Local Smoothing and Well-Posedness Results for KP-II Type Equations

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Contents

Chapter 1

Introduction

In this thesis, we study the qualitative properties of the solution of the Cauchy problem for the Kadomtsev-Petviashvili II (KP-II) equation

$$\partial_t u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u + 6u\partial_x u = 0,$$

and the well posedness of the Cauchy problem for the generalized Kadomtsev-Petviashvili II equation with cubical nonlinearity ($(\text{gKP-II})_3$)

$$\partial_t u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u - 6u^2\partial_x u = 0$$

that satisfy initial conditions with low regularity.

When the sign in front of $3\partial_x^{-1}\partial_y^2 u$ term is minus in the above two equations they are called the KP-I and the $(\text{gKP-I})_3$ equations respectively. Despite their formal similarity, the KP-I and the KP-II equations differ significantly with respect to their underlying mathematical structure. The KP-I, the KP-II and the $(\text{gKP-II})_3$ equations are integrable Hamiltonian systems and consequently possess infinitely many conservation laws. The KP-I and the $(\text{gKP-I})_3$ equations have conservation laws with positively defined quadratic parts. This allows the corresponding Sobolev type norms to be controlled by the KP-I flow and the use of energetic methods to analyze these equations. On the other hand, the KP-II equation has conservation laws that do not have positively defined quadratic parts. In order to study the KP-II and the $(\text{gKP-II})_3$ equation harmonic analysis methods have been used starting with [2].

The KP equation came as a natural generalization of the Korteweg-de Vries (KdV) equation from one to two spatial dimensions,

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

It was first introduced in 1970 by B. B. Kadomtsev and V. I. Petviashvili [14]. They derived the equation as a model to study the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. These equations were later derived by other researchers in other physical settings as well. The KP equations have been obtained as a reduced model in ferromagnetics [30], Bose-Einstein condensates [31] and string theory [7].

The KdV equation has remarkable solutions, called solitons. Solitons are solutions that are localised and maintain their form for long periods of time and depend upon variables x and t only through $x - ct$ where c is a fixed constant. Substituting $u(t,x) = Q(x - ct)$ into (1.1) one obtains the ordinary differential equation

$$-cQ' + Q^{(3)} + 6QQ' = 0$$

which is satisfied by the following family of solutions

$$Q = \frac{c}{2}\operatorname{sech}^2\Big(\frac{c^{1/2}}{2}x\Big).$$

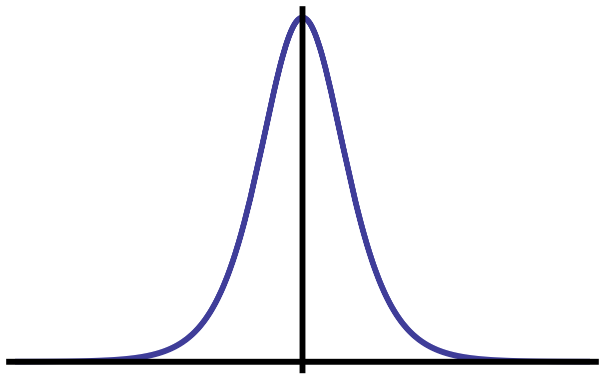


FIGURE 1.1: Graph of a soliton solution of the KdV equation.

Moreover the other solitons and radiations can pass through them without destroying their form, [35].

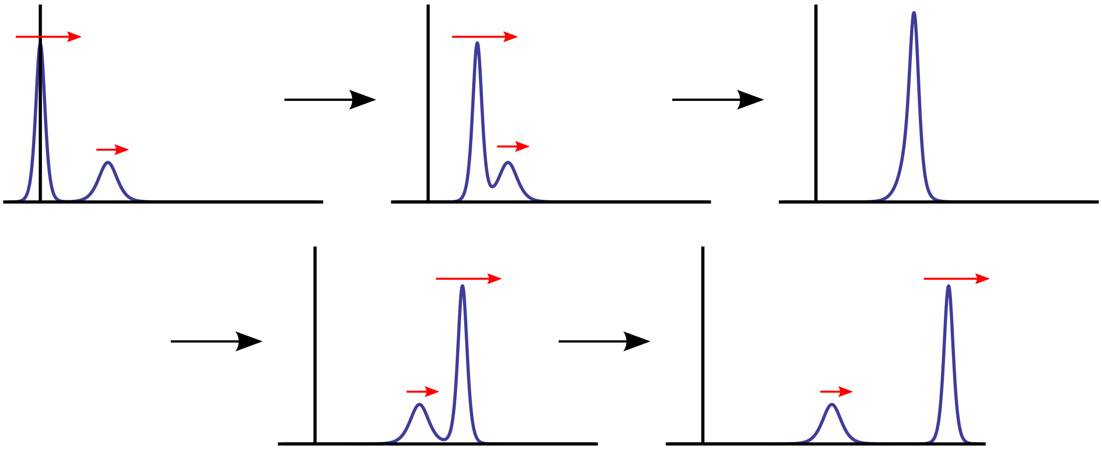


Figure 1.2: Interaction of two solitons.

The soliton solutions of the KdV equation considered as solutions of the KP equations are called the line solitons.

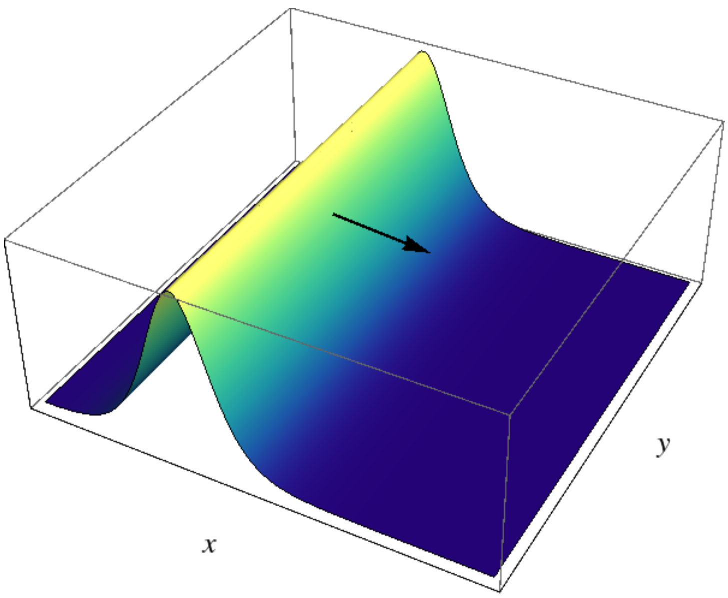


Figure 1.3: Graph of a line soliton.

The line solitons for KP-I are stable if they have small speed [27] and unstable if they have large speed [26], [36]. However, for the KP-II equation heuristic analysis [14] and inverse scattering [32] suggest that the line soliton is stable.

In Chapter 3, we present the results of our attempt to solve this problem. We conjectured a perturbed solution of the form

$$u(t,x,y) = Q(x-t,y) + \varepsilon w(t,x-t,y),$$

but T. Mizumachi in [23] showed that our hope was naive. The line soliton is more strongly perturbed than we hoped. In [23], T. Mizumachi proved the stability of line solitons for exponentially localized perturbations.

The $(\text{gKP-II})_3$ equation is a model for the evolution of sound waves in antiferromagnets [30]. The well posedness of this equation has been previously studied in [13], [15], [9] and in references therein. In Chapter 4, we prove global well posedness of the Cauchy problem for the $(\text{gKP-II})_3$ equation with initial condition in the space defined by the following norm

$$\|u\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)} := \sup_\lambda \lambda^{1/2}\Big(\sum_k \|u_{\lambda,k}\|^p_{L^2(\mathbb{R}^2)}\Big)^{1/p}.$$

This extends the result in [9]. The fundamental idea of the proof is due to J. Bourgain [2]. We construct function spaces based on the linear part of the dispersive equation we study. Instead of Bourgain spaces we use U^p (due to H. Koch-D. Tataru, [18]) and V^p (due to N. Wiener, [34]) function spaces, which are more useful in the analysis of nonlinear dispersive partial differential equations at critical regularity. This reduces our problem to proving multilinear estimates on the constructed spaces.

Chapter 2

Basic notions and function spaces

In this chapter, we review certain definitions and properties of the function spaces that are used throughout this work. The content of this chapter can be found in many sources. The author has consulted [20] and [16] for Section 2.1, [4], [16], [28] and [29] for Section 2.2, [16], [29] and [1] for Section 2.3, [28] for Section 2.4, and finally [10] and [17] for Section 2.5.

2.1 The Fourier Transform

Definition 2.1. Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}$, is defined as

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-i(x,\xi)} f(x)dx, \qquad \xi \in \mathbb{R}^n,$$

where

$$(x,\xi) := \sum_{i=1}^{n} x_i \xi_i.$$

We will use the notation $\mathcal{F}(f)$ and $\hat{f}$ interchangeably.

$\mathcal{F}$ is a bounded linear map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. The virtue of the Fourier transform is that it converts constant coefficient linear partial differential operators into multiplication with polynomials.

We summarize the fundamental properties of the Fourier transform in the following proposition.

Proposition 2.2. *If $f, g \in L^1(\mathbb{R}^n)$, then*

(i) $\mathcal{F}(f(\cdot - x_0))(\xi) = e^{-i(\xi,x_0)}\hat{f}(\xi)$,

(ii) $\mathcal{F}(e^{i(\cdot,\xi_0)}f(\cdot))(\xi) = \hat{f}(\xi - \xi_0)$,

(iii) $\mathcal{F}(\overline{f})(\xi) = \overline{\hat{f}}(-\xi)$,

(iv) For $(f * g)(y) = \int_{\mathbb{R}^n} f(y-x)g(x)dx$, *we have* $\widehat{f * g} = (2\pi)^{\frac{n}{2}}\hat{f}\hat{g}$,

(v) $\mathcal{F}(\partial_{x_j} f)(\xi) = i\xi_j \hat{f}(\xi)$,

(vi) $\mathcal{F}(x_j f)(\xi) = i\partial_{\xi_j}\hat{f}(\xi)$,

(vii) $\int f(x)\hat{g}(x)dx = \int \hat{f}(\xi)g(\xi)d\xi$.

Definition 2.3 (Schwartz function)**.** A function $\phi \in C^\infty(\mathbb{R}^n)$ is called rapidly decreasing or Schwartz function if for all multiindices α, β (i.e. $\alpha,\ \beta \in \mathbb{Z}_+^n$) there exist constants $c_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(\phi) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| \leq c_{\alpha,\beta}.$$

We call the Fréchet space of all Schwartz functions with the topology given by the family of semi-norms $\rho_{\alpha,\beta}$ the Schwartz space and denote it by $\mathcal{S}(\mathbb{R}^n)$. The natural topology on $\mathcal{S}(\mathbb{R}^n)$ is as follows: a sequence of functions ϕ_j converges to zero if for all multi-indices α, β, $x^\alpha \partial^\beta \phi_j$ converges uniformly to zero. A complete metric inducing the same topology on $\mathcal{S}(\mathbb{R}^n)$ can be defined by

$$d(\phi, \psi) = \sum_{\alpha,\beta} 2^{-|\alpha|-|\beta|} \frac{\rho_{\alpha,\beta}(\phi - \psi)}{1 + \rho_{\alpha,\beta}(\phi - \psi)}.$$

Note that $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ in the above defined metric topology.

Remark 2.4. The map $\phi \mapsto \hat{\phi}$ is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$ with the inverse

$$\check{\phi} = (2\pi)^{-\frac{n}{2}} \int e^{i(x,\xi)}\phi(\xi)d\xi, \qquad x \in \mathbb{R}^n.$$

Theorem 2.5 (Plancherel's Theorem)**.** *If ϕ and ψ are in $\mathcal{S}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \overline{\phi}(x)\psi(x)dx = \int_{\mathbb{R}^n} \overline{\hat{\phi}}(\xi)\hat{\psi}(\xi)d\xi.$$

Definition 2.6 (Tempered distributions)**.** We define the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ to be the dual space of the Schwartz space.

Note that for every tempered distribution u there exists $N \in \mathbb{N}$ and a constant $C = C_{\alpha,\beta}$ such that

$$|u(\phi)| \leq C \sum_{|\alpha|,|\beta| \leq N} \sup |x^\alpha \partial^\beta \phi|, \qquad \phi \in S(\mathbb{R}^n).$$

Then the definition of the Fourier transform can be further naturally extended to the tempered distributions by

$$\hat{u}(\phi) = u(\hat{\phi}), \qquad \phi \in \mathcal{S}(\mathbb{R}^n).$$

Theorem 2.7. *The Fourier transform $\mathcal{F}$ extends to a unitary map from $L^2(\mathbb{R}^n)$ to itself and thus the following identity of Parseval holds*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Furthermore since $L^p \subset S'(\mathbb{R}^n)$ the Fourier transform is also defined for all such spaces.

2.2 Sobolev Spaces

Definition 2.8. Let Ω be a nonempty open set in $\mathbb{R}^n$, $1 \leq p \leq \infty$ and s be a nonnegative integer. The Sobolev space $W^{s,p}$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq s$, $\partial^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$. $W^{s,p}$ is a normed space equipped with the norm

$$\|u\|_{W^{s,p}} := \begin{cases} \left(\sum_{|\alpha|\leq s} \int_\Omega |\partial^\alpha u|^p dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \ , \\ \sum_{|\alpha|\leq s} \operatorname{esssup}_\Omega |\partial^\alpha u| & \text{if } p = \infty. \end{cases}$$

Remark 2.9. Among the spaces $W^{s,p}$, particular importance is attached to $W^{s,2}$ because they are Hilbert spaces. We denote them by H^s.

Definition 2.10 (Fractional H^s-Sobolev spaces)**.** Let $s \in \mathbb{R}$. We say that $u \in H^s(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ has a locally integrable Fourier transform and

$$\|u\|^2_{H^s} := \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty.$$

In the following $X \hookrightarrow Y$ denotes a continuous embedding of X into Y, and $X \subset\subset Y$ denotes a compact embedding.

Proposition 2.11. *If*

$$1 < p \le q \le \infty \qquad \text{and} \qquad 0 \le t \le s < \infty$$

are such that

$$\frac{n}{p} - s \le \frac{n}{q} - t,$$

and such that at least one of the two inequalities

$$q \le \infty, \qquad \frac{n}{p} - s \le \frac{n}{q} - t$$

is strict, then

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow W^{t,q}(\mathbb{R}^n).$$

Next, we recall the definitions of the homogeneous Sobolev spaces which are commonly used, because of the symmetry properties they have.

Definition 2.12 (Homogeneous Sobolev Space)**.** We call the space $\dot{H}^s$ equipped with the following semi-norm

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty \tag{2.1}$$

the homogeneous Sobolev space.

Definition 2.13 (Non-isotropic Homogeneous Sobolev space)**.** Let $s_1, s_2 \in \mathbb{R}$. $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$ is the space of tempered distributions with

$$\|u\|_{\dot{H}^{s_1,s_2}} := \left(\int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} |\hat{u}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} < \infty. \tag{2.2}$$

2.3 Besov Spaces

The Littlewood-Paley theory is a method of decomposing a function into a sum of infinitely many frequency localised components, that have almost disjoint frequency supports. In the following we present one of the standard ways of setting up the Littlewood-Paley theory. We start with introducing a dyadic partition of unity. Let $\phi(\xi)$ be a real radial bump function such that

$$\phi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1, \\ 0 & \text{if } |\xi| > 2, \end{cases}$$

and $\chi(\xi) = \phi(\xi) - \phi(2\xi)$. Then $\chi(\xi)$ is supported on $\left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and satisfies

$$\sum_{k \in \mathbb{Z}} \chi(2^{-k}\xi) = 1.$$

We define the Littlewood-Paley projection P_k by

$$\widehat{P_k f}(\xi) = \chi(\xi/2^k)\hat{f}(\xi)$$

in frequency space, or equivalently in physical space by

$$P_k f = f_k = m_k * f,$$

where $m_k(x) = 2^{nk}m(2^k x)$ and $m(x)$ is the inverse Fourier transform of χ. Then $\forall f \in L^2(\mathbb{R}^n)$ we have

$$f = \sum_{k \in \mathbb{Z}} P_k f.$$

We sum up the crucial properties of the Littlewood-Paley projections in the following theorem.

Theorem 2.14. *The Littlewood-Paley projections have the following properties:*

(i) [Almost Orthogonality] The operators P_k are selfadjoint. Furthermore, the family $\{P_k f\}_k$ is almost orthogonal in $L^2(\mathbb{R}^n)$ in the following sense

$$P_{k_1}P_{k_2} = 0 \text{ whenever } |k_1 - k_2| \geq 2$$

and

$$\|f\|_{L^2} \approx \sum_k \|P_k f\|_{L^2}^2,$$

which is an easy consequence of Parseval's Identity.

(ii) *[L^p−boundedness] Let $J \subset \mathbb{Z}$ and $1 \leq p \leq \infty$. Then the following estimate holds true*

$$\|P_J f\|_{L^p} \lesssim \|f\|_{L^p}.$$

(iii) *[Finite band property] Let k be an integer. For any $1 \leq p \leq \infty$*

$$\|\partial P_k f\|_{L^p} \lesssim 2^k \|f\|_{L^p},$$
$$2^k \|P_k f\|_{L^p} \lesssim \|\partial f\|_{L^p}.$$

(iv) *[Bernstein inequalities] For any $1 \leq p \leq q \leq \infty$ we have*

$$\|P_k f\|_{L^q} \lesssim 2^{kn(1/p-1/q)} \|f\|_{L^p}, \quad \forall k \in \mathbb{Z},$$
$$\|P_{\leq 0} f\|_{L^q} \lesssim \|f\|_{L^p}.$$

Remark 2.15. The Bernstein inequality is a remedy for the failure of

$$W^{\frac{n}{p},p}(\mathbb{R}^n) \subset\subset L^\infty(\mathbb{R}^n).$$

The Littlewood-Paley theory has proven to be invaluable in studying partial differential equations. It allows us to decompose the data into pieces, solve the problem on each piece, and then "sum" these solution components.

Remark 2.16. The definitions of Sobolev norms can alternatively be given and extended to $s \in \mathbb{R}$ by using the Littlewood-Paley theory as follows

$$\|f\|_{\dot{W}^{s,p}} \approx \Big\| \sum_{k\in\mathbb{Z}} 2^{ks} P_k f \Big\|_{L^p},$$
$$\|f\|_{W^{s,p}} \approx \Big\| \sum_{k\in\mathbb{Z}} (1+2^k)^s P_k f \Big\|_{L^p}.$$

Definition 2.17 (Besov Spaces)**.** Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. The Besov space is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm defined by

$$\|f\|_{B^s_{p,q}} := \begin{cases} \left(\|P_{\le 0} f\|^q_{L^p} + \sum_{k=1}^\infty 2^{sqk} \|P_k f\|^q_{L^p}\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \\ \sup\{\|P_{\le 0} f\|_{L^p}, 2^{sk}\|P_k f\|_{L^p}\} & \text{if } q = \infty. \end{cases}$$

Definition 2.18 (Homogeneous Besov Spaces)**.** Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. The homogeneous Besov norm is defined by

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{sqk} \|P_k f\|^q_{L^p}\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \\ \sup_k 2^{sk}\|P_k f\|_{L^p} & \text{if } q = \infty. \end{cases}$$

We collect the main Besov space embeddings in the following proposition.

Proposition 2.19. *Assume that* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. *Then*

(i) $B^s_{p,q} \hookrightarrow B^{s_1}_{p_1,q_1}$, *if* $1 \le p \le p_1 \le \infty$, $1 \le q \le q_1 \le \infty$, $s, s_1 \in \mathbb{R}$,

(ii) $B^s_{p,p} \hookrightarrow W^{s,p} \hookrightarrow B^s_{p,2}$, *if* $s \in \mathbb{R}$, $1 < p \le 2$,

(iii) $B^s_{p,2} \hookrightarrow W^{s,p} \hookrightarrow B^s_{p,p}$, *if* $s \in \mathbb{R}$, $2 \le p < \infty$.

The anisotropic Besov spaces are called Besov-Nikol'skii spaces in literature.

Definition 2.20 (Besov-Nikol'skii Spaces)**.** Suppose $S = (s_1, s_2, \dots, s_n) \in \mathbb{R}^n$, $N = (N_1, N_2, \dots, N_n) \in \mathbb{Z}^n$ and $1 \le p, q \le \infty$. The linear space $B^S_{p,q}$ of tempered distributions equipped with the norm

$$\|f\|_{B^S_{p,q}} = \left(\|P_{(N_1\le 0, N_2 \le 0, \dots, N_n \le 0)} f\|^q_{L^p} + \sum_{N\in\mathbb{Z}^n_+} 2^{q(S\cdot N)} \|P_N f\|^q_{L^p} \right)^{1/q},$$

is called a Besov-Nikol'skii space.

2.4 Bourgain Spaces

In this section, we present Bourgain spaces (also known as Fourier restriction spaces, or $X^{s,b}$ spaces). The Bourgain spaces are constructed based on the linear part of the dispersive equation.

Let h be a real valued polynomial and $L = ih\left(\frac{1}{i}\nabla\right)$. We consider

$$\partial_t u - Lu = 0. \tag{2.3}$$

Taking the space-time Fourier transform of (2.3) we get

$$[\tau - \hat{h}(\xi)]\hat{u}(\tau, \xi) = 0.$$

Then $\hat{u}(\tau, \xi)$ is supported in $\{(\tau, \xi) : \tau = h(\xi)\}$ which is called the characteristic hypersurface of the space-time frequency space $\mathbb{R} \times \mathbb{R}^n$.

Hence

$$\hat{u}(\tau, \xi) = \delta(\tau - \hat{h}(\xi))\hat{u}_0(\xi),$$

where δ is the Dirac delta function defined by

$$\delta(\phi) = \phi(0).$$

Now we consider a nonlinear perturbation of (2.3)

$$\partial_t u - Lu - N(u) = 0. \tag{2.4}$$

Note that if one multiplies a solution of (2.4) by suitably short time cutoff function, then for many types of nonlinearities and initial data the localised Fourier transform concentrates near the characteristic hypersurface. Because Bourgain spaces are built on the linear parts of dispersive equations, they reflect this dispersive smoothing effect.

Definition 2.21 ($X^{s,b}$ spaces)**.** Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, and let $s, b \in \mathbb{R}$. The space $X^{s,b}_{\tau=h(\xi)}(\mathbb{R} \times \mathbb{R}^n)$, abbreviated $X^{s,b}(\mathbb{R} \times \mathbb{R}^n)$ or simply $X^{s,b}$, is then defined to be the closure of the Schwartz functions $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{R}^n)$ under the norm

$$\|u\|_{X^{s,b}_{\tau=h(\xi)}(\mathbb{R}\times\mathbb{R}^n)} := \|(1+|\xi|^2)^{s/2}(1+|\tau-h(\xi)|^2)^{b/2}\hat{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi(\mathbb{R}\times\mathbb{R}^n)}.$$

Observe that if we take $b=0$, then the $X^{s,b}$ space is $L^2_t H^s_x$, and if we take $h=0$ the $X^{s,b}$ space is simply $H^b_t H^s_x$.

Lemma 2.22 (The Basic Properties of $X^{s,b}$ spaces)**.**

(i) $X^{s,b}$ *spaces are Banach spaces,*

(ii) $X^{s',b'}_{\tau=h(\xi)} \hookrightarrow X^{s,b}_{\tau=h(\xi)}$ *whenever* $s' \geq s$ *and* $b' \geq b$*,*

(iii) $\left(X^{s,b}_{\tau=h(\xi)}\right)^* = X^{-s,-b}_{\tau=-h(-\xi)}$*,*

(iv) The $X^{s,b}$ *spaces are invariant under translations in space and time,*

(v) $\|\overline{u}\|_{X^{s,b}_{\tau=-h(-\xi)}} = \|u\|_{X^{s,b}_{\tau=h(\xi)}}$.

2.5 U^p and V^p spaces

In this section, we give a brief summary of the theory of U^p and V^p function spaces covered in detail in [10] and [17]. These spaces are useful in the analysis of nonlinear dispersive partial differential equations and have better properties than $X^{s,b}$ spaces especially at critical regularity. The U^p spaces have been introduced by H. Koch and D. Tataru in [18], [19] and the V^p spaces have been introduced by N. Wiener in [34].

2.5.1 U^p spaces

Let

$$\mathcal{Z} = \{(t_0,t_1,\ldots,t_K) \mid -\infty = t_0 < t_1 < \ldots < t_K = \infty\}$$

and

$$\mathcal{Z}_0 = \{(t_0,t_1,\ldots,t_K) \mid -\infty < t_0 < t_1 < \ldots < t_K < \infty\}$$

be the sets of finite partitions.

Definition 2.23. Let $1 \leq p < \infty$. Assume $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with

$$\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^p = 1 \quad \text{and} \quad \phi_0 = 0.$$

The function $a : \mathbb{R} \to L^2$ given by

$$a = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

is called a U^p-atom.

The atomic space U^p is defined as

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \mid a_j \ U^p - \text{atom}, \ \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\},$$

with norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \mid u = \sum_{j=1}^{\infty} \lambda_j a_j, \ \lambda_j \in \mathbb{C}, \ a_j \ U^p\text{-atom} \right\}.$$

Proposition 2.24 (Properties of U^p spaces)**.** *Let $1 \leq p < q < \infty$.*

(i) U^p is a Banach space,

(ii) $U^p \hookrightarrow U^q \hookrightarrow L^\infty(\mathbb{R}; L^2)$,

(iii) Every $u \in U^p$ is right continuous,

(iv) $\lim_{t \to -\infty} u(t) = 0, \ \lim_{t \to \infty} u(t)$ exists,

(v) The closed subspace of all continuous U^p functions, denoted by U_c^p, is a Banach space.

2.5.2 V^p spaces

Definition 2.25. The V^p space is the normed space of all functions $v : \mathbb{R} \to L^2$ such that $\lim_{t \to \pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{\frac{1}{p}}$$

is finite with $v(-\infty) = \lim_{t\to-\infty} v(t)$ and $v(\infty) = 0$.

V_-^p denotes the closed subspace of all $v \in V^p$ with $\lim_{t\to-\infty} v(t) = 0$.

Proposition 2.26 (Properties of V^p space)**.** *Let $1 \le p < q < \infty$.*

(i) Define

$$\|v\|_{V_0^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}_0} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right).$$

If $v : \mathbb{R} \to L^2$ and $\|v\|_{V_0^p} < \infty$, then v has left and right limits at every point. Moreover

$$\|v\|_{V^p} = \|v\|_{V_0^p}.$$

(ii) The closed subspaces of all right-continuous V^p and V_-^p functions are denoted by V_{rc}^p and $V_{-,rc}^p$, respectively.

(iii) $U^p \hookrightarrow V_{-,rc}^p$.

(iv) $V^p \hookrightarrow V^q$ and $V_-^p \hookrightarrow V_-^q$.

(v) $V_{-,rc}^p \hookrightarrow U^q$.

Proposition 2.27 (Duality)**.** *Let $u \in U^p$, $v \in V^{p'}$ and $\mathfrak{t} = \{t_k\}_{k=0}^K \in \mathcal{Z}$. Define*

$$B_{\mathfrak{t}}(u,v) := \sum_{k=1}^{K} \langle u(t_k) - u(t_{k-1}), v(t_k) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes L^2 inner product. There exists a unique number $B(u,v)$, such that for all $\varepsilon > 0$ there exists $\mathfrak{t} \in \mathcal{Z}$ such that for every $\mathfrak{t}' \supset \mathfrak{t}$

$$|B_{\mathfrak{t}'}(u,v) - B(u,v)| < \varepsilon.$$

is satisfied. Furthermore the associated bilinear form $B : (u,v) \mapsto B(u,v)$ satisfies

$$|B(u,v)| \le \|u\|_{U^p} \|v\|_{V^{p'}}.$$

Theorem 2.28. *Let* $1 < p < \infty$. *Then*

$$(U^p)^* = V^{p'},$$

in the sense that the operator

$$T : V^{p'} \to (U^p)^*,$$

defined by

$$T(v) := B(\cdot, v)$$

is an isometric isomorphism.

Proposition 2.29. *Let* $1 < p < \infty$, $u \in U^p$ *be continuous and* $v, v^* \in V^{p'}$. *Suppose that* $v(s) = v^*(s)$ *except for countably many points. Then*

$$B(u, v) = B(u, v^*).$$

Proposition 2.30. *Suppose that* $1 < p < \infty$, $v \in V^{p'}$ *and* $u \in V^1_-$ *is absolutely continuous on compact intervals. Then*

$$B(u, v) = -\int_{-\infty}^{\infty} \langle u'(t), v(t) \rangle dt.$$

Chapter 3

The Kadomtsev-Petviashvili II equation

In this chapter, we present the results of our attempt to solve the problem of the stability of line solitons

$$Q_c(x,y) = \frac{c}{2}\text{sech}^2\left(\frac{c^{1/2}x}{2}\right), \quad c > 0 \tag{3.1}$$

for the Kadomtsev-Petviashvili II (KP-II) equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u + 3\partial_x^{-1}\partial_y^2 u = 0, \tag{3.2}$$

where $u = u(t,x,y)$ is a real valued function and

$$(\partial_x^{-1}u)(x) := -\int_x^\infty u(s)ds. \tag{3.3}$$

The validity of the conserved quantities of the KP-II equation requires the following two constraints on the initial data

$$\int_{-\infty}^{\infty} u(x,y)dx = 0, \tag{3.4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{x} u(x',y)dx'dx = 0. \tag{3.5}$$

The solution that evolves from the initial data satisfying (3.4) and (3.5) preserves these constraints for all time, [33].

3.1 Linear Theory

In this section, we study the linear equation

$$\partial_t w + \partial_x^3 w - \partial_x w + 6\partial_x(Qw) + 3\partial_x^{-1}\partial_y^2 w = F, \tag{3.6}$$

where Q is the line soliton defined by (3.1) with $c = 1$.

The linear equation (3.6) results from linearization of (3.2) around Q in a moving coordinate system

$$x \to x - t.$$

First, we derive a local smoothing estimate for the solution of the linearized problem (3.6) without the potential term

$$\partial_t w + \partial_x^3 w - \partial_x w + \cancel{6\partial_x(Qw)} + 3\partial_x^{-1}\partial_y^2 w = F. \tag{3.7}$$

Next, we estimate the initial data in terms of the inhomogenous data using T^*T principle, [8]. Then, we prove estimates relating the solutions of the homogeneous linearized equation with and without potential term in L^2, L^∞ and L^1 spaces in $x-$direction using the mapping properties of Miura type transforms. Finally, we use properties of Miura maps and the local smoothing estimate obtained for (3.7) to prove the main result of this chapter, stated in the following theorem.

Theorem 3.1. *[A Local Smoothing Estimate]*

Let η be the Fourier variable corresponding to y and w be a solution of

$$\partial_t w + \partial_x^3 w - \partial_x w + 6\partial_x(Qw) + 3\partial_x^{-1}\partial_y^2 w = \underbrace{f + \partial_x g + \partial_x^{-1}\partial_y h}_{=F}, \tag{3.8}$$

where f, g and h have compact supports in $t \geq 0$.

Then we have the following local smoothing estimate

$$\|\mathcal{F}_y(w)\|_{L_x^\infty L_t^2} + \|\partial_x \mathcal{F}_y(w)\|_{L_x^\infty L_t^2} + \|\eta \partial_x^{-1} \mathcal{F}_y(w)\|_{L_x^\infty L_t^2} \lesssim \|\mathcal{F}_y(f)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(g)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(h)\|_{L_x^1 L_t^2} \quad (3.9)$$

provided that $\eta \neq 0$.

3.1.1 A Local Smoothing Estimate Part I

We study here the linear problem without the potential term $\partial_x(Qw)$, namely the equation (3.7). First, we prove the local smoothing estimate (3.9) for (3.7) with no restriction on η.

Theorem 3.2. *Let* w *be a solution of*

$$\partial_t w + \partial_x^3 w - \partial_x w + 3\partial_x^{-1}\partial_y^2 w = \underbrace{f + \partial_x g + \partial_x^{-1}\partial_y h}_{=F}, \quad (3.10)$$

then

$$\|\mathcal{F}_y(w)\|_{L_x^\infty L_t^2 L_\eta^2} + \|\partial_x \mathcal{F}_y(w)\|_{L_x^\infty L_t^2 L_\eta^2} + \|\eta \partial_x^{-1} \mathcal{F}_y(w)\|_{L_x^\infty L_t^2 L_\eta^2} \lesssim \|\mathcal{F}_y(f)\|_{L_x^1 L_t^2 L_\eta^2} + \|\mathcal{F}_y(g)\|_{L_x^1 L_t^2 L_\eta^2} + \|\mathcal{F}_y(h)\|_{L_x^1 L_t^2 L_\eta^2}, \quad (3.11)$$

where η *is the Fourier variable corresponding to* y *variable and* f, g, h *have compact supports in* $t \geq 0$.

Proof. We take the Fourier transform of (3.10) with respect to t, x and y

$$i\tau\hat{w} - i\xi^3\hat{w} - i\xi\hat{w} + 3i\frac{\eta^2}{\xi}\hat{w} = \hat{f} + i\xi\hat{g} + \frac{\eta}{\xi}\hat{h}. \quad (3.12)$$

Then we solve the above algebraic equation for $\hat{w}$ and take its inverse Fourier transform with respect to x which formally can be written as

$$\mathcal{F}_{ty}(w) = (2\pi)^{-\frac{1}{2}} \int \frac{\hat{f} + i\xi\hat{g} + \frac{\eta}{\xi}\hat{h}}{\tau - \xi^3 - \xi + 3\frac{\eta^2}{\xi}} e^{ix\xi} d\xi. \quad (3.13)$$

Then we calculate the $L_x^\infty L_t^2 L_\eta^2$ norms of the above expression, $\partial_x \mathcal{F}_{ty}(w)$ and $\eta\partial_x^{-1}\mathcal{F}_{ty}(w)$ which are simply the terms on the left hand side of the local smoothing estimate (3.11) that we want to prove. Before proceeding with the calculations we make the following two remarks which will help to make the integral on the right hand side of (3.13) well-defined.

Remark 3.3. Consider the Fourier transform of $F(= f + \partial_x g + \partial_x^{-1}\partial_y h)$ with respect to t

$$\hat{F}(\tau) = \int_0^\infty F(t)e^{-it\tau}dt.$$

Note that for $z = Re^{i\theta}$

$$\hat{F}(z) = \int_0^\infty F(t)e^{-iRt(\cos\theta + i\sin\theta)}dt$$

since $t \geq 0$ and $R \geq 0$ the above integral is bounded only if $\theta \in [\pi, 2\pi]$. Then $F(\tau)$ is nothing but the restriction of a holomorphic function defined on the lower half plane to the real axis.

Remark 3.4. Let $\mathcal{A}$ be the antiderivative operator defined by (3.3). Assume that $\phi \in \mathcal{S}(\mathbb{R})$.

If $\xi = 0$, then

$$\mathcal{F}[\mathcal{A}\phi](\xi) = -(2\pi)^{1/2}\int_{-\infty}^\infty \int_x^\infty \phi(s)dsdx.$$

If $\xi \neq 0$, then

$$\begin{aligned}
\mathcal{F}[\mathcal{A}\phi](\xi) &= -(2\pi)^{1/2}\int_{-\infty}^\infty e^{-ix\xi}\int_x^\infty \phi(s)dsdx \\
&= \frac{(2\pi)^{1/2}}{i\xi}e^{-ix\xi}\int_x^\infty \phi(s)ds\Big|_{x=-\infty}^{x=\infty} + (2\pi)^{1/2}\int_{-\infty}^\infty \frac{1}{i\xi}e^{-ix\xi}\phi(x)dx \\
&= \frac{1}{i\xi}\hat{\phi}(\xi).
\end{aligned}$$

Note that the right hand side is also defined for every complex ξ with positive imaginary part.

Thus we can write (3.12) as

$$i(\tau - i0)\hat{w} - i\xi^3\hat{w} - i\xi\hat{w} + 3i\frac{\eta^2}{\xi + i0}\hat{w} = \hat{f} + i\xi\hat{g} + \frac{\eta}{\xi + i0}\hat{h}.$$

Hence, in order to obtain (3.11) it is enough to show that the following 6 simpler integrals are uniformly bounded

$$\begin{aligned}
I_1 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi,\\
I_2 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\xi^2 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi,\\
I_3 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\eta e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi,\\
I_4 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi,\\
I_5 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\eta\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi,\\
I_6 &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\frac{\eta^2 e^{ix\xi}}{(\xi+i\varepsilon)(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi.
\end{aligned}$$

Note that I_4 only exists as improper Lebesgue integral, that is as the limit

$$\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\int_{-R}^{R}\frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi.$$

Let us denote the denominator of I_i for $i = 1\ldots, 5$ by

$$\overline{p}(\xi) = p(\xi) - i0\xi.$$

Observe that $p'(\xi) = 4\xi^3 + 2\xi - \tau$ and $p''(\xi) = 12\xi^2 + 2 > 0$ so p is a strictly convex function and $p(\xi)$ has one nonnegative and one nonpositive real root and 2 complex roots that are conjugates. Simple algebraic calculations show that adding $i0\xi$ to $p(\xi)$ pushes both real roots to the lower half plane. To be more precise the roots of $\overline{p}(\xi)$ which we denote by ξ_1, ξ_2, ξ_3 and ξ_4 have the following properties

$$\mathrm{Im}(\xi_1) = -\varepsilon_1 < 0, \tag{3.14}$$

$$\mathrm{Im}(\xi_2) = -\varepsilon_2 < 0, \tag{3.15}$$

$$\mathrm{Im}(\xi_3) > 0, \tag{3.16}$$

$$\mathrm{Im}(\xi_4) < 0, \tag{3.17}$$

where ε_1 and ε_2 are small positive numbers.

We continue by further analysing the polynomial $\overline{p}(\xi)$. We study the polynomial $\overline{p}(\xi)$ in the following 9 regions:

Region I:= $\{(\tau,\eta) : |\tau| \leq \frac{1}{2} \text{ and } |\eta| \leq \frac{1}{4}\}$,

Region II:= $\{(\tau,\eta) : |\tau| \leq \frac{1}{2} \text{ and } |\eta| > \frac{1}{4}\}$,

Region III:= $\{(\tau,\eta) : |\tau| \geq 10, |\eta| \geq 10 \text{ and } 3\frac{|\eta|^2}{|\tau|} \geq |\tau|^{1/3}\}$,

Region IV:= $\{(\tau,\eta) : |\tau| \geq 10, |\eta| \geq 10 \text{ and } 3\frac{|\eta|^2}{|\tau|} < |\tau|^{1/3}\}$,

Region V:= $\{(\tau,\eta) : |\tau| \geq 10, |\eta| \leq 10 \text{ and } |\tau| < 10|\eta|^2\}$,

Region VI:= $\{(\tau,\eta) : |\tau| \geq 10, |\eta| \leq 10 \text{ and } |\tau| \geq 10|\eta|^2\}$,

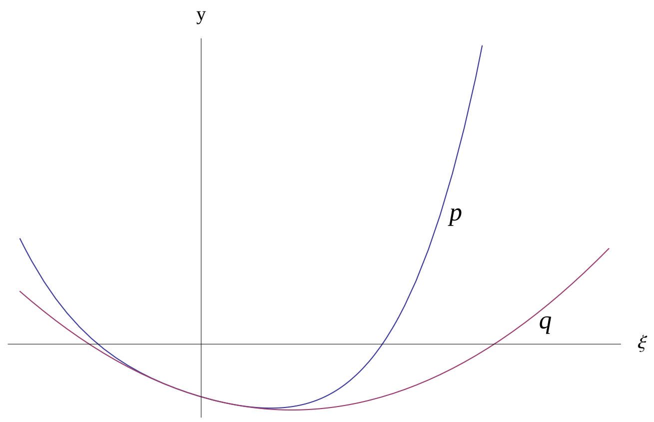
Region VII:= $\{(\tau,\eta) : \frac{1}{2} < |\tau| < 10, \frac{1}{4} < |\eta| \text{ and } |\tau| < |\eta|\}$,

Region VIII:= $\{(\tau,\eta) : \frac{1}{2} < |\tau| < 10, \frac{1}{4} < |\eta| \text{ and } |\tau| \geq |\eta|\}$,

Region IX:= $\{(\tau,\eta) : \frac{1}{2} < |\tau| < 10 \text{ and } |\eta| \leq \frac{1}{4}\}$.

In the region I, we approximate the real roots of $p(\xi)$ by the roots of the quadratic polynomial $q(\xi) = \xi^2 - \tau\xi - 3\eta^2$. Because

(i) $p'(0) = q'(0)$ and $p''(\xi) \geq q''(\xi)$ which suggest a picture as follows



(ii) As $\tau \to 0$ and $\eta \to 0$, the real roots of $p(\xi)$ approach to the roots of $q(\xi)$.

Since

$$p(\xi) = q(\xi)(\xi^2 + \tau\xi + 1 + \tau^2 + 3\eta^2) + (\tau^3 + 6\eta^2\tau)\xi + 3\eta^2\tau^2 + 9\eta^4$$

then the roots of $\overline{p}(\xi)$ are as follows:

$$\xi_1 \approx \frac{\tau - \sqrt{\tau^2 + 12\eta^2}}{2} - i\varepsilon_1,$$
$$\xi_2 \approx \frac{\tau + \sqrt{\tau^2 + 12\eta^2}}{2} - i\varepsilon_2,$$
$$\xi_3 \approx \frac{-\tau + \sqrt{-4 - 3\tau^2 - 12\eta^2}}{2},$$
$$\xi_4 \approx \frac{-\tau - \sqrt{-4 - 3\tau^3 - 12\eta^2}}{2}.$$

The $\approx$ sign above denotes a constant bound on the error which can be shown to be strictly less than $\frac{4}{5}|\eta|$ using the fact (ii) and the numerical data obtained by Mathematica 8.

Moreover

$$\mathrm{Im}(\xi_1) = \begin{cases} 0 \text{ if } \eta = 0 \text{ and } \tau \geq 0, \\ \text{strictly negative otherwise,} \end{cases}$$

$$\mathrm{Im}(\xi_2) = \begin{cases} 0 \text{ if } \eta = 0 \text{ and } \tau < 0, \\ \text{strictly negative otherwise,} \end{cases}$$

and

$$\left|\mathrm{Im}(\xi_{3,4})\right| \geq 1.$$

In regions II, III and VII one can approximate the roots of $p(\xi)$ by the roots of the simpler quartic polynomial $r(\xi) = \xi^4 + \xi^2 - 3\eta^2$. The numerical data obtained by Mathematica 8 suggests that

$$\begin{aligned}
&d(\{\text{roots of } r(\xi)\}, \{\text{roots of } p(\xi)\}) < 0.2 \quad \text{in Region II,}\\
&d(\{\text{roots of } r(\xi)\}, \{\text{roots of } p(\xi)\}) < 3^{1/4}\eta^{1/2} \quad \text{in Region III,}\\
&d(\{\text{roots of } r(\xi)\}, \{\text{roots of } p(\xi)\}) < 0.15 \quad \text{in Region VII,}
\end{aligned}$$

where d denotes the distance function defined by

$$d(\{\text{roots of } r(\xi)\}, \{\text{roots of } p(\xi)\}) := \inf_{\substack{\zeta_i:\ \text{root of } r(\xi),\\ \xi_i:\ \text{root of } p(\xi)}} d(\zeta_i, \xi_i).$$

Then we have

$$\begin{aligned}
\xi_1 &\approx -\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} - i\varepsilon_1, & \xi_2 &\approx \sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} - i\varepsilon_2,\\
\xi_3 &\approx i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}, & \xi_4 &\approx -i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}.
\end{aligned}$$

In analysing the regions IV, V, VIII and IX the theorem stated below will prove to be useful.

Theorem 3.5. *Let a_i, $x \in \mathbb{C}$ for $i = 1, \ldots, n$ and*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

(i) If there is a positive real number m such that

$$|a_0| \geq |a_1|m + |a_2|m^2 + \cdots + |a_n|m^n \tag{3.18}$$

then m is a lower bound for the size of all the roots of the polynomial $p(x)$. For example

$$m = \frac{|a_0|}{\max\{|a_0|, |a_1| + |a_2| + \ldots + |a_n|\}}$$

is a solution of the inequality (3.18).

(ii) If

$$|a_n|M^n \geq |a_0| + |a_1|M + \ldots + |a_{n-1}|M^{n-1} \tag{3.19}$$

then M is an upper bound for the size of all the roots of $p(x)$ and

$$M = \max\{1, \frac{1}{|a_n|}(|a_0| + |a_1| + \cdots + |a_{n-1}|)\}$$

is a solution of (3.19).

Proof. Let r be an arbitrary root of the polynomial $p(x)$.

(i) If $|r| < m$, then

$$|a_0| = \left|\sum_{j=1}^{n} a_j r^j\right| \leq \sum_{j=1}^{n} |a_j||r|^j < \sum_{j=1}^{n} |a_j| m^j,$$

which is the contrapositive form of the statement (i).

(ii) If $|r| > M$, then

$$\begin{aligned} 0 &= \left|\sum_{j=0}^{n} a_j r^j\right| = |r|^n \left|\sum_{j=0}^{n} a_j r^{j-n}\right| \\ &\geq |r|^n \left(|a_n| - \sum_{j=0}^{n-1} |a_j||r|^{j-n}\right) \\ &> |r|^n \left(|a_n| - \sum_{j=0}^{n-1} |a_j| M^{j-n}\right) \\ &= \frac{|r|^n}{M^n} \left(|a_n| M^n - \sum_{j=0}^{n-1} |a_j| M^j\right) \end{aligned}$$

which completes the proof, since it is the contrapositive of the statement we wanted to prove.

□

In region IV, $p(\xi)$ has one root that has size smaller than $3\frac{\eta^2}{|\tau|}$ and the remaining roots have sizes larger than $|\tau|^{1/3}$. Moreover

$$\min\left\{|\xi_i|\right\} \geq \frac{3}{2}\frac{\eta^2}{|\tau|} \quad \text{and} \quad \max\left\{|\xi_i|\right\} < 2|\tau|^{1/3}$$

due to the Theorem 3.5.

In region V again thanks to the Theorem 3.5,

$$\min\left\{|\xi_i|\right\} \geq 0.29 \quad \text{and} \quad \max\left\{|\xi_i|\right\} < 11.$$

In region VI, the roots of $\overline{p}(\xi)$ can be approximated as follows

$$\begin{aligned}
\xi_1 &\approx -\frac{3\eta^2}{\tau} - i\varepsilon_1, \\
\xi_2 &\approx \operatorname{sgn}(\tau)|\tau|^{1/3} - i\varepsilon_2, \\
\xi_3 &\approx \operatorname{sgn}(\tau)\left(-\frac{|\tau|^{1/3}}{2} + \frac{\sqrt{3}}{2}|\tau|^{1/3}i\right), \\
\xi_4 &\approx \operatorname{sgn}(\tau)\left(-\frac{|\tau|^{1/3}}{2} - \frac{\sqrt{3}}{2}|\tau|^{1/3}i\right).
\end{aligned}$$

Moreover

$$\left|\xi_1 - \operatorname{sgn}(-\tau)\frac{3\eta^2}{|\tau|}\right| < 0.01 \quad \text{and} \quad |\xi_i - \tau^{1/3}| < 0.2 \quad \text{for} \quad i = 2, 3, 4.$$

In region VIII we have

$$\min\left\{|\xi_i|\right\} > \frac{1}{54} \quad \text{and} \quad \max\left\{|\xi_i|\right\} < 5.$$

In region IX, $p(\xi)$ has one root that has the same size with

$$-3\frac{|\eta|^2}{|\tau|}+\alpha$$

where

$$\alpha \approx \frac{81\frac{|\eta|^8}{|\tau|^4}+9\frac{|\eta|^4}{|\tau|^2}}{|\tau|+108\frac{|\eta|^6}{|\tau|^3}+6\frac{|\eta|^2}{|\tau|}}.$$

Then we can decompose $p(\xi)$ into dominant parts and a small remainder as follows

$$\xi^4+\xi^2-\tau\xi-3\eta^2=\Big(\xi+\frac{3\eta^2}{\tau}-\alpha\Big)Q(\xi)-\alpha\tau+\Big(\frac{3\eta^2}{\tau}-\alpha\Big)^2+\Big(\frac{3\eta^2}{\tau}-\alpha\Big)^4,$$

where

$$Q(\xi)=\xi^3-\Big(\frac{3\eta^2}{\tau}-\alpha\Big)\xi^2+\left(1+\Big(\frac{3\eta^2}{\tau}-\alpha\Big)^2\right)\xi-\left(\tau+\frac{3\eta^2}{\tau}-\alpha+\Big(\frac{3\eta^2}{\tau}-\alpha\Big)^3\right).$$

It follows from the Theorem 3.5 that $m=0.25$ is a lower bound for the size of each root of $Q(\xi)$.

Also note that in each region all the roots are distinct and therefore all the poles of I_i for $i=1,\dots,5$ are simple.

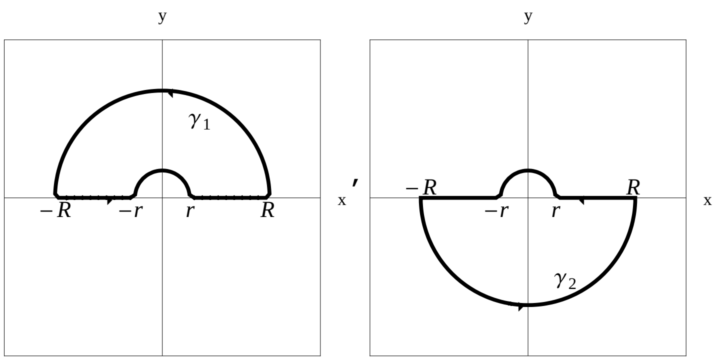
We summarise the analysis of the roots of $\overline{p}(\xi)$ or in other words the analysis of the poles of the integrand of I_i, $i=1,\dots,5$, in the following table

REGIONS	**Poles in upper half plane**	**Poles in lower half plane**
Region I	$\xi_3 \approx \frac{-\tau+\sqrt{-4-3\tau^2-12\eta^2}}{2}$	$\xi_1 \approx \frac{\tau-\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_1$ $\xi_2 \approx \frac{\tau+\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_2$ $\xi_4 \approx \frac{-\tau-\sqrt{-4-3\tau^3-12\eta^2}}{2}$
Regions II, III and VII	$\xi_3 \approx i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}$	$\xi_1 \approx -\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} - i\varepsilon_1$ $\xi_2 \approx \sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} - i\varepsilon_2$ $\xi_4 \approx -i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}$
Region IV	$\lvert\tau\rvert^{\frac{1}{3}} < \lvert\xi_3\rvert < 2\lvert\tau\rvert^{\frac{1}{3}}$	$\frac{3\eta^2}{2\lvert\tau\rvert} \leq \lvert\xi_1\rvert < 3\frac{\lvert\eta\rvert^2}{\lvert\tau\rvert}$ $\lvert\tau\rvert^{\frac{1}{3}} < \lvert\xi_2\rvert, \lvert\xi_4\rvert < 2\lvert\tau\rvert^{\frac{1}{3}}$
Region V	$0.29 \leq \lvert\xi_3\rvert < 11$	$0.29 \leq \lvert\xi_1\rvert, \lvert\xi_2\rvert, \lvert\xi_4\rvert < 11$
Region VI	$\xi_3 \approx \mathrm{sgn}(\tau)\left(-\frac{\lvert\tau\rvert^{1/3}}{2} + \frac{\sqrt{3}}{2}\lvert\tau\rvert^{1/3}i\right)$	$\xi_1 \approx -\frac{3\eta^2}{\tau} - i\varepsilon_1$ $\xi_2 \approx \mathrm{sgn}(\tau)\lvert\tau\rvert^{1/3} - i\varepsilon_2$ $\xi_4 \approx \mathrm{sgn}(\tau)\left(-\frac{\lvert\tau\rvert^{1/3}}{2} - \frac{\sqrt{3}}{2}\lvert\tau\rvert^{1/3}i\right)$
Region VIII	$1/54 \leq \lvert\xi_3\rvert < 5$	$1/54 \leq \lvert\xi_1\rvert, \lvert\xi_2\rvert, \lvert\xi_4\rvert < 5$
Region IX	$\lvert\xi_3\rvert \approx \lvert\tau\rvert^{\frac{1}{3}}$	$\lvert\xi_1\rvert \approx 3\lvert\eta\rvert^2/\lvert\tau\rvert + \alpha$ $\lvert\xi_2\rvert, \lvert\xi_4\rvert \approx \lvert\tau\rvert^{\frac{1}{3}}$

Now that we have the necessary information about the roots of the polynomial $p(\xi)$ we proceed to the calculations of bounds of integrals I_i, $i = 1, \ldots, 6$. Note that the boundedness of the integral I_2 follows from the application of Cauchy-Schwarz inequality and the boundedness of I_1 and I_4. Similarly, the boundedness of I_1 and I_6 imply boundedness of I_3 and the boundedness of I_4 and I_6 imply that I_5 is bounded due to Cauchy-Schwarz inequality. So it suffices to show that the integrals I_1, I_4 and I_6 are bounded.

Claim 1: $|I_1|$ is uniformly bounded.

Proof of Claim 1: We define closed curves γ_1 and γ_2 as illustrated below.



Note that if $x > 0$, then

$$
\begin{aligned}
\int_{\gamma_1} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi &= \int_{-R}^{-r} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi \\
&+ \underbrace{\int_{\pi}^{0} \frac{re^{i\theta}e^{ixr\cos\theta}e^{-xr\sin\theta}ire^{i\theta}d\theta}{(re^{i\theta})^4+(re^{i\theta})^2-(\tau-i\varepsilon)(re^{i\theta})-3\eta^2}}_{:=I_r} \\
&+ \int_{r}^{R} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi \\
&+ \underbrace{\int_{0}^{\pi} \frac{Re^{i\theta}e^{ixR\cos\theta}e^{-xR\sin\theta}iRe^{i\theta}d\theta}{(Re^{i\theta})^4+(Re^{i\theta})^2-(\tau-i\varepsilon)(Re^{i\theta})-3\eta^2}}_{:=I_R}.
\end{aligned}
$$

Since $x > 0$ we have $|e^{-xR\sin\theta}| \leq 1$ and hence

$$|I_R| \longrightarrow 0 \text{ as } R \longrightarrow \infty,$$

and

$$\begin{aligned} &\text{if } \eta = 0 \text{ and } \tau = 0, \quad |I_r| \longrightarrow \pi \text{ as } r \longrightarrow 0, \\ &\text{otherwise } |I_r| \longrightarrow 0 \text{ as } r \longrightarrow 0. \end{aligned}$$

If $x < 0$ we couldn't have argued this way. One alternative would be to choose the closed path γ_2, then we get

$$\begin{aligned} \int_{\gamma_2} \frac{\xi e^{ix\xi}}{\xi^4 + \xi^2 - (\tau - i\varepsilon)\xi - 3\eta^2} d\xi &= \int_R^r \frac{\xi e^{ix\xi}}{\xi^4 + \xi^2 - (\tau - i\varepsilon)\xi - 3\eta^2} d\xi \\ &+ \underbrace{\int_0^\pi \frac{re^{i\theta} e^{ixr\cos\theta} e^{-xr\sin\theta} ire^{i\theta} d\theta}{(re^{i\theta})^4 + (re^{i\theta})^2 - (\tau - i\varepsilon)(re^{i\theta}) - 3\eta^2}}_{=-I_r} \\ &+ \int_{-r}^{-R} \frac{\xi e^{ix\xi}}{\xi^4 + \xi^2 - (\tau - i\varepsilon)\xi - 3\eta^2} d\xi \\ &+ \underbrace{\int_{-\pi}^0 \frac{Re^{i\theta} e^{ixR\cos\theta} e^{-xR\sin\theta} iRe^{i\theta} d\theta}{(Re^{i\theta})^4 + (Re^{i\theta})^2 - (\tau - i\varepsilon)(Re^{i\theta}) - 3\eta^2}}_{:=\overline{I}_R}. \end{aligned}$$

where $|\overline{I}_R| \to 0$ as $R \to \infty$, since $\theta \in (-\pi, 0)$ and $x < 0$.

Finding a uniform bound on $|I_1|$ is thus equivalent to finding a uniform bound on

$$\int_{\gamma_i} \frac{\xi e^{ix\xi}}{\xi^4 + \xi^2 - (\tau - i\varepsilon)\xi - 3\eta^2} d\xi.$$

Tedious but simple estimates on the location of the roots of the polynomial show that the following estimates hold.

Let g denote the integrand of I_1 and $n(\gamma_1; \xi_k)$ denote the index of γ_1 with respect to ξ_k, then in the **Region I** if $x > 0$ we have

$$\begin{aligned}
\left|\int_{\gamma_1} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| &= 2\pi\left|\sum_{k=1}^{4} n(\gamma_1;\xi_k)\mathrm{Res}(g;\xi_k)\right| \\
&= 2\pi\left|\frac{\xi_3 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right| \\
&= \frac{2\pi\left|\frac{-\tau+\sqrt{-4-3\tau^2-12\eta^2}}{2}\right|}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2}+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq 1}\underbrace{\sqrt{4+3\tau^2+12\eta^2}}_{\geq 2}} \\
&\leq 2\pi,
\end{aligned}$$

and if $x<0$

$$\begin{aligned}
&\left|\int_{\gamma_2} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| = \\
&= \frac{2\pi|\xi_1|}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|}+\frac{2\pi|\xi_2|}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|}+\frac{2\pi|\xi_4|}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|} \\
&\quad= \frac{2\pi\left|\frac{\tau-\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_1\right|}{\left|-\sqrt{\tau^2+12\eta^2}-i\varepsilon_1+i\varepsilon_2\right|\underbrace{\left|\frac{2\tau-\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_1+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}} \\
&\quad+\frac{2\pi\left|\frac{\tau+\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_2\right|}{\left|\sqrt{\tau^2+12\eta^2}+i\varepsilon_1-i\varepsilon_2\right|\underbrace{\left|\frac{2\tau+\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_2+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}} \\
&\quad+\frac{2\pi\left|\frac{\tau+i\sqrt{4+3\tau^2+12\eta^2}}{2}\right|}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2}+i\varepsilon_1-i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}\sqrt{4+3\tau^2+12\eta^2}} \\
&\quad\leq 24\pi.
\end{aligned}$$

In **Regions II, III** and **VII** if $x>0$

$$\begin{aligned}
&\left|\int_{\gamma_1}\frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| = 2\pi\left|\frac{\xi_3 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right| \\
&= \frac{\pi}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_1+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_2+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|} \\
&\leq \pi,
\end{aligned}$$

and if $x<0$ then

$$\begin{aligned}
&\left|\int_{\gamma_2}\frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| = \frac{2\pi|\xi_1|}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|} \\
&\quad +\frac{2\pi|\xi_2|}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|}+\frac{2\pi|\xi_4|}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|} \\
&\quad = \frac{2\pi\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1\right|}{\left|-2\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1+i\varepsilon_2\right|\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2} \\
&\quad +\frac{2\pi\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2\right|}{\left|2\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2+i\varepsilon_1\right|\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2} \\
&\quad +\frac{\pi}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_1-i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_2-i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|} \\
&\quad \leq 3\pi.
\end{aligned}$$

In the **Region IV** if $x>0$ then we have

$$\begin{aligned}
\left|\int_{\gamma_1}\frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| &= 2\pi\left|\frac{\xi_3 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right| \\
&\leq \frac{4\pi|\tau|^{1/3}}{\underbrace{|\xi_3-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_3-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_3-\xi_4|}_{>|\tau|^{1/3}}} \\
&< 4\pi,
\end{aligned}$$

and if $x<0$ then

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{\xi e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| &= \frac{2\pi|\xi_1|}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|} \\
&+\frac{2\pi|\xi_2|}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|}+\frac{2\pi|\xi_4|}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|} \\
&\le \frac{4\pi|\tau|^{1/3}}{\underbrace{|\xi_1-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_1-\xi_3|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_1-\xi_4|}_{>\frac{1}{2}|\tau|^{1/3}}}+\frac{4\pi|\tau|^{1/3}}{\underbrace{|\xi_2-\xi_1|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_3|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_4|}_{>|\tau|^{1/3}}} \\
&+\frac{4\pi|\tau|^{1/3}}{\underbrace{|\xi_4-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_4-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_4-\xi_3|}_{>|\tau|^{1/3}}} \\
&< 12\pi.
\end{aligned}$$

In **Region V** $\min\left|\{\xi_i\}\right| > 0.29$ and in **Region VIII** $\min\left|\{\xi_i\}\right| > \frac{1}{54}$. Then

$$|\xi_1-\xi_2| > 0.58 \text{ in Region V}$$

and

$$|\xi_1-\xi_2| > \frac{2}{54} \text{ in Region VIII.}$$

Inserting the above information with (3.16) and (3.17) into the formula of the polynomial and using simple algebraic calculations we find that the minimum distance between any 2 roots of $\overline{p}(\xi)$ (other than $|\xi_1-\xi_2|$) is larger than 1 in both Regions V and VIII. Also since $\max\left|\{\xi_i\}\right| < 11$ in Region V and $\max\left|\{\xi_i\}\right| < 5$ in Region VIII, I_1 is uniformly bounded.

The boundedness of I_1 in **Regions VI** and **IX** follow from very similar calculations that let us to conclude the boundedness of I_1 in the region IV.

Claim 2: $|I_4|$ is uniformly bounded.

Proof of Claim 2:

First note that if $x > 0$

$$\begin{aligned}\int_{\gamma_1}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi &= \int_{-R}^{-r}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\\ &+ \int_{\pi}^{0}\frac{(re^{i\theta})^3e^{ixr\cos\theta}e^{-xr\sin\theta}ire^{i\theta}d\theta}{(re^{i\theta})^4+(re^{i\theta})^2-(\tau-i\varepsilon)(re^{i\theta})-3\eta^2}\\ &+ \int_{r}^{R}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\\ &+ \int_{0}^{\pi}\frac{(Re^{i\theta})^3e^{ixR\cos\theta}e^{-xR\sin\theta}iRe^{i\theta}d\theta}{(Re^{i\theta})^4+(Re^{i\theta})^2-(\tau-i\varepsilon)(Re^{i\theta})-3\eta^2}.\end{aligned}$$

Since

$$\left|\int_{0}^{\pi}\frac{(Re^{i\theta})^3e^{ixR\cos\theta}e^{-xR\sin\theta}iRe^{i\theta}d\theta}{(Re^{i\theta})^4+(Re^{i\theta})^2-(\tau-i\varepsilon)(Re^{i\theta})-3\eta^2}\right|\longrightarrow\pi\quad\text{as}\quad R\longrightarrow\infty,$$

and

$$\left|\int_{\pi}^{0}\frac{(re^{i\theta})^3e^{ixr\cos\theta}e^{-xr\sin\theta}ire^{i\theta}d\theta}{(re^{i\theta})^4+(re^{i\theta})^2-(\tau-i\varepsilon)(re^{i\theta})-3\eta^2}\right|\longrightarrow 0\quad\text{as}\quad r\longrightarrow 0,$$

we have

$$\left|\int_{-R}^{R}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-\eta^2}d\xi\right|\le\left|\int_{\gamma_1}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-\eta^2}d\xi\right|+\pi\ \text{ as }\ R\longrightarrow\infty.$$

If $x<0$ we choose the choose the closed curve γ_2 and repeat a similar calculation.

In **Region I** if $x>0$, we have

$$\begin{aligned}\left|\int_{\gamma_1}\frac{\xi^3e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| &= 2\pi\left|\frac{\xi_3^3e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right|\\ &= \frac{2\pi\left|\frac{-\tau+\sqrt{-4-3\tau^2-12\eta^2}}{2}\right|^3}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2}+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\ge 1}\underbrace{\sqrt{4+3\tau^2+12\eta^2}}_{\ge 2}}\\ &\le 2\pi,\end{aligned}$$

and if $x<0$, then

$$\left|\int_{\gamma_2} \frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2} d\xi\right| =$$

$$= \frac{2\pi|\xi_1|^3}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|} + \frac{2\pi|\xi_2|^3}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|} + \frac{2\pi|\xi_4|^3}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|}$$

$$= \frac{2\pi\left|\frac{\tau-\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_1\right|^3}{\left|-\sqrt{\tau^2+12\eta^2} - i\varepsilon_1 + i\varepsilon_2\right| \underbrace{\left|\frac{2\tau-\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_1 + i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}}$$

$$+ \frac{2\pi\left|\frac{\tau+\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_2\right|^3}{\left|\sqrt{\tau^2+12\eta^2} + i\varepsilon_1 - i\varepsilon_2\right| \underbrace{\left|\frac{2\tau+\sqrt{\tau^2+12\eta^2}}{2} - i\varepsilon_2 + i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}}$$

$$+ \frac{2\pi\left|\frac{\tau+i\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^3}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2} + i\varepsilon_1 - i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}} \sqrt{4+3\tau^2+12\eta^2}}$$

$$\leq 7\pi.$$

In **Regions II, III** and **VII** if $x > 0$

$$\left|\int_{\gamma_1} \frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2} d\xi\right| = 2\pi\left|\frac{\xi_3^3 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right|$$

$$= \frac{\pi\left(\frac{1+\sqrt{1+12\eta^2}}{2}\right)}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} + i\varepsilon_1 + i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right| \left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}} + i\varepsilon_2 + i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|}$$

$$\leq \pi$$

and if $x < 0$ then

$$\left|\int_{\gamma_2}\frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| = \frac{2\pi|\xi_1|^3}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|}$$

$$+\frac{2\pi|\xi_2|^3}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|}+\frac{2\pi|\xi_4|^3}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|}$$

$$=\frac{2\pi\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1\right|^3}{\left|-2\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1+i\varepsilon_2\right|\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2}$$

$$+\frac{2\pi\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2\right|^3}{\left|2\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2+i\varepsilon_1\right|\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2}$$

$$+\frac{\pi\left(\frac{1+\sqrt{1+12\eta^2}}{2}\right)}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_1-i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_2-i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|}$$

$$\leq 4\pi.$$

In the **Region IV** if $x>0$ then we have

$$\begin{aligned}\left|\int_{\gamma_1}\frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| &= 2\pi\left|\frac{\xi_3^3 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)}\right| \\ &\leq \frac{16\pi|\tau|}{\underbrace{|\xi_3-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_3-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_3-\xi_4|}_{>|\tau|^{1/3}}} \\ &< 32\pi,\end{aligned}$$

and if $x<0$ then

$$\left|\int_{\gamma_2} \frac{\xi^3 e^{ix\xi}}{\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2}d\xi\right| = \frac{2\pi|\xi_1|^3}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4|}$$
$$+\frac{2\pi|\xi_2|^3}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4|}+\frac{2\pi|\xi_4|^3}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3|}$$
$$\leq \frac{16\pi|\tau|}{\underbrace{|\xi_1-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_1-\xi_3|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_1-\xi_4|}_{>\frac{1}{2}|\tau|^{1/3}}}+\frac{16\pi|\tau|}{\underbrace{|\xi_2-\xi_1|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_3|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_4|}_{>|\tau|^{1/3}}}$$
$$+\frac{16\pi|\tau|}{\underbrace{|\xi_4-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_4-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_4-\xi_3|}_{>|\tau|^{1/3}}}$$
$$< 80\pi.$$

The calculations for I_4 in **Regions VI** and **IX** are essentially the same as above.

The same reasoning in calculation of a bound for integral I_1 in **Regions V** and **VIII** gives us the boundedness of I_4 as well.

Claim 3: $|I_6|$ is uniformly bounded.

Proof of Claim 3: I_6 has an integrand that has an additional fifth pole compared to I_i, $i = 1, \ldots, 5$. We denote it by ξ_5 and

$$\xi_5 = -i\varepsilon.$$

Note that if $\eta = 0$, which happens in Regions I, VI and IX, then

$$I_6 = 0.$$

If $x > 0$, then we have

$$\begin{aligned}\int_{\gamma_3} \frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-\tau\xi-3\eta^2+i\varepsilon)}d\xi &= \int_{-R}^{-r} \frac{\eta^2 e^{ix\xi}}{\xi^5+\xi^3-\tau\xi^2-3\eta^2\xi+i\varepsilon\xi}d\xi \\ &+ \int_{r}^{R} \frac{\eta^2 e^{ix\xi}}{\xi^5+\xi^3-\tau\xi^2-3\eta^2\xi+i\varepsilon\xi}d\xi \\ &+ \underbrace{\int_{\pi}^{0} \frac{e^{ixr\cos\theta}e^{-xr\sin\theta}\eta^2 ire^{i\theta}d\theta}{(re^{i\theta})^5+(re^{i\theta})^3-\tau(re^{i\theta})^2-3\eta^2 re^{i\theta}+i\varepsilon re^{i\theta}}}_{:=I_r'} \\ &+ \underbrace{\int_{0}^{\pi} \frac{e^{ixR\cos\theta}e^{-xR\sin\theta}\eta^2 iRe^{i\theta}d\theta}{(Re^{i\theta})^5+(Re^{i\theta})^3-\tau(Re^{i\theta})^2-3\eta^2 Re^{i\theta}+i\varepsilon Re^{i\theta}}}_{:=I_R'},\end{aligned}$$

where

$$|I_R| \longrightarrow 0 \text{ as } R \longrightarrow \infty,$$

$$|I_r| \longrightarrow \frac{\pi}{3} \text{ as } r \longrightarrow 0.$$

If $x < 0$, then we choose the closed curve γ_2 and repeat similar calculations.

In **Region I** if $x > 0$ then

$$\begin{aligned}&\left|\int_{\gamma_1} \frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| = 2\pi\left|\frac{\eta^2 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}\right| \\ &= \frac{2\pi\eta^2}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2}+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq 1}\underbrace{\sqrt{4+3\tau^2+12\eta^2}}_{\geq 2}\left|\frac{-\tau+\sqrt{-4-3\tau^2-12\eta^2}}{2}\right|} \\ &\leq \frac{\pi}{3}\end{aligned}$$

and if $x < 0$

$$\left|\int_{\gamma_2}\frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| = \frac{2\pi|\eta^2|}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4||\xi_1|}$$

$$+\frac{2\pi|\eta^2|}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4||\xi_2|}+\frac{2\pi\eta^2}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3||\xi_4|}+\frac{2\pi\eta^2}{|\xi_1||\xi_2||\xi_3||\xi_4|}$$

$$=\frac{2\pi\eta^2}{\sqrt{\tau^2+12\eta^2}\underbrace{\left|\frac{2\tau-\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_1+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}\left|\frac{\tau-\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_1\right|}$$

$$+\frac{2\pi\eta^2}{\sqrt{\tau^2+12\eta^2}\underbrace{\left|\frac{2\tau+\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_2+i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}\left|\frac{\tau+\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_2\right|}$$

$$+\frac{2\pi\eta^2}{\underbrace{\left|\frac{-2\tau+\sqrt{\tau^2+12\eta^2}}{2}+i\varepsilon_1-i\frac{\sqrt{4+3\tau^2+12\eta^2}}{2}\right|^2}_{\geq\frac{1}{2}}\sqrt{4+3\tau^2+12\eta^2}\left|\frac{\sqrt{4+4\tau^2+12\eta^2}}{2}\right|}$$

$$+\frac{2\pi\eta^2}{\left|\frac{\tau-\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_1\right|\left|\frac{\tau+\sqrt{\tau^2+12\eta^2}}{2}-i\varepsilon_2\right|\left|\frac{-\tau+i\sqrt{4+3\tau^2+12\eta^2}}{2}\right|\left|\frac{\tau+i\sqrt{4+3\tau^2+12\eta^2}}{2}\right|}$$

$$\leq 8\pi.$$

In **Regions II, III and VII** if $x>0$

$$\begin{aligned}\left|\int_{\gamma_1}\frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| &= 2\pi\left|\frac{\eta^2 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}\right| \\ &= \frac{\pi\eta^2}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2\left(\frac{1+\sqrt{1+12\eta^2}}{2}\right)} \\ &\leq 2\pi,\end{aligned}$$

and if $x<0$ then

$$\left|\int_{\gamma_2}\frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| = \frac{2\pi\eta^2}{|\xi_1-\xi_2||\xi_1-\xi_3||\xi_1-\xi_4||\xi_1|}$$

$$+\frac{2\pi\eta^2}{|\xi_2-\xi_1||\xi_2-\xi_3||\xi_2-\xi_4||\xi_2|}+\frac{2\pi\eta^2}{|\xi_4-\xi_1||\xi_4-\xi_2||\xi_4-\xi_3||\xi_4|}+\frac{2\pi\eta^2}{|\xi_1||\xi_2||\xi_3||\xi_4|}$$

$$=\frac{2\pi\eta^2}{(-1+\sqrt{1+12\eta^2})\left|-\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_1+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2}$$

$$+\frac{2\pi\eta^2}{(-1+\sqrt{1+12\eta^2})\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2+i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2}$$

$$+\frac{2\pi\eta^2}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_1-i\sqrt{\frac{1+\sqrt{1+12\eta^2}}{2}}\right|^2(1+\sqrt{1+12\eta^2})}$$

$$+\frac{2\pi\eta^2}{\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}+i\varepsilon_1\right|\left|\sqrt{\frac{-1+\sqrt{1+12\eta^2}}{2}}-i\varepsilon_2\right|\left(\frac{1+\sqrt{1+12\eta^2}}{2}\right)}$$

$$\leq 8\pi.$$

In **Regions V** and **VIII** the argument sequence that lead us to deduce the boundedness of integrals I_1 and I_4 and the fact that in these regions $|\eta| < 10$ imply the boundedness of I_6.

In **Regions IV, VI** and **IX** calculations are similar. Here we illustrate the calculations for the **Region IV**.

If $x > 0$ then we have

$$\begin{aligned}\left|\int_{\gamma_1}\frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| &= 2\pi\left|\frac{\eta^2 e^{ix\xi_3}}{(\xi_3-\xi_1)(\xi_3-\xi_2)(\xi_3-\xi_4)(\xi_3-\xi_5)}\right| \\ &\leq \frac{2\pi\eta^2}{\underbrace{|\xi_3-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_3-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_3-\xi_4|}_{>|\tau|^{1/3}}\underbrace{|\xi_3-\xi_5|}_{>\frac{3\eta^2}{2|\tau|}}} \\ &< 3\pi,\end{aligned}$$

and if $x < 0$ then

$$
\begin{aligned}
\left|\int_{\gamma_2} \frac{\eta^2 e^{ix\xi}}{\xi(\xi^4+\xi^2-(\tau-i\varepsilon)\xi-3\eta^2)}d\xi\right| \leq\ & \frac{2\pi\eta^2}{\underbrace{|\xi_1-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_1-\xi_3|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_1-\xi_4|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_1|}_{>\frac{3\eta^2}{2|\tau|}}} \\
&+\frac{2\pi\eta^2}{\underbrace{|\xi_2-\xi_1|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_3|}_{>|\tau|^{1/3}}\underbrace{|\xi_2-\xi_4|}_{>|\tau|^{1/3}}\underbrace{|\xi_2|}_{>|\tau|^{1/3}}} \\
&+\frac{2\pi\eta^2}{\underbrace{|\xi_4-\xi_1|}_{>\frac{1}{2}|\tau|^{1/3}}\underbrace{|\xi_4-\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_4-\xi_3|}_{>|\tau|^{1/3}}\underbrace{|\xi_4|}_{>|\tau|^{1/3}}} \\
&+\frac{2\pi\eta^2}{\underbrace{|\xi_1|}_{>\frac{3\eta^2}{2|\tau|}}\underbrace{|\xi_2|}_{>|\tau|^{1/3}}\underbrace{|\xi_3|}_{>|\tau|^{1/3}}\underbrace{|\xi_4|}_{>|\tau|^{1/3}}} \\
&< 9\pi.
\end{aligned}
$$

□

3.1.2 T^*T Principle

Theorem 3.6. *Assume w is a solution of*

$$\partial_t w + \partial_x^3 w - \partial_x w + 3\partial_x^{-1}\partial_y^2 w = \underbrace{f + \partial_x g + \eta\partial_x^{-1} h}_{=F}, \tag{3.20}$$

where f, g and h have compact supports in $t > 0$. Assume further that

$$\mathcal{F}_{x,y}(w(t,x,y)) \to 0 \text{ as } t \to -\infty.$$

Then

$$\|w(0,x,y)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^1_x L^2_{ty}} + \|g\|_{L^1_x L^2_{ty}} + \|h\|_{L^1_x L^2_{ty}}.$$

The proof of this theorem is an application of Lemma 2.2 of [8]. For the sake of completeness we state the lemma here.

Lemma 3.7. *Let $\mathscr{H}$ be a Hilbert space, X a Banach space, X^* the dual of X, and $\mathcal{D}$ a vector space densely contained in X. Assume that $T : \mathcal{D} \to \mathscr{H}$ is a linear map and $T^* : \mathscr{H} \to \mathcal{D}^*$ is its adjoint, defined by*

$$\langle T^* v, f\rangle_{\mathcal{D}} = \langle v, Tf\rangle, \forall f \in \mathcal{D}, \ \forall v \in \mathscr{H},$$

where $\mathcal{D}^*$ is the algebraic dual of $\mathcal{D}$, $\langle \phi, f\rangle_{\mathcal{D}}$ is the pairing between $\mathcal{D}^*$ and $\mathcal{D}$ (with $f \in \mathcal{D}$ and $\phi \in \mathcal{D}^*$), and $\langle \cdot, \cdot\rangle$ is the scalar product in $\mathscr{H}$ (conjugate linear in the first argument). Then the following conditions are equivalent:

(1) There exists $a \in [0, \infty)$, such that for all $f \in \mathcal{D}$

$$\|Tf\| \leq a\|f\|_X.$$

(2) $\mathscr{R}(T^*) \subset X^*$, and there exists $a \in [0, \infty)$, such that for all $v \in \mathscr{H}$,

$$\|T^* v\|_{X^*} \leq a\|v\|.$$

(3) $\mathscr{R}(T^*T) \subset X^*$, and there exists $a \in [0, \infty)$, such that for all $f \in \mathcal{D}$

$$\|T^*Tf\|_{X^*} \leq a^2\|f\|_X,$$

where $\|\cdot\|$ denotes the norm in $\mathscr{H}$. The constant a is the same in all three parts. If one of (all) those conditions is (are) satisfied, the operators T and T^*T extend by continuity to bounded operators from X to $\mathscr{H}$ and from X to X^*, respectively.

Proof of Theorem 3.6. The solution w of (3.20) can be written as

$$w = w_1 + w_2 + w_3$$

where w_i, for $i = 1, 2, 3$ are the solutions of the following inhomogeneous equations, respectively,

$$\partial_t w_1 + \partial_x^3 w_1 - \partial_x w_1 + 3\partial_x^{-1}\partial_y^2 w_1 = f, \tag{3.21}$$

$$\partial_t w_2 + \partial_x^3 w_2 - \partial_x w_2 + 3\partial_x^{-1}\partial_y^2 w_2 = \partial_x g, \tag{3.22}$$

$$\partial_t w_3 + \partial_x^3 w_3 - \partial_x w_3 + 3\partial_x^{-1}\partial_y^2 w_3 = \partial_x^{-1}\partial_y h, \tag{3.23}$$

where

$$\mathcal{F}_{xy}(w_i(t,x,y)) \to 0 \text{ as } t \to -\infty.$$

We start by studying (3.21). We take the Fourier transform of it with respect to the space variables x and y

$$\partial_t \hat{w}_1(t,\xi,\eta) - i\xi^3 \hat{w}_1 - i\xi \hat{w}_1 + 3i\frac{\eta^2}{\xi}\hat{w}_1 = \hat{f}(t,\xi,\eta). \tag{3.24}$$

Then we solve the resulted ordinary differential equation (3.24) for each fixed ξ and η.

$$\begin{aligned}
(3.24) \quad \Rightarrow \quad & \hat{w}_1(t,\xi,\eta) = \int_{-\infty}^{t} e^{(i\xi^3+i\xi-3i\frac{\eta^2}{\xi})(t-t')}\hat{f}(t',\xi,\eta)dt' \\
\Rightarrow \quad & w_1(t,x,y) \\
& = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{t} e^{(i\xi^3+i\xi-3i\frac{\eta^2}{\xi})(t-t')+i\xi x+i\eta y}\hat{f}(t',\xi,\eta)dt'd\xi d\eta \\
& =: \int_{-\infty}^{t} e^{(t-t')S} f(t',x,y)dt'.
\end{aligned}$$

We define the operator

$$T_1 : L_x^1 L_{ty}^2 \to L_{xy}^2,$$

by

$$T_1 f = \int_{-\infty}^{0} e^{-t'S} f(t',x,y)dt'.$$

We have

$$\begin{aligned}
\langle T_1^* v, f\rangle &= \langle v, T_1 f\rangle \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{v}\int_{-\infty}^{0} e^{-t'S} f(t',x,y)dt'dxdy \\
&\quad \text{(we use Plancherel's theorem)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{\hat{v}e^{(i\xi^3+i\xi-3i\frac{\eta^2}{\xi})t'}}\hat{f}(t',\xi,\eta)dt'd\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{e^{t'S}v} f(t',x,y)dt'dxdy.
\end{aligned}$$

This implies that

$$T_1^* : L^2_{xy} \to L^\infty_x L^2_{ty}$$

and it is defined by

$$T_1^* v = e^{tS} v.$$

Hence

$$T_1^* T_1 f = \int_{-\infty}^0 e^{(t-t')S} f(t', x, y) dt'.$$

The local smoothing estimate (3.11) proved in Theorem 3.2 implies the boundedness of the operator $T_1^* T_1$ and using the Lemma 3.7 we infer the boundedness of T_1 and T_1^*. The boundedness of T_1 gives us

$$\|w_1(0, x, y)\|_{L^2_{xy}} \lesssim \|f\|_{L^1_x L^2_{ty}}. \tag{3.25}$$

Next, we treat the equation (3.22) in a similar way.

We define the operator

$$T_2 : L^1_x L^2_{ty} \to L^2_{xy},$$

by

$$T_2 g = \int_{-\infty}^0 e{-}tS\partial_x g(t) dt.$$

Then

$$\begin{aligned}\langle T_2^* v, g\rangle &= \langle v, T_2 g\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{v} \int_{-\infty}^{0} e^{-tS}\partial_x g(t,x,y)dtdxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\hat{v}} \int_{-\infty}^{0} e^{-tS}\widehat{\partial_x g(t},x,y)dtdxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{-i\xi\hat{v}e^{(i\xi^3+i\xi-3i\frac{\eta^2}{\xi})t}}\hat{g}(t,\xi,\eta)dtd\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{-\partial_x(e^{tS}v)}g(t,x,y)dtdxdy.\end{aligned}$$

Hence

$$T_2^* v = -\partial_x(e^{tS}v),$$

and

$$T_2^* T_2 g = -\partial_x \int_{-\infty}^{0} e^{(t-t')S}\partial_x g(t',x,y)dt'.$$

As in the study of the operator $T_1^*T_1$, the boundedness of the operator $T_2^*T_2$ follows from the smoothing estimate (3.11), which implies the boundedness of the operator T_2. Thus we have

$$\|w_2(0,x,y)\| \lesssim \|g\|_{L_x^1 L_{ty}^2}. \tag{3.26}$$

Finally, we study the equation (3.23).

We define

$$T_3 : L_x^1 L_{ty}^2 \to L_{xy}^2,$$

by

$$T_3 h = \int_{-\infty}^{0} e^{-tS}\partial_x^{-1}\partial_y h(t,x,y)dt.$$

We have

$$\begin{aligned}\langle T_3^* v, h\rangle &= \langle v, T_3 h\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{v} \int_{-\infty}^{0} e^{-tS}\partial_x^{-1}\partial_y h(t,x,y)dtdxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\hat{v}} \int_{-\infty}^{0} e^{-tS}\widehat{\partial_x^{-1}\partial_y h}(t,x,y)dtdxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{\frac{\eta}{\xi}\hat{v}e^{(i\xi^3+i\xi-3i\frac{\eta^2}{\xi})t}}\hat{h}(t,\xi,\eta)dtd\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \overline{\partial_x^{-1}\partial_y(e^{tS}v)}h(t,x,y)dtdxdy.\end{aligned}$$

Then

$$T_3^* v = \partial_x^{-1}\partial_y e^{tS} v,$$

and

$$T_3^* T_3 h = \partial_x^{-1}\partial_y \int_{-\infty}^{0} e^{(t-t')S}\partial_x^{-1}\partial_y h(t')dt'.$$

Again (3.11) implies the boundedness of the operator $T_3^* T_3$, then by using the Lemma 3.7 we conclude the boundedness of the operator T_3, which gives us

$$\|w_3(0,x,y)\|_{L^2_{xy}} \lesssim \|h\|_{L^1_x L^2_{ty}}. \tag{3.27}$$

We combine (3.25), (3.26) and (3.27) and obtain the desired result

$$\begin{aligned}\|w(0,x,y)\|_{L^2_{xy}} &\leq \|w_1(0,x,y)\|_{L^2_{xy}} + \|w_2(0,x,y)\|_{L^2_{xy}} + \|w_3(0,x,y)\|_{L^2_{xy}} \\ &\lesssim \|f\|_{L^1_x L^2_{ty}} + \|g\|_{L^1_x L^2_{ty}} + \|h\|_{L^1_x L^2_{ty}}.\end{aligned}$$

□

Definition 3.8. We say $u \in U^2_S$ if and only if $e^{-\cdot S}u \in U^2$ and

$$\|u\|_{U^2_S} = \|e^{-\cdot S}u\|_{U^2}.$$

Proposition 3.9. *Assume $\psi \in U^2_S$. Then the following estimates hold true*

$$\|\psi\|_{L^\infty_x L^2_{ty}} \lesssim \|\psi\|_{U^2_S}, \tag{3.28}$$
$$\|\partial_x\psi\|_{L^\infty_x L^2_{ty}} \lesssim \|\psi\|_{U^2_S}, \tag{3.29}$$
$$\|\partial_x^{-1}\partial_y\psi\|_{L^\infty_x L^2_{ty}} \lesssim \|\psi\|_{U^2_S}. \tag{3.30}$$

Proof. Let ϕ be a U^2-atom. Then there exist $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2$ with

$$\sum_{k=0}^{K-1} \|\phi_k\|_{L^2}^2 = 1 \quad \text{and} \quad \phi_0 = 0$$

such that

$$\phi = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}.$$

Let

$$\psi = \sum_{k=0}^{K-1} \psi_k(t)$$

where

$$\psi_k(t) := e^{tS}\phi_k(t_k) \quad \text{on} \quad [t_k, t_{k+1}).$$

The boundedness of T_1^* defined in the proof of Theorem 3.6 gives us

$$\|\psi_k\|_{L^\infty_x L^2_{ty}} \le c\|\psi(t_k)\|_{L^2_{xy}} = c\|\phi_k(t_k)\|_{L^2_{xy}}$$

on $[t_k, t_{k+1})$.

Thus

$$\|\psi\|^2_{L^\infty_x L^2_{ty}} \le \sum_{k=0}^{K-1} \|\psi_k\|^2_{L^\infty_x L^2_{ty}} \le c^2 \sum_{k=0}^{K-1} \|\phi_k(t_k)\|^2_{L^2_{xy}} \le c^2,$$

which implies that (3.28) is true for $\psi = \sum_{k=0}^{K-1} \mathbb{1}_{[t_k,t_{k+1})} e^{tS}\phi_k(t_k)$, where ϕ is an arbitrary U^2 atom. Since the constant c is independent of ϕ then (3.28) holds for any $\phi \in U^2$.

Similarly the boundedness of operators T_2^* and T_3^* in the proof of Theorem 3.6 gives us

$$\|\partial_x \psi_j\|_{L_x^\infty L_{ty}^2} \lesssim \|\phi_j(t_j)\|_{L_{xy}^2} \tag{3.31}$$

and

$$\|\partial_x^{-1}\partial_y \psi_j\|_{L_x^\infty L_{ty}^2} \lesssim \|\phi_j(t_j)\|_{L_{xy}^2} \tag{3.32}$$

respectively.

Summing over all j's the estimates (3.31) and (3.32) we get the estimates (3.29) and (3.30), respectively. □

3.1.3 Miura Transformation

The Miura transformation is an explicit nonlinear transformation that relates solutions of the KdV equation and the mKdV equation, [22]:

If v is a solution of the mKdV equation

$$\partial_t v + \partial_x^3 v - 6v^2 \partial_x v = 0,$$

then u given by the Miura transformation

$$u = \pm\partial_x v - v^2$$

satisfies the KdV equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = 0.$$

In this section we use the idea of [24] and [21] of using the properties of the following generalisation of the Miura transformation

$$M_\pm^c(v) = \pm\partial_x v + \partial_x^{-1} v_y - v^2 + \frac{c}{2}$$

that exploits the Galilean invariance of the KP-II equation and maps the solution of the mKP-II equation

$$v_t + v_{xxx} + 3\partial_x^{-1} v_{yy} - 6v^2 v_x + 6v_x \partial_x^{-1} v_y = 0 \tag{3.33}$$

into the solution of the KP-II equation (3.2) by

$$u(t, x, y) = M_{\pm}^{c}(v)(t, x - 3ct, y).$$

As it is observed in [24], the kink Φ_c

$$\Phi_c(x, y) = \frac{c^{1/2}}{2} \tanh\left(\frac{c^{1/2}}{2} x\right)$$

is related to the line soliton of KP-II

$$Q_c(x, y) = \frac{c}{2} \mathrm{sech}^2\left(\frac{c^{1/2}}{2} x\right), \quad c > 0$$

through the following relation

$$M_{+}^{c}(\Phi_c) = Q_c.$$

One can also easily check that

$$M_{-}^{c}(\Phi_c) = 0.$$

In [3], the authors show that one can relate mKdV solutions near kink solutions to either KdV solutions near 0 or to KdV solutions near a soliton. As expected the same relations can be generalised to the case of KP-II and mKP-II equations' solutions.

Proposition 3.10. *Let v be a solution of the mKP-II equation linearized at $\Phi_c(x + \frac{c}{2}t)$ in a moving frame:*

$$\partial_t v + \frac{c}{2}\partial_x v + \partial_x^3 v + 3\partial_x^{-1}\partial_y^2 v + 6\partial_x \Phi_c \partial_x^{-1}\partial_y v - 6\partial_x(\Phi_c^2 v) = 0 \tag{3.34}$$

and u be a solution of the KP-II equation linearized at zero

$$\partial_t u - c\partial_x u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u = 0. \tag{3.35}$$

Then

$$\mathcal{F}_y(u) = M_0(\mathcal{F}_y(v)) := -\partial_x \mathcal{F}_y(v) - 2\Phi_c(x)\mathcal{F}_y(v) + i\eta\partial_x^{-1}\mathcal{F}_y(v).$$

transforms the Fourier transform (with respect to y) of a solution of (3.34) *to the Fourier transform (with respect to y) of a solution of* (3.35)*.*

Proposition 3.11. *If v is a solution of* (3.34)*, then*

$$\mathcal{F}_y(w) = M_Q(\mathcal{F}_y(v)) := \partial_x \mathcal{F}_y(v) - 2\Phi_c(x)\mathcal{F}_y(v) + i\eta\partial_x^{-1}\mathcal{F}_y(v)$$

is the Fourier transform (with respect to y) of a solution of

$$\partial_t w - c\partial_x w + \partial_x^3 w + 6\partial_x(Q_c w) + 3\partial_x^{-1}\partial_y^2 w = 0. \tag{3.36}$$

The proofs of the above two propositions follow from straightforward substitution and use of the following identities

$$\begin{aligned}
&c\Phi_{cx} + \Phi_{cxxx} - 6\Phi_c^2\Phi_{cx} = 0,\\
&\Phi_{cx} + \Phi_c^2 = \frac{c}{2},\\
&\Phi_{cxx} = -(\Phi_c^2)_x,\\
&\Phi_{cx} - \Phi_c^2 + \frac{c}{2} - Q_c = 0,\\
&\Phi_{cxx} - (\Phi_c^2)_x - Q_{cx} = 0.
\end{aligned}$$

Proposition 3.12. *Let w be a solution of*

$$\partial_t w - c\partial_x w + \partial_x^3 w + 6\partial_x(Q_c w) + 3\partial_x^{-1}\partial_y^2 w = 0.$$

Then

$$\|\mathcal{F}_y(w(t,\cdot,\eta))\|_{L^2} \le C(\eta)\|\mathcal{F}_y(w(0,\cdot,\eta))\|_{L^2} \quad \textit{for each} \quad \eta \neq 0,$$

where

$$C(\eta) = 3 + 6\left(2 + \frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4}\right).$$

Proof. Let $\partial_x V = \mathcal{F}_y(v)$. Note that

$$M_Q(\partial_x V) = M_0(\partial_x V) + 2\partial_x^2 V. \tag{3.37}$$

Multiplying both sides of

$$M_Q(\partial_x V) = \partial_x^2 V - 2\Phi_c \partial_x V + i\eta V \tag{3.38}$$

by $\overline{\partial_x^2 V}$ and integrating over $\mathbb{R}$ with respect to x and then adding its complex conjugate to the resulting equation, we obtain

$$2\int |\partial_x^2 V|^2 dx + \int Q_c |\partial_x V|^2 dx = \int M_Q(\partial_x V)\overline{\partial_x^2 V} dx + \int \overline{M_Q(\partial_x V)}\partial_x^2 V dx \tag{3.39}$$

which implies

$$\|\partial_x^2 V\|_{L_x^2} \leq \|M_Q(\partial_x V)\|_{L_x^2}. \tag{3.40}$$

The above estimate combined with (3.37) gives us

$$\|M_0(\partial_x V)\|_{L_x^2} \leq 3\|M_Q(\partial_x V)\|_{L_x^2}.$$

Similarly, we multiply both sides of

$$M_0(\partial_x V) = -\partial_x^2 V - 2\Phi_c \partial_x V + i\eta V \tag{3.41}$$

by $\overline{\partial_x^2 V}$ and integrate over $\mathbb{R}$ with respect to x. The real part of the resulting equation is

$$2\int |\partial_x^2 V|^2 dx = -2\mathrm{Re}\int M_0(\partial_x V)\overline{\partial_x^2 V} + \int Q_c |\partial_x V|^2 dx,$$

which implies

$$\int |\partial_x^2 V|^2 dx \leq \|M_0(\partial_x V)\|_{L^2}^2 + \int |\partial_x V|^2 dx. \tag{3.42}$$

We need to estimate $\|\partial_x V\|_{L^2}$. For this purpose we multiply (3.41) by $\overline{V}$. The real part of the resulting equation gives us

$$\int |\partial_x V|^2 dx \leq 2\|M_0(\partial_x V)\|_{L^2}\|V\|_{L^2} \tag{3.43}$$

and the imaginary part gives us

$$\eta \int |V|^2 dx = \mathrm{Im} \int M_0(\partial_x V)\overline{V} dx + 2\mathrm{Im} \int \Phi_c \partial_x V \overline{V} dx$$

$$\Rightarrow \|V\|_{L^2}^2 \leq \left(\frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4} \right) \|M_0(\partial_x V)\|_{L^2}^2. \tag{3.44}$$

Using the above inequality and (3.44), we get

$$\|\partial_x V\|_{L^2}^2 \leq \left(1 + \frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4} \right) \|M_0(\partial_x V)\|_{L^2}^2. \tag{3.45}$$

Combining (3.42) and (3.45), we obtain

$$\|\partial_x^2 V\|_{L^2} \leq \left(2 + \frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4} \right)^{1/2} \|M_0(\partial_x V)\|_{L^2}.$$

Hence

$$\begin{aligned} \|M_Q(\partial_x V)\|_{L^2} &\leq \left(1 + 2\sqrt{2 + \frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4}} \right) \|M_0(\partial_x V)\|_{L^2} \\ &\leq \left(3 + 6\sqrt{2 + \frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4}} \right) \|M_Q(\partial_x V)\|_{L^2} \end{aligned}$$

as desired.

□

Proposition 3.13. *Let η be a nonzero real number and w be a solution of* (3.36)

$$\partial_t w - \partial_x w + \partial_x^3 w + 6\partial_x(Qw) + 3\partial_x^{-1}\partial_y^2 w = 0,$$

and u be a solution of (3.35)

$$\partial_t u - \partial_x u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u = 0.$$

Then

(i)

$$\|\mathcal{F}_y(u(t,\cdot,\eta))\|_{L^\infty} \lesssim \|\mathcal{F}_y(w(t,\cdot,\eta))\|_{L^\infty},$$

(ii)

$$\|\mathcal{F}_y(w(t,\cdot,\eta))\|_{L^\infty} \lesssim \|\mathcal{F}_y(u(t,\cdot,\eta))\|_{L^\infty}.$$

Proof. (i) Let V be defined as in the proof of Proposition 3.12. We want to prove that

$$\|\partial_x^2 V\|_{L_x^\infty} \lesssim \|M_Q(\partial_x V)\|_{L_x^\infty}.$$

We first consider

$$\partial_x^2 y - \partial_x y + i\eta y = W, \tag{3.46}$$

where $W \in L_x^\infty$. The characteristic equation of the homogeneous equation corresponding to the ordinary differential equation (3.46) is

$$r^2 - r + i\eta = 0$$

and its roots are

$$r_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - i\eta}.$$

Then there is a convolution kernel

$$k(x) = \frac{e^{-\frac{1}{2}x - \sqrt{\frac{1}{4} - i\eta}|x|}}{-2\sqrt{\frac{1}{4} - i\eta}}$$

such that

$$y = k * W$$

is the unique C^2 solution of (3.46).

Let ϕ be a cutoff function that is identically equal to 1 in $x \geq 1$ and has a support in $x > -1$ such that

$$\phi(x) + \phi(-x) = 1 \quad \text{for each} \quad x \in \mathbb{R}. \tag{3.47}$$

We define

$$y^+ := k * (\phi(x)W(x))$$

and

$$y^- := k^- * (\phi(-x)W(x))$$

where

$$k^- := k(-x).$$

Then by Young's inequality it follows that

$$\|y^\pm\|_{L^\infty_x} + \|\partial_x y^\pm\|_{L^\infty_x} + \|\partial_x^2 y^\pm\|_{L^\infty_x} \lesssim \|W\|_{L^\infty_x}. \tag{3.48}$$

Moreover

$$|y^+(x)| + |\partial_x y^+(x)| + |\partial_x^2 y^+(x)| \leq e^{|\mathrm{Re}\ r_2|x}$$

and

$$|y^-(x)| + |\partial_x y^-(x)| + |\partial_x^2 y^-(x)| \leq e^{-|\mathrm{Re}\ r_1|x}.$$

We make the following ansatz

$$V = y^+ + y^- + Y.$$

Then Y satisfies

$$\partial_x^2 Y - 2\Phi\partial_x Y + i\eta Y = (1+2\Phi)\partial_x y^- + (-1+2\Phi)\partial_x y^+. \tag{3.49}$$

Let us denote the right hand side of the equation (3.49) by W_0. Note that W_0 decays exponentially as $x \to \pm\infty$. Let $Y_+ := \phi Y$ and $Y_- := \phi(-x)Y$. Note that

$$Y = Y_+ + Y_-$$

where Y_+ and Y_- satisfy

$$\begin{aligned}\partial_x^2 Y_+ - \partial_x Y_+ + i\eta Y_+ &= \phi(x)W_0 + 2\phi(x)\Phi\partial_x Y - \phi\partial_x Y - \partial_x\phi Y \\ &\quad + 2\partial_x\phi\partial_x Y + \partial_x^2\phi(x)Y\end{aligned}$$

and

$$\begin{aligned}\partial_x^2 Y_- + \partial_x Y_- + i\eta Y_- &= \phi(-x)W_0 + 2\phi(-x)\Phi\partial_x Y + \phi(-x)\partial_x Y - \partial_x\phi(-x)Y \\ &\quad - 2\partial_x\phi(-x)\partial_x Y + \partial_x^2\phi(-x)Y\end{aligned}$$

respectively.

Then for all R

$$\|\min\{e^{\frac{x}{2}}, \frac{1}{c(R)}\}Y_\pm\|_{L_x^\infty} \lesssim \|e^{\frac{x}{2}}W_0\|_{L^\infty} + \|Y\|_{L^\infty(-1,1)} + \|\partial_x Y\|_{L^\infty(-1,1)}, \tag{3.50}$$

where

$$c(R) = 3\sqrt{\frac{4}{3R^2} + \frac{256}{3R^4}}.$$

The same estimate holds for the derivatives of Y_+ and Y_- as well.

(3.39) implies

$$Q_c(1)\int_{-1}^{1} |\partial_x Y|^2 dx \le \int Q_c|\partial_x Y|^2 dx \le \|W_0\|_{L_x^2}^2$$

and (3.44) implies

$$\|Y\|_{L^2(-1,1)} \leq 3\sqrt{\frac{4}{3|\eta|^2} + \frac{256}{3|\eta|^4}}\|W_0\|_{L^2}.$$

Then

$$\|\partial_x Y\|_{L^\infty(-1,1)} + \|Y\|_{L^\infty(-1,1)} \lesssim c(\eta)\|W_0\|_{L^2}. \tag{3.51}$$

by Sobolev embedding theorem.

Combining (3.51) with (3.50) and its equivalents for the derivatives of $Y_\pm$ and (3.48) gives us the desired estimate. It remains to prove the existence of Y satisfying(3.49). If W_0 has a compact support then $Y_\pm \in C_0^2$ and hence $Y \in C_0^2$. Then (3.44), (3.45) and (3.40) imply the existence of solution of (3.49) in L^2 with derivatives in L^2.

(ii) Repeating the same argument in (i) with

$$k(x) = \frac{e^{-\frac{1}{2}x+\sqrt{\frac{1}{4}+i\eta}|x|}}{2\sqrt{\frac{1}{4}+i\eta}}$$

gives us the required estimate.

□

Corollary 3.14. *By duality*

(i)

$$\|\mathcal{F}_y(w(t,\cdot,\eta))\|_{L^1} \lesssim \|\mathcal{F}_y(u(t,\cdot,\eta))\|_{L^1},$$

(ii)

$$\|\mathcal{F}_y(u(t,\cdot,\eta))\|_{L^1} \lesssim \|\mathcal{F}_y(w(t,\cdot,\eta))\|_{L^1},$$

for each $\eta \neq 0$.

3.1.4 A Local Smoothing Estimate Part II

Proof of Theorem 3.1. We write the solution of (3.8) as

$$w = w_1 + w_2,$$

where w_1 satisfies

$$\partial_t w_1 + \partial_x^3 w_1 - \partial_x w_1 + 3\partial_x^{-1}\partial_y^2 w_1 = f + \partial_x g + \partial_x^{-1}\partial_y h, \tag{3.52}$$

and w_2 satisfies

$$\partial_t w_2 + \partial_x^3 w_2 - \partial_x w_2 + 3\partial_x^{-1}\partial_y^2 w_2 = -6\partial_x(Qw_1) - 6\partial_x(Qw_2). \tag{3.53}$$

Due to Theorem 3.2, we have

$$\begin{aligned}\|\mathcal{F}_y(w_1)\|_{L_x^\infty L_t^2 L_\eta^2} &+ \|\partial_x \mathcal{F}_y(w_1)\|_{L_x^\infty L_t^2 L_\eta^2} + \|\eta\partial_x^{-1}\mathcal{F}_y(w_1)\|_{L_x^\infty L_t^2 L_\eta^2} \\ &\lesssim \|\mathcal{F}_y(f)\|_{L_x^1 L_t^2 L_\eta^2} + \|\mathcal{F}_y(g)\|_{L_x^1 L_t^2 L_\eta^2} + \|\mathcal{F}_y(h)\|_{L_x^1 L_t^2 L_\eta^2},\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{F}_y(w_2)\|_{L_x^\infty L_t^2 L_\eta^2} + \|\partial_x \mathcal{F}_y(w_2)\|_{L_x^\infty L_t^2 L_\eta^2} &+ \|\eta\partial_x^{-1}\mathcal{F}_y(w_2)\|_{L_x^\infty L_t^2 L_\eta^2} \\ &\lesssim \|Q\mathcal{F}_y(w_1)\|_{L_x^1 L_t^2 L_\eta^2} + \|Q\mathcal{F}_y(w_2)\|_{L_x^1 L_t^2 L_\eta^2}.\end{aligned} \tag{3.54}$$

In order to estimate the first term on the right hands side of (3.54) we use Theorem 3.2 and the fact that

$$\int_{-\infty}^{\infty} Q(x)dx = 2. \tag{3.55}$$

Hence

$$\begin{aligned}\|Q\mathcal{F}_y(w_1)\|_{L_x^1 L_t^2} &\leq \|Q\|_{L_x^1 L_t^2}\|\mathcal{F}_y(w_1)\|_{L_x^\infty L_t^2} \\ &\lesssim \|\mathcal{F}_y(f)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(g)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(h)\|_{L_x^1 L_t^2}.\end{aligned} \tag{3.56}$$

It only remains to estimate the second term on the right hand side of (3.54). We rewrite (3.53) as

$$\partial_t \mathcal{F}_y(w_2) + \partial_x^3 \mathcal{F}_y(w_2) - \partial_x \mathcal{F}_y(w_2) - 3\eta^2 \partial_x^{-1} \mathcal{F}_y(w_2) + 6\partial_x(Q\mathcal{F}_y(w_2)) = -6\partial_x(Q\mathcal{F}_y(w_1)). \tag{3.57}$$

By Proposition 3.13 (ii) and Corollary 3.14 (i) it follows that

$$\|\mathcal{F}_y(w_2)\|_{L_x^\infty} \lesssim \|\mathcal{F}_y(w_1)\|_{L_x^\infty}$$

provided that $\eta \neq 0$.

Then by using, the above estimate, (3.55) and Minkowski inequality we achieve our goal

$$\begin{aligned}
\|Q\mathcal{F}_y(w_2)\|_{L_x^1 L_t^2} &\leq \|Q\|_{L_x^1 L_t^\infty} \|\mathcal{F}_y(w_2)\|_{L_x^\infty L_t^2} \\
&\lesssim \|\mathcal{F}_y(w_2)\|_{L_t^2 L_x^\infty} \\
&\lesssim \|\mathcal{F}_y(w_1)\|_{L_t^2 L_x^\infty} \\
&\lesssim \|\mathcal{F}_y(f)\|_{L_t^2 L_x^1} + \|\mathcal{F}_y(g)\|_{L_t^2 L_x^1} + \|\mathcal{F}_y(h)\|_{L_t^2 L_x^1} \\
&\lesssim \|\mathcal{F}_y(f)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(g)\|_{L_x^1 L_t^2} + \|\mathcal{F}_y(h)\|_{L_x^1 L_t^2}
\end{aligned}$$

only if $\eta \neq 0$.

□

3.2 Notes and References

Two important results have been published while this project was ongoing: [24] by T. Mizumachi and N. Tzvetkov and [23] by T. Mizumachi.

In [24], T. Mizumachi and N. Tzvetkov have proved the nonlinear stability of the line solitons with respect to periodic transverse perturbations. In [23], T. Mizumachi proved the stability of line solitons for exponentially localized perturbations. In his work, T. Mizumachi proved that solutions can be expressed as follows

$$u(t,x,y) = Q_{c(t,y)}(x - x(t,y)) - \psi_{c(t,y)}(x - x(t,y) + 4t) + v(t, x - x(t,y), y)$$

where $c(t,y)$ and $x(t,y)$ are the local amplitude and the local phase shift of the modulating line soliton, if the following assumptions are satisfied

(i)
$$\int_{\mathbb{R}} v(t,x,y)dx = \int_{\mathbb{R}} v(0,x,y)dx \quad \text{for any } t > 0$$

(ii) v satisfies

$$\lim_{M\to\infty}\int_{-M}^{M}\int_{\mathbb{R}} v(t,x-x(t,y),y)\overline{g_k^*(x-x(t,y),\eta,c(t,y))}e^{-iy\eta}dxdy = 0$$

in $L^2(-\eta_0,\eta_0)$ for $k=1,2$, where

$$\begin{aligned}
&\eta \in \mathbb{R}-\{0\},\\
&g_1^*(x,\eta,c) = cg_1^*(\sqrt{\frac{c}{2}}x,\eta),\\
&g_2^*(x,\eta,c) = \frac{c}{2}g_2^*(\sqrt{\frac{c}{2}}x,\eta)
\end{aligned}$$

and

$$\begin{aligned}
g_1^*(x,\eta) &= \frac{1}{2}(\partial_x(e^{\sqrt{1-i\eta}x}\mathrm{sech}x) + \partial_x(e^{\sqrt{1+i\eta}x}\mathrm{sech}x)),\\
g_2^*(x,\eta) &= \frac{i}{2\eta}(\partial_x(\partial_x(e^{\sqrt{1-i\eta}x}\mathrm{sech}x) - e^{\sqrt{1+i\eta}x}\mathrm{sech}x))
\end{aligned}$$

(iii) the sufficient smallness of the following expressions

$$\begin{aligned}
\mathbb{M}_1(T) &= \sup_{0\le t\le T}\{\sum_{k=0}^{1}(1+t)^{(2k+1)/4}(\|\partial_y^k\tilde{c}(t,\cdot)\|_{L^2} + \|\partial_y^{k+1}x(t,\cdot)\|_{L^2})\\
&\quad +(1+t)(\|\partial_y^2\tilde{c}(t,\cdot)\|_{L^2} + \|\partial_y^3 x(t,\cdot)\|_{L^2})\},\\
\mathbb{M}_2(T) &= (1+t)^{3/4}\|v(t,\cdot)\|_{L^2(\mathbb{R}^2;e^{2ax}dxdy)},\\
\mathbb{M}_3(T) &= \sup_{0\le t\le T}\|v(t,\cdot)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Under the above assumptions one of the main observations of this paper is that the local amplitude $c(t,y)$ and the y derivative of the local phase shift $x(t,y)$ of the modulating line soliton behave like self similar solution of the Burgers equation.

Chapter 4

The cubic generalized Kadomtsev-Petviashvili II equation

There are various generalizations of the Kadomtsev-Petviashvili II equation. In this chapter, we study the initial value problem for the generalized Kadomtsev-Petviashvili II equation (gKP-II) with nonlinearity $\partial_x(u^3)$ (gKP-II)$_3$

$$\begin{aligned} &\partial_t u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u - 6u^2\partial_x u = 0, \\ &u(0,x,y) = u_0(x,y), \quad (x,y)\in\mathbb{R}^2, \quad t\in\mathbb{R}. \end{aligned} \tag{4.1}$$

Note that (gKP-II)$_3$ equation is different from the mKP-II equation (3.33) discussed in Section 3.1.3.

The nonlinear term $u^2\partial_x u$ in (gKP-II)$_3$ is responsible for the weak nonlinearity, the $\partial_x^3 u$ term for the weak dispersion and the $\partial_y^2 u$ is for the diffractive divergence.

In [30], the (gKP-II)$_3$ equation is derived as a truncated equation that describes the evolution of sound waves in antiferromagnets. An antiferromagnet is a solid that has a weak magnetism, which is characterized by a small positive susceptibility. In their paper, S. K. Turitsyn and G. E. Fal'kovich use the experimental values for the antiferromagnet called hematite (Fe_2O_3) in their calculations. They study the problem in a coordinate system moving with the velocity of sound and keep the nonlinear and dispersive terms in the equation. This approach allows to study both the fast and the slow components of the evolution of the sound wave at a strain level that is not too high. The fast component is the transport of the initial perturbation at the velocity of sound and the slow component is the effect of weak nonlinearity and dispersion.

The local well posedness of the Cauchy problem (4.1) has been studied in a number of papers. In [13], R. J. Iorio and W. V. L. Nunes prove local well posedness of (gKP-II)$_3$ (and (gKP-I)$_3$) for both periodic and non-periodic initial data in $H^s(\mathbb{R}^2)$, $s > 2$. First the associated linear equation is studied. In the passage from linear to nonlinear theory in the case of periodic initial data Kato's quasilinear theory is used and in the case of non-periodic initial data the parabolic regularisation method is used. For the proof of continuous dependence of the solution on the initial data the Bona-Smith approximations method is used. The result of this paper on (gKP-II)$_3$ is improved in [15].

In [15], $L^4_x L^\infty_{yT}$ estimate for the solution of the linear initial value problem associated with the (gKP-II)$_3$ equation is proven. This estimate is sharp up to the endpoint. Using this maximal function type estimate C. E. Kenig and S. N. Ziesler prove via the contraction mapping principle that (gKP-II)$_3$ is locally well posed for initial data in the space with norm

$$\|(1+D_x)^{\frac{3}{4}+\varepsilon_1}(1+D_yD_x^{-1})^{\frac{1}{2}+\varepsilon_2}u_0\|_{L^2(\mathbb{R}^2)} + \|D_x^{\frac{5}{4}+\varepsilon_1+\varepsilon_2}u_0\|_{L^2(\mathbb{R}^2)},$$

where $\varepsilon_1, \varepsilon_2 > 0$ are small. Furthermore, it is shown that there can be no proof of local well posedness for (gKP-II)$_3$ with initial data in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 < \frac{1}{2}$ or $s_2 < 0$ via the contraction mapping argument. Considering almost the same data spaces A. Grünrock improves this result in [9] by $\frac{3}{4}$ derivatives. In [9], the author proves local well posedness of the Cauchy problem for the generalized KP-II equation with nonlinearity $\partial_x(u^l)$, $l \geq 3$, for initial data $u_0 \in H^{(s)}$ where $s = (s_1, s_2, \varepsilon)$, $s_1 > \frac{1}{2}$, $s_2 \geq \frac{l-3}{2(l-1)}$, $0 < \varepsilon \leq \min(s_1, 1)$,

$$\|u_0\|_{H^{(s)}} := \|u_0\|_{s_1+2s_2+\varepsilon,0,0} + \|u_0\|_{s_1,s_2,\varepsilon}$$

and

$$\|u_0\|_{\sigma_1,\sigma_2,\sigma_3} := \|\langle D_x\rangle^{\sigma_1}\langle D_y\rangle^{\sigma_2}\langle D_x^{-1}D_y\rangle^{\sigma_3}u_0\|_{L^2_{xy}},$$

in almost critical anisotropic Sobolev spaces $X_{(s),b(\delta)}$ where $\delta = \delta(\|u_0\|_{H^{(s)}})$, $b > \frac{1}{2}$,

$$\|u\|_{X_{(s),b}} := \|u\|_{X_{s_1+2s_2+\varepsilon,0,0;b}} + \|u\|_{X_{s_1,s_2,\varepsilon;b}},$$

and

$$\|u\|_{X_{\sigma_1,\sigma_2,\sigma_3;b}} := \|\langle D_x\rangle^{\sigma_1}\langle D_y\rangle^{\sigma_2}\langle D_x^{-1}D_y\rangle^{\sigma_3}u\|_{X_{0,b}}, \quad \|u\|_{X_{0,b}} := \|\langle \tau - \phi(\xi,\eta)\rangle^b \mathcal{F}u\|_{L^2_{\xi\tau}}.$$

In the above definition, $\phi(\xi,\eta) = \xi^3 - \frac{\eta^2}{\xi}$ is the phase function of the linearized KP-II equation. This local well posedness result is proved by the contraction mapping principle. The main ingredients of this proof are a local smoothing estimate, a maximal function estimate, Strichartz estimates and bilinear estimates. These are put together via Bourgain's Fourier restriction method.

In the following, we extend the above well posedness results to the space $\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)$ defined in Section 4.2. The function space $\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)$ is continuously embedded into anisotropic Besov space $\dot{B}^{(\frac{1}{2},0)}_{2,\infty}(\mathbb{R}_x)$ (also called Nikolskii-Besov spaces) provided that $p < 2$, which lets us to improve the well posedness result of [9] by extending s_1 to $\frac{1}{2}$.

4.1 The Symmetries of the (gKP-II)$_3$ Equation

The (gKP-II)$_3$ equation

$$\partial_t u + \partial_x^3 u + 3\partial_x^{-1}\partial_y^2 u - 6u^2\partial_x u = 0$$

possesses the following symmetries.

(i) The (gKP-II)$_3$ equation has **translational symmetry**. If $u(t,x,y)$ is a solution of the (gKP-II)$_3$ equation, it remains being a solution under the transformations

$$x \mapsto x + x_0, \quad \forall x_0 \in \mathbb{R},$$
$$y \mapsto y + y_0, \quad \forall y_0 \in \mathbb{R},$$
$$t \mapsto t + t_0, \quad \forall t_0 \in \mathbb{R}.$$

(ii) The (gKP-II)$_3$ equation has **scaling symmetry**. If $u(t,x,y)$ is a solution, then so is

$$u_\alpha(t,x,y) = \alpha u(\alpha^3 t, \alpha x, \alpha^2 y), \tag{4.2}$$

for $\alpha > 0$.

(iii) The (gKP-II)$_3$ equation has **Galilean symmetry**, which implies that if $u(t,x,y)$ is a solution

$$u_c(t,x,y) = u(t, x+cy-3c^2t, y-3ct), \tag{4.3}$$

will also satisfy the (gKP-II)$_3$ equation for all $c \in \mathbb{R}$.

Then it follows that the homogeneous space $\dot{H}^{s_1,s_2}$ is invariant with respect to the scaling symmetry of solutions of (gKP-II)$_3$ if $2s_1 + 4s_2 = 1$.

Furthermore it is invariant with respect to Galilean transform provided that $s_2 = 0$.

4.2 Function Spaces

Let $k \in \mathbb{Z}$ and $\lambda = 2^N$ for $N \in \mathbb{Z}$. We define

$$A_{\lambda,k} = \left\{ (\xi,\eta) \in \mathbb{R}^2 \mid \lambda \le |\xi| \le 2\lambda, \left| \frac{\eta}{\lambda\xi} - k \right| \le \frac{1}{2} \right\}, \tag{4.4}$$

and

$$u_{\lambda,k} = \mathcal{F}^{-1}(\chi_{A_{\lambda,k}} \hat{u}), \tag{4.5}$$

where $\chi_{A_{\lambda,k}}$ is the characteristic function of the set $A_{\lambda,k}$ (i.e. $\chi_{A_{\lambda,k}}$ is equal to 1 when $(\xi,\eta) \in A_{\lambda,k}$ and equal to 0 when (ξ,η) is in the complement of the set $A_{\lambda,k}$).

We define

$$\begin{aligned}
\|u\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(L^2)} &:= \sup_\lambda \lambda^{1/2} \Big(\sum_k \|u_{\lambda,k}\|^p_{L^2(\mathbb{R}^2)} \Big)^{1/p}, \\
\|u\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V^2_S)} &:= \sup_\lambda \lambda^{1/2} \Big(\sum_k \|u_{\lambda,k}\|^p_{V^2_S} \Big)^{1/p}.
\end{aligned}$$

Remark 4.1. If α is a dyadic number, i.e., $\alpha = 2^n$ for some $n \in \mathbb{Z}$, then

$$\|u_\alpha\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(L^2)} = \|u\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(L^2)}$$

where u_α is a scaled solution as described in (4.2).

Proof.

$$
\begin{aligned}
\|u_\alpha\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)} &= \sup_\lambda \lambda^{1/2}\Big(\sum_k \|(u_\alpha)_{\lambda,k}\|^p_{L^2(\mathbb{R}^2)}\Big)^{1/p} \\
&= \sup_\lambda \lambda^{1/2}\Big(\sum_k \|\chi_{A_{\lambda,k}}\hat{u}_\alpha\|^p_{L^2(\mathbb{R}^2)}\Big)^{1/p} \\
&= \sup_\lambda \lambda^{1/2}\Big(\sum_k \Big(\int_{A_{\lambda,k}}\Big|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(x\xi+y\eta)}\alpha u(\alpha x,\alpha^2 y)dxdy\Big|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&= \sup_\lambda \lambda^{1/2}\Big(\sum_k \Big(\int_{A_{\lambda,k}} \frac{1}{4\pi^2\alpha^4}\Big|\int_{\mathbb{R}^2} e^{-i\left(x\frac{\xi}{\alpha}+y\frac{y}{\alpha^2}\right)}u(x,y)dxdy\Big|^2 d\xi d\eta\Big)^{\frac{p}{2}}\Big)^{1/p} \\
\overset{\substack{\xi'=\frac{\xi}{\alpha} \\ \eta'=\frac{\eta}{\alpha^2}}}{=} & \sup_\lambda \lambda^{\frac{1}{2}}\Big(\sum_k \Big(\iint_{\substack{\frac{\lambda}{\alpha}\le|\xi'|\le\frac{2\lambda}{\alpha} \\ \left|\frac{\eta'}{(\frac{\lambda}{\alpha})\xi'}-k\right|\le\frac{1}{2}}} \frac{1}{4\pi^2\alpha}\Big|\int_{\mathbb{R}^2} e^{-i(x\xi'+y\eta')}u(x,y)dxdy\Big|^2 d\xi' d\eta'\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}} \\
&= \sup_\lambda \left(\frac{\lambda}{\alpha}\right)^{1/2}\Big(\sum_k \|\chi_{A_{\frac{\lambda}{\alpha},k}}\hat{u}\|^p_{L^2(\mathbb{R}^2)}\Big)^{\frac{1}{p}} \\
&= \|u\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)}.
\end{aligned}
$$

□

4.3 Multilinear Estimates

First we derive bilinear estimates that we need in order to handle the nonlinear term of (4.1). We follow the same argument that is presented in Chapter 5 of [17]. For the sake of completeness we start with a brief outline of this argument.

We recall the coarea formula

Theorem 4.2. *Let $U \subset \mathbb{R}^d$, $V \subset \mathbb{R}^n$ with $d \geq n$ and $\phi : U \to V$ be differentiable and surjective. Then*

$$\int_V \int_{\phi^{-1}(y)} f d\mathcal{H}^{d-n} dm^n(y) = \int_U f \det(D\phi D\phi^T)^{1/2} dm^d$$

where $\mathcal{H}^{d-n}$ denotes $d-n$ dimensional Hausdorff measure of $\phi^{-1}(y)$.

The following result on the convolution of two measures supported on the hypersurfaces Σ_1 and Σ_2 is an application of the coarea formula. For a detailed proof, see [17], page 54.

Theorem 4.3. *Let $d-1$ dimensional hypersurfaces $\Sigma_i \subset \mathbb{R}^d$, $i = 1,2$ be nondegenerate level sets of functions ϕ_i, $i = 1,2$ and f_i, $i = 1,2$ be square integrable functions on Σ_i with respect to δ_{ϕ_i}. Then*

$$\|f_1\delta_{\phi_1} * f_2\delta_{\phi_2}\|_{L^2(\mathbb{R}^d)} \leq L\|f_1|\nabla\phi_1|^{-1/2}\|_{L^2(\Sigma_1)}\|f_2|\nabla\phi_2|^{-1/2}\|_{L^2(\Sigma_2)}$$

where

$$L = \sup_{\substack{x\in\Sigma_1\\ y\in\Sigma_2}} \Big(\int_{\Sigma(x,y)} [|\nabla\phi_1(z-y)|^2|\nabla\phi_2(z-x)|^2 - \langle\nabla\phi_1(z-x), \nabla\phi_2(z-y)\rangle^2]^{-\frac{1}{2}} d\mathcal{H}^{d-2}\Big)^{\frac{1}{2}} \tag{4.6}$$

and

$$\Sigma(x,y) = \{y + \Sigma_1\} \cap \{x + \Sigma_2\}.$$

Lemma 4.4. *Let u be the solution of*

$$i\partial_t u - \psi(D)u = 0$$

with initial data u_0. Then the space-time Fourier transform of u is the measure $\sqrt{2\pi}\hat{u}_0\delta_\phi$.

Remark 4.5. Let ψ_1 and ψ_2 be real smooth functions. Consider the linear equations

$$i\partial_t u_i - \psi_i(D)u_i = 0, \quad i = 1,2,$$

where

$$\phi_i(\tau,\xi) = \tau - \psi_i(\xi) \quad i = 1,2$$

define the characteristic surfaces of above equations by their zero level sets.

Note that the product u_1u_2 of the solutions equals to the convolution of their Fourier transforms

$$\hat{u}_1(0)\delta_{\phi_1} * \hat{u}_2(0)\delta_{\phi_2}$$

which can be bounded by Theorem 4.3.

In the rest of this section we apply the argument summarised above to the linearized $(KP-II)_3$ equation.

First we introduce the following partition

$$\mathbb{R}^2 = \bigcup_{j=0}^{\infty} S(j)$$

where

$$S(0) = \bigcup_{|m-n|<8} Q_{0,m,n} \quad \text{with} \quad Q_{j,m,n} = 2^j([m,m+1) \times [n,n+1))$$

and

$$S(j) = 2^j S(0) \setminus 2^{j-1}S(0) \quad \text{for} \quad j = 1, 2, \dots .$$

Then we use this partition of $\mathbb{R}^2$ to make a partition of indices that correspond to the projections in η variable as described in (4.4) and (4.5), as follows

$$u_\mu u_\lambda = \sum_j \sum_{(l,k)\in\mathbb{Z}\times\mathbb{Z}\cap Q_{j,\frac{\lambda}{\mu}m,n}\subset S(j)} u_{\mu,l}u_{\lambda,k}.$$

We define

$$\begin{aligned} I^j(m,n) &:= \{l : \exists k, (l,k) \in \mathbb{Z}\times\mathbb{Z} \cap Q_{j,\frac{\lambda}{\mu}m,n} \subset S(j)\}, \\ J^j(m,n) &:= \{k : \exists l, (l,k) \in \mathbb{Z}\times\mathbb{Z} \cap Q_{j,\frac{\lambda}{\mu}m,n} \subset S(j)\}. \end{aligned}$$

Theorem 4.6. *If $\mu \leq \lambda$, then the following estimates hold true.*

(i)

$$\|u_{\mu,k}u_\lambda\|_{L^2(\mathbb{R}^3)} \leq c\frac{\mu}{\lambda}\|u_{\mu,k}\|_{U_S^2}\|u_\lambda\|_{U_S^2}, \tag{4.7}$$

(ii)

$$\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{-\frac{j}{2}(1-\varepsilon)} \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \Big\|_{V_S^2} \Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{V_S^2}. \quad (4.8)$$

Proof. (i) The bilinear estimate (4.7) is a special case of the bilinear estimate (5.22) of Theorem 5.7 in [17]. For the sake of completeness we reproduce the proof of (4.7) in Appendix B.

(ii) The strategy of this proof is the same as the proof of the bilinear estimate (4.7). In other words it is an application of Theorem 4.3. We take $\phi_1 = \phi_2 = \phi(\xi,\eta) := \tau - \xi^3 + \frac{\eta^2}{\xi}$. The curve of integration is

$$\begin{aligned}
&\Sigma((\tau_1,\xi_1,\eta_1),(\tau_2,\xi_2,\eta_2)) \\
&\qquad = \{(\tau_2,\xi_2,\eta_2) + \Sigma_1\} \cap \{(\tau_1,\xi_1,\eta_1) + \Sigma_2\} \\
&\qquad = \{(\tau,\xi,\eta) \mid (\tau-\tau_2,\xi-\xi_2,\eta-\eta_2) \in \Sigma_1 \text{ and } (\tau-\tau_1,\xi-\xi_1,\eta-\eta_1) \in \Sigma_2\}
\end{aligned}$$

where $(\tau_1,\xi_1,\eta_1) \in \Sigma_1$ and $(\tau_2,\xi_2,\eta_2) \in \Sigma_2$. Then we have

$$\begin{aligned}
\tau - \tau_2 - \psi_1(\xi-\xi_2,\eta-\eta_2) &= 0, & \tau - \tau_1 - \psi_2(\xi-\xi_1,\eta-\eta_1) &= 0, \\
\tau_1 - \psi_1(\xi_1,\eta_1) &= 0 \quad \text{and} & \tau_2 - \psi_2(\xi_2,\eta_2) &= 0,
\end{aligned}$$

which give us

$$\tau = \psi_2(\xi_2,\eta_2) + \psi_1(\xi-\xi_2,\eta-\eta_2) = \psi_1(\xi_1,\eta_1) + \psi_2(\xi-\xi_1,\eta-\eta_1)$$

or equivalently

$$\tau \overset{(1)}{=} \xi_2^3 - \frac{\eta_2^2}{\xi_2} + (\xi-\xi_2)^3 - \frac{(\eta-\eta_2)^2}{\xi-\xi_2} \overset{(2)}{=} \xi_1^3 - \frac{\eta_1^2}{\xi_1} + (\xi-\xi_1)^3 - \frac{(\eta-\eta_1)^2}{\xi-\xi_1}.$$

After rearranging the terms of the identity (2) above and adding the term $(\xi_2 - \xi_1)^3 - \frac{(\eta_2-\eta_1)^2}{\xi_2-\xi_1}$ to both sides of it we obtain

$$\xi_1^3 - \frac{\eta_1^2}{\xi_1} - \xi_2^3 + \frac{\eta_2^2}{\xi_2} + (\xi_2 - \xi_1)^3 - \frac{(\eta_2 - \eta_1)^2}{\xi_2 - \xi_1}$$
$$= (\xi - \xi_2)^3 - \frac{(\eta - \eta_2)^2}{\xi - \xi_2} - (\xi - \xi_1)^3 + \frac{(\eta - \eta_1)^2}{\xi - \xi_1} + (\xi_2 - \xi_1)^3 - \frac{(\eta_2 - \eta_1)^2}{\xi_2 - \xi_1}.$$

By the algebraic resonance identity, we have

$$\omega := \xi_1\xi_2(\xi_1 - \xi_2)\left[3 + \frac{\left|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\right|^2}{|\xi_2 - \xi_1|^2}\right]$$
$$= (\xi - \xi_2)(\xi - \xi_1)(\xi_1 - \xi_2)\left[3 + \frac{\left|\frac{\eta-\eta_1}{\xi-\xi_1} - \frac{\eta-\eta_2}{\xi-\xi_2}\right|^2}{|\xi_2 - \xi_1|^2}\right] \quad (4.9)$$

which implies

$$\operatorname{sgn}(\xi_1\xi_2) = \operatorname{sgn}((\xi - \xi_1)(\xi - \xi_2)). \quad (4.10)$$

It follows from the definitions (4.4) and (4.5) that

$$\mu \le |\xi_1| \le 2\mu, \qquad \lambda \le |\xi_2| \le 2\lambda, \quad (4.11)$$

and

$$-\frac{1}{2} + l \le \frac{\eta_1}{\mu\xi_1} < \frac{1}{2} + l, \qquad -\frac{1}{2} + k \le \frac{\eta_2}{\lambda\xi_2} < \frac{1}{2} + k. \quad (4.12)$$

Furthermore, since $(l, k) \in Q_{j,\frac{\lambda}{\mu}m,n}$ we have

$$-\frac{1}{2}\mu + 2^j m\lambda \le \frac{\eta_1}{\xi_1} < \frac{1}{2}\mu + 2^j(m+1)\lambda,$$
$$\left(-\frac{1}{2} + 2^j n\right)\lambda \le \frac{\eta_2}{\xi_2} < \left(\frac{1}{2} + 2^j(n+1)\right)\lambda,$$

where $|m - n| < 8$. Using these data we want to estimate L given by (4.6). First we estimate the denominator of the integrand

$$\begin{aligned}
&[|\nabla\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2)|^2|\nabla\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)|^2 \\
&\quad -\langle\nabla\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1),\nabla\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2)\rangle^2]^{1/2} \\
&\qquad = [|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2 \\
&\qquad + |\nabla\psi(\xi-\xi_2,\eta-\eta_2)|^2|\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2 \\
&\qquad - \langle\nabla\psi(\xi-\xi_2,\eta-\eta_2),\nabla\psi(\xi-\xi_1,\eta-\eta_1)\rangle^2]^{1/2} \\
&\qquad\qquad \geq [|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2]^{1/2}
\end{aligned}$$

which gives us

$$\begin{aligned}
&L^2((\tau_1,\xi_1,\eta_1),(\tau_2,\xi_2,\eta_2)) \\
&\qquad \leq \int_\Sigma \frac{d\mathcal{H}^{d-2}}{[|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2]^{1/2}}.
\end{aligned} \tag{4.13}$$

Next we provide more explicit determination of the interval of integration through a detailed study.

Without loss of generality we may assume that $\xi_1 < \xi_2$.

Note that if $\omega > 0$, then (4.9) implies that

$$\begin{aligned}
&0 < (\xi-\xi_2)(\xi-\xi_1)(\xi_1-\xi_2) \leq \frac{1}{3}\omega \\
&\Rightarrow (\xi-\xi_2)(\xi-\xi_1) < 0, \\
&\Rightarrow \xi \in (\xi_1,\xi_2).
\end{aligned} \tag{4.14}$$

Combining the above result with (4.10), we have $\xi_1 < 0 < \xi_2$. Since $\mu < |\xi-\xi_2| < 2\mu$ and $\lambda < |\xi-\xi_1| < 2\lambda$, in this case the interval of integration is restricted to $(\xi_2-\mu,\xi_2)$.

On the other hand if $\omega < 0$, then again from (4.9) it follows that

$$\begin{aligned}
&\frac{1}{3}\omega \leq (\xi-\xi_2)(\xi-\xi_1)\underbrace{(\xi_1-\xi_2)}_{<0} < 0 \\
&\Rightarrow (\xi-\xi_2)(\xi-\xi_1) > 0, \\
&\Rightarrow \xi \in \{\xi < \xi_1\} \cup \{\xi > \xi_2\}.
\end{aligned}$$

Thus in this case the interval of integration is $(\xi_2, \xi_2 + \mu)$. Substituting this information into (4.13) we get

$$
\begin{aligned}
(4.13) \quad &\leq \int_{\xi_2-\mu}^{\xi_2+\mu} \frac{\left[1 + \left(\frac{3(\xi-\xi_1)^2 - 3(\xi-\xi_2)^2 - \frac{(\eta-\eta_2)^2}{(\xi-\xi_2)^2} + \frac{(\eta-\eta_1)^2}{(\xi-\xi_1)^2}}{2\left(\frac{\eta-\eta_2}{\xi-\xi_2} - \frac{\eta-\eta_1}{\xi-\xi_1}\right)}\right)^2\right]^{1/2}}{[|\nabla\psi(\xi-\xi_2, \eta-\eta_2) - \nabla\psi(\xi-\xi_1, \eta-\eta_1)|^2]^{1/2}} d\xi \\
&\leq \frac{1}{2}\int_{\xi_2-\mu}^{\xi_2+\mu} \left| \frac{\eta-\eta_2}{\xi-\xi_2} - \frac{\eta-\eta_1}{\xi-\xi_1} \right|^{-1} d\xi \\
&\quad \text{(we use (4.9))} \\
&\leq \frac{1}{2}\int_{\xi_2-\mu}^{\xi_2+\mu} \left| \frac{(\xi-\xi_2)(\xi-\xi_1)}{(\xi_2-\xi_1)(\omega - 3(\xi_2-\xi_1)(\xi-\xi_2)(\xi-\xi_1))} \right|^{1/2} d\xi \\
&\leq c\frac{\mu}{\lambda} 2^{-j}.
\end{aligned}
$$

Hence we have the following estimate

$$
\begin{aligned}
\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} &= \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \\
&\leq c\left(\frac{\mu}{\lambda}\right)^{1/2} 2^{-\frac{j}{2}} \Big\| \sum_{l\in I^j(m,n)} u^0_{\mu,l} \Big\|_{L^2(\mathbb{R}^3)} \Big\| \sum_{k\in J^j(m,n)} u^0_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

where u^0 is the corresponding initial data. Applying Proposition 2.19 from [10], we get

$$
\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \leq c\left(\frac{\mu}{\lambda}\right)^{1/2} 2^{-\frac{j}{2}} \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \Big\|_{U^2_S} \Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{U^2_S}.
$$

On the other hand we have

$$
\begin{aligned}
\Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} &\leq \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \Big\|_{L^4(\mathbb{R}^3)} \Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{L^4(\mathbb{R}^3)} \\
&\leq \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \Big\|_{U^4_S} \Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{U^4_S}. \quad (4.15)
\end{aligned}
$$

Using the above two estimates and the embedding relation Proposition 2.24 (ii) via Proposition 2.20 in [10] we get

$$\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l}u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \leq$$
$$\frac{8\left(\frac{\mu}{\lambda}2^{-j}\right)^{\frac{1}{2}} \|\sum_{l\in I^j(m,n)} u_{\mu,l}\|_{U_S^2}}{\ln 2}\left(\ln c\left(\frac{\lambda}{\mu}2^j\right)^{\frac{1}{2}} + \ln 2 + 1\right)\Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{V_S^2}. \quad (4.16)$$

The estimate (v) in Proposition 2.26 and (4.15) give us

$$\Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{L^2(\mathbb{R}^3)} \leq \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l}\Big\|_{V_S^2}\Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{U_S^4}. \quad (4.17)$$

Applying Proposition 2.20 in [10] to (4.16) and (4.17) we obtain

$$\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l}u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)}$$
$$\leq c\Big(\frac{\mu}{\lambda}2^{-j}\Big)^{\frac{1}{2}}\left(\ln\left(\frac{\lambda}{\mu}2^j\right)^{\frac{1}{2}} + \ln 2 + 1\right)^2 \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l}\Big\|_{V_S^2}\Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{V_S^2} \quad (4.18)$$

where $\frac{\lambda}{\mu}2^j$ is large enough so that $\left(\ln c\left(\frac{\lambda}{\mu}2^j\right)^{\frac{1}{2}}\right)^2 \leq \left(\frac{\lambda}{\mu}2^j\right)^{\varepsilon}$. To be more precise we require $\frac{\lambda}{\mu}2^j \geq \left(\frac{1}{\varepsilon}\right)^{2/\varepsilon^2}$, since $\ln x \leq x^{\frac{1}{n}}$ provided that $x \geq n^{n^2}$.

□

Proposition 4.7 (Bilinear Estimates). *Let $p < 2$ and p' be the Hölder conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Assume further that $\mu \leq \lambda$. Then the following statements hold true.*

(i) If

$$0 < \epsilon_1 < \frac{1}{2} - \frac{1}{p'},$$

then

$$\|u_\mu u_\lambda\|_{L^2(\mathbb{R}^3)} \leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}+\epsilon_1} \|u_\mu\|_{\ell^p(V_S^2)}\|u_\lambda\|_{\ell^{p'}(V_S^2)}, \quad (4.19)$$

where the constant c depends on ϵ_1 and p,

(ii) If

$$\frac{1}{2} - \frac{1}{p'} < \epsilon_2$$

then

$$\|u_\mu u_\lambda\|_{L^2(\mathbb{R}^3)} \leq c \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\epsilon_2} \|u_\mu\|_{\ell^{p'}(V_S^2)} \|u_\lambda\|_{\ell^p(V_S^2)}. \tag{4.20}$$

where the constant c depends on ϵ_2 and p.

Proof. (i)

$$\begin{aligned}
\Big\| \sum_{j=0}^{M_1-1} \sum_{(l,k)\in S(j)} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} &\leq \Big\| \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \sum_{(l,k)\in \mathbb{Z}\times\mathbb{Z}\cap Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \\
&\leq \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \Big\| \sum_{(l,k)\in \mathbb{Z}\times\mathbb{Z}\cap Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \\
&\leq \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \sum_{k\in J^i(m,n)} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \\
&\leq \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \sum_{l\in I^j(m,n)} \Big\| u_{\mu,l} \sum_{k\in J^i(m,n)} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)}
\end{aligned} \tag{4.21}$$

we apply (4.7)

$$\begin{aligned}
&(4.21) \leq c \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \sum_{l\in I^j(m,n)} \left(\frac{\mu}{\lambda}\right)^{1-\varepsilon} \Big\| u_{\mu,l} \Big\|_{V_S^2} \Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k} \Big\|_{V_S^2} \\
&\leq c \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} \left(\frac{\mu}{\lambda}\right)^{1-\varepsilon} \left(\frac{\lambda}{\mu} 2^j\right)^{\frac{1}{p'}} \|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)} \|u_{\lambda,k}\|_{\ell^2(J^j(m,n);V_S^2)} \\
&\leq c \left(\frac{\mu}{\lambda}\right)^{1-\frac{1}{p'}-\varepsilon} \sum_{j=0}^{M_1-1} \sum_{|m-n|<8} 2^{\frac{j}{p'}} \|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)} 2^{j\left(\frac{1}{2}-\frac{1}{p'}\right)} \|u_{\lambda,k}\|_{\ell^{p'}(J^j(m,n);V_S^2)} \\
&\leq c \left(\frac{\mu}{\lambda}\right)^{1-\frac{1}{p'}-\varepsilon} \sum_{j=0}^{M_1-1} 2^{\frac{j}{2}} \sum_{m=-\infty}^{\infty} \|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)} \sum_{n=m-7}^{n=m+7} \|u_{\lambda,k}\|_{\ell^{p'}(J^j(m,n);V_S^2)} \\
&\leq c \left(\frac{\mu}{\lambda}\right)^{1-\frac{1}{p'}-\varepsilon} 2^{\frac{M_1-1}{2}} \|u_\mu\|_{\ell^p(V_S^2)} \|u_\lambda\|_{\ell^{p'}(V_S^2)}.
\end{aligned} \tag{4.22}$$

For $j \geq M_1$ by (4.8) we have

$$\Big\| \sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l}u_{\lambda,k}\Big\|_{L^2(\mathbb{R}^2)} = \Big\| \sum_{l\in I^j(m,n)} u_{\mu,l} \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{L^2(\mathbb{R}^3)}$$
$$\leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{-\frac{j}{2}(1-\varepsilon)}\Big\| \sum_{l\in I^j(m,n)} u_{\mu,l}\Big\|_{V_S^2}\Big\| \sum_{k\in J^j(m,n)} u_{\lambda,k}\Big\|_{V_S^2} \tag{4.23}$$

by almost L^2-orthogonality we get

$$(4.23) \leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{-\frac{j}{2}(1-\varepsilon)}\|u_{\mu,l}\|_{\ell^2(I^j(m,n);V_S^2)}\|u_{\lambda,k}\|_{\ell^2(J^j(m,n);V_S^2)}$$
$$\leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{-\frac{j}{2}(1-\varepsilon)}\|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)}2^{j\left(\frac{1}{2}-\frac{1}{p'}\right)}\|u_{\lambda,k}\|_{\ell^{p'}(J^j(m,n);V_S^2)}.$$

Summing over m's and n's the above estimate we obtain

$$\Big\| \sum_{|m-n|<8}\sum_{(l,k)\in Q_{j,\frac{\lambda}{\mu}m,n}} u_{\mu,l}u_{\lambda,k}\Big\|_{L^2(\mathbb{R}^3)}$$
$$\leq \sum_{|m-n|<8} c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{j\left(\frac{\varepsilon}{2}-\frac{1}{p'}\right)}\|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)}\|u_{\lambda,k}\|_{\ell^{p'}(J^j(m,n);V_S^2)}$$
$$\leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{j\left(\frac{\varepsilon}{2}-\frac{1}{p'}\right)}\sum_{m=-\infty}^{\infty}\|u_{\mu,l}\|_{\ell^p(I^j(m,n);V_S^2)}\sum_{n=m-7}^{m+7}\|u_{\lambda,k}\|_{\ell^{p'}(J^i(m,n);V_S^2)}$$
$$\leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{j\left(\frac{\varepsilon}{2}-\frac{1}{p'}\right)}\|u_\mu\|_{\ell^p(V_S^2)}\|u_\lambda\|_{\ell^{p'}(V_S^2)}.$$

We sum the above estimate over j's we get

$$\Big\| \sum_{j=M_1}^{\infty}\sum_{(l,k)\in S(j)} u_{\mu,l}u_{\lambda,k}\Big\|_{L^2(\mathbb{R}^3)}$$
$$\leq c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{\frac{M_1}{2}(\frac{\varepsilon}{2}-\frac{1}{p'})}\|u_\mu\|_{\ell^p(V_S^2)}\|u_\lambda\|_{\ell^{p'}(V_S^2)}. \tag{4.24}$$

Combining the estimates (4.22) and(4.24), we have

$$\|u_\mu u_\lambda\|_{L^2(\mathbb{R}^3)} \leq c\left(\frac{\mu}{\lambda}\right)^{1-\frac{1}{p'}-\varepsilon} 2^{\frac{M_1-1}{2}}\|u_\mu\|_{\ell^p(V_S^2)}\|u_\lambda\|_{\ell^{p'}(V_S^2)}$$
$$+c\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}-\varepsilon} 2^{M_1\left(\frac{\varepsilon}{2}-\frac{1}{p'}\right)}\|u_\mu\|_{\ell^p(V_S^2)}\|u_\lambda\|_{\ell^{p'}(V_S^2)}.$$

Then choosing

$$M_1 = \left\lfloor \frac{p'-2}{p'+2} \ln_2 \frac{\lambda}{\mu} \right\rfloor$$

and

$$\varepsilon < \frac{1}{p} - \frac{1}{2} - \epsilon_1,$$

gives us

$$\|u_\mu u_\lambda\|_{L^2(\mathbb{R}^3)} \leq c \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}+\epsilon_1} \|u_\mu\|_{\ell^p(V_S^2)} \|u_\lambda\|_{\ell^{p'}(V_S^2)}.$$

(ii) We use the estimate in Theorem 4.6 (ii) and skip the steps that are similar with the previous proof. Then we have

$$\Big\| \sum_{j=0}^{\infty} \sum_{(l,k)\in S(j)} u_{\mu,l} u_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \leq c \Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{p'}-\varepsilon} \sum_{j=0}^{\infty} 2^{j\left(\frac{\varepsilon}{2}-\frac{1}{p'}\right)} \|u_\mu\|_{\ell^{p'}(V_S^2)} \|u_\lambda\|_{\ell^p(V_S^2)}. \tag{4.25}$$

Selecting ε so that

$$\varepsilon < \min\{\frac{2}{p'}, \epsilon_2 - \frac{1}{2} + \frac{1}{p'}\}$$

in (4.25) we obtain the desired result.

□

Notation:

$$I(u_1, u_2, u_3)(t) := \int_0^t e^{(t-t')S} \partial_x (u_1 \overline{u}_2 u_3) dt'$$

where $S(t)$ is the solution operator for the linear $(gKP - II)_3$ equation.

Theorem 4.8. *[Multilinear Estimate] Let $u_1, u_2, u_3 \in \ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)$. Then there exists a constant C such that the following estimate holds*

$$\|I(u_1, u_2, u_3)\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \leq C \prod_{j=1}^{3} \|u_j\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \tag{4.26}$$

Proof.

$$\begin{aligned}\|I(u_1,u_2,u_3)\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} &= \sup_\lambda \lambda^{1/2}\Big(\sum_k \|P_{A_{\lambda,k}}I(u_1,u_2,u_3)\|^p_{V^2_S}\Big)^{1/p} \\ &= \sup_\lambda \lambda^{1/2}\Big(\sum_k \|e^{-\cdot S}P_{A_{\lambda,k}}I(u_1,u_2,u_3)\|^p_{V^2}\Big)^{1/p}. \quad (4.27)\end{aligned}$$

Due to the duality argument in Theorem 2 in [11], we have

$$\begin{aligned}
(4.27) &= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|B(e^{-\cdot S}P_{A_{\lambda,k}}I(u_1,u_2,u_3),v)\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_{-\infty}^{\infty} \langle (e^{-tS}P_{A_{\lambda,k}}I(u_1,u_2,u_3))', v(t)\rangle dt\Big| \\
&= \sup_\lambda \lambda^{\frac{1}{2}} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_{-\infty}^{\infty} \langle (e^{-tS}P_{A_{\lambda,k}}\int_0^t \chi_{[0,\infty)}(t)e^{(t-t')S}\partial_x(u_1\overline{u}_2u_3)dt')', v(t)\rangle dt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_{-\infty}^{\infty} \langle P_{A_{\lambda,k}}\chi_{[0,\infty)}(t)e^{-tS}\partial_x(u_1\overline{u}_2u_3)(t), v(t)\rangle dt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_0^{\infty} \chi_{A_{\lambda,k}}(\xi,\eta)\exp(-it(\xi^3-\frac{\eta^2}{\xi}))i\xi\widehat{u_1\overline{u}_2u_3}\widehat{\overline{v}}d\xi d\eta dt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_0^{\infty} \chi_{A_{\lambda,k}}(\xi,\eta)\widehat{u_1\overline{u}_2u_3}\overline{\exp(it(\xi^3-\frac{\eta^2}{\xi}))(-i\xi)\widehat{v}}d\xi d\eta dt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2)}=1} \Big|\int_0^{\infty}\int_{\mathbb{R}^2} P_{A_{\lambda,k}}(u_1\overline{u}_2u_3)\overline{\partial_x e^{tS}v}dxdydt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2_S)}=1} \Big|\int_0^{\infty}\int_{\mathbb{R}^2} P_{A_{\lambda,k}}(u_1\overline{u}_2u_3)\partial_x\overline{v}dxdydt\Big| \\
&= \sup_\lambda \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U^2_S)}=1} \Big|\int_0^{\infty}\sum_{k_i,\lambda_i} u_{1,\lambda_1,k_1}\overline{u}_{2,\lambda_2,k_2}u_{3,\lambda_3,k_3}\partial_x\overline{v}_{\lambda,k}dxdydt\Big|. \qquad (4.28)
\end{aligned}$$

We will control the term (4.28) by using the Cauchy-Schwarz inequality. Hence thanks to Plancherel identity we can ignore the complex conjugations. Then without loss of generality we may assume that $\lambda_1 \le \lambda_2 \le \lambda_3$. The nonzero contribution to the sum (4.28) comes in the following three cases:

Case I: $\lambda_1 \le \lambda_2 \le \lambda_3 \sim \lambda$,

Case II: $\lambda \le \lambda_1 \le \lambda_2 \sim \lambda_3$,

Case III: $\lambda_1 \le \lambda \le \lambda_2 \sim \lambda_3$.

In all three cases listed above the multilinear estimate (4.26) can be obtained by using the bilinear estimates in Proposition 4.7 and the basic fact about embedding of ℓ^p spaces:

- If $0 < p \leq q \leq \infty$ then $\|u\|_{\ell^q} \leq \|u\|_{\ell^p}$.

In the following we illustrate the calculations leading to (4.26) from (4.28) separately in each case.

Case I:

$$(4.28) \leq c \sup_{\lambda} \lambda^{1/2} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \sum_{\lambda_i} \Big\| \sum_{k_i} u_{1,\lambda_1,k_1} u_{3,\lambda,k_3} \Big\|_{L^2(\mathbb{R}^3)} \Big\| \sum_{k_2} u_{2,\lambda_2,k_2} \partial_x v_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)}. \tag{4.29}$$

We apply the estimate (i) of Proposition 4.7 to both factors of each summand of the sum above and use Theorem 2.14 (iii). Then rearranging the terms we obtain

$$\begin{aligned}
&\text{RHS of } (4.29) \leq \\
&c \sup_{\lambda} \lambda^{\frac{3}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \sum_{\lambda_1 \leq \lambda_2 \leq \lambda} \left(\frac{\lambda_1 \lambda_2}{\lambda^2}\right)^{\frac{1}{2}+\epsilon_1} \|u_{1,\lambda_1}\|_{\ell^p(V_S^2)} \|u_{3,\lambda}\|_{\ell^{p'}(V_S^2)} \|u_{2,\lambda_2}\|_{\ell^p(V_S^2)} \|v_\lambda\|_{\ell^{p'}(V_S^2)} \\
&\leq c \sup_{\lambda} \lambda \sum_{\lambda_1 \leq \lambda_2 \leq \lambda} \left(\frac{\lambda_1 \lambda_2}{\lambda}\right)^{\frac{1}{2}+\epsilon_1} \lambda_1^{-\frac{1}{2}} (\lambda_1^{\frac{1}{2}} \|u_{1,\lambda_1}\|_{\ell^p(V_S^2)}) \lambda_2^{-\frac{1}{2}} (\lambda_2^{\frac{1}{2}} \|u_{2,\lambda_2}\|_{\ell^p(V_S^2)}) (\lambda^{\frac{1}{2}} \|u_{3,\lambda}\|_{\ell^p(V_S^2)}) \\
&\leq c \sup_{\lambda} \prod_{i=1}^{3} \|u_i\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \sum_{\lambda_1 \leq \lambda_2 \leq \lambda} \left(\frac{\lambda_1}{\lambda}\right)^{\epsilon_1} \left(\frac{\lambda_2}{\lambda}\right)^{\epsilon_1} \\
&\leq c \left(\frac{1}{1 - \frac{1}{2^{\epsilon_1}}}\right)^2 \prod_{i=1}^{3} \|u_i\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)}.
\end{aligned}$$

Case II: Let ν be such that $\nu \sim \lambda_2 \sim \lambda_3$, then we have

$$(4.28) \leq c \sup_{\lambda} \lambda^{\frac{1}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \left| \int_{\mathbb{R}^3} \sum_{k_i,\lambda_1,\nu} u_{1,\lambda_1,k_1} u_{2,\nu,k_2} u_{3,\nu,k_3} \partial_x v_{\lambda,k} dxdydt \right|. \tag{4.30}$$

Next as in Case I we use the Cauchy-Schwarz inequality and we get

$$\text{RHS of (4.30)} \\ \leq c \sup_{\lambda} \lambda^{\frac{1}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \sum_{\lambda_1,\nu} \Big\| \sum_{k_i} u_{1,\lambda_1,k_1} u_{2,\nu,k_2} \Big\|_{L^2(\mathbb{R}^3)} \Big\| \sum_{k_3} u_{3,\nu,k_3} \partial_x v_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)}. \tag{4.31}$$

Applying the estimate (i) of Proposition 4.7 to the first factor and the estimate (ii) of Proposition 4.7 to the second factor of each summand of above sum and then using Theorem 2.14 (iii)

$$\text{RHS of (4.30)} \leq$$

$$c \sup_{\lambda} \lambda^{\frac{3}{2}} \sum_{\lambda \leq \lambda_1 \leq \nu} \frac{\lambda_1^{\frac{1}{2}+\epsilon_1} \lambda^{\frac{1}{2}-\epsilon_2}}{\nu} \frac{(\lambda_1^{\frac{1}{2}} \|u_{1,\lambda_1}\|_{\ell^p(V_S^2)})}{\lambda_1^{\frac{1}{2}}} \frac{(\nu^{\frac{1}{2}} \|u_{2,\nu}\|_{\ell^p(V_S^2)})}{\nu^{\frac{1}{2}}} \frac{(\nu^{\frac{1}{2}} \|u_{3,\nu}\|_{\ell^p(V_S^2)})}{\nu^{\frac{1}{2}}}$$

$$\leq c \sup_{\lambda} \left(\prod_{i=1}^{3} \|u_i\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \right) \sum_{\lambda \leq \lambda_1 \leq \nu} \frac{\lambda^{2-\epsilon_2} \lambda_1^{\epsilon_1}}{\nu^2}$$

$$\leq c \sup_{\lambda} \left(\prod_{i=1}^{3} \|u_i\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \right) \sum_{\lambda \leq \lambda_1} \frac{\lambda^{2-\epsilon_2}}{\lambda_1^{2-\epsilon_1}}$$

$$\leq c \frac{1}{1 - \frac{1}{2^{2-\epsilon_1}}} \left(\prod_{i=1}^{3} \|u_i\|_{\ell^\infty_{\frac{1}{2}} \ell^p_0(V_S^2)} \right).$$

Case III: As in Case II we again assume that ν is such that $\nu \sim \lambda_2 \sim \lambda_3$.

$$(4.28) \leq c \sup_{\lambda} \lambda^{\frac{1}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \Big| \int_{\mathbb{R}^3} \sum_{k_i,\lambda_i,\nu} u_{1,\lambda_1,k_1} u_{2,\nu,k_2} u_{3,\nu,k_3} \partial_x v_{\lambda,k} dxdydt \Big|$$
$$\leq c \sup_{\lambda} \lambda^{\frac{1}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \sum_{\lambda_i,\nu} \Big\| \sum_{k_i} u_{1,\lambda_1,k_1} u_{2,\nu,k_2} \Big\|_{L^2(\mathbb{R}^3)} \Big\| \sum_{k_3} u_{3,\nu,k_3} \partial_x v_{\lambda,k} \Big\|_{L^2(\mathbb{R}^3)} \tag{4.32}$$

Similar to Case II we apply the Proposition 4.7 (i) to the first factor and Proposition 4.7 (ii) to the second factor of each summand of above sum and use Theorem 2.14 (iii)

$$(4.32) \leq$$

$$c \sup_{\lambda} \lambda^{\frac{3}{2}} \sup_{\|v\|_{\ell^{p'}(U_S^2)}=1} \sum_{\lambda_1 \leq \lambda \leq \nu} \left(\frac{\lambda_1}{\nu}\right)^{\frac{1}{2}+\epsilon_1} \|u_{1,\lambda_1}\|_{\ell^p(V_S^2)} \|u_{2,\nu}\|_{\ell^{p'}(V_S^2)} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}-\epsilon_2} \|v_\lambda\|_{\ell^{p'}(V_S^2)} \|u_{3,\nu}\|_{\ell^p(V_S^2)}$$

$$\leq \left(\prod_{j=1}^{3} \|u_j\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)}\right) c \sup_\lambda \sum_{\lambda_1\leq\lambda\leq\nu} \lambda^{\frac{3}{2}} \left(\frac{\lambda_1}{\nu}\right)^{\frac{1}{2}+\epsilon_1} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}-\epsilon_2} \lambda_1^{-\frac{1}{2}}\nu^{-1}$$
$$\leq \left(\prod_{j=1}^{3} \|u_j\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)}\right) c \sup_\lambda \sum_{\lambda_1\leq\lambda\leq\nu} \frac{\lambda_1^{\epsilon_1}\lambda^{2-\epsilon_2}}{\nu^2}$$
$$\leq \left(\prod_{j=1}^{3} \|u_j\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)}\right) c \sup_\lambda \sum_{\lambda\leq\nu} \frac{\lambda^{2-\epsilon_1}}{\nu^{2-\epsilon_2}}$$
$$\leq c \left(\prod_{j=1}^{3} \|u_j\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)}\right).$$

□

4.4 Global well-posedness for small data

Theorem 4.9. *There exists $\delta > 0$ such that for any initial data satisfying*

$$\|u_0\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)} < \delta,$$

the Cauchy problem

$$u_t + u_{xxx} + 3\partial_x^{-1}\partial_y^2 u - 6u^2 u_x = 0 \tag{4.33}$$
$$u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^2 \tag{4.34}$$

has a unique global solution $u \in \ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)$ with $\|u\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} < \delta^{1/3}$, where $p < 2$.

Proof. We can rewrite the Cauchy problem (4.33)-(4.34) as an integral equation

$$u(t) = N(u(t)),$$

where

$$N(u(t)) = e^{tS}u_0 + \int_0^t e^{(t-t')S}\partial_x(u\bar{u}u)dt'.$$

We have

$$\|e^{\cdot S}u_0\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \leq \|u_0\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(L^2)}.$$

Let

$$\delta = \frac{1}{[5(C+1)^{\frac{1}{2}}]^3},$$

where the constant C is the same as the constant in the statement of Theorem 4.8.

Define

$$B_r := \{u \in \ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S) \mid \|u\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \leq r\}$$

with $r = \frac{1}{5(C+1)^{\frac{1}{2}}}$. Then for $u \in B_r$

$$\|e^{\cdot S}u_0 - 2I(u,u,u)\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \leq \delta + 2Cr^3 < r.$$

We have

$$\begin{aligned}
\|2I(u_1,u_1,u_1) \quad &- \quad 2I(u_2,u_2,u_2)\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \\
&= \quad \|2\int_0^\infty e^{(t-t')S}\partial_x \underbrace{(u_1^3 - u_2^3)}_{=(u_1^2+2u_1u_2+u_2^2-u_1u_2)(u_1-u_2)}(t')dt'\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \\
&\leq \quad \|2\int_0^\infty e^{(t-t')S}\partial_x((u_1+u_2)(u_1+u_2)(u_1-u_2))(t')dt'\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \\
&+ \quad 2\|\int_0^\infty e^{(t-t')S}\partial_x(u_1u_2(u_1-u_2))(t')dt'\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \\
&\leq \quad (2C2r2r + 2Crr)\|u_1 - u_2\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)} \\
&\leq \quad \frac{2}{5}\|u_1 - u_2\|_{\ell^\infty_{\frac{1}{2}}\ell^p_0(V^2_S)}.
\end{aligned}$$

Hence

$$\begin{aligned} N: \quad & B_r \to B_r \\ & u \mapsto e^{\cdot S} u_0 - 2I(u,u,u) \end{aligned}$$

is a strict contraction, and therefore it has a unique fixed point in B_r. □

Appendix A

KPII

A.1 Derivation of the explicit formula for the soliton Q

We are searching for solutions $u(t, x, y)$ of the KP-II equation (3.2) of the form $Q(\theta)$, where $\theta = x - ct$. Then $Q(\theta)$ satisfies the following ordinary differential equation

$$-cQ' + Q^{(3)} + 6QQ' = 0. \tag{A.1}$$

We integrate both sides of (A.1)

$$-cQ + Q'' + 3Q^2 = C, \tag{A.2}$$

where C is a constant of integration. Next we multiply both sides of (A.2) by $2Q'$

$$-2cQQ' + 2Q''Q' + 6Q^2Q' = 2CQ' \tag{A.3}$$

and then integrate

$$-cQ^2 + (Q')^2 + 2Q^3 = 2CQ + D. \tag{A.4}$$

We look for solutions Q such that Q, Q' tend to zero as $|\theta| \to \infty$. Thus $C = D = 0$. Then we have

$$
\begin{aligned}
& -cQ^2 + \left(\frac{dQ}{d\theta}\right)^2 + 2Q^3 = 0 \\
\Rightarrow \quad & \frac{dQ}{d\theta} = -Q\sqrt{c - 2Q}.
\end{aligned} \tag{A.5}
$$

We choose the negative square root on the right hand side of (A.5), which can be solved by separation of variables to give

$$
Q(\theta) = \frac{c}{2}\text{sech}^2\left(\frac{c^{1/2}\theta}{2}\right). \tag{A.6}
$$

Appendix B

(gKP-II)$_3$

Proof of Theorem 4.6. This proof is a simple application of Theorem 4.3 with

$$\Sigma_1 = \{(\tau,\xi,\eta) \mid \tau - \xi^3 + \frac{\eta^2}{\xi} = 0 \text{ with } \mu \le |\xi| \le 2\mu\}$$

and

$$\Sigma_2 = \{(\tau,\xi,\eta) \mid \tau - \xi^3 + \frac{\eta^2}{\xi} = 0 \text{ with } \lambda \le |\xi| \le 2\lambda\}.$$

Then the curve of integration is

$$\Sigma((\tau_1,\xi_1,\eta_1),(\tau_2,\xi_2,\eta_2)) = \\ \{(\tau,\xi,\eta) \mid \xi_2^3 - \frac{\eta_2^3}{\xi_2} + (\xi-\xi_2)^3 - \frac{(\eta-\eta_2)^2}{\xi-\xi_2} = \xi_1^3 - \frac{\eta_1^2}{\xi_1} + (\xi-\xi_1)^3 - \frac{(\eta-\eta_1)^2}{\xi-\xi_1}\}$$

with

$$(\tau_1,\xi_1,\eta_1),(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2) \in \Sigma_1$$

and

$$(\tau_2,\xi_2,\eta_2),(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1) \in \Sigma_2.$$

Note that due to the Galilean invariance under the following change of variables

$$x' = x + \frac{\eta_1}{\xi_1} y - 3\frac{\eta_1^2}{\xi_1^2} t$$
$$y' = y - 3\frac{\eta_1}{\xi_1} t,$$

where η_1 and ξ_1 are arbitrary but fixed, $u_{\mu,k}$ and u_λ still satisfy the $(gKP-II)_3$ equation. According to the definition of $u_{\mu,k}(t,x,y)$ the support of $\hat{u}_{\mu,k}$ is the following set

$$A_{\mu,k} = \{(\xi,\eta) \in \mathbb{R}^2 \mid \mu \leq |\xi| \leq 2\mu, \quad \left(k - \frac{1}{2}\right)\mu\xi \leq \eta < \left(k + \frac{1}{2}\right)\mu\xi\}.$$

Let's denote $u_{\mu,k}$ with changed variables by $u_{\tilde{\mu},\tilde{k}}$ and the Fourier variables corresponding to x' and y' by ξ' and η' respectively, then we have

$$|\hat{u}_{\tilde{\mu},\tilde{k}}(t,\xi,\eta)| = \frac{|e^{-3it\eta\eta_1/\xi_1}|}{2\pi} \left| \int_{\mathbb{R}^2} e^{-i(x'\xi + y'(\eta - \frac{\eta_1}{\xi_1}\xi))} u_{\mu,k}(t,x',y') dx'dy' \right|$$

which suggests that the support of the Fourier transform of $u_{\tilde{\mu},\tilde{k}}$ is the following set

$$A_{\tilde{\mu},\tilde{k}} = \{(\xi',\eta') \in \mathbb{R}^2 \mid \mu \leq |\xi'| \leq 2\mu, \quad -1 \leq \frac{\eta}{\mu\xi} \leq 1\} \subset A_{\mu,0} \cup A_{\mu,1}$$

and

$$\eta'_1 = 0.$$

Hence without loss of generality we may assume that $\eta_1 = 0$. Then we have $|\eta_1| < 3\mu^2$ and $|\eta - \eta_2| < 3\mu^2$. By rearranging the terms of (4.9) we get

$$3(\xi-\xi_2)^2(\xi-\xi_1)^2(\xi_2-\xi_1)^2 + \omega(\xi-\xi_1)(\xi-\xi_2)(\xi_2-\xi_1) \\ + \eta_2^2(\xi-\xi_1)^2 - 2\eta\eta_2(\xi_2-\xi_1)(\xi-\xi_1) + \eta^2(\xi_2-\xi_1)^2 = 0. \quad \text{(B.1)}$$

We assume $|\xi - \xi_1| \ll \mu|\omega|$ which gives us $|\omega| \gg 1$. Under this assumption the following estimate holds true

$$
\begin{aligned}
(\xi-\xi_2)^2(\xi-\xi_1)^2(\xi_2-\xi_1)^2 &= \left(\frac{|\xi_2-\xi_1||\xi-\xi_1||\xi-\xi_2|}{\omega}\right)\omega|\xi_2-\xi_1||\xi-\xi_1||\xi-\xi_2| \\
&\leq C\omega|\xi_2-\xi_1||\xi-\xi_1||\xi-\xi_2|. \qquad \text{(B.2)}
\end{aligned}
$$

We also have

$$
\begin{aligned}
\eta_2^2 \quad (\xi-\xi_1)^2 &= \frac{\eta_2^2(\xi-\xi_1)}{\omega(\xi_2-\xi_1)(\xi-\xi_2)}\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2) \\
&= \frac{\eta_2^2}{\omega}\left(\frac{1}{\xi_2-\xi_1}+\frac{1}{\xi-\xi_2}\right)\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2) \\
&\leq \frac{\eta_2^2}{|\xi_1||\xi_2||\xi_1-\xi_2|\left[3+\frac{\left|\frac{\eta_2}{\xi_2}\right|^2}{|\xi_2-\xi_1|^2}\right]}\left(\frac{1}{\xi_2-\xi_1}+\frac{1}{\xi-\xi_2}\right)\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2) \\
&\leq \frac{|\xi_2||\xi_2-\xi_1|}{|\xi_1|}\left(\frac{1}{\xi_2-\xi_1}+\frac{1}{\xi-\xi_2}\right)\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2) \\
&\leq C\omega(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2) \qquad \text{(B.3)}
\end{aligned}
$$

Using (B.2) and (B.3) from (B.1) we deduce

$$
|\xi-\xi_1| \leq C\frac{\eta^2}{\omega}.
$$

Next we want to calculate L that is formulated in (4.6). We have

$$
\begin{aligned}
&[|\nabla\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2)|^2|\nabla\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)|^2 \\
&\qquad -\langle\nabla\phi(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1),\nabla\phi(\tau-\tau_2,\xi-\xi_2,\eta-\eta_2)\rangle^2]^{1/2} \\
&\qquad = [|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2 \\
&\qquad + |\nabla\psi(\xi-\xi_2,\eta-\eta_2)|^2|\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2 \\
&\qquad - \langle\nabla\psi(\xi-\xi_2,\eta-\eta_2),\nabla\psi(\xi-\xi_1,\eta-\eta_1)\rangle^2]^{1/2} \\
&\qquad \geq \text{(by Hölder Inequality)} \\
&\qquad \geq |\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|
\end{aligned}
$$

which gives us

$$
\begin{aligned}
L^2 \quad ((\tau_1,\xi_1,\eta_1),(\tau_2,\xi_2,\eta_2)) &\leq \int_\Sigma \frac{d\mathcal{H}^1}{[|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2]^{1/2}} \\
= \quad & \int_{\eta_2-\mu^2}^{\eta_2+\mu^2} \frac{\sqrt{1+\left(\frac{d\xi}{d\eta}\right)^2}d\eta}{[|\nabla\psi(\xi-\xi_2,\eta-\eta_2)-\nabla\psi(\xi-\xi_1,\eta-\eta_1)|^2]^{1/2}} \\
= \quad & \int_{\eta_2-\mu^2}^{\eta_2+\mu^2} \frac{\sqrt{1+\left(\frac{-2\frac{\eta-\eta_2}{\xi-\xi_2}+2\frac{\eta-\eta_1}{\xi-\xi_1}}{3(\xi-\xi_2)^2+\left(\frac{\eta-\eta_2}{\xi-\xi_2}\right)^2-3(\xi-\xi_1)^2-\left(\frac{\eta-\eta_1}{\xi-\xi_1}\right)^2}\right)^2}d\eta}{\sqrt{(3(\xi-\xi_2)^2-3(\xi-\xi_1)^2+\left(\frac{\eta-\eta_2}{\xi-\xi_2}\right)^2-\left(\frac{\eta-\eta_1}{\xi-\xi_1}\right)^2)^2+\left(-2\frac{\eta-\eta_2}{\xi-\xi_2}+2\frac{\eta-\eta_1}{\xi-\xi_1}\right)^2}} \\
= \quad & \int_{\eta_2-\mu^2}^{\eta_2+\mu^2} \frac{d\eta}{\left|3(\xi-\xi_2)^2-3(\xi-\xi_1)^2+\left(\frac{\eta-\eta_2}{\xi-\xi_2}\right)^2-\left(\frac{\eta-\eta_1}{\xi-\xi_1}\right)^2\right|} \\
\leq \quad & C\frac{\mu^2}{\lambda^2}.
\end{aligned}
$$

Then we get

$$\|u_{\mu,k}u\|_{L^2(\mathbb{R}^3)} \leq c\left(\frac{\mu}{\lambda}\right)\|u^0_{\mu,k}\|_{L^2(\mathbb{R}^2)}\|u^0\|_{L^2(\mathbb{R}^3)}$$

where u^0 is the corresponding initial data. By applying Proposition 2.19 of [10] we get the desired result. □

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