Essays on Multidimensional Information

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[1] This thesis chapter is based on a paper under the same title that is joint work with Inga Deimen and Mark T. Le Quement, both from the University of Bonn.

[2] This thesis chapter is based on a paper under the same title that is joint work with Dezső Szalay from the University of Bonn and CEPR.

Chapter 1

Introduction

This thesis consists of three independent contributions in the area of microeconomic theory which constitute Chapters 2-4. All three contributions share the feature of agents holding multidimensional private information which motivates the title of the thesis.

Chapter 2, entitled "Consistency and Communication in Committees", is based on a working paper under the same title which is joint work with Inga Deimen and Mark T. Le Quement, both from the University of Bonn. In this paper, we generalize the classical binary Condorcet jury model by introducing a richer state and signal space, thereby generating a concern for consistency in the evaluation of aggregate information. Information is of multidimensional nature here as it does not only provide information about the defendant being guilty or innocent but also indicates a particular modality of guilt or innocence. As coherent evidence for a particular variant of either guilt or innocence is more meaningful than if that information were dispersed, the consistency of signals constitutes a second informative dimension besides the mere number of signals indicating guilt or innocence when deciding whether to acquit or convict the defendant. We use this approach to analyze truthtelling incentives in pre-vote communication in heterogeneous committees and find that full pooling of information followed by sincere voting is compatible with a positive probability of ex post conflict in the committee. This is in stark contrast to the benchmark finding for the binary signal model in Coughlan (2000). We furthermore characterize implementability conditions for a wide class of decision rules including the rule that maximizes committee welfare.

Chapters 3 and 4 analyze more classical instances of multidimensional information such as different production costs or different valuations for different goods which give rise to multidimensional screening problems. Both chapters have in common that they provide explicit characterizations of the optimal mechanisms, a feature that is often not achievable in the multidimensional screening literature.

Under the title "A Baseline Model of Multidimensional Screening", Chapter 3 generalizes upon a model of multidimensional screening from Armstrong and Rochet (1999) by allowing for interaction between the dimensions through the utility function. Identifying sets of binding incentive constraints with properties of the allocation I provide a full classification of the potential solutions to this model. While a large variety of solutions can be generated within the modeling approach, I show that a substantial combination of interaction and asymmetry between dimensions is necessary to distort allocations away from classical properties such as "no distortion at the top" and "no upward binding incentive constraints". Some fundamental properties of the optimal allocation are shown to carry over from the first best allocation to the case of private information. The chapter is framed as an analysis of multiproduct monopoly regulation. Its focus, however, is not so much on a particular application but on the general properties of the underlying model which applies to various contexts.

Chapter 4 is entitled "Pricing a Package of Services - When (not) to Bundle" and is based on a joint paper under the same title with Dezső Szalay from the University of Bonn and CEPR. In this paper, we study a tractable two-dimensional model of price discrimination. Consumers combine a rigid with a more flexible choice, such as choosing the location of a house and its quality or size. We show that the optimal pricing scheme involves no bundling, that is, additively separable prices if consumer types are affiliated. Conversely, if consumer types are negatively affiliated over some portion of types then some bundling occurs.

Chapter 2

Consistency and Communication in Committees[1]

2.1 Introduction

This thesis chapter considers a deliberation and voting model in which rich state and signal spaces combine with a binary action space. A committee consisting of privately informed agents with known heterogeneous preferences engages in simultaneous information exchange prior to voting. Our information structure generates a concern for consistency in the aggregation of individual signals; a given signal is interpreted differently depending on how it matches other available evidence. We find that in contrast to the classical model featuring binary state and signal spaces, full information sharing and sincere voting can constitute an equilibrium although agents with some probability disagree ex post.

Consider the example of a jury aiming at determining whether a defendant is guilty or innocent. If guilty, he must have committed the crime at some point in time, for example on any particular day of a given week. If innocent, at the moment of the crime he must have been somewhere else than at the crime scene; for example at home, at work, at the sports club, or at the grocery store. Conditional on the defendant being guilty, only one day of the week can be the day of the crime. Conditional on him being innocent, he can only have been in one particular place at the moment of the crime. Different days of the week constitute mutually exclusive variants of the guilty state while different locations constitute mutually exclusive variants of the innocent state.

Jurors gather evidence through a trial hearing which generates private signals. Before deciding whether to acquit or convict, jurors retire to deliberate and share their private signals. Consider two possible scenarios by the end of the deliberation. In the first scenario, half of the jurors have received a signal indicating Monday as the moment of the crime, while the other half has received a signal indicating Wednesday. In the second scenario, all jurors have received a signal indicating Monday. The difference

[1]This thesis chapter is based on a paper under the same title that is joint work with Inga Deimen and Mark T. Le Quement, both from the University of Bonn.

between these two scenarios is that the second is more consistent than the first and therefore provides more convincing evidence of guilt. In the first scenario, half of the signals must be wrong: the crime happened either on Monday or on Wednesday. In the second scenario, all signals are correct if the crime happened on Monday. Similarly, one could consider scenarios involving different alibis that vary in their consistency. Jurors do not as such care about the time at which the crime was committed or about where the defendant was if innocent. Jurors simply wish to establish with sufficient certainty whether the defendant is guilty or innocent and more consistent profiles yield stronger evidence.

The core elements of the above description apply to many other situations. A second example is that of a group of investment bankers that considers to invest in shares of a large manufacturer, for example Chrysler. In order to do so, committee members need to estimate whether Chrysler will avoid bankruptcy in the near future. This may happen if either the US Federal State provides a bailout package or if some private company (for example Fiat) decides to acquire a large part of the shares. On the other hand, if Chrysler does go bankrupt, this may happen according to a variety of scenarios, e.g. according to different chapters of the bankruptcy code. Another example is that of a board of directors that seeks to predict whether a Democrat or a Republican will win the next US presidential election, this fact affecting a variety of key variables of the US economy. Different Democratic (Republican) candidates constitute different variants of the Democratic (Republican) state.

We incorporate the key features of the above examples into a model of collective decision making. There are two basic states, each of which is split into a set of substates. Each signal is informative with respect to a basic state and a particular substate. It follows that when considering a set of signals, the consistency of this set matters. The more signals indicate a particular substate, the higher the evidence of the corresponding basic state. Members of a heterogeneous committee, call them hawks and doves, have the possibility to share their private signals via cheap talk before voting on a binary outcome. In contrast to results obtained in the classical binary signal setup,[2] we find that the truthful communication and sincere voting equilibrium (TS equilibrium) is virtually always compatible with a positive probability of ex post conflict among agents. Our main result, Theorem 1, provides necessary and sufficient conditions for the existence of the TS equilibrium that are satisfied for a large set of parameter values. Theorem 2 generalizes Theorem 1 to the case of more than two preference types. Theorem 3 provides necessary and sufficient conditions for the implementability of a wide class of decision rules including the rule that maximizes committee welfare. No rule belonging to this class is implementable in the classical model in the presence of substantial heterogeneity in preferences. Finally, we find that partially revealing equilibria can welfare dominate the TS equilibrium and that sequential communication protocols may serve as an equilibrium selection device.

The intuition for the existence of the TS equilibrium comes out clearly when com-

[2]See Coughlan (2000), Austen-Smith and Feddersen (2006), Meirowitz (2007), Van Weelden (2008).

pared to the dynamics of the classical binary signal model.[3] In the latter, in the putative TS equilibrium, pivotality in the communication stage pins down uniquely the information held by the remaining committee members. Disagreement among agents about the optimal decision rule furthermore implies that there is always at least one agent for whom this pivotal profile implies a suboptimal decision on the equilibrium path. Consequently, this agent profitably deviates and bends the decision rule in his favored direction. In our model, the set of pivotal profiles is not a singleton anymore. A multiplicity of signal profiles can yield similar conditional probabilities of guilt because the posterior probability of guilt depends on two aspects, the total number of signals indicating respectively guilt or innocence as well as the consistency among signals indicating any given basic state. A smaller total number of guilty signals can be compensated by a higher degree of consistency among guilty signals. The multiplicity of pivotal scenarios in turn allows two favorable effects to come into play. First, for a subset of pivotal signal profiles, all agents may agree with the decision following on the equilibrium path. We call this the *consensus effect*. Second, given deviations from truthtelling may overturn a conviction at some pivotal profiles while overturning an acquittal at others. In other words, deviations do not have a predictable impact on the outcome. We call this the *uncertainty effect*. These two effects independently generate incentives for truthtelling and thereby enable the TS equilibrium to exist.

We proceed as follows. Section 1 closes with a literature review. Section 2 presents the model. Section 3 presents an example that highlights the two effects mentioned above. Section 4 contains the equilibrium analysis and presents Theorems 1 and 2. Section 5 adopts a mechanism design approach. Section 6 discusses partially revealing equilibria and sequential communication. Section 7 concludes. Lengthy proofs are relegated to the Appendix.

Building on the theory of strategic voting as information aggregation (Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1998)), a milestone in the literature on cheap talk deliberation and collective decisions in heterogeneous committees is the impossibility result presented in Coughlan (2000).

This thesis chapter belongs to a class of contributions that modify the classical model and reestablish the TS equilibrium prediction under heterogeneous preferences. While our approach examines the role of informational consistency, Austen-Smith and Feddersen (2006) show that uncertainty about the preferences of jury members can render full pooling combined with sincere voting possible, as long as the voting rule is non-unanimous. In a complementary contribution, Meirowitz (2007) shows that the TS equilibrium exists if individual jurors are sufficiently confident that the majority of jurors shares their own preferences. Van Weelden (2008) adds an important caveat: when communication is sequential, uncertainty does not anymore suffice to ensure the existence of the TS equilibrium. Le Quement (2013) shows that only minimal disagreement is compatible with the TS equilibrium in large heterogeneous committees. These caveats do not generally apply to our model.

Another class of contributions approaches the communication problem from a mech-

[3] See in particular Coughlan (2000).

anism design perspective. In Gerardi et al. (2009), a mediator uses the correlation among signals to threaten heterogeneous individuals with punishment if their report does not match other experts' report. Gerardi and Yariv (2007) examine equilibria featuring cheap talk communication followed by weakly dominated voting and show that the set of equilibria is independent of the voting rule. In Wolinsky (2002), truthtelling requires the implementation of an ex post inefficient decision rule which generates pivotal scenarios in which lying is costly. In the spirit of this literature, we characterize conditions for the implementability of the welfare maximizing decision rule.

A third class of contributions maintains the classical model but examines different communication scenarios. In Hummel (2012) as well as Le Quement and Yokeeswaran (2014), a heterogeneous committee is split up into homogeneous subgroups in the deliberation phase. We show by means of an example that a partially revealing equilibrium can welfare dominate the TS outcome.

There is a set of positive and normative reasons to focus on the full pooling scenario. First, the experimental work of Goeree and Yariv (2011) documents extensive truthtelling in heterogeneous committees and finds that individuals assign substantial weight to the information revealed by others. Dickson et al. (2008) similarly find evidence of intense sharing of information among heterogeneous jurors. Second, full information equivalence, since Condorcet (1785), has been the main criterion for assessing the quality of collective decisions and the TS equilibrium satisfies this criterion asymptotically. Third, the philosophical literature on deliberation (e.g. Habermas (1992), Elster (1997), Manin (1987)) assigns an intrinsic value to exhaustive deliberations conducive to full exchange of information.

Finally, a set of contributions in the cheap talk literature bear a formal relation to our work. Battaglini (2002) considers a setup in which multidimensional state, signal and action spaces generically allow for full information extraction from multiple experts. Blume et al. (2007) show that exogenous noise in information transmission may improve the ex ante payoffs of players by increasing truthtelling incentives. In Deimen and Szalay (2014), the combination of a multidimensional state space and a one-dimensional action space allows for smooth communication (as opposed to partitional communication) in which the sender credibly announces his posterior despite relative disagreement. While all these approaches share certain formal features with our model, the mechanisms that drive our results are unrelated and, as far as we are aware, novel.

2.2 The Model

A jury of n agents, $n \in \mathbb{N}$, $n \geq 3$ is asked to decide whether to acquit (A) or convict (C) a defendant. The defendant is either innocent (I) or guilty (G). Both innocence and guilt occur in finitely many different variants $i_1, ..., i_{m_I}$ and $g_1, ..., g_{m_G}$, respectively, with $m_I, m_G \in \mathbb{N}$, $m_I, m_G \geq 2$.[5] The state space is hence given as $\Omega = I \cup G$ where $I = \{i_1, ..., i_{m_I}\}$ and $G = \{g_1, ..., g_{m_G}\}$. We denote the true state of the world by ω and say

[5]The case of $m_I = m_G = 1$ corresponds to the classical model analyzed in Coughlan (2000) and others.

that the defendent is innocent if $\omega \in I$ and guilty if $\omega \in G$. The true state of the world is drawn from a publicly known prior distribution f over Ω. The distribution f attaches different ex-ante probabilities to the events I and G, i.e. $f(\omega \in I), f(\omega \in G) \in (0,1)$ with $f(\omega \in I) + f(\omega \in G) = 1$. The distribution does not differentiate between variants within the set of I-states and G-states, respectively, i.e.

$$f(i_l) = \frac{f(\omega \in I)}{m_I} \quad \forall l \in \{1, ..., m_I\},$$
$$f(g_l) = \frac{f(\omega \in G)}{m_G} \quad \forall l \in \{1, ..., m_G\}.$$

The jury implements an action $a \in \{A, C\}$ by voting according to some prespecified voting rule $k \in \{1, \ldots, n\}$. Each agent $j \in \{1, \ldots, n\}$ casts a vote in favor of one of the two actions. If the number of votes cast for conviction is greater than or equal to k, the defendant is convicted while otherwise he is acquitted.

We assume that agents only care whether the defendant is innocent or guilty but are not interested per se in the particular variant of guilt or innocence that applies. The utility of agent j from action a conditional on state $\omega \in \Omega$ is given as

$$u_j(a, \omega) = \begin{cases} 0 & \text{if } (a, \omega) \in \{(A, I)\,(C, G)\} \\ -q_j & \text{if } (a, \omega) = (C, I) \\ -(1 - q_j) & \text{if } (a, \omega) = (A, G). \end{cases}$$

Utilities of correct decisions (acquitting an innocent or convicting a guilty defendant) are normalized to 0 and $q_j \in (0,1)$ is an individual and commonly known preference parameter that characterizes the relative importance assigned by an agent to the two types of errors.[7] As agent j maximizes expected utility, he prefers conviction over acquittal if and only if the probability of the defendant being guilty exceeds the cut-off q_j.

Prior to the voting stage, each agent receives a private signal $s \in S = \Omega$. Signals are drawn independently for each agent conditional on the state of the world ω according to the following distribution: If $\omega = i_l$ for some $l \in \{1, ..., m_I\}$, then

$$\begin{aligned} Pr(s = i_l | \omega = i_l) &= \lambda \cdot \frac{p}{\lambda + (m_I - 1)} \\ Pr(s = i_r | \omega = i_l) &= \frac{p}{\lambda + (m_I - 1)} \quad \forall r \in \{1, ..., m_I\}, r \neq l \\ Pr(s = g_r | \omega = i_l) &= \frac{1 - p}{m_G} \quad \forall r \in \{1, ..., m_G\}. \end{aligned}$$

If $\omega = g_l$ for some $l \in \{1, ..., m_G\}$, respective expressions apply after permutating i and g as well as I and G.[8] The parameters $\frac{1}{2} < p < 1$ and $\lambda > 1$ have the following

[7] Existing results (Austen-Smith and Feddersen (2006)) indicate that truthful communication is easier to achieve if there is uncertainty about preference types. By assuming observable preference types, we isolate the specific truthtelling incentives that are inherent to our model.

[8] Note that whenever $\lambda > 1$ our signal generating process is not reducible to a process that generates i.i.d signals conditional on I and G. It is, however, reducible to a process that generates correlated signals conditional on I and G.

interpretation: Applying Bayes' law, p measures up to priors the probability that the signal correctly reveals whether the defendant is innocent or guilty. The parameter $\lambda = \frac{Pr(s=i_l|\omega=i_l)}{Pr(s=i_l|\omega=i_r)} = \frac{Pr(s=g_l|\omega=g_l)}{Pr(s=g_l|\omega=g_r)}$ measures the relative informativeness of signals with respect to the particular variant of innocence or guilt. We finally assume that

$$\frac{Pr\left(s=i_l|\omega=i_r\right)}{Pr\left(s=i_l|\omega=g_t\right)} \geq 1 \quad \forall l,r \in \{1,...,m_I\}, \, t \in \{1,...,m_G\} \quad \text{and}$$
$$\frac{Pr\left(s=g_l|\omega=g_r\right)}{Pr\left(s=g_l|\omega=i_t\right)} \geq 1 \quad \forall l,r \in \{1,...,m_G\}, \, t \in \{1,...,m_I\}$$

to ensure that each i-signal (g-signal) renders each i-state (g-state) at least as likely as each g-state (i-state). Note that this is equivalent to

$$p \geq \max\left\{\frac{m_G - 1 + \lambda}{2m_G - 1 + \lambda}, \frac{m_I - 1 + \lambda}{2m_I - 1 + \lambda}\right\}. \tag{2.2.1}$$

A collection of signals constitutes a *signal profile* $\sigma = (x_1, ..., x_{m_I}, y_1, ..., y_{m_G})$ where x_l denotes the number of i_l-signals, $l \in \{1,...,m_I\}$ and y_l denotes the number of g_l-signals, $l \in \{1,...,m_G\}$. We write

$$x \equiv \sum_{l=1}^{m_I} x_l \quad \text{and} \quad y \equiv \sum_{l=1}^{m_G} y_l$$

for the total number of i-signals and g-signals in a given signal profile. Conditional on signal profile σ, the posterior probability of guilt is given as

$$\begin{aligned}\beta\left(\sigma\right) &= Pr\left(\omega \in G|\sigma\right)\\ &= \frac{f\left(\omega \in G\right) \cdot Pr\left(\sigma|\omega \in G\right)}{f\left(\omega \in G\right) \cdot Pr\left(\sigma|\omega \in G\right) + f\left(\omega \in I\right) \cdot Pr\left(\sigma|\omega \in I\right)}.\end{aligned}$$

In terms of utilities agents only care whether $\omega \in I$ or $\omega \in G$, hence the number $\beta\left(\sigma\right)$ is a sufficient statistic for the preferred action of each individual agent for any signal profile σ.

After having received their signals, agents engage in one round of simultaneous and public cheap talk.[9] As we focus on truthful equilibria, it is without loss of generality to assume that each agent sends a message $m \in M = S$.

To summarize, the timing of the game is as follows: Nature draws the true state of the world ω from the distibution f. Each agent receives a private signal s. Each agent simultaneously sends a public cheap talk message m. Each agent casts a vote. An action $a \in \{A, C\}$ is implemented according to the voting rule. Payoffs are realized.

Our equilibrium concept is Perfect Bayesian Equilibrium. For the main part of the paper we are concerned with the existence of the following particular equilibrium. Agents truthfully reveal their private information, sending a message $m = s$ at the communication stage. Agents (correctly) believe that other agents have revealed their private information truthfully, and agents vote sincerely, that is, they vote for conviction if and only if $\beta(\sigma) \geq q_j$. We call this the *TS equilibrium*.

[9]See Section 3 and Section 6 for some comments on sequential communication.

2.3 A Simple Example

In this section, we present a simple example that illustrates the key forces in our model and in particular highlights the two potential sources of truthtelling described in the introduction: the consensus and uncertainty effects.

Assume that the jury consists of only two preference types, doves and hawks, whose respective preference parameters are given by $q_D \in (0,1)$ and $q_H \in (0,1)$ with $q_D > q_H$. Consider a three persons committee consisting of one hawk and two doves with the voting rule given as simple majority, i.e. $k = 2$. Aggregate signal profiles can be ordered exhaustively with respect to the conditional probability of guilt that they induce, namely

$$\begin{matrix} \beta(3,0,0,0) \\ \beta(0,3,0,0) \end{matrix} < \dots < \begin{matrix} \beta(1,0,0,2), \beta(1,0,2,0) \\ \beta(0,1,0,2), \beta(0,1,2,0) \end{matrix} < \begin{matrix} \beta(0,0,1,2) \\ \beta(0,0,2,1) \end{matrix} < \begin{matrix} \beta(0,0,0,3) \\ \beta(0,0,3,0) \end{matrix}.$$

Suppose preference parameters q_H, q_D such that a dove favors conviction if and only if the aggregate signal profile is either $(0,0,0,3)$ or $(0,0,3,0)$ while a hawk favors conviction if and only if the aggregate signal profile is $(0,0,0,3), (0,0,3,0), (0,0,1,2)$ or $(0,0,2,1)$. In other words, while both groups require three g-signals to prefer conviction, doves furthermore require these g-signals to be consistent. There are thus two signal profiles for which hawks and doves disagree on the optimal decision, namely $(0,0,1,2)$ and $(0,0,2,1)$.

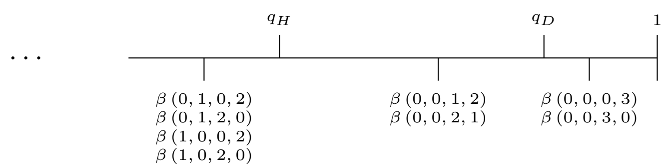

Figure 1: Conditional probabilities of guilt.

In this setting, TS strategies and beliefs always constitute an equilibrium, for any parameter values q_H and q_D consistent with the above preferences. We demonstrate this by verifying explicitly that in a putative TS equilibrium no individual agent has an incentive to deviate from truthtelling followed by sincere voting.

Note first that whatever the announcement made by an agent in the communication stage (truthful or not), the agent has no incentive to subsequently deviate from sincere voting as such a deviation decreases the probability that his favored decision ensues. Second, given that the voting rule is simple majority doves are always able to enforce their favored decision. Indeed, if doves favor an acquittal they can implement it simply by jointly voting for it. If doves prefer a conviction, hawks do so as well an the vote will be unanimous. Hence the doves have no incentive to deviate from truthtelling.

We now analyze the truthtelling incentives of the hawk in the putative TS equilibrium. The hawk's announcement is pivotal if the remaining two agents hold signal profiles $(0,0,2,0)$ or $(0,0,0,2)$. In the first (second) case, a g_1- (g_2-) announcement would trigger conviction while any of the remaining announcements would cause acquittal. To systematically analyze the hawk's incentive to deviate at each of his possible information sets i_1, i_2, g_1 and g_2, note that given the symmetry of the model conditions ensuring truthtelling of the hawk holding an i_1- or an i_2-signal respectively a g_1- or a g_2-signal are identical modulo an exchange of subscripts. We can therefore without loss restrict our analysis to deviations of the hawk holding an i_1-signal or a g_1-signal.

Assume that the hawk holds an i_1-signal and is pivotal at the communication stage. The signal profile of the entire committee is then $(1,0,2,0)$ or $(1,0,0,2)$. In either case, the decision that is taken by the committee given the true signal profile is acquittal and coincides with the decision favored by the hawk. Accordingly, he has no incentive to deviate from truthtelling. Here, the consensus effect is the sole source of truthtelling: despite heterogeneity and despite the existence of signal profiles generating conflict a hawk with an i_1-signal fully agrees with the doves on the preferred action in all pivotal scenarios.

Assume next that the hawk holds a g_1-signal and is pivotal in the communication stage. The signal profile of the entire committee is then either $(0,0,1,2)$ or $(0,0,3,0)$. Here, in contrast to the previous case the hawk disagrees with the acquittal decision ensuing from truthtelling at $(0,0,1,2)$ while he agrees with the conviction decision ensuing from truthtelling at $(0,0,3,0)$. If the hawk deviates to announcing some i-signal, the signal profile observed at the voting stage by other agents is either $(1,0,2,0)$, $(1,0,0,2)$, $(0,1,0,2)$ or $(0,1,2,0)$, thus leading to an undesired acquittal. The deviation to an i-report is therefore dominated by truthtelling. If the hawk deviates to a g_2-announcement, the signal profile observed at the voting stage by the remaining agents is given by $(0,0,0,3)$ or $(0,0,2,1)$. The deviation beneficially overturns an acquittal in the first case but adversely overturns a conviction in the second case. The hawk thus faces uncertainty about the impact of his statement. While for pivotal profile $(0,0,0,2)$ a g_2-report is harsher than a g_1-report and constitutes the only way to induce the desired conviction, the situation is exactly reversed for pivotal profile $(0,0,2,0)$.

Among the two pivotal profiles $(0,0,0,2)$ and $(0,0,2,0)$ faced by the hawk when holding a g_1-signal, profile $(0,0,0,2)$ thus incentivizes lying while profile $(0,0,2,0)$ incentivizes truthtelling. We call this the uncertainty effect. Which incentive dominates depends on the relative likelihood assigned to these two profiles, the latter itself depending on the probability assigned to the states g_1 and g_2. An agent holding a g_1-signal assigns a higher probability to state g_1 than to state g_2 and accordingly to profile $(0,0,2,0)$ than to profile $(0,0,0,2)$. The signal profile that incentivizes truthtelling is thus assigned a higher probability than the one that incentivizes lying. Hence the hawk when holding a g_1-signal never prefers to announce a g_2-signal. We conclude that the TS equilibrium exists despite the existence of signal profiles generating conflict.

We close the discussion of this example with two remarks. First, the TS equilibrium continues to exist under a sequential communication protocol where the single hawk speaks first. Indeed, the hawk's incentives when speaking first are identical to those under the simultaneous protocol while the doves still determine the outcome and hence have no incentives to deviate. Second, the TS equilibrium also exists under unanimity when $k = 3$ by exactly the same arguments.[11]

[11]The existence of the TS equilibrium under sequential communication and unanimity stands in

2.4 Analysis of the TS Equilibrium

The example of Section 3 shows that the TS equilibrium can exist despite potential disagreement after full pooling of information. In what follows, we provide an equilibrium analysis for arbitrary committee sizes, assuming first only two preference types and subsequently analyzing the case of individual preference parameters.

For any signal $s \in S$, let σ_s denote the signal profile that consists of one signal s only. Moreover, for a given agent j, we denote the signal profile of all other agents by σ_{-j}. The following lemma compares posterior probabilities of guilt for different signal profiles; it specifically addresses the effect of shifting mass from one entry of σ to another. This replicates the change in beliefs of other agents achievable by misreporting a signal in the putative TS equilibrium.

Lemma 0. *For any signal profile $\sigma = (x_1, ..., x_{m_I}, y_1, ..., y_{m_G})$, the function $\beta(\sigma)$ is invariant under any permutation of x-entries and any permutation of y-entries of σ. Moreover, the following inequalities hold:*

$$\beta(\sigma + \sigma_{g_r}) > \beta(\sigma + \sigma_{i_l}) \quad \forall l \in \{1, ..., m_I\}, r \in \{1, ..., m_G\}, \tag{2.4.1}$$

$$\beta(\sigma + \sigma_{g_l}) \geq \beta(\sigma + \sigma_{g_r}) \quad \forall l, r \in \{1, ..., m_I\}, y_r \leq y_l, \tag{2.4.2}$$

$$\beta(\sigma + \sigma_{i_l}) \leq \beta(\sigma + \sigma_{i_r}) \quad \forall l, r \in \{1, ..., m_I\}, x_r \leq x_l. \tag{2.4.3}$$

Condition (2.4.2) *holds with equality iff $y_l = y_r$ and condition* (2.4.3) *holds with equality iff $x_l = x_r$.*

Proof. See Appendix. □

Lemma 0 shows that three factors determine the posterior probability of guilt; an increase in the total number y of g-signals and in the consistency of the profile of g-signals leads to an increase in the posterior probability of guilt. An increase in the consistency of the profile of i-signals has the opposite effect.

Note that in the special case of $\lambda = 1$ which is excluded from our model, equations (2.4.2) and (2.4.3) always hold with equality. Indeed, if $\lambda = 1$ signals do not reveal any information that refers to a particular substate but only indicate whether $\omega \in I$ or $\omega \in G$. As a consequence, in the limit case of $\lambda = 1$ our model yields the same prediction as the classical model; the TS equilibrium is not compatible with a positive probability of ex post conflict.

Proposition 0 (Coughlan (2000)). *Suppose$\lambda = 1$ or, equivalently, $m_G = m_I = 1$. Then the TS equilibrium exists if and only if*
a) at least k agents ($n - k$ agents) favor conviction (acquittal) for any realization of signals.
b) all agents favor the same action for any realization of signals.

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contrast to the impossibility results of Van Weelden (2008) and Austen-Smith and Feddersen (2006).

2.4.1 A two-types committee: hawks and doves

Fix the committee size n. In this subsection, we assume that there are only two preference types q_H and q_D, referred to as hawks and doves, where $q_H < q_D$. For $j \in \{H, D\}$, we call a profile σ a *type q_j threshold profile* if $\beta(\sigma) \geq q_j$ and $\beta(\tilde{\sigma}) < q_j$ for all signal profiles $\tilde{\sigma}$ satisfying $\beta(\tilde{\sigma}) < \beta(\sigma)$. A juror of preference type q_j thus prefers conviction precisely for those signal profiles that yield at least as much evidence for the defendant being guilty as a type q_j threshold profile does.

We say that a signal profile σ is a *conflict profile* if conditional on signal profile σ hawks and doves disagree on the preferred action, that is, if

$$q_H \leq \beta(\sigma) < q_D.$$

Given that there are only two types of jurors, the non-unanimous voting rule k matters only in a binary sense; we say that hawks have *critical mass* if the number of hawks is weakly greater than k, so that hawks are sufficiently numerous to implement conviction whenever they wish. Otherwise, we say that doves have critical mass.

Finally, we say that preference types q and $\tilde{q}$ are equivalent, denoted by $q \sim \tilde{q}$, if and only if both preference types favor the same action for any signal profile σ consisting of n signals. Preference types are thus equivalent if and only if no conflict profiles exist between the two or, respectively, if they have identical threshold profiles.

Before stating our main result, we invoke simple arguments to rule out a subset of deviations from the putative TS equilibrium.

Lemma 1. *Agents of the type $j \in \{H, D\}$ that has critical mass never have an incentive to deviate in the putative TS equilibrium.*

Proof. In a putative TS equilibrium, the defendant will be convicted if and only if the thresholds of at least k agents satisfy $q_j \leq \beta(\sigma)$ for the revealed signal profile σ. Hawks will be able to convict the defendant whenever they prefer to if and only if there are at least k of them. Moreover, if hawks prefer to acquit, then doves do so as well since $q_H < q_D$ and the vote will be unanimous. Hence, if hawks have critical mass their favored action will always be implemented and they have no incentive to deviate. A similar reasoning holds if there are less than k hawks so that doves have critical mass. $\square$

Lemma 2. *Assume that type $j \in \{H, D\}$ does not have critical mass and assume that the voting rule is non-unanimous. Then the vote of an agent of type j will never influence the outcome in a putative TS equilibrium, irrespective of whether he reported truthfully or not at the communication stage.*

Proof. Agent j's vote can only influence the outcome if the group of agents that has critical mass votes "against their bias", i.e. if hawks vote for acquittal or doves vote for conviction, respectively. In both cases, however, *all* other agents will vote unanimously as the remaining doves (hawks) will also favor acquittal (conviction) if hawks (doves) do so. Hence, for any non-unanimous voting rule, the voting behavior of an agent that does not belong to the group having critical mass can never influence the outcome. $\square$

Lemma 2 allows us to restrict attention to incentives in the communication stage if the voting rule is non-unanimous. It furthermore provides a hint as to what will typically go wrong under unanimity, i.e. $k = n$ ($k = 1$). Hawks (doves) can misreport i-signals (g-signals) to trigger a conviction (an acquittal) and subsequently vote for acquittal (conviction) if conditional on the now revealed information of other jurors it turns out that their lie causes others to vote for an undesired outcome. Any downside from misreporting is hence absent under unanimity voting. As a consequence of Lemma 2, we exclude the unanimous voting rule from our analysis, i.e. we assume from now on that $k \in \{2, ..., n-1\}$.

Lemma 3. *Let $k \in \{2, ..., n-1\}$. In a putative TS equilibrium, a hawk never has an incentive to misreport a g-signal as an i-signal and a dove never has an incentive to misreport an i-signal as a g-signal.*

Proof. By Lemma 1 we may assume that the agent under consideration does not belong to the group that has critical mass. Suppose agent j is a hawk holding a signal $s = g_l$ for some $l \in \{1, ..., m_G\}$ but reports $m = i_r$ for some $r \in \{1, ..., m_I\}$. Then the true overall signal profile is given as $\sigma = \sigma_{-j} + \sigma_{g_l}$ while the reported signal profile is given as $\hat{\sigma} = \sigma_{-j} + \sigma_{i_r}$. As doves have critical mass, the defendant will be convicted if and only if $\beta(\hat{\sigma}) \geq q_D$. Agent j, being a hawk, would prefer conviction whenever $\beta(\sigma) \geq q_H$. So given that $q_D > q_H$, agent j can only profit from the proposed deviation if $\beta(\hat{\sigma}) > \beta(\sigma)$. However, by Lemma 0, Equation (2.4.1) we have

$$\beta(\sigma) = \beta(\sigma_{-j} + \sigma_{g_l}) > \beta(\sigma_{-j} + \sigma_{i_r}) = \beta(\hat{\sigma}).$$

Thus, for any signal profile σ_{-j} the proposed deviation triggers a lower Bayesian posterior probability of the defendant being guilty compared to a truthful report. Hence deviating in the proposed way is detrimental to agent j. The case of a dove misreporting an i-signal as a g-signal is alike. □

Lemma 4. *Let $k \in \{2, ..., n-1\}$. In a putative TS equilibrium, no agent has an incentive to misreport an i_l-signal as an i_r-signal, $l, r \in \{1, ..., m_I\}$, $r \neq l$, or to misreport a g_l-signal as a g_r-signal, $l, r \in \{1, ..., m_G\}$, $r \neq l$.*

Proof. See Appendix. □

The argument behind Lemma 4 relies on the uncertainty effect described in the example of Section 3. Deviations within the set of g-signals or i-signals have opposing effects on the outcome for different signal profils. As profiles that incentivize truthtelling are always more likely than those that incentivize deviating, truthtelling is always preferable.

The only deviations that remain to be excluded involve doves reporting an i-signal instead of a g-signal and hawks reporting a g-signal instead of an i-signal. Let q, up to equivalence, denote the threshold of the group that has critical mass and consider an agent j from the group that does not have critical mass. For a given threshold q of the group that has critical mass, we denote the set of signal profiles of others at which an i_r-report by agent j causes an acquittal while a g_l-report causes a conviction by

$$Piv_{i_r,g_l}(q) \equiv \{\sigma_{-j} : \beta(\sigma_{-j} + \sigma_{i_r}) < q \wedge \beta(\sigma_{-j} + \sigma_{g_l}) \geq q\}.$$

We finally impose the following minor assumption on preferences of hawks and doves.

Assumption 1 (No partisans).
a) The preferred action of each agent depends on the aggregate signal profile.
b) Hawks require less than the maximal possible evidence of guilt to prefer conviction and doves require less than the maximal possible evidence of innocence to prefer acquittal.

Part a) excludes partisan types whose optimal decision is independent of any available information. Part b) excludes preference types for hawks (doves) which imply either that the other type is partisan (in the sense of Part a)) or that preference types of hawks and doves are equivalent in which case the TS equilibrium trivially exists (cf. Proposition 0).

We now present our main result.

Theorem 1. *Let $k \in \{2, ..., n-1\}$.*
a) Assume hawks have critical mass. For any hawk type q_H the TS equilibrium exists if and only if the value of q_D lies below an upper bound $\hat{q}_D(q_H) > q_H$ given as

$$q_D \leq \hat{q}_D(q_H) \equiv \frac{1}{P(\sigma_{-j} \in Piv_{i_r,g_l}(q_H) | s_j = g_l)} \sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_H)} P(\sigma_{-j} | s_j = g_l) \cdot \beta(\sigma_{-j} + \sigma_{g_l}).$$

b) Assume doves have critical mass. For any dove type q_D the TS equilibrium exists if and only if the value of q_H lies above a lower bound $\hat{q}_H(q_D) < q_D$ given as

$$q_H \geq \hat{q}_H(q_D) \equiv \frac{1}{P(\sigma_{-j} \in Piv_{i_r,g_l}(q_D) | s_j = i_r)} \sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_D)} P(\sigma_{-j} | s_j = i_r) \cdot \beta(\sigma_{-j} + \sigma_{i_r}).$$

c) Impose Assumption 1. If hawks have critical mass, the pair $(q_H, \hat{q}_D(q_H))$ features at least one conflict profile. Similarly, if doves have critical mass, the pair $(\hat{q}_H(q_D), q_D)$ features at least one conflict profile.

Proof. See Appendix. □

Theorem 1 provides a general existence result for the TS equilibrium. Part a), assuming that hawks have critical mass, states the existence of a critical dove type $\hat{q}_D(q_H)$ such that the TS equilibrium exists if and only if $q_D \in (q_H, \hat{q}_D(q_H)]$. The threshold $\hat{q}_D(q_H)$ corresponds to the probability of guilt conditional on all pivotal profiles σ_{-j} where a truthful g_l-report of agent j leads to conviction while an i_r-report leads to acquittal. Part b) states the corresponding result for the case where doves have critical mass. Part c) yields the critical qualitative statement that the TS equilibrium is compatible with the existence of conflict profiles. It stands in stark contrast to Coughlan's impossibility result as stated in Proposition 0.

The existence of the TS equilibrium despite conflict is possible because the posterior probability of guilt is determined not only by the total number of g-signals but also by the consistency among signals indicating respectively guilt or innocence. Reports influence the posterior probability of guilt through both channels and hence change the outcome for more than one profile. As a direct consequence, the set of pivotal profiles that needs to be considered for the reporting decision is not a singleton.

Reports do not have a systematic impact on the posterior probability of guilt in the consistency dimension; clearly, whether a particular report is more or less consistent with other signals does not depend on the report itself but on the signals held by other jurors. As a result of the latter observation, agents always face some uncertainty as to which report to choose if they want to trigger a particular outcome. This uncertainty effect is at the core of Lemma 4 and rules out deviations that involve misreports within the set of i-reports or g-reports, respectively.

For deviations where doves report an i-signal instead of a g-signal or hawks report a g-signal instead of an i-signal, the strategic concern is different. Here, the deviation has a clear effect on the outcomes; g-reports unambiguously trigger additional convictions when compared to i-reports. The multiplicity of pivotal profiles here allows the consensus effect to positively affect truthtelling incentives. While hawks and doves have diverging interests for some pivotal profiles, they agree on the preferred outcome for others, so overturning the decision in the latter case is detrimental. This (partial) consensus between hawks and doves is more pronounced the smaller the gap between the two preference types. As a result, we get an upper bound for dove types with respect to a given hawk types in Part a) and lower bounds for hawk types with respect to a given dove type in Part b).

Given the above described consensus effect, Part c) of Theorem 1 becomes very intuitive. A type who is indifferent between truthtelling and lying necessarily faces pivotal profiles that incentivize truthtelling and pivotal profiles that incentivize deviating. Consider a dove holding a g_l-signal and let σ be a threshold profile of the hawks, the latter having critical mass. Whenever $q_D > \beta(\sigma)$, the profile σ is a conflict profile. Now consider a profile $\tilde{\sigma}$ that is either slightly less consistent with respect to its i-signals or slightly more consistent with respect to its g-signals, but has the same total number of i- and g-signals as σ. Then the dove is still pivotal for $\tilde{\sigma}$ as $\beta(\tilde{\sigma} - \sigma_{g_l} + \sigma_{i_r}) < \beta(\sigma)$ and $\beta(\tilde{\sigma}) > \beta(\sigma)$. Yet for q_D being sufficiently close to $\beta(\sigma)$, that is, sufficiently much smaller than $\beta(\tilde{\sigma})$, the truthtelling incentives from profile $\tilde{\sigma}$ dominate the deviation incentives from profile σ. Assumption 1 guarantees that such a profile $\tilde{\sigma}$ always exists.[15]

2.4.2 General committees

We next generalize our analysis to committees featuring more than two preference types. For each agent $j \in \{1, ..., n\}$ we denote the corresponding preference parameter by q_j.

[15] Part c) of Theorem 1 as well as its discussion in the text may suggest that the range of preference types as well as the number of conflict profiles compatible with the TS equilibrium are rather small. This is not the case. In contrast, numerical simulations show that the parameter area that is compatible with the existence of the TS equilibrium in our model is typically larger than in the classical binary model in Coughlan (2000). Moreover, the number of conflict profiles compatible with the TS equilibrium becomes large when committees increase, contrasting e.g. Le Quement (2013).

Without loss of generality, we assume that $q_1 \leq ... \leq q_n$.

Let $k \in \{2, ..., n-1\}$ denote the prespecified voting rule. In a putative TS equilibrium, whenever agent k prefers conviction after truthful revelation of information, so do all agents $j < k$ and hence the defendant will be convicted with at least k votes. Conversely, if agent k prefers acquittal, so do all agents $j > k$ and hence, by the votes of at least $n - k + 1$ agents, an acquittal is implemented. The implemented decision hence always coincides with the preferred action of agent k who therefore never has an incentive to deviate. For any juror $j < k$, the implemented decision rule is "dovish" as $q_j < q_k$ while for any juror $j > k$ the decision rule is "hawkish" as $q_j > q_k$.

With the insights from the two type case, the above considerations suggest a result of the following type. The TS equilibrium exists if and only if for all $j \in \{1, ..., n\}$ we have $q_j \in [\hat{q}_H(q_k), \hat{q}_D(q_k)]$ with $\hat{q}_H(q), \hat{q}_D(q)$ defined as in Theorem 1. Indeed, if agent k's vote were to always determine the outcome, the result would take this exact form. The caveat in the general jury model is that no pendant to Lemma 2 holds. To see this, consider the case where a juror $j < k$ holds an i_l-signal and considers reporting a g_r-signal instead to trigger additional convictions. Let σ_{-j} denote a signal profile held by other jurors. Four scenarios may occur.

If

$$\beta(\sigma_{-j} + \sigma_{i_l}) < \beta(\sigma_{-j} + \sigma_{g_r}) < q_k$$

then for both, the truthful report and the deviation, the defendant will be acquitted by the votes of agents $k, ..., n$. Similarly, if

$$q_k \leq \beta(\sigma_{-j} + \sigma_{i_l}) < \beta(\sigma_{-j} + \sigma_{g_r})$$

the defendant will be convicted by the votes of agents $1, ..., k$. Note that agent $j < k$ will always prefer a conviction here since $q_j \leq q_k \leq \beta(\sigma_{-j} + \sigma_{i_l})$. In both cases, the reporting decision does not affect the outcome.

If

$$\beta(\sigma_{-j} + \sigma_{i_l}) < q_k \leq q_{k+1} \leq \beta(\sigma_{-j} + \sigma_{g_r})$$

then the defendant will be acquitted when agent j reports truthful by the votes of agents $k, ..., n$. The defendant will be acquitted by the votes of agents $1, ..., j-1, j+1, ..., k+1$. Both outcomes are independent of the voting decision of agent j. The crucial case now occurs if

$$\beta(\sigma_{-j} + \sigma_{i_l}) < q_k \leq \beta(\sigma_{-j} + \sigma_{g_r}) < q_{k+1}.$$

Here, a truthful report again leads to an acquittal by the votes of agents $k, ..., n$. Deviating towards a g_r-report, however, makes agent j pivotal in the voting stage. Whenever agent j prefers a conviction as $\beta(\sigma_{-j} + \sigma_{i_l}) \geq q_j$ he can implement this outcome by his vote, together with agents $1, ..., j-1, j+1, ..., k$. Whenever agent j prefers an acquittal as $\beta(\sigma_{-j} + \sigma_{i_l}) < q_j$ he can also implement this outcome by his vote, together with agents $k+1, ..., n$. In other words, agent j can undo the adverse effect of his lie at the voting stage so that lying is not risky here.

These insights, which symmetrically apply to the reporting decision of agents $j > k$, are summarized in the following theorem.

Theorem 2. *Let* $k \in \{2, ..., n-1\}$.

a) The TS equilibrium exists if and only if

$$q_j \in \left[\underline{q}\left(q_k, q_{k+1}\right), \overline{q}\left(q_k, q_{k-1}\right)\right]$$

for all $j \in \{1, ..., n\}$. *The lower bound* $\underline{q}$ *is decreasing in* q_{k+1} *and increasing in* q_k. *The upper bound* $\overline{q}$ *is decreasing in* q_k *and increasing in* q_{k-1}.

b) If $q_{k+1} \sim q_k$ *then* $\underline{q}\left(q_k, q_{k+1}\right) = \hat{q}_H\left(q_k\right)$. *If* $q_{k-1} \sim q_k$ *then* $\overline{q}\left(q_k, q_{k-1}\right) = \hat{q}_D\left(q_k\right)$. *In particular, if* $q_{k-1} \sim q_k \sim q_{k+1}$ *the TS equilibrium exists if and only if*

$$q_j \in \left[\hat{q}_H\left(q_k\right), \hat{q}_D\left(q_k\right)\right] \quad \forall j \in \{1, ..., n\}.$$

Proof. In the Appendix we show that the only deviations in the communication stage that are potentially profitable involve an agent $j < k$ who holds an i_l-signal and considers a g_r-report or an agent $k > j$ who holds a g_r-signal and considers an i_l-report. Part a) of Theorem 2 is then a direct consequence of the previous analysis. Deviation incentives of an agent $j < k$ that holds an i-signal and considers a g-report are more pronounced the smaller q_j is since the deviation may only trigger additional convictions. Hence there exists a lower bound for q_j beyond which deviations become strictly advantageous. Moreover, by the above analysis, deviation incentives only depend on q_k and q_{k+1}. Finally, deviating is particularly attractive for profiles σ_{-j} of other jurors for which

$$\beta\left(\sigma_{-j} + \sigma_{i_l}\right) < q_k \leq \beta\left(\sigma_{-j} + \sigma_{g_r}\right) < q_{k+1}$$

as for these profiles agent j can always implement his preferred outcome based on the aggregate information. Clearly, the set of such profiles σ_{-j} shrinks when q_{k+1} becomes smaller or q_k becomes larger. For Part b) note that $q_k \sim q_{k+1}$ implies that no profile σ_{-j} of other jurors exist for which

$$q_k \leq \beta\left(\sigma_{-j} + \sigma_{g_r}\right) < q_{k+1}.$$

Agent j is never pivotal in the voting stage and hence incentives are identical to the two type model with $\underline{q}\left(q_k, q_{k+1}\right) = \hat{q}_H\left(q_k\right) = \hat{q}_H\left(q_{k+1}\right)$. The argument for jurors $j > k$ that hold a g_r-signal and consider an i_l-report is alike. □

Theorem 2 provides important insights on the optimal composition of heterogeneous committees with regard to maximizing truthtelling incentives. In many committees, heterogeneity is valued for exogenous reasons. A trial jury, for example, is supposed to represent society as a whole and hence should feature a substantial degree of heterogeneity. Similarly, a board of directors should feature some heterogeneity in order to avoid cut and dried decision making. To the extent that one insists on maintaining heterogeneity, Theorem 2 has two important implications. First, lying incentives by biased agents can be overcome by including moderate agents in the committee and assigning decision power to them through a suitably chosen voting rule. Second, a single moderate agent will not suffice to ensure truthtelling (and may even be detrimental) as the reporting decision is then not sufficiently decisive with regard to the final outcome.

2.5 General Mechanisms

We next consider the voting problem discussed in the previous subsection from a mechanism design perspective. Suppose a designer who wants to maximize the ex post expected utility among jurors according to

$$Eu = E\sum_{j=1}^{n}\alpha_j u_j = \sum_{j=1}^{n}\alpha_j Eu_j$$

for given Pareto-weights $\alpha_1, ..., \alpha_n \geq 0$ with $\sum_{j=1}^{n}\alpha_j = 1$. The expected utility of the committee conditional on signal profile σ is given by

$$Eu\left(\sigma, a\right) \quad = \quad \begin{cases} -P\left[\omega \in G|\sigma\right] \cdot \left(1 - \sum_{j=1}^{n}\alpha_j q_j\right) & a = A \\ -P\left[\omega = I|\sigma\right] \cdot \sum_{j=1}^{n}\alpha_j q_j & a = C. \end{cases}$$

Hence the optimal action coincides with the preferred action of a hypothetical juror with preference parameter $q^* \equiv \sum_{j=1}^{n}\alpha_j q_j$.

We refer to the decision rule that implements conviction whenever $\beta\left(\sigma\right) \geq q$ and implements acquittal whenever $\beta\left(\sigma\right) < q$ *as decision rule* q.[17] Note that whenever $q \in \left(\beta\left(n, 0, ..., 0\right), \beta\left(0, ..., 0, n\right)\right)$ there exist report profiles σ and $\tilde{\sigma}$ such that $a = C$ whenever σ is reported while $a = A$ whenever $\tilde{\sigma}$ is reported. We call such a decision rule *responsive*. Our main objective in this section is to identify conditions under which a given responsive decision rule q is implementable. Invoking the revelation principle, we may restrict to direct mechanisms where the designer commits to a policy conditional on reports and agents report their signals truthfully subject to incentive compatibility.

The following proposition states a negative benchmark result for the classical binary signal model.

Proposition 1. *Suppose $\lambda = 1$ or, equivalently, $m_G = m_I = 1$. Unless $q_1 \sim ... \sim q_n$, no responsive decision rule q is implementable.*[18]

Proof. See Appendix. □

In contrast, whenever $\lambda > 1$ as assumed in our model, this impossibility result does not apply as shown in the following theorem.

Theorem 3. *Assume $\lambda > 1$. A responsive decision rule q is implementable if and only if*

$$q_j \in \left[\hat{q}_H\left(q\right), \hat{q}_D\left(q\right)\right] \quad \forall j \in \{1, ..., n\}.$$

In particular, this applies to the welfare maximizing decision rule q^*.

[17] As reports are assumed to be truthful, we use the same notation for aggregated report profiles that we have used for aggregated signal profiles before.

[18] Gerardi et al. (2009) propose a sophisticated mechanism that under certain conditions implements a decision rule that is arbitrarily close to the welfare maximizing rule in the classical model with $m_I = m_G = 1$. The mechanism requires either highly accurate private signals or a large number of agents. Moreover, it involves a sophisticated elicitation scheme in which an agent is singled out for scrutiny with some probability; if his report is consistent with the majority's reports, the principal rewards the selected agent by taking his favorite action. Otherwise, the principal takes a random action.

Proof. By Lemmas 1 and 2, the reporting incentives of a juror j with $q_j < q$ in the direct mechanism that follows decision rule q are identical to the reporting incentives of juror j if he faces a group of doves with $q_D = q$ that has critical mass. Similarly, the reporting incentives of a juror j with $q_j > q$ in the direct mechanism that follows decision rule q are identical to the reporting incentives of juror j if he faces a group of hawks with $q_H = q$ that has critical mass. The result then follows directly from Theorem 1. □

Theorem 3 provides a precise and simple characterization of the circumstances under which a given responsive decision rule q is implementable in our model. Invoking Part c) of Theorem 1, implementability is always compatible with some conflict between agents whenever Assumption 1 applies to a virtual agent with preference parameter q.

If the assumptions of Theorem 3 are satisfied, implementation may take place through different mechanisms. The first possibility is to use a classical direct mechanism in which agents truthfully report to an external designer committed to choosing conviction if and only if $\beta(\sigma) \geq q$ for any reported signal profile σ. However, implementation is also possible without relying on an external designer. Consider simultaneous public communication followed by voting under a non-unanimous voting rule, and suppose a putative equilibrium where all agents communicate truthfully and subsequently vote according to decision rule q. Since voting is always unanimous on the equilibrium path, individual deviations in the voting stage never influence the outcome and incentives in the communication stage are as in the direct mechanism.[20] The gain of avoiding an external designer comes here at the cost of weakly dominated voting strategies. Finally, note that if there exist $j \in \{2, ..., n-1\}$ such that $q_{j-1} \sim q_j \sim q_{j+1} \sim q$ then decision rule q is implementable by setting $k = j$ as a TS equilibrium of the communication and voting game discussed in Theorem 2.

2.6 Comments on Partial and Sequential Communication

In this final section we show by means of an explicit example that an existing TS equilibrium may be welfare dominated by a sincere voting equilibrium that involves only partial revelation of information. We add a discussion of sequential communication as a natural context for such partially revealing equilibria.

Fix the committee size as $n = 3$ and suppose $m_I = m_G = 2$. Let $f(I) = f(G) = \frac{1}{2}$, $p = \frac{3}{5}$, and $\lambda = 2$. Say there are two hawks (agents 1 and 2) and one dove (agent 3) with preference parameters $q_H = \frac{5}{7}$ and $q_D = \frac{7}{9}$. We have

$$\begin{aligned}
\beta(1,0,0,2) &= \frac{5}{8} < q_H < \frac{6}{8} = \beta(0,0,1,2), \\
\beta(0,0,1,2) &= \frac{6}{8} < q_D < \frac{9}{11} = \beta(0,0,0,3),
\end{aligned}$$

[20] The prescribed mechanism as well as the independence of the (non-unanimous) voting rule are well-known from Gerardi and Yariv (2007). Note, however, that Gerardi and Yariv (2007) do not provide any results on implementability of particular outcomes as such, in stark contrast to our findings.

so preferences are as in Section 3. As demonstrated there, the TS equilibrium exists if doves have critical mass, i.e. $k = 3$. At the same time, the TS equilibrium does not exist if hawks have critical mass, that is $k = 1$ or $k = 2$, since $\hat{q}_D\left(q_H\right) = \frac{27}{35} < \frac{7}{9} = q_D$.[21] However, attaching equal Pareto weights $\alpha_j = \frac{1}{3}$ to all agents, it is easily seen that the socially optimal decision rule involves conviction for profiles $(0,0,2,1)$, $(0,0,1,2)$, so it coincides with the hawks' preferred decision rule. The existing TS equilibrium thus leads to suboptimal acquittals for profiles $(0,0,2,1)$ and $(0,0,1,2)$.

Consider now an alternative putative equilibrium under unanimity in which agents 1 and 3 behave as before but agent 2 only reveals whether he holds an i-signal or a g-signal, without specifying the variant, and where voting is sincere.[22] In such a putative equilibrium, it is immediate that the defendant will be acquitted whenever either at least one i-signal is reported or agents 1 and 3 report inconsistent g-signals. However, the conditional probability of guilt in case of agents 1 and 3 holding consistent g-signals and agent 2 holding some g-signal is given as

$$P\left[\omega \in G | s_1 = s_3 \in G, s_2 \in G\right] = \frac{15}{19} > \frac{7}{9} = q_D,$$

so the defendant will be unanimously convicted. It is straightforward to check that this reporting behavior together with correct Bayesian beliefs and sincere voting constitutes a partially revealing equilibrium.

The partially revealing equilibrium we just described improves welfare compared to the existing TS equilibrium as it involves conviction for some instances of signal profiles $(0,0,1,2)$ and $(0,0,2,1)$. Note that if both hawks would adopt the partially revealing reporting strategy of agent 2, this would not suffice to convince the dove to vote for conviction as

$$P\left[\omega \in G | s_1, s_2, s_3 \in G\right] = \hat{q}_D\left(q_H\right) = \frac{27}{35} < q_D.$$

This is in line with the characterization in Therorem 3; the welfare maximizing decision rule q_H cannot be implementable given that $q_D > \hat{q}_D\left(q_H\right)$.

The partially revealing equilibrium therefore essentially relies on asymmetric behavior of the two (ex ante identical) hawks. While this is an unattractive property in the simultaneous communication game analyzed so far, it is a very natural feature under sequential communication as we argue now.

Consider the same game as above under unanimity voting ($k = 3$) where agents speak in order of their numbers. First, look at a putative TS equilibrium. For agent 1, incentives are identical to the simultaneous game where the TS equilibrium exists. Agent 3 (the dove) has critical mass and hence does not have an incentive to deviate either. Agent 2 is pivotal if and only if both other agents hold consistent g-signals. However, in contrast to the simultaneous communication protocol, given the first agent's report of a particular g-signal there is no uncertainty left as to which variant of g-signal this could be. Hence agent 2, when holding some g-signal, will always imitate

[21] The value $\hat{q}_D\left(q_H\right)$ corresponds to the likelihood of guilt if it is known that all agents hold some g-signals but nothing is known about their consistency.

[22] For the given message space $M = S$, agent 2 might for example randomize with probability $\frac{1}{2}$ over both g-reports whenever he holds a g-signal.

the variant announced by agent 1 to trigger a conviction, whether or not their signals are consistent. Sequential communication therefore leads to a breakdown of the TS equilibrium. On the other hand, if agent 2 adopts the communication strategy just described as a deviation from the TS equilibrium path, the outcome will be precisely that of the partially revealing equilibrium discussed before. Again, it is easy to check that together with sincere voting these communication strategies form a Perfect Bayesian Equilibrium also under sequential communication.

Sequential communication is known to render full revelation of information under conflict more difficult as it reduces uncertainty of (late) speakers, see e.g. Van Weelden (2008). While this is typically considered an unfavorable property, our example highlights a potentially beneficial aspect of the very same feature. Under sincere voting and simultaneous communication, the TS equilibrium may be welfare dominated by partially revealing equilibria. Sequential communication can then serve as a selection device, eliminating the welfare dominated TS equilibrium while leaving the partially revealing equilibria intact.

2.7 Conclusions

In our collective decision model with pre-vote communication, a positive probability of ex post disagreement among agents is frequently compatible with the existence of the truthful communication and sincere voting equilibrium. The driving forces underlying our positive result are the consensus and uncertainty effects, both of which originate in the multiplicity of pivotal scenarios at the communication stage. The latter feature follows from the role played by consistency given our information structure. As a consequence of our equilibrium analysis we also obtain a positive result regarding the implementability of threshold decision rules in heterogenous committees including the decision rule that maximizes committee welfare. From a conceptual perspective, the key and novel feature of our information structure is that a given signal is interpreted differently depending on other available information; meaning is determined in context. We find this aspect worth exploring within other communication games.

2.8 Appendix

Proof of Lemma 0. The posterior probability of guilt for signal profile σ is given as

$$\begin{aligned}\beta(\sigma) &= \frac{f(\omega \in G) \cdot Pr(\sigma|\omega \in G)}{f(\omega \in G) \cdot Pr(\sigma|\omega \in G) + f(\omega \in I) \cdot Pr(\sigma|\omega \in I)} \\ &= \left[1 + \frac{P(\omega \in I)}{P(\omega \in G)} \cdot \frac{m_G}{m_I} \cdot \left(\frac{p \cdot m_I}{(1-p) \cdot (\lambda + m_I - 1)}\right)^x \cdot \left(\frac{(1-p) \cdot (\lambda + m_G - 1)}{p \cdot m_G}\right)^y \cdot \frac{\sum_{r=1}^{m_I} \lambda^{x_r}}{\sum_{l=1}^{m_G} \lambda^{y_l}}\right]^{-1}. \end{aligned} \tag{2.8.1}$$

The first claim is immediate from this formula. Note that if $\lambda = 1$ then $\beta(\sigma)$ only depends on x and y.

As $\frac{p \cdot m_I}{(1-p)\cdot(\lambda+m_I-1)} \geq 1$, $\frac{(1-p)\cdot(\lambda+m_G-1)}{p \cdot m_G} \leq 1$ by condition (2.2.1) and $\lambda > 1$, inequality (2.4.1) follows directly from equation (2.8.1). Similarly, $\lambda > 1$ together with equation

(2.8.1) implies (2.4.2) and (2.4.3). Equality holds if and only if $\lambda^{y_l} = \lambda^{y_r}$ or $\lambda^{x_l} = \lambda^{x_r}$, respectivly. □

Proof of Lemma 4. By Lemma 1 we may assume that the agent under consideration, agent j, is not part of the group that has critical mass. Suppose that agent j holds a signal $s = i_l$ and considers to report $m = i_r$ with $r \neq l$, $l, r \in \{1, ..., m_I\}$. Consider two candidates for σ_{-j}, namely $\hat{\sigma} = (x_1, ..., x_l, ..., x_r, ..., x_{m_I}, y_1, ..., y_{m_G})$ and $\hat{\sigma}_{x_l \longleftrightarrow x_r} = (x_1, ..., x_r, ..., x_l, ..., x_{m_I}, y_1, ..., y_{m_G})$ and assume without loss of generality that $x_l \geq x_r$. By Lemma 0 we have

$$\begin{aligned}
\beta(\hat{\sigma}) &= \beta(\hat{\sigma}_{x_l \longleftrightarrow x_r}), \\
\beta(\hat{\sigma} + \sigma_{i_l}) &= \beta(\hat{\sigma}_{x_l \longleftrightarrow x_r} + \sigma_{i_r}), \\
\beta(\hat{\sigma} + \sigma_{i_r}) &= \beta(\hat{\sigma}_{x_l \longleftrightarrow x_r} + \sigma_{i_l}), \\
\beta(\hat{\sigma} + \sigma_{i_l}) &\geq \beta(\hat{\sigma} + \sigma_{i_r}),
\end{aligned}$$

with equality in the last equation if and only if $x_l = x_r$. We can now compare the expected utility of the reports $m = i_l$ and $m = i_r$ conditional on $\sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}$. If both reports $m = i_l$ and $m = i_r$ trigger identical actions, the reporting decision does not matter. In particular, this is the case if $x_l = x_r$ where the two candidate profiles for σ_{-j} coincide. If the reports trigger different actions, then $m = i_l$ will trigger acquittal for $\sigma_{-j} = \hat{\sigma}$ and conviction for $\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r}$ while $m = i_r$ will trigger conviction for $\sigma_{-j} = \hat{\sigma}$ and acquittal for $\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r}$. Hence

$$\begin{aligned}
& Eu\left[m = i_l | \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}\right] - Eu\left[m = i_r | \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}\right] \\
= \; & -\sum_{t=1}^{m_G} P(\omega = g_t | s = i_l) \cdot P(\sigma_{-j} = \hat{\sigma} | \omega = g_t, \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}) \cdot (1-q) \\
& + \sum_{t=1}^{m_G} P(\omega = g_t | s = i_l) \cdot P(\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r} | \omega = g_t, \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}) \cdot (1-q) \\
& - \sum_{t=1}^{m_I} P(\omega = i_t | s = i_l) \cdot P(\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r} | \omega = i_t, \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}) \cdot q \\
& + \sum_{t=1}^{m_I} P(\omega = i_t | s = i_l) \cdot P(\sigma_{-j} = \hat{\sigma} | \omega = i_t, \sigma_{-j} \in \{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}) \cdot q \\
= \; & \frac{\frac{P(\omega \in I)}{m_I} \cdot \frac{p}{\lambda - 1 + m_I}}{\frac{P(\omega \in I)}{m_I} \cdot p + \frac{P(\omega \in G)}{m_G} \cdot (1-p)} \cdot \frac{-\lambda^{x_r+1} - \lambda^{x_l} + \lambda^{x_r} + \lambda^{x_l+1}}{\lambda^{x_r} + \lambda^{x_l}} \cdot q \\
> \; & 0.
\end{aligned}$$

As the set of signal profiles possibly held by the other agents splits into pairs of the form $\{\hat{\sigma}, \hat{\sigma}_{x_l \longleftrightarrow x_r}\}$ this shows the result. The proof for g-signals is alike. □

Proof of Theorem 1. a) Given Lemmas 1-4 it remains to analyze under which circumstances a dove holding some g-signal wants to deviate by reporting some i-signal instead. So assume that agent j is a dove holding signal $s_j = g_l$ for some $l \in \{1, ..., m_G\}$ and considers reporting $m = i_r$ for some $r \in \{1, ..., m_I\}$. Clearly, by ex-ante symmetry of the model, the particular specifications of the g-signal and the i-report do not matter.

Following the proof of Lemma 3, for any profile $\sigma_{-j} \in Piv_{i_r,g_l}(q_j)$ a truthful report $m = g_l$ will trigger conviction while reporting $m = i_r$ will trigger acquittal. Truthful reporting hence constitutes an equilibrium if and only if

$$\begin{aligned} 0 &\leq Eu(m_j = g_l) - Eu(m_j = i_r) \\ &= -\sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_H)} Pr(\omega \in I | s_j = g_l) \cdot Pr(\sigma_{-j} | \omega \in I) \cdot q_D \\ &\quad + \sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_H)} Pr(\omega \in G | s_j = g_l) \cdot Pr(\sigma_{-j} | \omega \in G) \cdot (1 - q_D). \end{aligned}$$

The above expression is decreasing in q_D, hence the inequality holds for any

$$q_D \leq \frac{1}{P(\sigma_{-j} \in Piv_{i_r,g_l}(q_H) | s_j = g_l)} \sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_H)} P(\sigma_{-j} | s_j = g_l) \cdot \beta(\sigma_{-j} + \sigma_{g_l}).$$

Equality yields $\hat{q}_D(q_H)$.

b) A symmetric argument as in Part a) yields that truthfully reporting a signal i_r rather than deviating to a signal g_l is incentive compatible for a hawk if and only if

$$\begin{aligned} 0 &\leq Eu(m_j = i_r) - Eu(m_j = g_l) \\ &= -\sum_{\sigma_{-j} \in Piv_{i_l,g_r}(q_D)} Pr(\omega \in G | s_j = i_r) \cdot Pr(\sigma_{-j} | \omega \in G) \cdot (1 - q_H) \\ &\quad + \sum_{\sigma_{-j} \in Piv_{i_l,g_r}(q_D)} Pr(\omega \in I | s_j = i_r) \cdot Pr(\sigma_{-j} | \omega \in I) \cdot q_H. \end{aligned}$$

The above expression is increasing in q_H, hence the inequality holds for any

$$q_H \geq \frac{1}{P(\sigma_{-j} \in Piv_{i_r,g_l}(q_D) | s_j = i_r)} \sum_{\sigma_{-j} \in Piv_{i_r,g_l}(q_D)} P(\sigma_{-j} | s_j = i_r) \cdot \beta(\sigma_{-j} + \sigma_{i_r}).$$

Equality yields $\hat{q}_H(q_D)$.

c) Consider first the case where hawks have critical mass. Let σ_H be a hawk threshold profile; it exists by Assumption 1. We need to show that $\hat{q}_D(q_H) > \beta(\sigma_H)$ in which case σ_H is a conflict profile. Suppose agent j is a dove holding a g_1-signal and considers deviating by reporting $m_j = i_1$. If $q_D = \beta(\sigma_H)$ then hawks and doves agree on the optimal action for any signal profile, so truthful revelation dominates any deviation. By continuity of $Eu(m_j = g_1) - Eu(m_j = i_1)$ in q_D it therefore suffices to find a pivotal profile $\sigma_{-j} \in Piv_{i_1,g_1}(q_H)$ for which dove type $q_D = \beta(\sigma_H)$ holding a g_1-signal strictly prefers conviction over acquittal. In other words, σ_{-j} has to satisfy

$$\beta(\sigma_{-j} + \sigma_{i_1}) < \beta(\sigma_H) < \beta(\sigma_{-j} + \sigma_{g_1}) \tag{2.8.2}$$

After reshuffling x- and y-entries, we may assume without loss of generality that σ_H satisfies $x_1 \geq ... \geq x_{m_I}$, $y_1 \geq ... \geq y_{m_G}$. First, assume that $y_2 > 0$. Then the profile $\sigma_{-j} = \sigma_H - \sigma_{g_2}$ satisfies (2.8.2). Similarly, if $x_2 > 0$ then profile $\sigma_{-j} = \sigma_H - \sigma_{i_2}$ satisfies

(2.8.2). So suppose $x_2 = y_2 = 0$. If $y_1 = 0$ then $x_1 = n$ and $\sigma_H = (n, 0, ..., 0)$, so hawks would want to convict irrespective of any information. If $x_1 = 0$ then $y_1 = n$ and $\sigma_H = (0, ..., 0, y_1 = n, 0, ..., 0)$. Both cases contradict Assumption 1. Finally, assume $x_1 \neq 0 \neq y_1$. Since $n \geq 3$ we must have $x_1 \geq 2$ or $y_1 \geq 2$. In the first case, $\sigma_{-j} = \sigma_H - \sigma_{i_1} - \sigma_{g_1} + \sigma_{i_2}$ satisfies (2.8.2) while in the latter case $\sigma_{-j} = \sigma_H - \sigma_{i_1} - \sigma_{g_1} + \sigma_{g_2}$ does. This finishes the proof for hawks having critical mass. The proof for the case of doves having critical mass is alike. □

Proof of Theorem 2. All we need to show is that no other deviations apart from those analyzed in the main text can be profitable. Clearly, as in Lemma 3, no agent $j < k$ with $q_j \leq q_k$ has an incentive to misreport a g-signal as an i-signal as this, if anything, will lead to additional (undesired) acquittals compared to the TS outcome. Similarly, no agent $j > k$ with $q_j \geq q_k$ has an incentive to misreport an i-signal as a g-signal as this will lead to additional (undesired) convictions.

What remains is to analyze deviations from a truthful i_l-report to an i_r-report as well as from a truthful g_l-report to a g_r-report. In particular, we need to show that the potential influence of juror j at the voting stage does not alter the incentives as compared to Lemma 4. Consider some juror j. If for a given signal profile σ_{-j} of other agents both reports result in the same voting behavior of agent k, then agent j is indifferent. If the action is conviction, he agrees if $j < k$ and he cannot change the outcome at the voting stage if $j > k$. Similarly, if the action is acquittal, he agrees if $j > k$ and he cannot change the outcome at the voting stage if $j < k$.

If for a given signal profile σ_{-j} of other agents both reports result in different voting behavior of agent k, we follow the proof of Lemma 4. Suppose that agent j holds signal $s = i_l$ and considers to report $m = i_r$. Consider candidates $\hat{\sigma} = (x_1, ..., x_l, ..., x_r, ..., x_{m_I}, y_1, ..., y_{m_G})$ and $\hat{\sigma}_{x_l \longleftrightarrow x_r} = (x_1, ..., x_r, ..., x_l, ..., x_{m_I}, y_1, ..., y_{m_G})$ for σ_{-j} with $x_l > x_r$ as in Lemma 4. Then truthfully reporting $m = i_l$ will trigger agent k to vote for acquittal if $\sigma_{-j} = \hat{\sigma}$ and for conviction if $\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r}$ while reporting $m = i_r$ will trigger agent k to vote for conviction if $\sigma_{-j} = \hat{\sigma}$ and for acquittal if $\sigma_{-j} = \hat{\sigma}_{x_l \longleftrightarrow x_r}$. If $j < k$ and hence $q_j \leq q_k$ agent j will either prefer conviction for both, $\hat{\sigma}$ and $\hat{\sigma}_{x_l \longleftrightarrow x_r}$, or will prefer acquittal for $\hat{\sigma}$ and conviction for $\hat{\sigma}_{x_l \longleftrightarrow x_r}$. In the former case, as agent $j < k$ can never overturn an acquittal vote from agent k, the action is implemented according to the vote of agent k and the same calculation as in Lemma 4 applies. In the latter case, truthtelling will ensure the preferred outcome of agent j so there is no incentive to deviate. Similarly, if $j > k$ and hence $q_j \geq q_k$, agent j will either prefer acquittal for both, $\hat{\sigma}$ and $\hat{\sigma}_{x_l \longleftrightarrow x_r}$, or will prefer acquittal for $\hat{\sigma}$ and conviction for $\hat{\sigma}_{x_l \longleftrightarrow x_r}$. In the former case, as agent $j > k$ can never overturn a conviction vote from agent k, the action is implemented according to the vote of agent k and again the same calculation as in Lemma 4 applies. In the latter case, truthtelling will again ensure the preferred outcome of agent j so there is no incentive to deviate. The proof when agent j holds signal $s = g_l$ and considers to report $m = g_r$ is alike. □

Proof of Proposition 1. As $\lambda = 1$, $\beta(\sigma)$ is a function of $x = \sum_{l=1}^{m_I} x_l$ and $y = \sum_{l=1}^{m_G} y_l$ only. Write l_j for the minimal number of g-signals required by agent j to prefer conviction over acquittal. If $q_1 \nsim q_n$ and $q_1 < q_n$, we have $l_1 < l_n$. Moreover, write l_d for the minimal number of reported g-signals for which the designer convicts the agent and note that, by responsiveness, $1 \leq l_d \leq n$. If $l_d < l_j$ for some j, then agent j has a strict incentive to misreport a g-signal as an i-signal. Indeed, the misreport will only influence the outcome if other agents hold and report precisely $l_d - 1$ g-signals, in which case the total number of g-signals is $l_d < l_j$, so agent j prefers to trigger an acquittal. Conversely, if $l_d > l_j$ for some j, then agent j has a strict incentive to misreport an i-signal as a g-signal. The misreport will only influence the outcome if other agents hold and report precisely $l_d - 1$ g-signals, in which case the total number of g-signals is $l_d - 1 \geq l_j$, so agent j prefers to trigger a conviction. But since $l_1 < l_n$, one of the two cases must apply to either agent 1 or agent n. Hence decision rule q is not implementable unless $q_1 \sim ... \sim q_n$. $\square$

Chapter 3

A Baseline Model of Multidimensional Screening

3.1 Introduction

Various economic contexts feature multidimensional allocation problems that come along with multidimensional preferences on the consumer's side or multidimensional costs on the producer's side. Instances of such problems are (multiproduct) monopoly regulation, (multiproduct) procurement, or pricing problems that involve more than one good sold at the same time. Economic modeling of these instances has given rise to essentially equivalent mathematical frameworks involving multidimensional screening. Rather than singling out and discussing a particular application, this thesis chapter presents a formal solution approach to such a multidimensional screening model which goes back to Armstrong and Rochet (1999).[1] Generalizing upon the special case of additively separable utilities analyzed by Armstrong and Rochet, I provide a full classification of all qualitatively distinct potential solutions for arbitrary interactions between the good dimensions. Moreover, I relate particular properties of the solution to particular properties of the designer's objective.

The screening model I consider is characterized by binary (cost) types in each of two good dimensions. While this is likely to be a simplification with respect to most applications it keeps the optimization problem finite-dimensional and thus allows for an explicit characterization of the optimal allocation. Nevertheless, the task remains intricate as various deviations must be taken into account for each type. A key step in my solution approach is to identify different combinations of binding incentive constraints with different properties of the (optimal) allocation. This equivalence is based on an explicit cost-rent analysis. First, optimal transfers are characterized as a function of the allocation, starting from a "no-rents-at-the-bottom" result for the fully inefficient firm type. As a consequence, rents left to different firm types are also a function of

[1] While this thesis chapter does not focus on one particular application, to fix terminology and notation I present the model in terms of a regulation problem and use the respective terminology also for the rest of the introduction.

the allocation only. It thus becomes a matter of direct comparison to check which allocations render which deviations particularly attractive, taking both costs and rents into account.

The identification of particular combinations of binding constraint with properties of the allocation allows me to break down the complex global optimization problem that features kinks at hyperplanes where one binding incentive constraint is replaced by another into several smooth subproblems. Each of these subproblems corresponds to a particular combination of binding incentive constraints whose characterizing allocational properties are imposed as additional constraints. In a second step, I provide explicit necessary and sufficient conditions under which circumstances a solution of a given subproblem constitutes the solution to the global optimization problem.

The classification theorem that results from the above process is at the core of this thesis chapter. It provides an algorithmic way to solve the global optimization problem in a limited number of steps; moreover, it serves as a starting point for all later refinements. The general classification renders a large set of qualitatively different allocations possible, including "exotic" properties that cannot arise in corresponding unidimensional problems. Solutions may feature distortions for fully efficient types, upward binding incentive constraints and higher production levels for inefficient types as compared to efficient types within the same dimension.

With the general classification theorem bearing a spirit of "anything goes" the key question arises how different properties of the optimal allocation relate to economic assumptions on the fundamentals. The model exhibits three main characteristics subject to specification, namely the interaction between the dimensions through the valuation function, asymmetry between the dimensions through the valuation function, and the underlying distribution of types.

Interaction between both dimensions through the valuation function gives rise to new effects compared to the unidimensional case. In the unidimensional case, production costs fully determine the allocation qualitatively − even when abstracting from optimality − as monotonicity of the allocation in costs is equivalent to implementability. In multiple dimensions, the interplay of the allocations in both dimensions affects efficiency and may even overturn the cost effect. To see this, consider two firm types, type 1 being efficent in dimension 1 (labeled x) and inefficient in the dimension 2 (labeled y) and type 2 vice versa. Suppose furthermore that the y-allocations of both types are ordered as efficiency with respect to costs would suggest, namely type 2 producing a larger quantity in y, being the more cost-efficient firm in that dimension. Yet, a higher quantity in y for type 2 may render it efficient to have type 2 also producing a higher quantity in x despite the higher production costs if goods are sufficiently complementary. A similar phenomenon may occur for strategic substitutes when considering a fully efficient and a fully inefficient firm type.

The above effects apply in purest form when types are common knowledge and no distortions through rent concerns must be taken into account. However, a main finding of this thesis chapter is that most of the basic first-best features related to interaction between the dimensions transfer to the private information case.[2] The nature of interaction thus imposes fundamental ex ante restrictions on the optimal allocation that cannot be overturned by other factors. Invoking the equivalence between allocational properties and combinations of binding constraints, this implies that the latter set is likewise restricted under, say, complementarity or substitutability. What is more, the sets of potential combinations of binding constraints under complementarity and substitutability, respectively, intersect precisely in those constraint combinations that are feasible for the additively separable benchmark. Independent of any additional effects arising from distributional properties or (a-)symmetry, complementarity and substitutability are thus shown to push the optimal allocation in distinctly different directions.

Nevertheless, distributional properties and asymmetry between the dimensions in the valuation have substantial influence on the shape of the optimal allocation within the limits imposed by the interaction. Consider first potential concerns arising from distributional properties. The more likely a particular type is to occur, the more costly it becomes to distort its allocation away from the first-best. In the unidimensional case these considerations are well-known to determine the classical rents vs. efficiency trade-off, see e.g. Baron and Myerson (1982). In the multidimensional framework which is much more flexible as to which constraints bind, a less (downward) distorted type may easily become attractive as a target for deviations and thus "attract" binding incentive constraints from other types.[3] Distributional effects are by definition inherent only to the private information case and thus are driven by rent concerns rather than efficiency concerns.

Finally, (a-)symmetries between the dimensions in the valuation function also have a strong impact on the optimal allocation. On the one hand, they clearly influence efficiency and thus the first-best allocation by determining the relative size of the cost effect in a given dimension compared to the effect resulting from interaction. On the other hand, asymmetry also matters when incorporating rent concerns as it determines which of the two allocation dimensions is less costly to distort.

Changing our perspective slightly, we can also consider asymmetry and interaction between the two good dimensions in general, whether induced by the valuation or by the type distribution.[4] In particular, the complete absence of either of the two forces drastically restricts the set of potential solutions as was shown for the case of missing interaction in Armstrong and Rochet (1999) and is shown for the case of complete symmetry in this thesis chapter. What is more, the resulting allocations feature the standard unidimensional properties such as "no distortions at the top" and monotonic allocations with respect to production costs. Conversely, pushing the optimal allocations beyond these standard features thus requires a substantial interplay of asymmetry and

[2] One important exception applies to the case of strategic substitutes where some monotonicity relations from the first-best case may fail to hold under private information.

[3] In Armstrong and Rochet (1999) where valuations are additively separable, correlation between preference types is the main determinant as to which incentive constraints bind.

[4] Interaction in the type distribution is captured by correlation while asymmetry enters via different probabilities for the two efficient-inefficient type combinations.

interaction between the good dimensions.[5]

There is a large and diverse literature on multidimensional screening. A seminal reference is Rochet and Choné (1998) who analyze the general problem of multidimensional preference types and multidimensional allocations in continuous dimensions. While Rochet and Choné obtain valuable insights such as the existence of bunching regions for the optimal allocation, a general explicit solution to their model seems out of bounds.

Several papers have taken a complementary approach to the present thesis chapter by combining a simple (binary) allocation choice and with continuous type spaces. This strand of literature has mainly been developed in the framework of pricing problems. Adams and Yellen (1976) were the first to analyze bundling incentives in multiproduct pricing. Formalizing the same approach, McAfee et al. (1989) show that bundling generically improves upon separable pricing schemes when buying behavior can be monitored. Moreover, they show that for uncorrelated types a discount for buying the bundle improves revenues upon pricing both goods separately. More recently, Manelli and Vincent (2007) analyze the problem from a more general design perspective and find that optimal pricing schemes are typically stochastic, thereby adding a flair of continuity to the allocation dimensions which again comes at the cost of explicit solvability. In a complementary approach, Armstrong (2013) analyzes optimal deterministic mechanisms allowing for complementarity and substitutability in consumers' utility. A mixed approach involving one continuous and one binary preference dimension is analyzed in Ketelaar and Szalay (2014).[6]

Apart from the aforementioned paper by Armstrong and Rochet, the paper most closely related to my contribution is Severinov (2008) who also adds complementarity or substitutability to the Armstrong-Rochet model. The difference between Severinov's and my contribution is twofold. First, Severinov assumes uncorrelated preference types and thus eliminates one source of interaction which may otherwise impact the optimal allocation. Second, Severinov's objective is different from mine; rather than solving the optimization problem, Severinov compares different organizational forms to decide, among others, whether a single producer of two goods can be procured more or less efficiently than two seperate firms producing one good each.[7] On the technical side, I use the homotopy method from Severinov's paper which involves the continuous deformation of first order conditions to analyze the dynamics of allocational properties.

Finally, Litterscheid and Szalay (2014) analyze an almost identical model with positive correlation in a sequential context where the firm learns its information in dimension 2 only after having reported its type in dimension 1. As in this thesis chapter, properties of the allocation are linked to binding incentive constraints and the underlying fundamentals. While the sequential problem is simpler in the sense that there are less

[5]Armstrong and Rochet show for the additively separable case that distortions at the top can only be optimal if substantial asymmetry between good dimensions and sufficiently strong negative correlation between type dimensions occur jointly.

[6]See also Chapter 4 of this thesis.

[7]Indeed, some of Severinov's informal comments on the general solution concerning e.g. the irrelevance of upward constraints seem dubious in the light of the analysis here which features some explicit examples for the latter phenomenon in the proofs of Section 7.

deviations available, due to the uncertainty in the first stage out of equilibrium behavior proves tricky to analyze.

3.2 The Model and the Maximization Problem

A designer wants to regulate or procure a firm that produces two goods whose quantity or quality characteristics are captured by parameters $x, y \in \mathbb{R}_+$. The firm has production costs $C(x,y) = \theta x + \eta y$ where $\theta \in \{\theta_l, \theta_h\}$ and $\eta \in \{\eta_l, \eta_h\}$, $\theta_h > \theta_l > 0$, $\eta_h > \eta_l > 0$ are private knowledge of the firm. The designer only knows the ex ante probabilities of the four possible cost types, denoted by p_{ij} for type (θ_i, η_j), $i, j \in \{l, h\}$.[8] I write $\Delta_\theta = \theta_h - \theta_l$, $\Delta_\eta = \eta_h - \eta_l$.

The designer aims to maximize a given demand or utility function $V(x,y)$ subject to participation of the firm. $V(x,y)$ is strictly concave and twice continuously differentiable with non-degenerate (and hence negative definite) Hessian for all $x, y \in \mathbb{R}_+$. To ensure inner solutions, I impose some additional Inada conditions such as $\lim_{x\to 0} V_x(x,y) = \infty$ for all $y \in \mathbb{R}_+$, $\lim_{y\to 0} V_y(x,y) = \infty$ for all $x \in \mathbb{R}_+$, $\lim_{x\to\infty} V_x(x,y) = 0$ for all $y \in \mathbb{R}_+$, and $\lim_{y\to\infty} V_y(x,y) = 0$ for all $x \in \mathbb{R}_+$. Examples may depart from these particular conditions.

Invoking the revelation principle (see e.g. Myerson (1982)) I restrict attention to direct mechanisms. For each firm type (θ_i, η_j), $i, j \in \{l, h\}$ the designer commits to a production policy (x_{ij}, y_{ij}) and fixes a transfer payment T_{ij}. Firms' profits when reporting truthfully are then given as

$$\pi_{ij} = T_{ij} - \theta_i x_{ij} - \eta_j y_{ij} \quad \forall (i,j) \in \{l,h\}^2$$

while profits of a firm of type $(i,j) \in \{l,h\}^2$ reporting type $(r,s) \neq (i,j) \in \{l,h\}^2$ are denoted as

$$\pi_{ijrs} = T_{rs} - \theta_i x_{rs} - \eta_j y_{rs}.$$

The designer aims to solve[9]

$$\max_{x_{ij}, y_{ij}, T_{ij},\, i,j \in \{l,h\}} \mathbb{E}_{(i,j)} \left[V(x_{ij}, y_{ij}) - T_{ij} \right]$$

subject to participation of the firm, i.e.

$$\pi_{ij} \geq 0 \quad \forall i,j \in \{l,h\},$$

[8]I will often write (i,j) to refer to type (θ_i, η_j) in what follows.

[9]The designer's objective can be rewritten as

$$\max_{x_{ij}, y_{ij}, t_{ij},\, i,j \in \{l,h\}} \mathbb{E}_{(i,j)} \left[V(x_{ij}, y_{ij}) - T_{ij} \right] = \max_{x_{ij}, y_{ij}, \pi_{ij},\, i,j \in \{l,h\}} \mathbb{E}_{(i,j)} \left[V(x_{ij}, y_{ij}) - C(x_{ij}, y_{ij}) - \pi_{ij} \right].$$

In some contexts, the designer may also care about the profits of the firm, weighted by some factor $\alpha \in (0,1)$, thereby adding a factor $(1-\alpha)$ in front of π_{ij} in the objective. As this additional factor does not qualitatively affect the analysis, I will assume $\alpha = 0$ for sake of simplicity and notational convenience.

and incentive compatibility of truthful reports for all firm types, i.e.

$$\pi_{ij} \quad \geq \quad \max_{(r,s)\in\{l,h\}^2,\,(r,s)\neq(i,j)} \pi_{ijrs} \quad \forall\,(i,j)\in\{l,h\}^2\,.$$

I write IR_{ij} to refer to the individual rationality (participation) constraint of the firm of type $(i,j) \in \{l,h\}^2$. The incentive compatibility constraint of firm $(i,j) \in \{l,h\}^2$ not to mimick firm $(r,s) \neq (i,j) \in \{l,h\}^2$ is referred to as IC_{ijrs}. If I refer to all three IC-constraints of type $(i,j) \in \{l,h\}^2$ simultaneously, I write IC_{ij}.

3.3 Implementability

In the unidimensional baseline model with discrete or continuous types it is a well-known result that implementability is equivalent to monotonicity of the solution in the reported parameter (see e.g. Baron and Myerson (1982)). This insight is in line with the intuition that the higher the production costs of a good are, the lower is the optimal output. With multiple goods, this intuition applies only partially as I demonstrate in the next proposition. While, to the best of my knowledge, the result for this particular setup has not been stated before explicitely, it can be derived as a direct implication of Theorem 1 in Rochet (1987). We may interpret our optimization problem as a network flow problem where each type $(i,j) \in \{l,h\}^2$ constitutes a node and is linked to any other type $(r,s) \in \{l,h\}^2$ by an arc of length $C(\theta_i,\eta_j,x_{ij},y_{ij}) - C(\theta_i,\eta_j,x_{rs},y_{rs})$ for a given allocation $(x,y) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$. Following Rochet (1987), (x,y) is implementable by some transfers $T(x,y)$ if and only if the network does not feature a cycle of positive length. In our setup this translates into the following proposition.

Proposition 1. *An allocation $(x,y) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$ is implementable through transfer* *payments $T = T(\theta, \eta)$ if and only if the following 16 inequalities hold:*

$$x_{ll} - x_{hl} \geq 0 \tag{I1}$$
$$y_{ll} - y_{lh} \geq 0 \tag{I2}$$
$$x_{lh} - x_{hh} \geq 0 \tag{I3}$$
$$y_{hl} - y_{hh} \geq 0, \tag{I4}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0 \tag{I5}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) \geq 0 \tag{I6}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) \geq 0 \tag{I7}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0 \tag{I8}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0 \tag{I9}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) \geq 0 \tag{I10}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) \geq 0 \tag{I11}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) \geq 0 \tag{I12}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) \geq 0 \tag{I13}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) \geq 0 \tag{I14}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0 \tag{I15}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) \geq 0. \tag{I16}$$

We denote the set of implementable allocations, i.e. all allocations $(x, y) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$ that satisfy the above 16 inequalities, by $\mathcal{I} \subset \mathbb{R}_+^4 \times \mathbb{R}_+^4$.

Proposition 1 can be read from two different points of view. First, it shows that one-dimensional monotonicity as stated in equations (I1)-(I4) “all else equal” is no longer sufficient for implementability but must be complemented by twelve additional conditions that link the two dimensions as stated in equations (I5)-(I16).

On the other hand, implementability in the multidimensional setup no longer implies that low-cost types in a particular dimension produce more than high-cost types in the same dimension. Indeed, equations (I1)-(I16) allow for the possibility that

$$x_{ll} - x_{hh} < 0,$$
$$x_{lh} - x_{hl} < 0,$$
$$y_{ll} - y_{hh} < 0,$$
$$y_{hl} - y_{lh} < 0,$$

while at most one of these inequalities can hold at a time. As I will demonstrate later on, and has partially been demonstrated by Severinov (Severinov (2008)) before, each of these inequalities may hold in an optimal mechanism if the interaction between both dimensions through $V(x, y)$ is sufficiently strong.

3.4 First Best Allocations

For later comparison it is useful to start with a close look at the first best scenario where the firm's type is known to the designer. In that case the designer leaves no profits to the firm and sets production quantities such that marginal valuations equal (constant) marginal costs of production, that is

$$V_x(x_{ij}, y_{ij}) = \theta_i$$
$$V_y(x_{ij}, y_{ij}) = \eta_j$$

for all $i, j \in \{l, h\}$. A central issue of this thesis chapter (cf. Section 6) is to analyze as to which extent properties of the first best allocation transfer to the second best allocation in the private information case. Since the relevant intuitions apply in most pure form to the undistorted first best case, I provide a brief discussion here.

Proposition 2. *The first best allocation has the following properties:*
a) $x_{ll} - x_{hl} > 0$, $x_{lh} - x_{hh} > 0$, $y_{ll} - y_{lh} > 0$, $y_{hl} - y_{hh} > 0$.
b) If goods are complements, then $x_{ll} > x_{lh}$, $x_{hl} > x_{hh}$, $y_{ll} > y_{hl}$, $y_{lh} > y_{hh}$.
c) If goods are substitutes, then $x_{ll} < x_{lh}$, $x_{hl} < x_{hh}$, $y_{ll} < y_{hl}$, $y_{lh} < y_{hh}$.
d) If $\frac{|V_{yy}(x,y)|}{\Delta_\eta} \geq \frac{|V_{xy}(x,y)|}{\Delta_\theta}$ *for all* $(x, y) \in \mathbb{R}^2_+$, *then* $x_{ll} > x_{hh}$, $x_{lh} > x_{hl}$. *If* $\frac{|V_{xx}(x,y)|}{\Delta_\theta} \geq \frac{|V_{xy}(x,y)|}{\Delta_\eta}$ *for all* $(x, y) \in \mathbb{R}^2_+$ *then* $y_{ll} > y_{hh}$, $y_{hl} > y_{lh}$.

Part a) of Proposition 2 seems straightforward. Production is cheaper for low cost types than for high cost types, so the efficient type should produce more. A bit more formally, decreasing the production costs should increase quantities due to decreasing marginal valuations in each dimension, captured by $V_{xx} < 0$ and $V_{yy} < 0$, respectively. While this essentially unidimensional intuition holds true in our multidimensional setup, it is nevertheless worth to take a slightly closer look. Indeed, gradually decreasing θ_h to θ_l, say, typically impacts the allocation in both dimensions unless valuations are additively separable. Taking derivatives with respect to t of the system of first order conditions given as

$$\begin{aligned} V_x(x, y) &= \theta_h - t \cdot \Delta_\theta \\ V_y(x, y) &= \eta \end{aligned}$$

yields

$$\begin{aligned} \frac{dx}{dt} \cdot V_{xx}(x, y) + \frac{dy}{dt} \cdot V_{xy}(x, y) &= -\Delta_\theta \\ \frac{dx}{dt} \cdot V_{xy}(x, y) + \frac{dy}{dt} \cdot V_{yy}(x, y) &= 0 \end{aligned}$$

which has a unique solution

$$\begin{aligned} \frac{dx}{dt} &= \frac{-\Delta_\theta \cdot V_{yy}(x, y)}{V_{xx}(x, y) \cdot V_{yy}(x, y) - V_{xy}(x, y)^2} \\ \frac{dy}{dt} &= \frac{\Delta_\theta \cdot V_{xy}(x, y)}{V_{xx}(x, y) \cdot V_{yy}(x, y) - V_{xy}(x, y)^2}. \end{aligned}$$

Note that it is indeed the negative sign of $V_{xx}(x, y)$ that determines the sign of the effect on x. The relative size of the effects on x and y, however, is measured by $|V_{yy}(x, y)|$ and $|V_{xy}(x, y)|$, respectively. This may be surprising at first sight, given that it is the cost parameter relating to the x-dimension that changes. However, it is the constant value of η in $V_y(x, y) = \eta$ that determines which two effects must outweigh each other, namely those coming from $|V_{yy}(x, y)|$ and $|V_{xy}(x, y)|$.

As just demonstrated, to keep both first order equations satisfied simultaneously a shift in x, say, coming from a change in θ will typically require the allocation in the second dimension to be adjusted as well. This is reflected in Parts b) and c) of Proposition 2. When gradually reducing θ_h to θ_l while keeping η fixed, under complementarity the increase in x must be compensated by an increase in y to keep equation $V_y(x, y) = \eta$ satisfied. Conversely, for the case of substitutes, y must decrease to compensate for an increase in x. Intuitively, under complementarity the value of producing more in dimension y is larger the more is produced in dimension x and vice versa. Since low cost types in θ will produce a larger amount x than high cost types, y is decreasing in θ. For substitutes the converse intuition applies.

Part d) of Proposition 2 is based on both previously described effects. Assume for the moment that $\Delta_\theta = \Delta_\eta$. Thinking unidimensionally, the same intuition as in Part a) should apply to the inequalities in Part d) as well. However, there is a caveat; this time, not only the cost parameter in the θ-x-dimension, say, is varied but the cost parameter in the η-y-dimension changes simultaneously, giving rise to a separate effect on the x-allocation as described in Parts b) and c), respectively. If the latter effect, measured by $|V_{xy}(x, y)|$ as shown above, is stronger than the effect through θ, measured by $|V_{yy}(x, y)|$, the intuition for Part a) may be overturned. The assumption in Part d) is therefore needed to rule out this case with certainty. If $\Delta_\theta \neq \Delta_\eta$, the same forces occur. However, the size of the two potentially opposing effects has to be adjusted by the inverse of the path length from one cost type to the other in the respective dimension. To get a qualitative intuition, consider a situation where both θ-types are almost identical so that Δ_θ is very small compared to Δ_η. In this case, the effect from changing θ (Part a) on x is very weak as there is hardly any difference in production costs for x conditional on θ. Hence the effect from changing η (Part b/c) dominates. Monotonicity of x in θ may hence fail to hold across η-types. Changing η also dominates changing θ with respect to y but has the reversed implication here; the effects driving Part a) dominate those from Part b/c), so low cost types in η will always produce a higher quantity of y than high cost types in η if Δ_θ is small.

3.5 A General Solution Approach

In this section I present the main structural results of this thesis chapter. I establish a characterization of the set of binding IC-constraints in terms of properties of the optimal allocation. This allows me to break down the non-smooth global optimization problem into eleven smooth subproblems that correspond to different properties of the allocation. All possible solutions of the global optimization problem can be characterized as solutions of these subproblems.

I start by a sequence of lemmas that simplify the original optimization problem with respect to transfers and constraints. As the arguments are easy to follow I omit formal proofs.

Lemma 1. *For each allocation $(x, y) \in \mathcal{I}$, optimal transfers are such that IR_{hh} binds while all other IR-constraints are slack. Moreover, IC_{hl}, IC_{lh} and IC_{ll} bind.*

The intuition for Lemma 1 is straightforward. Clearly, at least one IR-constraint must bind as otherwise all transfers could be reduced by an identical small amount. As any (partially or fully) efficient firm could produce the inefficient firm's quantities at lower costs, they would rather mimick the fully inefficient type than abstain from participation. Hence it must be the inefficient firm's IR-constraint that binds while all others are slack. But then the IC-constraint of any other firm type must bind as otherwise the designer could reduce the transfer to the respective type.

With IR_{hh} binding we have a standard no-rent-at-the-bottom scenario. In particular, whatever the optimal allocation $(x, y) \in \mathcal{I}$ looks like, the transfer payment to the fully inefficient firm is given as $T_{hh} = \theta_h x_{hh} + \eta_h y_{hh}$. As the next lemma states, the remaining optimal transfers are also given as a function of the allocation only.

Lemma 2. *Each type $(i, j) \neq (h, h)$ must be linked to type (h, h) via a path of binding IC-constraints. There exist a unique transfer function $T : \mathcal{I} \to \mathbb{R}_+^4$ such that $T(x, y)$ maximizes the objective for given $(x, y) \in \mathcal{I}$.*

Lemma 2 eliminates transfers as independent variables from the problem and thereby considerably simplifies the optimization task. As just argued the optimal transfer payment for type (h, h) is determined for any given implementable allocation through the binding participation constraint IR_{hh}. In turn, this consecutively determines the optimal transfers for any other type that is linked to (h, h) via a path of binding IC-constraints. But this must apply to all other types as otherwise one could reduce the transfers for all types that are not linked to (h, h) via a path of binding IC-constraint by an uniform small amount.

In what follows, when referring to an implementable allocation $(x, y) \in \mathcal{I}$ I will always silently assume that it comes along with optimal transfer payments $T(x, y) \in \mathbb{R}_+^4$.

All arguments to derive Lemma 1 and Lemma 2 have been made without any reference to the fully inefficient type's incentive compatibility constraint IC_{hh}. As a consequence, once we combine an implementable allocation with the optimal transfers according to Lemma 2, IC_{hh} should be satisfied automatically. Indeed, any type different from (h, h) is linked to the fully inefficient type (h, h) by a path of binding IC-constraints. So if IC_{hhij} was violated for some type $(i, j) \neq (h, h)$, this were to result in a cycle of positive length, following the path of binding IC-constraints from (i, j) to (h, h) and then closing the cycle along the link from (h, h) to (i, j). But the non-existence of cycles of positive length is precisely the necessary and sufficient condition for implementability as pointed out in Section 2. Hence we get

Lemma 3. *Let $(x, y) \in \mathcal{I}$ be an implementable allocation together with optimal transfers $T(x, y)$. Then IC_{hh} is implied by implementability.*

Lemma 2 has another direct implication which is at the core of our solution approach. Given that optimal transfers are uniquely determined by the allocation, the allocation also fully determines the set of binding IC-constraints. We can therefore identify particular sets of binding IC-constraints with particular subsets of the set of implementable allocations. Within each of these subsets, transfers are just linear functions of the allocation as they are always determined through the same set of binding IC-constraints. Moreover, incentive compatibility is no longer an issue as it is inherent in the definition of the optimal transfers $T(x, y)$. Altogether, we can split the complex original optimization problem that features non-differentiable IC-constraints into several smooth subproblems that correspond to particular combinations of binding IC-constraints. Each of this subproblems is then a standard Kuhn-Tucker-Problem with two types of constraints: implementability constraints ensuring $(x, y) \in \mathcal{I}$ as well as "regional" constraints ensuring that the optimum indeed lies in the subset of $\mathcal{I}$ that corresponds to the combination of binding IC-constraints under consideration. The next proposition formalizes these ideas and provides explicit definitions.

Proposition 3. *a) The set of implementable allocations is the union of eleven closed and convex sets* $\mathcal{I} = \bigcup_{i \in I} R_i$, $R_i \subset \mathcal{I}$, $i \in I = \{1a, 1b, 2, 3a, 3b, 4a, 4b, 5a, 5b, 6a, 6b\}$ *with disjoint interior of positive mass that have the following properties:*
$(x, y) \in R_{1a}$ *if and only if* IC_{lllh}, IC_{lhhh}, IC_{hlhh} *hold with equality if and only if*

$$\Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0 \tag{R1a,1b}$$
$$\Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \tag{R1a,2}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) - \Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \tag{R1a,4a}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hh}) - \Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \tag{R1a,5a}$$
$$\Delta_\eta \cdot (y_{hl} - y_{hh}) - \Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0. \tag{R1a,6b}$$

$(x, y) \in R_{1b}$ *if and only if* IC_{llhl}, IC_{lhhh}, IC_{hlhh} *hold with equality if and only if*

$$\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \tag{R1b,1a}$$
$$\Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0 \tag{R1b,2}$$
$$\Delta_\eta \cdot (y_{ll} - y_{hh}) - \Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0 \tag{R1b,4b}$$
$$\Delta_\eta \cdot (y_{hl} - y_{hh}) - \Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0 \tag{R1b,5b}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hh}) - \Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0. \tag{R1b,6a}$$

$(x, y) \in R_2$ *if and only if* IC_{llhh}, IC_{lhhh}, IC_{hlhh} *hold with equality if and only if*

$$\Delta_\eta \cdot (y_{hh} - y_{lh}) \geq 0 \tag{R2,1a}$$
$$\Delta_\theta \cdot (x_{hh} - x_{hl}) \geq 0 \tag{R2,1b}$$
$$\Delta_\theta \cdot (x_{ll} - x_{hh}) \geq 0 \tag{R2,3a}$$
$$\Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0. \tag{R2,3b}$$

$(x, y) \in R_{3a}$ if and only if IC_{llhh}, IC_{lhhh}, IC_{hlll} hold with equality if and only if

$$\Delta_\theta \cdot (x_{hh} - x_{ll}) \geq 0 \tag{R3a,2}$$
$$\Delta_\eta \cdot (y_{hh} - y_{lh}) \geq 0. \tag{R3a,4a}$$

$(x, y) \in R_{3b}$ if and only if IC_{llhh}, IC_{lhll}, IC_{hlhh} hold with equality if and only if

$$\Delta_\eta \cdot (y_{hh} - y_{ll}) \geq 0 \tag{R3b,2}$$
$$\Delta_\theta \cdot (x_{hh} - x_{hl}) \geq 0. \tag{R3b,4b}$$

$(x, y) \in R_{4a}$ if and only if IC_{lllh}, IC_{lhhh}, IC_{hlll} hold with equality if and only if

$$\Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{ll} - x_{hh}) \geq 0 \tag{R4a,1a}$$
$$\Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \tag{R4a,3a}$$
$$\Delta_\theta \cdot (x_{lh} - x_{ll}) \geq 0. \tag{R4a,5a}$$

$(x, y) \in R_{4b}$ if and only if IC_{llhl}, IC_{lhll}, IC_{hlhh} hold with equality if and only if

$$\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{ll} - y_{hh}) \geq 0 \tag{R4b,1b}$$
$$\Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0 \tag{R4b,3b}$$
$$\Delta_\eta \cdot (y_{hl} - y_{ll}) \geq 0. \tag{R4b,5b}$$

$(x, y) \in R_{5a}$ if and only if IC_{lllh}, IC_{lhhh}, IC_{hllh} hold with equality if and only if

$$\Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{lh} - x_{hh}) \geq 0 \tag{R5a,1a}$$
$$\Delta_\theta \cdot (x_{ll} - x_{lh}) \geq 0 \tag{R5a,4a}$$
$$\Delta_\theta \cdot (x_{lh} - x_{hl}) \geq 0. \tag{R5a,6a}$$

$(x, y) \in R_{5b}$ if and only if IC_{llhl}, IC_{lhhl}, IC_{hlhh} hold with equality if and only if

$$\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{hl} - y_{hh}) \geq 0 \tag{R5b,1b}$$
$$\Delta_\eta \cdot (y_{ll} - y_{hl}) \geq 0 \tag{R5b,4b}$$
$$\Delta_\eta \cdot (y_{hl} - y_{lh}) \geq 0. \tag{R5b,6b}$$

$(x, y) \in R_{6a}$ if and only if IC_{llhl}, IC_{lhhh}, IC_{hllh} hold with equality if and only if

$$\Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{lh} - x_{hh}) \geq 0 \tag{R6a,1b}$$
$$\Delta_\theta \cdot (x_{hl} - x_{lh}) \geq 0. \tag{R6a,5a}$$

$(x, y) \in R_{6b}$ if and only if IC_{lllh}, IC_{lhhl}, IC_{hlhh} hold with equality if and only if

$$\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{hl} - y_{hh}) \geq 0 \tag{R6b,1a}$$
$$\Delta_\eta \cdot (y_{lh} - y_{hl}) \geq 0. \tag{R6b,5b}$$

b) The optimal transfer function $T : \mathcal{I} \to \mathbb{R}^4_+$ is continuous, convex, and linear within any R_i, $i \in I$, with kinks precisely along the hyperplanes that separate the above regions.
c) The optimal mechanism is unique.

The central part of Proposition 3 is the characterization of combinations of binding IC-constraints in terms of the allocation as given in Part a). The inequalities are derived from an explicit comparison of firm rents. Consider for example region R_{1a} with binding IC-constraints IC_{lllh}, IC_{lhhh}, and IC_{hlhh} as depicted in Figure 1.

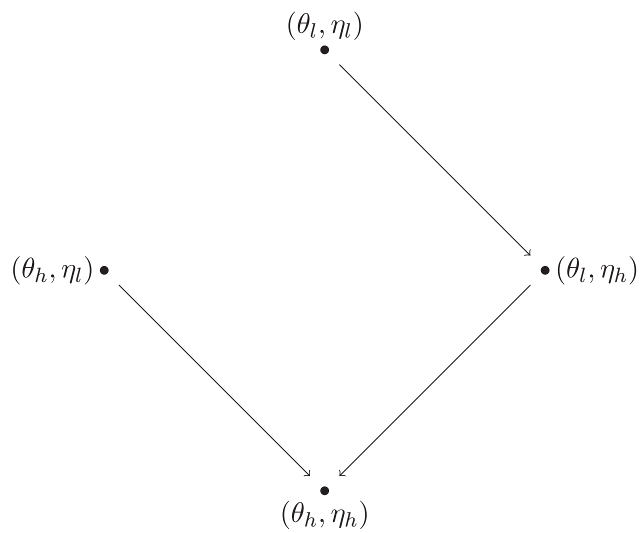

Figure 1: Region R_{1a}

The rents of types (h, l), (l, h) and (l, l) with the above combination of binding IC-constraints are given as follows. Type (h, l) is left a rent $\Delta_\eta y_{hh}$ for not imitating type (h, h). Likewise, type (l, h) is left a rent $\Delta_\theta x_{hh}$ for not imitating type (h, h). To ensure that type (l, l) does not imitate type (l, h) he must be left the rent that is left to type (l, h) plus an additional rent $\Delta_\eta y_{lh}$. The total rent of type (l, l) thus amounts to $\Delta_\theta x_{hh} + \Delta_\eta y_{lh}$.

Suppose now that type (l, l) considers to imitate type (h, l) rather than type (l, h). In that case he could claim a rent $\Delta_\eta y_{hh} + \Delta_\theta x_{hl}$. Imitating type (l, h) is thus more attractive than imitating type (h, l) if and only if $\Delta_\theta x_{hh} + \Delta_\eta y_{lh} - (\Delta_\eta y_{hh} + \Delta_\theta x_{hl}) \geq 0$, which constitutes constraint (R1a,1b). If type (l, l) considers imitating type (h, h), his rents would amount to $\Delta_\eta y_{hh} + \Delta_\theta x_{hh}$. As this is supposed to be less attractive than imitating type (l, h), one must have $\Delta_\eta y_{lh} - \Delta_\eta y_{hh} \geq 0$ as stated in (R1a,2). In the same way, one derives the remaining inequalities for R_{1a} as well as for all other regions. Note that some combinations of binding constraints have been excluded by Lemmas 1-2 and some others may violate implementability. Conversely, it is straightforward to check that all the eleven regions in Proposition 3 have non-empty interior within $\mathcal{I}$.

Part a) of Proposition 3 yields eleven convex regions that are separated by hyperplanes along which the constraint that sets the two regions apart binds. We call two regions adjacent if they are separated by a hyperplane of codimension 1. For any hyperplane bounding region R_i, the label of the corresponding hyperplane inequality indicates which region is adjacent to R_i along this hyperplane. A graphical illustration of Proposition 3, Part a) is given in Figure 1. Arcs are drawn between any pair of adjacent regions except for the inner part consisting of regions R_{1a}, R_{1b}, and R_2 where boundaries are depicted directly.[10]

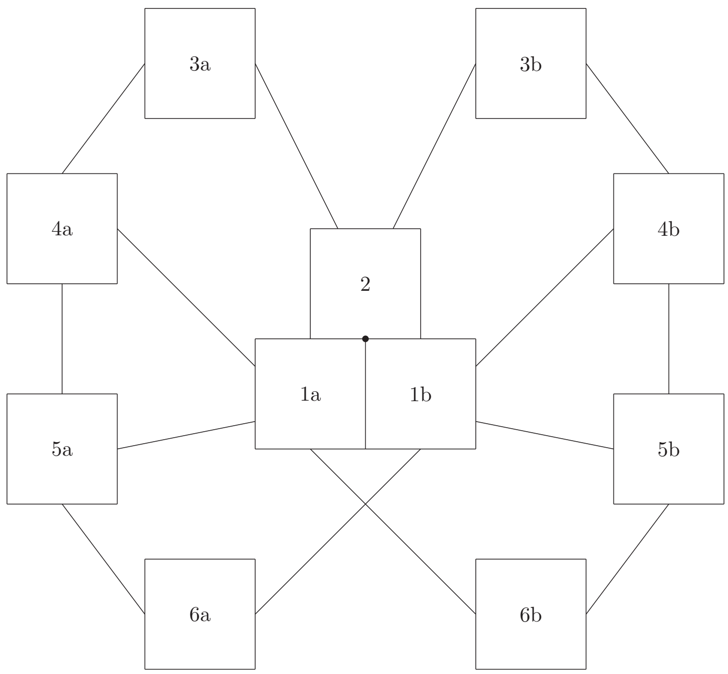

Figure 2: Regions

Figure 2 also highlights some structural symmetries across regions that stem from the fact that the model does not feature any prespecified asymmetries between the θ-x-dimension and the η-y-dimension. Apart from Region 2 which is symmetric with respect to both dimensions, each "a"-region has a "b"-region as a counterpart where the roles of the two dimensions are interchanged.

Finally, Figure 2 optically suggests that regions R_{1a}, R_{1b} and R_2 are somehow "central" within $\mathcal{I}$ whereas the other regions are more "peripheric". Despite the fact that the representation of an 8-dimensional structure in a plane is necessarily inadequate, in the course of this thesis chapter I will add to the idea that regions R_{1a}, R_{1b} and R_2 indeed play a central role for solving the problem in many cases.

[10] Clearly, Figure 1 is a vastly simplified representation of the set $\mathcal{I}$ where arcs connecting squares correspond to 7-dimensional hyperplanes separating 8-dimensional polyhedrons.

Transfers necessarily reflect the properties of the IC-constraints they need to keep satisfied as stated in Part b) of Proposition 3. The global problem thus consists of maximizing a continuous and strictly concave function $\mathbb{E}[V-T]$ over a convex set $\mathcal{I}$ which immediately yields uniqueness of the solution. The eleven subproblems arising from Proposition 3 involve the maximization of a strictly concave smooth objective $\mathbb{E}[V-T]$ over a convex eight-dimensional set of allocations R_i, $i \in I$, subject to various linear constraints. The solution to each subproblem is hence characterized by four pairs of first-order conditions (FOCs in what follows) of the following form

$$V_x(x_{ij}, y_{ij}) = \theta_i + \frac{1}{p_{ij}} \cdot \left[\sum_{(r,s)\in\{l,h\}^2} \left[p_{rs} \cdot \frac{\partial \pi_{rs}}{\partial x_{ij}} \right] + \lambda_{\text{implementability}} + \mu_{\text{region}} \right], \tag{3.5.1}$$

$$V_y(x_{ij}, y_{ij}) = \eta_j + \frac{1}{p_{ij}} \cdot \left[\sum_{(r,s)\in\{l,h\}^2} \left[p_{rs} \cdot \frac{\partial \pi_{rs}}{\partial y_{ij}} \right] + \lambda_{\text{implementability}} + \mu_{\text{region}} \right]. \tag{3.5.2}$$

As we see, there are three potential sources of distortion relative to the first best; rents of the firms, implementability concerns and the requirement to stay within the region under consideration. Note that implementability constraints (I1)-(I16) are global constraints and hence identical across all regions.[11] Moreover, all implementability constraints and all regional constraints as well as all transfers/profits are linear in (x, y) for a given region, hence the right hand side of the FOCs does not depend on (x, y) directly. To systematically label different multipliers attached to different constraints, let λ_j, $j \in \{1, ..., 16\}$ refer to implementability constraint (Ij) from (I1)-(I16). If it is helpful to additionally specify the region under consideration, I write $\lambda_{i,j}$, $i \in I$, $j \in \{1, ..., 16\}$. Similarly, $\mu_{i,j}$ refers to the hyperplane constraint separating region R_i from the adjacent region R_j, $i, j \in I$.

Proposition 3 and the insights from the previous paragraph provide an explicit way to solve the global optimization problem by standard methods: solve all eleven Kuhn-Tucker subproblems via the FOCs of the respective region and then compare which of the eleven solutions yields the highest objective. From an algorithmic point of view this raises three questions. First, can we know ex ante in which region(s) to expect the solution? Second, is it possible to directly decide whether a solution of a subproblem is globally optimal? Third, can we ease the regional problems, e.g. by excluding ex ante certain combinations of binding constraints?

Partial or full answers to any of these questions will be given in what follows. Economically, the first question clearly sticks out compared to the other two. Indeed, the subsequent two sections will be devoted almost exclusively to the question which economic or economically motivated properties of the optimization problem will imply the optimal allocation to feature particular properties associated with different regions. However, it is quite useful to proceed towards a map of the possible solutions in full generality beforehand, answering parts of Questions 2 and 3 en passant. We start with the following proposition.

[11] Different implementability constraints, however, will be relevant for different regions. In addition, as implementability constraints are not necessarily orthogonal to regional constraints, multipliers attached to implementability constraints may take different values for different regions if the optimum lies on the boundary between different regions as well as on the implementability frontier.

Proposition 4. *$(x, y) \in \mathcal{I}$ solves the global optimization problem if and only if it solves the subproblem for any region R_i, $i \in I$ for which $(x, y) \in R_i$.*

The necessity of the condition in Proposition 4 is trivial. Obviously, an allocation cannot be globally optimal if it is not even optimal within a given region R_i to which it belongs. The converse is a consequence of the concavity of the global problem. Suppose (x, y) satisfies the condition of Proposition 4 but does not solve the global problem. Then, due to concavity, the objective must strictly increase along the line between (x, y) and the global optimum. But as parts of this line must lie in some region R_i which contains (x, y), this contradicts the assumed optimality of (x, y) in the respective region.

As an immediate consequence of Proposition 4, any solution to some subproblem R_i that is interior in R_i solves the global problem. However, given the kinkiness of the global problem, we cannot generically expect the global solution to lie in the interior of some region R_i. The following classification theorem lists all possible loci for the optimal solution.

Theorem 1. *Let $(x, y) \in \mathcal{I}$ solve the global optimization problem. Then one of the following cases applies:*

a) (x, y) lies in the interior of $R_i \subset \mathcal{I}$ for some $i \in I$.[12] *(11 cases)*

b) $(x, y) \in R_i \cap R_j$ for $i, j \in I$ and $(x, y) \notin R_k$ for any $k \neq i, j \in I$. (17 cases)

c) One of the following nine cases applies:

c1) $(x, y) \in R_{1a} \cap R_{1b} \cap R_2$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 1b, 2\}$.

c2) $(x, y) \in R_{1a} \cap R_{4a} \cap R_{5a}$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 4a, 5a\}$.

c3) $(x, y) \in R_{1b} \cap R_{4b} \cap R_{5b}$ and $(x, y) \notin R_j$ for any $j \notin \{1b, 4b, 5b\}$.

c4) $(x, y) \in R_{1a} \cap R_{1b} \cap R_{5a} \cap R_{6a}$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 1b, 5a, 6a\}$.

c5) $(x, y) \in R_{1a} \cap R_{1b} \cap R_{5b} \cap R_{6b}$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 1b, 5b, 6b\}$.

c6) $(x, y) \in R_{1a} \cap R_2 \cap R_{3a} \cap R_{4a}$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 2, 3a, 4a\}$.

c7) $(x, y) \in R_{1b} \cap R_2 \cap R_{3b} \cap R_{4b}$ and $(x, y) \notin R_j$ for any $j \notin \{1b, 2, 3b, 4b\}$.

c8) $(x, y) \in R_{1a} \cap R_{4a} \cap R_{6b}$ and $(x, y) \notin R_j$ for any $j \notin \{1a, 4a, 6b\}$.

c9) $(x, y) \in R_{1b} \cap R_{4b} \cap R_{6a}$ and $(x, y) \notin R_j$ for any $j \notin \{1b, 4b, 6a\}$.

According to Theorem 1, the optimal solution always lies either in the interior of some region, or in the intersection (hyperplane) between exactly two regions, or within particular intersections of respectively three or four regions. The proof of Theorem 1 involves a thorough technical analysis of the FOCs for each of the eleven regions. On the one hand, any intersection of regions apart from those that are part of the classification must be excluded. On the other hand, I show that for any intersection of regions that is part of the classification there exist non-negative multipliers such that the FOCs of all involved regions are satisfied simultaneously.

Tying in with the latter point, in the Appendix I provide an extended version of Theorem 1, labeled Theorem 1*, which provides additional input to any of the 37 distinct loci (distinct in terms of intersections of regions R_i) that are potential global

[12] Interior refers to the metric space $\mathcal{I}$. In particular, any point on the boundary of $\mathcal{I}$ within $\mathbb{R}^8$ that belongs to only one region R_i is meant to be an interior point of R_i.

optima according to Theorem 1. Coming back to the previously formulated algorithmic questions, I specify the set of implementability constraints that may bind in addition at any of these loci, answering Question 3. Similarly, necessary and sufficient conditions for the regional multipliers in each region are provided that directly allow us to decide whether a solution of some regional subproblem solves the global problem as well or not, answering Question 2. Roughly speaking, while multipliers attached to implementability constraints may a priori become arbitrarily large, regional multipliers are bounded from above. This is quite intuitive; while implementability must be enforced at all costs, staying in a particular region will only be optimal if the additional distortions do not become too costly. Rather than letting some regional multiplier grow too large it will be more profitable to cross the boundary towards another region. The only exceptions from this intuition are cases c8) and c9) where $(x, y) \in R_{1a} \cap R_{4a} \cap R_{6b}$ or $(x, y) \in R_{1b} \cap R_{4b} \cap R_{6a}$ imply that respectively (I12) or (I11) hold with equality, mixing up regional and implementability concerns.

While these extensions of Theorem 1 are useful from the algorithmic perspective as well as for some later proofs, their flair is purely technical. I therefore only state a handy consequence here in the main text.

Proposition 5. *The optimal allocation features $x_{ll} > x_{hl}$, $y_{ll} > y_{lh}$, $x_{lh} > x_{hh}$, $y_{hl} > y_{hh}$.*

Proposition 5 states that strict one-dimensional monotonicity "all else equal" as stated in Proposition 2 continues to hold in the private information setup. As a consequence, monotonicity constraints I1-I4 do not play any role in our multidimensional framework; only those implementability constraints that link both dimensions are of potential relevance.

3.6 Adding Economic Structure

The solution approach of the previous section that culminates in the classification of potential solutions in Theorem 1 is useful on its own. It provides a general overview of what is possible and what is not, together with an algorithm how to ultimately determine the solution of any specified optimization problem covered by the model. Yet, making no additional structural assumptions whatsoever the algorithm as such does not help us to relate particular specifications of the general optimization problem to particular solution patterns. In this section, I complement the previous algorithmic analysis by demonstrating how additional economically motivated assumptions restrict and influence the qualitative features of the optimal allocation.

The underlying model offers two main characteristics for specification: interaction between the two dimensions and asymmetries between the two dimensions, both captured by $V(x, y)$ as well as the underlying distribution over types.[13] In this section I mainly focus on characteristics of $V(x, y)$ and their impact on the optimal allocation

[13] Armstrong and Rochet (1999) exlude interaction between dimensions through $V(x, y)$ while Severinov (2008) excludes interaction between dimensions in the type distribution.

while the parametrized example in Section 7 takes the interplay between distributional issues and properties of $V(x,y)$ into account.

I start with two benchmark cases, namely the absence of interaction through $V(x,y)$ and complete symmetry. First, recall the result in Armstrong and Rochet (1999) for the case of additively separable valuations $V(x,y)$.

Proposition 6, Armstrong and Rochet (1999). *Suppose $V(x,y) = V^1(x) + V^2(y)$ is additively separable.*
a) If types are strongly positively correlated in the sense that

$$p_{hh} \cdot p_{ll} - p_{lh} \cdot p_{hl} \geq \frac{p_{lh} p_{hl}}{p_{hh}},$$

then the optimal allocation features $(x,y) \in R_{1a} \cap R_{1b} \cap R_2$.
b) If types are uncorrelated or positively correlated with

$$p_{hh} \cdot p_{ll} - p_{lh} \cdot p_{hl} < \frac{p_{lh} p_{hl}}{p_{hh}}$$

then the optimal allocation lies in the interior of $R_{1a} \cap R_{1b}$.
c) Suppose types are negatively correlated. If correlation is sufficiently weak and dimensions are not too asymmetric, the solution lies in the interior of $R_{1a} \cap R_{1b}$ as in b). If correlation and asymmetry are sufficiently strong, the solution lies either in the interior of R_{1a} or R_{1b}, or in $R_{1a} \cap R_{4a} \cap R_{5a}$ or $R_{1b} \cap R_{4b} \cap R_{5b}$, respectively.

Armstrong and Rochet (1999) provide a detailed discussion as well as a full proof of Proposition 6 from which I abstain here. I give a strictly quantitative version of Proposition 6 and in particular of Part c) in the next section for the parametrized case of quadratic valuations.

With additively separable valuations, only six cases from Theorem 1 may apply. The solution is qualitatively determined by the remaining source of interaction betwewen dimensions, namely correlation of preference types, and by the degree of asymmetry between the dimensions. Note that if *all* interaction between the dimensions was eliminated, i.e. if types were additionally assumed to be uncorrelated, then Part b) applies and the optimal allocation necessarily lies in the interior of $R_{1a} \cap R_{1b}$. Moreover, the optimal allocation always lies in the interior of $R_{1a} \cup R_{1b} \cup R_2$ unless negative correlation and asymmetry are jointly large.

I next add the second benchmark where I allow for interaction between the dimensions but impose full symmetry across dimensions.

Proposition 7. *Assume $\theta_h = \eta_h$, $\theta_l = \eta_l$, $p_{hl} = p_{lh}$, and $V(x,y) = V(y,x)$. Then the solution satisfies $x_{ij} = y_{ji}$ for all $i,j \in \{h,l\}$ and is determined through the following FOCs*

$$\begin{aligned}
V_x(x_{hh}, x_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot \left(p_{lh} + p_{ll} - \frac{\alpha}{2} \cdot p_{ll}\right) \\
V_x(x_{hl}, x_{lh}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot \frac{\alpha}{2} \cdot p_{ll} \\
V_x(x_{lh}, x_{hl}) &= \theta_l \\
V_x(x_{ll}, x_{ll}) &= \theta_l
\end{aligned}$$

where

$$\alpha = \begin{cases} 0 & \textit{if for } \alpha = 0 \textit{ the solution of the system features } x_{hh} \geq x_{hl} \\ 1 & \textit{if for } \alpha = 1 \textit{ the solution of the system features } x_{hh} \leq x_{hl} \\ \alpha^* \in (0,1) & \textit{if for } \alpha = \alpha^* \textit{ the solution of the system features } x_{hh} = x_{hl}. \end{cases}$$

If $x_{hh} > x_{hl}$, the solution lies in the interior of R_2. If $x_{hh} < x_{hl}$, the solution lies in the interior of $R_{1a} \cap R_{1b}$. If $x_{hh} = x_{hl}$, the solution lies in $R_{1a} \cap R_{1b} \cap R_2$.

Full symmetry is even more restrictive than additive separability of $V(x,y)$ with respect to the set of possible optimal allocations and almost as restrictive as the full absence of interaction between the dimensions. The extensive consequences of full symmetry are already reflected in Proposition 3; all "a"- and "b"-regions require some amount of asymmetry in the allocation and no "a"- and "b"-regions of the same type are adjacent except for R_{1a} and R_{1b}. Indeed, together with Theorem 1 this directly implies that only the three cases captured by Proposition 7 can apply. These cases are characterized by a unique multiplier α. The value of α is necessarily determined by distributional properties and the interaction through V. As we see, for positive α increasing positive correlation (keeping $\frac{p_{ll}}{p_{hh}}$ fixed) increases $\frac{\Delta_\theta}{p_{hl}} \cdot \frac{\alpha}{2} \cdot p_{ll}$ relative to $\frac{\Delta_\theta}{p_{hh}} \cdot \left(p_{lh} + p_{ll} - \frac{\alpha}{2} \cdot p_{ll}\right)$. The multiplier α is hence "decreasing in the correlation", in line with Proposition 6. The impact of interaction through $V(x,y)$ on α in general is less easy to capture as it will typically depend on the allocation. However, if goods are global strategic complements (strategic substitutes), increasing the degree of complementarity (substitutability) will push α upwards (downwards), applying the same reasoning as explained after Proposition 2.

Just as in the additively separable case with moderate asymmetry and moderate (negative) correlation, under full symmetry the optimal allocation always lies in the interior of $R_{1a} \cup R_{1b} \cup R_2$. In the interior of $R_{1a} \cup R_{1b} \cup R_2$, some familiar features from the one-dimensional case apply. There is "no distortion at the top" in the sense that the allocation for the efficient type (l,l) coincides with the first best. Moreover, no upward or diagonal constraints ever bind, so no high-cost firm in either dimension will ever consider to imitate a low-cost firm. From Proposition 6 and 7 we may conclude that in any (continuously) parametrized version of our model the solution lies in the interior of $R_{1a} \cup R_{1b} \cup R_2$ if interaction and asymmetry between the two dimensions are sufficiently weak. A somewhat substantial combination of asymmetry and interaction is needed to get optimal allocations beyond the classical schemes.

I continue by considering less restrictive assumptions than the complete absence of interaction or asymmetry, namely those already touched upon in Proposition 2 for the case of observable costs − bounded interaction, strategic complementarity and strategic substitutability. As it is useful to discuss various aspects of these results in comparison to each other, I state all three propositions in a row and comment afterwards.

Proposition 8. *Assume that $\frac{|V_{xx}(x,y)|}{\Delta_\theta} \geq \frac{|V_{xy}(x,y)|}{\Delta_\eta}$ and $\frac{|V_{yy}(x,y)|}{\Delta_\eta} \geq \frac{|V_{xy}(x,y)|}{\Delta_\theta}$ for all $(x,y) \in \mathbb{R}^2_+$. Then*
a) $x_{ll} \geq x_{hh}$, $x_{lh} \geq x_{hl}$, $y_{ll} \geq y_{hh}$, $y_{hl} \geq y_{lh}$.

b) equality in Part a) only applies if the optimal allocation lies on a boundary hyperplane between two regions defined by the respective inequality.
c) the solution never lies in the interior of R_{3a}, R_{3b}, R_{6a}, or R_{6b}.
d) no implementability constraint binds, so in particular IC_{hh} is slack.

Proposition 9. *Suppose that $V_{xy}(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2_+$. Then*
a) $x_{ll} \geq x_{lh}$, $x_{hl} \geq x_{hh}$, $y_{ll} \geq y_{hl}$, $y_{lh} \geq y_{hh}$ and $x_{ll} > x_{hh}$, $y_{ll} > y_{hh}$.
b) equality in Part a) may only apply if the optimal allocation lies on a boundary hyperplane between two regions defined by the respective inequality or on the boundary of $\mathcal{I} \subset \mathbb{R}^8_+$.
c) the solution never lies in R_{3a} or R_{3b} or in the interior of R_{4a} or R_{4b}. If the solution lies in R_2, it lies in $R_{1a} \cap R_{1b} \cap R_2$.
d) no upward IC-constraints binds unless the optimal allocation lies in $R_{4a} \cap R_{5a}$ or $R_{4b} \cap R_{5b}$, respectively.

Proposition 10. *Suppose that $V_{xy}(x,y) < 0$ for all $(x,y) \in \mathbb{R}^2_+$. Then*
a) $x_{ll} \leq x_{lh}$, $y_{ll} \leq y_{hl}$ and $x_{lh} > x_{hl}$, $y_{hl} > y_{lh}$.
b) equality in Part a) may only apply if the optimal allocation lies on a boundary hyperplane between two regions defined by the respective inequality or on the boundary of $\mathcal{I} \subset \mathbb{R}^8_+$.
c) the solution never lies in R_{6a} or R_{6b} or in the interior of $R_{4a} \cup R_{5a}$ or $R_{4b} \cup R_{5b}$.
d) no diagonal IC-constraints binds unless the optimal allocation lies in $R_{1a} \cap R_{4a} \cap R_{5a}$ or $R_{1b} \cap R_{4b} \cap R_{5b}$, respectively.

The most noteworthy feature of Propositions 8-10 is that with one exception all properties from the case of observable costs as given in Proposition 2 carry over to the case of privately known costs. Strict inequalities may become weak inequalities due to the kinks or boundaries of the constraint objective (and may do so only at kinks or boundaries); yet, distortions through incentive compatibility constraints or transfers, respectively, are not sufficiently strong to overturn the efficiency concerns that apply in the unconstraint case. The only exception from this rule applies to Proposition 10. Substitutability does not imply the same monotonicity behaviour for high cost types in a given dimension with respect to the other cost parameter as in the first best; solutions may feature $x_{hh} < x_{hl}$ or $y_{hh} < y_{lh}$. Given Proposition 6, this does not come unexpected. If the solution in the no-interaction case may lie in the interior of $R_{1a} \cup R_{1b}$ and hence satisfy $x_{hh} < x_{hl}$ or $y_{hh} < y_{lh}$, then at least a small amount of substitutability should be compatible with these equations. The intuition for this particular inequalities to be potentially reversed compared to the first best is as follows. Both mixed types (l,h) and (h,l) claim rents $\Delta_\theta x_{hh}$ resp. $\Delta_\eta y_{hh}$ for not imitating type (h,h). The necessity of leaving these rents makes it unattractive to choose x_{hh} and y_{hh} too large, in particular if types (l,h) and (h,l) are likely to occur, i.e. if types are negatively correlated. This rent aspect not only drives the result by Armstrong and Rochet (1999) but may also outweigh the efficiency concern that is added in case of substitutes.

Complementarity and substitutability are the most common assumptions in almost any multiproduct context. Following Propositions 9 and 10, they allow for distinctly separate optimal allocations, the intersection of possible loci coinciding precisely with the feasible loci under additive separability. Substitutability allows for optimal allocations in regions $R_2 \cup R_{3a} \cup R_{3b}$ while complementarity allows for optimal allocations in regions $R_{5a} \cup R_{5b} \cup R_{6a} \cup R_{6b}$. Neither substitutability nor complementarity is compatible with an optimal allocation in the interior of R_{4a} or R_{4b}.[14] Hence already the assumption of a constant sign for $V_{xy}(x, y)$ as such is restrictive, even without further specification.

Bounded interaction as in Proposition 8 will often be a natural assumption. With cost differences being equal it simply says that marginal valuations in any given dimension should depend at least as much on the allocation in the very same dimension compared to the allocation in the other dimension. Note in addition that the assumptions underlying Proposition 8 do not exclusively relate to interaction. Indeed, in a very asymmetric model interaction must be very small to satisfy both assumptions simultaneously. Conversely, symmetry across dimensions implies $|V_{xx}(x, y)| = |V_{yy}(x, y)| > |V_{xy}(x, y)|$ due to concavity.

Any of the assumptions underlying Propositions 8-10 substantially restrict the feasible loci for the optimal allocation and even more so when combined. Substitutability and bounded interaction in the sense of Proposition 8 for example already imply $(x, y) \in R_{1a} \cup R_{1b} \cup R_2$. Moreover, the parallels to the properties of the first best solution show that the valuation $V(x, y)$ and in particular different kinds of interaction impose qualitative restrictions on the solution that are inherently based on efficiency concerns and cannot be overturned by rent concerns. In particular, apart from the exception for the case of substatibility discussed above, even an extremely imbalanced underlying type distribution cannot distort these fundamental properties as by definition it has no impact on the first-best solution.

3.7 The Quadratic Case – An Example

In this section I analyze the design problem under the specific assumption that valuations are quadratic, i.e.

$$V(x, y) = -\frac{1}{2}a_1x^2 - \frac{1}{2}a_2y^2 + bxy + c_1x + c_2y + d$$

with $a_1, a_2 > 0$, $b, c_1, c_2, d \in \mathbb{R}$. $V(x, y)$ is strictly concave if and only if $\det H(x, y) = a_1a_2 - b^2 > 0$ which I will assume throughout this section. Moreover, to avoid additional factors as in Proposition 2 and Proposition 8, in this section I assume $\Delta_\theta = \Delta_\eta = \Delta$ and normalize by setting $\Delta = 1$.

As I show in the Appendix, for all qualitative purposes it is sufficient to consider a reduced version of the quadratic problem where

$$V(x, y) = -\frac{1}{2}a_1x^2 - \frac{1}{2}a_2y^2 + bxy$$

[14] The same holds for allocations in $R_{1a} \cap R_{4a} \cap R_{6b}$ or $R_{1b} \cap R_{4b} \cap R_{6a}$ as can easily be seen from the proof of Proposition 9 and the statement of Proposition 10.

and $a_1 a_2 - b^2 = 1$, ignoring negativity of the optimal allocation and consumer valuations.[15]For any region R and any $r, s \in \{l, h\}$ the FOCs for the optimal allocation read

$$\begin{aligned} V_x\left(x_{rs}, y_{rs}\right) &= \gamma_{rs}^1, \\ V_y\left(x_{rs}, y_{rs}\right) &= \gamma_{rs}^2 \end{aligned}$$

where γ_{rs}^1, γ_{rs}^2 depend on $r, s \in \{l, h\}$, on the region, on the multipliers and on the model parameters but do not depend on (x, y). The solution of this system for our specification of $V(x, y)$ reads

$$x_{rs} = -b\gamma_{rs}^2 - a_2\gamma_{rs}^1 \tag{3.7.1}$$

$$y_{rs} = -b\gamma_{rs}^1 - a_1\gamma_{rs}^2. \tag{3.7.2}$$

The parameter $b = V_{xy}(x, y)$ measures the strength of interaction between the dimensions through $V(x, y)$. To measure the asymmetry between the dimensions induced by $V(x, y)$, I write $Q = \frac{a_1}{a_2}$. With this parametrization we have

$$\begin{aligned} a_1 &= \sqrt{Q} \cdot \sqrt{1 + b^2}, \\ a_2 &= \frac{1}{\sqrt{Q}} \cdot \sqrt{1 + b^2}. \end{aligned}$$

Equations (3.7.1) and (3.7.2) together with the results from the previous sections allow us to solve the quadratic optimization problem explicitly for any $b \in \mathbb{R}$, $Q > 0$ and any distribution of preference types. We use this to derive some results that specify and complement the results from Section 6 for additively separable preferences ($b = 0$), complements ($b > 0$), and substitutes ($b < 0$). In particular, while the main overall message from Section 6 was that minor interaction and asymmetry lead to "standard" allocation properties similar to those in the unidimensional case, in this section we demonstrate that strong interaction and large asymmetries may indeed distort the solution beyond these limits. Moreover, we comment on as to which extent the assumption of quadratic valuations is restrictive with respect to the set of possible loci compared to Proposition 6, Proposition 9, and Proposition 10.

We start with a full characterization for the additively separable case, specifying Part c) of Proposition 6.

Proposition 6E. *Let* $V_{xy}(x, y) = b = 0$.
a) If $p_{hh} \cdot p_{ll} - p_{lh} \cdot p_{hl} \geq \frac{p_{lh} p_{hl}}{p_{hh}}$ *then* $(x, y) \in R_{1a} \cap R_{1b} \cap R_2$.
b) If $\frac{p_{lh} p_{hl}}{p_{hh}} > p_{hh} \cdot p_{ll} - p_{lh} \cdot p_{hl} \geq 0$ *then* (x, y) *lies in the interior of* $R_{1a} \cap R_{1b}$.
c) If $p_{hh} \cdot p_{ll} - p_{lh} \cdot p_{hl} < 0$ *the following holds:*

[15] For any solution of the reduced version, there exist values c_1, c_2 and d such that the solution to the original problem has identical qualitative properties with respect to all constraints but features positive allocations and valuations. Fixing $a_1 a_2 - b^2 = 1$ is another normalization. See the Appendix for all technical details.

i) If negative correlation and asymmetry are weak in the sense that $\frac{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}{p_{hl}\cdot(p_{hl}+p_{ll})} \leq Q \leq \frac{p_{lh}\cdot(p_{lh}+p_{ll})}{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}$ *then* (x,y) *lies in the interior of* $R_{1a} \cap R_{1b}$.
ii) If negative correlation and asymmetry are intermediate in the sense that $\frac{p_{lh}\cdot(p_{lh}+p_{ll})}{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}} < Q < \frac{p_{lh}\cdot(1-p_{hl})}{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}$ *or* $\frac{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}{p_{hl}\cdot(1-p_{lh})} < Q < \frac{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}{p_{hl}\cdot(p_{hl}+p_{ll})}$ *then* (x,y) *lies in the interior of* R_{1a} *or* R_{1b}*, respectively.*
iii) If negative correlation and asymmetry are strong in the sense that $Q \geq \frac{p_{lh}\cdot(1-p_{hl})}{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}$ *or* $Q \leq \frac{p_{lh}\cdot p_{hl}-p_{hh}\cdot p_{ll}}{p_{hl}\cdot(1-p_{lh})}$ *then* (x,y) *lies in* $R_{1a}\cap R_{4a}\cap R_{5a}$ *or* $R_{1b}\cap R_{4b}\cap R_{5b}$*, respectively.*

Parts a-b) of Proposition 6E are identical to Proposition 6. In Part c), the parametrized model allows us to precisely determine the locus of the optimal allocation as a function of asymmetry and distributional features, with an emphasis on (negative) correlation. In particular, for any fixed distribution featuring negative correlation, increasing the asymmetry by means of $Q \to \infty$ or $Q \to 0$ will ultimately push the optimal allocation towards the non-standard cases $R_{1a}\cap R_{4a}\cap R_{5a}$ or $R_{1b}\cap R_{4b}\cap R_{5b}$, respectively, featuring distortions at the top and binding diagonal and upward incentive constraints.

Note that large values $Q \gg 1$, that is, a large weight on the x-dimension as compared to the y-dimension, tends to push the solution in “a”-type regions while small values $Q \ll 1$ have the opposite effect. Indeed, all “a”-type regions are characterized by requiring rather large differences in y-values for identical costs η. If the y-dimension is far less important than the x-dimension, such large differences are much easier tolerable despite their cost inefficiency. This scheme thus generally appears in all following results.

We next revisit the richer case of strategic complements.

Proposition 9E. *Let* $V_{xy}(x,y) = b > 0$.
a) The optimal allocation (x,y) *lies in* $R_{1a}\cup R_{1b}\cup R_{6a}\cup R_{6b}$.
b) No implementability constraint holds with equality.
c) The optimal allocation (x,y) *may only lie outside* $R_{1a}\cup R_{1b}$ *only if*

$$\begin{aligned} b^2 &\geq 4\cdot\frac{p_{hh}^2}{p_{lh}^2}\cdot(p_{hl}+p_{ll})\cdot\left(p_{hl}+p_{ll}+\frac{p_{lh}}{p_{hh}}\right), \\ \sqrt{Q} &\in \left(\frac{\sqrt{b^2+1}}{b}, \frac{b}{\sqrt{b^2+1}}\cdot\left(1+\frac{\frac{p_{lh}}{p_{hh}}+\sqrt{\left(\frac{p_{lh}}{p_{hh}}\right)^2-4\cdot\frac{1}{b^2}\cdot(p_{hl}+p_{ll})\cdot\left(p_{hl}+p_{ll}+\frac{p_{lh}}{p_{hh}}\right)}}{p_{hl}+p_{ll}}\right)\right). \end{aligned}$$

or

$$\begin{aligned} b^2 &\geq 4\cdot\frac{p_{hh}^2}{p_{hl}^2}\cdot(p_{lh}+p_{ll})\cdot\left(p_{lh}+p_{ll}+\frac{p_{hl}}{p_{hh}}\right), \\ \frac{1}{\sqrt{Q}} &\in \left(\frac{\sqrt{b^2+1}}{b}, \frac{b}{\sqrt{b^2+1}}\cdot\left(1+\frac{\frac{p_{lh}}{p_{hh}}+\sqrt{\left(\frac{p_{lh}}{p_{hh}}\right)^2-4\cdot\frac{1}{b^2}\cdot(p_{hl}+p_{ll})\cdot\left(p_{hl}+p_{ll}+\frac{p_{lh}}{p_{hh}}\right)}}{p_{hl}+p_{ll}}\right)\right). \end{aligned}$$

d) For any fixed values of b *and the underlying distribution as well as for sufficiently large (small)* Q *the solution lies in the interior of* $R_{1b}\cap R_{6a}$ $(R_{1a}\cap R_{6b})$.

Part a) of Proposition 9E shows that the assumption of quadratic valuations is qualitatively restrictive within the class of valuations that feature strategic complementarity.

Compared to Proposition 9, the solution cannot lie in the interior of R_{5a} (R_{5b}) or in the interior of $R_{4a} \cap R_{5a}$ ($R_{4b} \cap R_{5b}$). To get an intuition why the optimal allocation cannot lie in R_{5a}, recall the discussion of the first best allocation after Proposition 2 and consider R_{5a}. Whenever asymmetry becomes too large in the sense that $V_{yy} = a_2$ becomes too small relative to $V_{xy} = b$, the solution features $x_{hl} \geq x_{lh}$. The effect from costs θ on the x-allocation is dominated by the effect of the y-allocation on the x-allocation through complementarity. Hence constraint (R5a,6a) binds or is violated. On the other hand, to satisfy constraint (R5a,1a), namely $y_{lh} - y_{hh} - (x_{lh} - x_{hh}) \geq 0$, the effect of x on y measured by $V_{xy} = b$ must be sufficiently large relative to effect through costs within the x-dimension measured by $V_{yy} = a_2$. As we show in the Appendix, these two requirements cannot be satisfied simultaneously when there is no local variation in second derivatives. Similarly, constant second derivatives render binding implementability constraints impossible as stated in Part b), so in particular IC_{hh} is always slack.

Part c) formulates necessary conditions under which the solution may lie outside the core regions $R_{1a} \cup R_{1b}$. As a consequence of Part a) and Proposition 8, we must have $b \geq a_1$ or $b \geq a_2$ which is reflected in the lower bounds for $\sqrt{Q}$ and $\frac{1}{\sqrt{Q}}$, respectively. Moreover, a substantial degree of interaction through b is required, in line with the insights from the previous section. What may be surprising is the existence of an upper bound on the asymmetry. To see where this comes from, consider R_{6a} and note that from (3.7.1) and (3.7.2) we have

$$y_{lh} - y_{hh} - (x_{lh} - x_{hh}) = (b - a_1)\left(\gamma^2_{lh} - \gamma^2_{hh}\right) + (b - a_2)\left(\gamma^1_{hh} - \gamma^1_{lh}\right). \tag{3.7.3}$$

With constraints IC_{lhhh} binding, a rent $\Delta_\theta x_{hh}$ must be left to type (l, h) not to imitate type (h, h) while no compensation is needed in the y-dimension. Moreover, as constraints IC_{hllh} and IC_{llhl} bind, these rents (together with some other terms) must be left to types (l, l) and (h, l) as well. At the same time, with constraint IC_{hllh} binding, additional rents $\Delta_\eta y_{lh}$ must be left to type (h, l) while simultaneously type (h, l) can be charged an additional fee $\Delta_\theta x_{lh}$. Again, since IC_{llhl} binds, both effects also apply to type (l, l). As a result, a marginal increase in x_{hh} is very costly in the rent dimension compared to a marginal increase in x_{lh} and hence $\gamma^1_{hh} - \gamma^1_{lh} > 0$, while a marginal increase in y_{lh} is more costly than a marginal increase in y_{hh} in the rent dimension, so $\gamma^2_{lh} - \gamma^2_{hh} > 0$. With both γ-differences in equation (3.7.3) being positive, the existence of an upper bound on Q becomes obvious. If a_1 becomes very large relative to a_2, then $a_1 - b$ becomes very large relative to $b - a_2$ and hence (3.7.3) becomes negative.

Part d) finally shows that with constraint IC_{hlhh} binding in addition the above issue can be resolved for large values of Q such that (3.7.3) holds with equality. Indeed, suppose for a moment that IC_{hlhh} binds while IC_{hllh} is slack, so the solution lies in R_{1b}. Then, compared to the above discussion, the rents to type (h, l) (and, to gether with some other terms, to type (l, l)) are given as $\Delta_\eta y_{hh}$ rather than $\Delta_\eta y_{lh}$, so the sign of $\gamma^2_{lh} - \gamma^2_{hh}$ changes from positive to negative. As a conseqence, a large value of Q will cause expression (3.7.3) to be positive. By the intermediate value theorem, attaching a suitable multiplier $\mu_{6a,1b}$ to weight both concerns properly yields equality in (3.7.3) and hence shows the result.

We close with the case of strategic substitutes.

Proposition 10E. *Let* $V_{xy}(x, y) = b < 0$.
a) No implementability constraint holds with equality.
b) If

$$\begin{aligned} -b \cdot (1 - p_{hl}) - a_1 \cdot (p_{lh} + p_{ll}) &> 0, \quad (3.7.4) \\ -b \cdot (1 - p_{lh}) - a_2 \cdot (p_{hl} + p_{ll}) &> 0, \quad (3.7.5) \end{aligned}$$

then solution never lies in $R_{1a} \cup R_{1b}$. *In particular, if*

$$\max \left\{ \frac{1}{\sqrt{Q}} \cdot \frac{p_{lh} + p_{ll}}{p_{lh} + p_{ll} + p_{hh}}, \sqrt{Q} \cdot \frac{p_{hl} + p_{ll}}{p_{hl} + p_{ll} + p_{hh}} \right\} < 1 \quad (3.7.6)$$

then sufficiently strong interaction $|b| \gg 0$ *implies the above inequalities. Moreover, whenever the above inequalities hold the solution lies in the interior of* R_2 *if and only if* $a_1, a_2 > |b|$.
c) For any fixed values of b and the underlying distribution as well as for sufficiently large (small) Q the solution lies in the interior of $R_{3a} \cap R_{4a}$ $(R_{3b} \cap R_{4b})$.

Part a) of Proposition 10E is another consequence of constant second derivatives, similar to Part b) of Proposition 9E.

Part b) of Proposition 10E provides sufficient conditions under which the optimal allocation lies outside the regions that are feasible in the additively separable case. As expected, sufficiently large interaction is required. Moreover, a large probability p_{hh} to face a fully inefficient producer type relaxes condition (3.7.6). Again, this is very intuitive; the more likely type (h, h) is to occur, the less does a designer want to distort the allocation of this type. Thus it becomes more attractive to imitate this type for others, in particular for type (l, l). In addition, a high value p_{hh} is likely to come along with substantial positive correlation. As is known from Proposition 6, this pushes the solution towards region R_2 and causes constraint I_{llhh} to bind even in the additively separable case. Moreover, if (3.7.4) and (3.7.5) are satisfied, upward constraints become relevant whenever the interaction measured by the cross-derivative b dominates the intra-dimensional change of marginal utilities measured by the second derivatives a_1 or a_2. This is in line with the intuition for Proposition 2, Part d) as well as with Proposition 8.

Similar to what has been discussed for Part c) of Proposition 9E, large asymmetries are not compatible with both equations (3.7.4) and (3.7.2) simultaneously. On the other hand, as just argued, large asymmetries will push solutions towards regions R_{3a} or R_{3b}, respectively. Consider the case where Q becomes large, so the optimal allocation is driven towards R_{3a}. Constraint (R3a,4a) requires

$$y_{hh} - y_{lh} = -b\left(\gamma^1_{hh} - \gamma^1_{lh}\right) - a_1\left(\gamma^2_{hh} - \gamma^2_{lh}\right)$$

to be positive. With $a_1 \gg b$, the sign of the above expression crucially depends on the sign of $\gamma^2_{hh} - \gamma^2_{lh}$. In R_{3a}, constraints IC_{llhh}, IC_{lhhh}, and IC_{hlll} bind. As a consequence, type (l, l) is left a rent $\Delta_\eta y_{hh}$ in the y-dimension for not imitating type (h, h). Together with some additional term, this also applies to type (h, l) who otherwise would fake type

(l, l). So the marginal costs of raising y_{hh} exceed the marginal costs of raising y_{lh} due to the rent concern. Hence $\gamma_{hh}^2 - \gamma_{lh}^2 > 0$ and the above expression ultimately becomes negative for large Q. Considering the same expression in R_{4a} with constraint IC_{lllh} rather than constraint IC_{llhh} binding, the argument is exactly reversed. Type (l, l) is left a rent $\Delta_\eta y_{lh}$ in the y-dimension for not imitating type (l, h) and again this also applies to type (h, l). So $\gamma_{hh}^2 - \gamma_{lh}^2 < 0$ due to the rent concern and $y_{hh} - y_{lh}$ becomes positive. Invoking the intermediate value theorem again, attaching a suitable multiplier $\mu_{3a,4a}$ again provides the result.

Finally, apart from excluding binding implementability constraints, Proposition 10E does not impose any restrictions on the possible loci of the optimal allocation beyond Proposition 10. It is in fact not difficult to explicitly construct examples for all loci feasible according to Proposition 10 for quadratic valuations.

3.8 Conclusions

This thesis chapter classifies the set of potential solutions to a baseline model of multidimensional screening. Relating properties of the solution to properties of the fundamentals I show that a substantial joint degree of interaction and asymmetry between dimensions is necessary to push solutions away from the classical schemes of exclusively downward binding incentive constraints. Moreover, complementarity and substitutability are shown to shift the solution in opposing directions compared to the benchmark case of an additively separable objective. While this thesis chapter does not emphasize a particular application of the underlying model, related work has been applied to a large variety of questions concerning monopoly regulation, optimal pricing or institutional design. It is my best hope that this paper may serve as a useful tool or reference for future work related to multidimensional screening problems with a clear focus on applications and applicability rather than technical complexity.

3.9 Appendix

Lemma (Lemma A)**.** *Let $V : \mathbb{R}^2 \to \mathbb{R}^2$ be twice continuously differentiable and concave with invertible (negative definite) Hessian $H(x, y)$ everywhere. Consider a system of first order conditions*

$$\begin{aligned} V_x(x, y) &= c + t \cdot a \\ V_y(x, y) &= c + t \cdot b \end{aligned}$$

for $t \in [0, 1]$ with $a, b \in \mathbb{R}$. Then

$$\begin{aligned} \frac{dx}{dt} &= \frac{a \cdot V_{yy}(x, y) - b \cdot V_{xy}(x, y)}{\det H(x, y)} \\ \frac{dy}{dt} &= \frac{b \cdot V_{xx}(x, y) - a \cdot V_{xy}(x, y)}{\det H(x, y)}. \end{aligned}$$

Moreover, suppose the system of first order conditions

$$\begin{aligned} V_x(x,y) &= a \\ V_y(x,y) &= b \end{aligned}$$

has a solution $(x,y) \in \mathbb{R}^2$. *Then* (x,y) *is unique.*

Proof. To prove the first claim, totally differentiating

$$\begin{aligned} V_x(x,y) &= c + t \cdot a \\ V_y(x,y) &= c + t \cdot b \end{aligned}$$

with respect to t yields

$$\begin{aligned} \frac{dx}{dt} \cdot V_{xx}(x,y) + \frac{dy}{dt} \cdot V_{xy}(x,y) &= a \\ \frac{dx}{dt} \cdot V_{xy}(x,y) + \frac{dy}{dt} \cdot V_{yy}(x,y) &= b \end{aligned}$$

which has a unique solution given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{a \cdot V_{yy}(x,y) - b \cdot V_{xy}(x,y)}{\det H(x,y)} \\ \frac{dy}{dt} &= \frac{b \cdot V_{xx}(x,y) - a \cdot V_{xy}(x,y)}{\det H(x,y)}. \end{aligned}$$

For the second claim, suppose for contradiction that there exist two solutions $(x,y) \neq (\hat{x},\hat{y})$. Then the function $\tilde{V}(x,y) = V(x,y) - ax - by$ is still concave and has critical points at (x,y) and $(\hat{x},\hat{y})$. But by concavity of $\tilde{V}$ any critical point is a global maximum, hence $\tilde{V}(x,y) = \tilde{V}(\hat{x},\hat{y}) = \max_{(x,y)} \tilde{V}(x,y)$. This contradicts concavity as

$$\tilde{V}\left(\frac{1}{2}(x,y) + \frac{1}{2}(\hat{x},\hat{y})\right) > \frac{1}{2}\tilde{V}(x,y) + \frac{1}{2}\tilde{V}(\hat{x},\hat{y}) = \tilde{V}(x,y) = \tilde{V}(\hat{x},\hat{y}).$$

□

Proof of Proposition 1. By Theorem 1 in Rochet (1987) there exist payments that implement an allocation $(x,y) \in \mathbb{R}_+^4 \times \mathbb{R}_+^4$ in an incentive compatible way if and only if there exists no cycle of positive length. The network has six oriented 2-cycles, eight oriented 3-cycles and six oriented 4-cycles, providing the following inequalitiy conditions for the non-existence of positive length cycles:

$$
\begin{aligned}
\Delta_\theta \cdot (x_{ll} - x_{hl}) &\geq 0 && \text{(ll→hl→ll)}\\
\Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→lh→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) &\geq 0 && \text{(lh→hh→lh)}\\
\Delta_\eta \cdot (y_{hl} - y_{hh}) &\geq 0 && \text{(hl→hh→hl)}\\
\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) &\geq 0 && \text{(ll→hh→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) &\geq 0 && \text{(lh→hl→lh)}\\
\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) &\geq 0 && \text{(ll→hh→hl→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) &\geq 0 && \text{(ll→hh→lh→ll)}\\
\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) &\geq 0 && \text{(ll→hl→hh→ll)}\\
\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→lh→hh→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→hl→lh→ll)}\\
\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) &\geq 0 && \text{(ll→lh→hl→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) &\geq 0 && \text{(lh→hh→hl→lh)}\\
\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) &\geq 0 && \text{(hl→hh→lh→hl)}\\
\Delta_\theta \cdot (x_{lh} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{hh}) &\geq 0 && \text{(ll→hl→hh→lh→ll)}\\
\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) &\geq 0 && \text{(ll→lh→hh→hl→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→hh→hl→lh→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) &\geq 0 && \text{(ll→hh→lh→hl→ll)}\\
\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→lh→hl→hh→ll)}\\
\Delta_\theta \cdot (x_{lh} - x_{hh}) + \Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) &\geq 0 && \text{(ll→hl→lh→hh→ll)}
\end{aligned}
$$

The last four equations are redundant as they are implied by the first four equations. The first 16 inequalities correspond to the claim of the proposition. □

Proof of Proposition 2. To prove the proposition I use the homotopy method introduced in Severinov (2008).

a-c) Consider the following 1-parameter family of first order conditions:

$$
\begin{aligned}
V_x\left(x_{hh}(t), y_{hh}(t)\right) &= \theta_h\\
V_x\left(x_{lh}(t), y_{lh}(t)\right) &= \theta_h - t \cdot \Delta_\theta\\
V_y\left(x_{hh}(t), y_{hh}(t)\right) &= \eta_h\\
V_y\left(x_{lh}(t), y_{lh}(t)\right) &= \eta_h.
\end{aligned}
$$

At $t = 0$, the right hand side of the first two equations and the last two equations are equal. Hence, by Lemma A, we have $x_{hh}(0) = x_{lh}(0)$, $y_{hh}(0) = y_{lh}(0)$. At $t = 1$, however, the system of equations is precisely the system of first order conditions (FOCs) for the first best allocation for types (h, h) and (l, h). We next analyze how

$(x_{hh}(t), x_{lh}(t), y_{hh}(t), y_{lh}(t)) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$ evolves along the path from $t = 0$ to $t = 1$. Clearly, x_{hh} and y_{hh} are independent of t. For x_{lh} and y_{lh} Lemma A yields

$$\begin{aligned}
\frac{dx_{lh}(t)}{dt} &= \frac{-\Delta_\theta \cdot V_{yy}(x_{lh}(t), y_{lh}(t))}{\det H(x_{lh}(t), y_{lh}(t))} > 0, \\
\frac{dy_{lh}(t)}{dt} &= \frac{\Delta_\eta \cdot V_{xy}(x_{lh}(t), y_{lh}(t))}{\det H(x_{lh}(t), y_{lh}(t))},
\end{aligned}$$

which immediately implies

$$x^{fb}_{lh} = x_{lh}(1) > x_{lh}(0) = x_{hh}(0) = x_{hh}(1) = x^{fb}_{hh}.$$

Moreover, if goods are complements, i.e. $V_{xy}(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2_+$, then

$$y^{fb}_{lh} = y_{lh}(1) > y_{lh}(0) = y_{hh}(0) = y_{hh}(1) = y^{fb}_{lh},$$

while if goods are substitutes, i.e. $V_{xy}(x, y) < 0$ for all $(x, y) \in \mathbb{R}^2_+$, then

$$y^{fb}_{lh} = y_{lh}(1) < y_{lh}(0) = y_{hh}(0) = y_{hh}(1) = y^{fb}_{lh}.$$

The remaining claims of Parts a-c) are proven analogously.

d) Consider the following 1-parameter family of first order conditions:

$$\begin{aligned}
V_x(x_{hh}(t), y_{hh}(t)) &= \frac{\theta_h + \theta_l}{2} + t \cdot \frac{\Delta_\theta}{2} \\
V_x(x_{hl}(t), y_{hl}(t)) &= \frac{\theta_h + \theta_l}{2} + t \cdot \frac{\Delta_\theta}{2} \\
V_x(x_{lh}(t), y_{lh}(t)) &= \frac{\theta_h + \theta_l}{2} - t \cdot \frac{\Delta_\theta}{2} \\
V_x(x_{ll}(t), y_{ll}(t)) &= \frac{\theta_h + \theta_l}{2} - t \cdot \frac{\Delta_\theta}{2} \\
V_y(x_{hh}(t), y_{hh}(t)) &= \frac{\eta_h + \eta_l}{2} + t \cdot \frac{\Delta_\eta}{2} \\
V_y(x_{hl}(t), y_{hl}(t)) &= \frac{\eta_h + \eta_l}{2} - t \cdot \frac{\Delta_\eta}{2} \\
V_y(x_{lh}(t), y_{lh}(t)) &= \frac{\eta_h + \eta_l}{2} + t \cdot \frac{\Delta_\eta}{2} \\
V_y(x_{ll}(t), y_{ll}(t)) &= \frac{\eta_h + \eta_l}{2} - t \cdot \frac{\Delta_\eta}{2}.
\end{aligned}$$

We have $x_{ij}(0) = x_{rs}(0)$, $y_{ij}(0) = y_{rs}(0)$ for all $(i, j), (r, s) \in \{l, h\}^2$ as well as $x_{ij}(1) =$

x_{ij}^{fb}, $y_{ij}(1) = y_{ij}^{fb}$. Moreover, applying Lemma A yields

$$\begin{aligned}
\frac{dx_{hh}(t)}{dt} &= \frac{1}{2} \cdot \frac{\Delta_\theta \cdot V_{yy}(x_{hh}(t), y_{hh}(t)) - \Delta_\eta \cdot V_{xy}(x_{hh}(t), y_{hh}(t))}{\det H(x_{hh}(t), y_{hh}(t))} \leq 0, \\
\frac{dx_{hl}(t)}{dt} &= \frac{1}{2} \cdot \frac{\Delta_\theta \cdot V_{yy}(x_{hl}(t), y_{hl}(t)) + \Delta_\eta \cdot V_{xy}(x_{hl}(t), y_{hl}(t))}{\det H(x_{hl}(t), y_{hl}(t))} \leq 0, \\
\frac{dx_{lh}(t)}{dt} &= \frac{1}{2} \cdot \frac{-\Delta_\theta \cdot V_{yy}(x_{lh}(t), y_{lh}(t)) - \Delta_\eta \cdot V_{xy}(x_{lh}(t), y_{lh}(t))}{\det H(x_{lh}(t), y_{lh}(t))} \geq 0, \\
\frac{dx_{ll}(t)}{dt} &= \frac{1}{2} \cdot \frac{-\Delta_\theta \cdot V_{yy}(x_{ll}(t), y_{ll}(t)) + \Delta_\eta \cdot V_{xy}(x_{ll}(t), y_{ll}(t))}{\det H(x_{ll}(t), y_{ll}(t))} \geq 0, \\
\frac{dy_{hh}(t)}{dt} &= \frac{1}{2} \cdot \frac{\Delta_\eta \cdot V_{xx}(x_{hh}(t), y_{hh}(t)) - \Delta_\theta \cdot V_{xy}(x_{hh}(t), y_{hh}(t))}{\det H(x_{hh}(t), y_{hh}(t))} \leq 0, \\
\frac{dy_{hl}(t)}{dt} &= \frac{1}{2} \cdot \frac{-\Delta_\eta \cdot V_{xx}(x_{hl}(t), y_{hl}(t)) - \Delta_\theta \cdot V_{xy}(x_{hl}(t), y_{hl}(t))}{\det H(x_{hl}(t), y_{hl}(t))} \geq 0, \\
\frac{dy_{lh}(t)}{dt} &= \frac{1}{2} \cdot \frac{\Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) + \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t))}{\det H(x_{lh}(t), y_{lh}(t))} \leq 0, \\
\frac{dy_{ll}(t)}{dt} &= \frac{1}{2} \cdot \frac{-\Delta_\eta \cdot V_{xx}(x_{ll}(t), y_{ll}(t)) + \Delta_\theta \cdot V_{xy}(x_{ll}(t), y_{ll}(t))}{\det H(x_{ll}(t), y_{ll}(t))} \geq 0
\end{aligned}$$

and hence

$$\begin{aligned}
x_{ll}^{fb} &= x_{ll}(1) \geq x_{ll}(0) = x_{hh}(0) \geq x_{hh}(1) = x_{hh}^{fb}, \\
x_{lh}^{fb} &= x_{lh}(1) \geq x_{lh}(0) = x_{hl}(0) \geq x_{hl}(1) = x_{hl}^{fb}, \\
y_{ll}^{fb} &= y_{ll}(1) \geq y_{ll}(0) = y_{hh}(0) \geq y_{hh}(1) = y_{hh}^{fb}, \\
y_{hl}^{fb} &= y_{hl}(1) \geq y_{hl}(0) = y_{lh}(0) \geq y_{lh}(1) = y_{lh}^{fb}.
\end{aligned}$$

□

Proof of Proposition 3. a) By Lemma 2, each type $(i, j) \neq (h, h)$ must be linked to type (h, h) through a path of binding constraints. We next rule out some additional combinations of binding IC-constraints.

Step 1: Let (x, y) be an implementable allocation together with optimal transfers $T(x, y)$ such that IC_{llhh} binds while IC_{llhl} and IC_{lllh} are slack. Then IC_{hllh} and IC_{lhhl} are slack. To see this, note that IC_{llhh} binding and IC_{llhl} not binding together with IC_{hl} imply

$$\begin{aligned}
T_{hh} - \theta_l x_{hh} - \eta_l y_{hh} &> T_{hl} - \theta_l x_{hl} - \eta_l y_{hl} \\
&= T_{hl} - \theta_h x_{hl} - \eta_l y_{hl} + \Delta_\theta \cdot x_{hl} \\
&\geq T_{hh} - \theta_h x_{hh} - \eta_l y_{hh} + \Delta_\theta \cdot x_{hl}
\end{aligned}$$

and hence

$$x_{hh} > x_{hl}.$$

On the other hand, IC_{lhhl} binding together with IC_{hl} implies

$$\begin{aligned} T_{hl} - \theta_l x_{hl} - \eta_h y_{hl} &\geq T_{hh} - \theta_l x_{hh} - \eta_h y_{hh} \\ &= T_{hh} - \theta_h x_{hh} - \eta_h y_{hh} + \Delta_\theta \cdot x_{hh} \\ &\geq T_{hl} - \theta_h x_{hl} - \eta_h y_{hl} + \Delta_\theta \cdot x_{hh} \end{aligned}$$

and hence

$$x_{hl} \geq \Delta_\theta,$$

a contradiction. The proof for IC_{hllh} is alike.

Step 2: Let (x, y) be an implementable allocation together with optimal transfers $T(x, y)$ such that IC_{llhh} binds while IC_{llhl} and IC_{lllh} are slack. Then IC_{hlhh} or IC_{lhhh} must bind. Indeed, by Step 1 we know that IC_{hllh} and IC_{lhhl} are slack. Hence, by Lemma 1, IC_{hlll} or IC_{hlhh} resp. IC_{lhll} or IC_{lhhh} must bind. Suppose for contradiction that IC_{hlhh} and IC_{lhhh} are both slack. Then

$$\begin{aligned} T_{ll} - \theta_h x_{ll} - \eta_l y_{ll} &> T_{hh} - \theta_h x_{hh} - \eta_l y_{hh}, \\ T_{ll} - \theta_l x_{ll} - \eta_h y_{ll} &> T_{hh} - \theta_l x_{hh} - \eta_h y_{hh}, \\ T_{ll} - \theta_l x_{ll} - \eta_l y_{ll} &= T_{hh} - \theta_l x_{hh} - \eta_l y_{hh}, \end{aligned}$$

as IC_{llhh} binds by assumption. Substracting the third line twice from the sum of the first and the second line yields

$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta (y_{ll} - y_{hh}) < 0,$$

a contradiction to implementability constraint (I1).

Steps 1-2 together with Lemma 2 exclude all combinations of binding constraints that are not covered by the eleven combinations in Part a). We now derive the characterizing properties of the allocation that correspond to any particular set of binding constraints. For any particular combination of binding constraints, we need to check under which assumption on the allocation it is indeed this combination of constraints among the eleven candidates that binds. Note that, by Lemma 2, we always have $T_{hh} = \theta_h x_{hh} + \eta_h y_{hh}$. Moreover, as the "b"-cases are symmetric to the "a"-cases, by symmetry of the model it suffices to consider the "a"-cases as well as Case 2.

Case 1a: If IC_{lllh}, IC_{lhhh}, IC_{hlhh} bind, then

$$\begin{aligned} T_{hl} &= \theta_h x_{hl} + \eta_l y_{hl} + \Delta_\eta \cdot y_{hh} \\ T_{lh} &= \theta_l x_{lh} + \eta_h y_{lh} + \Delta_\theta \cdot x_{hh} \\ T_{ll} &= \theta_l x_{ll} + \eta_l y_{ll} + \Delta_\theta \cdot x_{hh} + \Delta_\eta \cdot y_{lh}. \end{aligned}$$

Given these transfers, the following conditions imply that among the eleven possible combinations it is indeed IC_{lllh}, IC_{lhhh}, IC_{hlhh} that bind:

$$\begin{array}{rcll}
T_{lh}-\theta_l x_{lh}-\eta_l y_{lh} & \geq & T_{hl}-\theta_l x_{hl}-\eta_l y_{hl} & \Rightarrow \quad \Delta_\eta\cdot(y_{lh}-y_{hh})-\Delta_\theta\cdot(x_{hl}-x_{hh})\geq 0\\
T_{lh}-\theta_l x_{lh}-\eta_l y_{lh} & \geq & T_{hh}-\theta_l x_{hh}-\eta_l y_{hh} & \Rightarrow \quad \Delta_\eta\cdot(y_{lh}-y_{hh})\geq 0\\
T_{hh}-\theta_h x_{hh}-\eta_l y_{hh} & \geq & T_{ll}-\theta_h x_{ll}-\eta_l y_{ll} & \Rightarrow \quad \Delta_\theta\cdot(x_{ll}-x_{hh})-\Delta_\eta\cdot(y_{lh}-y_{hh})\geq 0\\
T_{hh}-\theta_h x_{hh}-\eta_l y_{hh} & \geq & T_{lh}-\theta_h x_{lh}-\eta_l y_{lh} & \Rightarrow \quad \Delta_\theta\cdot(x_{lh}-x_{hh})-\Delta_\eta\cdot(y_{lh}-y_{hh})\geq 0\\
T_{hh}-\theta_l x_{hh}-\eta_h y_{hh} & \geq & T_{lh}-\theta_l x_{hl}-\eta_h y_{hl} & \Rightarrow \quad \Delta_\eta\cdot(y_{hl}-y_{hh})-\Delta_\theta\cdot(x_{hl}-x_{hh})\geq 0.
\end{array}$$

Case 2: If IC_{llhh}, IC_{lhhh}, IC_{hlhh} bind, then

$$\begin{array}{rcl}
T_{hl} & = & \theta_h x_{hl}+\eta_l y_{hl}+\Delta_\eta\cdot y_{hh}\\
T_{lh} & = & \theta_l x_{lh}+\eta_h y_{lh}+\Delta_\theta\cdot x_{hh}\\
T_{ll} & = & \theta_l x_{ll}+\eta_l y_{ll}+\Delta_\theta\cdot x_{hh}+\Delta_\eta\cdot y_{hh}
\end{array}$$

and

$$\begin{array}{rcll}
T_{hh}-\theta_l x_{hh}-\eta_l y_{hh} & \geq & T_{lh}-\theta_l x_{lh}-\eta_l y_{lh} & \Rightarrow \quad \Delta_\eta\cdot(y_{hh}-y_{lh})\geq 0\\
T_{hh}-\theta_l x_{hh}-\eta_l y_{hh} & \geq & T_{hl}-\theta_l x_{hl}-\eta_l y_{hl} & \Rightarrow \quad \Delta_\theta\cdot(x_{hh}-x_{hl})\geq 0\\
T_{hh}-\theta_h x_{hh}-\eta_l y_{hh} & \geq & T_{ll}-\theta_h x_{ll}-\eta_l y_{ll} & \Rightarrow \quad \Delta_\theta\cdot(x_{ll}-x_{hh})\geq 0\\
T_{hh}-\theta_l x_{hh}-\eta_h y_{hh} & \geq & T_{ll}-\theta_l x_{ll}-\eta_h y_{ll} & \Rightarrow \quad \Delta_\eta\cdot(y_{ll}-y_{hh})\geq 0.
\end{array}$$

Case 3a: If IC_{llhh}, IC_{lhhh}, IC_{hlll} bind, then

$$\begin{array}{rcl}
T_{hl} & = & \theta_h x_{hl}+\eta_l y_{hl}+\Delta_\theta\cdot(x_{hh}-x_{ll})+\Delta_\eta\cdot y_{hh}\\
T_{lh} & = & \theta_l x_{lh}+\eta_h y_{lh}+\Delta_\theta\cdot x_{hh}\\
T_{ll} & = & \theta_l x_{ll}+\eta_l y_{ll}+\Delta_\theta\cdot x_{hh}+\Delta_\eta\cdot y_{hh}
\end{array}$$

and

$$\begin{array}{rcll}
T_{ll}-\theta_h x_{ll}-\eta_l y_{ll} & \geq & T_{hh}-\theta_h x_{hh}-\eta_l y_{hh} & \Rightarrow \quad \Delta_\theta\cdot(x_{hh}-x_{ll})\geq 0\\
T_{hh}-\theta_l x_{hh}-\eta_l y_{hh} & \geq & T_{lh}-\theta_l x_{lh}-\eta_l y_{lh} & \Rightarrow \quad \Delta_\eta\cdot(y_{hh}-y_{lh})\geq 0.
\end{array}$$

Case 4a: If IC_{lllh}, IC_{lhhh}, IC_{hlll} bind, then

$$\begin{array}{rcl}
T_{hl} & = & \theta_h x_{hl}+\eta_l y_{hl}+\Delta_\theta\cdot(x_{hh}-x_{ll})+\Delta_\eta\cdot y_{lh}\\
T_{lh} & = & \theta_l x_{lh}+\eta_h y_{lh}+\Delta_\theta\cdot x_{hh}\\
T_{ll} & = & \theta_l x_{ll}+\eta_l y_{ll}+\Delta_\theta\cdot x_{hh}+\Delta_\eta\cdot y_{lh}
\end{array}$$

and

$$\begin{aligned} T_{ll} - \theta_h x_{ll} - \eta_l y_{ll} &\geq T_{hh} - \theta_h x_{hh} - \eta_l y_{hh} \quad \Rightarrow \quad \Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{ll} - x_{hh}) \geq 0 \\ T_{lh} - \theta_l x_{lh} - \eta_l y_{lh} &\geq T_{hh} - \theta_l x_{hh} - \eta_l y_{hh} \quad \Rightarrow \quad \Delta_\eta \cdot (y_{lh} - y_{hh}) \geq 0 \\ T_{ll} - \theta_h x_{ll} - \eta_l y_{ll} &\geq T_{lh} - \theta_h x_{lh} - \eta_l y_{lh} \quad \Rightarrow \quad \Delta_\theta \cdot (x_{lh} - x_{ll}) \geq 0. \end{aligned}$$

Case 5a: If IC_{lllh}, IC_{lhhh}, IC_{hllh} bind, then

$$\begin{aligned} T_{hl} &= \theta_h x_{hl} + \eta_l y_{hl} + \Delta_\theta \cdot (x_{hh} - x_{lh}) + \Delta_\eta \cdot y_{lh} \\ T_{lh} &= \theta_l x_{lh} + \eta_h y_{lh} + \Delta_\theta \cdot x_{hh} \\ T_{ll} &= \theta_l x_{ll} + \eta_l y_{ll} + \Delta_\theta \cdot x_{hh} + \Delta_\eta \cdot y_{lh} \end{aligned}$$

and

$$\begin{aligned} T_{lh} - \theta_h x_{lh} - \eta_l y_{lh} &\geq T_{hh} - \theta_h x_{hh} - \eta_l y_{hh} \quad \Rightarrow \quad \Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{lh} - x_{hh}) \geq 0 \\ T_{lh} - \theta_h x_{lh} - \eta_l y_{lh} &\geq T_{ll} - \theta_h x_{ll} - \eta_l y_{ll} \quad \Rightarrow \quad \Delta_\theta \cdot (x_{ll} - x_{lh}) \geq 0 \\ T_{lh} - \theta_l x_{lh} - \eta_l y_{lh} &\geq T_{hl} - \theta_l x_{hl} - \eta_l y_{hl} \quad \Rightarrow \quad \Delta_\theta \cdot (x_{lh} - x_{hl}) \geq 0. \end{aligned}$$

Case 6a: If IC_{llhl}, IC_{lhhh}, IC_{hllh} bind, then

$$\begin{aligned} T_{hl} &= \theta_h x_{hl} + \eta_l y_{hl} + \Delta_\theta \cdot (x_{hh} - x_{lh}) + \Delta_\eta \cdot y_{lh} \\ T_{lh} &= \theta_l x_{lh} + \eta_h y_{lh} + \Delta_\theta \cdot x_{hh} \\ T_{ll} &= \theta_l x_{ll} + \eta_l y_{ll} + \Delta_\theta \cdot (x_{hh} + x_{hl} - x_{lh}) + \Delta_\eta \cdot y_{lh} \end{aligned}$$

and

$$\begin{aligned} T_{lh} - \theta_h x_{lh} - \eta_l y_{lh} &\geq T_{hh} - \theta_h x_{hh} - \eta_l y_{hh} \quad \Rightarrow \quad \Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{lh} - x_{hh}) \geq 0 \\ T_{hl} - \theta_l x_{hl} - \eta_l y_{hl} &\geq T_{lh} - \theta_l x_{lh} - \eta_l y_{lh} \quad \Rightarrow \quad \Delta_\theta \cdot (x_{hl} - x_{lh}) \geq 0. \end{aligned}$$

b) This follows directly from Part a).

c) By continuity and convexity of $T : \mathcal{I} \to \mathbb{R}^4_+$ and continuity and strict concavity of V, the global optimization problem consists of maximizing a continuous and strictly concave function $\mathbb{E}[V - T]$ over a convex set $\mathcal{I}$. Together with the Inada conditions, this guarantees existence and uniqueness of a solution. □

Proof of Proposition 4. Clearly, it is a necessary condition for global optimality of an allocation $(x, y) \in \mathcal{I}$ that it is optimal within each region R_i for which $(x, y) \in R_i$. To prove sufficiency, assume that $(x, y) \in \mathcal{I}$ solves the subproblem for any region R_i for which $(x, y) \in R_i$ and suppose, for contradiction, that the globally optimal allocation is given by $(\hat{x},\hat{y}) \neq (x,y) \in \mathcal{I}$. Then, by concavity of the objective W, for any $\lambda > 0$ we have

$$\begin{aligned} W(\lambda(x,y) + (1-\lambda)(\hat{x},\hat{y})) &> \lambda W(x,y) + (1-\lambda) W(\hat{x},\hat{y}) \\ &> W(x,y). \end{aligned}$$

Moreover, as each R_j is closed, $(x,y) \notin R_j$ implies $\lambda(x,y) + (1-\lambda)(\hat{x},\hat{y}) \notin R_j$ for sufficiently small. But since $\mathcal{I} = \bigcup R_i$ and $\mathcal{I}$ is convex, this implies that there exists some $\lambda > 0$ and some $i \in I$ such that $(x,y) \in R_i$ and $\lambda(x,y) + (1-\lambda)(\hat{x},\hat{y}) \in R_i$, contradicting optimality of (x,y) within R_i. □

Theorem 1*. *An allocation $(x,y) \in \mathcal{I}$ solves the global optimization problem if and only if*

a) (x,y) lies in the interior of some region R_i, $i \in I$ and solves the associated FOCs with $\mu_i = 0$. All implementability constraints are slack with the possible exception of $\lambda_{3a,5}$, $\lambda_{3a,7}$, $\lambda_{3b,5}$, $\lambda_{3b,8}$, $\lambda_{4a,10}$, $\lambda_{4a,12}$, $\lambda_{4a,16}$, $\lambda_{4b,9}$, $\lambda_{4b,11}$, $\lambda_{4b,15}$, $\lambda_{5a,6}$, $\lambda_{5a,13}$, $\lambda_{5b,6}$, $\lambda_{5b,14}$, $\lambda_{6a,6}$, $\lambda_{6a,11}$, $\lambda_{6b,6}$, and $\lambda_{6b,12}$.

b) (x,y) lies in the interior of a hyperplane $R_i \cap R_j$ for two adjacent regions R_i and R_j, $i,j \in I$ and solves the associated FOCs such that all implementability constraints are slack with the possible exception of $\lambda_{3a,5} = \lambda_{4a,10}$, $\lambda_{3a,7} = \lambda_{4a,16}$, $\lambda_{3b,5} = \lambda_{4b,9}$, $\lambda_{3b,8} = \lambda_{4b,15}$, $\lambda_{4a,12} = \lambda_{5a,6}$, $\lambda_{4a,16} = \lambda_{5a,13}$, $\lambda_{4b,11} = \lambda_{5b,6}$, and $\lambda_{4b,15} = \lambda_{5b,14} \geq 0$ and $\mu_{i,j} = 1 - \mu_{j,i} \in [0,1]$ with the possible exception of

$$\begin{aligned} \mu_{3a,4a} &= 1 - \mu_{4a,3a} + \frac{\lambda_{4a,10} + \lambda_{4a,16}}{p_{ll} + p_{hl}} \in \left[0, 1 + \frac{\lambda_{4a,10} + \lambda_{4a,16}}{p_{ll} + p_{hl}}\right] \\ \mu_{3b,4b} &= 1 - \mu_{4b,3b} + \frac{\lambda_{4b,9} + \lambda_{4b,15}}{p_{ll} + p_{lh}} \in \left[0, 1 + \frac{\lambda_{4b,9} + \lambda_{4b,15}}{p_{ll} + p_{lh}}\right] \\ \mu_{4a,5a} &= 1 - \mu_{5a,4a} + \frac{\lambda_{5a,6} + \lambda_{5a,13}}{p_{hl}} \in \left[0, 1 + \frac{\lambda_{5a,6} + \lambda_{5a,13}}{p_{hl}}\right] \\ \mu_{4b,5b} &= 1 - \mu_{5b,4b} + \frac{\lambda_{5b,6} + \lambda_{5b,14}}{p_{lh}} \in \left[0, 1 + \frac{\lambda_{5b,6} + \lambda_{5b,14}}{p_{lh}}\right]. \end{aligned}$$

c1) $(x,y) \in R_{1a} \cap R_{1b} \cap R_2$ solves the associated FOCs with $\lambda = 0$ and

$$\begin{aligned} \mu_{1a,1b} &= 1 - \mu_{1b,1a} - \mu_{1b,2} = \mu_{2,1b} \in [0,1], \\ \mu_{1b,1a} &= 1 - \mu_{1a,1b} - \mu_{1a,2} = \mu_{2,1a} \in [0,1]. \end{aligned}$$

c2) $(x,y) \in R_{1a} \cap R_{4a} \cap R_{5a}$ solves the associated FOCs with $\lambda = 0$ and

$$\begin{aligned} \mu_{1a,4a} &= 1 - \mu_{4a,1a} - \mu_{4a,5a} = \mu_{5a,4a} \in [0,1], \\ \mu_{4a,1a} &= 1 - \mu_{1a,4a} - \mu_{1a,5a} = \mu_{5a,1a} \in [0,1]. \end{aligned}$$

c3) $(x,y) \in R_{1b} \cap R_{4b} \cap R_{5b}$ solves the associated FOCs with $\lambda = 0$ and

$$\begin{aligned} \mu_{1b,4b} &= 1 - \mu_{4b,1b} - \mu_{4b,5b} = \mu_{5b,4b} \in [0,1], \\ \mu_{4b,1b} &= 1 - \mu_{1b,4b} - \mu_{1b,5b} = \mu_{5b,1b} \in [0,1]. \end{aligned}$$

c4) $(x,y) \in R_{1a} \cap R_{1b} \cap R_{5a} \cap R_{6a}$ *solves the associated FOCs with* $\lambda = 0$ *and*

$$\begin{aligned}
\mu_{1a,1b} &= 1-\mu_{1b,1a} = 1-\mu_{6a,5a} = \mu_{5a,6a} \in [0,1]\,, \\
\mu_{1a,5a} &= \mu_{5a,6a} \cdot \frac{p_{ll}}{p_{hl}} + (1-\mu_{5a,1a}) \\
&= \mu_{1b,6a} \cdot \frac{p_{hl}+p_{ll}}{p_{hl}} \\
&= -\mu_{6a,5a} \cdot \frac{p_{ll}}{p_{hl}} + (1-\mu_{6a,1b}) \cdot \frac{p_{hl}+p_{ll}}{p_{hl}} \in \left[0, 1+\mu_{1a,1b} \cdot \frac{p_{ll}}{p_{hl}}\right].
\end{aligned}$$

c5) $(x,y) \in R_{1a} \cap R_{1b} \cap R_{5b} \cap R_{6b}$ *solves the associated FOCs with* $\lambda = 0$ *and*

$$\begin{aligned}
\mu_{1b,1a} &= 1-\mu_{1a,1b} = 1-\mu_{6b,5b} = \mu_{5b,6b} \in [0,1]\,, \\
\mu_{1b,5b} &= \mu_{5b,6b} \cdot \frac{p_{ll}}{p_{lh}} + (1-\mu_{5b,1b}) \\
&= \mu_{1a,6b} \cdot \frac{p_{lh}+p_{ll}}{p_{lh}} \\
&= -\mu_{6b,5b} \cdot \frac{p_{ll}}{p_{lh}} + (1-\mu_{6b,1a}) \cdot \frac{p_{lh}+p_{ll}}{p_{lh}} \in \left[0, 1+\mu_{1b,1a} \cdot \frac{p_{ll}}{p_{lh}}\right].
\end{aligned}$$

c6) $(x,y) \in R_{1a} \cap R_{2} \cap R_{3a} \cap R_{4a}$ *solves the associated FOCs with* $\lambda = 0$ *and*

$$\begin{aligned}
\mu_{1a,4a} &= 1-\mu_{4a,1a} = 1-\mu_{3a,2} = \mu_{2,3a} \in [0,1]\,, \\
\mu_{1a,2} &= \mu_{2,3a} \cdot \frac{p_{hl}}{p_{ll}} + (1-\mu_{2,1a}) \\
&= \mu_{4a,3a} \cdot \frac{p_{hl}+p_{ll}}{p_{ll}} \\
&= -\mu_{3a,2} \cdot \frac{p_{hl}}{p_{ll}} + (1-\mu_{3a,4a}) \cdot \frac{p_{hl}+p_{ll}}{p_{ll}} \in \left[0, 1+\mu_{1a,4a} \cdot \frac{p_{hl}}{p_{ll}}\right].
\end{aligned}$$

c7) $(x,y) \in R_{1b} \cap R_{2} \cap R_{3b} \cap R_{4b}$ *solves the associated FOCs with* $\lambda = 0$ *and*

$$\begin{aligned}
\mu_{1b,4b} &= 1-\mu_{4b,1b} = 1-\mu_{3b,2} = \mu_{2,3b} \in [0,1]\,, \\
\mu_{1b,2} &= \mu_{2,3b} \cdot \frac{p_{lh}}{p_{ll}} + (1-\mu_{2,1b}) \\
&= \mu_{4b,3b} \cdot \frac{p_{lh}+p_{ll}}{p_{ll}} \\
&= -\mu_{3b,2} \cdot \frac{p_{lh}}{p_{ll}} + (1-\mu_{3b,4b}) \cdot \frac{p_{lh}+p_{ll}}{p_{ll}} \in \left[0, 1+\mu_{1b,4b} \cdot \frac{p_{lh}}{p_{ll}}\right].
\end{aligned}$$

c8) $(x,y) \in R_{1a} \cap R_{4a} \cap R_{6b}$ *solves the associated FOCs with* $\lambda = 0$ *except for* $\lambda_{4a,12}$ *and* $\lambda_{6b,12}$ *and*

$$\begin{aligned}
\mu_{1a,4a} \cdot p_{hl} = (1-\mu_{4a,1a}) \cdot p_{hl} + \lambda_{4a,12} &= \lambda_{6b,12} \geq 0, \\
\mu_{1a,6b} \cdot (p_{ll}+p_{lh}) = (1-\mu_{6b,1a}) \cdot (p_{ll}+p_{lh}) + \lambda_{6b,12} &= \lambda_{4a,12} \geq 0.
\end{aligned}$$

c9) $(x, y) \in R_{1b} \cap R_{4b} \cap R_{6a}$ *solves the associated FOCs with* $\lambda = 0$ *except for* $\lambda_{4b,11}$ *and* $\lambda_{6a,11}$ *and*

$$\begin{aligned} \mu_{1b,4b} \cdot p_{lh} &= (1 - \mu_{4b,1b}) \cdot p_{lb} + \lambda_{4b,11} &= \lambda_{6a,11} \geq 0, \\ \mu_{1b,6a} \cdot (p_{ll} + p_{hl}) &= (1 - \mu_{6a,1b}) \cdot (p_{ll} + p_{hl}) + \lambda_{6a,11} &= \lambda_{4b,11} \geq 0. \end{aligned}$$

Proof of Theorem 1.* The proof of Theorem 1* proceeds in three steps. In Step 1, we identify for each region R_i the minimal set of implementabilty constraints λ_i that guarantees implementability of an allocation in R_i. All other implemetablity constraints can then be eliminated from the Kuhn-Tucker subproblem in region R_i. Step 2 is a lemma on convex sets. In Step 3, by ruling out all other potential options we prove that the optimal allocation always lies in one of the loci listed in Theorem 1*. The conditions on the multipliers then ensure that all multipliers are non-negative and the FOCs coincide for all regions in which the solution lies.

Step 1. In R_{1a}, constraints (I2), (I4) together with (R1a,1b), (R1a,2), (R1a,4a), (R1a,5a), and (R1a,6b) imply all other implementability constraints as[16]

$$\begin{aligned} I1 &= R1a, 1b + R1a, 4a \\ I3 &= R1a, 2 + R1a, 5a \\ I5 &= 2 \cdot R1a, 2 + R1a, 4a + I2 \\ I6 &= R1a, 5a + R1a, 6b \\ I7 &= R1a, 2 + R1a, 4a + I2 \\ I8 &= 2 \cdot R1a, 2 + R1a, 5a + I2 \\ I9 &= R1a, 1b + R1a, 2 + R1a, 4a + I2 \\ I10 &= R1a, 2 + R1a, 4a + I2 \\ I11 &= R1a, 1b + R1a, 5a + I2 \\ I12 &= R1a, 4a + R1a, 6b \\ I13 &= R1a, 5a + I4 \\ I14 &= R1a, 1b + R1a, 5a + I4 \\ I15 &= R1a, 1b + R1a, 2 + R1a, 5a + I2 \\ I16 &= R1a, 4a + I4. \end{aligned}$$

Likewise, constraints (I1) and (I3) together with (R1b,1a), (R1b,2), (R1b,4b), (R1b,5b), and (R1b,6a) imply all other implementability constraints in R_{1b}.

In R_2, constraints (I3), (I4) together with (R2,1a), (R2,1b), (R2,3a), (R2,3b) imply all other implementability constraints as

[16]Here as well as in later proofs I use references to inequality constraints without brackets when referring to their left hand side only, i.e. to the term required to be non-negative.

$$
\begin{aligned}
I1 &= R2,1b + R2,3a \\
I2 &= R2,1a + R2,3b \\
I1 &= R2,3a + R2,3b \\
I6 &= R2,1a + R2,1b + I3 + I4 \\
I7 &= R2,3a + I4 \\
I8 &= R2,3b + I3 \\
I9 &= R2,1b + R2,3a + R2,3b \\
I10 &= R2,1a + R2,3a + R2,3b \\
I11 &= R2,1a + R2,1b + R2,3b + I3 \\
I12 &= R2,1a + R2,1b + R2,3a + I4 \\
I13 &= R2,1a + I3 + I4 \\
I14 &= R2,1b + I3 + I4 \\
I15 &= R2,1b + R2,3a + I3 \\
I16 &= R2,1a + R2,3b + I4.
\end{aligned}
$$

In R_{3a}, constraints (I1), (I3), (I5), (I7) together with (R3a,2) and (R3a,4a) imply all other implementability constraints as

$$
\begin{aligned}
I2 &= R3a,2 + R3a,4a + I5 \\
I4 &= R3a,2 + I7 \\
I6 &= 2 \cdot R3a,2 + R3a,4a + I1 + I3 + I7 \\
I8 &= R3a,2 + I3 + I5 \\
I9 &= R3a,2 + I1 + I5 \\
I10 &= R3a,4a + I5 \\
I11 &= 2 \cdot R3a,2 + R3a,4a + I1 + I3 + I5 \\
I12 &= R3a,2 + R3a,4a + I1 + I7 \\
I13 &= R3a,2 + R3a,4a + I3 + I7 \\
I14 &= 2 \cdot R3a,2 + I1 + I3 + I7 \\
I15 &= 2 \cdot R3a,2 + I1 + I3 + I5 \\
I16 &= R3a,4a + I7.
\end{aligned}
$$

Likewise, constraints (I2), (I4), (I5), and (I8) together with (R3b,2) and (R3b,4b) imply all other implementability constraints in R_{3b}.

In R_{4a}, constraints I1, I2, I3, I10, I12, and I16 together with R4a,1a, R4a,3a, and

R4a,5a imply all other implementability constraints as

$$
\begin{aligned}
I4 &= R4a,1a + I16 \\
I5 &= R4a,3a + I10 \\
I6 &= R4a,5a + I12 \\
I7 &= R4a,3a + I16 \\
I8 &= R4a,3a + R4a,5a + I10 \\
I9 &= R4a,3a + I1 + I2 \\
I11 &= R4a,5a + I1 + I2 \\
I13 &= R4a,5a + I16 \\
I14 &= R4a,1a + R4a,5a + I1 + I16 \\
I15 &= R4a,1a + R4a,5a + I1 + I10.
\end{aligned}
$$

Likewise, constraints (I1), (I2), (I4), (I9), (I11), and (I15) together with (R4b,1b), (R4b,3b), and (R4b,5b) imply all other implementability constraints in R_{4b}.

In R_{5a}, constraints (I2), (I3), (I6), and (I13) together with (R5a,1a), (R5a,4a), and (R5a,6a) imply all other implementability constraints as

$$
\begin{aligned}
I1 &= R5a,4a + R5a,6a \\
I4 &= R5a,1a + I13 \\
I5 &= R5a,1a + R5a,4a + I2 + 2 \cdot I3 \\
I7 &= R5a,1a + 2 \cdot R5a,4a + I2 + I13 \\
I8 &= R5a,1a + I2 + 2 \cdot I3 \\
I9 &= R5a,1a + R5a,4a + R5a,6a + I2 + I3 \\
I10 &= R5a,4a + I2 + I3 \\
I11 &= R5a,6a + I2 \\
I12 &= R5a,4a + I6 \\
I14 &= R5a,1a + R5a,6a + I13 \\
I15 &= R5a,1a + R5a,6a + I2 + I3 \\
I16 &= R5a,4a + I13.
\end{aligned}
$$

Likewise, constraints (I1), (I4), (I6), and (I14) together with (R5b,1b), (R5b,4b), and (R5b,6b) imply all other implementability constraints in R_{5b}.

In R_{6a}, constraints (I1), (I3), (I6), and (I11) together with (R6a,1b) and (R6a,5a) imply all other implementability constraints as

$$\begin{aligned}
I2 &= R6a,5a + I11 \\
I4 &= R6a,1b + R6a,5a + I3 + I6 \\
I5 &= R6a,1b + 2 \cdot R6a,5a + I1 + I3 + I11 \\
I7 &= R6a,1b + 2 \cdot R6a,5a + I1 + 2 \cdot I3 + I6 \\
I8 &= R6a,1b + R6a,5a + 2 \cdot I3 + I11 \\
I9 &= R6a,1b + R6a,5a + I1 + I3 + I11 \\
I10 &= 2 \cdot R6a,5a + I1 + I3 + I11 \\
I12 &= R6a,5a + I1 + I6 \\
I13 &= R6a,5a + I3 + I6 \\
I14 &= R6a,1b + I3 + I6 \\
I15 &= R6a,1b + I3 + I11 \\
I16 &= 2 \cdot R6a,5a + I1 + I3 + I6.
\end{aligned}$$

Likewise, constraints (I2), (I4), (I6), and (I12) together with (R6b,1a) and (R6b,5b) imply all other implementability constraints in R_{6b}.

Assume that (I2) binds in R_{1a}, i.e. $y_{ll} = y_{lh}$. From the proof of Proposition 3 together with (3.5.1) and (3.5.2) we get FOCs in R_{1a} reading

$$\begin{aligned}
V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\mu_{1a,5a} \cdot p_{hl}] \\
V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\mu_{1a,4a} \cdot p_{hl}].
\end{aligned}$$

If $\mu_{1a,5a} > \mu_{1a,4a} = 0$, then $(x, y) \in R_{5a}$ and hence $x_{ll} \geq x_{lh}$. But from the above FOCs it then follows that $x_{ll} < x_{lh}$, a contradiction. If $\mu_{1a,4a} > \mu_{1a,5a} = 0$, then $(x, y) \in R_{4a}$ and hence $x_{ll} \leq x_{lh}$. But from the above FOCs it then follows that $x_{ll} > x_{lh}$, a contradiction. If $\mu_{1a,5a} > 0$ and $\mu_{1a,4a} > 0$, then $x_{ll} = x_{lh}$ as $(x, y) \in R_{4a} \cap R_{5a}$. If $\mu_{1a,5a} = \mu_{1a,4a} = 0$, then $x_{ll} = x_{lh}$ from the above FOCs. Hence $x_{ll} = x_{lh}$. But then, again invoking (3.5.1) and (3.5.2), FOCs in R_{1a} given as

$$\begin{aligned}
V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [p_{ll} - \mu_{1a,1b} \cdot p_{ll} - \mu_{1a,2} \cdot p_{ll} + \mu_{1a,4a} \cdot p_{hl} + \mu_{1a,5a} \cdot p_{hl} + \lambda_2] \\
V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_2]
\end{aligned}$$

imply that $(x, y) \in R_{1b}$ or $(x, y) \in R_2$. If $(x, y) \in R_{1b}$, then $x_{ll} = x_{lh}$ and $y_{ll} = y_{lh}$ together contradict the FOCs for R_{1b} given as

$$\begin{aligned}
V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{1b,1a} \cdot p_{ll} + \mu_{1b,6a} \cdot (p_{hl} + p_{ll})] \\
V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\mu_{1b,4b} \cdot p_{lh}]
\end{aligned}$$

If $(x, y) \in R_2$, then $x_{ll} = x_{lh}$ and $y_{ll} = y_{lh}$ together contradict the FOCs for R_2 given as

$$\begin{aligned} V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{2,1a} \cdot p_{ll}] \\ V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\mu_{2,3b} \cdot p_{lh} - \lambda_2] . \end{aligned}$$

Assume next that (I4) binds in R_{1a}, i.e. $y_{hl} = y_{hh}$. Then $x_{hl} \leq x_{hh}$ by (R1a,6b) and hence FOCs in R_{1a} given as

$$\begin{aligned} V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll}) - \mu_{1a,1b} \cdot p_{ll} + \mu_{1a,4a} \cdot p_{hl} + \mu_{1a,5a} \cdot p_{hl} - \mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\ V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\mu_{1a,1b} \cdot p_{ll} + \mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \end{aligned}$$

imply $(x, y) \in R_{1b}$ or $(x, y) \in R_{6b}$. If $(x, y) \in R_{1b}$ then $y_{hl} = y_{hh}$ implies $x_{hl} = x_{hh}$ through (R1b,2) and (R1b,5b). If $(x, y) \in R_{6b}$ then $y_{hl} = y_{hh}$ implies $x_{hl} = x_{hh}$ through (R1a,6b). Invoking the FOCs in R_{1a} given as

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [p_{hl} + \mu_{1a,1b} \cdot p_{ll} + \mu_{1a,2} \cdot p_{ll} - \mu_{1a,4a} \cdot p_{hl} - \mu_{1a,5a} \cdot p_{hl} + \mu_{1a,6b} \cdot (p_{lh} + p_{ll}) + \lambda_4] \\ V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\mu_{1a,6b} \cdot (p_{lh} + p_{ll}) - \lambda_4] \end{aligned}$$

this implies $(x, y) \in R_{4a}$ or $(x, y) \in R_{5a}$ which together with $y_{hl} = y_{hh}$ and $x_{hl} = x_{hh}$ contradicts the FOCs in R_{4a} given as

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [\mu_{4a,1a} \cdot p_{hl} + \mu_{4a,3a} \cdot (p_{hl} + p_{ll})] \\ V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_{12} - \lambda_{16}] \end{aligned}$$

or the FOCs in R_{5a} given as

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [\mu_{5a,1a} \cdot p_{hl}] \\ V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_6 - \lambda_{13}] . \end{aligned}$$

Likewise, (I1) and (I3) cannot bind in R_{1b}.

Suppose (I4) binds in R_2, i.e. $y_{hl} = y_{hh}$. Together with $x_{hl} \leq x_{hh}$ as given by (R2,1b), FOCs in R_2 given as

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{2,1a} \cdot p_{ll} + \mu_{2,3b} \cdot p_{lh} + \lambda_4] \\ V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_4] \end{aligned}$$

imply $(x, y) \in R_{1a}$, a contradiction to what we have already shown. Likewise, (I3) holding with equality implies $(x, y) \in R_{1b}$ and thus another contradiction.

Suppose I1 binds in R_{3a} , i.e. $x_{ll} = x_{hl}$. FOCs in R_{3a} read

$$V_y(x_{hl}, y_{hl}) = \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_7]$$

$$V_y(x_{ll}, y_{ll}) = \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_5].$$

If $\lambda_5 > \lambda_7 = 0$, then $y_{ll} > y_{hl}$ from the above FOCs, but $y_{ll} > y_{hl}$ together with (I5) binding contradicts (I7). If $\lambda_7 > \lambda_5 = 0$, then $y_{ll} < y_{hl}$ from the above FOCs, but $y_{ll} < y_{hl}$ together with (I7) binding contradicts (I1). If $\lambda_5 > 0$ and $\lambda_7 > 0$, then $y_{ll} = y_{hl}$ as both, (I5) and (I7) bind. If $\lambda_5 = \lambda_7 = 0$, then $y_{ll} = y_{hl}$ from the above FOCs. Thus $x_{ll} = x_{hl}$ and $y_{ll} = y_{hl}$, but this together with FOCs in R_{3a} given as

$$V_x(x_{hl}, y_{hl}) = \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\lambda_1]$$

$$V_x(x_{ll}, y_{ll}) = \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-p_{hl} + \mu_{3a,2} \cdot p_{hl} - \lambda_1 - \lambda_5 - \lambda_7]$$

implies $(x, y) \in R_2$ which in turn implies $(x, y) \in R_{1a}$, a contradiction. Next, assume (I3) binds in R_{3a}, i.e. $x_{lh} = x_{hh}$. Together with $y_{lh} \leq y_{hh}$ as implied by R3a,4a and FOCs in R_{3a} given as

$$V_y(x_{hh}, y_{hh}) = \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{3a,4a} \cdot (p_{hl} + p_{ll}) + \lambda_5 + \lambda_7]$$

$$V_y(x_{lh}, y_{lh}) = \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{3a,4a} \cdot (p_{hl} + p_{ll})]$$

this implies $(x, y) \in R_{4a} \cap R_{3a}$. But then $y_{lh} = y_{hh}$, and together with $x_{lh} = x_{hh}$ and FOCs in R_{3a} given as

$$V_x(x_{hh}, y_{hh}) = \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll} + p_{hl}) - \mu_{3a,2} \cdot p_{hl} + \lambda_3 + \lambda_5 + \lambda_7]$$

$$V_x(x_{lh}, y_{lh}) = \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\lambda_3]$$

this implies $(x, y) \in R_2$, a contradiction. Likewise, constraints (I2) and (I4) cannot bind in R_{3b}.

Assume (I2) binds in R_{4a}, i.e. $y_{ll} = y_{lh}$. Since we have $x_{ll} \leq x_{lh}$ from (R4a,5a) , FOCs in R_{4a} given as

$$V_x(x_{lh}, y_{lh}) = \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\mu_{4a,5a} \cdot p_{hl} - \lambda_3]$$

$$V_x(x_{ll}, y_{ll}) = \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-p_{hl} + \mu_{4a,1a} \cdot p_{hl} + \mu_{4a,5a} \cdot p_{hl} - \lambda_1 - \lambda_{10} - \lambda_{12} - \lambda_{16}]$$

imply either $(x, y) \in R_{1a}$ or $(x, y) \in R_{5a}$. The former being excluded already, $(x, y) \in R_{4a} \cap R_{5a}$ implies $x_{ll} = x_{lh}$. Together with $y_{ll} = y_{lh}$ and FOCs in R_{4a} given as

$$V_y(x_{lh}, y_{lh}) = \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) - \mu_{4a,1a} \cdot p_{hl} - \mu_{4a,3a} \cdot (p_{hl} + p_{ll}) + \lambda_2 + \lambda_{10} + \lambda_{12} + \lambda_{16}]$$

$$V_y(x_{ll}, y_{ll}) = \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_2 - \lambda_{10}]$$

this implies $(x,y) \in R_{1a}$ or $(x,y) \in R_{3a}$. The former case has been excluded already while the latter implies $(x,y) \in R_2$ which also has been excluded. Next, assume that (I1) binds in R_{4a}, i.e. $x_{ll} = x_{hl}$. FOCs of R_{4a} read

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_{12} - \lambda_{16}] \\ V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_{10}] . \end{aligned}$$

If $y_{ll} > y_{hl}$ then $\lambda_{10} > 0$, implying

$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) < \Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) = 0$$

and hence contradicting (I16). If $y_{hl} > y_{ll}$ then either $\lambda_{16} > 0$ contradicting (I10) as

$$\Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) < \Delta_\theta \cdot (x_{ll} - x_{hh}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) = 0$$

or $\lambda_{16} > 0$, contradicting (I1) or (I2) as

$$\Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{ll} - y_{lh}) < \Delta_\theta \cdot (x_{ll} - x_{hl}) + \Delta_\eta \cdot (y_{hl} - y_{lh}) = 0.$$

Thus $y_{ll} = y_{hl}$, and together with $x_{ll} = x_{hl}$ FOCs of R_{4a} reading

$$\begin{aligned} V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\lambda_1 + \lambda_{12}] \\ V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-p_{hl} + \mu_{4a,1a} \cdot p_{hl} + \mu_{4a,5a} \cdot p_{hl} - \lambda_1 - \lambda_{10} - \lambda_{12} - \lambda_{16}] \end{aligned}$$

imply $(x,y) \in R_{1a}$ or $(x,y) \in R_{5a}$. The former case being excluded already, $(x,y) \in R_{4a} \cap R_{5a}$ together with $x_{ll} = x_{hl}$, $y_{ll} = y_{hl}$ contradicts the FOCs of R_{5a} reading

$$\begin{aligned} V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\mu_{5a,6a} \cdot p_{ll} + \lambda_6] \\ V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\mu_{5a,4a} \cdot p_{hl}] . \end{aligned}$$

Finally, assume that I3 binds in R_{4a}, i.e. $x_{lh} = x_{hh}$. Together with $y_{hh} \geq y_{lh}$ as given by R4a,3a, FOCs in R_{4a} reading

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [\mu_{4a,1a} \cdot p_{hl} + \mu_{4a,3a} \cdot (p_{hl} + p_{ll})] \\ V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) - \mu_{4a,1a} \cdot p_{hl} - \mu_{4a,3a} \cdot (p_{hl} + p_{ll}) + \lambda_2 + \lambda_{10} + \lambda_{12} + \lambda_{16}] \end{aligned}$$

yield $(x,y) \in R_{1a}$ or $(x,y) \in R_{3a}$. The latter case has been excluded already while the former case implies $(x,y) \in R_2$ which has also been excluded. Likewise, constraints (I1), (I2), and (I4) cannot bind in R_{4b}.

Assume I2 binds in R_{5a} i.e. $y_{ll} = y_{lh}$. Together with $x_{ll} \geq x_{lh}$ as implied by (R5a,4a), FOCs in R_{5a} reading

$$V_x(x_{lh}, y_{lh}) = \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-p_{hl} + \mu_{5a,1a} \cdot p_{hl} + \mu_{5a,4a} \cdot p_{hl} - \mu_{5a,6a} \cdot p_{ll} - \lambda_3 - \lambda_6 - \lambda_{13}]$$
$$V_x(x_{ll}, y_{ll}) = \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\mu_{5a,4a} \cdot p_{hl}]$$

yield $(x, y) \in R_{1a}$ or $(x, y) \in R_{4a}$, both being excluded already. Next, suppose I3 binds in R_{5a}, i.e. $x_{lh} = x_{hh}$. Together with $y_{lh} \geq y_{hh}$ as implied by (R5a,1a), FOCs in R_{5a} given as

$$V_y(x_{hh}, y_{hh}) = \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [\mu_{5a,1a} \cdot p_{hl}]$$
$$V_y(x_{lh}, y_{lh}) = \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) - \mu_{5a,1a} \cdot p_{hl} + \lambda_2 + \lambda_6 + \lambda_{13}]$$

yields $(x, y) \in R_{1a}$ which implies $(x, y) \in R_2$ which has already been excluded. Likewise, constraints (I1) and (I4) cannot bind in R_{5b}.

Assume (I1) binds in R_{6a}, i.e. $x_{ll} = x_{hl}$. FOCs in R_{6a} yield

$$V_y(x_{hl}, y_{hl}) = \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_6]$$
$$V_y(x_{ll}, y_{ll}) = \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_{11}] .$$

If $\lambda_6 > \lambda_{11} = 0$ then $y_{hl} > y_{ll}$, but $y_{hl} > y_{ll}$ together with (I6) binding violates (I11). If $\lambda_{11} > \lambda_6 = 0$ then $y_{hl} < y_{ll}$, but $y_{hl} < y_{ll}$ together with (I11) binding violates (I6). If $\lambda_6 > 0$ and $\lambda_{11} > 0$, then $y_{ll} = y_{hl}$ as both (I6) and (I11) bind. If $\lambda_6 = \lambda_{11} = 0$, then $y_{ll} = y_{hl}$ from the above FOCs. Hence $x_{ll} = x_{hl}$ and $y_{ll} = y_{hl}$, but this together with FOCs of R_{5a} reading

$$V_x(x_{hl}, y_{hl}) = \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [p_{ll} - \mu_{6a,5a} \cdot p_{ll} + \lambda_1 + \lambda_{11} + \lambda_6]$$
$$V_x(x_{ll}, y_{ll}) = \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\lambda_1]$$

implies $(x, y) \in R_{5a}$ which in turn implies $(x, y) \in R_{4a}$, a contradiction. Next, suppose (I3) holds with equality, i.e. $x_{lh} = x_{hh}$. Together with $y_{lh} \geq y_{hh}$ as implied by R6a,1b implies and FOCs in R_{6a} reading

$$V_y(x_{hh}, y_{hh}) = \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [\mu_{6a,1b} \cdot (p_{ll} + p_{hl})]$$
$$V_y(x_{lh}, y_{lh}) = \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) - \mu_{6a,1b} \cdot (p_{ll} + p_{hl}) + \lambda_{11} + \lambda_6]$$

this yields$(x, y) \in R_{1b}$ which has already been excluded. Likewise, (I2) and (I4) cannot bind in R_{6b}. This finishes Step 1.

For later reference, we list the FOCs for each of the eleven subproblems, introducing some normalization of the multipliers μ_{region} to which the theorem refers.

$$\left.\begin{aligned}
V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll}) - \mu_{1a,1b} \cdot p_{ll} + \mu_{1a,4a} \cdot p_{hl} + \mu_{1a,5a} \cdot p_{hl} - \mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\
V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\mu_{1a,1b} \cdot p_{ll} + \mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\
V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\mu_{1a,5a} \cdot p_{hl}] \\
V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\mu_{1a,4a} \cdot p_{hl}] \\
V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [p_{hl} + \mu_{1a,1b} \cdot p_{ll} + \mu_{1a,2} \cdot p_{ll} - \mu_{1a,4a} \cdot p_{hl} - \mu_{1a,5a} \cdot p_{hl} + \mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\
V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\
V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [p_{ll} - \mu_{1a,1b} \cdot p_{ll} - \mu_{1a,2} \cdot p_{ll} + \mu_{1a,4a} \cdot p_{hl} + \mu_{1a,5a} \cdot p_{hl}] \\
V_y(x_{ll}, y_{ll}) &= \eta_l
\end{aligned}\right\} R_{1a}$$

$$\left.\begin{aligned}
V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [p_{lh} + \mu_{1b,1a} \cdot p_{ll} + \mu_{1b,2} \cdot p_{ll} - \mu_{1b,4b} \cdot p_{lh} - \mu_{1b,5b} \cdot p_{lh} + \mu_{1b,6a} \cdot (p_{hl} + p_{ll})] \\
V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [p_{ll} - \mu_{1b,1a} \cdot p_{ll} - \mu_{1b,2} \cdot p_{ll} + \mu_{1b,4b} \cdot p_{lh} + \mu_{1b,5b} \cdot p_{lh}] \\
V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\mu_{1b,6a} \cdot (p_{hl} + p_{ll})] \\
V_x(x_{ll}, y_{ll}) &= \theta_l \\
V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{1b,1a} \cdot p_{ll} + \mu_{1b,4b} \cdot p_{lh} + \mu_{1b,5b} \cdot p_{lh} - \mu_{1b,6a} \cdot (p_{hl} + p_{ll})] \\
V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\mu_{1b,5b} \cdot p_{lh}] \\
V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{1b,1a} \cdot p_{ll} + \mu_{1b,6a} \cdot (p_{hl} + p_{ll})] \\
V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\mu_{1b,4b} \cdot p_{lh}]
\end{aligned}\right\} R_{1b}$$

$$\left.\begin{aligned}
V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll}) - \mu_{2,1b} \cdot p_{ll} + \mu_{2,3a} \cdot p_{hl}] \\
V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\mu_{2,1b} \cdot p_{ll}] \\
V_x(x_{lh}, y_{lh}) &= \theta_l \\
V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-\mu_{2,3a} \cdot p_{hl}] \\
V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{2,1a} \cdot p_{ll} + \mu_{2,3b} \cdot p_{lh}] \\
V_y(x_{hl}, y_{hl}) &= \eta_l \\
V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{2,1a} \cdot p_{ll}] \\
V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\mu_{2,3b} \cdot p_{lh}]
\end{aligned}\right\} R_2$$

$$\left.\begin{aligned}
V_x\left(x_{hh},y_{hh}\right)&=\theta_h+\frac{\Delta_\theta}{p_{hh}}\cdot\left[\left(p_{lh}+p_{ll}+p_{hl}\right)-\mu_{3a,2}\cdot p_{hl}+\lambda_5+\lambda_7\right]\\
V_x\left(x_{hl},y_{hl}\right)&=\theta_h\\
V_x\left(x_{lh},y_{lh}\right)&=\theta_l\\
V_x\left(x_{ll},y_{ll}\right)&=\theta_l+\frac{\Delta_\theta}{p_{ll}}\cdot\left[-p_{hl}+\mu_{3a,2}\cdot p_{hl}-\lambda_5-\lambda_7\right]\\
V_y\left(x_{hh},y_{hh}\right)&=\eta_h+\frac{\Delta_\eta}{p_{hh}}\cdot\left[\left(p_{hl}+p_{ll}\right)-\mu_{3a,4a}\cdot\left(p_{hl}+p_{ll}\right)+\lambda_5+\lambda_7\right]\\
V_y\left(x_{hl},y_{hl}\right)&=\eta_l+\frac{\Delta_\eta}{p_{hl}}\cdot\left[-\lambda_7\right]\\
V_y\left(x_{lh},y_{lh}\right)&=\eta_h+\frac{\Delta_\eta}{p_{lh}}\cdot\left[\mu_{3a,4a}\cdot\left(p_{hl}+p_{ll}\right)\right]\\
V_y\left(x_{ll},y_{ll}\right)&=\eta_l+\frac{\Delta_\eta}{p_{ll}}\cdot\left[-\lambda_5\right]
\end{aligned}\right\}R_{3a}$$

$$\left.\begin{aligned}
V_x\left(x_{hh},y_{hh}\right)&=\theta_h+\frac{\Delta_\theta}{p_{hh}}\cdot\left[\left(p_{lh}+p_{ll}\right)-\mu_{3b,4b}\cdot\left(p_{lh}+p_{ll}\right)+\lambda_5+\lambda_8\right]\\
V_x\left(x_{hl},y_{hl}\right)&=\theta_h+\frac{\Delta_\theta}{p_{hl}}\cdot\left[\mu_{3b,4b}\cdot\left(p_{lh}+p_{ll}\right)\right]\\
V_x\left(x_{lh},y_{lh}\right)&=\theta_l+\frac{\Delta_\theta}{p_{lh}}\cdot\left[-\lambda_8\right]\\
V_x\left(x_{ll},y_{ll}\right)&=\theta_l+\frac{\Delta_\theta}{p_{ll}}\cdot\left[-\lambda_5\right]\\
V_y\left(x_{hh},y_{hh}\right)&=\eta_h+\frac{\Delta_\eta}{p_{hh}}\cdot\left[\left(p_{hl}+p_{ll}+p_{lh}\right)-\mu_{3b,2}\cdot p_{lh}+\lambda_5+\lambda_8\right]\\
V_y\left(x_{hl},y_{hl}\right)&=\eta_l\\
V_y\left(x_{lh},y_{lh}\right)&=\eta_h\\
V_y\left(x_{ll},y_{ll}\right)&=\eta_l+\frac{\Delta_\eta}{p_{ll}}\cdot\left[-p_{lh}+\mu_{3b,2}\cdot p_{lh}-\lambda_5-\lambda_8\right]
\end{aligned}\right\}R_{3b}$$

$$\left.\begin{aligned}
V_x\left(x_{hh},y_{hh}\right)&=\theta_h+\frac{\Delta_\theta}{p_{hh}}\cdot\left[\left(p_{lh}+p_{ll}+p_{hl}\right)-\mu_{4a,1a}\cdot p_{hl}+\lambda_{10}+\lambda_{16}\right]\\
V_x\left(x_{hl},y_{hl}\right)&=\theta_h+\frac{\Delta_\theta}{p_{hl}}\cdot\left[\lambda_{12}\right]\\
V_x\left(x_{lh},y_{lh}\right)&=\theta_l+\frac{\Delta_\theta}{p_{lh}}\cdot\left[-\mu_{4a,5a}\cdot p_{hl}-\lambda_3\right]\\
V_x\left(x_{ll},y_{ll}\right)&=\theta_l+\frac{\Delta_\theta}{p_{ll}}\cdot\left[-p_{hl}+\mu_{4a,1a}\cdot p_{hl}+\mu_{4a,5a}\cdot p_{hl}-\lambda_{10}-\lambda_{12}-\lambda_{16}\right]\\
V_y\left(x_{hh},y_{hh}\right)&=\eta_h+\frac{\Delta_\eta}{p_{hh}}\cdot\left[\mu_{4a,1a}\cdot p_{hl}+\mu_{4a,3a}\cdot\left(p_{hl}+p_{ll}\right)\right]\\
V_y\left(x_{hl},y_{hl}\right)&=\eta_l+\frac{\Delta_\eta}{p_{hl}}\cdot\left[-\lambda_{12}-\lambda_{16}\right]\\
V_y\left(x_{lh},y_{lh}\right)&=\eta_h+\frac{\Delta_\eta}{p_{lh}}\cdot\left[\left(p_{hl}+p_{ll}\right)-\mu_{4a,1a}\cdot p_{hl}-\mu_{4a,3a}\cdot\left(p_{hl}+p_{ll}\right)+\lambda_{10}+\lambda_{12}+\lambda_{16}\right]\\
V_y\left(x_{ll},y_{ll}\right)&=\eta_l+\frac{\Delta_\eta}{p_{ll}}\cdot\left[-\lambda_{10}\right]
\end{aligned}\right\}R_{4a}$$

$$\left.\begin{aligned}
V_x\left(x_{hh}, y_{hh}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot \left[\mu_{4b,1b} \cdot p_{lh} + \mu_{4b,3b} \cdot \left(p_{lh} + p_{ll}\right)\right] \\
V_x\left(x_{hl}, y_{hl}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot \left[\left(p_{lh} + p_{ll}\right) - \mu_{4b,1b} \cdot p_{lh} - \mu_{4b,3b} \cdot \left(p_{lh} + p_{ll}\right) + \lambda_9 + \lambda_{11} + \lambda_{15}\right] \\
V_x\left(x_{lh}, y_{lh}\right) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot \left[-\lambda_{11} - \lambda_{15}\right] \\
V_x\left(x_{ll}, y_{ll}\right) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot \left[-\lambda_9\right] \\
V_y\left(x_{hh}, y_{hh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot \left[\left(p_{hl} + p_{ll} + p_{lh}\right) - \mu_{4b,1b} \cdot p_{lh} + \lambda_9 + \lambda_{15}\right] \\
V_y\left(x_{hl}, y_{hl}\right) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot \left[-\mu_{4b,5b} \cdot p_{lh}\right] \\
V_y\left(x_{lh}, y_{lh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot \left[\lambda_{11}\right] \\
V_y\left(x_{ll}, y_{ll}\right) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot \left[-p_{lh} + \mu_{4b,1b} \cdot p_{lh} + \mu_{4b,5b} \cdot p_{lh} - \lambda_9 - \lambda_{11} - \lambda_{15}\right]
\end{aligned}\right\} R_{4b}$$

$$\left.\begin{aligned}
V_x\left(x_{hh}, y_{hh}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot \left[\left(p_{lh} + p_{ll} + p_{hl}\right) - \mu_{5a,1a} \cdot p_{hl} + \lambda_{13}\right] \\
V_x\left(x_{hl}, y_{hl}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot \left[\mu_{5a,6a} \cdot p_{ll} + \lambda_6\right] \\
V_x\left(x_{lh}, y_{lh}\right) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot \left[-p_{hl} + \mu_{5a,1a} \cdot p_{hl} + \mu_{5a,4a} \cdot p_{hl} - \mu_{5a,6a} \cdot p_{ll} - \lambda_6 - \lambda_{13}\right] \\
V_x\left(x_{ll}, y_{ll}\right) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot \left[-\mu_{5a,4a} \cdot p_{hl}\right] \\
V_y\left(x_{hh}, y_{hh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot \left[\mu_{5a,1a} \cdot p_{hl}\right] \\
V_y\left(x_{hl}, y_{hl}\right) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot \left[-\lambda_6 - \lambda_{13}\right] \\
V_y\left(x_{lh}, y_{lh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot \left[\left(p_{hl} + p_{ll}\right) - \mu_{5a,1a} \cdot p_{hl} + \lambda_6 + \lambda_{13}\right] \\
V_y\left(x_{ll}, y_{ll}\right) &= \eta_l
\end{aligned}\right\} R_{5a}$$

$$\left.\begin{aligned}
V_x\left(x_{hh}, y_{hh}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot \left[\mu_{5b,1b} \cdot p_{lh}\right] \\
V_x\left(x_{hl}, y_{hl}\right) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot \left[\left(p_{lh} + p_{ll}\right) - \mu_{5b,1b} \cdot p_{lh} + \lambda_6 + \lambda_{14}\right] \\
V_x\left(x_{lh}, y_{lh}\right) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot \left[-\lambda_6 - \lambda_{14}\right] \\
V_x\left(x_{ll}, y_{ll}\right) &= \theta_l \\
V_y\left(x_{hh}, y_{hh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot \left[\left(p_{hl} + p_{ll} + p_{lh}\right) - \mu_{5b,1b} \cdot p_{lh} + \lambda_{14}\right] \\
V_y\left(x_{hl}, y_{hl}\right) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot \left[-p_{lh} + \mu_{5b,1b} \cdot p_{lh} + \mu_{5b,4b} \cdot p_{lh} - \mu_{5b,6b} \cdot p_{ll} - \lambda_6 - \lambda_{14}\right] \\
V_y\left(x_{lh}, y_{lh}\right) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot \left[\mu_{5b,6b} \cdot p_{ll} + \lambda_6\right] \\
V_y\left(x_{ll}, y_{ll}\right) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot \left[-\mu_{5b,4b} \cdot p_{lh}\right]
\end{aligned}\right\} R_{5b}$$

$$\left.\begin{aligned}
V_x(x_{hh},y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}}\cdot\left[(p_{lh}+p_{ll}+p_{hl})-\mu_{6a,1b}\cdot(p_{ll}+p_{hl})\right]\\
V_x(x_{hl},y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}}\cdot\left[p_{ll}-\mu_{6a,5a}\cdot p_{ll}+\lambda_{11}+\lambda_6\right]\\
V_x(x_{lh},y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}}\cdot\left[-(p_{hl}+p_{ll})+\mu_{6a,1b}\cdot(p_{ll}+p_{hl})+\mu_{6a,5a}\cdot p_{ll}-\lambda_6-\lambda_{11}\right]\\
V_x(x_{ll},y_{ll}) &= \theta_l\\
V_y(x_{hh},y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}}\cdot\left[\mu_{6a,1b}\cdot(p_{ll}+p_{hl})\right]\\
V_y(x_{hl},y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}}\cdot\left[-\lambda_6\right]\\
V_y(x_{lh},y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}}\cdot\left[(p_{hl}+p_{ll})-\mu_{6a,1b}\cdot(p_{ll}+p_{hl})+\lambda_{11}+\lambda_6\right]\\
V_y(x_{ll},y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}}\cdot\left[-\lambda_{11}\right]
\end{aligned}\right\} R_{6a}$$

$$\left.\begin{aligned}
V_x(x_{hh},y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}}\cdot\left[\mu_{6b,1a}\cdot(p_{ll}+p_{lh})\right]\\
V_x(x_{hl},y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}}\cdot\left[(p_{lh}+p_{ll})-\mu_{6b,1a}\cdot(p_{ll}+p_{lh})+\lambda_6+\lambda_{12}\right]\\
V_x(x_{lh},y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}}\cdot\left[-\lambda_6\right]\\
V_x(x_{ll},y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}}\cdot\left[-\lambda_{12}\right]\\
V_y(x_{hh},y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}}\cdot\left[(p_{hl}+p_{ll}+p_{lh})-\mu_{6b,1a}\cdot(p_{ll}+p_{lh})\right]\\
V_y(x_{hl},y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}}\cdot\left[-(p_{lh}+p_{ll})+\mu_{6b,1a}\cdot(p_{ll}+p_{lh})+\mu_{6b,5b}\cdot p_{ll}-\lambda_6-\lambda_{12}\right]\\
V_y(x_{lh},y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}}\cdot\left[p_{ll}-\mu_{6b,5b}\cdot p_{ll}+\lambda_6+\lambda_{12}\right]\\
V_y(x_{ll},y_{ll}) &= \eta_l
\end{aligned}\right\} R_{6b}.$$

Step 2: We show the following lemma. Define $J \subset I$ via $(x,y) \in R_j \Leftrightarrow j \in J$ and assume that $\#J \geq 3$. Then at least one region R_{j_1} with $j_1 \in J$ is adjecent to two (or more) other regions R_{j_2}, R_{j_3} with $j_2, j_3 \in J$.

Proof: Choose $\epsilon > 0$ sufficiently small such that $B_\epsilon(x,y) \cap \bigcup_{i \in I \setminus J} R_i = \emptyset$ and fix $l, m \in J$. Let $(x,y)_l$ and $(x,y)_m$ be interior points in respectively R_l and R_m within $B_\epsilon(x,y)$. We can choose $(x,y)_l$ and $(x,y)_m$ such that the line $L = \{\lambda \cdot (x,y)_l + (1-\lambda)(x,y)_m, \lambda \in [0,1]\}$ intersects all hyperplanes separating regions R_i, $i \in I$ at most once and does not hit any intersection of two or more such hyperplanes. Then any point on L is either an interior point of some region R_j, $j \in J$ or is an interior point of a hyperplane $R_j \cap R_k$ between two adjacent regions R_j, R_k, $j,k \in J$. Hence there exist $0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1$, $n \leq \#I - 1$ such that $L = \bigcup_{i=1,\dots,n} L_i$, $L_i = \{\lambda \cdot (x,y)_l + (1-\lambda)(x,y)_m, \lambda \in [\lambda_{i-1}, \lambda_i]\}$ and $L_i \subset R_{j_i}$ for some $j_i \in J$ with $j_1 = l$ and $j_n = m$. Note that $n \geq 2$ by assumption. If $n = n(l,m) = 2$ then R_l and R_m are adjacent as $\lambda_1 \cdot (x,y)_l + (1-\lambda_1)(x,y)_m \in R_l \cap R_m$ is interior in $R_l \cap R_m$ by assumption. In this case we apply the same approach to R_l and some other region R_s, $s \neq m, l$, $s \in J$. If again $n = n(l, s) = 2$, then R_l is adjecent to R_m and R_s, proving the claim. So assume that $n \geq 3$. Then, choosing some $\lambda_{j_2} \in (\lambda_1, \lambda_2)$, $\lambda_{j_3} \in (\lambda_2, \lambda_3)$ applying the same approach to $R_{j_1} = R_l$ and R_{j_2} with points $(x, y)_l$ and $(x, y)_{j_2} = \lambda_{j_2} \cdot (x, y)_l + (1 - \lambda j_2)(x, y)_m$ as well as to R_{j_2} and R_{j_3} with points $(x, y)_{j_2} = \lambda_{j_2} \cdot (x, y)_l + (1 - \lambda_{j_2})(x, y)_m$ and $(x, y)_{j_3} = \lambda_{j_3} \cdot (x, y)_l + (1 - \lambda_{j_3})(x, y)_m$ yields $n(l, j_2) = 2 = n(j_2, j_3)$ and hence R_{j_2} is adjacent to R_{j_1} and R_{j_3}, proving the lemma.

The lemma from Step 2 implies that we only need to check intersections of three or more regions when at least one of these regions is adjacent to at least two of the other regions. Note first that for cases c4-c7) where (x, y) lies in the intersection of four regions, any triplet of regions implies the fourth in each case. It is therefore enough to show the following. The optimal allocation (x, y) cannot lie in the intersection of any three regions that are feasible according to Step 2 not covered by Theorem 1*. It then follows immediately by direct comparison that (x, y) cannot lie in any non-trivial intersection of any of the loci stated in Theorem 1*.

Step 3: Assume that $(x, y) \in R_{1a} \cap R_{1b} \cap R_{4a}$. Then (R1a,1b) and (R1a,4a) holding with equality imply $x_{ll} = x_{hl}$ in R_{1b}, contradicting Step 1. Likewise, $(x, y) \in R_{1a} \cap R_{1b} \cap R_{4b}$ is impossible.

Assume that $(x, y) \in R_{1a} \cap R_2 \cap R_{5a}$. Then (R1a,2) and (R1a,5a) holding with equality imply $x_{lh} = x_{hh}$ in R_2, contradicting Step 1. Likewise $(x, y) \in R_{1b} \cap R_2 \cap R_{5b}$ is impossible.

Assume that $(x, y) \in R_{1a} \cap R_2 \cap R_{6b}$. Then

$$\begin{aligned}
\Delta_\eta \cdot (y_{lh} - y_{hh}) &= 0, \\
\Delta_\eta \cdot (y_{lh} - y_{hl}) &\geq 0, \\
\Delta_\eta \cdot (y_{hl} - y_{hh}) &\geq 0
\end{aligned}$$

imply $y_{hh} = y_{hl}$ in R_{1a}, contradicting Step 1. Likewise, $(x, y) \in R_{1b} \cap R_2 \cap R_{6a}$ is impossible.

Assume that $(x, y) \in R_{1a} \cap R_{5a} \cap R_{6b}$. Then

$$\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{hl} - y_{hh}) = 0$$

implies $x_{hl} \geq x_{hh}$. Moreover, from

$$\begin{aligned}
\Delta_\eta \cdot (y_{lh} - y_{hh}) - \Delta_\theta \cdot (x_{lh} - x_{hh}) &= 0 \\
\Delta_\theta \cdot (x_{hl} - x_{hh}) - \Delta_\eta \cdot (y_{hl} - y_{hh}) &= 0
\end{aligned}$$

we get

$$\Delta_\theta \cdot (x_{hl} - x_{lh}) + \Delta_\eta \cdot (y_{lh} - y_{hl}) = 0.$$

As implementability implies

$$\Delta_\theta \cdot (x_{hh} - x_{lh}) + \Delta_\eta \cdot (y_{lh} - y_{hl}) \geq 0$$

it follows that $x_{hl} = x_{hh}$ in R_{5a}, contradicting Step 1. Likewise, $(x, y) \in R_{1a} \cap R_{5a} \cap R_{6b}$ is impossible.

Assume that $(x, y) \in R_{1a} \cap R_2 \cap R_{3b}$. Then $y_{ll} = y_{hh} = y_{lh}$ in R_{1a}, contradicting Step 1. Likewise, $(x, y) \in R_{1b} \cap R_2 \cap R_{3a}$ is impossible.

Assume that $(x, y) \in R_2 \cap R_{3a} \cap R_{3b}$. Then $x_{ll} = x_{hh}$, $y_{ll} = y_{hh}$, implying $(x, y) \in R_{1a}$ via the FOCs of R_2 given as

$$V_y(x_{hh}, y_{hh}) = \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{2,1a} \cdot p_{ll} + \mu_{2,3b} \cdot p_{lh} + \lambda_4]$$
$$V_y(x_{ll}, y_{ll}) = \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\mu_{2,3b} \cdot p_{lh}],$$

and thus contradicting $(x, y) \notin R_{1a} \cap R_2 \cap R_{3b}$.

Assume that $(x, y) \in R_{3a} \cap R_{4a} \cap R_{5a}$. Then $x_{lh} = x_{ll} \leq x_{hh}$ in R_{3a}, contradicting Step 1. Likewise, $(x, y) \in R_{3b} \cap R_{4b} \cap R_{5b}$ is impossible.

Finally, assume that $(x, y) \in R_{4a} \cap R_{5a} \cap R_{6b}$. Then $x_{ll} = x_{lh} = x_{hl}$ in R_{4a}, contradicting Step 1. Likewise, $(x, y) \in R_{4b} \cap R_{5b} \cap R_{6a}$ is impossible.

As any intersection of feasible regions according to Theorem 1* is either part of the classification or contains at least one of the intersections we just excluded, this completes Step 3. □

Proof of Proposition 5. This is a direct consequence of Step 1 of Theorem 1*. □

Proof of Proposition 7. We first prove the symmetry properties. Suppose there is a solution $(x, y) = (x_{hh}, x_{hl}, x_{lh}, x_{ll}, y_{hh}, y_{hl}, y_{lh}, y_{ll})$. Then, by symmetry between the x-dimension and the y-dimension, the vector $(\hat{x}, \hat{y})$ defined as $\hat{x}_{ij} = y_{ji}$, $\hat{y}_{ij} = x_{ji}$, $i, j \in \{l, h\}$ also constitutes a solution as

$$\begin{aligned} p_{ij} \cdot V(x_{ij}, y_{ij}) + p_{ji} \cdot V(x_{ji}, y_{ji}) &= p_{ij} \cdot V(\hat{y}_{ji}, \hat{x}_{ji}) + p_{ji} \cdot V(\hat{y}_{ij}, \hat{x}_{ij}) \\ &= p_{ij} \cdot V(\hat{x}_{ij}, \hat{y}_{ij}) + p_{ji} \cdot V(\hat{x}_{ji}, \hat{y}_{ji}) \end{aligned}$$

for all $i, j \in \{l, h\}$. Hence $(x, y) = (\hat{x}, \hat{y})$.

Next, note that $\frac{dx_{hh}}{d\alpha} > 0$ as differentiating the first FOC with respect to α yields

$$\begin{aligned} \frac{dx_{hh}}{d\alpha} &= \frac{-\frac{\Delta_\theta}{p_{hh}} \cdot \frac{p_{ll}}{2}}{(V_{xx}(x_{hh}, x_{hh}) + V_{xy}(x_{hh}, x_{hh}))} \\ &= -\frac{\Delta_\theta \cdot \frac{p_{ll}}{p_{hh}}}{\begin{pmatrix} 1 & 1 \end{pmatrix} \cdot H(x_{hh}, x_{hh}) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \\ &> 0 \end{aligned}$$

where the second equality follows from of V and positivity follows from negative definiteness of the Hessian H of V at (x_{hh}, x_{hh}). Hence precisely one of the three cases for α applies.

To check the defining inequalities of the respective regions as well as implementability it is enough to show that for any value of $\alpha \in [0,1]$ we have $x_{ll} \geq x_{hh}$, $x_{lh} \geq x_{lh}$, $x_{ll} \geq x_{hl}$, $x_{lh} \geq x_{hh}$. As $V_{xx}(x,y) = V_{yy}(xy)$ for all $(x,y) \in \mathbb{R}^8_+$ by symmetry and

$$H(x,y) = V_{xx}(x,y) \cdot V_{yy}(x,y) - V_{xy}(x,y)^2 > 0$$

by negative definiteness of $H(x,y)$ for all $(x,y) \in \mathbb{R}^8_+$, we have $|V_{xx}(x,y)| > |V_{xy}(x,y)|$ for all $(x,y) \in \mathbb{R}^8_+$, and hence the claim follows from Proposition 8 which is proven below as well as from Proposition 5. □

Proof of Proposition 8. a+b) We have

$$R_{1a}\begin{cases}\Delta_\theta \cdot (x_{ll} - x_{hh}) & = R1a,2 + R1a,4a \geq 0\\ \Delta_\theta \cdot (x_{lh} - x_{hl}) & = R1a,1b + R1a,5a \geq 0\\ \Delta_\eta \cdot (y_{ll} - y_{hh}) & = R1a,2 + I2 > 0\end{cases}$$

$$R_{1b}\begin{cases}\Delta_\theta \cdot (x_{ll} - x_{hh}) & = R1b,1a + I1 > 0\\ \Delta_\eta \cdot (y_{ll} - y_{hh}) & = R1b,2 + R1b,4b \geq 0\\ \Delta_\eta \cdot (y_{hl} - y_{lh}) & = R1b,1a + R1b,5b \geq 0\end{cases}$$

$$R_{2}\begin{cases}\Delta_\theta \cdot (x_{ll} - x_{hh}) & = R2,3a \geq 0\\ \Delta_\theta \cdot (x_{lh} - x_{hl}) & = R2,1b + I3 > 0\\ \Delta_\eta \cdot (y_{ll} - y_{hh}) & = R2,3b \geq 0\\ \Delta_\eta \cdot (y_{hl} - y_{lh}) & = R2,1a + I4 > 0\end{cases}$$

$$R_{3a}\begin{cases}\Delta_\theta \cdot (x_{lh} - x_{hl}) & = R3a,2 + I1 + I3 > 0\\ \Delta_\eta \cdot (y_{ll} - y_{hh}) & = R3a,2 + I1 > 0\\ \Delta_\eta \cdot (y_{hl} - y_{lh}) & = R3a,4a + I4 > 0\end{cases}$$

$$R_{3b}\begin{cases}\Delta_\theta \cdot (x_{ll} - x_{hh}) & = R3b,2 + I1 > 0\\ \Delta_\theta \cdot (x_{lh} - x_{hl}) & = R3b,4b + I3 > 0\\ \Delta_\eta \cdot (y_{hl} - y_{lh}) & = R3b,2 + I2 + I4 > 0\end{cases}$$

$$R_{4a}\begin{cases}\Delta_\theta \cdot (x_{lh} - x_{hl}) & = R4a,5a + I1 > 0\\ \Delta_\eta \cdot (y_{ll} - y_{hh}) & = R4a,3a + I2 > 0\end{cases}$$

$$R_{4b}\begin{cases}\Delta_\theta \cdot (x_{ll} - x_{hh}) & = R4b,3b + I1 > 0\\ \Delta_\eta \cdot (y_{hl} - y_{lh}) & = R4b,5b + I2 > 0\end{cases}$$

$$R_{5a}\begin{cases}\Delta_\theta\cdot(x_{ll}-x_{hh}) & = R5a, 4a + I3 > 0\\ \Delta_\theta\cdot(x_{lh}-x_{hl}) & = R5a, 6a \geq 0\\ \Delta_\eta\cdot(y_{ll}-y_{hh}) & = R5a, 1a + I2 + I3 > 0\end{cases}$$

$$R_{5b}\begin{cases}\Delta_\theta\cdot(x_{ll}-x_{hh}) & = R5b, 1b + I1 + I4 > 0\\ \Delta_\eta\cdot(y_{ll}-y_{hh}) & = R5b, 4b + I4 > 0\\ \Delta_\eta\cdot(y_{hl}-y_{lh}) & = R5b, 6b \geq 0\end{cases}$$

$$R_{6a}\begin{cases}\Delta_\theta\cdot(x_{ll}-x_{hh}) & = R6a, 5a + I1 + I3 > 0\\ \Delta_\eta\cdot(y_{ll}-y_{hh}) & = R6a, 1b + I2 + I3 > 0\\ \Delta_\eta\cdot(y_{hl}-y_{lh}) & = R6a, 5a + I6 > 0\end{cases}$$

$$R_{6b}\begin{cases}\Delta_\theta\cdot(x_{ll}-x_{hh}) & = R6b, 1a + I1 + I4 > 0\\ \Delta_\theta\cdot(x_{lh}-x_{hl}) & = R6b, 5b + I6 > 0\\ \Delta_\eta\cdot(y_{ll}-y_{hh}) & = R6b, 5b + I2 + I4 > 0.\end{cases}$$

All strict inequalities follow from Theorem 1* showing that not all constraints on the RHS can hold with equality simultaneously. Moreover, note that by Theorem 1* all weak inequalities above imply that the allocation lies on a separating hyperplane which is defined by the inequality under consideration, in line with Proposition 8, Part b). To prove the remaining (weak) inequalities, we apply the homotopy method extensively. Start with region R_{1a}. If $\mu_{1a,2} > 0$ then $(x, y) \in R_2$ and hence $y_{hl} > y_{lh}$. Similarly, if $\mu_{1a,1b} > 0$ then $(x, y) \in R_{1b}$ and hence $y_{hl} \geq y_{lh}$ and Part b) applies. So assume $\mu_{1a,2} = \mu_{1a,1b} = 0$. Consider

$$\begin{aligned}
V_x\left(x_{hl}(t), y_{hl}(t)\right) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta\cdot t\cdot a\\
V_x\left(x_{lh}(t), y_{lh}(t)\right) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta\cdot t\cdot b\\
V_y\left(x_{hl}(t), y_{hl}(t)\right) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta\cdot t\cdot c\\
V_y\left(x_{lh}(t), y_{lh}(t)\right) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta\cdot t\cdot d
\end{aligned}$$

with

$$\begin{aligned}
a &= \frac{1}{2} + \frac{\mu_{1a,6b}\cdot(p_{lh}+p_{ll})}{p_{hl}}\\
b &= \frac{1}{2} + \frac{\mu_{1a,5a}\cdot p_{hl}}{p_{lh}}\\
c &= \frac{1}{2} + \frac{\mu_{1a,6b}\cdot(p_{lh}+p_{ll})}{p_{hl}},\\
d &= \frac{1}{2} + \frac{p_{ll}+\mu_{1a,4a}\cdot p_{hl}+\mu_{1a,5a}\cdot p_{hl}}{p_{lh}}.
\end{aligned}$$

At $t = 1$ this yields the FOCs of R_{1a}. At $t = 0$ we have $x_{hl}(0) = x_{lh}(0)$, $y_{hl}(0) = y_{lh}(0)$.

Note that $a = c$, $d > b$. Hence, at any $t \in [0, 1]$ we have

$$\begin{aligned}
\frac{dy_{lh}}{dt} &= \Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot d + \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot b \leq 0, \\
\frac{dy_{hl}}{dt} &= -\Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot c - \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot a \geq 0,
\end{aligned}$$

implying $y_{hl} = y_{hl}(1) > y_{lh}(1) = y_{lh}$. Likewise, in R_{1b} we get $x_{lh} \geq x_{hl}$.

Suppose next that $(x, y) \in R_{3a}$. Consider

$$\begin{aligned}
V_x(x_{hh}(t), y_{hh}(t)) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta \cdot t \cdot a \\
V_x(x_{ll}(t), y_{ll}(t)) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta \cdot t \cdot b \\
V_y(x_{hh}(t), y_{hh}(t)) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta \cdot t \cdot c \\
V_y(x_{ll}(t), y_{ll}(t)) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta \cdot t \cdot d
\end{aligned}$$

with

$$\begin{aligned}
a &= \frac{1}{2} + \frac{(p_{lh} + p_{ll} + p_{hl}) - \mu_{3a,2} \cdot p_{hl} + \lambda_5 + \lambda_7}{p_{hh}} \\
b &= \frac{1}{2} + \frac{p_{hl} - \mu_{3a,2} \cdot p_{hl} + \lambda_5 + \lambda_7}{p_{ll}} \\
c &= \frac{1}{2} + \frac{(p_{hl} + p_{ll}) - \mu_{3a,4a} \cdot (p_{hl} + p_{ll}) + \lambda_5 + \lambda_7}{p_{hh}}, \\
d &= \frac{1}{2} + \frac{\lambda_5}{p_{ll}}.
\end{aligned}$$

At $t = 1$ this yields the FOCs of R_{3a}. At $t = 0$ we have $x_{hh}(0) = x_{ll}(0)$, $y_{hh}(0) = y_{ll}(0)$. Note that $a \geq c$, $b \geq d$. Hence, at any $t \in [0, 1]$ we have

$$\begin{aligned}
\frac{dx_{hh}}{dt} &= \Delta_\theta \cdot V_{yy}(x_{hh}(t), y_{hh}(t)) \cdot a - \Delta_\eta \cdot V_{xy}(x_{hh}(t), y_{hh}(t)) \cdot c \leq 0, \\
\frac{dx_{ll}}{dt} &= -\Delta_\theta \cdot V_{yy}(x_{ll}(t), y_{ll}(t)) \cdot b + \Delta_\eta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot d \geq 0
\end{aligned}$$

and hence $x_{ll} = x_{ll}(1) \geq x_{hh}(1) = x_{hh}$. Moreover, the inequality is strict unless $\mu_{3a,2} = 1$ implying $x_{ll} = x_{hh}$ and thus Part b) applies. Likewise, in R_{3b} we get $y_{ll} \geq y_{hh}$.

Suppose next that $(x, y) \in R_{4a}$. Consider first

$$\begin{aligned}
V_x(x_{hh}(t), y_{hh}(t)) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta \cdot t \cdot a \\
V_x(x_{ll}(t), y_{ll}(t)) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta \cdot t \cdot b \\
V_y(x_{hh}(t), y_{hh}(t)) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta \cdot t \cdot c \\
V_y(x_{ll}(t), y_{ll}(t)) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta \cdot t \cdot d
\end{aligned}$$

with

$$
\begin{aligned}
a &= \frac{1}{2} + \frac{(p_{lh}+p_{ll}+p_{hl}) - \mu_{4a,1a}\cdot p_{hl} + \lambda_{10} + \lambda_{16}}{p_{hh}} \\
b &= \frac{1}{2} + \frac{p_{hl} - \mu_{4a,1a}\cdot p_{hl} - \mu_{4a,5a}\cdot p_{hl} + \lambda_{10} + \lambda_{12} + \lambda_{16}}{p_{ll}} \\
c &= \frac{1}{2} + \frac{\mu_{4a,1a}\cdot p_{hl} + \mu_{4a,3a}\cdot (p_{hl}+p_{ll})}{p_{hh}} \\
d &= \frac{1}{2} + \frac{\lambda_{10}}{p_{ll}}.
\end{aligned}
$$

At $t=1$ this yields the FOCs of R_{4a}. At $t=0$ we have $x_{hh}(0) = x_{ll}(0)$, $y_{hh}(0) = y_{ll}(0)$. We have $a \geq c$, $b \geq d$. Hence, at any $t \in [0,1]$ we have

$$
\begin{aligned}
\frac{dx_{hh}}{dt} &= \Delta_\theta \cdot V_{yy}(x_{hh}(t), y_{hh}(t)) \cdot a - \Delta_\eta \cdot V_{xy}(x_{hh}(t), y_{hh}(t)) \cdot c \leq 0, \\
\frac{dx_{ll}}{dt} &= -\Delta_\theta \cdot V_{yy}(x_{ll}(t), y_{ll}(t)) \cdot b + \Delta_\eta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot d \geq 0,
\end{aligned}
$$

with at least one inequality being strict unless $(x,y) \in R_{1a} \cap R_2 \cap R_{3a} \cap R_{4a}$ so that Part b) is satisfied. So $x_{ll} = x_{ll}(1) \geq x_{hh}(1) = x_{hh}$. Likewise, one shows that $y_{ll} \geq y_{hh}$ in R_{4b}. Consider next

$$
\begin{aligned}
V_x(x_{hl}(t), y_{hl}(t)) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta \cdot t \cdot a \\
V_x(x_{lh}(t), y_{lh}(t)) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta \cdot t \cdot b \\
V_y(x_{hl}(t), y_{hl}(t)) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta \cdot t \cdot c \\
V_y(x_{lh}(t), y_{lh}(t)) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta \cdot t \cdot d
\end{aligned}
$$

with

$$
\begin{aligned}
a &= \frac{1}{2} + \frac{\lambda_{12}}{p_{hl}} \\
b &= \frac{1}{2} + \frac{\mu_{4a,5a}\cdot p_{hl}}{p_{lh}} \\
c &= \frac{1}{2} + \frac{\lambda_{12}+\lambda_{16}}{p_{hl}} \\
d &= \frac{1}{2} + \frac{(p_{hl}+p_{ll}) - \mu_{4a,1a}\cdot p_{hl} - \mu_{4a,3a}\cdot (p_{hl}+p_{ll}) + \lambda_{10} + \lambda_{12} + \lambda_{16}}{p_{lh}}.
\end{aligned}
$$

If $\mu_{4a,3a} > 0$ then $y_{hl} > y_{hh} = y_{lh}$, so assume $\mu_{4a,3a} = 0$. At $t=1$ this yields the FOCs of R_{4a} while at $t=0$ we have $x_{hl}(0) = x_{lh}(0)$, $y_{hl}(0) = y_{lh}(0)$. Note that $a \leq c$, $b < d$ if $\mu_{4a,3a} = 0$. Hence, at any $t \in [0,1]$ we have

$$
\begin{aligned}
\frac{dy_{lh}}{dt} &= \Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot d + \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot b \leq 0, \\
\frac{dy_{hl}}{dt} &= -\Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot c - \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot a \geq 0
\end{aligned}
$$

So $y_{hl} = y_{hl}(1) > y_{lh}(1) = y_{lh}$. Similarly, in R_{4b} we get $x_{lh} > x_{hl}$.

Suppose $(x, y) \in R_{5a}$. If $\mu_{5a,6a} > 0$ then $x_{lh} = x_{hl}$, then (I6) implies $y_{hl} \geq y_{lh}$ and $y_{hl} = y_{lh}$ contradicts FOCs

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\lambda_6 - \lambda_{13}] \\ V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) - \mu_{5a,1a} \cdot p_{hl} + \lambda_6 + \lambda_{13}] . \end{aligned}$$

Hence we may assume $\mu_{5a,6a} = 0$. Consider

$$\begin{aligned} V_x(x_{hl}(t), y_{hl}(t)) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta \cdot t \cdot a \\ V_x(x_{lh}(t), y_{lh}(t)) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta \cdot t \cdot b \\ V_y(x_{hl}(t), y_{hl}(t)) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta \cdot t \cdot c \\ V_y(x_{lh}(t), y_{lh}(t)) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta \cdot t \cdot d \end{aligned}$$

with

$$\begin{aligned} a &= \frac{1}{2} + \frac{\lambda_6}{p_{hl}} \\ b &= \frac{1}{2} + \frac{p_{hl} - \mu_{5a,1a} \cdot p_{hl} - \mu_{5a,4a} \cdot p_{hl} + \lambda_6 + \lambda_{13}}{p_{lh}} \\ c &= \frac{1}{2} + \frac{\lambda_6 + \lambda_{13}}{p_{hl}} \\ d &= \frac{1}{2} + \frac{(p_{hl} + p_{ll}) - \mu_{5a,1a} \cdot p_{hl} + \lambda_6 + \lambda_{13}}{p_{lh}} . \end{aligned}$$

At $t = 1$ this yields the FOCs of R_{5a}. At $t = 0$ we have $x_{hl}(0) = x_{lh}(0)$, $y_{hl}(0) = y_{lh}(0)$. Note that $a \leq c$, $b < d$. Hence, at any $t \in [0, 1]$ we have

$$\begin{aligned} \frac{dy_{lh}}{dt} &= \Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot d + \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot b < 0, \\ \frac{dy_{hl}}{dt} &= -\Delta_\eta \cdot V_{xx}(x_{lh}(t), y_{lh}(t)) \cdot c - \Delta_\theta \cdot V_{xy}(x_{lh}(t), y_{lh}(t)) \cdot a \geq 0, \end{aligned}$$

implying $y_{hl} = y_{hl}(1) > y_{lh}(1) = y_{lh}$. Similarly, in R_{5b} we get $x_{lh} > x_{hl}$.

Finally, suppose $(x, y) \in R_{6a}$. If $\mu_{6a,5a} > 0$ then $x_{lh} = x_{hl}$, so assume $\mu_{6a,5a} = 0$. Consider

$$\begin{aligned} V_x(x_{hl}(t), y_{hl}(t)) &= \theta_l + \frac{\Delta_\theta}{2} + \Delta_\theta \cdot t \cdot a \\ V_x(x_{lh}(t), y_{lh}(t)) &= \theta_l + \frac{\Delta_\theta}{2} - \Delta_\theta \cdot t \cdot b \\ V_y(x_{hl}(t), y_{hl}(t)) &= \eta_l + \frac{\Delta_\eta}{2} - \Delta_\eta \cdot t \cdot c \\ V_y(x_{lh}(t), y_{lh}(t)) &= \eta_l + \frac{\Delta_\eta}{2} + \Delta_\eta \cdot t \cdot d \end{aligned}$$

with

$$\begin{aligned}
a &= \frac{1}{2}+\frac{p_{ll}+\lambda_{11}+\lambda_6}{p_{hl}} \\
b &= \frac{1}{2}+\frac{p_{hl}+p_{ll}-\mu_{6a,1b}\cdot(p_{ll}+p_{hl})+\lambda_6+\lambda_{11}}{p_{lh}} \\
c &= \frac{1}{2}+\frac{\lambda_6}{p_{hl}} \\
d &= \frac{1}{2}+\frac{(p_{hl}+p_{ll})-\mu_{6a,1b}\cdot(p_{ll}+p_{hl})+\lambda_6+\lambda_{11}}{p_{lh}}.
\end{aligned}$$

At $t=1$ this yields the FOCs of R_{5a}. At $t=0$ we have $x_{hl}(0)=x_{lh}(0)$, $y_{hl}(0)=y_{lh}(0)$. Note that $a>c$, $b=d$. Hence, at any $t\in[0,1]$ we have

$$\begin{aligned}
\frac{dx_{lh}}{dt} &= -\Delta_\eta\cdot V_{xx}(x_{lh}(t),y_{lh}(t))\cdot b-\Delta_\theta\cdot V_{xy}(x_{lh}(t),y_{lh}(t))\cdot d\geq 0, \\
\frac{dx_{hl}}{dt} &= \Delta_\eta\cdot V_{xx}(x_{lh}(t),y_{lh}(t))\cdot a+\Delta_\theta\cdot V_{xy}(x_{lh}(t),y_{lh}(t))\cdot c<0,
\end{aligned}$$

implying $x_{lh}=x_{lh}(1)>x_{hl}(1)=x_{hl}$. But this contradicts (R6a,5a), so $\mu_{6a,5a}>0$, $x_{lh}=x_{hl}$ and $(x,y)\in R_{5a}\cap R_{6a}$. Similarly, in R_{6b} we get $y_{hl}=y_{lh}$ and $(x,y)\in R_{5b}\cap R_{6b}$.

c) This follows directly from (R3a,2), (R3b,2), (R6a,5a), and (R6b,5b).

d) By Part a) and Proposition 5 it suffices to check that (I5), (I6), (I15), and (I16) never hold with equality. However, by Part b) and Theorem 1* at most one of the inequalities in Part a) may hold with equality. This proves the result. □

Proof of Proposition 9. a-c) Suppose $(x,y)\in R_{1a}$. Then $y_{lh}\geq y_{hh}$ by (R1a,2). FOCs

$$\begin{aligned}
V_x(x_{hh},y_{hh}) &= \theta_h+\frac{\Delta_\theta}{p_{hh}}\cdot[(p_{lh}+p_{ll})-\mu_{1a,1b}\cdot p_{ll}+\mu_{1a,4a}\cdot p_{hl}+\mu_{1a,5a}\cdot p_{hl}-\mu_{1a,6b}\cdot(p_{lh}+p_{ll})] \\
V_x(x_{hl},y_{hl}) &= \theta_h+\frac{\Delta_\theta}{p_{hl}}\cdot[\mu_{1a,1b}\cdot p_{ll}+\mu_{1a,6b}\cdot(p_{lh}+p_{ll})]
\end{aligned}$$

together with $y_{hl}>y_{hh}$ imply $x_{hl}>x_{hh}$ unless $(x,y)\in R_{1b}$ or $(x,y)\in R_{6b}$, the former implying $x_{hl}\geq x_{hh}$ by (R1b,2) and the latter implying $x_{hl}>x_{hh}$ by (R6a,5a) and (I3). FOCs

$$\begin{aligned}
V_x(x_{lh},y_{lh}) &= \theta_l+\frac{\Delta_\theta}{p_{lh}}\cdot[-\mu_{1a,5a}\cdot p_{hl}] \\
V_x(x_{ll},y_{ll}) &= \theta_l+\frac{\Delta_\theta}{p_{ll}}\cdot[-\mu_{1a,4a}\cdot p_{hl}]
\end{aligned}$$

together with $y_{ll}>y_{lh}$ imply $x_{ll}>x_{lh}$ unless $\mu_{1a,5a}>0$ in which case $x_{ll}\geq x_{lh}$ is implied by (R5a,4a). Finally, if $\mu_{1a,6b}>0$ then $(x,y)\in R_{6b}$, implying $y_{ll}>y_{hl}$ via (R6b,5b) and (I2). Otherwise, if $\mu_{1a,6b}=0$ then FOCs

$$\begin{aligned}
V_y(x_{hl},y_{hl}) &= \eta_l \\
V_y(x_{ll},y_{ll}) &= \eta_l
\end{aligned}$$

together with $x_{ll} > x_{hl}$ and complementarity imply $y_{ll} > y_{hl}$. The proof for $(x, y) \in R_{1b}$ is alike.

Suppose $(x, y) \in R_2$. Then (R2,1a) and (R2,1b) must hold with equality. Indeed, if $\mu_{2,1a} = 0$ or $\mu_{2,1b} = 0$, respectively, then FOCs

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h + \frac{\Delta_\eta}{p_{hh}} \cdot [(p_{hl} + p_{ll}) - \mu_{2,1a} \cdot p_{ll} + \mu_{2,3b} \cdot p_{lh}] \\ V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [\mu_{2,1a} \cdot p_{ll}] \end{aligned}$$

and

$$\begin{aligned} V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll}) - \mu_{2,1b} \cdot p_{ll} + \mu_{2,3a} \cdot p_{hl}] \\ V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [\mu_{2,1b} \cdot p_{ll}] \end{aligned}$$

together with $x_{lh} > x_{hh}$ and $y_{hl} > y_{hh}$ and complementarity imply $y_{lh} > y_{hh}$ or $x_{hl} > x_{hh}$, respectively, contradicting (R2,1a) or (R2,1b). The remaining claims then follow from what was shown for R_{1a} and R_{1b}.

Suppose $(x, y) \in R_{5a}$. Then $x_{ll} \geq x_{lh}$ by (R5a,4a) and $y_{lh} > y_{hh}$ by (R5a,1a) and (I3). If (I6) or (I13) hold with equality, then (R5a,6a) or (I3) together with (I2) imply $y_{ll} > y_{hl}$. Otherwise $\lambda_6 = \lambda_{13} = 0$ and FOCs

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l \\ V_y(x_{ll}, y_{ll}) &= \eta_l \end{aligned}$$

together with $x_{ll} > x_{hl}$ and complementarity imply $y_{ll} > y_{hl}$. If (R5a,6a) holds with equality, then (I3) implies $x_{hl} > x_{hh}$. If $\lambda_6 > 0$ then (I6) holds with equality and $x_{hl} \geq x_{hh}$ as otherwise $I13 = I6 + \Delta_\theta \cdot (x_{hl} - x_{hh}) \geq 0$ was violated. If $\lambda_6 = \mu_{5a,6a} = 0$ then FOCs

$$\begin{aligned} V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll} + p_{hl}) - \mu_{5a,1a} \cdot p_{hl} + \lambda_{13}] \\ V_x(x_{hl}, y_{hl}) &= \theta_h \end{aligned}$$

together with $y_{hl} > y_{hh}$ and complementarity imply $x_{hl} > x_{hh}$. Analogous results follow for $(x, y) \in R_{5b}$.

If $(x, y) \in R_{4a}$ then (R4a,5a) must bind. Indeed, if $\mu_{4a,5a} = 0$ then FOCs

$$\begin{aligned} V_x(x_{lh}, y_{lh}) &= \theta_l \\ V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-p_{hl} + \mu_{4a,1a} \cdot p_{hl} - \lambda_1 - \lambda_{10} - \lambda_{12} - \lambda_{16}] \end{aligned}$$

together with $y_{ll} > y_{lh}$ and complementarity imply $x_{ll} < x_{lh}$, contradicting (R4a,5a). Similarly, if $(x, y) \in R_{4b}$ then (R4b,5b) must bind.

(x, y) cannot be an elemnt of R_{3a}. Indeed, if $(x, y) \in R_{3a}$ then $y_{ll} \geq y_{hh}$. But then FOCs

$$\begin{aligned} V_x(x_{hh}, y_{hh}) &= \theta_h + \frac{\Delta_\theta}{p_{hh}} \cdot [(p_{lh} + p_{ll} + p_{hl}) - \mu_{3a,2} \cdot p_{hl} + \lambda_5 + \lambda_7] \\ V_x(x_{ll}, y_{ll}) &= \theta_l + \frac{\Delta_\theta}{p_{ll}} \cdot [-p_{hl} + \mu_{3a,2} \cdot p_{hl} - \lambda_5 - \lambda_7] \end{aligned}$$

together with complementarity imply $x_{ll} > x_{hh}$, contradicting (R3a,2). Similarly, (x, y) cannot be an element of R_{3b}.

If $(x, y) \in R_{6a}$ then $x_{ll} > x_{lh}$ is implied by (R6a,5a) and (I1), $x_{hl} > x_{hh}$ is implied by (R6a,5a) and (I3), and $y_{lh} > y_{hh}$ is implied by (R6a,5a) and (I3). If (I6) holds with equality, then $y_{ll} \geq y_{hl}$ as otherwise $I11 = I6 + \Delta_\eta \cdot (y_{ll} - y_{hl}) \geq 0$ was violated. Otherwise $\lambda_6 = 0$ and FOCs

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l \\ V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_{11}] \end{aligned}$$

together with $x_{ll} > x_{hl}$ and complementarity imply $y_{ll} > y_{hl}$. Analogous results follow for R_{6b}. The second claim in Part a) follows directly from the first claim and Proposition 5.

d) This is an immediate consequence of Part c) given that $(x, y) \in R_{4a/b} \Rightarrow (x, y) \in R_{5a/b}$ as shown above. □

Proof of Proposition 10. a-c) Let $(x, y) \in R_{3a}$. Then (R3a,2) and (I2) imply $x_{lh} > x_{ll}$. If (I7) holds with equality, then $y_{hl} \geq y_{ll}$ as otherwise $I1 = I7 + \Delta_\eta \cdot (y_{hl} - y_{hh}) \geq 0$ was violated. Otherwise $\lambda_7 = 0$ and FOCs

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l \\ V_y(x_{ll}, y_{ll}) &= \eta_l + \frac{\Delta_\eta}{p_{ll}} \cdot [-\lambda_5] \end{aligned}$$

together with $x_{ll} > x_{hl}$ and substitutability imply $y_{ll} > y_{hl}$. The case $(x, y) \in R_{3b}$ is alike.

Suppose $(x, y) \in R_{1a}$. FOCs

$$\begin{aligned} V_y(x_{hl}, y_{hl}) &= \eta_l + \frac{\Delta_\eta}{p_{hl}} \cdot [-\mu_{1a,6b} \cdot (p_{lh} + p_{ll})] \\ V_y(x_{ll}, y_{ll}) &= \eta_l \end{aligned}$$

together with $x_{ll} > x_{hl}$ and substitutability imply $y_{hl} > y_{ll}$. If (R1a,4a) holds with equality then $x_{lh} \geq x_{ll}$ by (R4a,5a). Otherwise $\mu_{1a,4a} = 0$ and FOCs

$$\begin{aligned} V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-\mu_{1a,5a} \cdot p_{hl}] \\ V_x(x_{ll}, y_{ll}) &= \theta_l \end{aligned}$$

together with $y_{lh} < y_{ll}$ and substitutability imply $x_{ll} < x_{lh}$. Analogous implications follow for $(x, y) \in R_{1b}$.

If $(x, y) \in R_{4a}$ then (R4a,3a) or (R4a,1a) must bind. Indeed, if $\mu_{4a,3a} = \mu_{4a,1a} = 0$ then FOCs

$$\begin{aligned} V_y(x_{hh}, y_{hh}) &= \eta_h \\ V_y(x_{lh}, y_{lh}) &= \eta_h + \frac{\Delta_\eta}{p_{lh}} \cdot [(p_{hl} + p_{ll}) + \lambda_{10} + \lambda_{12} + \lambda_{16}] \end{aligned}$$

together with $x_{lh} > x_{hh}$ and substitutability imply $y_{lh} < y_{hh}$, contradicting (R4a,3a). Likewise, if $(x, y) \in R_{4b}$ then (R4b,3b) or (R4b,1b) must bind.

Suppose $(x, y) \in R_2$. If (R2,3a) or (R2,3b) hold with equality, then $x_{lh} > x_{hh} = x_{ll}$ or $y_{hl} > y_{hh} = y_{ll}$, respectively. Otherwise $\mu_{2,3a} = \mu_{2,3b} = 0$ and FOCs

$$\begin{aligned} V_x(x_{lh}, y_{lh}) &= \theta_l \\ V_x(x_{ll}, y_{ll}) &= \theta_l \\ V_y(x_{hl}, y_{hl}) &= \eta_l \\ V_y(x_{ll}, y_{ll}) &= \eta_l \end{aligned}$$

together with $y_{ll} > y_{lh}$ and $x_{ll} > x_{hl}$ and substitutability imply $x_{lh} > x_{ll}$ and $y_{hl} > y_{ll}$.

If $(x, y) \in R_{5a}$ then (R5a,4a) must bind. Otherwise, FOCs

$$\begin{aligned} V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-p_{hl} + \mu_{5a,1a} \cdot p_{hl} - \mu_{5a,6a} \cdot p_{ll} - \lambda_3 - \lambda_6 - \lambda_{13}] \\ V_x(x_{ll}, y_{ll}) &= \theta_l \end{aligned}$$

together with $y_{lh} < y_{ll}$ and substitutability imply $x_{ll} < x_{lh}$, contradicting (R5a,4a). Similarly, if $(x, y) \in R_{5b}$ then (R5b,4b) must bind.

The optimal allocation (x, y) cannot lie in R_{6a}. First, note that if $(x, y) \in R_{6a}$ then (R6a,5a) cannot hold with equality as this would imply $(x, y) \in R_{5a} \cap R_{6a}$ which, by our results on R_{5a}, would imply $(x, y) \in R_{4a} \cap R_{5a} \cap R_{6a}$, contradicting Theorem 1*. But if $\mu_{6a,5a} = 0$ then FOCs

$$\begin{aligned} V_x(x_{hl}, y_{hl}) &= \theta_h + \frac{\Delta_\theta}{p_{hl}} \cdot [p_{ll} + \lambda_{11} + \lambda_6] \\ V_x(x_{lh}, y_{lh}) &= \theta_l + \frac{\Delta_\theta}{p_{lh}} \cdot [-(p_{hl} + p_{ll}) + \mu_{6a,1b} \cdot (p_{ll} + p_{hl}) - \lambda_6 - \lambda_{11}] \end{aligned}$$

together with $x_{hl} > x_{lh}$ and substitutability imply $y_{hl} < y_{lh}$, contradicting (I6). Similarly, the optimal allocation (x, y) cannot lie in R_{6b}. The second claim in Part a) follows directly from the first claim and Proposition 5.

d) This is an immediate consequence of Part c). □

The quadratic case: Technicalities. Assume that

$$V(x,y) = -\frac{1}{2}a_1 x^2 - \frac{1}{2}a_2 y^2 + bxy + c_1 x + c_2 y + d$$

with $a_1, a_2 > 0$, $b, c_1, c_2, d \in \mathbb{R}$ and $\det H(x,y) = a_1 a_2 - b^2 > 0$, ensuring strict concavity of $V(x,y)$. Moreover, assume $\Delta_\theta = \Delta_\eta = 1$.

For any region R and any $r, s \in \{l, h\}$, FOCs read

$$\begin{aligned} V_x(x_{rs}, y_{rs}) &= \gamma^1_{rs} \\ V_y(x_{rs}, y_{rs}) &= \gamma^2_{rs} \end{aligned}$$

with solution

$$\begin{aligned} x_{rs} &= \frac{-b\left(\gamma^2_{rs} - c_2\right) - a_2\left(\gamma^1_{rs} - c_1\right)}{a_1 a_2 - b^2}, \\ y_{rs} &= \frac{-b\left(\gamma^1_{rs} - c_1\right) - a_1\left(\gamma^2_{rs} - c_2\right)}{a_1 a_2 - b^2}. \end{aligned}$$

The common factor $\frac{1}{a_1 a_2 - b^2} = [\det H(x,y)]^{-1}$ only rescales the solution and does not affect any implementability or regional constraints. It is therefore without loss of generality to assume that $a_1 a_2 - b^2 = 1$. Similarly, neither implementability nor regional constraints depend on c_1, c_2 as for any $r, s, l, m \in \{l, h\}$ we have

$$\begin{aligned} x_{rs} - x_{lm} &= -b\left(\gamma^2_{rs} - \gamma^2_{lm}\right) - a_2\left(\gamma^1_{rs} - \gamma^1_{lm}\right), \\ y_{rs} - y_{lm} &= -b\left(\gamma^1_{rs} - \gamma^1_{lm}\right) - a_1\left(\gamma^2_{rs} - \gamma^2_{lm}\right). \end{aligned}$$

Both c_1 and c_2 shift the solutions by a constant in all x-allocations and by another constant in all y-allocations, and these two constants only depends on a_1, a_2, b but not on any other primitives of the model as captured by γ^1, γ^2. A sufficiently large value for d guarantees positive consumer valuations without affecting the optimal allocation.

Lemma E. *Let*

$$\begin{aligned} \tilde{x}_{rs} &= -b\gamma^2_{rs} - a_2\gamma^1_{rs} \\ \tilde{y}_{rs} &= -b\gamma^1_{rs} - a_1\gamma^2_{rs} \end{aligned}$$

denote the (possibly negative) solutions of the first order conditions of the optimization problem for the case $c_1 = c_2 = 0$ (and $a_1 a_2 - b^2 = 1$), for any $r, s \in \{l, h\}$. Then

$$\begin{aligned} x_{rs} &= -b\left(\gamma^2_{rs} - c_2\right) - a_2\left(\gamma^1_{rs} - c_1\right), \\ y_{rs} &= -b\left(\gamma^1_{rs} - c_1\right) - a_1\left(\gamma^2_{rs} - c_2\right) \end{aligned}$$

solve the corresponding first order order conditions for any $c_1, c_2 \in \mathbb{R}$ and there exist $c_1, c_2 \in \mathbb{R}$ such that $x_{rs}, y_{rs} > 0$ for any $r, s \in \{l, h\}$.

Proof. We only need to show the statement concerning positivity of x_{rs}, y_{rs} for any $r, s \in \{l, h\}$. Note that

$$\begin{aligned} x_{rs} &= \tilde{x}_{rs} + bc_2 + a_2 c_1, \\ y_{rs} &= \tilde{y}_{rs} + bc_1 + a_1 c_2. \end{aligned}$$

If $b \geq 0$, choose $c_1 = c_2$ sufficiently large. If $b < 0$, define $z = \max \{ |\tilde{x}_{rs}|, |\tilde{y}_{rs}| \, | r, s \in \{l, h\}\}$, fix some $\epsilon > 0$ and set $c_2 = -\frac{a_2}{b} c_1 + \frac{(z+\epsilon)}{b}$. Then for c_1 sufficiently large we have

$$\begin{aligned} x_{rs} &= \tilde{x}_{rs} + (z + \epsilon) \\ &> 0 \\ y_{rs} &= \tilde{y}_{rs} + \frac{a_1}{b}(z+\epsilon) - \frac{1}{b} c_1 \\ &> 0. \end{aligned}$$

□

Proof of Proposition 6E. Part a-b) are an immediate consequence of Proposition 6. As for Part c), the FOCs for the interior of R_{1a} yield

$$\begin{aligned} x_{hh} &= -a_2 \cdot \left(\theta_h + \frac{p_{ll} + p_{lh}}{p_{hh}} \right) \\ y_{hh} &= -a_1 \cdot \left(\eta_h + \frac{p_{hl}}{p_{hh}} \right) \\ x_{hl} &= -a_2 \cdot \theta_h \\ y_{hl} &= -a_1 \cdot \eta_l \\ x_{lh} &= -a_2 \cdot \theta_l \\ y_{lh} &= -a_1 \cdot \left(\theta_h + \frac{p_{ll}}{p_{lh}} \right) \\ x_{ll} &= -a_2 \cdot \theta_l \\ y_{ll} &= -a_1 \cdot \eta_l \end{aligned}$$

and hence

$$\begin{aligned} y_{lh} - y_{hh} - (x_{hl} - x_{hh}) &= a_1 \cdot \left(\frac{p_{hl}}{p_{hh}} - \frac{p_{ll}}{p_{lh}} \right) - a_2 \cdot \left(\frac{p_{ll} + p_{lh}}{p_{hh}} \right) \\ x_{ll} - x_{hh} - (y_{lh} - y_{hh}) &= -a_1 \cdot \left(\frac{p_{hl}}{p_{hh}} - \frac{p_{ll}}{p_{lh}} \right) + a_2 \cdot \left(\frac{p_{ll} + p_{lh} + p_{hh}}{p_{hh}} \right) \\ x_{lh} - x_{hh} - (y_{lh} - y_{hh}) &= -a_1 \cdot \left(\frac{p_{hl}}{p_{hh}} - \frac{p_{ll}}{p_{lh}} \right) + a_2 \cdot \left(\frac{p_{ll} + p_{lh} + p_{hh}}{p_{hh}} \right). \end{aligned}$$

Symmetrically, for region R_{1b} we get

$$\begin{aligned}
x_{hl} - x_{hh} - (y_{lh} - y_{hh}) &= a_2 \cdot \left(\frac{p_{lh}}{p_{hh}} - \frac{p_{ll}}{p_{hl}}\right) - a_1 \cdot \left(\frac{p_{ll} + p_{hl}}{p_{hh}}\right) \\
y_{ll} - y_{hh} - (x_{hl} - x_{hh}) &= -a_2 \cdot \left(\frac{p_{lh}}{p_{hh}} - \frac{p_{ll}}{p_{hl}}\right) + a_1 \cdot \left(\frac{p_{ll} + p_{hl} + p_{hh}}{p_{hh}}\right) \\
y_{hl} - y_{hh} - (x_{hl} - x_{hh}) &= -a_2 \cdot \left(\frac{p_{lh}}{p_{hh}} - \frac{p_{ll}}{p_{hl}}\right) + a_1 \cdot \left(\frac{p_{ll} + p_{hl} + p_{hh}}{p_{hh}}\right).
\end{aligned}$$

Inserting $Q = \frac{a_1}{a_2}$ directly yields the result. □

Proof of Proposition 9E. a) Invoking Proposition 9 we only need to show that the optimal allocation (x, y) does not lie in the interior of R_{5a}, $R_{5a} \cap R_{4a}$, R_{5b}, or $R_{5b} \cap R_{4b}$. We focus on "a"-type regions. For R_{5a} we have

$$\begin{aligned}
\gamma^1_{hh} &= \theta_h + \frac{(p_{lh} + p_{ll} + p_{hl}) + \lambda_{13}}{p_{hh}} & \gamma^2_{hh} &= \eta_h \\
\gamma^1_{hl} &= \theta_h + \frac{\lambda_6}{p_{hl}} & \gamma^2_{hl} &= \eta_l + \frac{-\lambda_6 - \lambda_{13}}{p_{hl}} \\
\gamma^1_{lh} &= \theta_l - \frac{(1 - \mu_{5a,4a}) \cdot p_{hl} + \lambda_6 + \lambda_{13}}{p_{lh}} & \gamma^2_{lh} &= \eta_h + \frac{(p_{hl} + p_{ll}) + \lambda_6 + \lambda_{13}}{p_{lh}} \\
\gamma^1_{ll} &= \theta_l - \frac{\mu_{5a,4a} \cdot p_{hl}}{p_{ll}} & \gamma^2_{ll} &= \eta_l
\end{aligned}$$

and

$$\begin{aligned}
y_{lh} - y_{hh} - (x_{lh} - x_{hh}) = b\left(\gamma^1_{hh} - \gamma^1_{lh} + \gamma^2_{lh} - \gamma^2_{hh}\right) & \\
- \sqrt{Q}\sqrt{(1 + b^2)}\left(\gamma^2_{lh} - \gamma^2_{hh}\right) & \\
- \frac{1}{\sqrt{Q}}\sqrt{(1 + b^2)}\left(\gamma^1_{hh} - \gamma^1_{lh}\right) & \qquad (3.9.1)
\end{aligned}$$

$$x_{lh} - x_{hl} = -b\left(\gamma^2_{lh} - \gamma^2_{hl}\right) + \frac{1}{\sqrt{Q}}\sqrt{(1 + b^2)}\left(\gamma^1_{hl} - \gamma^1_{lh}\right). \qquad (3.9.2)$$

Note that (3.9.1) is concave in $\sqrt{Q}$. Taking derivatives with respect to $\sqrt{Q}$ yields a maximum at

$$\sqrt{Q} = \sqrt{\frac{\gamma^1_{hh} - \gamma^1_{lh}}{\gamma^2_{lh} - \gamma^2_{hh}}}. \qquad (3.9.3)$$

Inserting (3.9.3) in (3.9.1) yields

$$y_{lh} - y_{hh} - (x_{lh} - x_{hh}) \quad = \quad b\left(\gamma^1_{hh} - \gamma^1_{lh} + \gamma^2_{lh} - \gamma^2_{hh}\right) - 2\sqrt{(1 + b^2)} \cdot \sqrt{\gamma^2_{lh} - \gamma^2_{hh}}\sqrt{\gamma^1_{hh} - \gamma^1_{lh}}.$$

This expression is nonnegative if and only if

$$b^2 \geq \frac{4\left(\gamma^1_{hh} - \gamma^1_{lh}\right)\left(\gamma^2_{lh} - \gamma^2_{hh}\right)}{\left[\left(\gamma^1_{hh} - \gamma^1_{lh}\right) - \left(\gamma^2_{lh} - \gamma^2_{hh}\right)\right]^2} \qquad (3.9.4)$$

Now consider (3.9.2). Clearly, (3.9.2) is most relaxed if b is as small as possible. Inserting (3.9.3), rearranging and inserting (3.9.4) with equality requires

$$-4\left(\gamma_{hh}^1-\gamma_{lh}^1\right)^2\left(\gamma_{lh}^2-\gamma_{hl}^2\right)^2+\left[\left(\gamma_{hh}^1-\gamma_{lh}^1\right)+\left(\gamma_{lh}^2-\gamma_{hh}^2\right)\right]^2\left(\gamma_{hl}^1-\gamma_{lh}^1\right)^2 \quad \geq \quad 0. \qquad (3.9.5)$$

However, it is readily checked that this is a quadratic expression in $\gamma_{hh}^1-\gamma_{lh}^1$ with negative leading coefficient $-4\left(\gamma_{lh}^2-\gamma_{hl}^2\right)^2+\left(\gamma_{hl}^1-\gamma_{lh}^1\right)^2$ and negative discriminant

$$4\left(\gamma_{lh}^2-\gamma_{hh}^2\right)^2\left(\gamma_{hl}^1-\gamma_{lh}^1\right)^4-16\left(\gamma_{lh}^2-\gamma_{hl}^2\right)^2\left(\gamma_{lh}^2-\gamma_{hh}^2\right)^2\left(\gamma_{hl}^1-\gamma_{lh}^1\right)^2,$$

thus (3.9.5) is always negative. As a consequence, whenever b is sufficiently large for (3.9.1) to be non-negative at its maximum given by (3.9.3), constraint (3.9.2) is violated when imposing (3.9.3). Note that (3.9.2) is non-negative if and only if

$$\sqrt{Q} \leq \frac{\sqrt{(1+b^2)}}{b} \cdot \frac{\gamma_{hl}^1-\gamma_{lh}^1}{\gamma_{lh}^2-\gamma_{hl}^2}. \qquad (3.9.6)$$

To prove the claim of Part a) it is hence sufficient to show that inserting (3.9.6) with equality in (3.9.1) yields a negative expression. After some rearranging, the relevant term reads

$$b^2\underbrace{\left[\left(\gamma_{lh}^2-\gamma_{hl}^2\right)-\left(\gamma_{hl}^1-\gamma_{lh}^1\right)\right]}_{>0}\underbrace{\left[\left(\gamma_{hl}^1-\gamma_{lh}^1\right)\left(\gamma_{lh}^2-\gamma_{hh}^2\right)-\left(\gamma_{lh}^2-\gamma_{hl}^2\right)\left(\gamma_{hh}^1-\gamma_{lh}^1\right)\right]}_{<0}-\left(\gamma_{hl}^1-\gamma_{lh}^1\right)^2\left(\gamma_{lh}^2-\gamma_{hh}^2\right) \quad < \quad 0.$$

b) Invoking Theorem 1* it suffices to show that whenever the optimal solution lies in the interior of R_{6a} or R_{6b}, no implementability constraint holds with equality. Again it suffices to consider R_{6a}. In the interior of R_{6a} we have

$$\begin{aligned}
\gamma_{hh}^1 &= \theta_h+\frac{p_{lh}+p_{ll}+p_{hl}}{p_{hh}} & \gamma_{hh}^2 &= \eta_h \\
\gamma_{hl}^1 &= \theta_h+\frac{p_{ll}}{p_{hl}} & \gamma_{hl}^2 &= \eta_l \\
\gamma_{lh}^1 &= \theta_l-\frac{p_{hl}+p_{ll}}{p_{lh}} & \gamma_{lh}^2 &= \eta_h+\frac{p_{hl}+p_{ll}}{p_{lh}} \\
\gamma_{ll}^1 &= \theta_l & \gamma_{ll}^2 &= \eta_l
\end{aligned}$$

By Proposition 9 we have $y_{ll} \geq y_{hl}$ so we may focus on (I6) exclusively. Suppose (I6) is violated but (R6a,1b) is satisfied. Then

$$\begin{aligned}
x_{lh}-x_{hl}+y_{hl}-y_{lh} \quad &= \quad \left(\frac{1+Q}{\sqrt{Q}}\sqrt{(1+b^2)}-2b\right)\cdot\left(\frac{1-p_{lh}}{p_{lh}}\right) & (3.9.7)\\
&\quad\; +\left(\frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}-b\right)\left[\frac{p_{ll}}{p_{hl}}\right] \\
&< \quad 0, \\
x_{hl}-x_{lh} \quad &= \quad \left(b-\frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}\right)\left(\frac{1-p_{lh}}{p_{lh}}\right)-\frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}\left(\frac{p_{ll}}{p_{hl}}\right) & (3.9.8)\\
&\geq \quad 0.
\end{aligned}$$

But (3.9.7) implies $\sqrt{Q} < \frac{p_{hl}(1-p_{lh})+p_{ll}p_{lh}}{p_{hl}(1-p_{lh})}$ while (3.9.8) implies $\sqrt{Q} > \frac{p_{hl}(1-p_{lh})+p_{ll}p_{lh}}{p_{hl}(1-p_{lh})}$, a contradiction.

c) Consider region R_{6a}. To establish necessary conditions such that the optimal allocation may lie in the interior of $R_{5a} \cup R_{6a}$ or in the interior of R_{6a} we must ensure that constraint (R6a,1b) is satisfied with strict inequality. We have

$$\begin{aligned}
\gamma^1_{hh} &= \theta_h + \frac{p_{lh} + p_{ll} + p_{hl}}{p_{hh}} & \gamma^2_{hh} &= \eta_h \\
\gamma^1_{hl} &= \theta_h + \frac{(1-\mu_{6a,5a})\, p_{ll}}{p_{hl}} & \gamma^2_{hl} &= \eta_l \\
\gamma^1_{lh} &= \theta_l - \frac{p_{hl} + (1-\mu_{6a,5a})\, p_{ll}}{p_{lh}} & \gamma^2_{lh} &= \eta_h + \frac{p_{hl} + p_{ll}}{p_{lh}} \\
\gamma^1_{ll} &= \theta_l & \gamma^2_{ll} &= \eta_l
\end{aligned}$$

and

$$\begin{aligned}
y_{lh} - y_{hh} - x_{lh} + x_{hh} \;&=\; b\left(\gamma^1_{hh} - \gamma^1_{lh} + \gamma^2_{lh} - \gamma^2_{hh}\right) \\
&\quad - \sqrt{Q}\sqrt{(1+b^2)}\left(\gamma^2_{lh} - \gamma^2_{hh}\right) - \frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}\left(\gamma^1_{hh} - \gamma^1_{lh}\right)
\end{aligned} \tag{3.9.9}$$

By Proposition 8, $(x, y) \in R_{6a}$ is only possible if $b \geq a_2 = \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}$, providing the lower bound given in the Proposition. Moreover, this implies that the above expression is decreasing in $\mu_{6a,5a}$ as γ^1_{lh} is increasing in $\mu_{6a,5a}$. Being interested in necessary conditions only we may thus assume that $\mu_{6a,5a} = 0$ and neglect constraint (R6a,5a). Multiplying (3.9.9) with $\sqrt{Q} > 0$ yields a quadratic expression in $\sqrt{Q}$ with negative leading coefficient. The expression is hence positive if it has positive discriminant

$$b^2 \cdot \frac{1}{p^2_{hh}} - 4\left(\frac{p_{hl} + p_{ll}}{p_{lh}}\right)\left(\frac{1}{p_{hh}} + \frac{p_{hl} + p_{ll}}{p_{lh}}\right),$$

yielding the lower bound for b^2 stated in the Proposition, and if

$$\begin{aligned}
\frac{\sqrt{b^2+1}}{b} \cdot \sqrt{Q} \;&>\; 1 + \frac{\frac{p_{lh}}{p_{hh}} - \sqrt{\left(\frac{p_{lh}}{p_{hh}}\right)^2 - 4 \cdot \frac{1}{b^2} \cdot (p_{hl} + p_{ll}) \cdot \left(p_{hl} + p_{ll} + \frac{p_{lh}}{p_{hh}}\right)}}{p_{hl} + p_{ll}}, \\
\frac{\sqrt{b^2+1}}{b} \cdot \sqrt{Q} \;&<\; 1 + \frac{\frac{p_{lh}}{p_{hh}} + \sqrt{\left(\frac{p_{lh}}{p_{hh}}\right)^2 - 4 \cdot \frac{1}{b^2} \cdot (p_{hl} + p_{ll}) \cdot \left(p_{hl} + p_{ll} + \frac{p_{lh}}{p_{hh}}\right)}}{p_{hl} + p_{ll}}.
\end{aligned}$$

Together, these results prove Part c) with respect to R_{6a}. The proof for region R_{6b} is alike.

d) Consider again R_{6a}. Including constraint $\mu_{6a,1b}$ we have

$$\begin{aligned}
\gamma^1_{hh} &= \theta_h + \frac{(p_{lh} + p_{ll} + p_{hl}) - \mu_{6a,1b} \cdot (p_{ll} + p_{hl})}{p_{hh}} & \gamma^2_{hh} &= \eta_h + \frac{\mu_{6a,1b} \cdot (p_{ll} + p_{hl})}{p_{hh}} \\
\gamma^1_{hl} &= \theta_h + \frac{p_{ll}}{p_{hl}} & \gamma^2_{hl} &= \eta_l \\
\gamma^1_{lh} &= \theta_l - \frac{p_{hl} + p_{ll} - \mu_{6a,1b} \cdot (p_{ll} + p_{hl})}{p_{lh}} & \gamma^2_{lh} &= \eta_h + \frac{p_{hl} + p_{ll} - \mu_{6a,1b} \cdot (p_{ll} + p_{hl})}{p_{lh}} \\
\gamma^1_{ll} &= \theta_l & \gamma^2_{ll} &= \eta_l
\end{aligned}$$

and thus

$$\begin{aligned}
x_{hl} - x_{lh} &= \left(b - \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\right)\left[1 + \frac{(p_{hl}+p_{ll}) - \mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{lh}}\right] \\
&\quad - \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\left[1 + \frac{p_{ll}}{p_{hl}}\right] \\
y_{lh} - y_{hh} - x_{lh} + x_{hh} &= \left(2b - \frac{1+Q}{\sqrt{Q}}\sqrt{(1+b^2)}\right)\left(\frac{(p_{hl}+p_{ll}) - \mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{lh}}\right) \\
&\quad - \left(b - \sqrt{Q} \cdot \sqrt{(1+b^2)}\right)\left(\frac{\mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{hh}}\right) \\
&\quad + \left(b - \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\right)\left(1 + \frac{(p_{lh}+p_{ll}+p_{hl}) - 2\mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{hh}}\right).
\end{aligned}$$

Clearly, Q sufficiently large implies $x_{hl} - x_{lh} > 0$ irrespective of the value of $\mu_{6a,1b} \in [0,1]$. Moreover, as demonstrated in c), for $\mu_{6a,1b} = 0$ we have $y_{lh} - y_{hh} - x_{lh} + x_{hh} < 0$ whenever Q is sufficiently large while for $\mu_{6a,1b} = 1$ we have

$$\begin{aligned}
y_{lh} - y_{hh} - x_{lh} + x_{hh} &= \left(2b - \frac{1+Q}{\sqrt{Q}}\sqrt{(1+b^2)}\right)\left(\frac{(p_{hl}+p_{ll}) - \mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{lh}}\right) \\
&\quad - \left(b - \sqrt{Q} \cdot \sqrt{(1+b^2)}\right)\left(\frac{\mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{hh}}\right) \\
&\quad + \left(b - \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\right)\left(1 + \frac{(p_{lh}+p_{ll}+p_{hl}) - 2\mu_{6a,1b} \cdot (p_{ll}+p_{hl})}{p_{hh}}\right) \\
&= \left(\sqrt{Q} \cdot \sqrt{(1+b^2)} - b\right)\left(\frac{p_{ll}+p_{hl}}{p_{hh}}\right) \\
&\quad + \left(\frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)} - b\right)\left(\frac{p_{hh}+p_{lh} - (p_{ll}+p_{hl})}{p_{hh}}\right)
\end{aligned}$$

which, for any fixed vaues of b and the distribution, becomes positive for large Q. Hence, by the intermediate value theorem, for any Q sufficiently large there exists $\mu_{6a,1b} \in (0,1)$ such that

$$\begin{aligned}
x_{hl} - x_{lh} &> 0 \\
y_{lh} - y_{hh} - x_{lh} + x_{hh} &= 0.
\end{aligned}$$

This proves the claim. The statement for R_{6b} follows analogously. □

Proof of Proposition 10E. a) Invoking Theorem 1* and Proposition 10 it suffices to check that no implementability constraint holds with equality if the optimal allocation lies in the interior of R_{3a}, $R_{3a} \cap R_{4a}$, R_{3b}, or $R_{3b} \cap R_{4b}$. We focus on "a"-type regions. In R_{3a} we have

$$\begin{aligned}
\gamma^1_{hh} &= \theta_h + \frac{p_{lh}+p_{ll}+p_{hl}+\lambda_5+\lambda_7}{p_{hh}} & \gamma^2_{hh} &= \eta_h + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll}) + \lambda_5 + \lambda_7}{p_{hh}} \\
\gamma^1_{hl} &= \theta_h & \gamma^2_{hl} &= \eta_l - \frac{\lambda_7}{p_{hl}} \\
\gamma^1_{lh} &= \theta_l & \gamma^2_{lh} &= \eta_h + \frac{\mu_{3a,4a} \cdot (p_{hl}+p_{ll})}{p_{lh}} \\
\gamma^1_{ll} &= \theta_l - \frac{p_{hl}+\lambda_5+\lambda_7}{p_{ll}} & \gamma^2_{ll} &= \eta_l - \frac{\lambda_5}{p_{ll}}.
\end{aligned}$$

As $y_{ll} \le y_{hl}$ by Proposition 10, it suffices to check (I5) exclusively. Suppose (I5) is violated but (R3a,2) is satisfied. Then

$$\begin{aligned} x_{hh} - x_{ll} &= -\left(b + \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\right)\left(1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}}\right) \\ &\quad - \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\left(\frac{p_{lh} + \mu_{3a,4a} \cdot (p_{ll}+p_{hl})}{p_{hh}} + \frac{p_{hl}}{p_{ll}}\right) \qquad (3.9.10) \\ &> 0 \\ x_{ll} - x_{hh} + (y_{ll} - y_{hh}) &= \left(2b + \frac{1+Q}{\sqrt{Q}}\sqrt{(1+b^2)}\right)\left(1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}}\right) \\ &\quad + \left(b + \frac{1}{\sqrt{Q}} \cdot \sqrt{(1+b^2)}\right)\left(\frac{p_{lh} + \mu_{3a,4a} \cdot (p_{ll}+p_{hl})}{p_{hh}} + \frac{p_{hl}}{p_{ll}}\right) \qquad (3.9.11) \\ &< 0. \end{aligned}$$

But (3.9.10) implies

$$\sqrt{Q} > \frac{1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}} + \frac{p_{lh}+\mu_{3a,4a} \cdot (p_{ll}+p_{hl})}{p_{hh}} + \frac{p_{hl}}{p_{ll}}}{1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}}}$$

while (3.9.11) implies

$$\sqrt{Q} < \frac{1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}} + \frac{p_{lh}+\mu_{3a,4a} \cdot (p_{ll}+p_{hl})}{p_{hh}} + \frac{p_{hl}}{p_{ll}}}{1 + \frac{(1-\mu_{3a,4a}) \cdot (p_{hl}+p_{ll})}{p_{hh}}},$$

a contradiction.

b) Consider R_{2a}. Ignoring constraints towards regions R_{1a} and R_{1b} we have

$$\begin{aligned} \gamma^1_{hh} &= \theta_h + \frac{p_{lh} + p_{ll} + \mu_{2,3a} \cdot p_{hl}}{p_{hh}} & \gamma^2_{hh} &= \eta_h + \frac{p_{hl} + p_{ll} + \mu_{2,3b} \cdot p_{lh}}{p_{hh}} \\ \gamma^1_{hl} &= \theta_h & \gamma^2_{hl} &= \eta_l \\ \gamma^1_{lh} &= \theta_l & \gamma^2_{lh} &= \eta_h \\ \gamma^1_{ll} &= \theta_l - \frac{\mu_{2,3a} \cdot p_{hl}}{p_{ll}} & \gamma^2_{ll} &= \eta_l - \frac{\mu_{2,3b} \cdot p_{lh}}{p_{ll}}, \end{aligned}$$

yielding

$$\begin{aligned} y_{hh} - y_{lh} &= -b\left[1 + \frac{p_{lh} + p_{ll} + \mu_{2,3a} \cdot p_{hl}}{p_{hh}}\right] - a_1\left[\frac{p_{hl} + p_{ll} + \mu_{2,3b} \cdot p_{lh}}{p_{hh}}\right] \qquad (3.9.12) \\ x_{hh} - x_{hl} &= -b\left[1 + \frac{p_{hl} + p_{ll} + \mu_{2,3b} \cdot p_{lh}}{p_{hh}}\right] - a_2\left[\frac{p_{lh} + p_{ll} + \mu_{2,3a} \cdot p_{hl}}{p_{hh}}\right] \qquad (3.9.13) \\ x_{ll} - x_{hh} &= (b + a_2)\left[1 + \frac{p_{hl} + p_{ll}}{p_{hh}}\right] + 2b \cdot \frac{\mu_{2,3b} \cdot p_{lh}}{p_{ll}} + 2a_2 \cdot \frac{\mu_{2,3a} \cdot p_{hl}}{p_{ll}} \qquad (3.9.14) \\ y_{ll} - y_{hh} &= (b + a_1)\left[1 + \frac{p_{lh} + p_{ll}}{p_{hh}}\right] + 2b \cdot \frac{\mu_{2,3a} \cdot p_{hl}}{p_{ll}} + 2a_1 \cdot \frac{\mu_{2,3b} \cdot p_{lh}}{p_{ll}}. \qquad (3.9.15) \end{aligned}$$

In case of $\mu_{2,3a} = \mu_{2,3b} = 0$, equations (3.7.4) and (3.7.5) guarantee $y_{hh} - y_{lh} > 0$, $x_{hh} - x_{hl} > 0$ and the solution lies in the interior of R_2 if and only if in addition $a_1 > |b|$, $a_2 > |b|$. Note that at most one of (3.7.4) and (3.7.5) may fail to hold as

$a_1 a_2 - b^2 = 1$. Assume without loss of generality that $a_2 \leq |b|$ and consider R_{3a}. Ignoring constraint towards R_{4a} we have

$$\begin{aligned}
\gamma^1_{hh} &= \theta_h + \frac{p_{lh} + p_{ll} + (1-\mu_{3a,2}) \cdot p_{hl}}{p_{hh}} & \gamma^2_{hh} &= \eta_h + \eta_h + \frac{p_{hl} + p_{ll}}{p_{hh}} \\
\gamma^1_{hl} &= \theta_h & \gamma^2_{hl} &= \eta_l \\
\gamma^1_{lh} &= \theta_l & \gamma^2_{lh} &= \eta_h \\
\gamma^1_{ll} &= \theta_l - \frac{(1-\mu_{3a,2}) \cdot p_{hl}}{p_{ll}} & \gamma^2_{ll} &= \eta_l
\end{aligned}$$

and thus

$$\begin{aligned}
x_{hh} - x_{ll} &= -b\left[1 + \frac{p_{hl} + p_{ll}}{p_{hh}}\right] \\
&\quad - a_2\left[1 + \frac{p_{lh} + p_{ll} + (1-\mu_{3a,2}) \cdot p_{hl}}{p_{hh}} + \frac{(1-\mu_{3a,2}) \cdot p_{hl}}{p_{ll}}\right] \qquad (3.9.16) \\
y_{hh} - y_{lh} &= -b\left[1 + \frac{p_{lh} + p_{ll} + (1-\mu_{3a,2}) \cdot p_{hl}}{p_{hh}}\right] - a_1\left[\frac{p_{hl} + p_{ll}}{p_{hh}}\right]. \qquad (3.9.17)
\end{aligned}$$

Clearly, (3.9.17) holds by assumption for any $\mu_{3a,2} \in [0,1]$. But since $a_2 \leq |b|$ we either have that (3.9.16) is positive for $\mu_{3a,2} = 0$ in which case the optimal allocation lies in the interior of R_{3a} or there exists $\mu_{2,3a} \in [0,1]$ such that (3.9.16) holds with equality and the solution lies in the interior of $R_2 \cap R_{3a}$. If instead of $a_2 \leq |b|$ we assume $a_1 \leq |b|$, an analogous argument applies for R_{3b}.

c) Using the expressions listed in Part a) for R_{3a}, constraints (R3a,2) and (R3a,4a) are given as

$$\begin{aligned}
x_{hh} - x_{ll} &= -\left(b + \frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}\right)\left(1 + \frac{(1-\mu_{3a,4a})(p_{hl} + p_{ll})}{p_{hh}}\right) \\
&\quad - \frac{1}{\sqrt{Q}}\sqrt{(1+b^2)}\left(\frac{p_{lh} + \mu_{3a,4a}(p_{ll} + p_{hl})}{p_{hh}} + \frac{p_{hl}}{p_{ll}}\right) \qquad (3.9.18) \\
y_{hh} - y_{lh} &= -b\left[\frac{1}{p_{hh}}\right] - \sqrt{Q}\sqrt{(1+b^2)}\left[\frac{(1-\mu_{3a,4a})(p_{hl} + p_{ll})}{p_{hh}} - \frac{\mu_{3a,4a}(p_{hl} + p_{ll})}{p_{lh}}\right] \qquad (3.9.19)
\end{aligned}$$

Clearly, the value of (3.9.18) is strictly positive for any $\mu_{3a,4a} \in [0,1]$ if Q is sufficiently large. On the other hand, for any $Q \geq 1$ there exists some value $\mu_{3a,4a} \in (0,1)$ such that (3.9.19) is equal to zero. Hence, for sufficiently large Q, the solution lies in the interior of $R_{3a} \cap R_{4a}$. An analogous proof applies for Q very small, considering region R_{3b} instead. □

Chapter 4

Pricing a Package of Services - When (not) to Bundle[1]

4.1 Introduction

Many important decisions in our lives involve choices among bundles and trade-offs between several taste dimensions. Constructing a house is presumably one of the most important of such instances. First and foremost, the location of the house has to be chosen. Several choices ranging from the number of floors, the number of rooms on each floor, construction materials, and on to the very last details of the interior decor follow the first decision. Some of these choices are extremely flexible and so involve marginal adjustments. Other choices are arguably more rigid; typically, only a very limited number of alternative locations are available at any given time. We are interested in the design of pricing schemes in such situations featuring a combination of flexible and rigid choices. Moreover, as in our leading example, the rigid choice has important consequences both in terms of utility and in terms of costs.

A natural concern one may have when making such important choices is "not to give away too much". Will a real estate developer adjust the price for constructing our house condition on whether the house is located in a posh or a middle class area? Intuition - at least ours - suggests that this would make a lot of sense.

We study this question in a stylized model involving choices along two margins only: the location and the quality (or equivalently, the size) of a house. The answer contradicts our naïve intuition. In the very case where consumers with a taste for living in the posh area are likely to be ones who appreciate higher quality houses more in the sense of affiliated taste parameters, the real estate developer does not "exploit" consumers in the posh area at the optimum. The optimal marginal price for increases in the quality of the house is exactly the same, whether the house is located in the posh or the middle class area. In the terminology of the literature, there is no bundling.

The intuition is as follows. The seller wishes to extract informational rents from the

[1] This thesis chapter is based on a paper under the same title that is joint work with Dezső Szalay from the University of Bonn and CEPR.

consumer. The consumer has two pieces of information, his marginal valuation for moving to a posh area and his marginal valuation for a slightly nicer house. Unfortunately, from the seller's perspective, extracting rents from poshness tastes comes at the cost of extracting rents from quality tastes. Clearly, if the consumer could be forced to choose the location of his house based on his tastes for poshness only, then the seller would definitely adjust the marginal prices for constructing the house upwards in the posh area. Given consumers in the posh area are more likely to have higher valuations for nicer houses, the usual trade-off between extracting rents from quality tastes versus efficiency of the quality allocation is resolved more in favor of rent extraction and hence marginal prices for quality are higher. However, when the tastes for the area are unobservable, consumers with a taste for poshness facing the above marginal prices for quality would have a strict incentive to live in the middle class area. To make moving into the posh area more attractive, the seller has to lower the marginal prices for increases in the quality of houses. The surprising element is that the optimal selling procedure involves *no flexibility at all* to condition marginal prices on location choice.

To understand the complete absence of flexilibity, consider any type who is just indifferent between living in the posh and the middle class area. Affiliation implies that in an optimal pricing scheme, all consumers with higher tastes for quality must also be indifferent between living in either area. This indifference condition directly forces the quality choices of these consumers to be equal and hence requires that they face the same marginal prices. Thus, facing affiliated types, the seller's flexibility to condition marginal prices for quality on the tastes for poshness is confined to consumers with low tastes for quality if at all. To gain any flexibility even with that portion of consumers, the seller must leave rents to consumers buying the lowest quality house in the posh area, which implies an increase in rents to *all* consumers locating in the posh area. The gain from the adjustment of marginal prices for qualities simply does not outweigh this cost. In short, our result can be understood as a characterization of the set of consumers who are indifferent between entering the market and staying out: all consumers with the lowest taste for quality, both among those with and without a taste for poshness, are indifferent with respect to participating. Given this indifference at the optimum, any attempt to bundle that is desirable from the seller's point of view would violate incentive compatibility with respect to the location choice, i.e. give consumers with a taste for poshness a strict incentive to live in the middle class area.

We also establish a converse to our no-bundling result. If consumers' tastes are locally negatively affiliated - that is, tastes for living in the middle class area and for nicer houses are affiliated - then the optimum will necessarily display some bundling, at least for a strictly positive mass of consumers. Indeed, the seller now would ideally want to (locally) decrease marginal prices for higher qualities for consumers in the posh area relative to the middle class area, a change that consumers with a taste for poshness would welcome. On the other hand, consumers who don't value living in a posh area have a strict incentive to buy a house in the middle-class area when marginal prices for nicer houses are the same in both areas. Hence, there is now some flexibility to adjust marginal prices suitably.

This thesis chapter analyzes a model of multi-dimensional screening. The seminal references are Armstrong (1996) and Rochet and Choné (1998). Armstrong (1996) solves the multiproduct monopoly problem and shows that at the optimum typically some types are excluded if the type space is convex. Our problem involves convex types for the size of houses but large taste differences with respect to poshness of the area. As a result, we do not get exclusion. Rochet and Choné (1998) solve a very general problem and establish robust features of solutions; in particular, they confirm that optimal allocations generally feature exclusion of some types and show on top that optimal allocations also involve bunching over portions of types. Our problem is much simpler than these problems. The reason is that the second best optimal allocation of consumers to areas is immediate in our problem and so the name of the game is simply to choose optimal marginal prices for quality conditional on location choices. The optimum involves both bunching and separation: the allocation of consumers to areas separates consumers based on their tastes for poshness but is independent of their tastes for quality; the allocation of quality separates consumers with respect to their tastes for quality but is independent of their tastes for poshness.

Bundling was first analyzed by Adams and Yellen (1976), who showed by example that bundling can be profitable if tastes are negatively correlated. McAfee et al. (1989) establish sufficient conditions for bundling to increase profits in the Adams and Yellen (1976) model. Their conditions are consistent with weakly negative correlation, but they emphasize that the correlation of types is not the appropriate measure. In these approaches, the set of available mechanisms is restricted to prices of bundles, an approach that is extended by Manelli and Vincent (2007). Manelli and Vincent (2006) study more generally revenue maximizing mechanisms for a firm selling N objects. As in Thanassoulis (2004), the optimal mechanism may involve randomization. More recently, Hart and Reny (2014) show that revenue optimal pricing schemes may have surprising features; in particular, the seller's revenue can decrease if the buyer's multidimensional valuation increases - something that cannot happen in dimension one; moreover, they provide new examples where stochastic mechanisms are optimal. Armstrong and Rochet (1999) characterize the optimal mechanism in a model with two goods and two taste parameters on binary supports. They show that no bundling occurs for the case of strong positive correlation. We obtain no bundling even for slightly positive correlations - when types are affiliated. The reason lies precisely in the structure of our optimal location allocation, which is different from the one in Armstrong and Rochet (1999).

Our problem differs from the approaches taken in Manelli and Vincent (2006, 2007), Thanassoulis (2004), and Hart and Reny (2014) in two ways. First, these papers study revenue maximization whereas in our model there are substantial costs of production. In the presence of such costs - which seems reasonable in our introductory example - profit maximization and revenue maximization are different objectives. Though seemingly innocuous, this property allows us to reduce the dimensionality of our two-dimensional problem right away to one dimension. As a result, our optimum does not involve any randomization, which is not trivial to rule out in higher dimensional problems. The second difference is that we combine one inflexible choice (an either-or-choice) with a more flexible one. The inflexible choice is the same as the choices studied in the work by Manelli and Vincent (2006,2007) and, more recently, by Armstrong (2013).[3] We combine this choice of the seller with a more flexible one - any non-negative size of a house - as in the approaches of Armstrong (1996) and Rochet and Choné (1998).

Our model is so stylized that our problem becomes amenable to essentially uni-dimensional methods. In particular, our design problem boils down to choosing a pair of uni-dimensional schedules, where the consumer's choice of where to locate can be treated as a type dependent outside option. Type dependent outside options are studied in Jullien (2000). However, the difference to Jullien (2000) is that the outside option is endogenous, resulting from the optimal design of the scheme for consumers locating in the middle-class area. Similar techniques are also used in the countervailing incentives context by Maggi and Rodriguez-Clare (1995). Our approach is related to Kleven et al. (2009), who study the design of tax schemes for couples and singles and give conditions for bundling in the tax context: different marginal taxes at the same income level for couples and for singles. The institutional details as well as our approach and assumptions are quite different. [4] However, the common element is a combination of a discrete with a flexible choice. This gives rise to a design problem that remains nicely tractable despite its multidimensional nature.

4.2 The Model

A risk-neutral seller (she) wants to sell a package of two goods. The first good is divisible and its quantity (or quality) is labeled $x \in \mathbb{R}_{\geq 0}$. The second good is indivisible and we write $q \in [0, 1]$ for the probability of selling the second good. The seller faces a risk-neutral buyer (he) whose valuation for the bundle of goods is given as

$$V(x, q, \theta, \eta) = \theta x + \eta q$$

where $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, $0 < \underline{\theta} < \overline{\theta} < \infty$ and $\eta \in \left\{\underline{\eta}, \overline{\eta}\right\}$, $0 < \underline{\eta} < \overline{\eta}$ are preference parameters that are private knowledge of the buyer. The seller only knows the distribution $F(\theta, \eta)$ of the buyer's preference parameters. We assume that F has a continuously differentiable probability density function $f(\theta, \eta)$ which is strictly positive everywhere on $\left[\underline{\theta}, \overline{\theta}\right] \times \left\{\underline{\eta}, \overline{\eta}\right\}$. To shorten notation we write $\beta = Pr\left(\eta = \underline{\eta}\right)$ and $1 - \beta = Pr\left(\eta = \overline{\eta}\right)$. Moreover, the buyer has an outside option of buying nothing which earns him a utility of zero.

For given values $x \geq 0$ and $q \in [0, 1]$ the seller faces (expected) production costs

$$C(x, q) = C(x) + c \cdot q$$

where $C(x)$ denotes the costs for producing quantity x of good one and $c > 0$ denotes the constant marginal cost of producing the second good. We assume that $C(x)$ is

[3] Armstrong (2013) studies bundling when there are demand complementarities or substitutabilities and restricts his analysis to deterministic mechanisms. We allow for more flexible allocations and mechanisms but stick to additively separable valuations.

[4] The objective in the tax context (redistribution) differs from ours (profit maximization); while tax payers can be forced to participate, consumers cannot. Finally, we allow for statistical dependence among types and characterize direct mechanisms (also allowing for stochastic mechanisms).

increasing, twice continuously differentiable and strictly convex in x with $C(0) = 0$, $\lim_{x \searrow 0} \frac{\partial C}{\partial x}(x) = 0$ and $\lim_{x \nearrow \infty} \frac{\partial C}{\partial x}(x) = +\infty$. Production costs are known to the seller.

The seller aims to maximize his expected surplus from selling a bundle (or, more generally, a lottery over bundles) to the buyer given as

$$\Pi = \mathbb{E}\left[p(x,q) - C(x,q)\right]$$

where $p = p(x,q)$ denotes the (lottery over) prices for the bundle or the lottery (x,q).[6] Invoking the revelation principle (see e.g. Myerson (1982)) we can think of the seller's pricing problem as a direct mechanism where the buyer communicates his type (θ,η) and in return is offered a lottery over allocations $(x(\theta,\eta), q(\theta,\eta))$ together with a lottery over prices $p(\theta,\eta)$, subject to incentive compatibility and participation constraints.

A buyer (θ,η) who reports type $\left(\hat{\theta},\hat{\eta}\right)$ receives an expected utility of

$$U\left(\hat{\theta},\theta,\hat{\eta},\eta\right) = \theta x\left(\hat{\theta},\hat{\eta}\right) + \eta q\left(\hat{\theta},\hat{\eta}\right) - p\left(\hat{\theta},\hat{\eta}\right).$$

If reports coincide with true types we write $u(\theta,\eta) = U(\theta,\theta,\eta,\eta)$. Moreover, we write $\underline{u}(\theta)$, $\overline{u}(\theta)$, $\underline{x}(\theta)$, $\overline{x}(\theta)$ for $u\left(\theta,\underline{\eta}\right)$, $u(\theta,\overline{\eta})$, $x\left(\theta,\underline{\eta}\right)$, $x(\theta,\overline{\eta})$ respectively and $\underline{u} = \underline{u}(\underline{\theta})$, $\overline{u} = \overline{u}(\underline{\theta})$.

Writing

$$\Pi(\theta,\eta) = p(\theta,\eta) - C(x(\theta,\eta), q(\theta,\eta))$$

for her expected profit from type (θ,η), the seller seeks to maximize

$$\Pi = \mathbb{E}_{\theta,\eta}\Pi(\theta,\eta)$$

subject to incentive compatibility and participation[7]

$$u(\theta,\eta) \geq U\left(\hat{\theta},\theta,\hat{\eta},\eta\right) \quad \forall \theta,\hat{\theta} \in \left[\underline{\theta},\overline{\theta}\right], \eta,\hat{\eta} \in \left\{\underline{\eta},\overline{\eta}\right\}, \tag{1}$$

$$u(\theta,\eta) \geq 0 \quad \forall \theta \in \left[\underline{\theta},\overline{\theta}\right], \eta \in \left\{\underline{\eta},\overline{\eta}\right\}. \tag{2}$$

4.3 Analysis

To solve our problem, we proceed as follows. We begin with a discussion of the first-best and establish a connection between the first-best and the second-best that simplifies our problem dramatically. We then characterize implementable allocations and reformulate our problem in a more tractable way.

[6] Slightly abusing notation, we do not distinguish here notationswise between deterministic variables $x \geq 0$ and $p \geq 0$ and lotteries over these variables. As we shall prove in Lemma 1, only deterministic allocations will be relevant for our analysis.

[7] The seller may want to exclude certain types from participation. However, exclusion in the sense of buyers who prefer the outside option is outcome equivalent to offering the zero trade ($x = 0$, $q = 0$, $p = 0$) to these buyers and hence included in the optimization problem.

4.3.1 First-best Efficiency and the Optimal q-Allocation

When the buyer's preference parameters (θ, η) are common knowledge, the seller can perfectly discriminate between customers and would extract the full surplus by setting prices equal to

$$p(\theta, \eta) = \theta x(\theta, \eta) + \eta q(\theta, \eta)$$

where $x(\theta, \eta)$ and $q(\theta, \eta)$ are chosen efficiently, i.e.

$$C_x(x(\theta, \eta)) - \theta = 0$$

and

$$q(\theta, \eta) = \begin{cases} 0 & \eta < c \\ 1 & \eta \geq c. \end{cases}$$

In particular, $x(\theta, \eta) = x(\theta)$ is independent of η and $q(\theta, \eta) = q(\eta)$ is independent of θ.

In this paper we are interested in the case where production costs play a substantial role. We therefore assume that the efficient allocation involves allocating product q to the high preference type only and impose for the remainder of the paper

Assumption 1.

$$\underline{\eta} < c < \overline{\eta}.$$

The following lemma shows that with relevant production costs in the above sense the optimal mechanism is deterministic. Moreover, the efficient q-allocation determined by Assumption 3 will also be implemented in the optimal mechanism for the constrained problem with asymmetric information.

Lemma 1. *Under Assumption 1, the optimal mechanism is deterministic in x and q and separates η-types in q efficiently, that is $q(\theta, \overline{\eta}) = 1$, $q(\theta, \underline{\eta}) = 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.*

First best efficiency of the allocation in q translates into second best optimality for the following two reasons. Clearly, efficiency is desirable for the seller as it maximizes her profits for any given surplus of a buyer. At the same time the buyer's incentive constraints under the efficient allocation are as relaxed as they can possibly be. Indeed, the excess surplus of a buyer of type $\overline{\eta}$ mimicking type $\underline{\eta}$ compared to a buyer who truly is of type $\underline{\eta}$ is minimized, being equal to 0, while the excess loss of a buyer of type $\underline{\eta}$ mimicking type $\overline{\eta}$ compared to a buyer who truly is of type $\overline{\eta}$ is maximized, being equal to $\overline{\eta} - \underline{\eta}$. Clearly, the resulting allocation is deterministic in q. Since consumer valuations are linear in x and the seller has convex costs, the mechanism is deterministic in x by a standard result (cf. e.g. Fudenberg and Tirole (1991)).

Lemma 1 shows that with relevant production costs all potential distortions will occur in the x-dimension only. This finding has two important consequences. First, it makes the problem sufficiently tractable to derive explicit solutions, contrasting many other multidimensional setups. Secondly, it allows us to directly relate and compare all effects that arise from the presence of a second dimension to the well-known solution of the standard one-dimensional problem.

4.3.2 Implementable Allocations

In this section we bring the incentive constraints (1) into a more tractable form to solve the seller's problem. The key tool is the following lemma which allows us to split the two-dimensional incentive compatibility constraints into two one-dimensional constraints.

Lemma 2. *For any mechanism featuring $q(\theta,\overline{\eta}) = 1$, $q(\theta,\underline{\eta}) = 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$, thus in particular for the optimal mechanism in our maximization problem, the incentive constraint*

$$u(\theta,\eta) \quad \geq \quad U\left(\hat{\theta},\theta,\hat{\eta},\eta\right) \quad \forall \theta,\hat{\theta} \in [\underline{\theta},\overline{\theta}],\ \eta,\hat{\eta} \in \{\underline{\eta},\overline{\eta}\}$$

is equivalent to the pair of one-dimensional incentive constraints

$$u(\theta,\eta) \geq U\left(\hat{\theta},\theta,\eta,\eta\right) \quad \forall \theta,\hat{\theta} \in [\underline{\theta},\overline{\theta}],\ \eta \in \{\underline{\eta},\overline{\eta}\}, \tag{3}$$

$$u(\theta,\eta) \geq U(\theta,\theta,\hat{\eta},\eta) \quad \forall \theta \in [\underline{\theta},\overline{\theta}],\ \eta,\hat{\eta} \in \{\underline{\eta},\overline{\eta}\}. \tag{4}$$

The main insight behind Lemma 2 is that the buyer's incentive how to report his preference parameter θ does not depend on his preference parameter η. To see this, compare a buyer of type (θ,η) who considers reporting $\left(\hat{\theta},\hat{\eta}\right)$ with a buyer of type $(\theta,\hat{\eta})$ who considers the same report. The difference in utilities between type (θ,η) and type $(\theta,\hat{\eta})$ when reporting $\left(\hat{\theta},\hat{\eta}\right)$ is given as $q\left(\hat{\theta},\hat{\eta}\right)(\eta-\hat{\eta})$. But this term does not depend on $\hat{\theta}$ as $q\left(\hat{\theta},\hat{\eta}\right) = q(\hat{\eta})$. In particular, it takes the same value if $\hat{\theta} = \theta$, so it is non-positive by constraint (4). As a consequence, since type $(\theta,\hat{\eta})$ does not have an incentive to misreport as $\left(\hat{\theta},\hat{\eta}\right)$ by (3), neither does type (θ,η).

The optimal quality allocation characterized in Lemma 1 has the properties named in Lemma 2. We can therefore replace the general incentive compatibility constraint (1) in our maximization problem by the two one-dimensional constraints (3), (4). The following Lemma characterizes the implications of these two constraints for our maximization problem.

Lemma 3. *The incentive constraint* (3) *is satisfied if and only if*

$$u(\theta,\eta) = u(\underline{\theta},\eta) + \int_{\underline{\theta}}^{\theta} x(y,\eta)\,dy \tag{5}$$

and $x(\theta,\eta)$ is non-decreasing in θ for all $\eta \in \{\underline{\eta},\overline{\eta}\}$. The incentive constraint (4) *is satisfied if and only if*

$$q(\theta,\underline{\eta})(\overline{\eta}-\underline{\eta}) \leq \overline{u}(\theta) - \underline{u}(\theta) \leq q(\theta,\overline{\eta})(\overline{\eta}-\underline{\eta})$$

for any $\theta \in [\underline{\theta},\overline{\theta}]$.

Lemma 3 is a standard result. For the reader's convenience, we give a full proof in the Appendix. Note that Lemma 3 allows us to write prices as

$$\begin{aligned} p(\theta,\eta) &= \theta x(\theta,\eta)+\eta q(\theta,\eta)-u(\theta,\eta) \\ &= \theta x(\theta,\eta)+\eta q(\theta,\eta)-u(\underline{\theta},\eta)-\int_{\underline{\theta}}^{\theta} x(y,\eta)\,dy \end{aligned}$$

and thus to eliminate them from the optimization problem. Moreover, as an immediate consequence of Lemma 3 and non-negativity of $x \in \mathbb{R}_{\geq 0}$, all participation constraints $u(\theta,\eta) \geq 0$, $(\theta,\eta) \in \left[\underline{\theta},\overline{\theta}\right] \times \left\{\underline{\eta},\overline{\eta}\right\}$, are implied by $\underline{u} = u\left(\underline{\theta},\underline{\eta}\right) \geq 0$ and incentive compatibility.

4.3.3 The Pricing Problem

The results derived in the previous section enable us to state the seller's problem in a tractable way. We write

$$B(x,q,\theta,\eta) = \theta x + \eta q - C(x) - cq - x \cdot \frac{1-F(\theta|\eta)}{f(\theta|\eta)}$$

for the seller's virtual surplus conditional on η and

$$\rho(\theta,\underline{u},\overline{u}) = \overline{u}(\theta) - \underline{u}(\theta) = \overline{u} - \underline{u} + \int_{\underline{\theta}}^{\theta} \left[\overline{x}(y) - \underline{x}(y)\right] dy$$

for the excess rent of a type $(\theta,\overline{\eta})$ over a type $\left(\theta,\underline{\eta}\right)$. Substituting transfers, applying integration by parts and invoking Lemmas 1-3, the seller's optimization problem reads

$$\begin{aligned} \max_{\overline{x}(\cdot),\underline{x}(\cdot),\underline{u},\overline{u}} \quad & \Pi\left(\underline{x},\overline{x},\underline{u},\overline{u}\right) \\ &= \beta \int_{\underline{\theta}}^{\overline{\theta}} B\left(\underline{x}(\theta),0,\theta,\underline{\eta}\right) f\left(\theta|\underline{\eta}\right) d\theta - \beta\underline{u} \qquad (6) \\ &+ (1-\beta) \int_{\underline{\theta}}^{\overline{\theta}} B\left(\overline{x}(\theta),1,\theta,\overline{\eta}\right) f\left(\theta|\overline{\eta}\right) d\theta - (1-\beta)\,\overline{u} \end{aligned}$$

subject to

$$\underline{u} \geq 0 \qquad (7)$$
$$\rho(\theta,\underline{u},\overline{u}) \geq 0 \quad \forall\theta \in \left[\underline{\theta},\overline{\theta}\right] \qquad (8)$$
$$\rho(\theta,\underline{u},\overline{u}) \leq \overline{\eta} - \underline{\eta} \quad \forall\theta \in \left[\underline{\theta},\overline{\theta}\right] \qquad (9)$$
$$\overline{x}(\theta), \underline{x}(\theta) \quad \text{non-decreasing in } \theta \qquad (10)$$
$$\overline{x}(\theta), \underline{x}(\theta) \geq 0 \quad \forall\theta \in \left[\underline{\theta},\overline{\theta}\right]. \qquad (11)$$

We refer to this problem as Problem P. An immediate observation is that setting $\underline{u} = 0$ is optimal, avoiding any lump-sum rents for the low valuation types in the η-dimension.

Indeed, at $\underline{\theta}$ constraint (8) implies $\overline{u} \geq \underline{u}$, so a positive $\underline{u}$ implies a positive $\overline{u}$. Since both constraints (8) and (9) only depend on $\overline{u} - \underline{u}$ and both, $\overline{u}$ and $\underline{u}$, are costly for the seller, choosing $\underline{u}$ as small as possible is optimal. Thus, to simplify notation, we write $\rho(\theta, \underline{u}, \overline{u}) = \rho(\theta, \overline{u})$ henceforth.

In what follows we will approach the above optimization problem with control-theoretic methods. As we demonstrate in the next section, under the additional assumption of affiliated preference types this approach allows us to reduce the problem to a simple one-dimensional optimization task, maximizing the objective $\Pi(\overline{u})$ as a function of the lump-sum rents $\overline{u}$ for high η-types only. The reader who, when reading first, is mainly interested in the rents-vs-bundling trade-off involved in this maximization may skip through the following discussion and jump directly to Section 6.

4.4 Solution: a Control-Theoretic Characterization

To make Problem P more tractable, we first show that the optimal x-schedules are reasonably regular.

Lemma 4. *The schedules $(\underline{x}^*(\theta), \overline{x}^*(\theta))$ that solve Problem P are continuous.*

The main intuition for Lemma 4 is ubiquitous in economics: $B(x, q, \theta, \eta)$ is strictly concave in x and concavity favors smoothing. A formal proof which ensures that smoothing near a putative discontinuity is compatible with constraints (8) and (9) is provided in the Appendix.

To solve optimization problems like Problem P, it is a standard approach to consider a reduced problem with less constraints first and then impose (distributional) assumptions that guarantee the remaining constraints to be satisfied. Problem P without the monotonicity and non-negativity constraints (10) and (11) can be regarded as a control problem with state variables $\underline{u}(\theta)$, $\overline{u}(\theta)$, control variables $\underline{x}(\theta) = \dot{\underline{u}}(\theta)$, $\overline{x}(\theta) = \dot{\overline{u}}(\theta)$ and two inequality constraints that involve the state variables $\underline{u}(\theta)$, $\overline{u}(\theta)$ only. Call this reduced problem P'.

Problem P' is still relatively complex, yet solution techniques are available, e.g. from Seierstad and Sydsaeter (1987). Fixing some value $\overline{u}$ for the moment, Problem P' gives rise to a Hamiltonian

$$\begin{aligned} H &= H(\underline{u}(\theta), \overline{u}(\theta), \underline{x}(\theta), \overline{x}(\theta), \underline{\kappa}(\theta), \overline{\kappa}(\theta), \theta) \\ &= \beta B\left(\underline{x}(\theta), 0, \theta, \underline{\eta}\right) f\left(\theta|\underline{\eta}\right) + (1-\beta) B(\overline{x}(\theta), 1, \theta, \overline{\eta}) f(\theta|\overline{\eta}) \\ &\quad + \underline{\kappa}(\theta) \cdot \underline{x}(\theta) + \overline{\kappa}(\theta) \cdot \overline{x}(\theta). \end{aligned}$$

with two costate variables $\underline{\kappa}(\theta)$, $\overline{\kappa}(\theta)$ as well as an associated Lagrangian

$$L = H + \mu_1(\theta) \cdot (\overline{u}(\theta) - \underline{u}(\theta)) + \mu_2(\theta) \cdot \left(\left(\overline{\eta} - \underline{\eta}\right) - (\overline{u}(\theta) - \underline{u}(\theta))\right)$$

where $\mu_1(\theta)$ and $\mu_2(\theta)$ denote the multipliers associated to constraints (8) and (9).

A frequent issue in control-theoretic problems with constraints on the state variables lies in the fact that often neither costate variables nor control variables need to satisfy standard regularity conditions. However, since we know that the control schedules must be continuous to solve Problem P, we may restrict attention to continuous control schedules (which immediately implies continuous costate schedules) for Problem P' as well.

To proceed towards a solution of the control problem, it is helpful to make ourselves aware of some special features of constraints (8) and (9) as well as the Hamiltonian H. Indeed, note that H does not depend on $\overline{u}(\theta)$ and $\underline{u}(\theta)$ at all while (8) and (9) only depend on $\overline{u}(\theta) - \underline{u}(\theta)$ rather than each state variable individually. As a consequence, the Lagrange equations

$$\begin{aligned} \frac{\partial \underline{\kappa}(\theta)}{\partial \theta} &= -\frac{\partial L}{\partial \underline{u}}, \\ \frac{\partial \overline{\kappa}(\theta)}{\partial \theta} &= -\frac{\partial L}{\partial \overline{u}} \end{aligned}$$

rewrite as

$$\dot{\underline{\kappa}}(\theta) = \mu_1(\theta) - \mu_2(\theta) = -\dot{\overline{\kappa}}(\theta).$$

In the Appendix we show that $\mu_1(\theta)$ and $\mu_2(\theta)$ are sufficiently regular to apply the fundamental theorem of calculus, so together with transversality $\underline{\kappa}\left(\overline{\theta}\right) = 0 = \overline{\kappa}\left(\overline{\theta}\right)$ the above equations imply $\underline{\kappa}(\theta) = -\overline{\kappa}(\theta)$.[10]

The reduction from two costate variables to one is at the core of the following proposition which is based on a result in Seierstad and Sydsaeter (1987).

Proposition 1. *An optimal continuous allocation for the reduced problem P' is characterized by the following pair of equations*

$$\left(-C_x\left(\underline{x}^*(\theta)\right) + \theta - \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}\right) \cdot \beta f\left(\theta|\underline{\eta}\right) + \kappa^*(\theta) = 0 \tag{12}$$

$$\left(-C_x\left(\overline{x}^*(\theta)\right) + \theta - \frac{1 - F\left(\theta|\overline{\eta}\right)}{f\left(\theta|\overline{\eta}\right)}\right) \cdot (1-\beta) f\left(\theta|\overline{\eta}\right) - \kappa^*(\theta) = 0 \tag{13}$$

where $\kappa^*(\theta)$ denotes the optimal costate variable. The optimal costate variable has the following properties:

a) $\kappa^*(\theta)$ is continuous.

b) $\kappa^*(\theta)$ is locally constant around all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ at which neither (8) nor (9) binds.

c) At any $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$ where (8) or (9) binds,

$$\kappa^*(\theta) = \kappa_b(\theta) \equiv (1-\beta) f\left(\theta|\overline{\eta}\right) \cdot \left(\frac{1 - F(\theta)}{f(\theta)} - \frac{1 - F\left(\theta|\overline{\eta}\right)}{f\left(\theta|\overline{\eta}\right)}\right) \tag{14}$$

so that $\underline{x}^*(\theta) = \overline{x}^*(\theta)$.

d) $\kappa^*(\theta)$ is weakly increasing whenever (8) binds and weakly decreasing whenever (9) binds.

e) $\kappa^*\left(\overline{\theta}\right) = 0$.

[10]See the proof of Proposition 1 in the Appendix for the technical details.

Most of the properties of the optimal costate variable $\kappa(\theta)$ in Proposition 1 are either standard in control theory or easy to see.[11] Part a) is a direct consequence of Lemma 4. Part b) is the usual type of result in a control-theoretic problem: the costate variable is the dynamic "integrated" equivalent to a Lagrange multiplier in a static optimization problem. Part c) follows directly from Lemma 3: marginal utilities must be equal for both η-types whenever (8) or (9) bind at an interior point as otherwise the constraints would be violated "slightly to the left" or "slightly to the right". Part e) is the standard transversality condition for control problems with free endpoints, implying the classical "no-distortion-at-the-top".

Part d), finally, constitutes a powerful tool to determine the areas of binding constraints.[13] Indeed, whenever (8) or (9) bind at an interior value θ we must have $\underline{x}^*(\theta) = \overline{x}^*(\theta)$ according to Part c). The corresponding $\kappa(\theta)$-schedule easily computes from equations (12) and (13) as $\kappa_b(\theta)$ where the subscript "b" stands for "bunching" of x in η. For any interval where $\kappa_b(\theta)$ is monotonic, Part d) of Proposition 1 leaves only one of the two constraints (8) and (9) as potentially binding. Moreover, neither of the two constraints can bind around a local extremum of $\kappa_b(\theta)$. Note that $\kappa_b\left(\overline{\theta}\right) = 0$, so $\kappa_b(\theta)$ is compatible with transversality.

We find the following heuristic argument useful to understand the result. From the seller's point of view, the costate variable $\kappa(\theta)$ (or, more precisely, its absolute value) measures the additional distortion of the optimal x-allocation compared to the one-dimensional case with known η-types. This distortion is caused by the buyer's option to misreport his η-type, captured by constraints (8) and (9). The seller clearly prefers $\kappa(\theta)$ to be equal to zero (and $\overline{u} = 0$). In this case, both x-schedules maximize the values of the integrals in (6) pointwise and the schedules coincide with the solution of the two one-dimensional allocation problems in x conditional on $\eta = \underline{\eta}$ and $\eta = \overline{\eta}$, respectively. We therefore refer to this scenario as the case of (quasi-)observable η. It constitutes an upper bound on what is potentially achievable for the seller: she fully exploits the information over the θ-type contained in the η-type at no costs. Yet, the solution for observable η may not be incentive compatible as the resulting x- and u-schedules may violate constraints (8) or (9).

At the other extreme, consider the x-schedule that constitutes the solution to the one-dimensional problem unconditional on η defined by

$$-C_x(x(\theta)) + \theta - \frac{1 - F(\theta)}{f(\theta)} = 0.$$

Setting $\underline{x}(\theta) = \overline{x}(\theta) = x(\theta)$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ corresponds to setting $\kappa(\theta) = \kappa_b(\theta)$ everywhere. In this scenario, (8) and (9) are automatically satisfied as the utility of high and low η-types coincides for all θ and we may certainly set $\overline{u} = 0$. However, this

[11] We omit stars for κ- and x-schedules as well as values of $\overline{u}$ whenever we discuss potential candidates for the optimal mechanism. The relation between κ- and x-schedules is nonetheless assumed to be defined through equations (12) and (13).

[13] Its proof goes back to Neustadt (1976).

candidate solution comes at a cost: the seller does not exploit the information about θ that is contained in η at all. It thereby constitutes a lower bound on what is achievable for the seller.

We illustrate the previous discussion in Figure 1 and Figure 2. Both figures show a pair of x-schedules that corresponds to the case of observable η. In Figure 1, the schedules are clearly infeasible for our problem if $\overline{u} = 0$. Indeed, on $[\underline{\theta}, \theta_1]$ the x-schedule of the low η-type lies above the x-schedule of the high η-type, thereby violating (8) as

$$\rho\left(\theta, \overline{u} = 0\right) = \int_{\underline{\theta}}^{\theta} \left[\overline{x}\left(y\right) - \underline{x}\left(y\right)\right] dy < 0,$$

for any $\theta \in (\underline{\theta}, \theta_1)$. Hence some distortion through $\kappa\left(\theta\right)$ or some positive lump-sum rent $\overline{u}$ is necessary. On the other hand, if the roles of both schedules are reversed as depicted in Figure 2, constraint 8 is clearly satisfied everywhere as

$$A = \int_{\underline{\theta}}^{\theta_1} \left[\overline{x}\left(\theta\right) - \underline{x}\left(\theta\right)\right] d\theta > 0$$

and

$$A = \int_{\underline{\theta}}^{\theta_1} \left[\overline{x}\left(\theta\right) - \underline{x}\left(\theta\right)\right] d\theta > \int_{\theta_1}^{\overline{\theta}} \left[\underline{x}\left(\theta\right) - \overline{x}\left(\theta\right)\right] d\theta = B.$$

The allocation corresponding to the observable η-case is hence feasible if and only if constraint (9) holds everywhere, i.e. if

$$A = \rho\left(\theta_1, \overline{u} = 0\right) \leq \overline{\eta} - \underline{\eta}.$$

The level and shape of the optimal x-allocation depends crucially on the joint distribution of types. In the next section, we impose more structure in that respect.

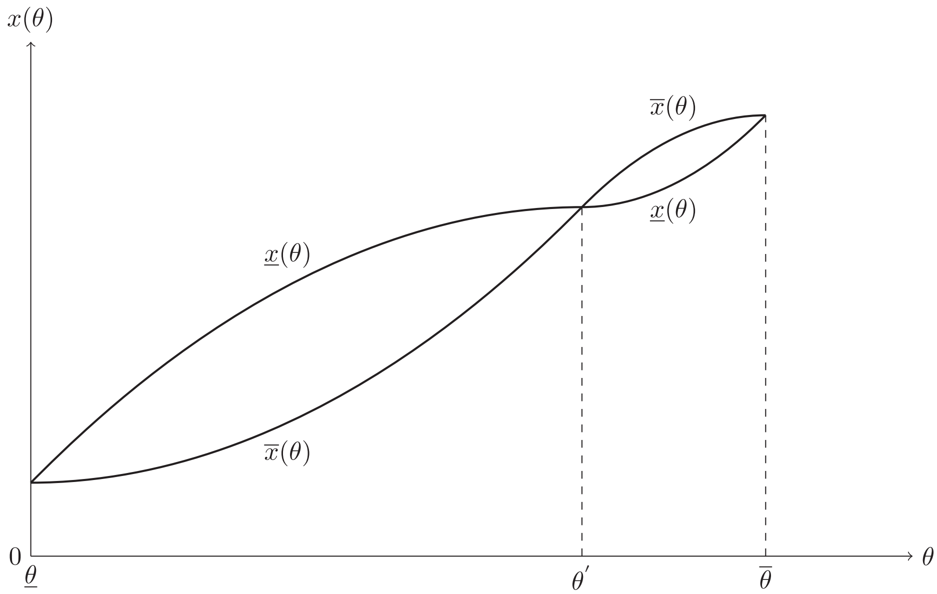

Figure 1: x-schedules for the observable-η-case are infeasible.

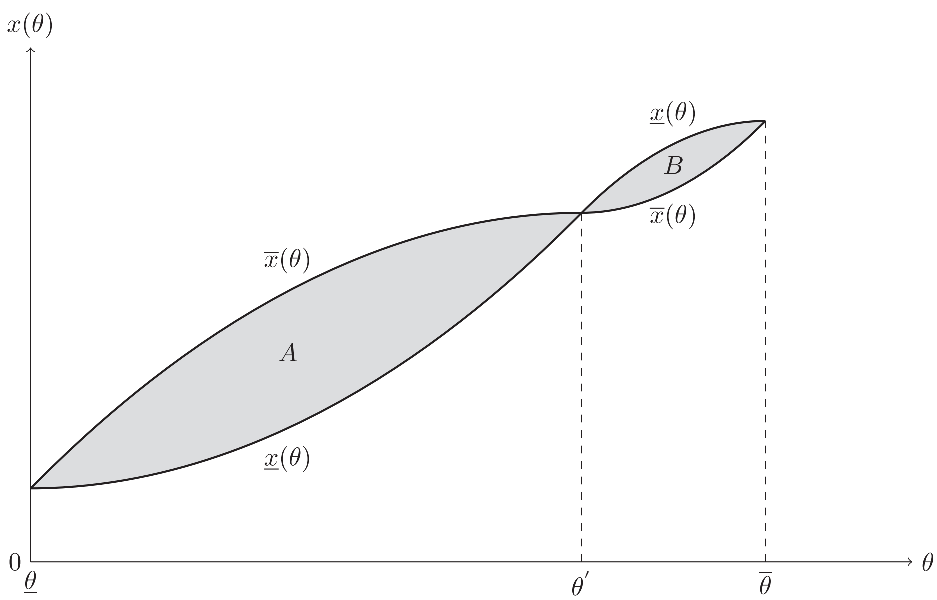

Figure 2: x-schedules for the observable-η-case are feasible iff $A \leq \overline{\eta} - \underline{\eta}$.

4.5 A Characterization for Affiliated Types

In this section we analyze the implications of Proposition 1 for the solution of Problem P' under the assumption that preference types are affiliated. We fully characterize the solution up to the choice of $\overline{u}$ which is studied in Section 6.

We consider continuously differentiable densities with full support. Hence affiliation is equivalent to

Assumption 2.

$$\frac{\partial}{\partial\theta}\left[\frac{f\left(\theta|\overline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}\right] > 0 \quad \forall\theta \in \left[\underline{\theta}, \overline{\theta}\right].$$

With affiliated types, it is easy to see that the seller cannot implement the schedules corresponding to observable η characterized by $\kappa\left(\theta\right) = 0$ and $\overline{u} = 0$. Affiliation implies the reversed hazard rate order

$$\frac{1 - F\left(\theta|\overline{\eta}\right)}{f\left(\theta|\overline{\eta}\right)} > \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}$$

for any $\theta < \overline{\theta}$, cf. Shaked and Shanthikumar (2007). Setting $\kappa\left(\theta\right) = 0$ for all θ then implies that $\underline{x}\left(\theta\right) > \overline{x}\left(\theta\right)$ for any $\theta < \overline{\theta}$ and hence, together with $\overline{u} = 0$, a violation of the constraint $\rho\left(\theta, \overline{u}\right) \geq 0$ for any $\theta > \underline{\theta}$. The seller therefore faces a trade-off. To relax the constraint $\rho\left(\theta, \overline{u}\right) \geq 0$ she could either leave higher rents to the high η-types by increasing $\overline{u}$ or she could distort the optimal schedules away from the observable-η-case by choosing $\kappa\left(\theta\right)$ different from zero.

Affiliation allows us to pin down the optimal bunching region for x in η as a function of $\overline{u}$.

Proposition 2. *For given $\overline{u} \in \left[0, \overline{\eta} - \underline{\eta}\right]$, under affiliation there exists $\theta^{'} \in \left[\underline{\theta}, \overline{\theta}\right]$ such that for all $\theta \geq \theta^{'}$ the optimal schedules satisfy $\overline{x}^{*}\left(\theta\right) = \underline{x}^{*}\left(\theta\right) = x^{*}\left(\theta\right)$ where $x^{*}\left(\theta\right)$ is defined by*

$$-C_x^1\left(x^{*}\left(\theta\right)\right) + \theta - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} = 0, \tag{15}$$

implying

$$\kappa^{*}\left(\theta\right) = \kappa_b\left(\theta\right)\ \forall\theta \geq \theta^{'}.$$

For all $\theta < \theta^{'}$ the optimal schedules $\underline{x}^{*}\left(\theta\right)$ and $\overline{x}^{*}\left(\theta\right)$ are defined by

$$\left(-C_x^1\left(\underline{x}^{*}\left(\theta\right)\right) + \theta - \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}\right) \cdot \beta f\left(\theta|\underline{\eta}\right) + \kappa^{*} = 0 \tag{16}$$

$$\left(-C_x^1\left(\overline{x}^{*}\left(\theta\right)\right) + \theta - \frac{1 - F\left(\theta|\overline{\eta}\right)}{f\left(\theta|\overline{\eta}\right)}\right) \cdot \left(1 - \beta\right) f\left(\theta|\overline{\eta}\right) - \kappa^{*} = 0 \tag{17}$$

for some constant $\kappa^* = \kappa^*(\overline{u})$ *such that*

$$\overline{u} + \int_{\underline{\theta}}^{\theta'} \left[\overline{x}^*(\theta, \kappa^*(\overline{u})) - \underline{x}^*(\theta, \kappa^*(\overline{u}))\right] d\theta = 0, \tag{18}$$

and

$$\kappa^* = \kappa_b\left(\theta'\right) \tag{19}$$

by continuity of $\kappa^*(\theta)$.

Proposition 2 states that there is a single point θ' at which constraint (8) switches from slack to binding and the x-schedules switch from separation in η to bunching in η. Heuristically, the argument is very simple. Consider any point $\tilde{\theta}$ at which (8) is binding, implying that $\overline{x}^*\left(\tilde{\theta}\right) = \underline{x}^*\left(\tilde{\theta}\right)$ at that point. Such a point must exist, since otherwise the seller could increase profits by reducing $\overline{u}$. Then, (8) is satisfied slightly to the right of $\tilde{\theta}$, say for $\tilde{\theta} + \varepsilon$ for ε small but positive if and only if $\overline{x}^*(\theta) \geq \underline{x}^*(\theta)$ for $\theta = \tilde{\theta} + \varepsilon$. However, this requires that $\kappa^*(\theta)$ increases at that point. To see this, suppose κ were constant around $\tilde{\theta}$. Then, totally differentiating (16) and (17) with respect to θ reveals that $\underline{x}^*$ increases faster in θ than $\overline{x}^*$ at points $\tilde{\theta}$ where $\overline{x}^*\left(\tilde{\theta}\right) = \underline{x}^*\left(\tilde{\theta}\right)$ if and only if types are locally affiliated around $\tilde{\theta}$.

We now relate this heuristic argument more formally to our previous analysis. Taking derivatives of (14) yields

$$\text{sign}\frac{\partial \kappa_b(\theta)}{\partial \theta} = \text{sign}\frac{\partial}{\partial \theta}\left[\frac{f(\theta|\overline{\eta})}{f(\theta|\underline{\eta})}\right] \quad \forall \theta < \overline{\theta}, \tag{20}$$

so together with $\kappa_b\left(\overline{\theta}\right) = 0$ this shows that under affiliation $\kappa_b(\theta)$ is strictly increasing and non-positive everywhere as depicted in Figure 3.

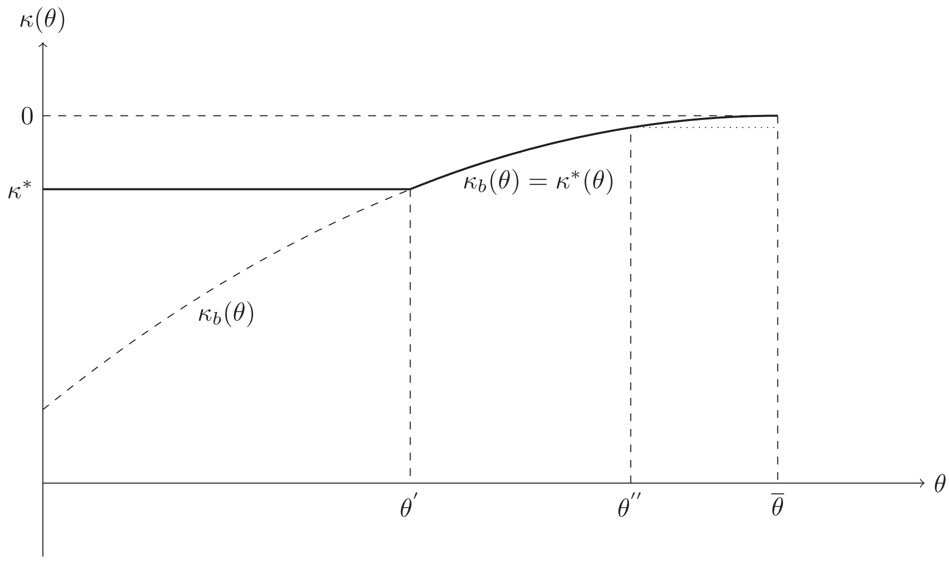

Figure 3: Following the schedule $\kappa_b(\theta)$ to the right of $\theta^{'}$ minimizes distortions.

Leaving the schedule $\kappa_b(\theta)$ in favor of a constant κ-schedule at some $\theta^{''} > \theta^{'}$ as indicated in Figure 3 through the dotted line is clearly suboptimal then. Informally speaking, it deliberately increases distortions through $\kappa(\theta)$ on the interval $\left[\theta^{''}, \overline{\theta}\right]$ relative to following the schedule $\kappa_b(\theta)$. Formally, this is reflected in a violation of the transversality condition $\kappa^*(\theta) = 0$ as stated in Proposition 1, Part e). For all technical details, we refer to the appendix.

4.6 A No-Bundling Result

Proposition 2 boils the complex control problem P' down to a one-dimensional optimization problem in the choice parameter $\overline{u}$. Reformulating Proposition 2 in that spirit, we get

Proposition 2*. *Under affiliation, Problem P' is equivalent to the following Problem*

P". Maximize

$$
\begin{aligned}
\Pi(\overline{u}) = \beta & \int_{\underline{\theta}}^{\theta'(\overline{u})} B\left(\underline{x}^*(\theta, \kappa^*(\overline{u})), 0, \theta, \underline{\eta}\right) f\left(\theta|\underline{\eta}\right) d\theta \\
& + (1-\beta) \int_{\underline{\theta}}^{\theta'(\overline{u})} B\left(\overline{x}^*(\theta, \kappa^*(\overline{u})), 1, \theta, \overline{\eta}\right) f\left(\theta|\overline{\eta}\right) d\theta \\
& + \beta \int_{\theta'(\overline{u})}^{\overline{\theta}} B\left(x^*(\theta), 0, \theta, \underline{\eta}\right) f\left(\theta|\underline{\eta}\right) d\theta \qquad (21) \\
& + (1-\beta) \int_{\theta'(\overline{u})}^{\overline{\theta}} B\left(x^*(\theta), 1, \theta, \overline{\eta}\right) f\left(\theta|\overline{\eta}\right) d\theta \\
& - (1-\beta)\,\overline{u}.
\end{aligned}
$$

subject to

$$\overline{u} \in \left[0, \overline{\eta} - \underline{\eta}\right]$$

where $\underline{x}^*(\theta)$, $\overline{x}^*(\theta)$, $x^*(\theta)$, $\kappa^*(\overline{u})$, θ' *are defined by equations* (15)-(19).

The trade-off underlying the optimal choice of $\overline{u} \in \left[0, \overline{\eta} - \underline{\eta}\right]$ is depicted in Figure 4.

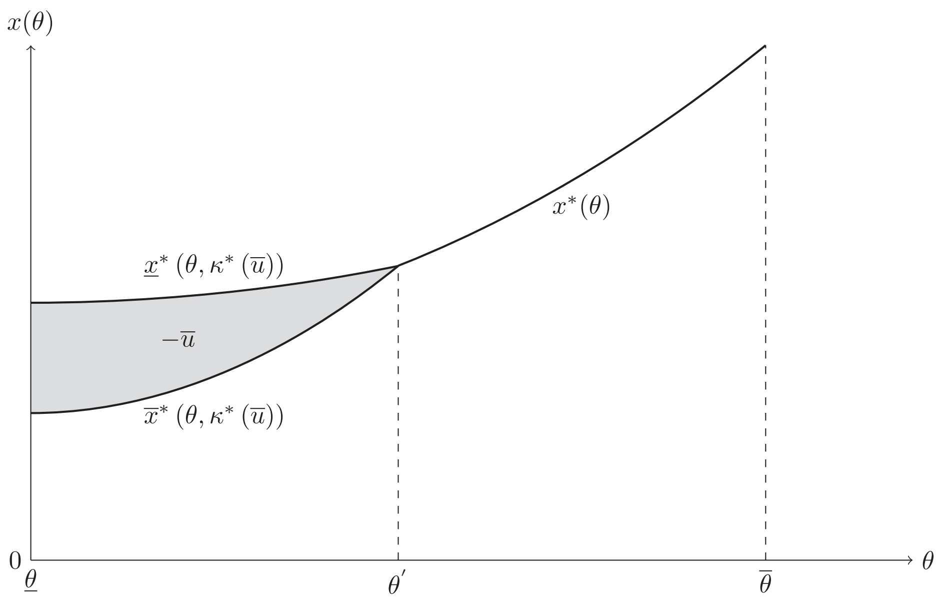

Figure 4: Raising $\overline{u}$ pushes θ' to the right and enables the seller to separate a larger portion of types.

Leaving higher rents $\overline{u}$ to the high η-types comes at a twofold gain. By moving κ^* upwards and hence closer towards zero, distortions of the x-schedules relative to the case of observable η in the separation region are reduced. By moving θ' to the right the separation region itself is enlarged. The first-order effect through marginally shifting θ', however, is zero by continuity of the optimal schedules at $\theta^{'}$ together with an envelope argument and hence negligible.

By (18), the increase of κ^* in $\overline{u}$ is measured by

$$\frac{d\kappa^*}{d\overline{u}}(\overline{u}) = -\frac{1}{\int_{\underline{\theta}}^{\theta^{'}}\left[\frac{\partial \overline{x}}{\partial \kappa^*}(\theta) - \frac{\partial \underline{x}}{\partial \kappa^*}(\theta)\right] d\theta} > 0. \tag{22}$$

Increasing κ^* shifts the x-schedules closer towards the case of observable η-types on $\left[\underline{\theta}, \theta^{'}\right]$ and reduces the excess rents of high η-types given by $\int_{\underline{\theta}}^{\theta^{'}}\left[\overline{x}^*(\theta) - \underline{x}^*(\theta)\right] d\theta$. Using the above formula (22) as well as equations (12) and (13), we get

$$\frac{d\Pi}{d\overline{u}}(\overline{u}) \quad = \quad -\kappa^*(\overline{u}) - (1-\beta),$$

hence these conducive effects are measured precisely by $|\kappa^*(\overline{u})| = -\kappa^*(\overline{u})$. Moreover, we have just argued that

$$\frac{d^2\Pi}{d\overline{u}^2}(\overline{u}) = -\frac{d\kappa^*}{d\overline{u}}(\overline{u}) < 0,$$

so our problem is concave in $\overline{u}$. Therefore, setting $\overline{u}^* > 0$ is optimal if and only if increasing $\overline{u}$ away from zero is optimal. However, (14) implies that

$$-\kappa^*(\overline{u} = 0) = -\kappa_b(\underline{\theta}) = (1-\beta) \cdot \left(1 - \frac{f(\underline{\theta}|\overline{\eta})}{f(\underline{\theta})}\right) < 1 - \beta.$$

Therefore the gain from moving the schedules closer to the observable-η-case can never compensate for the direct loss from leaving higher rents to high η-types through $\overline{u}$ measured by the share size $1-\beta$ of high η-types. Hence $\overline{u}^* = 0$ solves Problem P".

To ensure that the solution to Problem P' and P" is monotonic and non-negative, it suffices to impose standard assumptions on the inverse hazard rate:

Assumption 3. *The distribution $F = F(\theta)$ features strictly positive virtual valuations $\theta - \frac{1-F(\theta)}{f(\theta)}$ with strictly positive derivative $\frac{\partial}{\partial\theta}\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right] > 0$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$.*[17]

Our main result is now a direct consequence of the previous analysis.

Theorem 1. *Under Assumptions 1-3, the optimal mechanism for the seller involves no lump-sum rents to high η-types, i.e. $\overline{u}^* = 0$. The optimal allocations are given as*

$$q^*(\theta, \eta) = q^*(\eta) = \begin{cases} 0 & \eta = \underline{\eta} \\ 1 & \eta = \overline{\eta} \end{cases}$$

and $x^*(\theta, \eta) = x^*(\theta)$, where $x^*(\theta)$ solves

$$-C_x^1(x^*(\theta)) + \theta - \frac{1-F(\theta)}{f(\theta)} = 0.$$

[17] For Theorem 1, the slightly weaker condition of a non-negative and non-decreasing virtual valuation is sufficient. In the next section, however, it will be helpful to impose slightly stricter conditions as stated in Assumption 3.

Prices are given as

$$p^*(\theta,\eta) = \theta x^*(\theta) + \eta q^*(\eta) - \int_{\underline{\theta}}^{\theta} x^*(y)\,dy.$$

Conditioning prices on allocations rather than buyers' types by setting $p^*(x,q) = p^*(x(\theta), q(\eta))$, *optimal prices split into two additively separable price components*

$$p_1^*(x) + p_2^*(q)$$

for the two goods. No bundling occurs.

The seller is not willing to leave rents to high η-types in order to buy the ability to condition x-allocations on η under affiliation. Rather, the optimal mechanism involves complete bunching of the x-schedules with respect to η. Just as the optimal q-allocation only depends on η by Lemma 1, the optimal x-allocation only depends on θ. The quantity schedule $x^*(\theta)$ has the familiar features. There is no distortion for buyers with valuation $\overline{\theta}$, there is a downward distortion for all types with valuation below $\overline{\theta}$, and there is no (lump-sum) rent at $\underline{\theta}$ for either η-type. The schedule coincides with the solution for the one-dimensional problem unconditional on η.

The reformulation at the end of Theorem 1 is a direct implication of the taxation principle (see e.g. Rochet (1985)). It allows us to rewrite prices p^* as conditional on allocations rather than preference types. Optimal prices are given as the buyer's valuation for quantity x plus the buyer's valuation for good q minus rents of the buyer. The rents of the buyer, however, do not depend on his η-type but only on his θ-type given that the excess rents $\rho^*(\theta, \overline{u} = 0)$ of high η-types over low η-types are equal to zero for all θ. Hence, rents of the buyer do not depend on his q-allocation but only on his x-allocation and

$$p^*(x, q = 1) - p^*(x, q = 0) = \overline{\eta}$$

for any quantity x. Optimal prices can therefore be split into two additively separable components

$$p^*(x,q) = p_1^*(x) + p_2^*(q)$$

where p_1^* denotes the price for good one and p_2^* denotes the price for good two, the latter being equal to zero for $q = 0$ by normalization and equal to $\overline{\eta}$ for $q = 1$.

4.7 Beyond Affiliation

The previous two sections have been devoted to the analysis of the optimal pricing scheme for affiliated preference types, showing that no bundling is optimal. In this final section, we show the converse result: whenever types are not (weakly) affiliated, there exists an interval of positive mass where in the optimum x-schedules are separated in η and hence bundling occurs.

Building on our previous analysis we directly state our result.

Theorem 2. *Under Assumptions 1 and 3, the solution of Problem P features no bundling if and only if types are weakly affiliated.*

The "if"-part of Theorem 2 has been covered in the two previous sections, noting that all proofs go through for weak affiliation as well. Showing the opposite direction consists of two steps.

First, we argue that the solution of the reduced problem P' features bundling on an interval of positive mass whenever types are not weakly affiliated. To see this, note that no bundling implies $\kappa^*(\theta) = \kappa_b(\theta)$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ as otherwise, by continuity of $\kappa^*(\theta)$ and Proposition 1c), there will be an interval of positive length where $\underline{x}^*(\theta) \neq \overline{x}^*(\theta)$. Clearly, in an optimal mechanism (8) must bind for at least one $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ as otherwise $\overline{u} > 0$ can be reduced without violating any constraints. However, together with $\kappa^*(\theta) = \kappa_b(\theta)$ and $\underline{x}^*(\theta) = \overline{x}^*(\theta)$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ this implies that (8) must bind everywhere. But according to Proposition 1, Part d) this is only possible if $\kappa_b(\theta)$ is weakly increasing everywhere which cannot be the case unless types are weakly affiliated due to equation (20).

If the solution of Problem P' is feasible for Problem P, we are done. However, Assumption 3 in general will not guarantee that the solution schedules of Problem P' are non-negative and monotonic when $\kappa^*(\theta) \neq \kappa_b(\theta)$ on some interval. So suppose the solution to Problem P' is not feasible for Problem P. Note that Assumption 3 guarantees the no bundling schedule $x^*(\theta)$ being strictly positive with strictly positive first derivative everywhere. In other words, the no-bundling schedule $\underline{x}(\theta) = \overline{x}(\theta) = x^*(\theta)$ is bounded away from the boundaries of the convex set of implementable allocations that are defined by monotonicity and feasibility constraints (11) and (10). This allows us to form a non-trivial convex combination of the no-bundling schedules $\underline{x}(\theta) = \overline{x}(\theta) = x^*(\theta)$ and the solution to the reduced Problem P' that is feasible for Problem P and, due to concavity of the objective in x, strictly improves upon the no-bundling schedules.[19]

We qualitatively illustrate the reasoning of the previous paragraphs in Figure 5 and Figure 6 for the case of negatively affiliated preference types where

$$\frac{\partial}{\partial \theta}\left[\frac{f(\theta|\overline{\eta})}{f(\theta|\underline{\eta})}\right] < 0 \quad \forall \theta \in \left[\underline{\theta}, \overline{\theta}\right].$$

The solution for Problem P' is easy to derive from Proposition 1. Negatively affiliated preference types imply

$$\frac{1 - F(\theta|\overline{\eta})}{f(\theta|\overline{\eta})} < \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}$$

for all $\theta < \overline{\theta}$, so setting $\kappa^*(\theta) = 0$ would result in $\overline{x}(\theta) \geq \underline{x}(\theta)$ for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ with equality only at $\overline{\theta}$. The relevant constraint hence is given by (9). As a consequence, the seller clearly has no incentive to leave lump-sum rents to high η-types; setting $\overline{u} = 0$ maximally relaxes (9) and simultaneously maximizes her profits. The seller hence separates x-schedules in η as long as this is possible without violating constraint (9) for

[19] We refer to the Appendix for all technical details.

$\overline{u} = 0$, that is, up to some point $\theta' > \underline{\theta}$. Correspondingly, the schedule $\kappa_b(\theta)$ is positive and decreasing everywhere and the optimal costate schedule $\kappa^*(\theta)$ for Problem P' is shaped as in Figure 5, giving rise to schedules $\overline{x}^*(\theta)$ and $\underline{x}^*(\theta)$ such that

$$\int_{\underline{\theta}}^{\theta'} \left[\overline{x}^*(\theta) - \underline{x}^*(\theta)\right] d\theta = \overline{\eta} - \underline{\eta}.$$

This area corresponds to the grey-shaded area in Figure 6. However, the schedules $\overline{x}^*(\theta)$ and $\underline{x}^*(\theta)$ on $\left[\underline{\theta}, \theta'\right]$ do not necessarily constitute a feasible solution for Problem P, even though the bunching schedule $x^*(\theta)$ is positive and increasing everywhere. Indeed, for sufficiently small θ in Figure 6 the schedule $\overline{x}^*(\theta)$ is decreasing and the schedule $\underline{x}^*(\theta)$ becomes negative.[21] Yet, both these violations of constraints (10) or (11) can be resolved by forming a convex combination of $\overline{x}^*(\theta)$ or $\underline{x}^*(\theta)$, respectively, and the bunching schedule $x^*(\theta)$ as indicated by the dashed graphs in Figure 6 which still improves upon the full bunching schedule $x^*(\theta)$.

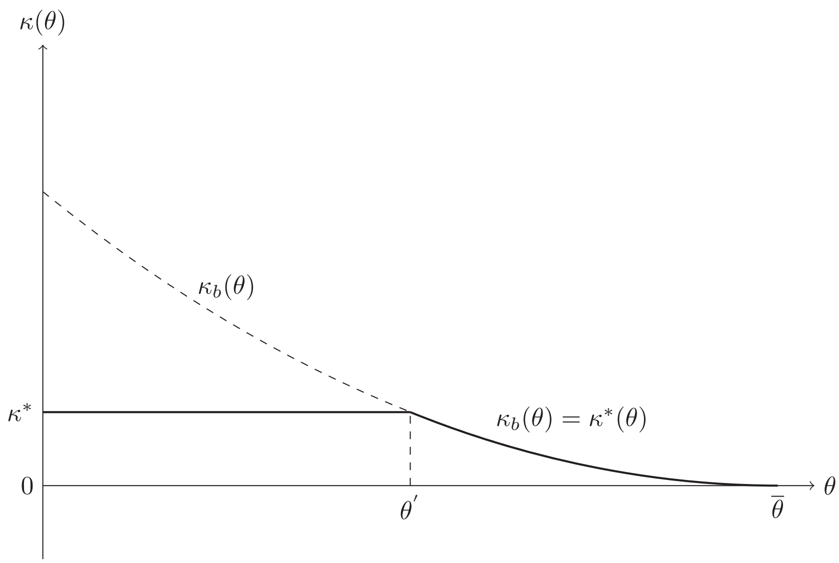

Figure 5: Optimal costate schedule for Problem P' if types are anti-affiliated.

[21] Schedules for $\overline{x}^*(\theta)$ and $\underline{x}^*(\theta)$ on $\left[\underline{\theta}, \theta'\right]$ are defined via (16) and (17). For positive κ^*, negative values of $\underline{x}^*(\theta)$ may occur if the optimal x-schedule for known $\eta = \underline{\eta}$ features exclusion of low θ-types while a (locally) decreasing schedule $\overline{x}^*(\theta)$ may occur if $f(\theta|\overline{\eta})$ is decreasing. Both phenomena are compatible with negative affiliation and Assumption 3.

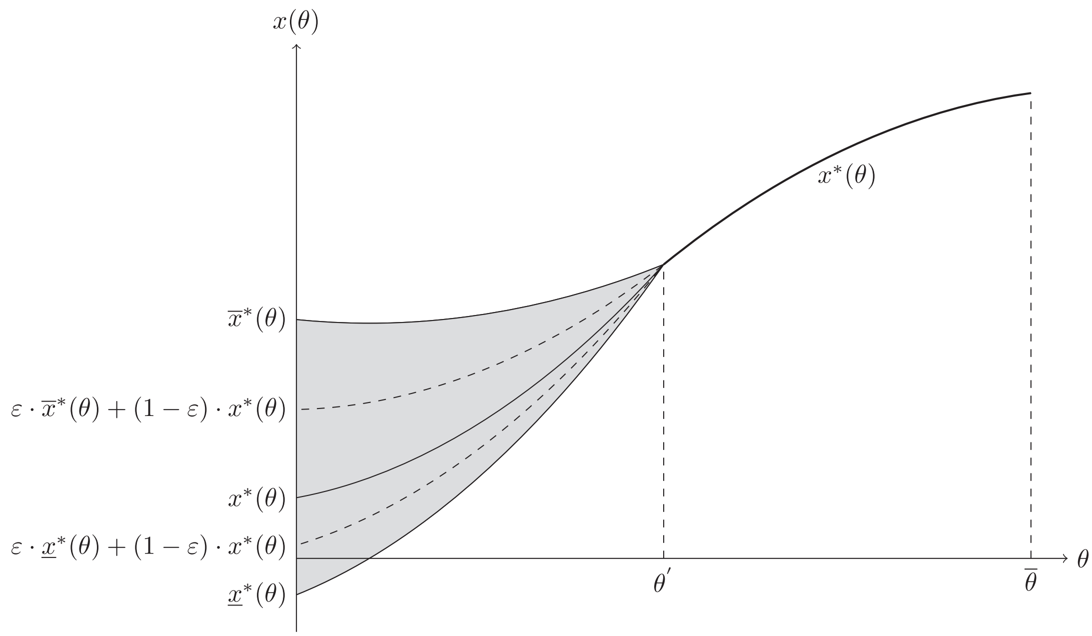

Figure 6: Convex combinations of schedules $\overline{x}^*(\theta)$ and $x^*(\theta)$ resp. $\underline{x}^*(\theta)$ and $x^*(\theta)$ render the solution feasible, that is non-negative and monotonic.

4.8 Conclusions

Adams and Yellen (1976) have shown that bundling increases profits if consumers' multi-dimensional tastes are negatively correlated. We study a related question in the context of a richer but still manageable allocation problem. We show that no bundling occurs if types are affiliated and conversely that some bundling does occur if they are not.

4.9 Appendix

Proof of Lemma 1:

For any report (θ, η), a feasible stochastic mechanism is characterized by a distribution $H(\theta, \eta)(x, q, p)$ over allocations and transfers such that expectations $\mathbb{E}_{H(\theta,\eta)}[\cdot]$ over x, q, p exist. Clearly, due to quasilinear utilities, seller and buyer are indifferent between any lottery over prices and the expected value of this lottery, so we may assume that prices are deterministic. Start with an arbitrary Bayesian incentive compatible mechanism H with deterministic transfers and change the q-allocation to first best, i.e. $q(\theta, \overline{\eta}) = 1$, $q(\theta, \underline{\eta}) = 0$, while adjusting prices such that expected equilibrium profits of the buyer remain constant. For any θ at which the original mechanism featured $q(\theta,\overline{\eta})<1$ this increases revenues from type $(\theta,\overline{\eta})$ by $(1-q(\theta,\overline{\eta}))\cdot[\overline{\eta}-c]>0$ while for any θ at which the original mechanism featured $q\left(\theta,\underline{\eta}\right)>0$ this increases revenues from type $\left(\theta,\underline{\eta}\right)$ by $q\left(\theta,\underline{\eta}\right)\cdot\left[c-\underline{\eta}\right]>0$. We need to show that the new mechanism is still incentive compatible. In the η-dimension, Bayesian incentive compatibility constraints of the original mechanism read

$$\begin{aligned} u^{old}(\theta,\overline{\eta}) &\geq U^{old}\left(\theta,\theta,\underline{\eta},\overline{\eta}\right)=u^{old}\left(\theta,\underline{\eta}\right)+q\left(\theta,\underline{\eta}\right)\cdot\left(\overline{\eta}-\underline{\eta}\right) \\ u^{old}\left(\theta,\underline{\eta}\right) &\geq U^{old}\left(\theta,\theta,\overline{\eta},\underline{\eta}\right)=u^{old}(\theta,\overline{\eta})-q(\theta,\overline{\eta})\cdot\left(\overline{\eta}-\underline{\eta}\right) \end{aligned}$$

at any $\theta\in\Theta$. They certainly imply the IC-constraints for the new mechanism given as

$$\begin{aligned} u^{new}(\theta,\overline{\eta}) &\geq U^{new}\left(\theta,\theta,\underline{\eta},\overline{\eta}\right)=u^{new}\left(\theta,\underline{\eta}\right), \\ u^{new}\left(\theta,\underline{\eta}\right) &\geq U^{new}\left(\theta,\theta,\overline{\eta},\underline{\eta}\right)=u^{new}(\theta,\overline{\eta})-\left(\overline{\eta}-\underline{\eta}\right). \end{aligned}$$

since the RHS is smaller for the new mechanism while the LHS hasn't changed. Bayesian incentive compatibility then follows from

$$\begin{aligned} U^{new}\left(\hat{\theta},\theta,\hat{\eta},\eta\right) &= U^{new}\left(\hat{\theta},\hat{\theta},\hat{\eta},\eta\right)+\left(\theta-\hat{\theta}\right)\cdot\mathbb{E}^{new}_{H\left(\hat{\theta},\hat{\eta}\right)}[x] \\ &= U^{new}\left(\hat{\theta},\hat{\theta},\hat{\eta},\eta\right)+\left(\theta-\hat{\theta}\right)\cdot\mathbb{E}^{old}_{H\left(\hat{\theta},\hat{\eta}\right)}[x] \\ &\leq U^{old}\left(\hat{\theta},\hat{\theta},\hat{\eta},\eta\right)+\left(\theta-\hat{\theta}\right)\cdot\mathbb{E}^{old}_{H\left(\hat{\theta},\hat{\eta}\right)}[x] \\ &= U^{old}\left(\hat{\theta},\theta,\hat{\eta},\eta\right) \\ &\leq u^{old}(\theta,\eta) \\ &= u^{new}(\theta,\eta). \end{aligned}$$

Here we use that neither buyer's utilities nor x-allocations were changed (2^{nd} and 6^{th} line), we use incentive compatibility of the original mechanism (5^{th} line) and we use that $U^{new}\left(\hat{\theta},\hat{\theta},\hat{\eta},\eta\right)\leq U^{old}\left(\hat{\theta},\hat{\theta},\hat{\eta},\eta\right)$ as shown above (3^{rd} line).

As a consequence, the optimal mechanism is non-stochastic in q. To show that the optimal mechanism is non-stochastic in x, note that for any report profile $\left(\hat{\theta},\hat{\eta}\right)$ and type profile (θ,η) expected buyers' utilities

$$U\left(\hat{\theta},\theta,\hat{\eta},\eta\right) = \theta\cdot\mathbb{E}_{H\left(\hat{\theta},\hat{\eta}\right)}[x]+\eta\cdot\mathbb{E}_{H\left(\hat{\theta},\hat{\eta}\right)}[q]-p(\theta,\eta)$$

only depend on expected values of x (and q). Hence assigning $x\left(\hat{\theta},\hat{\eta}\right)=\mathbb{E}_{H\left(\hat{\theta},\hat{\eta}\right)}[x]$ with probability 1 does not alter the incentive problem of the buyer but increases expected equilibrium profits of the firm from type $\left(\hat{\theta},\hat{\eta}\right)$ due to Jensen's inequality by

$$\mathbb{E}_{H\left(\hat{\theta},\hat{\eta}\right)}[C(x)]-C\left(\mathbb{E}_{H\left(\hat{\theta},\hat{\eta}\right)}[x]\right)\geq 0.$$

Proof of Lemma 2:

As $q(\theta,\eta) = q(\eta)$ is independent of θ for any $\eta \in \{\underline{\eta}, \overline{\eta}\}$, the one-dimensional constraints imply

$$\begin{aligned} U\left(\hat{\theta},\theta,\hat{\eta},\eta\right) &= U\left(\hat{\theta},\theta,\hat{\eta},\hat{\eta}\right) + q\left(\hat{\theta},\hat{\eta}\right)\cdot(\eta-\hat{\eta}) \\ &\leq u(\theta,\hat{\eta}) + q(\theta,\hat{\eta})\cdot(\eta-\hat{\eta}) \\ &= U(\theta,\theta,\hat{\eta},\eta) \\ &\leq u(\theta,\eta) \end{aligned}$$

for all $\theta,\hat{\theta} \in \left[\underline{\theta},\overline{\theta}\right]$, $\eta,\hat{\eta} \in \{\underline{\eta},\overline{\eta}\}$.

Proof of Lemma 3:

Monotonicity of x in θ is necessary as

$$\begin{aligned} u(\theta,\eta) &\geq U\left(\hat{\theta},\theta,\eta,\eta\right) = u\left(\hat{\theta},\eta\right) + x\left(\hat{\theta},\eta\right)\left(\theta-\hat{\theta}\right), \\ u\left(\hat{\theta},\eta\right) &\geq U\left(\theta,\hat{\theta},\eta,\eta\right) = u(\theta,\eta) - x(\theta,\eta)\left(\theta-\hat{\theta}\right) \end{aligned}$$

imply

$$\left[x\left(\hat{\theta},\eta\right) - x(\theta,\eta)\right]\cdot\left(\theta-\hat{\theta}\right) \leq 0.$$

To show necessity for the first part, suppose without loss of generality that $\theta > \hat{\theta}$. Then

$$x\left(\hat{\theta},\eta\right) \leq \frac{u(\theta,\eta) - u\left(\hat{\theta},\eta\right)}{\left(\theta-\hat{\theta}\right)} \leq x(\theta,\eta).$$

But as $x(\theta,\eta)$ is non-decreasing in θ, it is continuous but for at most countably many points and at any point of continuity of $x(\theta,\eta)$ in θ taking limits $\hat{\theta} \to \theta$ yields $u_\theta(\theta,\eta) = x(\theta,\eta)$, so integrating over θ yields (5). For sufficiency, note that by the fact that the allocation is non-negative and monotonic we have

$$\begin{aligned} u(\theta,\eta) - U\left(\hat{\theta},\theta,\eta,\eta\right) &= u(\theta,\eta) - u\left(\hat{\theta},\eta\right) - x\left(\hat{\theta},\eta\right)\left(\theta-\hat{\theta}\right) \\ &= \int_{\hat{\theta}}^{\theta}\left[x(y,\eta) - x\left(\hat{\theta},\eta\right)\right]dy \\ &\geq 0 \end{aligned}$$

for all $\theta,\hat{\theta} \in \left[\underline{\theta},\overline{\theta}\right]$. The second part of the Lemma follows from

$$\begin{aligned} U\left(\theta,\theta,\underline{\eta},\overline{\eta}\right) &= u\left(\theta,\underline{\eta}\right) + q\left(\theta,\underline{\eta}\right)\cdot\left(\overline{\eta}-\underline{\eta}\right), \\ U\left(\theta,\theta,\overline{\eta},\underline{\eta}\right) &= u\left(\theta,\overline{\eta}\right) - q\left(\theta,\overline{\eta}\right)\cdot\left(\overline{\eta}-\underline{\eta}\right). \end{aligned}$$

Proof of Lemma 4:

Suppose one of the solution schedules is not continuous at some $\theta^{'} \in \left(\underline{\theta},\overline{\theta}\right)$. By monotonicity of the optimal schedules, both schedules have left and right limits at $\theta^{'}$ so there must be a jump dicontinuity at $\theta^{'}$. Write $x_{l}^{*}\left(\theta\right)$ for the left limit points and. $x_{r}^{*}\left(\theta\right)$ for the right limit points. First, suppose that $\underline{x}^{*}$ is not continuous at $\theta^{'}$. For any $\delta > 0$ sufficiently small, define

$$\underline{x}_{\delta}\left(\theta\right) = \begin{cases} \underline{x}^{*}\left(\theta\right) & \theta \notin \left[\theta^{'}-\delta,\theta^{'}+\delta\right] \\ \underline{c}^{*}\left(\delta\right) = \frac{\int_{\theta^{'}-\delta}^{\theta^{'}+\delta}\underline{x}^{*}(\theta)d\theta}{2\delta} & \theta \in \left[\theta^{'}-\delta,\theta^{'}+\delta\right] \end{cases}.$$

Note that

$$\lim_{\delta\to 0}\underline{c}^{*}\left(\delta\right) = \frac{\left[\underline{x}_{l}^{*}\left(\theta^{'}\right)+\underline{x}_{r}^{*}\left(\theta^{'}\right)\right]}{2}$$

and that $\underline{x}_{\delta}\left(\theta\right)$ is non-decreasing and non-negative given that $\underline{x}^{*}$ is. By concavity of B in x, we have

$$\begin{aligned} & \frac{\partial}{\partial\delta}|_{\delta=0}\left[\Pi\left(\underline{x}_{\delta},\overline{x}^{*},\overline{u}\right)-\Pi\left(\underline{x}^{*},\overline{x}^{*},\overline{u}\right)\right] \\ = & \frac{\partial}{\partial\delta}|_{\delta=0}\left[\beta\int_{\theta^{'}-\delta}^{\theta^{'}}\left[B\left(\underline{x}_{\delta}\left(\theta\right),0,\theta,\underline{\eta}\right)-B\left(\underline{x}^{*}\left(\theta\right),0,\theta,\underline{\eta}\right)\right]f\left(\theta|\underline{\eta}\right)d\theta\right] \\ & +\frac{\partial}{\partial\delta}|_{\delta=0}\left[\beta\int_{\theta^{'}}^{\theta^{'}+\delta}\left[B\left(\underline{x}_{\delta}\left(\theta\right),0,\theta,\underline{\eta}\right)-B\left(\underline{x}^{*}\left(\theta\right),0,\theta,\underline{\eta}\right)\right]f\left(\theta|\underline{\eta}\right)d\theta\right] \\ = & \beta\cdot f\left(\theta^{'}|\underline{\eta}\right)\cdot\left[2B\left(\frac{\left[\underline{x}_{l}^{*}\left(\theta^{'}\right)+\underline{x}_{r}^{*}\left(\theta^{'}\right)\right]}{2},0,\theta^{'},\underline{\eta}\right)-B\left(\underline{x}_{l}^{*}\left(\theta^{'}\right),0,\theta^{'},\underline{\eta}\right)-B\left(\underline{x}_{r}^{*}\left(\theta^{'}\right),0,\theta^{'},\underline{\eta}\right)\right] \\ > & 0. \end{aligned}$$

Hence, for sufficiently small δ, the mechanism $(\underline{x}_{\delta},\overline{x}^{*})$ increases the value of the objective Π which would contradict optimality of $(\underline{x}^{*},\overline{x}^{*})$ if the mechanism $(\underline{x}_{\delta},\overline{x}^{*})$ were to satisfy all constraints. Note that $\rho\left(\theta,\overline{u}\right)$ takes the same values for the original mechanism and for $(\underline{x}_{\delta},\overline{x}^{*})$ outside $\left[\theta^{'}-\delta,\theta^{'}+\delta\right]$. Hence it suffices to check contraints (8) and (9) on $\left[\theta^{'}-\delta,\theta^{'}+\delta\right]$ for the mechanism $(\underline{x}_{\delta},\overline{x}^{*})$. If neither (8) nor (9) binds at $\theta^{'}$ for the original mechanism, then both constraints will also be satisfied for $(\underline{x}_{\delta},\overline{x}^{*})$ on $\left[\theta^{'}-\delta,\theta^{'}+\delta\right]$ if δ is chosen sufficiently small. Note that, at any θ,

$$\rho\left(\theta,\overline{u}\right)=\overline{u}+\int_{\underline{\theta}}^{\theta}\left[\overline{x}\left(y\right)-\underline{x}\left(y\right)\right]dy$$

is (weakly) larger for the original mechanism than for $(\underline{x}_{\delta},\overline{x}^{*})$. Hence constraint (9) will never be violated by $(\underline{x}_{\delta},\overline{x}^{*})$ given that it was satisfied by $(\underline{x}^{*},\overline{x}^{*})$. So the only case in which $(\underline{x}_{\delta}, \overline{x}^*)$ is not feasible for any $\delta > 0$ is when (8) binds at θ' for the original mechanism.

But then we must have $\overline{x}_l^*\left(\theta'\right) \leq \underline{x}_l^*\left(\theta'\right)$ as otherwise, if $\overline{x}_l^*\left(\theta'\right) > \underline{x}_l^*\left(\theta'\right)$, this inequality continues to hold on a small interval $\left[\theta' - \epsilon, \theta'\right]$ and hence $\rho\left(\theta' - \epsilon\right) < \rho\left(\theta'\right) = 0$, contradicting incentive compatibility of the mechanism $(\underline{x}^*, \overline{x}^*)$. Similarly we must have $\underline{x}_r^*\left(\theta'\right) \leq \overline{x}_r^*\left(\theta'\right)$ as otherwise, if $\underline{x}_r^*\left(\theta'\right) > \overline{x}_r^*\left(\theta'\right)$, this inequality continues to hold on a small interval $\left[\theta', \theta' + \epsilon\right]$ and hence $\rho\left(\theta' + \epsilon\right) < \rho\left(\theta'\right) = 0$, again contradicting incentive compatibility of the mechanism $(\underline{x}^*, \overline{x}^*)$. Thus we have $\overline{x}_l^*\left(\theta'\right) \leq \underline{x}_l^*\left(\theta'\right) < \underline{x}_r^*\left(\theta'\right) \leq \overline{x}_r^*\left(\theta'\right)$, so $\overline{x}^*$ also has a jump discontinuity at θ'.

Next, suppose that $\overline{x}^*$ is not continuous at θ'. Just as before, define the following schedule for $\delta > 0$ sufficiently small:

$$\overline{x}_{\delta}(\theta) = \begin{cases} \overline{x}^*(\theta) & \theta \notin \left[\theta' - \delta, \theta' + \delta\right] \\ \overline{c}^*(\delta) = \frac{\int_{\theta'-\delta}^{\theta'+\delta} \overline{x}^*(\theta) d\theta}{2\delta} & \theta \in \left[\theta' - \delta, \theta' + \delta\right] \end{cases}.$$

Note that

$$\lim_{\delta \to 0} \overline{c}^*(\delta) = \frac{\left[\overline{x}_l^*\left(\theta'\right) + \overline{x}_r^*\left(\theta'\right)\right]}{2}$$

and that $\overline{x}_{\delta}(\theta)$ is non-decreasing and non-negative given that $\overline{x}^*$ is. By concavity of B in x, we have

$$\begin{aligned}
& \frac{\partial}{\partial \delta}|_{\delta=0} \left[\Pi\left(\underline{x}^*, \overline{x}_{\delta}, \overline{u}\right) - \Pi\left(\underline{x}^*, \overline{x}^*, \overline{u}\right)\right] \\
= & \frac{\partial}{\partial \delta}|_{\delta=0} \left[(1-\beta) \int_{\theta'-\delta}^{\theta'} \left[B\left(\overline{x}_{\delta}(\theta), 1, \theta, \overline{\eta}\right) - B\left(\overline{x}^*(\theta), 1, \theta, \overline{\eta}\right)\right] f(\theta|\overline{\eta})\, d\theta\right] \\
& + \frac{\partial}{\partial \delta}|_{\delta=0} \left[(1-\beta) \int_{\theta'}^{\theta'+\delta} \left[B\left(\overline{x}_{\delta}(\theta), 1, \theta, \overline{\eta}\right) - B\left(\overline{x}^*(\theta), 1, \theta, \overline{\eta}\right)\right] f(\theta|\overline{\eta})\, d\theta\right] \\
= & (1-\beta) \cdot f\left(\theta'|\overline{\eta}\right) \cdot \left[2B\left(\frac{\left[\overline{x}_l^*\left(\theta'\right) + \overline{x}_r^*\left(\theta'\right)\right]}{2}, 1, \theta', \overline{\eta}\right) - B\left(\overline{x}_l^*\left(\theta'\right), 1, \theta', \overline{\eta}\right) - B\left(\overline{x}_r^*\left(\theta'\right), 1, \theta', \overline{\eta}\right)\right] \\
> & 0.
\end{aligned}$$

So again, for sufficiently small δ, the new mechanism $(\underline{x}^*, \overline{x}_{\delta})$ increases the value of the objective Π compared to the original mechanism. As before, $\rho(\theta, \overline{u})$ takes the same values for the original mechanism and for $(\underline{x}^*, \overline{x}_{\delta})$ outside $\left[\theta' - \delta, \theta' + \delta\right]$, so it suffices to check contraints (8 and (9) on $\left[\theta' - \delta, \theta' + \delta\right]$ for the mechanism $(\underline{x}^*, \overline{x}_{\delta})$. If neither

(8) nor (9) binds at $\theta^{'}$ for the original mechanism, then both constraints will also be satisfied for $(\overline{x}_{\delta}, \overline{x}^{*})$ on $\left[\theta^{'} - \delta, \theta^{'} + \delta\right]$ if δ is chosen sufficiently small. At any θ,

$$\rho\left(\theta, \overline{u}\right) = \overline{u} + \int_{\underline{\theta}}^{\theta} \left[\overline{x}\left(y\right) - \underline{x}\left(y\right)\right] dy$$

is (weakly) smaller for the original mechanism than for $(\underline{x}^{*}, \overline{x}_{\delta})$. Hence constraint (8) will never be violated by $(\underline{x}^{*}, \overline{x}_{\delta})$ given that it was not violated by $(\underline{x}^{*}, \overline{x}^{*})$. So the only case in which $(\underline{x}^{*}, \overline{x}_{\delta})$ is not feasible for any $\delta > 0$ is when (9) binds at $\theta^{'}$ for the original mechanism. In that case, however, we must now have $\underline{x}_{l}^{*}\left(\theta^{'}\right) \leq \overline{x}_{l}^{*}\left(\theta^{'}\right) < \overline{x}_{r}^{*}\left(\theta^{'}\right) \leq \underline{x}_{r}^{*}\left(\theta^{'}\right)$, so $\overline{x}^{*}$ also has a jump discontinuity at $\theta^{'}$.

Together, both cases yield a contradiction and hence prove the result on $\left(\underline{\theta}, \overline{\theta}\right)$. Either schedule $\underline{x}^{*}$ is continuous or, at any point where $\underline{x}^{*}$ is not continuous, both schedules are discontinuous and constraint (9) binds. At the same time, either $\overline{x}^{*}$ is continuous or, at any point where $\overline{x}^{*}$is not continuous, both schedules are discontinuous and constraint (8) binds. As (8) and (9) cannot both bind at a given $\theta^{'}$, the only possibility is that both schedules $\underline{x}^{*}$, $\overline{x}^{*}$ are continuous.

Finally, consider the the boundaries $\underline{\theta}$ and $\overline{\theta}$. By monotonicity, we have

$$\begin{array}{rcl} \underline{x}^{*}\left(\underline{\theta}\right) & \leq & \underline{x}_{r}^{*}\left(\underline{\theta}\right) \\ \underline{x}^{*}\left(\overline{\theta}\right) & \geq & \underline{x}_{l}^{*}\left(\overline{\theta}\right) \\ \overline{x}^{*}\left(\underline{\theta}\right) & \leq & \overline{x}_{r}^{*}\left(\underline{\theta}\right) \\ \overline{x}^{*}\left(\overline{\theta}\right) & \geq & \overline{x}_{l}^{*}\left(\overline{\theta}\right) \end{array}$$

so $\underline{x}_{r}^{*}\left(\underline{\theta}\right)$, $\underline{x}_{l}^{*}\left(\overline{\theta}\right)$, $\overline{x}_{r}^{*}\left(\underline{\theta}\right)$, $\overline{x}_{l}^{*}\left(\overline{\theta}\right)$ exist. Setting

$$\begin{array}{rcl} \underline{x}^{*}\left(\underline{\theta}\right) & = & \underline{x}_{r}^{*}\left(\underline{\theta}\right) \\ \underline{x}^{*}\left(\overline{\theta}\right) & = & \underline{x}_{l}^{*}\left(\overline{\theta}\right) \\ \overline{x}^{*}\left(\underline{\theta}\right) & = & \overline{x}_{r}^{*}\left(\underline{\theta}\right) \\ \overline{x}^{*}\left(\overline{\theta}\right) & = & \overline{x}_{l}^{*}\left(\overline{\theta}\right) \end{array}$$

neither changes Π or ρ nor violates monotonicity, and it guarantees continuity of $(\underline{x}^{*}, \overline{x}^{*})$ at the boundaries.

Proof of Proposition 1:

We invoke Seierstad and Sydsaeter (1987), Ch. 5, Thm. 2, p. 332 f.[25] For the optimal solution schedules $\left(\underline{x}^{*}\left(\theta\right), \overline{x}^{*}\left(\theta\right)\right)$ and fixed values $\underline{u} = \underline{u}\left(\underline{\theta}\right) = 0$, $\overline{u} = \overline{u}\left(\underline{\theta}\right)$ this theorem yields the existence of costate variables $\left(\underline{\kappa}\left(\theta\right), \overline{\kappa}\left(\theta\right)\right)$ with $\underline{\kappa}\left(\overline{\theta}\right) = \overline{\kappa}\left(\overline{\theta}\right) = 0$ such that

[25] Note that the roles of x and u are reversed in this thesis chapter as compared to the notation in Seierstad and Sydsaeter (1987)

$(\underline{x}^* (\theta), \overline{x}^* (\theta))$ maximizes

$$\begin{aligned} H\left(\underline{u}^*(\theta), \overline{u}^*(\theta), \underline{x}(\theta), \overline{x}(\theta), \underline{\kappa}(\theta), \overline{\kappa}(\theta), \theta\right) = {} & \beta B\left(\underline{x}(\theta), 0, \theta, \underline{\eta}\right) f\left(\theta | \underline{\eta}\right) \\ & + (1-\beta) B\left(\overline{x}(\theta), 1, \theta, \overline{\eta}\right) f\left(\theta | \overline{\eta}\right) \quad (23) \\ & + \underline{\kappa}(\theta) \cdot \underline{x}(\theta) + \overline{\kappa}(\theta) \cdot \overline{x}(\theta). \end{aligned}$$

In addition, there exist a componentwise non-decreasing function $\mu(\theta) = (\mu_1(\theta), \mu_2(\theta))$ with the following properties: μ_1 is constant on any interval where $\rho(\theta, \overline{u}) > 0$ and μ_2 is constant on any interval where $\left(\overline{\eta} - \underline{\eta}\right) - \rho(\theta, \overline{u}) > 0$. Moreover,

$$\varkappa(\theta) = \left(\underline{\varkappa}(\theta), \overline{\varkappa}(\theta)\right) \equiv \left(\underline{\kappa}(\theta), \overline{\kappa}(\theta)\right) + \mu^*(\theta) \cdot \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad (24)$$

is continuous, and at any point where $(\underline{x}^* (\theta), \overline{x}^* (\theta))$ and $\mu(\theta)$ are continuous, $\varkappa(\theta)$ is differentiable with

$$\begin{aligned} \frac{\partial\left(\underline{\varkappa}(\theta), \overline{\varkappa}(\theta)\right)}{\partial \theta} &= -\frac{\partial}{\partial\left(\overline{u}(\theta), \underline{u}(\theta)\right)} \\ & \left[H\left(\underline{u}^*(\theta), \overline{u}^*(\theta), \underline{x}^*(\theta), \overline{x}^*(\theta), \underline{\kappa}(\theta), \overline{\kappa}(\theta), \theta\right) - \mu(\theta) \cdot \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \underline{x}^*(\theta) \\ \overline{x}^*(\theta) \end{pmatrix}\right] \\ &= (0,0). \end{aligned}$$

Since $\underline{x}^* (\theta)$ and $\overline{x}^* (\theta)$ are assumed to be continuous and $\mu(\theta)$ is non-decreasing and hence continuous everywhere except for at most countably many points, $\varkappa(\theta)$ is continuous and differentiable everywhere except for at most countably many points, with derivative being equal to 0. But then, by standard calculus (see e.g. Königsberger (2004)), $\varkappa(\theta)$ is Lipschitz continuous with Lipschitz constant 0 everywhere, i.e. $\varkappa(\theta) = \varkappa$ is constant everywhere. Hence, evaluating (24) at $\theta = \overline{\theta}$ we get $\underline{\varkappa}\left(\overline{\theta}\right) = -\mu_1\left(\overline{\theta}\right) + \mu_2\left(\overline{\theta}\right)$, $\overline{\varkappa} = +\mu_1\left(\overline{\theta}\right) - \mu_2\left(\overline{\theta}\right)$ and hence

$$\begin{aligned} \underline{\kappa}(\theta) &= \mu_1(\theta) - \mu_1\left(\overline{\theta}\right) - \mu_2(\theta) + \mu_2\left(\overline{\theta}\right), \\ \overline{\kappa}(\theta) &= -\mu_1(\theta) + \mu_1\left(\overline{\theta}\right) + \mu_2(\theta) - \mu_2\left(\overline{\theta}\right) = -\underline{\kappa}(\theta). \end{aligned}$$

Define $\kappa^*(\theta) = \underline{\kappa}(\theta)$ and note the first-order conditions of (23) are precisely the equations charaterizing the optimal schedules. As H is concave in x the first-order conditions are sufficient. Moreover, as the constraints (8), (9) are linear (and hence quasi-concave) in $(\underline{u}(\theta), \overline{u}(\theta))$, the conditions in Seierstad and Sydsaeter (1987), Ch. 5, Thm. 2, p. 332 f. are sufficient, cf. Seierstad and Sydsaeter (1987), Ch. 5, Thm. 3, p. 337. The properties of $\kappa^*(\theta)$ are straightforward:

a) Continuity of $\kappa^*(\theta)$ follows directly from continuity of $(\underline{x}^* (\theta), \overline{x}^* (\theta))$.

b) This follows directly from $\kappa^*(\theta) = \underline{\kappa}(\theta) = \mu_1(\theta) - \mu_1\left(\overline{\theta}\right) - \mu_2(\theta) + \mu_2\left(\overline{\theta}\right)$ and the respective properties of μ_1 and μ_2.

c) Continuity of $\kappa^*(\theta)$ and$(\underline{x}^* (\theta), \overline{x}^* (\theta))$ implies that for any $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$ where (8) or (9) binds it must hold that $\underline{x}^*(\theta) = \overline{x}^*(\theta)$ as otherwise the respective constraint would be violated either at $\theta + \delta$ or at $\theta - \delta$ for $\delta > 0$ sufficiently small.

d) This is implied by the monotonicity properties of μ_1 and μ_2 via $\kappa^*(\theta) = \underline{\kappa}(\theta) = \mu_1(\theta) - \mu_1\left(\overline{\theta}\right) - \mu_2(\theta) + \mu_2\left(\overline{\theta}\right)$.

e) This follows from the transversality condition $\underline{\kappa}\left(\overline{\theta}\right) = 0$.

Proof of Proposition 2:

Note first that from equations (12) and (13) we get

$$\frac{\partial \underline{x}(\theta)}{\partial \kappa} = \frac{1}{\beta f(\theta|\underline{\eta}) \cdot C_{xx}^1(\underline{x}(\theta))} > 0,$$
$$\frac{\partial \overline{x}(\theta)}{\partial \kappa} = -\frac{1}{(1-\beta) f(\theta|\overline{\eta}) \cdot C_{xx}^1(\overline{x}(\theta))} < 0.$$

In an optimal mechanism, constraint (8) must bind for at least one $\theta \in [\underline{\theta}, \overline{\theta}]$. Otherwise the continuous function $\rho^*(\theta, \overline{u})$ attains its minimum $\epsilon > 0$ at some $\hat{\theta}$ on the compact interval $[\underline{\theta}, \overline{\theta}]$ and decreasing $\overline{u}$ by ϵ does not harm any constraints and simultaneously increases revenues. So assume $\rho^*(\hat{\theta}, \overline{u}) = 0$ for some $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$. We want to show that this implies $\rho^*(\hat{\theta}, \overline{u}^*) = 0$ for any $\theta \geq \hat{\theta}$. Suppose first that $\hat{\theta} = \underline{\theta}$. Since $\kappa^*(\theta)$ is continuous and locally either follows a constant schedule or $\kappa_b(\theta)$, and since $\kappa_b(\theta)$ is increasing, $\kappa^*(\underline{\theta}) < \kappa_b(\underline{\theta})$ would imply $\kappa^*(\theta) = \kappa^*(\underline{\theta}) < \kappa_b(\underline{\theta}) \leq \kappa_b(\overline{\theta}) = 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, contradicting transversality $\kappa^*(\overline{\theta}) = 0$ as stated in Proposition 1, Part e). On the other hand, $\kappa^*(\underline{\theta}) > \kappa_b(\underline{\theta})$ by continuity of $\kappa^*(\theta)$ and $\kappa_b(\theta)$ would imply $\kappa^*(\theta) > \kappa_b(\theta)$ on some interval $[\underline{\theta}, \underline{\theta} + \epsilon]$ of positive length $\epsilon > 0$, violating (8) at any $\theta > \underline{\theta}$ within this interval. Hence for $\hat{\theta} = \underline{\theta}$ we must have $\kappa^*(\hat{\theta}) = \kappa_b(\hat{\theta})$ just as for any other $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$, following Proposition 1, Part c) and e).

As a consequence, note that $\kappa^*(\theta) \leq \kappa_b(\theta)$ for any $\theta > \hat{\theta}$ as $\kappa_b(\theta)$ is increasing and $\kappa^*(\theta)$ is continuous and follows either a constant schedule or $\kappa_b(\theta)$. Suppose $\kappa^*(\tilde{\theta}) < \kappa_b(\tilde{\theta})$ for some $\hat{\theta} < \tilde{\theta} \leq \overline{\theta}$. Then $\kappa^*(\theta) = \kappa^*(\tilde{\theta}) < \kappa_b(\tilde{\theta}) \leq \kappa_b(\overline{\theta}) = 0$ for any $\theta \geq \tilde{\theta}$ by continuity of $\kappa^*(\theta)$ and Proposition 1, Part b)+c), contradicting $\kappa^*(\overline{\theta}) = 0$ as required by Proposition 1, Part e). Defining $\theta' = \inf\{\theta \in [\underline{\theta}, \overline{\theta}] : \rho^*(\theta, \overline{u}^*) = 0\}$ completes the proof.

Proof of Theorem 1:

By Proposition 2 we have

$$\overline{u} = \int_{\underline{\theta}}^{\theta'} [\overline{x}^*(y) - \underline{x}^*(y)]\, dy$$

which, as $\overline{x}^*(\theta') = \underline{x}^*(\theta') = x^*(\theta')$, implies

$$\frac{d\kappa^*}{d\overline{u}}(\overline{u}) = -\frac{1}{\int_{\underline{\theta}}^{\theta'} \left[\frac{\partial \overline{x}^*}{\partial \kappa^*}(y) - \frac{\partial \underline{x}^*}{\partial \kappa^*}(y)\right] dy} > 0.$$

Using this as well as equations (12) and (13) we get

$$\begin{aligned}
&\frac{d\Pi}{d\overline{u}}\left(\overline{u}\right) \\
= &\int_{\underline{\theta}}^{\theta'}\left[\beta f\left(\theta|\underline{\eta}\right)\cdot\frac{\partial B}{\partial\underline{x}}\left(\underline{x}^{*}\left(\theta\right),0,\theta,\underline{\eta}\right)\cdot\frac{\partial\underline{x}^{*}}{\partial\kappa^{*}}\left(\theta\right)\cdot\frac{d\kappa^{*}}{d\overline{u}}\left(\overline{u}\right)\right]d\theta \\
&+\int_{\underline{\theta}}^{\theta'}\left[\left(1-\beta\right)f\left(\theta|\overline{\eta}\right)\cdot\frac{\partial B}{\partial\overline{x}}\left(\overline{x}^{*}\left(\theta\right),1,\theta,\overline{\eta}\right)\cdot\frac{\partial\overline{x}^{*}}{\partial\kappa^{*}}\left(\theta\right)\cdot\frac{d\kappa^{*}}{d\overline{u}}\left(\overline{u}\right)\right]d\theta-\left(1-\beta\right) \\
= &\frac{\kappa^{*}\left(\overline{u}\right)\cdot\int_{\underline{\theta}}^{\theta'}\frac{\partial\underline{x}^{*}}{\partial\kappa^{*}}\left(\theta\right)d\theta}{\int_{\underline{\theta}}^{\theta'}\left[\frac{\partial\overline{x}^{*}}{\partial\kappa^{*}}\left(y\right)-\frac{\partial\underline{x}^{*}}{\partial\kappa^{*}}\left(y\right)\right]dy}-\frac{\kappa^{*}\left(\overline{u}\right)\cdot\int_{\underline{\theta}}^{\theta'}\frac{\partial\overline{x}^{*}}{\partial\kappa^{*}}\left(\theta\right)d\theta}{\int_{\underline{\theta}}^{\theta'}\left[\frac{\partial\overline{x}^{*}}{\partial\kappa^{*}}\left(y\right)-\frac{\partial\underline{x}^{*}}{\partial\kappa^{*}}\left(y\right)\right]dy}-\left(1-\beta\right) \\
= &-\kappa^{*}\left(\overline{u}\right)-\left(1-\beta\right),
\end{aligned}$$

implying $\overline{u}^{*}=0$ as demonstrated in the main text. The theorem then immediately follows from Proposition 2. The reformulation in terms of prices conditional on quantities rather than preference types is a direct implication of the taxation principle as explained in the main text.

Proof of Theorem 2:

As demonstrated in the main text, the solution schedules $\left(\underline{x}^{*}\left(\theta\right),\overline{x}^{*}\left(\theta\right),\overline{u}^{*}\right)$ for Problem P' deviate from the full bunching schedule $\left(x^{*}\left(\theta\right),x^{*}\left(\theta\right),\overline{u}=0\right)$ on an interval of positive mass. We are left to formally show that, under Assumption 3, there exists a convex combination

$$\left(\underline{x}_{\lambda}\left(\theta\right),\overline{x}_{\lambda}\left(\theta\right),\overline{u}_{\lambda}\right)=\lambda\cdot\left(\underline{x}^{*}\left(\theta\right),\overline{x}^{*}\left(\theta\right),\overline{u}^{*}\right)+\left(1-\lambda\right)\cdot\left(x^{*}\left(\theta\right),x^{*}\left(\theta\right),0\right)$$

with $\lambda>0$ such that the schedules $\left(\underline{x}_{\lambda}\left(\theta\right),\overline{x}_{\lambda}\left(\theta\right),\overline{u}_{\lambda}\right)$ satisfy constraints (8)-(11) and that

$$\Pi\left(\underline{x}_{\lambda}\left(\theta\right),\overline{x}_{\lambda}\left(\theta\right),\overline{\pi}_{\lambda}\right)>\Pi\left(x^{*}\left(\theta\right),x^{*}\left(\theta\right),0\right).$$

As constraints (8) and (9) are linear in $\left(\overline{x},\underline{x},\overline{u}\right)$ and, by construction, satisfied by $\left(\underline{x}^{*}\left(\theta\right),\overline{x}^{*}\left(\theta\right),\overline{u}^{*}\right)$ and $\left(x^{*}\left(\theta\right),x^{*}\left(\theta\right),\overline{u}=0\right)$, they are also satisfied by any convex combination of the two. Moreover, as $\left(\underline{x}^{*}\left(\theta\right),\overline{x}^{*}\left(\theta\right)\right)$ and $\left(x^{*}\left(\theta\right),x^{*}\left(\theta\right)\right)$ are continuous and hence bounded on $\left[\underline{\theta},\overline{\theta}\right]$ and $x^{*}\left(\theta\right)\geq x^{*}\left(\underline{\theta}\right)>0$ for all $\theta\in\left[\underline{\theta},\overline{\theta}\right]$ by Assumption 3, $\left(\underline{x}_{\lambda}\left(\theta\right),\overline{x}_{\lambda}\left(\theta\right)\right)$ are positive for sufficiently small values $\lambda>0$ as well and hence satisfy (11).

Concerning (10), note that the optimal costate variable $\kappa^{*}\left(\theta\right)$ that corresponds to $\left(\underline{x}^{*}\left(\theta\right),\overline{x}^{*}\left(\theta\right),\overline{u}^{*}\right)$ is bounded. Indeed, since $\kappa^{*}\left(\theta\right)$ is continuous on $\left[\underline{\theta},\overline{\theta}\right]$ with $\kappa^{*}\left(\overline{\theta}\right)=\kappa_{b}\left(\overline{\theta}\right)=0$ and is either locally constant or follows $\kappa_{b}\left(\theta\right)$, we have $\kappa^{*}\left(\theta\right)\in\left[\min_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}\kappa_{b}\left(\theta\right),\max_{\theta\in\left[\underline{\theta},\overline{\theta}\right]}\kappa_{b}\left(\theta\right)\right]$ which is bounded by compactness of $\left[\underline{\theta},\overline{\theta}\right]$ and continuity of $\kappa_{b}\left(\theta\right)$. Moreover, on any interval $\left[\theta',\theta''\right]\subset\left[\underline{\theta},\overline{\theta}\right]$, the costate variable $\kappa^{*}\left(\theta\right)$ is constant or lies within $\left[\min_{\theta\in\left[\theta',\theta''\right]}\kappa_{b}\left(\theta\right),\max_{\theta\in\left[\theta',\theta''\right]}\kappa_{b}\left(\theta\right)\right]$. Hence, for any $\theta\in\left[\underline{\theta},\overline{\theta}\right]$

the set of subdifferentials $\partial\kappa^*(\theta)$ is bounded by $\left[\min\left\{0, \frac{\partial\kappa_b(\theta)}{\partial\theta}\right\}, \max\left\{0, \frac{\partial\kappa_b(\theta)}{\partial\theta}\right\}\right]$ and therefore the subdifferentials of $(\underline{x}^*(\theta), \overline{x}^*(\theta))$ given as

$$\begin{aligned}
\partial\underline{x}^*(\theta) &= \frac{1}{C_{xx}(\underline{x}^*(\theta))} \cdot \left(\frac{\partial}{\partial\theta}\left[\theta - \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}\right] + \frac{\kappa^*(\theta) \cdot \frac{\partial f\left(\theta|\underline{\eta}\right)}{\partial\theta} - f\left(\theta|\underline{\eta}\right) \cdot \partial\kappa^*(\theta)}{\beta f\left(\theta|\underline{\eta}\right)^2}\right), \\
\partial\overline{x}^*(\theta) &= \frac{1}{C_{xx}(\overline{x}^*(\theta))} \cdot \left(\frac{\partial}{\partial\theta}\left[\theta - \frac{1 - F(\theta|\overline{\eta})}{f(\theta|\overline{\eta})}\right] - \frac{\kappa^*(\theta) \cdot \frac{\partial f(\theta|\overline{\eta})}{\partial\theta} - f(\theta|\overline{\eta}) \cdot \partial\kappa^*(\theta)}{(1-\beta) f(\theta|\overline{\eta})^2}\right)
\end{aligned}$$

are bounded as well. Since

$$\frac{\partial x^*(\theta)}{\partial\theta} = \frac{1}{C_{xx}(\underline{x}^*(\theta))} \cdot \left(\frac{\partial}{\partial\theta}\left[\theta - \frac{1 - F\left(\theta|\underline{\eta}\right)}{f\left(\theta|\underline{\eta}\right)}\right]\right)$$

is continuous on $\left[\underline{\theta}, \overline{\theta}\right]$ and strictly positive by Assumption 3, it is bounded away from zero by compactness of $\left[\underline{\theta}, \overline{\theta}\right]$. Hence, again, for sufficiently small $\lambda > 0$ we have

$$\begin{aligned}
\partial\underline{x}_\lambda(\theta) &\subset \mathbb{R}_+, \\
\partial\overline{x}_\lambda(\theta) &\subset \mathbb{R}_+,
\end{aligned}$$

showing (10).

Finally, from (6) and concavity of B in x together with

$$\Pi(\underline{x}^*(\theta), \overline{x}^*(\theta), \overline{u}^*) > \Pi(x^*(\theta), x^*(\theta), \overline{u} = 0)$$

we get

$$\begin{aligned}
\Pi(\underline{x}_\lambda(\theta), \overline{x}_\lambda(\theta), \overline{u}_\lambda) &= \Pi(\lambda \cdot (\underline{x}^*(\theta), \overline{x}^*(\theta), \overline{u}^*) + (1-\lambda) \cdot (x^*(\theta), x^*(\theta), 0)) \\
&> \lambda\Pi(\underline{x}^*(\theta), \overline{x}^*(\theta), \overline{u}^*) + (1-\lambda)\Pi(x^*(\theta), x^*(\theta), 0) \\
&> \lambda\Pi(x^*(\theta), x^*(\theta), 0) + (1-\lambda)\Pi(x^*(\theta), x^*(\theta), 0) \\
&= \Pi(x^*(\theta), x^*(\theta), 0)
\end{aligned}$$

which proves the theorem.

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