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ABSTRACT

Statistical Mechanics is considered as one of the most sound and confirmed theories in modern physics. In this thesis, we explore the possibility to view a large class of models under the point of view of statistical mechanics. The models are defined for simplicity on the standard lattice $\mathbb{Z}^d$. However, most of the results apply unchanged to very general lattices. The Hamiltonians considered are of gradient type. Namely, as a function of the field φ, they depend only on all the pair differences $\varphi(x) - \varphi(y)$, where x, y are elements of the lattice. Under suitable very general assumptions, we show that these models satisfy certain large deviation principles. The models considered contain in particular the typical models for Nonlinear Elasticity and Fracture Mechanics. Afterwards, we will concentrate on more specific models in which we show local properties of the free energy per particle. These models are sometimes known in the literature as mass-spring models. In particular, we will consider the space dependent case. For these models, we show the validity of the Cauchy-Born rule in a neighbourhood of the origin. The methods used to prove the Cauchy-Born rule are based on the Renormalization Group. We also show a new Finite Range Decomposition based on discrete L^p-theory.

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Introduction

In many instances, the physically relevant states come as the minimizers of some functional $\mathcal{F}$. This coincides with the fundamental problem in the Calculus of Variations. More precisely, given a functional $\mathcal{F} : X \to \bar{\mathbb{R}}$, where X is a topological space, one seeks to characterize its minimizers. A typical example is: *given a bounded open subset Ω of $\mathbb{R}^d$ and a free energy function $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \mapsto \mathbb{R}$, find all functions $u : \Omega \mapsto \mathbb{R}^m$ that (possibly subject to boundary conditions) minimize the free energy integral:*

$$\mathcal{F}(u, \Omega) := \int_\Omega g(x, \nabla u(x))dx.$$

The free-energy-minimizing approach has been successfully applied to many physical models. In particular, the above example is typical in Nonlinear Elasticity.

However, it is often unclear how to find the right functional $\mathcal{F}$ which should be minimized.

The approach of Statistical Physics is to start by postulating simple local interactions for particles and to show via some "thermodynamical limit" that with overwhelming high probability the configuration will be very close to the minimizer of some functional $\mathcal{F}$. In this way, it "justifies" the choice of the free energy functional $\mathcal{F}$ and the minimization procedure. Moreover, it also allows to determine how likely(or unlikely) particular configurations are.

In this thesis, we restrict ourselves to the Nonlinear Elasticity setting and very closely related ones. One of the features, we will be very interested in, is the so-called Cauchy-Born rule. The Cauchy-Born rule is a basic hypothesis used in the mathematical formulation of solid mechanics and relates the movement of atoms in a crystal to the overall deformation of the bulk solid. Namely, it says that in a crystalline solid subject to a small strain, the positions of the atoms within the crystal lattice follow the overall strain of the medium. Mathematically, the Cauchy-Born rule is closely related to the strict convexity of the free energy. The lack of some type of strict convexity gives rise to the pattern formation.

In Chapter 1, we will show that, if one starts with very general local interaction potentials, one obtains the physically relevant states concentrate with overwhelming high probability to the minimizers of the typical functionals considered in Nonlinear Elasticity. This setting has been considered before by R. Kotecký and S. Luckhaus in an important paper(cf. [19]). In Chapter 1, we present several extensions of their results, such as more general local interaction, an homogenization result as well as various technical improvements in the proof. For a more precise comparison see § 1.1.

In Chapter 2 and Chapter 3, we depart from the fairly general setting of Chapter 1 and consider a class of special local interactions. For these type of local interactions we show some local properties of the resulting free-energies and the corresponding rate functions. To do so we need to use the Renormalization Group theory developed by

Brydges *et al.*. In particular, we generalize some results of S. Adams, R. Kotecký and S. Müller with *non-translation invariant local interactions.* We will follow closely their strategy. However there are many technical problems that cannot be dealt with by modifying directly their proof. More precisely, a fundamental step is the construction of the Finite Range Decomposition, for which we need to apply a rather different strategy. For a more in-depth comparison see the corresponding introductory sections in Chapter 2 and Chapter 3.

1 Representation Theorems

1.1 Introduction

Recently, R. Kotecký and S. Luckhaus, have shown a remarkable result. They prove that in a fairly general setting, the limit of large volume equilibrium Gibbs measures for elasticity type Hamiltonians with clamped boundary conditions. The "zero"-temperature case was considered by R. Alicandro and M. Cicalese in [3].

Let us now briefly explain the results contained in [19]. The authors begin with the microscopic description and consider the space of microscopic configurations $X : \mathbb{Z}^d \to \mathbb{R}^m$. This includes the case of elasticity where $m = d$ and $X(i)$ denoting the vector of displacement of the atom labeled by i as well as the case of random interface with $m = 1$ and $X(i)$ denoting the height of interface above the lattice site i. For any fixed $Y : \mathbb{Z}^d \to \mathbb{R}^m$ and any finite $\Lambda \subset \mathbb{Z}^d$, the Gibbs measure $\mu_{\Lambda,Y}(dX)$ on $(\mathbb{R}^m)^\Lambda$ under the boundary conditions Y is defined in terms of a Hamiltonian H with a finite range interaction U.

Namely, let a finite $A \subset \mathbb{Z}^d$, a function $U : (\mathbb{R}^m)^A \to \mathbb{R}$ be given and let $R_0 = \operatorname{diam}(A)$ denote the range of potential U. The function U is also assumed to be invariant under rigid motions. In addition, natural growth conditions on U are imposed. Using X_A to denote the restriction of X to A for any $X : \mathbb{Z}^d \to \mathbb{R}^m$ and any $A \subset \mathbb{Z}^d$, the Hamiltonian is defined by

$$H_\Lambda(X) = \sum_{j\in\mathbb{Z}^d :\, \tau_j(A)\subset\Lambda} U(X_{\tau_j(A)})$$

with $\tau_j(A) = A + j = \{i\colon i - j \in A\}$. Moreover, they assume that

(A1) *There exist constants $p > 0$ and $c \in (0,\infty)$ such that*

$$U(X_A) \geq c|\nabla X(0)|^p$$

for any $X \in (\mathbb{R}^m)^{\mathbb{Z}^d}$.

(A2) *There exist constants $r > 1$ and $C \in (1,\infty)$ such that*

$$U(sX_A + (1-s)Y_A + Z_A) \leq C\big(1 + U(X_A) + U(Y_A) + \sum_{i\in A} |Z(i)|^r\big)$$

for any $s \in [0,1]$ and any $X, Y, Z \in (\mathbb{R}^m)^{\mathbb{Z}^d}$.

They introduce the clamped boundary conditions by considering a fixed configuration Y in the boundary layer

$$S_{R_0}(\Lambda) = \{i \in \Lambda |\, \operatorname{dist}(i, \mathbb{Z}^d \setminus \Lambda) \leq R_0\}$$

by restricting to the functions X which are contained in the set(whose indicator function will be denoted by $\mathbb{1}_{\Lambda,Y}(X)$,)

$$\{X \in (\mathbb{R}^m)^\Lambda : |X(i) - Y(i)| < 1 \text{ for all } i \in S_{R_0}(\Lambda)\}.$$

The Gibbs measure on $(\mathbb{R}^m)^\Lambda$ is defined by

$$\mu_{\Lambda,Y}(dX) = \frac{\exp\{-\beta H_\Lambda(X)\}}{Z_{\Lambda,Y}} \mathbb{1}_{\Lambda,Y}(X) \prod_{i\in\Lambda} dX(i)$$

with

$$Z_{\Lambda,Y} = \int_{(\mathbb{R}^m)^\Lambda} \exp\{-\beta H_\Lambda(X)\} \mathbb{1}_{\Lambda,Y}(X) \prod_{i\in\Lambda} dX(i).$$

For any $\varepsilon \in (0,1)$, let

$$\Omega_\varepsilon = \varepsilon\mathbb{Z}^d \cap \Omega \equiv (\mathbb{Z}^d \cap \tfrac{1}{\varepsilon}\Omega).$$

Naturally, $\frac{1}{\varepsilon}\Omega$ and $\varepsilon\mathbb{Z}^d$ denotes the rescaling of Ω and $\mathbb{Z}^d$ by $\frac{1}{\varepsilon}$ and ε, respectively.

With the above notation, in [19], the following theorem is proved:

Theorem 1.1.1. *Assume that U satisfies the assumptions (A1) and (A2) with $r \geq p > 1$, $\frac{1}{r} > \frac{1}{p} - \frac{1}{d}$ and let $v \in W^{1,p}(\Omega)$. Further, let*

$$F_{\kappa,\varepsilon}(v) = -\varepsilon^d |\Omega|^{-1} \log Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v,\kappa)),$$

and

$$F^+_\kappa(v) = \limsup_{\varepsilon\to 0} F_{\kappa,\varepsilon}(v) \tag{1.1}$$

$$F^-_\kappa(v) = \liminf_{\varepsilon\to 0} F_{\kappa,\varepsilon}(v) \tag{1.2}$$

Then:

(i) $\lim_{\kappa\to 0} F^-_\kappa(v) \geq \frac{1}{|\Omega|}\int_\Omega W(\nabla v(x))\,\mathrm{d}x$.

(ii) *If $v \in W^{1,r}(\Omega)$ then* $\lim_{\kappa\to 0} F^+_\kappa(v) \leq \frac{1}{|\Omega|}\int_\Omega W(\nabla v(x))\,\mathrm{d}x$.

The crucial step in the proof of the Large Deviation statement is based on the possibility to approximate with partition functions on cells of a triangulation given in terms of L^r-neighbourhoods of linearizations of a minimiser of the rate functional. An important tool that allows them to impose a boundary condition on each cell of the triangulation consists in switching between the corresponding partition function $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v,\kappa))$ and the version $Z_{\Omega_\varepsilon}(\mathcal{N}_{\Omega_\varepsilon,r}(v,2\kappa) \cap \mathcal{N}_{\Omega_\varepsilon,R_0,\infty}(Z))$ with an additional soft clamp $|X(i) - Z(i)| < 1$ enforced in the boundary strip of the width $R_0 > \mathrm{diam}(A)$ with $Z \in \mathcal{N}_{\Omega_\varepsilon,r}(v,\kappa)$ arbitrarily chosen.

We improve their result in the following manner:

(i) We consider Hamiltonians, where the interaction is *not* of finite range and is dependent[1] both on the scale ε and the position x. We are also able to give an homogenisation result.

[1] for a precise definition see the next section

(ii) By considering a different version of the interpolation argument we are able to consider "hard" boundary condition instead of the clamped ones. In our opinion this type of boundary conditions are more in line with the standard theory of Statistical Mechanics.

(iii) We simplify some of the arguments by relying on the representation formulas, hence avoiding the triangulation argument.

(iv) We are able to consider more general potentials, which "relax" in SBV.

1.2 Sobolev Representation Theorems

1.2.1 Preliminary results

Let Ω be an open set. We denote by $\mathcal{A}(\Omega)$ the family of all open sets contained in Ω. We now recall a well-known result in measure theory due to E. De Giorgi and G. Letta. The proof can be found in [4].

Theorem 1.2.1. *Let X be a metric space and let us denote by $\mathcal{A}$ its open sets. Let $\mu : \mathcal{A} \to [0, \infty]$ be an increasing set function such that*

(DL1) $\mu(\varnothing) = 0$*;*

(DL2) $A, B \in \mathcal{A}$ *then* $\mu(A \cup B) \leq \mu(A) + \mu(B)$*;*

(DL3) $A, B \in \mathcal{A}$*, such that* $A \cap B = \varnothing$ *then* $\mu(A \cap B) \geq \mu(A) + \mu(B)$

(DL4) $\mu(A) = \sup \{\mu(B) : \ B \Subset A\}$*. Then, the extension of μ to every $C \subset X$ given by*

$$\mu(C) = \inf \{\mu(A) : \ A \in \mathcal{A}, \ A \supset C\}$$

is an outer measure. In particular the restriction of μ to the Borel σ-algebra is a positive measure.

We recall the well-known integral representation formulas (see [12]).

Theorem 1.2.2. *Let $1 \leq p < \infty$ and let $F : W^{1,p} \times \mathcal{A}(\Omega) \to [0, +\infty]$ be a functional satisfying the following conditions:*

(i) (locality) F is local, i.e. $F(u, A) = F(v, A)$ if $u = v$ a.e.on $A \in \mathcal{A}(\Omega)$;

(ii) (measure property) for all $u \in W^{1,p}$ the set function $F(u, \cdot)$ is the restriction of a Borel measure to $\mathcal{A}(\Omega)$;

(iii) (growth condition) there exists $c > 0$ and $a \in L^1(\Omega)$ such that

$$F(u, A) \leq c \int_A (a(x) + |Du|^p) \ \mathrm{d}x$$

for all $u \in W^{1,p}$ and $A \in \mathcal{A}(\Omega)$;

(iv) (translation invariance in u) $F(u + z, A) = F(u, A)$ for all $z \in \mathbb{R}^d$, $u \in W^{1,p}$ and $A \in \mathcal{A}(\Omega)$;

(v) (lower semicontinuity) for all $A \in \mathcal{A}(\Omega)$ $F(\cdot, A)$ is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,p}$.

Then there exists a Carathéodory function $f : \Omega \times \mathbb{M}^{d\times N} \to [0, +\infty)$ *satisfying the growth condition*

$$0 \le f(x, M) \le c(a(x) + |M|^p)$$

for all $x \in \Omega$ *and* $M \in M^{d\times N}$*, such that*

$$F(u, A) = \int_A f(x, Du(x))\,\mathrm{d}x$$

for all $u \in W^{1,p}$ *and* $A \in \mathcal{A}(\Omega)$.
If in addition it holds

(vi) (translation invariance in x)

$$F(Mx, B(y, \varrho)) = F(Mx, B(z, \varrho))$$

for all $M \in M^{d\times N}$, $y, z \in \Omega$*, and* $\varrho > 0$ *such that* $B(y, \varrho) \cup B(z, \varrho) \subset \Omega$*, then* f *does not depend on* x.

1.2.2 Hypothesis and Main Theorem

For any $u \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^m)$, let $X_{u,\varepsilon} : \mathbb{Z}^d \to \mathbb{R}^m$ and $\varphi : \varepsilon\mathbb{Z}^d \to \mathbb{R}^m$ be defined by

$$\begin{aligned} X_{u,\varepsilon}(i) &= \frac{1}{\varepsilon} \fint_{\varepsilon i + Q(\varepsilon)} u(y)\ \mathrm{d}y \\ \varphi_{u,\varepsilon}(\varepsilon i) &= \frac{1}{\varepsilon} \fint_{\varepsilon i + Q(\varepsilon)} u(y)\ \mathrm{d}y \end{aligned} \tag{1.3}$$

for any $i \in \mathbb{Z}^d$. Here, $Q(\varepsilon) = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d$ and $\fint$ denotes the mean value,i.e., for every $f \in L^1(\mathbb{R}^d)$

$$\fint_A f(x)\,\mathrm{d}x = \frac{1}{|A|}\int_A f(x)\,\mathrm{d}x$$

Let $u \in W^{1,p}(\mathbb{R}^d)$, A is an open set and $p \ge 1$. Then it is not difficult to prove that

$$\lim_{\varepsilon \downarrow 0} \sum_{x \in A_\varepsilon} \varepsilon^d |\nabla \varphi_u(x)|^p = \int_A |\nabla u|^p. \tag{1.4}$$

On the other hand, let

$$\Pi_\varepsilon : (\mathbb{R}^m)_0^{\mathbb{Z}^d} \to W^{1,p}(\mathbb{R}^d) \tag{1.5}$$

be a canonical interpolation $X \to v$ such that $v(\varepsilon i) = \varepsilon X(i) = \varepsilon\varphi(\varepsilon i)$ for any $i \in \mathbb{Z}^d$. Here, $(\mathbb{R}^m)_0^{\mathbb{Z}^d}$ is the set of functions $X : \mathbb{Z}^d \to \mathbb{R}^m$ with finite support. To fix ideas, we can consider a triangulation of $\mathbb{Z}^d$ into simplexes with vertices in $\varepsilon\mathbb{Z}^d$, and choose v on each simplex as the linear interpolation of the values $\varepsilon X(i)$ on the vertices εi.

Let Ω be an open set with regular boundary. We denote by $\Omega_\varepsilon = \varepsilon\mathbb{Z}^d \cap \Omega$ and by $\mathcal{A}(\Omega)$ the set of all open sets contained in Ω with regular boundary. For every set $A \in \mathcal{A}(\Omega)$, we define

$$R_\varepsilon^\xi(A) := \{\alpha \in \varepsilon\mathbb{Z}^d \ [\alpha, \alpha + \varepsilon\xi] \subset A\},$$

where by $[x, y]$ we mean the segment connecting x and y, i.e., $\{\lambda x + (1-\lambda)y : \ \lambda \in [0,1]\}$.

The Hamilton H is defined by

$$H(\varphi, \varepsilon) := \sum_{\xi\in\mathbb{Z}^d} \sum_{x\in R_\varepsilon^\xi(\Omega)} f_{\xi,\varepsilon}(x, \nabla_\xi\varphi),$$

where $\xi \in \mathbb{Z}^d$, and

$$\nabla_\xi\varphi(x) := \frac{\varphi(x + \varepsilon\xi) - \varphi(x)}{|\xi|}.$$

We also define the Hamiltonian taking into account the contribution from the boundary as

$$H_\infty(\varphi, A, \varepsilon) := \sum_{\xi\in\mathbb{Z}^d} \sum_{x\in A_\varepsilon} f_{\varepsilon,\xi}(x, \nabla_\xi\varphi(x)).$$

The functions $f_{\xi,\varepsilon}$ will be specified later.

In order to apply the representation formulas, we shall need to localize. For this reason, for every $\varepsilon > 0$ and $A \subset \Omega$ open, set we introduce

$$H(\varphi, A, \varepsilon) := \sum_{\xi\in\mathbb{Z}^d} \sum_{x\in R_\varepsilon^\xi(A)} f_\xi\big(x, \nabla_{\xi,\varepsilon}\varphi(x)\big).$$

For simplicity of notation, we will also denote

$$H^\xi(\varphi, A, \varepsilon) := \sum_{x\in R_\varepsilon^\xi(A)} f_{\xi,\varepsilon}\big(x, \nabla_\xi\varphi(x)\big).$$

The localized version of H_∞ and H_∞^ξ are defined in the obvious way.

Moreover, let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^d$. In this section, the functions $f_{\xi,\varepsilon}$ will satisfy the followings

(C1) $f_{\xi,\varepsilon} > 0$;

(C2) there exist constants C_ξ such that

$$f_{\xi,\varepsilon}(x, s + t) \le f_{\xi,\varepsilon}(x, s) + C_\xi(|t|^p + 1);$$

where the constants C_ξ satisfy

$$\sum_{\xi\in\mathbb{Z}^d} C_\xi < +\infty;$$

(C3) there exists a constant C such that

$$f_{e_i,\varepsilon}(x,t) \geq C \max(|t|^p - 1, 0).$$

For every $A \in \mathcal{A}(\Omega)$, we define the free-energy as

$$\begin{aligned} F(u,A,\kappa,\varepsilon) &:= -\varepsilon^d \log \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\Big(- H(\varphi,A,\varepsilon)\Big)\,\mathrm{d}\varphi \\ F_\infty(u,A,\kappa,\varepsilon) &:= -\varepsilon^d \log \int_{\mathcal{V}_\infty(u,A,\kappa,\varepsilon)} \exp\Big(- H_\infty(\varphi,A,\varepsilon)\Big)\,\mathrm{d}\varphi, \end{aligned} \tag{1.6}$$

where

$$\mathcal{V}(u,A,\kappa,\varepsilon) = \left\{ \varphi : A_\varepsilon \to \mathbb{R}^m |\ \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon\varphi|^p \leq \kappa^p \right\}$$

$$\mathcal{V}_\infty(u,A,\kappa,\varepsilon) = \left\{ \varphi : \varepsilon\mathbb{Z}^d \to \mathbb{R}^m |\ \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon\varphi|^p \leq \kappa^p,\ \text{and } \varphi(x) = \varphi_{u,\varepsilon}(x)\ \forall x \notin A_\varepsilon \right\},$$

where $\varphi_{u,\varepsilon}$ is defined in (1.3).
Let us introduce the following notations:

$$\begin{aligned} F'(u,A,\kappa) &:= \liminf_{\varepsilon \downarrow 0} F(u,A,\kappa,\varepsilon) \\ F''(u,A,\kappa) &:= \limsup_{\varepsilon \downarrow 0} F(u,A,\kappa,\varepsilon) \\ F'(u,A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F(u,A,\kappa,\varepsilon) = \lim_{\kappa \downarrow 0} F'(u,A,\kappa) \\ F''(u,A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F(u,A,\kappa,\varepsilon) = \lim_{\kappa \downarrow 0} F''(u,A,\kappa) \\ F'_\infty(u,A,\kappa) &:= \liminf_{\varepsilon \downarrow 0} F_\infty(u,A,\kappa,\varepsilon) \\ F''_\infty(u,A,\kappa) &:= \limsup_{\varepsilon \downarrow 0} F_\infty(u,A,\kappa,\varepsilon) \\ F'_\infty(u,A) &:= \lim_{\kappa \downarrow 0} \liminf_{\varepsilon \downarrow 0} F_\infty(u,A,\kappa,\varepsilon) = \lim_{\kappa \downarrow 0} F'_\infty(u,A,\kappa) \\ F''_\infty(u,A) &:= \lim_{\kappa \downarrow 0} \limsup_{\varepsilon \downarrow 0} F_\infty(u,A,\kappa,\varepsilon) = \lim_{\kappa \downarrow 0} F''_\infty(u,A,\kappa) \end{aligned} \tag{1.7}$$

One of the main steps will be to show that $F'_\infty = F'$ and that $F''_\infty = F''$. The basic intuition behind is the so called interpolation lemma, which is well-known in the Γ-convergence community. Very informally, what it says that if one imposes "closeness" in $L^p(A)$ to some regular function u, then one can also impose the boundary condition by "paying a very small price in energy". More precisely, given a sequence $\{v_n\}$ such that $v_n \to u$ in $L^p(A)$, where A is an open set, then there exists a sequence $\{\tilde{v}_n\}$, such that $\tilde{v}_n \to u$ in $L^p(A)$, $\tilde{v}_n|_{\partial\Omega} = u|_{\partial\Omega}$ and such that

$$\liminf_n \int_A |\nabla \tilde{v}_n|^2 \leq \liminf_n \int_A |\nabla v_n|^2.$$

Remark 1.2.3. (i) *The functional $F(u,A,\kappa,\varepsilon)$ is monotonically decreasing in $\delta,\kappa>0$, i.e.*

$$F(u,A,\kappa,\varepsilon)\le F(u,A,\kappa+\delta,\varepsilon).$$

This justifies the outer limit in the formulas of (1.7). Moreover, the outer limit in the formulas in in (1.7) can be substituted with the supremum i.e.,

$$F'(u,A):=\sup_{\kappa>0}\liminf_{\varepsilon\downarrow 0}F(u,A,\kappa,\varepsilon)=\sup_{\kappa>0}F'(u,A,\kappa),$$
$$F''(u,A):=\sup_{\kappa>0}\limsup_{\varepsilon\downarrow 0}F(u,A,\kappa,\varepsilon)=\sup_{\kappa>0}F''(u,A,\kappa).$$

(ii) *Let A,B be two open sets such that $A\cap B=\varnothing$, then from the definitions it is not difficult to prove that*

$$F'(u,A)+F'(u,B)=F'(u,A\cup B)\qquad \text{and}\qquad F''(u,A)+F''(u,B)=F''(u,A\cup B).$$

(iii) *Whenever the function u is linear and the functions $f_{\xi,\varepsilon}$ do not depend on ε and the space variable x, it is well-known that $F'=F''$. In Theorem 1.2.18, we are going to prove a more general result, which contains as a particular case the previous claim.*

Proposition 1.2.4. *The maps F',F'' are lower semicontinuous with respect to the $L^p(A)$ convergence. Moreover, there exists a sequence $\{\varepsilon_n\}$ such that*

$$F'_{\{\varepsilon_n\}}(u)=F''_{\{\varepsilon_n\}}(u), \tag{1.8}$$

where

$$F'_{\{\varepsilon_n\}}(u):=\lim_{\kappa\downarrow 0}\liminf_{n\to\infty}F(u,A,\kappa,\varepsilon_n)\qquad \text{and}\qquad F''\{\varepsilon_n\}(u):=\lim_{\kappa\downarrow 0}\limsup_{n\to\infty}F(u,A,\kappa,\varepsilon_n).$$

Proof. Using $F(v,A,\kappa,\varepsilon)\ge F(u,A,\kappa+\delta,\varepsilon)$ where $\|u-v\|_{L^p(A)}<\delta$, one has that

$$F'(v,A,\kappa)=\liminf_{n\to\infty}F(u,A,\kappa,\varepsilon_n)\ge\liminf_{n\to\infty}F(v,A,\kappa+\delta,\varepsilon_n)=F(u,A,\kappa+\delta).$$

Thus,

$$\liminf_{v\to u}\sup_{\kappa>0}F'(u,A,\kappa)\ge\sup_{\kappa>0}F'(u,A,\kappa+\delta)$$

and finally passing also to the supremum in δ one has that F' is lower semicontinuous. The statement for F'' follows in a similar fashion.

Fix $\mathscr{D}$ a countable dense set in $L^p(A)$ and let $\mathcal{U}$ be the set of all balls centered in the elements of $\mathscr{D}$ with radii in $[0,1]\cap\mathbb{Q}$. Let us enumerate the balls in $\mathcal{U}$, namely $\mathcal{U}:=\{B_i:\ i\in\mathbb{N}\}$.

Let $u_1\in B_1$ be such that $F'(u_1,A)\le\inf_{B_1}F'+\operatorname{diam}(B_1)$. Let $\{\varepsilon_n^{(1)}\}$ be the sequence such that

$$F'(u_1,A)=\lim_{\kappa\downarrow 0}\lim_{n\to\infty}F(u_1,A,\kappa,\varepsilon_n^{(1)}).$$

In a similar way as for B_1, let $u_2 \in B_2$ be such that $F'_{\{\varepsilon_n^{(1)}\}}(u_2, A) \le \inf_{B_2} F'_{\{\varepsilon_n^{(1)}\}} + \mathrm{diam}(B_2)$. Moreover, let $\{\varepsilon_n^{(2)}\} \subset \{\varepsilon_n^{(1)}\}$ be such that

$$F'(u_2, A) = \lim_{\kappa \downarrow 0} \lim_{n\to\infty} F(u, A, \kappa, \varepsilon_n^{(2)}).$$

By an induction procedure it is possible to produce a sequence $\{\varepsilon_n^{(k+1)}\} \subset \{\varepsilon_n^{(k)}\}$ such that

$$F'(u_k, A) = \lim_{\kappa \downarrow 0} \lim_{n\to\infty} F(u_k, A, \kappa, \varepsilon_n^{(k)}),$$

where u_k is chosen such that

$$F'_{\{\varepsilon_n^{(k+1)}\}}(u_{k+1}, A) \le \inf_{B_{k+1}} F'_{\{\varepsilon_n^{(k)}\}} + \mathrm{diam}(B_{k+1}).$$

By a diagonal argument it is possible to chose a single sequence $\{\varepsilon_k\}$, such that all the above are satisfied. Because the second claim of the Proposition 1.2.4 consists in showing (1.8) for a particular sequence, one can assume without loss of generality that it satisfies the above relations.

Let us now show that $F'_{\{\varepsilon_n\}} = F''_{\{\varepsilon_n\}}$. From the definitions it is trivial that $F'_{\{\varepsilon_n\}} \le F''_{\{\varepsilon_n\}}$. Let us now show the opposite inequality. Fix u. For every i such that $u \in B_i$ we have that[2]

$$F'_{\{\varepsilon_n\}}(u, A) + \mathrm{diam}(B_i) \ge F'_{\{\varepsilon_n\}}(u_i, A) = F''_{\{\varepsilon_n\}}(u_i, A).$$

Passing to the limit for $i \to \infty$ and using the lower semicontinuity of $F''_{\{\varepsilon_n\}}$, we have the desired result. □

Fix Ω an open set, $\varepsilon > 0$ and $u \in W^{1,p}(\mathbb{R}^d)$ and let $\varphi_{u,\varepsilon}$ be defined by in (1.3). The Gibbs measure $\mu_{\Omega,\varepsilon,u}(\varphi)$ on $(\mathbb{R}^m)^{\Omega_\varepsilon}$ under the boundary conditions u is defined as the Borel measure such that

$$\mathrm{d}\mu_{\Omega,\varepsilon,u}(\varphi) = \frac{\exp\{-\beta H(\varphi, \Omega, \varepsilon)\}}{Z_{\Omega,\varepsilon,u}} \mathbb{1}(\varphi) \prod_{i\in\Omega_\varepsilon} d\varphi(i),$$

where $\mathbb{1}$ is the characteristic function of the set

$$\{\varphi \in (\mathbb{R}^m)^{\varepsilon\mathbb{Z}^d} : \ \varphi(x) = \varphi_{u,\varepsilon} \text{ for all } x \in \varepsilon\mathbb{Z}^d \setminus \Omega_\varepsilon\}$$

and

$$Z_{\Omega,\varepsilon,u} = \int_{(\mathbb{R}^m)^{\Omega_\varepsilon}} \exp\Big(-\beta H(\varphi, \Omega, \varepsilon)(\varphi)\Big) \mathbb{1}(\varphi) \prod_{x\in\Omega_\varepsilon} \mathrm{d}\varphi(i).$$

We are now able to write the main result in this section:

Theorem 1.2.5. *Assume the above hypothesis. Then for every infinitesimal sequence* (ε_n) *there exists a subsequence* (ε_{n_k}) *and there exists a function* $W : \Omega \times \mathbb{R}^{d\times m} \to \mathbb{R}$ *(depending on* $\{\varepsilon_{n_k}\}$*) such that*

$$F'_{\{\varepsilon_{n_k}\}}(u, A) = F''_{\{\varepsilon_{n_k}\}}(u, A) = \int_A W(x, \nabla u)\, \mathrm{d}x. \tag{1.9}$$

[2] by the above construction

1.2.3 Proofs

The next technical lemma asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions(see [3, Lemma 3.6]).

Lemma 1.2.6. *Let $A \in \mathcal{A}(\Omega)$ and set $A_\varepsilon = \{x \in A : \operatorname{dist}(x, \partial A) > 2\sqrt{N}\varepsilon\}$. Then for any $\xi \in \mathbb{Z}^d$ and $\varphi : A_\varepsilon \to \mathbb{R}^m$, it holds*

$$\sum_{x \in R_\varepsilon^\xi(A)} \Big| \frac{\varphi(x + \varepsilon\xi) - \varphi(x)}{|\xi|} \Big|^p \leq C \sum_{i=1}^{N} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_i \varphi(x)|^p, \tag{1.10}$$

where the constant C is independent of ξ.

Proof. Let $\xi \in \mathbb{Z}^d$. By decomposing it into coordinates, it is not difficult to notice that it can be written as

$$\xi = \sum_{k=1}^{N_\xi} \alpha_k(\xi) e_{i_k},$$

where $N_\xi \leq \delta|\xi|$ for some δ depending on the dimension d, and $\alpha_k(\xi) \in \{-1, 1\}$. Denote by

$$\xi_k = \sum_{j=1}^{k} \alpha_j(\xi) e_{i_j},$$

hence $\xi_k \leq |\xi|$ for all k. Thus,

$$\nabla_\xi u(x) = \frac{1}{|\xi|} \sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon\xi_k).$$

Moreover, by the convexity of the p-norm, we have

$$\Big| \frac{1}{N_\xi} \sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon\xi_k) \Big|^p \leq \frac{1}{N_\xi} \sum_{k=1}^{N_\xi} \Big| \nabla_{\alpha_k(\xi)e_i} u(x + \varepsilon\xi_k) \Big|^p.$$

Finally, by summing over all ξ, exchanging the sums and using the equivalence of the norms i.e., $|\xi| \leq N_\xi \leq d|\xi|$ one has the desired result.

□

Let also us recall a lemma found in [19]:

Lemma 1.2.7 ([19, Lemma A1])**.** *Let $a > 0$ and $\Lambda \subset \Omega_\varepsilon$ be connected (when viewed as a subgraph of $\mathbb{Z}^d$ with the set of edges consisting of all pairs of nearest neighbours $(i, j), |i - j| = 1$). Then:*

(i) *We have*

$$\int \mathbb{1}_{\{j\},y}(X) \exp\big(-a \sum_{i \in \Lambda} |\nabla X(i)|^p\big) \prod_{i \in \Lambda} \mathrm{d}X(i) \leq \omega(m) \big(a^{-m/p} c(p, m)\big)^{|\Lambda| - 1},$$

where $j \in \Lambda$ and $\mathbb{1}_{\{j\},y}$ is the indicator of the set $\{X \in (\mathbb{R}^m)^\Lambda \mid |X(j) - y| < 1\}$ and $\omega(m)$ is the volume of the unit ball in $\mathbb{R}^m$.

(ii) *For any $v \in L^r(\Omega, \mathbb{R}^m)$ and ε sufficiently small,*

$$\int_{\mathcal{N}_{\Lambda,r}(v,\kappa)} \exp\big(-a\sum_{i\in\Lambda}|\nabla X(i)|^p\big)\prod_{i\in\Lambda}\mathrm{d}X(i) \le \vartheta|\Lambda|^{1+\frac{m}{d}}\left(a^{-m/p}c(p,m)\right)^{|\Lambda|-1}, \quad (1.11)$$

where $\vartheta = \omega(m)\kappa^m$ and $c(p,m) = \int_{\mathbb{R}^m}\exp\big(-|\xi|^p\big)d\xi$.

Let G^λ be the free-energy (see (1.6) for the definition) induced by the Hamiltonian

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) := \lambda\sum_{i=1}^d \sum_{x\in R_\varepsilon^{e_i}(A)} |\nabla_i\varphi|^p.$$

Lemma 1.2.8. *There exists constants C_λ, D_λ, such that it holds*

$$C_\lambda \le G^\lambda(0, A, \kappa, \varepsilon) \le D_\lambda$$

Proof. Let us prove now the upper bound, namely

$$G^\lambda(0, A, \kappa, \varepsilon) \le D_\lambda. \quad (1.12)$$

Let us observe that

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) \le d\lambda\sum_{x\in A_\varepsilon}|\varphi(x)|^p, \quad (1.13)$$

hence

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\left(-\tilde{H}^\lambda(\varphi, A, \varepsilon)\right) \ge \int_{\{\varphi:\ |\varepsilon\varphi|\le\kappa\}} \exp\left(-\sum_{x\in A_\varepsilon}|\varphi(x)|^p\right).$$

Thus by using the Fubini Theorem, we have that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\left(-\tilde{H}^\lambda(\varphi, A, \varepsilon)\right) \ge \exp\left(-\varepsilon^{-d}D\right),$$

where

$$D := -\log\int_{\mathbb{R}}\exp\left(t^p\right).$$

Using the definition of the free-energy, one has the desired claim.

Let us now turn to the proof of the second inequality, namely there exits a constant C_λ such that

$$C_\lambda \le G(0, A, \kappa, \varepsilon). \quad (1.14)$$

Let us suppose that $|A| = 1$. By definition, for every $\varphi \in \mathcal{V}(0, A, \kappa, \varepsilon)$ it holds

$$\sum_{x\in A_\varepsilon}|\varphi(x)|^p \le \kappa^p/\varepsilon^{-d-p}.$$

Thus, it is immediate that for every $\varphi \in \mathcal{V}(0, A, \kappa, \varepsilon)$ there exists a $x \in A_\varepsilon$ such that

$$|\varphi(x)|^p \leq \kappa^p / \varepsilon^{-d-p}. \tag{1.15}$$

For ever $x \in A_\varepsilon$, let us denote with $\mathcal{N}_x$, the set of $\varphi \in \mathcal{V}(0, A, \kappa, \varepsilon)$ such that (1.15) holds. Thus $\bigcup_{x \in A_\varepsilon} \mathcal{N}_x \supset \mathcal{V}(0, A, \kappa, \varepsilon)$.

From Lemma 1.2.7 one has that

$$\int_{\mathcal{N}_x} \exp\left(-H^\lambda(\varphi, A, \varepsilon)\right) \leq \kappa \varepsilon^{-d-p} \exp\left(-(\varepsilon^d - 1)\tilde{C}_\lambda\right). \tag{1.16}$$

Hence, with simple calculations one has that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\left(-\tilde{H}^\lambda(\varphi, A, \varepsilon)\right) \leq \sum_{x \in A_\varepsilon} \varepsilon^{-d-p} \exp\left(-(\varepsilon^d - 1)\tilde{C}_\lambda\right),$$

thus because the exponential diverges “faster”, one can find another constant C_λ such that (1.14) holds. □

Lemma 1.2.9. *Let $\{f_{\xi,\varepsilon}\}$ satisfy our hypothesis. Then there exists a constant D such that for every $\kappa < 1$, one has that*

$$\exp\left(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon)\right) \leq \exp\left(D\varepsilon^{-d} + D \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi_{u,\varepsilon}(x)|^p\right), \tag{1.17}$$

where $\varphi_{u,\varepsilon}$.

Proof. Given that $\|b - a\|^p \geq 2^{1-p}\|a\|^p - \|b\|^p$ one has that there exists a constant C_1 such that

$$H(\varphi, A, \kappa, \varepsilon) \geq C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi(x)|^p \geq C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla \psi|^p - C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} (\varphi_{u,\varepsilon})(x)|^p,$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$ and $\varphi_{u,\varepsilon}$ is defined in (1.3). Hence, the estimate (1.17) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\Big(-C \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} |\nabla_{e_i} \varphi|^p\Big) \leq \exp\left(D\varepsilon^{-d}\right).$$

The above inequality was proved in Lemma 1.2.8.

□

Remark 1.2.10. *A simple consequence of the reasoning done in Lemma 1.2.9, is that there exists a constant C such that*

$$A \mapsto F'(u, A) + C(|\nabla u|_{L^p(A)} + 1) \qquad A \mapsto F''(u, A) + C(|\nabla u|_{L^p(A)} + 1)$$

are monotone with respect to the inclusion relation i.e., for every $A \subset B$ it holds that

$$F'(u, A) + C(|\nabla u|_{L^p(A)} + 1) \leq F'(u, B) + C(|\nabla u|_{L^p(B)} + 1).$$

Lemma 1.2.11. *Let $f_{\xi,\varepsilon}$ satisfy our hypothesis and let A be an open set. Then there exists a constant $D > 0$, such that*

$$\exp\left(-\varepsilon^{-d}F(u,A,\kappa,\varepsilon)\right) \geq \exp\left(-D\varepsilon^{-d} - \sum_{i=1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} |\nabla_{e_i}\varphi_{u,\varepsilon}(x)|\right), \tag{1.18}$$

where $\varphi_{u,\varepsilon}$ is defined in (1.3).

Proof. Using Lemma 1.2.6, one has that there exists a constant C such that

$$H(\varphi,A,\kappa,\varepsilon) \leq C\sum_{i=1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} |\nabla_{e_i}\varphi(x)|^p$$

Given that $\|a+b\|^p \leq 2^{p-1}\|a\|^p + 2^{p-1}\|b\|^{p-1}$, there exist a constant C_1 such that

$$H(\varphi,A,\varepsilon) \leq C_1\sum_{i=1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} (|\nabla_{e_i}\varphi_{u,\varepsilon}|^p + 1) + \sum_{i=1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} |\nabla_i\psi(x)|^p,$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence, the estimate (1.18) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\Big(-C\sum_{i=1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} |\nabla_{e_i}\varphi|^p\Big) \leq \exp\Big(D\varepsilon^{-d}\Big).$$

The above inequality was proved in Lemma 1.2.8.

□

Lemma 1.2.12 (exponential tightness)**.** *Let A be an open set and $K \geq 0$. Denote by*

$$\mathcal{M}_K := \Big\{\varphi : \; H(\varphi,A,\varepsilon) \geq K\varepsilon^{-d}|A|\Big\}.$$

Then there exist constants D, K_0, ε_0 such that for every $K \geq K_0,\ \varepsilon \leq \varepsilon_0$ and $u \in L^p(A)$ it holds

$$\int_{\mathcal{M}_K\cap\mathcal{V}(u,A,\kappa)} \exp\big(-H(\varphi,A,\varepsilon)\big) \leq \exp\Big(-\frac{1}{2}K\varepsilon^{-d} + D\varepsilon^{-d} - D\sum_{i-1}^{d}\sum_{x\in R_{\varepsilon}^{e_i}(A)} |\nabla_{e_i}\varphi_u|^p\Big)$$

Proof. For every $\varphi \in \mathcal{M}_K$ it holds

$$H(\varphi,A,\varepsilon) \geq K/2\varepsilon^{-d} + \frac{1}{2}H(\varphi,A,\varepsilon).$$

Hence, by using Lemma 1.2.9, we have the desired result.

□

We will now proceed to prove the hypothesis of Theorem 1.2.2.

Even though in the next two lemmas a very similar reasoning is used, they cannot be derived one from the other.

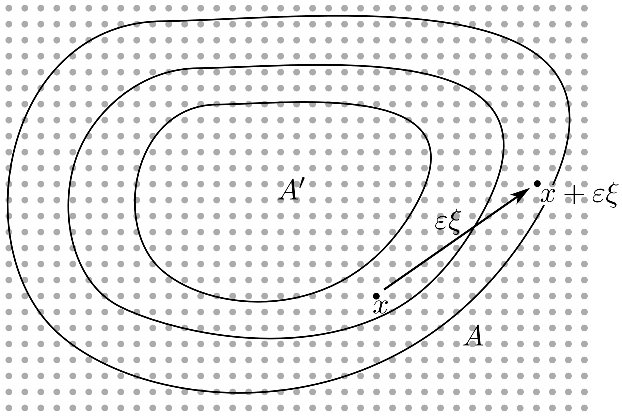


Figure 1.1

Lemma 1.2.13 (regularity). *Let f_ξ satisfy the usual hypothesis then*

$$\sup_{A'\Subset A} F''(u,A') = F''(u,A).$$

Proof. Let us fix $A' \Subset A$ and $N \in \mathbb{N}$ (to be chosen later). Let $\delta = \operatorname{dist}(A', A^C)$, and let $0 < t_1, \dots, t_N \le \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$. Without loss of generality, we may assume that there exists no $x \in A_\varepsilon$ such that $\operatorname{dist}(x, A^C) = t_i$. For every i we define

$$A_i := \left\{ x \in A_\varepsilon : \ \operatorname{dist}(x, A^C) \ge t_i \right\}$$

and

$$S_i^{\xi,\varepsilon} := \{ x \in (A_i)_\varepsilon : \ x + \varepsilon\xi \in A \setminus A_i \}.$$

With the above definitions it holds

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') + R_\varepsilon^\xi(A \setminus \bar{A}') + S_i^{\varepsilon,\xi},$$

thus

$$H^\xi(\varphi, A, \varepsilon) \le H^\xi(\varphi, A \setminus \bar{A}_i, \varepsilon) + H^\xi(\varphi, A_i, \varepsilon) + \sum_{x \in S_i^{\xi,\varepsilon}} f_{\xi,\varepsilon}(\nabla\varphi(x)).$$

Hence, by using hypothesis (C2) one has that,

$$H(\varphi, A, \varepsilon) \le H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus A_i, \varepsilon) + \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi,\varepsilon}} C_\xi(|\nabla_\xi \varphi(x)|^p + 1).$$

Let us now estimate the last term in the previous inequality.

We separate the sum into two terms

$$\sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi,\varepsilon}} |\nabla_\xi\varphi(x)|^p = \sum_{|\xi| \le M} \sum_{x \in S_i^{\xi,\varepsilon}} |\nabla_\xi\varphi(x)|^p + \sum_{|\xi| > M} \sum_{x \in S_i^{\xi,\varepsilon}} |\nabla_\xi\varphi(x)|^p, \tag{1.19}$$

where $M \in \mathbb{N}$. From hypothesis (C2) and by taking M sufficiently large, we may also assume without loss of generality that

$$\sum_{|\xi|\geq M} C_\xi \leq \delta_1,$$

hence using Lemma 1.2.6,

$$\sum_{|\xi|\geq M} \sum_{x\in S_i^{\xi,\varepsilon}} |\nabla_\xi \varphi(x)|^p \leq C\delta_1 \sum_{k=1}^{d} \sum_{x\in R_\varepsilon^{e_k}(A)} |\nabla_{e_k}\varphi(x)|^p \leq \tilde{C}\delta_1 H(\varphi, A, \varepsilon),$$

where in the last inequality we have used hypothesis (C3).
Let $|\xi| < M$. If $\varepsilon MN \leq 2\delta$, then

$$S_i^{\xi,\varepsilon} \cap S_j^{\xi,\varepsilon} = \varnothing \qquad \text{whenever } |i-j| \geq 2.$$

Without loss of generality we may assume the above condition as $\varepsilon \to 0$.
Given that

$$\frac{1}{N-2} \sum_{i=1}^{N-2} \sum_{|\xi|<M} \sum_{x\in S_i^{\varepsilon,\xi}} |\nabla\varphi(x)|^p \leq 2CH(\varphi, A, \varepsilon),$$

there exist $0 < i \leq N-2$ such that

$$\sum_{|\xi|<M} \sum_{x\in S_i^{\xi,\varepsilon}} |\nabla_\xi\varphi|^p < \frac{2}{N-2} H(\varphi, A, \varepsilon). \tag{1.20}$$

Let us denote by $\mathcal{N}_i$ the set of all $\varphi \in \mathcal{V}(u, A, \kappa, \varepsilon)$ such that (1.20) holds for the first time, namely for every $j \leq i$

$$\sum_{|\xi|<M} \sum_{x\in S_i^{\xi,\varepsilon}} |\nabla_\xi\varphi|^p \geq \frac{2}{N-2} H(\varphi, A, \varepsilon) \tag{1.21}$$

On one side, one has that

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \kappa, \varepsilon)\right) \leq \sum_{i=1}^{N} \int_{\mathcal{N}_i} \exp\left(-H(\varphi, A_i, \varepsilon) - H(\varphi, A\setminus \bar{A}_i, \varepsilon)\right). \tag{1.22}$$

On the other side, one has that

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \kappa, \varepsilon)\right) \geq \sum_{i=1}^{N} \int_{\mathcal{N}_i^K} \exp\left(-H(\varphi, A, \varepsilon)\right), \tag{1.23}$$

where $\mathcal{N}_i^K := N_i \setminus \mathcal{M}_K$. By using (1.21), one has that for every $\varphi \in \mathcal{N}_i^K$ it holds

$$H(\varphi, A, \varepsilon) + H(\varphi, A\setminus \bar{A}_i, \varepsilon) \leq H(\varphi, A, \varepsilon) \leq H(\varphi, A_i) + H(\varphi, A\setminus \bar{A}_i) + \frac{K}{N-2},$$

and for every φ it holds

$$H(\varphi, A, \varepsilon) \geq H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon). \tag{1.24}$$

Hence,

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right) \geq \sum_{i=1}^{N} \int_{\mathcal{N}_i^K} \exp\left(-H(\varphi, A_i) - H(\varphi, A \setminus \bar{A}_i) - \frac{K}{N-2}\right).$$

By using Lemma 1.2.12, i.e., the fact that there exist K_0, ε_0 and D such that for every $K > K_0$ and $\varepsilon \leq \varepsilon_0$ one has that

$$\int_{\mathcal{M}_K \cap \mathcal{V}(u,A,\kappa,\varepsilon)} \exp\big(- H(\varphi, A, \varepsilon)\big) \leq \exp\Big(- \frac{1}{2} K \varepsilon^{-d}|A| + D\varepsilon^{-d}|A|\Big),$$

and by using (1.22), one has that (1.23) can be further estimated as

$$\begin{aligned} \exp\Big(- \frac{K}{N-2} - \frac{1}{2} K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A|\Big) + \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right) \\ \geq \exp\left(-\frac{K}{N-2}\right) \sum_{i=1}^{N} \int_{\mathcal{N}_i} \exp\left(-H(\varphi, A_i, \varepsilon) - H(\varphi, A \setminus \bar{A}_i)\right). \end{aligned}$$

We also notice that by using (1.24) one has that

$$\sum_{i=1}^{N} \int_{\mathcal{N}_i} \exp\left(H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i)\right) \geq \int_{\mathcal{V}(u,A,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right),$$

thus there exists $1 \leq i_0 \leq N$ such that

$$\int_{\mathcal{N}_{i_0}} \exp\left(H(\varphi, A_{i_0}, \varepsilon) + H(\varphi, A \setminus \bar{A}_{i_0})\right) \geq \frac{1}{N} \int_{\mathcal{V}(u,A,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right). \tag{1.25}$$

Without loss of generality, we may assume that $i_0 = 1$. Hence, combining (1.25) with the previous estimates we have that

$$\begin{aligned} \exp\Big(- \frac{K}{N-2} - \frac{1}{2} K\varepsilon^{-d}|A| + D\varepsilon^{-d}|A|\Big) + \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right) \\ \geq \frac{1}{N} \exp\left(-\frac{K}{N-2}\right) \int_{\mathcal{N}_1} \exp\left(-H(\varphi, A_1, \varepsilon) - H(\varphi, A \setminus \bar{A}_1)\right). \end{aligned}$$

We notice that the variables $H(\varphi, A_1, \kappa, \varepsilon)$ and $H(\varphi, A \setminus \bar{A}_1, \kappa, \varepsilon)$ are "independent", thus by using the Fubini theorem one has that

$$\begin{aligned} \int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A_1, \varepsilon) - H(\varphi, A \setminus \bar{A}_1)\right) \geq \int_{\mathcal{V}(u,A_1,\kappa,\varepsilon)} \exp\left(-H(\varphi, A_1, \varepsilon)\right) \\ \times \int_{\mathcal{V}(u,A \setminus \bar{A}_1,\kappa,\varepsilon)} \exp\left(-H(\varphi, A \setminus \bar{A}_1)\right), \end{aligned}$$

where in the previous inequality we have also used that

$$\mathcal{V}(u, A \setminus \bar{A}_1, \kappa, \varepsilon) \cap \mathcal{V}(u, A_1, \kappa, \varepsilon) \subset \mathcal{V}(u, A, \kappa, \varepsilon).$$

To summarize, we have proved that for A_1

$$\varepsilon^{-d} \log\Big(\exp\big(F(u, A, \kappa, \varepsilon)\big) + \exp\big(- \frac{K}{N-2} - \frac{1}{2} K \varepsilon^{-d} |A| + D\varepsilon^{-d}|A|\big)\Big) \\ \leq -\varepsilon^{d} \log(\frac{K}{N-2}) + \varepsilon^{d} \log(N) + F(u, A_1, \kappa, \varepsilon) + F(u, A \setminus \bar{A}_1, \kappa, \varepsilon).$$

Finally, to conclude it is enough to pass to the limit in ε, then in N and then in κ, and use the “almost” monotonicity of the map $A \mapsto F''(u, A)$(see Remark 1.2.10) and Lemma 1.2.11 to estimate the term $F(u, A \setminus \bar{A}_1, \kappa, \varepsilon)$. □

Lemma 1.2.14. *For every open set A and $u \in W^{1,p}(\mathbb{R}^d)$ it holds*

$$F'(u, A) = F'_\infty(u, A) \qquad \text{and} \qquad F'(u, A) = F'_\infty(u, A)$$

Proof. Without loss of generality, we may assume that $u = 0$. Indeed, if it is possible to change the boundary condition to 0 it is possible to change the boundary condition for every $u \in W^{1,p}(A)$ as this would correspond to a translation in all the formulas, hence leaving the integrals unchanged.

Let us fix $A' \Subset A$. Let $\delta = \operatorname{dist}(A', A^C)$, and let $N = \left[\frac{1}{3\varepsilon}\right]$ $0 < t_1, \ldots, t_N \leq \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$. For every i we define

$$A_i := \big\{x \in A_\varepsilon : \ \operatorname{dist}(x, A^C) \geq t_i\big\}$$

and

$$S_i^{\xi,\varepsilon} := \{x \in (A_i)_\varepsilon : \ x + \varepsilon\xi \in A \setminus A_i\}.$$

With the above definitions it holds

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') + R_\varepsilon^\xi(A \setminus \bar{A}') + S_i^{\varepsilon,\xi} \ ,$$

Thus,

$$H(\varphi, A, \varepsilon) \leq H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon) + \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi,\varepsilon}} C_\xi f_{\xi,\varepsilon}(x, \varphi(x)/|\xi|) \\ + \sum_{\xi \in \mathbb{Z}^d} C_\xi \sum_{x \in S_i^{\xi,\varepsilon}} \Big|\frac{\varphi(x+\varepsilon\xi)}{|\xi|}\Big|^p + 1 \\ \leq H_\infty(\tilde{\varphi}, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \kappa, \varepsilon) \sum_{\xi \in \mathbb{Z}^d} C_\xi \sum_{x \in S_i^{\xi,\varepsilon}} \Big|\frac{\varphi(x+\varepsilon\xi)}{|\xi|}\Big|^p + 1,$$

where $\tilde{\varphi}$ is the function which coincides with φ in $(A_i)_\varepsilon$ and is equal to 0 outside of $(A_i)_\varepsilon$.

It is not difficult to verify that

$$S_i^{\xi,\varepsilon} \cap S_j^{\xi,\varepsilon} = \varnothing \qquad \text{whenever } |i-j| \geq |\xi|. \tag{1.26}$$

Fix $\delta_2 > 0$. Then for every ξ such that $\varepsilon|\xi| \geq \delta_2$ it holds

$$\frac{1}{N-1}\sum_{i=1}^{N-1}\sum_{x\in S_i^{\varepsilon,\xi}}\Big|\frac{\varphi(x+\varepsilon\xi)}{|\xi|}\Big|^p \leq \frac{1}{N-1}\sum_{i=1}^{N-1}\sum_{x\in S_i^{\xi,\varepsilon}}\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|^p. \tag{1.27}$$

Let us divide the last term in (1.19) into two terms

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{|\xi|>M}C_\xi\sum_{x\in S_i^{\xi,\varepsilon}}\Big|\frac{\varphi(x+\varepsilon\xi)-\psi(x+\varepsilon\xi)}{|\xi|}\Big|^p = \frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|>\delta_2}C_\xi\sum_{x\in S^{\xi,\varepsilon}}\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|^p$$
$$+\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|\leq\delta_2}C_\xi\sum_{x\in S^{\xi,\varepsilon}}\frac{|\varphi(x+\varepsilon\xi)|^p}{|\xi|^p}.$$

Because of (1.26), it holds

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|>\delta_2}C_\xi\sum_{x\in S^{\xi,\varepsilon}}\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|^p \leq C\frac{|\xi|}{N-1}\sum_{\xi\in\mathbb{Z}^d}C_\xi\kappa^p\varepsilon^{-d}/|A'|,$$

where in the last inequality we have used Lemma 1.2.6 and the fact that $\varphi \in \mathcal{V}(0, A, \kappa, \varepsilon)$.
For the second term

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|\leq\delta_2}C_\xi\sum_{x\in S^{\xi,\varepsilon}}\frac{|\varphi(x+\varepsilon\xi)|^p}{|\xi|^p} \leq \sum_{\xi\in\mathbb{Z}^d}C_\xi\sum_{x\in R_\varepsilon^\xi(A)}|\varphi|^p \leq \sum_{\xi\in\mathbb{Z}^d}C_\xi\sum_{x\in R_\varepsilon^\xi(A)}|\nabla\tilde{\varphi}|^p,$$

where in the first inequality we have used (1.26) and in the second inequality we have used the fact that the extension $\tilde{\varphi}$ has null boundary conditions.
Hence there exists there exist $0 < i \leq N-2$ such that

$$\sum_{|\xi|}\sum_{x\in S_i^{\xi,\varepsilon}}|\nabla_\xi\varphi|^p < \frac{2}{N-2}H(\tilde{\varphi}, A, \varepsilon).$$

After this step the proof continues in the same fashion as the proof of Lemma 1.2.13.

□

Lemma 1.2.15 (subadditivity)**.** *Let $A', A, B', B \subset \Omega$ be open sets such that $A' \Subset A$ and such that $B' \Subset B$. Then for every $u \in W^{1,p}$ one has that*

$$F''(u, A' \cup B') \leq F''(u, A) + F''(u, A).$$

Proof. The proof of this statement is very similar to Lemma 1.2.13.

□

Lemma 1.2.16 (locality)**.** *Let $u, v \in W^{1,p}(\Omega)$ such that $u \equiv v$ in A. Then*

$$F'(u, A) = F'(v, A) \qquad \textit{and} \qquad F''(u, A) = F''(v, A) \tag{1.28}$$

Proof. The statement follows from the definitions.

□

Proof of Theorem 1.2.5.
Let us suppose initially that there exists a sequence for which $F(\cdot,\cdot) = F'(\cdot,\cdot) = F''(\cdot,\cdot)$. Then to conclude it is enough to notice that F satisfies the conditions of Theorem 1.2.2. Indeed, in the previous Lemmas we prove that all the conditions (i)-(v) of Theorem 1.2.2 hold. □

Corollary 1.2.17. *Because of Lemma 1.2.14, the same statement holds true for F_∞. This in particular implies that for the sequence $\{\varepsilon_{n_k}\}$ in Theorem 1.2.5 there holds a large deviation principle with rate functional*

$$I(v) = \int_\Omega W(x, \nabla v)\,\mathrm{d}x - \min_{\bar v \in W_0^{1,p}(\Omega)+u} \int_\Omega W(\nabla \bar v(x))dx. \tag{1.29}$$

1.2.4 Homogenisation

In this section we will show that if the functions $f_{\xi,\varepsilon}$ are obtained by rescaling by ε in the space variable, then a LDP result holds true. This models the case when the arrangement of the "material points" presents a periodic feature, namely:

(H1) periodicity:

$$f_{\xi,\varepsilon}(x,t) = f^\xi\Big(\frac{x}{\varepsilon}, t\Big)$$

where the functions f^ξ are such that $f^\xi(x + Me_i, t) = f^\xi(x,t)$.

(H2) lower bound on the nearest neighbours:

$$f^{e_i}(x,t) \geq c_1(|t|^p - 1)$$

(H3) upper bound

$$f^\xi(x,t) \leq C_\xi(|t|^p + 1)$$

The main objective of this section is to prove the following homogenization result:

Theorem 1.2.18. *Let the functions $f^\xi_{\xi,\varepsilon}$ satisfy the above conditions. Then there exists a function f_{hom} such that for every $A \subset \Omega$ open set it holds*

$$F(u,A) = \begin{cases} \int_A f_{hom}(\nabla u) & \textit{if } u \in W^{1,p}(\Omega;\mathbb{R}^d) \\ +\infty & \textit{otherwise,} \end{cases} \tag{1.30}$$

where

$$f_{hom}(M) := \frac{1}{|A|}\lim_{\varepsilon \downarrow 0} F'(Mx, A, \kappa, \varepsilon). \tag{1.31}$$

Proof. Let (ε_n) be a sequence of positive numbers converging to 0. From Proposition 1.2.5 we can extract a subsequence (that we do not relabel for simplicity) such that

$$F'_{\{\varepsilon_n\}}(u,A) = F''_{\{\varepsilon_n\}}(u,A) = \int_A f_{\{\varepsilon_n\}}(x,\nabla u)\,\mathrm{d}x.$$

The theorem is proved if we show that f does not depend on the space variable x and on the chosen sequence ε_n. To prove the first claim, by Theorem 1.2.2, it suffices to show that, if one denotes by

$$F(u,A) = \int_A f(x,\nabla u)\,\mathrm{d}x,$$

then

$$F(Mx, B(y,\rho)) = F(Mx, B(z,\rho))$$

for all $M \in \mathbb{R}^{d\times m}$, $y,z \in \Omega$ and $\rho > 0$ such that $B(y,\rho)\cup B(z,\rho) \subset \Omega$. We will prove that

$$F(Mx, B(y,\rho)) \leq F(Mx, B(z,\rho)).$$

The proof of the opposite inequality is analogous.

Let $x, y \in \mathbb{R}^d$ and let $x_\varepsilon = \arg\min(\operatorname{dist}(y, x + (\varepsilon M)\mathbb{Z}^d))\big]$. Then $x_\varepsilon \to y$ as $\varepsilon \downarrow 0$. From the periodicity hypothesis, one has that

$$F\big(M, B(x,\rho,\kappa,\varepsilon)\big) = F\big(M, B(x_\varepsilon,\rho,\kappa,\varepsilon)\big) \leq F\big(M, B(y,\rho+\delta,\kappa,\varepsilon)\big)$$

where in the last inequality we have used the monotonicity with respect to the inclusion relation of $A \mapsto F(u,A,\kappa,\varepsilon)$ and δ is such that $|y - x_\varepsilon| \leq \delta$.

Let us now turn to the independence on the sequence on the chosen sequence. Let us initially notice that because of the LDP, whenever $u = Mx$ where M is a linear map it holds

$$F'(u,A,\kappa) = F'(u,A) \qquad \text{and} \qquad F''(u,A,\kappa) = F''(u,A). \tag{1.32}$$

Because of Theorem 1.2.2, it is enough to show that for every linear map M the following limit exists and

$$\frac{1}{|A|}\lim_{\varepsilon\downarrow 0} F'(Mx, A, \kappa, \varepsilon)$$

The existence of the above limit(and its independence on κ) follows easily by the standard methods with the help of an approximative subadditivity. A simple proof can be found in [19, Proposition 1.2]. □

1.3 SBV Representation Theorem

In this section we extend the results of the previous section to more general local interactions, where the problem relaxes naturally in SBV. The strategy will be very similar to the one used in § 1.2. However, we will need to use different tools and a different Representation Theorem. Repeating many of the arguments in the previous section is thus unavoidable, however we will refer to the previous section often when the repetition becomes pedantic.

1.3.1 A *very* short introduction to SBV

Before going into the details of our main Theorem of this section, let us define the functional spaces BV and SBV. For a general introduction on these spaces see [4]. However, please notice that the definitions given in this section differ slightly from the ones in [4]. More precisely, in the following, we additionally impose the finiteness of $(n-1)$-Hausdorff measure of the jump set. This technical modification is done in order to have at our disposal general representation theorems like the ones in the following section.

Let Ω be an open set. We say that $u \in L^1(\Omega)$ belongs to BV(Ω), if there exists a vector measure $Du = (D_1 u, \dots, D_n u)$ with finite total variation in Ω, such that

$$\int_\Omega u \partial_i \varphi \, \mathrm{d}x = - \int \varphi \, \mathrm{d}D_i u \qquad \forall \varphi \in C_0^1(\Omega)$$

Let $Du = D^a u + D^s u$ be the Radon-Nikodym decomposition of Du in absolutely continuous and singular part with respect to the $\mathcal{L}^n$ and let ∇u be the density of $D^a u$. It can be seen that u is approximately differentiable at x and the approximate differential equals to $\nabla u(x)$, i.e.,

$$\lim_{\rho \downarrow 0} \rho^{-n} \int_{B_\rho(x)} \frac{|u(y) - u(x) - \langle \nabla u, y - x \rangle|}{|y - x|} \, \mathrm{d}y = 0$$

for $\mathcal{L}^n$- a.e.$x \in \Omega$.

For the singular part, it is useful to introduce the upper and lower approximate limits u_+, u_-, defined by

$$u_-(x) = \inf \{ t \in [-\infty, +\infty] : \ \{ x \in \Omega : \ u(x) > t \} \text{ has density 0 at } x \}$$
$$u_+(x) = \sup \{ t \in [-\infty, +\infty] : \ \{ x \in \Omega : \ u(x) < t \} \text{ has density 0 at } x \} .$$

It is well-known that $u_+(x) \in \mathbb{R}$ for $\mathcal{H}^{d-1}$- a.e.$x \in \Omega$. The jump set S_u is defined by

$$S_u := \{ x \in : \ u_-(x) < u_+(x) \} .$$

We define the jump part Ju of the derivative as the restriction of $D^s u$ to the jump set S_u. We also recall that there exists a Borel map $\nu_u : S_u \to S^{d-1}$ such that

$$\nu_{E_t}(x) = \nu_u \qquad \text{for } \mathcal{H}^{d-1} \text{ a.e.} x \in \partial^* E_t \cap S_u$$

for any t such that $E_t := \{ x : u > t \}$.

Proposition 1.3.1. *Let $u \in BV(\Omega)$. Then, the jump part of the derivative is absolutely continuous with respect to $\mathcal{H}^{d-1}$ and*

$$Ju = (u_+ - u_-)\nu_u \mathcal{H}^{d-1} \llcorner S_u$$

Finally, we define the space $\mathrm{SBV}_p(\Omega)$ as the set of functions $u \in \mathrm{BV}(\Omega)$ such that $\nabla u \in L^p(\Omega)$ $D^s u = Ju$ and

$$\mathcal{H}^{d-1}(S_u) < +\infty. \tag{1.33}$$

Note that in [4] the condition (1.33) is not imposed.

1.3.2 Preliminary results

Let us now recall some well-known results, which will be useful in the sequel.
Theorem 1.3.2 ([4, Theorem 4.7])**.** *Let $\varphi : [0, +\infty) \to [0, +\infty]$, $\theta : [0, +\infty) \to [0, +\infty]$ be a lower semicontinuous function increasing functions and assume that*

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = +\infty \qquad \text{and} \qquad \lim_{t \to 0} \frac{\theta(t)}{t} = +\infty \tag{1.34}$$

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded and let $(u_h) \subset SBV(\Omega)$ such that

$$\sup \left\{ \int_\Omega \varphi(|\nabla u_h|) + \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|) \, \mathrm{d}\mathcal{H}^{n-1} \right\} < +\infty. \tag{1.35}$$

If (u_h) weakly converges in $BV(\Omega)$, then $u \in SBV(\Omega)$, the approximate gradients ∇u_h weakly converge to $\nabla u \in (L^1(\Omega))^N$. $D_j u_h$ weakly* converge to $D_j u \in \Omega$ and*

$$\int_\Omega \varphi(|\nabla u|) \, \mathrm{d}x \leq \liminf_{h \to +\infty} \int_\Omega \varphi(|\nabla u_h|) \, \mathrm{d}x \qquad \text{if } \varphi \text{ is convex}$$
$$\int_{J_u} \theta(|u^+ - u^-|) \, \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{h \to +\infty} \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|) \, \mathrm{d}\mathcal{H}^{n-1}$$

if θ is concave.
Theorem 1.3.3 (Compactness SBV [4, Theorem 4.8])**.** *Let φ, θ as in Theorem 1.3.2. Let (u_h) in $SBV(\Omega)$ satisfy* (1.35) *and assume in addition that $\|u_h\|_\infty$ is uniformly bounded in h. Then there exists a subsection (u_{h_k}) weakly* converging in $BV(\Omega)$ to $u \in SBV(\Omega)$.*
Let

$$\mathcal{F} : \mathrm{SBV}_p(\Omega, \mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0, +\infty]$$

such that the followings hold:

(H1) $\mathcal{F}(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure,

(H2) $\mathcal{F}(u, A) = \mathcal{F}(v, A)$ whenever $u = v$ $\mathcal{L}^n$ a.e.on $A \in \mathcal{A}(\Omega)$,

(H3) $\mathcal{F}(\cdot, A)$ is L^1 l.s.c., (H4) there exists a constant C such that

$$
\begin{aligned}
&\frac{1}{C}\Big(\int_A |\nabla u|^p \,\mathrm{d}x + \int_{S(u)\cap A} \big(1+|u^+-u^-|\big)\,\mathrm{d}\mathcal{H}^{n-1}\Big)\\
&\le \mathcal{F}(u,A) \qquad (1.36)\\
&\le C\Big(\int_A |\nabla u|^p \,\mathrm{d}x + \int_{S(u)\cap A} \big(1+|u^+-u^-|\big)\,\mathrm{d}\mathcal{H}^{n-1}\Big).
\end{aligned}
$$

Here, Ω is an open bounded set of $\mathbb{R}^n$. As before, $\mathcal{A}(\Omega)$ is the class of all open subsets of Ω and $\mathrm{SBV}_p(\Omega)$ is the space of functions $u \in \mathrm{SBV}(\Omega)$ such that $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{n-1}\big(J_u\big) < +\infty$. For every $u \in \mathrm{SBV}_p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ define

$$m(u;A) := \inf\{\mathcal{F}(u;A) : \ w \in \mathrm{SBV}_p(\Omega) \text{ such that } w = u \text{ in a neighbourhood of } \partial A\}$$

The role of Theorem 1.2.2, will be played by the following result, whose proof can be founded in [7].

Theorem 1.3.4. *Under hypotheses (H1)-(H4), for every $u \in SBV_p(\Omega)$ and $A \in \mathcal{A}(\Omega)$ there exists a function W_1 and W_2 such that W_1 is quasi-convex, W_2 is BV-elliptic and such that*

$$\mathcal{F}(u,A) := \int_A W_1(x,u,\nabla u)\,\mathrm{d}x + \int_{A\cap S_u} W_2(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathcal{H}^{n-1}.$$

Moreover, the functions W_1 and W_2 can be computed via

$$W_1(x_0,u_0,) := \limsup_{\varepsilon\to 0^+} \frac{\boldsymbol{m}\big(u_0+\xi(\cdot - x_0), Q(x_0,\varepsilon)\big)}{\varepsilon^n}$$

$$W_2(x_0,a,b,\nu) := \limsup_{\varepsilon\to 0^+} \frac{\boldsymbol{m}\big(u_{x_0,a,b,\nu}, Q_\nu(x_0,\varepsilon)\big)}{\varepsilon^{n-1}}$$

for all $x_0 \in \Omega$, u_0, $a,b \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $\nu \in S^{n-1}$ and where

$$u_{x_0,a,b,\nu}(x) := \begin{cases} a & \text{if } (x-x_0)\cdot\nu > 0,\\ b & \text{if } (x-x_0)\cdot\nu \le 0.\end{cases}$$

As $u_{x_0,b,a,\nu} = u_{x_0,a,b,\nu}$ $\mathcal{L}^n$ a.e. in $Q_\nu(x_0,\varepsilon) = Q_{-\nu}(x_0,\varepsilon)$, one has that

$$W_2(x_0,b,a,-\nu) = W_2(x_0,a,b,\nu),$$

for every $x_0 \in \Omega$, $a,b \in \mathbb{R}^d$ and $\nu \in \mathbb{R}^d$.

Remark 1.3.5. *Condition* (1.36), *can be softened to*

$$
\begin{aligned}
&\frac{1}{C}\Big(\int_A |\nabla u|^p \,\mathrm{d}x + \int_{S(u)\cap A} \big(|u^+-u^-|\big)\,\mathrm{d}\mathcal{H}^{n-1}\Big)\\
&\le \mathcal{F}(u,A) \qquad (1.37)\\
&\le C\Big(\int_A |\nabla u|^p \,\mathrm{d}x + \int_{S(u)\cap A} \big(|u^+-u^-|\big)\,\mathrm{d}\mathcal{H}^{n-1}\Big).
\end{aligned}
$$

Indeed, let us suppose that $\mathcal{F}$ satisfies only (1.37). *By the same theorem(Theorem 1.3.4) it is possible to represent $Fcal(u,A) + \mathcal{H}(S_u \cap A)$, thus by removing the subtracted part it is possible to represent $\mathcal{F}$.*

1.3.3 Hypothesis an Main Theorem

Given Theorem 1.3.2, it is natural to impose the following hypothesis.

Let $g^{(1)}$ a monotone convex functions such that there exists a constant C such that

$$g^{(1)}(t) \geq C \max(t^p - 1, 0)$$

and $g^{(2)}$ be a monotone concave function such that

$$g^{(2)}(t) \geq c > 0 \qquad \text{and} \qquad \lim_{t \uparrow \infty} \frac{g^{(1)}(t)}{t} = +\infty.$$

The typical example we have in mind is when $g^{(1)}(t) := t^p$ and $g^{(2)}(t) := 1 + t^\alpha$, where $0 < \alpha < 1$ and $p > 1$.

Let $T_\varepsilon \uparrow \infty$ be such that $\varepsilon T_\varepsilon \downarrow 0$. We denote

$$g_\varepsilon(x) = \begin{cases} g^{(1)}(\|x\|) & \text{if } \|x\| < T_\varepsilon, \\ \frac{1}{\varepsilon} g^{(2)}(\varepsilon \|x\|) & \text{if } \|x\| \geq T_\varepsilon. \end{cases}$$

We will also assume that there exists a constant C such that $g^{(1)}(T_\varepsilon) \leq \frac{C}{\varepsilon} g^{(2)}(T_\varepsilon \varepsilon)$, and that for every $M > 0$ there exists a constant C_M such that

$$g_\varepsilon(M|t|) \leq C_M g_\varepsilon(|t|).$$

The above

Let $(f_{\xi,\varepsilon})$ be a family of local interactions such that for every ξ, ε it holds

$$f_{\xi,\varepsilon}(x,t) \lesssim \big(g_\varepsilon(|t|) + 1\big) \tag{1.38}$$

and such that for every $1 \leq j \leq d$ it holds

$$f_{e_i,\varepsilon}(x,t) \gtrsim \big(g_\varepsilon(|t|) - 1\big) \tag{1.39}$$

We will assume also that there exists a constant $M < +\infty$ such that

$$\int_{\mathbb{R}} |t|^{d-1} \exp\left(-g_\varepsilon(t)\right) \, \mathrm{d}t \leq M.$$

Let us now define the Hamiltonians as

$$H(u, A, \varepsilon) = \sum_{\xi \in \mathbb{Z}^N} \sum_{x \in R_\varepsilon^\xi(A)} f_{\xi,\varepsilon}\Big(x, \frac{\varphi(x + \varepsilon\xi) - \varphi(x)}{|\xi|}\Big)$$

and

$$H_\infty(\varphi, A, \varepsilon) := \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in A_\varepsilon} f_{\varepsilon,\xi}(x, \nabla_\xi \varphi(x)).$$

Remark 1.3.6. *Let $u \in SBV_p(\Omega) \cap L^\infty(\Omega)$. Then like in 1.3, there exists an discretized $\varphi_{u,\varepsilon}$ such that*

$$\|u\|_{SBV_p} \lesssim \varepsilon^d \sum_{x \in \varepsilon\mathbb{Z}^d \cap \Omega} g_\varepsilon(\nabla \varphi_{u,\varepsilon}) \lesssim \|u\|_{SBV_p}.$$

Let us discuss very informally the above hypothesis. The function g_ε will play the role of $\|\cdot\|^p$ in § 1.2 and the conditions on $g^{(1)}$ and $g^{(2)}$ are in order to ensure the compactness and lower semicontinuity. Given that a discrete function can be interpolated by continuous functions, it does not make sense to talk about jump set. However, it makes sense to consider as a jump set, the set of points where the discrete gradient is bigger that a certain threshold T_ε. Indeed, if we were approximating a function with a jump, it is expected that the gradient would explode(in a neighbourhood of the jump set) like δ/ε, where δ is the amplitude of the jump and ε is the discretization parameter. Thus $T_\varepsilon \uparrow \infty$. Indeed, suppose that the function we are approximating is $\delta\chi_B$, where δ is a small parameter and B is the unit ball. Then the jump set would be the set of points where the gradient goes like $\frac{\delta}{\varepsilon}$. Thus in order to "catch" jumps of order δ one needs that the $\lim_{\varepsilon\downarrow 0} T_\varepsilon \varepsilon \leq \delta$. Thus $\lim_{\varepsilon\downarrow 0} T_\varepsilon\varepsilon = 0$.

As in the previous section, one of the main steps will be to show that $F'_\infty = F'$ and that $F''_\infty = F''$. The basic intuition behind, is again a version of the interpolation lemma. As before, we will show that if one imposes "closeness" v in $L^p(A)$ to some regular function u, then one can impose also the boundary condition by "paying a very small price in energy". More precisely, given a sequence $\{v_n\}$ such that $v_n \to u$ in $L^p(A)$, where A is an open set, then there exists a sequence $\{\tilde{v}_n\}$ such that $\tilde{v}_n \to u$, such that $\tilde{v}_n|_{\partial\Omega} = u|_{\partial\Omega}$ and

$$\liminf_n \|\tilde{v}_n\|_{\mathrm{SBV}_p(A)} \leq \liminf_n \|v_n\|_{\mathrm{SBV}_p(A)}.$$

Remark 1.3.7. *Let $f : [0, +\infty) \to [0, +\infty)$ be a monotone function. Then, it is immediate to have*

$$f\Big(\frac{1}{N}\sum_{i=1}^N t_i\Big) \leq \sum_i f(t_i),$$

where $t_i > 0$.

Similarly as in § 1.2, for every $A \in \mathcal{A}(\Omega)$, we define the free-energy as

$$F(u, A, \kappa, \varepsilon) := -\varepsilon^d \log \int_{\mathcal{V}(u,A,\kappa)} \exp\Big(- H(\varphi, A, \varepsilon)\Big)\, \mathrm{d}\varphi$$

$$F_\infty(u, A, \kappa, \varepsilon) := -\varepsilon^d \log \int_{\mathcal{V}_\infty(u,A,\kappa)} \exp\Big(- H_\infty(\varphi, A, \varepsilon)\Big)\, \mathrm{d}\varphi$$

where

$$\mathcal{V}(u, A, \kappa) = \left\{ \varphi : A_\varepsilon \to \mathbb{R}^m | \ \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon\varphi|^p \leq \kappa^p \right\}$$

$$\mathcal{V}_\infty(u, A, \kappa) = \left\{ \varphi : \varepsilon\mathbb{Z}^d \to \mathbb{R}^m | \ \frac{\varepsilon^d}{|A|^d} \sum_{x \in A_\varepsilon} |u - \varepsilon\varphi|^p \leq \kappa^p, \text{ and } \varphi(x) = \varphi_{u,\varepsilon}(x)\ \forall x \notin A_\varepsilon \right\},$$

where $\varphi_{u,\varepsilon}$ is defined in (1.3).

Similarly as in § 1.2, let us introduce the following notations:

$$\begin{aligned}
F'(u,A,\kappa) &:= \liminf_{\varepsilon\downarrow 0} F(u,A,\kappa,\varepsilon)\\
F''(u,A,\kappa) &:= \limsup_{\varepsilon\downarrow 0} F(u,A,\kappa,\varepsilon)\\
F'(u,A) &:= \lim_{\kappa\downarrow 0}\liminf_{\varepsilon\downarrow 0} F(u,A,\kappa,\varepsilon) = \lim_{\kappa\downarrow 0} F'(u,A,\kappa)\\
F''(u,A) &:= \lim_{\kappa\downarrow 0}\limsup_{\varepsilon\downarrow 0} F(u,A,\kappa,\varepsilon) = \lim_{\kappa\downarrow 0} F''(u,A,\kappa)\\
F'_\infty(u,A,\kappa) &:= \liminf_{\varepsilon\downarrow 0} F_\infty(u,A,\kappa,\varepsilon)\\
F''_\infty(u,A,\kappa) &:= \limsup_{\varepsilon\downarrow 0} F_\infty(u,A,\kappa,\varepsilon)\\
F'_\infty(u,A) &:= \lim_{\kappa\downarrow 0}\liminf_{\varepsilon\downarrow 0} F_\infty(u,A,\kappa,\varepsilon) = \lim_{\kappa\downarrow 0} F'_\infty(u,A,\kappa)\\
F''_\infty(u,A) &:= \lim_{\kappa\downarrow 0}\limsup_{\varepsilon\downarrow 0} F_\infty(u,A,\kappa,\varepsilon) = \lim_{\kappa\downarrow 0} F''_\infty(u,A,\kappa)
\end{aligned}$$

We are now able to write the main result of this section.

Theorem 1.3.8. *Assume the previous hypothesis and that $u \in SBV_p \cap L^\infty$. Then for every infinitesimal sequence (ε_n) there exists a subsequence ε_{n_k} and functions $W_1 : \Omega \times \mathbb{R}^{d\times m} \to \mathbb{R}$ and $W_2 : \Omega \times \mathbb{R}^m \times S^{d-1} \to \mathbb{R}$ such that*

$$F(u,A) := F'_{n_k}(u,A) = F''_{n_k}(u,A) = \int_A W_1(x,\nabla u)\,\mathrm{d}x + \int_{S_u} W_2(x,u^+(x)-u^-(x),\nu_u(x)),$$

where the function W_1 is a quasiconvex function and W_2 is a BV-elliptic function and depend on the chosen subsequence $\{\varepsilon_{n_k}\}$.

1.3.4 Proofs

The next technical lemma is a version of Lemma 1.2.6, that asserts that finite difference quotients along any direction can be controlled by finite difference quotients along the coordinate directions.

Lemma 1.3.9. *Let $A \subset \mathcal{A}(\Omega)$ and set $A_\varepsilon = \left\{x \in A : \operatorname{dist}(x,A) > 2\sqrt{N}\varepsilon\right\}$. Then there exists a dimensional constant $C := C(N)$ such that for any $\xi \in \mathbb{Z}^N$ there holds*

$$\sum_{x\in R^{e_i}_\varepsilon(A_\varepsilon)} g_\varepsilon\big(\nabla_\xi u(x)\big) \le C|\xi| \sum_{i=1}^N \sum_{x\in R^{e_i}_\varepsilon(A)} g_\varepsilon(\nabla_{e_i} u(x)).$$

Proof. As in the proof of Lemma 1.2.6, let $\xi \in \mathbb{Z}^d$. By decomposing it into coordinates, it is not difficult to notice that it can be written as

$$\xi = \sum_{k=1}^{N_\xi} \alpha_k(\xi) e_{i_k},$$

where $N_\xi \le \delta|\xi|$ and $\alpha_k(\xi) \in \{-1,1\}$. Denote by

$$\xi_k = \sum_{j=1}^{N_\xi} \alpha_k(\xi),$$

hence $|\xi_k| \le |\xi|$ for all k. Thus

$$\nabla_\xi u(x) = \frac{1}{|\xi|}\sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x+\varepsilon\xi_k)$$

Moreover, by the monotonicity of g_ε, we have

$$g_\varepsilon\Big(\frac{1}{N_\xi}\sum_{k=1}^{N_\xi} \nabla_{\alpha_k(\xi)e_i} u(x+\varepsilon\xi_k)\Big) \le \sum_{k=1}^{N_\xi} g_\varepsilon\Big(\nabla_{\alpha_k(\xi)e_i} u(x+\varepsilon\xi_k)\Big)$$

Finally by summing over all ξ, exchanging the sums and using the equivalence of the norms i.e., $|\xi| \le N_\xi \le d|\xi|$ one has the desired result.

□

As in the previous section, let G^λ be the free-energy (see (1.6) for the definition) induced by the Hamiltonian

$$\tilde{H}^\lambda(\varphi, A, \varepsilon) := \lambda \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_i \varphi|).$$

In a very similar fashion as in Lemma 1.2.8, one can prove
Lemma 1.3.10. *There exists constants C_λ, D_λ, such that it holds*

$$C_\lambda \le G^\lambda(0, A, \kappa, \varepsilon) \le D_\lambda$$

The next proof is the analog of Lemma 1.2.9.
Lemma 1.3.11. *Let $\{f_{\xi,\varepsilon}\}$ satisfy the usual hypothesis. Then there exists a constant $D > 0$ and $\varepsilon_0 > 0$ such that for every $\kappa < 1$ it holds*

$$\exp\Big(-\varepsilon^{-d} F(u, A, \kappa, \varepsilon)\Big) \le \exp\left(D\varepsilon^{-d} + D \sum_{\xi \in R_\xi^{e_i}(A)} \sum_{i=1}^{d} g_\varepsilon(\nabla_{e_i} \varphi_{u,\varepsilon}) \right), \tag{1.40}$$

where $\varphi_{u,\varepsilon}$ is defined in (1.3).

Proof. Given that $g_\varepsilon(|a|) \lesssim g_\varepsilon(|a-b|) + g_\varepsilon(|b|)$ one has that there exist constants C_1 such that

$$\begin{aligned} H(\varphi, A, \kappa, \varepsilon) &\ge C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i}\varphi(x)|) \\ &\ge C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla\psi|) - C_1 \sum_{i=1}^{d} \sum_{x \in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i}\varphi_{u,\varepsilon}(x)|) \end{aligned}$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence the estimate (1.40) reduces to prove that there exists a constant D such that

$$\int_{\{\|\varepsilon\varphi\|\leq\kappa\}} \exp\left(-C\sum_{i=1}^{d}\sum_{x\in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i}\varphi|)\right) \leq \exp\left(D\varepsilon^{-d}\right).$$

The above inequality was proved in Lemma 1.3.10.

□

As in Remark 1.2.10, we have the following:

Remark 1.3.12. *Let $u \in L^\infty \cap SBV_p$, then along the lines of Lemma 1.3.11 one can easily prove that there exists a constant C such that*

$$A \mapsto F'(u, A) + C(|u|_{SBV_p(A)} + 1) \qquad A \mapsto F''(u, A) + C(|u|_{SBV_p(A)} + 1)$$

are monotone with respect to the inclusion relation.

Lemma 1.3.13. *Let $f_{\xi,\varepsilon}$ satisfy our hypothesis and let A be an open set. Then there exists a constant $D > 0$ such that*

$$\exp\left(-\varepsilon^{-d}F(u, A, \kappa, \varepsilon)\right) \geq \exp\left(-D\varepsilon^{-d} - \sum_{i=1}^{d}\sum_{x\in R_\varepsilon^{e_i}} g_\varepsilon\big(|\nabla_{e_i}\varphi_{u,\varepsilon}(x)|\big)\right)$$

where $\varphi_{u,\varepsilon}$ is defined in (1.3)*.*

Proof. Using Lemma 1.3.9, one has that there exists a constant C such that

$$H(\varphi, A, \kappa, \varepsilon) \leq C\sum_{i=1}^{d}\sum_{x\in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i}\varphi|)$$

Given that $g_\varepsilon(a + b) \leq g_\varepsilon(2a) + g_\varepsilon(2b) \lesssim g_\varepsilon(a) + g_\varepsilon(b)$, there exist a constant C_1 such that

$$H(\varphi, A, \varepsilon) \leq C_1\sum_{i=1}^{d}\sum_{x\in R_\varepsilon^{e_i}(A)} (g_\varepsilon(|\nabla_{e_i}\varphi_{\varepsilon,u}|) + 1) + 2d\sum_{x\in A_\varepsilon} g_\varepsilon(|\psi(x)|),$$

where $\psi = \varphi - \varphi_{u,\varepsilon}$. Hence, the estimate (1.40) reduces to prove that there exists a constant D such that

$$\int_{\mathcal{V}(0,A,\kappa,\varepsilon)} \exp\left(-C\sum_{x\in A_\varepsilon} g_\varepsilon(|\psi(x)|)\right) \geq (\varepsilon\kappa)^{-d}\exp\left(D\varepsilon^{-d}\right).$$

The above inequality was proved in Lemma 1.2.8.

□

Lemma 1.3.14 (exponential tightness). *Let A be an open set and $K \geq 0$. Denote by*

$$\mathcal{M}_K := \Big\{\varphi : \ H(\varphi, A, \varepsilon) \geq K\varepsilon^{-d}|A|\Big\}.$$

Then there exists a constant D, K_0, ε_0 such that for every $K \geq K_0$, $\varepsilon \leq \varepsilon_0$ it holds

$$\int_{\mathcal{M}_K \cap \mathcal{V}(u,A,\kappa)} \exp\big(-H(\varphi, A, \varepsilon)\big) \leq \exp\Big(-\frac{1}{2}K\varepsilon^{-d} + D\varepsilon^{-d} - D\sum_{i-1}^{d}\sum_{x\in R_\varepsilon^{e_i}(A)} g_\varepsilon(|\nabla_{e_i}\varphi_u|)\Big)$$

Proof. For every $\varphi \in \mathcal{M}_K$ it holds

$$H(\varphi, A, \varepsilon) \geq K/2\varepsilon^{-d} + H(\varphi, A, \varepsilon).$$

Hence, by using Lemma 1.3.13 we have the desired result.

□

The proof of the following lemma is similar to Lemma 1.2.13.
Lemma 1.3.15 (regularity). *Let f_ξ satisfy the usual hypothesis then*

$$\sup_{A' \Subset A} F''(u, A') = F''(u, A).$$

Proof. Let us fix $A' \Subset A$ and $N \in \mathbb{N}$ (to be chosen later). Let $\delta = \operatorname{dist}(A', A^C)$, and let $0 < t_1, \ldots, t_N \leq \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$. Without loss of generality, we may assume that there exists no $x \in A_\varepsilon$ such that $\operatorname{dist}(x, A^C) = t_i$. For every i we define

$$A_i := \big\{x \in A_\varepsilon : \ \operatorname{dist}(x, A^C) \geq t_i\big\}$$

and

$$S_i^{\xi,\varepsilon} := \{x \in (A_i)_\varepsilon : \ x + \varepsilon\xi \in A \setminus A_i\}.$$

With the above definitions, it holds

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') + R_\varepsilon^\xi(A \setminus \bar{A}') + S_i^{\varepsilon,\xi}$$

thus

$$H^\xi(\varphi, A, \varepsilon) = H^\xi(\varphi, A \setminus \bar{A}_i, \varepsilon) + H^\xi(\varphi, A_i, \varepsilon) + \sum_{x \in S_i^{\xi,\varepsilon}} f_{\xi,\varepsilon}(\nabla_\xi\varphi(x)).$$

Hence,

$$H(\varphi, A, \varepsilon) = H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus A_i, \varepsilon) + \sum_{\xi\in\mathbb{Z}^d}\sum_{x\in S_i^{\xi,\varepsilon}} C_\xi\Big(g_\varepsilon\big(|\nabla_\xi\varphi(x)|\big) + 1\Big)$$

Let us now estimate the last term in the previous inequality.

We separate the sum into two terms

$$\sum_{\xi\in\mathbb{Z}^d}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi(x)|) = \sum_{|\xi|\leq M}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi(x)|) + \sum_{|\xi|>M}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi(x)|). \tag{1.41}$$

Let $M\in\mathbb{N}$. From the condition (1.38) and by taking M sufficiently large, we may also assume without loss of generality that

$$\sum_{|\xi|\geq M}|\xi|C_\xi \leq \delta_1.$$

Hence, by using Lemma 1.3.9 we have that

$$\sum_{|\xi|\geq M}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi(x)|) \leq C\delta_1\sum_{k=1}^{d}\sum_{x\in R_\varepsilon^{e_k}(A)} g_\varepsilon(|\nabla_{e_k}\varphi(x)|) \leq \tilde{C}\delta_1 H(\varphi,A,\varepsilon),$$

where in the last inequality we have used hypothesis (1.39).
Let $|\xi|<M$. If $\varepsilon MN\leq 2\delta$, then for every

$$S_i^{\xi,\varepsilon}\cap S_j^{\xi,\varepsilon}=\varnothing \qquad \text{whenever } |i-j|\geq 2.$$

Without loss of generality, we may assume the above condition as $\varepsilon\to 0$.
Given that

$$\frac{1}{N-2}\sum_{i=1}^{N-2}\sum_{|\xi|<M}\sum_{x\in S_i^{\varepsilon,\xi}} g_\varepsilon(|\nabla\varphi(x)|)\leq 2CH(\varphi,A,\varepsilon)$$

there exist $0<i\leq N-2$ such that

$$\sum_{|\xi|<M}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi|) < \frac{2}{N-2}H(\varphi,A,\varepsilon). \tag{1.42}$$

Let us denote by $\mathcal{N}_i$ the set of all $\varphi\in\mathcal{V}(u,A,\kappa,\varepsilon)$ such that (1.42) holds for the first time, namely for every $j\leq i$

$$\sum_{|\xi|<M}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi|) \geq \frac{2}{N-2}H(\varphi,A,\varepsilon). \tag{1.43}$$

On one side, we have that

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)}\exp\left(-H(\varphi,A,\kappa,\varepsilon)\right)\leq\sum_{i=1}^{N}\int_{\mathcal{N}_i}\exp\left(-H(\varphi,A_i,\varepsilon)-H(\varphi,A\setminus\bar{A}_i,\varepsilon)\right),$$

on the other side one has that

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)}\exp\left(-H(\varphi,A,\kappa,\varepsilon)\right)\geq\sum_{i=1}^{N}\int_{\mathcal{N}_i^K}\exp\left(-H(\varphi,A,\varepsilon)\right),$$

where $\mathcal{N}_i^K := \mathcal{N}_i \setminus \mathcal{M}_K$. By using (1.43), one has that for every $\varphi \in \mathcal{N}_i^K$ it holds

$$H(\varphi, A, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon) \leq H(\varphi, A, \varepsilon) \leq H(\varphi, A_i) + H(\varphi, A \setminus \bar{A}_i) + \frac{K}{N-2}$$

and for every φ it holds

$$H(\varphi, A, \varepsilon) \geq H(\varphi, A, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon).$$

Hence,

$$\int_{\mathcal{V}(u,A,\kappa,\varepsilon)} \exp\left(-H(\varphi, A, \varepsilon)\right) \geq \sum_{i=1}^{N} \int_{\mathcal{N}_i^K} \exp\left(-H(\varphi, A_i) - H(\varphi, A \setminus \bar{A}_i) - \frac{K}{N-2}\right).$$

From now on the proof follows as in Lemma 1.2.13. □

Lemma 1.3.16. *For every open set A and $u \in W^{1,p}(\mathbb{R}^d)$ it holds*

$$F'(u, A) = F'_\infty(u, A) \qquad \textit{and} \qquad F'(u, A) = F'_\infty(u, A).$$

Proof. As in Lemma 1.2.14, we may assume without loss of generality $u = 0$.
Let us fix $A' \subset\subset A$. Let $\delta = \operatorname{dist}(A', A^C)$, and let $N = \left[\frac{1}{3\varepsilon}\right]$ $0 < t_1, \ldots, t_N \leq \delta$ such that $t_{i+1} - t_i > \frac{\delta}{2N}$.
For every i we define

$$A_i := \left\{x \in A_\varepsilon : \ \operatorname{dist}(x, A^C) \geq t_i\right\}$$

and

$$S_i^{\xi,\varepsilon} := \left\{x \in (A_i)_\varepsilon : \ x + \varepsilon\xi \in A \setminus A_i\right\}.$$

With the above definitions, it holds

$$R_\varepsilon^\xi(A) = R_\varepsilon^\xi(A') + R_\varepsilon^\xi(A \setminus \bar{A}') + S_i^{\varepsilon,\xi}$$

thus,

$$\begin{aligned} H(\varphi, A, \varepsilon) &\leq H(\varphi, A_i, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \varepsilon) + \sum_{\xi \in \mathbb{Z}^d} \sum_{x \in S_i^{\xi,\varepsilon}} C_\xi f_{\xi,\varepsilon}(x, \varphi(x)/|\xi|) \\ &\quad + \sum_{\xi \in \mathbb{Z}^d} C_\xi \sum_{x \in S_i^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varphi(x + \varepsilon\xi)}{|\xi|}\Big|\Big) + 1 \\ &\leq H_\infty(\tilde{\varphi}, A, \varepsilon) + H(\varphi, A \setminus \bar{A}_i, \kappa, \varepsilon) \sum_{\xi \in \mathbb{Z}^d} C_\xi \sum_{x \in S_i^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varphi(x + \varepsilon\xi)}{|\xi|}\Big)\Big) + 1, \end{aligned}$$

where $\tilde{\varphi}$ is the function which coincides with φ in $(A_1)_\varepsilon$ and is equal to 0 outside of $(A_1)_\varepsilon$.

It is not difficult to verify that

$$S_i^{\xi,\varepsilon} \cap S_j^{\xi,\varepsilon} = \varnothing \qquad \text{whenever } |i-j| \geq |\xi|. \tag{1.44}$$

Fix $\delta_2 > 0$. Then for every ξ such that $\varepsilon|\xi| \geq \delta_2$ it holds

$$\frac{1}{N-1}\sum_{i=1}^{N-1}\sum_{x\in S_i^{\varepsilon,\xi}}\Big(\Big|\frac{\varphi(x+\varepsilon\xi)}{|\xi|}\Big|\Big) \leq \frac{1}{N-1}\sum_{i=1}^{N-1}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|\Big). \tag{1.45}$$

Let us divide the last term in (1.41) into two terms

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{|\xi|>M} C_\xi \sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varphi(x+\varepsilon\xi)-\psi(x+\varepsilon\xi)}{|\xi|}\Big|\Big) = \frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|>\delta_2} C_\xi \sum_{x\in S^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|\Big)$$
$$+\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|\leq\delta_2} C_\xi \sum_{x\in S^{\xi,\varepsilon}} g_\varepsilon\Big(\frac{|\varphi(x+\varepsilon\xi)|}{|\xi|}\Big)$$

Because of (1.44)

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|>\delta_2} C_\xi \sum_{x\in S^{\xi,\varepsilon}} g_\varepsilon\Big(\Big|\frac{\varepsilon\varphi(x+\varepsilon\xi)}{\delta_2}\Big|\Big) \leq C\frac{C}{N-1}\sum_{\xi\in\mathbb{Z}^d}|\xi|C_\xi\kappa^p\varepsilon^{-d}/|A'|$$

where in the last inequality we have used Lemma 1.3.9 and the fact that $\varphi \in \mathcal{V}(0,A,\kappa,\varepsilon)$.
For the second term

$$\frac{1}{N-1}\sum_{i=0}^{N}\sum_{\varepsilon|\xi|\leq\delta_2} C_\xi \sum_{x\in S^{\xi,\varepsilon}} g_\varepsilon\Big(\frac{|\varphi(x+\varepsilon\xi)|}{|\xi|}\Big) \leq \sum_{\xi\in\mathbb{Z}^d} C_\xi \sum_{x\in R_\varepsilon^\xi(A)} g_\varepsilon(|\varphi|) \leq \sum_{\xi\in\mathbb{Z}^d} C_\xi \sum_{x\in R_\varepsilon^\xi(A)} g_\varepsilon(|\nabla\tilde{\varphi}|)$$

where in the first inequality we have used (1.44) and in the second inequality we have used the fact that the extension $\tilde{\varphi}$ has null boundary conditions.
Hence, there exists there exist $0 < i \leq N-2$ such that

$$\sum_{|\xi|}\sum_{x\in S_i^{\xi,\varepsilon}} g_\varepsilon(|\nabla_\xi\varphi|) < \frac{2}{N-2}H(\tilde{\varphi},A,\varepsilon)$$

After this step the proof continues in the same fashion as the proof of Lemma 1.3.15

□

Lemma 1.3.17 (subadditivity)**.** *Let $A', A, B', B \subset \Omega$ be open sets such that $A' \Subset A$ and such that $B' \Subset B$. Then for every $u \in W^{1,p}$ one has that*

$$F''(u, A'\cup B') \leq F''(u,A) + F''(u,A)$$

Proof. The proof of this statement is very similar to Lemma 1.3.15 and Lemma 1.3.16
. □

Lemma 1.3.18 (locality)**.** *Let $u, v \in SBV_p(\Omega)$ such that $u \equiv v$ in A. Then*

$$F'(u, A) = F'(u, v) \qquad \text{and} \qquad F''(u, A) = F''(u, v)$$

Proof. The statement follows from the definitions.

□

Proof of Theorem 1.3.8. Let us suppose initially that there exists a sequence for which $F(\cdot,\cdot) = F'(\cdot,\cdot) = F''(\cdot,\cdot)$. Then to conclude it is enough to notice that F satisfies the conditions of Theorem 1.3.4, which are proved in the previous Lemmas. □

Corollary 1.3.19. *Because of Lemma 1.3.16, the same statement holds true for F_∞. This in particular implies that for the sequence $\{\varepsilon_{n_k}\}$ in Theorem 1.3.8 there holds a large deviation principle with rate functional*

$$I(v) = \int_\Omega W_1(x, \nabla v)\, \mathrm{d}x \int_{Ju} W_2(x, u_+(x) - u_-(x))\, \mathrm{d}\mathcal{H}^{d-1}(x) - \min_{\bar{v} \in W_0^{1,p}(\Omega)+u} \int_\Omega W(\nabla \bar{v}(x)) dx. \tag{1.46}$$

2 Finite Range Decomposition

2.1 Introduction

Recently, there has been some interest in the finite range decompositions of gradient Gaussian fields on $\mathbb{Z}^d$. In particular, in [1], S. Adams, R. Kotecký and S. Müller construct a finite range decomposition for a family of translation invariant gradient Gaussian fields on $\mathbb{Z}^d$ ($d \geq 2$) which depends real-analytically on the quadratic from that defines the Gaussian field: they consider a large torus $\mathbb{T}_N^d := (\mathbb{Z}/L^N\mathbb{Z})^d$ and obtain a finite range decomposition with estimates that do not depend on N.

More precisely, they show that the discrete Greens function $\mathcal{C}_A : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}^m$ of the (elliptic translation invariant) difference operator $\mathscr{A} = \nabla^* A \nabla$ can be written as a sum $\mathcal{C}_A = \sum_k \mathcal{C}_{A,k}$ of positive kernels $\mathcal{C}_{A,k}$ which are supported in cubes of size $\sim L^k$ with natural estimates for their discrete derivatives $\nabla^\alpha \mathcal{C}_{A,k}$ as well as for their derivatives with respect to A. The above results are obtained by via a careful analysis of the Fourier multipliers and combinatorics.

We extend their result in the following way: We consider non-translation invariant Gaussian gradient fields and show a similar result. Namely, we show that the discrete Green's function $\mathcal{C}_A(x, y) : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}^m$ of the elliptic difference operator $\mathscr{A} = \nabla^* A \nabla$, where $A = A(x)$ is a general elliptic operator (for detailed hypothesis see Section 2.3) can be written as the sum $\mathcal{C}_A = \sum_k \mathcal{C}_{A,k}$ of positive kernels $\mathcal{C}_{A,k}$ which are supported in cubes of size $\sim L^k$ with natural estimates for their discrete derivatives $\nabla^\alpha \mathcal{C}_{A,k}$ as well as for their derivatives with respect to A. Due to the general non-translation invariant setting the techniques used in [1], seem not to apply. In order to overcome these difficulties, we will use results from the well-known L^p-theory, which are extended to the discrete setting, and then approach the problem. Although this might not come as a surprise to the experts in regularity theory, we could not find in the literature suitable results. As a byproduct we are also able to prove the equivalent results in the continuous setting which are to our knowledge not known.

2.2 Preliminary Results

In this section we are going to describe *briefly* the results in [1].

Let $L \geq 3$ be a fixed odd integer and consider for any integer N the space

$$\boldsymbol{\mathcal{V}}_N = \{\varphi : \mathbb{Z}^d \to \mathbb{R}^m;\ \varphi(x+z) = \varphi(x) \text{ for all } z \in (L^N\mathbb{Z})^d\} = \left(\mathbb{R}^m\right)^{\mathbb{T}_N^d}$$

of functions on the torus $\mathbb{T}_N^d := \left(\mathbb{Z}/L^N\mathbb{Z}\right)^d$ equipped with with the scalar product

$$\langle \varphi, \psi \rangle = \sum_{x \in \mathbb{T}_N^d} \langle \varphi(x), \psi(x) \rangle_{\mathbb{R}^m}.$$

Notice that, a function on $\mathbb{T}_N^d$ can be identified with an L^N-periodic function on $\mathbb{Z}^d$. In the last section, it will denote the corresponding space of $\mathbb{C}^m$-valued function, equipped with the usual hermitian product.

Define

$$\rho(x, y) := \inf\{|x - y + z| \colon z \in (L^N\mathbb{Z})^d\}$$

and

$$\rho_\infty(x, y) := \inf\{|x - y + z|_\infty \colon z \in (L^N\mathbb{Z})^d\}.$$

Then, the torus can be represented by the lattice cube $\mathbb{T}_N^d = \{x \in \mathbb{Z}^d \colon |x|_\infty \leq \frac{1}{2}(L^N - 1)\}$ of side L^N, equipped with the metric ρ or ρ_∞.

Gradient Gaussian fields are naturally defined on

$$\boldsymbol{\mathcal{X}}_N := \{\varphi \in \boldsymbol{\mathcal{V}}_N : \sum_{x \in \mathbb{T}_N} \varphi(x) = 0\}. \tag{2.1}$$

For any set $M \subset \Lambda_N$, we define its closure by

$$\overline{M} = \{x \in \Lambda_N \colon \operatorname{dist}_\infty(x, M) \leq 1\}, \tag{2.2}$$

where

$$\operatorname{dist}_\infty(x, M) := \min\{\rho_\infty(x, y) \colon y \in M\}.$$

The forward and backward derivatives are defined as

$$\begin{aligned} (\nabla\varphi)_j^r(x) &:= \varphi^r(x + \mathrm{e}_j) - \varphi^r(x), \\ (\nabla^*\varphi)_j^r(x) &:= \varphi^r(x - \mathrm{e}_j) - \varphi^r(x), \quad r = 1, \ldots, m;\ j = 1, \ldots, d. \end{aligned} \tag{2.3}$$

Let $A \colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$ be a linear map that is symmetric with respect to the standard scalar product $(\cdot, \cdot)_{\mathbb{R}^{m\times d}}$ on $\mathbb{R}^{m\times d}$ and positive definite, that is, there exists a constant $c_0 > 0$ such that

$$(AF, F)_{\mathbb{R}^{m\times d}} \geq c_0 \|F\|^2_{\mathbb{R}^{m\times d}} \quad \text{for all } F \in \mathbb{R}^{m\times d},$$

with $\|F\|_{\mathbb{R}^{m\times d}} = (F, F)^{1/2}_{\mathbb{R}^{m\times d}}$. The corresponding Dirichlet form defines a scalar product on $\boldsymbol{\mathcal{X}}_N$,

$$(\varphi, \psi)_+ := \mathcal{E}(\varphi, \psi) = \sum_{x \in \mathbb{T}_N^d} \langle A(\nabla\varphi(x)), \nabla\psi(x) \rangle_{\mathbb{R}^{m\times d}}, \quad \text{where } \varphi, \psi \in \boldsymbol{\mathcal{X}}_N. \tag{2.4}$$

Skipping the index N, consider the triplet $\boldsymbol{\mathcal{H}}_- = \boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_+$ of (finite-dimensional) Hilbert spaces obtained by equipping the space $\boldsymbol{\mathcal{X}}_N$ with the norms $\|\cdot\|_-$, $\|\cdot\|_2$, and $\|\cdot\|_+$, respectively. Here, $\|\cdot\|_2$ denotes the ℓ_2-norm $\|\varphi\|_2 = \langle\varphi,\varphi\rangle^{1/2}$, $\|\varphi\|_+ = (\varphi,\varphi)_+^{1/2}$, and $\|\cdot\|_-$ is the dual norm

$$\|\varphi\|_- = \sup_{\psi:\|\psi\|_+\leq 1} \langle\psi,\varphi\rangle. \tag{2.5}$$

One easily checks that $\|\cdot\|_-$ is again induced in a unique way by a scalar product $(\cdot,\cdot)_-$. The linear map A defines an isometry

$$\mathscr{A} : \boldsymbol{\mathcal{H}}_+ \to \boldsymbol{\mathcal{H}}_-, \quad \varphi \mapsto \mathscr{A}\varphi = \nabla^*(A\nabla\varphi).$$

Indeed, it follows from the Lax-Milgram theorem that for each $f \in \boldsymbol{\mathcal{H}}_-$, the equation

$$(\varphi, v)_+ = \langle f, v\rangle \ \text{ for all } \ v \in \boldsymbol{\mathcal{H}}_+ \tag{2.6}$$

has a unique solution $\varphi \in \boldsymbol{\mathcal{H}}_+$. Hence $\mathscr{A}$ is a bijection from $\boldsymbol{\mathcal{H}}_+$ to $\boldsymbol{\mathcal{H}}_-$. Moreover,

$$\|\mathscr{A}\varphi\|_- = \sup\{\langle\mathscr{A}\varphi, v\rangle : \|v\|_+ \leq 1\} = \sup\{(\varphi, v)_+ : \|v\|_+ \leq 1\} = \|\varphi\|_+. \tag{2.7}$$

Hence, the map $\mathscr{A}$ is an isometry from $\boldsymbol{\mathcal{H}}_+$ to $\boldsymbol{\mathcal{H}}_-$. In view of the symmetry of $\mathscr{A}$, it follows that

$$(\varphi,\psi)_- = (\mathscr{A}^{-1}\varphi, \mathscr{A}^{-1}\psi)_+ = \langle\mathscr{A}^{-1}\varphi, \mathscr{A}\mathscr{A}^{-1}\psi\rangle = \langle\mathscr{A}^{-1}\varphi,\psi\rangle. \tag{2.8}$$

Consider now the inverse $\mathscr{C}_A = \mathscr{A}^{-1}$ of the operator $\mathscr{A}$ (or the Green function) and the corresponding bilinear form on $\boldsymbol{\mathcal{X}}_N$ defined by

$$G_A(\varphi,\psi) = \langle\mathscr{C}_A\varphi,\psi\rangle = (\varphi,\psi)_-, \quad \varphi,\psi \in \boldsymbol{\mathcal{X}}_N. \tag{2.9}$$

One writes $\mathcal{C}_A \in \boldsymbol{\mathcal{M}}_N$, using $\boldsymbol{\mathcal{M}}_N$ (in analogy with $\boldsymbol{\mathcal{X}}_N$) to denote the space of all matrix-valued maps on $\mathbb{T}_N$ with zero mean.

Given that the operator $\mathscr{A}$ and its inverse commutes with translations on $\mathbb{T}_N$, there exists a unique kernel $\mathcal{C}_A$ such that

$$(\mathscr{C}_A\varphi)(x) = \sum_{y\in\mathbb{T}_N} \mathcal{C}_A(x-y)\varphi(y). \tag{2.10}$$

It is easy to see that the function $G_{A,y}(\cdot) = \mathcal{C}_A(\cdot - y)$ is the unique solution of the equation

$$\mathscr{A}G_{A,y} = \big(\delta_y - \frac{1}{L^{Nd}}\big)\mathrm{Id}_\mathrm{m}, \tag{2.11}$$

where Id_m is the unit $m \times m$ matrix. Notice that for any $a \in \mathbb{R}^m$ one has:

$$(\mathscr{A}G_{A,y}) = \big(\delta_y - \frac{1}{L^{Nd}}\big) \in \boldsymbol{\mathcal{X}}_N.$$

In [1], among other things, the following result is proved:

Theorem 2.2.1. *Let $d \geq 2$ and let α be a multiindex. There exist constants $C_\alpha(d)$ and $\eta(\alpha, d)$ with the following properties. For each integer $N \geq 1$, each $k = 1, \ldots, N+1$ and each odd integer $L \geq 16$ there exist real-analytic maps $A \mapsto \mathcal{C}_{A,k}$ from U to $\boldsymbol{\mathcal{M}}_N$ such that the following three assertions hold.*

(i) *If* $\mathscr{C}_{A,k}$ *denotes the translation invariant operator on induced by* $\mathcal{C}_{A,k}$ *then*

$$\mathscr{C}_A = \sum_{k=1}^{N+1} \mathscr{C}_{A,k}. \tag{2.12}$$

(ii) *There exist constant* $m \times m$ *matrices* $C_{A,k}$ *such that*

$$\mathcal{C}_{A,k}(x) = C_{A,k} \quad \text{if } \rho_\infty(x,0) \geq \frac{1}{2}L^k. \tag{2.13}$$

(iii) *If* $(A_0F, F)_{\mathbb{R}^{m\times d}} \geq c_0\|F\|^2_{\mathbb{R}^{m\times d}}$ *for all* $F \in \mathbb{R}^{m\times d}$ *and* $c_0 > 0$ *then*

$$\sup_{\|\dot{A}\|\leq 1} \left\| \left(\nabla^\alpha D^j_A \mathcal{C}_{A_0,k}(x)(\dot{A},\dots,\dot{A})\right\| \leq C_\alpha(d) \left(\frac{2}{c_0}\right)^j j!\, L^{-(k-1)(d-2+|\alpha|)} L^{\eta(\alpha,d)}.$$

for all $x \in \mathbb{T}^d_N$ *and all* $j \geq 0$*. Here* $\nabla^\alpha = \prod_{i=1}^d \nabla_i^{\alpha_i}$*, we use* $\|\dot{A}\|$ *to denote the operator norm of a linear mapping* $\dot{A}\colon \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$*, and the* j*-th derivative with respect to* A *in the direction* $\dot{A}$ *is taken at* A_0*.*

2.3 Notation and Hypothesis

Let $\bar{A} : \mathbb{T}^d \to \mathcal{L}_{\text{sym}}(\mathbb{R}^{m\times d})$ be a C^3 function, where $\mathcal{L}_{\text{sym}}(\mathbb{R}^{m\times d})$ is the space of linear maps on $\mathbb{R}^{m\times d}$ such that $A = A^*$ and the associated operator is elliptic, namely there exists a constant $c_1, c_0 > 0$ such that

$$c_1|P|^2 \geq \bar{A}^{\alpha,\beta}_{i,j} P^i_\alpha P^j_\beta \geq c_0|P|^2 \qquad \forall P \in \mathbb{R}^{m\times d} \tag{2.14}$$

and there exists an $\varepsilon_0 > 0$ (small enough) such that

$$\sum_{|\gamma|\leq 3} \sup_{\mathbb{T}^d} |D^\gamma \bar{A}^{\alpha,\beta}_{i,j}| \leq \varepsilon_0, \tag{2.15}$$

where γ is a multi-index.

For every $N > 1$, we define the function $A_N : \mathbb{T}^d_N \to \mathcal{L}_{\text{sym}}(\mathbb{R}^{m\times d})$ in the following natural way:

$$A_N(x) = \bar{A}(x/L^N). \tag{2.16}$$

The condition (2.15), can be expressed in terms of A_N as

$$\sup_{|\gamma|\leq 3} \sup_{\mathbb{T}^d_N} L^{N|\gamma|} |\nabla^\gamma (A_N)^{\alpha,\beta}_{i,j}| \leq \varepsilon_0. \tag{2.17}$$

On the other hand, if there exists a A_N such that (2.17) holds, then by some elementary interpolation one can construct a $\bar{A}$ such that (2.16) holds.

Given that we will mainly work for N fixed, if it is clear from the context we will drop the N-subscript.

We denote by $\mathcal{E} \subset \{q : \mathbb{T}_N^d \to \mathcal{L}_{\text{sym}}(\mathbb{R}^{m\times d})\}$ such that there exist constants $c_0, c_1 \geq 0$ such that for every $x \in T_N^d$ and $F \in M_{\text{sym}}(\mathbb{R}^{m\times d})$, it holds

$$c_0\langle F, F\rangle \leq \langle q(x)F, F\rangle \leq c_1\langle F, F\rangle.$$

The space $\mathcal{E}$, is not a vector space. It will be endowed with the distance induced by the norm norm

$$\|q\|_{\mathcal{E}} = \sup_{x\in\mathbb{T}^d, |\beta|\leq 3} \|L^{|\beta|N}\nabla^\beta q(x)\|_{M_{\text{sym}}(\mathbb{R}^{m\times d})},$$

where β is a multiindex.

Similarly as before, we introduce the following notations:

$$\boldsymbol{\mathcal{X}}_N := \{\varphi \in \boldsymbol{\mathcal{V}}_N : \sum_{x\in\mathbb{T}_N} \varphi(x) = 0\}, \tag{2.18}$$

and

$$\mathscr{A} : \boldsymbol{\mathcal{H}}_+ \to \boldsymbol{\mathcal{H}}_-, \quad \varphi \mapsto \mathscr{A}\varphi := \nabla^*(A\nabla\varphi).$$

As in § 2.1, let $\mathcal{C}_A : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}^{m\times d}$ such that

$$\mathscr{A}\mathcal{C}_{A,y} = \big(\delta_y - \frac{1}{L^{Nd}}\big).$$

We will extend Theorem 2.2.1 in the following way:

Theorem 2.3.1. *Let $d \geq 3$, A_N be defined as above. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ the operator $\mathscr{C}_A \colon \boldsymbol{\mathcal{H}}_- \to \boldsymbol{\mathcal{H}}_+$, where $\|A\|_{\mathcal{E}} \leq \varepsilon$, admits a finite range decomposition, i.e., there exist positive-definite operators*

$$\mathscr{C}_{A,k} \colon \boldsymbol{\mathcal{H}}_- \to \boldsymbol{\mathcal{H}}_+,\ (\mathscr{C}_{A,k}\varphi)(x) = \sum_{y\in\mathbb{T}_N^d} \mathcal{C}_{A,k}(x,y)\varphi(y),\ k = 1,\dots,N+1, \tag{2.19}$$

such that

$$\mathscr{C}_A = \sum_{k=1}^{N+1} \mathscr{C}_{A,k},$$

and for associated kernel $\mathcal{C}_{A,k} \in \boldsymbol{\mathcal{M}}_N$, there exists a constant matrix $C_{A,k}$ such that

$$\mathcal{C}_{A,k}(x,y) = C_{A,k} \quad \textit{whenever} \quad \rho_\infty(x,y) \geq \frac{1}{2}L^k \qquad \textit{for } k = 1,\dots,N.$$

Moreover, if $(A_0F, F)_{\mathbb{R}^{m\times d}} \geq c_0\|F\|^2_{\mathbb{R}^{m\times d}}$ for all $F \in \mathbb{R}^{m\times d}$ and $c_0 > 0$ and if $\|A\|_{\mathcal{E}} \leq 1/2$ then

$$\sup_{\|\dot{A}\|\leq 1} \left\|\big(\nabla_y^\alpha D_A^j \mathcal{C}_{A_0,k}(x,y)(\dot{A},\dots,\dot{A})\right\| \leq C_\alpha(d)\left(\frac{2}{c_0}\right)^j j!\, L^{-(k-1)(d-2+|\alpha|)}L^{\eta(\alpha,d)}.$$

2.4 Outline

Before going to the discrete setting, we would like to briefly expose the basic idea in the continuous case.

In what follows, we will use the symbol $\lesssim$ to indicate an inequality is valid up to universal constants depending eventually on the dimensions d, m.

For the sake of simplicity, we take $A = A(x)$ be elliptic with A smooth.

Let B be a ball, $\Pi_B : W^{1,2}(\mathbb{R}^n) \to W_0^{1,2}(B)$ be the projection operator. Moreover, we define $P_B := \mathrm{Id} - \Pi_B$.

The construction technique is due to Brydges *et al.* (see [11, 8]) and consists in considering the operators

$$\mathscr{T}_B f := \frac{1}{|B|} \int_{\mathbb{T}^d} \Pi_{x+B} f \,\mathrm{d}x \qquad \text{and} \qquad \mathscr{R}_B := \mathrm{Id} - \mathscr{T}_B.$$

Let $r_1, \dots, r_k > 0$ and $B_{r_1}, \dots, B_{r_k}$ be the balls of radius r_k centered in 0. Whenever it is clear from the context, we will denote by $\mathscr{R}_k := \mathscr{R}_{B_k}$.

The operators $\mathscr{C}_k$ that appear in the Theorem 2.2.1 and Theorem 2.3.1, will be of the form

$$\mathscr{C}_k := (\mathscr{R}_1 \dots \mathscr{R}_{k-1})\mathscr{C}(\mathscr{R}'_{k-1} \dots \mathscr{R}'_1) - (\mathscr{R}_1 \dots \mathscr{R}_{k-1}\mathscr{R}_k)\mathscr{C}(\mathscr{R}'_k\mathscr{R}'_{k-1} \dots \mathscr{R}'_1),\ k = 1, \dots, N,$$

for a particular choice of $\{r_k\}$.

Then the proof of the finite range property will follow by abstract reasoning (see § 2.5).

In [15], among other things the authors show:

Theorem 2.4.1. *Let Ω be a regular domain and $A_{i,j}^{\alpha,\beta} \in C^{k,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ such that*

$$A_{i,j}^{\alpha,\beta} P_\alpha^i P_\beta^j > c|P|^2, \qquad \text{for some } c > 0 \text{ and every } P \in \mathbb{R}^{d\times m}.$$

Then there exists a matrix G_y such that

$$-D_\alpha(A_{i,j}^{\alpha,\beta} D_\beta (G_y)_k^j) = \delta_{i,k}\delta_j \qquad \text{in } \Omega$$

in the sense of distributions and

$$G_y = 0 \qquad \text{on } \partial\Omega.$$

Moreover, it holds

$$|D^\nu G(x,\cdot)| \leq C|x-y|^{2-d-|\nu|},$$

where ν is a multi-index such that $|\nu| \leq k$.

To simplify the notation we will write $\nabla^*(A\nabla u)$ instead of $D_\alpha(A_{\alpha,\beta}^{i,j} D_\beta u^j)$.

The above theorem is proven by using the following well-known L^p-estimates.

Lemma 2.4.2. *Suppose the same hypothesis as in Theorem 2.4.1 and let* $p \in (1, \infty)$, $q \in (1, n)$.

(i) *If* $f \in L^p(\Omega, \mathbb{R}^{m\times d}), F \in L^q(\Omega, \mathbb{R}^m)$, *then the system*

$$-D_\alpha(A_{i,j}^{\alpha,\beta} D_\beta u^j) = D_\alpha f_j^\alpha + F^i \qquad \text{in } \Omega,$$

with boundary condition

$$u = 0 \qquad \text{on } \partial\Omega,$$

has a weak solution in $W^{1,s}(\Omega; \mathbb{R}^m)$, *where*

$$s = \min(p, q^*), \qquad q^* = \frac{nq}{n-q},$$

and

$$\|u\|_{W^{1,s}} \leq C(\|f\|_{L^p} + \|F\|_{L^q}).$$

(ii) *If* $f \in L^{p,\infty},\ F \in L^{q,\infty}$ *then there exists a weak solution that satisfies*

$$\|u\|_{L^{s^*,\infty}} + \|Du\|_{L^{s,\infty}} \leq C(\|f\|_{L^{p,\infty}} + \|Du\|_{L^{q,\infty}}). \tag{2.20}$$

Lemma 2.4.3. *Suppose the same hypothesis as in Theorem 2.4.1. Let* B_{2r} *be a ball of radius* $2r$ *centered in* 0, $p > d$ *and let* u *be a solution to*

$$\nabla^*(A\nabla u) = 0 \qquad \text{in } B_{2r}.$$

Then

$$\sup_{B_r} |u| \leq r^{-n/q} M + r^{1-n/p} \|f\|_{B_{2r}},$$

where

$$M = \|Du\|_{L^{q,\infty}(B_{2r})} + \|u\|_{L^{q^*,\infty}(B_{2r})}.$$

Proposition 2.4.4. *Let* $B_1, \dots, B_k$ *be balls with radii* $r_1, \cdots, r_k$ *respectively. Then, there exists a dimensional constant* C_d, *such that*

$$\sup |\nabla^j u| \leq C_d^k \max\left(|x-y|, \operatorname{dist}(y, B_1^C), \dots, \operatorname{dist}(y, B_k^C)\right)^{2-d+j},$$

where $u = (P_{B_1} \cdots P_{B_k} C(x, \cdot))$ *and* $C(x,y)$ *is the Green's function and* $j < d-2$.

Proof. Let us sketch the proof of the above fact. In the discrete case it will be done in more detail.

The proof will follow by induction.

Let B_1 be a ball in generic position of size r_1. Given that $\nabla^*(A\nabla C_x(y)) = 0$, if $x \notin B_1$ then $\Pi_{B_1} C(x,y) = 0$, thus $P_{B_1} C(x,y) = C(x,y)$, hence the inequality follows from Theorem 2.4.1.

Let $\varepsilon := \operatorname{dist}(y, B_1^C) < r_1$. If $|x-y| > \varepsilon/2$, then by estimating the different terms $\Pi_{B_1} C(x,y)$ and $C(x,y)$ separately one has the desired result. Indeed, $C(x,y) \lesssim |x-y|^{2-d}$. Then by using an appropriate version of Lemma 2.4.3 one has that

$$|\Pi_{B_1} C(x,y)| \lesssim |x-y|^{2-d} M,$$

where

$$M = \|D\Pi_{B_1} C_x\|_{L^{d/d-2,\infty}(B_1)} + \|\Pi_{B_1} C_x\|_{L^{d/d-1,\infty}(B_1)}.$$

Then by using Lemma 2.4.2 one has that

$$\|D\Pi_{B_1} C_x\|_{L^{d/(d-2),\infty}} + \|\Pi_{B_1} C_x\|_{L^{d/(d-1),\infty}} \lesssim \|DC_x\|_{L^{d/(d-2),\infty}} + \|C_x\|_{L^{d/(d-1),\infty}} < \tilde{C}_d,$$

where $\tilde{C}_d$ is a constant depending only on the dimension d.

The inductive step is done in a very similar way and the higher derivative estimates follow similarly. □

Let $B_1, \ldots, B_k$ be k balls centered in 0, with radii $r_1, \ldots, r_k$ respectively and let $C(\cdot,\cdot)$ be the Green's function. We will denote by $C_k(x,\cdot) := \mathscr{R}_k \cdots \mathscr{R}_1 C(x,\cdot)$.

Let us now give a simple calculation that will be useful in Theorem 2.4.6.

Lemma 2.4.5. *Let $j > 1$ be an integer. Then*

$$\frac{1}{r^d} \int_0^r \max(\alpha, |r-\rho|)^{-j} \rho^{d-1} d\rho \lesssim \frac{\alpha^{1-j}}{r}.$$

Indeed, let us denote by I the right hand side of the previous equation. With a change of variables one has

$$\begin{aligned}
I &= \frac{1}{r^d} \int_0^{r-\alpha} |r-\rho|^{-j} \rho^{d-1} d\rho + \int_{r-\alpha}^r \alpha^{-j} \rho^{d-1}\, \mathrm{d}\rho \\
&= \frac{1}{r^j} \int_0^{1-\frac{\alpha}{r}} |1-t|^{-j} t^{d-1}\, \mathrm{d}t + \int_{1-\frac{\alpha}{r}}^1 \alpha^{-j} t^{d-1}\, \mathrm{d}t \\
&= \frac{1}{r^j} \int_0^{1-\frac{\alpha}{r}} |1-t|^{-j}\, \mathrm{d}t + \int_{1-\frac{\alpha}{r}}^1 \alpha^{-j}\, \mathrm{d}t \le r^{-j} \left(\frac{\alpha^{1-j}}{r^{1-j}} - 1 \right) + \frac{\alpha^{1-j}}{r} \\
&\le \frac{2\alpha^{1-j}}{r}.
\end{aligned}$$

If $j = 1$, then

$$\begin{aligned}
I &= \frac{1}{r^d} \int_0^{r-\alpha} |r-\rho|^{-1} \rho^{d-1} d\rho + \int_{r-\alpha}^r \alpha^{-1} \rho^{d-1}\, \mathrm{d}\rho \\
&= \frac{1}{r^1} \int_0^{1-\frac{\alpha}{r}} |1-t|^{-1} t^{d-1}\, \mathrm{d}t + \int_{1-\frac{\alpha}{r}}^1 \alpha^{-1} t^{d-1}\, \mathrm{d}t \\
&= \frac{1}{r^1} \int_0^{1-\frac{\alpha}{r}} |1-t|^{-1}\, \mathrm{d}t + \int_{1-\frac{\alpha}{r}}^1 \alpha^{-1}\, \mathrm{d}t \le \frac{1}{r} \Big(\big| \log\big(\frac{\alpha}{r}\big) \big| + 1 \Big).
\end{aligned}$$

Theorem 2.4.6. *Let C_k, B_i, r_i as above and such that $r_1 < \cdots < r_h < |x-y| < r_h+1 < \cdots < r_k$. Then,*

(i) *if $k-h < d-2$, then it holds*

$$|C_k(x,y)| \lesssim \frac{1}{r_{h+1}\cdots r_k}|x-y|^{2-d+k-h}\prod_{i=h+1}^{k}\left(\left|\log\left(\frac{|x-y|}{r_i}\right)\right|+1\right)$$
$$|\nabla_y^j C_k(x,y)| \lesssim \frac{1}{r_{h+1}\cdots r_k}|x-y|^{2-d+k-j-h},$$

(ii) *if $k-h \geq d-2$, it holds*

$$|C_k(x,y)| \lesssim \frac{1}{r_{k-d+3}\cdots r_k}\,|\log(|x-y|)|$$
$$|\nabla_y^j C_k(x,y)| \lesssim \frac{1}{r_{k-d+2-j}\cdots r_k}\prod_{i=h+1+j}^{k}\left(\left|\log\left(\frac{|x-y|}{r_i}\right)\right|+1\right).$$

Proof. We will prove only (i). The proof of (ii) is very similar.

Let us initially consider the case $k=1$. For simplicity we denote $\Pi_z := \Pi_{B_1+z}$. With simple computations, one has

$$\sup|C_1(x,y)| \leq \frac{1}{|B|}\int_{B_1+y}\sup|(\mathrm{Id}-\Pi_z)C(x,\cdot)| + \sup\left|\frac{1}{|B|}\int_{(y+B_1)^C}\Pi_z C(x,\cdot)\,\mathrm{d}z\right|. \tag{2.21}$$

Because of the fact that for every $t\in B_1+z$ the function $\Pi_z C_x$ is harmonic and has null boundary condition, one has that the second term in the right hand side of (2.21) is null. Hence it is enough to prove a bound only on the first term. Given that for every $z\in y+B$ it holds $\mathrm{dist}(y,z+B_1)=r_1-|z-y|$. Then, by using Proposition 2.4.4, one has that

$$\sup|(\mathrm{Id}-\Pi_z)C(x,\cdot)| \leq \begin{cases}(r_1-|z-y|)^{2-d} & \text{if } \ r_1-|y-z|\geq|x-y| \\ |x-y|^{2-d} & \text{otherwise.}\end{cases},$$

Thus,

$$\begin{aligned}\sup|C_1(x,y)| &\lesssim \int_0^{r_1-|y-x|}|r_1-\rho|^{2-d}\rho^{d-1}\,\mathrm{d}\rho + \int_{r_1-|x-y|}^{r_1}|x-y|^{2-d}\rho^{d-1}\,\mathrm{d}\rho \\ &\lesssim \frac{|x-y|^{3-d}}{r_1} - r_1^{2-d} + \frac{|x-y|^{3-d}}{r_1} \lesssim \frac{|x-y|^{3-d}}{r_1}.\end{aligned} \tag{2.22}$$

Let us now turn to the general case $k<d-2$, and let $B_1,\dots,B_k$ be balls of radii $r_1,\dots,r_k$ centered at the origin. From Proposition 2.4.4, we have that

$$\begin{aligned}&\sup|P_{z_1+B_1}\cdots P_{z_k+B_k}C(x,\cdot)| \leq \max\{|x-y|, r_1-|z_1-y|,\dots,r_k-|z_k-y|\}^{2-d} \\ &\leq \max\{|x-y|\}^{2-d+k}\cdot\max\{|x-y|,r_k-|z_k-y|\}^{-1}\cdots\max\{|x-y|,r_k-|z_k-y|\}^{-1} \\ &\qquad =: g(z_1,\dots,z_k).\end{aligned}$$

Thus,

$$\sup \mathcal{R}_1 \cdots \mathcal{R}_k C(x,\cdot) \leq \int_{B_1\times\cdots\times B_k} g(z_1,\ldots,z_k)\,\mathrm{d}z_1\cdots\,\mathrm{d}z_k.$$

From Lemma 2.4.5 we have that

$$\int_{B_1\times\cdots\times B_k} g(z_1,\ldots,z_k)\,\mathrm{d}z_1\cdots\,\mathrm{d}z_k \leq \frac{1}{r_1\cdots r_k}|x-y|^{2-d+k}\prod_i(|\log(|x-y|)|+\log(r_i)+1),$$

which proves the desired result.

□

Corollary 2.4.7. *Suppose that $|x-y|>1$ and let $B_1,\ldots,B_k$ and such that $r_i=L^i$ with $L>1$. Then there exists $\eta(j,d)$ such that*

$$\nabla^j C_k(x,y) \lesssim \frac{L^{\eta(j,d)}}{L^{k(d-2-j)}}.$$

Indeed, given that $\mathscr{R}_k'=\mathscr{A}\mathscr{R}_k\mathscr{C}$ one has that

$$\mathscr{R}_1\cdots\mathscr{R}_k\mathscr{C}\mathscr{R}_k'\cdots\mathscr{R}_1'=\mathscr{R}_1\cdots\mathscr{R}_k\cdot\mathscr{R}_k\cdots\mathscr{R}_1\mathscr{C}$$

hence by using Theorem 2.4.6, one has the desired result.

2.5 Construction of the finite range decomposition

In this section, we will briefly describe the construction of the finite range decomposition. Let us *stress* that main idea in the construction of the finite decomposition goes back to Brydges *et al.* (e.g., [11, 8]). Moreover, in this section we will follow closely and adapt the arguments given in [1].

Let Q be a cube of size l and let us denote for simplicity $\Pi_x := \Pi_{Q+x}$. We define

$$\mathscr{T}(\varphi) := \frac{1}{l^d}\sum_{x\in\mathbb{T}_N^d} \mathit{\Pi}_x\varphi, \tag{2.23}$$

for every $\varphi\in\boldsymbol{\mathcal{H}}_+$. The following result is the key estimate for construction the finite range decomposition.

Lemma 2.5.1 ([1, Lemma 3.1])**.** *For any $\varphi\in\boldsymbol{\mathcal{H}}_+$ we have*

(i) $\mathscr{A}(P_x\varphi)=\text{const.}$ *in* $Q+x$,

(ii) $P_x\varphi=\varphi$ *in* $\mathbb{T}_N\setminus(Q+x)$,

(iii) $\mathit{\Pi}_x\varphi=\varphi\mathbb{1}_{Q+x}$ *if* $\varphi=0$ *on* $\overline{(Q+x)}\setminus(Q+x)$.

Proof. (i): For all $\psi\in\boldsymbol{\mathcal{H}}_+(Q+x)$, we have that $(P_x\varphi,\psi)_+=0$ and hence $\langle\mathscr{A}(P_x\varphi),\psi\rangle=0$. Taking $\psi=\delta_v-\delta_z$ and for any pair of points $v,z\in Q+x$, we get $\mathscr{A}(P_x\varphi)(v)=\mathscr{A}(P_x\varphi)(z)$. This proves (i).
(ii): This follows from the fact that $\mathit{\Pi}_x\varphi$ belongs to $\boldsymbol{\mathcal{H}}_+(Q+x)$ and hence vanishes

outside $Q+x$.
(iii): It suffices to consider the case $x=0$ and we write Π for Π_0. Let $\tilde\varphi = \varphi 1_Q$. Then $\tilde\varphi \in \boldsymbol{\mathcal{H}}_+(Q)$ and hence $\Pi\tilde\varphi = \tilde\varphi$. Moreover $\varphi - \tilde\varphi$ vanishes in $\overline{Q}$. Thus $\nabla(\varphi - \tilde\varphi)$ vanishes in Q_-. Hence $(\varphi - \tilde\varphi, \psi)_+ = 0$ for all $\psi \in \boldsymbol{\mathcal{H}}_+(Q)$ since $\nabla\psi$ is supported in Q_-. Therefore $\Pi(\varphi - \tilde\varphi) = 0$ which yields the assertion.

□

Lemma 2.5.2 ([1, Lemma 3.3]).

(i) $\Pi_x \Pi_y = 0$ *whenever* $(Q_- + x) \cap (Q_- + y) = \varnothing$,

(ii) $\Pi_x \varphi = 0$ *whenever* $\operatorname{spt}\varphi \cap (\overline{Q} + x) = \varnothing$.

Proof. (i): For any $\varphi, \psi \in \boldsymbol{\mathcal{H}}_+$, the functions $\Pi_x\varphi$ and $\Pi_y\psi$ vanish on $\mathbb{T}_N \setminus (Q+x)$ and $\mathbb{T}_N \setminus (Q+y)$, respectively. Hence, $\nabla\Pi_x\psi$ and $\nabla\Pi_y\varphi$ vanish on $\mathbb{T}_N \setminus (Q_- + x)$ and on $\mathbb{T}_N \setminus (Q_- + y)$, respectively. Assuming now that $Q_- + x$ and $Q_- + y$ are disjoint and taking into account (2.4) we get

$$(\psi, \Pi_x \Pi_y \varphi)_+ = (\Pi_x\psi, \Pi_y\varphi)_+ = \sum_{z\in\mathbb{T}_N} \langle A(\nabla\Pi_x\psi)(z), (\nabla\Pi_y\varphi)(z)\rangle_{\mathbb{R}^{m\times d}} = 0. \tag{2.24}$$

(ii): For $\psi \in \boldsymbol{\mathcal{H}}_+(Q+x)$ we have $\mathscr{A}\psi = 0$ in $\mathbb{T}_N \setminus (\overline{Q}+x)$. Thus for any $\varphi \in \boldsymbol{\mathcal{H}}_+$ with $\operatorname{spt}\varphi \cap (\overline{Q}+x) = \varnothing$ we get $(\varphi,\psi)_+ = \langle \varphi, \mathscr{A}\psi\rangle = 0$. Thus by the definition of Π_x, we have that $\Pi_x\varphi = 0$. □

Next, consider the symmetric operator

$$\mathscr{T} = \frac{1}{l^d} \sum_{x\in\mathbb{T}_N} \Pi_x \tag{2.25}$$

on $\boldsymbol{\mathcal{H}}_+$.

Lemma 2.5.3 ([1, Lemma 3.4]). *For any* $\varphi \in \boldsymbol{\mathcal{H}}_+$ *we have*

(i) $0 \le (\Pi_x\varphi, \varphi)_+ \le \langle 1_{Q_-+x} A\nabla\varphi, \nabla\varphi\rangle$,

(ii) $0 \le (\mathscr{T}\varphi, \varphi)_+ \le (\varphi,\varphi)_+$ *and the inequalities are strict if* $\varphi \ne 0$,

(iii) $(\mathscr{T}\varphi, \mathscr{T}\varphi)_+ \le (\mathscr{T}\varphi, \varphi)_+$.

Proof. (i): We have $(\Pi_x\varphi,\varphi)_+ = (\varphi, \Pi_x\varphi)_+ = (\Pi_x\varphi, \Pi_x\varphi)_+ \ge 0$. For the other inequality we use that $\nabla\Pi_x\varphi$ is supported in $Q_- + x$. Thus

$$(\Pi_x\varphi, \varphi)_+ = \langle A\nabla\Pi_x\varphi, \nabla\varphi\rangle = \langle A\nabla\Pi_x\varphi, 1_{Q_-+x}\nabla\varphi\rangle. \tag{2.26}$$

Since A is symmetric and positive definite the expression $(F,G)_A := \langle AF, G\rangle$ is a scalar product on functions $\mathbb{Z}^d \to \mathbb{R}^{m\times d}$. Thus the Cauchy-Schwarz inequality yields

$$\begin{aligned}\langle A\nabla\Pi_x\varphi, 1_{Q_-+x}\nabla\varphi\rangle &\le \langle A\nabla\Pi_x\varphi, \nabla\Pi_x\varphi\rangle^{1/2} \langle A 1_{Q_-+x}\nabla\varphi, 1_{Q_-+x}\nabla\varphi\rangle^{1/2} \\ &= (\Pi_x\varphi, \Pi_x\varphi)_+^{1/2} \langle 1_{Q_-+x} A\nabla\varphi, \nabla\varphi\rangle^{1/2}. \end{aligned} \tag{2.27}$$

Together with (2.26) this yields the assertion since $(\Pi_x\varphi,\varphi)_+ = (\Pi_x\varphi, \Pi_x\varphi)_+$.

(ii): Since $\sum_{x\in\mathbb{T}_N} 1_{Q_-+x}(y) = l^d$ for all $y \in \mathbb{T}_N$ the inequalities follow by summing (i) over $x \in \mathbb{T}_N$. If $(\mathscr{T}\varphi,\varphi)_+ = 0$ then $(\Pi_x\varphi,\varphi)_+ = 0$ for all $x \in \mathbb{T}_N$ and thus $\Pi_x\varphi = 0$ and

$P_x\varphi = \varphi$. Lemma 2.5.1 implies that there exist constants c_x such that $(\mathscr{A}\varphi)(y) = c_x$ for all $y \in Q + x$. Since $l \geq 3$ the cubes $Q + x$ and $Q + (x + \mathrm{e}_i)$ overlap and this yields $c_x = c_{x+\mathrm{e}_i}$ for all $i = 1, \ldots, d$. Thus c_x is independent of x. Since $\mathscr{A}\varphi \in \boldsymbol{\mathcal{X}}_N$ this implies $c = 0$. Hence $\mathscr{A}\varphi = 0$ and therefore $\varphi = 0$.
Now suppose that $(\mathscr{T}\varphi, \varphi)_+ = (\varphi, \varphi)_+$. This implies that for all $x \in \mathbb{T}_N$ we have $(\Pi_x\varphi, \varphi)_+ = \langle 1_{Q_-+x} A\nabla\varphi, \nabla\varphi\rangle$. We claim that the last identity implies that $\nabla\varphi(x) = 0$. Indeed, if $1_{Q_-+x}\nabla\varphi = 0$ we are done. Otherwise the identity can only hold if the inequality in (2.27) is an identity. In particular we must have $\nabla\Pi_x\varphi = \lambda 1_{Q_-+x}\nabla\varphi$ and $\lambda = 1$. Now $\Pi_x\varphi$ vanishes outside $Q + x$ and in particular at the points x and $x + \mathrm{e}_i$. Thus $\nabla\Pi_x\varphi(x) = 0$ and hence $\nabla\varphi(x) = 0$. It follows that φ is constant on $\mathbb{T}_N$ and hence $\varphi = 0$ since φ has mean zero.

(iii): It follows from (ii) that $(\varphi, \psi)_* := (\mathscr{T}\varphi, \psi)_+$ defines a scalar product on $\boldsymbol{\mathcal{H}}_+$. Thus the Cauchy Schwarz inequality and (ii) yield

$$(\mathscr{T}\varphi, \psi)_+ \leq (\mathscr{T}\varphi, \varphi)_+^{1/2} (\mathscr{T}\psi, \psi)_+^{1/2} \leq (\mathscr{T}\varphi, \varphi)_+^{1/2} (\psi, \psi)_+^{1/2}. \tag{2.28}$$

Taking $\psi = \mathscr{T}\varphi$ we obtain the desired estimate. □

Consider the operator $\mathscr{T}' \colon \boldsymbol{\mathcal{H}}_- \to \boldsymbol{\mathcal{H}}_-$ dual with respect to $\mathscr{T}$ and defined by

$$\langle \mathscr{T}'\varphi, \psi\rangle = \langle \varphi, \mathscr{T}\psi\rangle, \quad \varphi \in \boldsymbol{\mathcal{H}}_-, \psi \in \boldsymbol{\mathcal{H}}_+. \tag{2.29}$$

Notice that

$$\mathscr{T}' = \mathscr{A}\mathscr{T}\mathscr{A}^{-1}, \quad (\mathscr{T}'\varphi, \psi)_- = (\varphi, \mathscr{T}'\psi)_-, \quad \text{and } (\mathscr{T}'\varphi, \varphi)_- = (\mathscr{T}\mathscr{A}^{-1}\varphi, \mathscr{A}^{-1}\varphi)_+. \tag{2.30}$$

Indeed, for any $\varphi \in \boldsymbol{\mathcal{H}}_+$, we have

$$\langle \mathscr{T}'\mathscr{A}\varphi, \psi\rangle = \langle \mathscr{A}\varphi, \mathscr{T}\psi\rangle = (\varphi, \mathscr{T}\psi)_+ = (\mathscr{T}\varphi, \psi)_+ = \langle \mathscr{A}\mathscr{T}\varphi, \psi\rangle, \tag{2.31}$$

and this yields the first identity in (2.30). Now

$$(\mathscr{T}'\varphi, \psi)_- = \langle \mathscr{A}^{-1}\mathscr{A}\mathscr{T}\mathscr{A}^{-1}\varphi, \psi\rangle = \langle \mathscr{T}\mathscr{A}^{-1}\varphi, \mathscr{A}\mathscr{A}^{-1}\psi\rangle = (\mathscr{T}\mathscr{A}^{-1}\varphi, \mathscr{A}^{-1}\psi)_+. \tag{2.32}$$

Since the last expression is symmetric in φ and ψ we get the second identity in (2.30) and taking $\psi = \varphi$ we obtain the third identity. Similarly, we have $\Pi_x' = \mathscr{A}\Pi_x\mathscr{A}^{-1}$ for the dual of Π_x. Notice that

$$\Pi_x'\varphi = 0 \quad \text{whenever } \operatorname{spt}\varphi \cap (Q + x) = \varnothing. \tag{2.33}$$

Indeed, considering any test function $\psi \in \boldsymbol{\mathcal{X}}_N$, we have $\langle \Pi_x'\varphi, \psi\rangle = \langle \varphi, \Pi_x\psi\rangle = 0$. We also consider the operator

$$\mathscr{R} := \mathrm{Id} - \mathscr{T} \quad \text{and its dual } \mathscr{R}' = \mathrm{Id} - \mathscr{T}'. \tag{2.34}$$

hence because of (2.30)

$$\mathscr{R}' = \mathscr{A}\mathcal{R}\mathscr{A}^{-1} \tag{2.35}$$

It follows from Lemma 2.5.3(ii) and (2.30) that

$$(\mathscr{T}'\varphi, \varphi)_- > 0, \quad (\mathscr{R}'\varphi, \varphi)_- > 0, \quad (\mathscr{T}'\varphi, \mathscr{T}'\varphi)_- \leq (\mathscr{T}'\varphi, \varphi)_- \ \text{ for all } \ \varphi \in \boldsymbol{\mathcal{H}}_- \setminus \{0\}. \tag{2.36}$$

Lemma 2.5.4. *Let B be a bilinear form on $\boldsymbol{\mathcal{X}}_N$. Then the following assertions hold.*

(i) There exists a unique linear operator $\mathscr{B}\colon \boldsymbol{\mathcal{X}}_N \to \boldsymbol{\mathcal{X}}_N$ such that

$$\langle \mathscr{B}\varphi, \psi\rangle = B(\varphi,\psi) \quad \textit{for all} \quad \varphi,\psi \in \boldsymbol{\mathcal{X}}_N. \tag{2.37}$$

(ii) There exists a unique matrix-valued kernel $\mathcal{B} \in \boldsymbol{\mathcal{M}}_N$ such that

$$(\mathscr{B}\varphi)(x) = \sum_{y\in\mathbb{T}_N} \mathcal{B}(x,y)\varphi(y) \quad \textit{for all} \quad x \in \mathbb{T}_N, \quad \textit{for all} \quad \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{2.38}$$

Moreover for $\tilde{\mathcal{B}}\colon \mathbb{T}_N \to \mathbb{R}^{m\times m}$ we have

$$(\mathscr{B}\varphi)(x) = \sum_{y\in\mathbb{T}_N} \tilde{\mathcal{B}}(x,y)\varphi(y) \quad \textit{for all} \quad x \in \mathbb{T}_N \quad \textit{for all} \quad \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{2.39}$$

if and only if

$$\tilde{\mathcal{B}} - \mathcal{B} = C \tag{2.40}$$

with a constant $m \times m$ matrix C.

(iii) If $\mathcal{B}' \in \boldsymbol{\mathcal{X}}_N$ denotes the kernel of the dual operator $\mathscr{B}'$ then

$$\mathcal{B}'(y,x) = \mathcal{B}(x,y). \tag{2.41}$$

Proof. The proof is a simple modification of the arguments in [1, Lemma 3.5]. □

For two sets $M_1, M_2 \subset \mathbb{T}_N$ we define

$$\operatorname{dist}_\infty(M_1,M_2) := \min\{\rho_\infty(x,y)\colon x \in M_1, y \in M_2\}. \tag{2.42}$$

Lemma 2.5.5. *Let B be a bilinear form on $\boldsymbol{\mathcal{X}}_N$ and let $\mathscr{B}$ and $\mathcal{B} \in \boldsymbol{\mathcal{M}}_N$ be the associated operator and the associated kernel, respectively. Let n be an integer and suppose that $L^N > 2n+3$. Then the following three statements are equivalent.*

(i) $B(\varphi,\psi) = 0$ *whenever* $\operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) > n$.

(ii) *There exists an $m \times d$ matrix C such that $\mathcal{B}(x,y) = C$ whenever $\rho_\infty(x,0) > n$.*

(iii) $\operatorname{spt}\mathscr{B}\varphi \subset \operatorname{spt}\varphi + \{-n,\dots,n\}^d$ *for all* $\varphi \in \boldsymbol{\mathcal{X}}_N$.

Proof. The proof follows by modifying the proof of [1, Lemma 3.6].

For the convenience of the user we sketch it.

The implication (ii) $\Longrightarrow$ (iii) is easy. Set $\tilde{\mathcal{B}}(x,y) = \mathcal{B}(x,y) - C$. Then $\tilde{\mathcal{B}}(z) = 0$ if $\rho_\infty(z) > n$ with $\rho_\infty(z) = \rho_\infty(z,0)$ and by Lemma 2.5.4(ii) we have

$$(\mathscr{B}\varphi)(x) = \sum_{y\in\mathbb{T}_N} \tilde{\mathcal{B}}(x,y)\varphi(y). \tag{2.43}$$

If $x \notin \operatorname{spt}\varphi + \{-n,\dots,n\}^d$ then either $y \notin \operatorname{spt}\varphi$ or $y \in \operatorname{spt}\varphi$ and $\rho_\infty(x,y,0) > n$. In either case $\mathscr{B}\varphi(x) = 0$.

The implication (iii) $\Longrightarrow$ (i) is also easy. Suppose that $\operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) > n$. Then (iii) implies that $\operatorname{dist}_\infty(\operatorname{spt}\mathscr{B}\varphi, \operatorname{spt}\psi) > 0$, i.e.,, $\mathscr{B}\varphi$ and ψ have disjoint support. Thus $B(\varphi,\psi) = \langle \mathscr{B}\varphi, \psi\rangle = 0$.

The implication (i) $\Longrightarrow$ (ii) follows in a similar way by using Lemma 2.5.5 □

Lemma 2.5.6. *Suppose that* $\operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) > l-1$. *Then*

$$\langle \mathscr{T}\varphi, \psi\rangle = 0, \quad \langle \mathscr{T}'\varphi, \psi\rangle = 0, \quad \langle \mathscr{R}\varphi, \psi\rangle = 0, \quad \langle \mathscr{R}'\varphi, \psi\rangle = 0. \tag{2.44}$$

Proof. The proof follows by modifying the proof of [1, Lemma 3.7]. For the convenience of the user we sketch it.

It suffices to prove the first identity. The second follows by exchanging φ and ψ and the third and fourth follow since $\mathscr{R} = \operatorname{Id} - \mathscr{T}$ and $\mathscr{R}' = \operatorname{Id} - \mathscr{T}'$. By Lemma 2.5.2 we have

$$\Pi_x \varphi = 0 \quad \text{if } \operatorname{spt}\varphi \cap (\overline{Q} + x) = \varnothing \tag{2.45}$$

and it follows from the definition of Π_x that $\operatorname{spt}\Pi_x\varphi \subset Q + x$. Assume $\langle \mathscr{T}\varphi, \psi\rangle \neq 0$. Then there exist $x \in \mathbb{T}_N$ such that $\langle \Pi_x\varphi, \psi\rangle \neq 0$. Thus $\operatorname{spt}\psi \cap (Q+x) \neq \varnothing$ and $\operatorname{spt}\varphi \cap (\overline{Q}+x) \neq \varnothing$. Therefore there exist $\xi \in \overline{Q}$ and $\zeta \in Q$ such that $x + \xi \in \operatorname{spt}\varphi$, $x + \zeta \in \operatorname{spt}\psi$. Thus

$$x + \xi - (x+\zeta) = \xi - \zeta \in \{-(l-1), \dots, l-1\}^d. \tag{2.46}$$

Hence $\operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) \le l-1$. □

Consider now the inverse $\mathscr{C} = \mathscr{A}^{-1}$.

The main step toward the decomposition, is to subtract a positive definite operator from $\mathscr{C}$ in such a way that the remnant is positive definite and of finite range. We define

$$\mathscr{C}_1 := \mathscr{C} - \mathscr{R}\mathscr{C}\mathscr{R}', \quad \text{which yields } \mathcal{C}_1 = \mathcal{C} - \mathcal{R} * \mathcal{C} * \mathcal{R}'. \tag{2.47}$$

Proposition 2.5.7 ([1, Proposition 3.8])**.** *Both* $\mathscr{C}_1$ *and* $\mathscr{R}\mathscr{C}\mathscr{R}'$ *are positive definite and* $\mathscr{C}_1$ *has finite range, i.e.,*

$$\langle \mathscr{C}_1\varphi, \psi\rangle = 0 \quad \textit{if } \operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) > 2l-3. \tag{2.48}$$

In particular, there exists an $m \times m$ *matrix* C *such that*

$$\mathcal{C}_1(z) = C \ \textit{if} \ \ \rho_\infty(z,0) > 2l-3. \tag{2.49}$$

Proof. For any $\varphi, \psi \in \boldsymbol{\mathcal{X}}_N$ by using (2.9), one obtains

$$\langle \mathscr{R}\mathscr{C}\mathscr{R}'\varphi, \varphi\rangle = (\mathscr{R}'\varphi, \mathscr{R}'\varphi)_- \ge 0. \tag{2.50}$$

If $\mathscr{R}'\varphi = 0$, then (2.36) implies that $\varphi = 0$. Thus $\mathscr{R}\mathscr{C}\mathscr{R}'$ is positive definite. Furthermore,

$$\begin{aligned} \langle \mathscr{C}_1\varphi, \psi\rangle &= \langle \mathscr{C}\varphi, \psi\rangle - \langle \mathscr{C}\mathscr{R}'\varphi, \mathscr{R}'\psi\rangle = (\varphi,\psi)_- - (\mathscr{R}'\varphi, \mathscr{R}'\psi)_- \\ &= (\mathscr{T}'\varphi, \psi)_- + (\varphi, \mathscr{T}'\psi)_- - (\mathscr{T}'\varphi, \mathscr{T}'\psi)_-. \end{aligned} \tag{2.51}$$

Thus (2.36) implies that $\mathscr{C}_1$ is positive definite.

To evaluate the range of the quadratic form $\langle \mathscr{C}_1\varphi, \psi\rangle$, we inspect the terms on the right hand side of (2.51). For the first (and similarly the second) term, we have

$$(\mathscr{T}'\varphi, \psi)_- = \frac{1}{l^d} \sum_{x \in \mathbb{T}_N} (\Pi'_x\varphi, \psi)_- = \frac{1}{l^d} \sum_{x \in \mathbb{T}_N} (\Pi'_x\varphi, \Pi'_x\psi)_-. \tag{2.52}$$

In view of (2.33), a term in the sum vanishes at x except when the supports of φ and ψ both intersect $Q + x$. Therefore, the scalar product is zero whenever the distance of the supports is strictly greater than $l - 1$. The second term of the bilinear form $G_1(\varphi, \psi) := \langle \mathscr{C}_1\varphi, \psi\rangle$ is the double sum

$$(\mathscr{T}'\varphi, \mathscr{T}'\psi)_- = \frac{1}{l^d}\sum_{y\in\mathbb{T}_N}\frac{1}{l^d}\sum_{x\in\mathbb{T}_N}(\Pi'_y\varphi, \Pi'_x\psi)_-. \tag{2.53}$$

By Lemma 2.5.2 we have $\Pi'_x\Pi'_y = \mathscr{A}\Pi_x\Pi_y\mathscr{A}^{-1} = 0$ whenever $(Q_- + x)\cap(Q_- + y) = \varnothing$, i.e., if $\rho_\infty(x, y) > l - 1$. Hence the double sum only contains a non-zero contribution if there exist x and y such that $\rho_\infty(x, y) \leq l-1$, $\operatorname{spt}\varphi \cap Q+x \neq \varnothing$, and $\operatorname{spt}\psi\cap Q+y \neq \varnothing$. Hence there must exist $\xi, \zeta \in Q$ such that $x + \xi \in \operatorname{spt}\varphi$ and $y + \zeta \in \operatorname{spt}\psi$. Hence

$$\operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) \leq \rho_\infty(x+\xi-(y+\zeta), 0) \leq \rho_\infty(x-y, 0)+\rho_\infty(\xi-\zeta, 0) \leq l-1+l-2 \leq 2l-3. \tag{2.54}$$

This proves (2.48), and (2.49) follows from Lemma 2.5.5. □

We construct a finite range decomposition by an iterated application of Proposition 2.5.7. Let $L \geq 16$ and consider

$$Q_j = \{1, \dots, l_j - 1\}^d \quad \text{with } l_j = \lfloor \tfrac{1}{8}L^j\rfloor + 1 \quad \text{for } j = 1, \dots, N. \tag{2.55}$$

Here $\lfloor a\rfloor$ denotes the integer part of a, i.e., the largest integer not greater than a. In particular we have

$$\tfrac{1}{8}L^j < l_j \leq \tfrac{1}{8}L^j + 1. \tag{2.56}$$

We define $\mathscr{T}_j$, $\mathscr{T}'_j$, and $\mathscr{R}'_j$ as before with Q replaced by Q_j and set

$$\mathscr{C}_k := (\mathscr{R}_1\dots\mathscr{R}_{k-1})\mathscr{C}(\mathscr{R}'_{k-1}\dots\mathscr{R}'_1) - (\mathscr{R}_1\dots\mathscr{R}_{k-1}\mathscr{R}_k)\mathscr{C}(\mathscr{R}'_k\mathscr{R}'_{k-1}\dots\mathscr{R}'_1),\ k = 1, \dots, N, \tag{2.57}$$

and

$$\mathscr{C}_{N+1} := (\mathscr{R}_1\dots\mathscr{R}_{N-1}\dots\mathscr{R}_N)\mathscr{C}(\mathscr{R}'_N\mathscr{R}'_{N-1}\dots\mathscr{R}'_1). \tag{2.58}$$

With these definitions, we show that the sequence $\{\mathscr{C}_k\}_{k=1,\dots,N+1}$ yields a finite range decomposition.

Proposition 2.5.8 ([1, Proposition 3.9]). *Suppose that $L \geq 16$. Then the operators $\mathscr{C}_k$ satisfy*

(i) $\mathscr{C} = \sum_{k=1}^{N+1}\mathscr{C}_k$.

(ii) $\mathscr{C}_k$ *is positive definite for* $k = 1, \dots, N + 1$.

(iii) For $k = 1, \dots, N$ *the range of* $\mathscr{C}_k$ *is bounded by* $\frac{1}{2}L^k$, *i.e.,*

$$\langle \mathscr{C}_k\varphi, \psi\rangle = 0 \quad \text{if } \operatorname{dist}_\infty(\operatorname{spt}\varphi, \operatorname{spt}\psi) > \tfrac{1}{2}L^k \tag{2.59}$$

and there exist $m \times m$ matrices C_k such that

$$\mathcal{C}_k(z) = C_k \quad \text{if } \rho_\infty(z, 0) > \tfrac{1}{2}L^k. \tag{2.60}$$

Proof. Assertion (i) follows directly from the definition. To prove (ii), set

$$\varphi_k := \mathscr{R}'_{k-1} \dots \mathscr{R}'_1 \varphi, \quad \psi_k := \mathscr{R}'_{k-1} \dots \mathscr{R}'_1 \psi, \quad k = 1, \dots, N+1. \tag{2.61}$$

Inductive application of (2.36) shows that $\varphi_k = 0$ implies $\varphi = 0$. Now, directly from definitions, $\langle \mathscr{C}_{N+1}\varphi, \varphi\rangle = (\varphi_{N+1}, \varphi_{N+1})_-$. Thus $\mathscr{C}_{N+1}$ is positive definite. For $k = 1, \dots, N$ we have

$$\langle \mathscr{C}_k \varphi, \varphi\rangle = \langle (\mathscr{C} - \mathscr{R}_k \mathscr{C} \mathscr{R}'_k)\varphi_k, \varphi_k\rangle. \tag{2.62}$$

Hence by Proposition 2.5.7 we get $\langle \mathscr{C}_k\varphi, \varphi\rangle \geq 0$ with equality only holding if $\varphi_k = 0$, which implies $\varphi = 0$. Thus $\mathscr{C}_k$ is positive definite.

(iii): In view of the equation $\langle \mathscr{C}_k\varphi, \psi\rangle = \langle (\mathscr{C} - \mathscr{R}_k \mathscr{C} \mathscr{R}'_k)\varphi_k, \psi_k\rangle$, Proposition 2.5.7 implies that

$$\langle \mathscr{C}_k\varphi, \psi\rangle = 0 \quad \text{if } \operatorname{dist}_\infty(\operatorname{spt}\varphi_k, \operatorname{spt}\psi_k) > 2l_k - 3. \tag{2.63}$$

Iterative application of Lemma 2.5.6 and Lemma 2.5.5 yields

$$\operatorname{spt}\varphi_k \subset \operatorname{spt}\varphi + \{-n_k, \dots, n_k\}^d, \quad \operatorname{spt}\psi_k \subset \operatorname{spt}\psi + \{-n_k, \dots, n_k\}^d, \quad n_k = \sum_{j=1}^{k-1}(l_j - 1). \tag{2.64}$$

Thus

$$\langle \mathscr{C}_k\varphi, \psi\rangle = 0 \quad \text{if } \operatorname{dist}_\infty(\operatorname{spt}\varphi_k, \operatorname{spt}\psi_k) > -1 + 2\sum_{j=1}^{k}(l_j - 1). \tag{2.65}$$

Now since $l_j - 1 \leq \frac{1}{8}L^j$ and $\sum_{n=0}^\infty L^{-n} \leq 2$ we get $2\sum_{j=1}^k (l_j - 1) \leq \frac{1}{2}L^k$. This finishes the proof. □

2.6 Discrete gradient estimates and L^p-regularity for elliptic systems

Let us now introduce some of the norms that will be used in the sequel. Let $Q = [0, n]^d \cap \mathbb{Z}^d$, be a generic cube. For $p > 0$ denote

$$\|f\|_{p,Q} = \Big(\frac{1}{|Q|}\sum_{x \in Q_n} |f(x)|^p\Big)^{1/p}, \tag{2.66}$$

where $|Q| := \#Q$.

To simplify notation, we will write $\sum_Q f := \sum_{i \in Q} f(i)$ and $f_Q := |Q|^{-1}\sum_Q f$.

Additionally, let us define

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|}\sum_Q \big|f - f_Q\big| \,\mathrm{d}x \qquad \text{and} \qquad \|f\|_{\mathrm{BMO}} := \sup_{x \in \mathbb{T}_N^d} |f^\#(x)|. \tag{2.67}$$

The Maximal Operator is defined by

$$\mathcal{M}f(x) := \sup_{Q \ni x} \frac{1}{|Q|}\sum_Q |f| \,\mathrm{d}x \tag{2.68}$$

Moreover, let

$$\|f\|_{p,\infty} = \inf\left\{\alpha : \frac{1}{\lambda}|\{f > \lambda\}|^{1/p} \le \alpha, \text{ for all } \lambda > 0\right\}$$

and

$$\|f\|_{p,\infty,Q} = |Q|^{-1/p} \inf\left\{\alpha : \frac{1}{\lambda}|\{f > \lambda\} \cap Q|^{1/p} \le \alpha, \text{ for all } \lambda > 0\right\}.$$

We now state a version of Sobolev inequality (see [21, 2]).

Proposition 2.6.1. *For every $p \ge 1$ and $m, M \in N$ there exists a constant $C = C(p, M, m)$ such that:*

(i) If $1 \le p \le d$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$, and $q \le p^*$, $q < \infty$, then

$$n^{-\frac{d}{q}}\|f\|_q \le Cn^{-\frac{d}{2}}\|f\|_2 + Cn^{1-\frac{d}{p}}\|\nabla f\|_p. \tag{2.69}$$

(ii) If $p > d$, then

$$\big|f(x) - f(y)\big| \le Cn^{1-\frac{d}{p}}\|\nabla f\|_p \qquad \text{for all } x, y \in Q_n. \tag{2.70}$$

(iii) If $m \in \mathbb{N}$, $1 \le p \le \frac{d}{m}$, $\frac{1}{p_m} = \frac{1}{p} - \frac{m}{d}$, and $q \le p_m$, $q < \infty$, then

$$n^{-\frac{d}{q}}\|f\|_q \le Cn^{-\frac{d}{2}}\sum_{k=0}^{M-1}\|(n\nabla)^k f\|_2 + Cn^{-\frac{d}{p}}\|(n\nabla)^M f\|_p. \tag{2.71}$$

(iv) If $M = \lfloor\frac{d+2}{2}\rfloor$, the integer value of $\frac{d+2}{2}$, then

$$\max_{x\in Q_n}|f(x)| \le Cn^{-\frac{d}{2}}\sum_{k=0}^{M}\|(n\nabla)^k f\|_2. \tag{2.72}$$

Lemma 2.6.2 (Caccioppoli inequality)**.** *Let v be such that $\nabla^*(A\nabla v) = 0$ for every $x \in Q_M$ then*

$$\sum_{Q_m}|\nabla v(x)|^2 \le \frac{c_0^4}{(M-m)^2}\sum_{Q_M}|v - \lambda|^2,$$

where c_0 is the constant defined in (2.14)*.*

Proof. Let $0 \le \eta \le 1$ be a that $|\nabla\eta| \le \frac{1}{M-m}$ and such that $\eta \equiv 1$ on Q_m and $\eta = 0$ on $\mathbb{T}_N^d \setminus \bar{Q}_M$. Then

$$\sum_{Q_M}(A\nabla u \cdot \nabla u)\eta^2 = \sum_{Q_M}A\nabla u \cdot \nabla(\eta^2(u-\lambda)) - \sum_{Q_M}A\nabla u \cdot 2\eta((u-\lambda)\otimes D\eta)$$

By hypothesis, the first term in the right hand side vanishes. Using the previous formula and the ellipticity, one has that

$$\sum_{Q_M}|\nabla u|^2\eta^2 \le c_0\sum_{Q_M}A\nabla u \cdot 2\eta((u-\lambda)\otimes D\eta) \le \frac{1}{2}\sum_{Q_M}|\nabla u|^2\eta^2 + \frac{c_0^4}{2}\sum_{Q_M}|D\eta|^2|u-\lambda|^2, \tag{2.73}$$

from which one has that

$$\sum_{Q_m} |\nabla u|^2 \le \sum_{Q_M} |\nabla u|^2 \eta^2 \le \frac{c_0^4}{(M-m)^2} \sum_{Q_M} |u-\lambda|^2.$$

□

Lemma 2.6.3 (Decay estimates)**.** *Let v be such that $\nabla^*(A\nabla v) = 0$ on Q_M, with $M, M/2 \in \mathbb{N}$ and $2m \le M$. Then,*

$$\begin{aligned} \sum_{Q_m} |u(x)|^2 &\lesssim (m/M)^d \sum_{Q_M} |u(x)|^2, \\ \sum_{Q_m} |u-(u)_m|^2 &\lesssim (m/M)^{d+2} |\sum_{Q_M} u-(u)_M|^2. \end{aligned} \tag{2.74}$$

Proof. From the Caccioppoli's inequality, one has that

$$\sum_{Q_{M/2}} |M\nabla u(x)|^2 \lesssim \sum_{Q_M} |u(x)|^2.$$

Noticing that if u is a solution then also ∇u is a solution, we have that

$$\sum_{Q_M} \|(M\nabla)^j u\| \lesssim \sum_{Q_M} |u(x)|^2,$$

hence

$$M^{-d} \sum_{j=0}^{k} \sum_{Q_{M/2}} \|(M/2\nabla)^j u\| \lesssim M^{-d} \sum_{Q_M} \|u\|^2.$$

Finally applying the Sobolev, inequality we have that

$$\sum_{Q_m} \|u\|^2 \le m^d \max_{Q_{M/2}} \|u\|^2 \le (\frac{m}{M})^d \sum_{Q_M} \|u\|^2. \tag{2.75}$$

Let us now prove the second inequality. Using the Poincaré inequality and than (2.75), we have that

$$\begin{aligned} \sum_{Q_M} |u-(u)_m|^2 \le m^2 \sum_{Q_m} |\nabla u|^2 &\lesssim m^2 \left(\frac{2m}{M}\right)^d \sum_{Q_{M/2}} |\nabla u|^2 \\ &\lesssim \left(\frac{m}{M}\right)^{d+2} \sum_{Q_M} |u-(u)_M|^2, \end{aligned}$$

where in the last step we have used the Caccioppoli inequality. □

Lemma 2.6.4. *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$, $p_1 \neq p_2$, $q_1 \neq q_2$. Let $\theta \in (0, 1)$ and define p, q by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \tag{2.76}$$

Suppose that T is a linear operator such that

$$\left(\frac{1}{|Q|} \sum_Q |Tf|^{q_i} \right)^{\frac{1}{q_i}} \leq C_i \left(\frac{1}{|Q|} \sum_Q |f|^{p_i} \right)^{\frac{1}{p_i}}$$

Then

$$\|Tf\|_{q,\infty,Q} \leq C_3 \|f\|_{p,\infty,Q},$$

where C_3 depends on θ, C_1, C_2.

Proof. The proof of this result is well-known (see e.g., [13, Theorem 3.3.1]). For completeness, we report an adapted elementary proof from [15, Lemma 1]. Let $p_1 < p_2$, $q_1 < q_2$ and p is as in (2.76). Assume that $\|Tf\|_{q_i} \leq C_i \|f\|_{p_i}$ with $i = 1, 2$. Let $\gamma > 0$ define

$$f_1 = \begin{cases} f & \text{if } |f| > \gamma \\ 0 & \text{if } |f| \leq \gamma \end{cases} \tag{2.77}$$

and

$$f_2 = \begin{cases} 0 & \text{if } |f| > \gamma \\ f & \text{if } |f| \leq \gamma. \end{cases} \tag{2.78}$$

Given that

$$\frac{1}{|Q|} \sum_Q |f_1|^{p_1} \leq \frac{p_1}{p - p_1} \gamma^{p_1 - p} \|f\|^p_{p,\infty,Q}$$

we have that

$$\begin{aligned} \left| \left\{ |Tf_1| > \frac{\alpha}{2} \right\} \right| &\leq A_1^{q_1} \left(\frac{2}{\alpha}\right)^{q_1} \|f_1\|_{p_1}^{q_1} \\ &\leq A_1^{q_1} \left(\frac{2}{\alpha}\right)^{q_1} \left(\frac{p_1}{p - p_1}\right)^{q_1/p_1} \gamma^{q_1 - pq_1/p_1} \|f\|_{p,\infty,Q}^{pq_1/p_1} \\ &= B_1 \alpha^{-q_1} \gamma^{q_1 - pq_1/p_1} \end{aligned}$$

and similarly

$$\left| \left\{ |Tf_2| \geq \frac{\alpha}{2} \right\} \right| \leq B_2 \alpha^{-q_2} \gamma^{q_2 - pq_2/p_2}. \tag{2.79}$$

Now

$$\|Tf\|_{q,\infty}^q = \sup_\alpha \alpha^q | \{ |Tf| > \alpha \} |$$

and now using the triangular inequality, we have

$$\begin{aligned}\alpha^q |\{|Tf| > \alpha/2\}| &\leq \alpha^q |\{|Tf_1| > \alpha/2\}| + \alpha^q |\{|Tf_2| > \alpha/2\}| \\ &\leq B_1 \alpha^{-q_1} \gamma^{q_1 - pq_1/p_1} + B_2 \alpha^{-q_2} \gamma^{q_2 - pq_2/p_2}.\end{aligned}$$

One can archive the desired result by choosing $\gamma = \alpha^\beta$ where $\beta = \left(\frac{q}{q_1} - \frac{q}{q_2}\right)\left(\frac{p}{p_1} - \frac{p}{p_2}\right)^{-1}$.

□

Theorem 2.6.5 (Marcinkiewicz interpolation theorem)**.** *Let $0 < p_0, p_1, q_0, q_1 \leq \infty$ and $0 < \theta < 1$ be such that $q_0 \neq q_1$, and $p_i \leq q_i$ for $i = 0, 1$. Let T be a sublinear operator which is of weak type (p_0, q_0) and of weak type (p_1, q_1). Then T is of strong type (p_θ, q_θ).*

Proof. The proof is well-known. □

Remark 2.6.6. *Let $K : \mathbb{T}_N^d \times \mathbb{T}_N^d \to \mathbb{R}^{d \times m}$ be such that $|K(x, y)| \leq |x - y|^{2-d}$. Then has that*

$$\|K(x, \cdot)\|_{L^{\frac{n}{n-2}, \infty}} \leq 1, \qquad \textit{and} \qquad \|K(x, \cdot)\|_{L^{\frac{n}{n-2}, Q, \infty}} \leq 1.$$

Indeed, fix $t > 0$ then

$$|\{y : \ |K(x, y)| > t\}| \leq |\left\{y : \ |x - y|^{2-d} > t\right\}| = |\left\{y : \ |x - y| < t^{-(2-d)}\right\}| \leq t^{-\frac{d}{d-2}}.$$

Let us recall the celebrated Hardy-Littlewood maximal theorem:

Theorem 2.6.7. *Let $f : \mathbb{T}_N^d \to \mathbb{R}^m$. Then*

$$|\mathcal{M} f|_p \leq |f|_p$$

Theorem 2.6.8 (Fefferman-Stein)**.** *Let Q be a cube and let $f : Q \to \mathbb{R}^m$ such that $\sum_Q f = 0$. Then there exists constants C_1, C_2 such that*

$$\|\mathcal{M} f\|_{p,Q} \leq C_1 \|f^\#\|_{p,Q} \qquad \textit{and} \qquad \|f^\#\|_{p,Q} \leq C_2 \|\mathcal{M} f\|_{p,Q}. \tag{2.80}$$

Proof. The proof follows from the classical Fefferman&Stein result after one does a piecewise linear interpolation of the function $f : Q \to \mathbb{R}^m$.

□

Corollary 2.6.9. *Let T be an linear operator such that for every $f : Q \to \mathbb{R}^m$. Then for every $q > p$, there exists a constant $C := C(p)$ such that for every $f : Q \to \mathbb{R}^m$ it holds*

$$\sum_{x \in Q} |Tf^\#(x)|^p \leq \sum_{x \in Q} |f(x)|^p.$$

Proof. The map $f \mapsto (Tf)^\#$ is a sublinear and a bounded map from $\mathrm{L}^\infty(\boldsymbol{\mathcal{X}}) \to \mathrm{L}^\infty(\boldsymbol{\mathcal{X}})$ which is of weak type (p, p) and of weak type (∞, ∞). Then for every $q \geq p$, it holds that $f \mapsto (Tf)^\#$ is bounded. This implies that $f \mapsto M(Tf)$ is bounded because Theorem 2.6.8 and hence $f \mapsto Tf$ is bounded. □

In the next lemma $A = A_0$ is a constant positive definite operator.

Let us now recall a classical result. We also provide a proof for completeness.

Lemma 2.6.10 ([18, Lemma V.3.1]). *Assume that $\phi(\rho)$ is a non-negative, real-valued, bounded function defined on an interval $[r, R] \subset \mathbb{R}^+$. Assume further that for all $r \leq \rho < \sigma \leq R$ we have*

$$\phi(\rho) \leq \big[A_1(\sigma-\rho)^{-\alpha_1} + A_2(\sigma-\rho)^{-\alpha_2} + A_3\big] + \vartheta\phi(\sigma)$$

for some non-negative constants A_1, A_2, A_3, non-negative exponents $\alpha_1 \geq \alpha_2$, and a parameter $\vartheta \in [0, 1)$. Then we have

$$\phi(r) \leq c(\alpha_1, \vartheta)\big[A_1(R-r)^{-\alpha_1} + A_2(R-r)^{-\alpha_2} + A_3\big]\,.$$

Proof. We proceed by iteration and start by defining a sequence $(\rho_i)_{i\in\mathbb{N}_0}$ via

$$\rho_i := r + (1-\lambda^i)(R-r)$$

for some $\lambda \in (0, 1)$. This sequence is increasing, converging to R, and the difference of two subsequent members is given by

$$\rho_i - \rho_{i-1} = (1-\lambda)\lambda^{i-1}(R-r)\,.$$

Applying the assumption inductively with $\rho = \rho_i$, $\sigma = \rho_{i-1}$ and taking into account $\alpha_1 > \alpha_2$, we obtain

$$\begin{aligned}\phi(r) &\leq A_1(1-\lambda)^{-\alpha_1}(R-r)^{-\alpha_1} + A_2(1-\lambda)^{-\alpha_2}(R-r)^{-\alpha_2} + A_3 + \vartheta\phi(\rho_1)\\ &\leq \vartheta^k\phi(\rho_k) + (1-\lambda)^{-\alpha_1}\sum_{i=0}^{k-1}\vartheta^i\lambda^{-i\alpha_1}\big[A_1(R-r)^{-\alpha_1} + A_2(R-r)^{-\alpha_2} + A_3\big]\end{aligned}$$

for every $k \in \mathbb{N}$. If we now choose λ in dependency of ϑ and α_1 such that $\vartheta\lambda^{-\alpha_1} < 1$, then the series on the right-hand side converges. Therefore, passing to the limit $k \to \infty$, we arrive at the conclusion with constant $c(\alpha_1, \vartheta) = (1-\lambda)^{-\alpha_1}(1-\vartheta\lambda^{-\alpha_1})^{-1}$. □

Lemma 2.6.11. *Let u be a solution to*

$$\begin{cases}\mathcal{A}_0 u = \nabla^* f, & \text{in } Q_M,\\ u = 0 & \text{in } \mathbb{T}_N^d \setminus \bar{Q}_M.\end{cases} \tag{2.81}$$

The map $f \mapsto \nabla u$ is a continuous map from $\mathrm{L}^\infty \to \mathrm{BMO}$

Proof. Let $m \leq [M/2]$ and let u_1 be such that

$$\begin{cases}\nabla^*(A\nabla u_1) = \nabla^* f & \text{in } Q_M\\ u_1 = 0 & \text{in } \mathbb{T}_N^d \setminus \bar{Q}_M\end{cases}$$

and $u_0 = u - u_1$. Notice that $\nabla^*(A\nabla u_0) = 0$ in Q_M. We have

$$\sum_{Q_M}|\nabla u_1|^2 \lesssim \sum_{Q_M} A\nabla u_1\cdot\nabla u_1 \leq \sum_{Q_M} f\nabla u_1 \leq |f|_\infty M^{d/2}\left(\sum_{Q_M}|\nabla u_1|^2\right)^{1/2}$$

from which we have that

$$\sum_{Q_M} |\nabla u_1|^2 \leq M^d |f|_\infty^2$$

Given that from Lemma 2.6.3 we have that

$$\sum_{Q_m} |\nabla u_0 - (\nabla u_0)_m|^2 \lesssim \left(\frac{m}{M}\right)^{d+2} \sum_{Q_M} |\nabla u_0 - (\nabla u_0)_M|^2$$

it follows that

$$\sum_{Q_m} |\nabla u - (\nabla u)_m|^2 \leq \left(\frac{m}{M}\right)^{d+2} \sum_{Q_M} |\nabla u - (\nabla u)_M|^2 + \sum_{Q_m} |\nabla u_1|^2 \leq \left(\frac{m}{M}\right)^{d+2} + M^d |f|_\infty^2$$

Finally using Lemma 2.6.10 we have the desired result.

□

From now on $A = A(x)$, namely depends on the space.

The next lemma is an adaption of [15, Lemma 2] to the discrete case. The original proof is based on an argument in [20]. We will rather use an argument based on Theorem 2.6.8.

In the continuous case, the analog version of the next lemma can be found in [15, Lemma 2].

Lemma 2.6.12 (Global estimate)**.** *Let $p \in (1, \infty)$ $q \in (1, n)$*

(i) *If $f : \mathbb{T}_N^d \to R^{md}$, $g : \mathbb{T}_N^d \to \mathbb{R}^m$ and let u be the solution of*

$$\begin{cases} -\nabla^*(A\nabla u) = \nabla^* f + g & \text{in } Q_M \\ u = 0 & \text{in } \mathbb{T}_N^d \setminus \bar{Q}_M \end{cases}$$

Then if

$$s = \min(p, q^*), \qquad q^* = \frac{dq}{d-q}$$

we have

$$\left(\sum_{Q_M} |\nabla u|^s\right)^{1/s} \lesssim \left(\sum_{Q_M} |f|^p\right)^{1/p} + \left(\sum_{Q_M} |Mg|^q\right)^{1/q}$$

(ii) *and*

$$\|u\|_{s^*,\infty} + \|\nabla u\|_{s,\infty} \leq C\left(\|f\|_{p,\infty,Q_M} + |g|_{q,\infty,Q_M}\right)$$

Proof. Let x_0 be the center of the cube Q_M. For simplicity of notation we will denote by $A_0 := A(x_0)$. With simple algebraic manipulations we have

$$\nabla^*(A_0 \nabla u) = \nabla^*(f + (A_0 - A)\nabla u)$$

Let η such that $\eta \equiv 0$ in $\mathbb{T}_N^d \setminus \bar{Q}_M$. Then we have

$$\nabla^* (A_0 \nabla(u\eta)) = \nabla^* \left((A_0 - A)\nabla(u\eta)\right) + G + \nabla^* F$$

where $G = g\eta + fD\eta + A(x)\nabla u D\eta$ and $F = f\eta + A(x)uD\eta$.
Let w be defined as

$$\begin{cases} \nabla^*(\nabla w) = -G & \text{in } Q_M \\ w = 0 & \text{in } \mathbb{T}_N^d \setminus \bar{Q}_M \end{cases}$$

Hence, from the constant coeficient case one has that

$$\left(\sum_{Q_M} \|M\nabla w\|^{r^*} \right)^{1/r^*} \lesssim \left(\sum_{Q_M} \|G\|^r \right)^{\frac{1}{r}}$$

Denoting with $\tilde{F} = F + \nabla w$ we have that

$$\nabla^*(A_0 \nabla(u\eta)) = \nabla^* (A - A_0)\nabla v) + \nabla^* \tilde{F} \qquad \text{in } Q_M.$$

We will now make a fixed point argument. Fix V and consider the linear operator $T : V \mapsto v$ where v is the solution of

$$\nabla^*(A_0 \nabla v) = \nabla^* (A - A_0)\nabla V) + \nabla^* \tilde{F}$$

The operator T is continuous, namely

$$\sum_{x\in Q_M} |\nabla T(V_1 - V_2)|^s \le c \sup_{x\in Q_M} |A(x) - A(x_0)|^s \sum_{x\in Q_M} |\nabla V_1(x) - \nabla V_2(x)|^s + c \sum_{x\in Q} |\tilde{F}|^s$$

If

$$\sup_{x\in Q_M} |A(x) - A_0| \le \frac{1}{2} A(x_0) \tag{2.82}$$

one can apply the fixed point theorem and deduce that the solution coincides with $u\eta$, and that

$$\left(\sum_{Q_M} |(M\nabla)u|^s \right)^{1/s} \le C \left(\sum_{Q_M} |\tilde{F}|^s \right)^{1/s}.$$

Finally the condition (2.82) is ensured by (2.17). □

For the continuous version of the following lemma see [15, Lemma 4]
Lemma 2.6.13. *Let $q \in (1,d)$ $p > d$. Let*

$$T = \|\nabla u\|_{\mathrm{L}^{q,\infty}(Q_{2M})} + \|u\|_{\mathrm{L}^{q^*,\infty}(Q_{2M})}. \tag{2.83}$$

Suppose that u satisfies

$$-\nabla^*(A\nabla u) = \nabla^* f \qquad \text{in } Q_{2M} \tag{2.84}$$

Then there exists $m_0 := m_0(p,q)$ *such that if* $M > m_0$ *then*

$$\sup_{Q_m} |u| \lesssim M^{-\frac{d}{q}} T + M^{1-\frac{d}{p}} \|f\|_{\mathrm{L}^p}, \tag{2.85}$$

where $m = \lceil M/d \rceil$

Proof. Let $\delta \in \mathbb{N}$ such that $\delta \leq M$. Set $\kappa = \lfloor \frac{M}{\delta} \rfloor$ and let φ be such that $\varphi \equiv 1$ in Q_M, $\varphi \equiv 0$ in $\mathbb{T}_N^d \setminus \bar{Q}_{M+\delta}$, and such that $|\nabla \varphi| \leq \frac{1}{\delta}$. Then for every $p_1 > 0$ one has that

$$\left(\frac{1}{|Q_M|} \sum_{Q_M} |\nabla u|^{p_1} \right)^{\frac{1}{p_1}} \leq \left(\frac{|Q_{M+\delta}|}{|Q_M|} \right)^{1/p_1} \left(\frac{1}{|Q_{M+\delta}|} \sum_{Q_{M+\delta}} |\nabla(\varphi u)|^{p_1} \right)^{\frac{1}{p_1}}$$

With simple calculations one has that

$$\begin{aligned} \nabla^*(A\nabla(\varphi u)) &= \sum_{i,j} \nabla_j^* \left(\varphi(x) A_{i,j}(x) \nabla_i u + A_{i,j}(x) \nabla_i \varphi \otimes u(x+e_j) \right) \\ = \sum_j \nabla_j^*(\varphi f_j) &+ \sum_{i,j} A_{i,j}(x) \left(\nabla_j u(x) - f_j(x) \right) \nabla_i \varphi(x) + \sum_{i,j} \nabla_j^* \left(A_{i,j} \nabla_i \varphi \otimes u(x+e_i) \right) \end{aligned} \tag{2.86}$$

Denote by

$$\begin{aligned} \tilde{f}_j &:= \varphi f_j + \sum_i A_{i,j} \nabla_i \varphi(x) \otimes u(x+e_i) \\ g &:= \sum_{i,j} A_{i,j} (\nabla_j u - f_j) \nabla_i \varphi(x) \end{aligned}$$

Equation (2.86) can be rewritten as

$$\nabla^*(A(\varphi u)) = \nabla^* \tilde{f} + \tilde{g}$$

Let $s = \min(p, t^*)$. One has that

$$\begin{aligned} \left(\frac{1}{(M+\delta)^d} \sum_{Q_{M+\delta}} \|\tilde{f}\|^s \right)^{1/s} &\leq \left(\frac{1}{(M+\delta)^d} \sum_{Q_{M+\delta}} |\varphi f|^p \right)^{1/p} \\ &+ \sum_{i,j} \left(\frac{1}{(M+\delta)^d} \sum_{Q_{M+\delta}} A_{i,j} |\nabla_i \varphi|^{t^*} |u|^{t^*} \right)^{1/t^*} \\ \lesssim \left(\frac{1}{(M+\delta)^d} \sum_{Q_{M+\delta}} |\varphi f|^p \right)^{1/p} &+ \left(\frac{1}{(M+\delta)^d} \sum_{Q_{M+\delta}} |u|^{t^*} \right)^{1/t^*} \end{aligned}$$

Using the Sobolev inequality, the last term in the previous equation can be bounded by

$$\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M\delta}}|u|^{t^*}\right)^{\frac{1}{t^*}} \leq \left[\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|u|^t\right)^{1/t} + \left(\frac{1}{(M+\delta)}\sum_{Q_{M+\delta}}|(M+\delta)\nabla u|^t\right)^{1/t}\right]$$

In a similar way one has

$$\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|g|^t\right)^{1/t} \lesssim \big(\sup_{i,j}|A_{i,j}|\big)\frac{1}{\delta}\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|\nabla u|^t\right)^{1/t} + \sup|A_{i,j}|\frac{1}{\delta}\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|f_j|^p\right)^p$$

Putting together all the previous inequalities and using Lemma 2.6.12, one has that

$$\left(\frac{1}{M^d}\sum_{Q_M}\|\nabla u\|^s\right)^{\frac{1}{s}} \lesssim \left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|u|^t\right)^{1/t} + \left(\frac{1}{(M+\delta)}\sum_{Q_{M+\delta}}|(M+\delta)\nabla u|^t\right)^{1/t} + \frac{M+\delta}{\delta}\left(\frac{1}{(M+\delta)^d}\sum_{Q_{M+\delta}}|f|^p\right)^{\frac{1}{p}}.$$

Applying the previous reasoning κ times, we have that

$$\left(\frac{1}{M^d}\sum_{Q_M}\|\nabla u\|^{t_\kappa}\right)^{\frac{1}{t_\kappa}} \leq C_\kappa\left(\frac{1}{(M+k\delta)^d}\sum_{Q_{M+k\delta}}|u|^t\right)^{1/t} + C_\kappa\left(\frac{1}{(M+k\delta)}\sum_{Q_{M+\delta}}|(M+\delta)\nabla u|^t\right)^{1/t} + C_\kappa\left(\frac{1}{(M+k\delta)^d}\sum_{Q_{M+k\delta}}|f|^p\right)^{\frac{1}{p}},$$

where t_κ is given by the recursive equation $t_j = \max(p, t^*_{j-1})$ and $t_1 = t$. It can be easily seen that for every $t > 1$, it holds that $t_j \geq d$ for some j which depends only on p and q.

□

Proposition 2.6.14. *Let $C(x,y)$ be the Green function,i.e., for every $x \in \mathbb{T}_N^d$ one has*

$$\nabla^*(A\nabla C(x,\cdot)) = \delta_x$$

where A satisfies the usual conditions.
Then

$$|\nabla^\alpha C(x,y)| \lesssim |x-y|^{2-d-|\alpha|}.$$

Proof. Let K be the solution of

$$\nabla^*(\nabla K) = \delta_x.$$

It is well-known that the following estimates hold

$$|(\nabla^\alpha K)(x-y)| \lesssim |x-y|^{2-d-|\alpha|}.$$

From Remark 2.6.6 we have that $|(\nabla^\alpha K)(x-y)|_{\frac{d}{d+|\alpha|-2},\infty} \leq C_{d,\alpha}$ where $C_{d,\alpha}$ is a constant depending only on the dimension d and the multiindex α.

Let us denote with $u(y) = C(x,y)$. Then from the definitions of K and C one has that

$$\nabla^*(A\nabla u) = \nabla^*(\nabla K(x-\cdot))$$

Let $|x-y| = R$. Without loss of generality we may assume that $M > 2m_0$, where m_0 is the constant in Lemma 2.6.13. Let $M = [\frac{R}{2}]$ and let Q_M be a cube such that $y \in Q_M$ and $x \notin Q_{2M}$. Given that $\mathcal{A}C(x,\cdot) = 0$ in Q_{2M}, using Lemma 2.6.13 we have that

$$C(x,y) \lesssim M^{2-d}C_d \leq |x-y|^{2-d}C_d.$$

Higher derivative follow in a similar way. For example to estimate $\nabla_i u$ it is enough to consider the equation

$$\nabla^*(A\nabla\nabla_i u) = \nabla^*((\nabla\nabla_i u)) - \nabla^*((\nabla_i A)\nabla u),$$

and apply the above reasoning, and hence using the global estimate one has that $|\nabla\nabla u|$

□

Proposition 2.6.15. *Let $Q_1,\dots,Q_k$ be cubes of length $l_1,\cdots,l_k$ respectively such that $y \in Q_i$. Then there exists a dimensional constants $C_{d,j}$ such that*

$$\sup|\nabla^j u| \leq 2^k C_{d,j} \max\Big(|x-y|, \operatorname{dist}(x, T_N^d \setminus Q_1),\dots,\operatorname{dist}(x,\mathbb{T}_N^d \setminus Q_k)\Big)^{2-d+j}, \quad (2.87)$$

where $u = (P_{Q_1}\cdots P_{Q_k}C(x,\cdot))$ and $C(x,y)$ is the Green's function.

Proof. Let Q_1 be a cube of size l_1 in generic position. Given that $\nabla^*(A\nabla C_x(y)) = 0$, if $x \notin \bar{Q}_1$ then $\Pi_{Q_1}C(x,y) = 0$, thus $P_{Q_1}C(x,y) = C(x,y)$, hence the inequality follows from Proposition 2.6.14.

Let $\varepsilon := \operatorname{dist}(y, \bar{Q}_1^C) < l_1$. If $|x-y| > \varepsilon/2$, then by estimating the different terms $\Pi_{Q_1}C(x,y)$ and $C(x,y)$ separately one has the desired result. Indeed, it is immediate that $C(x,y) \lesssim |x-y|^{2-d}$. On the other side it is not difficult to see that there exits a cube of size ε touching the boundary such that it does not contain x and such that twice the cube does not contain x. Then by using Lemma 2.4.3, one has that

$$|\Pi_{Q_1}C(x,y)| \lesssim |x-y|^{2-d}M,$$

where

$$M = \|D\Pi_{Q_1}C_x\|_{L^{d/d-2,\infty}(Q_1)} + \|\Pi_{Q_1}C_x\|_{L^{d/d-1,\infty}(Q_1)}.$$

Then by using Lemma 2.6.12 one has that

$$\|D\Pi_{B_1}C_x\|_{L^{d/(d-2),\infty}} + \|\Pi_{B_1}C_x\|_{L^{d/(d-1),\infty}} \lesssim \|DC_x\|_{L^{d/(d-2),\infty}} + \|C_x\|_{L^{d/(d-1),\infty}}$$

Suppose that $|x-y| \le \varepsilon/2$. Then one can find a cube of size $\lfloor \varepsilon/2 \rfloor$ such that double the cube is contained in Q_1. Finally by using Lemma 2.6.13 we have the desired result.

Let us now prove the inductive step. Let $Q_1, \dots, Q_k$ be k cubes cetered in 0. If the maximum in the right hand side of (2.87) is $|x-y|$ or $\mathrm{dist}(x, \mathbb{T}^d_n \setminus Q_1)$, then the same reasoning as above would apply. For simplicity let us suppose that

$$\max\Big(|x-y|, \mathrm{dist}(x, \mathbb{T}^d_N \setminus \bar{Q}_1), \dots, \mathrm{dist}(x, \mathbb{T}^d_N \setminus \bar{Q}_k)\Big) = \mathrm{dist}(x, \mathbb{T}^d_N \setminus \bar{Q}_1) =: \delta.$$

From the inductive step we know that

$$\sup |v| \lesssim \delta^{2-d} \qquad \sup |\nabla^\alpha v| \lesssim \delta^{2-d-|\alpha|},$$

where $v := P_2 \dots P_k C(x,\cdot)$. From the definition we have that $u = v - P_{Q_1}v$, hence $\sup|u| = \sup|v| + \sup|\Pi_{Q_1}v|$. Thus by using Lemma 2.6.13 and a very similar reasoning as above we have the desired result. □

Let $Q_1, \dots, Q_k$ be k cubes with radii $l_1, \dots, l_k$ respectively and let $\mathcal{C}$ be the Green's function. From now on we fix x and denote with $u(y) := (\mathscr{R}_1 \cdots \mathscr{R}_k \mathcal{C}(x,\cdot))(y)$, where for simplicity we will use $\mathcal{R}_i = \mathcal{R}_{Q_i}$.

The following simple calculation will be repeatedly used in the next theorem.

Remark 2.6.16. *Let $j > 1$ be an integer and Q be a cube of size l. Then*

$$\frac{1}{|Q|}\sum_{z\in Q} \max(\alpha, \mathrm{dist}(z, \mathbb{T}^d_N \setminus \bar{Q}))^{-j} \lesssim \frac{\alpha^{1-j}}{l} \tag{2.88}$$

and if $j = 1$ then

$$\frac{1}{|Q|}\sum_{z\in Q} \max(\alpha, \mathrm{dist}(z, \mathbb{T}^d_N \setminus \bar{Q}))^{-j} \lesssim \frac{\log(\alpha)}{l}. \tag{2.89}$$

To prove the above calculation, it is enough to view it as a discretization of the Lemma 2.4.5, hence use a similar process.

Theorem 2.6.17. *Let C_k, Q_i, r_i as above and such that $r_1 < \dots < r_h < |x-y| < r_h + 1 < \dots < r_k$. Then*

(i) *if $k - h < d - 2$*

$$\begin{aligned} |C_k(x,y)| &\lesssim \frac{1}{r_{h+1}\cdots r_k}|x-y|^{2-d+k-h} \prod_{i=h+1}^{k} (\log(|x-y|)+1) \\ |\nabla^j_y C_k(x,y)| &\lesssim \frac{1}{r_{h+1}\cdots r_k}|x-y|^{2-d+k-j-h} \end{aligned} \tag{2.90}$$

(ii) *if* $k - h \geq d - 2$

$$\begin{aligned} |C_k(x,y)| &\lesssim \frac{1}{r_{k-d+3}\cdots r_k} |\log(|x-y|)| \\ |\nabla^j_y C_k(x,y)| &\lesssim \frac{1}{r_{k-d+2-j}\cdots r_k} \prod_{i=h+1}^{k} (\log(|x-y|)+1) \end{aligned} \tag{2.91}$$

Proof. We will only prove the first part of (i). The proof of the other parts is similar. Let us initially consider the case $k = 1$. For simplicity we denote $\Pi_z := \Pi_{Q_1+z}$. With simple computations one has

$$\sup_y |u(y)| \leq \frac{1}{|Q|} \sum_{Q_1+y} \sup_y |(\mathrm{Id} - \Pi_z)u(y)| \tag{2.92}$$

Given that for every $z \in y + Q$ it holds $\mathrm{dist}(y, z + Q_1) = r_1 - |z - y|$, it holds

$$\sup |(\mathrm{Id} - \Pi_z)u| \leq \begin{cases} (r_1 - |z-y|)^{2-d} & \text{if } \ r_1 - |y-z| \geq |x-y| \\ |x-y|^{2-d} & \text{otherwise} \end{cases},$$

The above can be reformulated as $\sup |(\mathrm{Id} - \Pi_z)u| \leq \max(|x-y|, \mathrm{dist}(z, \mathbb{T}^d_N \setminus \bar{Q}))$. Hence using Remark 2.6.16 one immediately has

$$\sup_y |u_1(y)| \lesssim \frac{|x-y|^{3-d}}{r_1}. \tag{2.93}$$

Let us now turn to the general case $k < d - 2$. And let $Q_1, \ldots, Q_k$ be balls of radiusis $r_1, \ldots, r_k$ centered in 0. From Proposition 2.4.4 we have that

$$\begin{aligned} &\sup |P_{z_1+Q_1} \cdots P_{z_k+Q_k} C(x,\cdot)| \leq \max\{|x-y|, r_1 - |z_1 - y|, \ldots, r_k - |z_k - y|\}^{2-d} \\ &\leq \max\{|x-y|\}^{2-d+k} \cdot \max\{|x-y|, r_k - |z_k - y|\}^{-1} \cdots \max\{|x-y|, r_k - |z_k-y|\}^{-1} \\ &\qquad =: g(z_1, \ldots, z_k). \end{aligned}$$

$$\sup \mathcal{R}_1 \cdots \mathcal{R}_k C(x,\cdot) \leq \sum_{Q_1} \cdots \sum_{Q_k} g(z_1, \ldots, z_k)$$

From Remark 2.6.16 we have that

$$\sum_{Q_1} \cdots \sum_{Q_k} g(z_1, \ldots, z_k) \leq \frac{1}{r_1 \cdots r_k} |x-y|^{2-d+k} \prod_i (|\log(|x-y|)| + 1)$$

□

A direct consequence is the following corrollary:

Corollary 2.6.18. *Suppose that* $|x - y| > 1$ *and let* $Q_1, \ldots, Q_k$ *and such that* $r_i = L^i$ *with* $L > 1$. *Then there exists* $\eta(j, d)$ *such that*

$$|\nabla^j C_k(x,y)| \lesssim \frac{L^{\eta(j,d)}}{L^{k(d-2-j)}}.$$

Theorem 2.6.19 (Fixed A)**.** *Let*

$$\mathcal{C}_k := \mathcal{R}_1 \cdots \mathcal{R}_k \mathcal{C} \mathcal{R}_k^* \cdots \mathcal{R}_1^* - \mathcal{R}_1 \cdots \mathcal{R}_{k+1} C \mathcal{R}_{k+1}^* \cdots \mathcal{R}_1^*. \tag{2.94}$$

Then

$$\sup_{y \in \mathbb{T}_N^d} |\nabla^\alpha \tilde{C}_k(x,y)| \leq L^{\eta(d,|\alpha|)} L^{-(k-1)(d-2+|\alpha|)}$$

Proof. We will estimate the two term in right hand side of (2.94) separately. Given that $\mathcal{R}^* = \mathcal{A}\mathcal{R}\mathcal{A}^{-1}$, and denoting by $\mathcal{D}_k = \mathcal{R}_1 \cdots \mathcal{R}_k C \mathcal{R}_k^* \cdots \mathcal{R}_1^*$. one has that

$$\mathcal{D}_k = \mathcal{R}_1 \cdots \mathcal{R}_k \mathcal{R}_k \cdots \mathcal{R}_1 C. \tag{2.95}$$

Applying Theorem 2.6.17, we obtain that the supremum of $\mathcal{D}_k$ is bounded by

$$\prod_{j=1}^{d-2} L^{-k+j} \prod_{j=1}^{d-2} \log(L^{-k+j}) \leq L^{-k(d-2)} L^{\eta(d)}.$$

□

2.7 Analytic dependence on A

The proof of the analyticity is based on a very elegant argument using complex analysis, and it is originally found in [1]. In this section, we will make the appropriate modifications. Let $A : \mathbb{T}_N^d \to \mathcal{L}_{\mathbb{C}}(\mathbb{R}^{m\times d})$, where $\mathcal{L}_{\mathbb{C}}(\mathbb{R}^{m\times d})$ from $\mathbb{C}^{m\times d}$ to $\mathbb{C}^{m\times d}$ such that

$$A = A_0 + A_1 \tag{2.96}$$

with A_0 and A_1 such that for all $F, G \in \mathbb{C}^{m\times d}$,

$$\langle A_0(x)F, G\rangle_{\mathbb{C}^{m\times d}} = \langle F, A_0(x)G\rangle_{\mathbb{C}^{m\times d}}, \quad \langle A_0(x)F, F\rangle_{\mathbb{C}^{m\times d}} \geq c_0|F|^2, \tag{2.97}$$

and

$$\sup_{x\in\mathbb{T}_N^d} \|A_1(x)\| \leq \frac{c_0}{2}. \tag{2.98}$$

Here, $c_0 > 0$ is a fixed constant and, as before, $\langle\cdot,\cdot\rangle_{\mathbb{C}^{m\times d}}$ and $|\cdot|$ denote the standard scalar product and norm on $\mathbb{C}^{m\times d}$ and $\|A_1\|$ is the corresponding operator norm of A_1.

As before,

$$\mathscr{A} := \nabla^* A \nabla, \tag{2.99}$$

hence the sesquilinear form

$$(\varphi, \psi)_A := \langle A\nabla\varphi, \nabla\psi\rangle, \tag{2.100}$$

where $\langle\cdot,\cdot\rangle$ is the ℓ_2-scalar product on $\boldsymbol{\mathcal{X}}_N$, defining the adjoint $\mathscr{A}^*$ by

$$\langle \mathscr{A}\varphi, \psi\rangle = (\varphi,\psi)_A = \langle\varphi, \mathscr{A}^*\psi\rangle, \quad \text{with} \quad \mathscr{A}^* = \nabla^* A^* \nabla, \tag{2.101}$$

where A^* is the adjoint of A. Note that for real, symmetric A the form $(\cdot,\cdot)_A$ is a scalar product and agrees with $(\cdot,\cdot)_+$. In the following, we use the previous notation $\boldsymbol{\mathcal{H}}_+$ for the Hilbert space with the scalar product $(\cdot,\cdot)_{A_0}$ and define $\|\varphi\|_{A_0} := (\varphi,\varphi)_{A_0}^{1/2}$.

Using $\Re z$ and z^* to denote the real part and the complex conjugate of a complex number z, we summarize the main properties of the sesquilinear form $(\cdot,\cdot)_A$.

Lemma 2.7.1 ([1, Lemma 5.1]). *Assume that an operator A satisfies the conditions* (2.96), (2.97), *and* (2.98).

Then the sesquilinear form $(\cdot,\cdot)_A$ on $\boldsymbol{\mathcal{X}}_N$ satisfies

$$\Re(\varphi,\varphi)_A \geq \tfrac{1}{2}\|\varphi\|^2_{A_0}, \tag{2.102}$$

$$|(\varphi,\psi)_A| \leq \tfrac{3}{2}\|\varphi\|_{A_0}\|\psi\|_{A_0}, \tag{2.103}$$

$$(\psi,\varphi)_A = (\varphi,\psi)^*_{A^*}. \tag{2.104}$$

Proof. The first claim follows using the definition of the form $(\cdot,\cdot)_A$ and the lower bound

$$\Re\langle A(x)F,F\rangle_{\mathbb{C}^{m\times d}} \geq \langle A_0(x)F,F\rangle_{\mathbb{C}^{m\times d}} - \frac{c_0}{2}|F|^2 \geq \frac{1}{2}\langle A_0(x)F,F\rangle_{\mathbb{C}^{m\times d}} \tag{2.105}$$

implied by (2.97) and (2.98).

Using (2.98), the Cauchy-Schwarz inequality for the scalar product $\langle A_0F,G\rangle_{\mathbb{C}^{m\times d}}$, and the bound from (2.97), we also get

$$\begin{aligned}|\langle AF,G\rangle_{\mathbb{C}^{m\times d}}| \leq \langle A_0F,G\rangle_{\mathbb{C}^{m\times d}} + \frac{c_0}{2}|F||G| \leq \langle A_0F,F\rangle^{1/2}_{\mathbb{C}^{m\times d}}\langle A_0G,G\rangle^{1/2}_{\mathbb{C}^{m\times d}} + \\ + \tfrac{1}{2}\langle A_0F,F\rangle^{1/2}_{\mathbb{C}^{m\times d}}\langle A_0G,G\rangle^{1/2}_{\mathbb{C}^{m\times d}} \leq \tfrac{3}{2}\langle A_0F,F\rangle^{1/2}_{\mathbb{C}^{m\times d}}\langle A_0G,G\rangle^{1/2}_{\mathbb{C}^{m\times d}}\end{aligned} \tag{2.106}$$

implying the second claim.

The last identity follows from the relation

$$\langle AG,F\rangle_{\mathbb{C}^{m\times d}} = \langle G,A^*F\rangle_{\mathbb{C}^{m\times d}} = \langle A^*F,G\rangle^{*\mathbb{C}^{m\times d}}.$$

□

In view of the above Lemma, the complex version of the Lax-Milgram theorem can be used to ensure the existence of the bounded inverse operator $\mathscr{C}_A = \mathscr{A}^{-1}$.

In the following, similarly as in the case of the Hilbert space $\boldsymbol{\mathcal{H}}_+$, we use $\boldsymbol{\mathcal{H}}_+(Q+x)$ to denote the corresponding Hilbert space (of functions from $\boldsymbol{\mathcal{X}}_N$ with support in $Q+x$) with the scalar product $(\cdot,\cdot)_{A_0}$.

Next, we define an extension of the operators Π_x for a general complex A.

Lemma 2.7.2 ([1, Lemma 5.2]). *Assume that A satisfies* (2.96), (2.97), *and* (2.98). *Then, for each $\varphi \in \boldsymbol{\mathcal{X}}_N$, there exists a unique $v \in \boldsymbol{\mathcal{H}}_+(Q+x)$ such that*

$$(v,\psi)_A = (\varphi,\psi)_A \quad \textit{for all} \quad \psi \in \boldsymbol{\mathcal{H}}_+(Q+x). \tag{2.107}$$

Proof. The assertion follows from Lemma 2.7.1 and the Lax-Milgram theorem. □

Lemma 2.7.3 ([1, Lemma 5.3]). *Assume that A satisfies* (2.96), (2.97), *and* (2.98). *For any $\varphi \in \boldsymbol{\mathcal{X}}_N$, we set*

$$\Pi_{A,x}\varphi := v, \quad \Pi_A := \Pi_{A,0}, \tag{2.108}$$

with $v \in \boldsymbol{\mathcal{H}}_+(Q+x)$ defined by (2.107). *Using, as before, τ_x to denote the translation by x, 1_Q for the characteristic function of a set Q, and D for the open unit disc $D = \{w \in \mathbb{C}\colon |w| < 1\}$, we have The map $z \mapsto \Pi_{A_0+zA_1}\varphi$ is holomorphic for z in the open unit disc D.*

Proof. This follows from the complex inverse function theorem. Fix φ and consider the map R from $D \times \boldsymbol{\mathcal{H}}^+(Q)$ into the dual of $\boldsymbol{\mathcal{H}}^+(Q)$ given by

$$R(z,v)(\psi) = (v - \varphi, \psi)_{A_0 + zA_1}. \tag{2.109}$$

Then R is complex linear in z and v and hence complex differentiable. By the definition of Π_A we have $R(z,v) = 0$ if and only if $v = \Pi_{A_0+zA_1}\varphi$. Finally the derivative of R with respect to the second argument is given by the map L_z from $\boldsymbol{\mathcal{H}}^+(Q)$ into its dual with $L_z(\dot v)(\psi) = (\dot v, \psi)_{A_0+zA_1}$. By the Lax-Milgram theorem, L_z is invertible for $z \in D$. Hence the map $z \mapsto \Pi_{A_0+zA_1}\varphi$ is complex differentiable in z. □

We define, as before,

$$\mathcal{T}_A := l^{-d} \sum_{x \in \mathbb{T}_N^d} \Pi_{A,x}, \quad \mathscr{R}_A = \mathrm{Id} - \mathscr{T}_A. \tag{2.110}$$

Lemma 2.7.4 ([1, Lemma 5.4])**.** *Assume that A satisfies* (2.96)*,* (2.97)*, and* (2.98)*. Then*

$$\|\mathcal{T}_A\varphi\|_{A_0} \le 9\|\varphi\|_{A_0} \quad \textit{for all} \quad \varphi \in \boldsymbol{\mathcal{X}}_N. \tag{2.111}$$

Proof. This is an adaptation of the argument from [8] to the complex case. For the convenience, we include the details. We have

$$l^{2d}\|\mathcal{T}_A\varphi\|_{A_0}^2 \le 2\, l^{2d}\, |(\mathcal{T}_A\varphi, \mathcal{T}_A\varphi)_A| \le 2 \sum_{x,y \in \mathbb{T}_N^d} |(\Pi_{A,x}\varphi, \Pi_{A,y}\varphi)_A|. \tag{2.112}$$

Set $T_x := \nabla \Pi_{A,x}\varphi$. Then T_x vanishes outside $Q_- + x$ since $\Pi_{A,x}\varphi$ vanishes outside $Q + x$. Thus, in view of (2.100) and (2.103), we get, similarly as in (2.106),

$$\begin{aligned}\big|(\Pi_{A,x}\varphi, \Pi_{A,y}\varphi)_A\big| &= \big|\langle AT_x, T_y\rangle\big| = \big|\langle A\mathbb{1}_{Q_-+x}T_x, \mathbb{1}_{Q_-+y}T_y\rangle\big| = \big|\langle A\mathbb{1}_{Q_-+y}T_x, \mathbb{1}_{Q_-+x}T_y\rangle\big| \le \\ &\le \tfrac{3}{2}\langle A_0\mathbb{1}_{Q_-+y}T_x, \mathbb{1}_{Q_-+y}T_x\rangle^{1/2}\langle A_0\mathbb{1}_{Q_-+x}T_y, \mathbb{1}_{Q_-+x}T_y\rangle^{1/2} \le \\ &\le \tfrac{3}{4}\langle A_0\mathbb{1}_{Q_-+y}T_x, \mathbb{1}_{Q_-+y}T_x\rangle + \tfrac{3}{4}\langle A_0\mathbb{1}_{Q_-+x}T_y, \mathbb{1}_{Q_-+x}T_y\rangle = \\ &= \tfrac{3}{4}\langle A_0\mathbb{1}_{Q_-+y}T_x, T_x\rangle + \tfrac{3}{4}\langle A_0\mathbb{1}_{Q_-+x}T_y, T_y\rangle.\end{aligned} \tag{2.113}$$

Now $\sum_{y \in \mathbb{T}_N^d} 1_{Q+y}$ is the constant function l^d and thus

$$\begin{aligned}\sum_{x,y \in \mathbb{T}_N^d} \big|(\Pi_{A,x}\varphi, \Pi_{A,y}\varphi)_A\big| &\le \tfrac{3}{2}l^d \sum_{x \in \mathbb{T}_N^d} \langle A_0T_x, T_x\rangle = \tfrac{3}{2}l^d \sum_{x \in \mathbb{T}_N^d} (\Pi_{A,x}\varphi, \Pi_{A,x}\varphi)_{A_0} \le \\ &\le 3l^d \sum_{x \in \mathbb{T}_N^d} \Re(\Pi_{A,x}\varphi, \Pi_{A,x}\varphi)_A = 3l^d \sum_{x \in T_N} \Re(\varphi, \Pi_{A,x}\varphi)_A = \\ &= 3l^{2d}\Re(\varphi, \mathcal{T}_A\varphi)_A \le \tfrac{9}{2}l^{2d}\|\varphi\|_{A_0}\|\mathcal{T}_A\varphi\|_{A_0}.\end{aligned}$$

Combined with (2.112), this yields the assertion. □

Lemma 2.7.5 ([1, Lemma 5.4])**.** *Let $D = \{z \in \mathbb{C} : |z| < 1\}$.*

(i) Suppose that $f\colon D \to \mathbb{C}^{m\times m}$ *is holomorphic and*

$$\sup_{z\in D} \|f(z)\| \le M. \tag{2.114}$$

Then the j*-th derivative satisfies*

$$\|f^{(j)}(0)\| \le Mj!\ . \tag{2.115}$$

(ii) Suppose that $f\colon D \to \mathbb{C}^{m\times m}$ *and* $g\colon D \to \mathbb{C}^{m\times m}$ *are holomorphic and*

$$\sup_{z\in D} \|f(z)\| \le M_1, \quad \sup_{z\in D} \|g(z)\| \le M_2. \tag{2.116}$$

Then the function $h(t) = f(t)g^*(t)$ *is real-analytic in* $(-1,1)$ *and*

$$\|h^{(j)}(0)\| \le M_1 M_2 j!\ . \tag{2.117}$$

Here $g^*(t)$ *denotes the adjoint matrix of* $g(t)$*.*

Proof. Assertion (i) follows directly from the Cauchy integral formula. To show (ii), we note that $g(z) = \sum_j a_j z^j$ with $a_j \in \mathbb{C}^{m\times m}$. Define $G(z) := \sum_j a_j^* z^j$. Then $G(z) = g(z^*)^*$. Hence $\|G(z)\| = \|g(z^*)\|$. Thus $H := fG$ is holomorphic in D and satisfies $\sup_D \|H\| \le M_1 M_2$. Hence $H^{(k)}(0) \le k!\, M_1 M_2$. For $t \in (-1,1)$ we have $H(t) = h(t)$ and the assertion follows. □

Proof of 2.3.1 . The boundedness follows from the boundedness of the inverse $\mathcal{C}_A = \mathcal{A}^{-1}$ and Lemma 2.7.4.

□

3 Strict convexity of the surface tension for non-convex and space dependent potentials

In this chapter, we will extend some new results due to S. Adams, R. Kotecký and S. Müller in [2]. We will extensively use their general strategy and many of their results. As usual in the RG theory(which we will denote by RG from now on), there is a combinatoric part and a analytical part. The combinatoric part will apply *unchanged* to our setting. For the analytical part, one needs to find some appropriate norms that will capture the subtle growth in the RG step. Thus, because of the extra difficulty (due to the space dependence of the Hamiltonians), we need to find such appropriate norms, which generalize the ones in [2] and a space of relevant parameters so that we can get the RG machinery started. Following the general strategy proposed in [2], we will show the smoothness for the RG step.

Once this is done, the proof will follow by standard arguments.

In [2], a key ingredient in the recipe of the RG technique, is the use of the Finite Range Decomposition with optimal bounds. Because [2] is not available to the general public at the present moment when this thesis is being written, in many instances when a simple citation might be sufficient, we will include also the proof which is contained in [2]. *However, we will try to emphasise when such thing happens.*

3.1 Introduction

Let $\Lambda \subset \mathbb{Z}^d$ and real-valued height variables

$$x \in \Lambda \mapsto \varphi(x) \in \mathbb{R}.$$

We will consider the Hamiltonians of the form

$$H_\Lambda(\varphi) = \sum_{x\in\Lambda} \sum_{i=1}^{d} W(x, \nabla_i \varphi),$$

where $W : \mathbb{Z}^d \times \mathbb{R} \to \mathbb{R}$ is a perturbation of a quadratic function, i.e.,

$$W(x, \eta) = \frac{1}{2} a(x)\eta^2 + V(x, \eta), \qquad \text{with some perturbation } V : \mathbb{R} \to \mathbb{R}.$$

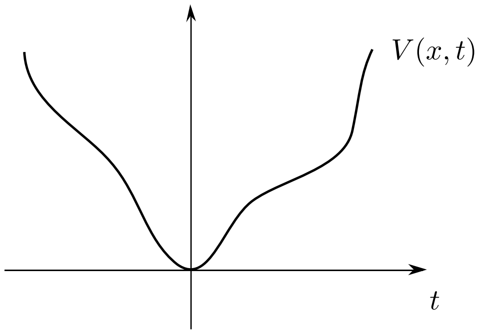


Figure 3.1: The graph of a typical function W.

The Gibbs distribution for a given boundary condition $\Psi \in \mathbb{R}^{\partial\Lambda}$, where

$$\partial\Lambda = \{z \in \mathbb{Z}^d \colon |z - x| = 1 \text{ for some } x \in \Lambda\},$$

at inverse temperature $\beta > 0$ is given by

$$\mu^{\Psi}_{\Lambda,\beta}(\,\mathrm{d}\varphi) = \frac{1}{Z_\Lambda(\beta,\Psi)} \exp\Big(-\beta H_\Lambda(\varphi)\Big) \prod_{x\in\Lambda} \mathrm{d}\varphi(x) \prod_{x\in\partial\Lambda} \delta_{\Psi(x)}(\,\mathrm{d}\varphi(x)),$$

where the normalisation constant $Z_\Lambda(\beta,\Psi)$ is the integral of the density and is called the partition function. As a direct consequence of the theory developed in Chapter 1, one is interested in linear boundary condition,

$$\Psi_u(x) = \langle x, \boldsymbol{u}\rangle, \quad \text{for some tilt } \boldsymbol{u} \in \mathbb{R}^d,$$

and in the *free energy*

$$\sigma(\boldsymbol{u}) = \lim_{\Lambda\uparrow\mathbb{Z}^d} -\frac{1}{\beta|\Lambda|} \log Z_\Lambda(\beta,\Psi).$$

Whenever the target space is one-dimensional (i.e., $m = 1$), it is called surface tension due to the fact that it appears naturally in the modelling of elastics sheets.

The surface tension $\sigma(\boldsymbol{u})$ can also be seen as the price to pay to tilt a totally flat interface. The existence of the above limit follows and the relation to the Gibbs measure was treated in Chapter 1. In case of *strictly* convex potential and no spatial dependence, Funaki and Spohn show in [17] that σ is convex as a function of the tilt. The simplest strictly convex potential is the quadratic one with $V = 0$, which corresponds to a Gaussian model, also called the gradient free field or *harmonic crystal. Notice that, as seen in Chapter 1 in general the surface tension is only quasi-convex, hence global convexity is not to be expected.* Models with non-quadratic potentials W are sometimes called *anharmonic crystals*. Strict convexity of the surface tension for strictly convex W with $0 < c_1 \le W'' \le c_2 < \infty$, was proved in [14]. Under the assumption of the bounds of the second derivative of W, a large deviations principle for the rescaled profile with rate function given in terms of the integrated surface tension has been derived in [14]. Both papers [16] and [14] use explicitly the conditions on the second derivative of W in their proof. In particular they rely on the Brascamp-Lieb inequality and on the random walk representation of Helffer and Sjöstrand, which requires a strictly convex potential W.

3.2 Preliminary results

Our starting point is [2], where the authors consider a Hamiltonian

$$H_\Lambda = \sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d W(\nabla_i\varphi).$$

The Mayer functions $K_{V,\beta,\boldsymbol{u}}$ are defined by

$$K_{V,\beta,\boldsymbol{u}}(\boldsymbol{z}) := \exp\bigl\{-\beta\sum_{i=1}^d V\bigl(\frac{z_i}{\sqrt{\beta}} - u_i\bigr)\bigr\} - 1. \tag{3.1}$$

Moreover, given any $h>0$, consider the Banach space $\boldsymbol{E}$ of functions $K:\mathbb{R}^d\to\mathbb{R}$ with the norm

$$\|K\|_h := \sup_{\boldsymbol{z}\in\mathbb{R}^d}\sum_{|\boldsymbol{\alpha}|\le r_0} h^{|\boldsymbol{\alpha}|}\bigl|\partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}}K(\boldsymbol{z})\bigr|\mathrm{e}^{-h^{-2}|\boldsymbol{z}|^2}.$$

In the above formula, the sum is over nonnegative integer multiindices $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\mathbb{N}, i=1,\dots,d$, with $|\boldsymbol{\alpha}| = \sum_{i=1}^d\alpha_i \le r_0\in\mathbb{N}$, and $\partial^{\boldsymbol{\alpha}} = \prod_{i=1}^d\partial_i^{\alpha_i}$. Denote by $B_\delta(0)\subset\mathbb{R}^d$ the ball $B_\delta(0)=\{\boldsymbol{u}\mid|\boldsymbol{u}|<\delta\}$.

The following is the main result in [2]:

Theorem 3.2.1 (Strict convexity of the surface tension)**.** *Let V be such that there exists $\delta>0$, $\varepsilon>0$, $h>0$, and $\beta_0<\infty$ such that the map $\mathbb{R}^d\supset B_\delta(0)\ni\boldsymbol{u}\mapsto K_{V,\beta,\boldsymbol{u}}\in\boldsymbol{E}$ is C^2 and*

$$\|K_{V,\beta,\boldsymbol{u}}\|_h + \sum_{i=1}^d\Bigl\|\frac{\partial}{\partial u_i}K_{V,\beta,\boldsymbol{u}}\Bigr\|_h + \sum_{i,j=1}^d\Bigl\|\frac{\partial^2}{\partial u_i\partial u_j}K_{V,\beta,\boldsymbol{u}}\Bigr\|_h \le \varepsilon$$

whenever $\boldsymbol{u}\in B_\delta(0)$ and $\beta\ge\beta_0$.

Then, the surface tension $\sigma_\beta(\boldsymbol{u}) := -\lim_{N\to\infty}\frac{1}{\beta L^{dN}}\log Z_{N,\beta}(\boldsymbol{u})$ exists and it is uniformly strictly convex in $\boldsymbol{u}$ for $\boldsymbol{u}\in B_\delta(0)$ and any $\beta\ge\beta_0$.

3.3 Hypothesis and Main Results

We will consider a potential $V:\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$, where V is sufficiently smooth. The Hamiltonians we will consider, are defined by

$$H_N(\varphi) := \sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d W(x/L^N,\nabla_i\varphi).$$

Whenever it is clear from the context we will drop the N and write H instead of H_N.

In a similar fashion as in (3.1), we denote

$$K_{V,\beta,u}(x,z) = \exp\Bigl(-\beta\sum_{i=1}^d U(x,\frac{z_i}{\beta},u_i)\Bigr) - 1,$$

where

$$U(x,s,t) = V(x,s-t) - V(x,-t) - s\,\partial_2 V(x,-t).$$

Let us recall the definition of the space $\mathscr{E} \subset \{\boldsymbol{q} : \mathbb{T}_N^d \to \mathcal{L}_{\text{sym}}(\mathbb{R}^{m\times d})\}$, such that there exists a constant $c_0, c_1 \geq 0$ such that for every $x \in \mathbb{T}_N^d$ and every F it holds

$$c_0|F|^2 \leq \langle \boldsymbol{q}(x)F, F\rangle \leq c_1|F|^2.$$

The above space is endowed with the distance induced by the norm

$$\|\boldsymbol{q}\|_{\mathscr{E}} = \sup_{x\in\mathbb{T}^d, j\leq d, |\beta|\leq 3} \|L^{|\beta|N}\nabla_i^\beta \boldsymbol{q}\|_{\mathbb{R}^{d\times d}},$$

where β is a multiindex.

Given $\boldsymbol{q} \in \mathscr{E}$, let us define

$$\mathcal{E}_{N,u}^{\boldsymbol{q}}(\varphi) := \frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{i,j=1}^d \boldsymbol{q}_{i,j}(x)(\nabla_i\varphi(x) - u_i)(\nabla_j\varphi(x) - u_j)$$

and

$$\mathcal{E}_N^{\boldsymbol{q}}(\varphi) := \frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{i,j=1}^d \boldsymbol{q}_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x),$$

where $\boldsymbol{u} \in \mathbb{R}^d$.

It is natural to consider the Banach space $\boldsymbol{E}$ which consists of functions $K : \mathbb{R}^d \to \mathbb{R}$ and the norm is defined by

$$\|K\|_h = \sup_{z\in\mathbb{R}^d}\sum_{|\boldsymbol{\beta}|\leq 2}\sum_{|\boldsymbol{\alpha}|\leq r_0} h^{|\boldsymbol{\alpha}|}\left|\partial_{\boldsymbol{z}}^{\boldsymbol{\alpha}}\partial_x^{\boldsymbol{\beta}} K(\boldsymbol{x},\boldsymbol{z})\right|\mathrm{e}^{-h^{-2}|\boldsymbol{z}|^2}$$

Here, the sum is over non-negative integer multiindices $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}$ where $i = 1, \ldots, d$.

With the above notations we can prove the analogous of Theorem 3.2.1, namely

Theorem 3.3.1. *Let $\delta > 0$, $\varepsilon > 0$, $h > 0$, and $\beta_0 < \infty$ such that the map $\mathbb{R}^d \supset B_\delta(0) \ni \boldsymbol{u} \mapsto K_{V,\beta,\boldsymbol{u}} \in \boldsymbol{E}$ is C^2 and*

$$\|K_{V,\beta,\boldsymbol{u}}\|_h + \sum_{i=1}^d\left\|\frac{\partial}{\partial u_i}K_{V,\beta,\boldsymbol{u}}\right\|_h + \sum_{i,j=1}^d\left\|\frac{\partial^2}{\partial u_i\partial u_j}K_{V,\beta,\boldsymbol{u}}\right\|_h \leq \varepsilon$$

whenever $\boldsymbol{u} \in B_\delta(0)$ and $\beta \geq \beta_0$.

Then the surface tension $\sigma_\beta(\boldsymbol{u}) := -\lim_{N\to\infty}\frac{1}{\beta L^{dN}}\log Z_{N,\beta}(\boldsymbol{u})$ exists and it is uniformly strictly convex in $\boldsymbol{u}$ for $\boldsymbol{u} \in B_\delta(0)$ and any $\beta \geq \beta_0$.

3.4 Outline of the Proof and Extension to Bonds

The strategy of the proof is based on [2] and uses the RG technique of Brydges *et al.*.

Our definitions deviate from [2] by enlarging the space of gradients to the functions on space of bonds(the precise definitions follow in the subsequent paragraph). This is done in order to keep track of the space dependence.

Each bond $b = (x, y)$ is directed from y to x. We also write $x_b = x$ and $y_b = y$ and we define $-b := (y, x)$. We say that $b \sim \tilde{b}$, if $x_b = x_{\tilde{b}}$. Moreover, we define the translation with respect to $e \in \mathbb{Z}^d$ as $\tau_e b := (x+e, y+e)$, where $b = (x, y)$. Note that each indirected bond appears twice in $(\mathbb{T}_N^d)^\star$. A sequence $C = \{b_1, \ldots, b_n\}$ is called a chain, if $y_{b_i} = x_{b_{i+1}}$ and in a similar way it is called closed if $y_{b_n} = x_{b_1}$. A plaquette is a closed loop consisting of four points such that $\{x_{b_i}\}$ consists of four different points. A field η is said to satisfy the plaquette condition if $\eta(-b) = -\eta(b)$ and $\sum_{\mathcal{P}} \eta(b) = 0$ for every plaquette $\mathcal{P}$. A particular example of a field satisfying the plaquette condition is the gradient field. In our setting, the plaquette condition characterizes being a gradient. Namely, it is not difficult to see that for the particular type of lattice we are considering, if a field η satisfies the plaquette conditions then there exists $\varphi : \mathbb{T}_N^d \to \mathbb{R}$ such that $\eta_{x,i} = \nabla_i \varphi$, where for simplicity of notation we denote by $\eta_{x,i} := \eta_{(x,x+e_i)}$ and by

$$\partial_{x,i} := \partial_{\eta_{x,i}}. \tag{3.2}$$

It will be also convenient to introduce $\delta_{x,i} : \mathbb{T}_N^d \to \mathbb{R}$ defined by

$$\delta_{x,i}(b) := \begin{cases} 1 & \text{if } b = (x,i) \\ 0 & \text{otherwise.} \end{cases}$$

The space of fields over the bonds will be denoted by $\mathcal{H}$, namely $\mathcal{H} := \{\eta : (\mathbb{T}_N^d)^\star \to \mathbb{R}\}$. Let us denote by $(\mathbb{T}_N^d)^\star$ the set of all directed bonds $b = (x, y)$ such that $|x - y|_\infty \leq 1$. The Hamiltonian H can be naturally extended on $\mathcal{H}$ by using the formula, i.e.,

$$H_N(\eta) := \sum_{x \in \mathbb{T}_N^d} \sum_{i=1}^{d} |\eta_{x,i}|^2 + V(\eta_{x,i}).$$

And hence the function $\exp(-H(\varphi))$ is also extended on $\mathcal{H}$.

Let us now consider the Gaussian measure ν_β on $\boldsymbol{\mathcal{X}}_N$ corresponding to the Dirichlet form $\beta \mathcal{E}_N(\varphi)$:

$$\nu_\beta(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}^{(0)}} \exp\bigl(-\beta \mathcal{E}_N(\varphi)\bigr) \lambda_N(\mathrm{d}\varphi),$$

with

$$Z_{N,\beta}^{(0)} = \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta \mathcal{E}_N(\varphi)\bigr) \lambda_N(\mathrm{d}\varphi).$$

Because we would like to consider the above as a measure on $\mathcal{H}$, we simply extend it by saying that the measure is defined on $\mathcal{H}$ but concentrated on the space of gradients. This can be easily done by decomposing $\mathcal{H} := V \oplus W$, where V is the space of gradients and W, and then extend $\tilde{\nu} := \nu \otimes \delta_0$.

The partition function can be rewritten as

$$
\begin{aligned}
Z_{N,\beta}(\boldsymbol{u}) = Z_{N,\beta}^{(0)} \exp\big(-\tfrac{\beta}{2} L^{Nd} \sum_{i,j=1}^{d} \langle \boldsymbol{q} \rangle_{i,j} u_i u_j\big) \int_{\boldsymbol{\mathcal{X}}_N} \exp\big(-\beta \sum_{x \in \mathbb{T}_N^d} \sum_{i=1}^{d} V\big(x, \nabla_i \varphi(x) - u_i\big) \\
+ \sum_{x \in \mathbb{T}_N^d} \sum_{i=1}^{d} q_{i,j} \nabla_i \varphi \boldsymbol{u}_i\big) \nu_\beta(\,\mathrm{d}\varphi)
\end{aligned}
\tag{3.3}
$$

where the last equation was obtained by rescaling the field φ by $\frac{1}{\sqrt{\beta}}$. Denoting by

$$
\langle \boldsymbol{q} \rangle_{i,j} = L^{-Nd} \sum_{x \in \mathbb{T}_N^d} \boldsymbol{q}_{i,j}(x)
$$

and by $\nu(\,\mathrm{d}\varphi) = \nu_{\beta=1}(\,\mathrm{d}\varphi)$ and $Z_N^{(0)} = Z_{N,\beta=1}^{(0)}$.

The term

$$
\sum_{x \in \mathbb{T}^d} \sum_{i=1}^{d} \boldsymbol{q}_{i,j}(x) \varphi_i(x) u_i,
$$

in (3.3) is harmless and does not change the above limit. Indeed, with simple computations one has that

$$
\sum_{i,j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi u_j = \sum_{i,j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \nabla_i^* \boldsymbol{q}_{i,j}(x) u_j \varphi(x).
$$

Because of the hypothesis on $\boldsymbol{q}$, we have that for every $x \in \mathbb{T}_N^d$ and $i,j \in \{1,\dots,d\}$, it holds $|\nabla_i^* q_{i,j}(x)| \leq \varepsilon_0 L^{-N}$. Thus, it is immediate to notice that

$$
\Big| \sum_{i,j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \boldsymbol{q}_{i,j} \nabla_i \varphi(x) u_j \Big| \lesssim \varepsilon_0 |u| L^{Nd} \left(L^{-Nd} \sum_{x \in \mathbb{T}_N^d} |\varphi(x)|^p \right)^{1/p}. \tag{3.4}
$$

In a similar way as in Chapter 1, when the boundary conditions are linear one can restrict by integrating over the factions which are close to the linear function in L^p(notice that for our case $p = 2$), namely all the functions φ such that the distance in L^2 is from the linear function which defines the boundary is less than every fixed κ. Thus, one can assume without loss of generality that

$$
\left(L^{-N(d+2)} \sum_{x \in \mathbb{T}_N^d} |\varphi(x)|^2 \right)^{1/2} \leq \kappa,
$$

and thus, the limit in the formula of the surface tension does not change.

However, because in Chapter 1 only the Dirichlet boundary conditions are considered and we have periodic boundary condition, the above claims need to be proven. In the following paragraph and Proposition, we fix this issue.

In the same spirit of Chapter 1, let us denote by

$$\begin{aligned}\sigma_N(\ell,\kappa) &:= -\frac{1}{L^{Nd}}\log\int_{\mathcal{V}(\ell,\kappa)}\exp\big(-H_N(\varphi)\big)\,\mathrm{d}\varphi\\ \sigma(\ell,\kappa) &:= -\lim_{N\uparrow\infty}\frac{1}{L^{Nd}}\log\int_{\mathcal{V}(\ell,\kappa)}\exp\big(-H_N(\varphi)\big)\,\mathrm{d}\varphi,\end{aligned}$$

where

$$\tilde{\mathcal{V}}_N(\ell,\kappa) := \{\varphi:\mathbb{T}_N^d\to\mathbb{R}\ :\ \sum_{x\in\mathbb{T}_N^d}|L^{-N}\varphi|^2\le\kappa L^{Nd}\}$$

Moreover, let us denote by $\tilde{\sigma}_N(\ell,\kappa)$ the same as before but instead of imposing periodic boundary conditions we impose linear boundary conditions. Namely fix $Q := [1, L^N]^d$ and denote by

$$\begin{aligned}\tilde{\sigma}_N(\ell,\kappa) &:= -\frac{1}{L^{Nd}}\log\int_{\tilde{\mathcal{V}}(\ell,\kappa)}\exp\big(-H_N(\varphi)\big)\,\mathrm{d}\varphi\\ \tilde{\sigma}(\ell,\kappa) &:= -\lim_{N\uparrow\infty}\frac{1}{L^{Nd}}\log\int_{\tilde{\mathcal{V}}(\ell,\kappa)}\exp\big(-H_N(\varphi)\big)\,\mathrm{d}\varphi,\end{aligned}$$

where

$$\mathcal{V}_N(\ell,\kappa) := \{\varphi:\mathbb{Z}^d\to\mathbb{R}\ :\ \sum_{x\in Q}|L^{-N}\varphi|^p\le\kappa L^{Nd}\text{ and }\varphi(x)=0\text{ for all }x\in\mathbb{Z}^d\setminus\bar{Q}\}$$

Because the zero boundary conditions are more restrictive than periodic boundary conditions, it is immediate to see that $\sigma\le\tilde{\sigma}$.

In the following proposition we prove that the two are equivalent.

Proposition 3.4.1. *For any linear boundary condition ℓ and every $\kappa>0$, it holds*

$$\tilde{\sigma}(\ell,\kappa)=\sigma(\ell,\kappa)=\sigma(\ell).$$

Proof. As we already observed in the comments above the statement proposition, we only need to prove that $\sigma\ge\tilde{\sigma}$.

Let $\{e_1,\dots,e_d\}$ be the coordinate directions. Denote by

$$Y_k := \{y\in\mathbb{T}_N^d:\ y\cdot e_1=k\}.$$

It is immediate to notice that

$$\mathbb{T}_N^d=\bigcup_{k=0}^{L^N-1}Y_k.$$

Hence, we have that

$$\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^{d} W(x,\nabla_i\varphi(x)) = \sum_{k=1}^{L^N-1}\sum_{y\in Y_k}\sum_{i=1}^{d} W(y,\nabla_i\varphi(y)),$$

thus for every fixed φ, there exists k such that

$$\sum_{y\in Y_k}\sum_{i=1}^{d} W(y,\nabla_i\varphi(y)) \leq \frac{1}{L^N}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^{d} W(x,\nabla_i\varphi(x)). \tag{3.5}$$

On the other side by using a version of Lemma 1.2.12, it is immediate to notice that there exists K large enough such that one can restrict oneself to the set

$$\mathcal{M}_K := \{\varphi : \sum_{x\in\mathbb{T}_N^d} W(x,\nabla\varphi(x)) \leq KL^{Nd}\}. \tag{3.6}$$

Combining (3.5) and (3.6), one has that there exists k_0 such that

$$\sum_{y\in Y_{k_0}}\sum_{i=1}^{d} W(y,\nabla_i\varphi(x)) \leq KL^{Nd-N}. \tag{3.7}$$

Let us denote by $\mathcal{N}_k$ the set of functions for which equation (3.7) holds for the first time at k. Namely for every $k' < k$, equation (3.7) does not hold.

We denote by $\bar{H}_{k,N}$ the Hamiltonian induced by unfolding the torus $\mathbb{T}_N^d$ at the hyperplane Y_k and extending outside with Dirichlet boundary conditions. Namely,

$$\bar{H}_{k,N}(\varphi) := \sum_{y\in Y_k} W(y,\varphi(y)) + \sum_{k'\neq k}\sum_{Y_k}\sum_{i=1}^{d} W(y,\nabla_i\varphi(x)).$$

Using the above observations, one has that for every $\varphi \in \mathcal{N}_k$

$$\bar{H}_{k,N}(\varphi) \geq H_N(\varphi) - KL^{Nd}/L^N.$$

Hence,

$$\sigma_N(\ell,\kappa) \leq -L^{-Nd}\log\left[\sum_k\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}\big)\,\mathrm{d}\varphi\right] + K/L^N.$$

Moreover notice that

$$-L^{-Nd}\log\left[\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}(\varphi)\big)\,\mathrm{d}\varphi\right]$$

corresponds to putting Dirichlet boundary conditions on the faces of the cube orthogonal to e_1 and periodic boundary condition on the other faces.

It is immediate to notice that

$$\sigma(\ell,\kappa) := -L^{-Nd}\log\Big[\sum_k \int_{\mathcal{N}_k} \exp\big(-\bar{H}_{k,N}\big)\,\mathrm{d}\varphi\Big].$$

With simple reasoning one has that

$$\begin{aligned}-L^{-Nd}\max_k \log\Big[\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}\big)\,\mathrm{d}\varphi\Big] &\le -L^{-Nd}\log\Big[\sum_k\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}\big)\,\mathrm{d}\varphi\Big]\\ &\le -L^{-Nd}\max_k\log\Big[L^N\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}\big)\,\mathrm{d}\varphi\Big],\end{aligned}$$

hence by passing to the limit for $N\uparrow\infty$ one has that

$$\sigma(\ell,\kappa) = \lim_{N\to\infty} -L^{-Nd}\max_k \log\int_{\mathcal{N}_k}\exp\big(-\bar{H}_{k,N}(\varphi)\big).$$

Finally we by passing to the limit in N and noticing that each of the terms in the above(after the max) is equal to $\tilde{\sigma}^1$, which corresponds to imposing Dirichlet boundary condition only on one of the axes. To conclude the proof one has to repeat the argument for each of the faces via an induction argument.

□

Let us now continue with the outline of the proof.

Define

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) := \int_{\boldsymbol{\mathcal{X}}_N}\exp\big(-\beta\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d V\big(x,\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x)-u_i\big)\big)\nu(\mathrm{d}\varphi). \tag{3.8}$$

One obtains that for the finite volume *surface tension*,

$$\sigma_{N,\beta}(\boldsymbol{u}) = -\frac{1}{\beta L^{Nd}}\log Z_{N,\beta}(\boldsymbol{u}) = -\frac{1}{\beta L^{Nd}}\log Z_N^{(0)} + \tfrac{1}{2}|\boldsymbol{u}|^2 - \frac{1}{\beta L^{Nd}}\log\mathcal{Z}_{N,\beta}(\boldsymbol{u}). \tag{3.9}$$

As in [2], to prove Theorem 3.3.1 we need to show that the surface tension $\sigma_{N,\beta}(\boldsymbol{u})$ in (3.9) is strictly convex in $\boldsymbol{u}$ uniformly in $N\in\mathbb{N}$.

Glancing at the formula (3.9), it is immediate to notice that in order to prove the strict convexity of the surface tension, one only needs to prove that the derivatives of the third term are sufficiently small.

In statistical mechanics, one of the ways of dealing with the perturbation and evaluating $\log\mathcal{Z}_{N,\beta}(\boldsymbol{u})$ and its derivatives, is to use some version of cluster expansion. Namely, one expands the integrand as

$$\prod_{x\in\mathbb{T}_N^d}\Big(1+\exp\{-\beta\sum_{i=1}^d V\big(x,\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x)-u_i\big)\}-1\Big)$$

in (3.8) and introduces, for any subset $X\subset\mathbb{T}_N^d$, the function

$$K(X,\varphi) = \prod_{x\in X}\big(\exp\{-\beta\sum_{i=1}^d V\big(x,\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x)-u_i\big)\}-1\big), \tag{3.10}$$

which allows to rewrite (3.8) as

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) = \int_{\boldsymbol{\mathcal{X}}_N} \sum_X K(X,\varphi)\nu(\,\mathrm{d}\varphi). \tag{3.11}$$

The function $K(X,\varphi)$ depends only on $\nabla\varphi(x)$ with x in the set X and its close neighborhood and for a disjoint union $X = X_1 \cup X_2$ one has $K(X,\varphi) = K(X_1,\varphi)K(X_2,\varphi)$.

We write $K(X,\varphi)$ instead of $K(X,\nabla\varphi)$ to keep the notation more compact and in line with the literature.

However, the Gaussian measure $\nu(\,\mathrm{d}\varphi)$ has a slowly decaying correlations and hence does not allow to separate the integral of $K(X,\varphi)$ into a product of integrals with the integrands $K(X_1,\varphi)$ and $K(X_2,\varphi)$, thus the classical methods used when one does a cluster expansion are not applicable.

As already mentioned the measure ν is extended on the space of functions defined on bonds[1] (see beginning of the section). Because we would like to some extent reformulate the considered quantities on the space of functions over bonds, one can rewrite the partition function as function

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) := \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta \sum_{b\in(\mathbb{T}_N^d)^\star} \tilde{V}(\eta_b)\bigr)\nu(\,\mathrm{d}\eta). \tag{3.12}$$

The functions $K(X,\varphi)$ can be rewritten in terms of bonds as

$$K(X,\varphi) = \prod_{b=(x,i):x\in X} \bigl(\exp\{-\beta \sum_{i=1}^d \tilde{V}(\eta_b)\} - 1\bigr), \tag{3.13}$$

thus

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) = \int_{\boldsymbol{\mathcal{X}}_N} \sum_X K(X,\eta)\nu(\,\mathrm{d}\eta). \tag{3.14}$$

The strategy (and the main idea of the RG) is to perform the integration in steps corresponding to increasing scales. Similarly as in [2], it is useful to introduce a parameter $\boldsymbol{q} \in \mathscr{E}$ that will be useful for fine-tuning so that the final integration will eventually yield a result with a straightforward bound.

Multiplying and dividing the integrand in (3.11) by $\exp\bigl\{\frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{i,j=1}^d \boldsymbol{q}_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x)\bigr\}$, one gets

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) = \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}} \int_{\boldsymbol{\mathcal{X}}_N} \exp\Bigl\{\frac{1}{2}\sum_{x\in\mathbb{T}_N^d}\sum_{i,j=1}^d \boldsymbol{q}_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x)\Bigr\} \sum_X K(X,\varphi)\mu(\,\mathrm{d}\varphi). \tag{3.15}$$

Here, μ is the Gaussian measure on $\boldsymbol{\mathcal{X}}_N$ with the Green function $\mathscr{C}^{(\boldsymbol{q})}$, the inverse of the operator $\mathscr{A}^{(\boldsymbol{q})} = \sum_{i,j=1}^d \nabla_i^*\bigl(\delta_{i,j} + \boldsymbol{q}_{i,j}(x)\bigr)\nabla_j$,

$$\mu(\,\mathrm{d}\varphi) = \frac{1}{Z_N^{(\boldsymbol{q})}} \exp\bigl\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\bigr\}\lambda_N(\,\mathrm{d}\varphi),$$

[1] which denoted by $\mathcal{H}$

with

$$\mathcal{E}_{\boldsymbol{q}}(\varphi) = \tfrac{1}{2}(\mathscr{A}^{(\boldsymbol{q})}\varphi, \varphi) = \tfrac{1}{2} \sum_{x \in \mathbb{T}_N^d} \sum_{i,j=1}^{d} \big(\delta_{i,j} + \boldsymbol{q}_{i,j}(x)\big) \nabla_i \varphi(x) \nabla_j \varphi(x),$$

and

$$Z_N^{(\boldsymbol{q})} = \int_{\mathcal{X}_N} \exp\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\} \lambda_N(\mathrm{d}\varphi) = \int_{\mathcal{X}_N} \exp\Big\{-\tfrac{1}{2} \sum_{x \in \mathbb{T}_N^d} \sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi(x) \nabla_j \varphi(x)\Big\} \nu(\mathrm{d}\varphi).$$

One of the main ingredients is the use of a version of the Finite Range Decomposition (Theorem 2.3.1) which allows (under suitable assumptions on the smallness of $\boldsymbol{q}$) to decompose the Gaussian measure μ into a convolution $\mu(\mathrm{d}\varphi) = \mu_1 * \cdots * \mu_{N+1}(\mathrm{d}\varphi)$, where $\mu_1, \ldots, \mu_{N+1}$ are Gaussian measures with a particular finite range property. Namely, the covariances of $\mathcal{C}_k^{(\boldsymbol{q})}(x, y)$ of the measures μ_k, $k = 1, \ldots, N+1$ vanish for $|y - x| \geq \frac{1}{2} L^k$ with a fixed parameter L with an additional bound on their derivatives with respect to $\boldsymbol{q}$ of the order $L^{-(k-1)(d-1)}$.

The integral in (3.15) can be symbolically written as

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H^{(\boldsymbol{q})}} \circ K^{(\boldsymbol{q})})(\varphi) \mu(\mathrm{d}\varphi), \tag{3.16}$$

where

$$H^{(\boldsymbol{q})} = -\tfrac{1}{2} \sum_{x \in \mathbb{T}_N^d} \sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi(x) \nabla_j \varphi(x),$$

the function $K^{(\boldsymbol{q})}$ is defined by

$$K^{(\boldsymbol{q})}(X, \varphi) := \exp\Big\{\tfrac{1}{2} \sum_{x \in X} \sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi(x) \nabla_j \varphi(x)\Big\} K(X, \varphi),$$

and $\circ$ is the *circle product* notation for the sum over subsets $X \subset \mathbb{T}_N^d$,

$$(\mathrm{e}^{-H^{(\boldsymbol{q})}} \circ K^{(\boldsymbol{q})})(\varphi) = \sum_{X \subset \mathbb{T}_N^d} \exp\Big\{\tfrac{1}{2} \sum_{x \in \mathbb{T}_N^d \setminus X} \sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi(x) \nabla_j \varphi(x)\Big\} K^{(\boldsymbol{q})}(X, \varphi).$$

The typical strategy of in the RG theory is to replace μ in (3.16) by the convolution $\mu_1 * \cdots * \mu_{N+1}(\mathrm{d}\varphi)$, and to proceed by integrating first over μ_1. A fundamental observation is that the form of the integral is conserved.

Namely, starting from $H_0 = H^{(\boldsymbol{q})}$ and $K_0 = K^{(\boldsymbol{q})}$, one defines H_1 and K_1 so that

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_0} \circ K_0)(\varphi + \xi)\, \mathrm{d}\mu_1(\xi) = (\mathrm{e}^{-H_1} \circ K_1)(\varphi).$$

Here, the function $K_1(X, \varphi)$ is defined (non-vanishing) only for sets X consisting of L^d-blocks and H_1 is again a quadratic form like H_0 but with modified coefficients $\boldsymbol{q}_{i,j}(x)$ and additional linear terms. Recursively, one can define a sequence of pairs

$(H_1, K_1), (H_2, K_2), \ldots, (H_N, K_N)$ with each H_k a quadratic form in $\nabla\varphi$ and $K_k(X, \varphi)$ defined for sets X consisting of L^{kd}-blocks so that

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_k} \circ K_k)(\varphi + \xi)\, \mathrm{d}\mu_{k+1}(\xi) = (\mathrm{e}^{-H_{k+1}} \circ K_{k+1})(\varphi). \tag{3.17}$$

The aim is to define of consecutive pairs of functions H_k, K_k so that not only (3.17) is valid, but also that the form of the quadratic function H_k is conserved, the coarse-grained dependence of K_k on blocks L^{dk} is maintained, and, most importantly, the size of the perturbation K_k in a conveniently chosen norm decreases (variable K_k is *irrelevant*).

Let $F : (\mathbb{T}_N^d)^\star \to \mathbb{R}$. Notice that the measures $\{\mu_i\}$ are extended on $\mathcal{H}$. Hence in a similar way one can perform the integration of F with respect to $\{\mu_i\}$ by the formula

$$\int_{\boldsymbol{\mathcal{X}}_N} F(\eta + \nabla\xi)\, \mathrm{d}\mu_i(\xi)$$

and show by induction that by performing these operations one obtains always functions defined on $\mathcal{H}$.

Thus it is not difficult see that if an expression as the one in (3.17) holds, then an similar one defined on $\mathcal{H}$ holds.

Using now sequentially the formula (3.17), we eventually get

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_0} \circ K_0)(\varphi)\mu(\,\mathrm{d}\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_N} \circ K_N)(\varphi)\mu_{N+1}(\,\mathrm{d}\varphi),$$

thus

$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) = \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}} \int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_N} \circ K_N)(\varphi)\mu_{N+1}(\,\mathrm{d}\varphi).$$

Because that all measures $\mu_1, \ldots, \mu_{N+1}$ and the map $\boldsymbol{T}_k$ itself will be shown to depend smoothly on the initial matrix $\boldsymbol{q}$, thus one can choose the value $\boldsymbol{q} = \boldsymbol{q}_0$ in such a way that $H_N = 0$. Given that the function $K_N(X, \cdot)$ is defined only for $X = \Lambda_N$ or $X = \varnothing$, one has that

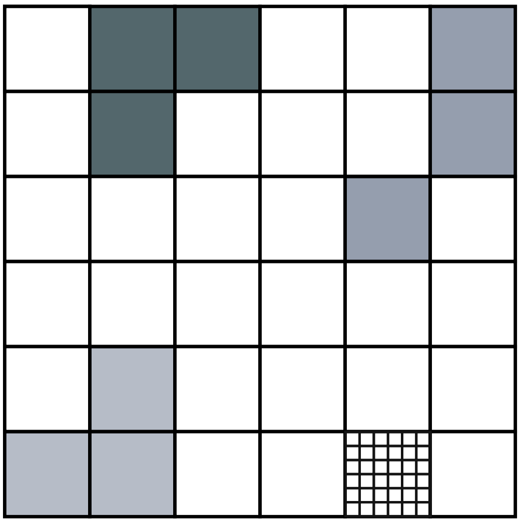
$$\mathcal{Z}_{N,\beta}(\boldsymbol{u}) = \frac{Z_N^{(\boldsymbol{q}_0)}}{Z_N^{(0)}} \int_{\boldsymbol{\mathcal{X}}_N} \big(1 + K_N(\Lambda_N, \varphi)\big)\mu_{N+1}(\,\mathrm{d}\varphi).$$

As in [2], one can compute $Z_N^{(\boldsymbol{q}_0)}$ explicitly by Gaussian calculus and its dependence on $\boldsymbol{u}$, and show that it depends smoothly $\boldsymbol{u}$. Moreover, the integral term, as well as its derivatives with respect to $\boldsymbol{u}$, can easily be bounded as a consequence of the iterative bound on K_N.

3.5 Definitions

3.5.1 Polymers

We now introduce some standard definitions and notations used in the RG theory.



For $k = 0, 1, 2, \ldots, N$, one paves the torus Λ_N by $L^{(N-k)d}$ disjoint cubes of side length L^k. These cubes are all translates (L is odd) of $\{x \in \Lambda_N \colon |x|_\infty \le \frac{1}{2}(L^k - 1)\}$ by vectors in $L^k\mathbb{Z}^d$. Such cubes are called *k-blocks* or blocks of k-th generation, and use $\mathcal{B}_k$ to denote the set of all k-blocks,

$$\mathcal{B}_k = \mathcal{B}_k(\Lambda_N) = \{B \colon B \text{ is a } k\text{-block}\}, \quad k = 0, 1, \ldots, N.$$

Single vertices of the lattice are 0-blocks, the starting generation for the RG transforms, $\mathcal{B}_0 = \Lambda_N$. The only N-block is the torus Λ_N itself, $\mathcal{B}_N = \{\Lambda_N\}$.

A union of k-blocks is called a *k-polymer*. We denote by $\mathcal{P}_k = \mathcal{P}_k(\Lambda_N)$ the set of all k-polymers in Λ_N.

As N is fixed through the major this chapter, we often skip Λ_N from the notation as indicated above.

Any subset $X \subset \mathbb{T}_N^d$ is said to be *connected* if for any $x, y \in X$ there exist a path $x_1 = x, x_2, \ldots, x_n = y$ such that $|x_{i+1} - x_i|_\infty = 1$, $i = 1, \ldots, n-1$. We use $\mathcal{C}(X)$ to denote the *set of connected components* of X. Two connected sets $X, Y \subset \Lambda_N$ are said to be strictly disjoint if their union is not connected. Notice that for any strictly disjoint $X, Y \in \mathcal{P}_k$, we have $\operatorname{dist}(X, Y) > L^k$.

We use $\mathcal{P}_k^{\mathrm{c}}$ to denote the set of all connected k-polymers. Let $X \in \mathcal{P}_k$. The set of all k-blocks in X will be denoted by $\mathcal{B}_k(X)$, and the number of the k-blocks of X will be denoted by $|X|_k = |\mathcal{B}_k(X)|$

The *closure* $\overline{X}$ of a polymer $X \in \mathcal{P}_k$ is the smallest polymer $Y \in \mathcal{P}_{k+1}$ of the next generation such that $X \subset Y$.

A polymer $X \in \mathcal{P}_k^{\mathrm{c}}$ is called *small* if $|X|_k \le 2^d$ and we denote $\mathcal{S}_k = \{X \in \mathcal{P}_k^{\mathrm{c}} \colon |X|_k \le 2^d\}$.

Let $B \in \mathcal{B}_k$. We define its *small set neighbourhood* B^* to be the cube of the side $(2^{d+1} - 1)L^k$ centered at B. The small set neighbourhood B^* can be equivalently defined as the smallest cube for which $B \subset Y$ and $Y \in \mathcal{S}_k$ implies $Y \subset B^*$. We will use X^* to denote its *small set neighbourhood*, $X^* = \cup\{B^* \colon B \in \mathcal{B}_k(X)\}$.

3.5.2 Polymer Functionals and Translation

Let us now introduce the space $M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ of all maps $F : \mathcal{P}_k \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ such that for all $X \in \mathcal{P}_k$ the map $F(X, \varphi)$ depends only on values of φ on X^*. Namely, for every $\varphi, \psi \in \boldsymbol{\mathcal{X}}$, one has that $\varphi\big|_{X^*} = \psi\big|_{X^*} \implies F(X, \varphi) = F(X, \psi)$ where $\varphi\big|_{X^*}$ denotes the restriction of φ to X^*.

The sets $M(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$ and $M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ are defined in an analogous way. We also consider the set $M^*(\mathcal{B}_k, \boldsymbol{\mathcal{X}}) \supset M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ of the maps $F \colon \mathcal{B}_k \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ with $F(B, \varphi)$ depending only on values of φ on the extended set $(B^*)^*$. For functions from $M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, one introduces the *circle product*,

$$F_1, F_2 \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}}),\ (F_1 \circ F_2)(X, \varphi) = \sum_{Y \subset X} F_1(Y, \varphi) F_2(X \setminus Y, \varphi).$$

Notice, that the product is defined pointwise in the variable φ. We often skip it and write $(F_1 \circ F_2)(X) = \sum_{Y \subset X} F_1(Y) F_2(X \setminus Y)$ instead. Observe that the circle product is commutative and distributive.

For $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ and $X \in \mathcal{P}_k$, we define

$$F^X(\varphi) := \prod_{B \in \mathcal{B}_k(X)} F(B, \varphi).$$

Extending any $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ to $M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ by taking

$$F(X, \varphi) := F^X(\varphi),$$

one gets

$$(F_1 + F_2)^X = \sum_{Y \subset X} F_1^Y F_2^{X \setminus Y} = (F_1 \circ F_2)(X).$$

For every $x \in \mathbb{T}_N^d$ and every $\varphi \in \boldsymbol{\mathcal{X}}$, we define the translation τ_x as $\tau_x \varphi(\cdot) := \varphi(\cdot - x)$. Given $F \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, we define the translated functional as $(\tau_x F)(B, \varphi) := F(B, \varphi, \tau_x \varphi)$.

Similarly as in [2], we introduce the space of ideal Hamiltonians $M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}}) \subset M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ defined as the family of all quadratic functions of the form

$$H(B, \varphi) = \lambda |B| + \ell(\varphi) + Q(\varphi),$$

where

$$\ell(\varphi) = \sum_{x \in B} \Big[\sum_{i=1}^{d} a_i(x) \nabla_i \varphi(x) + c_{i,j}(x) \nabla_i \nabla_j \varphi \Big] \tag{3.18}$$

and

$$Q(\varphi, \varphi) = \sum_{x \in B} \sum_{i,j=1}^{d} \boldsymbol{d}_{i,j}(x)\, \nabla_i \varphi(x)\, \nabla_j \varphi(x)$$

with coefficients $\lambda \in \mathbb{R}, a \in \mathbb{R}^d, \boldsymbol{c} \in \mathbb{R}^{d \times d}$ and $\boldsymbol{d} \in \mathcal{E}$. As we will see in the sequel, this space will consist of the space of the relevant parameters for the RG.

The space $M_0(\mathcal{B}_k, \mathcal{X})$ will be endowed with the norm

$$\|H\|_{k,0,B} = L^{kd} \sup_{x \in B} |\lambda(x)| + L^{kd/2} h \sum_{|\beta| \le 3} \sup_{x \in B} \sum_{i=i}^{d} |a_i(x)| + \sup_{x \in B} h^2 |L^{N|\beta|}| \sum_{i,j=1}^{N} |\nabla^\beta d_{i,j}(x)|$$

3.5.3 Norms

In order to carefully keep track of the contribution in the integration step when passing from one scale to the other, one needs to define some appropriate seminorms. If one restricts oneself to the subspace of $\mathcal{H}$ which is composed of all η which are gradient of some function φ, than the seminorms that will be considered in the sequel will be norms. For this reason and because of the nomenclature in the literature, we will refer to them as norms with a slight abuse of notation. If one restricts oneself to the subspace of gradients, the following norms generalize the ones employed in [2].

As in [2], we now introduce the norms $\|\cdot\|_{k,r}$ and $\|\cdot\|_{k+1,r}$ on $M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ and $M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ (with $r=1,\dots,r_0$, where r_0 is a fixed integer (to be chosen later) and a norm $\|\cdot\|_{k,0}$ on $M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$. For every $k\in\{0,1,\dots,N\}$ and $X\in\mathcal{P}_k$, one introduces the semi-norms $|\cdot|_{k,X}$ and $|\cdot|_{k+1,X}$ on $\boldsymbol{\mathcal{X}}$. Given $\varphi\in\boldsymbol{\mathcal{X}}$, one defines

$$|\varphi|_{k,X}:=\max_{1\le s\le 3}\sup_{x\in X^*}\frac{1}{h}L^{k(\frac{d-2}{2}+s)}\big|\nabla^s\varphi(x)\big|$$

and

$$|\varphi|_{k+1,X}=\max_{1\le s\le 3}\sup_{x\in X^*}\frac{1}{h}L^{(k+1)(\frac{d-2}{2}+s)}\big|\nabla^s\varphi(x)\big|,$$

where

$$|\nabla^s\varphi(x)|^2=\sum_{|\boldsymbol{\alpha}|=s}|\nabla^{\boldsymbol{\alpha}}\varphi(x)|^2.$$

Let S be an s-linear map on $(\mathbb{T}_N^d)^*$. Then it can be expressed as

$$S(\eta,\dots,\eta)=\sum_{b_1,\dots,b_n}c_{b_1,\dots,b_n}\prod_{i=1}^n\eta_{b_i}.$$

Let e_i be an element of the canonical basis. We define the translation $\tau_e C_{b_1,\dots,b_n}:=C_{\tau_e b_1,\dots,\tau_e b_n}$. Thus, for every $f:(\mathbb{T}_N^d)^\star\times\cdots\times(T_N^d)^\star\to\mathbb{R}$ we define

$$\nabla_i f(b_1,\dots,b_n):=\tau_{e_i}f(b_1,\dots,b_n)-f(b_1,\dots,b_n).$$

Moreover, for any s-linear function S on $\boldsymbol{\mathcal{X}}\times\cdots\times\boldsymbol{\mathcal{X}}$, we define

$$\begin{aligned}|S|^{j,X}&:=\sup_{|\dot\varphi|_{j,X}\le 1}\big|S(\dot\varphi,\dots,\dot\varphi)\big|,\qquad j=k,k+1,\\ |S|^{x,i,j,X}&=\sup_{|\dot\varphi|_{j,X}\le 1}\big|S(\eta_{x,i},\dot\varphi,\dots,\dot\varphi)/\eta_{x,i}\big|,\qquad j=k,k+1,\end{aligned}\tag{3.19}$$

and, for any $F\in C^r(\boldsymbol{\mathcal{X}})$, also

$$\begin{aligned}|F(\varphi)|^{j,X,r}=\sup_{x\in\mathbb{T}_N^d}\sup_{1\le i\le d}\sum_{s=1}^r\sum_{1\le|\beta|\le 3}L^{|\beta|N}\frac{1}{s!|\beta|!}|\nabla^\beta D^sF(\varphi)|^{x,i,j,X}\\ +\sup_{x\in\mathbb{T}_N^d}\sup_{1\le i\le d}\sum_{s=0}^r\frac{1}{s!}|D^sF(\varphi)|^{j,X},\end{aligned}\tag{3.20}$$

where whenever $s=0$, we mean

$$D^0F(\varphi)=F(\varphi).$$

Remark 3.5.1. *Notice that one can find examples of quadratic forms Q defined on $\mathcal{H}$ such that $Q = 0$ when one restricts oneself to the subspace of fields that are gradients and such that $\Pi(Q) \neq 0$ on the subspace of fields that are gradients. For this reason one needs to control also parts of the norms in the full space. This is why, differently from [2], we have in the definition of the norms extra terms like $|S|^{x,i,j,X}$.*

Let us introduce the weighted *strong* norm $\lvert\lvert\lvert F(X)\rvert\rvert\rvert_{k,X}$ as well as weighted *weak* norm $\|F(X)\|_{k,X,r}$, $r = 1, \dots, r_0$. The strong weight functions are defined by

$$W_k^X(\varphi) := \exp\Big\{\sum_{x\in X} G_{k,x}(\varphi)\Big\},$$

where

$$G_{k,x}(\varphi) = \frac{1}{h^2}\big(|\nabla\varphi(x)|^2 + L^{2k}\left|\nabla^2\varphi(x)\right|^2 + L^{4k}|\nabla^3\varphi(x)|^2\big).$$

We define the weighted strong norm by

$$\lvert\lvert\lvert F(X)\rvert\rvert\rvert_{k,X} := \sup_{\varphi} |F(X,\varphi)|^{k,X,r_0} W_k^{-X}(\varphi).$$

Notice that in contrast with [2], given any $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ the norm $\lvert\lvert\lvert F(B)\rvert\rvert\rvert_{k,B}$ does depend on B.

Moreover, let $B_x \in \mathcal{B}_k$ be the k-block containing x and let ∂X denote the boundary, namely

$$\partial X = \{y \notin X \mid \exists z \in X \text{ such that } |y-z|_2 = 1\} \cup \{y \in X \mid \exists z \notin X \text{ such that } |y-z|_2 = 1\}.$$

The weak weight functions are defined by

$$w_k^X(\varphi) := \exp\Big\{\sum_{x\in X} \omega\big(2^d g_{k,x}(\varphi) + G_{k,x}(\varphi)\big) + L^k \sum_{x\in\partial X} G_{k,x}(\varphi)\Big\}$$

with $G_{k,x}(\varphi)$ as above and

$$g_{k,x}(\varphi) = \frac{1}{h^2}\sum_{s=2}^{4} L^{(2s-2)k} \sup_{y\in B_x^*} |\nabla^s\varphi(y)|^2.$$

The weighted weak norm is defined by

$$\|F(X)\|_{k,X,r} := \sup_{\varphi} |F(X,\varphi)|^{k,X,r}\, w_k^{-X}(\varphi),\ \ r = 1, \dots, r_0.$$

One also introduces the norm $\|\cdot\|_{k:k+1,X,r}$ that can be viewed as being "halfway between" $\|\cdot\|_{k,X,r}$ and $\|\cdot\|_{k+1,U,r}$ with $U = \overline{X} \in \mathcal{P}_{k+1}$. Namely,

$$\|F(X)\|_{k:k+1,X,r} = \sup_{\varphi} |F(X,\varphi)|^{k+1,X,r}\, w_{k:k+1}^{-X}(\varphi),\ \ r = 1, \dots, r_0.$$

with

$$w_{k:k+1}^X(\varphi) = \exp\Big\{\sum_{x\in X}\big((2^d\omega - 1)g_{k:k+1,x}(\varphi) + \omega G_{k,x}(\varphi)\big) + 3L^k \sum_{x\in\partial X} G_{k,x}(\varphi)\Big\},$$

where

$$g_{k:k+1,x}(\varphi) = \frac{1}{h^2} \sum_{s=2}^{4} L^{(2s-2)(k+1)} \sup_{y \in B_x^*} |\nabla^s \varphi(y)|^2.$$

For any $r \leq r_0$, one has that

$$\|F(X)\|_{k,X,r} \leq |||F(X)|||_{k,X}.$$

Moreover is also easy to show that

$$\|F(X)\|_{k:k+1,X,r} \leq \|F(X)\|_{k,X,r}$$

whenever $\omega \geq 2^{d-1}$ (assuring that $2^d\omega(L^2-1) \geq L^2$), and, for any $U \in \mathcal{P}_{k+1} \subset \mathcal{P}_k$ and $F \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}}) \subset M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, also

$$\|F(U)\|_{k+1,U,r} \leq \|F(U)\|_{k:k+1,U,r} \leq \|F(U)\|_{k,U,r}. \tag{3.21}$$

As in [2], we introduce the weak norms

$$\|F\|_{k,r} = \sup_{X \in \mathcal{P}_k^{\mathrm{c}}} \|F(X)\|_{k,X,r} \Gamma_{k,A}(X), \;\; r = 1, \ldots, r_0,$$

where

$$\Gamma_{k,A}(X) = \begin{cases} A^{|X|} & \text{if } X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ 1 & \text{if } X \in \mathcal{S}_k. \end{cases}$$

Similarly, one also defines $\|F\|_{k:k+1,r}$. When comparing norms with different values of parameter A, we will denote explicitly $\|F\|_{k,r}^{(A)}$ and $\|F\|_{k:k+1,r}^{(A)}$. For $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$, one defines

$$\|F\|_{k,r} := \sup_{B \in \mathcal{B}_k} \|F(B)\|_{k,B,r}.$$

Let us fix k. For simplicity of notations it is convenient to write (H', K') instead of (H_{k+1}, K_{k+1}). Hence, it becomes

$$\boldsymbol{R}(\mathrm{e}^{-H} \circ K) = \mathrm{e}^{-H'} \circ K'.$$

3.5.4 Projection

Let $f : (\mathbb{T}_N^d)^* \to \mathbb{R}^m$ be a function. We denote by T_2 the Taylor expansion around zero up to the second order, namely

$$T_2 F(B, \dot{\eta}) = F(B, 0) + DF(B, 0)(\dot{\eta}) + \tfrac{1}{2} D^2 F(B, 0)(\dot{\eta}, \dot{\eta}),$$

where the functions can be represented as

$$DF(\dot{\eta}) = \sum_{b \in (T_N^d)^\star} c_b \eta_b$$

and

$$D^2F(B,0)(\dot{\eta},\dot{\eta}) = \sum_{b,b'} d_{b,b'}\eta_b\eta_{b'}.$$

Given that our functionals depend on $\nabla\varphi$, it is not difficult to see that

$$DF(B,\varphi) = \sum_{x\in\mathbb{T}_N^d} \tilde{c}_{x,i}\nabla_i\varphi(x)$$

and

$$\frac{1}{2}D^2F(B,\varphi) = \sum_{x\in\mathbb{T}_N^d} \tilde{d}_{i,j}(x,y)\nabla_i\varphi(x)\nabla_j\varphi(x).$$

Let us now define the "projection" on the space of Ideal Hamiltonians.

The elements of $d_{i,j}(x)$ in the formula for Q, will be defined by

$$d_{i,j}(x) := \sum_{y\in\mathbb{T}_N^d} \tilde{d}_{i,j}(x,y) \qquad \text{where } x \in B. \tag{3.22}$$

Notice that with this definition, we have that for every affine function φ on B^* it holds

$$\frac{1}{2}\sum_{x\in B} d_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x) = \frac{1}{2}D^2F(B,0)(\dot{\varphi},\dot{\varphi}).$$

and

$$d_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x) = \frac{1}{2}\partial_{x,i}DF(B,0)(\dot{\varphi},\dot{\varphi}),$$

where in the above equation the symbol $\partial_{x,i}$ we mean the derivative with respect to η_b and $b = (x, x+e_i)$.

The functional ℓ can be defined in the same way as in [2], namely such that for every quadratic function it holds

$$DF(B,0)(\dot{\varphi}) = \sum_{i=1}^{d} a_i\nabla_i\varphi(x) + \sum_{i,j=1}^{d} b_{i,j}\nabla_i\nabla_j\varphi(x).$$

In the following, we will need to apply the projection to functions of the form

$$\overline{F}(B,\varphi) = \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}F(X,\varphi),$$

for any $F \in M(\mathcal{S},\boldsymbol{\mathcal{X}})$. Hence, we extend the projection Π, by considering test functions $\dot{\varphi}$ on $(B^*)^*$ instead of B^*.

We define

$$H'(B',\varphi) := \sum_{B\subset B'} \Pi T_2\Big((\boldsymbol{R}H)(B,\varphi) - \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}(\boldsymbol{R}K)(X,\varphi)\Big). \tag{3.23}$$

Let us denote by $\tilde{H}(B,\varphi)$, the term in the right hand side of sum above, namely

$$\tilde{H}(B,\varphi) = \Pi T_2\Big((\boldsymbol{R}H)(B,\varphi) - \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}(\boldsymbol{R}K)(X,\varphi)\Big).$$

Denote by $\tilde{I}(B,\varphi) = \exp\big\{-\tilde{H}(B,\varphi)\big\}$ and by $\tilde{J} = 1 - \tilde{I}$. Moreover, we introduce

$$\tilde{K} := \tilde{J} \circ (I-1) \circ K. \tag{3.24}$$

For simplicity of notation, it is sometimes convenient to skip the polymer variable X and use $\tilde{K}(\varphi,\xi)$ for the mapping $\tilde{K}(\varphi,\xi) : \mathcal{P} \to \mathbb{R}$ defined by $\tilde{K}(\varphi,\xi)(X) := \tilde{K}(X,\varphi,\xi)$. Thus,

$$\tilde{K}(\varphi,\xi) = \tilde{J}(\varphi) \circ \big(I(\varphi+\xi) - 1\big) \circ K(\varphi+\xi).$$

Given that $I(\varphi+\xi) = \tilde{I}(\varphi) + \tilde{J}(\varphi) + \big(I(\varphi+\xi) - 1\big)$, one has that

$$I(\varphi+\xi) = \tilde{I}(\varphi) \circ \tilde{J}(\varphi) \circ \big(I(\varphi+\xi) - 1\big)$$

and thus

$$I(\varphi+\xi) \circ K(\varphi+\xi) = \tilde{I}(\varphi) \circ \tilde{J}(\varphi) \circ \big(I-1\big)(\varphi+\xi) \circ K(\varphi+\xi) = \tilde{I}(\varphi) \circ \tilde{K}(\varphi,\xi).$$

Hence,

$$\boldsymbol{R}(I \circ K)(\Lambda_N,\varphi) = (\tilde{I} \circ (\boldsymbol{R}\tilde{K}))(\Lambda_N,\varphi).$$

As usual in the RG theory, K' is defined by sorting the X-terms according to the next level closure U. One introduces the factor $\chi(X,U) = \frac{|\{B\in\mathcal{B}(X)\colon \overline{B^*}=U\}|}{|X|}$ for any $X \in \mathcal{S}(\Lambda_N)$ and $\chi(X,U) = \mathbb{1}_{U=\overline{X}}$ for $X \in \mathcal{P}(\Lambda_N) \setminus \mathcal{S}(\Lambda_N)$. Then, one has that

$$(\tilde{I} \circ \widetilde{K})(\Lambda_N,\varphi,\xi) = \sum_{U\in\mathcal{P}'} {I'}^{\Lambda_N\setminus U}(\varphi)\Big[\chi(X,U) \sum_{X\subset U} \tilde{I}^{U\setminus X}(\varphi)\widetilde{K}(X,\varphi,\xi)\Big]. \tag{3.25}$$

Where in (3.25), we used the fact that for any $X \in \mathcal{S}(\Lambda_N)$ contributing to several U's, one has that $\sum_{U\in\mathcal{P}'} \chi(X,U) = 1$ and also that $X \subset B^*$ and thus $\overline{X} \subset \overline{B^*}$.

Define

$$K'(U,\varphi) = \sum_{X\subset U} \chi(X,U)\tilde{I}^{U\setminus X}(\varphi) \int_{\boldsymbol{\mathcal{X}}} \widetilde{K}(X,\varphi,\xi)\,\mathrm{d}\mu_{k+1}(\xi) \tag{3.26}$$

for any connected $U \in \mathcal{P}'$. One has that

$$\boldsymbol{R}(I \circ K)(\Lambda_N,\varphi) = (I' \circ K')(\Lambda_N,\varphi).$$

The above transform conserves the factorisation property of the coordinate K, namely if K factors on the scale k,

$$X, Y \in \mathcal{P}, \text{ and } X \cap Y = \varnothing, \text{ then } K(X \cup Y,\varphi) = K(X,\varphi)K(Y,\varphi),$$

then K' factors on the scale $k+1$.

Proposition 3.5.2 ([2, Proposition 4.2]). *Let $k \in \{0, \dots, N-1\}$, $H_k \in M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$, and $K_k \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ that factors. Let $H_{k+1} \in M_0(\mathcal{B}_{k+1}, \boldsymbol{\mathcal{X}})$ be defined by*

$$H_{k+1}(B', \varphi) := \sum_{B \in \mathcal{B}_k(B')} \widetilde{H}_k(B, \varphi), \tag{3.27}$$

where

$$\widetilde{H}_k(B, \varphi) := \Pi T_2\Big((\boldsymbol{R}_{k+1} H_k)(B, \varphi) - \sum_{\substack{X \in \mathcal{S}_k \\ X \supset B}} \frac{1}{|X|_k} (\boldsymbol{R}_{k+1} K_k)(X, \varphi)\Big). \tag{3.28}$$

Let $\widetilde{K}_k(\varphi, \xi) := \big(1 - \mathrm{e}^{-\tilde{H}_k(\varphi)}\big) \circ \big(\mathrm{e}^{-H_k(\varphi+\xi)} - 1\big) \circ K_k(\varphi + \xi)$, and let $K_{k+1} \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$ be defined by

$$K_{k+1}(U, \varphi) := \sum_{X \in \mathcal{P}_k(U)} \chi(X, U) \exp\{- \sum_{B \in \mathcal{B}_k(U \setminus X)} \tilde{H}_k(B, \varphi)\} \int_{\boldsymbol{\mathcal{X}}} \widetilde{K}_k(X, \varphi, \xi) \, \mathrm{d}\mu_{k+1}(\xi) \tag{3.29}$$

for any connected $U \in \mathcal{P}'$, with

$$\chi(X, U) := \begin{cases} \frac{|\{B \in \mathcal{B}_k(X) \colon \overline{B^*} = U\}|}{|X|} & \text{if } X \in \mathcal{S}_k(\Lambda_N), \\ \mathbb{1}_{U = \overline{X}} & \text{if } X \in \mathcal{P}_k(\Lambda_N) \setminus \mathcal{S}_k(\Lambda_N), \end{cases}$$

and by the corresponding product over connected components for any non-connected U. Then $K_{k+1} \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$, it factors, and

$$\boldsymbol{R}_{k+1}(\mathrm{e}^{-H_k} \circ K_k)(\Lambda_N, \varphi) = (\mathrm{e}^{-H_{k+1}} \circ K_{k+1})(\Lambda_N, \varphi).$$

3.6 Auxiliary Results

The purpose of this section is to give some technical lemmas (without proof), that will be used in the sequel.

Proposition 3.6.1. *Suppose that*

$$\|D_1^k D_2^l F(x,y)(\dot{x}, \dots, \dot{x}, \dot{y}, \dots, \dot{y})\|_s \le C_1 \|\dot{x}\|_{s+2l}^k \|\dot{y}\|_Y^l$$
$$\|D_1^k D_2^l G(x,y)(\dot{x}, \dots, \dot{x}, \dot{y}, \dots, \dot{y})\|_s \le C_1 \|\dot{x}\|_{s+2l}^k \|\dot{y}\|_Y^l$$

for all $1 \le k + l \le m$. Let $H : \boldsymbol{X} \times \boldsymbol{Y} \to X$ be defined by $H(x,y) = G(F(x,y),y)$. Then there exists a constant C_3 such that

$$\|D_1^k D_2^l H(x,y)(\dot{x}, \dots, \dot{y}, \dot{y}, \dots, \dot{y})\|_s \le C_3 \|\dot{x}\|_{s+2l}^k \|\dot{y}\|_Y^l$$

for all $1 \le k + l \le m$.

Proof. A proof of the above claim can be found in [2]. □

Proposition 3.6.2 (Implicit Function Theorem with loss of regularity)**.** *Let $r \in \mathbb{N}, r \geq 9$ be fixed and let $\boldsymbol{\Xi}$ be a Banach space with a norm $\|\cdot\|$ and $\{\boldsymbol{X}_s\}_{s=r,r-2,r-4,r-6}$, a sequence of Banach spaces with norms $\|\cdot\|_{\boldsymbol{X}_s}$ such that $\boldsymbol{X}_s \subset \boldsymbol{X}_{s-2}$ (and $\|\boldsymbol{x}\|_{\boldsymbol{X}_s} \geq \|\boldsymbol{x}\|_{\boldsymbol{X}_{s-2}}$ for each $\boldsymbol{x} \in \boldsymbol{X}_s$), $s = r, r-2, r-4$. Further, let $\boldsymbol{F}\colon B_{\boldsymbol{\Xi}}(\rho_1) \times B_{\boldsymbol{X}_r}(\rho_2) \subset \boldsymbol{\Xi} \times \boldsymbol{X}_r \to \boldsymbol{X}_r$ be a C^3 map such that $\boldsymbol{F}(0,0) = 0$, and suppose that, with positive constants $C_0, C_{\ell,j} < \infty$, and $\gamma \in (0, 1/2)$, the following bounds are valid for each $\xi \in B_{\boldsymbol{\Xi}}(\rho_1)$ and $\boldsymbol{x} \in B_{\boldsymbol{X}_r}(\rho_2)$:*

$$\|\boldsymbol{F}(\xi, 0)\|_{\boldsymbol{X}_s} \leq C_0 \|\xi\|,$$

$$\Big\| \frac{\partial \boldsymbol{F}(\xi, \boldsymbol{x})}{\partial \boldsymbol{x}} \Big|_{\boldsymbol{x}=0} \Big\|_{\mathcal{L}(\boldsymbol{X}_s, \boldsymbol{X}_s)} \leq 1 - \gamma,$$

and

$$\Big\| \frac{\partial^{\ell+j} \boldsymbol{F}(\xi, \boldsymbol{x})}{\partial^\ell \xi \partial^j \boldsymbol{x}} (\dot{\xi}, \dots, \dot{\xi}, \dot{\boldsymbol{x}}, \dots, \dot{\boldsymbol{x}}) \Big\|_{\boldsymbol{X}_{s-2\ell}} \leq C_{\ell,j} \|\dot{\xi}\|^\ell \|\dot{\boldsymbol{x}}\|^j_{\boldsymbol{X}_s}$$

for $s = r, r-2, r-4, r-6$, $\ell = 0, 1, \dots, \min(3, \lfloor s/2 \rfloor)$ and $j = 0, 1, 2, 3$. Then there exists $\rho > 0$ and a unique $\boldsymbol{f}\colon B_{\boldsymbol{\Xi}}(\rho) \to \boldsymbol{X}_r$ so that

$$\boldsymbol{F}(\xi, \boldsymbol{f}(\xi)) = \boldsymbol{f}(\xi).$$

Moreover, $\boldsymbol{f} \in C^1(B_{\boldsymbol{\Xi}}(\rho), \boldsymbol{X}_{r-2}) \cap C^2(B_{\boldsymbol{\Xi}}(\rho), \boldsymbol{X}_{r-4}) \cap C^3(B_{\boldsymbol{\Xi}}(\rho), \boldsymbol{X}_{r-6})$ and

$$\|D\boldsymbol{f}(\xi)(\dot{\xi})\|_{\boldsymbol{X}_{r-2}} \leq C \|\dot{\xi}\|$$

and

$$\|D^2 \boldsymbol{f}(\xi)(\dot{\xi}, \dot{\xi})\|_{\boldsymbol{X}_{r-4}} \leq C \|\dot{\xi}\|^2$$

with a constant C depending only on the constants γ, C_0, and $C_{\ell,j}$.

Proof. The proof of the above claim can be found in [2]. □

Proposition 3.6.3 (Evaluation of the boundary terms)**.** *There exist a constant $c < 3\sqrt{2}$ such that for any $v\colon \mathbb{Z} \to \mathbb{R}$ and any $m \in \mathbb{N}$, $m > 1$, one has*

$$v(-m)^2 + v(m+1)^2 \leq \frac{c}{2m+1} \sum_{x=-m}^{m} v(x)^2 + c(2m+1) \sum_{x=-m}^{m} \partial v(x)^2.$$

Proof. The proof is contained in [2]. □

Proposition 3.6.4. *Let $X \in \mathcal{P}_k$ and $u : U_4(X) \to \mathbb{R}$. With the constant c from Proposition 3.6.3,*

(a) $L^k \sum_{x \in \partial X} |\nabla v(x)|^2 \leq 2c \Big(\sum_{x \in X} |\nabla v(x)|^2 + L^{2k} \sum_{x \in U_1(X)} |\nabla^2 v(x)|^2 \Big)$,

(b) $L^{3k} \sum_{x \in \partial X} |\nabla^2 v(x)|^2 \leq 2c \Big(L^{2k} \sum_{x \in X} |\nabla^2 v(x)|^2 + L^{4k} \sum_{x \in U_1(X)} |\nabla^3 v(x)|^2 \Big)$, *and*

(c) $L^{5k} \sum_{x \in \partial X} |\nabla^3 v(x)|^2 \leq 2c \Big(L^{4k} \sum_{x \in X} |\nabla^3 v(x)|^2 + L^{6k} \sum_{x \in U_1(X)} |\nabla^4 v(x)|^2 \Big)$

Proof. The proof is contained in [2].

□

Proposition 3.6.5. *Let $u, v : X \cup \partial X \to \mathbb{R}$ and $X \in \mathcal{P}_k$. With the constant c from Proposition 3.6.3 and any $\eta > 0$, we get*

$$\Big|\sum_{x\in X} \nabla u(x)\nabla v(x)\Big| \le \frac{\eta(1+cd)}{2L^{2k}} \sum_{x\in X\cup\partial^- X} v(x)^2 + \frac{L^k}{2\eta}\sum_{x\in\partial^- X}|\nabla u(x)|^2 + \frac{c\eta}{2}\sum_{x\in X}|\nabla v(x)|^2 + \frac{L^{2k}}{2\eta}\sum_{x\in X\cup\partial^- X}\big|\nabla^2 u(x)\big|^2. \tag{3.30}$$

3.7 Properties of the Renormalization Transformation

As in [2], we introduce the maps

$$\boldsymbol{T}_k\colon M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})\times\mathscr{E}\to M_0(\mathcal{B}_{k+1},\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_{k+1},\boldsymbol{\mathcal{X}}),$$

$k = 0, 1, \dots, N-1$, by $\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}) = (H_{k+1}, K_{k+1})$.

For any $\boldsymbol{q}$, the origin $(H, K) = (0, 0)$ is a fixed point of the transformation $\boldsymbol{T}$.

The parameters L, h, A in the definition of the norms will be chosen later. Let $\mathcal{U}_\delta \subset M_0(\mathcal{B},\boldsymbol{\mathcal{X}})\times M(\mathcal{P},\boldsymbol{\mathcal{X}})\times\mathcal{E}$ be defined by

$$\mathcal{U}_\delta := \{(H,K,\boldsymbol{q})\in M_0(\mathcal{B},\boldsymbol{\mathcal{X}})\times M(\mathcal{P},\boldsymbol{\mathcal{X}})\times\mathcal{E}\colon \|H\|_0<\delta, \|K\|_{r_0}<\delta, \|\boldsymbol{q}\|<\delta\},$$

and

$$\mathcal{O}_\delta := \{(H',K')\in M_0'(\mathcal{B},\boldsymbol{\mathcal{X}})\times M'(\mathcal{P},\boldsymbol{\mathcal{X}})\colon \|H'\|_0<\delta, \|K'\|_{r_0}<\delta\}.$$

For a linear operator $\boldsymbol{L}$ between Banach spaces, we denote by $\|\boldsymbol{L}\|$ the standard induced norm. The corresponding norms will be indicated as $\|\boldsymbol{L}\|_{k,r;k+1,0}$, or simply $\|\boldsymbol{L}\|_{r;0}$(whenever it is clear from the context), for a linear mapping $\boldsymbol{L} : M(\mathcal{P},\boldsymbol{\mathcal{X}})\to M_0(\mathcal{B},\boldsymbol{\mathcal{X}})$.

In the following, we extend [2, Proposition 4.3].

Proposition 3.7.1 (Linearization of $\boldsymbol{T}$). *Given the constants h, L, and sufficiently large A, there exist $\delta > 0$ such that $\boldsymbol{T}(\mathcal{U}_\delta)\subset\mathcal{O}_\delta$ and $\boldsymbol{T}$ is differentiable on $\mathcal{U}_\delta$. The first derivatives at $H = 0$ and $K = 0$ have a triangular form,*

$$D\boldsymbol{T}(0,0,\boldsymbol{q})(\dot H,\dot K) = \begin{pmatrix}\boldsymbol{A}^{(\boldsymbol{q})} & \boldsymbol{B}^{(\boldsymbol{q})}\\ \boldsymbol{0} & \boldsymbol{C}^{(\boldsymbol{q})}\end{pmatrix}\begin{pmatrix}\dot H\\ \dot K\end{pmatrix},$$

with

$$(\boldsymbol{A}^{(\boldsymbol{q})}\dot H)(B',\varphi) = \sum_{B\in\mathcal{B}(B')}\Big[\dot H(B,\varphi) + \sum_{x\in B}\sum_{i,j=1}^{d}\nabla_i\dot d_{i,j}(x)\nabla_j^*\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,x)\Big], \tag{3.31}$$

$$(\boldsymbol{B}^{(\boldsymbol{q})}\dot K)(B',\varphi) = -\sum_{B\in\mathcal{B}(B')}\Pi T_2\sum_{\substack{X\in\mathcal{S}\\ X\supset B}}\frac{1}{|X|}\Big(\int_{\boldsymbol{\mathcal{X}}}\dot K(X,\varphi+\xi)\,\mathrm{d}\mu_{k+1}^{(\boldsymbol{q})}(\xi)\Big), \tag{3.32}$$

and

$$(\boldsymbol{C}^{(\boldsymbol{q})}\dot{K})(U,\varphi) = \sum_{B:\overline{B^*}=U} \big(1-\Pi T_2\big) \sum_{\substack{Y\in\mathcal{S}\\ Y\supset B}} \frac{1}{|Y|}\Big(\int_{\boldsymbol{\mathcal{X}}} \dot{K}(Y,\varphi+\xi)\,\mathrm{d}\mu^{(\boldsymbol{q})}_{k+1}(\xi)\Big) + \sum_{\substack{X\in\mathcal{P}^c\setminus\mathcal{S}\\ \overline{X}=U}} \int_{\boldsymbol{\mathcal{X}}} \dot{K}(X,\varphi+\xi)\,\mathrm{d}\mu^{(\boldsymbol{q})}_{k+1}(\xi). \tag{3.33}$$

Moreover, there exist constants $\theta\in(0,1)$ and $M<\infty$ such that the following bounds on the norms of operators $\boldsymbol{A}^{(\boldsymbol{q})}$, $\boldsymbol{B}^{(\boldsymbol{q})}$, and $\boldsymbol{C}^{(\boldsymbol{q})}$ hold independently on N and k and for $\|\boldsymbol{q}\|<\delta$:

$$\|\boldsymbol{C}^{(\boldsymbol{q})}\|_{r;r}\le\theta,\, \|{\boldsymbol{A}^{(\boldsymbol{q})}}^{-1}\|_{0;0}\le\frac{1}{\sqrt{\theta}},\ \textit{and}\ \|\boldsymbol{B}^{(\boldsymbol{q})}\|_{r;0}\le M, \tag{3.34}$$

$r=1,\dots,r_0$. The operators $\boldsymbol{A}^{(\boldsymbol{q})}$, $\boldsymbol{B}^{(\boldsymbol{q})}$, and $\boldsymbol{C}^{(\boldsymbol{q})}$ are 3-times differentiable with respect to $\boldsymbol{q}$ and there exists a constant $C<\infty$ such that

$$\|\partial^\ell_{\boldsymbol{q}}\boldsymbol{A}^{(\boldsymbol{q})}\dot{H}\|_0\le C\|\dot{H}\|_0,\ \|\partial^\ell_{\boldsymbol{q}}\boldsymbol{B}^{(\boldsymbol{q})}\dot{K}\|_0\le C\|\dot{K}\|_r,\ \|\partial^\ell_{\boldsymbol{q}}\boldsymbol{C}^{(\boldsymbol{q})}\dot{K}\|_{r-2\ell}\le C\|\dot{K}\|_r, \tag{3.35}$$

for any $\ell=1,2,3$ and any $r\ge 2\ell$.

Proof. The bounds in (3.34) and (3.35) are very involved and will be the main purpose of § 3.9. Here, we only show the validity of the linearization formula, namely the validity of (3.31), (3.32) and (3.33). Starting from (3.27) and (3.28), one expands the linear and quadratic terms in $\dot{H}(B,\varphi+\xi)$ into the sum of the terms depending on φ, ξ, and the term proportional to $\dot{Q}(\varphi,\xi)$. Given that the integral of the terms linear in ξ vanishes, one has that

$$\begin{aligned}\int_{\boldsymbol{\mathcal{X}}}\sum_{x\in B}\sum_{i,j}\dot{d}_{i,j}(x)\nabla_i\xi\nabla_j\xi\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi) &= \sum_{x\in B}\sum_{i,j}\dot{d}_{i,j}\xi(x+e_i)\xi(x+e_j)+d_{i,j}(x)\xi(x)\xi(x)\\ &\quad-\sum_{x\in B}\sum_{i,j}d_{i,j}(x)\xi(x+e_i)\xi(x)-d_{i,j}(x)\xi(x+e_j)\xi(x)\\ &=\sum_{x\in B}\sum_{i,j}\dot{d}_{i,j}(C(x+e_i,x+e_j)-C(x+e_i,x)-(C(x,x)-C(x,x+e_j)))\\ &=\sum_{x\in B}\sum_{i,j}\dot{d}_{i,j}\nabla^{(1)}_{e_i}\nabla^{(2)*}_{e_2}C(x,x),\end{aligned}$$

where $\nabla^{(1)}$ and $\nabla^{(2)}$ denotes the derivative on the first variable and on the second variable, respectively.

The formula (3.32) follows directly from the linearization of the second term on the right hand side of (3.28). When computing $\boldsymbol{C}^{(\boldsymbol{q})}$, we first observe that only linear terms in $\widetilde{K}$ can contribute. Taking $\dot{H}=0$ and

$$\widetilde{H}(B,\varphi)=-\Pi T_2\sum_{\substack{X\in\mathcal{S}\\ X\supset B}}\frac{1}{|X|}(\boldsymbol{R}\dot{K})(X,\varphi)$$

and $\widetilde{K}(\varphi, \xi) = \big(1 - \mathrm{e}^{-\tilde{H}(\varphi)}\big) \circ K(\varphi + \xi)$, one has that

$$\boldsymbol{C}^{(\boldsymbol{q})}(\dot{K})(U, \varphi) = \sum_{Y \in \mathcal{S}} \chi(Y, U) \int_{\boldsymbol{\mathcal{X}}} D\widetilde{K}(0)(\dot{K})(Y, \varphi, \xi)\, \mathrm{d}\mu_{k+1}(\xi)$$
$$+ \sum_{\substack{X \in \mathcal{P}^c \setminus \mathcal{S} \\ \overline{X} = U}} \int_{\boldsymbol{\mathcal{X}}} D\widetilde{K}(0)(\dot{K})(X, \varphi, \xi)\, \mathrm{d}\mu_{k+1}(\xi).$$

Denoting by

$$\chi(Y, U) = \sum_{\substack{B \in Y \\ \overline{B^*} = U}} \frac{1}{|Y|}$$

and noticing that

$$D\widetilde{K}(0)(\dot{K})(B, \varphi, \xi) = \dot{K}(B, \varphi + \xi) - D\mathrm{e}^{-\tilde{H}(0)}(\dot{K})(B, \varphi) \text{ for } Y = B,$$
$$D\widetilde{K}(0)(\dot{K})(Y, \varphi, \xi) = \dot{K}(Y, \varphi + \xi) \text{ for } Y \neq B,$$

and

$$D\mathrm{e}^{-\tilde{H}(0)}(\dot{K})(B, \varphi) = \Pi T_2 \sum_{\substack{Y \in \mathcal{S} \\ Y \supset B}} \frac{1}{|Y|} \big(\boldsymbol{R}\dot{K}\big)(Y, \varphi),$$

one obtains (3.33). □

Given the linear dependence of $\boldsymbol{T}$ in H, one has that

$$\boldsymbol{T}(H, K, \boldsymbol{q}) = (\boldsymbol{A}^{(\boldsymbol{q})} H + \boldsymbol{B}^{(\boldsymbol{q})} K, S(H, K, \boldsymbol{q})), \tag{3.36}$$

with $DS(0,0,\boldsymbol{q})(\dot{K}) = \boldsymbol{C}^{(\boldsymbol{q})}\dot{K}$. Given that $(0,0)$ is a fixed point for each $\boldsymbol{q}$, namely $\boldsymbol{T}(0,0,\boldsymbol{q}) = (0,0)$, we have that

$$\frac{\partial \boldsymbol{T}(0,0,\boldsymbol{q})}{\partial \boldsymbol{q}} = 0$$

and thus

$$\frac{\partial S(0,0,\boldsymbol{q})}{\partial \boldsymbol{q}} = 0.$$

In order to apply the fixed point theorem, one needs to understand the smoothness of the nonlinear part S. The proof of the following Proposition, will be given in § 3.9 and is an extension of [2, Proposition 4.4].

Proposition 3.7.2 (Smoothness of expanding part S)**.** *Under the conditions of Proposition 3.7.1 and for any* $(H, K, \boldsymbol{q}) \in \mathcal{U}_\delta$*, we have*

$$\Big\| \frac{\partial^{\ell + j_1 + j_2} S}{\partial H^{j_1} \partial K^{j_2} \partial \boldsymbol{q}^{\ell}} (\dot{H}, \dots, \dot{K}, \dots, \dot{\boldsymbol{q}}, \dots) \Big\|_{r - 2\ell} \leq C \|\dot{H}\|_0^{j_1} \|\dot{K}\|_r^{j_2} \|\dot{\boldsymbol{q}}\|^{\ell}.$$

Here $r = 1, \dots, r_0$ *and* $\ell \leq 2r$*.*

3.8 Fine tuning of the initial conditions

As in [2], we are going to choose $\boldsymbol{q} \in \mathcal{E}$ such that the final Ideal Hamiltonian H_{N+1} vanishes.

We introduce the Banach spaces

$$\boldsymbol{Z}_r = \big\{Z = (H_0, H_1, K_1, \ldots, H_{N-1}, K_{N-1}, K_N) \colon H_k \in M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), K_k \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})\big\}$$

endowed with the norms

$$\|Z\|_{\boldsymbol{Z}_r} = \max_{k\in\{0,\ldots,N-1\}} \frac{1}{\eta^k}\|H_k\|_{k,0} \vee \max_{k\in\{1,\ldots,N\}} \frac{\alpha}{\eta^k}\|K_k\|_{k,r}$$

for $r = 1, \ldots, r_0$ and with suitable parameters $\eta \in (0,1)$ and $\alpha \geq 1$ to be chosen later. As in [2], the terms K_0 and H_N are not present in $Z \in \boldsymbol{Z}_r$, as the latter is set to be 0 and the former is singled out as an initial condition for a separate treatment.

Let us define the map

$$\boldsymbol{\mathcal{T}} \colon \boldsymbol{Y} \times \mathcal{E} \times \boldsymbol{Z}_r \to \boldsymbol{Z}_r$$

by

$$\boldsymbol{\mathcal{T}}(K, \boldsymbol{q}, Z) := \overline{Z},$$

where $\overline{Z}$ is given by

$$\begin{aligned} \overline{H}_k &:= \boldsymbol{A}_k^{-1}\big(H_{k+1} - \boldsymbol{B}_k K_k\big), \\ \overline{K}_{k+1} &:= \boldsymbol{C}_k K_k + g_{k+1}(H_k, K_k), \end{aligned}$$

where $g_{k+1}(H_k, K_k) := S(H_k, K_k, \boldsymbol{q}) - \boldsymbol{C}_k K_k$ is the nonlinear part of the map S and $k = 0, \ldots, N-1$, with initial $H_N = 0$ and K_0 given by

$$K_0(X, \varphi) := K^{(\boldsymbol{q})}(X, \boldsymbol{q}) := \exp\Big\{\tfrac{1}{2}\sum_{x\in X}\sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}(x)\nabla_i\varphi(x)\nabla_j\varphi(x)\Big\}K(X,\varphi).$$

Given K and $\boldsymbol{q}$, then the $2N$-tuple Z is a fixed point of $\boldsymbol{\mathcal{T}}$, namely it holds

$$\boldsymbol{\mathcal{T}}(K, \boldsymbol{q}, \widehat{Z}(K, \boldsymbol{q})) = \widehat{Z}(K, \boldsymbol{q}). \tag{3.37}$$

Theorem 3.8.1. *Let the constants h, L, A, and δ be chosen so that the Propositions 3.7.1 and Proposition 3.7.2 are valid. Then, there exist constants α and η determining the norm on $\boldsymbol{Z}_r$ and a constant $\rho > 0$ so that there exists a unique C^3-function $\widehat{Z} \colon B_{\boldsymbol{Y}\times\mathcal{E}}(\rho) \to \boldsymbol{Z}_r$ solving the equation* (3.37) *with bounds on derivatives that are uniform in N.*

The proof of the above statement will be given in § 3.9.

3.9 Proofs

In the following lemma, we extend [2, Lemma 5.1].

Lemma 3.9.1. *Let $\omega \geq 1 + 18\sqrt{2}$, $N \in \mathbb{N}$, $N \geq 1$, and $L \in \mathbb{N}$ odd, $L \geq 3$. Given $k \in \{0, \dots, N-1\}$, let $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, and let $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$. Then, the norms $\|\cdot\|_{k,X,r}$, $\|\cdot\|_{k:k+1,X,r}$, $r \in \{1, \dots, r_0\}$, and $|||\cdot|||_{k,X}$, $X \in \mathcal{P}_k$, satisfy the following conditions:*

(i) $\|K(X)\|_{k,X,r} \leq \prod_{Y \in \mathcal{C}(X)} \|K(Y)\|_{k,Y,r}$,

(iia) $\|F^X K(Y)\|_{k,X\cup Y,r} \leq \prod_{B \in \mathcal{B}_k(X)} |||F|||_k \|K(Y)\|_{k,Y,r}$

(iib) $\|F^X K(Y)\|_{k:k+1,X\cup Y,r} \leq \prod_{B \in \mathcal{B}_k(X)} |||F|||_k \|K(Y)\|_{k:k+1,Y,r}$ *for $X, Y \in \mathcal{P}_k$ disjoint,*

(iii) $|||\mathbb{1}(B)|||_{k,B} = 1$ *for $B \in \mathcal{B}_k$,*

(iv) There exist a constant $h_0 = h_0(d, \omega)$ depending only on the dimension d and value of the parameter ω, such that for any $h \geq L^{(d+\eta(d))/2} h_0$ and $X \in \mathcal{P}_k$, we have

$$\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r} \leq 2^{|X|_k} \|K(X)\|_{k,X,r}.$$

Proof. (i) For every $F_1, F_2 \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ and $X_1, X_2 \in \mathcal{P}_k$ (not necessarily disjoint), we have that

$$|F_1(X_1)(\varphi)F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r} \leq |F_1(X_1)(\varphi)|^{k,X_1,r} |F_2(X_2)(\varphi)|^{k,X_2,r}. \tag{3.38}$$

Indeed, let us remind the discrete product rule

$$\nabla_i(fg) = \nabla_i f\, \mathsf{S}_i\, g + \mathsf{S}_i\, f \nabla_i g,$$

where

$$(\mathsf{S}_i\, f)(x) := \tfrac{1}{2} f(x) + \tfrac{1}{2} f(x + \mathrm{e}_i).$$

The operations S_i commute with all discrete derivatives. Using multiindex notation

$$\nabla^{\boldsymbol{\alpha}} := \prod_{i=1}^{d} \nabla_i^{\alpha_i} \quad \text{and} \quad \mathsf{S}^{\boldsymbol{\alpha}} := \prod_{i=1}^{d} \mathsf{S}_i^{\alpha_i},$$

we get the Leibniz rule

$$\nabla^{\gamma}(fg) = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \big(\mathsf{S}^{\boldsymbol{\alpha}} \nabla^{\boldsymbol{\beta}} f\big)\big(\mathsf{S}^{\boldsymbol{\beta}}\, \nabla^{\boldsymbol{\alpha}} g\big),$$

with suitable constants $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Hence,

$$\nabla^{\alpha}(F_1(X_1,\varphi)F(X_2,\varphi)) = \sum_{\gamma_1+\gamma_2=\alpha} c_{\gamma_1,\gamma_2} (\mathsf{S}^{\gamma_1}\, \nabla^{\gamma_2} F(X,\varphi))(\mathsf{S}^{\gamma_2}\, \nabla^{\gamma_2} F_2(X,\varphi)).$$

Moreover, given that the Taylor series of the product is the product of the Taylor series, one has that

$$\begin{aligned} |F_1(X_1)(\varphi)F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r} &\leq \sum_{\gamma_1+\gamma_2=\beta} |\, \mathsf{S}^{\gamma_1}\, \nabla_{\gamma_1} F_1(X_1)(\varphi)|^{k,X_1\cup X_2,r} |\, \mathsf{S}^{\gamma_2}\, \nabla^{\gamma_2} F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r} \\ &\leq \Big(\sum_{\gamma_1 \leq \beta} |\nabla^{\gamma_1} F_1(X_1)(\varphi)|^{k,X_1\cup X_2,r}\Big)\Big(\sum_{\gamma_2 \leq \beta} |\nabla^{\gamma_2} F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r}\Big). \end{aligned} \tag{3.39}$$

Given that for any $\dot\varphi \in \boldsymbol{\mathcal{X}}_N$, one has that $|\dot\varphi|_{k,X_1} \le |\dot\varphi|_{k,X_1\cup X_2}$, then

$$\sup_{|\dot\varphi|_{k,X_1\cup X_2}\le 1} |D^s F_1(X_1)(\varphi)(\dot\varphi,\dots,\dot\varphi)| \le \sup_{|\dot\varphi|_{k,X_1}\le 1} |D^s F_1(X_1)(\varphi)(\dot\varphi,\dots,\dot\varphi)|.$$

Hence, given that $w_k^X(\tau_x\varphi) = w_k^X(\varphi)$ for every φ, one has that

$$|F_1(X_1)(\varphi)|^{k,X_1\cup X_2,r} \le |F_1(X_1)(\varphi)|^{k,X_1,r}$$

and similarly for F_2, thus (3.38).

Iterating (3.38), we can use it for $K(X,\varphi) = \prod_{Y\in\mathcal{C}(X)} K(Y)(\varphi)$, hence

$$|K(X,\varphi)|^{k,X,r} \le \prod_{Y\in\mathcal{C}(X)} |K(Y)(\varphi)|^{k,Y,r}.$$

To conclude, it suffices to observe that

$$w_k^X(\varphi) = \prod_{Y\in\mathcal{C}(X)} w_k^Y(\varphi).$$

In the above formula, we use the fact that the partition $X = \cup_{Y\in\mathcal{C}(X)} Y$ splits both X and its boundary ∂X into disjoint components: $Y_1, Y_2 \in \mathcal{C}(X)$, $Y_1 \ne Y_2$ implies that $\mathrm{dist}(Y_1,Y_2) > L^k$ and thus $Y_1\cap Y_2 = \varnothing$, $\partial Y_1 \cap \partial Y_2 = \varnothing$, and $\partial X = \cup_{Y\in\mathcal{C}(X)} \partial Y$.

(iia) Using (iterated) (3.38) for $\prod_{B\in\mathcal{B}_k(X)} F(B)(\varphi)K(Y)(\varphi)$, we have

$$|\big(F^X K(Y)\big)(\varphi)|^{k,X\cup Y,r} \le \prod_{B\in\mathcal{B}_k(X)} |F(B)(\varphi)|^{k,B,r} |K(Y)(\varphi)|^{k,Y,r}.$$

The right hand side in the above formula can be bounded by

$$\prod_{B\in\mathcal{B}_k(X)} |||F(B)|||_{k,B} \|K(Y)\|_{k,Y,r} \prod_{B\in\mathcal{B}_k(X)} W_k^B(\varphi) w_k^Y(\varphi).$$

Thus, in order to conclude the proof of (ii), it is enough to verify that

$$\prod_{B\in\mathcal{B}_k(X)} W_k^B(\varphi) w_k^Y(\varphi) \le w_k^{X\cup Y}(\varphi). \tag{3.40}$$

Using the definitions of the strong and weak functions, we have that (3.40) is satisfied if

$$L^k \sum_{x\in\partial Y} G_{k,x}(\varphi) \le \sum_{x\in X} \big(2^d \omega g_{k,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big) + L^k \sum_{x\in\partial(X\cup Y)} G_{k,x}(\varphi). \tag{3.41}$$

Indeed, it suffices to notice that each $y \in \partial Y \setminus \partial(X\cup Y)$ is necessarily contained in ∂B for some $B \in \mathcal{B}_k(X)$ (a block on the boundary of X touching Y). Thus, it suffices to show that for each such B one has

$$L^k \sum_{x\in\partial B} G_{k,x}(\varphi) \le \sum_{x\in B} \big(2^d \omega g_{k,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big). \tag{3.42}$$

Applying Proposition 3.6.4 (a), we have

$$h^2 L^k \sum_{x\in\partial B} G_{k,x}(\varphi) \le 2c\big(\sum_{x\in B} |\nabla\varphi(x)|^2 + L^{2k} \sum_{x\in U_1(B)} |\nabla^2\varphi(x)|^2\big) + L^k \sum_{x\in\partial B}\sum_{s=2}^{3} L^{(2s-2)k} |\nabla^s\varphi(x)|^2$$
$$\le h^2 2c \sum_{x\in B} G_{k,x}(\varphi) + h^2 2cL^k \sum_{z\in\partial B} g_{k,z}(\varphi),$$

where z is any point $z\in B$. Given that the size of the set ∂B is at most $(L^k+2)^d - (L^k-2)^d \le 2^d L^{(d-1)k}$ once $2 \le L$, one gets the desired bound once

$$2c \le \omega - 1.$$

Given that $c < 3\sqrt{2}$, the above condition is satisfied with the choice of ω.

(iib) The proof is similar, with (3.41) replaced by

$$3L^k \sum_{x\in\partial Y} G_{k,x}(\varphi) \le \sum_{x\in X} \big((2^d\omega - 1) g_{k:k+1,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big) + 3L^k \sum_{x\in\partial(X\cup Y)} G_{k,x}(\varphi) \tag{3.43}$$

that, in its turn, needs (3.42) in a slightly stronger version,

$$3L^k \sum_{x\in\partial B} G_{k,x}(\varphi) \le \sum_{x\in B} \big((2^d\omega - 1) g_{k:k+1,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big).$$

The above is true whenever

$$6c \le \omega - 1.$$

The claim (iii) follows immediately from the definition.

(iv) The integration and the differentiation implicitly contained in the norm $\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r}$ can be interchanged. Namely, recalling the definition (3.19), we have

$$\big|\nabla^\alpha D^s \int K(\varphi+\xi)\,\mathrm{d}\mu_{k+1}(\xi)\big|^{k+1,X} \le \int |\nabla^\alpha D^s K(\varphi+\xi)|^{k+1,X}\,\mathrm{d}\mu_{k+1}(\xi).$$

It follows directly from the definition that

$$|\dot\varphi|_{k,X} \le L^{-\frac{d}{2}} |\dot\varphi|_{k+1,X}, \tag{3.44}$$

we get

$$|\nabla^\alpha D^s K(X,\varphi+\xi)|^{k+1,X} \le |\nabla^\alpha D^s K(X,\varphi+\xi)|^{k,X}.$$

(For $s=0$ this is trivial since $|\nabla^\alpha K(X,\varphi+\xi)|^{k,X} = |\nabla^\alpha K(X,\varphi+\xi)|$ actually does not depend on k.) Thus,

$$\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r} \le \sum_{|\alpha|\le 3} \sup_{\varphi} \int |\nabla^\alpha K(X,\varphi+\xi)|^{k,X,r} w_{k:k+1}^{-X}(\varphi)\,\mathrm{d}\mu_{k+1}(\xi).$$

Evaluating now the integrand $|K(X,\varphi+\xi)|^{k,X,r}$ above by $\|K(X)\|_{k,X,r} w_k^X(\varphi+\xi)$, the proof of the needed bound amounts to showing that

$$\int_{\boldsymbol{\mathcal{X}}_N} w_k^X(\varphi+\xi)\,\mathrm{d}\mu_{k+1}(\xi) \le 2^{|X|} w_{k:k+1}^X(\varphi). \tag{3.45}$$

To conclude it is enough to apply Lemma 3.9.2

□

The next lemma is contained in [2, Lemma 5.2].

Lemma 3.9.2 ([2, Lemma 5.2])**.** *Let $\omega \geq 1+6\sqrt{2}$. There exist a constant $h_0 = h_0(d,\omega)$ such that for any $N \geq 1$, L odd, $L \geq 5$, $h \geq L^{(d+\eta(d))/2}h_0$, $k \in \{0,\dots,N-1\}$, $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, and any $X \in \mathcal{P}_k$, we have*

$$\int_{\boldsymbol{\mathcal{X}}_N} w_k^X(\varphi+\xi)\,\mathrm{d}\mu_{k+1}(\xi) \leq 2^{|X|_k} w_{k:k+1}^X(\varphi). \tag{3.46}$$

Proof. We follow the original proof and prove the bound (3.46) in three steps:

(1) Expanding the terms $(\nabla\varphi(x)+\nabla\xi(x))^2$ in $\sum_{x\in X} G_{k,x}(\varphi+\xi)$ and using the Cauchy's inequality $(a+b)^2 \leq 2a^2+2b^2$ for the remaining terms (those that are preceded by a power in L that allows to absorb the resulting pre-factors while passing to the next scale), we have

$$\begin{aligned} h^2\sum_{x\in X} G_{k,x}(\varphi+\xi) \leq& \sum_{x\in X}\big(|\nabla\varphi(x)|^2+|\nabla\xi(x)|^2\big)+2\Big|\sum_{x\in X}\nabla\varphi(x)\nabla\xi(x)\Big| \\ &+2\sum_{x\in X}\Big(L^{2k}|\nabla^2\varphi(x)|^2+L^{2k}|\nabla^2\xi(x)|^2+L^{4k}|\nabla^3\varphi(x)|^2+L^{4k}|\nabla^3\xi(x)|^2\Big). \end{aligned} \tag{3.47}$$

The other terms in $w_k^X(\varphi+\xi)$, can be estimated by

$$g_{k,x}(\varphi+\xi) \leq 2g_{k,x}(\varphi)+2g_{k,x}(\xi)$$

and

$$L^kG_{k,x}(\varphi+\xi) \leq 2L^kG_{k,x}(\varphi)+2L^kG_{k,x}(\xi).$$

(2) In view of Proposition 3.6.5, we bound the mixed term $2\big|\sum_{x\in X}\nabla\varphi(x)\nabla\xi(x)\big|$ by

$$L^{2k}\sum_{x\in X\cup\partial^-X}|\nabla^2\varphi(x)|^2+L^k\sum_{x\in\partial^-X}|\nabla\varphi(x)|^2+\frac{1+cd}{L^{2k}}\sum_{x\in X\cup\partial^-X}\xi(x)^2+c\sum_{x\in X}|\nabla\xi(x)|^2. \tag{3.48}$$

The sum over X in the first term above will be estimated by the regulator $g_{k:k+1,x}(\varphi)$ of the next generation. Namely, combining, for any $x\in X$, its terms with the corresponding φ-terms on the second line in (3.47), we have

$$\begin{aligned} 3L^{2k}|\nabla^2\varphi(x)|^2+2L^{4k}|\nabla^3\varphi(x)|^2 &\leq 3L^{-2}L^{2(k+1)}|\nabla^2\varphi(x)|^2+2L^{-4}L^{4(k+1)}|\nabla^3\varphi(x)|^2 \\ &\leq 3L^{-2}h^2g_{k:k+1,x}(\varphi), \end{aligned}$$

where we are assuming that

$$2L^{-2} \leq 3. \tag{3.49}$$

The remaining sum over $\partial^-X\setminus X$, together with the second term in (3.48), will be absorbed into the sum $\sum_{x\in\partial X} G_{k,x}(\varphi)$. Collecting now all the φ-terms in $\log w_k(\varphi+\xi)$ with expanded mixed term, we get

$$\sum_{x\in X}2^{d+1}\omega g_{k,x}(\varphi)+\sum_{x\in X}\omega G_{k,x}(\varphi)+3\omega L^{-2}\sum_{x\in X}g_{k:k+1,x}(\varphi)+3L^k\sum_{x\in\partial X}G_{k,x}(\varphi).$$

This is bounded by

$$\log w^X_{k:k+1}(\varphi) = \sum_{x\in X}\big((2^d\omega - 1)g_{k:k+1,x}(\varphi) + \omega G_{k,x}(\varphi)\big) + 3L^k \sum_{x\in\partial X} G_{k,x}(\varphi)$$

once

$$(3 + 2^{d+1})\omega \le (2^d\omega - 1)L^2. \tag{3.50}$$

This condition, including also (3.49), are satisfied once $L \ge 5$.

Turning now to the ξ-terms in $h^2 \log w_k(\varphi + \xi)$ with expanded mixed term, we get the bound

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) + \sum_{x\in X}\omega\Big((1+c)|\nabla\xi(x)|^2 + 2L^{2k}\big|\nabla^2\xi(x)\big|^2 + 2L^{4k}\big|\nabla^3\xi(x)\big|^2\Big) \\ + \omega(1+cd)L^{-2k}\sum_{x\in X\cup\partial^- X}\xi(x)^2 + 2L^k\sum_{x\in\partial X} h^2 G_{k,x}(\xi).$$

Bounding the last term with the help of Proposition 3.6.4, one gets

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) + \sum_{x\in U_1(X)}\Big(\omega(1+cd)L^{-2k}\xi(x)^2 + (\omega(1+c)+4c)|\nabla\xi(x)|^2 \\ + (2\omega + 8c)L^{2k}\big|\nabla^2\xi(x)\big|^2 + (2\omega+8c)L^{4k}\big|\nabla^3\xi(x)\big|^2 + 4cL^{6k}\big|\nabla^4\xi(x)\big|^2\Big).$$

Finally, the term $g_{k,x}(\xi)$ containing l_∞-norm of $\nabla^s\xi$, $s = 2,3,4$, is bounded with the help of the Sobolev inequality from Proposition 2.6.1. Taking B^* for the B_n with $n = (2^{d+1}-1)L^k$, one gets

$$\|\nabla^s\xi\|^2_{l_\infty(B^*)} \le C^2(2^{d+1}-1)^2\frac{1}{L^{kd}}\sum_{l=0}^{M} L^{2lk}\sum_{x\in B^*}|\nabla^l\nabla^s\xi|^2(x),$$

where $M = \lfloor\frac{d+2}{2}\rfloor$ is the integer value of $\frac{d+2}{2}$ and in computing the pre-factor we took into account that $2\lfloor\frac{d+2}{2}\rfloor - d \le 2$. Notice that the constant C depends (also through M) only on the dimension d. Thus, we have that

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) \le 2^{d+1}\omega\sum_{x\in X}\sum_{s=2}^{4} L^{(2s-2)k}C^2(2^{d+1}-1)^2\frac{1}{L^{kd}}\sum_{l=0}^{M} L^{2lk}\sum_{y\in B^*_x}|\nabla^l\nabla^s\xi|^2(x) \\ \le 2^{d+1}\omega 2^{d+1}C^2(2^{d+1}-1)^{d+2}3L^{-2k}\sum_{l=2}^{M+4}L^{2lk}\sum_{y\in X^*}|\nabla^l\xi|^2(x),$$

where in the last inequality we took into account that each point $y \in X^*$ may occur in B^*_x for at most $(2^{d+1}-1)^d L^{dk}$ points $x \in X$.

Thus we have shown that, if one supposes (3.49), (3.50), then it holds

$$\Big|\int_{\mathcal{X}_N} w^X_k(\varphi+\xi)\,\mathrm{d}\mu_{k+1}(\xi)\Big| \le w^X_{k:k+1}(\varphi)\int_{\mathcal{X}_N}\exp\Big(h^{-2}\frac{\overline{C}}{L^{2k}}\sum_{x\in X^*}\sum_{l=0}^{M+4}L^{2lk}|\nabla^l\xi(x)|^2\Big)\,\mathrm{d}\mu_{k+1}(\xi) \tag{3.51}$$

with the constant

$$\overline{C} = \max\{\omega(1+cd), \omega(1+c)+4c, 2(\omega+8c)+32^{d+1}\omega C^2(2^{d+1}-1)^{d+2}\}$$

that depends on ω is chosen and the dimension d.

(3) What remains is to bound the Gaussian integral in (3.51) by $2^{|X|}$. Let η_{X^*} be a cut-off function such that spt $\eta_{X^*} \subset (X^*)^*, \eta_{X^*} = 1$ on X^*, and

$$\left|\nabla^l \eta_{X^*}\right| \leq \Theta L^{-lk}.$$

Then the integral in (3.51) is bounded by

$$\int_{\boldsymbol{\mathcal{X}}_N} \exp\left(\frac{1}{2}\varkappa(\mathscr{B}_k\xi,\xi)\right) \, \mathrm{d}\mu_{k+1}(\xi),$$

where $\varkappa = 2\overline{C}h^{-2}$ and

$$(\mathscr{B}_k\xi,\xi) = \frac{1}{L^{2k}} \sum_{x\in\Lambda_N} \sum_{l=0}^{M+4} L^{2lk}\left|\eta_{X^*}(x)(\nabla^l\xi)(x)\right|^2. \tag{3.52}$$

Thus,

$$\mathscr{B}_k = \mathscr{B}_k^{(0)} + \sum_{l=1}^{M+4} \mathscr{B}_k^{(l)}$$

with

$$\mathscr{B}_k^{(l)}\xi = \frac{1}{L^{2k}}(\nabla^l)^*\eta_{X^*}^2\nabla^l\xi, \; l=1,\dots,M+4, \quad \text{and} \quad \mathscr{B}_k^{(0)}\xi = \frac{1}{L^{2k}}\Pi(\eta_{X^*}^2\xi),$$

where $\Pi\colon \boldsymbol{\mathcal{V}}_N \to \boldsymbol{\mathcal{X}}_N$ is the projection $(\Pi\varphi)(x) = \varphi(x) - \frac{1}{|\Lambda_N|}\sum_{y\in\Lambda_N}\varphi(y)$.

A formal Gaussian calculation with respect to the measure μ_{k+1} with the covariance operator $\mathscr{C}_{k+1}$ yields

$$\int_{\boldsymbol{\mathcal{X}}_N} \exp\left(\frac{1}{2}\varkappa(\mathscr{B}_k\xi,\xi)\right) \mathrm{d}\mu_{k+1}(\xi) = \left(\frac{\det(\mathscr{C}_{k+1}^{-1} - \varkappa\mathscr{B}_k)}{\det(\mathscr{C}_{k+1}^{-1})}\right)^{-\frac{1}{2}} = \det\left(\mathrm{Id} - \varkappa\mathscr{C}_{k+1}^{\frac{1}{2}}\mathscr{B}_k\mathscr{C}_{k+1}^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

In order to justify the above formula, one derives a bound on the spectrum $\sigma(\mathscr{C}_{k+1}^{\frac{1}{2}}\mathscr{B}_k\mathscr{C}_{k+1}^{\frac{1}{2}})$. The next lemma is contained in [2].

Lemma 3.9.3 ([2, Lemma 5.3])**.**

(i) The operators $\mathscr{C}_{k+1}^{\frac{1}{2}}\mathscr{B}_k\mathscr{C}_{k+1}^{\frac{1}{2}}$ are symmetric and positive definite.

There exist constants M_0 and M_1 that depend only on the dimension d such that for any N and any $k = 1,\dots,N$,

(ii) $\sup\sigma(\mathscr{C}_{k+1}^{\frac{1}{2}}\mathscr{B}_k\mathscr{C}_{k+1}^{\frac{1}{2}}) \leq M_0 L^{d+\eta(d)}$ and

(iii) $\mathrm{Tr}\left(\mathscr{C}_{k+1}^{\frac{1}{2}}\mathscr{B}_k\mathscr{C}_{k+1}^{\frac{1}{2}}\right) \leq M_1|X|_k L^{\eta(d)}$.

Postponing momentarily the proof of the Lemma, we observe that for $\varkappa < \frac{1}{2M_0 L^{d+\eta(d)}}$ the eigenvalues λ_j, $j = 1, \dots, L^{Nd} - 1$ of $\varkappa \mathscr{C}_{k+1}^{\frac{1}{2}} \mathscr{B}_k \mathscr{C}_{k+1}^{\frac{1}{2}}$ lie between 0 and $\frac{1}{2}$. The formal Gaussian calculation is thus justified and

$$\log\det\Big(\mathrm{Id} - \varkappa \mathscr{C}_{k+1}^{\frac{1}{2}} \mathscr{B}_k \mathscr{C}_{k+1}^{\frac{1}{2}}\Big) \geq \sum_i \log(1-\lambda_i) \geq \sum_i -2\lambda_i = -2\mathrm{Tr}\Big(\varkappa \mathscr{C}_{k+1}^{\frac{1}{2}} \mathscr{B}_k \mathscr{C}_{k+1}^{\frac{1}{2}}\Big)$$
$$\geq -2M_1 L^{\eta(d)} \varkappa |X|_k = -4\overline{C} M_1 L^{\eta(d)} h^{-2} |X|_k.$$

Hence

$$\det\Big(\mathrm{Id} - \varkappa \mathscr{C}_{k+1}^{\frac{1}{2}} \mathscr{B}_k \mathscr{C}_{k+1}^{\frac{1}{2}}\Big)^{-\frac{1}{2}} \leq \mathrm{e}^{\frac{4\overline{C} M_1 |X|_k}{h^2}} \tag{3.53}$$

and the Lemma 3.9.2 follows with $h_0(d,\omega)^2 = 4\overline{C} \max\big(M_0, M_1 \frac{\log 2}{5^d}\big)$. □

3.9.1 Smoothness

Let us now proceed by proving Proposition 3.7.2, which asserts the smoothness of the renormalization map

$$S\colon M_0(\mathcal{B}, \boldsymbol{\mathcal{X}}) \times M(\mathcal{P}^{\mathrm{c}}, \boldsymbol{\mathcal{X}}) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to M((\mathcal{P}')^{\mathrm{c}}, \boldsymbol{\mathcal{X}}).$$

To simplify notation, we denote by $\mathcal{B} = \mathcal{B}_k$, $\mathcal{P} = \mathcal{P}_k$, and $\mathcal{P}' = \mathcal{P}_{k+1}$ where k fixed. Let us recall the explicit formula (3.29) for $K_{k+1} = K'$,

$$K'(U,\varphi) = \sum_{X \in \mathcal{P}(U)} \chi(X,U) \widetilde{I}^{U\setminus X}(\varphi) \int_{\boldsymbol{\mathcal{X}}} \big(\widetilde{J}(\varphi) \circ L(\varphi+\xi)\big)(X)\, \mathrm{d}\mu_{k+1}(\xi)$$

with $\widetilde{I} = \mathrm{e}^{-\widetilde{H}}$, $\widetilde{J} = 1 - \widetilde{I}$, $L = (I-1)\circ K$, and $I = \mathrm{e}^{-H}$.

As in [2], we split the map S into a composition of a series of maps and prove the smoothness of each of them. Namely,

$$S_0\colon (M(\mathcal{B},\boldsymbol{\mathcal{X}}), |||\cdot|||_k) \times (M(\mathcal{B},\boldsymbol{\mathcal{X}}), |||\cdot|||_k) \times (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k:k+1,r}) \to (M((\mathcal{P}')^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A')}_{k+1,r}),$$
$$E\ \colon (M_0(\mathcal{B},\boldsymbol{\mathcal{X}}), \|\cdot\|_{k,0}) \to (M(\mathcal{B},\boldsymbol{\mathcal{X}}), |||\cdot|||_k),$$
$$S_1\colon (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k,r}) \times \mathscr{E}, \|\cdot\|) \to (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k:k+1,r}),$$
$$S_2\colon (M_0(\mathcal{B},\boldsymbol{\mathcal{X}}), \|\cdot\|_{k,0}) \times (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k,r}) \times (\mathscr{E}, \|\cdot\|) \to (M_0(\mathcal{B},\boldsymbol{\mathcal{X}}), \|\cdot\|_{k,0}),\ \text{and}$$
$$S_3\colon (M(\mathcal{B},\boldsymbol{\mathcal{X}}), |||\cdot|||_k) \times (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k,r}) \to (M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}}), \|\cdot\|^{(A)}_{k,r})$$

which are defined by

$$S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U,\varphi) = \sum_{\substack{X_1,X_2 \in \mathcal{P}(U)\\ X_1\cap X_2 = \varnothing}} \chi(X_1\cup X_2, U) \widetilde{I}^{U\setminus(X_1\cup X_2)}(\varphi) \widetilde{J}^{X_1}(\varphi) \widetilde{L}(X_2,\varphi),$$
$$E(\widetilde{H}) = \exp\{-\widetilde{H}\} = \widetilde{I},$$
$$S_1(L,\boldsymbol{q})(X,\varphi) = (\boldsymbol{R}^{(\boldsymbol{q})} L)(X,\varphi) = \int_{\boldsymbol{\mathcal{X}}} L(X,\varphi+\xi)\, \mathrm{d}\mu^{(\boldsymbol{q})}_{k+1}(\xi)), X \in \mathcal{P}^{\mathrm{c}}, \tag{3.54}$$
$$S_2(H,K,\boldsymbol{q})(B,\varphi) = \Pi T_2\Big((\boldsymbol{R}^{(\boldsymbol{q})} H)(B,\varphi) - \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \tfrac{1}{|X|} (\boldsymbol{R}^{(\boldsymbol{q})} K)(X,\varphi)\Big),\ \text{and}$$
$$S_3(I,K) = (I-1)\circ K.$$

Hence,

$$S(H,K,\boldsymbol{q}) = S_0\big(E(S_2(H,K,\boldsymbol{q})), 1-E(S_2(H,K,\boldsymbol{q})), S_1(S_3(E(H),K),\boldsymbol{q})\big).$$

In the next subsections, we will show that all maps considered are smooth and that they satisfy certain bounds, hence showing that the map S is smooth.

In the next proposition, we extend [2, Proposition 5.6].

Proposition 3.9.4. *Let $S_1 \colon M(\mathcal{P}^c, \boldsymbol{\mathcal{X}}) \times \mathcal{E} \to M((\mathcal{P})^c, \boldsymbol{\mathcal{X}})$ be the map defined in* (3.54)*, restricted to $B_{\rho_1} \times B_{\rho_2}$. Then, there exist a constant $h_0 = h_0(d,\omega)$ depending only on the dimension d and value of the parameter ω, such that for any $h \geq L^{(d+\eta(d))/2} h_0$ and $X \in \mathcal{P}_k$, one has that*

$$\|S_1(L,\boldsymbol{q})\|_{k:k+1,r}^{(A/2)} \leq 2^{2^d} \|L\|_{k,r}^{(A)}.$$

Moreover,

$$\|D_L^j D_{\boldsymbol{q}}^l S_1(L,\boldsymbol{q})(\dot{L},\ldots,\dot{L},\dot{\boldsymbol{q}},\ldots,\dot{\boldsymbol{q}})\|_{k:k+1,r-2l}^{A/2} \leq C\|\dot{L}\|^j \|\boldsymbol{q}\|^l, \qquad \text{for } r=1,\ldots,r_0 \quad \text{and} \quad 2l \leq r$$

for all $(L,\boldsymbol{q}) \in B_{\rho_1} \times B_{\rho_2}$.

Proof. The first part of the statement and the smoothness with respect to K are immediate consequences of the definitions and Lemma 3.9.1. Thus, we will only show the smoothness with respect to $\boldsymbol{q}$. Fix X and φ. With simple calculations, one has that

$$\partial_{\boldsymbol{q}} S_1(K,\boldsymbol{q})|_{t=0} = \frac{1}{2}\int_{\mathcal{X}} \mathrm{Tr}\Big[D_\xi^2 K(X,\varphi+\xi)\ \partial_{\boldsymbol{q}} C_{k+1}^{\boldsymbol{q}}(\dot{\boldsymbol{q}})\Big]\, \mathrm{d}\mu^{\boldsymbol{q}}(\xi),$$

where the symbol Tr is used to denote the usual trace operator. Using the finite range decomposition, one has that $\|\partial_{\boldsymbol{q}} C_{k+1}^{\boldsymbol{q}}(\dot{\boldsymbol{q}})\| \leq CL^{\tilde{\eta}(d)}\|\dot{\boldsymbol{q}}\|$.

Moreover, it is immediate to notice that $|\nabla^\alpha D_\varphi^s K(X,\varphi+\xi)|^{k+1,X} \leq |\nabla^\alpha D_\varphi^s K(X,\varphi+\xi)|^{k,X}$, thus

$$||\partial_{\boldsymbol{q}} S_1(K,\boldsymbol{q})(X)||_{k:k+1,X,r-2} \leq \sup_{|\varphi|_{k,X}\leq 1} \int_{\mathcal{X}} \Big|\mathrm{Tr}\Big(D_\xi^2 K(X,\varphi+\xi)\partial_{\boldsymbol{q}} C^{\boldsymbol{q}}(\dot{\boldsymbol{q}})\Big)\Big|^{k,X,r-2} w_{k:k+1,X}(\varphi)\, \mathrm{d}\mu^{\boldsymbol{q}}(\xi),$$

hence the right hand side of the previous formula can be estimated from above by

$$\begin{aligned} Cr(r-1)\int_{\mathcal{X}} \Big|K(X,\varphi)\Big|^{k,X,r} L^{\tilde{\eta}(d)}|\dot{\boldsymbol{q}}|^2 &\leq \|K\|_{k,X,r}\|\dot{\boldsymbol{q}}\|^2 \int_{\mathcal{X}} w_k^X(\varphi+\xi) w_{k:k+1}(\varphi)^{-X}\, \mathrm{d}\mu_k^{\boldsymbol{q}}(\xi) \\ &\leq Cr(r-1)2^{|X|_k} L^{\tilde{\eta}(d)}/h^2 \|K\|_{k,X,r}. \end{aligned}$$

From which we get the bound on the first derivative.

For the second and the third derivative, the calculations are similar. □

In the next lemma, we extend [2, Lemma 5.7].

Lemma 3.9.5. *The linear map $F(B,\cdot) \mapsto \Pi T_2 F(B,\cdot)$ is bounded, i.e., there exists $C > 0$ so that for any $F \in M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ one has*

$$\|\Pi T_2 F(B)\|_{k,0} \leq C\|F(B)\|_{k,B}.$$

Proof. Let $F \in M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ and let $H(B,\varphi) = \Pi T_2 F(B,\varphi) = H(B,0) + \ell(\dot\varphi) + Q(\dot\varphi)$. Then

$$L^{dk}|\lambda| = |H(B,0)| = |F(B,0)| \le \|F(B)\|_{k,B}.$$

Let $\dot\varphi$ be affine on $(B^*)^*$ ($*$-neighbourhood of B^*), i.e.,

$$\dot\varphi = \sum_{i=1}^{d} \eta_i \pi_i,$$

where $\eta = (\eta_i)_{i=1,\dots,d} \in \mathbb{R}^d$, and π_i is the coordinate projection $\pi_i(x) = x_i$ for $x \in \mathbb{Z}^d$. Then $\nabla_i \dot\varphi = \eta_i$ and $\nabla^\alpha \dot\varphi = 0$ on B^* if $|\alpha| \ge 2$.

Hence by the definition of the norms and the definition of the "projection" operator, one has that,

$$\begin{aligned}|\sum_{i,j=1}^{d} \boldsymbol{d}_{i,j}(x)\eta_i\eta_j| &= \Big|\sum_{i=1}^{d} D^2F(B,0)(\eta_{x,i},\dot\varphi)\Big| \le \sum_i \frac{1}{2}\left|D^2F(B,0)\right|^{x,i,k,B} |\dot\varphi|_{k,B}|\eta_i| \\ &= \frac{1}{2}\left|D^2F(B,0)\right|^{k,B} h^{-2} \max_{i\in\{1,\dots,d\}} |\eta_i|^2 L^{dk}.\end{aligned}$$

This implies

$$\sup_\eta h^2 \frac{|\sum_{i,j=1}^{d} \boldsymbol{d}_{i,j}(x)\eta_i\eta_j|}{\max_{i\in\{1,\dots,d\}}|\eta_i|^2} \le \frac{1}{2}\left|D^2F(B,0)\right|^{k,B},$$

thus

$$h^2 \sum_{i,j=1}^{d} |\boldsymbol{d}_{i,j}| \le \frac{d(d+1)}{2}\frac{1}{2}\left|D^2F(B,0)\right|^{k,B}.$$

In a similar way, one has that

$$L^{dk}\sum_{i=1}^{d} a_i\eta_i = \ell(\dot\varphi) = DF(B,0)(\dot\varphi) \le \left|DF(B,0)\right|^{k,B} h^{-1} \max_{i\in\{1,\dots,d\}} |\eta_i| L^{\frac{dk}{2}},$$

thus

$$hL^{\frac{dk}{2}} \sum_{i=1}^{d} |a_i| \le \left|DF(B,0)\right|^{k,B},$$

which proves the desired claim. □

In the next lemma, we extend [2, Lemma 5.8].

Lemma 3.9.6. *There exists $\delta > 0$ such that the mapping S_2 is smooth in*

$$\mathcal{U}_\delta := \left\{(H,K,\boldsymbol{q}) \in M_0(\mathcal{B},\mathcal{X}) \times M(\mathcal{P},\mathcal{X}) \times \mathcal{E}, \|H\|_{k,0} < \delta,\ \|K\|_{k,r}^{(A)} < \delta,\ \|\boldsymbol{q}\|_{\mathscr{E}} < \delta\right\}$$

uniformly in $k = 1,\dots,N$ and the derivatives up the order 3 are uniformly bounded in $\mathcal{U}_\delta$ i.e.,

$$\|D_H^i D_K^j D_{\boldsymbol{q}}^l S_2(\dot H,\dots,\dot H,\dot K,\dots,\dot K,\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k,0,B} \le \frac{C}{h^2} L^{\tilde\eta(d)}\delta \|\dot H\|_{k,0,B}^{j} \|\dot K\|_{k,r}^{j} \|\dot{\boldsymbol{q}}\|_{\mathcal{E}}^{l}$$

$j,k,l \in \{1,2,3\}$ with $j+k+l = 3$ and $r \ge 2l+3$.

Proof. Given that the map S_2 is linear in H and K it is immediate to notice that $D^2_H S_2 = 0$, $D^2_K S_2 = 0$ and $D_H D_K S_2(H, K, \boldsymbol{q}) = 0$.

Let us define

$$R(B,\varphi) := \sum_{X\in S_k, X\supset B} \frac{1}{|X|_k} \int_{\mathcal{X}} K(X,\varphi+\xi)\,\mathrm{d}\mu^{\boldsymbol{q}}_k(\xi)$$

From the definitions one has that

$$D_\varphi R(X,0) = \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\mathcal{X}} D_\varphi K(X,\varphi+\xi)\big|_{\varphi=0}(\dot\varphi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi),$$
$$D^2_\varphi R(X,0) = \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\mathcal{X}} D^2_\varphi K(X,\varphi+\xi)\big|_{\varphi=0}(\dot\varphi,\dot\varphi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi),$$

and

$$L^{N|\beta|}\nabla^\beta D_\varphi R(X,0) = \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\mathcal{X}} L^{N|\beta|}\nabla^\beta D_\varphi K(X,\varphi+\xi)\big|_{\varphi=0}(\dot\varphi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi),$$
$$L^{N|\beta|}\nabla^\beta D^2_\varphi R(X,0) = \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\boldsymbol{\mathcal{X}}} L^{N|\beta|}\nabla_\beta D^2_\varphi K(X,\varphi+\xi)\big|_{\varphi=0}(\dot\varphi,\dot\varphi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi).$$

Let us estimate only the second term. The first one follows in a similar fashion. For every φ such that $|\varphi|_{k,X} \leq 1$, one has that

$$\begin{aligned}\frac{1}{2}\Big|D^2_\varphi R(B,\varphi+\xi)\Big|^{k,X,r} &\leq \frac{1}{2} \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\boldsymbol{\mathcal{X}}} |K|^{k,X,r}\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi)\\ &\leq \frac{1}{2} \sum_{X\in \mathcal{S}_k, X\supset B} \frac{1}{|X|_k} \int_{\boldsymbol{\mathcal{X}}} |K|_{k,X,r} w^X_k(\xi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi).\end{aligned}$$

Recall that $\int_{\boldsymbol{\mathcal{X}}} w^X_k(\xi)\,\mathrm{d}\mu^{\boldsymbol{q}}(\xi) \leq 2^{|X|_k}$ and that $|\{X \in S_k :\ X \supset B\}| \leq (3^d-1)^{2^d}$. Hence it is enough to use Lemma 3.9.5 to obtain the desired result.

Let us now turn to the estimates with respect to $\boldsymbol{q}$. Let

$$F(B,\varphi) = \int_{\mathcal{X}} F(B,\varphi+\xi)\,\mathrm{d}\mu^{\boldsymbol{q}(t)}(\xi),$$

where the map $t \mapsto \boldsymbol{q}(t) \in \mathcal{E}$ is a C^3.

We need estimate

$$\|\frac{d^l}{\mathrm{d}t^l}\Big|_{t=0} \Pi T_2 F(B)\|_{k,0}, \qquad l = 1,2,3$$

Due to similarity in the calculations, we will estimate only the first derivative.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{X}} F(B,\varphi+\xi)\,\mathrm{d}\mu^{\boldsymbol{q}(t)}(\xi)\Big|_{t=0} = \frac{1}{2}\int_{\mathcal{X}} \Big(\mathrm{Tr}[D^2_\xi F(B,\varphi+\xi)\dot{\mathcal{C}}^{\boldsymbol{q}(t)}_{k+1}]\Big)\,\mathrm{d}\mu^{\boldsymbol{q}(t)}(\xi)$$

Finally, using the Finite Range Decomposition property (cf. Theorem 2.3.1), one has that

$$\|D^l_{\boldsymbol{q}}\Pi T_2 F(B)(\boldsymbol{q})(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})\|_{k,0} \le \frac{C}{h^2} L^{\tilde{\eta}(d)}\|F(B)\|_{k,B,r}$$

whenever $r \ge 2l+3$ and $l \le 3$ uniformly for $\|\boldsymbol{q}\|_{\mathcal{E}} \le 1/2$.

□

The following lemma is a contained in [2, Proposition 5.8].
Lemma 3.9.7 ([2, Proposition 5.8])**.** *Consider the map*

$$S_0 \colon M(\mathcal{B},\boldsymbol{\mathcal{X}}) \times M(\mathcal{B},\boldsymbol{\mathcal{X}}) \times M(\mathcal{P},\boldsymbol{\mathcal{X}}) \to M((\mathcal{P}')^{\mathrm{c}},\boldsymbol{\mathcal{X}})$$

defined in (3.54)*, restricted to* $B_{\rho_1}(1)\times B_{\rho_1}\times B_{\rho_2} \subset M(\mathcal{B},\boldsymbol{\mathcal{X}}) \times M(\mathcal{B},\boldsymbol{\mathcal{X}}) \times M(\mathcal{P}^{\mathrm{c}},\boldsymbol{\mathcal{X}})$ *with the balls* B_{ρ_1} *and* B_{ρ_2} *defined in terms of norms* $|||\cdot|||_k$ *and* $\|\cdot\|^{(A)}_{k:k+1,r}$*, and the target space* $M((\mathcal{P}')^{\mathrm{c}},\boldsymbol{\mathcal{X}})$ *equipped with the norm* $\|\cdot\|^{(A)}_{k+1,r}$*. For any* $A\ge 3$ *and* ρ_1,ρ_2 *such that*

$$\rho_1 < \tfrac{1}{2}A^{-1}, \ \textit{and}\ \rho_2 < \tfrac{1}{2}A^{-2^d},$$

the map S_0 *is smooth and, for any* $j_1,j_2,j_3\in\mathbb{N}, j_1,j_2,j_3\le r_0$*, satisfies the bound*

$$\frac{1}{j_1!}\frac{1}{j_2!}\frac{1}{j_3!}\Big\|D_1^{j_1}D_2^{j_2}D_3^{j_3}S_0(\widetilde{I},\widetilde{J},\widetilde{L})(\dot{\widetilde{I}},\dots,\dot{\widetilde{I}},\dot{\widetilde{J}},\dots,\dot{\widetilde{J}},\dot{\widetilde{L}},\dots,\dot{\widetilde{L}})\Big\|^{(A/3)}_{k+1,r} \le$$
$$\le (2A)^{j_1+j_2}\big(2A^{2^d}\big)^{j_2}|||\dot{I}|||_k^{j_1}\,|||\dot{J}|||_k^{j_2}\,\|\dot{\widetilde{L}}\|^{j_3}_{k:k+1,r}. \quad (3.55)$$

$$\frac{1}{j_1!}\frac{1}{j_2!}\frac{1}{j_3!}\|D_1^{j_1}D_2^{j_2}D_3^{j_3}S_0(\tilde{I},\tilde{J},\tilde{K})\|^{A/3}_{k+1,r} \le (2A)^{j_1+j_2}(A^{2^d})^{j_3}|||\dot{\tilde{I}}|||^{j_1}|||\dot{\tilde{J}}|||_k^{j_2}|||\dot{K}|||_{k:k+1,r}$$

Proof. Pick $U\in\mathcal{P}^{\mathrm{c}}_{k+1}$. Then

$$\begin{aligned}\|S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U)\|_{k+1,U,r} &\le \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\X_1\cap X_2=\varnothing}}\chi(X_1\cup X_2,U)|||\widetilde{I}|||_k^{|U\setminus(X_1\cup X_2)|}|||\widetilde{J}|||_k^{|X_1|}\big(\|\widetilde{L}\|_{k:k+1,r}\big)^{|\mathcal{C}(X_2)}A^{2^d|\mathcal{C}(X_2)|}A^{-|X_2|}\\
&\le \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\X_1\cap X_2=\varnothing}}\chi(X_1\cup X_2,U)2^{|U\setminus(X_1\cup X_2)|}\big(\frac{1}{2}A^{-1}\big)^{|X_1|}\prod_{Y\in\mathcal{C}(X_2)}\|\widetilde{L}(Y)\|_{k:k+1,Y,r}\\
&\le 2^{|U|}\sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\X_1\cap X_2=\varnothing}}\chi(X_1\cup X_2,U)2^{-|X_1|-|X_2|-|\mathcal{C}(X_2)|}A^{-|X_1|-|X_2|} =: 2^{|U|_k}k(A,U).\end{aligned}$$

Thus

$$\begin{aligned}\|S_0(\widetilde{I},\widetilde{J},\widetilde{L})\|^{(A/3)}_{k+1,r} &= \sup_{U\in\mathcal{P}^{\mathrm{c}}_{k+1}} 2^{|U|_k}\|S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U)\|_{k+1,U,r}\Gamma_{A/3}(U)\\
&\le \sup_{U\in\mathcal{P}^{\mathrm{c}}_{k+1}} 2^{(L^d+1)|U|_{k+1}}\big(\frac{A}{3}\big)^{|U|_{k+1}}k(A,U).\end{aligned} \quad (3.56)$$

For any $c>1$ we have $\lim_{A\to\infty}2^{(L^d+1)|U|_{k+1}}\big(\frac{A}{3}\big)^{(1-c)|U|_{k+1}}=0$, and by an adaption of [10, Lemma 6.17] we get the claim, i.e. $\lim_{A\to\infty}\|S_0(\widetilde{I},\widetilde{J},\widetilde{L})\|^{(A/3)}_{k+1,r}=0$.

$$\frac{1}{j_1!}\frac{1}{j_2!}\frac{1}{j_3!}D_1^{j_1}D_2^{j_2}D_3^{j_3}S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U)(\dot{\widetilde{I}},\ldots,\dot{\widetilde{I}},\dot{\widetilde{J}},\ldots,\dot{\widetilde{J}},\dot{\widetilde{L}},\ldots,\dot{\widetilde{L}}) =$$
$$= \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing}} \chi(X_1\cup X_2,U) \sum_{\substack{Y_1\in\mathcal{P}(U\setminus(X_1\cup X_2)),|Y_1|=j_1\\ Y_2\in\mathcal{P}(X_1),|Y_2|=j_2\\ \mathcal{J}\subset\mathcal{C}(X_2),|\mathcal{J}|=j_3}} \widetilde{I}^{(U\setminus(X_1\cup X_2))\setminus Y_1}(\dot{\widetilde{I}})^{Y_1}\widetilde{J}^{X_1\setminus Y_2}(\dot{\widetilde{J}})^{Y_2} \prod_{Z\in\mathcal{C}(X_2)\setminus\mathcal{I}} \widetilde{L}(Z)\prod_{Z\in\mathcal{I}}\dot{\widetilde{L}}$$

Applying Lemma 3.9.1 (iia) and (i) as well as the obvious bounds (3.21) and $\chi(X_1\cup X_2,U)\le 1$, we get

$$\frac{1}{j_1!}\frac{1}{j_2!}\frac{1}{j_3!}\Big\|D_1^{j_1}D_2^{j_2}D_3^{j_3}S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U)(\dot{\widetilde{I}},\ldots,\dot{\widetilde{I}},\dot{\widetilde{J}},\ldots,\dot{\widetilde{J}},\dot{\widetilde{L}},\ldots,\dot{\widetilde{L}})\Big\|_{k+1,U,r} \le$$
$$\le \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing}} \binom{|U\setminus(X_1\cup X_2)|}{j_1}\binom{|X_1|}{j_2}\binom{|\mathcal{C}(X_2)|}{j_3}|||\widetilde{I}|||_k^{|U\setminus(X_1\cup X_2)|-j_1}|||\dot{\widetilde{I}}|||_k^{j_1}|||\widetilde{J}|||_k^{|X_1|-j_2}|||\dot{\widetilde{J}}|||^{j_2}\times$$
$$\times(\|\widetilde{L}\|_{k:k+1,r}^{(A)})^{|\mathcal{C}(X_2)|-j_3}(\|\dot{\widetilde{L}}\|_{k:k+1,r}^{(A)})^{j_3}\prod_{Z\in\mathcal{C}(X_2)}\Gamma_A(Z)^{-1}. \quad (3.58)$$

Hence,

$$\frac{1}{j_1!}\frac{1}{j_2!}\frac{1}{j_3!}\Big\|D_1^{j_1}D_2^{j_2}D_3^{j_3}S_0(\widetilde{I},\widetilde{J},\widetilde{L})(U)(\dot{\widetilde{I}},\ldots,\dot{\widetilde{I}},\dot{\widetilde{J}},\ldots,\dot{\widetilde{J}},\dot{\widetilde{L}},\ldots,\dot{\widetilde{L}})\Big\|_{k+1,U,r} \le$$
$$\le 3^{|U|_k}A^{-|U|_k}\Big(\frac{A}{1-A|||\widetilde{I}|||_k}\Big)^{j_1}|||\dot{\widetilde{I}}|||_k^{j_1}\Big(\frac{A}{1-A|||\widetilde{J}|||_k}\Big)^{j_2}|||\dot{\widetilde{J}}|||^{j_2}\Big(\frac{A^{2^d}}{1-A^{2^d}\|\widetilde{L}\|_{k:k+1,r}^{(A)}}\Big)^{j_3}(\|\dot{\widetilde{L}}\|_{k:k+1,r}^{(A)})^{j_3} \quad (3.59)$$

which proves the desired claim. ☐

3.9.2 Contraction

Here we prove the contraction property from Proposition 3.7.1.

We need to show that the following holds:
Lemma 3.9.8. *Let $\theta<1$ and $\omega\ge 2(d^2 2^{2d+1}+1)$. There exist constants $h_0=h_0(d,\omega)$, $L_0=L_0(d,\omega)$, and $A_0=A_0(d,\omega)$ such that*

$$\|\boldsymbol{C}^{(\boldsymbol{q})}\|_r = \sup_{\|K\|_{k,r}\le 1}\|\boldsymbol{C}^{(\boldsymbol{q})}K\|_{k+1,r}\le\theta.$$

for any $\|\boldsymbol{q}\|_{\mathcal{E}}\le\frac{1}{2}$, any $k=1,\ldots,N$, $r=1,\ldots,r_0$, and any $L\ge L_0$, $A\ge A_0$ and $h\ge L^{\frac{d+\eta(d)}{2}}h_0$.

Proof. The proof will be divided in several lemmas. In Lemma 3.9.10, we will estimate the first term on the right had side of (3.33), and in Lemma 3.9.9, we will estimate the last term on the right had side of (3.33). ☐

The next lemma is a generalization of [2, Lemma 5.11].

Lemma 3.9.9. *Let $L \geq 2^d + 1$ and $\omega \geq 18\sqrt{2} + 1$. There exist a constant $h_0 = h_0(d, \omega)$ and a function $\varepsilon_d : \mathbb{R}_+ \to \mathbb{R}_+$ depending only on the dimension d such that $\lim_{A\to\infty} \varepsilon_d(A) = 0$ and*

$$\|F\|_{k+1,r} \leq \varepsilon_d(A)\|K\|_{k,r}$$

for any $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ and any $h \geq h_0$. Here, the function $F \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$ is defined by

$$F(U, \varphi) := \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \overline{X} = U}} \int_{\boldsymbol{\mathcal{X}}} K(X, \varphi + \xi)\, \mathrm{d}\mu_{k+1}(\xi).$$

Proof. For any $X \subset U$, because of (3.20), we have that

$$\sup_{\varphi} |(\boldsymbol{R}_{k+1}K)(X, \varphi)|^{k+1,U,r} w_{k+1}^{-U} \leq \sup_{\varphi} |(\boldsymbol{R}_{k+1}K)(X, \varphi)|^{k+1,X,r} w_{k:k+1}^{-X}. \tag{3.60}$$

Indeed, by noticing that

$$|(\boldsymbol{R}_{k+1}K)(X, \varphi)|^{k+1,U,r} \leq |(\boldsymbol{R}_{k+1}K)(X, \varphi)|^{k+1,X,r}$$

and that

$$w_{k+1}^{-U}(\varphi) \leq w_{k:k+1}^{-X},$$

one has that

$$\begin{aligned}\sum_{x \in X} \big((2^d\omega - 1) g_{k:k+1,x}(\varphi) + \omega G_{k,x}(\varphi)\big) + 3L^k \sum_{x \in \partial X} G_{k,x}(\varphi) \\ \leq \sum_{x \in U} \omega\big(2^d g_{k+1,x}(\varphi) + G_{k+1,x}(\varphi)\big) + L^{k+1} \sum_{x \in \partial U} G_{k+1,x}(\varphi).\end{aligned} \tag{3.61}$$

Where in the above formula we have used that $g_{k:k+1,x}(\varphi) \leq g_{k+1,x}(\varphi)$, $G_{k,x}(\varphi) \leq G_{k+1,x}(\varphi)$, and that any $x \in \partial X \setminus \partial U$ is necessarily contained in ∂B for some $B \in \mathcal{B}_k(U \setminus X)$. For each such B one has that

$$3L^k \sum_{x \in \partial B} G_{k,x}(\varphi) \leq \sum_{x \in B} \omega\big(2^d g_{k+1,x}(\varphi) + G_{k+1,x}(\varphi)\big)$$

whenever $\omega \geq 6c + 1$.

Combining (3.60) with the bound from Lemma 3.9.1 (iv), one has that

$$\begin{aligned}&\Gamma_{k+1,A}(U)\|F(U)\|_{k+1,U,r} \leq \\ &\leq A^{|U|_{k+1}} \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \overline{X} = U}} 2^{|X|_k} \|K(X)\|_{k,X,r} \leq \|K\|_{k,r} \sup_{Y \in \mathcal{P}_{k+1}^c} \Big\{ A^{|Y|_{k+1}} \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \overline{X} = Y}} (\tfrac{A}{2})^{-|X|_k} \Big\}.\end{aligned} \tag{3.62}$$

To conclude it is enough to use [10, Lemma 6.18], which asserts that, whenever $L \geq 2^d + 1$, one has that

$$\lim_{A \to \infty} \sup_{Y \in \mathcal{P}_{k+1}^c} \big\{ A^{|Y|_{k+1}} \sum_{\substack{X \in \mathcal{P}_k^c \setminus \mathcal{S}_k \\ \overline{X} = Y}} (\tfrac{A}{2})^{-|X|_k} \big\} = 0.$$

□

The following lemma is contained [2, Lemma 5.13].

Lemma 3.9.10. *Let $L \geq 7$, $\omega \geq 2(d^2 2^{2d+1}+1)$, $h \geq h_0$ (with $h_0 = h_0(\omega, d)$ from Lemma 3.9.1(iv)), and $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ with $G \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$ defined by*

$$G(U,\varphi) := \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} \big(1-\Pi T_2\big) \sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}} \frac{1}{|X|_k} (\boldsymbol{R}_{k+1}K)(X,\varphi).$$

Then

$$\|G\|_{k+1,r} \leq 2^{d+2^d}(3^d-1)^{2^d}\big(5L^{-\frac{d}{2}} + 2^{d+3}L^{\frac{d}{2}-2} + 9L^{-1}\big)\|K\|_{k,r}. \tag{3.63}$$

Proof. It is not difficult to see that the sum vanishes $U \notin \mathcal{S}_{k+1}$. Thus, the norms in (3.63) contain only the contributions of small sets and do not depend on A according to the definition of the factor $\Gamma_{j,A}(X)$, $j = k, k+1$. Defining $R(B,\varphi) := \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}(\boldsymbol{R}_{k+1}K)(X,\varphi)$ and replacing $1-\Pi T_2$ by $(1-T_2)+(T_2-\Pi T_2)$, we trivially have

$$G_1(U,\varphi) := \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} (1-T_2)R(B,\varphi),$$

$$G_2(U,\varphi) := \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} (T_2-\Pi T_2)R(B,\varphi), \tag{3.64}$$

and $G(U,\varphi) = G_1(U,\varphi) + G_2(U,\varphi)$.

We will evaluate them separately in Lemma 3.9.12 and Lemma 3.9.13.

□

The following proof is a generalization of [10, Lemma 6.8] and [2, Lemma 5.14].

Lemma 3.9.11. *Let $F \in \mathcal{M}(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, $X \in \mathcal{P}_k$, $r = 1,\ldots,r_0$, and $j = k, k+1$. Then*

$$|F(X,\varphi) - T_2F(X,\varphi)|^{j,X,r} \leq (1+|\varphi|_{j,X})^3 \sup_{t\in(0,1)} \sum_{s=3}^{r} \frac{1}{s!}|D^sF(X,t\varphi)|^{j,X}. \tag{3.65}$$

Proof. Let us denote by $f(\varphi) = (1-T_2)F(X,\varphi)$ and $f_{x,i,s,\beta}(\varphi) = \nabla^\beta D^s F(X,\varphi)(\delta_{x,i},\dot\varphi,\ldots,\dot\varphi)$ for any $s \geq 1$. The terms on the left hand side of (3.65) can be rewritten via Taylor reminders as

$$f(\varphi) = \int_0^1 \frac{(1-t)^2}{2} D^3F(X,t\varphi)(\varphi,\varphi,\varphi)\,\mathrm{d}t,$$

$$\nabla^\beta Df(\varphi)(\dot\varphi) = \nabla^\beta \int_0^1 (1-t)\nabla^\beta D^3F(X,t\varphi)(\dot\varphi,\varphi,\varphi)\,\mathrm{d}t,$$

$$\nabla^\beta Df(\varphi)(\delta_{x,i}) = \nabla^\beta \int_0^1 (1-t)\nabla^\beta D^3F(X,t\varphi)(\delta_{x,i},\varphi,\varphi)\,\mathrm{d}t,$$

$$\frac{1}{2}\nabla^\beta D^2 f(\varphi)(\delta_{x,i},\dot\varphi) = \int_0^1 D^3F(X,t\varphi)(\dot\varphi,\dot\varphi,\varphi)\,\mathrm{d}t,$$

and, for $s \geq 3$,

$$\frac{1}{s!} D^s f(\varphi)(\dot\varphi, \dots, \dot\varphi) = \frac{1}{s!} D^s F(X, \varphi)(\dot\varphi, \dots, \dot\varphi).$$

$$\frac{1}{s!} D^s f(\varphi)(\delta_{x,i}, \dots, \dot\varphi) = \frac{1}{s!} D^s F(X, \varphi)(\delta_{x,i}, \dot\varphi, \dots, \dot\varphi).$$

To conclude it is sufficient to sum the above equations and use

$$|\nabla^\beta D^{s+m} F(X, t\varphi)(\dot\varphi, \dots, \dot\varphi, \varphi, \dots, \varphi)| \leq |\nabla^\beta D^{s+m} F(X, t\varphi)|^{j,X} |\dot\varphi|^s_{j,X} |\varphi|^m_{j,X},$$

$$|\nabla^\beta D^{s+m} F(X, t\varphi)(\delta_{x,i}, \dot\varphi, \dots, \dot\varphi, \varphi, \dots, \varphi)| \leq |\nabla^\beta D^{s+m} F(X, t\varphi)|^{j,X} |\dot\varphi|^{s-1}_{j,X} |\varphi|^m_{j,X},$$

as well as the fact that

$$|\varphi|^3_{j,X} \int_0^1 \frac{(1-t)^2}{2}\, \mathrm{d}t + |\varphi|^2_{j,X} \int_0^1 (1-t)\, \mathrm{d}t + \frac{1}{2} |\varphi|_{j,X} + \frac{1}{3!} = \frac{1}{3!} (1 + |\varphi|_{j,X})^3.$$

□

The following lemma is a generalization of [2, Lemma 5.15]

Lemma 3.9.12. *Let $K \in \mathcal{M}(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$, $X \in \mathcal{S}_k$, $B \in \mathcal{B}_k(X)$, and $U = \overline{B^*}$. Assume also that $L \geq 7$, $\omega \geq 2(d^2 2^{2d+1} + 1)$, and $h \geq h_0$. Then*

$$\sup_\varphi |(\boldsymbol{R}_{k+1} K)(X, \varphi) - T_2(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,X,r} w_{k+1}^{-U}(\varphi) \leq 5 L^{-\frac{3d}{2}} 2^{|X|_k} \|K(X)\|_{k,X,r}. \tag{3.66}$$

Moreover, one also has

$$\|G_1(U)\|_{k+1,U,r} \leq 5\, 2^{d+2^d} (3^d - 1)^{2^d} L^{-\frac{d}{2}} \|K\|_{k,r}, \tag{3.67}$$

where G_1 is defined in (3.65).

Proof. By using Lemma 3.9.11, for any $\varphi \in \boldsymbol{\mathcal{X}}$ one has that

$$\begin{aligned} &\big|(\boldsymbol{R}_{k+1} K)(X, \varphi) - T_2(\boldsymbol{R}_{k+1} K)(X, \varphi)\big|^{k+1,X,r} \\ &\qquad \leq (1 + |\varphi|_{k+1,X})^3 \sup_{t \in (0,1)} \sum_{s=3}^r \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1} K)(X, t\varphi)|^{k+1,X}. \end{aligned} \tag{3.68}$$

Moreover, interchanging differentiation and integration, one gets

$$\begin{aligned} \sum_{s=3}^r \frac{1}{s!} \big|D^s(\boldsymbol{R}_{k+1} K)(X, t\varphi)\big|^{k+1,X} &\leq \sum_{s=3}^r \frac{1}{s!} \sup_{\dot\varphi \neq 0} \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi) \left| \frac{D^s K(X, t\varphi + \xi)(\dot\varphi, \dots, \dot\varphi)}{|\dot\varphi|^s_{k+1,X}} \right| \\ &= \sum_{s=3}^r \frac{1}{s!} \sup_{\dot\varphi \neq 0} \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi) \left| \frac{D^s K(X, t\varphi + \xi)(\dot\varphi, \dots, \dot\varphi)}{|\dot\varphi|^s_{k,X}} \frac{|\dot\varphi|^s_{k,X}}{|\dot\varphi|^s_{k+1,X}} \right| \\ &\leq L^{-\frac{3d}{2}} \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, |K(X, t\varphi + \xi)|^{k,X,r}, \end{aligned} \tag{3.69}$$

where in the last inequality we used (3.44). Given that

$$|K(X, t\varphi + \xi)|^{k,X,r} \leq \|K(X)\|_{k,X,r} w_k^X(t\varphi + \xi)$$

and (3.46), one has that

$$\sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X, t\varphi)|^{k+1,X} \leq 2^{|X|_k} L^{-\frac{3d}{2}} \|K(X)\|_{k,X,r} \frac{w^X_{k:k+1}(\varphi)}{w^U_{k+1}(\varphi)} w^U_{k+1}(\varphi), \quad (3.70)$$

where in the above inequality, one uses fact that $w^X_{k:k+1}(t\varphi)$ is monotone in t.
Bounding $(1 + |\varphi|_{k+1,X})^3$ via

$$(1+u)^3 \leq 5\mathrm{e}^{u^2}, \quad (3.71)$$

it is not difficult to show that

$$|\varphi|^2_{k+1,X} \leq \log \frac{w^U_{k+1}(\varphi)}{w^X_{k:k+1}(\varphi)}. \quad (3.72)$$

Indeed, notice that

$$\begin{aligned} \log \frac{w^U_{k+1}(\varphi)}{w^X_{k:k+1}(\varphi)} &\geq \sum_{x \in U \setminus X} \big((2^d\omega - 1) g_{k+1,x}(\varphi) + \omega G_{k+1,x}(\varphi)\big) + \sum_{x \in U} g_{k:k+1,x}(\varphi) \\ &\quad + L^k(L-3) \sum_{x \in \partial U} G_{k+1,x}(\varphi) - 3L^k \sum_{x \in \partial X \setminus \partial U} G_{k,x}(\varphi) \\ &\geq \sum_{x \in U \setminus X} (2^d\omega - 1) g_{k+1,x}(\varphi) + L^k(L-3) \sum_{x \in \partial U} G_{k+1,x}(\varphi). \end{aligned} \quad (3.73)$$

To verify the last inequality, we show that

$$3L^k \sum_{x \in \partial X \setminus \partial U} G_{k,x}(\varphi) \leq \sum_{x \in U} g_{k:k+1,x}(\varphi) + \sum_{x \in U \setminus X} \omega G_{k+1,x}(\varphi)$$

in analogy with (3.43). Indeed, arguing that any $x \in \partial X \setminus \partial U$ is contained in ∂B for $B \in \mathcal{B}_k(U \setminus X)$, and applying again Proposition 3.6.4 (a), we have

$$\begin{aligned} h^2 L^k \sum_{x \in \partial B} G_{k,x}(\varphi) &\leq 2c\Big(\sum_{x \in B} |\nabla\varphi(x)|^2 + L^{2k} \sum_{x \in U_1(B)} |\nabla^2\varphi(x)|^2\Big) + L^k \sum_{x \in \partial B} \sum_{s=2}^{3} L^{(2s-2)k} |\nabla^s\varphi(x)|^2 \\ &\leq h^2 2c \sum_{x \in B} G_{k,x}(\varphi) + h^2 2c L^k \sum_{x \in \partial B} L^{-2} g_{k:k+1,z}(\varphi), \end{aligned} \quad (3.74)$$

where z is any point $z \in B$. Using $|\partial B| \leq 2^d L^{(d-1)k}$, we get the desired bound once $\omega \geq 18\sqrt{2}$ and $L \geq 5$ (when $6c \leq \omega$ and $6cL^{-2} \leq 1$).
In view of (3.73) and using that $|\varphi|^2_{k+1,X} \leq |\varphi|^2_{k+1,U}$, it suffices to show that

$$|\varphi|^2_{k+1,U} \leq \sum_{x \in U \setminus X} (2^d\omega - 1) g_{k+1,x}(\varphi) + L^k(L-3) \sum_{x \in \partial U} G_{k+1,x}(\varphi). \quad (3.75)$$

Given that,

$$h^2 |\varphi|^2_{k+1,U} \leq \sum_{1 \leq s \leq 3} L^{(k+1)(d-2+2s)} \max_{x \in U^*} |\nabla^s\varphi(x)|^2$$

and applying [10, Lemma 6.20], one has

$$L^{(k+1)d}\max_{x\in U^*}|\nabla\varphi(x)|^2 \le \frac{2L^{(k+1)d}}{|\partial U|}\sum_{x\in\partial U}|\nabla\varphi(x)|^2 + 2L^{(k+1)d}(\mathrm{diam}U^*)^2\max_{x\in U^*}\left|\nabla^2\varphi(x)\right|^2.$$

Given that $|\partial U|\ge 2dL^{(k+1)(d-1)}$, the first term above is covered by the second term on the right hand side of (3.75) once $L\ge 7$,

$$\frac{2L^{(k+1)d}}{|\partial U|}\le\frac{2L^{(k+1)d}}{2dL^{(k+1)(d-1)}}=\frac{1}{d}L^{k+1}\le L^k(L-3).$$

Given that $\mathrm{diam}(U^*)\le d2^dL^{k+1}$, the second term is bounded by

$$d^2 2^{2d+1}L^{(k+1)(d+2)}\max_{x\in U^*}\left|\nabla^2\varphi(x)\right|^2$$

and will be treated together with the remaining terms $\max_{x\in U^*}\left|\nabla^s\varphi(x)\right|^2$, $s=2,3$, contained in $|\varphi|^2_{k+1,U}$.

Given that the number of $(k+1)$-blocks in U is at most 2^d, one has that

$$\max_{x\in U^*}\left|\nabla^s\varphi(x)\right|^2\le 2^d\sum_{B\in\mathcal{B}_{k+1}(U)}\max_{x\in B^*}\left|\nabla^s\varphi(x)\right|^2,$$

hence

$$(d^2 2^{2d+1}L^{(k+1)(d+2)}+L^{(k+1)(d+2)})\max_{x\in U^*}\left|\nabla^2\varphi(x)\right|^2\le 2^d(d^2 2^{2d+1}+1)L^{(k+1)(d+2)}\times$$
$$\times\sum_{B\in\mathcal{B}_{k+1}(U)}\max_{x\in B^*}\left|\nabla^2\varphi(x)\right|^2$$

and

$$L^{(k+1)(d+4)}\max_{x\in U^*}\left|\nabla^3\varphi(x)\right|^2\le 2^dL^{(k+1)(d+4)}\sum_{B\in\mathcal{B}_{k+1}(U)}\max_{x\in B^*}\left|\nabla^3\varphi(x)\right|^2.$$

Each of the terms on the right hand sides of the above formula will be bounded by the corresponding term in

$$h^2\sum_{x\in B\setminus X}(2^d\omega-1)g_{k+1,x}(\varphi)=(2^d\omega-1)\sum_{x\in B\setminus X}\sum_{s=2}^{4}L^{(2s-2)(k+1)}\sup_{y\in B_x^*}|\nabla^s\varphi(y)|^2.$$

Indeed, given that $g_{k+1,x}(\varphi)$ is constant over each $(k+1)$-block $B\subset U$, and the volume of $B\setminus X$ is at least $L^{kd}(L^d-2^d)=L^{(k+1)d}(1-(\frac{2}{L})^d)$since the number of k-blocks in X is at most 2^d, while B consists of L^d of them, one needs

$$2^d(d^2 2^{2d+1}+1)L^{(k+1)(d+2)}\le(2^d\omega-1)L^{(k+1)d}(1-(\tfrac{2}{L})^d)L^{2(k+1)}$$

and

$$2^dL^{(k+1)(d+4)}\le(2^d\omega-1)L^{(k+1)d}(1-(\tfrac{2}{L})^d)L^{4(k+1)}.$$

These conditions are satisfied if $\omega\ge 2(d^2 2^{2d+1}+1)$.

Combining (3.70), (3.71), and (3.72), we have that

$$(1+|\varphi|_{k+1,X})^3 \sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X,t\varphi)|^{k+1,X} \leq 5L^{-\frac{3d}{2}} 2^{|X|_k} \|K(X)\|_{k,X,r}\, w_{k+1}^U(\varphi),$$

for any $\varphi \in \boldsymbol{\mathcal{X}}$ and any $t \in (0,1)$, which proves of the inequality (3.66).

To prove (3.67), one uses that $|\mathcal{B}_k(U)| \leq (2L)^d$ and the obvious bound

$$|\{X \in \mathcal{S}_k \mid X \supset B\}| \leq (3^d-1)^{2^d}.$$

Hence,

$$\|G_1(U)\|_{k+1,U,r} \leq 5\, L^{-\frac{3d}{2}} \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} 2^{|X|_k} \|K(X)\|_{k,X,r} \leq$$
$$\leq 5\, L^{-\frac{3d}{2}} (2L)^d (3^d-1)^{2^d} \|K\|_{k,r} 2^{2^d} \leq 5\, 2^{d+2^d} (3^d-1)^{2^d} L^{-\frac{d}{2}} \|K\|_{k,r}. \quad (3.76)$$

□

By using the above, we have the following which is adapted from [2, 5.16].

Lemma 3.9.13. *Let $K \in \mathcal{M}(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$, $U = \overline{B^*}$, and assume that $L \geq 7$ and $\omega \geq 2(d^2 2^{2d+1}+1)$. For G_2 defined in (3.64) we have*

$$\|G_2(U)\|_{k+1,U,r} \leq 2^{2^d+d+1} (3^d-1)^{2^d} \big((2^{d+2}-1) L^{\frac{d}{2}-2} + (8L^{-1}+2L^{-2})\big) \|K\|_{k,r}.$$

Proof. Given that $G_2(U,\varphi) = \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} (T_2 - \Pi T_2) R(B,\varphi)$ with $R \in M^*(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ defined by

$$R(B,\varphi) := \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} (\boldsymbol{R}_{k+1}K)(X,\varphi),$$

one has that the polynomial $\Pi T_2 R(B,\varphi) = \lambda|B| + \ell(\varphi) + Q(\varphi,\varphi)$ is characterised by taking a unique linear function $\ell(\varphi)$ of the form (3.18), $\ell(\varphi) = \sum_{x\in(B^*)^*} \big[\sum_{i=1}^d a_i \nabla_i\varphi(x) + \sum_{i,j=1}^d \boldsymbol{c}_{i,j} \nabla_i\nabla_j\varphi(x)\big]$, Regularization by noise for transport and kinetic equations that agrees with $DR(B,0)(\varphi)$ on all quadratic functions φ on $(B^*)^*$ and a unique quadratic function $Q(\varphi,\varphi)$ of the form (3.71),

$$Q(\varphi,\varphi) = \sum_{x\in(B^*)^*} \sum_{i,j=1}^d \boldsymbol{d}_{i,j}\, \nabla_i\varphi(x)\, \nabla_j\varphi(x),$$

that agrees with $\frac{1}{2}D^2R(B,0)(\varphi,\varphi)$ on all affine functions φ on $(B^*)^*$.

Observing that

$$D(\boldsymbol{R}_{k+1}K)(X,0)(\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, DK(X,\xi)(\varphi)$$
$$D^2(\boldsymbol{R}_{k+1}K)(X,0)(\varphi,\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, D^2K(X,\xi)(\varphi,\varphi),$$

and introducing, similarly as above,

$$\Pi T_2 R_\xi(B,\varphi) = \lambda_\xi |B| + \ell_\xi(\varphi) + Q_\xi(\varphi,\varphi),$$

the uniqueness implies that $\ell(\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, \ell_\xi(\varphi)$ and $Q(\varphi,\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, Q_\xi(\varphi,\varphi)$. Given that $G_2(B,\varphi) = (T_2 - \Pi T_2)R(B,\varphi)$ is a polynomial of second order, we have

$$|G_2(B,\varphi)|^{k+1,U,r} = |G_2(B,\varphi)|^{k+1,U,2}.$$

Let us initially evaluate separately the absolute value of the linear and quadratic terms $P_1(\varphi)$ and $P_2(\varphi)$ in $G_2(B,\varphi)$.

Observing that for any affine function φ_1 and any quadratic function φ_2 on $(B^*)^*$ we have $P_1(\varphi - \varphi_1 - \varphi_2) = P_1(\varphi)$, we get

$$\begin{aligned}
\big|P_1(\varphi)\big| &= \Big|\int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi)\, \big(DR_\xi(B,0)(\varphi-\varphi_1-\varphi_2) - \ell_\xi(\varphi-\varphi_1-\varphi_2)\big)\Big| \le \\
&\le (2^{d+2}-1) \sum_{\substack{X\in\mathcal{S}_k \\ X\supset B}} \frac{1}{|X|_k} \|K(X)\|_{k,X,r} |\varphi-\varphi_1-\varphi_2|_{k,B^*} \int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi) w_k^X(\xi) \le \\
&\le 2^{2^d}(3^d-1)^{2^d}(2^{d+2}-1)\|K\|_{k,r}|\varphi-\varphi_1-\varphi_2|_{k,B^*}. \qquad (3.77)
\end{aligned}$$

Here, we first used the inequalities

$$|\ell_\xi(\varphi)| \le (2^{d+2}-2) \sum_{\substack{X\in\mathcal{S}_k \\ X\supset B}} \frac{1}{|X|_k} |K(X,\xi)|^{k,X,r} |\varphi|_{k,B^*} \qquad (3.78)$$

and

$$|DR_\xi(B,0)(\varphi)| \le \sum_{\substack{X\in\mathcal{S}_k \\ X\supset B}} \frac{1}{|X|_k} |K(X,\xi)|^{k,X,r} |\varphi|_{k,X}$$

combined with the bounds $|K(X,\xi)|^{k,X,r} \le \|K(X)\|_{k,X,r} w_k^X(\xi)$ and $|\varphi|_{k,X} \le |\varphi|_{k,B^*}$, and then the bounds $\int_{\boldsymbol{\mathcal{X}}} \mathrm{d}\mu_{k+1}(\xi) w_k^X(\xi) \le 2^{|X|_k}$, and, as in (3.76), $|\{X\in\mathcal{S}_k \mid X\supset B\}| \le (3^d-1)^{2^d}$.

To verify (3.78), we first observe that for every quadratic $\tilde\varphi$ one has that

$$P_1(\varphi) = D(RK)(X,0)(\varphi - \tilde\varphi)$$

where $\tilde\varphi$ is the projection of φ on the subspace of all the φ which are quadratic. Thus by using the Poincaré inequalities,

$$\inf_{\varphi_1 \text{ affine}} |\varphi-\varphi_1|_{k,B^*} \le \frac{1}{h} L^{k(\frac{d}{2}+1)} \sup_{x\in(B^*)^*} |\nabla^2\varphi(x)| \le L^{-(\frac{d}{2}+1)} |\varphi|_{k+1,B^*} \qquad (3.79)$$

and

$$\inf_{\substack{\varphi_1 \text{ affine,} \\ \varphi_2 \text{ quadratic}}} |\varphi-\varphi_1-\varphi_2|_{k,B^*} \le \frac{1}{h} L^{k(\frac{d}{2}+2)} \sup_{x\in(B^*)^*} |\nabla^3\varphi(x)| \le L^{-(\frac{d}{2}+2)} |\varphi|_{k+1,B^*},$$

we get

$$\big|P_1(\varphi)\big| \le L^{-(\frac{d}{2}+2)} 2^{2^d} (3^d-1)^{2^d} (2^{d+2}-1) \|K\|_{k,r} |\varphi|_{k+1,B^*}. \tag{3.80}$$

A similar claim follows for the quadratic part.

Moreover applying (3.79), one gets

$$|P_2(\varphi,\varphi)| \le \big(4L^{-(d+1)} + L^{-(d+2)}\big) 2^{2^d+1} (3^d-1)^{2^d} \|K\|_{k,r} |\varphi|^2_{k+1,B^*}. \tag{3.81}$$

By combining (3.80) and (3.81), one gets

$$\begin{aligned} &|\big(T_2 - \Pi T_2\big) R(B,\varphi)| \le \\ &\le 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)} + (8L^{-(d+1)} + 2L^{-(d+2)})|\varphi|_{k+1,B^*}\big)|\varphi|_{k+1,B^*}\|K\|_{k,r}. \end{aligned} \tag{3.82}$$

For the first and second the derivatives, notice that

$$D\big(P_1(\varphi) + P_2(\varphi,\varphi)\big)(\dot\varphi) = P_1(\dot\varphi) + 2P_2(\varphi,\dot\varphi)$$

and

$$D^2\big(P_1(\varphi) + P_2(\varphi,\varphi)\big)(\dot\varphi,\dot\varphi) = 2P_2(\dot\varphi,\dot\varphi)$$

hence, by (3.80) and (3.81) one has that

$$\begin{aligned} &\big|D\big(P_1(\varphi) + P_2(\varphi,\varphi)\big)\big|^{k+1,B^*} \le \\ &\le 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)} + (16L^{-(d+1)} + 4L^{-(d+2)})|\varphi|_{k+1,B^*}\big)\|K\|_{k,r}. \end{aligned} \tag{3.83}$$

Using (3.81), one has that

$$\big|D^2\big(P_1(\varphi) + P_2(\varphi,\varphi)\big)\big|^{k+1,B^*} \le 2^{2^d}(3^d-1)^{2^d}(8L^{-(d+1)} + 2L^{-(d+2)})\|K\|_{k,r}.$$

Combining last two inequalities with (3.82), one has that

$$\begin{aligned} &|\big(T_2 - \Pi T_2\big) R(B,\varphi)|^{k+1,B^*,r} \le \\ &\le 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)} + (8L^{-(d+1)} + 2L^{-(d+2)})(1+|\varphi|_{k+1,B^*})\big)(1+|\varphi|_{k+1,B^*})\|K\|_{k,r}. \end{aligned} \tag{3.84}$$

With $(1+u)^2 \le 2e^{u^2}$ and (3.72), we get

$$\|G_2(U)\|_{k+1,U,r} \le 2^{2^d+1}(3^d-1)^{2^d}(2L)^d\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)} + (8L^{-(d+1)} + 2L^{-(d+2)})\big)\|K\|_{k,r}$$

which gives the desired bound.

Lemma 3.9.8 is then proven by combining the claims of Lemma 3.9.9 and Lemma 3.9.10.

□

We next lemma is generalizes in [2, Lemma 5.17].

Lemma 3.9.14. *Let $\theta < 1$ and $\omega \geq 2(d^2 2^{2d+1} + 1)$. There exist constants $h_0 = h_0(d,\omega)$, $L_0 = L_0(d,\omega)$, and $A_0 = A_0(d,\omega)$ such that*

$$\|\boldsymbol{A}^{(\boldsymbol{q})^{-1}}\|_{0;0} \leq \frac{1}{\sqrt{\theta}}$$

and

$$\|\boldsymbol{B}^{(\boldsymbol{q})}\|_{r;0} \leq M$$

for any $\|\boldsymbol{q}\| \leq \frac{1}{2}$, any $k = 1,\dots,N$, $r = 1,\dots,r_0$, and any $L \geq L_0$, $A \geq A_0$ and $h \geq L^{\frac{d+\eta(d)}{2}} h_0$.

Proof. When expressed in the coordinates $\dot{\lambda}, \dot{a}, \dot{\boldsymbol{c}}, \dot{\boldsymbol{d}}$ of $\dot{H}$, the linear map $\boldsymbol{A}$ according to (3.31) keeps $\dot{a}$, $\dot{\boldsymbol{c}}$, and $\dot{\boldsymbol{d}}$ unchanged and only shifts $\dot{\lambda}$ by $\sum_{x\in B}\sum_{i,j=1}^{d} \dot{d}_{i,j}\nabla_i^2\nabla_j^{1*}\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,x)$. Hence, $\boldsymbol{A}^{-1}$ only makes the opposite shift and thus

$$\|\boldsymbol{A}^{-1}\dot{H}\|_{k,0} = L^{dk}|\dot{\lambda}| + L^{\frac{dk}{2}} h\sum_{i=1}^{d}|\dot{a}_i| + L^{\frac{(d-2)}{2}k} h\sum_{i,j=1}^{d}|\dot{\boldsymbol{c}}_{i,j}| + h^2\sum_{i,j=1}^{d}|\dot{\boldsymbol{d}}_{i,j}| + L^{dk}\sum_{i,j=1}^{d}|\dot{\boldsymbol{d}}_{i,j}|\big|\nabla_i^2\nabla_j^{1*}\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,x)\big|.$$

Using

$$\sum_{i,j=1}^{d}|\dot{\boldsymbol{d}}_{i,j}| \leq \frac{1}{h^2}\|\dot{H}\|_{k,0}, \tag{3.85}$$

we get

$$\|\boldsymbol{A}^{-1}\dot{H}\|_{k,0} \leq (1 + c_{2,0}L^{d+\eta(d)}h^{-2})\|\dot{H}\|_{k+1,0}$$

using that $\max_{i,j=1}^{d}\big|\nabla_i^2\nabla_j^{1*}\mathcal{C}_{k+1}^{(\boldsymbol{q})}(x,x)\big| \leq c_{2,0}L^{-(k-1)d}L^{\eta(d)}$ according to Theorem 2.3.1.

For the second bound, According to Lemma 3.9.5,

$$\begin{aligned}
\|\boldsymbol{B}K\|_{k+1,0} &\leq \sum_{B\in\mathcal{B}_k(B')} \Big\|\Pi T_2 \sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}} \frac{1}{|X|_k}(\boldsymbol{R}_{k+1}K)(X)\Big\|_{k+1,0} \leq \\
&\leq \sum_{B\in\mathcal{B}_k(B')} C \sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}} \frac{1}{|X|_k}\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r} \leq \sum_{B\in\mathcal{B}_k(B')}\sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}} \frac{C2^{|X|_k}}{|X|_k}\|K(X)\|_{k,X,r} \leq \\
&\leq \sum_{B\in\mathcal{B}_k(B')}\sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}} \frac{C2^{|X|_k}}{|X|_k}A^{-|X|_k}\|K_k\|_k \leq CL^dS\big(\frac{2}{A}\big)\|K_k\|_k,
\end{aligned} \tag{3.86}$$

for any $B' \in \mathcal{B}_{k+1}$ and $A > 2$. This implies $\|\boldsymbol{B}^{(\boldsymbol{q})}\| \leq M < \infty$. □

The following proof is an adaptation of the proof contained in [2].

Proof of the strict Convexity. Once we have proved all the analogues bounds, we can finally give a proof of the strict convexity, by following [2].

Chose all parameters according to Proposition 3.7.1, Proposition 3.7.2 and define the renormalization mapping $K \in \boldsymbol{E}$. According to Theorem 3.8.1, there exists a unique C^3

mapping $\tilde{h} : B_{\boldsymbol{E}}(\varepsilon) \times \mathcal{E}$ and a unique $\hat{\lambda} : B_{\boldsymbol{E}}(\varepsilon) \to \mathbb{R}$ such that $\tilde{h}(K)$ is quadratic and $\hat{\lambda}$ is the constant part of H_0 for all $K \in \boldsymbol{E}$ with $\|K\|_h \le \varepsilon$ and $H_N = H_N^{\tilde{h}(K)} = 0$.

With simple calculations we have that

$$\sigma_{N,\beta}(\boldsymbol{u}) = \frac{1}{2}|\boldsymbol{u}|^2 - \frac{1}{\beta L^{dN}} \log Z_N^{(\boldsymbol{q})} + \widehat{\lambda}(K_{\boldsymbol{u}}, \boldsymbol{q}) + \frac{1}{\beta L^{dN}} \log\Big(\int_{\boldsymbol{\mathcal{X}}_N} \Big(1 + K_N(\Lambda_N, \varphi)\Big) \mu_{N+1}^{(\boldsymbol{q})}(\,\mathrm{d}\varphi)\Big), \tag{3.87}$$

We will show that the derivatives with respect to u up to the third order are independent of N. To do so we will consider the different terms in (3.87) independently.

For the first term it is sufficient to differentiate the kernel $\mathcal{C}^{\boldsymbol{q}}$ of the covariance with respect to $\boldsymbol{q} = \boldsymbol{q}(u)$ which in turn gives the smoothness with respect to the tilt. Indeed, using standard Gaussian calculus one has that the first term is

$$-\frac{1}{L^{dN}} \log \left(\frac{Z_N^{(\boldsymbol{q})}}{Z^{(0)}} \right) = \frac{1}{2L^{dN}} \log\big(\det \mathcal{C}^{\boldsymbol{q}}\big).$$

Then, using the smoothness of the kernel $\mathcal{C}^{\boldsymbol{q}}$ with respect to $\boldsymbol{q}$ given by the Finite Range decomposition one has the desired result.

The second term, is a C^3 function of the tilt via the dependence of K_u. Taking into account

$$\|K_N^{\boldsymbol{q}}\|_{N,r} \le \alpha^{-1}\eta^N \|\hat{Z}(\tau(u), \tilde{h}(\tau(u)))\|_{\boldsymbol{Z}_r} \le C_0 \alpha^{-1}\eta^N$$

for all $u \in B_\delta(0)$ and from Proposition 3.7.2, the chain rule

$$\partial_u^\alpha K_N^{\boldsymbol{q}}(\dot{u}, \dots, \dot{u})_{r-\alpha} \le C|\dot{u}|^\alpha$$

we finally obtain the desired result. □

Proof of Theorem 3.8.1. The proof is basically contained in [2].

(i) Let us estimate norm of K_0 in terms of the norm of the initial perturbation and the tuning parameter $\boldsymbol{q}$. Recall that

$$K_0(X, \varphi) = \exp\left(\frac{1}{2} \sum_{x \in X} \sum_{i=1}^{d} \boldsymbol{q}_{i,j}(x) \nabla_i \varphi(x) \nabla_j \varphi(x) \right) \prod_{x \in X} K(x, \nabla\varphi)$$

and

$$|K_0(X, \varphi)|^{0,X} \le \|K\|_h^{|X|} \exp\left([\frac{1}{h^2} + \frac{1}{2}\|\boldsymbol{q}\|] \sum_{x \in X} |\nabla\varphi(x)| \right)$$

Moreover, observe that

$$|D^s K_0(X, \varphi)|^{0,X} \le \|K\|_h^{|X|} \sup_{|(\nabla_i \varphi)| \le 1} |D^s K_0(\dot{\varphi}, \dots, \dot{\varphi})|$$

With simple computations, one has that

$$DK_0(X,\varphi)(\dot\varphi) = \exp\left(\frac{1}{2}\sum_{x\in X}\langle \boldsymbol{q}\nabla\varphi,\nabla\varphi\rangle K(y,\nabla\varphi(y))\right)\Bigg[\sum_{x\in X}\prod_{y\in X\setminus\{x\}}\langle \nabla K(\varphi)(x),\dot\varphi(x)\rangle + \prod_{x\in X}K(x,\nabla\varphi(x))\sum_{x\in X}\langle \boldsymbol{q}\nabla\varphi(x),\nabla\dot\varphi(x)\rangle\Bigg].$$

In a very similar way, one has that

$$|D^sK_0(X,\varphi)|^{0,X} \le \exp\Big((h^{-2}+\frac{1}{2}\|\boldsymbol{q}\|\Big)\|K\|_h^{|X|}(d^s|X|^s + \mathrm{Pol}_s(h,|X|,\|\boldsymbol{q}\|,|X|^{1/2}\sum_{x\in\mathbb{T}_N^d}\sum_{i=1}^d|\nabla_i\varphi(x)|^2))\Big) \tag{3.88}$$

where Pol_s denotes a polynomial of order s in the arguments. Let $\tilde h$ be such that

$$\frac{1}{h^2}+\frac{1}{2}\|\boldsymbol{q}\|^2 \le \frac{1}{2}\frac{1}{\tilde h^2} \tag{3.89}$$

The first volume term which comes without $\sum_{x\in X}|\eta_b|^2$ is taken care by $\|K\|_h$. If $\|K\|_h \le 1/A$, we get the norm $\|K_0\|_{0,r} \le \varepsilon_1$ sufficiently small where $\varepsilon_1 = \varepsilon_1(\|K\|_h,\|\boldsymbol{q}\|)$. Having the norm $\|K_0\|_{0,r}\le\varepsilon_1$ small the statement follows with the remaining parts.

(ii) $\mathcal{T}(K,\boldsymbol{q},0)=\bar Z$ with $\bar H_k = 0$, $\bar K_{k+1}=0$ for $k=1,\dots,N-1$ and $\bar H_0 = -A_1B_1K_0$ and $\bar K_1 = C_0K_0 + g_1(0,K_0)$. Hence,

$$\|\mathcal{T}(K,\boldsymbol{q},0)\|_{Z_r} \le \Big(\frac{1}{\sqrt{\theta}}M\|K_0\|_r \vee \frac{\alpha}{\eta}(\theta\|K\|_r + |g_1(0,K_0)|)\Big).$$

From Proposition 3.7.2, we have that $g_1(0,0)=0$ and that $|g_1(0,K_0)|\le c\varepsilon_1 \le c\|K\|_h\|\boldsymbol{q}\|$. Hence,

$$\|\mathcal{T}(K,\boldsymbol{q},0)\|_{Z_r} \le c\|K\|_r\|\boldsymbol{q}\|\Big((\frac{1}{\sqrt{\theta}}M)\vee\frac{\alpha}{\eta}(\theta+1)\Big).$$

(iii) Let us estimate the operator norm of the Jacobian of the mapping $F: Z_r\to Z_r$, where $F: Z\mapsto \bar Z$ and $\bar Z = \mathcal{T}(K,\boldsymbol{q},Z)$. We compute

$$\frac{\partial \bar H_k}{\partial H_j} = \begin{cases} 0 & k=N-1 \text{ or } j\ne k+1 \text{ for all } j=0,\dots,N-1\\ \mathbf{A}_k^{-1} & j=k+1\end{cases}$$

$$\frac{\partial \bar H_k}{\partial K_j} = \begin{cases} \mathbf{A}_k^{-1} & j=k+1\\ 0 & \text{otherwise}\end{cases}$$

for $k, j = 0, \dots, N-1$ and

$$\frac{\partial \bar{H}_k}{\partial H_j} = \begin{cases} 0 & j \neq k \\ \frac{\partial g_{k+1}(H_k,K_k)}{\partial H_k} & j = k \end{cases}$$

$$\frac{\partial \bar{H}_k}{\partial K_j} = \begin{cases} 0 & j = k+1 \\ \mathbf{C}_k + \frac{\partial g_{k+1}(H_k,K_k)}{\partial K_k} & j = k \end{cases}$$

Writing $\bar{Z} = (\bar{H}_0, \bar{H}_1, \dots, \bar{H}_{N-1}, \bar{K}_1, \dots, \bar{K}_N)$ and estimating the norm of the image $\bar{\bar{Z}} = DF(0)(\bar{Z})$ with $\|\bar{Z}\|_{Z_r} \leq 1$, we have that the vector $\bar{\bar{Z}}_{Z_r}$ is

$$\begin{aligned} \bar{\bar{Z}}_{Z_r} = \Big(& \boldsymbol{A}_0^{-1}\bar{H}_1 - \boldsymbol{A}_0^{-1}\boldsymbol{B}_0\bar{K}_0; \boldsymbol{A}_1^{-1}\bar{H}_2 - \boldsymbol{A}_1^{-1}\boldsymbol{B}_1\bar{K}; \dots, \boldsymbol{A}_{N-2}\bar{H}_{N-1} - \boldsymbol{A}_{N-2}^{-1}\boldsymbol{B}_{N-2}\bar{K}_{N-2}; \\ & - \boldsymbol{A}_{N-1}^{-1}\boldsymbol{B}_{N-1}\bar{K}_{N-1}; \bar{H}_0\frac{\partial g_1(H_0,K_0)}{\partial H_0}\big|Z = 0 + (\boldsymbol{C}_0 + \frac{g_1(H_0,K_0)}{\partial K_0})\bar{K}_1; \dots \\ & \dots; \bar{H}_{N-1}\frac{\partial g_N(H_{N-1},K_{N-1})}{\partial H_{N-1}}\big|_{Z=0} + (\boldsymbol{C}_{N-1} + \frac{\partial g_{N-1}(H_{N-1},K_{N-1})}{\partial K_{N-1}}\big|_{Z=0})\bar{K}_N\Big). \end{aligned}$$

From Proposition 3.7.2, one has that

$$\|D_{H_k}g_{k+1}(H_k,K_k)\big|_{Z=0}(\dot{H}_k)\|_r \leq \bar{\varepsilon}\|\dot{H}_k\|_o \text{ and } \|D_{K_k}g_{k+1}(H_k,K_k)\Big|_{Z=0}(\dot{K}_k)\| \leq \bar{\varepsilon}\|\dot{K}_k\|_r$$

Given that $\|\bar{Z}\|_{Z_r} \leq 1$, we have that $\|\bar{H}_k\|_{k,0} \leq \eta^k$ for $k = 0, \dots, N-1$ and $\|\bar{K}_k\|_{k,r} \leq \frac{\eta^k}{\alpha}$ for $k = 1, \dots, N$. Hence,

$$\|\bar{\bar{H}}_k\|_{k,0} \leq \|\mathbf{A}_k^{-1}\|\eta^{k+1} + \|\mathbf{A}_k^{-1}\|\|B_k\|\frac{\eta^k}{\alpha} \leq \frac{\eta^k}{\sqrt{\theta}}(\eta + \frac{M}{\alpha}), \ k = 0, \dots, N-2$$

$$\|\bar{\bar{H}}_N\|_{N-1,0} \leq \|\mathbf{A}_{N-1}^{-1}\|\|\mathbf{B}_{N-1}\| \leq \frac{\eta^{N-1}M}{\alpha\sqrt{\theta}}$$

$$\|\bar{\bar{K}}_k\|_{k,r} \leq \eta^{k-1}\bar{\varepsilon} + \frac{\eta^\kappa}{\alpha}\big(\|C_{k-1}\| + \bar{\varepsilon}\big) \leq \eta^{k-1}(\bar{\varepsilon} + \frac{\eta}{\alpha}(\theta + \bar{\varepsilon}))\ k = 1, \dots, N,$$

thus

$$\|\bar{\bar{Z}}\|_{Z_r}\big(\frac{1}{\sqrt{\theta}}(\eta + \frac{M}{\alpha})\big) \vee \big(\frac{\alpha}{\eta}(\bar{\varepsilon}\frac{\eta}{\alpha}(\theta + \bar{\varepsilon}))\big).$$

Choosing the parameters η and α such that $\eta + \frac{M}{\alpha} \leq \theta^{3/2}$, we have that

$$\|DF(0)\|_{\mathcal{L}(Z_s,Z_s)} = \Big\|\frac{\mathcal{T}(K,\boldsymbol{q},Z)}{Z}\Big|_{Z=0}\Big\|_{\mathcal{L}(Z_s,Z_s)} \leq (\frac{\alpha}{\eta} + 1)\bar{\varepsilon} + \theta \leq 1$$

(iv) The bounds for the derivatives with respect to H_k and K_k for the first component follow immediately from the linearity, i.e., $\bar{H}_k = \mathbf{A}_{k+1}^{-1}(H_{k+1} - \mathbf{B_k}K_k)$, whereas the second component one uses Proposition 3.7.2. Let us now check the bounds for the derivatives with respect to the two parameters $\boldsymbol{q}$ and initial perturbation $K \in E$. The images $\bar{Z} = \mathcal{T}(K, \boldsymbol{q}, Z)$ depend on the initial perturbation K only through the coordinates $\bar{H}_0 = A_1^{-1}(H_1 - B_0K_0)$ and $\bar{K}_1 = C_0K_0 + g_1(H_0, K_0)$. Let us estimate

the norm $\|\frac{\partial^l}{\partial K(K_0)(\dot K,\dots,\dot K)}\|_{0,X,r}$ for $l=1,2,3$. We only sketch the first derivative here as the second and the third follow analogously. Pick $X\subset\Lambda$, then

$$\frac{\partial}{\partial K}K_0(X,\varphi)=\exp\left(\frac{1}{2}\sum\langle\boldsymbol{q}\nabla\varphi,\nabla\varphi\rangle\right)\sum\prod_{y\in X\setminus\{x\}}K(y,\nabla\varphi(y))\dot K(x,\nabla\varphi(x)).$$

Proceeding as above, we have that

$$\frac{\partial}{\partial K}K_0(X,\varphi)\le|X|\|K\|_h^{|X|-1}\|\dot K\|_h\exp\left((\frac{1}{h^2}+\frac{1}{2}\|\boldsymbol{q}\|)\sum_x|\nabla\varphi(x)|^2\right)$$

and for the derivative, one has an extra volume factor

$$\begin{aligned}|D\frac{\partial}{\partial K}K_0(X,\varphi)(\dot K)|^{0,X}\le&\exp\left((\frac{1}{h^2}+\frac{1}{2}\|\boldsymbol{q}\|)\sum_x|\nabla\varphi(x)|^2\right)\\&\times\left(d|X|+h\|\boldsymbol{q}\||X|^{1/2}\sum_x|\nabla\varphi(x)|^2\right)|X|\|K\|_h^{|X|-1}\|\dot K\|_h.\end{aligned}$$

Hence, we have a similar estimate to (3.88) and thus the bounds for the derivatives with respect to the perturbation K. The derivatives with respect to $\boldsymbol{q}$ for the linear parts are bounded by Proposition 3.7.1 whereas the derivatives of $\exp\left(\frac{1}{2}\sum_x\langle\boldsymbol{q}\nabla\varphi,\nabla\varphi\rangle\right)$ gives only polynomials in $\boldsymbol{q}$ which are taken care of by the condition (3.89) above for the weight function for the norm. The derivatives with respect to $\boldsymbol{q}$ for the nonlinear part are taken care in the nonlinear parts we differentiate Gaussian expectations with the respect to the parameter $\boldsymbol{q}$ of its covariance operator. This follows due to the well-known formula

$$\frac{\mathrm{d}}{\mathrm{d}q}E_{\mathcal{C}^{\boldsymbol{q}}}[F(X)]=\frac{1}{2}E_{\mathcal{C}^{\boldsymbol{q}}}[\mathrm{Tr}D^2F(X)\dot{\mathcal{C}}^{\boldsymbol{q}}].$$

The differentiability of the solution map $\hat Z$ and the bounds follow with Proposition 3.8.1.

□

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