lambda}{24} = \operatorname{ord}_{\frac{\alpha}{\lambda}}(\eta^{v_\lambda}) = \operatorname{ord}_{\frac{\alpha}{\lambda}}(F_N). \tag{5.15}$$

Let h be a holomorphic eta quotient on $\Gamma_0(N)$ such that $F_N \,|\, h$. Then h/F_N is a holomorphic eta quotient on $\Gamma_0(N)$, which is true if and only if $\operatorname{ord}_s(h) \geq \operatorname{ord}_s(F_N)$ at each cusp s of $\Gamma_0(N)$. Therefore from (5.15), it follows that $\operatorname{ord}_s(h) \geq \operatorname{ord}_s(\eta^Z)$ for all $Z \in \mathcal{Z}_N$ at each cusp s of $\Gamma_0(N)$. In particular, for any holomorphic eta quotient g which is irreducible on $\Gamma_0(N)$ and for any cusp s of $\Gamma_0(N)$, we have $\operatorname{ord}_s(h) \geq \operatorname{ord}_s(g)$, which is equivalent to say that $g \,|\, h$.

Conversely, let h, a holomorphic eta quotient on $\Gamma_0(N)$ be divisible by each holomorphic eta quotient g which is irreducible on $\Gamma_0(N)$. That holds if and only if for any holomorphic eta quotient g which is irreducible on $\Gamma_0(N)$ and for any cusp s of $\Gamma_0(N)$, we have $\operatorname{ord}_s(h) \geq \operatorname{ord}_s(g)$. Since any cusp s of $\Gamma_0(N)$ could be represented by some $\frac{\alpha}{\lambda} \in \mathbb{Q}$ (see Proposition 1.7), where $\lambda|N$ and $(\alpha, \lambda) = 1$, taking $g = \eta^{v_\lambda}$, we have $\operatorname{ord}_{\frac{\alpha}{\lambda}}(h) \geq \operatorname{ord}_{\frac{\alpha}{\lambda}}(\eta^{v_\lambda}) = \operatorname{ord}_{\frac{\alpha}{\lambda}}(F_N)$, where the last equality holds by (5.15). In other words, the order of h at each cusp is greater than or equal to the order of F_N at that cusp, which is equivalent to say that $F_N \,|\, h$. $\square$

Remark 5.11. Let N and F_N be as in the proof of the last theorem. Then, given any holomorphic eta quotient g on $\Gamma_0(N)$, the eta quotient F_N/g on $\Gamma_0(N)$ is holomorphic if and only if g corresponds to some point in $\mathcal{Z}_N \cap \mathbb{Z}^{\mathcal{D}_N}$.

5.5 Comparison of weights

In the tables below, we compare $k_{\max}(N)$ with $\kappa(N)$, where $k_{\max}(N)/2$ is the maximum weight of a holomorphic eta quotient which is irreducible on $\Gamma_0(N)$ and $\kappa(N)/2$ (see (5.3)) is the weight of F_N, where F_N is as defined in Theorem 5.10. In the first table, we list only prime power levels, whereas the levels listed in the second are products of small primes or products of small powers of such primes.

N	$k_{\max}(N)$	$\kappa(N)$
4	1	3
8	1	4
9	4	8
16	2	5
25	16	24
27	4	12
32	2	6
49	36	48
64	3	7
81	9	16
121	100	120
125	16	40
128	3	8
169	144	168
243	10	20
256	4	9
289	256	288
343	36	84
361	324	360
512	5	10
529	484	528
625	41	56
841	784	840
1024	6	11
1331	100	220

N	$k_{\max}(N)$	$\kappa(N)$
6	2	8
10	4	16
12	3	12
14	6	24
15	8	32
18	5	16
20	5	24
21	12	48
22	10	40
26	12	48
34	16	64
35	24	96
38	18	72
39	24	96
46	44	176
51	32	128
55	40	160
69	44	176
77	60	240
85	64	256
87	56	224
91	72	288
95	72	288
115	88	352
119	96	384

In particular, we note that in the last table, whenever N is a product of two primes, $k_{\max}(N)$ equals $\kappa(N)/4$. Again, for a prime p, we have $\kappa(p) = 2(p-1)$ (see

(5.3)) and from the discussion preceding Section 5.1, we know that $k_{\max}(p) = p-1$. So, we end this chapter with the following conjecture:

Conjecture 5.12. *For all squarefree values of N, the following equality holds:*

$$k_{\max}(N) = \frac{\kappa(N)}{2^n},$$

where n is the number of distinct prime divisors of N.

Chapter 6

The levels of simple holomorphic eta quotients

We recall that a *simple* holomorphic eta quotient is a holomorphic eta quotient that is both primitive and irreducible. For example, η^2/η_2 is a simple holomorphic eta quotient of level 2, $\eta_2^5/(\eta^2\eta_4^2)$ is a simple holomorphic eta quotient of level 4 but there are no simple holomorphic eta quotients of level 8, i. e. any irreducible holomorphic eta quotient of level 8 is a rescaling of some eta quotient of levels 1, 2 or 4. Given a positive integer N, can one decide a priori whether or not a simple holomorphic eta quotient of level N exist? The general question reamins open. However, in this chapter we shall see examples of families of simple holomorphic eta quotients of various levels.

6.1 Extension of levels

Given a simple holomorphic eta quotients of a suitable level, we can construct up to thirteen new simple holomorphic eta quotients of the same weight but of higher levels from it:

Proposition 6.1. *Let there be a simple holomorphic eta quotient of an odd level* N.

(a) *Then there are at least two simple holomorphic eta quotients of level* $2N$ *and and at least three simple holomorphic eta quotients of level* $4N$. *Moreover, if* $3 \nmid N$, *then there are also four simple holomorphic eta quotients of levels* $6N$ *and four simple holomorphic eta quotients of* $12N$.

(b) *In particular, if there exists a simple holomorphic eta quotient of an odd level N that is also irreducible, then there exists at least as many simple holomorphic eta quotients of the respective levels as mentioned in (a), all of which are irreducible.*

Proof. Let f be a simple holomorphic eta quotient of level N. Let g be a simple holomorphic eta quotient of weight $1/2$ from Zagier's list (see Section 4.1) such that N is coprime to the level of g. Since both f and g are primitive, so is $f \otimes g$. Again, since g is of weight $1/2$ from Lemma 2.29 and from Corollary 2.14, it follows that $f \otimes g$ is strongly reducible (resp. reducible) if and only if so is f. □

6.2 Cubefree levels

In this section, we show that if N is cubefree, then there is a simple holomorphic eta quotient of level N:

Theorem 6.2. *If $N \in \mathbb{N}$ is cubefree, then there is a simple holomorphic eta quotient of level N and weight $\kappa_1(N)/2$, where*

$$\kappa_1(N) := \varphi\left(\operatorname{rad}(N)\right) \cdot \varphi\left(\frac{N}{\operatorname{rad}(N)}\right). \tag{6.1}$$

In fact, for a cubefree N, the following is a simple holomorphic eta quotient of level N and weight $\frac{1}{2}\kappa_1(N)$:

$$\prod_{d|N} \eta(dz)^{(-1)^{\nu(N,d)}\,\Psi(N,d)}, \tag{6.2}$$

where $\nu(N,d) := \omega(N) - \omega(d) + \omega(d/\operatorname{rad}(d))$ and $\Psi(N,d) := N \cdot \operatorname{rad}(N) \cdot \sigma_2((d, N/d))/\left(d \cdot \operatorname{rad}(N/d)^2\right)$. Here for a positive integer n, by $\omega(n)$ and $\sigma_2(n)$ we denote respectively the number of prime divisors of n and the sum of squares of the divisors of n.

Proof. For $d|N$, let $v_d \in \mathbb{Z}^{\mathcal{D}_N}$ is the column of A_N^{-1} indexed by d, multiplied with the least positive integer such that the resulting vector has integer entries. Then from Theorem 5.8, we know that the holomorphic eta quotient η^{v_d} is not reducible on $\Gamma_0(N)$. In particular, $\eta^{v_{\operatorname{rad}(N)}}$ is irreducible on $\Gamma_0(N)$. Let $N = \prod_{j=1}^{m} p_j^{e_j}$, where

p_j is prime for all $j \in \{1, \ldots, m\}$. Since N is cubefree, $e_j \leq 2$ for all j. From Proposition 1.41, we have

$$A_N^{-1} = A_{p_1^{e_1}}^{-1} \otimes \cdots \otimes A_{p_m^{e_m}}^{-1}.$$

Hence, it follows that for $d = \prod_{j=1}^{m} p_j^{i_j} \in \mathcal{D}_N$, we have $v_d = v_{p_1,i_1} \otimes \cdots \otimes v_{p_m,i_m}$, where $v_{p_j,i_j} \in \mathbb{Z}^{D(p^{e_j})}$ is the column of $A_{p_j^{e_j}}^{-1}$ indexed by $p_j^{i_j}$, multiplied with the least positive integer such that the resulting vector has integer entries. In particular,

$$v_{\mathrm{rad}(N)} = v_{p_1,1} \otimes \cdots \otimes v_{p_m,1}, \quad \text{where} \quad v_{p_j,1} = \begin{cases} \begin{pmatrix} -1 \\ p_j \end{pmatrix} & \text{if } e_j = 1, \\ \\ \begin{pmatrix} -p_j \\ p_j^2 + 1 \\ -p_j \end{pmatrix} & \text{if } e_j = 2. \end{cases} \tag{6.3}$$

It follows that $\sigma(v_{p_j,1}) = (p_j - 1)^{e_j}$ for all $j \in \{1, \ldots, m\}$. Therefore, the weight of $\eta^{v_{\mathrm{rad}(N)}}$ is

$$\frac{\sigma\left(v_{\mathrm{rad}(N)}\right)}{2} = \frac{\sigma(v_{p_1,1}) \cdots \sigma(v_{p_m,1})}{2} = \frac{1}{2}\prod_{j=1}^{m}(p_j - 1)^{e_j} = \frac{1}{2}\varphi\left(\mathrm{rad}(N)\right) \cdot \varphi\Big(\frac{N}{\mathrm{rad}(N)}\Big). \tag{6.4}$$

Since for all $j \in \{1, \ldots, m\}$, none of the entries of $v_{p_j,1}$ are zero, $v_{\mathrm{rad}(N)}$ has no zero entries. So, the level of $\eta^{v_{\mathrm{rad}(N)}}$, i. e. the l.c.m. of the scaling factors corresponding to its nonzero exponents is the l.c.m. of all the divisors of N and hence is equal to N. Similarly, the g.c.d. of the scaling factors corresponding to the nonzero exponents of $\eta^{v_{\mathrm{rad}(N)}}$ is the g.c.d. of all the divisors of N and hence is equal to 1. Therefore, $\eta^{v_{\mathrm{rad}(N)}}$ is not a rescaling of another eta quotient. In other words, the holomorphic eta quotient $\eta^{v_{\mathrm{rad}(N)}}$ is primitive and not reducible over $\Gamma_0(N)$, hence it is a simple holomorphic eta quotient of level N and weight $\frac{1}{2}\kappa_1(N)$. Using (6.3), it is quite straightforward to check that $\eta^{v_{\mathrm{rad}(N)}}$ is the eta quotient in (6.2). □

Corollary 6.3. *If $N \in \mathbb{N}$ is a perfect square, then there is a simple holomorphic eta quotient f of level N and weight $\varphi(\mathrm{rad}(N))^2$ such that f is irreducible.*

Proof. Follows from Theorem 6.2 and Theorem 5.8. □

6.3 Level p^n for $n \leq 3$

We recall that by Theorem 2.36, any simple holomorphic eta quotient of a prime power level is irreducible. Also, from Corollary 2.39, it follows that for a prime p, the holomorphic eta quotients η_p^p/η and η^p/η_p are simple and it is easy to check that these two are the only simple holomorphic eta quotients of level p. In the next section, we shall show that there are exactly $2p-1$ simple holomorphic eta quotients of level p^2:

Theorem 6.4. *Let p be a prime. The only simple holomorphic eta quotients of level p^2 are*

$$\frac{\eta^r \eta_{p^2}^{p-r}}{\eta_p}, \ 1 \leq r \leq p-1 \quad \textit{and} \quad \frac{\eta_p^{sp+1}}{\eta^s \eta_{p^2}^s}, \ 1 \leq s \leq p.$$

Corollary 6.5. *For any prime p and $m \in \mathbb{N}$, the holomorphic eta quotients*

$$\frac{\eta_m^r \eta_{mp^2}^{p-r}}{\eta_{mp}} \quad \textit{and} \quad \frac{\eta_{mp}^{sp+1}}{\eta_m^s \eta_{mp^2}^s}$$

are irreducible for all $r, s \in \mathbb{N}$ with $1 \leq r \leq p-1$ and $1 \leq s \leq p$.

Proof. Follows from Theorem 6.4, Theorem 2.36 and Corollary 2.38. □

Unlike the cases of p and p^2 and also unlike the cases of the higher powers of p (see Section 6.4) , a strange thing occurs in the case of the holomorphic eta quotients of level p^3: A great many numerical evidences suggest that all the holomorphic eta quotients of level p^3 are either rescalings or products of holomorphic eta quotients of levels $1, p$ or p^2:

Conjecture 6.6. *For any prime p, there does not exist a simple holomorphic eta quotient of level p^3.*

Proof of Theorem 6.4. Let $\eta^X := \eta(z)^{X_1}\eta(pz)^{X_2}\eta(p^2z)^{X_3}$ be a simple holomorphic eta quotients of level p^2. As η^X is holomorphic, from Proposition 1.41 and from (5.9) we get

$$X_1 + \frac{1}{p}X_2 + \frac{1}{p^2}X_3 \geq 0,$$
$$\frac{1}{p}X_1 + X_2 + \frac{1}{p}X_3 \geq 0,$$
$$\frac{1}{p^2}X_1 + \frac{1}{p}X_2 + X_3 \geq 0.$$

Upto Fricke involution, we have only the following four cases to consider:

Case 1. $(X_1 < 0, X_2 \geq 0, X_3 > 0)$

Let $X_1 = -a$, $X_2 = b$ and $X_3 = c$, where a, b, c are nonnegative integers. As

$$-a + \frac{1}{p}b + \frac{1}{p^2}c \geq 0,$$

we have $b = ap - \frac{c}{p} + \varepsilon$ for some $\varepsilon \geq 0$ such that $-\frac{c}{p} + \varepsilon \in \mathbb{Z}$. Let

$$X' = \begin{pmatrix} -a \\ ap \\ 0 \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} 0 \\ -\frac{c}{p} + \varepsilon \\ c \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic, $\eta^X = \eta^{X'} \cdot \eta^{X''}$ is reducible. So, there does not exist any simple holomorphic eta quotient of this form.

Case 2. $(X_1 < 0, X_2 < 0, X_3 > 0)$

Let $X_1 = -a$, $X_2 = -b$ and $X_3 = c$, where a, b, c are positive integers. As

$$-a - \frac{1}{p}b + \frac{1}{p^2}c \geq 0,$$

we have $c = ap^2 + bp + \varepsilon$ for some $\varepsilon \geq 0$. Let

$$X' = \begin{pmatrix} -a \\ 0 \\ ap^2 \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} 0 \\ -b \\ bp + \varepsilon \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic, $\eta^X = \eta^{X'} \cdot \eta^{X''}$ is reducible. So, there does not exist any simple holomorphic eta quotient of this form.

Case 3. $(X_1 \leq X_3 < 0, X_2 \geq 0)$

Let $X_1 = -a$, $X_2 = b$ and $X_3 = -c$, where a, b, c are nonnegative integers such that $a \geq c$. As

$$-a + \frac{1}{p}b - \frac{1}{p^2}c \geq 0,$$

we have $b = ap + \frac{c}{p} + \varepsilon$ for some $\varepsilon \geq 0$ such that $\frac{c}{p} + \varepsilon \in \mathbb{Z}$. Let

$$X' = \begin{pmatrix} -(a-c) \\ (a-c)p \\ 0 \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} -c \\ cp + \frac{c}{p} + \varepsilon \\ -c \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic and $\eta^{X'} \cdot \eta^{X''} = \eta^X$ is irreducible, we must have $X' = 0$, i. e. $a = c$.

Suppose if possible, $a > p$. Then there exists an integer $a' > 0$ such that $a = p + a'$. Let

$$X' = \begin{pmatrix} -p \\ p^2 + 1 \\ -p \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} -a' \\ a'p + \frac{a'}{p} + \varepsilon \\ -a' \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic, $\eta^X = \eta^{X'} \cdot \eta^{X''}$ is reducible. Thus we get a contradiction! So, we must have $a \leq p$.

It follows that

$$\frac{\eta(pz)^{sp+1}}{\eta(z)^s \eta(p^2 z)^s}, \; 1 \leq s \leq p$$

are the only simple holomorphic eta quotients in this case.

Case 4. $(X_1 > 0, X_2 < 0, X_3 > 0)$

Let $X_1 = a$, $X_2 = -b$ and $X_3 = c$, where a, b, c are nonnegative integers. If $a \geq bp$, then both of $\eta(z)^a \eta(pz)^{-b}$ and $\eta(p^2 z)^c$ are holomorphic, which contradicts the fact that $\eta^x = \eta(z)^a \eta(pz)^{-b} \eta(p^2 z)^c$ is irreducible. So, we must have $a < bp$. Let $\lceil \frac{a}{p} \rceil = b_0$. Then there exists an integer $r \geq 0$ such that $b = b_0 + r$.

As

$$\frac{1}{p}a - b + \frac{1}{p}c \geq 0,$$

$c = bp - a + \varepsilon$ for some $\varepsilon \geq 0$. Let

$$X' = \begin{pmatrix} a \\ -b_0 \\ b_0 p - a + \varepsilon \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} 0 \\ -r \\ rp \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic and $\eta^{X'} \cdot \eta^{X''} = \eta^X$ is irreducible, we must have $X'' = 0$, i. e. $r = 0$. So, we get $b = b_0$.

Suppose if possible, $a \geq p$. Then there exists an integer $a' \geq 0$ such that $a = p + a'$. Let $\lceil \frac{a'}{p} \rceil = b'$ and let

$$X' = \begin{pmatrix} p \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad X'' = \begin{pmatrix} a' \\ -b' \\ b'p - a' + \varepsilon \end{pmatrix}.$$

Then $X = X' + X''$. Since both $\eta^{X'}$ and $\eta^{X''}$ are holomorphic, either $\eta^X = \eta^{X'} \cdot \eta^{X''}$ is reducible or
$X'' = 0$ and so, $X_3 = 0$. But we have have started with the assumption that $X_3 > 0$, (since otherwise η^X is not an eta quotient of level p^2). Thus we get a contradiction! Hence, we must have $a < p$ and therefore, $b = \lceil \frac{a}{p} \rceil = 1$. It follows that

$$\frac{\eta(z)^r \eta(p^2 z)^{p-r}}{\eta(pz)}, \quad 1 \leq r \leq p-1$$

are the only simple holomorphic eta quotients in this case. □

6.4 Level p^n for $n > 3$

For a prime p and an integer $n > 3$, we define the eta quotient $f_{p,n}$ by

$$f_{p,n} := \begin{cases} \dfrac{\eta_p^p \, \eta_{p^{n-1}}^{(p-1)^2} \displaystyle\prod_{s=1}^{n/2-1} \eta_{p^{2s-1}}^{p^2-3p+1} \, \eta_{p^{2s}}^{p^2-2p+2}}{(\eta \, \eta_{p^n})^{p-1}} & \text{if } n \text{ is even.} \\[2em] \dfrac{(\eta_p \, \eta_{p^{n-1}})^p \displaystyle\prod_{s=1}^{n-1} \eta_{p^s}^{p^2-3p+2}}{(\eta \, \eta_{p^n})^{p-1}} & \text{if } n \text{ is odd and } p \neq 2, \end{cases} \tag{6.5}$$

Clearly, $f_{p,n}$ is invariant under the Fricke involution W_{p^n}. We shall show that:

Theorem 6.7. *For any integer $n > 3$, $f_{p,n}$ is a simple holomorphic eta quotient of level p^n.*

In particular, the above theorem together with Theorem 2.36 implies:

Corollary 6.8. *For any integer $n > 3$, the eta quotient $f_{p,n}$ is irreducible.*

With a good amount of numerical evidence, we conjecture that

Conjecture 6.9. *For any integer $n > 3$ and for any odd prime p, there are no simple holomorphic eta quotients of level p^n and of weight greater that of $f_{p,n}$.*

Proof of Theorem 6.7. We shall only prove the theorem for the case where n is even. The proof for the case where n is odd is similar. Let $\widetilde{A}_N$ denote the normalized valuation matrix of level N (see (5.7)) and let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be defined by

$$B_N := \widetilde{A}_N^{-1} = \bigotimes_{p^n \| N} \widetilde{A}_{p^n}^{-1} = \bigotimes_{p^n \| N} \begin{pmatrix} p & -p & & & & \\ -1 & p^2+1 & -p & & & \\ & -p & p^2+1 & -p & & \\ & & \ddots & \ddots & \ddots & \\ & & & -p & p^2+1 & -1 \\ & & & & -p & p \end{pmatrix} \tag{6.6}$$

In Lemma 5.3, we saw that if η^X is an irreducible holomorphic eta quotient on $\Gamma_0(N)$, then $X \in \mathcal{Z}_N \cap \mathbb{Z}^{\mathcal{D}_N}$, where

$$\mathcal{Z}_N = \left\{ \sum_{d|N} C_d v_d \,\middle|\, C_d \in [0,1] \text{ for all } d|N \right\} \tag{6.7}$$

and for all $d \in \mathcal{D}_N$, $v_d = m_d u_d$, where u_d is the column of B_N indexed by d and m_d is the smallest positive integer such that $v_d \in \mathbb{Z}^{\mathcal{D}_N}$. Let $v := \sum_{d|N} v_d$ and let $F_N = \eta^v$. Remark 5.11 says that given a holomorphic eta quotient g on $\Gamma_0(N)$, the eta quotient F_N/g on $\Gamma_0(N)$ is holomorphic if and only if g corresponds to some point in $\mathcal{Z}_N \cap \mathbb{Z}^{\mathcal{D}_N}$.

In particular, for $N = p^{2m}$, the eta quotient $F_{p^{2m}} = \eta^v$ is given by

$$F_{p^{2m}}(z) = \begin{cases} \eta(pz)^{p^2-1} & \text{if } m = 1, \\ \left(\eta(pz)\,\eta(p^{2m-1}z)\right)^{p(p-1)} \prod_{r=2}^{2m-2} \eta(p^r z)^{(p-1)^2} & \text{if } m > 1. \end{cases} \tag{6.8}$$

Let $f_{p,2m}$ be the eta quotient defined in (6.5). Then we have

$$\frac{F_{p^{2m}}(z)}{f_{p,2m}(z)} = \begin{cases} \eta(z)^{p-1}\eta(pz)^{p-2}\eta(p^2z)^{p-1} & \text{if } m = 1, \\ \eta(z)^{p-1}\eta(pz)^{p-2}\eta(p^{2m}z)^{p-1} \prod_{r=1}^{m-1} \frac{\eta(p^{2r-1}z)\,\eta(p^{2r+1}z)^{p-1}}{\eta(p^{2r}z)} & \text{if } m > 1. \end{cases} \tag{6.9}$$

Since $\frac{\eta(z)\eta(p^2z)^{p-1}}{\eta(pz)}$ is a holomorphic eta quotient of level p^2, it follows that $\frac{F_{p^{2m}}(z)}{f_{p,2m}(z)}$ is a holomorphic eta quotient of level p^{2m} for all $m \in \mathbb{N}$. Let $X \in \mathbb{Z}^{\mathcal{D}_N}$ be such that $f_{p,2m} = \eta^X$. From Remark 5.11 and from (6.9), we conclude that $X \in \mathcal{Z}_N$. In other words, $Y := \widetilde{A}_N X$ has all its entries in the interval $[0, 1]$. From (6.5), it easily follows that $\mathrm{ord}_\infty(f_{2m,p}) = 1/24$. Since $f_{2m,p}$ is invariant under the Fricke involution on $\Gamma_0(p^{2m})$, we also have $\mathrm{ord}_0(f_{2m,p}) = 1/24$, since the Fricke involution interchanges the cusps 0 and ∞ of $\Gamma_0(p^{2m})$. Since 0 and 1 (resp. ∞ and $1/p^{2m}$) represent the same cusp of $\Gamma_0(p^{2m})$, from (1.80), (1.84) and (5.7), we get that both the first and the last entries of Y are equal to $\frac{1}{p^{2m-1}(p^2-1)}$.

There exists $U_N, V_N \in GL_{\sigma_0(N)}(\mathbb{Z})$ and a diagonal matrix D_N such that $D_N = U_N \times B_N \times V_N$. We shall see in the next section that if $N = p^n$ for some prime p and some integer $n > 2$, then

$$D_N = \mathrm{diag}(1, 1, \ldots, 1, p^{n-1}, p^{n-1}(p^2 - 1)) \tag{6.10}$$

and the last two columns of V_N are respectively

$$\mathcal{C}_{n,1} := \begin{cases} (1,0)^t & \text{if } n = 1, \\ (-1,0,1)^t & \text{if } n = 2, \\ (1,1,p,p^2,\ldots,p^{n-3},p^{n-2},0)^t & \text{if } n > 2 \end{cases} \tag{6.11}$$

and

$$\mathcal{C}_{n,2} := \begin{cases} (p,1)^t & \text{if } n = 1, \\ (p^2,1,1)^t & \text{if } n = 2, \\ (p^n,p^{n-2},p^{n-3}\ldots,p,1,1)^t & \text{if } n > 2. \end{cases} \tag{6.12}$$

Next we briefly recall an useful tool from Linear Algebra:

By elementary row and column operations [3], one can reduce any matrix $B \in \mathrm{GL}_n(\mathbb{Z})$ to a diagonal matrix D. In other words, there exists $U, V \in \mathrm{GL}_n(\mathbb{Z})$ and $D = \mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathrm{GL}_n(\mathbb{Z})$ such that $D = U \cdot B \cdot V$. Since $U, V \in \mathrm{GL}_n(\mathbb{Z})$, we have $U^{-1} \cdot \mathbb{Z}^n = \mathbb{Z}^n$ and $V \cdot \mathbb{Z}^n = \mathbb{Z}^n$. Therefore,

$$\mathbb{Z}^n/(B\cdot\mathbb{Z}^n) = U^{-1}\cdot\mathbb{Z}^n/(B\cdot V\cdot\mathbb{Z}^n) \simeq \mathbb{Z}^n/(U\cdot B\cdot V\cdot\mathbb{Z}^n) = \mathbb{Z}^n/(D\cdot\mathbb{Z}^n) = \bigoplus_{i=1}^{n} \mathbb{Z}^n/d_i\mathbb{Z}^n.$$

The above isomorphism maps the element $\underline{\ell} := (\ell_1 \ldots \ell_n)^t$ of $\bigoplus_{i=1}^{n} \mathbb{Z}^n/d_i\mathbb{Z}^n$ to the element

$$U^{-1} \cdot \underline{\ell} \pmod{B \cdot \mathbb{Z}^n}$$

of $\mathbb{Z}^n/(B \cdot \mathbb{Z}^n)$. Since B is invertible, there is a bijection between $\mathbb{Z}^n/(B \cdot \mathbb{Z}^n)$ and $[0,1)^n \cap B^{-1} \cdot \mathbb{Z}^n$, given by

$$X \pmod{B \cdot \mathbb{Z}^n} \ \mapsto \ B^{-1} \cdot X \pmod{\mathbb{Z}^n}.$$

Composing this bijection with the isomorphism above, we get a bijection between $\bigoplus_{i=1}^{n} \mathbb{Z}^n/d_i\mathbb{Z}^n$ and $[0,1)^n \cap B^{-1} \cdot \mathbb{Z}^n$, given by

$$\underline{\ell} \ \mapsto \ B^{-1} \cdot U^{-1} \cdot \underline{\ell} \pmod{\mathbb{Z}^n} \ = \ V \cdot D^{-1} \cdot \underline{\ell} \pmod{\mathbb{Z}^n}.$$

Now multiplication by B maps $[0,1)^n \cap B^{-1} \cdot \mathbb{Z}^n$ bijectively to $B \cdot [0,1)^n \cap \mathbb{Z}^n$.

Let $N = p^{2m}$ and suppose, $\eta^X = f_{2m,p}$ is reducible. Let $Y = A_N X$. Since η^X is reducible there exists $Y', Y'' \in \mathbb{Z}^{\mathcal{D}_N} \smallsetminus \{0\}$ with $Y' \geq 0$ and $Y'' \geq 0$ such that $Y = Y' + Y''$ and both $B_N Y'$ and $B_N Y''$ have integer entries. Since B_N is an integer matrix with determinant $d_N := p^{2m-1}(p^2-1)$, we see that $\frac{1}{d_N}$ is the least possible entry for Y' and Y''. Since $Y' + Y'' = Y$ has $\frac{1}{d_N}$ as its first entry, either the first entry of Y' or that of Y'' is zero. Similarly, either the last entry of Y' or that of Y'' is zero. But it is easy to show* that if both the first and the last entries of Y' (resp. Y'') is zero, then Y' (resp. Y'') is entirely zero. So, without loss of generality, we may assume that the first entry of Y' is $\frac{1}{d_N}$ and the last entry of Y' is 0. From the previous section and from the entries of the diagonal matrix D_N, we know that there exists $\ell_1 \in \{0, 1, \ldots, p^{2m-1} - 1\}$ and $\ell_2 \in \{0, 1, \ldots, p^{2m-1}(p^2-1) - 1\}$ such

*From the congruence relation (6.13) (resp. replacing Y' with Y'' in (6.13)).

that
$$\frac{\ell_1}{p^{2m-1}} \cdot \mathcal{C}_{2m,1} + \frac{\ell_2}{p^{2m-1}(p^2-1)} \cdot \mathcal{C}_{2m,2} \equiv Y' \pmod{\mathbb{Z}^n}. \tag{6.13}$$

Case 1. $(m = 1)$

Equating only the first and the last entries from both sides of (6.13), we obtain

$$\frac{\ell_1}{p} + \frac{p\ell_2}{p^2-1} \equiv \frac{1}{d_N} \pmod{\mathbb{Z}} \quad \text{and} \quad \frac{\ell_1}{p} + \frac{\ell_2}{p(p^2-1)} \equiv 0 \pmod{\mathbb{Z}},$$

which together implies that

$$\frac{\ell_1}{p} \equiv \frac{1}{d_N} \pmod{\mathbb{Z}}.$$

But this modular equation has no solution in $\ell_1 \in \{0, 1, \ldots, p-1\}$. Thus we get a contradiction!

Case 2. $(m > 1)$

Since the last entries of Y' and $\mathcal{C}_{2m,1}$ are 0, whereas the last entry of $\mathcal{C}_{2m,2}$ is 1, it follows that $\ell_2 = 0$. Since the first entry of $\mathcal{C}_{2m,1}$ is 1, we get

$$\frac{\ell_1}{p^{2m-1}} \equiv \frac{1}{d_N} \pmod{\mathbb{Z}}$$

as in the previous case. Since as before, this has no solution in $\ell_1 \in \{0, 1, \ldots, p^{2m-1} - 1\}$, we get a contradiction.

Hence, $f_{2m,p} = \eta^X$ is irreducible. □

6.5 The matrix identities

We continue to prove that the matrix identities $B = UDV$, $UU' = 1$ and $VV' = 1$ with $B = B_{p^n}$ as defined in (6.6) and $D = D_{p^n}$ as defined in (6.10) holds if we define $U = U_{p^n}$, $V = V_{p^n}$, $U' = U'_{p^n}$ and $V' = V'_{p^n}$ as follows for $n = 1, 2, 3$ or $n \geq 4$:

For $n = 1$, we define

$$U := \begin{pmatrix} 0 & -1 \\ 1 & p \end{pmatrix}, \; V := \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \; U' := \begin{pmatrix} p & 1 \\ -1 & 0 \end{pmatrix} \text{ and } V' := \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}.$$

For $n = 2$, we define

$$U := \begin{pmatrix} 0 & 1 & 0 \\ 0 & p & 1 \\ 1 & p & 1 \end{pmatrix}, \; V := \begin{pmatrix} 0 & -1 & p^2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \; U' = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -p & 1 & 0 \end{pmatrix} \text{ and}$$

$$V' = \begin{pmatrix} -1 & p^2+1 & -1 \\ -1 & p^2 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $n = 3$, we define

$$U := \begin{pmatrix} 0 & -1 & -p & -p^2 \\ 0 & 0 & -1 & -p \\ 0 & 0 & -p & -(p^2+1) \\ 1 & p & p^2 & p^3 \end{pmatrix}, \qquad V := \begin{pmatrix} 1 & 0 & 1 & p^3 \\ 0 & 0 & 1 & p \\ 0 & -1 & p & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U' := \begin{pmatrix} p & 0 & 0 & 1 \\ -1 & p & 0 & 0 \\ 0 & -(p^2+1) & p & 0 \\ 0 & p & -1 & 0 \end{pmatrix} \text{ and } V' := \begin{pmatrix} 1 & -1 & 0 & -p(p^2-1) \\ 0 & p & -1 & -(p^2-1) \\ 0 & 1 & 0 & -p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $n > 3$:

We define $U = (U_{i,j})_{0 \le i,j \le n}$ by

	$j = 0$	$j > 0$
$i < n-1$	0	$\begin{cases} -p^{j-i-1} & \text{if } j > i, \\ 0 & \text{otherwise.} \end{cases}$
$i = n-1$	0	$-p^{n-j} \cdot \frac{p^{2(j-1)}-1}{p^2-1}$
$i = n$	1	p^j

(6.14)

We define $V = (V_{i,j})_{0 \le i,j \le n}$ by

	$j = 0$	$0 < j < n-1$	$j = n-1$	$j = n$
$i = 0$	1	0	1	p^n
$0 < i < n$	0	$\begin{cases} -p^{i-j-1} & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$	p^{i-1}	p^{n-i-1}
$i = n$	0	0	0	1

(6.15)

We define $U' = (U'_{i,j})_{0 \le i,j \le n}$ by

	$j=0$	$0<j<n-2$	$j=n-2$	$j=n-1$	$j=n$
$i=0$	p	0	0	0	1
$0<i<n-1$	-1 0 $\vdots$ 0	$\begin{cases} p & \text{if } i=j, \\ -1 & \text{if } i=j+1, \\ 0 & \text{otherwise.} \end{cases}$	0 $\vdots$ 0 p	0	0
$i=n-1$	0	$-p^{n-j}$	$-(p^2+1)$	p	0
$i=n$	0	p^{n-j-1}	p	-1	0

(6.16)

We define $V' = (V'_{i,j})_{0 \le i,j \le n}$ by

	$j=0$	$0<j<n$	$j=n$
$i=0$	1	$-1 \quad 0 \quad \cdots \quad 0$	$-p^{n-2}(p^2-1)$
$0<i<n-1$	0	$\begin{cases} p & \text{if } i=j, \\ -1 & \text{if } i=j-1, \\ 0 & \text{otherwise.} \end{cases}$	$-p^{n-i-2}(p^2-1)$
$i=n-1$	0	$1 \quad 0 \quad \cdots \quad 0$	$-p^{n-2}$
$i=n$	0	0	1

(6.17)

Proposition 6.10. *Given $n \in \mathbb{N}$, let U, V, U' and V' be the matrices as defined above. For $N = p^n$, we set the matrices $B = B_N$ and $D = D_N$ as in equations equations (6.6) and (6.10). Then we have*

$$UU' = I,\ VV' = I, \text{ and } D = UBV.$$

Proof. If $n \le 3$, these identities hold trivially. If $n > 3$, the proofs of the equalities of the corresponding matrix entries in each of these matrix relations involve (at most) summation of some geometric series. For example, consider the identity $D = UBV$. It is equivalent to $DV' = UB$, assuming $VV' = I$. Now from (6.10) and (6.17), we see that for $i, j \in \{0, \ldots, n\}$, the (i,j)-th entry of DV' is given by

	$j=0$	$0<j<n$	$j=n$
$i=0$	1	$-1 \quad 0 \quad \ldots \quad 0$	$-p^{n-2}(p^2-1)$
$0<i<n-1$	0	$\begin{cases} p & \text{if } i=j, \\ -1 & \text{if } i=j-1, \\ 0 & \text{otherwise.} \end{cases}$	$-p^{n-i-2}(p^2-1)$
$i=n-1$	0	$p^{n-1} \quad 0 \quad \ldots \quad 0$	$-p^{2n-3}$
$i=n$	0	0	$p^{n-1}(p^2-1)$

(6.18)

If we consider the case $0 < i < n-1$ and $0 < j < n$, then from (6.14) and (6.6) it follows that the product of the i-th row of U and the j-th column of B is

$$\begin{aligned}
-\sum_{k=i+1}^{n} p^{k-i-1}\, B_{k,j} &= \sum_{k=i+1}^{n} p^{k-i-1}(p\,\delta_{|k-j|,1} - (p^2+1)\,\delta_{k,j}) \\
&= \sum_{k=\max\{i+1,j-1\}}^{j+1} p^{k-i-1}(p\,\delta_{|k-j|,1} - (p^2+1)\,\delta_{k,j}) \\
&= \begin{cases} 0 & \text{if } j < i, \\ p & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{if } j > i+1, \end{cases}
\end{aligned}$$

where δ is the usual Kronecker delta function. So, the claim holds in this case.

Again, if we consider the case $i = n-1$ and $0 < j < n$, then the product of the i-th row of U and the j-th column of B is

$$\begin{aligned}
-\sum_{k=1}^{n} p^{n-k}\left(p^{2(k-1)}-1\right) B_{k,j} &= \sum_{k=2}^{n} p^{n-k}\left(p^{2(k-1)}-1\right)\left(p\,\delta_{|k-j|,1} - (p^2+1)\,\delta_{k,j}\right) \\
&= \sum_{k=\max\{2,j-1\}}^{j+1} p^{n-k}\left(p^{2(k-1)}-1\right)\left(p\,\delta_{|k-j|,1} - (p^2+1)\,\delta_{k,j}\right) \\
&= \begin{cases} p^{n-1} & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where the first equality holds since $p^{2(k-1)} = 1$ for $k = 1$. Thus, the claim also holds in this case. The rest of the proof is quite similar and only requires some more straightforward checks as above. □

Appendices

Appendix A

Table of simple holomorphic eta quotients of weight 1

In the following table, we give examples of simple holomorphic eta quotients weight 1 and of various levels. For each level N that appears in the tables below, there are three entries on the left column: the first one being N, the second one (in round brackets) being the number of simple holomorphic eta quotients of level N modulo Atkin-Lehner involutions of level N and the third one [in square brackets] being the total number of distinct simple holomorphic eta quotients of level N. Also, for level N, each formal expressions of the type $\prod_{d|N} d^{X_d}$ on the right column denotes a simple holomorphic eta quotient: $\prod_{d|N} \eta_d^{X_d}$ of level N and weight 1. This table is complete up to Atkin-Lehner involutions for the levels $N < 24$ and for $N =$ 30, 32, 42, 45, 50, 54, 56, 63, 64, 70, 84, 88, 90, 100, 112, 126, 140, 150, 168, 176. Also, in this table, the list of levels below 260 is complete, i.e., if a level $N < 260$ does not appear in the table, then this is because there are no simple holomorphic eta quotients of weight 1 and level N.

	Simple holomorphic eta quotients
3 (1) [2]	$1^{-1} \cdot 3^{3}$
6 (1) [4]	$1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 6^{-3}$
9 (2) [3]	$1^{-1} \cdot 3^{4} \cdot 9^{-1}$ $1^{1} \cdot 3^{-1} \cdot 9^{2}$
10 (2) [8]	$1^{-1} \cdot 2^{1} \cdot 5^{3} \cdot 10^{-1}$ $1^{-1} \cdot 2^{1} \cdot 5^{4} \cdot 10^{-2}$
12 (17) [60]	$1^{4} \cdot 2^{-2} \cdot 3^{-2} \cdot 6^{3} \cdot 12^{-1}$ $1^{-1} \cdot 2^{-1} \cdot 3^{3} \cdot 4^{3} \cdot 6^{-1} \cdot 12^{-1}$ $1^{-1} \cdot 3^{1} \cdot 4^{3} \cdot 12^{-1}$ $1^{-1} \cdot 3^{3} \cdot 4^{1} \cdot 6^{-1}$ $1^{-1} \cdot 3^{3} \cdot 4^{1} \cdot 6^{-2} \cdot 12^{1}$ $1^{-1} \cdot 3^{4} \cdot 4^{1} \cdot 6^{-2}$ $1^{-1} \cdot 2^{1} \cdot 3^{-1} \cdot 4^{2} \cdot 6^{3} \cdot 12^{-2}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 4^{-1} \cdot 6^{-3} \cdot 12^{3}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 6^{-1} \cdot 12^{3}$ $1^{-1} \cdot 2^{3} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-3} \cdot 12^{4}$ $1^{-2} \cdot 2^{7} \cdot 3^{2} \cdot 4^{-3} \cdot 6^{-3} \cdot 12^{1}$ $1^{-3} \cdot 2^{8} \cdot 3^{1} \cdot 4^{-3} \cdot 6^{-3} \cdot 12^{2}$ $1^{-3} \cdot 2^{8} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-4} \cdot 12^{4}$ $1^{-3} \cdot 2^{9} \cdot 3^{1} \cdot 4^{-3} \cdot 6^{-3} \cdot 12^{1}$ $1^{-3} \cdot 2^{9} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-3} \cdot 12^{2}$ $1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-3} \cdot 12^{2}$ $1^{-6} \cdot 2^{15} \cdot 3^{2} \cdot 4^{-6} \cdot 6^{-5} \cdot 12^{2}$
15 (1) [2]	$1^{-1} \cdot 3^{2} \cdot 5^{2} \cdot 15^{-1}$
16 (7) [12]	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{1} \cdot 16^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-2} \cdot 16^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 16^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-3} \cdot 8^{4} \cdot 16^{-1}$ $1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 16^{1}$ $1^{1} \cdot 2^{-2} \cdot 4^{4} \cdot 8^{-2} \cdot 16^{1}$ $1^{1} \cdot 4^{-1} \cdot 8^{1} \cdot 16^{1}$
18 (17) [52]	$1^{-2} \cdot 2^{1} \cdot 3^{7} \cdot 6^{-4} \cdot 9^{-3} \cdot 18^{3}$ $1^{-1} \cdot 3^{3} \cdot 6^{1} \cdot 9^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 9^{-1} \cdot 18^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{3}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 18^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{5} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{2}$ $1^{-1} \cdot 2^{2} \cdot 3^{-1} \cdot 9^{4} \cdot 18^{-2}$ $1^{-1} \cdot 2^{2} \cdot 6^{-1} \cdot 9^{1} \cdot 18^{1}$ $1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{4}$ $1^{-1} \cdot 2^{3} \cdot 6^{-1} \cdot 9^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{7} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{8} \cdot 6^{-4} \cdot 9^{-2} \cdot 18^{1}$ $1^{-2} \cdot 2^{2} \cdot 3^{5} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{2}$ $2^{1} \cdot 3^{-1} \cdot 6^{1} \cdot 9^{1}$ $2^{1} \cdot 3^{1} \cdot 6^{-1} \cdot 18^{1}$
20 (9) [26]	$1^{-2} \cdot 2^{4} \cdot 4^{-1} \cdot 5^{2} \cdot 10^{-2} \cdot 20^{1}$ $1^{-1} \cdot 2^{1} \cdot 4^{1} \cdot 5^{1} \cdot 10^{1} \cdot 20^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{1} \cdot 5^{2} \cdot 10^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 20^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{2} \cdot 10^{-2} \cdot 20^{2}$ $1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 10^{-1} \cdot 20^{2}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{2}$ $1^{-3} \cdot 2^{8} \cdot 4^{-3} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{1}$ $1^{-4} \cdot 2^{10} \cdot 4^{-4} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{1}$

	Simple holomorphic eta quotients
24 (81) [312]	$1^{2} \cdot 2^{-1} \cdot 3^{-2} \cdot 6^{4} \cdot 12^{-2} \cdot 24^{1}$ $1^{2} \cdot 2^{-2} \cdot 3^{-1} \cdot 4^{1} \cdot 6^{1} \cdot 8^{1} \cdot 12^{1} \cdot 24^{-1}$ $1^{2} \cdot 2^{-2} \cdot 3^{-1} \cdot 4^{2} \cdot 6^{1} \cdot 8^{-1} \cdot 24^{1}$ $1^{2} \cdot 2^{-2} \cdot 3^{-1} \cdot 4^{2} \cdot 6^{2} \cdot 8^{-1} \cdot 12^{-2} \cdot 24^{2}$ $1^{2} \cdot 2^{-3} \cdot 3^{-1} \cdot 4^{4} \cdot 6^{2} \cdot 8^{-1} \cdot 12^{-1}$ $1^{2} \cdot 2^{-3} \cdot 3^{-1} \cdot 4^{4} \cdot 6^{2} \cdot 8^{-1} \cdot 12^{-2} \cdot 24^{1}$ $1^{3} \cdot 2^{-1} \cdot 3^{-1} \cdot 4^{-1} \cdot 8^{1} \cdot 12^{1}$
30 (8) [44]	$1^{1} \cdot 2^{-2} \cdot 3^{-2} \cdot 5^{-2} \cdot 6^{4} \cdot 10^{4} \cdot 15^{1} \cdot 30^{-2}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{1} \cdot 15^{-1} \cdot 30^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{2} \cdot 10^{-1} \cdot 15^{-1} \cdot 30^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{2} \cdot 10^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{3} \cdot 6^{-1} \cdot 10^{-2} \cdot 15^{-2} \cdot 30^{3}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 5^{1} \cdot 6^{-2} \cdot 10^{-1} \cdot 15^{-2} \cdot 30^{3}$ $1^{-2} \cdot 2^{2} \cdot 3^{3} \cdot 5^{3} \cdot 6^{-2} \cdot 10^{-2} \cdot 15^{-2} \cdot 30^{2}$
32 (4) [8]	$1^{-1} \cdot 2^{2} \cdot 4^{1} \cdot 8^{-2} \cdot 16^{3} \cdot 32^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{1} \cdot 8^{-1} \cdot 32^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{2} \cdot 16^{-1} \cdot 32^{1}$ $1^{1} \cdot 2^{-1} \cdot 4^{2} \cdot 8^{-1} \cdot 32^{1}$
36 (178) [529]	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 6^{-1} \cdot 9^{2} \cdot 18^{-2} \cdot 36^{2}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 6^{-2} \cdot 9^{2} \cdot 12^{2} \cdot 18^{-1}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 6^{-1} \cdot 9^{1} \cdot 12^{1} \cdot 18^{-1} \cdot 36^{1}$ $1^{-2} \cdot 2^{5} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-3} \cdot 9^{-1} \cdot 12^{1} \cdot 18^{4} \cdot 36^{-1}$ $1^{-4} \cdot 2^{10} \cdot 3^{2} \cdot 4^{-4} \cdot 6^{-5} \cdot 9^{-2} \cdot 12^{2} \cdot 18^{5} \cdot 36^{-2}$ $1^{-2} \cdot 2^{5} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-3} \cdot 9^{-1} \cdot 12^{2} \cdot 18^{2}$ $1^{-3} \cdot 2^{5} \cdot 3^{4} \cdot 4^{-1} \cdot 6^{-3} \cdot 9^{-1} \cdot 36^{1}$
40 (23) [88]	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 5^{1} \cdot 8^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 5^{2} \cdot 8^{-1} \cdot 10^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 8^{1} \cdot 10^{-1} \cdot 20^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 8^{1} \cdot 10^{-2} \cdot 20^{3} \cdot 40^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{2} \cdot 8^{1} \cdot 10^{-1}$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{3} \cdot 40^{-1}$ $2^{-1} \cdot 4^{2} \cdot 5^{1} \cdot 20^{-1} \cdot 40^{1}$
42 (2) [16]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 7^{2} \cdot 14^{-1} \cdot 21^{-1} \cdot 42^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 7^{2} \cdot 14^{-1}$
45 (1) [2]	$1^{1} \cdot 3^{-1} \cdot 5^{-1} \cdot 9^{1} \cdot 15^{3} \cdot 45^{-1}$
48 (135) [496]	$1^{2} \cdot 2^{-1} \cdot 3^{-1} \cdot 6^{2} \cdot 12^{-1} \cdot 48^{1}$ $1^{2} \cdot 2^{-1} \cdot 3^{-1} \cdot 6^{2} \cdot 12^{-2} \cdot 24^{3} \cdot 48^{-1}$ $1^{2} \cdot 2^{-2} \cdot 3^{-1} \cdot 6^{3} \cdot 8^{2} \cdot 12^{-1} \cdot 16^{-1} \cdot 24^{-1} \cdot 48^{1}$ $1^{2} \cdot 2^{-2} \cdot 3^{-1} \cdot 4^{1} \cdot 6^{3} \cdot 8^{-1} \cdot 12^{-2} \cdot 16^{1} \cdot 24^{2} \cdot 48^{-1}$ $1^{2} \cdot 3^{-1} \cdot 4^{-1} \cdot 6^{1} \cdot 8^{2} \cdot 16^{-1} \cdot 24^{-1} \cdot 48^{1}$ $1^{2} \cdot 3^{-1} \cdot 6^{1} \cdot 8^{-1} \cdot 12^{-1} \cdot 16^{1} \cdot 24^{2} \cdot 48^{-1}$ $1^{3} \cdot 2^{-1} \cdot 3^{-1} \cdot 8^{-1} \cdot 16^{1} \cdot 24^{2} \cdot 48^{-1}$
50 (3) [8]	$1^{-1} \cdot 2^{1} \cdot 5^{4} \cdot 10^{-2} \cdot 25^{-1} \cdot 50^{1}$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 10^{-1} \cdot 25^{-1} \cdot 50^{2}$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 10^{-1} \cdot 50^{1}$
54 (6) [20]	$1^{-1} \cdot 2^{2} \cdot 3^{-1} \cdot 6^{1} \cdot 9^{3} \cdot 18^{-2} \cdot 27^{-1} \cdot 54^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{2} \cdot 27^{2} \cdot 54^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{3} \cdot 27^{1} \cdot 54^{-1}$ $1^{-1} \cdot 2^{2} \cdot 9^{1} \cdot 18^{-1} \cdot 54^{1}$ $1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{1} \cdot 27^{1}$ $2^{1} \cdot 3^{1} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{1} \cdot 27^{1}$
56 (11) [40]	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 7^{1} \cdot 8^{-1} \cdot 28^{-1} \cdot 56^{1}$ $1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 7^{1} \cdot 8^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 7^{2} \cdot 8^{-1} \cdot 14^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 8^{1} \cdot 14^{-1} \cdot 28^{2} \cdot 56^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 8^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{2} \cdot 8^{1} \cdot 14^{-1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 7^{1} \cdot 14^{-1} \cdot 56^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 7^{1} \cdot 14^{-2} \cdot 28^{3} \cdot 56^{-1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 14^{-2} \cdot 28^{5} \cdot 56^{-2}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 7^{1} \cdot 8^{1} \cdot 14^{-2} \cdot 28^{3} \cdot 56^{-1}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 7^{1} \cdot 14^{-1} \cdot 56^{1}$

	Simple holomorphic eta quotients
60 (75) [478]	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{1} \cdot 10^{1} \cdot 30^{-1} \cdot 60^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{2} \cdot 10^{-1} \cdot 12^{-1} \cdot 20^{2}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 4^{-1} \cdot 5^{2} \cdot 6^{-1} \cdot 10^{-1} \cdot 15^{-1} \cdot 60^{1}$ $1^{-4} \cdot 2^{10} \cdot 3^{2} \cdot 4^{-4} \cdot 5^{2} \cdot 6^{-5} \cdot 10^{-5} \cdot 12^{2} \cdot 15^{-4} \cdot 20^{2} \cdot 30^{10} \cdot 60^{-4}$ $1^{-2} \cdot 2^{4} \cdot 3^{2} \cdot 4^{-2} \cdot 5^{2} \cdot 6^{-3} \cdot 10^{-3} \cdot 12^{2} \cdot 15^{-2} \cdot 20^{2} \cdot 30^{4} \cdot 60^{-2}$ $1^{-2} \cdot 2^{4} \cdot 3^{1} \cdot 4^{-1} \cdot 5^{-1} \cdot 6^{-1} \cdot 10^{4} \cdot 20^{-2} \cdot 30^{-1} \cdot 60^{1}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 10^{-1} \cdot 15^{1} \cdot 20^{1} \cdot 30^{-1} \cdot 60^{1}$
63 (1) [2]	$1^{-1} \cdot 3^{3} \cdot 7^{1} \cdot 9^{-1} \cdot 21^{-1} \cdot 63^{1}$
64 (7) [12]	$1^{-1} \cdot 2^{2} \cdot 4^{1} \cdot 8^{-1} \cdot 16^{-1} \cdot 32^{3} \cdot 64^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{1} \cdot 8^{-1} \cdot 64^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 8^{-1} \cdot 16^{2} \cdot 32^{-1} \cdot 64^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{2} \cdot 16^{-1} \cdot 64^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{2} \cdot 16^{-2} \cdot 32^{3} \cdot 64^{-1}$ $1^{1} \cdot 2^{-1} \cdot 4^{2} \cdot 8^{-1} \cdot 64^{1}$ $1^{1} \cdot 4^{-1} \cdot 8^{2} \cdot 16^{-1} \cdot 64^{1}$
70 (2) [12]	$1^{2} \cdot 2^{-1} \cdot 7^{-1} \cdot 14^{1} \cdot 35^{2} \cdot 70^{-1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 7^{2} \cdot 10^{-1} \cdot 14^{-1} \cdot 35^{-1} \cdot 70^{1}$
72 (205) [708]	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{1} \cdot 8^{1} \cdot 12^{-1} \cdot 36^{2} \cdot 72^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-2} \cdot 9^{1} \cdot 12^{-2} \cdot 18^{-1} \cdot 24^{2} \cdot 36^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 4^{3} \cdot 6^{-2} \cdot 8^{-2} \cdot 9^{-2} \cdot 12^{-1} \cdot 18^{5} \cdot 24^{2} \cdot 36^{-2}$ $1^{-2} \cdot 2^{5} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-4} \cdot 9^{-1} \cdot 12^{5} \cdot 18^{3} \cdot 24^{-2} \cdot 36^{-3} \cdot 72^{2}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{1} \cdot 8^{1} \cdot 24^{-1} \cdot 36^{-1} \cdot 72^{2}$ $1^{-1} \cdot 2^{2} \cdot 6^{-2} \cdot 9^{1} \cdot 12^{4} \cdot 24^{-2} \cdot 36^{-2} \cdot 72^{2}$ $1^{-1} \cdot 2^{2} \cdot 9^{1} \cdot 12^{-2} \cdot 18^{-2} \cdot 24^{2} \cdot 36^{4} \cdot 72^{-2}$
80 (23) [76]	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 16^{1}$ $2^{-2} \cdot 4^{5} \cdot 5^{1} \cdot 8^{-2} \cdot 20^{-1} \cdot 80^{1}$ $5^{-1} \cdot 8^{-1} \cdot 10^{3} \cdot 16^{2} \cdot 20^{-2} \cdot 40^{2} \cdot 80^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 8^{1} \cdot 10^{-1} \cdot 20^{1} \cdot 40^{-1} \cdot 80^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 8^{2} \cdot 10^{-1} \cdot 16^{-1} \cdot 20^{1} \cdot 40^{-1} \cdot 80^{1}$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 8^{-1} \cdot 10^{-1} \cdot 16^{1} \cdot 40^{2} \cdot 80^{-1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 10^{-2} \cdot 16^{1} \cdot 20^{5} \cdot 40^{-2}$ $1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 5^{1} \cdot 8^{3} \cdot 16^{-1}$
84 (13) [88]	$1^{2} \cdot 2^{-1} \cdot 7^{-1} \cdot 14^{2} \cdot 28^{-1} \cdot 84^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 4^{1} \cdot 6^{-1} \cdot 7^{1} \cdot 14^{-1} \cdot 21^{-1} \cdot 42^{3} \cdot 84^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{-1} \cdot 14^{3} \cdot 28^{-1} \cdot 42^{-1} \cdot 84^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 21^{-1} \cdot 28^{1} \cdot 42^{2} \cdot 84^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{2} \cdot 12^{1} \cdot 14^{-2} \cdot 21^{-1} \cdot 28^{1} \cdot 42^{2} \cdot 84^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{1} \cdot 7^{1} \cdot 14^{-1} \cdot 21^{-1} \cdot 28^{1} \cdot 42^{2} \cdot 84^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 6^{1} \cdot 7^{1} \cdot 14^{-1} \cdot 28^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 6^{-1} \cdot 12^{1} \cdot 21^{2} \cdot 42^{-1}$ $1^{-1} \cdot 2^{3} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-2} \cdot 12^{1} \cdot 21^{-2} \cdot 42^{5} \cdot 84^{-2}$ $1^{-1} \cdot 2^{3} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-2} \cdot 7^{1} \cdot 12^{1} \cdot 14^{-2} \cdot 21^{-2} \cdot 28^{1} \cdot 42^{5} \cdot 84^{-2}$ $1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 6^{-1} \cdot 12^{1} \cdot 14^{-1} \cdot 21^{1} \cdot 28^{1}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 14^{-1} \cdot 21^{1} \cdot 28^{1}$ $1^{-2} \cdot 2^{5} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-2} \cdot 12^{1} \cdot 14^{-1} \cdot 21^{1} \cdot 28^{1}$
88 (4) [16]	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{-1} \cdot 11^{2} \cdot 22^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 11^{2} \cdot 22^{-1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 22^{-2} \cdot 44^{5} \cdot 88^{-2}$ $1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 11^{1} \cdot 22^{-1} \cdot 88^{1}$
90 (10) [44]	$1^{1} \cdot 2^{-2} \cdot 3^{-3} \cdot 5^{-1} \cdot 6^{6} \cdot 9^{1} \cdot 10^{2} \cdot 15^{1} \cdot 18^{-2} \cdot 30^{-2} \cdot 45^{-1} \cdot 90^{2}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{2} \cdot 6^{-1} \cdot 10^{-1} \cdot 18^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 9^{-1} \cdot 15^{1} \cdot 18^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 9^{-1} \cdot 15^{2} \cdot 18^{1} \cdot 30^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 5^{1} \cdot 6^{-1} \cdot 9^{-1} \cdot 10^{-1} \cdot 15^{-1} \cdot 18^{1} \cdot 30^{2} \cdot 45^{1} \cdot 90^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 5^{1} \cdot 6^{-1} \cdot 9^{-1} \cdot 10^{-1} \cdot 15^{-2} \cdot 18^{1} \cdot 30^{3} \cdot 45^{1} \cdot 90^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 5^{1} \cdot 6^{-1} \cdot 10^{-1} \cdot 15^{-2} \cdot 30^{3} \cdot 45^{1} \cdot 90^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-1} \cdot 15^{-1} \cdot 18^{1} \cdot 30^{2}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-1} \cdot 18^{1} \cdot 30^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 5^{1} \cdot 6^{-2} \cdot 9^{-1} \cdot 10^{-1} \cdot 15^{-2} \cdot 18^{1} \cdot 30^{3} \cdot 45^{1} \cdot 90^{-1}$

	Simple holomorphic eta quotients
96 (80) [304]	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 24^{2} \cdot 32^{1} \cdot 48^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 16^{1} \cdot 24^{1} \cdot 48^{-1} \cdot 96^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 16^{2} \cdot 24^{1} \cdot 32^{-1} \cdot 48^{-1} \cdot 96^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 8^{1} \cdot 12^{1} \cdot 16^{-2} \cdot 24^{-2} \cdot 32^{2} \cdot 48^{4} \cdot 96^{-2}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 8^{1} \cdot 16^{-1} \cdot 24^{1} \cdot 32^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 8^{1} \cdot 24^{1} \cdot 48^{-1} \cdot 96^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 8^{2} \cdot 16^{-1} \cdot 96^{1}$
100 (10) [20]	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 10^{-1} \cdot 20^{1} \cdot 25^{2} \cdot 50^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{2} \cdot 10^{-2} \cdot 20^{2} \cdot 25^{-1} \cdot 50^{2} \cdot 100^{-1}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 10^{-1} \cdot 20^{1} \cdot 25^{1}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{1} \cdot 25^{-2} \cdot 50^{5} \cdot 100^{-2}$
	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 5^{1} \cdot 10^{-2} \cdot 20^{1} \cdot 25^{-2} \cdot 50^{5} \cdot 100^{-2}$
	$2^{-1} \cdot 4^{2} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{2} \cdot 50^{-1}$
	$4^{1} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{1}$
	$2^{1} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{1} \cdot 50^{-1} \cdot 100^{1}$
	$1^{1} \cdot 2^{-1} \cdot 4^{1} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{1} \cdot 50^{-1} \cdot 100^{1}$
	$1^{1} \cdot 2^{-2} \cdot 4^{1} \cdot 5^{-4} \cdot 10^{10} \cdot 20^{-4} \cdot 25^{1} \cdot 50^{-2} \cdot 100^{1}$
108 (34) [104]	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{2} \cdot 36^{-1} \cdot 108^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 6^{-1} \cdot 18^{-1} \cdot 27^{-1} \cdot 36^{1} \cdot 54^{3} \cdot 108^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{2} \cdot 4^{-1} \cdot 6^{-3} \cdot 9^{-1} \cdot 12^{2} \cdot 18^{2} \cdot 27^{-2} \cdot 36^{-1} \cdot 54^{5} \cdot 108^{-2}$
	$1^{-1} \cdot 2^{2} \cdot 3^{2} \cdot 4^{-1} \cdot 6^{-3} \cdot 9^{-1} \cdot 12^{2} \cdot 18^{4} \cdot 36^{-2} \cdot 54^{-1} \cdot 108^{1}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 6^{-1} \cdot 12^{1} \cdot 27^{2} \cdot 54^{-1}$
	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 18^{-1} \cdot 27^{1} \cdot 36^{1}$
112 (6) [20]	$1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 7^{2} \cdot 8^{3} \cdot 14^{-1} \cdot 16^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 8^{2} \cdot 14^{-1} \cdot 16^{-1} \cdot 28^{1} \cdot 56^{-1} \cdot 112^{1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{2} \cdot 14^{-1} \cdot 16^{1}$
	$1^{-1} \cdot 2^{2} \cdot 7^{1} \cdot 8^{-1} \cdot 14^{-1} \cdot 16^{1} \cdot 56^{2} \cdot 112^{-1}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 8^{-1} \cdot 16^{1} \cdot 28^{-2} \cdot 56^{5} \cdot 112^{-2}$
	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 7^{1} \cdot 14^{-1} \cdot 112^{1}$
120 (49) [384]	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 8^{1} \cdot 10^{-1} \cdot 60^{2} \cdot 120^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 6^{1} \cdot 10^{-2} \cdot 12^{-1} \cdot 20^{3} \cdot 24^{1} \cdot 40^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{2} \cdot 6^{-1} \cdot 10^{-2} \cdot 12^{3} \cdot 15^{-1} \cdot 20^{1} \cdot 24^{-1} \cdot 30^{2} \cdot 60^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 10^{-1} \cdot 15^{1} \cdot 20^{2} \cdot 40^{-1} \cdot 60^{-1} \cdot 120^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 5^{-1} \cdot 6^{-1} \cdot 10^{3} \cdot 12^{1} \cdot 20^{-2} \cdot 30^{-1} \cdot 40^{1} \cdot 60^{2} \cdot 120^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 10^{-1} \cdot 20^{3} \cdot 24^{1} \cdot 30^{1} \cdot 40^{-1} \cdot 60^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 20^{1} \cdot 24^{1}$
126 (7) [28]	$1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 6^{-3} \cdot 7^{2} \cdot 9^{-2} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-2} \cdot 42^{1} \cdot 63^{2} \cdot 126^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{1} \cdot 21^{2} \cdot 42^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-1} \cdot 42^{2} \cdot 63^{1} \cdot 126^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-2} \cdot 42^{3} \cdot 63^{1} \cdot 126^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 14^{-1} \cdot 21^{-2} \cdot 42^{3} \cdot 63^{1} \cdot 126^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 9^{-1} \cdot 18^{1} \cdot 21^{-1} \cdot 42^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-2} \cdot 42^{3} \cdot 63^{1} \cdot 126^{-1}$
140 (4) [12]	$1^{1} \cdot 2^{-1} \cdot 4^{1} \cdot 7^{-1} \cdot 14^{2} \cdot 28^{-1} \cdot 35^{1} \cdot 70^{-1} \cdot 140^{1}$
	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{1} \cdot 7^{1} \cdot 10^{-1} \cdot 14^{-1} \cdot 20^{1} \cdot 28^{1} \cdot 35^{-1} \cdot 70^{2} \cdot 140^{-1}$
	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 7^{1} \cdot 14^{-2} \cdot 28^{1} \cdot 35^{-2} \cdot 70^{5} \cdot 140^{-2}$
	$1^{-2} \cdot 2^{5} \cdot 4^{-2} \cdot 5^{1} \cdot 7^{1} \cdot 10^{-2} \cdot 14^{-2} \cdot 20^{1} \cdot 28^{1} \cdot 35^{-2} \cdot 70^{5} \cdot 140^{-2}$
144 (217) [736]	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-1} \cdot 8^{3} \cdot 16^{-1} \cdot 36^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-1} \cdot 8^{3} \cdot 12^{1} \cdot 16^{-1} \cdot 24^{-1} \cdot 144^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-1} \cdot 8^{3} \cdot 12^{1} \cdot 16^{-1} \cdot 24^{-2} \cdot 36^{-2} \cdot 48^{1} \cdot 72^{5} \cdot 144^{-2}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 6^{-1} \cdot 8^{-1} \cdot 12^{-1} \cdot 16^{1} \cdot 24^{2} \cdot 36^{1} \cdot 48^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-2} \cdot 6^{-1} \cdot 8^{3} \cdot 16^{-1} \cdot 36^{2} \cdot 72^{-1}$
150 (2) [6]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 5^{3} \cdot 6^{-1} \cdot 10^{-2} \cdot 15^{-2} \cdot 25^{-1} \cdot 30^{3} \cdot 50^{1} \cdot 75^{1} \cdot 150^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 5^{1} \cdot 6^{-1} \cdot 10^{-1} \cdot 15^{-1} \cdot 25^{-1} \cdot 30^{1} \cdot 50^{2} \cdot 75^{1} \cdot 150^{-1}$
168 (10) [80]	$1^{1} \cdot 2^{-1} \cdot 4^{2} \cdot 7^{1} \cdot 8^{-1} \cdot 12^{-1} \cdot 14^{-1} \cdot 21^{-1} \cdot 24^{1} \cdot 42^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 4^{2} \cdot 6^{-1} \cdot 8^{-1} \cdot 14^{-1} \cdot 28^{2} \cdot 42^{1} \cdot 56^{-1} \cdot 84^{-1} \cdot 168^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 4^{2} \cdot 6^{-1} \cdot 8^{-1} \cdot 42^{1} \cdot 84^{-1} \cdot 168^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 4^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 8^{-1} \cdot 14^{-1} \cdot 21^{-1} \cdot 42^{2} \cdot 84^{-1} \cdot 168^{1}$
	$1^{-1} \cdot 2^{2} \cdot 14^{-1} \cdot 21^{1} \cdot 28^{2} \cdot 56^{-1} \cdot 84^{-1} \cdot 168^{1}$
	$1^{-1} \cdot 2^{2} \cdot 21^{1} \cdot 28^{-1} \cdot 42^{-1} \cdot 56^{1} \cdot 84^{2} \cdot 168^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 7^{-1} \cdot 12^{1} \cdot 14^{3} \cdot 28^{-2} \cdot 42^{-1} \cdot 56^{1} \cdot 84^{2} \cdot 168^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 24^{1} \cdot 28^{2} \cdot 56^{-1} \cdot 84^{-1} \cdot 168^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-1} \cdot 7^{1} \cdot 14^{-2} \cdot 21^{-1} \cdot 24^{1} \cdot 28^{3} \cdot 42^{2} \cdot 56^{-1} \cdot 84^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-2} \cdot 12^{3} \cdot 14^{1} \cdot 24^{-1} \cdot 28^{-1} \cdot 56^{1}$

	Simple holomorphic eta quotients
176 (2) [4]	$1^{-1}\cdot 2^{2}\cdot 4^{-1}\cdot 8^{2}\cdot 11^{1}\cdot 16^{-1}\cdot 22^{-1}\cdot 44^{1}\cdot 88^{-1}\cdot 176^{1}$ $1^{-1}\cdot 2^{2}\cdot 8^{-1}\cdot 11^{1}\cdot 16^{1}\cdot 22^{-1}\cdot 88^{2}\cdot 176^{-1}$
180 (42) [166]	$1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 9^{-1}\cdot 12^{1}\cdot 18^{2}\cdot 30^{1}\cdot 36^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 9^{-1}\cdot 12^{1}\cdot 15^{1}\cdot 18^{2}\cdot 30^{-1}\cdot 36^{-1}\cdot 60^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 5^{1}\cdot 6^{-2}\cdot 9^{-1}\cdot 10^{-1}\cdot 12^{1}\cdot 18^{3}\cdot 20^{1}\cdot 36^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 5^{1}\cdot 6^{-2}\cdot 9^{-1}\cdot 10^{-1}\cdot 12^{2}\cdot 15^{-1}\cdot 18^{2}\cdot 30^{2}\cdot 36^{-1}\cdot 45^{1}\cdot 90^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{2}\cdot 4^{-1}\cdot 6^{-3}\cdot 9^{-1}\cdot 10^{-1}\cdot 12^{2}\cdot 18^{2}\cdot 20^{1}\cdot 30^{3}\cdot 36^{-1}\cdot 60^{-2}\cdot 90^{-1}\cdot 180^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{2}\cdot 4^{-1}\cdot 6^{-3}\cdot 9^{-1}\cdot 12^{2}\cdot 15^{-2}\cdot 18^{2}\cdot 30^{5}\cdot 36^{-1}\cdot 60^{-2}$
192 (62) [228]	$1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 6^{-1}\cdot 8^{-1}\cdot 16^{2}\cdot 32^{-1}\cdot 192^{1}$ $1^{-1}\cdot 2^{2}\cdot 24^{1}\cdot 32^{-1}\cdot 48^{-1}\cdot 64^{1}\cdot 96^{2}\cdot 192^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 16^{1}\cdot 24^{1}\cdot 48^{-1}\cdot 192^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 8^{1}\cdot 16^{-1}\cdot 32^{2}\cdot 48^{1}\cdot 64^{-1}\cdot 96^{-1}\cdot 192^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 8^{1}\cdot 48^{1}\cdot 96^{-1}\cdot 192^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 32^{2}\cdot 64^{-1}\cdot 96^{-1}\cdot 192^{1}$
216 (20) [72]	$1^{-1}\cdot 2^{1}\cdot 3^{1}\cdot 4^{2}\cdot 6^{-1}\cdot 8^{-1}\cdot 54^{1}\cdot 108^{-1}\cdot 216^{1}$ $1^{-1}\cdot 2^{2}\cdot 18^{-1}\cdot 27^{1}\cdot 36^{2}\cdot 72^{-1}\cdot 108^{-1}\cdot 216^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 8^{1}\cdot 18^{-1}\cdot 27^{-1}\cdot 36^{2}\cdot 54^{3}\cdot 72^{-1}\cdot 108^{-2}\cdot 216^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 8^{1}\cdot 36^{1}\cdot 72^{-1}\cdot 108^{-1}\cdot 216^{2}$ $1^{-1}\cdot 2^{3}\cdot 4^{-2}\cdot 6^{-1}\cdot 8^{1}\cdot 12^{2}\cdot 24^{-1}\cdot 27^{1}\cdot 54^{-1}\cdot 108^{1}$
240 (≥ 26) [≥ 200]	$1^{2}\cdot 2^{-1}\cdot 5^{-1}\cdot 10^{1}\cdot 15^{1}\cdot 20^{-1}\cdot 40^{2}\cdot 80^{-1}\cdot 120^{-1}\cdot 240^{1}$ $1^{2}\cdot 2^{-1}\cdot 3^{-1}\cdot 5^{-1}\cdot 6^{1}\cdot 10^{1}\cdot 15^{1}\cdot 40^{-1}\cdot 60^{-1}\cdot 80^{1}\cdot 120^{2}\cdot 240^{-1}$ $3^{-2}\cdot 5^{1}\cdot 6^{5}\cdot 12^{-2}\cdot 20^{-1}\cdot 30^{-1}\cdot 40^{2}\cdot 60^{1}\cdot 80^{-1}\cdot 120^{-1}\cdot 240^{1}$ $1^{2}\cdot 2^{-1}\cdot 5^{-1}\cdot 10^{1}\cdot 15^{1}\cdot 40^{-1}\cdot 60^{-1}\cdot 80^{1}\cdot 120^{2}\cdot 240^{-1}$ $1^{2}\cdot 2^{-1}\cdot 10^{-1}\cdot 15^{-1}\cdot 30^{3}\cdot 40^{2}\cdot 60^{-1}\cdot 80^{-1}\cdot 120^{-1}\cdot 240^{1}$
252 (≥ 9) [≥ 20]	$1^{-1}\cdot 3^{3}\cdot 4^{1}\cdot 6^{-1}\cdot 7^{1}\cdot 9^{-1}\cdot 12^{-1}\cdot 21^{-1}\cdot 28^{-1}\cdot 36^{1}\cdot 42^{-1}\cdot 63^{1}\cdot 84^{3}\cdot 252^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 9^{-1}\cdot 12^{1}\cdot 18^{2}\cdot 21^{1}\cdot 36^{-1}\cdot 42^{-1}\cdot 84^{1}$ $3^{-2}\cdot 6^{5}\cdot 7^{-1}\cdot 12^{-2}\cdot 14^{2}\cdot 21^{2}\cdot 28^{-1}\cdot 42^{-3}\cdot 63^{-1}\cdot 84^{2}\cdot 126^{2}\cdot 252^{-1}$

⋮

288 (≥ 1) [≥ 4]	$1^{-1}\cdot 2^{2}\cdot 8^{-1}\cdot 16^{2}\cdot 18^{1}\cdot 32^{-1}\cdot 36^{-1}\cdot 48^{-1}\cdot 72^{1}\cdot 96^{1}$

⋮

384 (≥ 14) [≥ 48]	$1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 12^{-1}\cdot 16^{-1}\cdot 48^{1}\cdot 64^{2}\cdot 128^{-1}\cdot 192^{-1}\cdot 384^{1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 12^{-1}\cdot 16^{-1}\cdot 32^{1}\cdot 48^{1}\cdot 64^{-1}\cdot 96^{-1}\cdot 128^{1}\cdot 192^{2}\cdot 384^{-1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 12^{-1}\cdot 32^{-1}\cdot 48^{1}\cdot 64^{2}\cdot 128^{-1}\cdot 192^{-1}\cdot 384^{1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 16^{-1}\cdot 24^{-1}\cdot 48^{1}\cdot 64^{2}\cdot 128^{-1}\cdot 192^{-1}\cdot 384^{1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 16^{-1}\cdot 24^{-1}\cdot 32^{1}\cdot 48^{1}\cdot 64^{-1}\cdot 96^{-1}\cdot 128^{1}\cdot 192^{2}\cdot 384^{-1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 6^{2}\cdot 8^{1}\cdot 24^{-1}\cdot 48^{1}\cdot 64^{-1}\cdot 96^{-1}\cdot 128^{1}\cdot 192^{2}\cdot 384^{-1}$ $1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 4^{1}\cdot 6^{2}\cdot 8^{-1}\cdot 12^{-1}\cdot 16^{1}\cdot 24^{1}\cdot 32^{-1}\cdot 48^{-1}\cdot 64^{2}\cdot 96^{1}\cdot 128^{-1}\cdot 192^{-1}\cdot 384^{1}$

⋮

432 (≥ 10) [≥ 40]	$1^{-1}\cdot 2^{2}\cdot 24^{-1}\cdot 27^{1}\cdot 36^{-1}\cdot 48^{1}\cdot 54^{-1}\cdot 72^{2}\cdot 108^{1}\cdot 144^{-1}$ $3^{-1}\cdot 4^{1}\cdot 6^{2}\cdot 8^{-1}\cdot 9^{1}\cdot 12^{-1}\cdot 16^{1}\cdot 18^{-1}\cdot 72^{1}\cdot 144^{-1}\cdot 216^{-1}\cdot 432^{2}$ $1^{-1}\cdot 2^{2}\cdot 9^{1}\cdot 18^{-2}\cdot 24^{-1}\cdot 27^{-2}\cdot 48^{1}\cdot 54^{5}\cdot 72^{2}\cdot 108^{-2}\cdot 144^{-1}$ $3^{-1}\cdot 6^{2}\cdot 8^{-1}\cdot 9^{1}\cdot 12^{-1}\cdot 16^{2}\cdot 18^{-1}\cdot 24^{1}\cdot 48^{-1}\cdot 108^{1}\cdot 216^{-1}\cdot 432^{1}$

⋮

576 (≥ 16) [≥ 44]	$1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 16^{1}\cdot 36^{1}\cdot 48^{-1}\cdot 96^{2}\cdot 192^{-1}\cdot 288^{-1}\cdot 576^{1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 4^{-1}\cdot 6^{-1}\cdot 8^{1}\cdot 32^{-1}\cdot 36^{1}\cdot 48^{-1}\cdot 64^{1}\cdot 72^{-1}\cdot 96^{2}\cdot 144^{1}\cdot 192^{-1}$ $1^{-1}\cdot 2^{2}\cdot 3^{1}\cdot 6^{-1}\cdot 8^{-2}\cdot 12^{-2}\cdot 16^{1}\cdot 24^{6}\cdot 36^{1}\cdot 48^{-2}\cdot 72^{-2}\cdot 96^{-1}\cdot 192^{1}\cdot 288^{2}\cdot 576^{-1}$ $3^{-1}\cdot 4^{-1}\cdot 6^{2}\cdot 8^{3}\cdot 9^{1}\cdot 16^{-1}\cdot 18^{-1}\cdot 24^{-2}\cdot 32^{-1}\cdot 36^{-1}\cdot 64^{1}\cdot 72^{3}\cdot 96^{2}\cdot 144^{-1}\cdot 192^{-1}$ $3^{-1}\cdot 4^{-2}\cdot 6^{2}\cdot 8^{5}\cdot 9^{1}\cdot 16^{-2}\cdot 18^{-1}\cdot 24^{-2}\cdot 72^{1}\cdot 96^{2}\cdot 192^{-1}\cdot 288^{-1}\cdot 576^{1}$

⋮

768 (≥ 1) [≥ 4]	$1^{1}\cdot 2^{-1}\cdot 3^{-1}\cdot 4^{1}\cdot 6^{2}\cdot 12^{-1}\cdot 16^{-1}\cdot 24^{-1}\cdot 48^{3}\cdot 96^{-1}\cdot 128^{2}\cdot 256^{-1}\cdot 384^{-1}\cdot 768^{1}$

The level of a simple holomorphic eta quotient of weight 1 may go up much higher than 768. However, numerically we have also found that there exist no simple holomorphic eta quotients of weight 1 and of the levels 324, 375, 648, 864, 875, 896, 972, 1000, 1152, 1280, 1458, 1536 and 3072.

In particular, considering the highest powers of different primes which divide the levels of the eta quotients appearing in the above list, we get:

Corollary A.1. *Let M_2 be defined as in Theorem 2.40$'$. Then M_2 is divisible by*

$$2^8 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11.$$

Appendix B

Table of holomorphic eta quotients of weight $3/2$

In the following table, we give examples of simple holomorphic eta quotients weight 3/2 and of various levels. As before, for each level N that appears in the tables below, there are three entries on the left column: the first one being N, the second one (in round brackets) being the number of simple holomorphic eta quotients of level N modulo Atkin-Lehner involutions of level N and the third one [in square brackets] being the total number of distinct simple holomorphic eta quotients of level N. Also, for level N, each formal expressions of the type $\prod_{d|N} d^{X_d}$ on the right column denotes a simple holomorphic eta quotient: $\prod_{d|N} \eta_d^{X_d}$ of level N and weight 1. This table is complete up to Atkin-Lehner involutions for the levels $N < 24$ and for $N = 44, 45, 50, 45, 50, 66, 75, 81, 98, 102, 105, 114, 130, 135, 138, 182., 189$. Also, in this table, the list of levels below 132 is complete, i.e., if a level $N < 132$ does not appear in the table, then this is because there are no simple holomorphic eta quotients of weight 3/2 and level N.

	Simple holomorphic eta quotients
9 (1) [1]	$1^{-2} \cdot 3^{7} \cdot 9^{-2}$
12 (3) [12]	$1^{3} \cdot 2^{3} \cdot 3^{-1} \cdot 4^{-2} \cdot 6^{-1} \cdot 12^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{2} \cdot 4^{-2} \cdot 6^{-3} \cdot 12^{5}$
	$1^{-3} \cdot 2^{12} \cdot 3^{1} \cdot 4^{-5} \cdot 6^{-4} \cdot 12^{2}$
14 (3) [12]	$1^{-1} \cdot 2^{1} \cdot 7^{4} \cdot 14^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 7^{5} \cdot 14^{-2}$
	$1^{-1} \cdot 2^{1} \cdot 7^{6} \cdot 14^{-3}$
18 (18) [58]	$1^{-4} \cdot 2^{2} \cdot 3^{13} \cdot 6^{-6} \cdot 9^{-4} \cdot 18^{2}$
	$1^{-1} \cdot 2^{-1} \cdot 3^{3} \cdot 6^{4} \cdot 9^{-1} \cdot 18^{-1}$
	$1^{-1} \cdot 3^{2} \cdot 6^{1} \cdot 9^{3} \cdot 18^{-2}$
	$1^{-1} \cdot 3^{4} \cdot 6^{-1} \cdot 9^{-2} \cdot 18^{3}$
	$1^{-1} \cdot 3^{4} \cdot 6^{-1} \cdot 9^{-3} \cdot 18^{4}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 6^{-1} \cdot 9^{4} \cdot 18^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-2} \cdot 9^{1} \cdot 18^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{5}$
	$1^{-1} \cdot 2^{3} \cdot 3^{1} \cdot 6^{-2} \cdot 9^{-1} \cdot 18^{3}$
	$1^{-1} \cdot 2^{3} \cdot 3^{3} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{3}$
	$1^{-2} \cdot 2^{2} \cdot 3^{9} \cdot 6^{-5} \cdot 9^{-3} \cdot 18^{2}$
	$1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 6^{-2} \cdot 9^{2}$
	$1^{-2} \cdot 2^{5} \cdot 3^{1} \cdot 6^{-2} \cdot 18^{1}$
	$1^{-3} \cdot 2^{1} \cdot 3^{10} \cdot 6^{-3} \cdot 9^{-3} \cdot 18^{1}$
	$1^{-3} \cdot 2^{2} \cdot 3^{10} \cdot 6^{-5} \cdot 9^{-3} \cdot 18^{2}$
	$1^{-3} \cdot 2^{2} \cdot 3^{9} \cdot 6^{-4} \cdot 9^{-3} \cdot 18^{2}$
	$1^{-3} \cdot 2^{5} \cdot 3^{3} \cdot 6^{-3} \cdot 18^{1}$
	$1^{-4} \cdot 2^{2} \cdot 3^{14} \cdot 6^{-7} \cdot 9^{-4} \cdot 18^{2}$
20 (18) [72]	$1^{-6} \cdot 2^{15} \cdot 4^{-6} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{2}$
	$1^{-1} \cdot 4^{1} \cdot 5^{4} \cdot 20^{-1}$
	$1^{-1} \cdot 4^{1} \cdot 5^{5} \cdot 10^{-2}$
	$1^{-1} \cdot 4^{1} \cdot 5^{5} \cdot 10^{-3} \cdot 20^{1}$
	$1^{-1} \cdot 4^{1} \cdot 5^{6} \cdot 10^{-3}$
	$1^{-1} \cdot 4^{2} \cdot 5^{3} \cdot 20^{-1}$
	$1^{-1} \cdot 4^{4} \cdot 5^{-1} \cdot 10^{3} \cdot 20^{-2}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 10^{-2} \cdot 20^{5}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 10^{-3} \cdot 20^{6}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{5}$
	$1^{-1} \cdot 2^{6} \cdot 4^{-3} \cdot 10^{-2} \cdot 20^{3}$
	$1^{-2} \cdot 2^{3} \cdot 4^{1} \cdot 5^{3} \cdot 10^{-2}$
	$1^{-2} \cdot 2^{6} \cdot 4^{-3} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{4}$
	$1^{-3} \cdot 2^{7} \cdot 4^{-3} \cdot 5^{2} \cdot 10^{-3} \cdot 20^{3}$
	$1^{-3} \cdot 2^{9} \cdot 4^{-4} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{3}$
	$1^{-4} \cdot 2^{12} \cdot 4^{-5} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{2}$
	$1^{-4} \cdot 2^{12} \cdot 4^{-5} \cdot 5^{2} \cdot 10^{-3} \cdot 20^{1}$
	$1^{-5} \cdot 2^{13} \cdot 4^{-5} \cdot 5^{1} \cdot 10^{-3} \cdot 20^{2}$
21 (1) [4]	$1^{-1} \cdot 3^{2} \cdot 7^{3} \cdot 21^{-1}$
24 (196) [784]	$1^{5} \cdot 2^{-2} \cdot 3^{-2} \cdot 6^{1} \cdot 12^{2} \cdot 24^{-1}$
	$1^{-1} \cdot 2^{-1} \cdot 3^{-1} \cdot 4^{8} \cdot 6^{4} \cdot 8^{-3} \cdot 12^{-4} \cdot 24^{1}$
	$1^{-1} \cdot 2^{2} \cdot 3^{2} \cdot 4^{-4} \cdot 6^{-3} \cdot 8^{5} \cdot 12^{5} \cdot 24^{-3}$
	$1^{-1} \cdot 2^{2} \cdot 3^{2} \cdot 4^{2} \cdot 6^{-1} \cdot 8^{-2} \cdot 12^{-2} \cdot 24^{3}$
	$1^{-1} \cdot 2^{2} \cdot 3^{3} \cdot 4^{-2} \cdot 6^{-4} \cdot 8^{1} \cdot 12^{7} \cdot 24^{-3}$
	$1^{-5} \cdot 2^{12} \cdot 3^{2} \cdot 4^{-5} \cdot 6^{-4} \cdot 8^{1} \cdot 12^{2}$
	$1^{-5} \cdot 2^{12} \cdot 3^{3} \cdot 4^{-5} \cdot 6^{-5} \cdot 12^{2} \cdot 24^{1}$
	$1^{-5} \cdot 2^{13} \cdot 3^{2} \cdot 4^{-5} \cdot 6^{-4} \cdot 12^{1} \cdot 24^{1}$
28 (33) [120]	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 28^{2}$
	$1^{-1} \cdot 4^{4} \cdot 14^{1} \cdot 28^{-1}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 14^{-1} \cdot 28^{3}$
	$1^{-1} \cdot 2^{1} \cdot 7^{5} \cdot 14^{-3} \cdot 28^{1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{-1} \cdot 7^{1} \cdot 14^{2} \cdot 28^{-1}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 14^{-2} \cdot 28^{4}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 7^{1} \cdot 14^{-1} \cdot 28^{2}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 7^{1} \cdot 14^{-2} \cdot 28^{3}$
	$1^{-1} \cdot 2^{7} \cdot 4^{-3} \cdot 14^{-1} \cdot 28^{1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{1} \cdot 7^{1} \cdot 14^{-1}$

	Simple holomorphic eta quotients
30 (90) [720]	$1^{-2} \cdot 2^{1} \cdot 3^{5} \cdot 5^{1} \cdot 6^{-2}$ $1^{-1} \cdot 5^{5} \cdot 6^{1} \cdot 10^{-1} \cdot 15^{-1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 10^{-1} \cdot 15^{4} \cdot 30^{-2}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 6^{1} \cdot 10^{-1} \cdot 30^{1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 6^{1} \cdot 10^{-1} \cdot 15^{1}$ $1^{1} \cdot 2^{-2} \cdot 3^{-1} \cdot 5^{-4} \cdot 6^{2} \cdot 10^{8} \cdot 15^{1} \cdot 30^{-2}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 6^{1} \cdot 10^{-1} \cdot 15^{2} \cdot 30^{-1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 6^{1} \cdot 15^{-1} \cdot 30^{1}$ $1^{-2} \cdot 2^{2} \cdot 3^{5} \cdot 6^{-3} \cdot 30^{1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 6^{1}$
36 (1096) [4013]	$1^{-2} \cdot 3^{7} \cdot 4^{1} \cdot 9^{-2} \cdot 12^{-2} \cdot 36^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{5} \cdot 12^{-1} \cdot 18^{-1} \cdot 36^{1}$ $1^{-2} \cdot 3^{7} \cdot 4^{2} \cdot 6^{-3} \cdot 9^{-2} \cdot 12^{-1} \cdot 36^{2}$ $1^{-2} \cdot 3^{7} \cdot 4^{2} \cdot 6^{-4} \cdot 9^{-4} \cdot 18^{6} \cdot 36^{-2}$ $1^{-2} \cdot 2^{1} \cdot 3^{4} \cdot 4^{1} \cdot 6^{1} \cdot 12^{-2} \cdot 18^{-1} \cdot 36^{1}$ $1^{1} \cdot 2^{-1} \cdot 3^{-2} \cdot 4^{1} \cdot 6^{4} \cdot 9^{2} \cdot 12^{-2} \cdot 18^{-3} \cdot 36^{3}$ $2^{1} \cdot 3^{-3} \cdot 6^{6} \cdot 9^{3} \cdot 12^{-3} \cdot 18^{-4} \cdot 36^{3}$ $1^{1} \cdot 3^{-1} \cdot 4^{1} \cdot 6^{1} \cdot 9^{1} \cdot 12^{-1} \cdot 36^{1}$ $1^{1} \cdot 2^{-1} \cdot 4^{2} \cdot 9^{1} \cdot 12^{-1} \cdot 36^{1}$
40 (144) [576]	$1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 8^{1}$ $1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-2} \cdot 40^{2}$ $1^{-1} \cdot 2^{1} \cdot 4^{6} \cdot 8^{-3} \cdot 20^{-1} \cdot 40^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{3} \cdot 10^{-2} \cdot 20^{1} \cdot 40^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 5^{3} \cdot 10^{-3} \cdot 20^{4} \cdot 40^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 5^{1} \cdot 8^{2} \cdot 10^{-1} \cdot 20^{4} \cdot 40^{-2}$ $1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 5^{1} \cdot 8^{2} \cdot 20^{2} \cdot 40^{-1}$ $1^{-1} \cdot 2^{2} \cdot 5^{-1} \cdot 8^{1} \cdot 10^{4} \cdot 20^{-2}$ $1^{-1} \cdot 2^{2} \cdot 8^{1} \cdot 10^{-1} \cdot 20^{4} \cdot 40^{-2}$ $1^{-2} \cdot 2^{4} \cdot 4^{-2} \cdot 5^{2} \cdot 8^{1}$
42 (40) [320]	$1^{-1} \cdot 2^{1} \cdot 6^{1} \cdot 7^{3} \cdot 14^{-1}$ $1^{-1} \cdot 3^{3} \cdot 7^{1} \cdot 21^{-1} \cdot 42^{1}$ $1^{-1} \cdot 2^{1} \cdot 6^{1} \cdot 7^{4} \cdot 14^{-2} \cdot 21^{-2} \cdot 42^{2}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 6^{-1} \cdot 7^{4} \cdot 14^{-2} \cdot 21^{-3} \cdot 42^{4}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 6^{-1} \cdot 7^{4} \cdot 14^{-3} \cdot 21^{-3} \cdot 42^{5}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 14^{2} \cdot 21^{1} \cdot 42^{-1}$ $1^{-1} \cdot 2^{1} \cdot 6^{1} \cdot 7^{4} \cdot 14^{-2}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 7^{1} \cdot 42^{1}$
44 (3) [12]	$1^{-2} \cdot 2^{4} \cdot 4^{-1} \cdot 11^{4} \cdot 22^{-2}$ $1^{-1} \cdot 2^{1} \cdot 4^{1} \cdot 11^{4} \cdot 22^{-2}$ $1^{-4} \cdot 2^{10} \cdot 4^{-4} \cdot 11^{1} \cdot 22^{-2} \cdot 44^{2}$
45 (2) [4]	$1^{1} \cdot 3^{-1} \cdot 9^{1} \cdot 15^{2}$ $1^{-1} \cdot 3^{3} \cdot 9^{-1} \cdot 15^{2}$
48 (2132) [8448]	$1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 8^{-1} \cdot 16^{1}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 6^{-2} \cdot 12^{2} \cdot 16^{1} \cdot 24^{-1}$ $1^{-2} \cdot 2^{3} \cdot 3^{1} \cdot 4^{2} \cdot 8^{-2} \cdot 12^{-2} \cdot 24^{2} \cdot 48^{1}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 4^{1} \cdot 6^{-2} \cdot 8^{-2} \cdot 16^{1} \cdot 24^{4} \cdot 48^{-2}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 4^{1} \cdot 6^{-1} \cdot 8^{-3} \cdot 12^{-2} \cdot 16^{1} \cdot 24^{7} \cdot 48^{-3}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 4^{2} \cdot 6^{-1} \cdot 8^{-2} \cdot 12^{-1} \cdot 16^{1} \cdot 48^{1}$ $1^{-2} \cdot 2^{3} \cdot 3^{2} \cdot 4^{2} \cdot 6^{-2} \cdot 8^{-2} \cdot 16^{1} \cdot 48^{1}$ $1^{-2} \cdot 2^{3} \cdot 3^{3} \cdot 4^{-1} \cdot 6^{-2} \cdot 8^{1} \cdot 16^{1}$ $1^{-2} \cdot 2^{3} \cdot 3^{3} \cdot 4^{-1} \cdot 12^{-1} \cdot 48^{1}$ $1^{-2} \cdot 2^{6} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-2} \cdot 48^{1}$
50 (2) [8]	$1^{-1} \cdot 2^{1} \cdot 5^{5} \cdot 10^{-3} \cdot 25^{-2} \cdot 50^{3}$ $2^{1} \cdot 5^{-1} \cdot 10^{1} \cdot 25^{2}$
54 (124) [496]	$1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 9^{-1} \cdot 54^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 9^{-1} \cdot 27^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{4} \cdot 6^{-2} \cdot 9^{-1} \cdot 54^{2}$ $1^{-1} \cdot 2^{1} \cdot 3^{4} \cdot 6^{-2} \cdot 9^{-1} \cdot 27^{-1} \cdot 54^{3}$ $1^{-1} \cdot 2^{1} \cdot 3^{5} \cdot 6^{-2} \cdot 9^{-2} \cdot 18^{1} \cdot 27^{2} \cdot 54^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{6} \cdot 6^{-3} \cdot 9^{-2} \cdot 18^{1} \cdot 54^{1}$ $1^{-1} \cdot 2^{2} \cdot 3^{-1} \cdot 6^{2} \cdot 18^{-1} \cdot 27^{2}$ $1^{-1} \cdot 2^{2} \cdot 6^{-1} \cdot 18^{2} \cdot 27^{3} \cdot 54^{-2}$ $1^{-2} \cdot 2^{4} \cdot 9^{1} \cdot 18^{-1} \cdot 27^{1}$

	Simple holomorphic eta quotients
56 (36) [144]	$1^{-1} \cdot 2^{1} \cdot 4^{1} \cdot 7^{2} \cdot 28^{-1} \cdot 56^{1}$
	$2^{3} \cdot 4^{-2} \cdot 7^{1} \cdot 8^{1} \cdot 14^{-1} \cdot 56^{1}$
	$1^{-2} \cdot 2^{6} \cdot 4^{-3} \cdot 7^{1} \cdot 8^{1} \cdot 14^{-2} \cdot 28^{2}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-3} \cdot 8^{3} \cdot 14^{-1} \cdot 28^{4} \cdot 56^{-2}$
	$1^{-3} \cdot 2^{7} \cdot 4^{-3} \cdot 7^{2} \cdot 8^{1} \cdot 14^{-2} \cdot 28^{1}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 14^{-1} \cdot 28^{2} \cdot 56^{1}$
	$2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 8^{1} \cdot 14^{-1} \cdot 56^{1}$
60 (≥ 24) [≥ 172]	$1^{-5} \cdot 2^{12} \cdot 3^{2} \cdot 4^{-5} \cdot 5^{1} \cdot 6^{-4} \cdot 10^{-2} \cdot 12^{2} \cdot 20^{1} \cdot 30^{1}$
	$1^{-5} \cdot 2^{12} \cdot 3^{2} \cdot 4^{-5} \cdot 5^{2} \cdot 6^{-4} \cdot 10^{-4} \cdot 12^{2} \cdot 15^{-2} \cdot 20^{2} \cdot 30^{4} \cdot 60^{-1}$
	$1^{-8} \cdot 2^{20} \cdot 3^{2} \cdot 4^{-8} \cdot 5^{2} \cdot 6^{-5} \cdot 10^{-5} \cdot 12^{2} \cdot 15^{-2} \cdot 20^{2} \cdot 30^{5} \cdot 60^{-2}$
	$1^{-5} \cdot 2^{12} \cdot 3^{1} \cdot 4^{-4} \cdot 5^{1} \cdot 6^{-3} \cdot 10^{-3} \cdot 12^{1} \cdot 15^{1} \cdot 20^{2} \cdot 30^{1} \cdot 60^{-1}$
	$1^{-5} \cdot 2^{12} \cdot 3^{1} \cdot 4^{-5} \cdot 5^{2} \cdot 6^{-3} \cdot 10^{-2} \cdot 12^{2} \cdot 20^{1}$
	$1^{-5} \cdot 2^{12} \cdot 3^{2} \cdot 4^{-5} \cdot 6^{-3} \cdot 10^{-1} \cdot 12^{1} \cdot 20^{2}$
64 (29) [52]	$1^{-1} \cdot 2^{3} \cdot 8^{-1} \cdot 32^{1} \cdot 64^{1}$
	$1^{-1} \cdot 2^{4} \cdot 4^{-1} \cdot 8^{-1} \cdot 16^{1} \cdot 64^{1}$
	$1^{-1} \cdot 2^{3} \cdot 4^{-3} \cdot 8^{3} \cdot 16^{2} \cdot 32^{-2} \cdot 64^{1}$
	$1^{-1} \cdot 2^{3} \cdot 8^{-1} \cdot 16^{-1} \cdot 32^{4} \cdot 64^{-1}$
	$1^{1} \cdot 2^{-2} \cdot 4^{3} \cdot 8^{-1} \cdot 16^{3} \cdot 32^{-2} \cdot 64^{1}$
	$1^{1} \cdot 2^{-2} \cdot 4^{4} \cdot 8^{-2} \cdot 32^{1} \cdot 64^{1}$
	$1^{1} \cdot 4^{1} \cdot 8^{-1} \cdot 32^{1} \cdot 64^{1}$
66 (15) [120]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{3} \cdot 22^{-2} \cdot 33^{-2} \cdot 66^{3}$
	$1^{-1} \cdot 3^{3} \cdot 11^{1} \cdot 33^{-1} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 6^{1} \cdot 11^{6} \cdot 22^{-3} \cdot 33^{-2} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{2} \cdot 22^{-1} \cdot 33^{1} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{2} \cdot 22^{-1} \cdot 33^{2} \cdot 66^{-1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{2} \cdot 33^{-1} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{2} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{3} \cdot 22^{-1} \cdot 33^{-1} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{3} \cdot 22^{-1} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{3} \cdot 22^{-2} \cdot 33^{-1} \cdot 66^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{4} \cdot 22^{-2} \cdot 33^{-1} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{4} \cdot 22^{-2} \cdot 33^{-2} \cdot 66^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{4} \cdot 22^{-2} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 11^{6} \cdot 22^{-3} \cdot 33^{-2} \cdot 66^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{3} \cdot 6^{-2} \cdot 11^{1} \cdot 22^{-2} \cdot 33^{-3} \cdot 66^{6}$
70 (≥ 7) [≥ 56]	$1^{4} \cdot 2^{-2} \cdot 7^{-1} \cdot 14^{1} \cdot 35^{1} \cdot$
	$2^{2} \cdot 5^{1} \cdot 10^{-1} \cdot 70^{1}$
	$1^{1} \cdot 2^{-1} \cdot 7^{-1} \cdot 10^{1} \cdot 14^{3} \cdot 35^{1} \cdot 70^{-1}$
	$1^{1} \cdot 2^{-1} \cdot 5^{1} \cdot 7^{-2} \cdot 14^{4} \cdot$
	$1^{1} \cdot 2^{-1} \cdot 5^{1} \cdot 7^{-2} \cdot 14^{4} \cdot 35^{1} \cdot 70^{-1}$
	$1^{1} \cdot 5^{-1} \cdot 7^{-1} \cdot 10^{1} \cdot 14^{1} \cdot 35^{4} \cdot 70^{-2}$
	$1^{1} \cdot 5^{-1} \cdot 10^{1} \cdot 35^{3} \cdot 70^{-1}$
72 (≥ 20) [≥ 80]	$1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-4} \cdot 12^{4} \cdot 24^{-1} \cdot 36^{2} \cdot 72^{-1}$
	$1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-4} \cdot 12^{5} \cdot 24^{-2} \cdot 36^{-1} \cdot 72^{2}$
	$1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-6} \cdot 12^{9} \cdot 18^{2} \cdot 24^{-3} \cdot 36^{-3} \cdot 72^{1}$
	$1^{-4} \cdot 2^{8} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{1} \cdot 8^{-1} \cdot 12^{-1} \cdot 18^{-1} \cdot 72^{1}$
	$1^{-4} \cdot 2^{8} \cdot 3^{1} \cdot 4^{1} \cdot 6^{-2} \cdot 8^{-2} \cdot 12^{-1} \cdot 24^{1} \cdot 36^{2} \cdot 72^{-1}$
	$1^{-4} \cdot 2^{8} \cdot 3^{1} \cdot 4^{1} \cdot 6^{-2} \cdot 8^{-2} \cdot 36^{-1} \cdot 72^{2}$
	$1^{-4} \cdot 2^{9} \cdot 4^{-2} \cdot 6^{1} \cdot 8^{-1} \cdot 9^{1} \cdot 12^{-2} \cdot 18^{-2} \cdot 24^{2} \cdot 36^{2} \cdot 72^{-1}$
	$1^{-4} \cdot 2^{9} \cdot 3^{1} \cdot 4^{-1} \cdot 6^{-2} \cdot 8^{-2} \cdot 12^{-2} \cdot 18^{-1} \cdot 24^{4} \cdot 36^{3} \cdot 72^{-2}$
75 (3) [8]	$1^{-1} \cdot 3^{1} \cdot 5^{4} \cdot 15^{-1} \cdot 25^{-1} \cdot 75^{1}$
	$1^{-1} \cdot 3^{3} \cdot 5^{1} \cdot 15^{-1} \cdot 75^{1}$
	$3^{1} \cdot 5^{2} \cdot 15^{-1} \cdot 75^{1}$
78 (≥ 7) [≥ 56]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{3} \cdot 26^{-1} \cdot 39^{-1} \cdot 78^{1}$
	$1^{-1} \cdot 3^{3} \cdot 13^{1} \cdot 39^{-1} \cdot 78^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{3} \cdot 26^{-1} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{4} \cdot 26^{-2} \cdot 39^{-1} \cdot 78^{1}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{4} \cdot 26^{-2} \cdot 39^{-2} \cdot 78^{2}$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{4} \cdot 26^{-2} \cdot$
	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 13^{6} \cdot 26^{-3} \cdot 39^{-2} \cdot 78^{1}$
80 (≥ 19) [≥ 72]	$1^{-1} \cdot 2^{2} \cdot 5^{-1} \cdot 8^{-1} \cdot 10^{3} \cdot 16^{1} \cdot 20^{-1} \cdot 80^{1}$
	$1^{-1} \cdot 2^{2} \cdot 5^{-1} \cdot 8^{-1} \cdot 10^{3} \cdot 16^{1} \cdot 20^{-2} \cdot 40^{3} \cdot 80^{-1}$
	$1^{-1} \cdot 2^{2} \cdot 5^{-1} \cdot 8^{-2} \cdot 10^{3} \cdot 16^{3} \cdot 20^{-1} \cdot 40^{1} \cdot 80^{-1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{-1} \cdot 5^{2} \cdot 8^{-1} \cdot 10^{-1} \cdot 16^{1} \cdot 20^{-1} \cdot 40^{3} \cdot 80^{-1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{-1} \cdot 5^{2} \cdot 8^{-1} \cdot 10^{-2} \cdot 16^{1} \cdot 20^{2} \cdot 40^{1} \cdot 80^{-1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{-2} \cdot 5^{2} \cdot 8^{2} \cdot 10^{-1} \cdot 16^{-1} \cdot 80^{1}$
	$1^{-2} \cdot 2^{4} \cdot 4^{-2} \cdot 5^{2} \cdot 8^{2} \cdot 10^{-2} \cdot 16^{-1} \cdot 20^{3} \cdot 40^{-2} \cdot 80^{1}$
81 (1) [1]	$1^{1} \cdot 3^{-1} \cdot 9^{3} \cdot 27^{-1} \cdot 81^{1}$

	Simple holomorphic eta quotients
84 (≥ 20) $[\geq 160]$	$1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-1} \cdot 14^{1} \cdot 84^{1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-1} \cdot 14^{1} \cdot 42^{2} \cdot 84^{-1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-1} \cdot 14^{1} \cdot 21^{1} \cdot 28^{-1} \cdot 42^{-2} \cdot 84^{3}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-1} \cdot 14^{1} \cdot 21^{1} \cdot 42^{-1} \cdot 84^{1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-2} \cdot 14^{3} \cdot 28^{-1} \cdot 42^{2} \cdot 84^{-1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-2} \cdot 14^{3} \cdot 21^{1} \cdot 28^{-1} \cdot 42^{-1} \cdot 84^{1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{1} \cdot 7^{-2} \cdot 14^{3} \cdot 21^{1} \cdot 28^{-1}.$
88 (≥ 8) $[\geq 32]$	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{-1} \cdot 11^{1} \cdot 44^{2} \cdot 88^{-1}$ $1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{-1} \cdot 11^{1} \cdot 22^{1} \cdot 44^{-1} \cdot 88^{1}$ $1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-1} \cdot 11^{-1} \cdot 22^{3} \cdot 44^{-2} \cdot 88^{1}$ $1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-1} \cdot 11^{1} \cdot 44^{-1} \cdot 88^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 11^{1} \cdot 44^{2} \cdot 88^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 11^{1} \cdot 22^{1} \cdot 44^{-1} \cdot 88^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 11^{1} \cdot 22^{-2} \cdot 44^{4} \cdot 88^{-1}$ $1^{-1} \cdot 2^{4} \cdot 4^{-2} \cdot 8^{1} \cdot 11^{1} \cdot 22^{-1} \cdot 88^{1}$
90 (≥ 20) $[\geq 136]$	$1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 6^{-3} \cdot 9^{-1} \cdot 15^{1} \cdot 18^{1} \cdot 45^{-1} \cdot 90^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 5^{1} \cdot 6^{-3} \cdot 9^{-2} \cdot 15^{-1} \cdot 18^{1} \cdot 45^{1} \cdot 90^{1}$ $1^{-2} \cdot 2^{1} \cdot 3^{6} \cdot 5^{1} \cdot 6^{-3} \cdot 9^{-2} \cdot 15^{-2} \cdot 18^{1} \cdot 30^{1} \cdot 45^{4} \cdot 90^{-2}$ $1^{-2} \cdot 2^{1} \cdot 3^{7} \cdot 6^{-4} \cdot 9^{-2} \cdot 15^{-2} \cdot 18^{1} \cdot 30^{4}.$ $2^{-1} \cdot 3^{-1} \cdot 6^{4} \cdot 9^{2} \cdot 15^{1} \cdot 18^{-2} \cdot 45^{-1} \cdot 90^{1}$
96 (≥ 20) $[\geq 80]$	$1^{4} \cdot 2^{-1} \cdot 3^{-1} \cdot 16^{-1} \cdot 32^{1} \cdot 48^{2} \cdot 96^{-1}$ $1^{4} \cdot 2^{-1} \cdot 3^{-1} \cdot 12^{1} \cdot 16^{-1} \cdot 24^{-1} \cdot 32^{1} \cdot 48^{2} \cdot 96^{-1}$ $1^{4} \cdot 2^{-1} \cdot 3^{-2} \cdot 6^{2} \cdot 8^{-1} \cdot 12^{-1} \cdot 16^{2} \cdot 24^{1} \cdot 32^{-1} \cdot 48^{-1} \cdot 96^{1}$ $1^{4} \cdot 2^{-1} \cdot 3^{-2} \cdot 6^{2} \cdot 8^{-1} \cdot 16^{2} \cdot 32^{-1} \cdot 48^{-1} \cdot 96^{1}$ $1^{4} \cdot 2^{-1} \cdot 3^{-2} \cdot 6^{2} \cdot 12^{-1} \cdot 16^{-1} \cdot 32^{1} \cdot 48^{2} \cdot 96^{-1}$ $1^{4} \cdot 2^{-1} \cdot 3^{-2} \cdot 6^{2} \cdot 16^{-1} \cdot 24^{-1} \cdot 32^{1} \cdot 48^{2} \cdot 96^{-1}$ $1^{5} \cdot 2^{-3} \cdot 3^{-2} \cdot 6^{2} \cdot 16^{2} \cdot 32^{-1} \cdot 48^{-1} \cdot 96^{1}$
98 (8) [28]	$1^{-1} \cdot 2^{1} \cdot 7^{5} \cdot 14^{-2} \cdot 49^{-1} \cdot 98^{1}$ $1^{-1} \cdot 2^{1} \cdot 7^{6} \cdot 14^{-3} \cdot 49^{-1} \cdot 98^{1}$ $1^{-1} \cdot 2^{2} \cdot 7^{1} \cdot 14^{-1} \cdot 49^{-1} \cdot 98^{3}$ $1^{-1} \cdot 2^{2} \cdot 7^{1} \cdot 14^{-1} \cdot 49^{-2} \cdot 98^{4}$ $1^{-1} \cdot 2^{2} \cdot 7^{1} \cdot 14^{-1} \cdot 98^{2}$ $1^{-1} \cdot 2^{3} \cdot 7^{1} \cdot 14^{-1} \cdot 98^{1}$ $1^{-2} \cdot 2^{4} \cdot 7^{1} \cdot 14^{-1} \cdot 98^{1}$ $1^{-2} \cdot 2^{4} \cdot 7^{1} \cdot 14^{-1} \cdot 49^{1}.$
100 (199) [734]	$1^{-1} \cdot 2^{2} \cdot 4^{1} \cdot 10^{1} \cdot 20^{-1} \cdot 25^{1} \cdot 50^{-1} \cdot 100^{1}$ $1^{-1} \cdot 2^{3} \cdot 4^{-1} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{1} \cdot 50^{-1} \cdot 100^{2}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 10^{-2} \cdot 20^{5} \cdot 25^{-1} \cdot 50^{3} \cdot 100^{-2}$ $1^{-1} \cdot 2^{3} \cdot 5^{-1} \cdot 10^{2} \cdot 20^{-1} \cdot 25^{2} \cdot 50^{-1}.$ $1^{-1} \cdot 2^{3} \cdot 10^{-1} \cdot 20^{1} \cdot 25^{2} \cdot 50^{-1}.$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 10^{1} \cdot 20^{-1} \cdot 100^{1}$ $1^{-1} \cdot 2^{2} \cdot 5^{1} \cdot 10^{3} \cdot 20^{-2} \cdot 50^{-1} \cdot 100^{1}$
102 (3) [24]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 17^{4} \cdot 34^{-2} \cdot 51^{-1} \cdot 102^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 17^{4} \cdot 34^{-2}.$ $1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 17^{6} \cdot 34^{-3} \cdot 51^{-2} \cdot 102^{1}$
104 (≥ 19) $[\geq 76]$	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 13^{2}.$ $1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{-1} \cdot 13^{2}.$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 13^{1} \cdot 26^{2} \cdot 52^{-1}.$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 13^{2} \cdot 26^{-1} \cdot 104^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 13^{2} \cdot 26^{-1} \cdot 52^{2} \cdot 104^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 13^{2} \cdot 52^{-1} \cdot 104^{1}$
105 (2) [16]	$1^{3} \cdot 3^{-1} \cdot 5^{-1} \cdot 15^{1} \cdot 35^{1}.$ $1^{-1} \cdot 3^{2} \cdot 5^{1} \cdot 7^{1}.$
108 (≥ 20) $[\geq 80]$	$1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-2} \cdot 9^{-1} \cdot 27^{1} \cdot 36^{3} \cdot 108^{-1}$ $1^{-4} \cdot 2^{10} \cdot 3^{2} \cdot 4^{-4} \cdot 6^{-5} \cdot 9^{-4} \cdot 12^{2} \cdot 18^{10} \cdot 27^{1} \cdot 36^{-4} \cdot 54^{-3} \cdot 108^{2}$ $1^{-4} \cdot 2^{10} \cdot 3^{1} \cdot 4^{-4} \cdot 6^{-2} \cdot 18^{-2} \cdot 27^{-1} \cdot 36^{4} \cdot 54^{3} \cdot 108^{-2}$ $1^{-4} \cdot 2^{8} \cdot 3^{3} \cdot 4^{-3} \cdot 6^{-1} \cdot 9^{-1} \cdot 27^{1} \cdot 54^{-1} \cdot 108^{1}$ $1^{-4} \cdot 2^{10} \cdot 4^{-4} \cdot 6^{-2} \cdot 9^{3} \cdot 12^{1} \cdot 18^{-1} \cdot 27^{-1} \cdot 108^{1}$
112 (≥ 20) $[\geq 80]$	$1^{-1} \cdot 2^{2} \cdot 8^{-1} \cdot 16^{1} \cdot 28^{1} \cdot 56^{2} \cdot 112^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 14^{1} \cdot 16^{1} \cdot 28^{-1} \cdot 112^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 7^{1} \cdot 14^{1} \cdot 16^{1} \cdot 28^{-2} \cdot 56^{3} \cdot 112^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 8^{4} \cdot 14^{1} \cdot 16^{-1} \cdot 56^{-1} \cdot 112^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-2} \cdot 7^{1} \cdot 8^{3} \cdot 14^{1} \cdot 16^{-1} \cdot 28^{-1} \cdot 112^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-3} \cdot 8^{7} \cdot 14^{1} \cdot 16^{-3} \cdot 56^{-1} \cdot 112^{1}$
114 (1) [8]	$1^{1} \cdot 2^{-2} \cdot 3^{-3} \cdot 6^{6} \cdot 38^{1} \cdot 57^{1} \cdot 114^{-1}$

	Simple holomorphic eta quotients
120 (≥ 20) $[\geq 160]$	$1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-1} \cdot 6^{1} \cdot 10^{1} \cdot 20^{-1} \cdot 40^{1} \cdot 60^{1} \cdot$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-1} \cdot 6^{1} \cdot 10^{1} \cdot 60^{2} \cdot 120^{-1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-1} \cdot 6^{1} \cdot 10^{1} \cdot 30^{1} \cdot 60^{-1} \cdot 120^{1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-2} \cdot 6^{1} \cdot 10^{2} \cdot 15^{2} \cdot 20^{-1} \cdot 30^{-1} \cdot 40^{1} \cdot$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-2} \cdot 6^{1} \cdot 10^{3} \cdot 20^{-1} \cdot 60^{2} \cdot 120^{-1}$ $1^{6} \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-1} \cdot 6^{1} \cdot 15^{1} \cdot 20^{2} \cdot 30^{-1} \cdot 40^{-1} \cdot 60^{1} \cdot$
126 (≥ 24) $[\geq 176]$	$1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 9^{-1} \cdot 18^{1} \cdot 21^{1} \cdot 126^{1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-2} \cdot 42^{2} \cdot 63^{2} \cdot$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-3} \cdot 42^{3} \cdot 63^{5} \cdot 126^{-3}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-2} \cdot 7^{1} \cdot 14^{-1} \cdot 18^{2} \cdot 21^{-2} \cdot 42^{3} \cdot 63^{1} \cdot 126^{-1}$ $1^{-1} \cdot 2^{1} \cdot 3^{2} \cdot 6^{-1} \cdot 7^{1} \cdot 9^{-1} \cdot 14^{-1} \cdot 18^{1} \cdot 21^{-1} \cdot 42^{1} \cdot 63^{-1} \cdot 126^{3}$ $1^{-2} \cdot 2^{1} \cdot 3^{5} \cdot 6^{-2} \cdot 7^{2} \cdot 14^{-1} \cdot 21^{-1} \cdot 63^{-1} \cdot 126^{2}$ $2^{-1} \cdot 3^{-1} \cdot 6^{5} \cdot 9^{1} \cdot 18^{-2} \cdot 63^{-1} \cdot 126^{2}$
128 (≥ 23) $[\geq 46]$	$1^{-1} \cdot 2^{1} \cdot 4^{3} \cdot 8^{-3} \cdot 16^{3} \cdot 64^{-1} \cdot 128^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{3} \cdot 16^{-2} \cdot 64^{3} \cdot 128^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{3} \cdot 16^{-3} \cdot 32^{4} \cdot 64^{-2} \cdot 128^{1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{4} \cdot 16^{-2} \cdot 32^{-1} \cdot 64^{3} \cdot 128^{-1}$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{4} \cdot 16^{-2} \cdot 128^{1}$ $1^{-1} \cdot 2^{2} \cdot 8^{1} \cdot 16^{-1} \cdot 32^{-1} \cdot 64^{4} \cdot 128^{-1}$ $1^{-1} \cdot 2^{2} \cdot 8^{1} \cdot 16^{-1} \cdot 64^{1} \cdot 128^{1}$
130 (3) [24]	$5^{-2} \cdot 10^{4} \cdot 13^{-1} \cdot 26^{2} \cdot 65^{1} \cdot 130^{-1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 10^{-1} \cdot 13^{4} \cdot 26^{-2} \cdot 65^{-1} \cdot 130^{1}$ $1^{-1} \cdot 2^{1} \cdot 5^{2} \cdot 13^{2} \cdot 26^{-1} \cdot$

⋮

135 (3) [12]	$1^{1} \cdot 5^{-1} \cdot 15^{3} \cdot 45^{-1} \cdot 135^{1}$ $1^{-1} \cdot 3^{4} \cdot 5^{1} \cdot 9^{-1} \cdot 15^{-1} \cdot 135^{1}$ $3^{1} \cdot 5^{1} \cdot 9^{-1} \cdot 15^{-1} \cdot 27^{1} \cdot 45^{2} \cdot$
136 (≥ 14) $[\geq 56]$	$1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 17^{3} \cdot 34^{-1} \cdot$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 17^{3} \cdot 34^{-2} \cdot 68^{1} \cdot$ $1^{-1} \cdot 2^{2} \cdot 4^{-1} \cdot 8^{1} \cdot 17^{4} \cdot 34^{-2} \cdot$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 34^{-3} \cdot 68^{7} \cdot 136^{-2}$ $1^{-1} \cdot 2^{3} \cdot 4^{-2} \cdot 8^{1} \cdot 34^{-4} \cdot 68^{10} \cdot 136^{-4}$ $1^{-3} \cdot 2^{7} \cdot 4^{-2} \cdot 17^{1} \cdot 34^{-1} \cdot 136^{1}$
138 (1) [8]	$1^{-1} \cdot 2^{1} \cdot 3^{1} \cdot 23^{6} \cdot 46^{-3} \cdot 69^{-2} \cdot 138^{1}$
140 (≥ 1) $[\geq 1]$	$1^{4} \cdot 2^{-2} \cdot 7^{-1} \cdot 14^{2} \cdot 28^{-1} \cdot 70^{-1} \cdot 140^{2}$

⋮

154 (≥ 1) $[\geq 1]$	$7^{-1} \cdot 11^{-2} \cdot 14^{2} \cdot 22^{4} \cdot 77^{1} \cdot 154^{-1}$

⋮

162 (≥ 1) $[\geq 1]$	$2^{-1} \cdot 3^{-1} \cdot 6^{5} \cdot 9^{1} \cdot 18^{-2} \cdot 27^{-1} \cdot 54^{1} \cdot 81^{2} \cdot 162^{-1}$

⋮

182 (2) [16]	$1^{4} \cdot 2^{-2} \cdot 7^{-1} \cdot 14^{1} \cdot 91^{2} \cdot 182^{-1}$ $1^{-1} \cdot 2^{1} \cdot 7^{4} \cdot 13^{1} \cdot 14^{-2} \cdot$
184 (≥ 1) $[\geq 1]$	$1^{-1} \cdot 2^{1} \cdot 4^{2} \cdot 8^{-1} \cdot 23^{4} \cdot 46^{-2} \cdot$
189 (1) [4]	$1^{-1} \cdot 3^{3} \cdot 7^{1} \cdot 9^{-1} \cdot 27^{1}$

	Simple holomorphic eta quotients
192 (≥ 25) [≥ 100]	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 12^2 \cdot 24^{-1} \cdot 192^1$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 4^{-1} \cdot 6^1 \cdot 8^3 \cdot 16^{-2} \cdot 24^{-1} \cdot 32^2 \cdot 48^1 \cdot 64^{-1} \cdot 96^{-1} \cdot 192^1$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 12^2 \cdot 24^{-1} \cdot 48^{-1} \cdot 96^3 \cdot 192^{-1}$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 8^1 \cdot 12^1 \cdot 16^{-1} \cdot 32^2 \cdot 64^{-1} \cdot 96^{-1} \cdot 192^1$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 4^1 \cdot 6^1 \cdot 12^{-1} \cdot 16^{-1} \cdot 24^1 \cdot 32^2 \cdot 64^{-1} \cdot 96^{-1} \cdot 192^1$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 8^1 \cdot 12^1 \cdot 32^{-1} \cdot 48^{-1} \cdot 64^1 \cdot 96^2 \cdot 192^{-1}$
	$1^4 \cdot 2^{-2} \cdot 3^{-1} \cdot 6^1 \cdot 12^{-1} \cdot 24^2 \cdot 48^{-1} \cdot 192^1$

$\vdots$

196 (≥ 21) [≥ 75]	$1^{-1} \cdot 2^2 \cdot 14^1 \cdot 28^{-1} \cdot 98^{-2} \cdot 196^4$
	$1^{-1} \cdot 2^1 \cdot 7^4 \cdot 28^{-1} \cdot 49^{-1} \cdot 98^1 \cdot$
	$1^{-1} \cdot 2^2 \cdot 7^1 \cdot 14^{-2} \cdot 28^1 \cdot 49^{-4} \cdot 98^{10} \cdot 196^{-4}$
	$1^{-2} \cdot 2^5 \cdot 4^{-2} \cdot 14^{-1} \cdot 28^1 \cdot 49^2 \cdot 98^{-1} \cdot 196^1$
	$1^{-2} \cdot 2^5 \cdot 4^{-2} \cdot 14^{-1} \cdot 28^1 \cdot 49^2 \cdot$
	$1^{-2} \cdot 2^5 \cdot 4^{-2} \cdot 14^{-1} \cdot 28^1 \cdot 49^2 \cdot 98^1 \cdot 196^{-1}$
	$1^{-2} \cdot 2^5 \cdot 4^{-2} \cdot 14^{-1} \cdot 28^1 \cdot 49^3 \cdot 98^{-1} \cdot$

$\vdots$

200 (≥ 24) [≥ 96]	$1^{-1} \cdot 2^2 \cdot 4^{-1} \cdot 8^1 \cdot 10^1 \cdot 20^1 \cdot 25^{-1} \cdot 40^{-1} \cdot 50^2 \cdot$
	$1^{-1} \cdot 2^2 \cdot 4^{-1} \cdot 8^1 \cdot 10^1 \cdot 20^1 \cdot 40^{-1} \cdot 50^{-1} \cdot 100^2 \cdot$
	$1^{-1} \cdot 2^2 \cdot 5^{-1} \cdot 10^1 \cdot 20^3 \cdot 40^{-2} \cdot 100^{-1} \cdot 200^2$
	$1^{-1} \cdot 2^2 \cdot 5^{-1} \cdot 10^3 \cdot 20^{-3} \cdot 40^2 \cdot 50^{-2} \cdot 100^5 \cdot 200^{-2}$
	$1^{-1} \cdot 2^2 \cdot 5^{-2} \cdot 10^5 \cdot 20^{-3} \cdot 25^1 \cdot 40^1 \cdot 50^{-2} \cdot 100^3 \cdot 200^{-1}$
	$1^{-1} \cdot 2^2 \cdot 10^{-1} \cdot 20^2 \cdot 25^{-1} \cdot 40^{-1} \cdot 50^4 \cdot 100^{-2} \cdot 200^1$
	$1^{-1} \cdot 2^2 \cdot 20^{-1} \cdot 25^{-1} \cdot 40^1 \cdot 50^3 \cdot 100^1 \cdot 200^{-1}$

Though the above list ends at level 200, but there are levels higher than 200 for which simple holomorphic eta quotients of weight 3/2 exist.

In particular, considering the highest powers of different primes which divide the levels of the eta quotients appearing in the above list, we get:

Corollary B.1. *Let M_3 be defined as in Theorem 2.40$'$. Then M_3 is divisible by*

$$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23.$$

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Summary

This thesis is about factorization and irreducibility of holomorphic eta quotients. We observe that whenever an eta quotient of level N is reducible, it has a special type of factor, viz. a factor whose level is a divisor of N. We call this observation the Reducibility Conjecture. We prove this conjecture for prime power levels. As a corollary to it, we obtain that rescalings and Atkin-Lehner involutions of irreducible holomorphic eta quotients of prime power levels are irreducible. We show that if the Reducibility Conjecture holds, then similar consequences regarding rescalings and Atkin-Lehner involutions of irreducible holomorphic eta quotients of arbitrary levels should also hold. We prove two partial results towards the Reducibility conjecture, one of which can always be used to trim the levels of the factors of a reducible holomorphic eta quotient off the divisors those are prime to N and the other one is a generalization of the method with which we prove the theorem for prime power levels. In particular, the Reducibility Conjecture implies an irreducibility-checking algorithm. We show that even without assuming the Reducibility Conjecture, irreducibility of a holomorphic eta quotient can be checked – as we show using Mersmann's finiteness theorem on simple holomorphic eta quotients that the level of any factor of a holomorphic eta quotient f can be bounded w. r. t. the weight and the level of f. Then we give respectively simplified and short proofs of the Mersmann's finiteness theorem on simple holomorphic eta quotients of a fixed weight and Mersmann's Second Theorem which asserts that Zagier's list of simple holomorphic eta quotients of weight $1/2$ is exhaustive. We also prove that there exists only finitely many simple holomorphic eta quotients of a fixed level and we give an explicit bound on the weights of such eta quotients. Next we classify all simple holomorphic eta quotients of prime square levels and conjecture that there are no simple holomorphic eta quotient of any prime cube level. Finally, we construct examples of infinite families of simple holomorphic eta quotients of cubefree levels and of higher prime power levels.