Shock Fluctuations in KPZ Growth Models

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Abstract

The Kardar-Parisi-Zhang (KPZ) universality class is a class of stochastic growth models which has attracted much interest, especially since the discovery about 15 years ago that the Tracy-Widom distributions from random matrix theory arise in it. Since then, more and more subclasses of the KPZ class have been studied, and experimental evidence for the soundness of KPZ scalings and statistics has been given.

The aims of this thesis are the following. First, we introduce the KPZ class and discuss its conjectured universal scaling properties, limiting distributions and processes. As examples of growth models belonging to the KPZ class where these aspects have been studied, we treat in particular the (totally) asymmetric simple exclusion process ((T)ASEP) and last passage percolation (LPP). We describe the Tracy-Widom distributions, and the Airy processes which appear in these models. As a first result, we obtain the limiting distribution of certain particle positions in TASEP with particular initial data.

Second, we focus on the study of shocks. After introducing the main concepts, we prove the emergence of an independence structure, which appears on a general level in LPP. With this independence, we provide the limiting distributions of shock positions in concrete cases in TASEP and show that they are given by products of Tracy-Widom distributions. We also show that the correlation length in KPZ models, which in all settings considered so far was $t^{2/3}$ (t being the observation time), degenerates at the shock to $t^{1/3}$.

Finally, we consider a critical scaling, which, depending on the choice of the parameter, interpolates between shocks, flat profiles, and rarefaction fans. We prove that the fluctuations of particle positions in this critical scaling are, in the large time limit, given by a new transition process. The correlation length is shown to be $t^{2/3}$ again. We perform a numerical study which suggests that we recover the product structure of shocks by letting the scaling parameter tend to infinity.

Contents

Introduction

One of the most widely used results in probability theory is the central limit theorem. It provides an apt description for a large class of mathematical and real world phenomena - *the Gaussian universality class*. However, this class is not all-encompassing.

Triggered by the seminal paper [48] of Kardar, Parisi and Zhang in 1986, a class of stochastic growth models has been introduced and studied which does not belong to the Gaussian class. In recognition of the authors of [48] it is called the KPZ universality class. Beginning with the work [9] of 1999, more and more rigorous results became available which show that the KPZ class has indeed very different scaling properties and limit laws than the Gaussian class. These scalings and statistics have experimentally been proven to give a sound description of e.g. turbulence in liquid crystals, see [66]. To illustrate the difference between the Gaussian and the KPZ class, consider the *random deposition model* (Figure 1 (a)). Take a familiy of independent, rate 1 Poisson processes and assign one to each integer $i \in \mathbb{Z}$. Whenever its Poisson process jumps, a unit box rains down on i and is added to the (initially empty) column of boxes having already rained down on i. The height at i at time t is the column height, equivalently, the position of the heighest box located over i. Due to the independence of the Poisson processes and the abscence of interaction between columns, the heights of different columns are independent, and the height of a fixed column at time t has gaussian fluctuations of order $\sqrt{t}$ around its mean. In contrast, consider now the *ballistic deposition model* (Figure 1 (b)). Boxes rain down as before, but now a spatial correlation is introduced: A box raining down sticks to the edge of the first box it encounters. The height at i is again the position of the heighest box over i. So if the height at $i+1$ is e.g. three times as large as the height at i, it suffices one box to rain down on i, to have the same height at i and $i+1$. The independence of column heights is now broken, and the model has the following features: It is *smoothing* (large gaps are filled quickly due to the sticking), its *growth velocity depends on the slope of the interface* (a higher absolute value of the slope leads to faster growth), and a *random, local growth rule* (the Poisson processes are independent and each one has only local influence). These features are the defining properties of the KPZ class, see the very beginning of Chapter 1 for a precise formulation of them.

Given this class, one asks: What is the order of fluctuations in the KPZ class, is it $t^{1/2}$

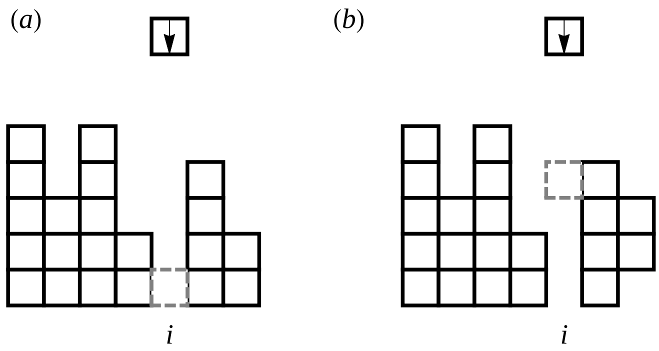

Figure 1: The Gaussian versus the KPZ universality class: In the random deposition model (a), columns of boxes grow independently of each other as boxes (driven by independent Poisson processes) rain down and are added on top of the initially empty column on i. The dashed box indicates the final position of the box raining down. The height at a given site is asymptotically gaussian under rescaling. In the ballistic deposition model (b), growth is again driven by independent Poisson processes, but a spatial correlation is introduced by letting a new box stick to the first box it encounters. Thanks to Sunil Chhita for helping make this figure.

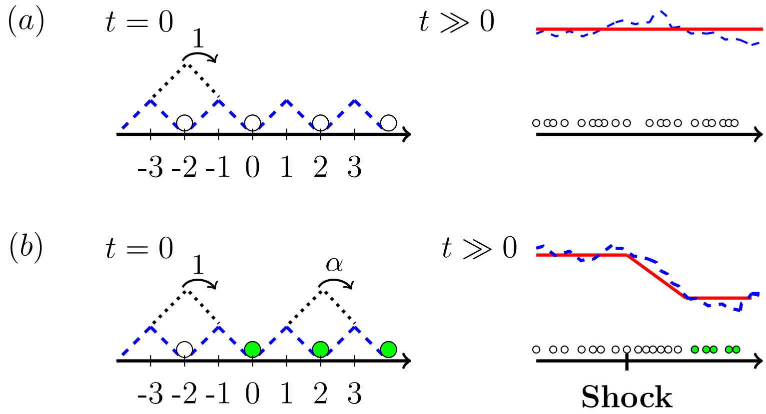

Figure 2: TASEP and associated interface (dashed line) with particles initially occupying $2\mathbb{Z}$. Particles jump from their position $i \in \mathbb{Z}$ to $i+1$ after an exponential waiting time if $i+1$ is empty, the height at i increases by 2 after each jump (indicated by dotted lines). In (a), all particles have speed 1, the interface at $t = 0$ is a hat function. For $t \gg 0$, the limit shape is a horizontal line (solid) and the interface (dashed) fluctuates around it on the $t^{1/3}$ scale. In (b), particles also initially occupy $2\mathbb{Z}$, and the initial interface is the same. But particles starting from a nonnegative number have speed $\alpha < 1$, particles behind them speed 1. The speed 1−particles are blocked behind the last slower particle, the interface (dashed) fluctuates on the $t^{1/3}$ scale around the limit shape, which has a point where it is not smooth (shock).

like in the gaussian class ? Is the KPZ class governed by a single limit law ? What is the order of spatial correlations ? There are some KPZ models for which results are available, which are then conjectured by universality to hold for the entire KPZ class.

The first KPZ models for which partial answers have been given are Poisson last passage percolation and (the interface associated to) the totally asymmetric simple exclusion process (TASEP) (see [9], [45]). Let us deal with TASEP here. Consider an initial interface given by a hat function on $\mathbb{R}$, as shown in Figure 2 (a), (b) (left), linearly interpolating between the values 0 and 1 taken on consecutive integers. Imagine now that initially every even integer is occupied by a particle. Let now each particle wait, independently of all other particles, an exponential time. If the particle is located at i, and $i+1$ is empty by the time the exponential clock rings the particle jumps to $i+1$, otherwise nothing happens. As the particle jumps, the height of the interface at i increases by two. If now the exponentials have, additionally to their independence, the same parameter, say 1, there is flat limit shape, around which the interface fluctuates, see Figure 2 (a). In this case, the order of fluctuations has been proven to be $t^{1/3}$, the correlations are $t^{2/3}$ and fluctuations are governed by the GOE Tracy-Widom distribution, see [19]. The situation is very different if the particles initially located on $2\mathbb{N}_0$ are slow (parameter $\alpha < 1$), and the particles behind them have parameter 1. What happens is that the faster particles (which cannot overtake) get jammed behind the last slower particle, leading to a different limit shape which at a point is not smooth. This corresponds to a discontinuity in the particle density, which is called a *shock*. These shocks are the topic of this thesis.

Again we ask: What limit laws does one obtain at the shock ? What is the correlation length ? Such shocks in the exclusion process have been an object of study for a long time without any motivation coming from the KPZ picture. All previous results, however, on shocks are about random initial data - and the shock fluctuations one observes then are the ones of the initial data, not of the KPZ model itself. We thus study deterministic initial data which lead to shocks. We provide a full description of such shocks. For the first time, we show that the correlation length in the KPZ class can degenerate to $t^{1/3}$ when the limit shape is not smooth. We also determine the limit law of fluctuations at the shock, which is a product of two Tracy-Widom distributions. This product form is based on a key idea of this thesis: An asymptotic independence, which holds in a general setting in TASEP and last passage percolation under some assumptions.

Having thus described the shock behavior, we are interested how the transition from the flat case (Figure 2 (a)) to the shock case (Figure 2 (b)) occurs. We prove that a new transition process $\mathcal{M}_a$ arises, which not only interpolates between the shock and the flat case, but also the flat case and a rarefaction fan (linearly decreasing density). We show that in this critical regime, the correlation length is again $t^{2/3}$. Finally, we perform a numerical study, which indicates that we recover the asymptotic independence observed at the shock if we send the parameter tuning the criticality of the scaling to infinity.

The thesis is organized as follows. In Chapter 1, we give an overview on the KPZ universality class, with a special focus on TASEP and last passage percolation, and state the first result of this thesis about particle fluctuations, Theorem 1.27. Theorem 1.27 will be used in Chapter 3 to prove the limit law of the shock fluctuations in a special case.

In Chapter 2, we introduce the concept of shocks and give the known result for random initial data. Then we state and explain the main results about shock fluctuations we obtained : Theorems 2.4, 2.5, 2.6 give the limit law of the shock in three different, concrete cases, and the respective limit law is a product of Tracy-Widom distributions; and Theorem 2.7, which establishes the product form of the shock in a general model, given some assumptions. We also give the main result we obtained about critical scalings, Theorem 2.8, which states that the process of rescaled particle positions converges to $\mathcal{M}_a$ when looking at the shock.

In Chapter 3, we prove Theorems 1.27, 2.4, 2.5, 2.6 and 2.7.

In Chapter 4, we prove Theorem 2.8 and perform the numerical study. Chapter 3 is based on the article [38], and Chapter 4 is based on [34], both written with Patrik Ferrari.

Chapter 1

Kardar-Parisi-Zhang Universality

1.1 KPZ growth and equation

A universality class of growth models is named after Kardar, Parisi and Zhang (abbreviated KPZ in the following), who, in [48], introduced a SPDE modeling interface growth (see (1.2)) which belongs to this class. Let $x \in \mathbb{R}^d$ be the spatial coordinate and $t \geq 0$ be the observation time. Then we denote by $h(x,t)$ the height of the interface in x at time t. A model is said to belong to the KPZ universality class if it has the following three properties.

1 **Smoothing:** The growth has an intrinsic smoothing mechanism, caused by the surface tension. This amounts to the existence of a *deterministic* limit shape

$$\lim_{t\to\infty} \frac{h(t\xi, \tau t)}{t} = h_{\mathrm{ma}}(\xi, \tau) \tag{1.1}$$

($t\xi$ is the scalar multiplication in $\mathbb{R}^d$).

2 **Slope-dependant macroscopic speed:** The macroscopic speed $v_{\mathrm{ma}} = \partial_\tau h_{\mathrm{ma}}$ is a function of the macroscopic slope $u_{\mathrm{ma}} = \nabla_x h_{\mathrm{ma}}$ only and satisfies $\Delta_x v_{\mathrm{ma}}(u_{\mathrm{ma}}) \neq 0$.

3 **Random, local growth:** There is a random growth rule which is local in space and time.

We do not mathematically formalize property 3, but it will be clear by the examples what we mean by it. In this thesis, we will restrict us to growth in one dimension, i.e. $d = 1$, since KPZ growth models in higher dimensions are substantially less well understood.

KPZ equation in dimension one

In [48], KPZ introduce the equation, nowadays called *KPZ equation*, which in dimension one ($x \in \mathbb{R}$) reads

$$\partial_t h = \lambda \partial_x^2 h + \mu \left(\partial_x h\right)^2 + D\Xi \tag{1.2}$$

for $\lambda, \mu, D \in \mathbb{R}$. Here Ξ is space-time white noise, for which $\Xi(x,t)$ and $\Xi(y,s)$ are correlated if and only if $x = y$ and $t = s$ (see [27], p. 29 and the references therein for more on space time white noise). This reflects property 3 of a KPZ model. The Laplacian in (1.2) accounts for the smoothing mechanism 1. The second term on the right-hand side (R.H.S.) of (1.2) is the slope dependance. One could replace it by a more general term $v(\partial_x h)$, however, Taylor-developping v around zero we obtain

$$v(u) = v(0) + v'(0)u + \frac{v''(0)u^2}{2} + \mathcal{O}(u^3). \tag{1.3}$$

The first term in (1.3) can be removed from the equation by a time-shift, and the second one is often assumed to be zero since v often can be assumed to be symmetric, but anyway it can also be removed from the equation. Hence the quadratic term is the first non-trivial contribution, and it is the only one we keep. Note we have not commented on the limit shape (1.1) yet. The reason is that the KPZ equation is a priori ill-defined, it is e.g. not clear what $\left(\partial_x h\right)^2$ is supposed to mean. Recently a theory has been developed by Martin Hairer (see [42]) to solve, among others, the KPZ equation. It is an extension of the Cole-Hopf solution, which we now present. We specialize our parameters to $\lambda = 1/2$, $\mu = -1/2$ and $D = 1$, which is no loss of generality.

The Cole-Hopf solution is based on the Cole-Hopf transform, given by

$$Z(x,t) = e^{-h(x,t)}. \tag{1.4}$$

Of course, nothing is won in terms of well-definedness from this. But, as was already noted by KPZ in [48], formally Z solves the well-posed stochastic heat equation (SHE) with multiplicative noise

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z - Z\Xi. \tag{1.5}$$

We then define the Cole-Hopf solution of the KPZ equation via the solution of (1.5),

$$h(x,t) := -\log(Z(x,t)). \tag{1.6}$$

A deeper vindication of this procedure is that then the solution of the KPZ equation arises from a model in the KPZ class - the ASEP defined in Section 1.3, with a so-called weak asymmetry, see [15]. First results about the law of the solution of the KPZ equation, especially its large time behavior, were obtained by Amir, Corwin and Quastel in [3] and, independently, by Sasamoto and Spohn in [60]. They consider the narrow-wedge initial data $Z(x,0) = \delta_{x=0}$, provide an explicit formula for

$$F_t(s) = \mathbb{P}(h(x,t) - x^2/2t - t/24 \geq -2^{-1/3}t^{1/3}s) \tag{1.7}$$

and show that F_t is independent of x. As a corollary of their formula for (1.7), they obtain the following result.

Theorem 1.1 (Corollary 1.3 in [3], see also [60]). *Consider F_t as defined in* (1.7). *Then*

$$\lim_{t\to\infty} F_t(s) = F_{\mathrm{GUE}}(s), \tag{1.8}$$

where F_{GUE} is the GUE Tracy-Widom distribution which we define in (1.18).

From the convergence in (1.8) we obtain as deterministic limit shape

$$h_{\mathrm{ma}}^{\mathrm{KPZeq,wedge}}(\xi,\tau) = \xi^2/2\tau + \tau/24. \tag{1.9}$$

Especially, we have property 2 of a KPZ model since $v_{\mathrm{ma}}^{\mathrm{KPZeq,wedge}} = -(u_{\mathrm{ma}}^{\mathrm{KPZeq,wedge}})^2/2 + 1/24$, hence the KPZ equation (at least with narrow wedge initial data) is in the KPZ universality class. Recently, formulas analogous to (1.7) have been derived in [17] for the initial data $Z(x,0) = e^{B_x}$, where $(B_x, x \in \mathbb{R})$ is two-sided brownian motion.

1.1.1 Conjectured scaling and statistics in KPZ growth

Recall that in this thesis we restrict ourselves to KPZ growth in one dimension, the interface $h(x,t)$ will always take $x \in \mathbb{R}$ and $t \geq 0$ as arguments. Along with the KPZ universality class comes a conjecture about scalings and limit laws within the class. Some authors even characterize the KPZ class by these scalings and statistics. We do not take this route here and rather give the conjectured behavior of all KPZ models. These conjectures are based on known results for *some* KPZ model, which by universality are then believed to hold for *all* KPZ models. KPZ models for which the scaling properties and limiting statistics are (at least partially) known are called exactly solvable. We will present in detail one solvable model called TASEP, see Section 1.3.

Scalings: One conjectures that the order of fluctuations of $h(\xi t, t)$ around the limit shape $th_{\mathrm{ma}}(\xi, 1)$ is $t^{1/3}$, and, if h_{ma} is smooth in ξ, the length of correlations is $t^{2/3}$: One defines the rescaled interface

$$h^{\mathrm{resc}}(u,t) = \frac{h(\xi t + ut^{2/3}, t) - th_{\mathrm{ma}}(\xi + ut^{-1/3}, 1)}{t^{1/3}} \tag{1.10}$$

and $h^{\mathrm{resc}}(u,t)$ converges in the $t \to \infty$ limit to a non-trivial process $\mathcal{A}(u)$ (the convergence is usually in the sense of finite dimensional distributions). The fact that the correlation length in KPZ models may be different if h_{ma} is not smooth was already expected in e.g. [37], but the first proof for this phenomenon is given by the results in this thesis, see e.g. Theorem 2.4.

Statistics: While the scaling (1.10) is believed to be universal, the limit process $\mathcal{A}(u)$ is conjectured to depend on the geometry, i.e. the initial data/the limit shape. One thus divides the KPZ class in a few subclasses depending on the geometry. The geometries which have attracted the most attention are the following three, and crossovers between them.

> **Curved geometries:** $\partial_\xi^2 h_{\rm ma}(\xi,\tau) \neq 0$ and deterministic initial data. The corresponding limit process is the Airy_2 process defined in Section 1.3.5.
>
> **Flat geometries (deterministic and stationary)**: For these $\partial_\xi^2 h_{\rm ma}(\xi,\tau) = 0$.
>
> For deterministic initial data the corresponding limit process is the Airy_1 process defined in Section 1.3.4.
>
> For stationary initial data, the evolution of the interface h is determined by the evolution of a stochastic process η (as we have seen in the introduction with the particle process determining the interface) which is started from a stationary, translation invariant measure. Here one sees the $\mathrm{Airy}_{\rm stat}$ process, see Section 1.3.7.

One also considers crossovers between any of these three subclasses. The crossover which will appear most often later is the following.

> **Flat (deterministic) - Curved**: Given deterministic initial data, there is a transition from a flat profile to a curved one; in the transition region one observes the $\mathrm{Airy}_{2\to 1}$ process which interpolates between the Airy_1 and the Airy_2 process, see Section 1.3.6.

Crossovers from flat (deterministic) to flat (stationary) have also been considered, see [23]. The transition from one flat region to another flat region with deterministic initial data has not been considered before, and this is one of the contributions of this thesis, see Figure 2.2 and Theorem 2.4.

1.2 Tracy-Widom Distributions

The following two definitions are from [4], except that we alter the moment condition. By $Z_{i,j} \sim \mathcal{N}(0,1)$ below we mean that the random variable $Z_{i,j}$ has standard normal distribution, we will use the $\sim$ symbol for other distributions in the following too.

Definition 1.2 (See [4]). *Let $(Z_{i,j})_{j>i\geq 1}$ and $(Y_{i,i})_{i\geq 1}$ be two independent families of random variables each of which consists of i.i.d., zero mean, real-valued random variables.*

Assume that $Z_{1,2}$ *has variance one. A real* $N \times N$ *Wigner matrix* H^N *is a symmetric matrix with*

$$H^N_{i,j} = \begin{cases} \frac{Z_{i,j}}{\sqrt{N}} & \text{if} \quad i < j \\ \frac{Y_{i,i}}{\sqrt{N}} & \text{if} \quad i = j. \end{cases} \tag{1.11}$$

If additionaly $Z_{i,j} \sim \mathcal{N}(0,1)$ *and* $Y_{i,i} \sim \mathcal{N}(0,2)$*, then* H^N *is a real Wigner matrix belonging to the Gaussian Orthogonal Ensemble (GOE), equivalently we say it is a GOE matrix.*

The complex case is defined analogously.

Definition 1.3 (See [4])**.** *Let* $(\tilde{Z}_{i,j})_{j>i\geq 1}$ *and* $(\tilde{Y}_{i,i})_{i\geq 1}$ *be two independent families of random variables each of which consists of i.i.d. random variables.* $\tilde{Z}_{1,2}$ *is complex-valued and satisfies* $\mathbb{E}\left(|\tilde{Z}_{1,2}|^2\right) = 1$ *and we have* $\mathbb{E}(|Y_{1,1}|) < \infty$*. A complex* $N \times N$ *Wigner matrix* M^N *is a Hermitian matrix which satisfies*

$$M^N_{i,j} = \begin{cases} \frac{\tilde{Z}_{i,j}}{\sqrt{N}} & \text{if} \quad i < j \\ \frac{\tilde{Y}_{i,i}}{\sqrt{N}} & \text{if} \quad i = j. \end{cases} \tag{1.12}$$

If additionaly $\mathrm{Re}(\tilde{Z}_{1,2}) \sim \mathcal{N}(0,1/2)$ *and* $\mathrm{Re}(\tilde{Z}_{1,2}), \mathrm{Im}(\tilde{Z}_{1,2})$ *are i.i.d. and* $\tilde{Y}_{i,i} \sim \mathcal{N}(0,1)$*, then* M^N *is a complex Wigner matrix belonging to the Gaussian Unitary Ensemble (GUE), equivalently we say it is a GUE matrix.*

A fundamental object is the spectrum of a $N \times N$ Wigner matrix. Denote $\lambda_1^N, \dots, \lambda_N^N$ its eigenvalues. We define the associated (random) empirical measure

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^N}. \tag{1.13}$$

The semicircle law is the probability measure on $\mathbb{R}$ (equipped with the Borel sigma-algebra and the Lebesgue measure) with density

$$\sigma(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{[-2,2]}(x). \tag{1.14}$$

We now have that μ_N converges to $\sigma(x)\mathrm{d}x$ in the following sense. See Theorem 2.1.21 in the book [4] for a proof, a version of this result was first proved by Wigner in [69].

Theorem 1.4 (Theorems 2.1.1 and 2.2.1 in [4])**.** *Let* μ_N *be the the measure* (1.13) *of a real or complex Wigner matrix. We assume for all* $k \in \mathbb{N}$

$$\mathbb{E}\left(|Z_{1,2}|^k\right), \mathbb{E}\left(|\tilde{Z}_{1,2}|^k\right), \mathbb{E}\left(|Y_{1,2}|^k\right), \mathbb{E}\left(|\tilde{Y}_{1,2}|^k\right) < \infty.$$

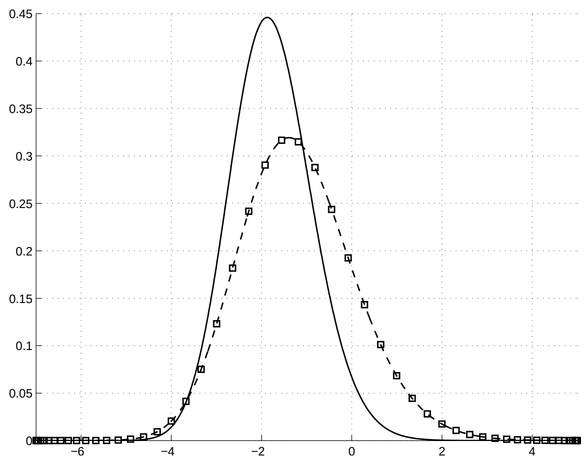

Figure 1.1: The densities of the GUE Tracy-Widom distribution F_{GUE} (solid line) and the GOE Tracy-Widom Distribution F_{GOE} (dashed line with boxes) on $[-7, 5]$.

Then for all $\varepsilon > 0$ and $f \in C_{\mathrm{b}}(\mathbb{R})$ (the space of bounded continuous functions on $\mathbb{R}$) we have

$$\lim_{N\to\infty} \mathbb{P}\left(\left| \int_{\mathbb{R}} \mathrm{d}\mu_N(x) f(x) - \int_{\mathbb{R}} \mathrm{d}x f(x)\sigma(x) \right| > \varepsilon \right) = 0. \tag{1.15}$$

The moment assumption can be substantially weakend, see Theorem 2.1.21 in [4]. However the finiteness of second moments needs to be assumed, see [6]. There are almost-sure versions of (1.15), see e.g. [5].

Our aim is to rescale the largest eigenvalue λ_N^N of a $N \times N$ Wigner matrix in such a way that its limiting behavior is exhibited. From (1.15) we already know that λ_N^N is located around $2N$. Let $\varepsilon > 0$ be small. For $0 < x < \varepsilon$ we have $\sqrt{4-(2-x)^2} = \sqrt{x}\sqrt{4-x}$ with $\sqrt{4-x} = \mathcal{O}(1)$. Hence approximately we have for the eigenvalues of a Wigner matrix

$$\#\{\text{eigenvalues in } (2N - \varepsilon N, 2N)\} \approx N \int_0^{\varepsilon} \mathrm{d}x \sqrt{4-(2-x)^2} \approx N \int_0^{\varepsilon} \mathrm{d}x \sqrt{x} \approx N\varepsilon^{3/2}.$$

Hence in order for this quantity to be of order one, we need to take $\varepsilon = N^{-2/3}$. This leads to the rescaling

$$\lambda_N^N \approx 2N + N^{1/3}\zeta_N, \tag{1.16}$$

where ζ_N is random. The correctness of this scaling and the limit law of the ζ_N were shown by Tracy and Widom for GUE matrices in [67] and for GOE matrices in [68].

Theorem 1.5 ([67], [68])**.** *Let $\lambda_{N,GOE}^N$ be the largest eigenvalue of a $N \times N$ GOE matrix. Then*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda_{N,GOE}^N - 2N}{N^{1/3}} \le s \right) \tag{1.17}$$

	Expectation	Variance	Skewness	Kurtosis
F_{GOE}	-1.2065	1.6078	0.2935	0.1652
F_{GUE}	-1.7711	0.8132	0.2241	0.0934

Table 1.1: Data of the basic statistics of the GOE and GUE Tracy-Widom distributions $F_{\mathrm{GOE}}, F_{\mathrm{GUE}}$.

exists and is denoted by $F_{\mathrm{GOE}}(s)$. F_{GOE} *is a probability distribution function, and the associated law is called the GOE Tracy-Widom law.*

Equally, let $\lambda^N_{N,GUE}$ *be the largest eigenvalue of a* $N \times N$ *GUE matrix. Then*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\lambda^N_{N,GUE} - 2N}{N^{1/3}} \leq s\right) \tag{1.18}$$

exists and is denoted by $F_{\mathrm{GUE}}(s)$. F_{GUE} *is a probability distribution function and the associated law is called the GUE Tracy-Widom law.*

The densities $F'_{\mathrm{GOE}}, F'_{\mathrm{GUE}}$ are plotted in Figure 1.1, the basic statistics are in Table 1.1.

1.3 The (Totally) Asymmetric Simple Exclusion Process

We now turn to a model in the KPZ class for which in a special case detailed information about the rescaled interface (1.10) is available. This process is the exclusion process (EP) in the asymmetric simple case, or ASEP for short. This is a Markov process of particles moving on $\mathbb{Z}$ which was introduced by Spitzer in [64]. In the exclusion process on $\mathbb{Z}$, there is at most one particle on each $i \in \mathbb{Z}$. For ASEP, the dynamics are as follows.

i) Independently of all other particles, a particle on i waits an exponential time.

ii) After that time, it chooses the site $i+1$ with probability $p \neq 1/2$ and the site $i-1$ with probability $1-p=q$.

iii) If the chosen site is empty, the particle jumps to it, if not, nothing happens.

In fact, all new results concerning the EP of this thesis are restricted to the totally ASEP (TASEP for short) for which $p=1$. The symmetric simple EP $p=1/2$ has very different behavior than ASEP, see e.g. [49], Chapter 4.

1.3.1 Construction of ASEP

ASEP is a continuous-time Markov process with state space $X = \{0,1\}^{\mathbb{Z}}$. For an $\eta \in X$ and $z \in \mathbb{Z}$ we denote by $\eta(z)$ the zth coordinate of η. The space X can be equipped with the product topology: For the projection maps $\pi_z(\eta) = \eta(z)$ the topology on X is the unique topology with subbase $\mathcal{S}_X = \cup_{z\in\mathbb{Z}}\pi_z^{-1}(A)$ with $A \subseteq \{0,1\}$. X is then equipped with the sigma-algebra generated by the open sets. We denote by $C(X)$ the set of $\mathbb{R}$-valued continuous functions on X. There are two classical and equivalent ways of constructing a Markov process with dynamics given by i), ii), iii). The first is a graphical construction which goes back to Harris in [44], see also [54]. The other is the Hille-Yosida construction which is described in full detail in [53]. Both have been presented in the author's master thesis [56], from which we borrow some of the notation and an earlier version of Figure 1.2 to give a short description of the construction of ASEP. Assume that the exponential waiting times are i.i.d. with parameter 1. If not stated otherwise, we always refer to ASEP as the process where this is the case. To construct ASEP graphically, one starts with an $\eta_0 \in X$ as initial value of the process. We take a family of independent Poisson processes $\mathcal{N} = (T^{i,j}, (i,j) \in \mathbb{Z}^2, |i-j| = 1)$ defined on some space $(\Omega, \mathcal{B}, \mathbb{P})$ with parameter p if $i = j-1$ and parameter q if $i = j+1$, i.e.

$$T_t^{i,j} = \sum_{N=1}^{\infty} \mathbf{1}_{(0,t]}\left(T_{i,j,N}\right)$$

where $T_{i,j,N} = \sum_{k=1}^{N} Y_{i,j,k}$ and the $Y_{i,j,k}$ are i.i.d. exponentials with parameter either p (if $i = j-1$) or q (if $i = j+1$). The idea is that whenever $T^{i,j}$ has a jump, a particle on i jumps to j if that is possible. Let $t > 0$ be fixed, and we construct the process up to time t first. Note that with probability 1 no two Poisson processes jump at the same time. Furthermore, for the events

$$\begin{aligned} B_{i,n} &= \{T_{i-1,i,1}, T_{i,i-1,1} > nt\} \quad i \in \mathbb{N}, n \in \mathbb{N} \\ B_{j,n} &= \{T_{j-1,j,1}, T_{j,j-1,1} > nt\} \quad j \in -\mathbb{N}_0, n \in \mathbb{N} \end{aligned}$$

we have $\mathbb{P}(\limsup_{i\to\infty} B_{i,1}) = \mathbb{P}(\limsup_{j\to-\infty} B_{j,1}) = 1$ by the Borel-Cantelli Lemma. Hence for almost every $\omega \in \Omega$ there are finite boxes $A_{k,1}$ such that no particle jumps in or out of the box $A_{k,1}$ before time t. Within a given box A_{k_0}, there are finitely many jumps at times $\tau_1^{k_0} < \tau_2^{k_0} < \cdots < \tau_r^{k_0}$ before t. We can construct $(\eta_s^{\eta_0}, s \le t) = (\eta_s^{\eta_0}(\mathcal{N}, \omega), s \le t)$ as an a.s. well-defined, deterministic function of $\mathcal{N}$ by postulating that the process starts in η_0, and within each A_{k_0} $(\eta_s^{\eta_0}, s \le t)$ is constant on the intervals $[0, \tau_1^{k_0}), [\tau_i^{k_0}, \tau_{i+1}^{k_0}), i = 1, \ldots, r-1$ and that if the jump at time τ_l belongs to $T^{i,j}$, then if $\eta^{\eta_0}_{\tau^{k_0}_{l-1}}(i) = 1, \eta^{\eta_0}_{\tau^{k_0}_{l-1}}(j) = 0$, we have $\eta^{\eta_0}_{\tau^{k_0}_{l}}(j) = 1, \eta^{\eta_0}_{\tau^{k_0}_{l}}(i) = 0$ and $\eta^{\eta_0}_{\tau^{k_0}_{l}}(z) = \eta^{\eta_0}_{\tau^{k_0}_{l-1}}(z)$ for $z \neq i, j$. See Figure 1.2 for an illustration. We sometimes suppress the dependence on $\mathcal{N}$ and/or ω, i.e. we write $\eta_t^{\eta_0}(\mathcal{N}, \omega) = \eta_t^{\eta_0}(\omega) = \eta_t^{\eta_0}$. If it leads not to confusion, we also suppress the initial value of the process later.

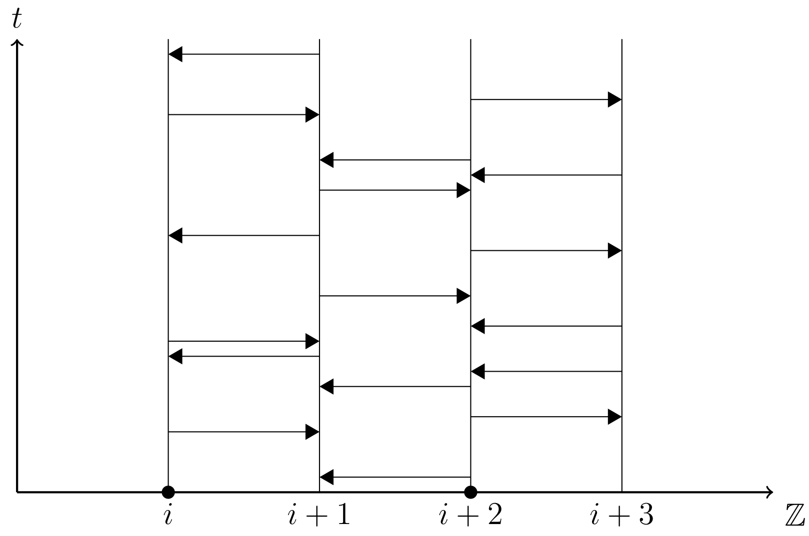

Figure 1.2: Graphical construction of the ASEP: Initially the sites i and $i+2$ are occupied. Each arrow corresponds to a jump of a Poisson processes. Particles move upwards the time axis and cross every arrow they encounter, unless the site the arrow points to is occupied.

Having constructed the process up to time t, we now iterate this procedure by the same Borell-Cantelli Argument for $n > 1$. The ASEP $(\eta_t^{\eta_0}, t \geq 0)$ starts in η_0 and consists of $\eta_t^{\eta_0} \in X$ for which for all $i \in \mathbb{Z}$ $\eta_t^{\eta_0}(i) = 1$ if i is occupied at time t, and $\eta_t^{\eta_0}(i) = 0$ if i is empty at time t. Now one has to prove that this indeed induces a Markov semigroup, see [44].

The Hille-Yosida construction is more abstract but gives a Markov process more directly. For $f : X \to \mathbb{R}$ depending only on finitely many coordinates (i.e. there is a finite set $A_f \subset \mathbb{Z}$ such that $f(\eta) = f(\rho)$ if ρ and η are equal on A_f), we define the operator

$$L^{\text{pre}} f(\eta) = \sum_{i \in \mathbb{Z}} \left(p\eta(i)(1-\eta(i+1)) + q\eta(i+1)(1-\eta(i))\right)\left(f(\eta^{i,i+1}) - f(\eta)\right), \qquad (1.19)$$

where $\eta^{i,i+1}(z) = \eta(z)$ if $z \neq i, i+1$ and $\eta^{i,i+1}(i) = \eta(i+1), \eta^{i,i+1}(i+1) = \eta(i)$. One starts with the operator L^{pre} and shows that it has a closure L. One shows that L is a Markov generator (Theorem 3.9 in [53], Chapter 1), it is the generator of a Semigroup S_t of a Feller process on X, and this Feller process is the ASEP $(\eta_t^{\eta_0}, t \geq 0)$.

So far the initial value η_0 is deterministic. To allow for random initial data, we can consider a probability space $(\Omega_0, \mathcal{F}, \mathbb{P}_0)$ (e.g. $\Omega_0 = X$) and consider the product space $\tilde{\Omega} = \Omega_0 \times \Omega$ together with the product sigma-algebra and the product measure $\tilde{\mathbb{P}} = \mathbb{P}_0 \otimes \mathbb{P}$. If then $\eta_0 : \Omega_0 \to X$ is a random variable and $\tilde{\omega} = (\omega_0, \omega) \in \tilde{\Omega}$, we can consider $\eta_t^{\eta_0}(\tilde{\omega}) = \eta_t^{\eta_0(\omega_0)}(\omega)$.

Later we will study TASEP whith two groups of particles, one with an exponential waiting time with parameter 1, and another with parameter α. This case is not directly covered by what we did so far, but can be included. Formally, one can consider the state space $\{0,1,2\}^{\mathbb{Z}} = \tilde{X}$ with the understanding that $\eta(i) = 2$ if it is occupied by a speed α particle, $\eta(i) = 1$ if occupied by a speed 1 particle and $\eta(i) = 0$ if it is empty. The transition rates for this TASEP then are

$$\eta \to \eta^{i,i+1} \text{ at rate } \begin{cases} 1 & \text{if } \eta(i) = 1, \eta(i+1) = 0 \\ \alpha & \text{if } \eta(i) = 2, \eta(i+1) = 0 \end{cases} \tag{1.20}$$

and then one writes down the operator (1.19) according to (1.20), and one readily checks that this operator satisfies the assumptions needed for the Hille-Yosida construction to work, namely (3.3) and (3.8) of [53], Chapter I, thus yielding a Feller process.

It will be important for us to have some information about the stationary measures of the exclusion process.

Definition 1.6 (Definitions 1.6 and 1.7 in [53], Chapter I)**.** *Consider the exclusion process $(\eta_t, t \geq 0)$ with initial distribution μ and denote S_t its semigroup. The measure μS_t is the unique measure satisfying*

$$\int [\mathrm{d}\mu]\, S_t f = \int [\mathrm{d}\mu S_t]\, f$$

for all $f \in C(X)$. The measure μ is called stationary if $\mu = \mu S_t$ for all $t \geq 0$. We denote by $\mathcal{I}$ the set of all stationary measures.

Stationary measures are also often called *invariant*. A useful characterization of invariance is the following.

Theorem 1.7 (Proposition 2.13 in [53], Chapter II)**.** *Let $\mathcal{I}$ be the set of all stationary measures. Let L be the generator of ASEP and $\mathcal{D}(L)$ its domain. Let D be a linear subspace of $\mathcal{D}(L)$ such that L is the closure of its restriction to D. We then have*

$$\mathcal{I} = \left\{ \mu : \int \mathrm{d}\mu L f = 0 \ \forall f \in D \right\}.$$

A special family of stationary measures will be of particular interest to us.

Definition 1.8 (See [53], p. 380)**.** *The homogeneous Bernoulli product measures ν_λ for $0 \leq \lambda \leq 1$ are the product measures on X with the Borel sigma-algebra with marginal*

$$\nu_\lambda(\{\eta : \eta(i) = 1\}) = \lambda \tag{1.21}$$

for all $i \in \mathbb{Z}$.

The importance of the ν_λ comes from the following fact.

Theorem 1.9 (Theorem 2.1 in [53], Chapter VIII, and [51])**.** *The ν_λ are stationary measures for ASEP. Furthermore, they are the only translation invariant stationary measures for ASEP.*

Furthermore, we will later want to follow the motion of certain particles. For this, we first fix an initial configuration $\eta_0(\omega_0)$. Suppressing the ω_0, we let $A = \{i : \eta_0(i) = 1\}$. A labeling is a map from A to $\mathbb{Z}$. All labelings we consider are from right to left, i.e. such that if $i \in A$ has received label n, then the largest $j \in A$ with $j < i$ gets label $n + 1$ (if such a j exists) and the smallest $l \in A$ with $l > i$ (if such an l exists) gets label $n - 1$. This way, it suffices to define the label of one $i \in A$, and furthermore, all labelings from right to left are just shifts from each other. Given a labeling from right to left, we denote by $x_n(0)$ the initial position of the particle with label n. From the construction we have

$$x_{n+1}(0) < x_n(0). \tag{1.22}$$

This order is preserved in time, i.e., denoting by $x_n(t)$ the position of particle n at time t, we have for all $t > 0$ that

$$x_{n+1}(t) < x_n(t). \tag{1.23}$$

1.3.2 Hydrodynamics for ASEP, ASEP as growth model

1.3.2.1 Hydrodynamics

The hydrodynamic limit is a law of large number limit for some density field of a particle system. Often the limiting density satisfies a PDE, which is then called the hydrodynamic equation. For the exclusion process, we can associate to a particle configuration $\eta \in X$ a measure on $\mathbb{R}$

$$\pi^n(\eta) = \frac{1}{n} \sum_{i \in \mathbb{Z}} \eta(i) \delta_{i/n}.$$

This definition already contains the rescaling i/n. For asymmetric systems such as ASEP, the correct scaling of time is $n\tau$. So we will look for each $\tilde{\omega} = (\omega_0, \omega) \in \tilde{\Omega}$ at the measure

$$\pi^n(\eta_{n\tau}^{\eta_0(\omega_0)}(\omega)) = \frac{1}{n} \sum_{i \in \mathbb{Z}} \eta_{n\tau}^{\eta_0(\omega_0)}(\omega)(i) \delta_{i/n}.$$

($\eta_{n\tau}^{\eta_0(\omega_0)}(\omega)(i)$ is the ith coordinate of the configuration $\eta_{n\tau}^{\eta_0(\omega_0)}(\omega)$) and ask for the convergence, as n goes to infinity, of $\pi^n(\eta_{n\tau}^{\eta_0(\omega_0)}(\omega))$ (we suppress the ω_0, ω dependance in the following). The convergence of measures we consider here is that of vague convergence.

Definition 1.10 (See e.g. [7], p. 4). *Let $(\mu_n)_{n\in\mathbb{N}}, \mu$ be measures on $\mathbb{R}$. The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to μ, denoted by $\mu_n \to_V \mu$, if*

$$\int_{\mathbb{R}} \mathrm{d}\mu_n f \to \int_{\mathbb{R}} \mathrm{d}\mu f \quad \forall f \in C_c(\mathbb{R}) \tag{1.24}$$

where $C_c(\mathbb{R})$ are the continuous, compactly supported functions on $\mathbb{R}$.

One distinguishes between weak hydrodynamic limits, where one proves the convergence

$$\forall f \in C_c(\mathbb{R}) \quad \forall \varepsilon > 0 \quad \tilde{\mathbb{P}}\left(\left| \int_{\mathbb{R}} \mathrm{d}\pi^n(\eta^{\eta_0}_{n\tau}) f - \int_{\mathbb{R}} \mathrm{d}\mu f \right| \geq \varepsilon \right) \to_{n\to\infty} 0,$$

with some limiting probability measure μ, and strong hydrodynamics, where one shows

$$\pi^n(\eta^{\eta_0}_{n\tau}) \to_V \mu \quad \tilde{\mathbb{P}} - a.s. \tag{1.25}$$

For the particular case of so-called step initial data, i.e. $\eta^{\eta_0}_0(i) = \eta_0(i) = \mathbf{1}_{-\mathbb{N}}(i)$ Rost in 1981 proved an explicit hydrodynamic limit, see [58]. Later Seppäläinen in [61] showed the existence of an hydrodynamic limit for TASEP with general initial data. In fact, he considers more generally a so-called K-TASEP, which is like TASEP except that up to $K \in \mathbb{N}$ particles are allowed on one site. Finally, in [7], Saada et al. proved strong hydrodynamics for a large class of particle systems including in particular ASEP with $p \neq 1$. All results refer to ASEP with i.i.d. exponential waiting times, since different speeds are not visible in the initial density. The strong hydrodynamic behavior of ASEP is as follows.

Theorem 1.11 (Theorem 1 in [61], Theorem 2.1 in [7]). *Consider the ASEP $(\eta^{\eta_0}_t, t \geq 0)$ with i.i.d. parameter 1 exponential waiting times. Assume there is a measurable, $[0,1]$ valued function $\rho_0(\cdot)$ on $\mathbb{R}$ such that*

$$\pi^n(\eta_0) \to_V \rho_0(\xi)\mathrm{d}\xi \quad \mathbb{P}_0 - a.s. \tag{1.26}$$

Let $\rho(\xi,\tau)$ be the unique entropy solution to the Burger's equation

$$\partial_\tau \rho + \partial_\xi[(p-q)\rho(1-\rho)] = 0 \tag{1.27}$$

with initial data $\rho(\xi,0) = \rho_0(\xi)$. Then, as n goes to infinity,

$$\pi^n(\eta^{\eta_0}_{n\tau}) \to_V \rho(\xi,\tau)\mathrm{d}\xi \quad \tilde{\mathbb{P}} - a.s. \tag{1.28}$$

Loosely speaking, the entropy solution of (1.27) is the physically relevant solution, of which we give several examples in the following, see Section 2.1 for the general entropy solution of the Burger's equation with Riemann initial data. Note that the equation (1.27) is invariant under the scaling $(\xi,\tau) \to (\xi C, \tau C)$, and in case this is true for the initial data also, we can write the self-similar entropy solution of (1.27) by

$$u(\xi,\tau) = u(\xi/\tau, 1) \equiv u(v,1), \quad v = \xi/\tau. \tag{1.29}$$

Self-similarity holds especially for Riemann initial data, see Section 2.1.

1.3.2.2 ASEP as growth model

So far we have considered ASEP only as an interacting particle system, not as a growth model. We associate to ASEP an interface in the following way.

Definition 1.12 (see e.g. (1.11) in [40]). *Let $(\eta_t^{\eta_0}, t \geq 0)$ be the ASEP from Section 1.3.1. Let $x \in \mathbb{Z}$ and $t \geq 0$. We define the interface*

$$h(x,t) = \begin{cases} 2N_t + \sum_{i=1}^{x} 1 - 2\eta_t^{\eta_0}(i) & \text{for } x \geq 1 \\ 2N_t & \text{for } x = 0 \\ 2N_t - \sum_{i=x+1}^{0} 1 - 2\eta_t^{\eta_0}(i) & \text{for } x \leq -1, \end{cases} \tag{1.30}$$

where N_t is the number of particles that have jumped from 0 to 1 during the time interval $[0,t]$ minus the number of particles that have jumped from 1 to 0 in the time interval $[0,t]$. We extend h to $x \in \mathbb{R}$ by linear interpolation.

Note this is precisely the growth mecanism we informally introduced in the Introduction, see Figure 2. The link between particle positions and the interface $h(x,t)$ is as follows. Note that $h(j,t) - h(j,0) = 2N_t^j$ where

$$\begin{aligned} N_t^j =& \,\#\text{particles that have jumped from}\, j \,\text{to}\, j+1 \,\text{before or at time}\, t \\ &- \#\text{particles that have jumped from}\, j+1 \,\text{to}\, j \,\text{before or at time}\, t. \end{aligned} \tag{1.31}$$

In particular $N_t^0 = N_t$, and for TASEP $h(j,\cdot)$ is constant if $\eta_0(i) = 0$ for all $i \leq j$. If there is initially a particle at or to the left of j, let Z_j be the label of the rightmost particle initially to the left or at j, and consider as always a labeling from right to left. Then we have for any $k \in \mathbb{N}$

$$\{h(j,t) - h(j,0) \geq 2(k+1)\} = \{x_{Z_j+k}(t) \geq j+1\}. \tag{1.32}$$

Especially, we may study the rescaled interface (1.10) for TASEP via particle positions, which we will do often in the following.

We are again interested in the limit shape

$$\lim_{t\to\infty} \frac{h(t\xi, \tau t)}{t} =: h_{\rm ma}^{\rm ASEP}(\xi,\tau). \tag{1.33}$$

Given the form of h, we first take $\xi \geq 0$ and then can deal with $\xi < 0$ in analogous way.

Note first,

$$\lim_{t\to\infty} \frac{h(\xi t, t\tau) - 2N_{t\tau}}{t} = \xi - 2\lim_{t\to\infty} \sum_{i=1}^{[\xi t]} \frac{\eta_{t\tau}^{\eta_0}(i)}{t} \tag{1.34}$$

due to the fact that $|h(\lceil \xi t \rceil, t\tau) - h(\xi t, t\tau)| \leq 1$ and $\lim_{t\to\infty} \frac{\lceil \xi t \rceil}{t} = \xi$. Next define the function $\varphi(x) = \mathbf{1}_{(0,\xi]}(x)$. Let $\varepsilon > 0$ and denote by φ^ε a continuous, $[0,1]$ valued function with support $[-\varepsilon, \xi+\varepsilon]$ and $\varphi^\varepsilon(x) = \varphi(x)$ for $0 < x \leq \xi$. In particular $\varphi^\varepsilon \in C_c(\mathbb{R})$. Note now that

$$\lim_{t\to\infty} -2\sum_{i=1}^{\lceil \xi t \rceil} \frac{\eta_{t\tau}^{\eta_0}(i)}{t} = \lim_{t\to\infty} -2\sum_{i\in\mathbb{Z}} \frac{\eta_{t\tau}^{\eta_0}(i)\varphi(i/t)}{t}. \tag{1.35}$$

Finally we have

$$\begin{aligned} \left| \sum_{i\in\mathbb{Z}} \frac{-2\eta_{t\tau}^{\eta_0}(i)}{t}\varphi(i/t) - \sum_{i\in\mathbb{Z}} \frac{-2\eta_{t\tau}^{\eta_0}(i)}{t}\varphi^\varepsilon(i/t) \right| &\leq \frac{2\#\{i\in\mathbb{Z} : -\lceil \varepsilon t \rceil \leq i \leq 0\}}{t} \\ &+ \frac{2\#\{i\in\mathbb{Z} : \lfloor \xi t \rfloor \leq i \leq \lceil \xi t \rceil + \lceil \varepsilon t \rceil\}}{t} \\ &\leq 4\varepsilon + 8/t. \end{aligned} \tag{1.36}$$

By Theorem 1.11,

$$\lim_{t\to\infty} \sum_{i\in\mathbb{Z}} \frac{\eta_{t\tau}^{\eta_0}(i)}{t}\varphi^\varepsilon(i/t) = \int_0^\xi \mathrm{d}x\rho(x,\tau) + \int_{-\varepsilon}^0 \mathrm{d}x\rho(x,\tau)\varphi^\varepsilon(x) + \int_\xi^{\xi+\varepsilon} \mathrm{d}x\rho(x,\tau)\varphi^\varepsilon(x), \tag{1.37}$$

from which follows

$$\lim_{\varepsilon\to 0}\lim_{t\to\infty} \sum_{i\in\mathbb{Z}} \frac{\eta_{t\tau}^{\eta_0}(i)}{t}\varphi^\varepsilon(i/t) = \int_0^\xi \mathrm{d}x\rho(x,\tau). \tag{1.38}$$

Putting together (1.34), (1.35) and (1.36) we obtain

$$\left| \lim_{t\to\infty} \frac{h(\xi t, t\tau) - 2N_{t\tau}}{t} - \left(\xi - 2\lim_{t\to\infty} \sum_{i\in\mathbb{Z}} \frac{\eta_{t\tau}^{\eta_0}(i)\varphi^\varepsilon(i/t)}{t} \right) \right| \leq 4\varepsilon. \tag{1.39}$$

Hence using (1.38), and an analogous computation for $\xi < 0$, we obtain the limiting function $\tilde{U}$

$$\tilde{U}(\xi,\tau) := \lim_{t\to\infty} \frac{h(\xi t, t\tau) - 2N_{t\tau}}{t} = \begin{cases} \xi - 2\int_0^\xi \mathrm{d}x\rho(x,\tau) & \text{if } \xi > 0 \\ \xi + 2\int_\xi^0 \mathrm{d}x\rho(x,\tau) & \text{if } \xi \leq 0. \end{cases} \tag{1.40}$$

The entropy solution $\rho(\xi,\tau)$ of Theorem 1.11 for $\tau > 0$ can be defined (see (2.9), (2.11) in [61] for TASEP) via

$$\rho(\xi,\tau) = \partial_\xi U(\xi,\tau) \tag{1.41}$$

where U is the unique so-called viscosity solution of

$$\partial_\tau U + (p-q)\partial_\xi U(1-\partial_\xi U) = 0 \quad U(\xi,0) = U_0(\xi) \tag{1.42}$$

where U_0 is a fixed function that satisfies $U_0(b) - U_0(a) = \int_{(a,b]} \mathrm{d}x\rho(x,0)$ for all $a,b\in\mathbb{R}$.

As for the limit of $\frac{N_t}{t}$, let us outline how it can be deduced from the hydrodynamic limit. The precise argument for the identity (1.47) is presented in detail in [7], p. 15 -18 (it is (61) there)[1]. We have

$$|N_t - N_t^x| \leq 2|x|, \tag{1.43}$$

which implies for $\varepsilon > 0$

$$\lim_{t\to\infty} \left| \frac{N_t - \sum_{|x|\leq \varepsilon t} \frac{N_t^x}{2\varepsilon t}}{t} \right| \leq 2\varepsilon \tag{1.44}$$

and hence

$$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{\varepsilon\to\infty} \lim_{t\to\infty} \frac{\sum_{|x|\leq \varepsilon t} \frac{N_t^x}{2\varepsilon t}}{t}. \tag{1.45}$$

Next one observes that

$$M_t^x := N_t^x - p \int_0^t \mathrm{d}s \eta_s(x)(1-\eta_s(x+1)) + q \int_0^t \mathrm{d}s \eta_s(x+1)(1-\eta_s(x)) \tag{1.46}$$

is a mean zero martingale (see also [54], p.240 for a proof of this) for which one has a large deviation estimate $\mathbb{P}(|M_t^x| \geq y) \leq e^{-t\mathcal{I}(y)}$, with $y > 0$ and $\mathcal{I}(y) > 0$ the rate function. From this, one deduces that

$$\lim_{t\to\infty} t^{-1} \frac{1}{2\varepsilon t} \sum_{|x|\leq \varepsilon t} M_t^x = 0 \quad \tilde{\mathbb{P}} - \text{a.s.} \tag{1.47}$$

Consequently, using (1.45), we get with $\zeta^\varepsilon(x) = \mathbf{1}_{[-\varepsilon,\varepsilon]}(x)$

$$\lim_{t\to\infty} \frac{N_t}{t} = \lim_{\varepsilon\to 0} \lim_{t\to\infty} t^{-1} \frac{p}{2\varepsilon t} \int_0^t \mathrm{d}s \sum_{|x|\leq \varepsilon t} \eta_s(x)(1-\eta_s(x+1)) \tag{1.48}$$

$$- \lim_{\varepsilon\to 0} \lim_{t\to\infty} t^{-1} \frac{q}{2\varepsilon t} \int_0^t \mathrm{d}s \sum_{|x|\leq \varepsilon t} \eta_s(x+1)(1-\eta_s(x)) \tag{1.49}$$

$$= \lim_{\varepsilon\to 0} \lim_{t\to\infty} \frac{p}{2\varepsilon} \int_0^1 \mathrm{d}\tilde{\tau} \frac{\sum_{x\in\mathbb{Z}} \eta_{\tilde{\tau}t}(x)(1-\eta_{\tilde{\tau}t}(x+1))\zeta^\varepsilon(x/t)}{t} \tag{1.50}$$

$$- \lim_{\varepsilon\to 0} \lim_{t\to\infty} \frac{q}{2\varepsilon} \int_0^1 \mathrm{d}\tilde{\tau} \frac{\sum_{x\in\mathbb{Z}} \eta_{\tilde{\tau}t}(x+1)(1-\eta_{\tilde{\tau}t}(x))\zeta^\varepsilon(x/t)}{t}. \tag{1.51}$$

[1] Our case corresponds to $\nu = 0$ and $f(\eta) = p\eta(0)(1-\eta(1)) - q\eta(1)(1-\eta(0))$ in [7] p. 16. In [7], they establish for more general systems the limit N_t/t for Bernoulli initial data (Lemma 3.1 in [7]) as a step towards Theorem 1.11. Here we sketch how one can obtain $\lim_{t\to\infty} N_t/t$ from their argument given one already knows the hydrodynamic limit.

So far we did not need any knowledge on the hydrodynamics. However, given that we know

$$(1.50) - (1.51) = \lim_{\varepsilon\to 0}\int_0^1 \mathrm{d}\tilde\tau \int_{-\varepsilon}^{\varepsilon} \mathrm{d}z \frac{(p-q)\rho(z,\tilde\tau)(1-\rho(z,\tilde\tau))}{2\varepsilon}, \tag{1.52}$$

we may proceed as follows. Let us first consider the case where $\rho(\cdot,1)$ is continuous at zero and we have self-similarity (1.29), we then obtain

$$\begin{aligned}(1.52) &= \int_0^1 \mathrm{d}\tilde\tau \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \mathrm{d}z \frac{(p-q)\rho(z/\tilde\tau,1)(1-\rho(z/\tilde\tau,1))}{2\varepsilon} &(1.53)\\ &= \int_0^1 \mathrm{d}\tilde\tau (p-q)\rho(0,1)(1-\rho(0,1)) &(1.54)\\ &= (p-q)\rho(0,1)(1-\rho(0,1)). &(1.55)\end{aligned}$$

Finally, since

$$\lim_{t\to\infty}\frac{N_{t\tau}}{t} = \tau \lim_{t\to\infty}\frac{N_{t\tau}}{t\tau} = \tau(p-q)\rho(0,1)(1-\rho(0,1)) \tag{1.56}$$

we get as limit shape

$$h_{\mathrm{ma}}^{\mathrm{ASEP}}(\xi,\tau) = \begin{cases} 2\tau(p-q)\rho(0,1)(1-\rho(0,1)) + \xi - 2\int_0^\xi \mathrm{d}x\rho(x,\tau) & \text{if } \xi > 0 \\ 2\tau(p-q)\rho(0,1)(1-\rho(0,1)) + \xi + 2\int_\xi^0 \mathrm{d}x\rho(x,\tau) & \text{if } \xi \le 0. \end{cases} \tag{1.57}$$

Especially, for $\xi > 0$ by (1.57) and (1.42)

$$\begin{aligned}\partial_\tau h_{\mathrm{ma}}^{\mathrm{ASEP}}(\xi,\tau) &= 2(p-q)\rho(0,1)(1-\rho(0,1)) - 2\partial_\tau U(\xi,\tau) \\ &= 2(p-q)\rho(0,1)(1-\rho(0,1)) + \frac{p-q}{2}(1-(\partial_\xi h_{\mathrm{ma}}^{\mathrm{ASEP}}(\xi,\tau))^2)\end{aligned} \tag{1.58}$$

hence ASEP has all properties 1, 2, 3 of a KPZ model (see beginning of Section 1.1): 1 and 2 we just proved and 3 is clear from the definition of ASEP. For general ρ, we make the assumption (fullfilled for all ρ appearing in this thesis) that the limits

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \mathrm{d}z\rho(z,\tilde\tau)(1-\rho(z,\tilde\tau)) &= \rho(0^+,\tilde\tau)(1-\rho(0^+,\tilde\tau) \\ \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\varepsilon}^0 \mathrm{d}z\rho(z,\tilde\tau)(1-\rho(z,\tilde\tau)) &= \rho(0^-,\tilde\tau)(1-\rho(0^-,\tilde\tau)\end{aligned}$$

exist for all $\tilde\tau \in [0,1]$. Then

$$(1.52) = \int_0^1 \mathrm{d}\tilde\tau \frac{p-q}{2}(\rho(0^+,\tilde\tau)(1-\rho(0^+,\tilde\tau) + \rho(0^-,\tilde\tau)(1-\rho(0^-,\tilde\tau)), \tag{1.59}$$

and the limit shape for $\xi > 0$ is

$$
\begin{aligned}
h_{\text{ma}}^{\text{ASEP}}(\xi,\tau) =& \tau(p-q)\int_0^1 \mathrm{d}\tilde{\tau}\rho(0^+,\tilde{\tau})(1-\rho(0^+,\tilde{\tau}))+\rho(0^-,\tilde{\tau})(1-\rho(0^-,\tilde{\tau})) \\
&+\xi-2\int_0^{\xi}\mathrm{d}x\rho(x,\tau)
\end{aligned}
\tag{1.60}
$$

and for $\xi \leq 0$

$$
\begin{aligned}
h_{\text{ma}}^{\text{ASEP}}(\xi,\tau) =& \tau(p-q)\int_0^1 \mathrm{d}\tilde{\tau}\rho(0^+,\tilde{\tau})(1-\rho(0^+,\tilde{\tau}))+\rho(0^-,\tilde{\tau})(1-\rho(0^-,\tilde{\tau})) \\
&+\xi+2\int_{\xi}^{0}\mathrm{d}x\rho(x,\tau).
\end{aligned}
\tag{1.61}
$$

1.3.3 Solvability of TASEP via Fredholm determinants

Here we present a general Theorem of Borodin and Ferrari in [18], which gives a Fredholm determinant formula for the joint distribution of particle positions in TASEP. Such a theorem is not available for ASEP with $p \neq 1$. Let us first briefly recall the notion of a Fredholm determinant. They are treated in full detail in [63], for integral operators (the only ones that will appear in the following) they are treated in [4], Section 3.4. Let H be a separable Hilbert space and $T : H \to H$ be compact. Let $T^\star$ be its adjoint. One can show that there is a unique operator $|T|$ such that $|T|^2 = T^\star T$ and $\langle \phi, |T|\phi\rangle \geq 0$ for all $\phi \in H$. Let $(s_n(T))_{n\in\mathbb{N}}$ be the sequence of eigenvalues of $|T|$, counted up to their multiplicity. We can define a determinant for the following class of operators.

Definition 1.13 (See [63], p. 18)**.** *A compact operator T on H is called trace-class if $(s_n(T))_{n\in\mathbb{N}} \in \ell^1$. Especially, one can define a trace* Tr *for T, which satisfies $|\mathrm{Tr}(T)| \leq \|s_n\|_{\ell^1}$.*

For a trace class operator one defines the Fredholm determinant in terms of an infinite product, see Theorem 3.7 in [63]. Rather than pursuing this general functional analytic approach, we define the Fredholm determinant for certain integral operators as a series. Let $(\Lambda, \mathcal{A}, \mu)$ be a measurable space. In our context often $\Lambda = \{c_1, \ldots, c_k\} \times \mathbb{R}$ or $\Lambda = \{c_1, \ldots, c_k\} \times \mathbb{Z}$ for some $\{c_1, \ldots, c_k\} \subset \mathcal{S}$ with $\mathcal{S}$ defined (definition from [18]) by

$$\mathcal{S} = \{(n_k, t_k) \in \mathbb{Z} \times [0, \infty), k \in \mathbb{N} : (n_k, t_k) \prec (n_{k+1}, t_{k+1})\}$$

where, by definition,

$$(n_k, t_k) \prec (n_j, t_j) \,\text{if}\, n_j \geq n_k, t_k \geq t_j \quad \text{and} \quad (n_k, t_k) \neq (n_j, t_j).$$

The sets $\{c_1, \ldots, c_k\}, \mathbb{Z}$ are equipped with the counting measure ζ, $\mathbb{R}$ with the Lebesgue measure λ, and on $\Lambda = \{c_1, \ldots, c_k\} \times \mathbb{R}$ we take the product measure $\zeta \otimes \lambda$ and on

$\Lambda = \{c_1, \ldots, c_k\} \times \mathbb{Z}$ the product measure $\zeta \otimes \zeta$. In the following, by an abuse of notation, for an integral operator with kernel K, we say that K is trace class (or not), meaning the associated integral operator is trace-class. By the same token, $\det(1-K)_{L^2(\Lambda)}$ will denote the Fredholm determinant of the integral operator with kernel K.

Definition 1.14 (Definition 3.4.3 in [4], see also Theorem 3.10 in [63])**.** *Let $K : \Lambda^2 \to \mathbb{C}$ be a kernel function that is measurable with respect to the product sigma-algebra, and for which there is a function $Y \in L^2(\Lambda)$ with $Y > 0$ and $\sup_{x,y\in\Lambda} \frac{|K(x,y)|}{Y(y)Y(x)} < \infty$. We then define the Fredholm determinant of an integral operator whose kernel K has the aforementioned properties by*

$$\det(1-K)_{L^2(\Lambda)} := \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\Lambda^n} \mathrm{d}\mu(y_n) \ldots \mathrm{d}\mu(y_1) \det(K(y_i, y_j)_{1\le i,j\le n}). \tag{1.62}$$

Finally, if we consider the kernel $\tilde{K}(x,y) = \frac{e^{f(x)}}{e^{f(y)}} K(x,y)$ for some $f : \Lambda \to \mathbb{C}$, then $\det(K(y_i,y_j)_{1\le i,j\le n}) = \det(\tilde{K}(y_i,y_j)_{1\le i,j\le n})$ and hence the Fredholm determinant is unchanged. We speak of $\tilde{K}$ as a conjugated kernel, and of e^f as the conjugation.

The convergence of the series follows from the fact $\sup_{x,y\in\Lambda} \frac{|K(x,y)|}{Y(y)Y(x)} < \infty$ and that, for a $n \times n$ matrix A whose entries $a_{i,j}$ satisfy $|a_{i,j}| \le 1$ one has $|\det(A)| \le n^{n/2}$ (this is called Hadamard's bound). Some of the kernels we present later are trace-class only after a conjugation, but we do not consider this issue in the following.

We need a space of functions V_n. Let $\{v_1, \ldots, v_n\}$ be positive numbers (v_i will be the jump rate of particle number i) and let $u_1 < \cdots < u_\nu$ be their different values, and α_k the multiplicity of u_k. We define (definition from [18]) the function space

$$V_n = \mathrm{span}\{x^l u_k^x, 1 \le k \le \nu, 0 \le l \le \alpha_k - 1\}. \tag{1.63}$$

Theorem 1.15 (Special case of Proposition 3.1 in [18])**.** *Consider a system of TASEP particles with indices $n = 1, 2, \ldots$ starting from positions $y_1 > y_2 > y_3 \ldots$. Denote by $x_n(t)$ the position of particle with label n at time t. Then the joint distribution of particle positions is given by the Fredholm determinant*

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{x_{n_k}(t_k) \ge s_k\}\right) = \det(1 - \tilde{\chi}_s K \tilde{\chi}_s)_{\ell^2(\{(n_1,t_1),\ldots,(n_m,t_m)\}\times\mathbb{Z})} \tag{1.64}$$

with $((n_1,t_1),\ldots,(n_m,t_m)) \in \mathcal{S}$, and $\tilde{\chi}_s((n_k,t_k))(x) = \mathbf{1}_{(-\infty,s_k)}(x)$. The kernel K is given by

$$K((n_1,t_1),x_1;(n_2,t_2),x_2) = -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) \tag{1.65}$$

where

$$\Psi^{n,t}_{n-l}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y_l-1} e^{t/z} \frac{(1-v_1 z)\cdots(1-v_n z)}{(1-v_1 z)\cdots(1-v_l z)}, \quad l = 1, 2, \dots \tag{1.66}$$

and the functions $\{\Phi^{n,t}_{n-k}\}_{k=1}^{n}$ *are uniquely determined by the orthogonality relations*

$$\sum_{x\in\mathbb{Z}} \Psi^{n,t}_{n-l}(x)\Phi^{n,t}_{n-k}(x) = \delta_{k,l}, \quad 1 \leq k, l \leq n, \tag{1.67}$$

and by the requirement $\mathrm{span}\{\Phi^{n,t}_{n-l}(x), l = 1, \dots, n\} = V_n$*. The first term in* (1.65) *is given by*

$$\phi^{((n_1,t_1),(n_2,t_2))}(x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z^{y-x+1}} \frac{e^{(t_1-t_2)/z}}{(1-v_{n_1+1}z)\cdots(1-v_{n_2}z)} \mathbf{1}_{\{(n_1,t_1)\prec(n_2,t_2)\}}. \tag{1.68}$$

The notation Γ_0 *stands for any anticlockwise oriented simple loop including only the pole at* 0*.*

Often, Theorem 1.15 is used as follows. Given an explicit initial configuration, one determines the functions Φ in (1.67) and from this the kernel K. Then one shows that, as t goes to infinity, K (suitably conjugated and rescaled) converges pointwise to some limiting kernel K_∞. Finally, one needs to show that one may take this limit inside the series expansion (1.62). This is usually done by providing an integrable bound for K and using dominated convergence. This way, one can show that the rescaled particle positions converge in distribution, as t goes to infinity, to the law given by the Fredholm determinant of K_∞. In the following sections, we present this use of Theorem 1.15 for different subclasses of KPZ. If not stated otherwise, we always refer to TASEP where all particles have jump rate 1.

1.3.4 TASEP with flat geometries (deterministic) : Airy_1 process

We start with flat geometries coming from deterministic initial configurations. More specifically, we consider so called $k-$*periodic initial data* ($k \in \mathbb{N}$), for which

$$\eta_0(i) = \mathbf{1}_{k\mathbb{Z}}(i), \tag{1.69}$$

with the labeling such that particle number n starts in

$$x_n(0) = -kn, \quad k \in \mathbb{N}, n \in \mathbb{Z}. \tag{1.70}$$

It is easy to see that (1.69) satisfies (1.26) with $\rho_0(\xi) = \frac{1}{k}$. Since constants are entropy solutions (see e.g. [49], p. 370), the limiting density is given by $\rho(\xi,\tau) = \frac{1}{k}$. Hence the function $\tilde{U}$ of (1.40) is given by $\xi(1-2/k)$, especially its second derivative vanishes. The kernel function K from Theorem 1.15 and its $t\to\infty$ limit has been obtained in [20] for the 2-periodic case, where K is obtained by considering a system with N particles first and then turning to the $N\to\infty$ limit. General k-periodic initial data have been considered in [19], and the convergence of the Fredholm determinants of the kernels they induce has been proven in [19] for a discrete time TASEP. The authors of [20] note that the results for continuous time TASEP follow along the same lines as in [19]. Finally, a full analysis of 2-periodic TASEP (even a generalization of TASEP) has been given in [18]. In the remainder of this section we just treat the result for the 2-periodic case.

We are interested in the long time behavior of particle positions $x_n(t)$. By the link (1.32), this is equivalent to studying (1.10). As we have seen, the density is 1/2 everywhere ($k=2$). Since particles move with speed one, particle n_i at time t is approximately at position $\bar{x}_i=-2n_i+t/2$ plus some fluctuations, which are conjectured by universality (see Section 1.1.1) to be of order $t^{1/3}$. The correlation length is by universality conjectured to be $t^{2/3}$. We focus here on particles that are located around the origin at time t, i.e. the rescaling for n_i is (the u_i being some real numbers)[2]

$$n_i=\left[t/4+u_it^{2/3}\right]\quad \bar{x}_i=\left[-2u_it^{2/3}\right]. \tag{1.71}$$

The result is then as follows.

Theorem 1.16 (Theorems 2.2, 2.3 in [20], Theorem 2.5 in [19])**.** *Consider TASEP with 2-periodic initial data given by* (1.69) *with $k=2$ and labeling* (1.70)*. Let $n_i,\bar{x}_i$ be rescaled as in* (1.71)*. Then*

$$\lim_{t\to\infty}\mathbb{P}\left(\bigcap_{i=1}^{m}\{\frac{x_{n_i}(t)-\bar{x}_i}{-t^{1/3}}\leq s_i\}\right)=\det(1-\chi_sK_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}, \tag{1.72}$$

where $\chi_s(u_i,x)=\mathbf{1}_{(s_i,\infty)}(x)$. The kernel $K_{\mathcal{A}_1}$ is called the Airy$_1$ *kernel, given by*

$$\begin{aligned}K_{\mathcal{A}_1}(u_1,x_1;u_2,x_2)=&-\frac{1}{\sqrt{4\pi(u_2-u_1)}}\exp\left(-\frac{(x_2-x_1)^2}{4(u_2-u_1)}\right)\mathbf{1}_{\{u_2>u_1\}}(u_1,u_2)\\&+\mathrm{Ai}(x_1+x_2+(u_2-u_1)^2)\exp\left((u_2-u_1)(x_1+x_2)+\frac{2}{3}(u_2-u_1)^3\right).\end{aligned} \tag{1.73}$$

Here Ai *denotes the* Airy *function, see* (4.78) *for an integral representation.*

Theorem 1.16 is proven as outlined after Theorem 1.15: One obtains an explicit formula for K (Theorem 2.2 in [20]), shows that $Kt^{1/3}2^{x_2-x_1}$ converges pointwise to $K_{\mathcal{A}_1}$ (Theorem 2.3 in [20]), and finally proves convergence of the Fredholm determinant (corollary of Theorem 2.5 in [19]). The Fredholm determinant (1.72) is, according to the series development (1.62)

[2] Here, and in the following, we will not always write the integer parts.

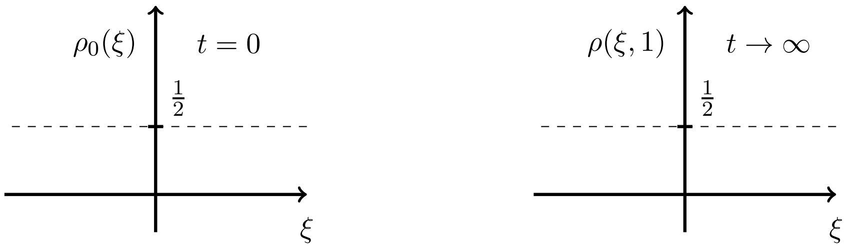

Figure 1.3: Density profile for TASEP started with 2-periodic initial data. On the left we see the initial density $\rho_0(\xi)$ (dashed) from (1.26), on the right the large time density $\rho(\xi,1)$ (dashed) from (1.27), both ρ_0 and ρ are constant 1/2.

(recall we equip $\{u_1,\dots,u_m\}$ with the counting, $\mathbb{R}$ with the Lebesgue and $\{u_1,\dots,u_m\}\times\mathbb{R}$ with the product measure) given by

$$\begin{aligned}&\det(1-\chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}\\&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{i_1=1}^{m}\cdots\sum_{i_n=1}^{m}\int_{s_{i_1}}^{\infty}\mathrm{d}x_1\cdots\int_{s_{i_n}}^{\infty}\mathrm{d}x_n\det[K_{\mathcal{A}_1}(u_{i_r},x_r;u_{i_p},x_p)_{1\le r,p\le n}].\end{aligned}\tag{1.74}$$

1.3.4.1 The Airy$_1$ process

We now define the process with marginals given by (1.74).

Definition 1.17 (Definition 2 in [21]). *The* Airy$_1$ *process is the process* $(\mathcal{A}_1(u),u\in\mathbb{R})$ *with marginals given by*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}_1(u_k)\le s_k\}\right)=\det(1-\chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})},\tag{1.75}$$

where $\chi_s(u_k,x)=\mathbf{1}_{(s_k,\infty)}(x)$ *and* $K_{\mathcal{A}_1}$ *is given in* (1.73).

The Airy$_1$ process was first discovered by Sasamoto in [59]. It is known to appear in other KPZ growth models with flat geometries, e.g. the polynuclear growth model (PNG), see e.g. Theorem 6 in [21]. As a first property of the Airy$_1$ process we can note that the kernel $K_{\mathcal{A}_1}$ depends on u_1,u_2 only through u_2-u_1. Given the Fredholm determinant form (1.74) of the marginals, this implies that the Airy$_1$ process is stationary. It turns out that we have already encountered the distribution of $\mathcal{A}_1(u)$.

Theorem 1.18 (Proposition 1 in [39]). *Let* F_{GOE} *be the GOE Tracy-Widom distribution defined in Theorem 1.5. Then we have*

$$\mathbb{P}(\mathcal{A}_1(0)\le s)=F_{\mathrm{GOE}}(2s).\tag{1.76}$$

By stationarity, the same statement holds for $\mathcal{A}_1(u), u \in \mathbb{R}$.

This Theorem is proven by showing that $F_{\rm GOE}$ can be expressed as Fredholm determinant of the kernel $K_{\mathcal{A}_1}(0, x_1; 0, x_2)$. Next we look at the covariance function

$$g_1(u) = \mathrm{Cov}(\mathcal{A}_1(u), \mathcal{A}_1(0)). \tag{1.77}$$

In [16], the authors perform numerical computations which clearly show an superexponential decay of $g_1(u)$ as u goes from 0 to 2.5 and provide the reason for this behavior, see [16], p. 413. As for the short time behavior of g_1, it is known (see [16]) that we have $g_1'(0) = -1$ and $g_1(0) = 0.402\ldots$ (of course, this is just the variance of $F_{\rm GOE}(2\cdot)$, compare with Table 1.1).

1.3.5 TASEP with curved geometries: Airy_2 process

Here we study TASEP with deterministic initial data resulting in a curved geometry. The initial configuration we consider is the *step initial condition*, for which η_0 is the step function

$$\eta_0(i) = \mathbf{1}_{\{i \leq -1\}}(i) \tag{1.78}$$

with labeling

$$x_n(0) = -n, \quad n \in \mathbb{N}. \tag{1.79}$$

Clearly, the initial density (1.26) is given by $\rho_0(\xi) = \mathbf{1}_{\{\xi \leq 0\}}(\xi)$. The entropy solution ρ of the Burger's equation was shown by Rost in [58] in 1981 to be the hydrodynamic limit. The function ρ is given by (see Figure 1.4)

$$\rho(\xi, 1) = \begin{cases} 1 & \text{if } \xi \leq -1 \\ \frac{1-\xi}{2} & \text{if } -1 \leq \xi \leq 1 \\ 0 & \text{if } \xi \geq 1. \end{cases} \tag{1.80}$$

From the density it follows that, macroscopically, $x_1(t)$ is located at position t, and $x_t(t)$ is located at position $-t$, hence there are t many particles in $[-t, t]$ and for $\xi \in [-1, 0]$ there are $t \int_\xi^0 \mathrm{d}x \rho(x, 1) = t(\xi^2/4 - \xi/2)$ many particles in $[\xi t, 0]$ and hence $t/4$ many in $[0, t]$ so that particle number $t/4$ is macroscopically located at the origin. More generally, particle number nt is located at $(1 - 2\sqrt{n})t$ at time t. By universality, we assume $t^{2/3}$ correlations and $t^{1/3}$ fluctuations. This explains the scaling in (1.81).

The analogue of Theorem 1.16 for step initial data is as follows.

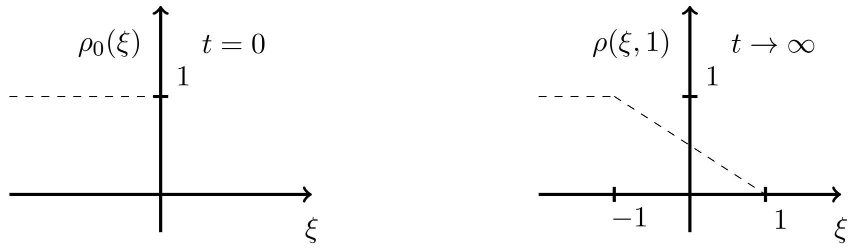

Figure 1.4: Density profile for TASEP started with step initial data. On the left we see the initial density $\rho_0(\xi)$ (dashed) from (1.26), on the right the large time density $\rho(\xi, 1)$ (dashed) from (1.27), ρ has a linear decreasing part from from 1 to 0 in the interval $[-1, 1]$.

Theorem 1.19 (Theorem 1.6 in [45], Theorem 1.1 in [47], Proposition 3.4, Section 5.2 in [18])**.** *Consider TASEP with step initial data* (1.78) *and labeling* (1.79)*. Then we have*

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{m} \{ \frac{x_{\left[t/4+u_i(t/2)^{2/3}\right]} + 2u_i(t/2)^{2/3} - u_i^2(t/2)^{1/3}}{-(t/2)^{1/3}} \le s_i \}\right) \\ &\quad = \det(1 - \chi_s K_{\mathcal{A}_2} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}, \end{aligned} \tag{1.81}$$

where $\chi(u_k, x) = \mathbf{1}_{\{x>s_k\}}(x)$*. The kernel* $K_{\mathcal{A}_2}$ *is the* Airy_2 *kernel, given by*

$$K_{\mathcal{A}_2}(u_1, x_1; u_2, x_2) = \begin{cases} \int_{\mathbb{R}_+} \mathrm{d}\lambda e^{-\lambda(u_2-u_1)} \mathrm{Ai}(x_1+\lambda)\mathrm{Ai}(x_2+\lambda) & \text{if } u_2 \ge u_1 \\ -\int_{\mathbb{R}_-} \mathrm{d}\lambda e^{(u_2-u_1)\lambda} \mathrm{Ai}(x_1+\lambda)\mathrm{Ai}(x_2+\lambda) & \text{if } u_2 < u_1. \end{cases} \tag{1.82}$$

Theorem 1.19 was first proved by Johansson in [45], Theorem 1.6, with $m = 1$ in (1.81), i.e. for the one-point-distribution, which Johansson gives explicitly, and which we identify below. In [45], the author does not use the kernel approach of Theorem 1.15, but formulates the problem in an equivalent model, we will come back to this result in Section 1.4.2.1. Later, Johansson proved in [47] the convergence (1.81) for the discrete polynuclear growth model (even a sharpened functional limit theorem), see Theorem 1.1 in [47]. Theorem 1.81 has also been dealt with using Theorem 1.15. In [18], the authors compute explicitly the kernel (1.65) (Proposition 3.4 in [18]) for step initial data. In Section 5.2 of the same paper, the authors show that this kernel (multiplied by $t^{1/3}$) converges pointwise to the Airy_2 kernel. What is not carried out (but is similar to other estimates done in [18]) is to provide an integrable function as upper bound for the rescaled kernel, so that one can take the established limit inside the series development (1.62).

Next we turn to the process whose finite dimensional distributions are given by (1.19).

Definition 1.20 (Definition 4.2 in [57])**.** *The* Airy_2 *process is the process* $(\mathcal{A}_2(u), u \in \mathbb{R})$

with finite dimensional distributions given by

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{A}_2(u_i) \leq s_i\}\right) = \det(1 - \chi_s K_{\mathcal{A}_2} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}, \tag{1.83}$$

where $\chi_s(u_k, x) = \mathbf{1}_{(s_k,\infty)}(x)$ *and the* $K_{\mathcal{A}_2}$ *is given in* (1.82).

The Airy$_2$ process was first discovered in the PNG model of the KPZ class (see [57]). Just like for the Airy$_1$ process, we can deduce stationarity of the Airy$_2$ process from the fact that $K_{\mathcal{A}_2}(u_1, x_1; u_2, x_2)$ depends on u_1, u_2 only through $u_2 - u_1$. We have also already encountered the one-point-distribution of $\mathcal{A}_2$.

Theorem 1.21 (Theorem 4.3 in [57])**.** *Let* F_{GUE} *be the GUE Tracy-Widom distribution defined in Theorem 1.5. Then we have*

$$\mathbb{P}(\mathcal{A}_2(0) \leq s) = F_{\mathrm{GUE}}(s). \tag{1.84}$$

By stationarity, the same statement holds for $\mathcal{A}_2(u), u \in \mathbb{R}$.

Furthermore, there is a continuous version of the Airy$_2$ process (Theorem 4.3 in [57]). Additionally, Hägg proves in [41] that the Airy$_2$ process behaves locally like a Brownian motion. Finally, we can again look at the covariance function

$$g_2(u) = \mathrm{Cov}(\mathcal{A}_2(u), \mathcal{A}_2(0)). \tag{1.85}$$

It behaves quite differently from the function g_1 from the Airy$_1$ process. Namely, we have (see [1], [57]) that

$$g_2(u) = u^{-2} + \mathcal{O}(u^{-4}) \quad \text{as}\, u \to \infty \tag{1.86}$$

and ($\mathbb{V}$ designating variance)

$$\mathbb{V}(\mathcal{A}_2(u) - \mathcal{A}_2(0)) = 2|u| + \mathcal{O}(u^2) \quad \text{as}\, u \to 0. \tag{1.87}$$

1.3.6 TASEP with flat-curved geometries: Airy$_{2\to1}$ process

Here we deal with a crossover geometry, having both a flat and a curved region. The initial data is half 2-periodic, i.e.

$$\eta_0(i) = \mathbf{1}_{-2\mathbb{N}_0}(i) \tag{1.88}$$

and labeling

$$x_n(0) = -2n, \quad n \in \mathbb{N}_0. \tag{1.89}$$

This initial data has been fully studied in [22], and we follow [22] closely in our presentation.

Clearly, the initial density is given by $\rho_0(\xi) = \frac{1}{2}\mathbf{1}_{\{\xi\le 0\}}(\xi)$. The limit density ρ is given by (see Figure 1.5)

$$\rho(\xi,1) = \begin{cases} 1/2 & \text{if } \xi \le 0 \\ \frac{1-\xi}{2} & \text{if } 0 \le \xi \le 1 \\ 0 & \text{if } \xi \ge 1. \end{cases} \tag{1.90}$$

From the density, we can deduce that, for $a \ge -1/4$ we have

$$\lim_{t\to\infty} \frac{x_{t/4+at}}{t} = \begin{cases} 1-\sqrt{1+4a} & \text{for } a \in [-1/4, 0] \\ -2a & \text{for } a \ge 0. \end{cases} \tag{1.91}$$

So for $a > 0$ and $a/t = \mathcal{O}(1)$ we are in the flat region where we expect to see the Airy$_1$ process, and for $-1/4 < a < 0$ and $a/t = \mathcal{O}(1)$ we are in a curved region where we see the Airy$_2$ process. Hence we are in the transition region for $a = 0$, and by universality we conjecture that for $at = \mathcal{O}(t^{2/3})$ we are still in the transition region, leading finally to take $a(u_i,t)t = u_i(t/2)^{2/3}$, with $u_i \in \mathbb{R}$. Since $1-\sqrt{1+4a} = -2a + 2a^2 + \mathcal{O}(a^3)$ we get that particle number

$$n_i = \left[t/4 + u_i(t/2)^{2/3}\right] \tag{1.92}$$

is approximately located at

$$\bar{x}_i = \left[-2u_i(t/2)^{2/3} + \min\{0,u_i\}^2(t/2)^{1/3}\right]. \tag{1.93}$$

We then have the following Theorem.

Theorem 1.22 (Theorem 2 in [22])**.** *Consider TASEP with the half-flat initial data* (1.88) *and labeling* (1.89)*. Let $n_i, \bar{x}_i$ be as in* (1.92)*,* (1.93)*. Then*

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{i=1}^m \left\{\frac{x_{n_i}(t) - \bar{x}_i}{-(t/2)^{1/3}} \le s_i\right\}\right) = \det(1 - \chi_s K_{\mathcal{A}_{2\to 1}} \chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}, \tag{1.94}$$

where $\chi_s(u_i,x) = \mathbf{1}_{(s_i,\infty)}(x)$. The kernel $K_{\mathcal{A}_{2\to1}}$ is called the Airy$_{2\to1}$ *transition kernel, with*

$$K_{\mathcal{A}_{2\to1}} = K^0_{\mathcal{A}_{2\to1}} + K^1_{\mathcal{A}_{2\to1}} + K^2_{\mathcal{A}_{2\to1}}. \tag{1.95}$$

To define the $K^i_{\mathcal{A}_{2\to1}}$, $i = 0,1,2$ we write (given some x_i, u_i) $\tilde{x}_i = x_i - \min\{0,u_i\}^2$ and $\hat{x}_i = x_i + \max\{0,u_i\}^2$. Then,

$$K^0_{\mathcal{A}_{2\to1}}(u_1,x_1;u_2,x_2) = -\frac{e^{2u_1^3/3+u_1\tilde{x}_1}}{e^{2u_2^3/3+u_2\tilde{x}_2}} \frac{1}{\sqrt{4\pi(u_2-u_1)}} \exp\left(-\frac{(\tilde{x}_2-\tilde{x}_1)^2}{4(u_2-u_1)}\right) \mathbf{1}_{\{u_2>u_1\}}(u_1,u_2) \tag{1.96}$$

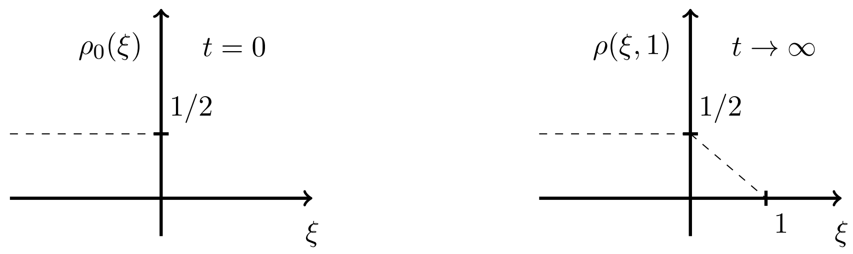

Figure 1.5: Density profile for TASEP started with half-flat initial data. On the left we see the initial density $\rho_0(\xi)$ (dashed) from (1.26), on the right the large time density $\rho(\xi,1)$ (dashed) from (1.27), ρ has a linear decreasing part from from 1/2 to 0 in the interval $[0,1]$.

and

$$K^1_{\mathcal{A}_{2\to1}}(u_1,x_1;u_2,x_2)=\int_0^\infty \mathrm{d}\lambda e^{\lambda(u_1+u_2)}\mathrm{Ai}(\hat{x}_2+\lambda)\mathrm{Ai}(\hat{x}_1-\lambda) \tag{1.97}$$

and finally

$$K^2_{\mathcal{A}_{2\to1}}(u_1,x_1;u_2,x_2)=\int_0^\infty \mathrm{d}\lambda e^{\lambda(-u_1+u_2)}\mathrm{Ai}(\hat{x}_2+\lambda)\mathrm{Ai}(\hat{x}_1+\lambda). \tag{1.98}$$

We can now define the process with marginals given by (1.94).

Definition 1.23 (Definition 2 in [22]). *The* $\mathrm{Airy}_{2\to1}$ *process* $(\mathcal{A}_{2\to1}(u),u\in\mathbb{R})$ *is the process with finite dimensional distributions given by*

$$\mathbb{P}\left(\bigcap_{i=1}^m\{\mathcal{A}_{2\to1}(u_i)\le s_i\}\right)=\det(1-\chi_s K_{\mathcal{A}_{2\to1}}\chi_s)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}, \tag{1.99}$$

where $\chi_s(u_i,x)=\mathbf{1}_{(s_i,\infty)}(x)$ *and* $K_{\mathcal{A}_{2\to1}}$ *is defined in* (1.95).

The $\mathrm{Airy}_{2\to1}$ process is a transition process ($\mathcal{A}_{2\to1}(t+u)$ becomes $2^{1/3}\mathcal{A}_1(2^{-2/3}u)$ as $t\to+\infty$, and $\mathcal{A}_2(u)$ as $t\to-\infty$, see Section 5 of [22]). As such, it cannot be stationary, rather its one-point distribution interpolates between F_{GOE} and F_{GUE}.

1.3.7 TASEP with flat geometries (stationary): $\mathrm{Airy}_{\mathrm{stat}}$ process

Here we consider TASEP with the initial data η_0 random and distributed according to the stationary Bernoulli product measure ν_λ defined in (1.21), such that $\eta_0(i),i\in\mathbb{Z}$ are i.i.d. random variables with

$$\mathbb{P}_0(\eta_0(i)=1)=\lambda=1-\mathbb{P}_0(\eta_0(i)=0). \tag{1.100}$$

The labeling is as always from right to left with the convention

$$x_1(0) = \max\{i : i \leq -1, \eta_0(i) = 1\}. \tag{1.101}$$

By the law of large numbers, we have $\rho_0(\xi) = \lambda$. By stationarity (see Theorem 1.9) we also get $\rho(\xi, \tau) = \lambda$. For $c > 0$, $x_{[ct]}(0)$ is distributed as the sum of $[ct]$ i.i.d. geometric random variables on $-\mathbb{N}$ with parameter $1 - \lambda$. Hence $\mathbb{E}(x_{[ct]}(0)) = -ct/\lambda$ and since the density is λ everywhere, we get as approximative location for $x_{[ct]}(t)$

$$-ct/\lambda + (1 - \lambda)t. \tag{1.102}$$

We can now, as in the previous sections ask for the fluctuations of $x_{[ct]}(t)$ around its macroscopic position (1.102). However, there is a difference due to the randomness in the initial data: To see the fluctuations from the TASEP dynamics (which are of order $t^{1/3}$) and not of the initial data (which are of order $t^{1/2}$ by the central limit theorem), we need to look at a characteristic speed. This can be conviently explained in the framework of last passage percolation (we introduce this model in Section 1.4), see Appendix D of [10] for how gaussian limits arise. Here we just state that to see the $t^{1/3}$- fluctuations of TASEP, one needs to take $c = \lambda^2$ so that, assuming as usual a $t^{2/3}$ correlation length and putting the constant $\chi = \lambda(1 - \lambda)$ so that it does not appear in the limit, the scaling we consider is (with $u_i \in \mathbb{R}$)

$$n_i = \left[\lambda^2 t - 2u_i \lambda \chi^{1/3} t^{2/3}\right] \tag{1.103}$$

and

$$\bar{x}_i = \left[(1 - 2\lambda)t + 2u_i \chi^{1/3} t^{2/3}\right]. \tag{1.104}$$

To state the result, we need to define some functions.

Definition 1.24 (Definition 1.1 in [10])**.** *Fix $m \in \mathbb{N}$, and let $u_1 < u_2 < \cdots u_m$ and $s_1, \ldots, s_m$ be real numbers. We define*

$$\begin{aligned}
\mathcal{R} &= s_1 + e^{-\frac{2}{3}u_1^3} \int_{s_1}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y \mathrm{Ai}(x + y + u_1^2) e^{-u_1(x+y)}, \\
\Psi_{u_j}(y) &= e^{\frac{2}{3}u_j^3 + u_j y} - \int_0^{\infty} \mathrm{d}x \mathrm{Ai}(x + y + u_j^2) e^{-u_j x}, \\
\Phi_{u_i}(x) &= e^{-\frac{2}{3}u_1^3} \int_0^{\infty} \mathrm{d}\theta \int_{s_1}^{\infty} \mathrm{d}y e^{-\theta(u_1 - u_i)} e^{-u_1 y} \mathrm{Ai}(x + u_i^2 + \theta) \mathrm{Ai}(y + u_1^2 + \theta) \\
&\quad + \mathbf{1}_{\{i \geq 2\}}(i) \frac{e^{-\frac{2}{3}u_i^3 - u_i x}}{\sqrt{4\pi(u_i - u_1)}} \int_{-\infty}^{s_1 - x} \mathrm{d}y e^{-\frac{y^2}{4(u_i - u_1)}} - \int_0^{\infty} \mathrm{d}y \mathrm{Ai}(y + x + u_i^2) e^{u_i y}.
\end{aligned} \tag{1.105}$$

The result is then as follows.

Theorem 1.25 (Theorem 1.6 in [10])**.** *Consider TASEP with stationary initial data* (1.21)*, labeling* (1.101) *and let n_i and $\bar{x}_i$ be given as in* (1.103) *and* (1.104)*. Then we*

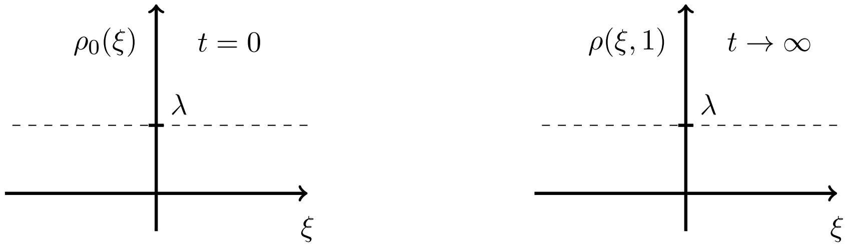

Figure 1.6: Density profile for TASEP started with Bernoulli initial data ν_λ. On the left we see the initial density $\rho_0(\xi)$ (dashed) from (1.26), on the right the large time density $\rho(\xi,1)$ (dashed) from (1.27), both ρ_0 and ρ are constant λ.

have

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}\left(\bigcap_{i=1}^{m}\{\frac{x_{n_i}(t)-\bar{x}_i}{-(1-\lambda)t^{1/3}\chi^{-1/3}}\le s_i\}\right)\\ =\sum_{i=1}^{m}\frac{\partial}{\partial s_i}\left(g_m(u,s)\det\left(1-\chi_s\hat{K}_{\mathrm{Ai}}\chi_s\right)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}\right),\end{aligned}\tag{1.106}$$

where $\chi_s(u_k,s)=\mathbf{1}_{\{x>s_k\}}(x)$. *The function* $g_m(u,s)$ *is given by*

$$\begin{aligned}g_m(u,s)&=\mathcal{R}-\langle\zeta\chi_s\Phi,\chi_s\Phi\rangle\\ &=\mathcal{R}-\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{s_i}^{\infty}\mathrm{d}x\int_{s_j}^{\infty}\mathrm{d}y\Psi_{u_j}(y)\zeta_{u_j,u_i}(y,x)\Phi_{u_i}(x),\end{aligned}\tag{1.107}$$

where

$$\zeta:=(1-\chi_s\hat{K}_{\mathrm{Ai}}\chi_s)^{-1},\quad \zeta_{u_j,u_i}(y,x):=\zeta((u_j,y),(u_i,x)),\tag{1.108}$$

and $\Phi((u_i,x)):=\Phi_{u_i}(x)$, $\Psi((u_j,y)):=\Psi_{u_j}(y)$. *Finally,* $\hat{K}_{\mathrm{Ai}}$ *is a shifted* Airy_2 *kernel* $K_{\mathcal{A}_2}$ *(see* (1.82)*)*

$$\hat{K}_{\mathrm{Ai}}(u_1,x_1;u_2,x_2)=K_{\mathcal{A}_2}(u_1,x_1+u_1^2;u_2,x_2+u_2^2).$$

The invertibility of $1-\chi_s\hat{K}_{\mathrm{Ai}}\chi_s$ follows from the fact that it is trace class (see [47]) and that $\det(1-\chi_s\hat{K}_{\mathrm{Ai}}\chi_s)>0$.

The limit process is now determined by the marginals given in (1.106).

Definition 1.26 (Theorem 1.6 in [10], Definition 2.1 in [35])**.** *The* $\mathrm{Airy}_{\mathrm{stat}}$ *process* $(\mathcal{A}_{\mathrm{stat}}(u),u\in\mathbb{R})$ *is the process with marginals given by*

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{A}_{\mathrm{stat}}(u_i)\le s_i\}\right)=\sum_{i=1}^{m}\frac{\partial}{\partial s_i}\left(g_m(u,s)\det\left(1-\chi_s\hat{K}_{\mathrm{Ai}}\chi_s\right)_{L^2(\{u_1,\dots,u_m\}\times\mathbb{R})}\right),\tag{1.109}$$

where $\chi_s(u_k,x)=\mathbf{1}_{\{x>s_k\}}(x)$ *and* $u_1<u_2<\cdots<u_m$.

A different representation of the $\mathrm{Airy}_{\mathrm{stat}}$ process has been recently given in [35]. There, in another KPZ-like model with stationary initial data, the $\mathrm{Airy}_{\mathrm{stat}}$ process has also been found, thus strengthening the conjecture of its universality. Finally, the $\mathrm{Airy}_{\mathrm{stat}}$ process also arises in the KPZ equation itself, see Theorem 2.17 in [17].

1.3.8 TASEP with two speeds and step-flat initial data: Convergence to F_{GOE}

Here, unlike in the preceeding Sections 1.3.4 - 1.3.7, we consider TASEP with two different speeds. We will use Theorem 1.27 to study shocks in Chapter 3, see Section 3.3.3. We consider the initial configuration

$$\eta_0(i) = \mathbf{1}_{-\mathbb{N}_0}(i) + \mathbf{1}_{2\mathbb{N}}(i) \tag{1.110}$$

with the labeling

$$x_n(0) = -n,\, n \in \mathbb{N} \quad x_n(0) = -2n,\, n \in -\mathbb{N}_0 \tag{1.111}$$

and speeds v_n of particle n (i.e. the parameter of the exponential time particle n waits before it tries to jump) given by

$$v_n = 1,\, n \in \mathbb{N} \quad v_n = \alpha,\, n \in -\mathbb{N}_0. \tag{1.112}$$

See Section 2.3 on how to obtain the density profile in the presence of two speeds. In particular, the following theorem gives the fluctuations of particle positions in a flat density region. Hence the limit law is expected by universality to be F_{GOE}, and the following Theorem confirms this. It is the first result by the author we present.

Theorem 1.27 (Proven in Section 3.3.1.3)**.** *Consider TASEP with initial data given by* (1.110)*, labeling* (1.111) *and speeds* (1.112)*. Then we have for* $\alpha \in (0,1)$ *and* $\kappa \in (0,1)$ *that*

$$\lim_{t\to\infty} \mathbb{P}\left(x_{\left[\kappa \frac{2-\alpha}{4} t\right]}(t) \geq \frac{\alpha-\kappa}{2} t - \sigma^{-1} s t^{-1/3} \right) = F_{\mathrm{GOE}}(2s), \tag{1.113}$$

where $\sigma = \frac{(2-\alpha)^{2/3}}{(\alpha((2-\alpha)^2 - 2(1-\alpha)\kappa))^{1/3}}$.

The proof of Theorem 1.27 is an application of Theorem 1.15 and the procedure outlined after it. See Section 3.3.1.3.

1.4 Last passage percolation

1.4.1 LPP on $\mathbb{Z}^2$

Here we come to a model which can be seen as a generalization of TASEP: Last passage percolation (LPP). In the work of Rost [58] from 1981 we already encountered, he also gives a result about the limit shape of a growth process on $\mathbb{Z}^2$, which is precisely the limit shape of an LPP model, which we now define. The counterpart of LPP is first passage percolation (take min instead of max in (1.114), for which an early reference is [43].

Definition 1.28 (See [43] for first passage percolation, also [58]). *Let $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}$ be independent, nonnegative random variables. Let $\mathcal{L},\mathcal{A}\subseteq\mathbb{Z}^2$ be disjoint. Let $\pi=(\pi(1),\dots,\pi(n))\in\mathbb{Z}^{2n}$ be an up-right path from $\mathcal{L}$ to $\mathcal{A}$, i.e. $\pi(1)\in\mathcal{L}$ and $\pi(n)\in\mathcal{A}$ and $\pi(i+1)-\pi(i)\in\{(0,1),(1,0)\}$. Denote $||\pi||=n$ the number of points of π. If the number of up-right paths from $\mathcal{L}$ to $\mathcal{A}$ is finite and positive, we define the last passage percolation time from $\mathcal{L}$ to $\mathcal{A}$ as*

$$L_{\mathcal{L}\to\mathcal{A}} := \max_{\pi:\mathcal{L}\to\mathcal{A}} \sum_{(i,j)\in\pi\setminus\mathcal{L}} \omega_{i,j} \tag{1.114}$$

where the maximum is taken over all up-right paths from $\mathcal{L}$ to $\mathcal{A}$. We denote by $\pi^{\max}$ any up-right path for which the maximum in (1.114) *is attained.*

If there are infinitely many or no up-right paths, we may replace max by sup in (1.114), but this will not be needed here. One often refers to the $\omega_{i,j}$ as weights. If $\mathcal{L}=(k,l)$ and $\mathcal{A}=(m,n)$ are points, we speak of a *point-to-point* problem, if $\mathcal{L}$ is more like a line, we often speak of a line-to-point problem, see Figure 1.7. Analogous to hydrodynamics in TASEP, we are interested in the shape function [3]

$$\Psi_{\mathcal{L}}(x,y) := \lim_{N\to\infty} \frac{L_{\mathcal{L}\to([Nx],[Ny])}}{N}. \tag{1.115}$$

Here we will present the existence and some properties of the shape function for point-to-point problems with i.i.d. weigths $(\omega_{i,j})_{(i,j)\in\mathbb{Z}^2}$. While we will consider more general $\mathcal{L}$ and not identically distributed weights later, often there will be points $Z_N\in\mathcal{L}$ such that $\Psi_{\mathcal{L}}(x,y)=\lim_{N\to\infty}\frac{L_{Z_N\to([xN],[yN])}}{t}$ and this limit can be computed in the concrete situation at hand. We mention that some cases of not identically distributed geometric and exponential $\omega_{i,j}$ have been considered and the shape function been computed, see e.g. [30].

[3] As in TASEP, we will not always write the integer parts.

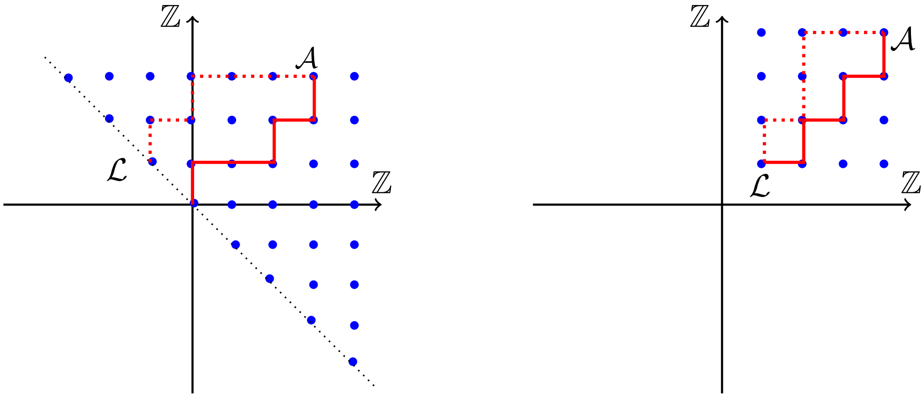

Figure 1.7: Left: Example of a line-to-point problem: $\mathcal{L} = \{(-k, k) : k \in \mathbb{Z}\}$ and $\mathcal{A} = \{(3, 3)\}$. Up-right paths π (dotted and solid) collect independent random weights $\omega_{i,j}$ on each point (i, j) they pass. Right: A point-to-point problem with $\mathcal{L} = \{(1, 1)\}$ and $\mathcal{A} = \{(4, 4)\}$.

1.4.1.1 Shape function for point-to-point problems with i.i.d. weights

Here we consider point-to-point problems with i.i.d. weights. By the i.i.d. assumption we may take $\mathcal{L} = (1, 1)$ (or any other point) as starting point without loss of generality. Let $m < n$ and $(x, y) \in \mathbb{N}^2$. Define

$$Z_{m,n} := L_{(mx,my)\to(nx,ny)}, \tag{1.116}$$

which satisfies

$$Z_{0,n} \geq Z_{0,m} + Z_{m,n}. \tag{1.117}$$

One can guarentee the existence of the (possibly infinite) shape function $\Psi_{(1,1)}(x, y)$ by a (superadditive version) of the subadditive ergodic theorem of Liggett (Theorem 1.10 in [52]); the version we need can be found in Corollary A.3 of [62]. We give the existence result based on the ergodic theorem and some simple properties of the shape function.

Theorem 1.29 (Theorem 2.1 in [62])**.** *Consider the LPP model* (1.114) *with* $\mathcal{L} = (1, 1)$ *and i.i.d. weights. Then there exists a deterministic function* $\Psi_{(1,1)} : (0, \infty)^2 \to [0, \infty]$ *such that for all* $(x, y) \in (0, \infty)^2$

$$\Psi_{(1,1)}(x, y) = \lim_{N\to\infty} \frac{L_{(1,1)\to([Nx],[Ny])}}{N} \quad \text{a.s.} \tag{1.118}$$

Either $\Psi_{(1,1)} = \infty$ *or* $\Psi_{(1,1)} < \infty$ *on all* $(0, \infty)^2$*. In the latter case,* $\Psi_{(1,1)}$ *is superadditive, i.e.*

$$\Psi_{(1,1)}(x_1, y_1) + \Psi_{(1,1)}(x_2, y_2) \leq \Psi_{(1,1)}(x_1 + x_2, y_1 + y_2), \tag{1.119}$$

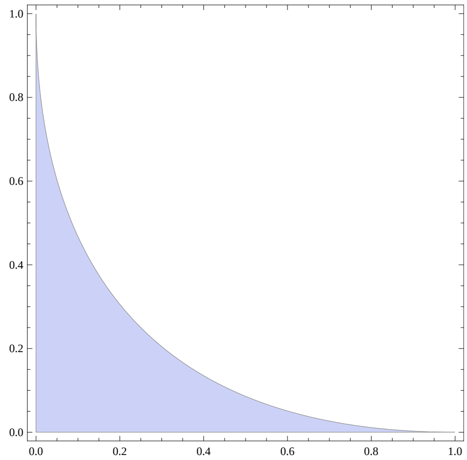

Figure 1.8: The limit shape (1.123) $\{(x,y)\in\mathbb{R}_+^2 : \sqrt{x}+\sqrt{y}\le 1\}$ of point-to-point LPP for i.i.d. exponential weights with parameter one.

and satisfies for $c>0$ $\Psi_{(1,1)}(cx_1,cy_1)=c\Psi_{(1,1)}(x_1,y_1)$. *Furthermore it is concave, i.e. for* $s\in(0,1)$ *we have*

$$s\Psi_{(1,1)}(x_1,y_1)+(1-s)\Psi_{(1,1)}(x_2,y_2)\le\Psi_{(1,1)}(s(x_1,y_1)+(1-s)(x_2,y_2)) \tag{1.120}$$

and finally, $\Psi_{(1,1)}$ *is symmetric.*

We may ask when $\Psi_{(1,1)}$ is finite. By the ergodic theorem, this will be the case if $\mathbb{E}(L_{(1,1)\to(Nx,Ny)})$ grows linearly in N. In Proposition 2.2. of [55], the author shows that if $\mathbb{E}(\omega_{1,1})<\infty$ and if

$$\int_0^\infty \mathrm{d}s\sqrt{1-\mathbb{P}(\omega_{1,1}\le s)}<\infty, \tag{1.121}$$

then $\Psi_{(1,1)}$ is a.s. finite everywhere.

Of course, we would like to compute explicitly the shape function. However, this has essentially only been achieved in the case of geometric and exponential weights. For the exponential weights, Rost proves (a reformulation of) the following.

Theorem 1.30 (Remark 1 in [58])**.** *Consider the LPP model* (1.114) *with* $\mathcal{L}=(1,1)$ *and i.i.d. weights with* $\omega_{1,1}$ *an exponential with parameter one. Then*

$$\Psi_{(1,1)}(x,y)=(\sqrt{x}+\sqrt{y})^2. \tag{1.122}$$

What Rost precisely considers is (for the LPP model defined in Theorem 1.30) the set $B_t=\{(x,y)\in\mathbb{R}_+^2 : L_{(1,1)\to(x,y)}+\omega_{1,1}\le t\}$ (actually he uses the different starting point $(0,0)$) and he remarks that B_t/t converges to the limit shape

$$B=\{(x,y)\in\mathbb{R}_+^2 : \Psi_{(1,1)}(x,y)\le 1\}, \tag{1.123}$$

(see Figure 1.8) in the sense that for all $\varepsilon > 0$ there is a t_0 such that for all $t > t_0$

$$(1-\varepsilon)B \subseteq B_t/t \subseteq (1+\varepsilon)B \tag{1.124}$$

with probability 1. As we will see in Section 1.4.2, this is closely related to the hydrodynamic behavior of TASEP with step initial data. The convergence of B_t/t to (1.123) holds under certain moment assumptions for more general i.i.d. weights, see Theorem 5.1 in [55].

1.4.2 Linking TASEP and LPP

Here we explain the link between TASEP and LPP, which will allow us to obtain limit laws for TASEP random variables (e.g. particle positions) in LPP and vice-versa. Consider TASEP with initial configuration η_0 and labeling $(x_k(0), k \in \mathbb{Z})$; as usual, the labeling is from right to left, and if η_0 has a left- or rightmost particle, k does not run over all $\mathbb{Z}$. Let particle j have an exponential waiting time with parameter v_j. Take now

$$\mathcal{L} = \{(k + x_k(0), k) \in \mathbb{Z}^2 : k \in \mathbb{Z}\} \tag{1.125}$$

and

$$\omega_{i,j} \sim \exp(v_j) \tag{1.126}$$

(if the label j has not been attributed, we set $\omega_{i,j} = 0$). Then the link between TASEP and LPP is

$$\mathbb{P}(x_n(t) \geq m - n) = \mathbb{P}(L_{\mathcal{L}\to(m,n)} \leq t). \tag{1.127}$$

In this section, for the sake of clarity we introduce various superscripts in our notation as various LPP times and TASEP random variables will appear simultaneously. Hence denote the step initial data with all jump rates $v_j = 1$

$$x_j^{\rm step}(0) = -j, \quad v_j = 1, \quad j \in \mathbb{N}. \tag{1.128}$$

Denote by $L^{\exp(1)}_{(1,1)\to(m,n)} = L_{(1,1)\to(m,n)} + \omega_{1,1}$ the LPP model where the weights are given by (1.126) for step initial data (1.128) plus an extra weight coming from the starting point. Then, (1.127) becomes

$$\mathbb{P}(x_n^{\rm step}(t) \geq m - n) = \mathbb{P}(L^{\exp(1)}_{(1,1)\to(m,n)} \leq t). \tag{1.129}$$

We will deal with (1.129) only, but our arguments are general and do apply to the general case (1.127) too. We are going to obtain (1.129) in a discrete time version, from which (1.129) will follow by taking a suitable limit. We closely follow [62] for the discrete time result. We start by formally defining the cluster B_t for geometric weights.

Definition 1.31 (From (1.4) in [62]). *Consider the LPP model* $L^{\text{geom(p)}}_{(1,1)\to(m,n)} = L_{(1,1)\to(m,n)} + \omega_{1,1}$ *with weights* $w_{i,j} = 0$ *if* $(i,j) \notin \mathbb{N}^2$ *and otherwise*

$$\mathbb{P}(\omega_{i,j} = k) = p(1-p)^{k-1} \tag{1.130}$$

with k *a positive integer, and* $p \in (0,1)$. *Then define the cluster process* $(B^{\text{geom(p)}}_t, t \in \mathbb{N}_0)$

$$B^{\text{geom(p)}}_t = \{(m,n) : L^{\text{geom(p)}}_{(1,1)\to(m,n)} \le t\}. \tag{1.131}$$

Thus $(B^{\text{geom(p)}}_t, t \in \mathbb{N}_0)$ is a discrete time process with state space

$$\Gamma = \{U \subseteq \mathbb{N}^2 : U \text{ is finite and } (i,j) \in U \text{ implies that } \{1,\dots,i\} \times \{1,\dots,j\} \subseteq U\}, \tag{1.132}$$

and $B^{\text{geom(p)}}_0 = \emptyset$ and $B^{\text{geom(p)}}_t \subseteq [0,t]^2$. We will link $B^{\text{geom(p)}}_t$ to so called discrete time TASEP with parallel update, denoted by $(\eta^{\text{disc(p)}}_t, t \in \mathbb{N}_0)$, which can be described as follows. Let $(\xi^{i,i+1}_k)_{(k,i)\in\mathbb{N}\times\mathbb{Z}}$ be i.i.d. Bernoulli random variables with $(p \in (0,1))$

$$\mathbb{P}(\xi^{i,i+1}_k = 1) = p = 1 - \mathbb{P}(\xi^{i,i+1}_k = 0). \tag{1.133}$$

The state space is $X = \{0,1\}^{\mathbb{Z}}$ and time $t = 0,1,2,\dots$ is discrete. Let $\eta^{\text{disc(p)}}_0 \in X$ be an initial configuration and denote by $x^{\text{disc(p)}}_n(0)$ the initial position of particle n. Then the dynamics are as follows: As in continuous TASEP, there is at most one particle at each site. The process starts in $\eta^{\text{disc(p)}}_0$. Given $(x^{\text{disc(p)}}_n(t-1), n \in \mathbb{Z})$ the configuration at time t is obtained as follows. If $x^{\text{disc(p)}}_n(t-1)+1$ is occupied at time $t-1$, then $x^{\text{disc(p)}}_n(t-1) = x^{\text{disc(p)}}_n(t)$. If $x^{\text{disc(p)}}_n(t-1)+1$ is empty at time $t-1$, then if $\xi^{x^{\text{disc(p)}}_n(t-1), x^{\text{disc(p)}}_n(t-1)+1}_{t-1} = 1$ $x^{\text{disc(p)}}_n(t-1)+1 = x^{\text{disc(p)}}_n(t)$, otherwise $x^{\text{disc(p)}}_n(t-1) = x^{\text{disc(p)}}_n(t)$.

Consider now again the step initial configuration

$$x^{\text{disc(p),step}}_j(0) = -j, j \in \mathbb{N}.$$

We associate to this a cluster process $(A_t, t \in \mathbb{N}_0)$ by setting

$$A^{\text{disc(p)}}_t = \{(i,j) \in \mathbb{N}^2 : i - j \le x^{\text{disc(p),step}}_j(t)\}. \tag{1.134}$$

By showing that $A^{\text{disc(p)}}_t$ and $B^{\text{geom(p)}}_t$ are Markov chains with the same transition probabilities (note both processes start in the empty set), Seppäläinen in [62] shows the following.

Theorem 1.32 (Proposition 1.2 in [62]). *Let* $(A^{\text{disc(p)}}_t, t \in \mathbb{N}_0)$ *be the cluster process* (1.134) *of discrete time TASEP with step initial data and* $(B^{\text{geom(p)}}_t, t \in \mathbb{N}_0)$ *be the cluster process* (1.131) *of* $L^{\text{geom(p)}}_{(1,1)\to(m,n)}$. *Then* $(B^{\text{geom(p)}}_t, t \in \mathbb{N}_0)$ *and* $(A^{\text{disc(p)}}_t, t \in \mathbb{N}_0)$ *are equal in distribution.*

Now, it is an imminent corollary of Theorem 1.32 that we have

$$\mathbb{P}(x_n^{\text{disc(p),step}}(t) \geq m-n) = \mathbb{P}(L^{\text{geom(p)}}_{(1,1)\to(m,n)} \leq t). \tag{1.135}$$

So, to obtain (1.129), we need to take the appropriate limit in (1.135). Let $L>0$. If $\omega_{i,j}$ is distributed as (1.130) with parameter $p=1/L$, then $\omega_{i,j}/L$ converges in distribution as $L\to\infty$ to the exponential distribution with parameter 1. From this it easily follows that

$$\lim_{L\to\infty} \mathbb{P}(L^{\text{geom(1/L)}}_{(1,1)\to(m,n)} \leq \lceil Lt\rceil) = \mathbb{P}(L^{\exp(1)}_{(1,1)\to(m,n)} \leq t). \tag{1.136}$$

Hence it remains to see

$$\lim_{L\to\infty} \mathbb{P}(x_n^{\text{disc(1/L),step}}(\lceil Lt\rceil) \geq m-n) = \mathbb{P}(x_n^{\text{step}}(t) \geq m-n). \tag{1.137}$$

For (1.137), we couple the $(\xi_k^{i,i+1})_{(k,i)\in\mathbb{N}\times\mathbb{Z}}$ with a family of independent Poisson processes $(T^{i,i+1}, i\in\mathbb{Z})$ with rate one and construct $x_n^{\text{step}}(t)$ graphically with the $(T^{i,i+1}, i\in\mathbb{Z})$. Fix $t>0$ and $\tilde{L}>0$ and define $t_k=\frac{k}{\tilde{L}}, k=0,1,\dots,\lceil\tilde{L}t\rceil$. Then define

$$\frac{1}{L} = (t_{k+1}-t_k)e^{-(t_{k+1}-t_k)} = \frac{1}{\tilde{L}} + \mathcal{O}(\tilde{L}^{-2})$$

and set

$$\xi_k^{i,i+1} = \begin{cases} T^{i,i+1}_{t_k} - T^{i,i+1}_{t_{k-1}} & \text{if} \quad T^{i,i+1}_{t_k} - T^{i,i+1}_{t_{k-1}} \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases} \tag{1.138}$$

Then the $(\xi_k^{i,i+1})_{(k,i)\in\mathbb{N}\times\mathbb{Z}}$ are i.i.d. Bernoulli on $\{0,1\}$ with $\mathbb{E}(\xi_k^{i,i+1})=1/L$. Let $\varepsilon>0$ and take $M>m-n$ such that

$$D_M^n = \bigcup_{i=x_n^{\text{step}}(0)}^{M} \{T_t^{i,i+1}=0\} \tag{1.139}$$

satisfies $\mathbb{P}(D_M^n)>1-\varepsilon$. Define furthermore

$$\mathcal{E}_k = \{\sum_{i=x_n^{\text{step}}(0)}^{M} T^{i,i+1}_{t_k} - T^{i,i+1}_{t_{k-1}} \leq 1\}, \quad \mathcal{E} = \bigcap_{k=1}^{\lceil\tilde{L}t\rceil} \mathcal{E}^k. \tag{1.140}$$

For $(\mathcal{E}_k)^c$ to hold, there either has to be an $i\in\{x_n^{\text{step}}(0),\dots,M\}$ such that

$$F_k^i = \{T^{i,i+1}_{t_k} - T^{i,i+1}_{t_{k-1}} \geq 2\} \tag{1.141}$$

holds or there are $i,l\in\{x_n^{\text{step}}(0),\dots,M\}, i\neq l$, such that

$$F_k^{i,l} = \{T^{i,i+1}_{t_k} - T^{i,i+1}_{t_{k-1}} \geq 1\} \cap \{T^{l,l+1}_{t_k} - T^{l,l+1}_{t_{k-1}} \geq 1\} \tag{1.142}$$

holds. Now $\mathbb{P}(F_k^{i,l}) = (1-e^{-1/L})^2 = L^{-2} + o(L^{-2})$ and $\mathbb{P}(F_k^i) = 1 - e^{-1/L} - e^{-1/L}/L = \mathcal{O}(L^{-2})$ too. From this it follows that we may bound $\mathbb{P}(\mathcal{E}) \geq 1 - M^3/L$ for M large enough. Note now that on $\mathcal{E} \cap D_M^n$ we have

$$x_n^{\mathrm{disc}(1/\mathrm{L}),\mathrm{step}}(\tilde{L}t_k) = x_n^{\mathrm{step}}(t_k), \quad k = 0, \ldots, \lceil \tilde{L}t \rceil, \tag{1.143}$$

from which (1.137) follows by sending L, M to infinity.

We can use now the link (1.129) and (1.32), to get, for $m - n > 0$ and the height function $h^{\mathrm{TASEP,step}}$ of step TASEP

$$\mathbb{P}(h^{\mathrm{TASEP,step}}(m-n,t) \geq m+n) = \mathbb{P}(L_{(1,1)\to(m+1,n)} \leq t). \tag{1.144}$$

Hence we should have $\Psi_{(1,1)}(x,y) = 1$ if $h_{\mathrm{ma}}^{\mathrm{TASEP,step}}(x-y,1) = \frac{1+(x-y)^2}{2}$ equals $x+y$; due to Theorem 1.30, this is indeed the case, see [58].

1.4.2.1 Step initial data revisited

Now we can give the original version of Theorem 1.19 from Johansson. He considers the LPP time $L_{(1,1)\to(\eta N,N)}$ and shows that it converges, properly rescaled, to F_{GUE}.

Theorem 1.33 (Theorem 1.6 in [45])**.** *Let $\eta \geq 1$ and consider the LPP model* (1.114) *with $\mathcal{L} = (1,1)$, $\mathcal{A} = (\eta N, N)$ and the $\omega_{i,j}$ are i.i.d. with $\omega_{1,1} \sim \exp(1)$. Then we have*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{(1,1)\to(\eta N,N)} - (1+\sqrt{\eta})^2 N}{\eta^{-1/6}(1+\sqrt{\eta})^{4/3}N^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s). \tag{1.145}$$

Johansson proves that for the LPP model of Theorem 1.33

$$\mathbb{P}(L_{(1,1)\to(m,n)} \leq t) = \mathbb{P}(\lambda_{m,n}^{\mathrm{Lag,max}} \leq t),$$

where $\lambda_{m,n}^{\mathrm{Lag,max}}$ is the largest eigenvalue of the $m \times n$ *Laguerre ensemble*, built from a $m \times n$ $(n \leq m)$ matrix A with complex gaussian entries with zero mean and variance $1/2$. Then $\lambda_{m,n}^{\mathrm{Lag,max}}$ is the largest eigenvalue of $A\bar{A}^T$, where $\bar{A}^T$ is the transposed matrix of A with conjugated entries. The proof then proceeds by giving a Fredholm determinant formula for the law of $\lambda_{m,n}^{\mathrm{Lag,max}}$, of which one then takes asymptotics. Furthermore, Johansson provides large deviation estimates, modifications of these will appear as Propositions 3.9, 3.10 in Chapter 3.

Theorem 1.34 (Theorem 1.6 in [45])**.** *Consider the LPP model of Theorem 1.33. Then there are functions $i_\star$, $l_\star : (0,\infty) \to (0,\infty)$ such that for all $\varepsilon > 0$*

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(L_{(1,1)\to(\eta N,N)} \leq N((1+\sqrt{\eta})^2 - \varepsilon)\right) = -l_\star(\varepsilon) \\ &\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(L_{(1,1)\to(\eta N,N)} \geq N((1+\sqrt{\eta})^2 + \varepsilon)\right) = -i_\star(\varepsilon). \end{aligned} \tag{1.146}$$

1.4.3 Poisson LPP and transversal fluctuations

Here we turn to a model of last passage percolation on $\mathbb{R}^2$, Poisson LPP. It is linked to the polynuclear growth model, a model in the KPZ class we mentioned in passing earlier, see [24], Section 4 for details. It is a continuous space analogue of LPP on $\mathbb{Z}^2$ using Poisson points, see Figure 1.9 (left), and can be obtained from LPP on $\mathbb{Z}^2$ by taking i.i.d. Bernoulli weights on $\{0,1\}$with vanishing probability for 1.

Definition 1.35 (Taken from [46]). *Consider the space Ω of locally finite, simple point measures on $\mathbb{R}^2$,*

$$\omega = \sum_i \delta_{\zeta_i} \in \Omega$$

where $\zeta_i = (x_i, y_i)$ are the points in ω. Write for two points in $\mathbb{R}^2$ $(x,y) \precsim (x',y')$ if $x < x'$ and $y < y'$. Given an ω and two points $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \precsim z_2$ we define a north-east path π from z_1 to z_2 in ω to be a collection of points $(\zeta_{i_k})_{k=1}^M$ in ω such that

$$z_1 \precsim \zeta_{i_1} \precsim \cdots \zeta_{i_M} \precsim z_2.$$

The length of such a path is M, the number of Poisson points on π, and denoted by $|\pi|$. We then define the Poisson last passage percolation time to be

$$\ell_{z_1 \to z_2}(\omega) := \max\{|\pi| : \pi \text{ is a north} - \text{east path from } z_1 \text{ to } z_2 \text{ in } \omega\}. \tag{1.147}$$

We denote by $\Pi^{\max}(z_1, z_2, \omega)$ the set of all north- east paths π from z_1 to z_2 in ω for which $\ell_{z_1 \to z_2}(\omega) = |\pi|$. We suppress the ω often in the following.

The limit law of $\ell_{(0,0) \to (N,N)}$ under suitable scaling was determined in [9] and turned out to be F_{GUE}. This is equivalent to the asymptotic distribution of the rescaled length of a longest increasing subsequence of a random permution on $\{0, \ldots, N\}$, see [9]. We do not treat this issue here, though, and deal with the geometric properties of the maximizing paths, as they are relevant to our study of shocks in the following.

We give a result of Johansson (obtained in [46]) about the transversal fluctuations of maximizing paths. To put this into perspective, let us (non-rigorously, but see (1.155) below) introduce the two scaling exponents χ and ξ, which are conjectured to have some universality properties. Roughly speaking (see also [26]) χ gives the order of length fluctuations of $\ell_{(0,0) \to z_1}$ around its mean, i.e.

$$\ell_{(0,0) \to z_1} - \mathbb{E}(\ell_{(0,0) \to z_1}) \text{ is of order } \|z_1\|^{\chi}, \tag{1.148}$$

whereas ξ gives the order of transversal fluctuations of the maximal deviation of a maximizing path from the straight line $\overline{0, z_1} = \{\lambda \cdot z_1 : \lambda \in [0,1]\}$:

$$\text{the maximal deviation of } \pi \in \Pi^{\max}(0, z_1) \text{ from } \overline{0, z_1} \text{ is of order } \|z_1\|^{\xi}. \tag{1.149}$$

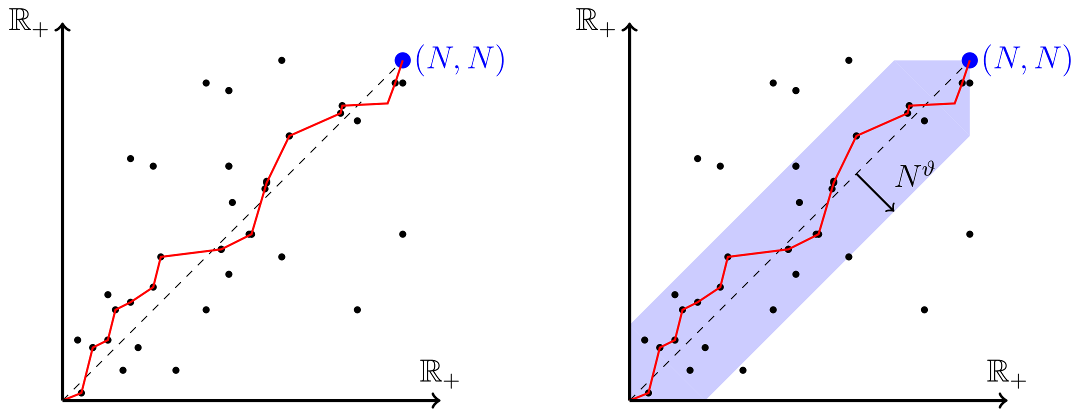

Figure 1.9: Poisson LPP on $\mathbb{R}_+^2$: Paths π move north-east from the origin to (N,N) along poisson points (left, π is solid), and a maximizing path stays within a cylinder of width $\mathcal{O}(N^\vartheta)$, for all $\vartheta > 2/3$ (right) with probability going to 1.

It is now a conjecture (see e.g. the introduction of [26] or [50] and the references therein) that, irrespective of the dimension in which the percolation occurs (here we consider percolation in dimension 2, but the model generalizes easily to higher dimensions) we have the relation

$$2\xi - 1 = \chi. \tag{1.150}$$

The relation (1.150) has been proven for *first-passage* percolation in $\mathbb{Z}^d$ by Chatterjee in [26] (first passage percolation is simply the model one obtains if one replaces max by min in (1.114)) under some assumptions on the weights and assuming that the exponents exist in a certain sense, see Theorem 1.1 of [26] for details. For Poisson LPP, and more generally in the KPZ class, we have $\chi = 1/3$ (note however that in last passage percolation it can happen that $\chi = 1/2$, namely if on $\mathbb{Z}^2$ the weights and/or the sets $\mathcal{L}, \mathcal{A}$ are such that the LPP time is essentially a sum of fixed i.i.d. random variables, e.g. for $L_{(0,0)\to(N,0)}$.) To obtain the value of ξ, we have the following heuristics of Johansson [46].

One knows that the leading order of $\ell_{(0,0)\to(x,y)}$ is $\sqrt{xy}$, see [2]. Consider a maximizing path π_1 from $(0,0)$ to (N,N), and another north-east path π_2 from $(0,0)$ to (N,N) that passes through $(N(t-\delta), N(t+\delta))$, where $t \in (0,1)$ and δ is small. Then typically π_2 is shorter than π_1 by the amount

$$2N(\sqrt{(t-\delta)(t+\delta)} + \sqrt{(1-t-\delta)(1-t+\delta)} - 1) = 2N\mathcal{O}(\delta^2). \tag{1.151}$$

Now (1.151) should have the same order as the length fluctuations, which are N^χ, leading to $\delta^2 = \mathcal{O}(N^{\chi-1})$. Thus

$$N^\xi \approx N\delta \approx N^{\chi/2+1/2}, \tag{1.152}$$

yielding (1.150). We now define precisely the scaling exponent ξ and give the result which proves the correctness of the preceeding heuristics. Define the cylinder

$$C(\vartheta, N) = \{(x,y) : 0 \le x+y \le 2N,\ -\sqrt{2}N^\vartheta \le y-x \le \sqrt{2}N^\vartheta\}, \tag{1.153}$$

see Figure 1.9, and consider the event

$$A_N^\vartheta = \{\omega \in \Omega :\ \text{for all } \pi \in \Pi^{\max}((0,0),(N,N),\omega) \text{ we have } \pi \subseteq C(\vartheta, N)\}. \tag{1.154}$$

With this we can now define the scaling exponent of the transversal fluctuations as

$$\xi := \inf\{\vartheta > 0 : \liminf_{N\to\infty} \mathbb{P}(A_N^\vartheta) = 1\}. \tag{1.155}$$

Then, the relation (1.150) is established in the next Theorem.

Theorem 1.36 (Theorem 1.1 in [46])**.** *Consider the model of Poisson LPP defined in* (1.147) *and ξ defined in* (1.155). *Then $\xi = 2/3$.*

In Section 3.3.2.1 we will prove statements about maximizing paths in LPP on $\mathbb{Z}^2$ which follow the same line of argumentation as the proof of the preceeding Theorem. Theorem 1.36 was later refined in [12]. Namely, they show that the probability of having transversal fluctuations larger than $kt^{2/3}, k > 0$ decays exponentially in k, see Theorem 9.13 in [12].

Chapter 2

Shocks in (T)ASEP

2.1 Characteristics of the Burger's equation

Here we come back to the Burger's equation (1.27) and its entropy solution. An important special case of initial data are *Riemann* initial data

$$\rho_0(\xi) = \begin{cases} \lambda & \text{if } \xi < 0 \\ \varsigma & \text{if } \xi \geq 0 \end{cases} \tag{2.1}$$

with $\lambda, \varsigma \in [0,1]$. Riemann initial data cover all the examples of TASEP with one speed we considered so far. To obtain information about solutions of the Burger's equation we use the method of characteristics. This is a classical tool in partial differential equations, and covered in many textbooks, see e.g. [31], § 3.2. We refer to the same source for the notion of entropy solutions. The basic idea to obtain the solution at some point (ξ, τ) is to construct a curve $(\xi(s), \tau(s))$ (the *characteristcs*) beginning in some point $(\xi_0, 0)$ (where the solution is known) along which one can compute the solution and which ends in (ξ, τ). For the Burger's equation (more generally, for conservation laws) the solution, whenever smooth, is constant along the characteristics. If we make the ansatz

$$\rho(\xi(s), \tau(s)) = \text{ const.} \tag{2.2}$$

then since $\partial_s \xi \partial_\xi \rho + \partial_s \tau \partial_\tau \rho = 0$ and as ρ solves the Burger's equation we get the ordinary differential equations

$$\begin{aligned} &\partial_s \tau = 1 \\ &\partial_s \xi = (p-q)(1 - 2\rho(\xi, \tau)) = \text{ const.} \end{aligned} \tag{2.3}$$

Hence the characteristic starting in $(\xi_0, 0)$ is a line with slope $((p-q)(1-2\rho(\xi_0, 0)))^{-1}$ along which the solution should be constant $\rho(\xi_0, 0)$. One problem with this ansatz is

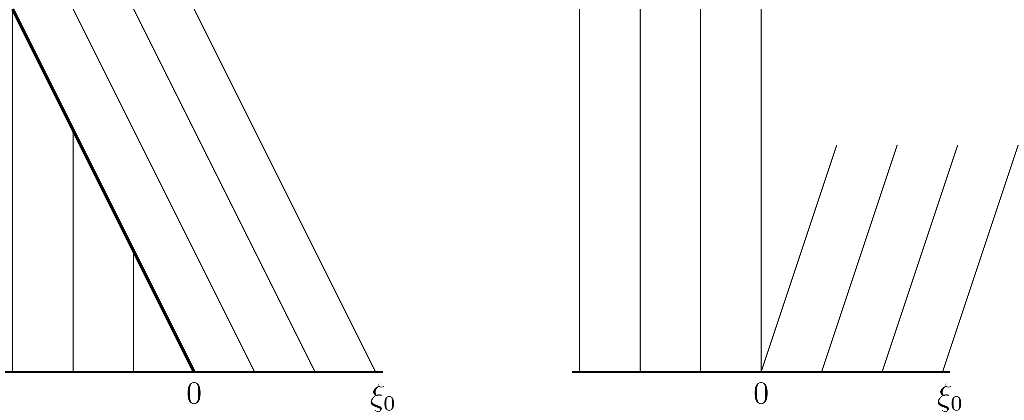

Figure 2.1: Characteristics of the Burger's equation for $p = 1$ and Riemann initial data. Left: For $\lambda = 1/2 < 3/4 = \varsigma$ the characteristics cross and create a shock. Right: For $\lambda = 1/2 > 1/3 = \varsigma$ the characteristics do not cross, but the method of characteristics fails to provide information within the wedge $\{(\xi, \tau) : \xi > 0, \tau > 3\xi\}$. The entropy solution picks out the relevant solution.

that the characteristic lines may cross: For Riemann initial data, and $p > q$, this happens when $\lambda < \varsigma$. For $\lambda > \varsigma$ the characteristics never cross, see Figure 2.1. The answer to this problem is that the entropy solution has a discontinuity, called **shock**, if $\lambda < \varsigma$, whereas it has a linearly decreasing region, called rarefaction fan, if $\lambda > \varsigma$. Note that we have already encountered rarefaction fans for the step and half-flat initial data, see Figures 1.4 and 1.5.

Theorem 2.1 (Theorem 4 in § 3.4.4 in [31])**.** *Consider the Burger's equation* (1.27) *with Riemann initial data* (2.1) *and* $p > q$*. If* $\lambda < \varsigma$*, the unique entropy solution of* (1.27) *is given by*

$$\rho(\xi, \tau) = \begin{cases} \lambda & \text{if } \xi \leq (p-q)(1-\lambda-\varsigma)\tau \\ \varsigma & \text{if } \xi > (p-q)(1-\lambda-\varsigma)\tau. \end{cases} \tag{2.4}$$

We speak of a shock wave seperating the states λ, ς*.*

If $\lambda > \varsigma$*, the unique entropy solution is given by*

$$\rho(\xi, \tau) = \begin{cases} \lambda & \text{if } \xi \leq \tau(p-q)(1-2\lambda) \\ \frac{1}{2}(1 - \frac{\xi}{\tau(p-q)}) & \text{if } (1-2\lambda)\tau(p-q) < \xi \leq \tau(p-q)(1-2\varsigma) \\ \varsigma & \text{if } \xi > \tau(p-q)(1-2\varsigma). \end{cases} \tag{2.5}$$

We speak of a rarefaction wave separating the states λ, ς*.*

2.2 Shocks with random initial data

Let us summarize the different types of density regions created by Riemann initial data. For a rarefaction fan, the density regions that occur are constant density (hence a flat geometry), decreasing density (hence a curved geometry) and transitions between the two. Note that we have seen the conjecturally universal limit processes for flat, curved and flat-curved geometries.

For the shock, there are two constant density regions, hence flat geometries. But what happens at the discontinuity, i.e. the shock, itself ? In view of what we have done before, it is natural to take ct such that

$$\lim_{t\to\infty}\frac{x_{ct}}{t}=(p-q)(1-\lambda-\varsigma). \tag{2.6}$$

From the density profile, c is easily computed to be $\lambda\varsigma$ for TASEP. We then ask for the fluctuations of x_{ct} around $t(p-q)(1-\lambda-\varsigma)$. Studying these fluctuations is one of the main contributions of this thesis, see Section 2.3 and Section 2.4 for the results we will prove in Chapters 3 and 4. However, there is a different way of defining the random shock location via a particle position, which also satisfies the law of large numbers (2.6). This particle is a second class particle, initially placed at the origin, and defined as follows (definition taken from [36]).

Consider two initial configurations $\eta_0,\eta_0'\in X=\{0,1\}^{\mathbb{Z}}$. We can construct $(\eta_t^{\eta_0}(t),t\geq 0)$ and $(\eta_t^{\eta_0'}(t),t\geq 0)$ graphically using the same Poisson processes and then speak of a basic coupling. Suppose now we have $\eta_0(0)=0\neq 1=\eta_0'(0)$ and $\eta_0(i)=\eta_0'(i)$ for $i\neq 0$. Then in the basic coupling, for all t $\eta_t^{\eta_0}$ and $\eta_t^{\eta_0'}$ differ exactly at one site. We denote by

$$X(t)=\sum_{x\in\mathbb{Z}}x\mathbf{1}_{\{\eta_t^{\eta_0}(x)\neq\eta_t^{\eta_0'}(x)\}}(x) \tag{2.7}$$

its position. To see this is as the position of a particle, imagine particles in η_0' all belong to the first class except for one second class particle at the origin. The evolution of particles is the same as in ASEP, except that a second class particle cannot jump to a site occupied by a first class particle, whereas when a first class particle jumps to the position of a second class particle, they exchange positions. Now the movement of first class particles is exactly the same as any pair of particles $(\eta_0(i),\eta_0'(i))=(1,1)$ in the basic coupling, whereas the second class particles moves like the pair $(\eta_0(0),\eta_0'(0))=(0,1)$. Hence we can think of the coupled process $((\eta_t^{\eta_0},\eta_t^{\eta_0'}),t\geq 0)$ as the process $(\eta_t^{\eta_0},t\geq 0)$ with the only difference that initially the site 0 is not empty, but occupied by a second class particle. Especially the process of first class particles has the same law as ASEP started from η_0. The proces $X(t)$ is not Markov, its motion depends on $\eta_t^{\eta_0'}$ in its neighboring sites.

It now turns out that the second class particle initially placed at zero follows the shock. For *random* initial data, its fluctuations have been discovered, and many results have been reviewed in the book [54], Part III. We consider two sided Bernoulli initial data $\nu^{\lambda,\varsigma}$, this is a product measure on X with marginals (the definition can e.g. be found in [54], p. 221)

$$\nu^{\lambda,\varsigma}(\{\eta : \eta(i) = 1\}) = \begin{cases} \lambda & \text{for } i < 0 \\ \varsigma & \text{for } i \geq 0. \end{cases} \tag{2.8}$$

If now the initial data η_0 has a second class particle initially at 0 and is otherwise distributed according $\nu^{\lambda,\varsigma}$ with $\lambda < \varsigma$, this creates a shock. The result is now that the second class particle follows the shock and has gaussian fluctuations around it.

Theorem 2.2 (Theorem 2.90 in [54], Part III)**.** *Consider ASEP with $p > q$. Suppose η_0 has distribution $\nu^{\lambda,\varsigma}$ (see* (2.8)*) with $\lambda < \varsigma$ on $\mathbb{Z} \setminus \{0\}$, with a second class particle placed at the origin. Let $X(t)$ be the location of the second class particle at time t. Then*

$$\frac{X(t) - (p-q)(1-\lambda-\varsigma)t}{\sqrt{t}} \tag{2.9}$$

converges in distribution, as $t \to \infty$ to the normal distribution with mean zero and variance

$$D = (p-q)\frac{\varsigma(1-\varsigma) + \lambda(1-\lambda)}{\varsigma - \lambda}. \tag{2.10}$$

As the fluctuations of TASEP built up at time t are $\mathcal{O}(t^{1/3})$ and not $\sqrt{t}$ one could guess these fluctuations come from the initial data, and not from the dynamics of the process. And indeed, in [54] Theorem 2.2 is obtained from the following result.

Theorem 2.3 (Proposition 2.74 in [54], Part III)**.** *Let η_0 and $X(t)$ be as in Theorem 2.2. Then, in L^2 we have*

$$\lim_{t\to\infty} \frac{(\varsigma-\lambda)X(t) - (p-q)(\varsigma-\lambda)t + \sum_{|x|\leq(p-q)(\varsigma-\lambda)t} \eta_0(x)}{\sqrt{t}} = 0. \tag{2.11}$$

Since the $\eta_0(i), i < 0$ and $\eta_0(i), i > 0$ are two independent families of i.i.d. random variables, Theorem 2.2 follows from Theorem 2.3 by the central limit theorem.

If the initial randomness is only at one side of the shock, a similar picture still holds. For example, in [23] the initial condition is Bernoulli-ρ to the right and periodic with density 1/2 to the left of the origin. When $\rho > 1/2$ there is a shock with Gaussian fluctuations in the scale $t^{1/2}$. In that work, the fluctuations of the shock position are derived from the ones of the particle positions. The result fits in with the heuristic argument in [65] (Section 5). The Gaussian form of the distribution function is not robust (see for instance Remark 17 in [23]).

We may now ask what the difference between the second class particle $X(t)$ and the normal particle x_{ct} interpretation is. We explain this briefly in Section 3.1.1.1 .

2.3 Shocks with deterministic initial data

We have seen that for random initial data, the shock (interpreted via the second class particle) has gaussian fluctuations, which are precisely the fluctuations of the initial data (Theorem 2.3). This naturally leads to the question what happens in the absence of such fluctuations, i.e. what are the shock fluctuations for deterministic initial data ? Answering this question (for usual, not second-class particles) is the main contribution of this thesis. As we will see, the answer will unravel a generic independence structure in LPP, and will lead to previously unobserved correlation lengths and limit laws in the KPZ class. The results are the basis for obtaining results about second class particles too, this is however still ongoing work. Since ASEP with $p \neq 1$ is not linked to LPP, our results are restricted to TASEP.

We will give three theorems about shock fluctuations - Theorem 2.4, Theorem 2.5 and Theorem 2.6. They give the fluctuations of particles at the shock for three different deterministic initial data. However, they are all instances of a general phenomenon: Asymptotic independence in last passage percolation. Under three general assumptions, this asymptotic independence is established in a very general setting (Theorem 2.7).

The Theorems 2.4 and 2.5 treat shocks created by the presence of particles with smaller speed $\alpha < 1$. The asymptotic independence can also be applied to shocks created by TASEP where all particles have identical speed, e.g. by the initial data $\eta_0 = \mathbf{1}_{-3\mathbb{N}} + \mathbf{1}_{2\mathbb{N}}$. Due to the presence of two speeds and/or non-Riemann initial data, however, shocks can be created that do not always are as in (2.4), but are discontinuities going from a linearly decreasing density region to a flat/another decreasing region, see Figure 2.2.

In the presence of two speeds, such discontinuities can occur since the full density profile is no longer the solution of a single Burger's equation. Let us explain this in the setup of Theorem 2.4 (the only difference for Theorem 2.5 and Theorem 1.27 is the initial data in (2.14)). There are particles initially occupying $2\mathbb{N}_0$ with speed $\alpha < 1$. The particles to their left have speed 1. Let us call particles with speed α $\alpha-$particles, and particles with speed 1 normal particles. To obtain the density profile for the $\alpha-$particles one has to solve the Burger's equation

$$\begin{aligned} &\partial_\tau \rho + \alpha \partial_\xi [\rho(1-\rho)] = 0 \\ &\rho(\xi, 0) = \frac{1}{2}\mathbf{1}_{[0,\infty)}(\xi). \end{aligned} \tag{2.12}$$

Note this equation is identical to the Burger's equation for ASEP with $p - q = \alpha$. The entropy solution is thus

$$\rho(\xi, \tau) = \begin{cases} 0 & \text{if } \xi \leq \tau\alpha/2 \\ \frac{1}{2} & \text{if } \xi > \tau\alpha/2. \end{cases} \tag{2.13}$$

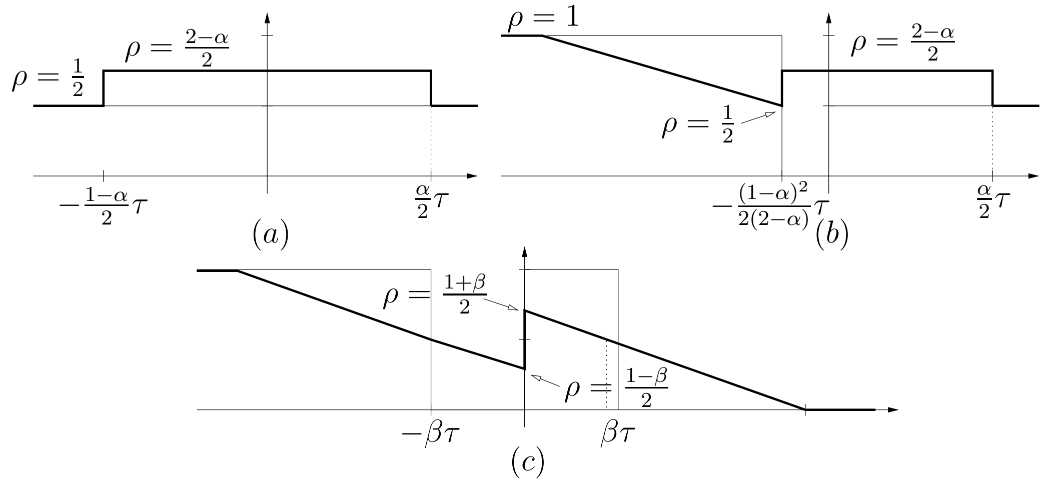

Figure 2.2: The thick lines are the density profiles ρ of (a) Theorem 2.4, (b) Theorem 2.5, and (c) Theorem 2.6, for $\alpha = 1/2$. The thin lines are the initial conditions. The dotted vertical lines indicate the macroscopic position of the particle that started from the origin.

The density profile created by the normal particles in Theorem 2.4 is given by the entropy solution of the Burger's equation

$$\begin{aligned} &\partial_\tau \rho + \partial_\xi[\rho(1-\rho)] = 0 \\ &\rho(\xi, 0) = \frac{1}{2}\mathbf{1}_{(-\infty,0]}(\xi) \end{aligned} \tag{2.14}$$

under a boundary condition. Namely, the first normal particle is blocked by the last α-particle, which starts at the origin and moves with speed $\alpha/2$. Therefore the macroscopic density profile of the normal particles can be obtained by solving (2.14) for $\xi \in (-\infty, \tau\alpha/2]$ under the boundary condition $\rho(\tau\alpha/2, \tau) = 1-\alpha/2$. Hence the density profile of the normal particles is given by

$$\rho(\xi\tau, \tau) = \begin{cases} 1/2 & \text{if } \xi \leq \tau(\alpha-1)/2 \\ 1-\alpha/2 & \text{if } \tau(\alpha-1)/2 < \xi \leq \tau\alpha/2, \end{cases} \tag{2.15}$$

see Figure 2.2 (a).

Let us sketch however that one can obtain the density profile of the normale particles from solving a Burger's equation without boundary condition. Namely, the blocking of the first normal particle is the same if we replace all $\alpha-$particles by a single particle initially placed at the origin that has speed $\alpha/2$. By Burke's property (as explained and used in e.g. [23]) having a single particle with speed $\alpha/2$ initially at the origin is equivalent to have Bernoulli initial data with density $1-\alpha/2$ on $\mathbb{R}_+$. Hence, to obtain the density profile of the normal particles in Theorem 2.4, we might as well solve

$$\partial_\tau \rho + \partial_\xi[\rho(1-\rho)] = 0 \tag{2.16}$$

with initial data

$$\rho(\xi,0)=\begin{cases}\frac{1}{2} & \text{if } \xi\leq 0\\ 1-\alpha/2 & \text{if } \xi>0.\end{cases} \tag{2.17}$$

This is now a usual shock for Riemann initial data, and the solution is given in Theorem 2.1.

In addition to the fluctuations of a particle located at the shock we are interested in the correlation length. All correlation lengths we saw so far were $\mathcal{O}(t^{2/3})$. We have seen in Section 1.3.6 on the $\mathrm{Airy}_{2\to 1}$ process that the correlation length is still $\mathcal{O}(t^{2/3})$ even at points where the density is no longer smooth. Is this still true when the density is not continuous ? As we shall see, the answer is no: The correlations are $t^{1/3}$, see Theorems 2.4, 2.5 and 2.6. These are the first results where such a correlation length has been observed in the KPZ class.

Choosing (as in (2.6)) $c\in\mathbb{R}$ such that x_{ct} is located at the shock, the obvious ansatz to obtain the fluctuations of x_{ct} around the shock is to apply Theorem 1.15 and do asymptotics. However, it turns out that the kernel one obtains e.g. in the situation of Theorem 2.4 does not converge pointwise under any (meaningful) conjugation and rescaling. Hence Theorem 1.15 is not applicable, at least not directly. What is required is a *whole new idea.* This idea is the asymptotic independence in LPP, given in Theorem 2.7. Let us now state the results.

Theorem 2.4 (At the F_{GOE}–F_{GOE} shock, proven in Section 3.3.3)**.** *Let $x_n(0)=-2n$ for $n\in\mathbb{Z}$ and let $\alpha\in(0,1)$. Let the exponential waiting times of particle n have parameter*

$$v_n=\begin{cases}\alpha & \text{if } n\leq 0\\ 1 & \text{if } n>0.\end{cases} \tag{2.18}$$

Let $\nu=\frac{2-\alpha}{4}$ and $v=-\frac{1-\alpha}{2}$. Then it holds

$$\lim_{t\to\infty}\mathbb{P}\left(x_{\nu t+\xi t^{1/3}}(t)\geq vt-st^{1/3}\right)=F_{\mathrm{GOE}}\left(\frac{s-\xi/\rho_1}{\sigma_1}\right)F_{\mathrm{GOE}}\left(\frac{s-\xi/\rho_2}{\sigma_2}\right), \tag{2.19}$$

with $\rho_1=\frac{1}{2}$, $\rho_2=\frac{2-\alpha}{2}$, $\sigma_1=\frac{1}{2}$, and $\sigma_2=\frac{\alpha^{1/3}(2-2\alpha+\alpha^2)^{1/3}}{2(2-\alpha)^{2/3}}$.

As one can see from (2.19) the shock moves with speed v. When ξ is very large we are in the region before the shock, where the density of particles is 1/2. Indeed, by replacing $s\to s+2\xi$ and taking the $\xi\to\infty$ limit, then (2.19) converges to $F_{\mathrm{GOE}}(s/\sigma_1)$. Similarly, when $-\xi$ is very large we are already to the right of the shock, where the density of particles is $(2-\alpha)/2$. Indeed, by replacing $s\to s+2\xi/(2-\alpha)$ and taking $\xi\to-\infty$, then (2.19) converges to $F_{\mathrm{GOE}}(s/\sigma_2)$. This is the reason why we call this situation a F_{GOE}–F_{GOE} shock.

Theorem 2.5 (At the $F_{\rm GUE}$–$F_{\rm GOE}$ shock, proven in Section 3.3.3)**.** *For $\alpha<1$ let $\nu=1/4$ and $v=-\frac{(1-\alpha)^2}{2(2-\alpha)}$. Let $x_n(0)=v\ell-n$ for $n\geq 1$ and $x_n(0)=-2n$ for $n\leq 0$. Let v_n be as in Theorem 2.4. Then it holds*

$$\lim_{\ell\to\infty}\mathbb{P}\left(x_{\nu\ell+\xi\ell^{1/3}}(t=\ell)\geq v\ell-s\ell^{1/3}\right)=F_{\rm GUE}\left(\frac{s-\xi/\rho_1}{\sigma_1}\right)F_{\rm GOE}\left(\frac{s-\xi/\rho_2}{\sigma_2}\right),\qquad(2.20)$$

with $\rho_1=\frac{1}{2}$, $\rho_2=\frac{2-\alpha}{2}$, $\sigma_1=2^{-1/3}$, and $\sigma_2=\frac{\alpha^{1/3}(6-10\alpha+6\alpha^2-\alpha^3)^{1/3}}{2(2-\alpha)}$.

Theorem 2.6 (At the $F_{\rm GUE}$–$F_{\rm GUE}$ shock, proven in Section 3.3.3)**.** *For a fixed $\beta\in(0,1)$, consider the initial condition given by $x_n(0)=-n-\lfloor\beta\ell\rfloor$ for $n\geq 1$ and $x_n(0)=-n$ for $-\lfloor\beta\ell\rfloor\leq n\leq 0$. Then, with all particles having speed 1, $\nu=\frac{(1-\beta)^2}{4}$ it holds*

$$\lim_{\ell\to\infty}\mathbb{P}\left(x_{\nu\ell+\xi\ell^{1/3}}(t=\ell)\geq -s\ell^{1/3}\right)=F_{\rm GUE}\left(\frac{s-\xi/\rho_1}{\sigma_1}\right)F_{\rm GUE}\left(\frac{s-\xi/\rho_2}{\sigma_2}\right)\qquad(2.21)$$

with $\rho_1=\frac{1-\beta}{2}$, $\rho_2=\frac{1+\beta}{2}$, $\sigma_1=\frac{(1+\beta)^{2/3}}{2^{1/3}(1-\beta)^{1/3}}$, and $\sigma_2=\frac{(1-\beta)^{2/3}}{2^{1/3}(1+\beta)^{1/3}}$.

As expected by KPZ universality, if we move away from the shock, the distribution function considered above becomes a single GOE or GUE distribution, with GOE whenever the particles density is constant and GUE whenever the particle density is decreasing, e.g., in the $F_{\rm GUE}$-$F_{\rm GUE}$ shock, the particle density is decreasing both to the left and to the right of the shock.

The reason of the product form of the distribution function is that (1) at the shock two characteristics merge and (2) along the characteristics decorrelation is slow [29, 37]. More precisely, if we look at the history of a particle close to the shock at time t, it has non-trivial correlations with a region of width $\mathcal{O}(t^{2/3})$ around the characteristics. At the shock the two characteristics come together with a positive angle so that at time $t-t^\nu$, $2/3<\nu<1$, their distance will be farther away than $\mathcal{O}(t^{2/3})$. This implies that the fluctuations built up along the two characteristics before time $t-t^\nu$ will be (asymptotically) independent. But if we stay on a characteristic, then the dynamical fluctuations created between time $t-t^\nu$ and time t are only $o(t^{1/3})$, which are irrelevant with respect to the total fluctuations present at time $t-t^\nu$ that are of order $t^{1/3}$ (this is also known as the slow-decorrelation phenomenon [29, 37]).

2.3.1 General Asymptotic independence

Here we establish the product form we observed in Theorems 2.4 - 2.6 in a more general setting in LPP. So let us consider an LPP model (see (1.114))[1] where the end set is one

[1] One can also apply our arguments to Poisson LPP. The general arguments are unchanged. Only a minor modification in the proofs in Section 3.3.2 is needed, namely the discretization used in Johansson's argument [46].

point and the starting set is a union of sets, namely

$$\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-, \quad \mathcal{A} = E = (\lfloor \eta t \rfloor, \lfloor t \rfloor), \tag{2.22}$$

where $\mathcal{L}^+ \subseteq \{(v,n) \in \mathbb{Z}^2 : v \leq 0, n \geq 0\}$, $\mathcal{L}^- \subseteq \{(v,n) \in \mathbb{Z}^2 : n \leq 0, v \geq 0\}$. Note that, by putting some of the $\omega_{i,j}$ to zero, it is always possible to choose $\mathcal{L}^+ = (\mathbb{Z}_-, 0)$ and $\mathcal{L}^- = (0, \mathbb{Z}_-)$.

With this choice it follows from the definition of the last passage time that

$$L = L_{\mathcal{L} \to \mathcal{A}} = \max \left\{ L_{\mathcal{L}^+ \to (\eta t, t)}, L_{\mathcal{L}^- \to (\eta t, t)} \right\}. \tag{2.23}$$

The two random variables $L_1 = L_{\mathcal{L}^+ \to (\eta t,t)}$ and $L_2 = L_{\mathcal{L}^- \to (\eta t,t)}$ are not independent. However, under some assumptions they are essentially independent as $t \to \infty$, in the sense that the random last passage time $L = \max\{L_1, L_2\}$ properly rescaled has asymptotically the law of the product of the two rescaled random variables. This is due to the fact that the fluctuations present in the region where the maximizers of the two LPP problems tend to come together are on a smaller scale than the typical fluctuations. This is by virtue of the slow-decorrelation phenomenon [29, 37].

From Theorem 1.29 and Section 1.4.3 we have a law of large numbers $L_i/t \to \mu_i$ as $t \to \infty$ and expect a fluctuation result $L_i - \mu_i t = \mathcal{O}(t^{\chi_i})$ with $\chi_i = 1/3$ or $\chi_i = 1/2$. If L_1 and L_2 have different leading orders μ_1, μ_2, the result is quite easy since only the largest of the two random variables is relevant in the $t \to \infty$ limit. This situation can be treated directly with coupling arguments as in [14]. If $\mu_1 = \mu_2 = \mu$ but for instance $\chi_1 < \chi_2$, then the natural scaling is $(L - \mu t)/t^{\chi_2}$, under which scaling $(L_1 - \mu t)/t^{\chi_2}$ degenerates to the trivial random variable 0 and acts as a cut-off. This situation occured for instance in [23] (Proposition 1).

In Chaper 3 we consider the case where L_1 and L_2 have the same leading order μ and both fluctuations live in the scale $t^{1/3}$. This is our first assumption for the asymptotic independence to hold.

Assumption 1. *Assume that there exists some μ such that*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}^+ \to (\eta t,t)} - \mu t}{t^{1/3}} \leq s \right) = G_1(s), \tag{2.24}$$

and

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}^- \to (\eta t,t)} - \mu t}{t^{1/3}} \leq s \right) = G_2(s), \tag{2.25}$$

where G_1 and G_2 are some distribution functions.

Secondly, we assume that there is a point E^+ at distance of order t^ν, for some $1/3 < \nu < 1$, which lies on the characteristic line from $\mathcal{L}^+$ to E and that there is slow-decorrelation as in Theorem 2.1 of [29].

Assumption 2. *Assume that we have a point* $E^+ = (\eta t - \kappa t^\nu, t - t^\nu)$ *such that for some* μ_0*, and* $\nu \in (1/3, 1)$ *it holds*

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}\left(\frac{L_{E^+\to(\eta t,t)} - \mu_0 t^\nu}{t^{\nu/3}} \leq s\right) &= G_0(s), \\ \lim_{t\to\infty} \mathbb{P}\left(\frac{L_{\mathcal{L}^+\to E^+} - \mu t + \mu_0 t^\nu}{t^{1/3}} \leq s\right) &= G_1(s), \end{aligned} \tag{2.26}$$

where G_0 *and* G_1 *are distribution functions.*

Then, provided (2.24) and (2.26) hold, Theorem 2.1 of [29] implies that for any $M > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(|L_{\mathcal{L}^+\to(\eta t,t)} - L_{\mathcal{L}^+\to E^+} - \mu_0 t^\nu| \geq M t^{1/3}\right) = 0. \tag{2.27}$$

This means that the fluctuations of $L_{\mathcal{L}^+\to(\eta t,t)}$ are the same as the ones of $L_{\mathcal{L}^+\to E^+}$ up to $o(t^{1/3})$. Thus, we have to determine the maximum of $L_{\mathcal{L}^+\to E^+}$ and $L_{\mathcal{L}^-\to E}$. The final assumption ensures that these two random variables are asymptotically independent.

Assumption 3. *Let* ν *be as in Assumption 2. Consider the points* $D_\gamma = (\lfloor \gamma\eta t\rfloor, \lfloor \gamma t\rfloor)$ *with* $\gamma \in [0, 1 - t^{\beta-1}]$*. Assume that there exists a* $\beta \in (0, \nu)$*, such that*

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma \\ \gamma\in[0,1-t^{\beta-1}]}} \left\{ D_\gamma \in \pi^{\max}_{L_{\mathcal{L}^+\to E^+}} \right\} \Bigg) &= 0, \\ \lim_{t\to\infty} \mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma \\ \gamma\in[0,1-t^{\beta-1}]}} \left\{ D_\gamma \in \pi^{\max}_{L_{\mathcal{L}^-\to(\eta t,t)}} \right\} \Bigg) &= 0. \end{aligned} \tag{2.28}$$

Under these assumptions, we have the following Theorem.

Theorem 2.7 (Proven in Section 3.2)**.** *Under Assumptions 1–3 we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\max\left\{L_{\mathcal{L}^+\to(\eta t,t)}, L_{\mathcal{L}^-\to(\eta t,t)}\right\} - \mu t}{t^{1/3}} \leq s\right) = G_1(s) G_2(s), \tag{2.29}$$

whenever G_1, G_2 *are continuous at* s.

2.4 Critical Scaling

In the setting of Theorem 2.4 we are interested how the transition from the flat case ($\alpha = 1$) and the shock ($\alpha < 1$) occurs. Especially, we would like to know how the statistics behave as α moves away from 1. For this, we take

$$\alpha = 1 - 2a(t/2)^{1/3}, \quad a \in \mathbb{R}. \tag{2.30}$$

Of course, the case $a > 0$ corresponds to having a (microscopic) shock, whereas $a < 0$ corresponds to a rarefaction fan. We refer to (2.30) as a critical scaling, since we have a strong correlation of the LPP times which are independent for $1 - \alpha = \mathcal{O}(1)$ but at the same time, the situation is different from the flat case. As discussed in Section 4.1.1, a similar critical scaling, where particles start from $\mathbb{Z}_-$ but the first n particles have jump rate α has been considered in the context of last passage percolation. In the large time t limit, the distribution function of a particle that is around the origin at time t has a BBP distribution function [8, 25]. The results we obtain are the following.

In the critical scaling, as we will prove, the correlation length is again $\mathcal{O}(t^{2/3})$. So (compare with the scaling in Theorem 2.4) scaling so as to be at the microscopic shock with the correct correlation length leads to

$$n(u,t) = \left\lfloor \frac{t}{4} + (a+u)(t/2)^{2/3} \right\rfloor, \quad x(u,t) = \left\lfloor -2(a+u)(t/2)^{2/3} \right\rfloor, \tag{2.31}$$

We define the accordingly scaled particle position process by

$$u \mapsto X_t(u) = \frac{x_{n(u,t)} - x(u,t)}{-(t/2)^{1/3}}. \tag{2.32}$$

We show that $X_t(u)$ converges to a new limit process $\mathcal{M}_a$.

Theorem 2.8. *Let $\mathcal{M}_a$ be the limit process given in Definition 4.1 of Chapter 4. It holds*

$$\lim_{t \to \infty} X_t(u) = \mathcal{M}_a(u) \tag{2.33}$$

in the sense of finite dimensional distributions.

We then perform a numerical study to obtain information about the $a \to +\infty$ limit of $\mathbb{P}(\mathcal{M}_a(0) \leq s)$. When $a \to \infty$ one expects to recover the macroscopic shock picture, i.e. that we have

$$\lim_{a \to +\infty} \mathbb{P}(\mathcal{M}_a(0) \leq s) = F_{\mathrm{GOE}}(2^{2/3}s)^2. \tag{2.34}$$

The numerics strongly suggest that this is indeed the case, note the R.H.S. of (2.34) is the $\alpha \to 1$ limit (with $\xi = 0$, and up to a factor $2^{1/3}$ in the argument) of (2.19). A proof of this will be given in an upcoming work of the author.

Chapter 3

Emergence of Independence

In this chapter we will prove the Theorems 2.4, 2.5, 2.6 in Section 3.3.3, the general Theorem 2.7 in Section 3.2 and Theorem 1.27 in Section 3.3.1.3.

3.1 Shocks in LPP, translation to TASEP

Let us start by reformulating Theorems 2.4, 2.5 and 2.6 in terms of LPP. Recall that the general LPP model considered for Theorem 2.7 is given by

$$\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-, \quad \mathcal{A} = E = (\lfloor \eta t \rfloor, \lfloor t \rfloor), \tag{3.1}$$

where $\mathcal{L}^+ \subseteq \{(v,n) \in \mathbb{Z}^2 : v \leq 0, n \geq 0\}$, $\mathcal{L}^- \subseteq \{(v,n) \in \mathbb{Z}^2 : n \leq 0, v \geq 0\}$ and general weights $\omega_{i,j}$.

Let us consider now $\omega_{i,j}$ to be exponentially distributed random variables, that will become waiting times for TASEP particles. Let the waiting times be given by

$$\begin{aligned} \omega_{i,j} &\sim \exp(1), \quad j \geq 1, \\ \omega_{i,j} &\sim \exp(\alpha), \quad j \leq 0, \end{aligned} \tag{3.2}$$

for some $\alpha > 0$. We are going to consider the scaling

$$\eta = \eta_0 + u t^{-2/3}. \tag{3.3}$$

Then the following results hold true and will be proven in Section 3.3.3, see Figure 3.1 for an illustration of the geometry in the following three results.

Theorem 3.1 (Two point-to-line problems, LPP version of Theorem 2.4). *Let*

$$\mathcal{L}^+ = \{(-v,v), v \in \mathbb{Z}_+\}, \quad \mathcal{L}^- = \{(-v,v), v \in \mathbb{Z}_-\}, \tag{3.4}$$

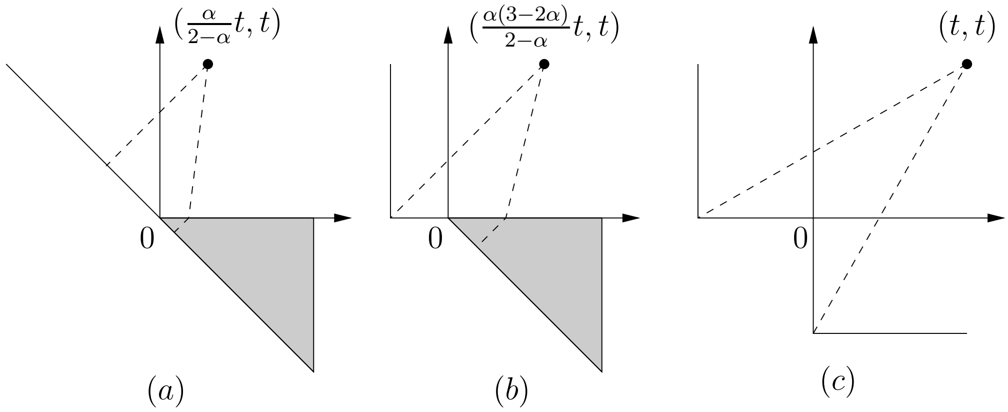

Figure 3.1: Illustration of the geometry considered in (a) Theorem 3.1, (b) Theorem 3.2, and (c) Theorem 3.3, for $u = b = 0$ and $\alpha = 1/2$. The random variables in the gray (resp. white) regions are $\exp(\alpha)$ (resp. $\exp(1)$) distributed. The dashed lines represents the typical trajectories of the maximizers for the two LPP problems.

with $\eta_0 = \frac{\alpha}{2-\alpha}$ *and* $\alpha < 1$. *Then, Theorem 2.7 holds with* $\mu = 4/(2-\alpha)$ *and*

$$G_1(s) = F_{\rm GOE}\left(\frac{s-2u}{\sigma_1}\right), \quad G_2(s) = F_{\rm GOE}\left(\frac{s-2u/\alpha}{\sigma_2}\right), \tag{3.5}$$

where $\sigma_1 = \frac{2^{2/3}}{(2-\alpha)^{1/3}}$ *and* $\sigma_2 = \frac{2^{2/3}(2-2\alpha+\alpha^2)^{1/3}}{\alpha^{2/3}(2-\alpha)}$.

Theorem 3.2 (One point-to-point and one point-to-line problem, LPP version of Theorem 2.5)**.** *Let*

$$\mathcal{L}^+ = ([-\lfloor \beta t\rfloor, 0], 0) \cup (-\lfloor \beta t\rfloor, \mathbb{Z}_+), \quad \mathcal{L}^- = \{(-v, v), v \in \mathbb{Z}_-\}, \tag{3.6}$$

with $\beta = \beta_0 + bt^{-2/3}$, $\beta_0 = 1 - \eta_0$, $\eta_0 = \frac{\alpha(3-2\alpha)}{2-\alpha}$ *and* $\alpha \in (0,1)$. *Then, Theorem 2.7 holds with* $\mu = 4$ *and*

$$G_1(s) = F_{\rm GUE}\left(\frac{s-2(u+b)}{\sigma_1}\right), \quad G_2(s) = F_{\rm GOE}\left(\frac{s-2u/\alpha}{\sigma_2}\right), \tag{3.7}$$

where $\sigma_1 = 2^{4/3}$, *and* $\sigma_2 = \frac{2^{2/3}(6-10\alpha+6\alpha^2-\alpha^3)^{1/3}}{\alpha^{2/3}(2-\alpha)}$.

Theorem 3.3 (Two point-to-point problems, LPP version of Theorem 2.6)**.** *Let us fix a* $\beta > 0$ *and consider*

$$\mathcal{L}^+ = (-\lfloor \beta t\rfloor, \mathbb{Z}_+) \cup ([-\lfloor \beta t\rfloor, 0], 0), \quad \mathcal{L}^- = (0, [0, -\lfloor \beta t\rfloor]) \cup (\mathbb{Z}_+, -\lfloor \beta t\rfloor), \tag{3.8}$$

with $\eta_0 = 1$ *and* $\alpha = 1$. *Then, Theorem 2.7 holds with* $\mu = (1+\sqrt{1+\beta})^2$ *and*

$$G_1(s) = F_{\rm GUE}\left(\frac{s-u(1+1/\sqrt{1+\beta})}{\sigma}\right), \quad G_2(s) = F_{\rm GUE}\left(\frac{s-u(1+\sqrt{1+\beta})}{\sigma}\right), \tag{3.9}$$

where $\sigma = (1+\sqrt{1+\beta})^{4/3}/(1+\beta)^{1/6}$.

3.1.1 Application to the totally asymmetric simple exclusion process

Let us shortly explain how to obtain Theorems 2.4 - 2.6 from Theorems 3.1 -3.3. Recall the link from TASEP to LPP: We assign as usual to each particle a number and do it from right to left, i.e.

$$\ldots < x_2(0) < x_1(0) < 0 \le x_0(0) < x_{-1}(0) < \cdots .$$

If we take $\mathcal{L} = \{(u,k) \in \mathbb{Z}^2 : u = k + x_k(0), k \in \mathbb{Z}\}$ and let $\omega_{i,j}$ be the exponential waiting time of particle j, then

$$\mathbb{P}\left(L_{\mathcal{L}\to(m,n)} \le t\right) = \mathbb{P}\left(x_n(t) + n \ge m\right). \tag{3.10}$$

This will be used several times to verify that Assumptions 1–3 of Theorem 2.7 hold in special cases.

Thus, the particular choice of the $\omega_{i,j}$ in (3.2) means that particles with label $n \ge 1$ have jump rate 1, while particles with label $n \le 0$ have jump rate α. The choice (3.3) implies that we look at particle number t at different times. Indeed, if

$$\lim_{t\to\infty} \mathbb{P}\left(L_{\mathcal{L}\to(\eta_0 t + u t^{1/3}, t)} \le \mu t + s t^{1/3}\right) = F(u,s), \tag{3.11}$$

then by (3.10) we have that

$$\lim_{t\to\infty} \mathbb{P}\left(x_t(\mu t + \tau t^{1/3}) \ge (\eta_0 - 1)t - s t^{1/3}\right) = F(-s,\tau). \tag{3.12}$$

Since this relation is straightforward we did not restate Theorems 3.1 - 3.3 for the tagged particle problem. Instead, we stated them in Theorems 2.4 - 2.6 so that they give the distribution function at a fixed time t of particles around the shock, which is achieved as follows.

In the case of Theorems 3.2 - 3.3, the boundaries of the LPP problem to $(\eta t, t)$ also depend on the variable t. This has to be taken in account here too. Therefore, let us write explicitly this dependence in the measure and just write $L_{m,n}$ for the last passage time. For the case of Theorem 3.1, the boundary condition does not depend on the observation time parameter t. For this case, one can just set $\beta = 0$ in the computations below. Assume that we have, as in the previous section,

$$\lim_{t\to\infty} \mathbb{P}_{\beta t}(L_{\eta_0 t + u t^{1/3}, t} \le \mu(\beta)t + s t^{1/3}) = F(\beta, u, s). \tag{3.13}$$

By (3.10) we have

$$\mathbb{P}_{\beta t}(x_{\nu t + \xi t^{1/3}}(t) \ge v t - s t^{1/3}) = \mathbb{P}_{\beta t}(L_{(\nu+v)t + (\xi - s)t^{1/3}, \nu t + \xi t^{1/3}} \le t) \tag{3.14}$$

Let us define $\tilde{t}$, η, and $\tilde{\beta}$ by the equations

$$\tilde{t} = \nu t + \xi t^{1/3}, \quad \eta\tilde{t} = (\nu + v)t + (\xi - s)t^{1/3}, \quad \tilde{\beta}\tilde{t} = \beta t. \tag{3.15}$$

This gives, $t = \tilde{t}/\nu - \xi\nu^{-4/3}\tilde{t}^{1/3} + \mathcal{O}(\tilde{t}^{-1/3})$, from which

$$\begin{aligned} \eta &= (1 + v/\nu) - (s + \xi v/\nu)\nu^{-1/3}\tilde{t}^{-2/3}, \\ \tilde{\beta} &= \beta/\nu - \xi\beta\nu^{-4/3}\tilde{t}^{-2/3}, \end{aligned} \tag{3.16}$$

up to $\mathcal{O}(\tilde{t}^{-4/3})$. By plugging this in (3.14) one readily obtains

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}_{\beta t}(x_{\nu t+\xi t^{1/3}}(t) \geq vt - st^{1/3}) &= \lim_{\tilde{t}\to\infty} \mathbb{P}_{\tilde{\beta}\tilde{t}}(L_{\eta_0\tilde{t}+u\tilde{t}^{1/3},\tilde{t}} \leq \tilde{t}/\nu - \xi\nu^{-4/3}\tilde{t}^{1/3}) \\ &= \lim_{\tilde{t}\to\infty} \mathbb{P}_{\tilde{\beta}\tilde{t}}(L_{\eta_0\tilde{t}+u\tilde{t}^{1/3},\tilde{t}} \leq \mu(\tilde{\beta})\tilde{t} + \tilde{s}\tilde{t}^{1/3}) \end{aligned} \tag{3.17}$$

with

$$\eta_0 = 1 + v/\nu, \quad u = -(s + \xi v/\nu)\nu^{-1/3}, \quad \tilde{s} = \xi(\beta\mu'(\beta/\nu) - 1)\nu^{-4/3}. \tag{3.18}$$

provided that it holds $\mu(\beta/\nu) = 1/\nu$. This condition sets which particles are around the shock position at time t. Then, by (3.14) we have

$$\lim_{t\to\infty} \mathbb{P}_{\beta t}(x_{\nu t+\xi t^{1/3}}(t) \geq vt - st^{1/3}) = F\left(\frac{\beta}{\nu}, -\frac{s + \xi v/\nu}{\nu^{1/3}}, \frac{\xi(\beta\mu'(\beta/\nu) - 1)}{\nu^{4/3}}\right). \tag{3.19}$$

3.1.1.1 Second class particles and Competition Interface

Let us describe shortly, based on [33], how the alternative interpretation (for TASEP) of the shock via second class particles fits in the framework of LPP . For precise definitions and results, see [33]. Consider $\mathcal{L}$ as in (3.1). Put a point z in the cluster Γ^1_∞ if $L_{\mathcal{L}^+\to z} < L_{\mathcal{L}^-\to z}$ and in cluster Γ^2_∞ if $L_{\mathcal{L}^-\to z} < L_{\mathcal{L}^+\to z}$. We assume all used LPP times exist (i.e. there is finite positive number of up-right paths to z) and are a.s. not identical. The two clusters are seperated by a line $(\phi_n)_{n\in\mathbb{N}}$, the competition interface, defined to start in the origin. Note the difference between $L_{\mathcal{L}^-\to\phi_n}$ and $L_{\mathcal{L}^+\to\phi_n}$ is very small. Define now $\tau_n = L_{\mathcal{L}\to\phi_n}$ and set

$$(I(t), J(t)) := \phi_n \quad \text{if } t \in [\tau_n, \tau_{n+1}). \tag{3.20}$$

Consider now TASEP with initial data η_0 with $\eta_0(0) = 0, \eta_0(1) = 1$. Choose now $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^+$ and the weights according to η_0 (see (2.20) in [33]). In Proposition 2.2 of [33], the authors show that there is a coupling under which the second class particle $X(t)$ (initially placed at the origin, all other sites initially occupied according to η_0) has the same trajectory as $I(t) - J(t)$. The second class particle, i.e. $I(t) - J(t)$ has the same speed as the shock for Riemann initial data (Theorem 3 in [33]). Note now that by Assumption 1 we consider η for which $L_{\mathcal{L}^+\to(\eta t,t)}$ and $L_{\mathcal{L}^-\to(\eta t,t)}$ have the same leading order. Hence the competition interface needs to have the same asymptotic direction $(\eta t, t)$, but it has random fluctuations on the $o(t)$ scale around it.

3.2 Proof of Theorem 2.7

In the following, we will several times use the following two lemmas from [14]. By " $\Rightarrow$ " we designate convergence in distribution.

Lemma 3.4 (Lemma 4.1 in [14]). *Let D be a probability distribution and $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. If $X_n \geq \tilde{X}_n$ and $X_n \Rightarrow D$ and $X_n - \tilde{X}_n$ converges to zero in probability, then $\tilde{X}_n \Rightarrow D$ as well.*

Lemma 3.5 (Lemma 4.2 in [14]). *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$, $(\tilde{X}_n)_{n\in\mathbb{N}}$, $(\tilde{Y}_n)_{n\in\mathbb{N}}$ be sequences of random variables and D_1, D_2, D_3 be probability distributions. Assume $X_n \geq \tilde{X}_n$ and $X_n \Rightarrow D_1$ as well as $\tilde{X}_n \Rightarrow D_1$; and similarly $Y_n \geq \tilde{Y}_n$ and $Y_n \Rightarrow D_2$ as well as $\tilde{Y}_n \Rightarrow D_2$. Let $Z_n = \max\{X_n, Y_n\}$ and $\tilde{Z}_n = \max\{\tilde{X}_n, \tilde{Y}_n\}$. Then if $\tilde{Z}_n \Rightarrow D_3$, we also have $Z_n \Rightarrow D_3$.*

We denote

$$L^{\text{resc}}_{\mathcal{L}^+\to E} = \frac{L_{\mathcal{L}^+\to(\eta t,t)} - \mu t}{t^{1/3}}, \tag{3.21}$$

i.e. the last passage time $L_{\mathcal{L}^+\to E}$ rescaled as required by Assumption 1, we define analogously $L^{\text{resc}}_{\mathcal{L}^-\to E}, L^{\text{resc}}_{E^+\to(\eta t,t)}$ and $L^{\text{resc}}_{\mathcal{L}^+\to E^+}$ as the last passage times rescaled as required by Assumption 1 resp. 2. We first note the following.

Proposition 3.6. *If $\max\{L^{\text{resc}}_{\mathcal{L}^+\to E}, L^{\text{resc}}_{\mathcal{L}^-\to E}\} \Rightarrow D$ as $t \to \infty$, then*

$$\frac{L_{\mathcal{L}\to E} - \mu t}{t^{1/3}} \Rightarrow D. \tag{3.22}$$

Proof. Simply note that $L_{\mathcal{L}\to E} = \max\{L_{\mathcal{L}^+\to E}, L_{\mathcal{L}^-\to E}\}$. □

Thus it suffices to determine the limiting distribution of $\max\{L^{\text{resc}}_{\mathcal{L}^+\to E}, L^{\text{resc}}_{\mathcal{L}^-\to E}\}$. We can actually reduce our problem a bit more.

Proposition 3.7. *Under Assumptions 1 and 2,*

$$\max\left\{\frac{L_{\mathcal{L}^+\to E^+} + L_{E^+\to E} - \mu t}{t^{1/3}}, L^{\text{resc}}_{\mathcal{L}^-\to E}\right\} \Rightarrow D \tag{3.23}$$

implies

$$\frac{L_{\mathcal{L}\to E} - \mu t}{t^{1/3}} \Rightarrow D. \tag{3.24}$$

Proof. We have

$$L^{\text{resc}}_{\mathcal{L}^+\to E} \geq \frac{L_{\mathcal{L}^+\to E^+} - \mu t + \mu_0 t^\nu}{t^{1/3}} + \frac{L_{E^+\to E} - \mu_0 t^\nu}{t^{1/3}} = L^{\text{resc}}_{\mathcal{L}^+\to E^+} + L^{\text{resc}}_{E^+\to(\eta t,t)}. \tag{3.25}$$

By Assumption 2, $L^{\mathrm{resc}}_{\mathcal{L}^+\to E^+}$ converges to G_1. Also by Assumption 2, $L_{E^+\to E}$ has fluctuations of order $t^{\nu/3}$, thus one can write

$$L^{\mathrm{resc}}_{E^+\to E} = \frac{1}{t^{(1-\nu)/3}} X_t, \tag{3.26}$$

where X_t is a random variable converging to G_0. In particular, (3.26) vanishes as $t\to\infty$. Applying Lemma 3.5 to $X_n = L^{\mathrm{resc}}_{\mathcal{L}^+\to E}$, $\tilde{X}_n = \left(L_{\mathcal{L}^+\to E^+} + L_{E^+\to E} - \mu t\right)/t^{1/3}$ and $Y_n = \tilde{Y}_n = L^{\mathrm{resc}}_{\mathcal{L}^-\to E}$ finishes the proof. □

Using the preceeding Propositions we can now prove Theorem 2.7.

Proof of Theorem 2.7. In the proof, we assume throughout that G_1, G_2 are continuous at s. Define, for some set B and point C, $\tilde{L}_{B\to C}$ to be the last passage time of all paths from B to C conditioned not to contain any point $\bigcup_{\gamma\in[0,1-t^{\beta-1}]} D_\gamma$ with D_γ as in Assumption 3. Then,

$$\mathbb{P}\left(\left|\frac{L_{\mathcal{L}^+\to E^+} - \tilde{L}_{\mathcal{L}^+\to E^+}}{t^{1/3}}\right| > \varepsilon\right) \le \mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma\\ \gamma\in[0,1-t^{\beta-1}]}} \{D_\gamma \in \pi^{\max}_{\mathcal{L}^+\to E^+}\}\Bigg) \to 0 \tag{3.27}$$

as $t\to\infty$, so that

$$\mathbb{P}\left(\frac{\tilde{L}_{\mathcal{L}^+\to E^+} + \tilde{L}_{E^+\to E} - \mu t}{t^{1/3}} \le s\right) \to G_1(s) \tag{3.28}$$

by the vanishing of (3.26) and Lemma 3.4. Using Assumptions 1 and 3, an analogous argument shows

$$\mathbb{P}\big(\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\big) \to G_2(s). \tag{3.29}$$

Let $\varepsilon > 0$ and recall X_t from (3.26). We take $R > 0$ such that with $A_R = \{|\tilde{X}_t| < R\}$ $\mathbb{P}(A_R^c) \le \varepsilon$ for all t large enough. This implies that

$$\begin{aligned}\big|&\mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} + t^{(\nu-1)/3}\tilde{X}_t \le s\} \cap A_R \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\})\\ &-\mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} + t^{(\nu-1)/3}\tilde{X}_t \le s\} \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\})\big| \le \varepsilon.\end{aligned} \tag{3.30}$$

Then,

$$\mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} + t^{(\nu-1)/3}R \le s\} \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\}) - \varepsilon \tag{3.31}$$

$$\le \mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} + t^{(\nu-1)/3}\tilde{X}_t \le s\} \cap A_R \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\}) \tag{3.32}$$

$$\le \mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} - t^{(\nu-1)/3}R \le s\} \cap A_R \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\}) \tag{3.33}$$

$$\le \mathbb{P}(\{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^+\to E^+} - t^{(\nu-1)/3}R \le s\} \cap \{\tilde{L}^{\mathrm{resc}}_{\mathcal{L}^-\to E} \le s\}). \tag{3.34}$$

Finally, by construction, $\tilde{L}^{\text{resc}}_{\mathcal{L}^+\to E^+}$ and $\tilde{L}^{\text{resc}}_{\mathcal{L}^-\to E}$ are independent random variables, since $\beta < \nu$ and $\pi^{\max}_{\mathcal{L}^-\to E}$ has to pass to the right of $D_{1-t^{\beta-1}}$ by conditioning. Due to this independence, the fact that $\nu < 1$ and the convergence in (3.28), (3.29), there is a t_0 such that for $t > t_0$

$$G_1(s)G_2(s) - 2\varepsilon \leq (3.31) \leq (3.32) \leq (3.34) \leq G_1(s)G_2(s) + \varepsilon. \tag{3.35}$$

Thus applying (3.30) to (3.32) yields

$$\left| \mathbb{P}\left(\{\tilde{L}^{\text{resc}}_{\mathcal{L}^+\to E^+} + t^{(\nu-1)/3}\tilde{X}_t \leq s\} \cap \{\tilde{L}^{\text{resc}}_{\mathcal{L}^-\to E} \leq s\} \right) - G_1(s)G_2(s) \right| \leq 3\varepsilon, \tag{3.36}$$

for all t large enough. Therefore

$$\lim_{t\to\infty} \mathbb{P}\left(\max\left\{ \frac{\tilde{L}_{\mathcal{L}^+\to E^+} + \tilde{L}_{E^+\to E} - \mu t}{t^{1/3}}, \tilde{L}^{\text{resc}}_{\mathcal{L}^-\to E} \right\} \leq s \right) = G_1(s)G_2(s). \tag{3.37}$$

Applying Lemma 3.5 to $X_n = \left(L_{\mathcal{L}^+\to E^+} + L_{E^+\to E} - \mu t\right)/t^{1/3}$, $Y_n = L^{\text{resc}}_{\mathcal{L}^-\to E}$, $\tilde{X}_n = \left(\tilde{L}_{\mathcal{L}^+\to E^+} + \tilde{L}_{E^+\to E} - \mu t\right)/t^{1/3}$, $\tilde{Y}_n = \tilde{L}^{\text{resc}}_{\mathcal{L}^-\to E}$, and using Proposition 3.7 finishes the proof. □

3.3 Results on specific LPP

In this section we derive some results on the LPP model with

$$\begin{aligned} \omega_{i,j} &\sim \exp(1), \quad j \geq 1, \\ \omega_{i,j} &\sim \exp(\alpha), \quad j \leq 0, \end{aligned} \tag{3.38}$$

and with two half-lines given by

$$\mathcal{L}^+ = \{(-v, v) | v \in \mathbb{Z}_+\} \text{ and } \mathcal{L}^- = \{(-v, v) | v \in \mathbb{Z}_-\}. \tag{3.39}$$

Assumptions 1-2 will be verified by using the results of Section 3.3.1. After that, in Section 3.3.2 we determine the no-crossing results corresponding to Assumption 3.

3.3.1 Deviation Results for LPP

3.3.1.1 Point-to-point LPP results

First we restate Proposition 1.33 in a slightly more general way and introduce some notation. Namely, by symmetry of the LPP one easily extends Proposition 1.33 to any $\eta > 0$ and thus obtains the following.

Proposition 3.8 (Point-to-point LPP: convergence to $F_{\rm GUE}$, Theorem 1.6 in [45])**.** *Let $0<\eta<\infty$. Then,*

$$\lim_{\ell\to\infty}\mathbb{P}\left(L_{0\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)}\leq\mu_{\rm pp}\ell+s\sigma_\eta\ell^{1/3}\right)=F_{\rm GUE}(s)\tag{3.40}$$

where $\mu_{\rm pp}=(1+\sqrt{\eta})^2$, and $\sigma_\eta=\eta^{-1/6}(1+\sqrt{\eta})^{4/3}$.

The distribution function of $L_{0\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)}$ has the following known decay, the following two propositions are modifications of Theorem 1.34 (which is part of Theorem 1.6 in [45]). [1]

Proposition 3.9 (Point-to-point LPP: upper tail)**.** *Let $0<\eta<\infty$. Then for given $\ell_0>0$ and $s_0\in\mathbb{R}$, there exist constants $C,c>0$ only dependent on ℓ_0,s_0 such that for all $\ell\geq\ell_0$ and $s\geq s_0$ we have*

$$\mathbb{P}\left(L_{0\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)}>\mu_{\rm pp}\ell+\ell^{1/3}s\right)\leq C\exp(-cs),\tag{3.41}$$

where $\mu_{\rm pp}=(1+\sqrt{\eta})^2$.

Proof. By symmetry, it is enough to consider $\eta\in(0,1]$. Also, we will (re)derive the statement for the complementary event. As stated in Proposition 6.1 of [8], we have

$$\mathbb{P}(\lambda_1(m-d,m+d)\leq u)=\mathbb{P}\left(L_{0\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)}\leq u\right),\tag{3.42}$$

where λ_1 is the largest eigenvalue of a $(m-d)\times(m+d)$ Laguerre Unitary Ensemble (LUE), i.e., the largest eigenvalue of $\frac{1}{m-d}XX^*$, where X is a $(m-d)\times(m+d)$ matrix with i.i.d. standard complex Gaussian entries; the choice of parameters is so that $m+d=\lfloor\eta\ell/\mu_{\rm pp}\rfloor$ and $m-d=\lfloor\ell/\mu_{\rm pp}\rfloor$ (explicitly, one might take $m=\lfloor\frac{\ell(\eta+1)}{2\mu_{\rm pp}}\rfloor$ and $d=\lfloor\frac{\ell(1-\eta)}{2\mu_{\rm pp}}\rfloor$, but then these identites might only hold with an error ±1). Take $K_{m,d}$ to be the kernel (3.13) of [40] (with $w=0$), which, according to Proposition C.1 of [40], is a conjugated correlation kernel for the LUE. Then, with $\chi_u=\mathbf{1}_{(u,+\infty)}$

$$F(u):=\det(1-\chi_uK_{m,d}\chi_u)=\mathbb{P}(\lambda_1(m-d,m+d)\leq u).\tag{3.43}$$

Define the function $u(s,\ell)=\ell-s\ell^{1/3}$. The decay of $F(u)$ is known, see (37) in [11]; more precisely we have with $C,d>0$ dependent on $s_0\in\mathbb{R}$ and $\ell_0>0$

$$1-Ce^{-ds}\leq F(u(s,\ell))\tag{3.44}$$

for $\ell>\ell_0$ and $s>s_0$. Making the change of variable $\ell\to\mu_{\rm pp}\ell$, (3.41) follows with $c=d/\mu_{\rm pp}^{1/3}$. ☐

[1] One could improve the decay of Proposition 3.9 to $\exp(-cs^{3/2})$ and of Proposition 3.10 to $\exp(-c|s|^3)$, but it is not needed for our purposes.

Proposition 3.10 (Point-to-point LPP: lower tail)**.** *Let* $0 < \eta < \infty$ *and* $\mu_{\rm pp} = (1+\sqrt{\eta})^2$. *There exist positive constants* s_0, ℓ_0, C, c *such that for* $s \leq -s_0$, $\ell \geq \ell_0$,

$$\mathbb{P}\left(L_{0\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} \leq \mu_{\rm pp}\ell + s\ell^{1/3}\right) \leq C\exp(-c|s|^{3/2}). \tag{3.45}$$

Proof. Take the functions $F, u(s,t)$ and the parameters w, m, d as in the proof of Proposition 3.9. Proposition 3 of [11] (to be found in the proof of Proposition 2 of [11]) and the inequality (56) of the same paper imply that there exist positive constants s_0, t_0, C, c such that

$$F(u(s,t)) \leq C\exp(-c|s|^{3/2}), \tag{3.46}$$

for all $s \leq -s_0$ and $t \geq t_0$. □

3.3.1.2 Half-line $\mathcal{L}^+$-to-point LPP results

To obtain the results for the LPP from the half-line $\mathcal{L}^+$ to a point $(\eta\ell,\ell)$, we use the correspondence of LPP and TASEP, namely

$$\mathbb{P}\left(L_{\mathcal{L}^+\to(m,n)} \leq t\right) = \mathbb{P}\left(x_n(t)+n \geq m\right), \tag{3.47}$$

where $x_n(t)$ is the position at time t of the TASEP particle that started from $x_n(0) = -2n$ in the initial configuration where particles occupy $-2\mathbb{N}_0$. TASEP particle have all jump rate 1. The latter distribution function is expressed as a Fredholm determinant of a kernel $\hat{K}_{n,t}$, as is shown in [22]. This is the finite time kernel for the half-flat initial data (1.88), here we will look at the region where the fluctuations of particle positions are governed by the $F_{\rm GOE}$ distribution.

Proposition 3.11 (Proposition 3 in [22])**.** *Let particle number* $n \in \mathbb{N}_0$ *start in* $-2n$ *at time* $t = 0$. *Denote by* $x_n(t)$ *the position of particle number* n *at time* t. *We then have*

$$\mathbb{P}(x_n(t) > s) = \det(1-\chi_s\hat{K}_{n,t}\chi_s)_{\ell^2(\mathbb{Z})} \tag{3.48}$$

where $\chi_s = \mathbf{1}_{(-\infty,s]}$ *and* $\hat{K}_{n,t}$ *is given by*[2]

$$\begin{aligned}\hat{K}_{n,t}(x_1,x_2) = \frac{1}{(2\pi i)^2}\oint_{\Gamma_1}\mathrm{d}v\oint_{\Gamma_{0,1-v}}\frac{\mathrm{d}w}{w}\frac{e^{tw}(w-1)^n}{w^{x_1+n}}\frac{v^{x_2+n}}{e^{tv}(v-1)^n}\\ \times\frac{2v-1}{(w+v-1)(w-v)}.\end{aligned} \tag{3.49}$$

To get a bound for the upper tail we need to have the following estimate of the decay of the kernel.

[2] For a set S, the notation Γ_S means a path anticlockwise oriented enclosing only poles of the integrand belonging to the set S.

Proposition 3.12 (Exponential decay $\hat{K}_{n,t}$)**.** *Consider the scaling*

$$n(t) = \left[\frac{r}{4}t\right] \quad x_i = \left[\frac{1-r}{2}t - s_i t^{1/3}\right], \tag{3.50}$$

for some $r > 1$. With this choice, there exists a constant C and a t_0 such that for $t > t_0$ and $s_1, s_2 \geq 0$

$$|\hat{K}_{n,t}(x_1, x_2) t^{1/3} 2^{x_2 - x_1} e^{-(s_2 - s_1)/2}| \leq C\, e^{-(s_1+s_2)/2}. \tag{3.51}$$

Proof of Proposition 3.12. Below we will show that for t large enough, there are constants $C, \mu(r) > 0$ such that we have uniformly in $s_1, s_2 \geq 0$

$$|\hat{K}_{n,t}(x_1, x_2) t^{1/3} 2^{x_2 - x_1}| \leq C e^{-(s_1+s_2)} + C t^{1/3} e^{-\mu(r)t} e^{s_1 t^{1/3} \ln(2-r)}. \tag{3.52}$$

From this then follows that

$$|\hat{K}_{n,t}(x_1, x_2) t^{1/3} 2^{x_2 - x_1} e^{-(s_2 - s_1)/2}| \leq 2C e^{-(s_1+s_2)/2} \tag{3.53}$$

since $t^{1/3} e^{-\mu(r)t} \leq 1$ and $e^{s_1(t^{1/3}\ln(2-r)+1/2)} \leq 1$ for t large enough (because $\ln(2-r) < 0$).

Therefore, below we need to bound $\hat{K}_{n,t}(x_1, x_2) t^{1/3} 2^{x_2 - x_1}$. We can divide the kernel $\hat{K}_{n,t}$ into the contribution coming from the residue at $w = -v+1$ and the rest. The contribution of this residue is

$$(-1)^{x_1+1} 2^{x_2 - x_1} \frac{t^{1/3}}{2\pi i} \oint_{\Gamma_1} \mathrm{d}v \frac{v^{x_2+2n}}{(1-v)^{x_1+2n+1}} e^{(1-2v)t} \tag{3.54}$$

This kernel was already analyzed in [18]. Indeed, (3.54) is the kernel from Proposition 5.3 in [18] for the special choice of parameters $t_1 = t_2 = T = t$, $L = 0$, and $R = 1$. Our scaling also fits in the one from (2.9) in [18]; take $\pi(\theta) = r/4 + \theta$ and θ to be the solution of $r/4 + 2\theta = 1$, i.e., $\theta = 1/2 - r/8$. Then (2.9) in [18] equals exactly (3.50). Said Proposition yields now that for any $(s_1, s_2) \in [-l, \infty)^2$ we have

$$|(3.54)| \leq \mathrm{const}\, e^{-(s_1+s_2)}. \tag{3.55}$$

Let us deal now with the remaining part. Taking $\tilde{s}_i = s_i t^{-2/3}$, we have to bound the kernel

$$\begin{aligned} &2^{x_2 - x_1} \frac{t^{1/3}}{(2\pi i)^2} \oint_{\Gamma_1} \mathrm{d}v \oint_{\Gamma_0} \frac{\mathrm{d}w}{w} \frac{e^{tw}(w-1)^n}{w^{x_1+n}} \frac{v^{x_2+n}}{e^{tv}(v-1)^n} \frac{2v-1}{(w+v-1)(w-v)} \\ =& \frac{t^{1/3}}{(2\pi i)^2} \oint_{\Gamma_1} \mathrm{d}v \oint_{\Gamma_0} \mathrm{d}w \frac{e^{t f_0(w, \tilde{s}_1)}}{e^{t f_0(v, \tilde{s}_2)}} \frac{2v-1}{(w+v-1)(w-v)} \end{aligned} \tag{3.56}$$

with

$$f_0(w, s) = \frac{r}{4}\ln(w-1) + w - \frac{2-r}{4}\ln(w) + s\ln(2w). \tag{3.57}$$

We first note that for $r \geq 2$ the pole at $w = 0$ disappears and thus (3.56) vanishes. We therefore assume $1 < r < 2$ in the following. We now claim that

$$\Gamma_0(t) = \lambda e^{it}, \quad t \in [0, 2\pi) \tag{3.58}$$

is a steep descent path of f_0 for $\lambda = 1 - r/2$. To check the steep descent condition, note

$$\begin{aligned} \mathrm{Re}(f_0(\Gamma_0(t), \tilde{s}_1)) &= \tilde{s}_1 \ln(2\lambda) + \lambda \cos(t) - \frac{2-r}{4} \ln(\lambda) + \frac{r}{4} \ln(|\lambda e^{it} - 1|) \\ &= \tilde{s}_1 \ln(2\lambda) + \lambda \cos(t) - \frac{2-r}{4} \ln(\lambda) + \frac{r}{8} \ln\left(\lambda^2 + 1 - 2\lambda \cos(t)\right). \end{aligned} \tag{3.59}$$

Thus we have

$$\frac{\partial}{\partial t} \mathrm{Re}\left(f_0(\Gamma_0(t), \tilde{s}_1)\right) = -\lambda \sin(t) \left(1 - \frac{r/4}{|\lambda e^{it} - 1|^2}\right), \tag{3.60}$$

which is strictly negative for all $t \in (0, \pi)$ (and strictly positive for $t \in (\pi, 2\pi)$). Indeed, $|\lambda e^{it} - 1| \geq r/2$, from which $1 - \frac{r/4}{|\lambda e^{it}-1|^2} \geq 1 - 1/r > 0$. Thus Γ_0 as chosen above is a steep descent path for f_0 with maximum at $t = 0$.

For Γ_1, we choose

$$\Gamma_1(t) = 1 - \frac{1}{2} e^{it}, \quad t \in [0, 2\pi) \tag{3.61}$$

and we want to show that it is a steep descent path for $-f_0$. We have

$$\begin{aligned} \mathrm{Re}(-f_0(\Gamma_1(t), \tilde{s}_2)) = &-\frac{r}{4} \ln(1/2) + \frac{2-r}{8} \ln\left(5/4 - \cos(t)\right) + \frac{1}{2} \cos(t) \\ &- \tilde{s}_2 \ln(|2 - e^{it}|). \end{aligned}$$

The term $-\tilde{s}_2 \ln(|2 - e^{it}|)$ reaches clearly its maximum at $t = 0$ for any $\tilde{s}_2 \geq 0$. Thus we can focus on the $\tilde{s}_2 = 0$ case. We have

$$\frac{\partial}{\partial t} \mathrm{Re}\left(f_0(\Gamma_1(t), 0)\right) = -\frac{\sin(t)}{2} \left(1 - \frac{2-r}{8} \frac{1}{|1 - \frac{1}{2} e^{it}|^2}\right), \tag{3.62}$$

which is strictly negative for $t \in (0, \pi)$ and strictly positive for $t \in (\pi, 2\pi)$. This follows from $|1 - \frac{1}{2} e^{it}| \geq 1/2$, so that $1 - \frac{2-r}{8} |1 - \frac{1}{2} e^{it}|^{-2} \geq r/2 > 0$. Thus Γ_1 is a steep descent path for $-f_0$ attaining its maximum at $t = 0$.

The paths Γ_0 and Γ_1 are such that the factor $\frac{2v-1}{(w+v-1)(w-v)}$ in (3.56) is uniformly bounded and the length of the paths is also bounded. Therefore, since Γ_0 and Γ_1 are steep descent paths, we get the easy bound

$$|(3.56)| \leq t^{1/3} e^{t(f_0(1-r/2, \tilde{s}_1) - f_0(1/2, \tilde{s}_2))} = t^{1/3} e^{-\mu(r) t} e^{t^{1/3} \ln(2-r) s_1}, \tag{3.63}$$

with $\mu(r) = -\frac{r}{4} \ln(r) - \frac{1-r}{2} + \frac{2-r}{4} \ln(2 - r) > 0$ for all $1 < r < 2$. □

Proposition 3.13. *Fix an* $0 < \eta < 1$ *and let* $\mu = 2(1+\eta)$*. Then, for any* $\varepsilon \in [0, 2(1-\eta))$*, there exists constants* $C, \tilde{c} > 0$ *and* $\ell_0 > 0$ *such that for all* $\ell > \ell_0$

$$\mathbb{P}\left(L_{\mathcal{L}^+\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} > (\mu+\varepsilon/2)\ell\right) \leq C\exp\left(-\tilde{c}\,\varepsilon\ell^{2/3}\right). \tag{3.64}$$

Proof of Proposition 3.13. We follow along the lines of the proof of Theorem 2.5 in Section 5 of [19]. We use the relation (3.47) between LPP and TASEP, in which we set $t := (\mu+\varepsilon/2)\ell$ and denote by $\ell(t) = t/(\mu+\varepsilon/2)$ its inverse function. Then, using this relation and Proposition 3.11, we see that

$$(3.64) = 1 - \mathbb{P}\left(x_{\ell(t)}(t) \geq (\eta-1)\ell(t)\right). \tag{3.65}$$

Let us denote

$$X_t^{\mathrm{resc}} = \frac{x_{\ell(t)}(t) - (-2\ell(t)+t/2)}{-t^{1/3}}. \tag{3.66}$$

Then,

$$\begin{aligned}(3.65) &= 1 - \mathbb{P}\left(X_t^{\mathrm{resc}} \leq \frac{(\eta+1)\ell(t)-t/2}{-t^{1/3}}\right)\\ &= -\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\int \mathrm{d}s_1\cdots\int \mathrm{d}s_m \det[t^{1/3}\hat{K}_{\ell(t),t}([x(s_i)],[x(s_j)])]_{1\leq i,j\leq m}\end{aligned} \tag{3.67}$$

where $x(s) = (-2\ell(t)+t/2) - st^{1/3}$ and the integration domain of the s_i's is $(\varepsilon t^{2/3}/4(\mu+\varepsilon/2),\infty)$. On (3.67) we apply Proposition 3.12 with $r = 4/(\mu+\varepsilon/2)$.

We can thus single out a product $\prod_{i=1}^m e^{-s_i}$ of the determinant, so that the absolute value of all entries in the matrix is bounded by a constant C, so using Hadamard's bound, we get

$$\begin{aligned}|(3.67)| &\leq \sum_{m=1}^{\infty}\frac{C^m m^{m/2}}{m!}\int_{\varepsilon t^{2/3}/4(\mu+\varepsilon/2)} \mathrm{d}s_1\cdots\int_{\varepsilon t^{2/3}/4(\mu+\varepsilon/2)} \mathrm{d}s_m\prod_{i=1}^m e^{-s_i}\\ &= \sum_{m=1}^{\infty}\frac{(2C)^m m^{m/2}\exp\left(-m\varepsilon t^{2/3}/4(\mu+\varepsilon/2)\right)}{m!}\\ &\leq \tilde{C}\exp\left(-\varepsilon t^{2/3}/4(\mu+\varepsilon/2)\right) \leq \tilde{C}\exp\left(-\tilde{c}\varepsilon\ell^{2/3}\right)\end{aligned} \tag{3.68}$$

for some constants $\tilde{C}, \tilde{c}$ (uniform in ℓ). ☐

Proposition 3.14 (Half-line $\mathcal{L}^+$-to-point LPP: convergence to F_{GOE})**.** *For any fixed* $0 < \eta < 1$*, it holds*

$$\lim_{\ell\to\infty}\mathbb{P}\left(L_{\mathcal{L}^+\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} \leq \mu\ell + s\tilde{\sigma}_\eta\ell^{1/3}\right) = F_{\mathrm{GOE}}(2s) \tag{3.69}$$

where $\mu = 2(1+\eta)$, $\tilde{\sigma}_\eta = 2^{4/3}(1+\eta)^{1/3}$.

Proof of Proposition 3.14. As in the proof of Proposition 3.13 we use the relation (3.47) between LPP and TASEP, in which we set $t := \mu\ell + s\tilde{\sigma}_\eta \ell^{1/3}$ and denote by

$$\ell(t) = \frac{t}{\mu} - 2s\frac{t^{1/3}}{\mu} + o(1) \tag{3.70}$$

its inverse function. Thus,

$$\mathbb{P}\left(L_{\mathcal{L}^+\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} \leq \mu\ell + s\tilde{\sigma}_\eta \ell^{1/3}\right) = \mathbb{P}\left(x_{\ell(t)}(t) \geq (\eta-1)\ell(t)\right). \tag{3.71}$$

Let us denote

$$X_t^{\rm resc} = \frac{x_{\ell(t)}(t) - (-2\ell(t)+t/2)}{-t^{1/3}}. \tag{3.72}$$

Then,

$$\begin{aligned}(3.71) &= \mathbb{P}\left(X_t^{\rm resc} \leq \frac{(\eta+1)\ell(t)-t/2}{-t^{1/3}}\right)\\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_s^\infty \mathrm{d}s_1\cdots\int_s^\infty \mathrm{d}s_m \det[t^{1/3}\hat{K}_{\ell(t),t}([x(s_i)],[x(s_j)])]_{1\leq i,j\leq m}\end{aligned} \tag{3.73}$$

where $x(s) = (-2\ell(t)+t/2) - st^{1/3}$. The bound of Proposition 3.12 allows us to apply dominated convergence and take the $t\to\infty$ (i.e., $\ell\to\infty$) inside the Fredholm series. Thus it remains to show that the rescaled kernel $t^{1/3}\hat{K}_{\ell(t),t}([x(s_i)],[x(s_j)])$, or a conjugation of it, converges pointwise to the Airy$_1$ kernel $\mathcal{A}_1(s_i,s_j) = \mathrm{Ai}(s_i+s_j)$.

As in Proposition 3.12, we consider the kernel conjugated by the factor $2^{x(s_j)-x(s_i)}$. We can divide the kernel $\hat{K}_{n,t}$ into the contribution coming from (a) the residue at $u=-v+1$ and (b) the rest. The contribution coming from the residue is (3.54), that is, the kernel for the flat initial configuration (all even sites are initially occupied by a particle). It was shown in Theorem 2.3 of [20] (see also Proposition 5.1 of [18]) that the kernel converges pointwise to the Airy$_1$ kernel. The control of the contribution of (b) is already made in the proof of Proposition 3.12. Indeed, the estimate (3.63) implies that this contribution goes to 0 as $t\to\infty$ for all fixed $s\in\mathbb{R}$. This ends the proof of Proposition 3.14, since $\det(1-\mathcal{A}_1)_{L^2(s,\infty)} = F_{\rm GOE}(2s)$ by Proposition 1.18 and the phrase after it. □

A simple corollary of Proposition 3.13 adapted to the problem we are looking at is the following.

Corollary 3.15. *Fix an $0<\eta<1$, a $\beta\in(1/3,1]$ and define*

$$\gamma\in[0,1-t^{\beta-1}],\quad \varepsilon = t^{-\chi} \textit{ with } \chi\in(0,2/3). \tag{3.74}$$

Then there exists constants $C,\tilde{c}>0$ and $t_0>0$ such that for all $t>t_0$

$$\mathbb{P}\left(L_{\mathcal{L}^+\to D_\gamma} > \left(\mu_\gamma + \frac{\varepsilon}{2}\right)t\right) \leq C\exp\left(-\tilde{c}\,t^{2/3-\chi}\right). \tag{3.75}$$

Proof. It is a straightforward consequence of Proposition 3.13. Indeed, setting $\ell = \gamma t$,

$$\begin{aligned}\mathbb{P}\left(L_{\mathcal{L}^+\to D_\gamma} > (\mu_\gamma + \varepsilon/2)\, t\right) &= \mathbb{P}\left(L_{\mathcal{L}^+\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} > (\mu + \varepsilon/(2\gamma))\ell\right) \\ &\leq \mathbb{P}\left(L_{\mathcal{L}^+\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} > (\mu + \varepsilon/2)\ell\right)\end{aligned} \tag{3.76}$$

since $\gamma \in [0,1]$. Then the result is the bound (3.64). ☐

3.3.1.3 Half-line $\mathcal{L}^-$-to-point LPP results, Proof of Theorem 1.27

In this Section we will prove Theorem 1.27. To obtain the results for the LPP from the half-line $\mathcal{L}^-$ to a point $(\eta\ell, \ell)$, we use the correspondence of LPP and TASEP, namely

$$\mathbb{P}\left(L_{\mathcal{L}^-\to(m,n)} \leq t\right) = \mathbb{P}\left(x_n(t) + n \geq m\right), \tag{3.77}$$

where $x_n(t)$ is the position at time t of the TASEP particle with label n. The initial condition is

$$x_n(0) = -n, n \geq 1, \quad x_n(0) = -2n, n \leq 0, \tag{3.78}$$

and the jump rates v_n of particles are given by

$$v_n = 1, n \geq 1, \quad v_n = \alpha, n \leq 0. \tag{3.79}$$

Proposition 3.16. *Let us consider TASEP with jump rates (3.79) and initial condition (3.78). Denote $x_n(t)$ the position of particle number n at time t. We then have*

$$\mathbb{P}(x_n(t) > s) = \det(1 - \chi_s \tilde{K}_{n,t} \chi_s)_{\ell^2(\mathbb{Z})} \tag{3.80}$$

where $\chi_s = \mathbf{1}_{(-\infty,s]}$ and $\tilde{K}_{n,t} = K^{(1)}_{n,t} + K^{(2)}_{n,t}$ with

$$\begin{aligned}K^{(1)}_{n,t}(x_1,x_2) &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \oint_{\Gamma_{0,\alpha-2-w}} \mathrm{d}z \frac{e^{t(w+1)} w^n}{(w+1)^{x_1+n}} \\ &\qquad\qquad \times \frac{(z+1)^{x_2+n}}{e^{t(z+1)} z^n} \frac{1}{z - (\alpha - 2 - w)}, \\ K^{(2)}_{n,t}(x_1,x_2) &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)} w^n}{(w+1)^{x_1+n}} \frac{(z+1)^{x_2+n}}{e^{t(z+1)} z^n} \frac{1}{w - z}.\end{aligned} \tag{3.81}$$

The proof of this proposition is not so short and it is given in Section 3.4 below.

Next we show the point-wise convergence and get bounds for the properly rescaled kernel. Consider the scaling

$$n = \left[\frac{\kappa(2-\alpha)}{4} t\right] \quad x_i = \left[\frac{\alpha - \kappa}{2} t - s_i t^{1/3}\right], \tag{3.82}$$

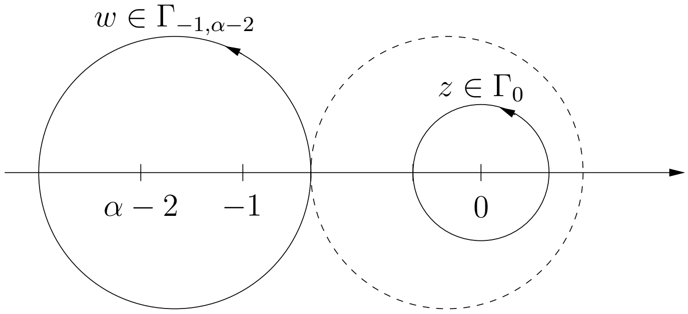

Figure 3.2: Illustration of the paths used in the kernel $K^{(1,a)}_{t,\mathrm{resc}}$. The dashed line is the image of $\alpha-2-w$.

for $\alpha\in[0,1)$ and $\kappa\in[0,1)$. Then, we define the rescaled and conjugated kernels by

$$K^{(i)}_{t,\mathrm{resc}}(s_1,s_2)=t^{1/3}(\alpha/2)^{x_1-x_2}K^{(i)}_{n,t}(x_1,x_2),\quad i=1,2, \tag{3.83}$$

with x_i and n as in (3.82). Before stating the results, let us manipulate the kernel slightly. Denote by $\tilde{s}_i=s_it^{-2/3}$. In particular, we can assume $0\leq\tilde{s}_1\leq\alpha(2-\kappa)/4$, since otherwise the kernel is identically equal to zero. Because of that, the Fredholm determinant in (3.80) is identically equal to zero for $s>\alpha(2-\kappa)t^{2/3}/4$. Therefore, below we can restrict our estimates to $s_1,s_2\leq\alpha(2-\kappa)t^{2/3}/4$ only.

Let us introduce the function

$$f_0(w,\tilde{s})=w+1+\frac{\kappa(2-\alpha)}{4}\ln(w)-\left(\frac{\alpha(2-\kappa)}{4}-\tilde{s}\right)\ln(2(w+1)/\alpha). \tag{3.84}$$

We have

$$K^{(2)}_{t,\mathrm{resc}}(s_1,s_2)=\frac{t^{1/3}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}z\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\frac{e^{tf_0(w,\tilde{s}_1)}}{e^{tf_0(z,\tilde{s}_2)}}\frac{1}{w-z} \tag{3.85}$$

and, separating the contribution of the simple pole at $z=\alpha-2-w$ in $K^{(1)}_{n,t}$,

$$K^{(1)}_{t,\mathrm{resc}}(s_1,s_2)=K^{(1,a)}_{t,\mathrm{resc}}(s_1,s_2)+K^{(1,b)}_{t,\mathrm{resc}}(s_1,s_2) \tag{3.86}$$

where

$$\begin{aligned}K^{(1,a)}_{t,\mathrm{resc}}(s_1,s_2)&=\frac{t^{1/3}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{-1,\alpha-2}}\frac{\mathrm{d}w}{w+1}\oint_{\Gamma_0}\mathrm{d}z\frac{e^{tf_0(w,\tilde{s}_1)}}{e^{tf_0(z,\tilde{s}_2)}}\frac{1}{z-(\alpha-2-w)},\\ K^{(1,b)}_{t,\mathrm{resc}}(s_1,s_2)&=\frac{t^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_{-1,\alpha-2}}\frac{\mathrm{d}w}{w+1}e^{t[f_0(w,\tilde{s}_1)-f_0(\alpha-2-w,\tilde{s}_2)]}.\end{aligned} \tag{3.87}$$

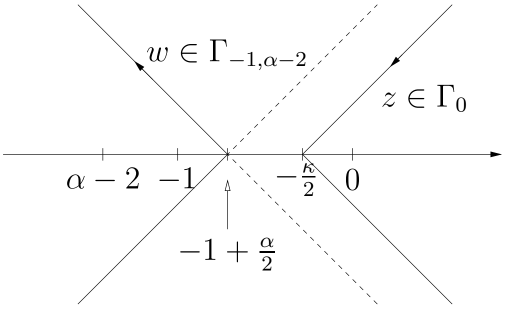

Figure 3.3: Paths used for the asymptotic analysis in Proposition 3.19 and Proposition 3.20. The dashed line is the image of $\alpha - 2 - w$.

Remark 3.17. *$\alpha - 2$ is not a pole for the double integral, but the reason why we have chosen the path for w to encircle also $\alpha-2$ is the following. The function $-f_0(\alpha-2-w, \tilde{s}_2)$ has a pole at $w = \alpha - 2$. Therefore, if, before computing the residue at $z = \alpha - 2 - w$, we choose the path w so that it goes around $\alpha - 2$ too, then, its image by $\alpha - 2 - w$ goes around the origin too, see Figure 3.2. This means that, the path for z in the first term of (3.87) will have to be chosen to stay inside the image of $\alpha - 2 - w$. We could have also chosen to have $\alpha - 2$ outside the path for w, but this is not adequate to get the bounds on the kernel.*

Remark 3.18. *For large $|w|$, the leading term in $f_0(w, \tilde{s})$ is given simply the linear term w. So, we can as well consider (open) contours $\Gamma_{-1,\alpha-2}$ such that the real part of w goes to $-\infty$, and similarly Γ_0 such that the real part of z goes to ∞, see Figure 3.3.*

Proposition 3.19 (Bounds for $K^{(1,a)}_{t,\text{resc}}$ and $K^{(2)}_{t,\text{resc}}$)**.** *For any $\ell_0 > 0$, there exists a t_0 such that for $t > t_0$ and $s_1, s_2 \in [-\ell_0, \frac{\alpha(2-\kappa)}{4} t^{2/3}]$,*

$$\begin{aligned} |K^{(1,a)}_{t,\text{resc}}(s_1, s_2)| &\leq e^{-tF(\alpha,\kappa)/2}, \\ |K^{(2)}_{t,\text{resc}}(s_1, s_2)| &\leq e^{-tF(\alpha,\kappa)/2}, \end{aligned} \tag{3.88}$$

where

$$F(\alpha, \kappa) = -\frac{\alpha + \kappa - 2}{2} - \frac{\kappa(2 - \alpha)}{4} \ln\left(\frac{2 - \alpha}{\kappa}\right) + \frac{\alpha(2 - \kappa)}{4} \ln\left(\frac{2 - \kappa}{\alpha}\right) > 0 \tag{3.89}$$

for all $\alpha, \kappa \in [0, 2)$ and $\kappa \in [0, 2 - \alpha)$.

Proof. To get the result we need to choose the paths for z, w so that they will be steep descent. Let us consider the following paths:

$$\begin{aligned} \Gamma_{-1,\alpha-2} &= \left\{ w = -1 + \frac{\alpha}{2} + \mathrm{i}y - |y|, y \in \mathbb{R} \right\}, \\ \Gamma_0 &= \left\{ z = -\frac{\kappa}{2} + \mathrm{i}y + |y|, y \in \mathbb{R} \right\}. \end{aligned} \tag{3.90}$$

With this choice, Γ_0 stays on the right of $\alpha-2-\Gamma_{-1,\alpha-2}$ since we assumed $\kappa<2-\alpha$, see Figure 3.3. Now we verify the steep descent property of the paths. By symmetry it is enough to consider the portion of the paths in the upper-half plane.

Path $\Gamma_{-1,\alpha-2}$*:* Consider $w=-1+\frac{\alpha}{2}+\mathrm{i}y-y$ for $y\geq 0$, $\tilde{s}\in[0,\alpha(2-\kappa)/4]$. Then,

$$\mathrm{Re}(f_0(w,\tilde{s}))=\mathrm{const}-y+\frac{\kappa(2-\alpha)}{8}\ln(|w|^2)-\frac{1}{2}\left(\frac{\alpha(2-\kappa)}{4}-\tilde{s}\right)\ln(|w+1|^2), \tag{3.91}$$

with $|w|^2=\frac{(2-\alpha)^2}{4}+(2-\alpha)y+2y^2$ and $|w+1|^2=\frac{\alpha^2}{4}-\alpha y+2y^2$. Thus,

$$\frac{\partial\,\mathrm{Re}(f_0(w,\tilde{s}))}{\partial y}=-1+\frac{\kappa(2-\alpha)}{8|w|^2}(4y+2-\alpha)-\left(\frac{\alpha(2-\kappa)}{4}-\tilde{s}\right)\frac{4y-\alpha}{2|w+1|^2}. \tag{3.92}$$

Now we consider two cases:

Case a: $0<y\leq\alpha/4$. In this case,

$$\begin{aligned}(3.92)&\leq -1+\frac{\kappa(2-\alpha)}{8|w|^2}(4y+2-\alpha)-\frac{\alpha(2-\kappa)}{8}\frac{4y-\alpha}{|w+1|^2}\\ &=-y^2\frac{8y^2+(4y+1-\alpha)(2-\alpha-\kappa)+2-\alpha}{2|w|^2|w+1|^2}<0\end{aligned} \tag{3.93}$$

for all $0<\alpha<2$ and $0\leq\kappa<2-\alpha$.

Case b: $y\geq\alpha/4$. In this case,

$$\begin{aligned}(3.92)&\leq -1+\frac{\kappa(2-\alpha)}{8|w|^2}(4y+2-\alpha)\\ &=-\frac{(2-\kappa)\left(\frac{(2-\alpha)^2}{4}+(2-\alpha)y\right)+4y^2}{2|w|^2}<0\end{aligned} \tag{3.94}$$

for all $\kappa<2$.

Further, as $y\to\infty$, $\frac{\partial\,\mathrm{Re}(f_0(w,\tilde{s}))}{\partial y}\to -1$, i.e., $\mathrm{Re}(f_0(w,\tilde{s}))\simeq -y$. This implies that the estimates of the integrand in w will have an exponential decay as e^{-yt}. Thus our chosen path $\Gamma_{-1,\alpha-2}$ is steep descent.

Path Γ_0*:* Consider $z=-\frac{\kappa}{2}+\mathrm{i}y+y$ for $y\geq 0$. Then

$$\mathrm{Re}(-f_0(z,\tilde{s}))=\mathrm{const}-y-\frac{\kappa(2-\alpha)}{8}\ln(|z|^2)+\frac{1}{2}\left(\frac{\alpha(2-\kappa)}{4}-\tilde{s}\right)\ln(|z+1|^2), \tag{3.95}$$

with $|z|^2 = \frac{\kappa^2}{4} - \kappa y + 2y^2$ and $|z+1|^2 = \frac{(2-\kappa)^2}{4} + (2-\kappa)y + 2y^2$. Thus, using $\tilde{s} \geq 0$,

$$\begin{aligned} \frac{\partial \operatorname{Re}(-f_0(z,\tilde{s}))}{\partial y} &= -1 - \frac{\kappa(2-\alpha)}{8|z|^2}(4y-\kappa) + \left(\frac{\alpha(2-\kappa)}{4} - \tilde{s}\right)\frac{4y+2-\kappa}{2(|z+1|^2)} \\ &\leq -1 - \frac{\kappa(2-\alpha)}{8|z|^2}(4y-\kappa) + \frac{\alpha(2-\kappa)}{8}\frac{4y+2-\kappa}{(|z+1|^2)} \\ &= -y^2\frac{8y^2+(4y+2-\kappa)(2-\alpha-\kappa)+\alpha\kappa}{2|z|^2|z+1|^2} < 0 \end{aligned} \tag{3.96}$$

for all $\kappa > 0$, $y > 0$, since we assume $0 < \alpha < 2$ and $0 \leq \kappa < 2 - \alpha < 2$.

By these two results on the steep descent property, the exponential decay for large y, and the fact that $|z - w|$ remains bounded away from 0, we get the bound

$$\begin{aligned} \left|K^{(2)}_{t,\text{resc}}(s_1,s_2)\right| &\leq \text{const}\, t^{1/3} e^{t\operatorname{Re}(f_0((\alpha-2)/2,\tilde{s}_1)) - t\operatorname{Re}(f_0(-\kappa/2,\tilde{s}_2))} \\ &= \text{const}\, t^{1/3} e^{t[\frac{\alpha+\kappa-2}{2} + \frac{\kappa(2-\alpha)}{4}\ln(\frac{2-\alpha}{\kappa}) - \frac{\alpha(2-\kappa)}{4}\ln(\frac{2-\kappa}{\alpha})]} e^{-s_2\ln((2-\kappa)/\alpha)t^{1/3}}. \end{aligned} \tag{3.97}$$

Since $(2-\kappa)/\alpha > 1$ and $s_2 \geq -\ell_0$, the last term is at worse $e^{c\ell_0 t^{1/3}}$ with $c = \ln((2-\kappa)/\alpha) > 0$. Further one can verify that $F(\alpha,\kappa) > 0$ for all $\alpha \in [0,2)$ and $\kappa \in [0, 2-\alpha)$. Thus $\text{const}\, t^{1/3} e^{-tF(\alpha,\kappa)} e^{c\ell_0 t^{1/3}} \leq e^{-tF(\alpha,\kappa)/2}$ for t large enough. We have obtained that

$$\left|K^{(2)}_{t,\text{resc}}(s_1,s_2)\right| \leq e^{-tF(\alpha,\kappa)/2} \tag{3.98}$$

for t large enough.

By exactly the same argument, but using that $|z - (\alpha - 2 - w)|$ remains bounded away from zero, we can bound $K^{(1,a)}_{t,\text{resc}}$, namely

$$\left|K^{(1,a)}_{t,\text{resc}}(s_1,s_2)\right| \leq e^{-tF(\alpha,\kappa)/2}. \tag{3.99}$$

□

Proposition 3.20 (Convergence for $K^{(1,b)}_{t,\text{resc}}$)**.** *For any s_1, s_2 in a bounded set,*

$$\lim_{t\to\infty} K^{(1,b)}_{t,\text{resc}}(s_1,s_2) = \sigma \operatorname{Ai}(\sigma(s_1+s_2)) \tag{3.100}$$

with $\sigma = \frac{(2-\alpha)^{2/3}}{(\alpha((2-\alpha)^2 - 2(1-\alpha)\kappa))^{1/3}}$.

Proof. We have

$$K^{(1,b)}_{t,\text{resc}}(s_1,s_2) = \frac{t^{1/3}}{2\pi \mathrm{i}} \oint_{\Gamma_{-1,\alpha-2}} \frac{\mathrm{d}w}{w+1} e^{t[f_0(w,0) - f_0(\alpha-2-w,0)]} e^{t^{1/3}[s_1 f_2(w) - s_2 f_2(2-\alpha-w)]} \tag{3.101}$$

with $f_2(w) = \ln(2(w+1)/\alpha)$.

First we show that $\Gamma_{-1,\alpha-2}$ as in (3.90) is steep descent for

$$g_0(w, \tilde{s}_1, \tilde{s}_2) := f_0(w, \tilde{s}_1) - f_0(\alpha - 2 - w, \tilde{s}_2), \tag{3.102}$$

for $\tilde{s}_1, \tilde{s}_2 \in [0, \alpha(2-\kappa)/4]$. It is a little bit more than what we need for this proposition, but we will use it in Proposition 3.21 again. From the proof of Proposition 3.19 we already know that the path is steep descent for $f_0(w, \tilde{s}_1)$. Now consider $z = \alpha - 2 - w = -1 + \frac{\alpha}{2} + \mathrm{i}y + y$, $y \geq 0$. Then, $|z|^2 = \frac{(2-\alpha)^2}{4} - (2-\alpha)y + 2y^2$ and $|z+1|^2 = \frac{\alpha^2}{4} + \alpha y + 2y^2$. The same computation as in (3.96) given, for $\tilde{s} \geq 0$,

$$\begin{aligned} \frac{\partial \operatorname{Re}(-f_0(z, \tilde{s}))}{\partial y} &\leq -1 - \frac{\kappa(2-\alpha)}{8|z|^2}(4y - 1 + \alpha/2) + \frac{\alpha(2-\kappa)}{8} \frac{4y + 1 + \alpha/2}{(|z+1|^2)} \\ &= -y^2 \frac{8y^2 + (4y + 1 - \alpha)(2 - \alpha - \kappa) + 2 - \alpha}{2|z|^2|z+1|^2} < 0 \end{aligned} \tag{3.103}$$

for all $y > 0$ under our assumptions $0 < \alpha < 2$ and $0 \leq \kappa < 2 - \alpha$. Moreover, as $y \to \infty$, $\operatorname{Re}(-f_0(z, \tilde{s})) \simeq -y$. Putting together the two results, we have that the chosen path $\Gamma_{-1,\alpha-2}$ is steep descent for $g_0(w, \tilde{s}_1, \tilde{s}_2)$ and for $y \to \infty$ we have $\operatorname{Re}(g_0(w, \tilde{s}_1, \tilde{s}_2)) \lesssim -2y$.

Therefore, the contribution to $K^{(1,b)}_{t,\mathrm{resc}}(s_1, s_2)$ coming from $|y| \geq \delta$ is of order $\mathcal{O}(t^{1/3} e^{-c(\delta)t})$ for some $c(\delta) > 0$. It remains to control the contribution for $|y| \leq \delta$. By Taylor series we have

$$g_0(w, 0, 0) = -Q(\alpha, \kappa) \frac{(2(\mathrm{i}-1)y/\alpha)^3}{3} + \mathcal{O}(y^4), \tag{3.104}$$

with

$$Q(\alpha, \kappa) = \frac{\alpha\left((2-\alpha)^2 - 2(1-\alpha)\kappa\right)}{(2-\alpha)^2} \tag{3.105}$$

and

$$s_1 f_2(w) - s_2 f_2(2 - \alpha - w) = (s_1 + s_2) 2(\mathrm{i}-1)y/\alpha + \mathcal{O}(y^2). \tag{3.106}$$

So, the contribution from $0 \leq y \leq \delta$ is given by

$$\frac{t^{1/3}}{2\pi \mathrm{i}} \frac{2(\mathrm{i}-1)}{\alpha} \int_0^\delta \mathrm{d}y e^{-tQ(\alpha,\kappa)(2(\mathrm{i}-1)y/\alpha)^3/3 + t^{1/3}(s_1+s_2)2(\mathrm{i}-1)y/\alpha} e^{\mathcal{O}(ty^4, t^{1/3}y^2)}. \tag{3.107}$$

The cubic term has a prefactor with negative real part, so that it dominates all the error terms. Consider first (3.107) without the error terms. Then, by the change of variables $W := -t^{1/3} Q(\alpha, \kappa)^{1/3} 2(\mathrm{i}-1)y/\alpha$, we get

$$\frac{Q(\alpha, \kappa)^{-1/3}}{2\pi \mathrm{i}} \int_{-t^{1/3}Q(\alpha,\kappa)^{1/3}2(1-\mathrm{i})\delta/\alpha}^{0} \mathrm{d}W e^{W^3/3 - (s_1+s_2)Q(\alpha,\kappa)^{-1/3}W}. \tag{3.108}$$

Extending the contour to $(\mathrm{i}-1)\infty$ the error term is only $\mathcal{O}(e^{-c(\delta)t})$ and adding the contribution of $y \leq 0$ we finally get that the main contribution is given by

$$\frac{Q(\alpha,\kappa)^{-1/3}}{2\pi\mathrm{i}} \int_{-(1-\mathrm{i})\infty}^{-(1+\mathrm{i})\infty} \mathrm{d}W e^{W^3/3-(s_1+s_2)Q(\alpha,\kappa)^{-1/3}W} = \sigma \operatorname{Ai}(\sigma(s_1+s_2)) \tag{3.109}$$

where we set $\sigma = Q(\alpha,\kappa)^{-1/3}$. Finally, to control the error terms in (3.107), one uses as usual the identity $|e^{|x|}-1| \leq |x|e^{|x|}$ with x replaced by the error terms, and obtains a contribution of order $\mathcal{O}(t^{-1/3})$. □

Proposition 3.21 (Bounds for $K^{(1,b)}_{t,\mathrm{resc}}$)**.** *For any $\ell_0 > 0$, there exists a t_0 such that for $t > t_0$ and $s_1, s_2 \in [-\ell_0, \frac{\alpha(2-\kappa)}{4} t^{2/3}]$*

$$|K^{(1,b)}_{t,\mathrm{resc}}(s_1,s_2)| \leq C e^{-(s_1+s_2)/2}, \tag{3.110}$$

for some finite constant C.

Proof. The proof is very similar to the one in previous papers, see e.g. Proposition 5.3 of [18]. We will skip some algebraic details and focus on the strategy and the key points. First, for any t-independent $\tilde{\ell}$ the result for $(s_1,s_2) \in [-\ell_0,\tilde{\ell}]^2$ follows from the proof of Proposition 3.20. The constant $\tilde{\ell}$ can be chosen later and, for instance, if $(s_1,s_2) \in [-\ell_0,\infty)^2 \setminus [-\ell_0,\tilde{\ell}]^2$, it can be chosen such that s_1+s_2 is large enough.

As before, we denote $\tilde{s}_i = s_i t^{-2/3}$. The integral we have to estimate is then

$$\frac{t^{1/3}}{2\pi\mathrm{i}} \oint_{\Gamma_{-1,\alpha-2}} \frac{\mathrm{d}w}{w+1} e^{t g_0(w,\tilde{s}_1,\tilde{s}_2)} \tag{3.111}$$

with g_0 given in (3.102). We have seen in the first part of the proof of Proposition 3.20 that the path $\Gamma_{-1,\alpha-2}$ as in (3.90) is steep descent for general values of s_1, s_2 in our domain. The idea is now to consider a minor modification of this path around $w_c = -1+\alpha/2$ as follows, see Figure 3.4.

Consider

$$w = w_c - \rho(1-\mathrm{i}y), \quad |y| \leq 1, \tag{3.112}$$

where ρ is chosen as follows:

$$\rho = \begin{cases} \frac{\alpha}{2\sqrt{Q(\alpha,\kappa)}}\sqrt{\tilde{s}_1+\tilde{s}_2}, & \text{for } 0 \leq \tilde{s}_1+\tilde{s}_2 \leq \varepsilon, \\ \frac{\alpha}{2\sqrt{Q(\alpha,\kappa)}}\sqrt{\varepsilon}, & \text{for } \tilde{s}_1+\tilde{s}_2 \geq \varepsilon, \end{cases} \tag{3.113}$$

with $Q = Q(\alpha,\kappa)$ given in (3.105). For the asymptotic analysis, $\varepsilon > 0$ can be chosen as small as needed (but independent of t). This piece of contour joins the original path

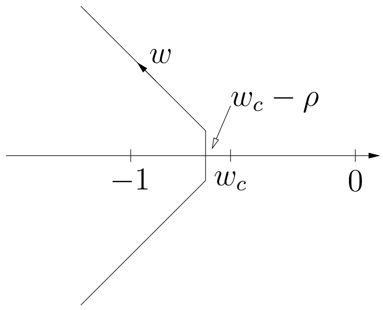

Figure 3.4: Paths used for the asymptotic analysis in Proposition 3.21.

(3.90). Now one has to control the real part of g_0 only in a neighborhood of $-1+\alpha/2$ (at a distance $\mathcal{O}(\varepsilon)$ only). Taylor series at w_c gives

$$g_0(w,\tilde{s}_1,\tilde{s}_2) = -Q\frac{2^3}{\alpha^3}\frac{(w-w_c)^3}{3}+(\tilde{s}_1+\tilde{s}_2)\frac{2}{\alpha}(w-w_c)+\mathcal{O}\left((w-w_c)^4,\tilde{s}_i(w-w_c)^2\right). \quad (3.114)$$

For the choice in (3.112)-(3.113), one looks for the minimal w of (3.114) without the error terms and gets the first choice. However, in order to have enough control through Taylor approximation, we have to stay in a small neighborhood of w_c. This is the reason for the ε cut-off in (3.113).

Replacing (3.112) into the main part of (3.114) one gets, for $0\leq\tilde{s}_1+\tilde{s}_2\leq\varepsilon$,

$$\mathrm{Re}\left(-Q\frac{2^3}{\alpha^3}\frac{(w-w_c)^3}{3}+(\tilde{s}_1+\tilde{s}_2)\frac{2}{\alpha}(w-w_c)\right) = -\frac{(\tilde{s}_1+\tilde{s}_2)^{3/2}(2+3y^2)}{3\sqrt{Q}}, \quad (3.115)$$

while for $\tilde{s}_1+\tilde{s}_2\geq\varepsilon$,

$$\begin{aligned}\mathrm{Re}\left(-Q\frac{2^3}{\alpha^3}\frac{(w-w_c)^3}{3}+(\tilde{s}_1+\tilde{s}_2)\frac{2}{\alpha}(w-w_c)\right) &= -\frac{3(\tilde{s}_1+\tilde{s}_2)\sqrt{\varepsilon}+(3y^2-1)\varepsilon^{3/2}}{3\sqrt{Q}} \\ &\leq -\frac{2(\tilde{s}_1+\tilde{s}_2)\sqrt{\varepsilon}+3y^2\varepsilon^{3/2}}{3\sqrt{Q}}.\end{aligned} \quad (3.116)$$

The two key properties in (3.115) and (3.116) are: (1) the quadratic decay of $e^{tg_0(w,\tilde{s}_1,\tilde{s}_2)}$ due the y^2 term, and (2) at $y=0$ one would have the bound

$$e^{t\,\mathrm{Re}(g_0(w,\tilde{s}_1,\tilde{s}_2))} \lesssim \begin{cases} e^{-\frac{2}{3}(s_1+s_2)^{3/2}Q^{-1/2}}, & \text{for } 0\leq\tilde{s}_1+\tilde{s}_2\leq\varepsilon, \\ e^{-\frac{2}{3}(s_1+s_2)\sqrt{\varepsilon}t^{1/3}Q^{-1/2}}, & \text{for } \tilde{s}_1+\tilde{s}_2\geq\varepsilon, \end{cases} \quad (3.117)$$

by ignoring the error terms in (3.114). For s_1+s_2 large enough and t large enough, in both cases (3.116) is bounded by $e^{-c(s_1+s_2)}$ for any choice of $c>0$. By choosing ε small enough, it is not so difficult (but a bit lengthy) to control the error terms in (3.114) too. This can be made in exactly the same way as in the proof of Proposition 5.3 of [18] (see the argument between equations (5.40) and (5.47) in [18]). As a result, one obtains for instance a bound for the rescaled kernel (3.111) like (3.116) with the prefactor $\frac{2}{3}$ replaced by $\frac{1}{3}$. This estimate is good enough and leads to the bound (3.110). □

Proposition 3.22. *Let $\eta > \frac{\alpha^2}{(2-\alpha)^2}$ and $\tilde{\mu} = 2\left(\frac{\eta}{\alpha} + \frac{1}{2-\alpha}\right)$. Then, for any $\epsilon \geq 0$, there exist constants $C, \tilde{c}$ such that*

$$\mathbb{P}\left(L_{\mathcal{L}^- \to (\lfloor \eta\ell \rfloor, \lfloor \ell \rfloor)} > (\tilde{\mu} + \epsilon/2)\ell\right) \leq C \exp(-\tilde{c}\epsilon\ell^{2/3}). \tag{3.118}$$

Proof. It is quite similar to the one of Proposition 3.13. We use again the correspondance (3.47) between TASEP and LPP. We set $t := (\tilde{\mu}+\epsilon/2)\ell$, $\ell(t) = t/(\tilde{\mu}+\epsilon/2)$, Proposition 3.16 tells us

$$(3.118) = 1 - \mathbb{P}(x_{\ell(t)}(t) \geq (\eta - 1)\ell(t)). \tag{3.119}$$

We denote

$$X_t^{\text{resc}} = \frac{x_{\ell(t)}(t) - \frac{(\alpha-\kappa)t}{2}}{-t^{1/3}} \tag{3.120}$$

with $\kappa = \frac{4}{2-\alpha}\left(\tilde{\mu} + \frac{\epsilon}{2}\right)^{-1}$ so that $\ell(t) = \kappa\frac{2-\alpha}{4}t$. Then,

$$\begin{aligned} (3.118) &= 1 - \mathbb{P}(X_t^{\text{resc}} \leq \frac{(\eta-1)\ell(t) - \frac{\alpha-\kappa}{2}t}{-t^{1/3}}) \\ &= -\sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int \mathrm{d}s_1 \cdots \int \mathrm{d}s_m \det[t^{1/3}\tilde{K}_{\ell(t),t}(x(s_i), x(s_j)]_{1\leq i,j\leq m}, \end{aligned} \tag{3.121}$$

where $x(s) = \frac{\alpha-\kappa}{2}t - st^{1/3}$ and the integration domain of the s_i is $(\alpha\epsilon t^{2/3}/4(\tilde{\mu}+\epsilon/2), \alpha(2-\kappa)t^{2/3}/4]$. This comes from the fact that with $x = (\eta-1)\ell(t)$ we have

$$s = \frac{x - \frac{\alpha-\kappa}{2}t}{-t^{1/3}} = \frac{\alpha\epsilon t^{2/3}}{4(\tilde{\mu}+\epsilon/2)} \tag{3.122}$$

together with the fact that the original kernel $K_{n,t}$ is identically equal to zero for $x(s)+n < 0$.

A straightforward consequence of Proposition 3.19 is that, for $s_1, s_2 \in [-\ell_0, \alpha(2-\kappa)t^{2/3}/4]$ it holds

$$|K_{t,\text{resc}}^{(1,a)}(s_1,s_2)| + |K_{t,\text{resc}}^{(2)}(s_1,s_2)| \leq e^{-F(\alpha,\kappa)t/4}e^{-(s_1+s_2)/2} \tag{3.123}$$

for t large enough. This together with the exponential bound of Proposition 3.21 implies that we can thus single out a factor $\prod_{i=1}^m C^m e^{-s_i}$ so that using Hadamard's bound, we get

$$\begin{aligned} |(3.118)| &\leq \sum_{m=1}^{\infty} \frac{C^m m^{m/2}}{m!} \int_{\varepsilon\alpha t^{2/3}/4(\tilde{\mu}+\varepsilon/2)}^{\alpha(2-\kappa)t^{2/3}/4} \mathrm{d}s_1 \cdots \int_{\varepsilon\alpha t^{2/3}/4(\tilde{\mu}+\varepsilon/2)}^{\alpha(2-\kappa)t^{2/3}/4} \mathrm{d}s_m \prod_{i=1}^m e^{-s_i} \\ &\leq \tilde{C}\exp\left(-\tilde{c}\varepsilon\ell^{2/3}\right) \end{aligned} \tag{3.124}$$

for some constants $\tilde{C}, \tilde{c}$ (uniform in ℓ), where the last steps are identical to the ones of Proposition 3.13. □

The following Proposition is the LPP version of Theorem 1.27.

Proposition 3.23 (Half-line $\mathcal{L}^-$-to-point LPP: convergence to $F_{\rm GOE}$)**.** *For any fixed $\eta > \frac{\alpha^2}{(2-\alpha)^2}$, it holds*

$$\lim_{\ell\to\infty} \mathbb{P}\left(L_{\mathcal{L}^-\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} \leq \tilde\mu\ell + s\hat\sigma_\eta \ell^{1/3}\right) = F_{\rm GOE}(2s) \tag{3.125}$$

where $\tilde\mu = 2(\frac{\eta}{\alpha} + \frac{1}{2-\alpha})$, $\hat\sigma_\eta = \frac{2^{4/3}}{\alpha}\left(\eta + \frac{\alpha^3}{(2-\alpha)^3}\right)^{1/3}$, and $F_{\rm GOE}$ is the GOE Tracy-Widom distribution function.

Proof of Theorem 1.27 (and Proposition 3.23). First, with σ as in Proposition 3.20, it holds

$$\begin{aligned}\mathbb{P}\left(L_{\mathcal{L}^-\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} \leq \tilde\mu\ell + s\hat\sigma_\eta \ell^{1/3}\right) &= \mathbb{P}\left(x_\ell(\tilde\mu\ell + s\hat\sigma_\eta\ell^{1/3}) \geq (\eta-1)\ell\right)\\ &= \mathbb{P}\left(x_{[\kappa(2-\alpha)t/4]}(t) \geq \frac{\alpha-\kappa}{2}t - \sigma^{-1}st^{1/3}\right)\end{aligned} \tag{3.126}$$

if we choose

$$\begin{aligned} t = \tilde\mu\ell + s\hat\sigma_\eta\ell^{1/3} &\Leftrightarrow \ell = \frac{t}{\tilde\mu} - \frac{s\hat\sigma_\eta t^{1/3}}{\tilde\mu^{4/3}} + o(1),\\ \frac{\kappa(2-\alpha)}{4}t = \ell &\Leftrightarrow \kappa = \frac{4}{2-\alpha}\left(\frac{1}{\tilde\mu} - \frac{s\hat\sigma_\eta t^{-2/3}}{\tilde\mu^{4/3}}\right),\end{aligned} \tag{3.127}$$

and finally $\frac{\alpha-\kappa}{2}t - \sigma^{-1}st^{1/3} = (\eta-1)\ell$, which fixes the values of $\tilde\mu$ and $\hat\sigma_\eta$ as given in the statement. Now, the R.H.S. of (3.126) is given by a Fredholm determinant like in (3.121), with the minor difference that now the lower integration bound is simply given by s and that the scaling of the kernel has the extra σ^{-1} in front. From Propositions 3.19 and 3.21 we know that the kernel is uniformly bounded (in t) by a function so that its Fredholm series is bounded. Thus we can apply dominated convergence to take the limit inside the Fredholm series. Finally, Proposition 3.20 tells us that the pointwise limit of the rescaled kernel (including the extra σ^{-1} factor in the spatial scaling) converges pointwise to $\mathrm{Ai}(s_1+s_2)$. Thus,

$$\lim_{t\to\infty}\mathbb{P}\left(x_{[\kappa(2-\alpha)t/4]}(t) \geq \frac{\alpha-\kappa}{2}t - \sigma^{-1}st^{1/3}\right) = F_{\rm GOE}(2s), \tag{3.128}$$

which ends the proof. □

Corollary 3.24. *Fix an $\eta > \alpha^2/(2-\alpha)^2$, a $\beta\in(1/3,1]$ and define*

$$\gamma\in[0,1-t^{\beta-1}],\quad \varepsilon = t^{-\chi} \text{ with } \chi\in(0,2/3). \tag{3.129}$$

Then there exists constants $C,\tilde c>0$ and $t_0>0$ such that for all $t>t_0$

$$\mathbb{P}\left(L_{\mathcal{L}^-\to D_\gamma} > \left(\tilde\mu_\gamma + \frac{\epsilon}{2}\right)t\right) \leq C\exp\left(-\tilde c\, t^{2/3-\chi}\right), \tag{3.130}$$

where $\tilde\mu_\gamma = 2\gamma\left(\frac{\eta}{\alpha}+\frac{1}{2-\alpha}\right)$.

Proof. It is a straightforward consequence of Proposition 3.22. Indeed, with $\ell = \gamma t$, we have

$$\begin{aligned}\mathbb{P}\left(L_{\mathcal{L}^-\to D_\gamma} > (\tilde{\mu}_\gamma + \epsilon/2)\, t\right) &= \mathbb{P}\left(L_{\mathcal{L}^-\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} > (\tilde{\mu} + \epsilon/(2\gamma))\,\ell\right)\\ &\leq \mathbb{P}\left(L_{\mathcal{L}^-\to(\lfloor\eta\ell\rfloor,\lfloor\ell\rfloor)} > (\tilde{\mu} + \epsilon/2)\,\ell\right)\\ &\leq C\exp(-\tilde{c}t^{2/3-\chi}),\end{aligned} \tag{3.131}$$

where the second inequality holds since $\gamma \leq 1$. □

3.3.2 No-crossing results

In this section we collect the non-crossing results, which are proven below.

Proposition 3.25. *Consider the point $E = (\lfloor\eta t\rfloor, \lfloor t\rfloor)$ for $0 < \eta < 1$ (see Figure 3.5). For some fixed $\beta \in (1/3, 1]$, consider the points $D_\gamma = (\lfloor\gamma\eta t\rfloor, \lfloor\gamma t\rfloor)$ with $\gamma \in [0, 1 - t^{\beta-1}]$. Then, for all t large enough*

$$\mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma\\ \gamma\in[0,1-t^{\beta-1}]}} \{D_\gamma \in \pi^{\max}_{\mathcal{L}^+\to E}\}\Bigg) \leq C\exp(-ct^{\beta-1/3}), \tag{3.132}$$

for some t-independent constants $C, c > 0$.

Proposition 3.26. *Consider the point $E^+ = (\lfloor\eta t - t^\nu\rfloor, \lfloor t - t^\nu\rfloor)$ for $0 < \eta < 1$ and $1/3 < \nu < 1$ (see Figure 3.5). For some fixed $\beta \in (1/3, 1]$, consider the points $D_\gamma = (\lfloor\gamma\eta t)\rfloor, \lfloor\gamma t\rfloor)$ with $\gamma \in [0, 1 - t^{\beta-1}]$. Then, for all t large enough*

$$\mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma\\ \gamma\in[0,1-t^{\beta-1}]}} \{D_\gamma \in \pi^{\max}_{\mathcal{L}^+\to E^+}\}\Bigg) \leq C\exp(-ct^{\beta-1/3}), \tag{3.133}$$

for some t-independent constants $C, c > 0$.

Proposition 3.27. *Let $\alpha \in (0, 2)$. Consider, for some $\eta > \alpha^2/(2-\alpha)^2$, the point $E = (\lfloor\eta t\rfloor, \lfloor t\rfloor)$ as in Figure 3.5. For some fixed $\beta \in (1/3, 1]$, consider the points $D_\gamma = (\lfloor\gamma\eta t\rfloor, \lfloor\gamma t\rfloor)$ with $\gamma \in [0, 1 - t^{\beta-1}]$. Then, for all t large enough*

$$\mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma\\ \gamma\in[0,1-t^{\beta-1}]}} \{D_\gamma \in \pi^{\max}_{\mathcal{L}^-\to E}\}\Bigg) \leq C\exp(-ct^{\beta-1/3}), \tag{3.134}$$

for some t-independent constants $C, c > 0$.

Similarly, for the point-to-point geometry we have:

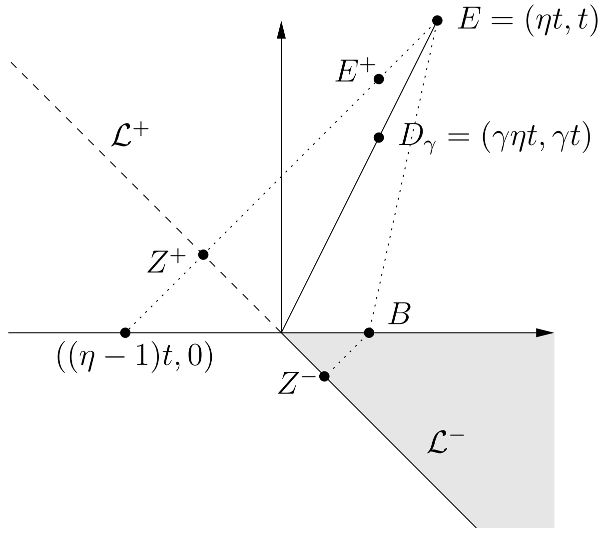

Figure 3.5: Illustration of the geometry for the LPP of Propositions 3.25–3.28. The half-line $\mathcal{L}^-$ is the solid one, while the half-line $\mathcal{L}^+$ is the dashed one. Further, $E^+ = (\eta t - t^\nu, t - t^\nu)$, $B = (\eta - \alpha^2/(2-\alpha)^2)(t, 0)$, $Z^+ = (1-\eta)(-t/2, t/2)$, and $Z^- = (\eta - \alpha^2/(2-\alpha)^2)(t/2, -t/2)$. In the grey regions, the exponential random variables have parameter $\alpha \in (0, 2)$, while in the white regions, they have parameter 1.

Proposition 3.28. *Consider the point $E = (\lfloor \eta t \rfloor, \lfloor t \rfloor)$ for $0 < \eta < 1$. For some fixed $\beta \in (1/3, 1]$, consider the points $D_\gamma = (\lfloor \gamma\eta t \rfloor, \lfloor \gamma t \rfloor)$ with $\gamma \in [0, 1 - t^{\beta-1}]$. Then, for all t large enough*

$$\mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma \\ \gamma\in[0,1-t^{\beta-1}]}} \{ D_\gamma \in \pi^{\max}_{(\lfloor(\eta-1)t\rfloor,0)\to E} \} \Bigg) \leq C \exp(-ct^{\beta-1/3}), \tag{3.135}$$

for some t-independent constants $C, c > 0$.

Proposition 3.29. *For some fixed $\beta \in (1/3, 1]$, consider the points $D_\gamma = (\lfloor \gamma t \rfloor, \lfloor \gamma t \rfloor)$ with $\gamma \in [0, 1 - t^{\beta-1}]$. Then, for all t large enough*

$$\mathbb{P}\Bigg(\bigcup_{\substack{D_\gamma \\ \gamma\in[0,1-t^{\beta-1}]}} \{ D_\gamma \in \pi^{\max}_{(-t,0)\to(t,t)} \} \Bigg) \leq C \exp(-ct^{\beta-1/3}), \tag{3.136}$$

for some t-independent constants $C, c > 0$.

3.3.2.1 Proof of Propositions 3.25, 3.26, 3.28, 3.29

In order to prove Proposition 3.25, we will adopt the notation and line of argumentation of the proof by Johansson of Theorem 1.36, namely the Lemmas 3.1, 3.2 and 3.3 in [46].

Using the deviation results from the previous section, we first show that the probability that for some γ the LPP-times $L_{\mathcal{L}^+\to D_\gamma}$ and $L_{D_\gamma\to E}$ exceed by $\varepsilon t/2$ their leading orders converges to zero.

Proposition 3.30. *Fix an $0<\eta<1$, a $\beta\in(1/3,1]$, a $\chi\in(0,2/3)$. Let us set $\varepsilon=t^{-\chi}$. We define a finite*[3] *family of events $\left\{E_{D_\gamma}\right\}_{\gamma\in[0,1-t^{\beta-1}]}$ via*

$$E_{D_\gamma} := \{\omega : L_{\mathcal{L}^+\to D_\gamma}(\omega) \le (\mu_\gamma+\varepsilon/2)t\} \cap \{L_{D_\gamma\to E}(\omega)\le(\mu_{\mathrm{pp},\gamma}+\varepsilon/2)t\}, \tag{3.137}$$

where

$$\mu_\gamma = 2(1+\eta)\gamma, \quad \mu_{\mathrm{pp},\gamma} = (1-\gamma)(1+\sqrt{\eta})^2. \tag{3.138}$$

Then

$$\mathbb{P}\Big(\bigcup_{D_\gamma}\Omega\setminus E_{D_\gamma}\Big) \le C'\exp(-c't^{2/3-\chi}) \tag{3.139}$$

for some constants $C',c'>0$.

Proof. To get the result, notice that there are $\mathcal{O}(t)$ many points D_γ, $\gamma\in[0,1-t^{\beta-1}]$, so that it is enough to get a good bound (uniform in γ) of $\mathbb{P}\left(\Omega\setminus E_{D_\gamma}\right)$. We have

$$\mathbb{P}\left(\Omega\setminus E_{D_\gamma}\right) \le \mathbb{P}(L_{\mathcal{L}^+\to D_\gamma}\ge(\mu_\gamma+\varepsilon/2)t) + \mathbb{P}(L_{D_\gamma\to E}\ge(\mu_{\mathrm{pp},\gamma}+\varepsilon/2)t). \tag{3.140}$$

According to Corollary 3.15 there is a t_0 such that for $t>t_0$ we get

$$\mathbb{P}(L_{\mathcal{L}^+\to D_\gamma}\ge(\mu_\gamma+\varepsilon/2)t) \le C\exp(-\tilde{c}t^{2/3-\chi}). \tag{3.141}$$

Remark that (with $\stackrel{d}{=}$ designating equality in distribution)

$$L_{D_\gamma\to E} \stackrel{d}{=} L_{0\to(\lfloor(1-\gamma)\eta t\rfloor,\lfloor(1-\gamma)t\rfloor)}. \tag{3.142}$$

Furthermore, Proposition 3.9 with $\ell=(1-\gamma)t$ and $s=\frac{\varepsilon t^{2/3}}{(1-\gamma)^{1/3}}$ gives

$$\mathbb{P}(L_{D_\gamma\to E}\ge(\mu_{\mathrm{pp},\gamma}+\varepsilon/2)t) \le C\exp\left(-\varepsilon t^{2/3}\frac{c}{(1-\gamma)^{1/3}}\right) \le C\exp(-ct^{2/3-\chi}). \tag{3.143}$$

The bounds (3.141) and (3.143) imply that, for some constants C',c',

$$\mathbb{P}\left(\Omega\setminus E_{D_\gamma}\right) \le C'\exp(-c't^{2/3-\chi}). \tag{3.144}$$

Being the number of D_γ of order t only, the claimed bound holds true. $\square$

Now we know that if a path goes through a point D_γ, then its typical last passage time is smaller than $(\mu_\gamma+\mu_{\mathrm{pp},\gamma}+2\varepsilon)t$. However, the typical last passage time of the maximizing paths is μt, which is much larger.

[3] The family is finite even if γ is uncountable, since the number of different D_γ is finite.

Proposition 3.31. *Fix an $0 < \eta < 1$, a $\beta \in (1/3, 1]$, and $\gamma \in [0, 1 - t^{\beta-1}]$. Let us set $\varepsilon = Ct^{\beta-1}$. Then for all $t > 0$ it holds*

$$\frac{(\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \mu)t}{t^{1/3}} \leq -Ct^{\beta-1/3}, \tag{3.145}$$

with $C = (1 - \sqrt{\eta})^2/2$, and

$$\mu = 2(1+\eta), \quad \mu_\gamma = 2(1+\eta)\gamma, \quad \mu_{\mathrm{pp},\gamma} = (1-\gamma)(1+\sqrt{\eta})^2. \tag{3.146}$$

Proof. A simple computations gives, for $0 < \eta < 1$,

$$\begin{aligned} \frac{(\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \mu)t}{t^{1/3}} &= t^{\beta-1/3}(1-\sqrt{\eta})^2/2 - (1-\gamma)(1-\sqrt{\eta})^2 t^{2/3} \\ &\leq -t^{\beta-1/3}(1-\sqrt{\eta})^2/2, \end{aligned} \tag{3.147}$$

where we used $1 - \gamma \geq t^{\beta-1}$. ☐

We can now proceed to the final Proposition.

Proposition 3.32. *Fix an $0 < \eta < 1$, a $\beta \in (1/3, 1]$ and $\gamma \in [0, 1 - t^{\beta-1}]$. Then, there exists a $t_0 > 0$ such that for all $t \geq t_0$ it holds*

$$\mathbb{P}(\{\omega : D_\gamma \in \pi^{\max}_{\mathcal{L}^+ \to E}(\omega)\}) \leq C\exp(-c\,t^{\beta-1/3}), \tag{3.148}$$

for some t-independent constants $C, c > 0$.

Proof of Proposition 3.32. Denote by I_{D_γ} the event that the maximizer from $\mathcal{L}^+$ to E passes by the point D_γ, namely

$$I_{D_\gamma} = \{\omega : D_\gamma \in \pi^{\max}_{\mathcal{L}^+ \to E}(\omega)\}. \tag{3.149}$$

Let us choose $\varepsilon = t^{\beta-1}(1-\sqrt{\eta})^2/2$. Then,

$$\mathbb{P}(I_{D_\gamma}) \leq \mathbb{P}\Big(I_{D_\gamma} \cap \Big(\bigcap_{D_\gamma} E_{D_\gamma}\Big)\Big) + \mathbb{P}\Big(\Big(\bigcap_{D_\gamma} E_{D_\gamma}\Big)^c\Big). \tag{3.150}$$

The second term is exactly (3.139) with $\chi = 1 - \beta$ (the extra coefficient in the definition of ε is irrelevant, since it just modifies the value of the constant c'). Thus, the decay of the second term is as $\exp(-c' t^{\beta-1/3})$.

To bound the first term, notice that if $\omega \in I_{D_\gamma}$ and at the same time in each of the E_{D_γ}'s, then by Propositions 3.30 and 3.31,

$$\begin{aligned} L_{\mathcal{L}^+ \to E}(\omega) &\leq (\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon)t = \mu t + (\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \mu)t \\ &\leq \mu t - (Ct^{\beta-1/3})t^{1/3}. \end{aligned} \tag{3.151}$$

Therefore,

$$\mathbb{P}\Big(I_{D_\gamma} \cap \Big(\bigcap_{D_\gamma} E_{D_\gamma}\Big)\Big) \leq \mathbb{P}(L_{\mathcal{L}^+\to E} \leq \mu t - (Ct^{\beta-1/3})t^{1/3}). \tag{3.152}$$

Further, denote by Z^+ the orthogonal projection of E on $\mathcal{L}^+$, i.e., $Z^+ = \lfloor \frac{1-\eta}{2} \rfloor(-1,1)$. Then, since $L_{\mathcal{L}^+\to E} \geq L_{Z^+\to E}$, it follows that

$$(3.152) \leq \mathbb{P}(L_{Z^+\to E} \leq \mu t - (Ct^{\beta-1/3})t^{1/3}). \tag{3.153}$$

Moreover, since $L_{Z^+\to E} \stackrel{d}{=} L_{0\to(\lfloor \frac{1+\eta}{2}t\rfloor,\lfloor \frac{1+\eta}{2}t\rfloor)}$ we can apply the bound of Proposition 3.10 (with $\ell \to (1+\eta)t/2$, $\eta \to 1$, and $s\ell^{1/3} \to Ct^\beta$) to obtain

$$(3.153) \leq \tilde{C}\exp(-\tilde{c}t^{3\beta/2-1/2}) \tag{3.154}$$

for some constants $\tilde{C}, \tilde{c} > 0$.

Since for $\beta \in (1/3, 1]$ and $\beta - 1/3 \leq 3\beta/2 - 1/2$, then for all t large enough

$$\mathbb{P}(I_{D_\gamma}) \leq C\exp(-ct^{\beta-1/3}), \tag{3.155}$$

for some t-independent constants $C, c > 0$, which is the claimed result. □

Proof of Proposition 3.25. The proof is a straightforward consequence of Proposition 3.32, since the cardinality of the family of points $\{D_\gamma\}_{\gamma\in[0,1-t^{\beta-1}]}$ is only of order t. □

Proof of Proposition 3.26. The proof is very similar to the one of Proposition 3.25. Note first that for $\gamma > 1 - t^{\nu-1}$ then $\mathbb{P}(D_\gamma \in \pi^{\max}_{\mathcal{L}^+\to E^+}) = 0$. For the analogue of Proposition 3.30, one only has to replace E by E^+ in (3.137), which amounts to replace η by $\tilde{\eta} = \frac{(1-\gamma)\eta t - t^\nu}{(1-\gamma)t - t^\nu} \to_{t\to\infty} \eta$ in (3.142), $\mu_{\mathrm{pp},\gamma}$ by $\mu^+_{\mathrm{pp},\gamma} = (1-\gamma-t^{\nu-1})\left(1+\sqrt{\frac{\eta-t^{\nu-1}}{1-t^{\nu-1}}}\right)^2 \to_{t\to\infty} \mu_{\mathrm{pp},\gamma}$ and apply Proposition 3.9 to this new point-to-point LPP. The following analogue of Proposition 3.31 is a bit different.

Proposition 3.33. *Fix an $0 < \eta < 1$, a $\nu, \beta \in (1/3, 1)$, and $\gamma \in [0, 1-t^{\beta-1}]$. Let us set $\varepsilon = Ct^{\beta-1}$. Then for all t large it holds*

$$\frac{(\mu^+_\gamma + \mu^+_{\mathrm{pp},\gamma} + \varepsilon - \mu^+)t}{t^{1/3}} \leq -Ct^{\beta-1/3}, \tag{3.156}$$

with $C = (1-\sqrt{\eta})^2/4$, and

$$\mu^+ = 2(1+\eta) - 4t^{\nu-1}, \quad \mu^+_\gamma = \gamma\mu^+, \quad \mu^+_{\mathrm{pp},\gamma} = (1-\gamma-t^{\nu-1})\left(1+\sqrt{\frac{\eta-t^{\nu-1}}{1-t^{\nu-1}}}\right)^2. \tag{3.157}$$

Proof. Using $\sqrt{\frac{\eta-t^{\nu-1}}{1-t^{\nu-1}}} < \sqrt{\eta}$ for $\eta < 1$, we have $\mu^+_{\mathrm{pp},\gamma} \leq (1-\gamma)(1+\sqrt{\eta})^2$ so that

$$\frac{(\mu^+_\gamma + \mu^+_{\mathrm{pp},\gamma} + \varepsilon - \mu^+)t}{t^{1/3}} \leq Ct^{\beta-1/3} - (1-\gamma)\left(t^{2/3}(1-\sqrt{\eta})^2 - 4t^{\nu-1/3}\right). \tag{3.158}$$

Then, using $\nu < 1$ and $1-\gamma \geq t^{\beta-1}$ we have, for t large enough,

$$(3.158) \leq Ct^{\beta-1/3} - t^{\beta-1/3}(1-\sqrt{\eta})^2/2 = -Ct^{\beta-1/3}. \tag{3.159}$$

□

With these two analogous statements at hand, we can adopt the proof of Proposition 3.32, simply replace again E by E^+ in (3.153), and then again apply Proposition 3.10 with $\ell \to \frac{1+\eta}{2}t - t^\nu$ to obtain a bound analogous to (3.154), which finishes the proof. □

Proof of Proposition 3.28. The proof of Proposition 3.28 is almost identical, so let us indicate just the minor modifications. What we have to do is to replace $\mathcal{L}^+$ with the point $(\lfloor(\eta-1)t\rfloor, 0)$, now $\mu = 4$ and $\mu_\gamma = 4\gamma$. Further, there is one simplification, namely, the step (3.153) is not needed (we would have equality in there). □

Proof of Proposition 3.29. The analogue of Proposition 3.30 can be proven almost identically, one has $\mu_\gamma = \left(1+\sqrt{\frac{1+\gamma}{\gamma}}\right)^2\gamma, \mu_{\mathrm{pp},\gamma} = 4(1-\gamma)$ and uses twice Proposition 3.9.

The analogue of Proposition 3.31 is again a bit different.

Proposition 3.34. *Fix a $\beta \in (1/3, 1]$, and $\gamma \in [0, 1-t^{\beta-1}]$. Let us set $\varepsilon = Ct^{\beta-1}$. Then for all t large it holds*

$$\frac{(\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \mu)t}{t^{1/3}} \leq -Ct^{\beta-1/3}, \tag{3.160}$$

with $C = (3-2\sqrt{2})/4$, and

$$\mu_\gamma = \left(1+\sqrt{\frac{1+\gamma}{\gamma}}\right)^2\gamma, \quad \mu = (1+\sqrt{2})^2, \quad \mu_{\mathrm{pp},\gamma} = 4(1-\gamma). \tag{3.161}$$

Proof of Proposition 3.34. We have

$$\mu_\gamma + \mu_{\mathrm{pp},\gamma} - \mu = 2\left(\sqrt{\frac{1}{\gamma}+1} - 1\right)\gamma - 2\sqrt{2} + 2 \tag{3.162}$$

that is increasing in γ. Further, it holds

$$\mu_\gamma + \mu_{\mathrm{pp},\gamma} - \mu = (\gamma-1)\frac{3-2\sqrt{2}}{\sqrt{2}} + \mathcal{O}((\gamma-1)^2). \tag{3.163}$$

Thus by choosing $\gamma = 1 - t^{\beta-1}$ we get

$$\frac{(\mu_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \mu)t}{t^{1/3}} \leq -t^{\beta-1/3}\left(\frac{3-2\sqrt{2}}{\sqrt{2}} - C\right) + \mathcal{O}(t^{2(\beta-1/3)}) \leq -Ct^{\beta-1/3} \tag{3.164}$$

for t large enough. □

The analogue of Proposition 3.32 can be proven almost identically, the only difference being that the step (3.153) is not needed. □

3.3.2.2 Proof of Proposition 3.27

The proof is very close to the one of Proposition 3.25, therefore we will skip some of the details, focusing more on the differences.

Proposition 3.35. *Fix an $\eta > \alpha^2/(2-\alpha)^2$, a $\beta \in (1/3, 1]$ and a $\chi \in (0, 2/3)$. Let us set $\varepsilon = t^{-\chi}$. We define a finite family of events $\{\tilde{E}_{D_\gamma}\}_{\gamma\in[0,1-t^{\beta-1}]}$ via*

$$\tilde{E}_{D_\gamma} := \{\omega : L_{\mathcal{L}^-\to D_\gamma}(\omega) \leq (\tilde{\mu}_\gamma + \varepsilon/2)t\} \cap \{L_{D_\gamma\to E}(\omega) \leq (\mu_{\mathrm{pp},\gamma} + \varepsilon/2)t\}, \tag{3.165}$$

where

$$\tilde{\mu}_\gamma = 2\gamma\left(\frac{\eta}{\alpha} + \frac{1}{2-\alpha}\right), \quad \mu_{\mathrm{pp},\gamma} = (1-\gamma)(1+\sqrt{\eta})^2. \tag{3.166}$$

Then

$$\mathbb{P}\Big(\bigcup_{D_\gamma} \Omega \setminus \tilde{E}_{D_\gamma}\Big) \leq C' \exp(-c' t^{2/3-\chi}) \tag{3.167}$$

for some constants $C', c' > 0$.

Proof. The proof is like the one of Proposition 3.30, with the only difference that we employ Corollary 3.24 instead of Corollary 3.15 to control the decay of $\mathbb{P}(L_{\mathcal{L}^-\to D_\gamma} \geq (\tilde{\mu}_\gamma + \varepsilon/2)t)$. □

Now we know that if a path goes through the a point D_γ, then its typical last passage time is smaller than $(\tilde{\mu}_\gamma + \mu_{\mathrm{pp},\gamma} + 2\varepsilon)t$. However, the typical last passage time of the maximizing path is $\tilde{\mu}t$ which is much larger.

Proposition 3.36. *Fix $\eta > \alpha^2/(2-\alpha)^2$, $\beta \in (1/3, 1]$, and $\gamma \in [0, 1-t^{\beta-1}]$. Let us set $\varepsilon = Ct^{\beta-1}$. Then for all $t > 0$ it holds*

$$\frac{(\tilde{\mu}_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \tilde{\mu})t}{t^{1/3}} \leq -\tilde{C}t^{\beta-1/3}, \tag{3.168}$$

with $C = \frac{(\alpha-(2-\alpha)\sqrt{\eta})^2}{2\alpha(2-\alpha)}$*, and*

$$\tilde{\mu} = 2\left(\frac{\eta}{\alpha} + \frac{1}{2-\alpha}\right), \quad \tilde{\mu}_\gamma = \gamma\tilde{\mu}, \quad \mu_{\mathrm{pp},\gamma} = (1-\gamma)(1+\sqrt{\eta})^2. \tag{3.169}$$

Proof. A simple computations gives,

$$\begin{aligned} \frac{(\tilde{\mu}_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \tilde{\mu})t}{t^{1/3}} &= (\gamma - 1)\frac{(\alpha-(2-\alpha)\sqrt{\eta})^2}{\alpha(2-\alpha)} t^{2/3} + Ct^{\beta-1/3} \\ &\leq -t^{\beta-1/3}\left(\frac{(\alpha-(2-\alpha)\sqrt{\eta})^2}{\alpha(2-\alpha)} - C\right) \leq -Ct^{\beta-1/3} \end{aligned} \tag{3.170}$$

where we used $\alpha < 1$ and $\gamma - 1 \leq -t^{\beta-1}$ and the fact that $\eta > \alpha^2/(2-\alpha)^2$. □

We can now proceed to the final proposition.

Proposition 3.37. *Fix an* $\eta > \alpha^2/(2-\alpha)^2$*, a* $\beta \in (1/3, 1]$ *and let* $\gamma \in [0, 1 - t^{\beta-1}]$*. Then, there exists a* $t_0 > 0$ *such that for all* $t \geq t_0$ *it holds*

$$\mathbb{P}(\{\omega : D_\gamma \in \pi^{\max}_{\mathcal{L}^- \to E}(\omega)\}) \leq \tilde{C}\exp(-c\, t^{\beta-1/3}), \tag{3.171}$$

for some t*-independent constants* $\tilde{C}, c > 0$*.*

Proof of Proposition 3.37. This proof is very close to the one of Proposition 3.32. This time we choose $\varepsilon = \frac{C}{2}t^{\beta-1}$ with $C = \frac{(\alpha-(2-\alpha)\sqrt{\eta})^2}{2\alpha(2-\alpha)}$ and denote by $\tilde{I}_{D_\gamma}$ the events such that the maximizers from $\mathcal{L}^-$ to E passes by the point D_γ. Then,

$$\mathbb{P}(\tilde{I}_{D_\gamma}) \leq \mathbb{P}\biggl(\tilde{I}_{D_\gamma} \cap \Bigl(\bigcap_{D_\gamma} \tilde{E}_{D_\gamma}\Bigr)\biggr) + \mathbb{P}\biggl(\Bigl(\bigcap_{D_\gamma} \tilde{E}_{D_\gamma}\Bigr)^c\biggr). \tag{3.172}$$

Using Corollary 3.24 we can bound the second term as $\exp(-c' t^{\beta-1/3})$. By Propositions 3.35 and 3.36 we obtain

$$\begin{aligned} L_{\mathcal{L}^- \to E}(\omega) &\leq (\tilde{\mu}_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon)t = \tilde{\mu}t + (\tilde{\mu}_\gamma + \mu_{\mathrm{pp},\gamma} + \varepsilon - \tilde{\mu})t \\ &\leq \tilde{\mu}t - (\tilde{C}t^{\beta-1/3})t^{1/3} \end{aligned} \tag{3.173}$$

for $\omega \in \tilde{I}_{D_\gamma}$ and at the same time in each of the $\tilde{E}_{D_\gamma}$'s. Therefore,

$$\mathbb{P}\biggl(\tilde{I}_{D_\gamma} \cap \Bigl(\bigcap_{D_\gamma} \tilde{E}_{D_\gamma}\Bigr)\biggr) \leq \mathbb{P}\left(L_{\mathcal{L}^- \to E} \leq \mu t - (\tilde{C}t^{\beta-1/3})t^{1/3}\right). \tag{3.174}$$

The following is slightly different from the previous proof. Denote by

$$Z^- = (\kappa t/2, -\kappa t/2), \quad B = (\kappa t, 0), \tag{3.175}$$

where $\kappa = \eta - \alpha^2/(2-\alpha)^2$. Then, since $L_{\mathcal{L}^-\to E} \geq L_{Z^-\to B} + L_{B\to E}$, it follows that

$$\begin{aligned}(3.174) &\leq \mathbb{P}\left(L_{Z^-\to B} + L_{B\to E} \leq \tilde{\mu}t - (\tilde{C}t^{\beta-1/3})t^{1/3}\right)\\ &\leq \mathbb{P}\bigg(L_{Z^-\to B} \leq \tilde{\mu}_1 t - \frac{\tilde{C}t^{\beta-1/3}}{2}t^{1/3}\bigg) + \mathbb{P}\bigg(L_{B\to E} \leq \tilde{\mu}_2 t - \frac{\tilde{C}t^{\beta-1/3}}{2}t^{1/3}\bigg),\end{aligned} \tag{3.176}$$

where $\tilde{\mu}_1 = 2\kappa/\alpha$ and $\tilde{\mu}_2 = \tilde{\mu} - \tilde{\mu}_1 = 4/(2-\alpha)^2$. We can finally apply the bound of Proposition 3.10 to the two point-to-point problems and finish the proof as in Proposition 3.32. □

Proof of Proposition 3.27. The proof is a straightforward consequence of Proposition 3.37, since the cardinality of the family of points $\{D_\gamma\}_{\gamma\in[0,1-t^{\beta-1}]}$ is only of order t. □

3.3.3 Proof of Theorems 2.4, 2.5 and 2.6, Verification of Assumptions 1–3

Proof of Theorem 2.4 / Theorem 3.1. Assumption 1 is fulfilled through Propositions 3.14 and 3.23. Note that taking $\hat{\sigma}_\eta, \tilde{\sigma}_\eta$ or $\hat{\sigma}_{\eta_0}, \tilde{\sigma}_{\eta_0}$ yields the same limits. Let $\tilde{\mu}_\eta = 2(\eta/\alpha + 1/(2-\alpha))$ and $\mu_\eta = 2(1+\eta)$ be the leading order terms of the two LPP problems for η. The shift in G_2 comes from the fact that $\frac{(\mu-\tilde{\mu}_\eta)t}{t^{1/3}} = -\frac{2u}{\alpha}$ and $\frac{(\mu-\mu_\eta)t}{t^{1/3}} = -2u$. Assumption 2 is directly satisfied via Propositions 3.8 and 3.14 with $E^+ = (\eta t - t^\nu, t - t^\nu)$. Finally, Assumption 3 is precisely the content of Propositions 3.25 and 3.27. □

Proof of Theorem 2.5 / Theorem 3.2. Clearly any maximizing path $\pi^{\max}_{\mathcal{L}^+\to(\eta t,t)}$ starts off at $(-\lfloor \beta_0 t + bt^{1/3}\rfloor, 0)$. Let $\tilde{\mu}_\eta = 2(\eta/\alpha + 1/(2-\alpha))$ and $\mu_{\mathrm{pp},\eta} = 4 + 2(u+b)t^{-2/3}$ be the leading order terms of the two LPP problems for η.Then we have $\frac{(4-\tilde{\mu}_\eta)t}{t^{1/3}} = -\frac{2u}{\alpha}$, $\frac{(4-\mu_{\mathrm{pp},\eta})t}{t^{1/3}} = -2(u+b)$. Assumption 1 is fulfilled through Propositions 3.8 and 3.23. The requirement $\alpha < 1$ comes from the requirement $\eta_0 > \alpha^2/(2-\alpha)^2$ from Proposition 3.23. Assumption 2 is directly satisfied via Propositions 3.8 and 3.14. Finally, Assumption 3 is precisely the content of Propositions 3.28 and 3.27. □

Proof of Theorem 2.6 / Theorem 3.3. Any maximizing path $\pi^{\max}_{\mathcal{L}^+\to(\eta t,t)}$ starts off from $(-\lfloor \beta t\rfloor, 0)$. Let $\mu_{\mathrm{pp},\eta} = (1+\sqrt{1+\beta})^2 + (1+\frac{1}{\sqrt{1+\beta}})ut^{-2/3}$ be the leading order of $L_{\mathcal{L}^+\to(\eta t,t)}$, i.e. $\frac{(\mu-\mu_{\mathrm{pp},\eta})t}{t^{1/3}} = (1+\frac{1}{\sqrt{1+\beta}})u$, so Assumption 1 is fulfilled through Proposition 3.8 with $G_1(s) = F_{\mathrm{GUE}}(s/\sigma - u(1+1/\sqrt{1+\beta})/\sigma)$. Note now $L_{\mathcal{L}^-\to(\eta t,t)} \overset{d}{=} L_{0\to(\eta t,(1+\beta)t)}$, implying that the leading order of this LPP is $\mu_{\mathrm{pp},\gamma} = (1+\sqrt{1+\beta})^2 + (1+\sqrt{1+\beta})ut^{-2/3}$ so that $\frac{(\mu-\mu_{\mathrm{pp},\gamma})t}{t^{1/3}} = -u(1+\sqrt{1+\beta})$, which shows $G_2(s) = F_{\mathrm{GUE}}(s/\sigma - u(1+\sqrt{1+\beta})/\sigma)$.

Assumption 2 is directly satisfied via Proposition 3.8. Finally, Assumption 3 holds by Proposition 3.29.

□

3.4 Derivation of the kernel for TASEP with α-particles

In order to prove Proposition 3.16 we first study the system with only M $\alpha-$particles. We denote by $\mathbb{P}^{(M)}$ the probability measure for this system. The system we are considering is then recovered by taking the $M \to \infty$. We first recall the generic theorem (Theorem 1.15) for joint distributions in TASEP, specialized to our jump rates and initial configuration.

Proposition 3.38 (Proposition 3.1 in [18])**.** *Let us consider particles starting from*

$$x_j(0) = 2(M-j), 1 \le j \le M, \quad x_j(0) = -j + M, j > M \tag{3.177}$$

and having jump rates v_j given by

$$v_j = \alpha, 1 \le j \le M, \quad v_j = 1, j > M. \tag{3.178}$$

Denote $x_j(t)$ the position of particle j at time t. Then

$$\mathbb{P}^{(M)}(x_n(t) > s) = \det(1 - \chi_s K_{n,t} \chi_s)_{\ell^2(\mathbb{Z})}, \tag{3.179}$$

where $\chi_s = \mathbf{1}_{(-\infty,s)}$. The kernel $K_{n,t}$ is given by

$$K_{n,t}(x_1, x_2) = \sum_{k=1}^{n} \Psi^{n,t}_{n-k}(x_1) \Phi^{n,t}_{n-k}(x_2). \tag{3.180}$$

The functions $\Psi^{n,t}_{n-j}$ are given by

$$\Psi^{n,t}_{n-j}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w} \frac{e^{tw}}{w^{x - x_j(0) + n - j}} \prod_{k=j+1}^{n} (w - v_k). \tag{3.181}$$

The functions $\{\Phi^{n,t}_{n-j}\}_{1 \le j \le n}$ are characterized by the two conditions:

$$\langle \Psi^{n,t}_{n-j}, \Phi^{n,t}_{n-k} \rangle := \sum_{x \in \mathbb{Z}} \Psi^{n,t}_{n-j}(x) \Phi^{n,t}_{n-k}(x) = \delta_{j,k}, \quad 1 \le j, k \le n, \tag{3.182}$$

and

$$\mathrm{span}\{\Phi^{n,t}_{n-j}(x), 1 \le j \le n\} = \mathrm{span}\{1, x, \ldots, x^{n-M-1}, \alpha^x, x\alpha^x, \ldots, x^{M-1}\alpha^x\}. \tag{3.183}$$

The following lemma gives explicit formulas for the orthogonal functions Φ, Ψ defined in the preceeding proposition. We only give them for $n \geq M+1$, since these are the ones we need.

Lemma 3.39. *Let $n \geq M+1$. We then have two cases:*

(a) for $j = M+1, \ldots, n$,

$$\begin{aligned} \Psi^{n,t}_{n-j}(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)}}{(w+1)^{x-M+n}} w^{n-j} \\ \Phi^{n,t}_{n-j}(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-j+1}} \end{aligned} \tag{3.184}$$

(b) for $j = 1, \ldots, M$,

$$\begin{aligned} \Psi^{n,t}_{n-j}(x) &= \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{w^{n-M}(w+1-\alpha)^{M-j}}{(w+1)^{x-2M+n+j}} e^{t(w+1)} \\ \Phi^{n,t}_{n-j}(x) &= \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_{0,v}} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-M}} \\ &\qquad \times \frac{2v+2-\alpha}{((v+1)(v+1-\alpha))^{M-j+1}} \frac{1}{z-v} \end{aligned} \tag{3.185}$$

Proof. The formulas for $\Psi^{n,t}_{n-j}$ are easily obtained by plugging (3.177),(3.178) into (3.181).

In case (a), using the derivative formula for the residue, one sees that $\Phi^{n,t}_{n-j}$ is a polynomial of degree $n-j$ and thus

$$\operatorname{span}\{\Phi^{n,t}_{n-j}(x), j = M+1, \ldots, n\} = \operatorname{span}\{1, x, \ldots, x^{n-M-1}\}. \tag{3.186}$$

In case (b), taking the residue at $z = v$, one gets

$$\Phi^{n,t}_{n-j}(x) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \frac{(2v+2-\alpha)(v+1)^{x-2M+j-1}}{e^{t(v+1)} v^{n-M} (v+1-\alpha)^{M-j+1}} \tag{3.187}$$

$$+ \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-M}} \frac{2v+2-\alpha}{((v+1)(v+1-\alpha))^{M-j+1}} \frac{1}{z-v}. \tag{3.188}$$

Now, (3.187) $= \alpha^x p_{M-j}(x)$, where p_{M-j} is a polynomial of degree $M-j$. For (3.188), we choose the integration paths such that $|v| > |z|$, apply the identity $(z-v)^{-1} = -v^{-1} \sum_{\ell \geq 0} (z/v)^\ell$, and obtain

$$(3.188) = \sum_{\ell \geq 0} \frac{-1}{(2\pi\mathrm{i})^2} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \frac{(2v+2-\alpha) v^{-(\ell+1)}}{((v+1)(v+1-\alpha))^{M-\ell+1}} \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-M-\ell}}, \tag{3.189}$$

which for $\ell = 0, \dots, n-M-1$ is a polynomial of degree $n-M-1-\ell$, and is 0 for larger ℓ. Therefore (3.183) holds.

Next we check the biorthogonality relations (3.182). We shall recurrently use

$$\sum_{x \geq M-n} \left(\frac{z+1}{w+1}\right)^{x-M+n} = \frac{w+1}{w-z}, \tag{3.190}$$

which holds if $|w+1| > |z+1|$.

Case $M+1 \leq j, k \leq n$:

$$\begin{aligned} \langle \Psi^{n,t}_{n-j}, \Phi^{n,t}_{n-k} \rangle &= \sum_{x \in \mathbb{Z}} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)} w^{n-j}}{(w+1)^{x-M+n}} \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-j+1}} \\ &= \sum_{x \geq M-n} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)} w^{n-j}}{(w+1)^{x-M+n}} \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-j+1}} \end{aligned} \tag{3.191}$$

since for $x < M-n$ the functions $\Psi^{n,t}_{n-j}(x) = 0$. We can now choose the integration paths such that $|w+1| > |z+1|$. Applying (3.190), the pole at $w=-1$ disappears and instead there is a simple pole at $w = z$,

$$\begin{aligned} (3.191) &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \frac{1}{e^{t(z+1)} z^{n-j+1}} \oint_{\Gamma_z} dw \frac{e^{t(w+1)} w^{n-j}}{w-z} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \frac{1}{z^{j-k+1}} = \delta_{j,k}. \end{aligned} \tag{3.192}$$

Case $M+1 \leq j \leq n$ and $1 \leq k \leq M$: Also in this case we first restrict the sum over $x \geq M-n$, use (3.190), and integrate out the remaining simple pole at $w = z$, with the result

$$\begin{aligned} \langle \Psi^{n,t}_{n-j}, \Phi^{n,t}_{n-k} \rangle &= \sum_{x \in \mathbb{Z}} \frac{1}{(2\pi \mathrm{i})^3} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)} w^{n-j}}{(w+1)^{x-M+n}} \\ &\times \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_{0,v}} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-M}} \frac{2v+2-\alpha}{((v+1)(v+1-\alpha))^{M-k+1}} \frac{1}{z-v} \\ &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_{0,v}} \mathrm{d}z \frac{1}{z^{j-M}} \frac{(2v+2-\alpha)}{(z-v)((v+1)(v+1-\alpha))^{M-k+1}}. \end{aligned} \tag{3.193}$$

Since $j > M$, for $|z| \to \infty$, the integrand in z goes to zero at least as fast as $1/z^2$ and it does not contain any other poles than $z = 0, v$. Therefore, the integrand in z has no pole at infinity and consequently (3.193) = 0.

Case $1 \leq j,k \leq M$: Also in this case we first restrict the sum over $x \geq M-n$, use (3.190), and integrate out the remaining simple pole at $w=z$. This gives

$$\langle \Psi^{n,t}_{n-j}, \Phi^{n,t}_{n-k} \rangle = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_{0,v}} \mathrm{d}z \frac{(2v+2-\alpha)((z+1)(z+1-\alpha))^{M-j}}{((v+1)(v+1-\alpha))^{M-k+1}(z-v)}. \tag{3.194}$$

Now, the pole at $z=0$ disappeared and the only contribution comes from the simple pole $z=v$, i.e.,

$$(3.194) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \frac{2v+2-\alpha}{((v+1)(v+1-\alpha))^{j-k+1}} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}u \frac{1}{u^{j-k+1}} = \delta_{j,k}, \tag{3.195}$$

where we used the change of variables $u=(v+1)(v+1-\alpha)$.

Case $1 \leq j \leq M$ and $M+1 \leq k \leq n$: Doing the first steps as in the three other cases above, we get

$$\begin{aligned} \langle \Psi^{n,t}_{n-j}, \Phi^{n,t}_{n-k} \rangle &= \sum_{x\in\mathbb{Z}} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \frac{w^{n-M}(w+1-\alpha)^{M-j}}{(w+1)^{x+n-2M+j}} e^{t(w+1)} \\ &\quad \times \oint_{\Gamma_0} \mathrm{d}z \frac{(z+1)^{x-M+n}}{e^{t(z+1)} z^{n-k+1}} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \frac{((z+1)(z+1-\alpha))^{M-j}}{z^{M-k+1}} = 0, \end{aligned} \tag{3.196}$$

since, for $k>M$ the pole at $z=0$ disappears. □

Later, we will take the $M \to \infty$ limit with $n-M$ finite. To this end we give a compact form of $K_{n,t}$.

Corollary 3.40. *Let $K_{n,t}$ be the kernel defined in* (3.180)*. Then*

$$K_{n+M,t} = K^{(0)}_{n,M,t} + K^{(1)}_{n,t} + K^{(2)}_{n,t}, \tag{3.197}$$

where $K^{(1)}_{n,t}$ and $K^{(2)}_{n,t}$ are given in (3.81) and

$$\begin{aligned} K^{(0)}_{n,M,t}(x_1,x_2) = \frac{-1}{(2\pi \mathrm{i})^3} \oint_{\Gamma_{-1}} \frac{\mathrm{d}w}{w+1} \oint_{\Gamma_{\alpha-1}} \mathrm{d}v \oint_{\Gamma_{0,v}} \mathrm{d}z \frac{e^{t(w+1)} w^n}{(w+1)^{x_1+n}} \frac{(z+1)^{x_2+n}}{e^{t(z+1)} z^n} \\ \times \frac{1}{z-v} \frac{2v+2-\alpha}{(v-w)(v+w+2-\alpha)} \left(\frac{(w+1)(w+1-\alpha)}{(v+1)(v+1-\alpha)} \right)^M. \end{aligned} \tag{3.198}$$

Proof. We first show that

$$K^{(0)}_{n,M,t}(x_1,x_2) + K^{(1)}_{n,t}(x_1,x_2) = \sum_{k=1}^{M} \Psi^{n+M,t}_{n+M-k}(x_1) \Phi^{n+M,t}_{n+M-k}(x_2). \tag{3.199}$$

We have

$$\begin{aligned}
&\sum_{k=1}^{M}\Psi_{n+M-k}^{n+M,t}(x_1)\Phi_{n+M-k}^{n+M,t}(x_2)\\
&\quad=\sum_{k=1}^{M}\frac{1}{(2\pi\mathrm{i})^3}\oint_{\Gamma_{\alpha-1}}\mathrm{d}v\oint_{\Gamma_{0,v}}\mathrm{d}z\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\frac{e^{t(w+1)}w^n}{(w+1)^{x_1+n}}\frac{(z+1)^{x_2+n}}{e^{t(z+1)}z^n}\\
&\qquad\times\frac{2v+2-\alpha}{(v+1)(v+1-\alpha)}\left(\frac{(w+1-\alpha)(w+1)}{(v+1)(v+1-\alpha)}\right)^{M-k}\frac{1}{z-v}.
\end{aligned}\tag{3.200}$$

We apply a finite geometric sum formula to $q=\frac{(w+1-\alpha)(w+1)}{(v+1)(v+1-\alpha)}$. For this the contours need to satisfy $q\neq 1$. We take the contours such that

$$-\Gamma_{-1}-2+\alpha\subset\Gamma_{\alpha-1},\Gamma_{-1}\not\subset\Gamma_{\alpha-1},\Gamma_{\alpha-1}\subset\Gamma_{0,v},\quad\text{and}\quad q\neq 1.\tag{3.201}$$

Note that none of these conditions alter (3.200). An explicit choice of paths satisfying (3.201) is later given in (3.208). Using the linearity of the integral, we get

$$\begin{aligned}
(3.200)=&\frac{1}{(2\pi\mathrm{i})^3}\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\oint_{\Gamma_{\alpha-1,-w-2+\alpha}}\mathrm{d}v\oint_{\Gamma_{0,v}}\mathrm{d}z\frac{e^{t(w+1)}w^n}{(w+1)^{x_1+n}}\frac{(z+1)^{x_2+n}}{e^{t(z+1)}z^n}\\
&\times\frac{2v+2-\alpha}{(v-w)(v+w+2-\alpha)}\left(1-\left(\frac{(w+1-\alpha)(w+1)}{(v+1)(v+1-\alpha)}\right)^{M}\right)\frac{1}{z-v}\\
=&\frac{1}{(2\pi\mathrm{i})^3}\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\oint_{\Gamma_{-w-2+\alpha}}\mathrm{d}v\oint_{\Gamma_{0,v}}\mathrm{d}z\frac{e^{t(w+1)}w^n}{(w+1)^{x_1+n}}\frac{(z+1)^{x_2+n}}{e^{t(z+1)}z^n}\\
&\times\frac{2v+2-\alpha}{(v-w)(v+w+2-\alpha)}\frac{1}{z-v}\\
&-\frac{1}{(2\pi\mathrm{i})^3}\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\oint_{\Gamma_{\alpha-1}}\mathrm{d}v\oint_{\Gamma_{0,v}}\mathrm{d}z\frac{e^{t(w+1)}w^n}{(w+1)^{x_1+n}}\frac{(z+1)^{x_2+n}}{e^{t(z+1)}z^n}\\
&\times\frac{2v+2-\alpha}{(v-w)(v+w+2-\alpha)}\left(\frac{(w+1-\alpha)(w+1)}{(v+1)(v+1-\alpha)}\right)^{M}\frac{1}{z-v}.
\end{aligned}\tag{3.202}$$

Here we used that in the first triple integral the pole $v=\alpha-1$ is no longer present. Plugging in the remaining residue at $v=-w-2+\alpha$ yields then

$$(3.202)=K_{n,M,t}^{(0)}(x_1,x_2)+K_{n,t}^{(1)}(x_1,x_2).\tag{3.203}$$

Next we define

$$K_{n,t}^{(2)}(x_1,x_2):=\sum_{k=M+1}^{n+M}\Psi_{n+M-k}^{n+M-k,t}(x_1)\Phi_{n+M-k}^{n+M-k,t}(x_2).\tag{3.204}$$

Note that Φ_{n+M-k} is zero for $k\geq n+M+1$, thus

$$K_{n,t}^{(2)}(x_1,x_2)=\sum_{k=M+1}^{\infty}\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}z\oint_{\Gamma_{-1}}\frac{\mathrm{d}w}{w+1}\frac{e^{t(w+1)}w^{n+M-k}}{(w+1)^{x_1+n}}\frac{(z+1)^{x_2+n}}{e^{t(z+1)}z^{n+M-k+1}}.\tag{3.205}$$

Assuming the contours are such that $|w| > |z|$, taking geometric series yields

$$K^{(2)}_{n,t}(x_1, x_2) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{-1,z}} \frac{\mathrm{d}w}{w+1} \frac{e^{t(w+1)} w^n}{(w+1)^{x_1+n}} \frac{(z+1)^{x_2+n}}{e^{t(z+1)} z^n} \frac{1}{w-z}. \tag{3.206}$$

Finally, it is straightforward to check that the contribution of the simple pole at $w = z$ is zero, so that we can drop it in the final expression of $K^{(2)}_{n,t}$. □

Proposition 3.41. *Let* $K^{(0)}_{n,M,t}, K^{(1)}_{n,t}, K^{(2)}_{n,t}$ *be as in* (3.81) *and* (3.198)*. Then, for* $x_1, x_2 \leq \ell$*, we have the following bounds.*

$$\begin{aligned} |K^{(0)}_{n,M,t}(x_1, x_2)| &\leq C\, e^{cx_2} q^M \\ |K^{(1)}_{n,t}(x_1, x_2)| &\leq C\, e^{cx_2} \qquad , \\ |K^{(2)}_{n,t}(x_1, x_2)| &\leq C\, e^{cx_2} \end{aligned} \tag{3.207}$$

with $q \in [0, 1)$*,* $c > 0$ *a constant, and* C *depends only on* ℓ, n, t*.*

Proof. To bound $|K^{(0)}_{n,M,t}(x_1, x_2)|$, we set

$$\Gamma_{-1} = -1 + r_1 e^{\mathrm{i}s_1} \quad \Gamma_{\alpha-1} = \alpha - 1 + r_2 e^{\mathrm{i}s_2} \quad \Gamma_{0,v} = r_3 e^{\mathrm{i}s_3} \tag{3.208}$$

with $r_1 = \frac{\alpha^2}{10}, r_2 = \frac{\alpha}{\sqrt{1.5}}, r_3 = 1 - \alpha + r_2 + \frac{|r_1+r_2-\alpha|}{2}$. It is straightforward to check that (3.208) satisfy (3.201). We will bound the different parts of $K^{(0)}_{n,M,t}$. First we note

$$\begin{aligned} q &:= \frac{\max_{\Gamma_{-1}} |(w+1)(w+1-\alpha)|}{\min_{\Gamma_{-1}} |(v+1)(v+1-\alpha)|} < \frac{\sqrt{1.5}|\alpha(-1-\alpha/10)|}{10(1-1/\sqrt{1.5}))} < 1 \\ &\frac{\max_{\Gamma_{0,v}} |(z+1)^{x_2+n}|}{\min_{\Gamma_{0,v}} |e^{t(z+1)} z^n|} \leq C\,(1+r_3)^{x_2} \leq Ce^{cx_2} \end{aligned} \tag{3.209}$$

The remaining parts can now be bounded by a constant:

$$\begin{aligned} &\frac{\max_{\Gamma_{\alpha-1}} |2v+2-\alpha|}{\min_{\Gamma_{-1},\Gamma_{\alpha-1}} |(v-w)(v+w+2-\alpha)|} \leq \frac{\alpha+2r_2}{(r_2-r_1)(\alpha-\alpha/\sqrt{1.5}-\alpha^2/10)} < C \\ &\frac{1}{\min_{\Gamma_{-1},\Gamma_{0,v}} |z-v|} < C, \\ &\frac{\max_{\Gamma_{-1}} |e^{t(w+1)} w^n|}{\min_{\Gamma_{-1}} |(w+1)^{x_1+n}|} \leq \tilde{C}\, r_1^{-x_1} \leq C, \end{aligned} \tag{3.210}$$

where the last estimate in (3.210) holds since $0 < r_1 < 1$ and $x_1 \leq \ell$. Putting these bounds together gives the estimate for $K^{(0)}_{n,M,t}$. Note that the contour for z contains $\alpha - 2 - w$.

Therefore, in $K^{(1)}_{n,t}$ we can choose the same contours for z, w as before and use the estimates from (3.209), (3.210). Noting

$$\min_{\Gamma_{-1},\Gamma_{0,\alpha-2-w}} |z-(\alpha-2-w)|^{-1} \leq C, \tag{3.211}$$

one gets the same bound as for $K^{(0)}_{n,M,t}$, only without the q^M.

As for $K^{(2)}_{n,t}$, we can again choose the same contours for z, w as before. Since $|w-z|$ is bounded from below, we get the same estimate as for $K^{(1)}_{n,t}$. □

Now we are ready to proof Proposition 3.16.

Proof of Proposition 3.16. Denote for clarity by $x^M_{n+M}(t)$ the position of particle number $n+M$ at time t in the system with M slow particles (defined via (3.177) and (3.178)), and by $x_n(t)$ the position of particle n at time t in the system with infinitely many slow particles (defined via (3.78) and (3.79)). First we note that

$$\lim_{M\to\infty} \mathbb{P}^{(M)}(x^M_{n+M}(t) > s) = \mathbb{P}(x_n(t) > s). \tag{3.212}$$

This follows since $x^M_{n+M}(0) = x_n(0)$ and by the fact that in TASEP the position of a particle up to a fixed time t depends only on finitely many other particles with probability one, as is seen from a graphical construction of it. Therefore, by Corollary 3.40, it remains to prove

$$\lim_{M\to\infty} \det(1-\chi_s K_{n+M,t}\chi_s)_{\ell^2(\mathbb{Z})} = \det(1-\chi_s \tilde{K}_{n,t}\chi_s)_{\ell^2(\mathbb{Z})}, \tag{3.213}$$

where we used the notation $K_{n+M,t} = K^{(0)}_{n,M,t} + K^{(1)}_{n,t} + K^{(2)}_{n,t}$.

By the bounds in (3.207), we know that $K^{(0)}_{n,M,t}$ converges pointwise to 0. Thus, it remains to show that also the Fredholm determinant converges. Consider the Fredholm series expansion

$$\det(1-\chi_s K_{n+M,t}\chi_s)_{\ell^2(\mathbb{Z})} = \sum_{m\geq 0} \frac{(-1)^m}{m!} \sum_{x_1\leq s} \dots \sum_{x_m\leq s} \det[K_{n+M,t}(x_i,x_j)]_{1\leq i,j\leq m}. \tag{3.214}$$

By (3.207), we have

$$\left| \frac{(-1)^n}{n!} \det\left(K_{n+M,t}(x_k,x_l)\right)_{1\leq k,l\leq n} \right| \leq \frac{1}{n!} e^{c(x_1+\dots+x_n)} C^n (2+q^M)^n n^{n/2}, \tag{3.215}$$

where $n^{n/2}$ is the Hadamard bound for matrices with entries of absolute value less or equal than 1. Since $q<1$, we may replace $2+q^M$ by 3 to get a summable uniform bound. Thus we may apply dominated convergence to (3.214) to take the $M\to\infty$ inside the sum, which proves the result. □

Chapter 4

Critical Scaling

In this Chapter we prove the main result we obtained in the critical scaling, Theorem 2.8 (proven in Section 4.2) and perform a numerical study (done in Section 4.1.2). Furthermore, we explain that the limit process we obtain also appears in last passage percolation and state the corresponding result, Theorem 4.3.

4.1 Model and limit process

Recall that in this chapter we consider the following specialization of TASEP:

$$x_n(0) = -2n, n \in \mathbb{Z}, \quad v_n = \begin{cases} 1, & n > 0, \\ \alpha, & n \leq 0., \end{cases} \tag{4.1}$$

with

$$\alpha = 1 - 2a(t/2)^{-1/3}. \tag{4.2}$$

Now we define the limit process $\mathcal{M}_a = \lim_{t\to\infty} X_t$ of Theorem 2.8, recall that the scaling we consider is

$$n(u,t) = \left\lfloor \frac{t}{4} + (a+u)(t/2)^{2/3} \right\rfloor, \quad x(u,t) = \left\lfloor -2(a+u)(t/2)^{2/3} \right\rfloor, \tag{4.3}$$

and the process of rescaled particle positions is defined as

$$u \mapsto X_t(u) = \frac{x_{n(u,t)} - x(u,t)}{-(t/2)^{1/3}}. \tag{4.4}$$

Definition 4.1 (The limit process $\mathcal{M}_a$)**.** *Define the extended kernel*

$$
\begin{aligned}
K_a(u_1,\xi_1;u_2,\xi_2) &= -\frac{1}{\sqrt{4\pi(u_2-u_1)}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(u_2-u_1)}\right)\mathbf{1}_{(u_2>u_1)} \\
&+\frac{-1}{(2\pi \mathrm{i})^2}\int_{\gamma_+}\mathrm{d}w\int_{\gamma_-}\mathrm{d}z\frac{e^{w^3/3+(u_2+a)w^2-\xi_2 w}}{e^{z^3/3+(u_1+a)z^2-\xi_1 z}}\frac{2w}{(z-w)(z+w)} \\
&+\frac{1}{(2\pi \mathrm{i})^2}\int_{\Gamma_+}\mathrm{d}w\int_{\Gamma_-}\mathrm{d}z\frac{e^{w^3/3+(u_2-a)w^2-w(\xi_2+4au_2)+4u_2a^2}}{e^{z^3/3+(u_1-a)z^2-z(\xi_1+4au_1)+4u_1a^2}}\frac{2(w-2a)}{(z+w)(z-w+4a)}.
\end{aligned}
\tag{4.5}
$$

The curves can be chosen as follows. Let $\theta=\max\{|u_1|+|a|,|u_2|+|a|\}$. For any choice of $r_\pm, R_\pm$ satisfying $r_+>-r_->\theta$ and $-R_->R_+>\theta+4|a|$, we can set $\gamma_\pm=r_\pm+\mathrm{i}\mathbb{R}$ and $\Gamma_\pm=R_\pm+\mathrm{i}\mathbb{R}$ (oriented with increasing imaginary parts). The limit process $\mathcal{M}_a$ is defined by its finite-dimensional distribution: for any given $u_1<u_2<\cdots<u_m$,

$$
\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{M}_a(u_k)\le s_k\}\right)=\det(1-\chi_s K_a\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}
\tag{4.6}
$$

where $\chi_s(u_k,x)=\mathbf{1}_{(s_k,\infty)}$. An explicit expression of K_a in terms of Airy functions is given in Section 4.3.

Remark 4.2. *In some special cases or limits we recover previous known processes. For example:*

(a) For $a=0$ we have the flat TASEP and see the Airy$_1$ process (Definition 1.17) : $\mathcal{M}_0(u)=2^{1/3}\mathcal{A}_1(2^{-2/3}u)$.

(b) When $a\to-\infty$ a rarefaction fan is created. At his left edge, $\mathcal{M}_a$ becomes the Airy$_{2\to1}$ process (Definition 1.23) : $\lim_{a\to-\infty}\mathcal{M}_a(u-a)=\mathcal{A}_{2\to1}(u)$. Inside the rarefaction fan $\mathcal{M}_a$ becomes the Airy$_2$ process (Definition 1.20). For instance, in the middle of the rarefaction fan: $\lim_{a\to-\infty}\mathcal{M}_a(u)+(u+a)^2=\mathcal{A}_2(u)$.

(c) For $a>0$, there is a shock and $\mathcal{M}_a$ is a transition process between two $\mathcal{A}_1$ processes. Indeed, $\lim_{M\to\infty}\mathcal{M}_a(u\pm M)=2^{1/3}\mathcal{A}_1(2^{-2/3}u)$.

4.1.1 Last passage percolation

The limit process $\mathcal{M}_a$ occurs in a related last passage percolation (LPP) model as well. As usual, to each site (i,j) of $\mathbb{Z}^2$ we assign an independent random variable $\omega_{(i,j)}$ with

$$
\omega_{(i,j)}\sim\exp(v_j).
\tag{4.7}
$$

The connection between TASEP and LPP gives

$$\mathbb{P}\Big(\bigcap_{k=1}^{r}\{x_{n_k}(t)\geq m_k-n_k\}\Big)=\mathbb{P}\Big(\bigcap_{k=1}^{r}\{L_{\mathcal{L}\to(m_k,n_k)}\leq t\}\Big). \tag{4.8}$$

with $\mathcal{L}=\{(-n,n),n\in\mathbb{Z}\}$. Consider the critical scaling

$$\alpha=1-2a(2\ell)^{-1/3}, \tag{4.9}$$

and focus at the position

$$m(v,\ell)=\ell-2(v+a)(2\ell)^{2/3},\quad n=\ell. \tag{4.10}$$

Define the rescaled LPP time by

$$L_\ell^{\rm resc}(v):=\frac{L_{\mathcal{L}\to(m(v,\ell),\ell)}-\big[4\ell-4(v+a)(2\ell)^{2/3}\big]}{2(2\ell)^{1/3}}. \tag{4.11}$$

Theorem 4.3. *It holds*

$$\lim_{\ell\to\infty}L_\ell^{\rm resc}(v)=\mathcal{M}_a(v) \tag{4.12}$$

in the sense of finite dimensional distributions.

In short, to prove Theorem 4.3, one starts with the relation (4.8) that gives the joint distributions of $L_\ell^{\rm resc}$ in terms of positions of TASEP particles at different times, varying around $t=4\ell$ on a $\ell^{2/3}$ scale only. By the slow-decorrelation phenomenon [29, 37] the fluctuations at different times are asymptotically the same as the fixed time fluctuations for points lying on special space-time directions (the characteristics). At fixed time, the result is exactly given by Theorem 2.8. The details of this procedure have been worked out for instance in [10, 28].

4.1.2 Numerical study

Here we numerically compute the distribution function of the process $\mathcal{M}_a$, given by a Fredholm determinant of the kernel K_a, as well as some of its basic statistics: Expectation, Variance, Skewness, Kurtosis. For the computation, we use the formula for K_a given in the Section 4.3, since Airy functions are already implemented functions in Matlab, and apply Bornemann's method for the evaluation of the Fredholm determinants, see [32], which is well-adapted for analytic kernels. Bornemann's algorithm also comes with an error control that we used, see Section 4.4 of [32].

For simplicity, we study the validity of (2.34) (in the form (4.13)), i.e., we set $u=0$. In principle, one could look also at general u, but then it has to be taken as a function of a

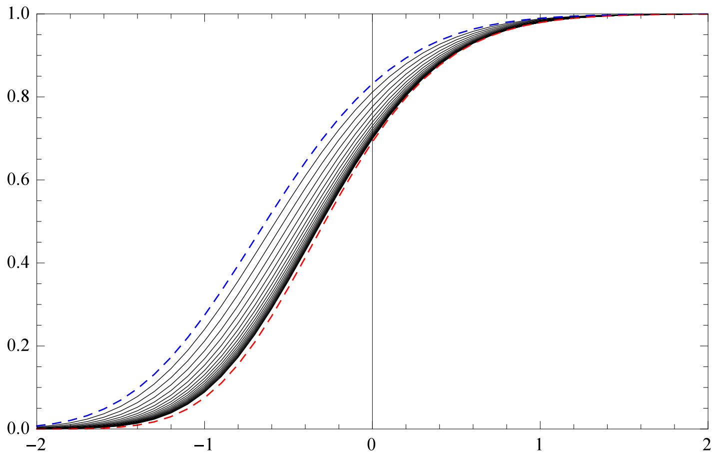

Figure 4.1: The dashed lines are $s \mapsto F_{\rm GOE}(2s)$ (the left one) and $s \mapsto F_{\rm GOE}(2s)^2$ (the right one), $s \in [-2, 2]$. The solid lines are the functions $s \mapsto G_a(s)$ for $a = 0.1, 0.2, \ldots, 1.8$. For $a = 0$, $G_0(s) = F_{\rm GOE}(2s)$, when a grows larger, $G_a(s)$ approximates the macroscopic shock distribution $F_{\rm GOE}(2s)^2$ as conjectured, see (4.13).

too (since in the unscaled process the correlation scale of the process changes from $t^{2/3}$ to $t^{1/3}$ as α varies from 1 to a value strictly less then 1).

To avoid to carry around a lot of $2^{1/3}$ factors, we rescale space by a factor $2^{1/3}$ so that (2.34) writes

$$\lim_{a\to\infty} \mathbb{P}(\mathcal{M}_a(0) \leq s2^{1/3}) = (F_{\rm GOE}(2s))^2. \tag{4.13}$$

We denote $\widetilde{K}_a(\xi_1, \xi_2) := 2^{1/3} K_a(0, 2^{1/3}\xi_1; 0, 2^{1/3}\xi_2)$. Remark that in the special case $a = 0$ we have the Airy$_1$ kernel, $\widetilde{K}_0(\xi_1, \xi_2) = \mathrm{Ai}(\xi_1 + \xi_2)$. By (4.6) we have

$$G_a(s) := \mathbb{P}(\mathcal{M}_a(0) \leq s2^{1/3}) = \det(1 - \chi_s^c \widetilde{K}_a \chi_s^c), \quad \chi_s^c = \mathbf{1}_{(s,\infty)}. \tag{4.14}$$

By Theorem 2.8, this is the $t \to \infty$ limit of the rescaled position of a particle in the microscopic shock. As mentioned earlier, we let a grow large so as to recover the macroscopic shock distribution, which for large but finite time t would correspond to the choice $a = \frac{(1-\alpha)t^{1/3}}{2^{4/3}}$. Due to the numerical limitations discussed below, we will compute G_a and its basic statistics up to $a = 1.8$. Surprisingly, already for this relatively small value of a, one is already quite close to the asymptotic behavior, see Figure 4.1. The reason for this is the following. At first approximation, from the KPZ scalings, we know that the randomness that influences the statistical properties of particle positions around the shock lives in a $t^{2/3}$ neighborhood of the characteristic lines that come together at the shock (for a proof in a special case, see [45]) and the neighborhood should be quite tight to provide the super-exponential decay of the covariance for the Airy$_1$ process, recall (1.77). Further, by a closer inspection near the end-points, we discover that at $t^{1/3}$ distance from the shock, the neighborhood is only of order $t^{1/3}$ as well (Chapter 3). These two phenomena imply that the convergence will happen on a of order 1.

Numerical Limitations

The limitation to $a \leq 1.8$ is due to the numerical difficulty of evaluating G_a for a large. As a grows, $\widetilde{K}_a$ has some terms which are of order 1 and one term which is (super-) exponentially diverging. More precisely, one has

$$\begin{aligned}
(4.76) &= e^{4a^3/3-2a(\xi_1+\xi_2)-(\xi_2-\xi_1)^2/16a-\ln(4\sqrt{\pi a})} + \varepsilon_0(a) \\
(4.75) &= 2^{-1/3}\mathrm{Ai}(2^{-1/3}(\xi_1+\xi_2))e^{-a(\xi_2-\xi_1)} + \varepsilon_1(a) \\
(4.77) &= 2^{-1/3}\mathrm{Ai}(2^{-1/3}(\xi_1+\xi_2))e^{-a(\xi_1-\xi_2)} + \varepsilon_2(a) \\
|(4.74)| &\leq c_i \max_{\lambda\geq 0} \mathrm{Ai}(\lambda+\xi_i+a^2), \quad i=1,2.
\end{aligned}, \tag{4.15}$$

where $|\varepsilon_0(a)| \leq 1/4a$, and, for $i=1,2$, $|\varepsilon_i(a)| \leq \max_{\lambda\geq 0}\mathrm{Ai}(\lambda+\xi_i+a^2)/a$ and $c_i = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(\xi_{3-i}+a^2+\lambda)$. This implies that when a increases, the ratio between the bounded terms and the large term becomes smaller than 10^{-16} machine precision and no reliable numerical evaluation is possible. In our case, already for $a \geq 4$, $\widetilde{K}_a$ (much less G_a) can no longer be computed in Matlab. For instance, Matlab computes $G_3(s) = \mathrm{NaN}$ for all tested s, $G_{2.5}(-1) = 0.0838$ with an error 0.0044, whereas $G_{1.8}(-2) = 1.4879 \cdot 10^{-4}$, with an error $5.6831 \cdot 10^{-9}$. We present the numerical computations until $a = 1.8$, since for higher values the error term in the Kurtosis becomes visible.

Generally, the computational error of $G_a(s)$ decreases as s increases since then $\widetilde{K}_a(\xi_1,\xi_2)$ needs to be computed only for small entries and the evaluation of G_a is easier (namely, the matrix whose determinant approximates G_a gets closer to the Identity matrix as s increases, see (4.3) in [32]). The statistics of G_a were computed using the `chebfun` package (see [13]), in which G_a is represented by its polynomial interpolant in Chebyshev points, for our choice in $n = 4096$ points.

In Figure 4.1 we plot $F_{\mathrm{GOE}}(2s) = G_0(s)$, $G_a(s)$ for $a \in \{0.1, 0.2, \ldots, 1.8\}$ and the conjectured $a \to \infty$ limit, namely $F_{\mathrm{GOE}}(2s)^2$. A property which is apparent from Figure 4.1 is that G_a monotonically decreases towards $F_{\mathrm{GOE}}(2s)^2$ as a grows. Indeed, for all $a, a' \in \{0, \ldots, 1.8\}$, and $s \in \{-2, -1.9, \ldots, 2\}$ we have

$$G_a(s) > F_{\mathrm{GOE}}(2s)^2, \quad G_{a'}(s) < G_a(s) \quad \text{if} \quad a < a'. \tag{4.16}$$

An analytic proof of this property does not seem to be trivial and is not available so far.

To further quantify the difference between $G_{1.8}$ and $F_{\mathrm{GOE}}(2\cdot)^2$ we computed

$$D(a) := \max_{s=-2,-1.9,\ldots,2} |F_{\mathrm{GOE}}(2s)^2 - G_{1.8}(s)|. \tag{4.17}$$

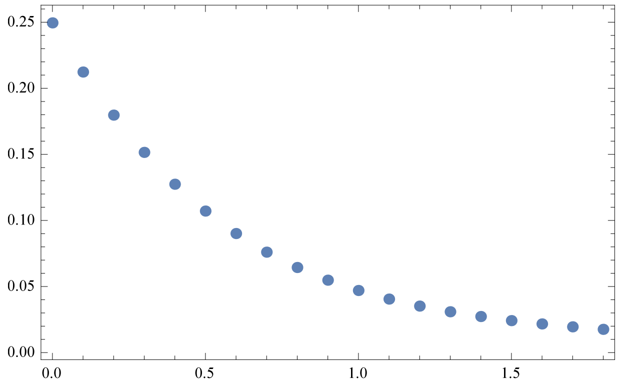

Figure 4.2: Plot of the function $a \mapsto D(a)$ defined by (4.17). On this limited interval width for a the convergence of the difference of the distribution functions is almost exponentially fast.

	Expectation	Variance	Skewness	Kurtosis
G_0	$-0.6033; 144\%$	$0.4019; 31\%$	$0.2935; -25\%$	$3.1652; -4.3\%$
$G_{0.3}$	$-0.4524; 83\%$	$0.3816; 24\%$	$0.3028; -23\%$	$3.1811; -3.9\%$
$G_{0.6}$	$-0.3632; 47\%$	$0.3624; 17\%$	$0.3127; -20\%$	$3.1988; -3.3\%$
$G_{0.9}$	$-0.3145; 27\%$	$0.3466; 13\%$	$0.3240; -17\%$	$3.2184; -2.7\%$
$G_{1.2}$	$-0.2889; 17\%$	$0.3353; 8.9\%$	$0.3359; -14\%$	$3.2377; -2.1\%$
$G_{1.5}$	$-0.2751; 11\%$	$0.3277; 6.4\%$	$0.3469; -11\%$	$3.2540; -1.6\%$
$G_{1.8}$	$-0.2670; 8.2\%$	$0.3226; 4.7\%$	$0.3561; -9.1\%$	$3.2658; -1.3\%$
$\mathbf{F}_{\mathrm{GOE}}(\mathbf{2}\cdot)^{\mathbf{2}}$	-0.2468	0.3080	0.3917	3.3086

Table 4.1: Data of the basic statistics and their relative difference to the conjectured limit distribution $F_{\mathrm{GOE}}(2\,\cdot)^2$ for a few values of a.

(4.16) and (4.17) are compatible with the conjecture (4.13), but to have a further more reliable verification we study numerically the basic statistics too. The reason is that the distribution functions might be optically close but still be different. For example, the plots of the GUE and GOE Tracy-Widom distribution functions scaled to have both average 0 and variance 1, are almost indistinguishable. However, by looking at their skewness and kurtosis one can clearly differentiate between them.

In Figure 4.3 we plot the basic statistics of G_a and compare them with those of $F_{\mathrm{GOE}}(2\,\cdot)^2$. The approximation is fastest for the expectation, and slowest for the kurtosis (though the observation window for a is too small to quantify the different rates of convergence). Finally, let us resume in Table 4.1 the basic statistics of G_a in comparison to $F_{\mathrm{GOE}}(2\cdot)^2$.

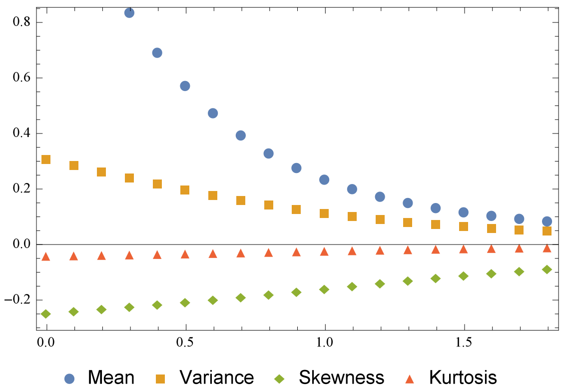

Figure 4.3: Relative difference between the basic statistics of G_a and of the conjectured limiting distribution, $F_{\rm GOE}(2\cdot)^2$.

4.2 Asymptotic analysis - Proof of Theorem 2.8

In Section 4.2.1 we derive the finite time kernel, whose Fredholm determinant gives us the joint distributions of TASEP particle positions, see Proposition 4.4. For the derivation we first need to consider the case of a finite number M of α-particles and then take the $M\to\infty$ limit. In Section 4.2.2 we then perform the asymptotic analysis and complete the proof of Theorem 2.8.

4.2.1 Finite time formula

Taking the limit of the situation with finitely many slow particles we obtain the following result.

Proposition 4.4. *Consider Two-Speed TASEP as defined in* (4.1). *Then the joint distribution of the positions of m normal particles with labels $0<n_1<n_2<\ldots<n_m$ at time t is given by*

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{x_{n_k}(t)>s_k\}\right)=\det(\mathbb{1}-\chi_s K\chi_s)_{\ell^2(\{n_1,\ldots,n_m\}\times\mathbb{Z})},\tag{4.18}$$

with $K=-\phi+K^1+K^2$, where $\chi_s(n_k,x)=\mathbf{1}_{(-\infty,s_k]}(x)$ and[1]

$$\phi(n_1,x_1;n_2,x_2)=\frac{1}{2\pi{\rm i}}\oint_{\Gamma_0}\frac{{\rm d}w}{w}\frac{(w-1)^{n_1-n_2}}{w^{x_1-x_2+n_1-n_2}}\mathbf{1}_{\{n_1<n_2\}},\tag{4.19}$$

[1]Recall that, for a set S, the notation Γ_S means a simple path anticlockwise oriented enclosing only poles of the integrand belonging to the set S.

$$\begin{aligned} K^1(n_1,x_1;n_2,x_2) =& \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}v \oint_{\Gamma_{0,-v}} \frac{\mathrm{d}w}{w}\frac{e^{tw}(w-1)^{n_1}}{w^{x_1+n_1}} \\ &\times \frac{(1+v)^{x_2+n_2}}{e^{t(v+1)}v^{n_2}}\frac{1+2v}{(w+v)(w-v-1)}, \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} K^2(n_1,x_1;n_2,x_2) =& \frac{-1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w \oint_{\Gamma_{0,\alpha-1-w}} \mathrm{d}v \frac{e^{tw}(w-1)^{n_1}}{w^{x_1+n_1+1}}\frac{(1+v)^{x_2+n_2}}{e^{t(v+1)}v^{n_2}} \\ &\times \frac{1+2v}{(v+w+1-\alpha)(w-v-\alpha)}. \end{aligned} \tag{4.21}$$

The system with finitely many slow particles has been already partially studied in [23]. There, it was shown that the distribution function of particles positions is given by a Fredholm determinant and the kernel was given. For a fixed $M \in \mathbb{N}$, consider TASEP with initial conditions and jump rates given by

$$x_n^M(0) = 2(M-n), \quad n\in\mathbb{N}, \qquad v_n^M = \begin{cases} 1, & \text{for } n > M, \\ \alpha, & \text{for } 1 \le n \le M. \end{cases} \tag{4.22}$$

To distinguish this system with the one we want to study, i.e., $M=\infty$, we will index all quantities by a M. Proposition 6 of [23] tells us that

$$\mathbb{P}\left(\bigcap_{k=1}^m \{x^M_{n_k+M}(t) > s_k\}\right) = \det(1-\chi_s K^M \chi_s)_{\ell^2(\{n_1,\ldots,n_m\}\times\mathbb{Z})}, \tag{4.23}$$

where the kernel kernel K^M has the decomposition

$$K^M = -\phi + K^1 + K^{2,M}. \tag{4.24}$$

Here ϕ and K^1 are as in Proposition 4.4, while $K^{2,M}$ is given by

$$\begin{aligned} K^{2,M}(n_1,x_1;n_2,x_2) =& \frac{1}{(2\pi \mathrm{i})^3}\oint_{\Gamma_{\alpha-1}}\mathrm{d}v \oint_{\Gamma_{0,v}}\mathrm{d}z \oint_{\Gamma_{0,\alpha-1-v}} \frac{\mathrm{d}w}{w}\frac{e^{tw}(w-1)^{n_1}}{w^{x_1+n_1}} \\ &\times \frac{(1+z)^{x_2+n_2}}{e^{tz}z^{n_2}}\left(\frac{w(w-\alpha)}{(v+1)(v+1-\alpha)}\right)^M \\ &\times \frac{(1+2z)(2v+2-\alpha)}{(z-v)(z+v+1)(w-1-v)(w+1-\alpha+v)}. \end{aligned} \tag{4.25}$$

Proof of Proposition 4.4. First we note that

$$\lim_{M\to\infty}\mathbb{P}\left(\bigcap_{k=1}^m \{x^M_{n_k+M}(t) > s\}\right) = \mathbb{P}\left(\bigcap_{k=1}^m \{x_{n_k}(t) > s\}\right). \tag{4.26}$$

This follows since $x^M_{n+M}(0) = x_n(0)$ for all $n \geq -M$ and by the fact that in TASEP the positions of the normal particles up to a fixed time t depend only on finitely many other particles on the right with probability one, as is seen from a graphical construction of it. So it remains to show that the convergence in (4.27) holds also on the level of fredholm determinants.

First of all, as shown already in Corollary 8 of [23]), it holds

$$\lim_{M\to\infty} K^M(n_1+M, x_1; n_2+M, x_2) = K(n_1, x_1; n_2, x_2) \tag{4.27}$$

pointwise. The reason being that for any $M > (n_1 + x_1 + 1)$ the pole at $w = 0$ in (4.25) vanishes, in the limit of large M we can integrate out explicitly the simple pole at $w = \alpha - 1 - v$ of the kernel $K^{2,M}$ and it results in the kernel K^2 given in Proposition 4.4.

To show the convergence of Fredholm determinants we use their series expansion expression, namely

$$\begin{aligned}\det(1 - \chi_s K^M \chi_s)_{\ell^2(\{n_1,\dots,n_m\}\times\mathbb{Z})} \\ = \sum_{n\geq 0} \frac{(-1)^n}{n!} \sum_{i_1,\dots,i_n=1}^{m} \sum_{x_1\leq s_1} \cdots \sum_{x_n\leq s_n} \det[K^M(n_{i_k}, x_k; n_{i_l}, x_l)]_{1\leq k,l\leq n}.\end{aligned} \tag{4.28}$$

It is easy to see that $K^1(n_1, x_1; n_2, x_2) = 0$ for $x_1 < -2n_1$ since the pole at $w = 0$ vanishes after computing the residue at $w = -v$ the pole at $v = 0$ vanishes. Similarly, $K^{2,M}(n_1, x_1; n_2, x_2) = 0$ for $x_1 < -n_1$ since the pole at $w = 0$ vanishes. Further, in the term $\phi(n_1, x_1; n_2, x_2)$, if x_2 is bounded from below, then for x_1 small enough this term is also zero. This implies that the $n \times n$ determinant in (4.28) is strictly equal to zero if $x_i < -2n_m$. The physical reason for this is that if we consider the system with particle numbers bounded from above by n_m, then by TASEP dynamics particles can be present only in the region on the right of $x_{n_m}(0) = -2n_m$. Consequently, the sums are finite and the by Hadamard bound $|\det[K^M(n_{i_k}, x_k; n_{i_l}, x_l)]_{1\leq k,l\leq n}| \leq C^n n^{n/2}$ for some finite constant C. Thus by dominated convergence we can take the limit $M \to \infty$ inside the sum and the proof is completed. □

4.2.2 Scaling limit and asymptotics

With the finite time formula of Proposition 4.4 at hand, we can now proceed to prove the main result of this Chapter.

Proof of Theorem 2.8. The proof is identical to the one of Theorem 2.5 in [19], given that the we have convergence of the (properly rescaled) kernel in a bounded set (Proposition 4.5), and good enough bounds to control the convergence of the Fredholm determinant by the use of dominated convergence. The bounds are contained in Propositions 4.6, 4.7, and 4.8 below. This is precisely the procedure outlined after Theorem 1.15. □

From now on, the u_i are some fixed real values. We first prove convergence to the limit kernel K_a and then provide integrable bounds. We consider the scaling

$$\begin{aligned} n_i(u,t) &= t/4 + (u_i + a)\,(t/2)^{2/3}\,, \\ x_i(u,t) &= -2(u_i+a)\,(t/2)^{2/3} - \xi_i\,(t/2)^{1/3}\,. \end{aligned} \tag{4.29}$$

The ξ_i measure the fluctuations in the $(t/2)^{1/3}$ scale with respect to the macroscopic approximation given in (4.3). Accordingly, we define the rescaled kernel

$$K^{\mathrm{resc}}(u_1,\xi_1;u_2,\xi_2) = 2^{x_2-x_1}(-1)^{n_1-n_2}\,(t/2)^{1/3}\,K(n_1,x_1;n_2,x_2) \tag{4.30}$$

and similarly for each component of the kernel.

Proposition 4.5 (Convergence on bounded sets)**.** *For any fixed $L>0$, we have*

$$\lim_{t\to\infty} K^{\mathrm{resc}}(u_1,\xi_1;u_2,\xi_2) = K_a(u_1,\xi_1;u_2,\xi_2), \tag{4.31}$$

uniformly for ξ_1,ξ_2 in $[-L,L]$. Here K_a is the kernel from Definition 4.1.

Proof. We start with ϕ. The residue at 0 can be easily computed expanding $(w-1)^{n_1-n_2}$ with the binomial formula and one readily obtains that $\phi(n_1,x_1;n_2,x_2) = (-1)^{n_1-n_2}\binom{x_1-x_2-1}{n_2-n_1-1}$. It is then an easy computation to show that (see e.g. Proposition 7 of [22])

$$\phi^{\mathrm{resc}}(u_1,\xi_1;u_2,\xi_2) \to \frac{\mathbf{1}_{\{u_2>u_1\}}}{\sqrt{4\pi(u_2-u_1)}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(u_2-u_1)}\right). \tag{4.32}$$

Next we consider K^1. We make the change of variables $w\to w+1$, rename $u=w$, and set $\tau_i = u_1 + a$ and $\tilde{s}_i = \xi_i$. Then, $K^{1,\mathrm{resc}}$ equals the kernel $\widehat{K}_t^{\mathrm{resc}}$ in (3.7) of [22]. The convergence of $\widehat{K}_t^{\mathrm{resc}}$ to the $\mathcal{A}_{2\to 1}$ transition kernel is proven in Proposition 4 of [22], giving the first double integral of K_a, i.e., (4.74)+(4.75) in the integral representation of Section 4.3.

Finally consider K^2. We have

$$\begin{aligned} &K^{2,\mathrm{resc}}(u_1,\xi_1;u_2,\xi_2) \\ &\quad= -\frac{(t/2)^{1/3}}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{-1}}\mathrm{d}u\oint_{\Gamma_{0,\alpha-2-u}}\mathrm{d}v\frac{1+2v}{(v+u+2-\alpha)(u+1-v-\alpha)} \\ &\qquad\times\frac{e^{tf_0(v)+(t/2)^{2/3}(a+u_2)f_1(v)+(t/2)^{1/3}\xi_2 f_2(v)}}{e^{tf_0(u)+(t/2)^{2/3}(a+u_1)f_1(u)+(t/2)^{1/3}\xi_1 f_2(u)+f_3(u)}} \end{aligned} \tag{4.33}$$

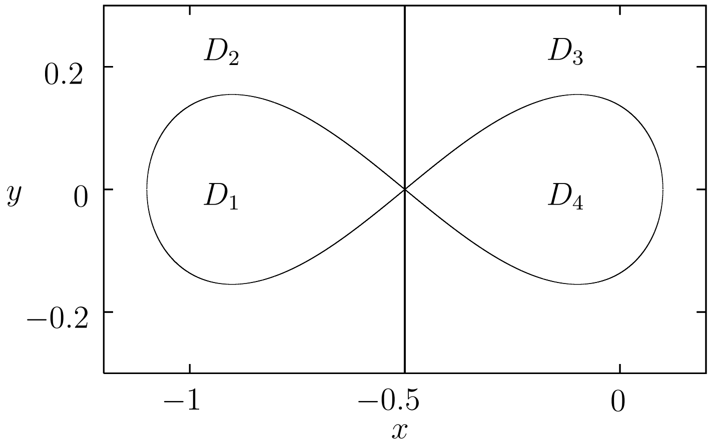

Figure 4.4: The signum of $\mathrm{Re}(f_0(x+\mathrm{i}y)-f_0(-1/2))$ is positive in D_2 and D_4 and negative in D_1 and D_3.

with

$$\begin{aligned} f_0(v) &= -v + \frac{1}{4}\ln((1+v)/v), \\ f_1(v) &= -\ln(-4v(1+v)), \\ f_2(v) &= -\ln(2(1+v)), \\ f_3(v) &= \ln(1+v). \end{aligned} \tag{4.34}$$

The poles and order of integration are different, but the exponential part (4.33) equals again the exponential part of $\widehat{K}_t^{\mathrm{resc}}$ in (3.7) of [22], so let us focus on the differences. The critical point of f_0 is $-1/2$, and in Proposition 4 of [22] $\mathbb{C}$ is divided in four regions D_i depending on the sign of $\mathrm{Re}(v)+1/2$ and of $\mathrm{Re}(f_0(v)-f_0(-1/2)))$, see Figure 4.4. For $\Gamma_{0,\alpha-2-u}$ we may choose any simple anticlockwise oriented closed path passing through $-1/2$ and staying in D_3. Γ_{-1} is restricted to stay in D_2 except for a local modification in a $t^{-1/3}$–neighborhood of the critical point in order to satisfy $\alpha-2-\Gamma_{-1}\subset\Gamma_{0,\alpha-2-u}$. More precisely, Γ_{-1} passes through $-1/2-\kappa/t^{1/3}$ for some $\kappa>2^{4/3}a$, see Figure 4.5. We will take $\Gamma_{0,\alpha-2-u}$ to arrive in $-1/2$ with an angle $\varphi\in(\pi/6,\pi/3)$.

Define for $\delta>0$ the segments $\Gamma^{\delta}_{0,\alpha-2-u}=\{v\in\Gamma_{0,\alpha-2-u}:|1/2+v|<\delta\}$ and $\Gamma^{\delta}_{-1}=\{u\in\Gamma_{-1}:|1/2+u|<\delta\}$. Denote by Σ the part of the contours where $v\notin\Gamma^{\delta}_{0,\alpha-2-u}$ and/or $u\notin\Gamma^{\delta}_{-1}$. Then the integral is on

$$\Sigma+(\Gamma^{\delta}_{0,\alpha-2-u}\cup\Gamma^{\delta}_{-1})=\Gamma_{0,\alpha-2-u}\cup\Gamma_{-1}. \tag{4.35}$$

On Σ there exists a $c_0>0$ that $\mathrm{Re}(f_0(v)-f_0(-1/2))\le -c_0$ and/or $\mathrm{Re}(-f_0(u)+f_0(-1/2)\le -c_0$. Further $\exp(t(f_0(-1/2-\kappa/t^{1/3})-f_0(-1/2)))=\mathcal{O}(1)$. Hence the contribution coming from $\Gamma_{0,\alpha-2-u}\setminus\Gamma^{\delta}_{\alpha-2-u}$ and $\Gamma_{-1}\setminus\Gamma^{\delta}_{-1}$ is bounded by $e^{-c_0t+\mathcal{O}(t^{2/3})}$. Furthermore, on Σ, $\left|\frac{1+2v}{(v+u+2-\alpha)(u+1-v-\alpha)}\right|\le C(\delta)$ with $C(\delta)$ depending only on δ. Hence

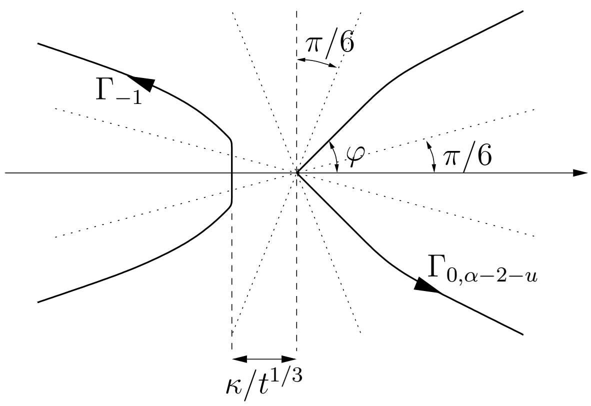

Figure 4.5: The contours Γ_{-1} and $\Gamma_{0,\alpha-2-u}$ used for the pointwise convergence. The point in the middle is $(-1/2,0)$. The vertical piece in Γ_{-1} is of length of order $t^{-1/3}$.

we may bound the overall contribution of Σ by

$$\left|\int_\Sigma \cdots\right| \leq c_1 t^{1/3} C(\delta) e^{-tc_0/4} \tag{4.36}$$

for some finite constant c_1. As we will show below, the contribution coming from $\Gamma^\delta_{0,\alpha-2-u}$ and Γ^δ_{-1} is of order one, therefore the contribution of the integrals over Σ is negligible in the $t\to\infty$ limit.

Next consider the contribution from the integral over $\Gamma^\delta_{0,\alpha-2-u}\cup\Gamma^\delta_{-1}$. Consider the change of variables

$$u = -1/2 + (U-2a)/(4t)^{1/3}, \quad v = -1/2 + (V-2a)/(4t)^{1/3} \tag{4.37}$$

and denote $F_i(v) = e^{tf_0(v)+(t/2)^{2/3}(a+u_i)f_1(v)+(t/2)^{1/3}\xi_i f_2(v)+(2-i)f_3(v)}$. Then by Taylor expansion we obtain

$$\frac{F_2(v)}{F_1(u)} = 2\frac{e^{V^3/3+(u_2-a)V^2-(\xi_2+4au_2)V+4u_2a^2}}{e^{U^3/3+(u_1-a)U^2-(\xi_1+4au_1)U+4u_1a^2}}\frac{e^{\mathcal{O}(V^2/t^{1/3})+\mathcal{O}(V^3/t^{1/3})+\mathcal{O}(V^4/t^{1/3})}}{e^{\mathcal{O}(U/t^{1/3})+\mathcal{O}(U^2/t^{1/3})+\mathcal{O}(U^3/t^{1/3})+\mathcal{O}(U^4/t^{1/3})}} \tag{4.38}$$

The control of the error term in (4.38) is (almost) identical to the one given in Proposition 4 of [22], we therefore omit it. The error term is of order $\mathcal{O}(t^{-1/3})$. For the remaining part, denote $\gamma^\delta_+ = (4t)^{1/3}(\Gamma^\delta_{0,\alpha-2-u}+1/2)+2a$ and $\gamma^\delta_- = (4t)^{1/3}(\Gamma^\delta_{-1}+1/2)+2a$. Any extension of finite length $\gamma^\delta_+,\gamma^\delta_-$ gives an error of order (4.36). For $|v|$ large, $\mathrm{Re}(f_0(v)-f_0(-1/2))$ (resp. $\mathrm{Re}(-f_0(v)+f_0(-1/2))$) decays linearly along γ^δ_+ (resp. γ^δ_-). Therefore also extending the curves to infinity creates an error of order e^{-ct} for some $c>0$. We denote the resulting curves by γ_+,γ_- and we are thus left with

$$\frac{-1}{(2\pi\mathrm{i})^2}\int_{\gamma_-}\mathrm{d}U\int_{\gamma_+}\mathrm{d}V\frac{e^{V^3/3+(u_2-a)V^2-(\xi_2+4au_2)V+4u_2a^2}}{e^{U^3/3+(u_1-a)U^2-(\xi_1+4au_1)U+4u_1a^2}}\frac{2(V-2a)}{(V+U)(U-V+4a)}. \tag{4.39}$$

The integration paths can be deformed as in Definition 4.1 without errors (the minus factors come from the change of orientation of one of the paths).

□

For ϕ, an integrable bound was already obtained in [22] (with ϕ as binomial coefficient, see the beginning of the proof of Proposition 4.5).

Proposition 4.6 (Proposition 8 in [22]). *For any ξ_1, ξ_2 in $\mathbb{R}$ and $u_2 - u_1 > 0$ fixed, there exist a finite constants C and t_0, such that for all $t \geq t_0$,*

$$0 \leq \phi^{\mathrm{resc}}(u_1, \xi_1; u_2, \xi_2) \leq C\, e^{-|\xi_2 - \xi_1|}. \tag{4.40}$$

Proposition 4.7 (Moderate deviations for K^1, K^2). *For any L large enough, there are ε_0, t_0 such that for all $0 < \varepsilon \leq \varepsilon_0$, $t \geq t_0$, there exists a finite constant C such that*

$$\left|K^{1,\mathrm{resc}}(u_1, \xi_1; u_2, \xi_2) + K^{2,\mathrm{resc}}(u_1, \xi_1; u_2, \xi_2)\right| \leq Ce^{-(\xi_1+\xi_2)/2} \tag{4.41}$$

for all $\xi_1, \xi_2 \in [-L, \varepsilon t^{2/3}] \setminus [-L, L]$.

Proof. For $K^{1,\mathrm{resc}}$ the statement is Proposition 5 in [22]. For $K^{2,\mathrm{resc}}$, we follow a similar strategy, but let us give the details. Define $\sigma_i = \xi_i t^{-2/3} 2^{-1/3} \in (0, \varepsilon]$ and denote the integrand by $G_{\sigma_1,\sigma_2}(u, v) := \frac{F_2(v)}{F_1(u)} \frac{(t/2)^{1/3}(1+2v)}{(v+u+2-\alpha)(u+1-v-\alpha)}$. Let $\mathcal{I}$ be an interval on which $\Gamma_{-1}, \Gamma_{0,\alpha-2-u}$ are parametrized. The analysis of Proposition 4.5 shows that for a constant C

$$\begin{aligned}&|K^{2,\mathrm{resc}}(n_1, 0; n_2, 0)|\\ &\qquad \leq \int_{\mathcal{I}^2} \mathrm{d}s \mathrm{d}r |\Gamma'_{-1}(s)\Gamma'_{0,\alpha-2-u}(r) G_{0,0}(\Gamma_{-1}(s), \Gamma_{0,\alpha-2-u}(r))| \leq C.\end{aligned} \tag{4.42}$$

If $\sigma_i > 0$, we have an additional factor

$$\exp(-t\sigma_2 \ln(2 + 2v)) \exp(t\sigma_1 \ln(2 + 2u)) \tag{4.43}$$

in the integrand of (4.42). As we shall show in (b), (c), (e), (f) below, if we are not close to $-1/2$, then $|(4.43)| \leq e^{-(\xi_1+\xi_2)/2}$ and thus get the bound $Ce^{-(\xi_1+\xi_2)/2}$. Close to $-1/2$, we do a modification of one of the contours, depending on whether $\sigma_1 \leq \sigma_2$ or $\sigma_1 \geq \sigma_2$, and then get the needed decay for (4.43).

In the $\sigma_1 \geq \sigma_2$ case, we modify Γ_{-1} near the critical point $-1/2$ and show that in the unmodified region the decay is the same as in the case $\sigma_1 = \sigma_2 = 0$ case times an integrable factor. We then deal with the modified region and provide the needed decay there too. If $\sigma_1 \leq \sigma_2$, we integrate out the residue at $v = \alpha - 2 - u$, and show the needed decay for it by modifying Γ_{-1}. In the remaining integral we may then deform the contour $\Gamma_{0,\alpha-2-u}$ to get the desired decay.

Case $\sigma_1 \geq \sigma_2$. The paths $\Gamma_{0,\alpha-2-u}$ and Γ_{-1} are as in Figure 4.5 except that the distance of the vertical piece of Γ_{-1} with respect to $-1/2$ is $\sqrt{\sigma_1}/2 + \kappa/t^{1/3}$ instead of $\kappa/t^{1/3}$. Near $-1/2$ we modify Γ_{-1} by a vertical part Γ_{vert} that passes through $-1/2-\mu$ with $\mu \ll 1$ (see (4.44)) and which is symmetric w.r.t. the real line. As in Proposition 4.5 let $\varphi \in (\pi/6, \pi/3)$ be the angle with which $\Gamma_{0,\alpha-2-u}$ leaves $-1/2$ and let $\kappa > 2^{4/3}a$. The region D_1 in Figure 4.4 leaves $-1/2$ with angle $\pm 5\pi/6$. Consequently, for Γ_{vert} to end outside D_1 and satisfy $\alpha - 2 - \Gamma_{\text{vert}} \subset \Gamma_{\alpha-2-u}$ we can choose (for t large enough) its length as μb for some $b \in (\tan(\pi/6), \tan(\varphi))$. Hence we define

$$\Gamma_{\text{vert}} = \{-1/2 - (\sqrt{\sigma_1}/2 + \kappa/t^{1/3})(1+\mathrm{i}\rho), \rho \in [-b, b]\}. \tag{4.44}$$

(a) The choice of contours is such that

$$\begin{aligned} \operatorname{dist}(\Gamma_{0,\alpha-2-u}, \Gamma_{-1} + 1 - \alpha) &\geq c_3\sqrt{\sigma_1}, \\ \operatorname{dist}(-\Gamma_{0,\alpha-2-u}, \Gamma_{-1} + 2 - \alpha) &\geq c_3\sqrt{\sigma_1}. \end{aligned} \tag{4.45}$$

for some constant $c_3 = c_3(b,\varphi) > 0$. This is at least the same order as for the contours in Proposition 4.5 where we had

$$\begin{aligned} \operatorname{dist}(\Gamma_{0,\alpha-2-u}, \Gamma_{-1} + 1 - \alpha) &\leq (\kappa - 2^{4/3}a)/t^{1/3}, \\ \operatorname{dist}(-\Gamma_{0,\alpha-2-u}, \Gamma_{-1} + 2 - \alpha) &\leq (\kappa - 2^{4/3}a)/t^{1/3}. \end{aligned} \tag{4.46}$$

Hence (as in the $\sigma_1 = \sigma_2 = 0$ case) the $\left|\frac{1+2v}{(v+u+2-\alpha)(u+1-v-\alpha)}\right|$ term does not create problems .

(b) The contour $\Gamma_{0,\alpha-2-u}$ can be chosen such that $|1+v|$ reaches its minimum at $v = -1/2$ so we can simply bound

$$|e^{-t\sigma_2 \ln(2(1+v))}| \leq 1. \tag{4.47}$$

(c) Let $u \in \Gamma_{-1} \setminus \Gamma_{\text{vert}}$. In the following, we set $\widehat{\sigma}_1 := (\sqrt{\sigma_1} + 2\kappa/t^{1/3})^2$, which is just a shift in the variable $\sqrt{\xi_1}$. Γ_{-1} can be chosen such that on $\Gamma_{-1} \setminus \Gamma_{\text{vert}}$ the maximum of $|1+u|$ is reached at $\rho = \pm b$. For ε small enough

$$(2|1+u|)^2 = 1 - 2\sqrt{\widehat{\sigma}_1} + (b^2+1)\widehat{\sigma}_1 \leq 1 - \sqrt{\sigma_1}. \tag{4.48}$$

Therefore it holds

$$|e^{t\sigma_1 \ln(2(1+u))}| \leq e^{t\sigma_1 \ln(1-\sqrt{\sigma_1})/2} \leq e^{-\xi_1^{3/2}/2^{3/2} + \mathcal{O}(t\sigma_1^2)} \leq e^{-\xi_1^{3/2}/4} \tag{4.49}$$

for ε small enough.

(d) For the integral on $\Gamma_{\rm vert}$, it is an integral on $[-b,b]$ in the variable ρ. Since $\Gamma'_{\rm vert}(\rho) = (\sqrt{\sigma_1}/2+\kappa/t^{1/3})\mathrm{i}$, this term multiplied by the $t^{1/3}$ prefactor gives a term $\mathcal{O}(\xi_1^{1/2})$. So it suffices to have a bound on the integrand that controls it. On $\Gamma_{\rm vert}$ we use Taylor expansion around $-1/2$ (from which $\Gamma_{\rm vert}$ is at most $\mathcal{O}(\sqrt{\varepsilon})$ far away). The $u-$dependant part of the exponential term becomes

$$e^{-tf_0(-1/2)+t\widehat{\sigma}_1^{3/2}(1+\mathrm{i}\rho)^3/6-u_1(t/2)^{2/3}\widehat{\sigma}_1(1+\mathrm{i}\rho)^2}e^{-t\sigma_1\sqrt{\widehat{\sigma}_1}(1+\mathrm{i}\rho)+\mathcal{O}(t\widehat{\sigma}_1^2)}. \tag{4.50}$$

Now we take real parts in the exponent. We see that for L large and ε small enough we have $\xi_1^{3/2}\gg t\widehat{\sigma}_1^2$ and $\xi_1^{3/2}\gg(2^{-1/6}\sqrt{\xi_1}+2\kappa)^2$. We get the upper bound

$$\begin{aligned}|(4.50)| &\leq e^{-tf_0(-1/2)+\xi_1^{3/2}(-5/6-\rho^2/2)}e^{-u_1\frac{(2^{-1/6}\sqrt{\xi_1}+2\kappa)^2}{2^{2/3}}(1-\rho^2)}e^{-\xi_1\kappa 2^{1/3}}e^{\mathcal{O}(t\widehat{\sigma}_1^2)}\\ &\leq e^{-tf_0(-1/2)-\xi_1^{3/2}/4}.\end{aligned} \tag{4.51}$$

The $e^{-tf_0(-1/2)}$ cancels exactly with the contribution coming from the integrand in the v variable. Finally note that for L large enough

$$e^{-\xi_1^{3/2}/4}\leq e^{-\xi_1\sqrt{L}/4}\leq e^{-(\xi_1+\xi_2)/2}. \tag{4.52}$$

Case $\sigma_1\leq\sigma_2$. Here we integrate out the residue $w=\alpha-2-u$ and obtain

$$\begin{aligned}&K^{2,\rm resc}(n_1,\xi_1;n_2,\xi_2)\\ &\quad=\mathcal{I}_1-(t/2)^{1/3}\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{-1}}\mathrm{d}u\oint_{\Gamma_0}\mathrm{d}v\frac{1+2v}{(v+u+2-\alpha)(u+1-v-\alpha)}\\ &\qquad\qquad\times\frac{e^{tf_0(v)+(t/2)^{2/3}(a+u_2)f_1(v)+(t/2)^{1/3}\xi_2 f_2(v)}}{e^{tf_0(u)+(t/2)^{2/3}(a+u_1)f_1(u)+(t/2)^{1/3}\xi_1 f_2(u)+f_3(u)}},\end{aligned} \tag{4.53}$$

where

$$\begin{aligned}\mathcal{I}_1=(t/2)^{1/3}\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{-1}}&\mathrm{d}u e^{t(f_0(\alpha-2-u)-f_0(u))}e^{(t/2)^{2/3}((a+u_2)f_1(\alpha-2-u)-(a+u_1)f_1(u))}\\ &\times e^{(t/2)^{1/3}(\xi_2 f_2(\alpha-2-u)-\xi_1 f_2(u))}e^{-f_3(u)}.\end{aligned} \tag{4.54}$$

The contours Γ_0 and Γ_{-1} in the double integral in (4.53) satisfy $\alpha-2-\Gamma_{-1}\supset\Gamma_0$ and Γ_{-1} passes through the critical point $-1/2$. To provide the integrable bound for the double integral in (4.53), one does the same analysis as in the $\sigma_1\geq\sigma_2$ case, except that the roles of Γ_{-1} and Γ_0, and σ_1 and σ_2 are reversed: We modify Γ_0 by a vertical part with distance $(\sqrt{\sigma}_2+2\kappa/t^{1/3})/2$ to $-1/2$ and then go through the steps (a) to (d).

We have for Γ_{-1} as in Figure 4.5, $\sigma_1=\sigma_2=0$ and t large enough the bound

$$\mathcal{I}_1=\frac{(t/2)^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_{-1}}|\mathrm{d}u|\left|\frac{e^{tf_0(\alpha-2-u)}e^{(t/2)^{2/3}(a+u_2)f_1(\alpha-2-u)}}{e^{tf_0(u)}e^{(t/2)^{2/3}(a+u_1)f_1(u)}e^{f_3(u)}}\right|\leq c_2. \tag{4.55}$$

The bound (4.55) follows from the identity in (4.53) and the fact that respective bounds hold for $K^{2,\text{resc}}$ and the double integral in (4.53).

For $\mathcal{I}_1$ we modify Γ_{-1} near $-1/2$ by a vertical piece

$$\Gamma_{\text{vert}} = \{-1/2 - \sqrt{\sigma_2}(1+\mathrm{i}\rho)/2, \rho \in [-b,b]\} \tag{4.56}$$

where $b > \tan(\pi/6)$.

Compared to $\sigma_1 = \sigma_2 = 0$, the integrand has the additional factor

$$\exp(t\sigma_2 f_2(\alpha - 2 - u)) \exp(-t\sigma_1 f_2(u)) \tag{4.57}$$

(e) We can choose the contour Γ_{-1} such that $|1+u| < 1/2$ for all $u \in \Gamma_{-1}$. In particular for $u \in \Gamma_{-1} \setminus \Gamma_{\text{vert}}$ we may simply bound

$$|\exp(-t\sigma_1 f_2(u))| = \exp(t\sigma_1 \ln(2|1+u|) \leq 1. \tag{4.58}$$

(f) Furthermore, Γ_{-1} may be chosen such that for $u \in \Gamma_{-1} \setminus \Gamma_{\text{vert}}$ the minimum of $|\alpha - 1 - u| = |u + 2^{4/3}a/t^{1/3}|$ is reached at $\rho = \pm b$. For this u we have

$$(2|u + 2^{4/3}a/t^{1/3}|)^2 = (1 + 2^{7/3}/t^{1/3} + \sqrt{\sigma_2})^2 + \sigma_2 b^2 \geq 1 + \sqrt{\sigma_2},$$

so that we get the bound for L large and ε small

$$e^{t\sigma_2 f_2(\alpha-2-u)} = e^{-t\sigma_2 \ln((2|u+2^{4/3}a/t^{1/3}|)^2)/2} e^{\mathcal{O}(t\sigma_2^2)} \leq e^{-t\sigma_2^{3/2}/4} \leq e^{-\xi_2^{3/2}/6} \leq e^{-(\xi_1+\xi_2)}.$$

Now we deal with Γ_{vert}. We write

$$\alpha - 2 - u = -1/2 + V_1 \quad u = -1/2 + V_2 \tag{4.59}$$

with $V_2 = -\sqrt{\sigma_2}(1+\mathrm{i}\rho)/2$ and $V_1 = -V_2 - 2^{4/3}/t^{1/3}$. Next we do Taylor around $-1/2$ in f_0 and we first obtain

$$e^{t(f_0(\alpha-2-u)-f_0(u))} = e^{4t(V_1^3 - V_2^3)/3} e^{\mathcal{O}(t\sigma_2^2)}. \tag{4.60}$$

We compute

$$\mathrm{Re}(4tV_1^3/3) = -2^6/3 + 2^{11/3}\sqrt{\sigma_2}t^{1/3} - 2^{4/3}\sigma_2 t^{2/3}(1-\rho^2) + t\sigma_2^{3/2}(1-3\rho^2)/6. \tag{4.61}$$

For L large, the last term in (4.61) dominates, thus (4.61) reaches its maximum at $\rho = 0$. So we may bound

$$|(4.60)| \leq e^{\xi_2^{3/2}(1/(3\sqrt{2}) + c_4\sqrt{\varepsilon})}, \tag{4.62}$$

with $c_4 > 0$ a constant.

As for f_1, by Taylor expansion around $-1/2$ we obtain

$$e^{(t/2)^{2/3}((a+u_2)f_1(\alpha-2-u)-(a+u_1)f_1(u))} = e^{(t/2)^{2/3}\sigma_2(1+\mathrm{i}\rho)^2(u_2-u_1)} e^{\mathcal{O}(\sqrt{\xi_2})} e^{\mathcal{O}(t^{2/3}(V_1^3 - V_2^3))}. \tag{4.63}$$

Thus, for L large we can bound

$$|(4.63)| \leq e^{c_5 \xi_2}, \tag{4.64}$$

for some constant $c_5 > 0$.

Finally, for f_2 we obtain

$$\begin{aligned} e^{(t/2)^{1/3}(\xi_2 f_2(\alpha-2-u)-\xi_1 f_2(u))} &= e^{t\sigma_2(-2V_1)} e^{t\sigma_1 2V_2} e^{\mathcal{O}(t\sigma_2 V_1^2)} e^{(t\sigma_1 V_2^2)} \\ &\leq e^{-t\sqrt{\sigma_2}(1+\mathrm{i}\rho)(\sigma_2+\sigma_1)} e^{c_7\xi_2} e^{c_8\sqrt{\varepsilon}\xi_2^{3/2}}, \end{aligned} \tag{4.65}$$

with $c_7 > 0$ and $c_8 > 0$ some constants. Therefore, for L large and ε small enough we obtain

$$|(4.65)| \leq e^{-\xi_2^{3/2}/(2\sqrt{2})} \tag{4.66}$$

Now (4.66) dominates (4.62), (4.64) for L large and ε small enough. So, putting together (e), (f) and (4.55), (4.62), (4.64), (4.66) we obtain the desired bound for $|K^{2,\text{resc}}|$. □

Proposition 4.8 (Large deviations for K^1, K^2)**.** *Let $\varepsilon > 0$. Then, there is a finite constant C such that for t large enough we have*

$$\left|K^{1,\text{resc}}(u_1,\xi_1;u_2,\xi_2) + K^{2,\text{resc}}(u_1,\xi_1;u_2,\xi_2)\right| \leq Ce^{-(\xi_1+\xi_2)/2}. \tag{4.67}$$

for $\xi_1, \xi_2 \geq \varepsilon t^{2/3}$.

Proof. The estimate for K_1^{resc} is contained in Proposition 6 of [22]. As in Proposition 4.7 we denote $\sigma_i = \xi_i 2^{-1/3} t^{-2/3}$ and distinguish the cases $\sigma_1 \geq \sigma_2$ and $\sigma_1 \leq \sigma_2$.

Case $\sigma_1 \geq \sigma_2$. We choose the same contours as in the moderate deviations regime for $\sigma_1 \geq \sigma_2$, here with $\sigma_1 = \sigma_2 = \varepsilon/2$ (the additional shift by $2\kappa/t^{1/3}$ in Γ_{vert} is however unnecessary for t large enough). We write $f_{0,\sigma}(v) = f_0(v) - \sigma\ln(2(1+v))$ so that

$$f_{0,\sigma} = f_{0,\varepsilon/2}(v) - (\sigma_2 - \varepsilon/2)\ln(2(1+v)). \tag{4.68}$$

Thus, compared to the $\sigma_1 = \sigma_2 = \varepsilon/2$ case we have the additional factor

$$e^{-t(\sigma_2-\varepsilon/2)\ln(2(1+v))} e^{t(\sigma_1-\varepsilon/2)\ln(2(1+u))}. \tag{4.69}$$

It suffices to bound $|(4.69)|$ because the integrand for $\sigma_1 = \sigma_2 = \varepsilon/2$ is (uniformely for t large enough) bounded in L^1. The choice of contours is such that $|1+v|$ reaches its minimum at $v = -1/2$ and $|1+u|$ its maximum at $u = -1/2 - \sqrt{\varepsilon/2}/2$. Using further $\sigma_1 - \varepsilon/2 \geq \sigma_1/2$, we may bound

$$|(4.69)| \leq e^{t\sigma_1 \ln(1-\sqrt{\varepsilon/2})/2} \leq e^{-c_9 t^{1/3}\xi_1} \leq e^{-(\xi_1+\xi_2)}. \tag{4.70}$$

for some constant $c_9 > 0$.

Case $\sigma_1 \leq \sigma_2$. We again choose the same contours as in the moderate deviations regime for $\sigma_1 \leq \sigma_2$, with $\sigma_2 = \sigma_1 = \varepsilon/2$ (again the additional shift by $2\kappa/t^{1/3}$ is unnecessary for t large enough). We again integrate out the residue at $v = \alpha - 2 - u$. In the double integral (4.53), with respect to $\sigma_1 = \sigma_2 = \varepsilon/2$ we get the same additional factor, which can now be bounded

$$e^{-t(\sigma_2-\varepsilon/2)\ln(2(1+v))}e^{t(\sigma_1-\varepsilon/2)\ln(2(1+u))} \leq e^{-t(\sigma_2-\varepsilon/2)\ln(1+\sqrt{\varepsilon/2})} \leq e^{-(\xi_1+\xi_2)}. \tag{4.71}$$

As for the residue (4.54), compared to $\sigma_1 = \sigma_2 = \varepsilon/2$ we have the additional term

$$e^{t(\sigma_1-\varepsilon/2)\ln(2(1+u))}e^{-t(\sigma_2-\varepsilon/2)\ln(2(-u-2^{4/3}/t^{1/3}))} \leq e^{-(\xi_1+\xi_2)}, \tag{4.72}$$

where the inequality holds since $|2(-u - 2^{4/3}/t^{1/3})| \geq 1 + \sqrt{\varepsilon/2}$ and $|2(1+u)| \leq 1$. $\square$

4.3 Kernel K_a in terms of Airy functions

Here we give the explicit form of K_a that we used for the numerical evaluation of G_a and its statistics.

Lemma 4.9. *Denote $u_{i,a} = u_i + a$ we have (with the conjugation transferred to the diffusion part)*

$$K_a(u_1, \xi_1; u_2, \xi_2) \stackrel{\text{conj}}{=} -\frac{e^{\frac{2}{3}u_{1,a}^3 + u_{1,a}\xi_1}}{e^{\frac{2}{3}u_{2,a}^3 + u_{2,a}\xi_2}} \frac{e^{-(\xi_2-\xi_1)^2/(4(u_2-u_1))}}{\sqrt{4\pi(u_2-u_1)}} \mathbf{1}_{(u_2>u_1)} \tag{4.73}$$

$$+ \int_0^\infty \mathrm{d}\lambda \, \mathrm{Ai}(\xi_1 + u_{1,a}^2 + \lambda) \, \mathrm{Ai}(\xi_2 + u_{2,a}^2 + \lambda) e^{\lambda(u_2-u_1)} \tag{4.74}$$

$$+ \int_0^\infty \mathrm{d}\lambda \, \mathrm{Ai}(\xi_1 + u_{1,a}^2 - \lambda) \, \mathrm{Ai}(\xi_2 + u_{2,a}^2 + \lambda) e^{\lambda(2a+u_1+u_2)} \tag{4.75}$$

$$- \int_0^\infty \mathrm{d}\lambda \, \mathrm{Ai}(\xi_1 + u_{1,a}^2 + \lambda) \, \mathrm{Ai}(\xi_2 + u_{2,a}^2 + \lambda) e^{\lambda(4a+u_2-u_1)} \tag{4.76}$$

$$+ \int_0^\infty \mathrm{d}\lambda \, \mathrm{Ai}(\xi_1 + u_{1,a}^2 + \lambda) \, \mathrm{Ai}(\xi_2 + u_{2,a}^2 - \lambda) e^{\lambda(2a-u_1-u_2)}. \tag{4.77}$$

Proof. The result is an easy computation that uses the identities

$$\begin{aligned} &\frac{-1}{2\pi\mathrm{i}} \int_{\delta+\mathrm{i}\mathbb{R}} \mathrm{d}v e^{v^3/3 + xv^2 + yv} = \mathrm{Ai}(x^2 - y) e^{\frac{2}{3}x^3 - xy}, \\ &\frac{1}{z} = \int_0^\infty \mathrm{d}\lambda e^{-\lambda z} \qquad (z \in \mathbb{C}, \, \mathrm{Re}(z) > 0), \end{aligned} \tag{4.78}$$

for any $\delta > \max\{0, x\}$. ☐

Remark 4.10. *Alternatively, via the identity (A.6) of [22], one has*

$$\begin{aligned} (4.75) = &- \int_{-\infty}^0 \mathrm{d}\lambda e^{\lambda(u_{2,a}+u_{1,a})} \, \mathrm{Ai}(\xi_1 + u_{1,a}^2 - \lambda) \, \mathrm{Ai}(\xi_2 + u_{2,a}^2 + \lambda) \\ &+ 2^{-1/3} \, \mathrm{Ai}\left(2^{-1/3}(\xi_1+\xi_2) + 2^{-4/3}(u_1-u_2)^2\right) e^{-\frac{1}{2}(u_{1,a}+u_{2,a})(\xi_2 + u_{2,a}^2 - \xi_1 - u_{1,a}^2)}, \end{aligned} \tag{4.79}$$

with an analogous formula for (4.77)*.*

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